Numerical Determination Of The Transition Boundary Between Regular and Mach Reflection For Planar Shocks Striking Wedges and Cones in Air

by

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A thesis submitted in conformity with the requirements for the degree of Masters of Applied Science
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Abstract

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A numerical investigation of the interaction of a planar shock wave with a rigid wedge and cone in an air-filled shock tube is performed by computing the unsteady flow field of the interaction process. The Euler and Navier-Stokes equations are solved in two dimensions to produce flow solutions for regular and Mach reflections with and without the viscous and thermal boundary layer on the inclined surface. The transition boundary between these two patterns is determined by changing both the shock strength and the angle of the inclined surface so that the simulations are perpendicular to the theoretical transition boundary. The numerically determined boundaries are compared to the theoretical boundaries predicted by two- and three- shock theories and with results obtained from experiments. The results show that the transition boundary between regular and Mach reflection is different not only for wedges and cones but also for inviscid and viscous numerical solutions.
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Nomenclature

\( \alpha \) perpendicular distance from detachment criterion line (in the \( M_s-\theta_w \) plane)

\( \beta \) tangential distance from detachment criterion reference point (in the \( M_s-\theta_w \) plane)

\( \gamma \) specific heat capacities ratio

\( \theta_w \) angle of incline of wedge or cone surface

\( \theta_1 \) flow deflection angle after incident oblique shock, counterclockwise is positive

\( \theta_2 \) flow deflection angle after reflected oblique shock, counterclockwise is positive

\( \kappa \) thermal conductivity

\( \mu \) dynamic viscosity

\( \rho \) density

\( \tau \) fluid stress tensor

\( \phi \) rotation angle between the \( M_s-\sin(\omega) \) and \( \alpha-\beta \) axes

\( \omega \equiv 90^\circ - \theta \)

\( e \) internal energy

\( k \) symbol identifying a side of a quadrilateral cell used in the finite-volume scheme

\( l_i \) horizontal distance from wedge corner to incident shock

\( l_k \) length of side \( k \) of quadrilateral cell

\( l_m \) horizontal distance from wedge corner to Mach stem
$m_\beta$ slope of the $\beta$ axis

$n_k$ outward normal vector of side $k$ of quadrilateral cell

$p$ pressure

$p_l$ left state pressure

$p_r$ right state pressure

$p^*$ average of $p_l$ and $p_r$

$q$ heat flux vector

$t$ time

$\delta t$ change in time

$u$ velocity in the $x$-direction

$v$ velocity in the $y$-direction

$A_{ij}$ area of the cell $(i,j)$

$C_v$ constant volume specific heat constant

$E$ total energy per unit mass

$\mathbf{F}, \mathbf{G}$ inviscid flux vectors

$\mathbf{F}_v, \mathbf{G}_v$ viscous flux vector

$\mathbf{H}$ sum of inviscid and viscous fluxes

$L$ non-dimensional Mach stem length

$M_l$ left state Mach number

$M_s$ shock strength, Mach number

$M_0$ Mach number ahead of shock

$P$ point of reflection
\( R \) specific gas constant

\( S \) source term used in Navier-Stokes equations

\( S_i \) incident shock

\( S_r \) reflected shock

\( T \) temperature

\( T_i \) left state temperature

\( U \) vector of conserved variables

\( V_i \) incident shock velocity

\( V_m \) Mach stem velocity in the \( x \)-direction

\( V_w \) Mach stem velocity up the wedge or cone surface

AMR adaptive mesh refinement

CFD computational fluid dynamics

CFL Courant–Friedrichs–Lewy (criterion)

CPU central processing unit

HLLE Harten, Lax, van Leer and Einfeldt

MR Mach reflection

RP reference point

RR regular reflection

UTIAS University of Toronto Institute for Aerospace Studies
Chapter 1

Introduction

1.1 Preliminary Remarks

A uniform flow interacting with an inclined surface, such as a wedge or a cone, can result in many different types of flow patterns depending on the Mach number (and more importantly, whether it is subsonic or supersonic), the angle of the incline, and if the flow is steady or unsteady. Figure 1.1 shows some different types of steady flows interacting with an inclined surface. In the subsonic case, the flow anticipates the corner and makes a smooth turn up the wedge. For the supersonic case, a shock wave appears that either instantly turns the flow through an oblique shock that starts at the corner, or subsonically turns the flow after a bow shock that starts before the corner. Steady flow reflections from inclined surfaces is not considered in this thesis.

When the initial flow is driven by a shock wave the flow is unsteady. Thus we need to first consider an initial geometry, such as the one shown in Figure 1.2. Once the shock strikes the inclined surface, the shock strength, $M_s$, and the angle of the incline, $\theta_w$, will...
determine the type of reflection that occurs. For any $M_s - \theta_w$ combination, the resulting reflection pattern will have two or more shocks. A two-shock regular reflection (RR), as shown in Figure 1.3(a), occurs when a planar shock encounters a surface inclined at a large $\theta_w$. The incident ($S_i$) and reflected ($S_r$) shocks meet at a reflection point on the surface of the wedge. By decreasing $\theta_w$, the reflection point moves away from the wedge surface and a Mach stem shock appears that connects the point of reflection with the wedge, as shown in Figure 1.3(b). The point at which all three shocks meet is called the triple point. A slipstream emanates from the triple point and separates two different flows. These two different flows have the same pressure but different velocities along the slip stream. At higher Mach numbers, complex reflection patterns occur with more than three shocks, as shown in Figure 1.3(c). Even though there are many different complex types of reflection patterns, the use of the term “Mach reflection” in this work simply refers to any reflection that includes a Mach stem. Shock wave reflections with unsteady flows over an inclined surface are the primary focus of this thesis.

