A Configurable B-Spline Parameterization Method for Structural Optimization of Wing Boxes

by

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A dissertation submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Aerospace Science and Engineering
University of Toronto

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Abstract

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This dissertation presents a synthesis of methods for structural optimization of aircraft wing boxes. The optimization problem considered herein is the minimization of structural weight with respect to component sizes, subject to stress constraints. Different aspects of structural optimization methods representing the current state-of-the-art are discussed, including sequential quadratic programming, sensitivity analysis, parameterization of design variables, constraint handling, and multiple load treatment. Shortcomings of the current techniques are identified and a B-spline parameterization representing the structural sizes is proposed to address them. A new configurable B-spline parameterization method for structural optimization of wing boxes is developed that makes it possible to flexibly explore design spaces. An automatic scheme using different levels of B-spline parameterization configurations is also proposed, along with a constraint aggregation method in order to reduce the computational effort. Numerical results are compared to evaluate the effectiveness of the B-spline approach and the constraint aggregation method. To evaluate the new formulations and explore design spaces, the wing box of an airliner is optimized for the minimum weight subject to stress constraints under multiple load conditions. The new approaches are shown to significantly reduce the computational time required to perform structural optimization and to yield designs that are more realistic than existing methods.
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Chapter 1

Introduction

1.1 Background

What is optimization? According to Merriam–Webster’s Collegiate Dictionary, optimization is defined as:

“an act, process, or methodology of making something (as a design, system, or decision) as fully perfect, functional, or effective as possible; specifically: the mathematical procedures (as finding the maximum of a function) involved in this”

In mathematics, optimization consists in finding the maximum or minimum of a function with respect to a set of variables. In engineering, optimization is supposed to find the best (i.e., the optimal) design, but in a practical context, optimization tools are often used simply to find an improved or better design.

With the rapid development of computer hardware, numerical analysis methods in many disciplines became mature and were accepted by industry. For example, computational fluid dynamics (CFD), which mainly employs the finite-difference method (FDM) and the finite-volume method (FVM), has been used widely for aircraft and automobile design. Computational structural mechanics (CSM), which mainly applies the finite-element method (FEM), has been used widely for civil, aerospace, and automobile engineering [1, 2]. In the meantime, numerical optimization came to the stage and started to attract increasing interest from industry [3–5]. Consequently, design optimization was spread to different sectors and it caused
Section 1.1. Background

the emergence of new disciplines such as structural optimization (SO) [6], aerodynamic shape optimization (ASO) [7], and multidisciplinary design optimization (MDO) [8–14].

In particular, structural optimization includes sizing, shape, and topology optimization [13–21]. Sizing optimization consists in optimizing a structure with respect to finite-element thicknesses or cross-sectional areas with a fixed shape and layout; shape optimization consists in optimizing the shape of a structure with a fixed layout; and topology optimization consists in optimizing the layout. A given structural design optimization problem can involve a combination of one or more of these types of optimization techniques. Schmit [6] first presented a comprehensive methodology that integrated the finite-element method with mathematical programming methods to solve a nonlinear inequality-constrained structural design problem.

Because of the evolving complexity of modern aircraft, aircraft design inherently consists of many disciplines such as aerodynamics, structural analysis, propulsion, control and flight mechanics, as well as marketing and cost analysis. With the growth of the aerospace industry, much research in numerical analysis and optimization has been conducted. As a result, numerical analysis methods now usually play a more important role in design stages than experimental methods. For example, using CFD instead of wind tunnels, or using FEM instead of structural experiments, has become more commonplace. Furthermore, numerical optimization methods have become popular in academia and many attempts have been conducted to serve industry by developing the next generation of methods.

As stated by Nocedal and Wright [3], “people optimize and nature optimizes”. This statement indicates that human beings have a natural tendency to optimize. The history of commercial transport jets allows one to see what has been achieved so far, and to anticipate what might take place in the future. Kroo [22] showed a comparison of two aircraft models, an Airbus 340 and a Boeing 707, which allows one to see that the two configurations look very similar, although the A340’s performance is much better than the B707’s. The reason for not changing the basic configuration can be explained by risk.

However, upon comparison of the state-of-the-art commercial airliners, A380 and B787, with the first modern commercial airliner, DC-3 in 1933, drastic improvements have been made with respect to comfort, safety, convenience, efficiency, cost, performance, materials, etc. Therefore, developing design tools for aircraft design is the long-term mission of aero-
nautical researchers and engineers, and numerical optimization represents a powerful tool that can help fulfill this mission.

This dissertation, which focuses on the structural optimization of wing boxes, intends to develop an innovative method for the structural design optimization within a high-fidelity aero-structural design optimization framework, which has been developed at the University of Toronto Institute for Aerospace Studies (UTIAS) [23].

A wing box is a structure that takes most of the loads of an aircraft wing, and usually consists of spar shear webs, ribs, stringers, spar caps, rib caps, and skins. Figure 1.1 illustrates a finite-element model of a wing box with five spars, eight top stringers, bottom skin, spar caps, rib caps and fifteen ribs. Figures 1.2 to 1.5 show the spar shear webs and caps, ribs and caps, stringers, and skins, respectively. In these figures, the green color is used for spar caps, rib caps, and stringers.

There are two kinds of finite elements in this model: three-dimensional frames that are used for stringers, spar caps, and rib caps, and three-dimensional shells that are used for spars, ribs and skins. Thus, the design variables consist of thicknesses of shell elements and cross-sectional areas of three-dimensional frame elements. Usually, design variables are grouped by component such as a spar shear web or a spar cap.

Although the structural layout of the wing of commercial transport jets was reported by Sensmeier and Samareh to be mature [24], engineers are driven by their optimizing nature to work on improving design tools for pursuing better designs. Hence, the following questions are raised: What is the significance of this topic? How does one model the design of wing boxes? What are the challenges involved in the design? How is an optimum found? What kind of costs are involved? How does one measure the effectiveness of the optimization? How does one evaluate the methodology used? In order to answer the above questions as well as others, this dissertation starts with a literature review.

1.2 Literature Review

Structural optimization has been the target of extensive research in the last few decades, generating a vast body of literature. This literature review concentrates on structural optimization of wing boxes and other related structures as well as the use of B-spline paramet-
Section 1.2. Literature Review

Characterization in optimization. The time frame spans from 1960s to the present.

Figure 1.1: Wing box model.
Figure 1.2: Spar webs and spar caps.
Figure 1.3: Ribs and rib caps.
Figure 1.4: Stringers.
Figure 1.5: Top and bottom skins.
1.2.1 Optimization of Wing Boxes and Related Structures

The first phase of the development of structural optimization was in the 1960s and 1970s. Schmit [6] first optimized a three bar truss problem to illustrate the formulation of a structural optimization in 1960. In order to demonstrate the capability of structural optimization in more practical designs, Schmit [25] presented a more complex and realistic example consisting of the optimization of integrally stiffened waffle plates. The problem was formulated to minimize weight with respect to design variables and subject to inequality constraints. Orthotropic plate theory was employed to perform analysis, which could be viewed as an eigenvalue problem. The design variables were the thicknesses of the plate and stiffeners, as well as the spacing of the stiffeners. The inequality constraints were stress constraints, local buckling constraints, and lower and upper bounds of design variables.

Emero and Spunt [26] presented an engineering method for multi-rib wing box optimization subject to vertical shear and a unidirectional bending moment in 1966. Various rib configurations were studied and it was the first application of structural optimization to wing box design. Olhoff [27, 28] used a variational method as the analysis method to optimize vibrating circular plates in 1970 and rectangular plates in 1974. More plate bending problems and approaches can be found in the survey paper of Haftka and Prasad [29].

Schmit and Miura [30] proposed a new structural optimization system called ACCESS, and implemented it in wing structural problems. Truss idealization was used for representing the wing structure. Several approximation concepts were introduced, which included the design variable linking, reciprocal design variables, constraint deletion, explicit approximation for stresses and displacements, and the selective sensitivity concept. Due to the complexity of problems and limited computational resources, very few design variables (less than 10) were used. The major issue is the computational cost, which meant that these problems fell short of representing realistic engineering applications.

The second phase in the evolution of structural optimization took place in the 1980s and 1990s. During this period, approximation techniques were developed in order to tackle the challenge of reducing computational cost and more complex problems were studied. Schmit and Mehrinfar [31] presented a multilevel approach to minimize the weight of wing box structures with fiber-composite stiffened-panel components in 1982. Botkin [32] used
a combination of predetermined basis functions with coefficients, which were the design variables, to represent the shape of plate and shell structures. Refined triangle meshes of specific regions were generated for detailed finite-element analyses. This was the first implementation of design variable parameterization in shape optimization. Bennett and Botkin [33] developed an adaptive mesh generation and incorporated it with the shape optimization of plate and shell structures in 1985.

In 1990, Berkes [34] presented a variable reduction method using mathematical functions to represent the thickness distributions of wing boxes. Although the details of the mathematical functions were not disclosed, different combinations of mathematical functions were used and compared the results with those of the variable linking method. Bartholomew and Wellen [35] employed a finite-element model for structural optimization of the wing box of a modern airliner in 1990. As stated in the paper, the idea of design variable linking was used to reduce the size of the optimization problem. Other techniques, such as the stress-ratio method (SRM), Newton method, and reciprocal variable method were used to solve the problem. A software, STARS, was developed by the Royal Aerospace Establishment (RAE). There were 294 design variables in the problem, but the computational cost was not disclosed. Later, Bartholomew [36] presented the progress of MDO of aircraft wings in 1998. Other research and applications are cited in the survey paper of Vanderplaats [19].

The third phase, which is currently in progress, started in the 2000s. Schuhmacher et al. [37] applied MDO to the sizing of the wing box of a regional aircraft wing in 2002 using a panel model and a finite-element model with the MSC.Nastran SOL 200 software. The design variables were combined by a variable linking scheme that took into account constructive, manufacturing and numerical considerations. Some details of the design variables and constraints were presented in the paper. However, the details of the linking scheme were not disclosed. In the same year, Klimmek et al. [38] proposed a parametric thickness model for a rib-spar-skin wing box representing the thicknesses of skins, spars, and ribs. Bézier splines were used and the control points at the same spanwise stations were combined into a single design variable. A cubic distribution and a linear distribution in the spanwise direction were compared for the layout of two wing boxes. Few design variables were used in the course of optimization.

Martins et al. [23] used a rib-spar-skin wing box for the aero-structural optimization
of a supersonic business jet in 2004. The structural design variables were the thicknesses of the top and bottom skins in the spanwise direction using plate finite elements. Maute and Allen [39] implemented a three-dimensional rib-spar wing box for aero-structural optimization using a topology optimization approach. Kapania and Chun [40] optimized a structural wing box under a twist constraint that used a beam-type model and a conjugate gradient method. Krog and Tucker [41] presented the implementation of structural topology optimization of Airbus 380 wing box ribs, which provided an example of the improvement in practical designs. Moore et al. [42] presented an analysis driven design and optimization method for aircraft structures and implemented it on a wing box design which used a genetic algorithm and a coupled-hierarchical analysis method in 2006. Taylor et al. [43] proposed an evolutionary design process for structural optimization of wing boxes using finite element models. Three levels of design phase were supported by corresponding FE models. The design process was tested and implemented in a real design company.

In 2007, Piperni et al. [44] used MDO technologies for the design of a business jet at Bombardier Aerospace, which incorporated a CFD code, called KTRAN, and a FEM code, called TWSAP. KTRAN solves a modified transonic small disturbance (TSD) equation coupled with boundary layer calculations on the lifting surface and uses embedding Cartesian grids. TWSAP is a thin-walled model finite element analysis program. The detailed wing box structural model included ribs, spars, skins, stringers, and spar caps. The structural design variables were the skin thickness, stringer pitch, stringer area, front and rear spar thickness and cap area. The details of the design variables were not clearly described. Lee [45] et al. proposed an equivalent static load method to optimize a joint wing. Hüttner and Grosspietsch [46] presented an example wing box optimized by an approach to solve very large-scale problems. In 2008, Gomes and Suleman [47] optimized a reinforced wing box using the spectral level set method.

As part of the work towards this dissertation, Yu and Martins [48] proposed a B-spline parameterization method for structural optimization of wing boxes, which included examples of plate structures, plates with stiffeners, and rib-spar-skin wing boxes. This is the first implementation of B-spline parameterization in structural optimization towards wing box design. More recently, a new configurable B-spline parameterization method for structural optimization of wing boxes was also presented [49].
1.2.2 Optimization with B-spline Parameterization

A spline is a piecewise polynomial function. In its most general form, it consists of polynomial pieces of different orders. B-spline is an acronym for basis spline [50], which is used to describe a combination of weighted piecewise polynomial.

The B-spline technique [51] has been applied for several decades to define geometric shapes, and has been adopted in the parameterization for optimization problems. Examples of the successful implementation of B-splines in optimization problems include: Nemec and Zingg [52] whose work used a B-spline to represent the geometry of an airfoil with B-spline control points as design variables in aerodynamic shape optimization; Painchaud–Ouellet et al. [53], who used a non-uniform rational B-spline (NURBS) parameterization for airfoil shape optimization subject to thickness constraints; Schramm et al. [54], who used a nonuniform rational B-spline curves to model the geometry of thin-walled beam sections in structural shape optimization; Bailey [55], who employed B-splines to construct the shape functions of finite elements in structural topology optimization; and Kimmek et al. [38], who used Bézier splines — which are specific B-splines without internal knots — to model the thickness of spars, ribs and skins in aero-structural optimization. However, the advantages of B-splines were inadequately explored.

Other research that has used B-splines in high-fidelity multidisciplinary shape optimization can be found in the survey paper of Samareh [56]. The details of Bézier spline and B-spline techniques can be found in de Boor [50] and Prautzsch et al. [51]

1.2.3 Remarks on Current Status and Directions

Based on the above literature review, several remarks are in order. At the component level (ribs, spars, skins, stringers, caps), structural optimization has been implemented in real world design, for example, the leading edge ribs were optimized for the inboard outer fixed leading edge of the A380 wing box [41].

However, the solution of such optimization problems is in its infancy, which motivates the application of structural optimization to other components, and more importantly, the overall structure. The industrial requirements for the solution of such problems is leading to an increasing number of research topics in this area, which are likely to be conducted by
commercial software companies and academia.

At the complete wing box level, structural or aero-structural optimization is in the developing stage, although an implementation of aero-structural optimization in industry was reported \[44\]. According to the statements of Renton et al. \[57\], the future perspective of airframes has three options: improved derivative models, all new conventional models, and revolutionary models. In each case, the methodology of structural optimization is fundamentally in demand.

Thus, the current status of wing box structural optimization shows that the methodology has not evolved enough to address the future industry requirements. The drawbacks of the current state-of-the-art include: small numbers of design variables, incomplete wing box models, high computational costs, lack of robustness of the algorithms, difficult and inflexible implementation, and lack of verification and validation.

### 1.3 Objectives

The goal of this dissertation is to present an innovative methodology for structural design optimization of wing boxes, which will facilitate the preliminary design of aircraft.

According to Mohaghegh \[58\] from Boeing, the challenge in design of airframe structures could be summarized as finding an optimal solution satisfying the competing requirements for safety, performance, and cost.

Due to the nature of academic work, this research concentrates on the major requirements, which are the minimum weight for performance and stress constraints for safety. Although structural materials have evolved from aluminum alloys to composite materials \[59\], the future role of these materials in aerospace applications is still uncertain \[57\]. However, the layout of structures is dominantly a spar-rib-skin-stringer type. Thus, the design problems concentrate on wing boxes and their components or this type of layout.

