HYBRID MAGNETIC ATTITUDE CONTROLLER FOR LOW EARTH ORBIT SATELLITES USING THE TIME-VARYING LINEAR QUADRATIC REGULATOR

by

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A thesis submitted in conformity with the requirements for the degree of Masters of Applied Science Graduate Department of Aerospace Science and Engineering University of Toronto

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Abstract

Hybrid Magnetic Attitude Controller for Low Earth Orbit Satellites using the Time-Varying Linear Quadratic Regulator

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The following is a study of an attitude control system (ACS) for a low earth orbit nanosatellite. Control actuation is applied using three reaction wheels and three mutually orthogonal current-driven magnetorquers which produce torques by interacting with the earth’s magnetic field. Control torques are distributed amongst the actuators allowing them to work together in concert. This type of control is referred to as hybrid magnetic attitude control. To account for the nearly periodic behavior of the earth’s magnetic field, control torques are assigned using periodic and optimal control theory. The primary focus is to apply the time-varying Linear Quadratic Regulator controller to test the stability and energy consumption of the ACS when reaction wheels are removed from the control law, or are simulated to be missing. Other situations studied include the effects of control saturation, introducing uncertainty in the orbital inclination, and observing performance as the number of magnetic coils is increased.
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Chapter 1

Introduction

On February 4th 2008, The National Aeronautics and Space Administration (NASA) announced that their 2009 fiscal budget was to be $17.6 billion, with $5.78 billion towards the Space Shuttle and Space Station programs, and another $3.5 billion for the development of new manned spacecraft systems [1]. With such gigantic investments being made, it is of great interest that the technology being sent out into space performs well and is able to successfully complete its mission goals. One vital factor that plays a major role in the success of a spacecraft is how well the spacecraft can point to its intended target. A communications satellite, telescope, or satellite used for GPS will only be useful to us if we are able to make it point towards the object we wish to look at or communicate with. In other cases, we might require that its solar panels point to the sun so that it can continue to function properly [2].

By mathematically modeling spacecraft, we have been able to study, analyze, and control their behavior in space. The ability to do this has provided us with great technologies, capabilities, and continued learning which shall no doubt lead us to greater achievements.
1.0.1 Literature Review

Advances in scientific fields such as computer technology, electronics, and material science have found ways to use smaller satellites that are now capable of performing tasks usually considered for larger more expensive satellites [3]. This has generated a lot of research interest in these smaller satellites for scientific study. One important feature for these satellites is attitude control. Correct attitude is mandatory for not only the successful completion of the satellite’s mission goals, but also for its survival [4]. The system is responsible for accurately pointing the satellite to its desired targets while managing external disturbances that may occur during orbit [5]. Satellite control can be achieved through different means such as exploiting the gravity gradient, employing spin stabilization or three-axis stabilization [5]. The most accurate attitude control systems, which usually include momentum wheels or reaction wheels, have the ability to obtain pointing accuracies of $10^{-4}$ degrees or better [6]. These devices are known to be costly, with rotation elements that may be subject to failure [7].

Studies also show that electromagnetic actuators, or magnetorquers, are an effective and reliable form of actuation for satellites in Low Earth Orbit (LEO) [4]. These are coils of wire mounted on the satellite such that they are mutually perpendicular or orthogonal to one another. The control currents flowing through the coils can be used to create a magnetic dipole moment to interact with the earth’s magnetic field. This interaction between the magnetic field of the earth and the magnetic field generated in the magnetorquers produces a mechanical torque which can be used to correct the satellite’s attitude [8]. Their ability to exploit the earth’s magnetic field and provide a simple method to generate torques on board the satellite is deemed as being very advantageous [4], [9], [6], [10]. Active magnetorquers can be considered as the most reliable and lightweight form of an attitude control system (ACS) [3]. Thus, this option is beneficial for smaller LEO satellites as the magnetorquers weigh significantly less than wheel-based system, consume less power, and are comparatively inexpensive [9], [6], [10].
Generally, to achieve precise stabilization through magnetic control, the magnetorquers in the ACS are working together with other actuators such as momentum or reaction wheels [7]. The benefit of this is that the satellite sees a reduction in power consumption as it is lighter in weight and less dependent upon its other means of actuation [9], [6], [10]. It is common that magnetic control is implemented with the intent of reducing power [6]. This is of great importance to the satellite as the energy used on board is self-obtained [11].

There is, however, a well documented and very significant limitation associated with magnetic control. It is clear that the control torques that are produced follow the cross product law of vectors [12]:

\[ T_m = m \times B_b \]

where \( T_m \) is the magnetic control torque, \( m \) is the magnetic control dipole moment, and \( B_b \) is the geomagnetic field vector. It is clear that the control torques are constrained to always lie perpendicular to the earth’s magnetic field vector [8],[4]. This is significant as it shows that the direction parallel to the local geomagnetic field vector is not controllable [9]. Clearly, it is not possible for the magnetorquers to provide three independent control torques at each instant in time [13]. Since a rigid spacecraft has three rotational degrees of freedom, but the magnetorquers can only torque about two axes at a time, the spacecraft is underactuated [6], [4]. The only case where magnetic control can provide three-axis stability is if there is sufficient variability of the magnetic field vector along the considered orbit [14]. This variation allows for the pointwise uncontrollable direction to change over time allowing for control torques to be applied on each axis [14],[15]. Due to this limitation, magnetorquers are most commonly used for detumbling or momentum dumping of a satellite to desaturate a wheel-based ACS [8], [15], [6].

For a circular orbit, a satellite following an equatorial orbit with magnetic control will yield almost uncontrollable dynamics [14]. Conversely, the inclination of a polar orbit provides the maximum variability in the Earth’s magnetic field vector, thus allowing
Chapter 1. Introduction

the system to be controllable [14], [6]. It is intuitive that the majority of magnetic control studies deal with satellites at polar or near-polar orbital inclinations. Due to the nature of the geomagnetic field, a spacecraft following a polar or near-polar orbit that is controlled through magnetorquers is approximately time periodic and can be modeled as a time-varying periodic system when linearized [4], [8], [6], [14], [16].

For the attitude control problem, linear and nonlinear control are both active topics of research [16]. Linear attitude control studies are often from a Linear Quadratic (LQ) perspective, while nonlinear control is often investigated using Lyapunov methods. Exploiting the periodicity of the earth’s magnetic field, stability analysis can be done by applying the Krasovski-La Salle Theorem for the nonlinear control case, or a Floquet analysis for the linearized control case [4], [2].

The attitude of the satellite is most often modeled using Euler parameters, or quaternions, as they are an efficient and singularity-free way to describe rotations [2], [9], [16]. This attitude representation has been successfully used in several spacecraft and is preferred over other representations such as a Gibbs vector or Euler angles [11]. Quaternions also provide simple calculations and attitude representations that the ACS can easily implement along with angular velocity readings to control the satellite. Selecting an Earth-centered, or geocentric, inertial frame, an inertial pointing satellite will require that the rotation between the satellite’s body frame with respect to the inertial frame is zero [16], [12]. This implies that the satellite’s rotational motion with respect to its body axes are also equal to zero [9]. The linearized model provides a good approximation of the nonlinear system in the region of state space that is near this equilibrium over a wide range of conditions [6]. Considering small displacements from the nominal values of the angular velocity and the vector part of the attitude quaternion, linearized attitude dynamics and local linear dynamics can be defined [14].

As previously mentioned, most linearized magnetic systems can be formulated with a Linear Quadratic Regulator (LQR) [17], [18]. Determining the LQR control inputs of
the Linear Time-Varying (LTV) system requires the solution of the Riccati equation [17], [6]. With a terminal condition \( P(t_f) \), and an input matrix \( B(t) \) that is assumed to be periodic, it is well known that the steady-state solution of the Riccati equation is \( P(t) \) which is periodic with period \( T \). In the literature, great measures have been taken to approximate a constant steady-state solution or constant gain for the periodically time-varying matrix Riccati equation by imposing the boundary condition: \( P(0) = P(T) \) [6], [16]. The solution is calculated using backward integration of the Riccati equation for a given final condition. As opposed to computing and keeping a periodic gain matrix time history, it is considered advantageous to store a single constant \( P_{ss} \) [6]. This constant would not need to have its time variations synchronized with the actual time variations in \( B(t) \), a familiar problem with periodic control [6]. Furthermore, the true \( B(t) \) is not exactly periodic, therefore, synchronization of a periodic \( P(t) \) with the quasi-periodic \( B(t) \) could be difficult [6],[4]. Constant gain values have been shown to be successful in simulation and in implementation on the Danish Ørstead satellite, a small magnetically controlled satellite with loose requirements for pointing accuracy [8].

Other investigations include integrators in their controller to counteract steady-state effects of disturbances and eliminate any offset that may be present in the system [6]. Integral control can be added to the controller by augmenting the state vector to include the integral of the attitude error [6].

In order to represent the earth’s magnetic field, it is common practice to take a periodic approximation according to a time history of the IGRF model of the earth’s magnetic field along five orbits [14]. It is also possible to obtain a magnetic field approximation using a tilted dipole model given by Wertz (1978) [17].

Disturbances modeled depend on the application and orbital element of the satellite. Some studies model aerodynamic drag, and almost all include the effect of the gravity gradient. A residual magnetic dipole disturbance torque is also taken into consideration. This disturbance arises by the electronics in the satellite creating an unwanted magnetic
dipole that interacts with the earth’s magnetic field, and can be modeled by applying a small magnetic dipole onto each of the spacecraft’s body axes [19], [13].

Other points of interest related to modeling the system include the electrical transient as being negligible and a moment of inertia matrix as being proportional to the identity matrix [20].