Research in the field of shock wave reflection phenomena has focused on many points of interest since different reflection patterns were first discovered and studied by Ernst Mach [22] in the last quarter of the 19th century [24]. The transition point between these different patterns has been one of these points of interest. Since $M_s$ and $\theta_w$ are the two

![Figure 1.2: The initial condition of an unsteady flow interaction with a inclined surface within a shock tube.](image)

![Figure 1.3: Diagrams of two- and three-shock regular and Mach reflections, as well as a complex Mach reflection showing the incident ($S_i$) reflected ($S_r$) and Mach stem shocks that intersect at the wedge and triple point.](image)
1.1. Preliminary Remarks

Figure 1.4: The $M_s$-$\theta_w$ plane is formed by the two main parameters that determine the type of reflection pattern.

main parameters that determine the type of reflection pattern, a region for RR and MR can be visualized in the $M_s$-$\theta_w$ plane, as depicted in Figure 1.4. As mentioned before, RR occurs at a larger $\theta_w$ compared to that for MR. When a transition point between RR and MR is determined for each value of $M_s$ in this plane, a transition boundary will result. This type of boundary can be found theoretically by using the conventional equations for oblique shocks that are straight (not curved). By considering flow states only in the near vicinity of the reflection point on the wedge, three theoretical transitions

Figure 1.5: The theoretical transition boundaries for air determined by two- and three-shock theories that divide the $M_s$-$\theta_w$ plane into a RR, MR and dual region.
Chapter 1. Introduction

separating RR and MR regions can be determined and are shown in Figure 1.5. These transition boundaries correspond to shock reflection occurring in air. The dual region is an overlap region in which RR or MR can occur.

1.2 Theoretical RR to MR Transition Boundaries

The detachment, sonic and mechanical equilibrium transitions are theoretical transition lines that have been known for some time and date back to the work of von Neumann [29]. Griffith [13] states the assumptions made by von Neumann: the fluid (air in this case) is an inviscid perfect gas and has a constant specific heat ratio, $\gamma$; the shocks are discontinuous and have finite curvature; near the point of intersection, two-dimensional steady flow theory can be used; there is no net flow deflection after passing through the incident and regular shocks in RR; and that in a MR, the downstream pressure and flow angle behind both the Mach stem and the incident-reflected shock pair are equal, which means that there must be a slipstream.

The detachment criterion was proposed by von Neumann [29] and gives a boundary in the $M_s-\theta_w$ plane below which RR can not occur. It considers the maximum flow deflection through an oblique shock. Figure 1.6 shows a close-up of the point of reflection of a RR. Notice that the area being considered is actually a very small region since the incident and reflected shocks are considered straight. In the diagram, point P is non-accelerating and is therefore a location of an inertial frame of reference. The incident, I, and reflected, R, shocks separate three different thermodynamic states. For RR to occur, the flow

![Figure 1.6: The vicinity of the point of reflection. Point P is a point of an inertial frame of reference that is not accelerating.](image-url)
deflection angles, $\theta_1$ and $\theta_2$, must be equal in magnitude and opposite in direction so the net flow deflection is zero. The transition point from RR to MR occurs when the maximum flow deflection angle across the oblique shock is less than that required for the flow after the reflected shock to be parallel to the wall.

Mechanical equilibrium was first proposed by von Neumann [29] and refers to a system of gas particles in this case that have no resulting external forces or moments acting on it. When considering the transition from RR to MR, Henderson and Lozzi [16] later proposed that for the mechanical equilibrium criterion, there will be no pressure discontinuity during the transition process. The Mach stem exists and is everywhere normal to the flow, but it also has zero length and therefore at this point both the RR and the MR exist.

The sonic criterion was also first proposed by von Neumann [29] but later reintroduced by Hornung, Oertel and Sanderman [18] and Lock and Dewey [21]. It considers the signal from the corner (the beginning of the inclined surface), and argues that transition occurs when this signal can interact (catch up) with the reflection point. The signal speed is the flow speed along the surface plus the local sound speed. Transition can occur when the flow speed behind the reflected shock is supersonic.

Von Neumann applied the conservation of mass, momentum and energy equations across the straight incident, reflected and Mach stem shocks, and arrived at a set of equations that describe the flow states near the reflection point for the two-shock RR and three-shock MR patterns. These theoretical results by von Neumann have been reworked into a special exact form by Henderson [17]. The resulting criteria involve complicated and higher-order polynomials with multiple roots and require great care in order to determine the correct transitions. Henderson also provides the necessary equations for determining a point of intersection between the mechanical equilibrium line and the detachment criterion. For air and perfect diatomic gases ($\gamma = 7/5$) the point occurs at a shock strength of $M_s = 1.457$ and an incline $\theta_w = 48.588^\circ$, but many diagrams do not show the mechanical equilibrium line before this point. However, it is shown in Figure 1.5, and it is interesting to note that it curves away from the detachment and sonic criterion boundaries.

The mechanical transition applies in steady flows only, whereas the other two are for moving shocks over inclined surfaces. Figure 1.7 shows the theoretical transition lines as well as experimental transition points determined by Bleakney & Taub [6], Smith [27], Griffith and Bleakney [12] and Kawamura and Saito [19]. Although the experimental
points are close to the detachment and sonic boundaries, they don’t agree exactly, and this discrepancy has been the source of much interest and research in the past.

Also of interest is the bifurcated shock tube experiment by Barbosa and Skews [1]. Using a carefully designed shock tube shown in Figure 1.8(a), an initial shock wave is split into two planar shocks that travel along separate tubes and subsequently interact. They reflect off each other with no boundary layer because a nonmoving inclined surface is replaced by a virtual reflecting plane. Figure 1.8(b) shows an actual experimental photo of a mutual reflection and a resulting MR occurring in the test section.