The challenges inherent in the structural optimization of wing boxes arise from the complexity of the design space. Of these challenges, one of the most prominent is the existence of multiple local optima. Also, the complexity of component configurations leads to a large dimensionality of designs, and the need to consider multiple load conditions corresponding to various flight conditions and maneuvers, add significant complexity to the problem.
Sizing optimization is chosen to address these challenges because it is the main design procedure in wing box design. Topology optimization and shape optimization are beyond the scope of this dissertation. The objectives of this dissertation were decided based on the careful consideration of the balance between relevance, complexity and feasibility, and are as follows:

1. Formulate a general structural optimization problem for wing boxes
2. Build up an efficient and robust methodology for structural optimization of wing boxes
3. Exploit the design method as well as design space to achieve the best design
4. Optimize wing boxes effectively under multiple load cases

1.4 Dissertation Layout

In this chapter, the background, literature review, and objectives have been presented. The rest of this dissertation starts with the description of related theory in Chapter 2, which includes the optimization algorithm, sensitivity analysis method, and B-spline parameterization method. Chapter 3 contains the description of the methodology from an optimization point of view, including the problem statement, the configurable B-spline parameterization method, the constraint aggregations and the handling of multiple load conditions. Chapter 4 presents numerical optimization results of selected structural design problems and discusses the results. Chapter 5 includes conclusions, contributions and recommendations.
Chapter 2

Background Theory

This chapter describes the background theory corresponding to the methods used in this dissertation which include: the sequential quadratic programming (SQP) optimization algorithm, the sensitivity analysis method, and the B-spline parameterization.

2.1 Sequential Quadratic Programming

Optimization — which is often referred to as mathematical programming — can be classified with respect to different aspects, such as unconstrained optimization versus constrained optimization; gradient-based optimization versus gradient-free optimization; continuous optimization versus discrete optimization; linear programming versus nonlinear programming; and deterministic versus stochastic [3]. Sometimes a mixed optimization method is used to find optima for problems being considered, for instance, if some variables are integers and other variables are not, the specific optimization is called a mixed integer optimization. Due to the nature of structural optimization of wing boxes, optimization employed in this dissertation focuses on numerical, continuous, gradient-based, nonlinear, constrained optimization.

There are many gradient-based optimization algorithms available, including conjugate gradient methods, quasi-Newton methods, and interior-point methods. When an optimization problem employs any gradient-based method, gradients or sensitivities are required.

Sequential quadratic programming (SQP) is a quasi-Newton method that can be used to
solve nonlinear constrained programming problems, i.e., optimization problems in which at least one of the functions involved is nonlinear with respect to the design variables. SQP solves problems with equality constraints, inequality constraints, or both. For example, the weight of a plate structure is a linear function with respect to the element thicknesses and the element stresses are nonlinear functions with respect to the element thicknesses. Therefore, a minimum weight problem subject to yield stress constraints is a nonlinear programming problem with inequality constraints.

SQP consists in solving the Karush–Kuhn–Tucker (KKT) conditions, which are the optimality conditions for constrained problems. SQP can be described as follows:

1. Employ slack variables to transform inequality-constraint conditions into equality-constraint conditions
2. Use Lagrangian multipliers to transform an equality-constrained problem into an unconstrained problem
3. Apply the KKT conditions to form a nonlinear system with respect to design variables
4. Use the Newton method to linearize this nonlinear system with respect to an approximation of design variables in an iterative fashion
5. Use a quasi-Newton method to solve the linear system by approximating the Hessian matrix of this linear system
6. Use a merit function to control the line search direction

Problem Statement

A general optimization problem is defined as

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{with respect to} & \quad x \\
\text{subject to} & \quad c_m(x) = 0, \quad m = 1, 2, \ldots, \hat{m} \\
& \quad c_n(x) \leq 0, \quad n = \hat{m} + 1, \hat{m} + 2, \ldots, \hat{m} + \hat{n}
\end{align*}$$

(2.1)

where $\hat{m}$ is the number of equality constraints and $\hat{n}$ is the number of inequality constraints.
Step 1

Denoting slack variables as \( s \), which are used to turn the inequality constraints into equality constraints, the constraints of the above problem are transformed into a form

\[
\begin{align*}
\text{\( s \) = \begin{bmatrix} 0 \\ s_n^2 \end{bmatrix}, \quad \text{\( c(x) = \begin{bmatrix} c_m(x) \\ c_n(x) \end{bmatrix} \)}}
\end{align*}
\]

where \( s_n^2 \) is used to ensure it is greater than or equal to zero.

Step 2 and 3

Defining Lagrangian function as \( \mathcal{L}(x, \lambda) = f(x) - \lambda^T (c(x) - s) \) and Lagrange multipliers \( \lambda \), the KKT conditions are

\[
\begin{align*}
\nabla_x \mathcal{L} &= 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial x} = \nabla_x f(x) - \nabla_x c^T(x) \lambda = 0 \\
\nabla_s \mathcal{L} &= 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial s_n} = 2\lambda_n s_n = 0, \quad n = \hat{m} + 1, ..., \hat{m} + \hat{n} \\
\nabla_\lambda \mathcal{L} &= 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \lambda_j} = -c_j(x) + s_j = 0, \quad j = 1, 2, ..., \hat{m} + \hat{n}
\end{align*}
\]

where \( \nabla_x c^T(x) = [\nabla_x c_1(x), ..., \nabla_x c_{\hat{m}+\hat{n}}(x)] \) is the transpose of the Jacobian and \( \nabla_x c_j \) is a column vector. The Lagrange multipliers are the rates of the changes of the optimal function value with respect to the changes in constraints, respectively.

Then, a nonlinear system is represented in matrix form as

\[
\mathcal{F}(x, \lambda) = \begin{bmatrix}
\nabla_x \mathcal{L} \\
\nabla_s \mathcal{L} \\
\nabla_\lambda \mathcal{L}
\end{bmatrix} = 0 \tag{2.4}
\]

Note that slack variables are appended to design variables to form a column vector that will be solved by the following step.
Section 2.1. Sequential Quadratic Programming

Step 4

Using the Newton’s method to linearize the above nonlinear system with \( x \) and \( \lambda \) as the variables, the following linear system can be obtained:

\[
\begin{bmatrix}
\nabla_{xx} \mathcal{L}_k & \nabla_{x\lambda} \mathcal{L}_k \\
\nabla_{\lambda x} \mathcal{L}_k & \nabla_{\lambda\lambda} \mathcal{L}_k \\
\end{bmatrix}
\begin{bmatrix}
x_{k+1} - x_k \\
\lambda_{k+1} - \lambda_k \\
\end{bmatrix}
= - \begin{bmatrix}
\nabla_x \mathcal{L}(x_k) \\
\nabla_{\lambda} \mathcal{L}(x_k) \\
\end{bmatrix}
\tag{2.5}
\]

It yields,

\[
\begin{bmatrix}
\nabla_{xx} \mathcal{L}_k & - \nabla_x c(x_k) \\
- \nabla_x c^T(x_k) & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_{k+1} - x_k \\
\lambda_{k+1} - \lambda_k \\
\end{bmatrix}
= \begin{bmatrix}
- \nabla_x f(x_k) + \nabla_x c^T(x_k) \lambda_k \\
- \nabla_x f(x_k) + \nabla_x c^T(x_k) \lambda_k - s \\
\end{bmatrix}
\tag{2.6}
\]

where \( \nabla_{xx} \mathcal{L}_k \) is the Hessian matrix of second-order partial derivatives of the Lagrangian function.

A quadratic problem (QP) approximation is used to model the above problem (2.1) as follows:

\[
\begin{align*}
\text{minimize} & \quad f_k + \nabla_x f_k^T p + \frac{1}{2} p^T \nabla_{xx} \mathcal{L}_k p \\
\text{with respect to} & \quad p \\
\text{subject to} & \quad \nabla_x c^T_m(x_k) p + c_m(x) = 0, \quad m = 1, 2, \ldots, \hat{m} \\
& \quad \nabla_x c^T_n(x_k) p + c_n(x) \leq 0, \quad n = \hat{m} + 1, \hat{m} + 2, \ldots, \hat{m} + \hat{n}
\end{align*}
\tag{2.7}
\]

where the Lagrangian function involved in the quadratic term is \( \mathcal{L}_k(x, \lambda) = f_k(x_k) - \lambda_k^T c(x_k) \) and \( x_k \) and \( \lambda_k \) are the \( k^{th} \) iteration of the design variables \( x \) and the Lagrange multipliers \( \lambda \), respectively.

The Lagrangian function of this QP is denoted with the following operator,

\[
\hat{\mathcal{L}} = f_k + \nabla_x f_k^T p + \frac{1}{2} p^T \nabla_{xx} \mathcal{L}_k p - \hat{\lambda}^T (\nabla_x c^T(x_k) p + c - s)
\tag{2.8}
\]

Using the above equation, and applying optimality conditions, a linear system is formed as follows:

\[
\nabla_p \hat{\mathcal{L}} = 0 \Rightarrow \frac{\partial \hat{\mathcal{L}}}{\partial p} = \nabla_x f_k(x_k) + \nabla_{xx} \mathcal{L}_k p - \nabla_x c^T(x_k) \hat{\lambda} = 0
\]

\[
\nabla_s \hat{\mathcal{L}} = 0 \Rightarrow \frac{\partial \hat{\mathcal{L}}}{\partial s_n} = \hat{\lambda}_n s_n = 0, \quad n = \hat{m} + 1, \ldots, \hat{m} + \hat{n}
\tag{2.9}
\]

\[
\nabla_\lambda \hat{\mathcal{L}} = 0 \Rightarrow \frac{\partial \hat{\mathcal{L}}}{\partial \lambda_j} = - \nabla_x c_j^T(x_k) p - (c_j - s_j) = 0, \quad j = 1, 2, \ldots, \hat{m} + \hat{n}
\]
which is identical to that obtained by applying the Newton–KKT approach where \( p \) is \( x_{k+1} - x_k \) and \( \hat{\lambda} \) is \( \lambda_{k+1} \) in equation (2.6).

## 2.2 Sensitivity Analysis Methods

A sensitivity is the rate of change of the output of a system with respect to the input, at a specific state. Since the state varies, sensitivity is a function that represents the values of the sensitivities at different states. Furthermore, whether the input is a single variable or a vector of variables, or whether the output is a scalar function or a vector of functions depends on the problem at hand. There are several ways of performing sensitivity analyses: analytic differentiation, numerical differentiation, and automatic differentiation.

In some cases sensitivities can be calculated analytically, which means the exact explicit function is known and it allows one to derive the sensitivity of that known function with respect to one or more design variables. This can be facilitated by software capable of symbolic differentiation. Without an explicit expression for the function, it is impossible to obtain the sensitivities through analytical differentiation.

Most commonly, numerical differentiation methods, such as the finite-difference methods, are used to obtain sensitivities with a truncation error at the discretization. However, these methods are subject to subtractive cancellation, which makes the accuracy of these methods unpredictable.

Semi-analytical-numerical methods achieve higher accuracy and efficiency compared to numerical differentiation. There are two main types of semi-analytical methods: direct methods and adjoint methods.

Automatic differentiation is based on the assumption that all the functions calculated in a computer code is a sequence of basic functions and the chain rule is applied to those basic functions. Instead of calculating sensitivities of a function of interest with respect to a variable at the top level, it calculates sensitivities of the basic functions involved in the function of interest with respect to the variable at a low level and makes a lookup table to reduce computational cost. It can also avoid numerical errors such as round-off errors for a complicated function under some circumstances.
2.2.1 Finite-Difference Method

The simplest and most popular way of estimating the total sensitivity of any function with respect to design variables is the finite-difference method. This method is convenient because it is not necessary for one to know the form of the function; that is, the analysis can be considered to be a black box. For example, the weight of a structure is a simple function and the stress of a structure is only available through the discretization of the continuous problem using the finite-element method, solving governing equations of the finite-element nodes for each degree of freedom and then calculating the stresses.

Defining \( \mathbf{x} \) as a vector of the design variables, \( x_n \) as the \( n^{th} \) design variable of \( \mathbf{x} \), and \( I \) as a function of interest, the simplest finite-difference formula to approximate the sensitivity of \( I \) is given by the forward difference formulation

\[
\frac{dI(x)}{dx_n} = \frac{I(x + h_n) - I(x)}{h} + O(h) \tag{2.10}
\]

or by the centered difference formulation

\[
\frac{dI(x)}{dx_n} = \frac{I(x + h_n) - I(x - h_n)}{2h} + O(h^2) \tag{2.11}
\]

where \( h_n \) represents the vector of stepsizes corresponding to \( x_n \) as follows:

\[
h_n = [0, ..., h, ..., 0]^T, \quad n = 0, 1, ..., \hat{n}
\]

and \( h \) is the step size of design variables and \( \hat{n} + 1 \) is the number of design variables.

2.2.2 Complex Step Method

A relatively new method that can also be conveniently implemented is the complex-step derivative approximation \[^{[60][62]}\]. Using this method, a second-order accurate estimate of the first derivative is given by

\[
\frac{dI(x)}{dx_n} = \frac{\text{Im}\{I(x + i h_n)\}}{h} + O(h^2) \tag{2.13}
\]

where \( i = \sqrt{-1} \) and \( \text{Im} \) represents the imaginary part of the function of interest \( I \).
Section 2.2. Sensitivity Analysis Methods

2.2.3 Direct Method

Due to the high computational cost and iterative nature of aero-structural analysis, it is not computationally efficient to use the above methods, especially when a large number of design variables are required. Since a new function evaluation is necessary for each design variable, the overall computational cost of performing a sensitivity analysis can be prohibitive. Performing gradient-based optimization is then not computationally practical.

In order to decrease the computational cost of calculating total sensitivities, analytic methods are employed. This class of methods applies analytic partial derivatives and the chain rule to obtain sensitivity equations.

The governing equations for a system’s analysis can be written as

\[ R(x, y(x)) = 0 \]  \hspace{1cm} (2.14)

where \( y \) is a state variable vector and \( R \) is a vector of disciplinary governing equations.

The total sensitivity of the function of interest \( I \) is

\[ \frac{dI}{dx_n} = \frac{\partial I}{\partial x_n} + \frac{\partial I}{\partial y} \frac{dy}{dx_n} \]  \hspace{1cm} (2.15)

According to the chain rule and assuming the governing equations are continuous, the total sensitivity of the governing equations can be written as

\[ \frac{dR}{dx_n} = \frac{\partial R}{\partial x_n} + \frac{\partial R}{\partial y} \frac{dy}{dx_n} = 0 \]  \hspace{1cm} (2.16)

In order to be solved for the total derivative of \( I \), the above equation (2.16) can be rewritten as

\[ \frac{\partial R}{\partial y} \frac{dy}{dx_n} = - \frac{\partial R}{\partial x_n} \]  \hspace{1cm} (2.17)

The direct method consists in solving equation (2.17) and then substituting the result into equation (2.15), i.e.,

\[ \frac{dI}{dx_n} = \frac{\partial I}{\partial x_n} - \frac{\partial I}{\partial y} \left[ \frac{\partial R}{\partial y} \right]^{-1} \frac{\partial R}{\partial x_n} \]  \hspace{1cm} (2.18)

2.2.4 Adjoint Method

The adjoint method consists in solving the following equations for the adjoint vector, \( \psi \), which is

\[ \left[ \frac{\partial R}{\partial y} \right]^T \psi = - \left[ \frac{\partial I}{\partial y} \right]^T \]  \hspace{1cm} (2.19)
Solving the above equation and substituting the solution into total sensitivity (2.15), the following equation is obtained:

\[
\frac{dI}{dx_n} = \frac{\partial I}{\partial x_n} + \psi^T \frac{\partial R}{\partial x_n}
\] (2.20)

If the number of functions of interest is much larger than the number of design variables, the direct method is more efficient than the adjoint method because \(d\mathbf{y}/dx\) can be used for all functions of interest. On the other hand, if the number of functions of interest is much smaller than the number of design variables, the adjoint method is more useful because the adjoint vector \(\psi\) can be used for any design variable.

However, if both the number of functions of interest and the number of design variables are large, there is no known efficient way of computing sensitivities. For further details on structural sensitivity analysis the reader is referred to Chio and Kim [63].