It is clear that there is a lot of research interest in the field of magnetic control. The role of magnetorquers has clearly expanded from just momentum desaturation, through proportional momentum feedback, and is now a serious consideration as an active ACS for specific satellite applications. Many satellites carry both sets of actuation, magnetorquers as well as an alternative three-axis actuation system such as wheels [15]. It is now becoming of interest to consider the use of these forms of actuation together with a control law distributing corrective control torques simultaneously with each other [15], [3]. This can be termed hybrid magnetic attitude control [15]. There have been studies into using the two control forms together in concert with one another with the intent of reducing the overall power consumption of the system [10]. Such has been found to be true with an LQR approach [10]. The addition of three magnetorquers will also provide a layer of redundancy in the event that failure occurs with one or more of the satellite’s reaction wheels. This is significant as the satellite will be capable of effective attitude correction despite the lost reaction wheel through reconfiguration of the control law.

1.1 Subject Material

It has been well documented that magnetic control is an effective low power form of attitude control for LEO nanosatellites. Furthermore, reaction wheels are well known for their abilities to provide highly accurate attitude control. The application of optimal control theory to a hybrid magnetic attitude control system has the potential to lead to positive results.
1.1.1 Purpose

The purpose of this thesis is to study the behavior of a hybrid magnetic control system for the ACS of a LEO nanosatellite. Specifically, this thesis will cover the regulator problem of an inertial pointing satellite in a near-polar orbit. From optimal control theory, a Linear Quadratic Regulator (LQR) can be applied to control a satellite and appropriately allocate control torques between the two sets of actuation. There will be a specific focus of the satellite’s energy consumption and Root Mean Squared (RMS) error as a measure for performance. Other studies will include controller performance when:

- additional magnetorquer coils are added
- wheel failure occurs
- control law reconfiguration accommodates missing wheel(s)
- the effects of control saturation are included
- the orbital inclination is inaccurate

1.1.2 Thesis Outline

The different aspects will be presented as follows.

Modeling Spacecraft Motion

This chapter will outline the spacecraft dynamics and kinematics needed to model the spacecraft. This chapter will also discuss attitude representation and introduces the linearized system.

Modeling Orbital Motion

This chapter will outline and present the orbital mechanics involved with modeling a spacecraft in a circular Keplerian orbit. This chapter will also briefly touch upon modeling
the earth’s magnetic field and disturbances.

**Spacecraft Control**

This chapter will outline the optimal control theory applied to determine the appropriate control torques. Specifically, more details will be presented on the Matrix Riccati equation and the LQR.

**Investigation Results**

The different investigations and results will be presented in multiple chapters. The tests performed focus on the energy consumed, attitude errors, missing wheel scenarios, control law re-configuration, the effect of extra magnetorquer coils on the energy, the effects of control saturation, and the effect of inclination perturbations or inaccuracies.
Chapter 2

Modeling Spacecraft Motion

2.1 Introduction

To study and develop an understanding of the spacecraft’s behavior and how we can control it, we must first model it mathematically. Using appropriate reference frames and equations of motion, we are able to describe the spacecraft’s attitude and how its motion changes with respect to time. By understanding how angular momentum might affect the satellite, and by modeling the spacecraft as a rigid body, we can develop the governing motion equation. Linearizing the motion equations allows us to develop a state-space model.

2.2 Reference Frames

For our purposes of modeling and controlling spacecraft, a reference frame is a set of mutually perpendicular geometrical axes that span a three-dimensional space and can be denoted by the character $\mathbf{F}$. The two important frames of interest in this study are the inertial reference frame, and the spacecraft’s body reference frame.
2.2.1 Inertial Reference Frame

The inertial frame is a fixed and non-spinning frame that is commonly used for locating orbits [21]. For our purposes, the inertial frame used will be the *Geocentric Equatorial Coordinate System* where the origin is located at the centre of the earth [21],[22]. The \( \mathbf{i}_1 \) basis vector lies in the equatorial plane and points towards Aries, \( \Upsilon \), the direction of the vernal equinox. The \( \mathbf{i}_3 \) basis vector points towards the geographical north pole, and the \( \mathbf{i}_2 \) vector also lies in the equatorial plane completing the dextral orthonormal triad [21], [2]. It is common to use the geocentric equatorial coordinate system as the inertial reference frame for spacecraft control analysis.

2.2.2 Body Reference Frame

The body reference frame is fixed to the spacecraft with the origin at the centre of mass [21]. Measurements taken by instruments on-board the satellite are often taken with respect to the body frame [23]. For an inertial-pointing satellite, we wish for the body frame to be aligned with the inertial reference frame. To be able to enforce this, we require methods to rotate between reference frames.

2.3 Spacecraft Kinematics

2.3.1 Rotations

Two reference frames that share a common vector but different basis vectors will have different components to describe the shared vector. A rotation matrix can be applied to map the components of a vector in one frame, to the components of the same vector in the other. Rotations about the object’s principal axes are known as principal rotations [2]. Subscripts can be used to signify a rotation matrix that maps the coordinates from one frame to another, hence \( C_{21} \) rotates the vector in \( \mathbf{F}_1 \) into \( \mathbf{F}_2 \).
2.3.2 Angular Velocities

Letting $\omega_{12}$ represent the angular velocity between two reference frames with respect to one another, it can be seen that

$$\omega_{12} = -\omega_{21}$$

where $\omega_{12}$ is the angular velocity of frame 1 with respect to frame 2. It is important to note that an observer in a rotating frame does not see the same motion as an observer in another frame. Letting $\mathbf{r} = [r_1 \ r_2 \ r_3]^T$ represent the components of a position vector, we can determine its time derivative:

$$\dot{\mathbf{r}} = \mathbf{F}_{T1} \dot{r}_1$$

$$\dot{\mathbf{r}} = \mathbf{F}_{T2}(\dot{r}_2 + \omega_{21} \times \mathbf{r}) \quad (2.1)$$

We can signify the derivative seen in the rotating frame, $\mathbf{F}_{T2} \dot{\mathbf{r}}$, as $\dot{\mathbf{V}}$, so that

$$\dot{\mathbf{r}} = \mathbf{V} = \dot{\mathbf{r}} + \omega_{21} \times \mathbf{r} \quad (2.2)$$

where $\dot{\mathbf{r}}$ and $\mathbf{V}$ can represent the vector’s velocity.

2.3.3 Accelerations

To obtain an equation for acceleration, we can apply Eq. (2.2) to $\mathbf{V}$:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{V}}$$

$$= \ddot{\mathbf{r}} + \omega_{21} \times \dot{\mathbf{V}}$$

$$= \ddot{\mathbf{r}} + 2\omega_{21} \times \dot{\mathbf{r}} + \omega_{21} \times \ddot{\mathbf{r}} + \omega_{21} \times (\omega_{21} \times \mathbf{r}) \quad (2.3)$$

2.4 Quaternion Attitude Representation

To represent the attitude of the satellite, we use four-dimensional column matrices known as Euler’s Parameters or quaternions, which are derived from Euler’s Theorem:
Theorem 1 (Euler’s Theorem) The most general motion of a rigid body with one point fixed is a rotation about an axis through that point.

Using a unit vector $\mathbf{a} = [a_1 \ a_2 \ a_3]^T$ to describe the axis, and angle $\varphi$ to describe the angle of rotation, a rotation matrix that maps from $\mathcal{F}_1$ to $\mathcal{F}_2$ can be expressed as

$$C^2_1 = \cos \varphi \mathbf{1} + (1 + \cos \varphi) \mathbf{a} \mathbf{a}^T - \sin \varphi \mathbf{a}^\times$$  \hspace{1cm} (2.4)

From this theorem, we can define our attitude parameterization.

2.4.1 Definitions

We select the parameters $\mathbf{\epsilon}$ and $\eta$ according to:

$$\mathbf{\epsilon} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = a \sin \frac{\varphi}{2}, \quad \eta = q_4 = \cos \frac{\varphi}{2}$$ \hspace{1cm} (2.5)

Substituting Eq. (2.5) into Eq. (2.4) we obtain

$$C^2_1 = (\eta^2 - \mathbf{\epsilon}^T \mathbf{\epsilon}) \mathbf{1} + 2 \mathbf{\epsilon} \mathbf{\epsilon}^T - 2 \eta \mathbf{\epsilon}^\times = C_{b_i}$$ \hspace{1cm} (2.6)

where $C_{b_i}$ is the rotation matrix from $\mathcal{F}_i$ to $\mathcal{F}_b$.

Euler Parameters or quaternions consist of a three-dimensional vector, $\mathbf{\epsilon}$, and a scalar parameter $\eta$. Their two-norm can be shown to have unit length

$$|\mathbf{q}| = \sqrt{\mathbf{q}^T \mathbf{q}} = \sqrt{\mathbf{\epsilon}^T \mathbf{\epsilon} + \eta^2} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} = 1$$ \hspace{1cm} (2.7)

where $\mathbf{q} = [q_1 \ q_2 \ q_3 \ q_4]^T$. We can refer to this kind of quaternion as the unit quaternion. They are often used to parameterize attitude as they can effectively describe a rotation between two frames and have no singularities associated with them [2].
2.4.2 Quaternion Kinematics

One of the benefits of representing our attitude like this is the ease of calculations for the rate kinematics [2],[24]:

\[
\dot{\epsilon} = -\frac{1}{2} \omega \times \epsilon + \frac{1}{2} \omega \eta \\
\dot{\eta} = -\frac{1}{2} \omega^T \epsilon
\] (2.8)

where \( \omega \) are the components of the spacecraft’s angular velocity expressed in the body frame.

2.5 Spacecraft Dynamics

The spacecraft is modeled as a rigid body that moves without deformation. If we are to treat the spacecraft as a collection of particles, then the distance between each of the particles would remain constant regardless of the external forces. From the equations of motion of a particle or point mass, we can see how a change in momentum can affect the satellite.

2.5.1 Angular Momentum

Letting vector \( \vec{r} = [r_1 \ r_2 \ r_3]^T \in \mathbb{F}_1 \) represent the position of a particle in frame 1, we can define

\[
\vec{p} = m \dot{\vec{r}} \\
\vec{f} = m \ddot{\vec{r}}
\] (2.9)

where \( \vec{p} \) is the momentum of the particle, \( m \) is a constant mass, and \( \vec{f} \) is the force applied. We can define the angular momentum of the particle about the origin \( O \) as

\[
\vec{h}_o \triangleq \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}}
\] (2.10)
where $\mathbf{h}$ is angular momentum.