Any experimental setup will have factors that degrade the quality of the results. For example, the roughness of the surface of the wedge or the resolution of the photographic equipment are variables that increase the complexity of the experiment. These make it difficult to pinpoint the exact transition between RR and MR. One of the difficulties unique to the Barbosa and Skews experiment was ensuring that both planar shock waves arrived at the test section (and more importantly the corner) at the same time and with the same strength. The authors claim that the transition result determined from the bifurcated shock tube confirms von Neumann’s transition boundary theory. However, it can also be argued that their experimental results show some slight discrepancy with von Neumann’s theory since it does not lie exactly on the theoretical transition line. A more
1.2. Theoretical RR to MR Transition Boundaries

Figure 1.8: A schematic diagram of the bifurcated shock tube used by Barbosa and Skews\cite{1} is shown in (a). Actual experimental photos of their MR pattern occurring in the test section is shown in (b).
systematic mapping of the transition boundary by accurate numerical computational fluid dynamics methods undertaken in this thesis is aimed at explaining such discrepancies.

1.3 Planar and Axisymmetric Shock Wave Reflections

A key feature of the shock reflection phenomena from wedges and cones considered in this work is that the reflecting flows are self-similar. This means that after the incident planar shock encounters the corner of the inclined surface, the reflection pattern grows in size but does not change in shape. As time evolves, the distance between any two points in the pattern simply scales with time. Any two pictures of the reflection pattern taken at different times will look exactly the same with the only difference being the size of the reflection pattern.

For self-similar flows, the only two geometries that can be considered are the planar and axisymmetric scenarios, and these are illustrated in Figure 1.9. The theoretical transition boundaries shown in previous sections do not consider whether the inclined surface is planar or axisymmetric. Formally, the transition boundaries do not depend on the azimuthal angle of the inclined surface.

![Figure 1.9: The only two geometries that can be considered with the self-similar shock reflection process.](image)

1.4 Objective

The primary objective of this research is to verify the discrepancy in the theoretical transition boundaries with the experimentally determined boundaries. Computational fluid dynamics (CFD) simulations are performed in order to probe the boundary between RR
and MR, and innovative methods of estimating the transition are used that results in a numerically determined transition boundary. The numerical simulations use the latest CFD and Propulsion group code at the University of Toronto Institute for Aerospace Studies (UTIAS). This versatile code allows, unlike experiments, for the combined viscous and thermal boundary layer to be turned on and off by switching between the solution of the Navier-Stokes and Euler equations, respectively. This work will show that the numerically determined transition boundaries are different for numerical calculations with and without the presence of a combined viscous and thermal boundary layer. In addition, numerical simulations will be performed to determine the transition boundaries for the only two possible self-similar reflecting geometries, that is for planar (wedge) and axisymmetric (cone) surfaces. The results will show that von Neumann theory does provide reasonable transition values, but they are not necessarily exact or equally applicable to both wedge and cone geometries.
Chapter 2

Numerical Solution Procedure

2.1 Governing Equations

The interactions of planar shocks and their flow fields with wedges and cones in air filled shock tubes are predicted by solving the compressible forms of the Euler and Navier-Stokes equations in two dimensions, for shock induced flow-fields for both planar and axisymmetric geometries. The set of partial differential equations that form the Navier-Stokes equations that describe laminar flow contain terms involving the viscosity and heat conduction. When these terms are omitted we are left with the Euler equations. The Navier-Stokes equations for a two-dimensional Cartesian coordinate system are

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \frac{\partial \mathbf{F}_v}{\partial x} + \frac{\partial \mathbf{G}_v}{\partial y} + \mathbf{S},$$

(2.1)

where the conserved variables $\mathbf{U}$ are given by

$$\mathbf{U} = [\rho, \rho u, \rho v, \rho E]^T,$$

(2.2)

the inviscid flux vectors $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{F} = \left[ \begin{array}{c} \rho u, \rho u^2 + p, \rho uv, \rho u \left( E + \frac{p}{\rho} \right) \end{array} \right]^T,$$

(2.3)

$$\mathbf{G} = \left[ \begin{array}{c} \rho v, \rho uv, \rho v^2 + p, \rho v \left( E + \frac{p}{\rho} \right) \end{array} \right]^T,$$

(2.4)
and the viscous flux vectors $\mathbf{F}_v$ and $\mathbf{G}_v$ are given by

$$
\mathbf{F}_v = [0, \tau_{xx}, \tau_{xy}, -q_x + u\tau_{xx} + v\tau_{xy}]^T, \quad (2.5)
$$

$$
\mathbf{G}_v = [0, \tau_{xy}, \tau_{yy}, -q_y + u\tau_{xy} + v\tau_{yy}]^T. \quad (2.6)
$$

The conserved solution variables of $\mathbf{U}$ in equation (2.2) are expressed in terms of the density, $\rho$, the $x$ and $y$ velocity components, $u$ and $v$, respectively, and the total energy per unit mass, $E$, and the pressure, $p$, is a variable in the inviscid fluxes, $\mathbf{F}$ and $\mathbf{G}$. The fluid stress tensor elements, $\tau_{xx}$, $\tau_{xy}$ and $\tau_{yy}$, are given by

$$
\tau_{xx} = \frac{\mu}{3} \left( 4\frac{\partial u}{\partial x} - 2\frac{\partial v}{\partial y} \right), \quad (2.7)
$$

$$
\tau_{yy} = \frac{\mu}{3} \left( -2\frac{\partial u}{\partial x} + 4\frac{\partial v}{\partial y} \right), \quad (2.8)
$$

$$
\tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau_{yx}, \quad (2.9)
$$

where $\mu$ is dynamic viscosity. The $x$ and $y$ components of the heat flux vector $q_x$ and $q_y$ are given by

$$
q_x = -\kappa \frac{\partial T}{\partial x}, \quad (2.10)
$$

$$
q_y = -\kappa \frac{\partial T}{\partial y}. \quad (2.11)
$$

They are used in defining elements of the flux vectors. The thermal conductivity is given by $\kappa$. When considering only the Euler equations (inviscid solution), the $\mathbf{F}_v$ and $\mathbf{G}_v$ vectors vanish and the following results remains as

$$
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}. \quad (2.12)
$$

Assuming a perfect gas, the equation of state is given by $p = \rho RT$, and the internal energy $e$ is given by $e = C_v T$, where the constant volume specific heat constant is given by $C_v = \frac{R}{\gamma - 1}$. Here, $T$ is the temperature, and $R$ is the gas specific constant. Note that in the axisymmetric case, an additional source term $\mathbf{S}$ incorporates the influence of the
axisymmetric geometry. It must be taken into account when solving equation (2.1) and (2.12) using the finite-volume scheme described below.