### 2.2.5 Selection of Sensitivity Analysis Methods

When selecting a sensitivity analysis method, the following considerations are applied: the availability of the method, accuracy, implementation cost, and computational cost. The selection depends on the situation. A rule of thumb is to select a method which can be easily implemented in an affordable manner. For example, if the major concern is to obtain optimization results with short development times, then the finite-difference method and complex-step methods are favorable. However, if the major concern is computational cost, the direct or adjoint methods is a better choice. Automatic differentiation is very useful for a large well-organized code. It is important to note that the accuracy of the sensitivity analysis is no better than the accuracy of the analysis code in general.

In this dissertation, the finite-difference and adjoint methods are employed to perform the sensitivity analysis, partly because the primary interest of the research is the optimization not the sensitivity analysis, and also because the implementation of these methods is readily available and affordable.
2.3 B-Spline

A B-spline is a generalized Bézier spline formulated in terms of basis splines and knot vectors \[50, 51\]. It can be written as

\[ f(u) = \sum_{i=0}^{n} C_i N_i^p(u) \]  

(2.21)

where \( C_i \) is a control point or de Boor point, \( n + 1 \) is the number of control points, and \( N_i^p(t) \) is the \( i^{th} \) basis spline function defined recursively as

\[ N_0^0(u) = \begin{cases} 1 & \text{if } u \in [a_i, a_{i+1}) \\ 0 & \text{otherwise} \end{cases} \]  

(2.22)

\[ N_i^p(u) = \frac{u - a_i}{a_{i+p} - a_i} N_{i-1}^p(u) + \frac{a_{i+p+1} - u}{a_{i+p+1} - a_{i+p}} N_{i+1}^{p-1}(u) \]  

(2.23)

where \( p \) is the degree of B-spline, \( \mathbf{a} = [a_0, \ldots, a_i, \ldots, a_m]^T \) is the knot vector, and \( a_i \) is a knot. The degree of B-spline must be lower than the number of control points, i.e., \( p \leq n \). Knots can be any vector of real numbers with a strictly increasing sequence. If multiple knots, or coincided knots exist, the basis B-splines \( N_i^p(u) \) are still defined by equation (2.22) and the convention is as follows:

\[ N_i^{r-1} = \frac{N_i^{r-1}}{a_{i+r} - a_i} = 0, \quad \text{if } a_i = a_{i+r} \]  

(2.24)

Figure 2.1 shows two examples of basis B-splines with a uniformly spaced knot vector for four control points. The basis B-splines of degree two and three are shown in Figure 2.1(a) and Figure 2.1(b), respectively. The basis B-splines are positive and symmetric. From Figure 2.1(a), it is observed that at the left half interval, the curve is defined by the first three control points and at the right half interval, the curve is determined by the last three control points. This means that B-splines hold a property of local modifications in contrast with Bézier splines which provides a more flexible control in design. From Figure 2.1(b), the basis B-spline of degree three for four control points is identical to the Bézier spline of degree three that is to say, B-splines contain Bézier splines.

The knot vector \( \mathbf{a} \) for an open B-spline is

\[ \mathbf{a} = [0, \ldots, 0, a_{p+1}, \ldots, a_{p+j}, \ldots, a_n, 1, \ldots, 1]^T, \quad a_{p+j} \in [0, 1] \]  

(2.25)
This can be specified to construct a non-periodic and uniformly spaced B-spline by the following equation:

\[ a_{p+j} = \frac{j}{n-p+1}, \quad \text{for} \quad j = 1, 2, \ldots, n-p \] (2.26)

Alternatively, it can also be generated by the de Boor averaged method to construct a non-periodic and non-uniform B-spline as follows:

\[ a_{p+j} = \frac{1}{p} \sum_{i=j}^{j+p-1} t_i, \quad \text{if} \quad j = 1, 2, \ldots, n-p \] (2.27)

The global parameter vector \( \mathbf{u} \) of control points for a one-dimensional B-spline is used for calculating the local parameter vector \( \mathbf{t} \) as follows:

\[ t_i = \frac{u_i - u_0}{u_n - u_0}, \quad i = 0, 1, \ldots, n \] (2.28)

A two-dimensional B-spline is a B-spline with respect to two parameters, which is often used as a surface representation. For two-dimensional B-splines, the global parameter vectors, \( \mathbf{u} \) and \( \mathbf{v} \), are used and the coefficient matrix \( \mathbf{N} \) is obtained by multiplying two scalar coefficients \( N_p^i(\mathbf{u}) \) and \( N_q^i(\mathbf{v}) \) for each component.

A non-uniform rational B-spline (NURBS) is a generalized B-spline with weighted control points. The original motivation for NURBS was to provide the capability of representing circles, ellipses and other curves that cannot be represented by polynomials. A NURBS uses homogeneous coordinates instead of Cartesian coordinates to construct a curve and can be written as follows:

\[ f(u) = \frac{\sum_{i=0}^{n} w_i C_i N_p^i(u)}{\sum_{i=0}^{n} w_i N_p^i(u)} \] (2.29)

where \( w_i \) is the weight of control point \( i \). If \( w_i = 1 \) for \( i = 0, 1, \ldots, n \), a NURBS is identical to a B-spline because the denominator is 1.

Two examples are given to illustrate the use of B-splines. Example A relates to a known function defined as

\[ f(x) = \frac{1}{30} x^3 + \sin(3x - 2), \quad x \in [1, 4] \] (2.30)
Section 2.3. B-Spline

Figure 2.1: Basis B-splines.

Figure 2.2 compares the B-spline and control polygonal curve for this function with the uniformly spaced and de Boor averaged knot vectors. There are ten control points used to represent 50 data points for the function. The red curve represents the given function \( f(x) \), the green curve represents the B-spline of degree three and the dots are the data points being represented, the blue line and squares represent the control polygonal curve and the control points, respectively. The left plot is for the uniformly spaced knot vector and the right plot is for the de Boor averaged knot vector. From Figure 2.2, it is noted that the B-spline curve is very close to the known function in both cases. It is worth mentioning that the quality of approximation is based on a specific known function.

Example B is given to explore the behavior of B-splines, namely the well-known Runge’s phenomenon, which is a problem that occurs when using polynomial interpolation with polynomials of high degree. The curve considered is

\[
    f(x) = \frac{1}{1 + 25x^2}
\]  

(2.31)

Figures 2.3 and 2.4 show three B-splines of different degrees with 100 data points using 14
Section 2.3. B-Spline

Figure 2.2: Example A: Comparison of B-spline representations for function (2.30).

(a) Uniformly spaced knot vector
(b) de Boor averaged knot vector

Figure 2.3: Example B: B-spline representation for function (2.31) using 14 control points.
Section 2.3. B-Spline

It is observed that a better fit is obtained with the increase in the number of control points. But the oscillation exists in both cases. It is also noticed that the quadratic B-spline fits better than the linear or cubic B-spline for this specific function.

Therefore, the representation of a B-spline depends on the selection of the degree, control points, and knot vector. The quality of approximation of a B-spline and a represented function depends on the case. When applying B-splines to structural optimization, it is worth exploring the effects of various options because the quality of approximation of B-spline representations is unknown unless optima are found. The details of using B-splines are described in the following chapter.
Chapter 3

Methodology

In this chapter, a mathematical model is presented that provides a formal description of the design optimization problem in its most general sense. Different aspects of optimization techniques are investigated and several approaches are presented, including the configurable B-spline parameterization, the automatic B-spline scheme, constraint aggregation and element stress aggregation techniques, as well as the handling of multiple load conditions.

3.1 Optimization Model Construction

3.1.1 Problem Statement

A common structural optimization problem consists in minimizing the weight of a structure subject to stress constraints and sizing bounds. The problem can be written as follows:

\[
\begin{align*}
\text{minimize} & \quad W(x) \\
\text{w.r.t.} & \quad x \\
\text{s.t.} & \quad g \leq 0 \\
& \quad x \geq x_l \\
& \quad x \leq x_u \\
& \quad Kd_j = f_j, \quad j = 1, 2, ..., \hat{j}
\end{align*}
\]
where \( W(\mathbf{x}) \) represents the weight of the structure; \( \mathbf{x} \) represents the vector of design variables, \( \mathbf{x}_l \) is the vector of lower bounds for these variables and \( \mathbf{x}_u \) is the vector of upper bounds; \( \mathbf{K} \) is the stiffness matrix given by the finite element method; \( \mathbf{f}_j \) is the external force vector for the \( j^{th} \) load case; and \( \mathbf{d}_j \) is the corresponding deformation of the structure, where \( \hat{j} \) is the maximum number of considered load cases; \( \mathbf{g} \) is a set of constraints used to represent the stress constraints, which is discussed in the following sections.

### 3.1.2 Analysis Models

The objective function in the present work is the weight of the structure. In aircraft design, the weight calculation can be so complicated that a group of people are employed to make it as accurate as possible. Here, a simple model is used to calculate the weight by multiplying the density of material and the volume of the structure as follows:

\[
W = \hat{\rho}V \tag{3.2}
\]

where \( \hat{\rho} \) is the density of material and \( V \) is the volume of the structure.

The structural analysis is performed by a finite-element method that solves the following governing equations:

\[
\mathbf{K}\mathbf{d}_j = \mathbf{f}_j, \quad j = 1, 2, ..., \hat{j} \tag{3.3}
\]

The finite element models use three-dimensional shell elements for plates such as spars shear webs, ribs shear webs, and skin. Three-dimensional frame elements are used to model longitudinal stringers, vertical stringers, spar caps and rib caps.

The stress in the optimization problem constraints is the von Mises stress of each element. The details of the stress computation are presented in Appendix A.

### 3.1.3 Design Variables

The design variables are divided into two groups: the original design variables, which include shell thicknesses and frame cross-sectional areas, and the B-spline control point design variables, which represent the original design variables. Several design variable handling approaches are used:
**Individual design variable (IDV) method:** This treats each original design variable separately and is a one-to-one association of the design variables to the finite element sizes.

**Variable linking method:** This links the individual design variables in groups. For example, one might associate only one variable to the thickness of a rib, by linking the values for all the thicknesses corresponding to the elements of that rib.

**B-spline parameterization method:** This uses B-spline to represent the variation of the element sizes. The size of a particular element is usually the value of the B-spline at the midpoint of the element. The control points become the design variables. This makes the number of design variables independent of the finite-element discretization, and also tends to smooth out the variation of sizes across elements.

For a given finite-element discretization, the individual design variable method provides the highest degree of design freedom but is also the most computationally expensive. The variable linking method is more constrained, but reduces the computational cost. The B-spline parameterization method has the capability of compromising between high degree of design freedom and computational cost. Furthermore, the B-spline parameterization method is equivalent to the individual design variable method when using constant B-splines. Figure 3.1 compares the three methods in different levels of finite-element models.

![Figure 3.1: Comparison of design variable handling methods.](image-url)
3.1.4 Constraints

There are three types of constraints in the stated optimization problem \(3.1\). These are
design variable bounds (lower bounds and upper bounds of element sizes), stress constraints
that are required to prevent structural failure, and the governing equations.

The design variable bounds are obtained from manufacturing limits or reasonable di-

mensional limits. These bounds do not incur a significant computational cost, since the
optimization algorithm is able to handle these easily.

The stress constraints can be handled with different approaches:

1. Individual stress constraints (ISC): This applies a constraint to the stress of each
   element.

2. Maximum stress constraint (MSC): This constrains the maximum stress of a group of
   elements.

3. Kreisselmeier–Steinhauser (KS) aggregation: This aggregates the stresses of a group
   of elements using a KS function. A parameter \(\rho\) determines how close the aggregation
   is to the original constraints. An adaptive KS function (AKS) employing a strategy to
   automatically control the selection of \(\rho\) is also possible.

The governing equations are solved for all loading conditions at each optimization it-
eration. The structural analyses in this work are performed using FEAP, a finite-element
analysis program developed by Taylor \[64\]. FEAP is a general purpose finite-element analy-
sis program, which was developed for research and educational use. There are many useful
features such as: support for multiple operating systems (Windows, Linux/Unix, Mac), in-
terface to MATLAB, interfaces to different linear equation solvers, and it is well documented.
The details of FEAP can be found on its official website \[64\].

3.1.5 Optimization Method

A sequential quadratic programming (SQP) method package, called SNOPT, is employed
to solve the above structural optimization problem \[65\]. SNOPT, which was developed
by Gill, Murray and Saunders, is a software package for solving large-scale optimization
problems (linear and nonlinear programs). It is especially effective for nonlinear problems whose functions and gradients are computationally expensive. The details of SNOPT can be found on the Stanford Optimization Laboratory webpage [66].

3.2 Configurable B-Spline Parameterization

3.2.1 Introduction

Data from any source is usually represented in three forms: graphical, tabular, and functional. A mathematical model defines a representation of such forms using a convenient algebraic formulation or a complicated system, which can also be viewed as a black box and implemented as a set of computer codes.

For example, an \( n^{th} \) degree polynomial is a mathematical model that represents data points. Theoretically, the coefficients are uniquely defined if and only if there are \( n + 1 \) data points. However, finding the coefficients is difficult for high degree polynomials, as it requires solving a large system of linear equations, as well as computing expensive high-degree terms. Even for a small degree polynomial, numerical difficulties exist for many real world data that are caused by solving the ill-conditioned linear systems.

Curve fitting aims to find the “best-fit” that can be realized by using methods such as splines and B-splines, which split the data points into small segments with low-degree splines and constrain the known data points. Least squares fit can also be used. This minimizes the sum of squared difference between the known data points and unknown data points. In a least squares fit, the number of known data points usually is larger than that of the unknown data points. However, least squares fit inherently consists of an optimization problem that may incur complexity when being implemented in design optimization.

The challenges of large scale optimization problems are related to the computational cost and the uncertainty of finding a better optimum, assuming the problems themselves are well-posed.

Parameterization of design variables is a common method used to address this challenge. There are two origins for the parameterization of design variables. One is variable linking, which was previously mentioned. This is an engineering-oriented approach to reduce the
number of design variables in order to reduce the computational cost. The other origin relies on data fitting.

There are many methods to represent the design variables such as polynomials, splines, Bézier splines, B-splines, and NURBS curves. Among these methods, only the B-spline technique strikes a balance between power and convenience for optimization applications.

Instead of using individual design variables separately, the design variables are determined by the value of a B-spline at a particular location. The main advantage of this approach is that it allows one to vary the number of control points to decide on how much fidelity to have in the parameterization, independent of the discretization of the finite-element model.

However, the optimization results vary with the configuration of B-splines. The questions are: how to determine a suitable B-spline, what is the effect of a predetermined B-spline on optimal design, and what is the best selection of B-splines for a complex problem such as wing box optimization. The methods are described in the following sections and the results are presented in Chapter 4.

### 3.2.2 B-Spline Parameterization

#### One-Dimensional Formula

A B-spline which represents a group of design variables can be written in matrix form as the following:

\[
\mathbf{x} = \mathbf{N} \mathbf{c}
\]  

(3.4)

where \( \mathbf{c} = [c_0, c_1, ..., c_i]^T \) is the vector of control points, \( \mathbf{x} = [x_0, x_1, ..., x_j]^T \) is the vector of element thicknesses or element cross-sectional areas, and \( \mathbf{N} \) is the coefficient matrix as

\[
\mathbf{N} = \begin{bmatrix}
N_{00} & N_{01} & ... & N_{0}\hat{i} \\
N_{10} & N_{11} & ... & N_{1}\hat{i} \\
... & ... & ... & ... \\
N_{j0} & N_{j1} & ... & N_{j}\hat{i}
\end{bmatrix}
\]  

(3.5)

For a one-dimensional B-spline,

\[
N_{ji} = N_i^p(u_j), \quad j = 0, 1, ..., \hat{j} \quad \text{and} \quad i = 0, 1, ..., \hat{i}
\]  

(3.6)
where \( p \) is the degree of B-spline. Usually the number of control points \( \hat{i} + 1 \) is less than the number of design variables \( \hat{j} + 1 \). The global parameter vector \( \mathbf{u} \) of control points for a one-dimensional B-spline is used for calculating the local parameter vector \( \mathbf{t} \) as follows:

\[
t_i = \frac{u_i - u_0}{u_i - u_0}, \quad i = 0, 1, ..., \hat{i} \tag{3.7}
\]

where \( \mathbf{u} = [u_0, u_1, ..., u_{\hat{i}}]^T \) uses the Cartesian coordinates of elements.