Taking the time derivative of the angular momentum we obtain:

$$\dot{\mathbf{h}} = m \dot{\mathbf{r}} \times \mathbf{r} + m \mathbf{r} \times \ddot{\mathbf{r}}$$

$$= \mathbf{r} \times \mathbf{f}$$

$$= \mathbf{G}_o$$

where $\mathbf{G}_o$ is the torque produced by $\mathbf{f}$. Hence the derivative of the angular momentum is the external torque applied to the system [21].

Since the internal forces of the body acting between the particles themselves cancel out from Newton’s Third Law, we can model the total external force as the mass multiplied by the acceleration of the body’s centre of mass [21]. This can be extended to show that the sum of torques about $O$ of the body equals the time derivative of the system’s angular momentum about $O$ [21].

The angular momentum with respect to the centre of mass can be shown to equal

$$\mathbf{h}_c = \mathbf{F}_b^T \mathbf{I} \mathbf{\omega}$$

(2.11)

where $\mathbf{h}_c$ is the angular momentum with respect to the mass centre, and $\mathbf{I}$ is the moment of inertia matrix.

### 2.5.2 Euler’s Equation for Rigid Body Motion

One of the simplest ways to correct the satellite’s attitude is to rotate it about its mass centre. This suggests the presence of angular momentum and control torques which leads to the governing equation of motion for the satellite. This is significant as the instruments we use to correct the satellite’s attitude apply corrective torques that rotate the satellite to the desired attitude.

From Eq. (2.1) and Eq. (2.11), it can be seen that:

$$\dot{\mathbf{h}}_c = \dot{\mathbf{h}}_c + \mathbf{\omega} \times \mathbf{h}_c = \mathbf{G}_c$$

(2.12)
If we wish to express $G_c$ in the body-fixed frame, using Eq. (2.11) and Eq. (2.12), we can see that:

$$\dot{h}_c = G_c$$
$$\dot{h}_c + \omega \times h_c = G_c$$
$$I\dot{\omega} + \omega \times I\omega = G_c$$ (2.13)

Isolating $\dot{\omega}$, we obtain:

$$\dot{\omega} = -I^{-1}\omega \times I\omega + I^{-1}G_c$$ (2.14)

Understanding this relationship is important as it allows us to determine the appropriate control torques required to correct for the satellite’s attitude errors. The total torque about the mass centre $G_c$ can be broken up into two distinct torques

$$G_c = u_d + u_c$$ (2.15)

where $u_d$ is a disturbance torque, and $u_c$ is the control torque that is allocated to control the satellite. For a hybrid magnetic attitude control system, the $u_c$ control torque alone can be broken up into control torques being provided by two distinct sets of actuation provided by magnetic torques and reaction wheels:

$$u_c = u_m + \tau$$ (2.16)

Here, $u_m$ is the control torques produced by the magnetorquers, and $u_r$ is the control torque produced by the reaction wheels. By creating a magnetic dipole moment, the magnetorquers can act against the earth’s magnetic field to produce control torques following the cross product law of vectors

$$u_m = m \times B_b$$ (2.17)

where $m$ is the magnetic dipole moment produced by the magnetorquers and $B_b$ is the magnetic field vector as seen in the satellite’s body frame. Note that $B_b = C_{bi}B_i$ where
**Chapter 2. Modeling Spacecraft Motion**

\( \mathbf{B}_i \) are the magnetic field components expressed in the inertial frame and \( \mathbf{C}_{bi} \) describes the attitude of \( \mathbf{F}_k \) with respect to \( \mathbf{F}_i \). Torques provided by the reaction wheels can be applied directly about the appropriate axes. This allocation of control torques between the actuators will be discussed in greater detail in later chapters.

### 2.6 Linearized Motions

For an inertial pointing spacecraft, the performance of the control system is analyzed near the equilibrium reference point [9], [24]:

\[
\begin{align*}
\mathbf{\omega} &= 0 \\
\mathbf{q} &= [0 \ 0 \ 0 \ 1]
\end{align*}
\] (2.18)

We desire to keep the satellite pointing towards the vernal equinox point such that Eq. (2.18) is true. By studying a linearized model we can assume small angles and rates [16]:

\[
\begin{align*}
\mathbf{\omega} &= \dot{\theta} \\
\theta &= 2\epsilon
\end{align*}
\]

Linearizing Eq. (2.13) by substituting our small angles and rates we obtain

\[
\begin{align*}
\mathbf{I} \ddot{\theta} &= \mathbf{G}_e \\
\dot{\theta} &= \mathbf{I}^{-1} \mathbf{G}_e \\
\end{align*}
\] (2.19)

Hence for the controller, our linearized state-space equation following from Eq. (2.19), Eq. (2.16), and Eq. (2.17) is as follows:

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\theta
\end{bmatrix} +
\begin{bmatrix}
-I^{-1} \mathbf{B}_i^x & I^{-1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{m} \\
\tau
\end{bmatrix}
\] (2.20)

or

\[
\dot{x} = \mathbf{Ax} + \mathbf{Bu}
\] (2.21)
where $x = [\dot{\theta}^T \theta^T]^T$ is the state vector, $u = [m^T \tau^T]^T$ are the control inputs, and $A$ and $B$ can be inferred from Eq. (2.20). Determining the appropriate $u$ vector to control the satellite will be discussed in later chapters. Although the control is designed using the linearized motions, the simulated motion is done using the full nonlinear equations shown in Eq. (2.14) and Eq. (2.8).
Chapter 3

Modeling Orbits

To develop a control system for a satellite in orbit, we need methods to model the satellite’s path around the planet. This can be done by modeling the dynamics of two objects in space, starting with point masses to establish the two-body problem. From the two-body problem, it is seen that the motion of the mass of interest is planar and that we require equations of motions to describe the orbit. We can then introduce the classical orbital elements to help describe the shape of the orbit, its orientation, and the position of the mass of interest as it orbits. Lastly, we introduce the tilted dipole model to describe the Earth’s magnetic field, followed by the disturbance torques we need to model the spacecraft’s motion.

3.1 Orbital Dynamics

3.1.1 The Two-Body Problem

Starting with two point masses, $m_1$ and $m_2$, and their positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to the same frame, let $\mathbf{r}$ represent position vector between the two masses. Newton’s law of universal gravitation between the two point masses is as follows:

$$\mathbf{f}_{12} = -\frac{Gm_1m_2}{r^3} \mathbf{r} = -\mathbf{f}_{21}$$  \hspace{1cm} (3.1)
where $\mathbf{f}_{12}$ is the force of attraction of $m_1$ acting on $m_2$, $G$ is the Universal Gravitational Constant, $m$ represents the mass of the object specified by its subscript, and $r$ represents the magnitude of the distance separating the two masses. From Newton’s second law, we can also observe that $\mathbf{f}_{12} = m_1 \ddot{\mathbf{r}}_1$. Substituting this into Eq. (3.1), we can observe that:

$$m_1 \ddot{\mathbf{r}}_1 = -\frac{Gm_1m_2}{r^3} \mathbf{r},$$

$$\ddot{\mathbf{r}}_1 = -\frac{Gm_2}{r^3} \mathbf{r}$$ \hspace{1cm} (3.2)

Similarly:

$$\ddot{\mathbf{r}}_2 = -\frac{Gm_1}{r^3} \mathbf{r}$$ \hspace{1cm} (3.3)

Letting $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2$, we obtain an equation for a reference acceleration as a result of the individual accelerations of the point masses:

$$\ddot{\mathbf{r}} = -\frac{G(m_1 + m_2)}{r^3} \mathbf{r}$$ \hspace{1cm} (3.4)

To provide further context to this equation, let $m_1$ represent the mass of a man-made satellite and $m_2$ represents the mass of the planet it is orbiting. Clearly, $m_1$ is small and almost negligible compared to the planet. Applying this reasoning to the point masses, we can form the assumption that $m_1 \ll m_2$. We can then define the following two parameters:

$$M \triangleq m_1 + m_2, \quad m \triangleq \frac{m_1m_2}{m_1 + m_2}$$ \hspace{1cm} (3.5)

These assumptions also imply that $M \approx m_2$ and $m = m_1$. Substituting Eq. (3.5) into Eq. (3.4) we obtain:

$$m \ddot{\mathbf{r}} = -\frac{GMm}{r^3} \mathbf{r}$$

$$= -\frac{Gm_1m_2}{r^3} \mathbf{r}$$

$$= \mathbf{f}_{12}$$

This is significant as the resulting force of attraction between the two masses is equivalent to the force of attraction of $m_1$ on $m_2$. Replacing $m_1$ with the reduced mass
Chapter 3. Modeling Orbits

20

$m$ and using the resulting reference acceleration $\vec{r}$ from Eq. (3.4) allows us to treat $m_2$ as being stationary relative to $m$.

We can conclude that modeling the above two-body problem for a reduced mass allows us to treat $m_2$ as a fixed point. Having this point established allows us to now focus on $m_1$, which is treated as the only object in the system with motion. Before discussing methods and equations to model the spacecraft’s translational motion and position, we must first investigate some rotational kinematics.

3.1.2 Angular Motion

The equation for angular momentum about a point $O$ is defined by Eq. (2.10). Dividing by $m$ provides us with the angular momentum of the system per unit mass:

$$\vec{h} = \vec{r} \times \vec{v} \quad \text{(3.6)}$$

Taking the time derivative, we obtain:

$$\dot{\vec{h}} = \dot{\vec{r}} \times \vec{v} + \vec{r} \times \dot{\vec{v}}$$

$$= -\vec{r} \times \vec{r} \frac{G(m_1 + m_2)}{r^3}$$

$$= 0$$

Therefore, $\vec{h}$ is a constant vector that does not change with time. From the cross product law of vectors, we can see in Eq. (3.6) that $\vec{h}$ is perpendicular to $\vec{r}$ and $\vec{v}$.