### 2.2 Numerical Method

The governing equations given above are solved using an explicit higher-order Godunov-type finite-volume scheme over a structured multi-block quadrilateral mesh. The semi-discrete form of these equations is solved and given by

\[
\frac{dU_{ij}}{dt} = -\frac{1}{A_{ij}} \sum_{k=0}^{l_k} (H \cdot n_k)_{ij}.
\]

In the above equation, \(A_{ij}\) is the area of the quadrilateral cell, such as the one shown in Figure 2.1, \(l_k\) is the length of a side \(k\), and \(n_k\) is its outward normal vector. In equation (2.13), \(H\) is the sum of the inviscid and viscous fluxes of the Navier-Stokes equations in two dimensions. A second-order predictor-corrector time marching method is applied to the time derivative and the finite-volume scheme becomes fully discrete. The HLLE [9, 15] approximate flux function is used to solve the Riemann problem and obtain the inviscid fluxes at the cell interfaces. Godunov showed that a constant-coefficient scheme cannot be both second order accurate and preserve monotonicity[10]. Thus, the slope limiters of Venkatakrishnan [28] or Barth-Jespersen [2] are used to ensure monotonicity of the solution near discontinuities while maintaining a higher order of accuracy in the

![Figure 2.1: The two-dimensional quadrilateral cell for the semi-discrete approach.](image)
smooth regions. The viscous fluxes in the Navier-Stokes equations are calculated using
the diamond path reconstruction method described by Coirier and Powell [7, 8].

2.3 Grid, Initial and Boundary Conditions

Simulations are performed on a grid that begins as 2 blocks, each consisting of 10 or 14
cells in each of the $x$ and $y$ directions. Parallel implementation of the solution method
is used to help reduce computational times in conjunction with eight or ten levels of
adaptive mesh refinement (AMR), which is employed to accurately capture complicated
flow features with large gradients, such as shock fronts, triple shock confluence points
and shear layers. The AMR procedure increases the number of cells near these features
without having to apply the same high level of mesh refinement to the entire grid. This
helps to reduce memory and central processing unit (CPU) requirements. Berger and
Collela [5] demonstrated this approach to be successful for hyperbolic conservation laws in
two dimensions. The code used to perform simulations is based on the method developed
by Groth, De Zeeuw, Powell, Gombosi, and Stout [14] which uses a parallel AMR scheme
to solve a hyperbolic set of partial differential equations. The multi-block body-fitted
mesh adapts to the flow field depending on the physics-based criteria. Figure 2.2 shows
an example of an early stage of a simulation before the incident shock interacts with the
inclined surface. This early stage is similar to the initial condition shown in Figure 1.2
in chapter 1. Details of the numerical scheme and parallel adaptive mesh refinement are
available in previous papers by Sachdev, Groth and Gottlieb [25, 26].

2.3.1 Grid for Inviscid Flow

The geometry is setup for a simulation to allow the incident shock wave to propagate
approximately 0.9 meters up the wedge, which means that in the $x$-direction after the
corner, the shock wave will propagate less than 0.9 meters. Care must be taken to
ensure that by the end of a simulation, the computational effort is mostly focused on
the reflection pattern and the incident shock and Mach stem which should occupy a
large portion of the computational domain. The first block of cells (for the non-inclined
surface) has a length of 1.0 meter in the $x$-direction, and the second block (for the wedge
surface) has a length of 1.0 meter along the wedge surface.
Figure 2.2: The grid near the beginning of a Navier-Stokes numerical simulation. The shock is traveling to the right in the positive $x$-direction and located at approximately $x = -0.03$ m. Adaptive mesh refinement at this early stage of the simulation occurs near the shock and near the lower boundary behind the shock. The level of refinement was reduced in this case to clearly show the computational cells.
### 2.3.2 Grid for Viscous Flow

When considering the presence of a boundary layer, it is important to ensure that the grid contains an adequate number of cells and that the boundary layer is properly resolved. Solving the Navier-Stokes equations however is much more computationally intensive. Therefore, to capture the boundary layer while reducing the computational times, the size of the domain is reduced to one-tenth the domain of the inviscid computations. This reduction in domain size means that the incident shock waves will propagate approximately 0.09 meters up the wedge. The first block of cells (the non-inclined surface) has a length of 0.10 meter in the $x$-direction and the second block (the inclined surface) has a length of 0.10 meters along the wedge surface. It should be noted that although there is a great deal of refinement in the vicinity of any shock, the shock structure is not resolved.

### 2.3.3 Initial and Boundary Conditions

The input file used to run these simulations requires some initial and boundary conditions along with some CFD and grid parameters. Once air as a perfect gas has been specified as the gas used in the simulation, and standard atmospheric conditions have been specified as the right state (the state ahead of the shock) and these are given in Table 2.1. The input file then requires manual input of the left state. The left Mach number $M_l$, pressure $p_l$ and temperature $T_l$ are needed as boundary conditions. For viscous computations, the wall temperature is set to 288.16°. Since the amount of interaction with the wall is desired to be constant for all simulations, the distance a shock travels up a wedge is kept constant. Therefore, the time that a simulation is allowed to progress is different for each run and depends on the angle of the incline.