Two-Dimensional Formula

For a two-dimensional B-spline, two global parameter vectors, \( \mathbf{u} \) and \( \mathbf{v} \), are used and the coefficient matrix \( \mathbf{N} \) is obtained by multiplying two scalar coefficients \( N_i^p(u) \) and \( N_j^q(v) \) for each component. Assuming the following conditions:

1. a set of \( \hat{i} + 1 \) rows and \( \hat{j} + 1 \) columns control points \( c_{i+j(i+1)} \) where \( i = 0, 1, ..., \hat{i} \) and \( j = 0, 1, ..., \hat{j} \)

2. a global parameter vector \( \mathbf{u} = [u_0, u_1, ..., u_{\hat{r}}]^T \), a degree \( p \) in the \( u \)-direction, and a knot vector of \( \hat{r} + p + 2 \) knots in the \( u \)-direction

3. a global parameter vector \( \mathbf{v} = [v_0, v_1, ..., v_{\hat{k}}]^T \), a degree \( q \) in the \( v \)-direction, and a knot vector of \( \hat{k} + q + 2 \) knots in the \( v \)-direction

the detailed form of a two-dimensional B-spline is defined as

\[
\begin{bmatrix}
    x_0 \\
    x_1 \\
    ... \\
    x_{\hat{r}+\hat{k}(\hat{r}+1)} \\
\end{bmatrix} =
\begin{bmatrix}
    N_i^p(u_0) N_j^q(v_0) & N_i^p N_j^q & ... & N_i^p N_{i-1}^q N_j^q & N_i^p(u_0) N_j^q(v_0) \\
    N_i^p(u_1) N_j^q(v_0) & N_i^p N_j^q & ... & N_i^p N_{i-1}^q N_j^q & N_i^p(u_1) N_j^q(v_0) \\
    ... \\
    N_i^p(u_{\hat{r}}) N_j^q(v_k) & N_i^p N_j^q & ... & N_i^p N_{i-1}^q N_j^q & N_i^p(u_{\hat{r}}) N_j^q(v_k) \\
\end{bmatrix}
\begin{bmatrix}
    c_0 \\
    c_1 \\
    ... \\
    c_{i+j(i+1)} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    c_0 \\
    c_1 \\
    ... \\
    c_{i+j(i+1)} \\
\end{bmatrix}
\]

(3.8)
3.2.3 Sensitivity Analysis for B-Spline Parameterization

When using the control points as the design variables, a function of interest can be represented as

\[ I = I(\mathbf{x}) = I(N\mathbf{c}) \quad (3.9) \]

where the B-Spline coefficient matrix \( N \) is independent of the vector of control points \( \mathbf{c} \) and depends on the global parameters, knot vector and degree of B-spline. Then, the sensitivities of \( I \) with respect to the control points are

\[
\left[ \frac{dI}{dc} \right]^T = \left[ \frac{dx}{dc} \right]^T \left[ \frac{dI}{dx} \right]^T = N^T \left[ \frac{dI}{dx} \right]^T \quad (3.10)
\]

Since the number of control points is usually much less than the number of individual design variables, the above equation can be replaced by

\[
\left[ \frac{dI}{dc} \right]^T = \left[ \frac{dI(N\mathbf{c})}{dc} \right]^T \quad (3.11)
\]

This is employed in order to decrease computational cost. For example, the sensitivities of the stresses of elements with respect to the \( i^{th} \) control point \( \mathbf{c}_i \) can be calculated as

\[
\frac{d\mathbf{\sigma}}{dc_i} = \frac{d\mathbf{\sigma}(N\mathbf{c})}{dc_i} = \frac{\mathbf{\sigma}(N\mathbf{c} + Nh_i) - \mathbf{\sigma}(N\mathbf{c})}{h} \quad (3.12)
\]

where \( \mathbf{\sigma} \) is the vector of von Mises stresses for the elements in the finite-element model, \( h \) is the step size, and \( h_i \) is denoted as \( h_i = [0, ..., h, ..., 0]^T \).

In the adjoint method, the sensitivity of function of interest \( I \) is

\[
\frac{dI}{dc_i} = \frac{\partial I}{\partial c_i} + \psi^T \frac{\partial R}{\partial c_i} \quad (3.13)
\]

where \( \psi \) is the adjoint vector.

3.2.4 Configurable B-Spline Parameterization

The configurable B-spline parameterization aims to provide all possible options for B-splines and combines various B-splines for specific implementations. More specifically, an implementation could be different for each component (such as a rib, a spar, or a stringer); for parts of a group of components (two out of four stringers, for example); for groups of components (for example, the front spar and the rear spar); or for a complete wing box.
The configurable B-spline parameterization consist of: multiple B-spline options that include the selection of B-spline dimensions (one-dimensional or two-dimensional), number of control points, location of control points, global parameters \((x, y, z)\), degree of B-spline (constant, linear, quadratic, cubic, etc.), and knot vectors (uniform or non-uniform knot vectors); the selection of B-splines for each group of components, which means various 2D B-splines for spars, ribs, and skin, as well as 1D B-splines for spar caps, rib caps, stringers; and the combination of grouped B-splines.

In this method, a transformation is defined between the vector of design variables and the grouped B-spline control points as follows:

\[
\tilde{x} = \tilde{N} \tilde{c}
\]  

where \(\tilde{N}\) is a block diagonal matrix which can be calculated once and stored as a dictionary for a specific configuration. For example, assuming the \(\tilde{N}\) is a 5 \(\times\) 5 block matrix, it can be written as follows:

\[
\tilde{N} = \begin{bmatrix}
N_1 & 0 & 0 & 0 & 0 \\
0 & N_2 & 0 & 0 & 0 \\
0 & 0 & I_3 & 0 & 0 \\
0 & 0 & 0 & N_4 & 0 \\
0 & 0 & 0 & 0 & I_5
\end{bmatrix}
\]  

where \(N_1, N_2,\) and \(N_4\) represent the B-spline coefficient matrices for the three different groups of control points while \(I_3\) and \(I_5\) represent the identity matrices for the two different groups of control points of constant B-splines, which indeed are equal to individual design variables. To demonstrate the flexibility of implementation, the 3\(^{rd}\) and 5\(^{th}\) group of design variables are treated separately. Thus, \(\tilde{c}\) can be denoted as

\[
\tilde{c} = [c_1, c_2, x_3, c_4, x_5]^T
\]  

and \(\tilde{x}\) is

\[
\tilde{x} = [x_1, x_2, x_3, x_4, x_5]^T
\]  

In addition, the above configurable B-spline equation is extended to handle the uniform design variable method, which usually is used for reducing details of some wing box components
in practical design. It is defined as

\[ x_6 = N_6 \bar{x}_6, \quad N_6 = [1, \ldots, 1]^T \]  \hspace{1cm} (3.18)

where the \( \bar{x}_6 \) represents a linked design variable, for example, the linked design variable for a rib.

### 3.3 Automatic Scheme for Configurable B-Spline Parameterization

When using B-spline control points as the design variables, the size distribution in the finite-element model is smoothed out. An automatic scheme is devised to utilize the property of smoothness. The idea is to start optimization from few control points on a design space \( Z \), then switch to more control points (twice as many, for example) which reflect on a new design space \( 2Z \), and so on. Once a converged optimum is obtained on \( Z \), it is applied to construct a starting point for the optimization on \( 2Z \). Because the computational cost of optimization on the design space \( Z \) for a given problem is much lower than that of optimization on the design space \( 2Z \), the automatic scheme may reduce the total computational cost. This is especially useful for large scale optimization problems.

This idea is similar to the refinement of element meshes but is applied on the refinement of control points. For example, a spar with two caps is optimized using an automatic scheme with the following steps: first, start optimization from a B-spline configuration with four control points in the spanwise direction and four control points in the vertical direction, optimize the structure up to a certain convergence tolerance, then, reconstruct a new B-spline configuration by doubling the control points in the spanwise direction to eight and maintaining four control points in the vertical direction, and optimize the structure using the new B-spline configuration. The above steps are then repeated for increasing the number of control points. An example of the application of this approach is shown in Chapter 4. Figure 3.2 illustrates the process.
3.4 Constraint Handling Approaches

Stress constraint aggregations play a significant role in reducing computational cost in a gradient-based optimization method using adjoint sensitivity analysis. This is because computational efficiency of the adjoint method is particularly favorable when computing the gradients of a few functions with respect to a large number of design variables.

Constraint aggregation is a technique that reduces the number of constraints in a composite function. For instance, taking the maximum of a set of constraints is a constraint aggregation method. However, taking the maximum value results in a non-smooth constraint that is difficult to be handled by the SQP method.
The Kreisselmeier–Steinhauser (KS) function, is an aggregation function, which was first used for controller design [67–69]. In this section, the KS function and its variants are studied and especially, an adaptive KS (AKS), which was originally developed by Poon and Martins [69], is incorporated into the structural optimization of wing boxes.

In order to justify various constraint aggregation methods, a benchmark result is necessary. The appropriate benchmark in this case is the problem with individual stress constraints (ISC). The maximum stress constraint (MSC) approach represents the most conservative stress constraint and subsequently, the most conservative optimum. The KS function is a compromise between ISC and MSC and it narrows the gap of the stress constraints by a predetermined parameter. The modified KS function (MKS) is used in order to partly overcome the numerical difficulties because it reduces the magnitude of the power of the constant $e$. The AKS approach is an attempt to automatically control the process of selecting a sequence of suitable parameters that converges to a pre-specified tolerance as efficiently as possible.

The mathematical expressions for each of these methods are as follows:

1. Individual stress constraints (ISC)

\[ g(x) = \frac{\sigma(x)}{\sigma_{ys}} - 1.0 \]  

where $\sigma(x)$ are element stresses and $\sigma_{ys}$ is the material yield stress.

2. Maximum stress constraint (MSC)

\[ g_{max}(x) = \frac{\max[\sigma(x)]}{\sigma_{ys}} - 1.0 \]  

3. KS function (KS)

\[ KS[g(x)] = \frac{1}{\rho} \ln \left[ \sum_{i=1}^{m} e^{\rho g_i(x)} \right] \]  

where $m$ is the number of constraints, and $\rho$ is the KS parameter.

4. Modified KS function (MKS)

\[ MKS[g(x)] = g_{max}(x) + \frac{1}{\rho} \ln \left[ \sum_{i=1}^{m} e^{\rho(g_i(x)-g_{max}(x))} \right] \]
Section 3.5. Elemental Maximum Stress and Aggregated Stress

5. Adaptive KS (AKS) function, refer to Poon and Martins [69] for details.

Items 2 to 5 are constraint aggregation techniques, which are often used with the adjoint method to decrease the number of constraints and thus reduce the computational cost in each optimization iteration. When increasing the value of the aggregation parameter, the KS function gives a closer approximation of constraints until numerical problems occurs due to overflow, or the Hessian becomes ill-conditioned. However, the process of selecting a suitable aggregation parameter is highly problem-dependent. AKS selects the aggregation parameter automatically as necessary.

3.5 Elemental Maximum Stress and Aggregated Stress

The elemental stress refers to the stress within an element. Theoretically, stress is a concept of measuring the average amount of force exerted per unit area, which is based on the continuum assumption. When using a numerical method such as the FEM, a method is required to calculate the stress. Usually, one of two approaches is used: nodal stress, which is computed at a node, or elemental stress, which is based on an element. Because of the discontinuity of stress on nodes in the FEM, Gauss points are used to measure the nodal stress, which smooth the stress function. However, the element-based stress is more reasonable for structural design.

In the following work, the elemental maximum stress refers to the maximum stress within an element, which is the maximum of selected nodal stresses. During the course of optimization, sensitivities of elemental maximum stresses may cause oscillations due to the use of the maximum function. In order to overcome this issue, a smooth aggregation function, elemental KS (eKS), is introduced. This function can be written as follows:

\[
\sigma_{\text{max}} = \max(\sigma_i), \quad i = 1, 2, ..., \hat{i}
\]  

\[
\bar{\sigma}_i = \frac{\sigma_i}{\sigma_{\text{max}}}
\]  

\[
\sigma_{\text{KS}} = \frac{\sigma_{\text{max}}}{\rho_c} \ln \left[ \sum_{i=1}^{\hat{i}} e^{\rho_c \bar{\sigma}_i} \right]
\]
where $\hat{i}$ represents the number of nodes within an element, which is 2 for 3D-frame elements, and 4 for 3D-shell elements. In addition, the following treatment is added to avoid numerical problems:

$$\sigma_{KS} = 0, \quad \text{if} \quad \sigma_{\text{max}} = 0$$

(3.26)

### 3.6 Handling of Multiple Load Conditions

In structural airframe design, there are numerous load cases, corresponding to the various flight conditions, which include maneuvers, taxiing, gust loads, control surface deflection loads, landing, and maneuvers. More details on load conditions can be found in the books by Raymer [70] and Lomax [71].

Structural design optimization has to be carried out under multiple load conditions that take into account all the critical load cases. If the number of critical single loads is relatively small, it is acceptable to perform the structural analysis corresponding to each single load consecutively. However, if the number of critical single loads is very large ($O(10^2)$, for example), it is recommended that parallel structural analyses must be performed.

![Diagram of multiple load treatment.](image)

Figure 3.3: Diagram of multiple load treatment.

The process of the treatment for multiple load conditions is illustrated in Figure 3.3. The analysis model constraints are multiple load conditions being considered simultaneously and the algorithm can be described as follows:
1. Set the design variables

2. Solve displacements for all the load conditions being considered

3. Calculate the stresses of all the elements under all the single loads given by the elemental KS functions

4. Collect the stresses for multiple load conditions by one of the three means: all individual, maximum, and KS of the stresses from all the load conditions for each element

5. Select one of the stress constraint methods to calculate stress constraints: ISC, MKS, or AKS

6. Evaluate sensitivities with respect to control points

7. Iterate once and modify the design variables

8. Repeat above steps until the optimization converges

The results of the application of this method are presented in the next chapter.
Chapter 4

Results and Discussion

4.1 Optimization of Plates and Plates with Stiffeners

When optimizing a plate structure analyzed using the FEM, a *hinge problem* can occur at the connecting nodes and the solution exhibits alternating elements with the minimum thickness and normal thickness. This is called *checkerboarding*, and it refers to the pattern that appears in the optimal plate. This hinge problem also occurs in a plate with stiffeners. This problem has been investigated by several researchers using topology optimization approaches, such as Diaz and Sigmund [72], Sigmund and Petersson [73], Poulsen [74], and Zhou et al. [75]. Since the hinge problem results in an unrealistic design, it is of significance to understand this kind of phenomenon.

One of the most basic structures is a four-element plate when using a quadrilateral finite-element mesh. From a topological viewpoint, there are 16 basic combinations of a solid and a hole. Figure 4.1 illustrates the 16 solid-hole patterns in which solids and holes are represented by the yellow color (non-zero) and gray color (zero or null), respectively. The holes do not actually have zero thickness in the FEM, but some pre-specified minimum that is several orders of magnitude smaller than the solid thicknesses.

Among the combinations shown in Figure 4.1, two patterns, labeled as “1-3” and “2-4”, are hinged and determined to represent the fundamental patterns of checkerboards. The question is how many of these patterns are optima and which one of them is the global optimum. From a structural finite-element viewpoint, only two restrictions are necessary to
form a structure. First, there is no existence of isolated solid elements, which means a solid element exists to connect itself with the rest of the structure or with a boundary. Second, at least one solid element must exist to take the load. Satisfaction of the two conditions eliminates any singularities when solving the governing equation in the FEM. However, even if these two conditions are satisfied, it does not mean the plate structure is physically realizable nor does it prevent the occurrence of a hinge problem.

In this section, a cantilevered square plate and the plate with web stiffeners is selected as test cases to be optimized for minimum weight problem subject to stress constraints, as specified in § 3.1.1. The plate is modeled with different meshes and optimized under various load conditions using the individual design variable and B-spline parameterization methods. The plate-stiffener structure is optimized to illustrate the effects of using stiffeners.