We now require a position equation to describe the planar motion.

Using polar coordinates, and allowing frame 2 to act as a frame that rotates with the satellite, the position can be described as:

$$r = \frac{1}{\frac{\mu}{h^2} + C \cos(\theta - \theta_0)}$$

$$= \frac{h^2}{\mu} \frac{1}{1 + \frac{h^2C}{\mu} \cos(\theta - \theta_0)} \quad \text{(3.7)}$$
Eq. (3.7) provides the position of the mass $m_1$ with respect to $m_2$ being the point it is orbiting about. The value of $r$ will only remain constant for a circular orbit and will be subject to change for more eccentric orbits. It is clear, that $r$ takes on minimum values when $\theta = \theta_0 + 2n\pi$, where $n$ is an integer representing the number of complete orbits that have occurred. As a starting point, we allow $\theta_0 = 0$ and assign $t = t_0$ for when $r = r_{\text{min}}$.

We can define the constants

$$p = \frac{h^2}{\mu}, \quad e = \frac{h^2 C}{\mu}$$

which we can use to rewrite Eq. (3.7) as:

$$r = \frac{p}{1 + e \cos \theta}$$

It is clear that aside from the angle $\theta$ formed between the reference frame’s axes and the position of $m_1$, the parameters that affect the magnitude of $r$ are $p$ and $e$. The value of $p$ has been clearly defined. However, Eq. (3.8) for $e$ still has a constant of integration $C$ present. Solving for $C$ by looking at the total energy, $E$, in the system we obtain:

$$C = \sqrt{\frac{2E}{h^2} + \frac{\mu^2}{h^4}}$$

Substituting Eq. (3.10) into Eq. (3.8), we can obtain an equation for $e$:

$$e = \frac{h^2 C}{\mu} = \sqrt{1 + \frac{2Ep}{\mu}}$$

The equations for $p$, $e$, and $r$ can determine the shape of the orbit of $m_1$ about $m_2$.

### 3.1.3 Orbit Shapes

Eq. (3.9) describes the polar equation of a conic section [21]. The constants $e$ and $p$ are known as the eccentricity and semilatus rectum respectively. Both parameters describe
the shape of the orbit. The value $r$, which represents the distance between $m_1$ and $m_2$, will change unless a circular orbit is followed.

We are primarily interested in ellipses and circles where the trajectory of $m_1$ forms a closed path. With $m_2$ being held at a focus referred to as the occupied focus, we can see that when $0 \leq e < 1$, the minimum value of $r$ occurs when $\theta = 0$ and the maximum value occurs when $\theta = \pi$. The point at which these critical points occurs are called periapsis (perigee for an earth orbit) for $r_{\text{min}}$ and apoapsis (apogee) for $r_{\text{max}}$. Referring back to our previous equations, when modeling the orbit, we can take perigee to take place at time $t_0 = 0$.

For an eccentricity $e = 0$, we can see from Eq. (3.9) that $r$ will remain constant, and thus this orbit with a constant radius is a circle. In this instance we can see that $p = a$ where $a$ is the semi-major axis. The constants $a$ and $e$ are of special significance as they can be used to help describe the motion of the satellite in a plane. They are two of the six classical orbital elements.

### 3.1.4 Classical Orbital Elements

To describe the path of an spacecraft in orbit around the earth, we can use the classical orbital elements. These are a set of six parameters that can be used to accurately describe the motion, shape, and orientation of a satellite orbit about the earth for the case being studied.

To describe the motion of the spacecraft in the orbital plane, we can use the set of parameters $\{a, e, t_0\}$ using the equations listed in Section 3.1.2. We still require three parameters to specify the orientation of the orbital plane. Two parameters are used to orient the plane of the orbit, and one is needed to locate the orbit within that plane.

One secondary reference parameter used is the ascending node, where the spacecraft crosses the earth’s equatorial plane from south to north. A vector of interest is the unit vector $\mathbf{n}$ representing the line of nodes, the line from the centre of the earth through the
ascending node. Recall that in a Geocentric Equatorial Inertial Frame, the basis vector \( \mathbf{i}_1 \) always points toward the vernal equinox direction \( \Upsilon \). The angle between \( \mathbf{i}_1 \) and \( \mathbf{n} \) is \( \Omega \), known as the right ascension of the ascending node, where \( 0 \leq \Omega < 2\pi \). For the case of the earth, it is also known as the longitude of the ascending node.

The angle between the equatorial plane and the orbital plane measured at the ascending node is \( i \), known as the inclination, where \( 0 \leq i \leq \pi \). The last element that orients the orbit is the argument of perigee, \( \omega \), where \( 0 \leq \omega < 2\pi \). This parameter is the angle measured in the plane of the orbit between \( \mathbf{n} \) and perigee. Collectively, the classical orbital elements are \( \{a, e, \Omega, i, \omega, t_0\} \). The elements values used for the constructed simulation are listed in Table 3.1. The element \( a \) is calculated as the sum of the radius of the earth and the simulated altitude of the satellite. Since \( e = 0 \), the orbit is circular and \( r \) remains constant.

<table>
<thead>
<tr>
<th>( a )</th>
<th>6378 km + 450 km</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>0</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>0</td>
</tr>
<tr>
<td>( i )</td>
<td>87°</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0</td>
</tr>
<tr>
<td>( t_0 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.1: Orbital elements used for simulation purposes.

of the earth and the simulated altitude of the satellite. Since \( e = 0 \), the orbit is circular and \( r \) remains constant.

### 3.2 Orbital Description

#### 3.2.1 Near-Polar Orbit and Tilted Dipole Model

Magnetic control requires a sufficient variation in the earth’s magnetic field for controllability. This can generally only occur, as literature suggests, using a near-polar orbit. A near-polar orbit occurs when the satellite traverses over an area on earth relatively
close to the poles which suggests a high value for $i$, the inclination. For this simulation study, the inclination used is $87^\circ$. The geomagnetic field has been computed using a tilted dipole model [25]. The geomagnetic components in the inertial frame are calculated as

\[
\mathbf{B}_i = \begin{bmatrix}
(B_r \cos \delta + B_\theta \sin \delta) \cos \alpha - B_\phi \sin \alpha \\
(B_r \cos \delta + B_\theta \sin \delta) \sin \alpha + B_\phi \cos \alpha \\
(B_r \sin \delta - B_\theta \cos \delta)
\end{bmatrix}
\]

where $\alpha$ and $\delta$ are the spacecraft’s right ascension and declination [15],[25]. From them, we can calculate $\phi$, the East Longitude from Greenwich, and $\theta$, the coelevation. $B_r$, $B_\theta$, $B_\phi$ are the geomagnetic field components in spherical coordinates and are given as follows:

\[
B_r = 2 \left( \frac{R_e}{R_b} \right)^3 [g_0^1 \cos \theta + (g_1^1 \cos \phi + h_1^1 \sin \phi) \sin \theta]
\]

\[
B_\theta = \left( \frac{R_e}{R_b} \right)^3 [g_0^1 \sin \theta - (g_1^1 \cos \phi + h_1^1 \sin \phi) \cos \theta]
\]

\[
B_\phi = \left( \frac{R_e}{R_b} \right)^3 [g_1^1 \sin \phi - h_1^1 \cos \phi]
\]

where $R_e$ is the Earth’s mean equatorial radius, $R_b$ is the spacecraft position, and the coefficients $g_0^1$, $g_1^1$, and $h_1^1$ are taken from the 1995 International Geomagnetic Reference Field (IGRF) [15]:

\[
g_0^1 = -29682 \ nT, \quad g_1^1 = -1789 \ nT, \quad h_1^1 = 5310 \ nT
\]

The components calculated from the magnetic field model are shown in Fig. 3.1 for a circular orbit at an inclination of $87^\circ$ and an altitude of 450 km.

### 3.2.2 Low Earth Orbit and Disturbance Torques

As previously mentioned, the simulation tests represent a low earth orbit with an altitude of 450 km, a value that is consistent with literature. The altitude of the orbit has an impact on the disturbances the satellite observes.
Figure 3.1: Magnetic field vector measured in the Inertial frame.

From [2] we can see that the given altitude is too low to significantly feel the effects of solar pressure and too high to be effected by aerodynamic torques acting from the earth’s atmosphere. Instead, the major disturbances are the effects due to gravity and the disturbance due to the spacecraft interacting with the earth’s magnetic field.

The disturbances are significant as they will act against the control and will have a greater presence as the control effort decreases. The disturbance torque applied by the gravity-gradient is given by

\[ G_g = \frac{3\mu}{R_0^5} R_0^5 I R_0 \]  \hspace{1cm} (3.12)

where, \( \mu \) is the geocentric gravitational parameter, and \( R_0 \) is a position vector in the body frame [21],[26].

The other disturbance torque included is a magnetic disturbance caused by current flowing through the satellite’s electrical components. This torque will be represented by

\[ G_d = m_d^x B_b \]  \hspace{1cm} (3.13)
where, $\mathbf{m}_d$ is the disturbance magnetic dipole moment and $\mathbf{B}_b$ is the geomagnetic field components in the body frame [26].
Chapter 4

Spacecraft Control

From Eq. (2.14), we can see that an input torque, \( u \), will create a change in the output angular velocity \( \omega \). Furthermore, from Eq. (2.8), we can see that a change in the angular velocity will impact the attitude of the spacecraft we wish to control. It is important to note that these kinematic equations are simultaneous first-order differential equations. Since these system variables that appear as first derivatives can collectively define the state of the system at any instant, we can define them to be our state variables \([27]\). Our state variables are the angular velocity, the unit quaternion representing attitude, and the integral of the attitude error to provide I control.

Our two methods of actuation are the magnetorquers interacting with the earth’s magnetic field through the cross product law of vectors, and reaction wheels apply torques directly about the principal axes. We wish to see convergence toward our desired equilibrium point while not allowing for the control actuation to be significantly large. More desirably, we seek a method such that the two forms of control authority can act together in concert in some optimal way to reduce the control action so that we can see some way to reduce power consumption.