There are several user-set parameters related to the AMR process that affect the

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</tr>
<tr>
<td>flow velocity $v$</td>
<td>0 m/s</td>
</tr>
</tbody>
</table>

Table 2.1: Right state boundary conditions are the standard atmospheric conditions.
resolution of the final result and the computational time required to run the inviscid and viscous simulations. AMR is performed every six time steps and, as mentioned at the beginning of section 2.3, a maximum of eight refinement levels are used for the inviscid simulations, whereas ten are used for the viscous simulations. Each step increase in the level of refinement quadruples the number of cells in a block. This makes a $10 \times 10$ block containing 100 cells into a $20 \times 20$ with 400 cells. These are then split into quarters and the result is four new blocks of $10 \times 10$. Refinement or coarsening take place when the gradient of a user-selected variable exceeds a threshold. In this work, the user-selected variable is the gradient of the density. These thresholds can changes depending on whether weak or strong shocks are being considered. As an example, in some cases when the gradient of density is greater than 0.10, refinement is performed, whereas if it is less than 0.05 coarsening occurs. It should also be mentioned that the simulation begins with at least 2 mesh refinements. Also, the CFL number is 0.50 and since these shock reflection simulations are unsteady, the problem is solved time accurately. The simulations described in this thesis were performed mostly on a parallel cluster of 4-way Integrity rx4640 SMP HP Itanium servers with 1500 MHz Itanium 2 processors and with 8 GBytes of main memory per node. The total CPU time and number of mesh points for any given simulation varied since each run had a unique set of initial conditions. Generally speaking, the total CPU time varied between 10 000 and 40 000 minutes for simulations solving the Navier-Stokes equations, and 1000 and 3000 minutes for those solving the Euler equations. The total number of mesh points varied between 500 000 and 1 000 000 for Navier-Stokes simulations, and 200 000 and 350 000 for Euler simulations.

Figure 2.3 shows the boundary conditions that are applied on the entire grid for the inviscid and viscous simulations. Notice that both grids have the same conditions applied on the north, west and east boundaries. The difference occurs on the south boundary where the inviscid grid has a reflection surface where slip is allowed, whereas the viscous grid has a no-slip isothermal surface.

The combined viscous and thermal boundary layer forms and evolves on the entire south surface. For all viscous simulations, the incident shock is located four centimeters before the corner at $t = 0$ and the boundary layer begins to develop behind this shock immediately afterwards.
Chapter 2. Numerical Solution Procedure

2.4 Selected Numerical Results

A sample inviscid numerical result of an RR pattern is shown in Figure 2.4. The incident shock wave in this case has a Mach number of \( M_s = 1.84906 \), a pressure ratio of about 3.8 and is striking a 1 meter long wedge inclined at an angle of 50.8268°. The contours shown correspond to lines of constant density (isopycnics). The incident and reflected shocks are

Figure 2.4: Numerical inviscid result of a planar shock wave with an incident shock number of 1.84906 striking a wedge at an angle of 50.8268°.
very thin and thus very well captured by the combination of AMR and the finite-volume methods. The reflected shock near the shock tube floor, before the corner, is actually a distributed compression wave that is also very well captured and in agreement with experimental observation.

Selected results for inviscid and viscous axisymmetric simulations for RR and MR patterns at low and high Mach numbers are shown in Figures 2.5, 2.6, 2.7 and 2.8. These results are constant density flood-filled contours instead of isopycnic lines. The incident shock strength and the angle of incline are given under each simulation. Notice that the higher Mach number reflection patterns are actually more complicated double Mach reflections. Figure 2.9 shows a series of constant density flood-filled contours at different times of a particular viscous simulation. The shock wave has a Mach number of about 1.3 and it strikes a wedge inclined at an angle of about 45.47°. Starting from the top left and moving horizontally towards the bottom right, each plot shows the resulting reflection pattern after 0.04 ms have elapsed. The total length of the simulation was about 0.24 ms. Since this type of unsteady reflection phenomenon is initially self-similar, the same reflection pattern appears at every time step after the incident shock strikes the corner.

A typical mesh near the point of reflection during a viscous computations shown in Figure 2.10. Mesh refinement occurs about the incident and reflected shocks as well as along the boundary layer near the wedge surface.
Chapter 2. Numerical Solution Procedure

Figure 2.5: Constant density contours of RR patterns over a cone for a low Mach number inviscid (a) and viscous (b) flow.

(a) $M_s = 1.1742, \theta_w = 40.3531^\circ$

(b) $M_s = 1.1747, \theta_w = 40.2757^\circ$
2.4. Selected Numerical Results

Figure 2.6: Constant density contours of MR patterns over a wedge for a low Mach number inviscid (a) and viscous (b) flow.

(a) $M_s = 1.2063, \theta_w = 35.5308^\circ$

(b) $M_s = 1.2063, \theta_w = 35.5183^\circ$
Figure 2.7: Constant density contours of RR patterns over a cone for a higher Mach number inviscid (a) and viscous (b) flow.