Figure 4.1: Sixteen basic patterns and checkerboard patterns.
In addition, convergence studies of using the elemental KS stress and maximum stress (as described in § 3.5) are conducted.

### 4.1.1 Optimization of Plates Using Individual Design Variables

A cantilevered square plate $10m \times 10m$ in size is optimized by solving the weight minimization problem (3.1) using individual thicknesses as the design variables, $x$. The process starts with a four-element model and then the number of elements is increased to nine and more. Many local optima occur with the increase in the number of elements. Moreover, some specific patterns of global optima are observed for this problem. In the following problems, the magnitude of the applied nodal force is $10^6N$. The lower bound for the thickness is $10^{-5}m$, and the optimality convergence tolerance is $10^{-6}$. When the lower bound is reached for a given element, it is assumed that the plate disappears and a hole is formed. FEAP [76] is used for the structural analysis and SNOPT [77] is the optimizer. The 3D-shell elements are 4-node elements using the linear Lagrangian shape functions with six nodal DOF. The discrete Kirchhoff plate model is used in FEAP 3D-shell element. The verification of FEAP in analysis and optimization is presented in Appendix B.

**Four-element case**

Figure 4.2 shows the ground structure, which is the starting layout, and optimal topologies of the four-element plate in seven rows corresponding to seven different transversal loads, labeled as L1 to L7. All of the local optima are found for the four-element plate. The global optimum under load L1, the nodal force at the right corner, is shown in the first row labeled as P4L1-1, which is symmetric to its counterpart under load L3, labeled as P4L3-1, in the second row. The two global optima, P4L2-1 and P4L2-2, are shown in the third row for load L2, a nodal force applied at the center point. The global optima of the other rows are P4L4-1 and P4L4-2, P4L5-1, P4L6-1, and P4L7-1 and P4L7-2, respectively. The results imply a symmetric pattern existing for the optima considering symmetric load conditions, which is expected. It is also noticed that the number of local optima decreases when the number of nodal forces applied increases. Table 4.1 shows the minimum weight and optimal thicknesses for all the single loads, respectively. The global optimum is highlighted.
Figure 4.2: Complete global and local optimal topologies for a cantilevered four-element plate under different transversal loads.
Section 4.1. Optimization of Plates and Plates with Stiffeners

<table>
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<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>$x_1$ (m)</th>
<th>$x_2$ (m)</th>
<th>$x_3$ (m)</th>
<th>$x_4$ (m)</th>
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<td>1.6301E-01</td>
<td>1.0315E-01</td>
<td>1.2504E-01</td>
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<tr>
<td>P4L4-1</td>
<td>33425.08</td>
<td>1.0000E-05</td>
<td>2.2420E-01</td>
<td>1.2529E-01</td>
<td>1.3312E-01</td>
</tr>
<tr>
<td>P4L4-2</td>
<td>33425.08</td>
<td>2.2420E-01</td>
<td>1.0000E-05</td>
<td>1.3312E-01</td>
<td>1.2529E-01</td>
</tr>
<tr>
<td>P4L4-3</td>
<td>35926.96</td>
<td>1.5259E-01</td>
<td>1.0678E-01</td>
<td>1.0678E-01</td>
<td>1.0678E-01</td>
</tr>
<tr>
<td>P4L5-1</td>
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<td>1.4351E-01</td>
</tr>
<tr>
<td>P4L5-2</td>
<td>35073.81</td>
<td>2.7411E-01</td>
<td>1.0000E-05</td>
<td>1.0000E-05</td>
<td>2.3229E-01</td>
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<tr>
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<td>36305.66</td>
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<td>1.0000E-05</td>
<td>1.6158E-01</td>
<td>1.2410E-01</td>
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<tr>
<td>P4L5-4</td>
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<td>1.8787E-01</td>
<td>1.3428E-01</td>
<td>1.0841E-01</td>
<td>1.0998E-01</td>
</tr>
<tr>
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<td>35073.81</td>
<td>1.0000E-05</td>
<td>2.7411E-01</td>
<td>2.3229E-01</td>
<td>1.0000E-05</td>
</tr>
<tr>
<td>P4L6-3</td>
<td>36305.66</td>
<td>1.0000E-05</td>
<td>2.3852E-01</td>
<td>1.2410E-01</td>
<td>1.6158E-01</td>
</tr>
<tr>
<td>P4L6-4</td>
<td>37437.01</td>
<td>1.3428E-01</td>
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<td>1.0841E-01</td>
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<td>1.7430E-01</td>
</tr>
<tr>
<td>P4L7-2</td>
<td>38931.78</td>
<td>2.7446E-01</td>
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<td>1.7430E-01</td>
<td>1.1335E-01</td>
</tr>
<tr>
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<td>43601.56</td>
<td>1.8847E-01</td>
<td>1.8847E-01</td>
<td>1.2630E-01</td>
<td>1.2630E-01</td>
</tr>
</tbody>
</table>

Table 4.1: Minimum weight and optimal thicknesses of the four-element plate under single loads L1 to L7.
The handling of multiple loads, explained in §3.6, is also performed for this four-element plate to provide a benchmark for the more complex problems, as well as a comparison to the above optimization under various loads. The multiple load conditions are labeled as ML1, ML2, and ML3, which take into account the single loads L1 and L2, L3 and L2, and L1 and L3, respectively.

Table 4.2 shows the optimal weight and thicknesses for multiple load conditions ML1, ML2, and ML3. An interesting observation is that the number of multiple optima increases when compared with the single load. Also, the observed symmetric property of optima is applicable for the symmetric load conditions.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>(x_1) (m)</th>
<th>(x_2) (m)</th>
<th>(x_3) (m)</th>
<th>(x_4) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P4M1-1</td>
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<td>1.0000E-05</td>
<td>1.5843E-01</td>
<td>1.0000E-05</td>
<td>1.1398E-01</td>
</tr>
<tr>
<td>P4M1-2</td>
<td>27305.24</td>
<td>2.0715E-01</td>
<td>1.0000E-05</td>
<td>1.0000E-05</td>
<td>1.8708E-01</td>
</tr>
<tr>
<td>P4M1-3</td>
<td>29225.65</td>
<td>1.7853E-01</td>
<td>7.9317E-02</td>
<td>1.0000E-05</td>
<td>1.6412E-01</td>
</tr>
<tr>
<td>P4M1-4</td>
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<td>1.0000E-05</td>
<td>1.1933E-01</td>
<td>1.3796E-01</td>
</tr>
<tr>
<td>P4M1-5</td>
<td>31586.63</td>
<td>1.6301E-01</td>
<td>6.4867E-02</td>
<td>1.2504E-01</td>
<td>1.0315E-01</td>
</tr>
<tr>
<td>P4M2-1</td>
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<td>1.0000E-05</td>
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</tr>
<tr>
<td>P4M2-2</td>
<td>27305.24</td>
<td>1.0000E-05</td>
<td>2.0715E-01</td>
<td>1.8708E-01</td>
<td>1.0000E-05</td>
</tr>
<tr>
<td>P4M2-3</td>
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<td>7.9317E-02</td>
<td>1.7853E-01</td>
<td>1.6412E-01</td>
<td>1.0000E-05</td>
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<td>P4M2-4</td>
<td>30333.71</td>
<td>1.0000E-05</td>
<td>1.8067E-01</td>
<td>1.3796E-01</td>
<td>1.1933E-01</td>
</tr>
<tr>
<td>P4M2-5</td>
<td>31586.63</td>
<td>6.4867E-02</td>
<td>1.6301E-01</td>
<td>1.0315E-01</td>
<td>1.2504E-01</td>
</tr>
<tr>
<td>P4M3-1</td>
<td>30333.59</td>
<td>1.8067E-01</td>
<td>1.0000E-05</td>
<td>1.1933E-01</td>
<td>1.3796E-01</td>
</tr>
<tr>
<td>P4M3-2</td>
<td>30333.59</td>
<td>1.0000E-05</td>
<td>1.8067E-01</td>
<td>1.3797E-01</td>
<td>1.1933E-01</td>
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<tr>
<td>P4M3-3</td>
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<td>1.8143E-01</td>
<td>1.0000E-05</td>
<td>1.0602E-01</td>
<td>1.5054E-01</td>
</tr>
<tr>
<td>P4M3-4</td>
<td>30335.50</td>
<td>1.0000E-05</td>
<td>1.8143E-01</td>
<td>1.5054E-01</td>
<td>1.0602E-01</td>
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<td>P4M3-5</td>
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<td>1.6301E-01</td>
<td>1.0315E-01</td>
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<td>1.6301E-01</td>
<td>6.4867E-02</td>
<td>1.2504E-01</td>
<td>1.0315E-01</td>
</tr>
</tbody>
</table>

Table 4.2: Minimum weight and optimal thicknesses of the four-element plate under multiple load ML1, ML2, and ML3.

Three different ground structures corresponding to the baseline layouts are studied to compare the optima and illustrate the significance of the selection of ground structures.
Figure 4.3 shows the three ground structures, which are labeled as P4G1, P4G2, and P4G3, and the optimal topologies under load L1. The global optimum is labeled as P4G1L1-1 for ground structure P4G1, which is identical to P4L1-1 shown in Figure 4.2. Similarly, the global optimum of ground structure P4G2 is labeled as P4G2L1-1, which is different from the global optimum of ground structure P4G1. Note that there is only one optimum P4G3L1-1 found of ground structure P4G3 and P4G3L1-1 is identical to P4L1-1 as well as P4G1L1-1. This indicates that if an inappropriate ground structure is selected such as P4G2, it is impossible to obtain the global optimum. Therefore, in practical optimization, different ground structures need to be optimized because it is impossible to determine, a priori, a complete ground structure in general.

Table 4.3 compares the optimal weight and design variables for the three ground structures under load L1. The global optimum is identical for using the ground structures P4G1 and P4G3 and is highlighted in Table 4.3.
Table 4.3: Minimum weight and optimal thicknesses of ground structures P4G1, P4G2, and P4G3 under load L1.

Nine-element case

Mathematically, there are $2^9$ possible solid-hole combinations for a nine-element case. To simplify the problem, two nodal forces at the two corners of the free edge are considered to reduce the number of possible combinations to $2^7$ topologies because there must be solids to support the load. Furthermore, since a path of solids must exist between the applied external forces and the support, it is concluded that 78 different topologies are physically realizable in this case.

The minimum number of solid elements for this problem is four. If categorizing the feasible topologies into six classes according to the number of solid elements from 4 to 9, the number of feasible topologies for each class forms the set \{3, 17, 29, 21, 7, 1\}. Only some of these feasible topologies are found to be local optima. For example, although all the three feasible topologies of four solid elements are found, the number of local optima of the five solid elements is 5, which is less than 17, the number of feasible topologies. The issue here is that it cannot be stated that an optimum exists or not unless the optimum itself has been found numerically. Therefore, it is difficult to claim the completion of the optimization.

In the following figures, the color represents the stress, the yellow color for the yield stress and the blue color for the zero stress, and the dimension represents the design variables. Figure 4.4 shows the determined optimal topologies. In general, the minimum weight of the plate increases with increasing number of solid elements. The determined global optimum corresponds to the first optimal structure shown in Figure 4.4. It is identified as a symmetric distribution, and the minimum weight is 26149.89 kg.
Figure 4.4: Optimal topologies using 9 elements under the load of two vertical nodal forces.
It is also observed that the symmetry of optimal structures appears in two ways: either a single optimum with a symmetric shape or a pair of mirrored asymmetric optima.

**Twelve-element case**

Figure 4.5 shows some of the optimal topologies that are found for a twelve element case with the same load of two nodal forces. The global optimum is found to be the first optimal structure in Figure 4.5 and the minimum weight is 24509.72 kg. The symmetry of optima is applicable for this case.

![Optimal topologies using 12 elements under the load of two vertical nodal forces.](image)

The significance of the symmetric property for optimal structures under a symmetric load condition is that when optimizing a symmetric geometry with symmetric loads, the implementation of symmetric condition on half of the geometry inherently restricts the design
space which could affect the performance of the design negatively. Furthermore, it is only possible to determine the global optimality for low-dimensional problems, and it is almost impossible to determine the global optimality for problems with a large number of design variables.

**256-element case**

Figure 4.6 shows an optimal plate found using 256 elements under the same load of two nodal forces. The optimization takes 46 major iterations with 5369 function evaluations. It is observed that many of the optimal element thicknesses are at the lower bounds for this local optimum. The total CPU time of the optimization process is 142 minutes.

Figure 4.6: An optimal deformed plate of 256 elements under the load of two nodal forces.

Figure 4.7: Comparison of two optimal deformed plates of 256 elements under the load of 17 nodal forces at the front edge.
Figure 4.7 and Figure 4.8 show the comparison of two optimal topologies using 256 elements under the load of 17 nodal vertical forces at the front edge and the load of uniform nodal forces at each node, respectively. The results illustrate that the checkerboard phenomena exists regardless of the load conditions.

Figure 4.8: Comparison of two optimal deformed plates of 256 elements under the load of uniform nodal forces at each node.

4.1.2 Optimization of Plates Using B-Splines

256-element case

Different B-splines with $16 \times 16$, $8 \times 8$, and $4 \times 4$ control points are studied. Table 4.4 lists the minimum weight and B-spline degree. The benchmark is the minimum weight of optimal plate P256L4-DV1 obtained by using one design variable for the 256-element plate under load L4 in order to maintain the accuracy of structural analysis at the same finite-element model level.

It is observed that the minimum weight decreases with increasing number of control points because there are more design freedoms. The last optimal structure obtained by using $16 \times 16$ control points is equivalent to using the individual design variable method. Although the improvement of the minimum weight is very high, it is an unrealistic design, as shown in Figure 4.6.
### Table 4.4: Comparison of minimum weight improvement for 256-element cases.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>Improvement(%)</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>P256L4-DV1</td>
<td>44592.89</td>
<td>0</td>
<td>NA</td>
</tr>
<tr>
<td>P256L4BS4x4-1</td>
<td>31696.56</td>
<td>28.92</td>
<td>1</td>
</tr>
<tr>
<td>P256L4BS4x4-2</td>
<td>32277.06</td>
<td>27.62</td>
<td>1</td>
</tr>
<tr>
<td>P256L4BS4x4-3</td>
<td>31983.44</td>
<td>28.28</td>
<td>3</td>
</tr>
<tr>
<td>P256L4BS8x8-1</td>
<td>26203.77</td>
<td>41.24</td>
<td>3</td>
</tr>
<tr>
<td>P256L4BS16x16-1</td>
<td>17304.90</td>
<td>61.19</td>
<td>NA</td>
</tr>
</tbody>
</table>

Figure 4.9 shows the local optimal plates found using the B-spline configurations with $4 \times 4$ and $8 \times 8$ control points. The thickness distribution of the optimal plate becomes smoother with fewer control points, but the optimal weight increases. However, the B-spline approach eliminates the unrealistic design obtained by using the individual thicknesses with the large number of design variables. Furthermore, the efficiency of the B-spline approach is much higher than the individual thicknesses approach through reducing the number of design variables, which could be very useful for the preliminary design stage.
4.1.3 Optimization of Plate-Stiffener Structures Using Individual Design Variables

In structural design, stiffeners are commonly used to increase the performance of plates. A web stiffener of 21 3D-frame elements is used as a sample to study the effects of using stiffeners. The geometry of the plate is the same as that of the plate used in previous sections and the load is consistent as well. In the following figures, the yellow color represents the yield stress and the blue color represents the lowest stress.

Figure 4.10: Symmetric optimal web-stiffener plates under the load of two vertical nodal forces.
Figure 4.11: Asymmetric optimal web-stiffener plates under the load of two vertical nodal forces.

Figure 4.10 shows the shape and stress distribution for two symmetric optimal plates with stiffeners found using 9 3D-shell and 21 3D-frame elements. In this figure, the upper left structure represents the global optimum and the minimum weight is 12445.91 kg. Figure 4.11 shows the shape and stress distribution for ten non-symmetric optimal plates. Many local optima are found for this problem and some of them are shown in Figure 4.10 and Figure 4.11.
The symmetry of optimal structures is noticed. It is observed that unrealistic designs are encountered for many of the local optima. It is worthwhile to mention that more local optima of the plate-stiffener problems exist than those of the plate problems because of the larger number of design variables.