We select a performance index that is a function of both the states, and the control inputs. Through applying the calculus of variations to the performance integral, we can
see that the solution to the optimal $u$ can be provided through a linear control law with state feedback. Determining the correct control law requires solving the matrix Riccati equation. Solving this equation requires numerical methods where we backwards integrate to obtain the matrix $P(t)$. Since the system is almost periodic, our solution for $P(t)$ converges to one that is nearly periodic.

### 4.1 State-Space Representation

Our linearized state-space equation following from Eq. (2.19), Eq. (2.16), and Eq. (2.17) is as follows

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (4.1)

or:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\theta} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \theta \\ z \end{bmatrix} + \begin{bmatrix} -I^{-1}B_i^x & I^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m \\ \tau \end{bmatrix}$$  \hspace{1cm} (4.2)

To provide integral control, we can introduce a new state variable $z(t) = \int_0^t \theta(T) dT$ which will help eliminate the offset errors at steady state:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\theta} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \theta \\ z \end{bmatrix} + \begin{bmatrix} -I^{-1}B_i^x & I^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m \\ \tau \end{bmatrix}$$  \hspace{1cm} (4.3)

### 4.2 Linear Quadratic Regulator

We select a minimization criteria containing both the state vector $x$ and control input $u$:

$$J[x, u] = \psi[x(t_f)] + \int_0^{t_f} \phi(x, u) dt$$  \hspace{1cm} (4.4)

Since the case being studied consists of linear plant dynamics in Eq. (4.1), we select a quadratic performance index

$$J = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2} \int_0^{t_f} (x^TQx + u^TRu) dt$$  \hspace{1cm} (4.5)
where $t_f$ is the final or terminal time and the matrices $S$, $Q$ and $R$ are symmetric positive definite. We can define the Hamiltonian as:

$$H(x, u, \lambda) = \phi + \lambda^T f$$

$$= \frac{1}{2}(x^T Q x + u^T R u) + \lambda^T (A x + B u)$$

(4.6)

By minimizing $H$ with respect to $u$, and solving for $\lambda$ from $\dot{\lambda}$ and a $\lambda(t_f)$ boundary condition, we can obtain the optimal control law

$$u = -R^{-1} B^T(t) P(t) x(t)$$

(4.7)

$$= F(t) x(t)$$

where $x$ is the linearized state vector and $P(t)$ is the solution to the matrix differential Riccati equation:

$$-\dot{P}(t) = P A + A^T P - PB(t) R^{-1} B(t)^T P + Q, \quad P(t_f) = S$$

(4.8)

Integrating backwards from $t_f$ allows to calculate a solution for $P(t)$. Since the system is nearly periodic, the solution obtained is nearly periodic. An example of this is shown in Fig. 4.1. Taking the data from the first orbit only, we can create a periodic representation using a Fourier series that the controller can use for the duration of the mission. An example of this is shown in Fig. 4.2.

The control performance is dependent upon the appropriate selection of weighting or penalty matrices $Q$ and $R$. Once weightings have been found to produce stability, control performance can be studied by appropriate adjustments of $Q$ or $R$.

Applying the control law from Eq. (4.8) to our satellite model, we obtain

$$u = -R^{-1} \begin{bmatrix} B_i^*(t) I^{-1} & 0 & 0 \\ I^{-1} & 0 & 0 \end{bmatrix} P(t) \begin{bmatrix} \dot{\theta} \\ \theta \\ Z \end{bmatrix}$$
Figure 4.1: Nearly periodic solution of $P_{11}$ from backwards integration.

Figure 4.2: Periodic solution of $P_{11}$ used by controller.
which when extended to the nonlinear case gives

\[
\mathbf{u} = -\mathbf{R}^{-1} \left[ \begin{bmatrix} \mathbf{B}_i^x(t) & 0 & 0 \\ \mathbf{I}^{-1} & 0 & 0 \end{bmatrix} \mathbf{P}(t) \right] \begin{bmatrix} \omega(t) \\ 2\epsilon(t) \\ \mathbf{Z}(t) \end{bmatrix}
\]

(4.9)

where \(\omega\) and \(\epsilon\) are taken from the simulated state vector at each time instant and \(\mathbf{Z}(t)\) can be calculated from the equation

\[
\mathbf{Z}(t) = 2 \int_0^t \epsilon(T) \, dT,
\]

(4.10)

and \(\mathbf{B}_i\) is the magnetic field vector from the tilted dipole model taken at each time instant.
Chapter 5

Hybrid Magnetic Control: Primary Investigation

This chapter discusses various tests and results found for investigating the hybrid magnetic ACS using a time-varying LQR control scheme. Specifically, this chapter covers cases that exploit both the magnetorquers and all three reaction wheels working together in concert. To begin, we first present the control from a reaction wheel-based LQR. This is done to establish a baseline and understanding of how the magnetic hybrid controller is performing by comparison. Special attention is specifically paid to the total energy being consumed by the system as well as the Root Mean Squared (RMS) error.

Since energy is important to the satellite, we seek a lower energy consumption with the magnetic hybrid controller. One way that might aid in minimizing energy consumption is by incorporating additional coils with the magnetorquers. This allows for the required magnetic control dipole to be achieved with a lower current.

5.1 Parameters of Interest

Specific attention is paid to the total energy and RMS errors as measures of the controller’s performance. First, we will focus our attention on the energy consumption. It
is intuitive that the energy consumed can be determined from the power consumed. The rates of power consumed at each instance of time can be easily calculated if specific parameters for the satellite being simulated are provided.

### 5.1.1 Energy Consumption

The energy consumed is calculated from the power consumed as power can be considered to be the energy transferred per unit time:

$$ E = \int_0^t \tilde{P} dt $$

Since there are two forms of control actuation, the total power consumed can be categorized into the power consumed by the magnetorquers and power consumed by the wheel system

$$ \tilde{P}_{tot} = \tilde{P}_m + \tilde{P}_r \tag{5.1} $$

where $\tilde{P}_{tot}$ is the total power consumed, $\tilde{P}_m$ is the power consumed by the magnetorquers, and $\tilde{P}_r$ is the power consumed by the reaction wheels. The power consumed by each is dependent upon certain design parameters of the satellite’s physical components as well as the amount of torque the actuator is required to apply. A summary of the parameters of interest is shown in Table 5.1 for the magnetorquers and Table 5.2 for wheel actuators.

Using the power formula

$$ \tilde{P}_A = \tilde{I}_A R_A \tilde{I}_A $$

### Table 5.1: Actuator Parameters for magnetorquers

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loop Area</td>
<td>0.00611</td>
<td>$m^2$</td>
</tr>
<tr>
<td>Number of Turns</td>
<td>283</td>
<td>–</td>
</tr>
<tr>
<td>Maximum Current</td>
<td>0.072</td>
<td>A</td>
</tr>
<tr>
<td>Resistance</td>
<td>48.9</td>
<td>$\Omega$</td>
</tr>
</tbody>
</table>
we can see that the power of the actuator $\tilde{P}_A$ can be calculated from the corresponding resistance $R_A$ in the actuator’s components and the required control current $\tilde{I}$ which is determined by the required control torque. Provided the required magnetic dipole given by the control law, the control current can be calculated as:

$$\tilde{I}_m = (NA)^{-1}m$$

(5.2)

where $\tilde{I}_m$ is the control current required for the magnetorquers, $m$ is the required magnetic dipole, $N$ is the scalar number of coils or turns of wire in the magnetorquer, and $A$ is the loop area. For the wheels, the power is determined from the calculated control torques

$$\tilde{I}_r = (K_r)^{-1}u_r$$

(5.3)

where $\tilde{I}_r$ is the control current required for the wheels, $u_r$ is the required torques provided by the wheels, and $K_r$ is the torque constant from Table 5.2.

### 5.1.2 Root Mean Squared Error

The set point for the vector part of the quaternions is $0$, hence the values for $\epsilon$ at each time instance can be considered as the error. We can then define our root mean squared error as

$$\rho = \sqrt{\int_0^{t_f} \epsilon^T \epsilon dt}$$

(5.4)

where $\rho$ is the total RMS error, $t_f$ is the final terminal time, and $\epsilon$ is the vector portion of the quaternions representing the attitude or attitude error.
5.1.3 Simulation Initial Conditions

The initial conditions and other parameters of importance with regards to the simulation are listed in Table 5.3.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Orbits</td>
<td>7</td>
<td>–</td>
</tr>
<tr>
<td>$I_{xx}$</td>
<td>0.1</td>
<td>kg·m²</td>
</tr>
<tr>
<td>$I_{yy}$</td>
<td>0.2</td>
<td>kg·m²</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>0.3</td>
<td>kg·m²</td>
</tr>
<tr>
<td>$\omega(0)$</td>
<td>[0.01 0.01 0.01]^T</td>
<td>rad/s</td>
</tr>
<tr>
<td>$q(0)$</td>
<td>[0 0 0 1]^T</td>
<td>–</td>
</tr>
<tr>
<td>$Q$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 51 &amp; 0 \ 0 &amp; 0 &amp; 101 \end{bmatrix}$</td>
<td>–</td>
</tr>
<tr>
<td>Step size $h$</td>
<td>0.1</td>
<td>s</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$3.986 \times 10^{14}$</td>
<td>m³/s</td>
</tr>
<tr>
<td>$m_d$</td>
<td>$[0.1 \ 0.1 \ 0.1]^T$</td>
<td>A·m²</td>
</tr>
</tbody>
</table>

Table 5.3: Initial conditions used for simulation

Similar to Eq. (4.9), the control law implemented for the hybrid magnetic ACS for three reaction wheels is

$$u = -R^{-1} \begin{bmatrix} B^T(t)I^{-1} & 0 & 0 \\ I^{-1} & 0 & 0 \end{bmatrix} P(t) \begin{bmatrix} \omega(t) \\ 2\epsilon(t) \\ Z(t) \end{bmatrix}$$

where $Z(t)$ is calculated from the equation:

$$Z(t) = \int_0^t \epsilon(T)dT,$$

When this is compared with Eq. (4.10), we see that the integral gain is effectively halved.
5.2 Reaction Wheel LQR System

For the reaction wheel ACS, there is only one form of actuation consisting of three reaction wheels. Hence $R$ is proportional to a $3 \times 3$ identity matrix with a scalar parameter, $R_2$, that is adjusted manually. Testing is done by varying $R_2$ and observing the changes in behavior in an attempt to understand the fundamental relationships where:

$$R = R_2I$$

5.2.1 Reaction Wheel LQR Control Results

![Graph showing $E(J)$ consumed versus $R_2$ parameter for reaction wheels LQR control scheme.](image)

Figure 5.1: $E(J)$ consumed versus $R_2$ parameter for reaction wheels LQR control scheme.