(a) $M_s = 2.9990, \theta_w = 52.5307^\circ$

(b) $M_s = 2.9989, \theta_w = 50.7033^\circ$
2.4. Selected Numerical Results

(a) $M_s = 2.9988$, $\theta_w = 48.8269^\circ$

(b) $M_s = 2.9988$, $\theta_w = 47.6750^\circ$

Figure 2.8: Constant density contours of MR patterns over a wedge for a higher Mach number inviscid (a) and viscous (b) flow.
Figure 2.9: Series of constant density filled contours (isopycnics) showing the propagation of a shock wave with a strength of $M_s = 1.3$ striking a wedge (with a boundary layer) inclined at an angle of about $\theta_w = 45.47^\circ$. 
Figure 2.10: Viscous numerical result near the point of reflection on the wedge showing the adaptive mesh refinement. The mesh refinement that occurs near the boundary layer can be seen near the bottom of the image.
2.5 Calculation of Non-dimensional Mach Stem Length

Post processing of each numerical solution is performed to determine whether the simulation of a shock reflection has resulted in a RR or a MR. This is done by considering the position of the incident shock and the Mach stem, which are recorded at every time step of a given simulation. The position of these shocks can be found by probing each cell along the top horizontal reflection surface and bottom wedge surface of the computational boundary for a given pressure. This pressure $p^*$ being probed for on both surfaces is equal to the average of the left and right initial pressures, thus $p^* = (p_l + p_r)/2$. Generally speaking, by considering only the points recorded after the corner, the position $x$ of these mid-pressure points can be plotted in position-time graphs as shown in Figures 2.11 and 2.12 for computations without a boundary layer. In Figure 2.11, the Mach stem data points fall on top of the incident shock data points, meaning that they have the same position in the $x$-direction at every time step. The Mach stem therefore has a length of zero, and this simulation corresponds to a RR. In contrast, Figure 2.12 shows a Mach stem that is always ahead of the incident shock at every time step. The Mach stem in this case has a non-zero length and therefore this simulation corresponds to a MR.

The velocities of any of the shocks in Figures 2.11 and 2.12 can be found by deter-

![Figure 2.11: Position-time graph of a RR obtained from a numerical simulation showing identical locations of the incident shock and Mach stem in the $x$-direction.](image1)

![Figure 2.12: Position-time graph of a MR obtained from a numerical simulation showing separate locations of the incident shock and Mach stem in the $x$-direction.](image2)
mining the slope of a linear fit to the points on this graph. Knowing the velocities of both the shocks, a non-dimensional length of the Mach stem can be determined using the equation

\[ L = \frac{l_m - l_i}{l_i} = \frac{(l_m/\delta t) - (l_i/\delta t)}{(l_i/\delta t)} = \frac{V_m - V_i}{V_i}, \tag{2.14} \]

in which the lengths \(l_i\) and \(l_m\) are defined as the lengths shown in Figure 2.13. When these lengths are divided by a change in time, the resulting equation involves only the horizontal velocities of the incident and Mach stem shocks, \(V_i\) and \(V_m\) respectively. These velocities are obtained by finding the slope of the data points in the position-time graphs shown by Figures 2.11 and 2.12. RR occurs when \(L = 0\), MR occurs when \(L > 0\), and the transition boundary occurs when \(L\) just exceeds zero. See chapter 3 for more information on the use of the \(L\) to obtain the numerical transition boundary.

### 2.6 Boundary Layer

The combined viscous and thermal boundary layer on the bottom horizontal and inclined surface is an important part of the flow for the Navier-Stokes cases considered herein, because it is a contributing factor to transition boundary between RR and MR, being different from the theoretical boundary predicted by von Neumann. Normally in a sub-sonic or supersonic steady flow, the boundary layer on the inclined surface begins at the leading edge. In the case of the simulations performed for unsteady shock reflection, the boundary layer begins wherever the shock (either incident or Mach stem) makes contact with the inclined surface. This makes it difficult to define a boundary layer, since it is also growing with time. At the surface of the wedge or cone, the fluid velocity is zero and
the velocity gradient is large initially and this gradient becomes smaller as the freestream velocity is approached.

The boundary layer thickness is often defined as the distance in the normal direction from the surface to the point where the velocity reaches 99% of the freestream velocity. The Blasius equation for the boundary layer thickness for a laminar flow on a flat plate is given by

$$\delta = 5.0x \frac{x}{Re_x^{1/2}}.$$  \hspace{1cm} (2.15)

This equation, however, corresponds to a boundary layer that has developed over a flat plate and is in a steady state condition, which is somewhat unlike the unsteady situation studied in this research that involves a moving shock. This makes it difficult to define a good position where the boundary layer profile can be illustrated.

Figure 2.14 shows a close-up of the grid on a wedge of a Navier-Stokes simulation.

![Figure 2.14: The density variation and grid from a Navier-Stokes simulation of a planar shock wave with an incident shock number of 2.424 striking a wedge at an angle of 48.7662°.](image_url)
2.6. Boundary Layer

Also shown is a line perpendicular to the wedge surface along which a boundary layer profile can be determined by finding the flow velocity parallel to the wall. Notice the higher concentration of cells near the surface, within the boundary layer.

The location of several boundary layer profiles taken for a RR pattern whose incident shock has a strength of 1.4711 and is striking a wedge inclined at an angle of 47.9492° is shown in Figure 2.15. The percentages shown correspond to the ratio of the distance from the corner to the point where the profile is taken to the overall distance from the corner to the location of the Mach stem. Figure 2.16 shows the boundary layer profiles at these locations. The horizontal axis represents the flow velocity parallel to the wall (which can be quite high) and the vertical axis is the perpendicular distance from the wall. The points shown in these figures are determined by extracting 20 points along the line shown in Figure 2.14 and each of these points lie approximately in the center of the cells.

Another RR pattern for a stronger shock with a strength of 2.424 striking a wedge inclined at angle of 50.0243° is shown in Figure 2.17. The corresponding boundary layer profiles are shown in Figure 2.18. Note that the disturbed reflection pattern on the south boundary where the inclined surface begins stems from the boundary layer on the horizontal plate interacting with the corner.