### 4.1.4 Convergence Studies of Elemental KS Stress and Elemental Maximum Stress

Because the constraint handling (see § 3.1.4) is very important in the minimum weight problem, this section compares the use of elemental maximum stress versus elemental KS stress both for ISC and KS. In the practice of structural optimization, a challenge usually encountered is the complexity of the unknown design spaces which makes it difficult to compare and verify proposed methods. There are many factors that affect the optimization which include the problem statement, the analysis, the sensitivity analysis, the design variable handling, the constraint handling, the dependency of starting points, as well as the accuracy and convergence criteria of algorithms.

The existence of multiple optima makes the availability of comparisons between different methods difficult because it is only meaningful to compare the same design. It is wise to select a problem as a benchmark to compare with different methods. Based on the above studies for the four-element plate using IDV and elemental maximum stress under various loads, the multiple optima for each case are completely found and reasonably well understood. Hence, this four-element plate is selected as a benchmark.

The benchmark optimization is that of the four-element problem under load L1 using ISC and elemental maximum stress. Figure 4.12 shows the curve of accuracy and computational cost of optimization using elemental KS stress. The computational cost is normalized by the CPU time of the benchmark optimization with the same convergence requirement and from the same starting point. It is observed that the accuracy of optimal weight smoothly increases with the increase in the value of the elemental KS parameter $\rho_e$ until it reaches the maximum of 0.0495% at the upper bound of parameter $\rho_e = 705.0$, while the computational cost stays practically identical. It is concluded that the elemental KS stress is an accurate approximation of the maximum stress and without any extra computational cost.
Section 4.1. Optimization of Plates and Plates with Stiffeners

Figure 4.12: Accuracy and efficiency of the optimization using elemental KS.

Figure 4.13: Comparison of accuracy and efficiency of the optimization using MKS.
Section 4.1. Optimization of Plates and Plates with Stiffeners

It is worthy mentioning the value of the maximum elemental KS parameter \( \rho_e \) depends on the range of floating-point numbers, for example, in Python, it should be less than \( \ln(10^{308}) \).

Figure 4.13 compares the accuracy and efficiency of MKS with the elemental KS and the elemental maximum stress. As KS parameter \( \rho \) reaches 200, the accuracy of MKS-eMax is 0.17% and that of MKS-eKS is 0.22% at \( \rho_e = 700.0 \).

A more complex benchmark is the optimization of 256-element under the load of front line nodal forces. Figure 4.14 compares the convergence history of weight of using elemental KS stress with that of using elemental maximum stress. In the optimization, a bicubic B-spline configuration, which consists of \( 8 \times 8 \) control points, is used and the optimum is found for each case.

Figure 4.15 compares the convergence history of weight using IDV. The optimum is found by using elemental KS stress, while it reached a sub-optimal stagnation and the design point would not converge for using elemental maximum stress. It indicates that the element KS stress approach both reduces the computational cost and increases the robustness of the algorithm.
Figure 4.15: Comparison of convergence history for the 256-element plate under the load of 17 nodal forces at the front edge using IDV, $W_{e\text{Max}} = 15062.05$ kg vs. $W_{e\text{KS}} = 14951.08$ kg.

4.2 Optimization of Spar-Cap Structures

In order to demonstrate the effect of simultaneously considering multiple load conditions, a spar-cap structure is optimized under different load conditions individually, and multiple load conditions simultaneously. The optimization problem is the weight minimization of a spar-cap structure subject to stress constraints, as originally stated in § 3.1.1. Both individual design variables and B-spline control points are used, and the results are compared.
Section 4.2. Optimization of Spar-Cap Structures

The elemental KS stress and elemental maximum stress method — described in §3.5 — are used in this optimization problem.

4.2.1 Finite-Element Model and Load Conditions

The spar-cap structure contains a spar with two caps and the structure is clamped at one end. The total number of elements is 240 — 40 3D-frame elements for the top spar cap, 40 3D-frame elements for the bottom spar cap, and 160 3D-shell elements for the spar shear web. Figure 4.16 shows the FE model of the spar with two caps. There are six single loads labeled as L1 to L6.

![Six single load conditions.](image)

Figure 4.17: Six single load conditions.

Figure 4.17 shows the load distributions for the six load conditions. In the figure, L4 and L5 represent elliptical distributions, and L6 represents a parabolic distribution. The
multiple load condition is labeled as ML16. This case simultaneously takes into account the six single load conditions L1 to L6.

4.2.2 Optimization Using Individual Design Variables

Single Load Conditions

This spar-cap structure is optimized under each of the six single loads using the individual design variables. Table 4.5 compares the minimum weight, number of major optimization iterations, and CPU times for all the single load conditions. The convergence tolerance of the optimization is set to $10^{-5}$. The weight of optimal structures B and C are the top two among the optimal structures under different loads. Optimal structure B2 is another local optimum under load L2.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Load</th>
<th>Weight (kg)</th>
<th>Major Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>L1</td>
<td>45.34</td>
<td>43</td>
<td>1258</td>
</tr>
<tr>
<td>B</td>
<td>L2</td>
<td>83.62</td>
<td>85</td>
<td>1927</td>
</tr>
<tr>
<td>B2</td>
<td>L2</td>
<td>78.64</td>
<td>85</td>
<td>3386</td>
</tr>
<tr>
<td>C</td>
<td>L3</td>
<td>74.80</td>
<td>166</td>
<td>5328</td>
</tr>
<tr>
<td>D</td>
<td>L4</td>
<td>62.20</td>
<td>43</td>
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<tr>
<td>E</td>
<td>L5</td>
<td>54.32</td>
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<td>1461</td>
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<tr>
<td>F</td>
<td>L6</td>
<td>62.70</td>
<td>203</td>
<td>4253</td>
</tr>
</tbody>
</table>

Table 4.5: Comparison of optimization status for the single load condition.

Figures 4.18 to 4.24 show the thickness, cross-sectional area and stress distributions of the optimal structures under the six single loads. The red color represents the yield stress of the material. Each optimal structure under the corresponding single load condition shows a very high percentage of fully-stressed elements.
Figure 4.18: Optimal structure A under load L1.

Figure 4.19: Optimal structure B under load L2.

Figure 4.20: Optimal structure B2 under load L2.
Figure 4.21: Optimal structure C under load L3.

Figure 4.22: Optimal structure D under load L4.

Figure 4.23: Optimal structure E under load L5.
Multiple Load Conditions

This spar-cap structure is optimized under the multiple load case (ML16) using the individual design variables. Table 4.6 shows the comparison of two local optima, labeled as G and H, for the multiple load condition taking into account all the six single loads. Figures 4.25 to 4.30 show the thickness, cross-sectional area and stress distributions of the optimal structure G under the six single load conditions. The stress distributions of G under load L2 and load L3 have a very high percentage of fully-stressed elements, which means that the dominant load conditions are L2 and L3. Figure 4.31 to 4.36 shows the stress distributions of the optimal structure H under the six single load conditions.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Load</th>
<th>Weight (kg)</th>
<th>Major Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>ML16</td>
<td>80.59</td>
<td>76</td>
<td>14087</td>
</tr>
<tr>
<td>H</td>
<td>ML16</td>
<td>83.87</td>
<td>73</td>
<td>13368</td>
</tr>
</tbody>
</table>

Table 4.6: Comparison of optimization status for the multiple load condition.
Section 4.2. Optimization of Spar-Cap Structures

Figure 4.25: Optimal structure G under load L1.

Figure 4.26: Optimal structure G under load L2.

Figure 4.27: Optimal structure G under load L3.
Section 4.2. Optimization of Spar-Cap Structures

Figure 4.28: Optimal structure G under load L4.

Figure 4.29: Optimal structure G under load L5.

Figure 4.30: Optimal structure G under load L6.
Figure 4.31: Optimal structure H under load L1.

Figure 4.32: Optimal structure H under load L2.

Figure 4.33: Optimal structure H under load L3.
Section 4.2. Optimization of Spar-Cap Structures

Figure 4.34: Optimal structure H under load L4.

Figure 4.35: Optimal structure H under load L5.

Figure 4.36: Optimal structure H under load L6.
The weight of optimal structure H is greater than those of optimal structures B and B2 under load condition L2. The weight of optimal structure G is lower than that of optimal structure B but it is greater than that of optimal structure B2. This is because of the existence of local optima. In general, the optimal structure under the multiple load is at least as heavy as the heaviest optimal structure under each of the single loads involved.

Convergence Studies of eKS

In the above presented results, the elemental KS stress method is used. The reason is that the use of the elemental KS stress method eliminates oscillations that often occur in the process of optimization using the elemental maximum stress method, especially after achieving a certain convergence level. This is clearly illustrated in the following two cases: the first case is the optimization under single load L4 and the second case is the optimization
under multiple load ML16.

Figure 4.37 compares the convergence history of the optimum found with the elemental KS stress method versus the elemental maximum stress method under single load condition L4.

Figure 4.38 compares the convergence history of the optimum found with the elemental KS stress method versus the elemental maximum stress method under multiple load ML16. From Figures 4.37 and 4.38, it is observed that the convergence of using the elemental KS function is much faster than that of using the elemental maximum stress in both cases and more importantly, the elemental KS method shows a stable performance, drastically improving the computational efficiency, corresponding to 80% and 85% savings in terms of the number of iterations, respectively.
4.2.3 Optimization Using B-spline Configurations

From the above section, it is observed that the optimal structures obtained by using the individual design variable method are not smooth and are unrealistic for manufacturing. The configurable B-spline parameterization method described in §3.2 is used to solve this minimum weight problem. The two-dimensional B-splines are used for the spar and one-dimensional B-splines are used for the caps. Four B-spline configurations are studied and the details of B-spline configurations are listed in Table 4.7.

First, the elemental KS stress method described in §3.5 is used to improve the efficiency of using the configurable B-spline parameterization method based on the previous experience with the individual design variables. Second, the automatic scheme approach of B-spline configurations described in §3.3 is used to further reduce the computational cost by starting from a B-spline configuration with a few control points, and then switching to a B-spline configuration with more control points. This scheme gradually improves the minimum weight and reduces computational cost.

Two automatic schemes are devised and tested to illustrate the capability of reducing the computational cost while improving the optimal design. The first scheme, labeled as “Auto1”, starts from BSC1 and switches to BSC4 (BSC1-BSC4) by increasing control points from 4 to 20 in the spanwise direction and the second scheme (Auto2) starts from BSC2 and switches to BSC4 by increasing control points from 5 to 20 in the spanwise direction.

<table>
<thead>
<tr>
<th>Name</th>
<th>Design Variables</th>
<th>Spar Option</th>
<th>Cap Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>BSC1</td>
<td>24 × 4 2D Bicubic</td>
<td>4 1D Cubic</td>
<td></td>
</tr>
<tr>
<td>BSC2</td>
<td>30 × 4 2D Bicubic</td>
<td>5 1D Cubic</td>
<td></td>
</tr>
<tr>
<td>BSC3</td>
<td>60 × 4 2D Bicubic</td>
<td>10 1D Cubic</td>
<td></td>
</tr>
<tr>
<td>BSC4</td>
<td>120 × 4 2D Bicubic</td>
<td>20 1D Cubic</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.7: Details of B-spline configurations.

Single Load Condition L4

The spar-caps structure is optimized using these methods under load L4. This is the same load condition used in the previous section, for which optimal structures were obtained.
Table 4.8: Comparison of optimization status for B-spline configurations under load L4.

Figures 4.39 to 4.42 compare the thickness, cross-sectional area and stress distributions of the optimal structures D1 to D4 using different B-spline configurations under load condition L4. From these figures, it is observed that the optimal structures are smoothened compared with the optimal structure D which is obtained using IDV. It is also observed that D1 has the fewest fully-stressed elements, because D1 uses the least number of control points. D3 and D4 are quite similar. From D1 to D4, the more control points used, the better the optima obtained. It is noted that the optimal structures D1 to D4 are obtained by using the elemental KS method.

Figure 4.43 shows the sizing and stress distributions of optimal structure D5 using the elemental maximum stress method and BSC4. D5 is observed almost as the same as D4, which illustrates the effectiveness of using the elemental KS method instead of the elemental maximum stress method.

Figures 4.44 and 4.45 illustrate optimal structures DA1 and DA2 which are obtained by using the automatic schemes, Auto1 (BSC1-BSC4) and Auto2 (BSC2-BSC4), respectively. It turns out that DA1, DA2, and D4 are the same optimum.

Table 4.8 compares the optimal weight and computational cost for different methods under load L4. From Table 4.8, it is concluded that using eKS with the B-spline approach is much better than using eMax with B-spline approach in terms of computational cost: there is a 70% savings in computational cost for a slight increase in minimum weight (0.11%).

It is noticed that using more design variables in the B-spline configurations results in lower
Section 4.2. Optimization of Spar-Cap Structures

optima, but it also incurs greater computational cost. Optimal structure DA1, obtained by using automatic scheme Auto1, is identical to D4 but the computational time is 30% lower than that of D4. It is also observed that DA2, obtained by using automatic scheme Auto2, is identical to D4, but the computational time is 42% lower. This is significant because the improvement from using the automatic scheme is based on using the elemental KS stress method, which has already contributed 70% improvement in computational time when compared to the elemental stress method.

Figure 4.39: Optimal structure D1 under load L4 using eKS and BSC1.

Figure 4.40: Optimal structure D2 under load L4 using eKS and BSC2.
Section 4.2. Optimization of Spar-Cap Structures

Figure 4.41: Optimal structure D3 under load L4 using eKS and BSC3.

Figure 4.42: Optimal structure D4 under load L4 using eKS and BSC4.

Figure 4.43: Optimal structure D5 under load L4 using eMax and BSC4.
Section 4.2. Optimization of Spar-Cap Structures

The convergence history of optimal structures using BSC4 under single load L4 is compared in Figure 4.46 for reference. In this figure, the blue line is for D5 and the magenta line is for D4.
Multiple Load ML16

The same B-spline configurations and automatic schemes are implemented in optimization under multiple load ML16.

Figures 4.47 to 4.52 show the thickness, cross-sectional area and stress distributions corresponding to the six single load conditions of the optimal structure I5 obtained under multiple load ML16. It illustrates clearly that the B-spline parameterization method gives a very smooth optimal structure under multiple load conditions.

Table 4.9 lists the optimization results for different B-spline configurations under multiple load ML16. With the increase in the complexity of B-spline configurations, the same pattern
Section 4.2. Optimization of Spar-Cap Structures

<table>
<thead>
<tr>
<th>Structure</th>
<th>Method</th>
<th>Weight (kg)</th>
<th>Major Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>BSC1</td>
<td>97.53</td>
<td>20</td>
<td>385</td>
</tr>
<tr>
<td>I2</td>
<td>BSC2</td>
<td>93.02</td>
<td>21</td>
<td>502</td>
</tr>
<tr>
<td>I3</td>
<td>BSC3</td>
<td>89.05</td>
<td>49</td>
<td>2154</td>
</tr>
<tr>
<td>I4</td>
<td>BSC4</td>
<td>87.77</td>
<td>96</td>
<td>7770</td>
</tr>
<tr>
<td>I5</td>
<td>BSC4</td>
<td>86.36</td>
<td>98</td>
<td>6769</td>
</tr>
<tr>
<td>I6</td>
<td>BSC1-BSC4</td>
<td>87.77</td>
<td>51</td>
<td>4554</td>
</tr>
<tr>
<td>I7</td>
<td>BSC2-BSC4</td>
<td>87.77</td>
<td>64</td>
<td>5897</td>
</tr>
</tbody>
</table>

Table 4.9: Comparison of optimization status for B-spline configurations under multiple load ML16.

of the performance is observed, that is to say, a lower weight design is achieved with a higher computational cost. It is also observed that the two automatic schemes reduce computational cost: 41% for Auto1, which starts from BSC1 and switches to BSC4, and 24% for Auto2, which starts from BSC2 and switches to BSC4. It is concluded that Auto1 outperforms Auto2 under multiple load conditions. Furthermore, using the automatic scheme method increases the computational efficiency regardless of the difference in the B-spline configurations both in optimization for multiple load conditions and single load conditions.