The performance results are shown in Fig. 5.1 for $E(J)$ and Fig. 5.2 for error. From the $R$ parameters tested for reaction wheel control, it was found that the lowest $E$ value was approximately 23.820 J occurring at $R_2 = 1.0 \times 10^{14}$. The corresponding $\rho$ was found to be 0.183.
Figure 5.2: $\rho$ versus $R_2$ parameter for reaction wheels LQR control scheme.

5.2.2 Reaction Wheel Control Discussion

Referring to Fig. 5.1, we see that the total amount of energy required for the reaction wheel ACS to correct the satellite decreases as $R_2$, the penalty against the reaction wheels control, is increased. This may strengthen the hypothesis that we may obtain a savings in energy through the magnetic hybrid control scheme by lowering the dependence on the reaction wheels. It is important to note that a great reduction in the energy consumption with the reaction wheel ACS clearly results in a drastic degradation of the control quality. There is clear instability shown in Fig. 5.2 for large $R_2$ values. The value of $\rho$ is considerably low for a wide range of $R_2$ and confirms that reaction wheels can provide very accurate and precise attitude control when used abundantly.
5.3 Hybrid Magnetic Control

The following are the results for the hybrid magnetic controller. Now that there are three magnetorquers acting in concert with three reaction wheels, $R$ is given by:

$$
R = \begin{bmatrix}
R_1 & 0 \\
0 & R_2
\end{bmatrix} = \begin{bmatrix}
R_1 & 0 \\
0 & R_2
\end{bmatrix}
$$

Several simulation tests have been taken by holding $R_2$ constant for a specific value, and then varying $R_1$. This process is then repeated for a wide range of $R_2$ values.

5.3.1 Hybrid Control Results

For the purpose of clarity, the results have been split up amongst the figures in accordance to their $R_2$ values. Results for $\rho$ have been shown in Fig. 5.3 and Fig. 5.5 while results for their corresponding $E$ have been shown in Fig. 5.4 and Fig. 5.6. From the values

Figure 5.3: $\rho$ versus $R$ parameters at low $R_2$ values.
Figure 5.4: $E(J)$ consumed versus $R$ parameters at low $R_2$ values.

Figure 5.5: $\rho$ versus $R$ parameters at high $R_2$ values.
Figure 5.6: $E(J)$ consumed versus $R$ parameters at high $R_2$ values.

Figure 5.7: $E(J)$ consumed versus $R$ parameters at high $R_2$ values.
Figure 5.8: Angular velocities for $R_1 = 1.0 \times 10^8$ and $R_2 = 1.0 \times 10^{14}$.

Figure 5.9: Quaternions for $R_1 = 1.0 \times 10^8$ and $R_2 = 1.0 \times 10^{14}$. 
tested, it was found that the lowest $E$ value was 23.783 $J$ occurring at $R_1 = 1.0 \times 10^8$ and $R_2 = 1.0 \times 10^{14}$. The corresponding $\rho$ was found to be 0.182. The angular velocities and attitude behavior for these parameters is shown in Fig. 5.8 and Fig. 5.9. To get a better understanding of the control torques that are applied through each of the actuators, their behaviors can be seen in Fig. 5.10 and Fig. 5.11 for magnetorquers and wheels respectively.

Figure 5.10: Control dipole moments for $R_1 = 1.0 \times 10^8$ and $R_2 = 1.0 \times 10^{14}$.

5.3.2 Hybrid Control Discussion

We can see that for each $R_2$ data set, the values correspond to those values found in Fig. 5.1 for $E$ and Fig. 5.2 for $\rho$ as we increase $R_1$ to a significantly high value. This is a positive outcome and helps demonstrate the the simulation is working correctly. As the impact of the magnetics is reduced to a point where they can be considered negligible, it is intuitive that the reaction wheel ACS becomes prominent and hence the values match.
Figure 5.11: Control torques from wheels for $R_1 = 1.0 \times 10^6$ and $R_2 = 1.0 \times 10^{14}$.

It was consistently found that when $R_2$ is greater than $R_1$ by a magnitude of $1.0 \times 10^6$, the amount of energy being consumed will be lower compared to the reaction wheels energy consumption for the same $R_2$ value. It should be noted, however, that the energy savings found are marginal and perhaps just offer a starting point where further optimization can be performed.

**Comments on Control Saturation**

Taking into account control saturation did not yield any major developments as the main points of interest with regards to energy consumption were all when the controller did not heavily rely on the magnetorquers. As a result, the parameters that yielded low $E$ were not subject to control saturation. A second ratio of interest is perhaps when $R_2$ is greater than $R_1$ by a magnitude of $1.0 \times 10^{10}$. Although this data is not quite as consistent for all the tests, it is noticeable that when $R_2 > 1.0 \times 10^{10}$, then the lowest $\rho$ values always occurs when $R_1$ is greater than $R_2$ by a magnitude of $1.0 \times 10^{10}$. This pattern is more
visible in Fig. 5.5. Closer examination of the values obtained shows that through these trials, hybrid magnetic control does have the ability to achieve better performance. The energy savings, however, are marginal and may not provide the same conclusion if finer tuning is performed. Most of the energy is consumed within the first couple of seconds of the simulation to correct for initial conditions. Therefore, the savings are not expected to grow when more orbits are added to the simulation. With the current specifications, the hybrid controller does not appear to stand out or provide any substantial benefit compared to a reaction wheel controlled system.

5.4 Additional Coils

Previously, we saw that putting a great amount of dependence on the magnetorquers in a magnetic hybrid control scheme does not necessarily translate into a lower energy consumption. This may be partially attributed to high current demands required to provide the necessary magnetic dipoles. One way in which we can reduce the current demand and energy consumption is to add extra coils onto the magnetorquers.

From Eq. (5.2) we can see that increasing the number of coils will reduce the current demand for the required control dipole. Although the extra coils will also increase the total resistance proportionally, from Eq. (5.2) it is clear that a reduction in the current has a greater effect than an increase in the coils. To test additional coil schemes we can introduce a Coil Factor (CF) parameter where $CF = 1$ is the original amount of coils used from Table 5.1, $CF = 2$ will double the number of coils, and $CF = 10$ is ten times, etc.

5.4.1 Results

To help demonstrate the CF effect on the control performance, Fig. 5.12 shows a comparison of the energy consumed between the original hybrid magnetic controller and the
Chapter 5. Hybrid Magnetic Control: Primary Investigation

Table 5.4: The $E(J)$ values listed for $R_2 = 1.0 \times 10^{14}$ magnetic hybrid controller.

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$CF = 1$</th>
<th>$CF = 2$</th>
<th>$CF = 4$</th>
<th>$CF = 8$</th>
<th>$CF = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \times 10^3$</td>
<td>302802.</td>
<td>151401.</td>
<td>75701.</td>
<td>37851.</td>
<td>30281.</td>
</tr>
<tr>
<td>$1.0 \times 10^4$</td>
<td>16316.</td>
<td>8159.</td>
<td>4080.</td>
<td>2041.</td>
<td>1633.</td>
</tr>
<tr>
<td>$1.0 \times 10^5$</td>
<td>5127.</td>
<td>2566.</td>
<td>1286.</td>
<td>646.</td>
<td>517.</td>
</tr>
</tbody>
</table>

controllers with $CF > 1$. In addition to this, values are listed in Table 5.4 and Table 5.5.

5.4.2 Extra Coils Discussion

By comparing the $CF = 1$ column to the $CF = 2$ or $CF = 10$ column in either Table 5.4 or Table 5.5, it is clear that there is an inverse relationship between the amount of energy consumed and the $CF$. Trials with a relatively low $R_1$ value showed a
greater savings in energy as these parameter sets have a greater reliance on the modified magnetorquers. Hence, the amount of energy saved then decreases as the control system becomes less and less dependent on them.

It is important to note, however, that placing great reliance on the reaction wheels with a low $R_2$ does not yield noticeable savings. This is shown in Fig. 5.12 where savings become significant as $R_2$ increases. Recall from Fig. 5.7 that the minimum $E$ takes place at a moderately sized $R_1$. The significance behind this is that the point that holds the minimum $E$ does not place a high demand on the magnetorquers. Therefore, the energy savings of multiple coil factors are present but are still marginal. It is important to keep in mind however, that although the energy consumed varies, the $\rho$ will remain constant for each distinct $R$ regardless of any changes in the CF. For the specific performances where the magnetorquers are active but are consuming a lot of power, it is clear that through the inverse relationship demonstrated in Table 5.4 and Table 5.5, we can establish the power level to the value we want by adding extra coils proportionally. This can be shown in Table 5.6 and Table 5.7 where by using a $CF = 700$, we can proportionally reduce the energy consumed to much smaller levels.

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$CF = 1$</th>
<th>$CF = 2$</th>
<th>$CF = 4$</th>
<th>$CF = 8$</th>
<th>$CF = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \times 10^2$</td>
<td>331337.</td>
<td>165669.</td>
<td>82835.</td>
<td>41418.</td>
<td>33135.</td>
</tr>
<tr>
<td>$1.0 \times 10^3$</td>
<td>17517.</td>
<td>8759.</td>
<td>4381.</td>
<td>2192.</td>
<td>1754.</td>
</tr>
<tr>
<td>$1.0 \times 10^4$</td>
<td>5307.</td>
<td>2657.</td>
<td>1332.</td>
<td>669.</td>
<td>537.</td>
</tr>
</tbody>
</table>

Table 5.5: The $E(J)$ values listed for $R_2 = 1.0 \times 10^{13}$ magnetic hybrid controller.
Table 5.6: The $E(J)$ and $\rho$ values listed for $R_2 = 1.0 \times 10^{14}$ magnetic hybrid controller with $CF = 700$.