Although the boundary layer profiles shown in Figures 2.16 and 2.18 occur for shocks of different strengths, they are similar. The profiles begin with a zero velocity near at wall and increase asymptotic-like toward the freestream velocity as expected. Also, notice how the profiles are nicely separated, never crossing each other, which is what is expected with RR. At the 95% point along the wedge, the closest profile to the reflection point, both the strong and weak shock profiles show a rapid increase in speed and reach the 99% freestream velocity within less than 0.1 mm above the inclined surface. Moving further behind, the 90%, 85%, 80% and 75% profiles show gradual increases in the boundary layer thickness and a reduced freestream velocity. This is also seen in the 50% profile, halfway between the corner and the reflection point. However, the boundary layer thickness is larger and no longer contained within the 20 points sampled from the first 20 cells closest to the wedge surface. These boundary layer results illustrate that the grid with AMR captures the boundary layer within twenty or more cells.
Figure 2.15: A complete view of a RR reflection pattern ($M_s = 1.4711$, $\theta_w = 47.9492^\circ$), along with the locations of the boundary layer profiles that are shown in Figure 2.16. The percentages represent the distance along the wedge from the corner compared to the distance along the wedge from the corner to the Mach stem.
Figure 2.16: The boundary layer profiles from the Navier-Stokes simulation shown in Figure 2.15. The flow velocity parallel to the wall shown on the horizontal axis is determined from the $x$ and $y$ velocity components taken from the line perpendicular to the wedge surface shown in figure 2.14.
Figure 2.17: A complete view of a MR reflection pattern with a stronger incident shock ($M_s = 2.424$, $\theta_w = 50.0243^\circ$), along with the locations of the boundary layer profiles that are shown in Figure 2.18.
Figure 2.18: The boundary layer profiles from the Navier-Stokes simulation of the higher Mach number RR shown in Figure 2.17.
Chapter 3

Method for Determining the Transition Boundary

3.1 The $M_s$-$\sin(\omega)$ Plane and the $\alpha$-$\beta$ Transformation

The two main parameters introduced in chapter 1 that determine the reflection pattern when a planar shock strikes an inclined surface are the shock strength $M_s$ and the wedge angle $\theta_w$. The theoretical transition boundaries introduced previously and plotted in the plane created by the two critical parameters are shown in Figure 3.1. Starting from the RR region in the $M_s$-$\theta_w$ plane where $\theta_w$ is large, the value of the non-dimensional Mach stem length $L$ will be zero since there is no Mach stem. Moving towards the MR region, $L$ will increase in different ways. When $M_s$ is low, the trend of $L$ going from RR to MR is concave downwards and seems parabolic-like. While in the sudden transition region, however, the trend is not as simple and likely further complicated by the fact that the reflections involve more complicated wave patterns, such as double or transitional MR. $L$ does not increase indefinitely, eventually it will reach a maximum and begin to decrease back to zero. This corresponds to scenario where $\theta_w$ is equal to zero and the incident shock simply propagates with no shock reflection occurring at all.

A transition point between RR and MR can be found by starting from the MR region and incrementally changing $M_s$ and/or $\theta_w$ so that the Mach stem is decreased. Eventually the height will just vanish and indicate the transition point between RR to MR. As described in chapter 2, CFD is used to simulate the reflection process given by any $M_s$ and $\theta_w$ combination. Although the theoretical detachment line does not agree with experiment, it is used as a reference line, since it is close to the transition point.
Chapter 3. Method for Determining the Transition Boundary

This helps to concentrate the computational effort near the transition boundary and avoid performing time-consuming simulations that will not help in the extrapolation to a zero Mach stem length.

Previous work on this subject by Nam [23] and Gottlieb, Nam and Groth [11] incrementally changed only the strength of the shock or the wedge angle — not both at the same time. Although this did yield transition values, the extrapolation to zero Mach stem length required a second degree polynomial fitted to the data. Belhimeur [4] and Belhimeur, Gottlieb and Groth [3] have shown that performing simulations perpendicular to the detachment criterion line produces more consistent and linear results for extrapolation, and therefore this affords a better estimate of the transition boundary. Thus, at each of the preselected points on the detachment criterion line, a transformation is performed creating a perpendicular line along which $M_s$ and $\theta_w$ combinations are determined.

Belhimeur [4] and Belhimeur, Gottlieb and Groth [3] also showed that working in the $M_s\sin(\omega)$ plane, where $\omega = 90 - \theta$, was a good approach. The $M_s\sin(\omega)$ plane is shown in Figure 3.2, with the $\alpha$- and $\beta$-axes that are perpendicular and parallel, respectively, to the theoretical detachment criterion line. The figure is drawn with an equal grid spacing so that the transformed $\alpha$- and $\beta$-axes correctly appear perpendicular to each other and are not skewed.
3.1. The $M_s$-sin($\omega$) Plane and the $\alpha$-$\beta$ Transformation

Figure 3.2: An example of how the theoretical transition boundary can be used to create new $\alpha$-$\beta$ coordinates. Performing simulations with values corresponding to points on the $\alpha$-axis has been shown to produce good results.

The preselected reference points on the theoretical detachment criterion line are given in Table 3.1, and they also include the transformed values in the $M_s$-sin($\omega$) plane. These points are shown graphically in Figure 3.3. The transformation of $\theta_w$ to sin($\omega$) has a physical significance. Figure 3.4 shows that sin($\omega$) is also equal to $V_i/V_w$, where $V_i$ is the speed of the incident shock in the $x$-direction and $V_w$ is the speed of the incident shock travelling up the wedge. Both of these physically measurable quantities are inherently important near the reflection point.

The new $\alpha$- and $\beta$-axes are determined by a rotation through $\phi$ degrees of the original $M_s$ and sin($\omega$) axes at any particular reference point. The following method shows how any particular value of $M_s$ or $\theta_w$ used as the initial condition in a simulation can be determined given a value of $\alpha$. Note that since simulations are performed along the

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<th>sin($\omega$)</th>
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Table 3.1: The preselected reference points in both the $M_s$-$\theta_w$ and $M_s$-sin($\omega$) planes that lie on the theoretical detachment criterion line.
α-axis, the value of β is always zero.