Figure 4.47: Optimal structure I5 under load L1.
Section 4.2. Optimization of Spar-Cap Structures

Figure 4.48: Optimal structure I5 under load L2.

Figure 4.49: Optimal structure I5 under load L3.

Figure 4.50: Optimal structure I5 under load L4.
4.3 Optimization of Wing Boxes

A wing similar to that of the B737-200 Advanced is chosen as the test case for full wing box optimization. Considering the typical commercial airplane wing box configurations, a wing box component configuration is selected that consists of two spars, fifteen ribs, top and bottom skins, three top stringers and three bottom stringers, top and bottom spar caps, and top and bottom rib caps. The front spar is located at 20% chord, and the rear spar is located at 70% chord. The Cartesian coordinates are $X$ in the chordwise direction, $Y$ in the spanwise direction, and $Z$ in the vertical direction. This wing box is labeled as WB6.
4.3.1 Models and Load Conditions

The FE model of WB6 is shown in Figure 4.53. The spars, ribs, and skins are modeled by 3D shell elements while stringers, spar caps, and rib caps are modeled by 3D frame elements. The total number of elements is 1320 and the number of the degrees of freedom (DOF) of this structure is 4110. The details of the FE model of WB6 are listed in Table 4.10.

<table>
<thead>
<tr>
<th>Group</th>
<th>Components</th>
<th>Elements per Component</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spar</td>
<td>2</td>
<td>56</td>
<td>112</td>
</tr>
<tr>
<td>Rib</td>
<td>15</td>
<td>16</td>
<td>240</td>
</tr>
<tr>
<td>Skin</td>
<td>2</td>
<td>224</td>
<td>448</td>
</tr>
<tr>
<td>Stringer</td>
<td>6</td>
<td>28</td>
<td>168</td>
</tr>
<tr>
<td>Spar cap</td>
<td>4</td>
<td>28</td>
<td>112</td>
</tr>
<tr>
<td>Rib cap</td>
<td>30</td>
<td>8</td>
<td>240</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>59</strong></td>
<td></td>
<td><strong>1320</strong></td>
</tr>
</tbody>
</table>

Table 4.10: Details of the finite-element model of wing box WB6.
Section 4.3. Optimization of Wing Boxes

Figure 4.54: Load conditions.

(a) LA1.

(b) LA2.

(c) LA3.

(d) LA4.
Section 4.3. Optimization of Wing Boxes

There are four single loads labeled as LA1, LA2, LA3 and LA4, which are taken from aerodynamic loads representing one cruise condition and three maneuver conditions. Figure 4.54 shows the four single loads.

The multiple load case is labeled as MLA14, and takes into account the four single loads, L1 to L4. The control points are used as the design variables and three different B-spline configurations, BSC1, BSC2, and BSC3, are studied. A total of 81, 117, and 171 control points are used for these configurations respectively. The number of constraints is 2640, which includes 1320 stress constraints and 1320 additional variable constraints, as mentioned in §3.3. The details of the B-spline configurations are listed in Tables 4.11 to 4.13.

In the case studies for each B-spline configuration, each rib and its corresponding top and bottom rib caps are treated as a single design variable respectively because these components are not of the primary interest. The number of design variables of BSC2 is double that of BSC1 in the spanwise direction for spars, top and bottom skins, and top and bottom spar caps. BSC3 uses cubic 1D B-splines with seven control points in the spanwise direction for spar caps and stringers as well as cubic-linear 2D B-splines with seven control points in the spanwise direction and two control points in the other direction.

WB6 is optimized under all the single loads and the multiple load condition corresponding to the above B-spline configurations.

<table>
<thead>
<tr>
<th>Group</th>
<th>Components</th>
<th>Design Variables</th>
<th>Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spar</td>
<td>2</td>
<td>8</td>
<td>2D Bilinear</td>
</tr>
<tr>
<td>Rib</td>
<td>15</td>
<td>15</td>
<td>Uniform</td>
</tr>
<tr>
<td>Skin</td>
<td>2</td>
<td>8</td>
<td>2D Bilinear</td>
</tr>
<tr>
<td>Stringer</td>
<td>6</td>
<td>12</td>
<td>1D Linear</td>
</tr>
<tr>
<td>Spar cap</td>
<td>4</td>
<td>8</td>
<td>1D Linear</td>
</tr>
<tr>
<td>Rib cap</td>
<td>30</td>
<td>30</td>
<td>Uniform</td>
</tr>
</tbody>
</table>

Table 4.11: Details of B-spline configuration 1 (BSC1) for wing boxes.

In this section, the color in the figures represents stress: the red color represents the yield stress and the blue color represents the zero value of stress. Figure 4.55 shows the color bar. The thicknesses of spars, ribs and skins, as well as the cross-sectional areas of stringers, spar caps, and rib caps, are shown in dimensions.
### 4.3. Optimization of Wing Boxes

#### Table 4.12: Details of B-spline configuration 2 (BSC2) for wing boxes.

<table>
<thead>
<tr>
<th>Group</th>
<th>Components</th>
<th>Design Variables</th>
<th>Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spar</td>
<td>2</td>
<td>16</td>
<td>2D Bilinear</td>
</tr>
<tr>
<td>Rib</td>
<td>15</td>
<td>15</td>
<td>Uniform</td>
</tr>
<tr>
<td>Skin</td>
<td>2</td>
<td>16</td>
<td>2D Bilinear</td>
</tr>
<tr>
<td>Stringer</td>
<td>6</td>
<td>24</td>
<td>1D Linear</td>
</tr>
<tr>
<td>Spar cap</td>
<td>4</td>
<td>16</td>
<td>1D Linear</td>
</tr>
<tr>
<td>Rib cap</td>
<td>30</td>
<td>30</td>
<td>Uniform</td>
</tr>
</tbody>
</table>

#### Table 4.13: Details of B-spline configuration 3 (BSC3) for wing boxes.

<table>
<thead>
<tr>
<th>Group</th>
<th>Components</th>
<th>Design Variables</th>
<th>Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spar</td>
<td>2</td>
<td>28</td>
<td>2D Y Cubic-Z Linear</td>
</tr>
<tr>
<td>Rib</td>
<td>15</td>
<td>15</td>
<td>Uniform</td>
</tr>
<tr>
<td>Skin</td>
<td>2</td>
<td>28</td>
<td>2D Y Cubic-X Linear</td>
</tr>
<tr>
<td>Stringer</td>
<td>6</td>
<td>42</td>
<td>1D Y Cubic</td>
</tr>
<tr>
<td>Spar cap</td>
<td>4</td>
<td>28</td>
<td>1D Y Cubic</td>
</tr>
<tr>
<td>Rib cap</td>
<td>30</td>
<td>30</td>
<td>Uniform</td>
</tr>
</tbody>
</table>

#### 4.3.2 Optimization Using B-Spline Configurations

**BSC1 for Single and Multiple Load Conditions**

WB6 is optimized under each of the four single load conditions for BSC1. The results presented correspond to two cases: optimal wing boxes WB601 to WB604 are optimized from the same starting point as the first case, and optimal wing boxes WB601A to WB604A are optimized from another starting point as the second case. The same convergence criteria ($10^{-5}$) are applied for both cases and B-spline configuration BSC1 is used. These two cases correspond to two sets of local optima for each single load condition.

Table 4.14 summarizes the optimization status of each wing box for the single loads. From Table 4.14, it is observed that the dominant load conditions are LA2 and LA4 in both cases, and the second case contains best optima. The second case converges much faster than the first case because the starting point for the second case is an obtained optimum from previous results.
Table 4.14: Summary of minimum weight of optimal wing boxes under four single loads using BSC1.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>Load</th>
<th>Method</th>
<th>Major Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WB601</td>
<td>5284.11</td>
<td>LA1</td>
<td>ISC-eKS</td>
<td>60</td>
<td>4193</td>
</tr>
<tr>
<td>WB602</td>
<td>8742.16</td>
<td>LA2</td>
<td>ISC-eKS</td>
<td>36</td>
<td>2580</td>
</tr>
<tr>
<td>WB603</td>
<td>8375.78</td>
<td>LA3</td>
<td>ISC-eKS</td>
<td>47</td>
<td>3374</td>
</tr>
<tr>
<td>WB604</td>
<td>8823.46</td>
<td>LA4</td>
<td>ISC-eKS</td>
<td>45</td>
<td>3196</td>
</tr>
<tr>
<td>WB601A</td>
<td>5158.17</td>
<td>LA1</td>
<td>ISC-eKS</td>
<td>11</td>
<td>1110</td>
</tr>
<tr>
<td>WB602A</td>
<td>8086.36</td>
<td>LA2</td>
<td>ISC-eKS</td>
<td>8</td>
<td>828</td>
</tr>
<tr>
<td>WB603A</td>
<td>7780.41</td>
<td>LA3</td>
<td>ISC-eKS</td>
<td>11</td>
<td>1094</td>
</tr>
<tr>
<td>WB604A</td>
<td>8101.40</td>
<td>LA4</td>
<td>ISC-eKS</td>
<td>12</td>
<td>1125</td>
</tr>
</tbody>
</table>

Figures 4.56 and 4.57 show the optimal wing boxes under each single load condition for the two cases respectively. From these two figures, it is observed that the optimal wing boxes are different for each single load condition. It is also observed that fully-stressed elements exist in those optimal wing boxes and more fully-stressed elements appear in the second case than the first case, which implies that more active constraints result in a better design that is the least minimum weight.

BSC1 is also used to optimize WB6 under multiple load ML14. The optimal wing boxes, labeled as WB605 and WB605A, are two local optima for this multiple load condition. There are 81 control points used to represent 1320 individual design variables which are thicknesses of spars, ribs and skins, and cross-sectional areas of spar caps, rib caps and stringers, see Table 4.11.
Section 4.3. Optimization of Wing Boxes

(a) WB601.

(b) WB602.

(c) WB603.

(d) WB604.

Figure 4.56: Optimal wing boxes under each single load conditions (Case A.)
Section 4.3. Optimization of Wing Boxes

Figure 4.57: Optimal wing boxes under each single load conditions (Case B)
Section 4.3. Optimization of Wing Boxes

Figures 4.58 and 4.59 show the thickness, cross-sectional area and stress distributions of optimal wing box WB605 and WB605A obtained under multiple load MLA14 for each single load, respectively.

BSC2, BSC3 and an Automatic Scheme for Multiple Load Conditions

Two B-spline configurations, BSC2 and BSC3, are applied and the performance of these configurations is compared through results in terms of the objective function, as well as the computational cost associated with those configurations. In addition, an automatic scheme, labeled as Auto1, is devised to switch automatically from BSC2 to BSC3 once the process of optimization using BSC1 converges to predetermined convergence criteria.

Figures 4.60 and 4.61 show the thickness, cross-sectional area and stress distributions of optimal wing box WB606A obtained by using BSC2 and WB609 obtained from the automatic scheme using BSC2 and BSC3.

Figure 4.62 compares the convergence history of the automatic scheme that is BSC2-BSC3 and the direct BSC3. This shows the decrease of the computational cost of the optimization when using the automatic scheme: a 55% reduction in the number of iterations to achieve the optimal design. In the plot, the green dot represents the automatic switching point according to the convergence criteria, which is equivalent to 64 iterations of BSC3.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>Configuration</th>
<th>Method</th>
<th>Major Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WB605</td>
<td>8805.21</td>
<td>BSC1</td>
<td>ISC-eKS-Max</td>
<td>38</td>
<td>11379</td>
</tr>
<tr>
<td>WB605A</td>
<td>8126.57</td>
<td>BSC1</td>
<td>ISC-eKS-Max</td>
<td>35</td>
<td>11559</td>
</tr>
<tr>
<td>WB606</td>
<td>5759.27</td>
<td>BSC2</td>
<td>ISC-eKS-Max</td>
<td>57</td>
<td>27407</td>
</tr>
<tr>
<td>WB607</td>
<td>5168.27</td>
<td>BSC3</td>
<td>ISC-eKS-Max</td>
<td>132</td>
<td>162341</td>
</tr>
<tr>
<td>WB609</td>
<td>5102.38</td>
<td>BSC2-BSC3</td>
<td>ISC-eKS-Max</td>
<td>43</td>
<td>64284</td>
</tr>
</tbody>
</table>

Table 4.15: Summary of minimum weight of optimal wing boxes under multiple load MLA14.

Finally, Table 4.15 presents the summary of optimization results for the multiple load case (MLA14). As expected, with the increase in the complexity in the design space, it takes much more computational effort to achieve the optimum. For the case with the most complexity (BSC3), it takes roughly 45 hours to find the optimum.
Section 4.3. Optimization of Wing Boxes

(a) Under LA1.
(b) Under LA2.
(c) Under LA3.
(d) Under LA4.

Figure 4.58: Comparison of stress distributions for optimal wing box WB605 using BSC1.
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(a) Under LA1.
(b) Under LA2.
(c) Under LA3.
(d) Under LA4.

Figure 4.59: Comparison of stress distributions for optimal wing box WB605A using BSC1.
Section 4.3. Optimization of Wing Boxes

Figure 4.60: Comparison of stress distributions for optimal wing box WB606A using BSC/2.
Section 4.3. Optimization of Wing Boxes

(a) Under LA1.
(b) Under LA2.
(c) Under LA3.
(d) Under LA4.

Figure 4.61: Comparison of stress distributions for optimal wing box WB609 using BSC3.
However, the use of the automatic scheme results in a reduction in the computational cost of 60% in terms of the CPU time, i.e., less than 18 hours to achieve the optimum. The implementation of the automatic scheme method in wing boxes is significant: not only does it succeed in reducing the computational cost, but also it provides opportunities for the use of the configurable B-spline parameterization method in a practical environment.

Furthermore, if the structural analysis for each critical load can be parallelized in each iteration of optimization, the computational cost can be reduced to less than 5 hours, which fulfills the expectation of the computational cost for optimizations at this stage of the design process [18]. This projection is based on the current use of Intel Itanium 2 with 1.66 GHz only.

Figure 4.62: Convergence history comparison: BSC2-BSC3 versus BSC3.
Chapter 5

Conclusions, Contributions and Recommendations

In this chapter, final remarks are given on conclusions resulting from the numerical results, a summary of contributions, and recommendations of future research.

5.1 Conclusions

The conclusions are ordered by the solved optimization problems. First, the B-spline parameterization method was developed and incorporated with the individual design variable method for the existing high-fidelity aero-structural design framework. Both methods were implemented for structural optimization of plate and plate-stiffener problems to determine the minimum weight of the structure with stress constraints under various single load conditions and simultaneous multiple load conditions. The aggregation methods and especially, the elemental KS stress method are used to solve the problems that were considered.

It was found that sizing optimization leads to topologically optimal designs by using nearly zero lower bounds. This is because when using the finite-element method, the one-node hinge problem occurs, resulting in optimal design that exhibit a checkerboard pattern. Around any hinged node, the optimizer automatically selects a combination of the connected elements. However, the combinations have many possibilities, for example, as shown in §4.1, sixteen basic topologies exist for plate problems using quadrilateral elements, and only two
of those topologies cause the checkerboard pattern. The total number of combinations for a given finite-element model of the structure varies depending on the number of nodes, the number of hinged elements, the type of elements, boundary conditions, and load conditions. Once the optimizer allows nearly zero lower bounds without upper bounds, which means it has more freedom in the design, the situation becomes worse. This is why the optimum shows a mesh-dependent property and on a given mesh, many local optima exist. The existence of multiple local optima makes it very challenging to determine the global optimum for even small sized problems and prohibitive to determine the global optimum for large scale problems.