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$E(J)$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \times 10^3$</td>
<td>433.6</td>
<td>0.097</td>
</tr>
<tr>
<td>$1.0 \times 10^4$</td>
<td>24.9</td>
<td>0.101</td>
</tr>
<tr>
<td>$1.0 \times 10^5$</td>
<td>12.7</td>
<td>0.148</td>
</tr>
</tbody>
</table>

Table 5.7: The $E(J)$ and $\rho$ values listed for $R_2 = 1.0 \times 10^{13}$ magnetic hybrid controller with $CF = 700$.

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$E(J)$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \times 10^2$</td>
<td>475.</td>
<td>0.057</td>
</tr>
<tr>
<td>$1.0 \times 10^3$</td>
<td>27.3</td>
<td>0.059</td>
</tr>
<tr>
<td>$1.0 \times 10^4$</td>
<td>14.1</td>
<td>0.086</td>
</tr>
</tbody>
</table>
Chapter 6

Hybrid Magnetic Control: Actuator Redundancy Results

This chapter investigates cases where not all three reaction wheels are providing control torques. The first half of the chapter looks into wheel failures that prevent a required control torque from being applied. The second half of the chapter looks at a control law reconfiguration to accommodate for the missing or removed reaction wheel. For each case, it is assumed that all three magnetorquers are used.

6.1 Magnetic Hybrid Control with Missing Wheels Scenarios

One way that magnetic hybrid control can be advantageous is providing redundancy in the event of actuator failure. As previously discussed, reaction wheels and other devices with rotating parts may be subject to failure resulting in an unactuated axis. This problem might be remedied with the application of hybrid magnetic control. Provided that there is sufficient variability in the Earth’s magnetic field, the magnetorquers may be capable of mitigating the effects of the lost wheel and provide the necessary control
for the satellite to fulfill its mission goals.

### 6.1.1 Missing Wheels Results

If a reaction wheel fails, it can be expected that the satellite may go unstable due to missing control actuation. This is especially true if there is a heavy reliance on the wheels when $R_2$ is relatively low. There were, however, specific instances when stability through magnetic hybrid control was achieved. The results of interest are presented.

**Wheel 1: missing**

Cases were found where stability was achieved despite the fact that the control torques required from wheel one were removed. The $\rho$ and energy consumption are presented in Fig. 6.1 and Fig. 6.2 respectively.

![Figure 6.1: Selected $\rho$ values for magnetic hybrid control scheme with wheel from axis 1 missing.](image-url)
Figure 6.2: Selected energy values for hybrid magnetic control scheme with wheel from axis 1 missing.

Wheel 2: missing

Similarly promising results were found when the required control torquers for wheel two were removed. The corresponding $\rho$ and energy consumption values are presented in Fig. 6.3 and Fig. 6.4 respectively.

Wheel 3: missing

Finally, stability results were found when the required control torques for wheel three were removed. The corresponding $\rho$ and energy consumption are presented in Fig. 6.5 and Fig. 6.6 respectively.
6.1.2 Missing Wheels Discussion

It is clear that with the appropriate selection of $R_1$ and $R_2$, the magnetic hybrid ACS can mitigate the missing wheel torques and disturbances to provide adequate control capabilities. There is, however, a high cost in energy consumption to achieve stability. It was found that missing more than one wheel always resulted in instability.

6.2 Magnetic Hybrid Control Reconfigured Control Law

In the event of wheel failure, another method to mitigate the problem may be to reconfigure the control law such that it takes into account that there are only one or two wheels and three magnetorquers present.
Figure 6.4: Selected energy values for hybrid magnetic control scheme with wheel from axis 2 missing.

### 6.2.1 Reconfigured Control Law Results

#### Wheel 1 Removed

There were several instances of low $\rho$ very similar to the values shown in Fig. 6.7. These instances appeared to overlap directly over top of one another and have been omitted for the purpose of clarity. The corresponding $E$ values are shown in Fig. 6.8.

#### Wheel 2 Removed

Similar results can be found when wheel two is removed from the control law. There were several cases of stable and bounded behavior, of which a select set are shown in Fig. 6.9 for $\rho$ and Fig. 6.10 for $E$. 
Wheel 3 Removed

Similar stable and bounded results are shown in Fig. 6.11 for $\rho$ and Fig. 6.12 for $E$.

Wheels 1 and 2 Removed

Removing two wheels is also handled well by the hybrid controller as shown in Fig. 6.13 for $\rho$ and Fig. 6.14 for $E$.

6.2.2 Reconfigured Control Law Discussion

The results shown are quite promising as for each case tested, stability can be achieved whether one wheel or two wheels have been removed. This is of great significance as it shows that in the event of failure, reconfiguring the control law will stabilize the satellite and mitigate disturbances effectively.
Figure 6.6: Selected energy values for magnetic hybrid control scheme with wheel from axis 3 missing.

Similar to the missing reaction wheel case, there is a higher energy consumption cost to achieve stability. One reason for the higher energy expenditure is due to the fact that more control torques are required now that a reaction wheel has been removed. Reaction wheels are known to be one of the most accurate methods of applying a torque, and removing one requires that the remaining actuators provide more control to make up for this.
Figure 6.7: Selected $\rho$ values for hybrid magnetic control scheme with wheel from axis 1 removed.

Figure 6.8: Selected $E(J)$ values for hybrid magnetic control scheme with wheel from axis 1 removed.
Figure 6.9: Selected $\rho$ values for hybrid magnetic control scheme with wheel from axis 2 removed.

Figure 6.10: Selected $E(J)$ values for hybrid magnetic control scheme with wheel from axis 2 removed.
Figure 6.11: Selected $\rho$ values for hybrid magnetic control scheme with wheel from axis 3 removed.

Figure 6.12: Selected $E(J)$ values for hybrid magnetic control scheme with wheel from axis 3 removed.
Figure 6.13: Selected $\rho$ values for hybrid magnetic control scheme with wheels from axes 1 and 2 removed.

Figure 6.14: Selected $E(J)$ values for hybrid magnetic control scheme with wheels from axes 1 and 2 removed.
Chapter 7

Control Saturation Effects

Through simulating various scenarios, there were several cases found when the calculated magnetic control dipole will require a current that exceeds the maximum limits the magnetorquers can provide. This is especially true in cases when less than three wheels are used as there is a heavier reliance on the magnetorquers as opposed to having all three wheels available. It is intuitive that limiting the control dipole by enforcing the maximum current may have adverse effects on the spacecraft as it is not receiving the complete set of control torques that it may require at each time instant. There were also cases of interest that proved to be stable even though the control dipoles were limited. Section 7.1 will examine a case where one wheel has been removed from the control law, thus increasing the instances where control saturation will take effect as opposed to the three wheel case. Section 7.2 will examine another specific case where two control wheels have been removed further increasing the reliance on the magnetorquers.
7.1 Single Wheel Removed with Control Saturation Results

Including the effects of control saturation, we can see that it is possible to achieve stability even when wheels are removed. For the case shown, \( R_1 = 5.0 \times 10^5 \) and \( R_2 = 5.0 \times 10^5 \). The corresponding angular velocities are shown in Fig. 7.1, the quaternions in Fig. 7.2, the magnetic dipole in Fig. 7.3, and the control torques provided by the wheels in Fig. 7.4. It is clear from Fig. 7.4 that wheel 2 is not providing control as \( \tau_2 \) remains constantly at zero. From Fig. 7.3, we can clearly see the maximum and minimum control dipoles being enforced on axes 1 and 2 to compensate for wheel 2 being removed. These bounds are determined from the maximum control currents taken from the spacecraft’s specifications.

![Graphs showing control saturation effects](image-url)

Figure 7.1: Angular velocities with the effects of control saturation shown for hybrid magnetic attitude control with wheel from axis 2 removed.
Figure 7.2: Quaternions with the effects of control saturation shown for hybrid magnetic attitude control with wheel from axis 2 removed.

Figure 7.3: Control dipoles with the effects of control saturation shown for hybrid magnetic attitude control with wheel from axis 2 removed.
Figure 7.4: Control torques with the effects of control saturation shown for hybrid magnetic attitude control with wheel from axis 2 removed.

7.1.1 Single Wheel Removed with Control Saturation Discussion

This is significant as we have a controller that no longer has as many wheels as we would desire, but with careful selection of the $R$ parameters, we can achieve stability even though there are often times when the required magnetic dipole cannot be provided due to the physical constraints of the system. The corresponding $E$ and $\rho$ for this case are approximately $3.628 \times 10^5$ joules and 0.279 respectively. This energy value is considered to be quite high. This is understandable however, as the spacecraft can be seen in Fig. 7.1 and Fig. 7.2 to have tumbling motion within the first orbit before settling.
7.2 Two Wheels Removed with Control Saturation

Results

Removing two wheels will place an even higher dependence on the magnetorquers and hence the effects of control saturation may play a more significant role. Stability can however be achieved as shown in the presented figures. For the case shown, $R_1 = 1.0 \times 10^5$ and $R_2 = 1.0 \times 10^{13}$. The corresponding angular velocities are shown in Fig. 7.5, the quaternions in Fig. 7.6, the magnetic dipole in Fig. 7.7, and the control torques provided by the wheels in in Fig. 7.8. It is clear from Fig. 7.8 that wheels 2 and 3 are not providing control as $\tau_2$ and $\tau_3$ remains constantly at zero.

![Angular velocities](image)

Figure 7.5: Angular velocities with the effects of control saturation shown for hybrid magnetic attitude control with wheels from axes 2 and 3 removed.
Chapter 7. Control Saturation Effects

Figure 7.6: Quaternions with the effects of control saturation shown for hybrid magnetic attitude control with wheels from axes 2 and 3 removed.