Furthermore, either one symmetric optimum (local or global) or a pair of mirrored non-symmetric optima (local or global) exists for the optimization of plate as well as plate-stiffener structures under symmetric load conditions, see §4.1.1 and §4.1.3. Complete optimal patterns for the selected problems such as the four-element problem have been found and presented as a reference for future research. The pursuit of complete optimal patterns for large scale problems is a mathematical concern. From an engineering viewpoint, the pursuit of a best optimum within a time frame is more important, therefore, the B-spline parameterization method has been tested for these problems and resolves the one-node hinge problem by smoothing the design space and reduces the number of design variables when compared with the individual design variable method. The elemental KS method provides a high quality approximation of the elemental maximum stress and drastically reduces the computational cost for the plate problems in both the B-spline parameterization and individual design variable methods.

The configurable B-spline parameterization method was developed to facilitate the B-spline parameterization method and implemented in the structural optimization of a spar-cap structure and compared with the individual design variable method. The results illustrate two advantages of using the configurable B-spline parameterization method: the smoothing of the design space to achieve realistic designs, and the flexibility provided by the B-splines, as shown in the results presented in §4.2.

In addition, an automatic scheme was proposed for utilizing the configurable B-spline parameterization method, which improved the computational efficiency, as demonstrated in §3.3. It is concluded that although the improvement in computational efficiency depends
on the B-spline configurations, and differs between single load conditions to multiple load conditions, the optimization involving the use of the automatic scheme method converges at least 1.3 times as fast as directly using the finer B-spline configuration. More importantly, this improvement is calibrated by the use of the elemental KS stress method, which already saved at least approximately 70% in computational cost. It is also concluded that multiple optima exist for the spar-cap optimization problems regardless of the particular single load or multiple load conditions. For all the different cases, the best optimum for each method has been presented.

The configurable B-spline parameterization method was extended to handle different grouped components and applied to complete wing boxes consisting of spars, ribs, skin, stringers, spar caps, and rib caps. Different wing box configurations were studied for a similar commercial plane and optima were found. The results illustrate the successful implementation in structural optimization of wing boxes and it is concluded that the configurable B-spline parameterization method is a practical method for the design optimization of wing boxes. The use of the automatic scheme reduced the computational cost, saving 60% (equivalent to 2.52 times fast) in terms of the CPU time, which illustrates the benefit of increasing computational efficiency in large scale problems. Again, this improvement in computational efficiency is based on the ground of using the elemental KS stress method.

Thus, through the integration of the methods developed in this dissertation, namely the elemental KS stress, the configurable B-spline parameterization, and the automatic scheme method, the objective of solving a large scale nonlinear structural optimization of wing boxes under multiple load conditions within an affordable time frame has been achieved.

## 5.2 Contributions

In work towards this dissertation, many areas of structural optimization were explored in depth. The major contributions are summarized as follows:

1. A new configurable B-spline parameterization method was developed and presented as a practical tool for structural optimization of wing boxes. Implementation of this method shows that it is a promising solution of solving large optimization problems in wing box design.
2. An elemental stress aggregation method for structural analysis is employed for the first time in structural optimization for different problems which benefits structural optimization through a much faster convergence and greater robustness of the optimization process.

3. An automatic scheme method was devised for the configurable B-spline parameterization method to further reduce the computational cost, which is a new implementation of refinement on B-spline control points.

4. Plate and stiffened plate problems were explored thoroughly and the pattern of optima was found to be either a symmetric optimum or a pair of non-symmetric optima under symmetric load conditions. More significantly, all local optima were found for the selected problems, which provide a benchmark for further research. The reason for multiple optima and one-node hinge problems is explained.

5. A method for handling multiple load conditions was presented and extensively studied for different problems.

6. An integration of the configurable B-spline parameterization, the automatic scheme, and the elemental stress aggregation method, was developed to address the challenges in solving large scale structural optimization problems of wing boxes under multiple load requirements.

### 5.3 Recommendations

As a large-scale optimization problem, structural optimization of wing boxes is a significant topic that has an obvious application in industry. This problem also motivates academic research, since it constitutes a challenging problem. However, the challenges of this topic have not been fully addressed.

Through the completion of this dissertation, there are several considerations for the future research:

First, structural optimization needs to be incorporated with computational fluid dynamics to pursue aero-structural optimization.
Second, a parallel structural analysis for multiple load conditions needs to be implemented in order to increase the computational efficiency in the analysis level to match the same pace of performing an analysis for one single load condition.

For more practical applications, more constraints need to be considered such as thermal effect constraints, bulking constraints, etc. Composite materials could be included in the optimization formulation and solved at the component design level. A test model for validation needs to be built and relevant testing methods are required.

The configurable B-spline parameterization method could be implemented with knowledge-based engineering to optimize the method itself as well as specific designs. The idea of configurable B-spline parameterization may be useful to other optimization fields. Finally, the automatic scheme could be extended to handle more complex situations to exploit the optimization process.
References


REFERENCES


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Appendix A

Stress Analysis for Wing Boxes

The term stress used in this dissertation means the von Mises stress. For the frame element, the von Mises stress $\sigma_v$ is calculated by the normal stress and shear stress\[78\]. For the shell element, the stress is calculated by the in-plane stress and bending stress\[79\].

The stress analysis for a wing box depends on the structural model used and the loads applied. The thin-walled beam model can be used to estimate a rough analysis and the FE model can be used to provide a more accurate analysis. In any case, the stresses can be divided into four types subject to axial, bending, shear and torsional loads. For the frame element, the von Mises stress $\sigma_v$ is calculated as

$$\sigma_v = \sqrt{\sigma_n^2 + 3\sigma_t^2}$$  \hspace{1cm} (A.1)

where $\sigma_n$ is the stress resulting from axial and bending loads, $\sigma_t$ is the stress resulting from the torsional load\[78\]. For the shell element, the stress can be calculated with the in-plane stresses and bending stresses\[79\] as follows:

$$\sigma_v = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$  \hspace{1cm} (A.2)

where $\sigma_1$ and $\sigma_2$ are the plane principal stresses.

Stress Analysis for 3D-Frame Elements

Normal stress: tensional and two-dimensional bending normal stresses

$$\sigma_n = \frac{f_x}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$  \hspace{1cm} (A.3)
Maximum stress at the surface of solid circle

\[ z = r \cos \theta, \ y = r \sin \theta, \ I_y = I_z \] (A.4)

\[ \sigma_n(\theta) = \frac{f_x}{A} + \frac{M_y r \cos \theta}{I_y} - \frac{M_z r \sin \theta}{I_z} \] (A.5)

\[ \frac{d\sigma_n(\theta)}{d\theta} = -\frac{r}{I_y} (M_z \cos \theta + M_y \sin \theta) = 0 \] (A.6)

\[ \tan \theta = -\frac{M_z}{M_y}, \ M_y \neq 0 \] (A.7)

\[ \tan \theta = -\frac{\pi}{2}, \ M_y = 0 \] (A.8)

Shear stress: torsional shear stress only

\[ \sigma_t = \frac{Mr}{I_p} = 2 \frac{M_z}{\pi r^3} \] (A.9)

**Stress Analysis for 3D-Shell Elements**

In-plane stresses

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
D
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}, \quad
D =
\frac{E}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\] (A.10)

Bending stresses

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\frac{6}{t^2}
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}, \quad
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\frac{t^3}{12}
D
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
2\kappa_{xy}
\end{bmatrix}
\] (A.11)

2D Principal Stresses and Plane Stresses

\[ \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \] (A.12)

\[ \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \] (A.13)

\[ \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \ \sigma_x \neq \sigma_y \] (A.14)

\[ \tan 2\theta = 45^\circ, \ \sigma_x = \sigma_y \] (A.15)
Appendix B

Verification

B.1 FEAP Verification

In order to verify the finite element analysis code FEAP [64, 76], a plate with four cantilevered edges is selected and analysed by FEAP. The results are compared with the exact solution of the same problem shown in the book by Shames and Dym [80].

The dimensions of the plate are: 10 feet in length, 10 feet in width, and 1 inch in height. The physical properties include: Young’s modulus, $E = 3.0 \times 10^7$ psi, Poisson’s ratio, $\mu = 0.3$, and the shear factor $\kappa = 1$. The uniformly distributed pressure $q_0 = 20$ psi. The four edges of this square plate are cantilevered. The theoretical maximum displacement $w_{ref}$ is obtained from the analytical method of Timoshenko [80] and equals to 0.1588 inch. The convergence of the maximum displacement is shown in Figure B.1. It is concluded that FEAP is a reliable analysis tool.
B.2 Optimization Verification

A square plate with one cantilevered edge is selected as the test sample for optimization verification. The length and width are 10m. The optimization problem is the minimum weight of the plate with respect to the thickness and subject to the allowable yield stress. The magnitude of the nodal force applied is $10^6$N. The single load conditions are labeled as L1, L2, and L4 which correspond to one nodal force at the right corner, at the left corner, and two nodal forces at the right and left corners. The multiple load conditions take into account different single loads and labeled as ML1 for L1 and L3, ML2 for L1 and L4, ML3 for L3 and L4, and ML4 for L1, L2, and L4. The lower bound is $10^{-5}$m for thicknesses and the convergence criteria are $10^{-6}$. The yield stress is 431MPa and the density is 2770kg/m$^3$.

Figure B.2 shows the ground structure with a right-handed coordinate system as well as the load conditions which are perpendicular to the plate. The red line represents the fixed edge of each structure.
Section B.2. Optimization Verification

First, this plate is optimized under the three single load conditions and the minimum weight and optimal thickness are listed in Table B.1 for reference. It is observed that the optimal structure P1L1-1 which is obtained under load L1 is identical to the optimal structure P1L3-1 which is found under load L3. For each load, there is only one optimal structure found and it has one fully-stressed element, which is caused by the use of the elemental stress in the analysis.

![Figure B.2: Shape and load conditions of a cantilevered square plate.](image)

Second, this plate is optimized under the four multiple load conditions and the minimum weight and optimal thickness are listed in Table B.2. For this one-element plate, the optimum of a multiple load case is the maximum optimum of each single load involved.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>Optimal thickness (m)</th>
<th>Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1L1-1</td>
<td>31441.30</td>
<td>1.1349E-01</td>
<td>L1</td>
</tr>
<tr>
<td>P1L3-1</td>
<td>31441.30</td>
<td>1.1349E-01</td>
<td>L3</td>
</tr>
<tr>
<td>P1L4-1</td>
<td>38755.99</td>
<td>1.3989E-01</td>
<td>L4</td>
</tr>
</tbody>
</table>

Table B.1: Minimum weight and optimal thickness of the one-element plate under single loads.
Section B.2. Optimization Verification

<table>
<thead>
<tr>
<th>Structure</th>
<th>Weight (kg)</th>
<th>Optimal thickness (m)</th>
<th>Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1ML1-1</td>
<td>31441.30</td>
<td>1.1349E-01</td>
<td>ML1 (L1, L3)</td>
</tr>
<tr>
<td>P1ML2-1</td>
<td>38755.99</td>
<td>1.3989E-01</td>
<td>ML2 (L1, L4)</td>
</tr>
<tr>
<td>P1ML3-1</td>
<td>38755.99</td>
<td>1.3989E-01</td>
<td>ML3 (L3, L4)</td>
</tr>
<tr>
<td>P1ML4-1</td>
<td>38755.99</td>
<td>1.3989E-01</td>
<td>ML4 (L1, L3, L4)</td>
</tr>
</tbody>
</table>

Table B.2: Minimum weight and optimal thickness of the one-element plate under multiple loads.

Figure B.3: Optimal topologies for a cantilevered one-element plate under different loads.

A refinement of elements is applied and the optimal structures obtained under the single load conditions are shown in Figure B.3. In this figure, the columns correspond to the three loads, L1, L2 and L4, respectively. It is observed that there is only one optimum obtained for each refined finite element model regardless of the load conditions because there is only one design variable used.
B.3 B-Spline Parameterization Verification

In order to verify the B-spline parametrization method, the same plate problem is selected and optimized under load L4 using both the B-spline parametrization and individual design variable method. In the run, the convergence criteria and starting design point are the same for both methods, which are $10^{-6}$. But the variable lower bounds that is $10^{-5}$ m are used for the IDV method and the equivalent variable lower bounds are used as additional constraints for the B-spline approach. The two-dimensional bicubic B-spline with $4 \times 4$ control points is used for the 16-element model to match the 16 individual design variables.

The comparison of the optimal thicknesses is listed in Table B.3 for reference.

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>IDV (m)</th>
<th>Bicubic B-spline (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>2.2383040149110695e − 01</td>
<td>2.238304016337429E − 01</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.0000000000003243e − 05</td>
<td>1.000000000000000306E − 05</td>
</tr>
<tr>
<td>$x_3$</td>
<td>9.999999999999754e − 06</td>
<td>1.00000000001000E − 05</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2.2383040163354073e − 01</td>
<td>2.238304016660555E − 01</td>
</tr>
<tr>
<td>$x_5$</td>
<td>2.0007565714363082e − 01</td>
<td>2.00075657897118E − 01</td>
</tr>
<tr>
<td>$x_6$</td>
<td>9.999999999992656e − 06</td>
<td>9.9999999650193E − 06</td>
</tr>
<tr>
<td>$x_7$</td>
<td>1.000000000004827e − 05</td>
<td>9.9999999565261E − 06</td>
</tr>
<tr>
<td>$x_8$</td>
<td>2.000756578965158e − 01</td>
<td>2.000756534817370E − 01</td>
</tr>
<tr>
<td>$x_9$</td>
<td>1.5918601200340615e − 01</td>
<td>1.591860852620878E − 01</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>9.9999999999973276e − 06</td>
<td>1.0000000000161354E − 05</td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>1.0000000000003321e − 05</td>
<td>1.000000000020256E − 05</td>
</tr>
<tr>
<td>$x_{12}$</td>
<td>1.5918608526210970e − 01</td>
<td>1.591860835015038E − 01</td>
</tr>
<tr>
<td>$x_{13}$</td>
<td>1.1301304070965493e − 01</td>
<td>1.130130434320368E − 01</td>
</tr>
<tr>
<td>$x_{14}$</td>
<td>1.000000000002693e − 05</td>
<td>1.00000000000133E − 05</td>
</tr>
<tr>
<td>$x_{15}$</td>
<td>9.999999999995666e − 06</td>
<td>9.99999999996123E − 06</td>
</tr>
<tr>
<td>$x_{16}$</td>
<td>1.1301304343204298e − 01</td>
<td>1.130130434017252E − 01</td>
</tr>
</tbody>
</table>

Table B.3: Comparison of optimal design variables.

From Table B.3 it is concluded that the optimal thicknesses are identical. The identical
results verify the correctness of the B-spline approach because the control points of this B-spline configuration are identical to the individual design variables. Figure B.4 compares the optimal thicknesses and stress distributions with the two methods.

![Figure B.4: Comparison of optimal plates of 16 elements using B-spline thicknesses versus individual thicknesses.](image)

### B.4 Optimization Verification with Beam Theory

A spar with two caps is shown in Figure B.5. The load is a nodal force, $P$, applied at the upper-right node. The minimum weight problem is studied. In the course of optimization, the classical beam theory is implemented and compared with the finite-element method.

The contour plot of weight function and stress constraint are plotted in Figure B.6 for using the classical beam theory. The blue circle represents the optimum found for the spar-cap structure with the classical beam theory as the analysis tool. The red circle is the optimum found for the spar only structure using the classical beam theory. The green squares represent the optima found by using the finite-element method.
Section B.4. Optimization Verification with Beam Theory

Figure B.5: Problem layout.

Figure B.6: Contour plot of weight and stress constraint using beam theory.

Figure B.7 shows the contour plot of weight and constraints for the spar-cap structure using the finite-element method. Two optima exist for this spar-cap structure problem: one optimum corresponds to an optimal spar with two caps and another one corresponds to an optimal spar without caps. Figure B.8 shows the contour plot for the spar-cap problem using the KS function and modified KS function as the constraint. These is no discrepancy in optima of using two kinds of KS function. The optima are very close to those of the individual stress constraints.
Figure B.7: Contour plot of weight and individual stress constraints using FE analysis.

Figure B.8: Contour plot of weight and aggregated stress constraint using FE analysis.