Figure 7.7: Control dipoles with the effects of control saturation shown for hybrid magnetic attitude control with wheels from axes 2 and 3 removed.
Chapter 7. Control Saturation Effects

7.2.1 Two Wheels Removed with Control Saturation Discussion

It is clear that stability can be achieved while considering the effects of control saturation when two wheels are removed from the control law. The corresponding $E$ and $\rho$ for this case are approximately $14.715 \times 10^3$ joules and 0.316 respectively. Similar to the results in Section 7.1.1, these numbers are high but are acceptable as the spacecraft undergoes some tumbling motion and must cope with the effects of control saturation before settling toward the equilibrium. It also appears that reaction wheels are more effective at mitigating disturbances as opposed to magnetorquers, so removing two wheels from the control law will also greatly affect the accuracy of the control system. This is most noticeable when comparing any of the graphs showing the behavior of $\tau$ to their corresponding $\epsilon$ or $\omega$.
Chapter 8

Hybrid Magnetic Control: Inclination Tests Results

This chapter reviews the tests and results of the hybrid magnetic controller at different and inaccurate inclinations. Tests will specifically look at performance near the desired 87° inclination, and as far as 10° away from the desired inclination. Section 8.1 will first look at the controller’s performance away from the 87° orbital inclination. Section 8.2 will then look at the controller’s performance in situations when the controller has incorrect information about the spacecraft’s orbital inclination.

8.1 Hybrid Magnetic Control: Preliminary Inclination Tests

Hybrid control, like magnetic control, is most effective at high near-polar orbits. A commonly used orbital inclination with magnetic control is 87° as it allows for sufficient variability in the earth’s magnetic field. The following presents the results for how hybrid magnetic control performs near 87° inclination. It is a promising result that with the regular hybrid magnetic controller, the performance of the inclinations near the vicinity
of 87° were identical to those found at 87°.

8.1.1 Magnetic Hybrid Control at 77° and 97° Inclination

The following shows the performance of the hybrid controller when the desired orbit is set to be far away from the regularly used 87° point. This inclination change will affect the variability of the magnetic field. From Fig. 8.1 and Fig. 8.4, we observe the same general trend for error that the hybrid magnetic controller provides us at 87°. From the perspective of energy, Fig. 8.2 and Fig. 8.5 also confirm the similar performance. Taking a closer look at the energy values of interest, we can see from Fig. 8.3 and Fig. 8.6 that accurate control can be provided at a low energy cost.
Figure 8.2: Selected $E(J)$ values for hybrid magnetic control at a 77$^\circ$ inclination.

Figure 8.3: $E(J)$ values of interest for hybrid magnetic control at a 77$^\circ$ inclination.
Figure 8.4: Selected $\rho$ values for hybrid magnetic control at a 97° inclination.

Figure 8.5: Selected $E(J)$ values for hybrid magnetic control at a 97° inclination.
Chapter 8. Hybrid Magnetic Control: Inclination Tests Results

8.2 Hybrid Magnetic Control: Perturbed Inclination Tests

An interesting problem to look at is how the controller performs provided with the incorrect inclination. This is significant as the control action and Riccati solution are calculated assuming the $B_i$ is found using the magnetic field vector and position variables of the $87^\circ$ inclination even though the position and tilted dipole model are calculated with completely different orbital inclination. This will provide a good test as to how robust the controller is.

8.2.1 Perturbed Inclination Results

For an incorrect inclination that is in the vicinity of $87^\circ$, the $\rho$ and $E$ can be seen in Fig. 8.7 and Fig. 8.8 respectively. The results from these graphs are also similar for
For a more drastic case, we can look at the performance of the controller when the assumed and actual orbital inclinations are off by 10°. In Fig. 8.9 and Fig. 8.10, we can see the $\rho$ and $E$ when the actual inclination is 77° whereas the Riccati equation was solved for an inclination of 87°. Similarly, in Fig. 8.11 and Fig. 8.12, the $\rho$ and $E$ are shown when the actual inclination is 97° instead of the calculated 87°. It is observed that stability and good performance has been found for each case.

### 8.3 Inclination Tests Discussion

For each test case with either a selected or perturbed inclination, it was found that stability could be achieved. Furthermore, it was seen that accurate control was obtained with a low energy expenditure. Equally good control performance was found for cases
Figure 8.8: Selected $E(J)$ values for hybrid magnetic control at $84^\circ$ inclination with the $87^\circ$ controller.

near the vicinity of $87^\circ$, and cases when the inclination was inaccurate with a $10^\circ$ error. This study helps establish a wider basis of inclinations where hybrid magnetic control can be successfully used. This also demonstrates that the controller possesses a certain level of robustness and is capable of handling uncertainties in the earth’s magnetic field and inclination.
Chapter 8. Hybrid Magnetic Control: Inclination Tests Results

Figure 8.9: Selected $\rho$ for hybrid magnetic control at $77^\circ$ inclination with the $87^\circ$ controller.

Figure 8.10: Selected $E(J)$ for hybrid magnetic control at $77^\circ$ inclination with the $87^\circ$ controller.
Figure 8.11: Selected $\rho$ for hybrid magnetic control at $97^\circ$ inclination with the $87^\circ$ controller.

Figure 8.12: Energy for hybrid magnetic control at $97^\circ$ with the $87^\circ$ controller.
Chapter 9

Conclusions and Recommendations

In an effort to reduce the power and energy consumption of a LEO nanosatellite in a near-polar orbit, a hybrid magnetic attitude controller was investigated. Power conservation is known to be a point of interest along with proper attitude control. Both aspects are of great importance for a satellite to fulfill its mission goals. Using optimal control theory, this study implemented a time-varying LQR to account for the nearly periodic behavior of the Earth’s magnetic field to obtain the optimal control inputs. Appropriate selection and adjustment of weighting matrices $Q$ and $R$ were done to observe the controller’s behavior over a wide range of weights.

9.1 Three-axis Hybrid Magnetic Attitude Control

A hybrid magnetic control consisting of three magnetorquers working together in concert with three-axis reaction wheel control was simulated to observe if a savings in energy was present when compared to reaction wheel control. The results showed that energy savings were present, but the effects were quite marginal. A factor which may contribute to this conclusion is the performance of the magnetorquers when they are relied heavily upon to provide control torques. Delegating control away from the reaction wheels led to increases in attitude error. When the $R$ weightings were set to moderate values with
respect to one another, savings in energy were observed as compared to a primarily reaction wheel control. It was also found that when $R_2$ was greater than $R_1$ by $10^6$, the energy level for that specific $R$ would be minimized. This perhaps suggests an initial starting point where further optimization can be applied.

9.1.1 Extra Coils

In another effort to reduce energy consumption, extra coils were added to the magnetorquers. This allowed for the required control dipole to be achieved at a lower current, thus lowering the power consumption. Using coil factors, it was found that an inverse relationship existed between the energy consumed and the CF for low values of $R_1$. This relationship could be used to greatly reduce the power consumption to desired levels, even at extremely large $CF$ values. This effect was only observed when a low control penalty was applied to the magnetorquers. Delegating more control to the reaction wheel system effectively removed the large savings that could be achieved with Coil Factors. It was still found however, that some small energy savings could be made at the $R$ matrices that already produced low energy. Extra coils may be exceptionally useful when the abilities and efficiency of the magnetorquer’s designs improve over time.

9.1.2 Missing Wheels

In an effort to test the redundancy that hybrid magnetic attitude control can provide, select wheels were simulated as if a failure had occurred. The results found that stability and disturbance mitigation could be achieved with a single wheel failure and the appropriate selection of weightings for $Q$ and $R$. This is of great significance as it proves that redundancy is possible. When more than one wheel fails, however, the system was found to become unstable.
9.1.3 Control Law Reconfiguration

In an effort to further study the capabilities of a hybrid magnetic control system, simulations were performed that took into account when a wheel or two wheels were removed. The reconfiguration of the control law appropriately redistributed the control torques to accommodate for the wheel that was not present. It was found that stability could be achieved but the energy consumption was higher. This perhaps suggests that extra coils may prove to be more useful in these scenarios at reducing the energy consumption as now they are required to provide more control to achieve stability.

9.1.4 Control Saturation

These tests implemented the effects of control saturation that would be seen if the physical constraints of the spacecraft’s actuators were enforced. Restricting the magnitude of the control dipole was often found to degrade control performance. Careful weighting of the $\mathbf{R}$ matrix can be found to achieve stability even with control saturation taking effect. Stability results are shown with specific cases where one wheel and two wheels have been removed from the control law, thus increasing the reliance on the magnetorquers and the effect of control saturation. In both cases the spacecraft was able to recover from tumbling motion, a possible consequence of the maximum control dipole being enforced, and was effectively able to mitigate disturbances.

9.1.5 Orbital Inclination Tests

This section tested the abilities of the hybrid magnetic attitude controller at different inclinations. The controller performed well as far away as $10^\circ$ away from an ideal near-polar orbit. It was clear that for the inclinations tested, that there was sufficient variability in the Earth’s magnetic field so that the satellite was able to be stabilized and mitigate disturbances. Furthermore, in an effort to test how robust the controller is, incorrect
inclinations were simulated. The controller was given an ideal orbital inclination which it used to solve the Riccati equation and determine $P(t)$. The satellite’s orbit was then placed as far as $10^\circ$ away from the desired inclination and stability was still achieved. This allows us to conclude that the controller is quite robust and is able to handle these uncertainties with the magnetic field and inclination without compromising the quality of control.

9.1.6 Final Thoughts

Although the hybrid magnetic attitude controller did not show significant energy savings, it does show promise in its ability to provide redundancy and robustness with respect to uncertainties in orbital inclination and the magnetic field. Its ability to provide stability despite a single wheel failure provides a simple solution to a familiar and difficult issue. Furthermore, the addition of extra coils to reduce energy may be more useful in the reconfiguration case where there is a higher demand on the actuators to make up for the missing control torques.
Bibliography