The general coordinate transformation used to determine any particular \(M_s\)-\(\sin(\omega)\) combination for a given value of \(\alpha\) is

\[
\begin{bmatrix}
\Delta M_s \\
\Delta \sin(\omega)
\end{bmatrix} = \begin{bmatrix}
M_s - M_{s,RP} \\
\sin(\omega) - \sin(\omega)_{RP}
\end{bmatrix} = \begin{bmatrix}
\cos(\phi) & -\sin(\phi) \\
\sin(\phi) & \cos(\phi)
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}.
\]

Starting with the \(M_s\)-\(\sin(\omega)\) coordinates, the β-axis is determined by finding the line tangent to the detachment criterion. The rotation angle, \(\phi\), is the angle that the β-axis makes with the \(M_s\) coordinate axis. The angle \(\phi\) is found using the slope of the β-axis as

\[
\phi = \sin^{-1}(m_\beta),
\]

where \(m_\beta\) is the slope. Finally, with \(\beta = 0\), the values of \(M_s\) and \(\sin(\omega)\) are determined from equations 3.3 and 3.4 in the form

\[
M_s = M_{s,RP} + \cos(\phi)\alpha,
\]

\[
\sin(\omega) = \sin(\omega)_{RP} + \sin(\phi)\alpha.
\]
3.1. The $M_s$-$\sin(\omega)$ Plane and the $\alpha$-$\beta$ Transformation

Figure 3.3: The reference points in the $M_s$-$\theta_w$ and $M_s$-$\sin(\omega)$ planes. This figure is used as an inset later on to roughly show the position of the reference point in both these planes.

Figure 3.4: $\sin(\omega)$ is physically significant because it is equal to $V_i/V_w$, which is the ratio of incident shock velocity to this same velocity up the wedge.
3.2 Displaying Simulation Results with the $L$-$\alpha$ Graph

Various coordinates in the $M_s$-sin($\omega$) plane along the $\alpha$-axis described in section 3.1 are used to determine initial conditions for CFD shock reflection simulations. Using the analysis described in chapter 2, a position-time graph can be made for each of these simulations and a resulting non-dimensional Mach stem length $L$ can be determined. Every value of $\alpha$ now has a corresponding value of $L$, which can be graphed in a plane created by these two parameters. This $L$-$\alpha$ plane is used to present simulations results and determine transition points. One might expect that the length of the Mach stem always gradually decreases to zero at transition. Although this smooth transition is observed at lower Mach numbers, higher Mach number $L$-$\alpha$ graphs feature virtually sudden transitions. In these cases, the length $L$ decreases slowly when in the MR region, but suddenly drops to zero at the transition boundary.

3.2.1 Smooth transition at low Mach numbers

An example of the change in the non-dimensional length of the Mach stem that occurs when performing simulations in both the MR and RR regions along the $\alpha$-axis are displayed in Figure 3.5. This $L$-$\alpha$ graph corresponds to a preselected reference point on the theoretical detachment criterion line, with a shock strength of $M_s = 1.16421$, and $\theta_w = 41.75951^\circ$ ($\alpha = 0$), from where the $\alpha$-axis extends. The inset diagram to the upper right of the transition results in Figure 3.5 depicts the location of the reference point in the $M_s$-$\theta_w$ and $M_s$-sin($\omega$) planes. The simulation results given in this figure correspond to reflections occurring with a boundary layer. Also included is a linear fit to some of the data that, when extrapolated to zero Mach stem length ($L = 0$), gives the transition point and is displayed with a black dot. For this particular reference point the wedge transition point occurs at wedge angle of $\theta_w = 40.5040^\circ$ and a shock strength of $M_s = 1.17315$, and the cone transition occurs at $\theta_w = 40.3236^\circ$ and $M_s = 1.17442$.

As the data from simulations approach the transition boundary between MR and RR, the size of the cells makes it difficult to differentiate between the incident shock and Mach stem positions, and thus data below a certain value of $L$, is not included in the extrapolation and transition calculation. This procedure is done for all the preselected points along the detachment criterion line from a shock strength of $M_s = 1.01323$ to $M_s = 2.998$, so that a transition boundary is created across the entire $M_s$-$\theta_w$ plane. When the incident Mach number is low (less than about 1.6), the results on the $L$-$\alpha$
3.2. Displaying Simulation Results with the $L$-$\alpha$ Graph

The lower limit of $L$ is determined by considering when the data on the $L$-$\alpha$ graph becomes concave upwards. This is where the small but finite cell-size of the mesh begins to affect the accuracy of the results, as mentioned before. The upper limit of $L$ used in the extrapolation to a zero Mach stem length is found when the data becomes concave downwards. As shown in Figure 3.5 and the results are given in the appendix, the extrapolation is composed only of the linear part of the data near the transition boundary. For some reference points (such as those without a boundary layer), the $L$-$\alpha$ graph is larger and the number of data points used in the linear fits is larger than those shown in Figure 3.5.

The top half of Figure 3.6 shows a series of constant density contoured reflection patterns from Navier-Stokes simulations for a reference point of $M_s = 1.16421$ and $\theta_w = 41.75951^\circ$, which is a relatively weak incident shock and within the smooth transition region. As the simulations approach the transition boundary, the length of the Mach stem gradually decreases from 2.051% to 0.029%. The bottom half of Figure 3.6 shows a series of position-time graphs (as described in chapter 2) corresponding to the same

Figure 3.5: Numerical simulation results showing the change in non-dimensional Mach stem length $L$ along the $\alpha$-axis, as well as the calculated transition points. The inset diagrams show the position of the reference point with $M_s = 1.16421$ and $\theta_w = 41.75951^\circ$ in the $M_s$-$\theta_w$ and $M_s$-$\sin(\omega)$ planes.
Figure 3.6: A series of constant density contours (top) and their corresponding position-time graphs (bottom) from Navier-Stokes simulations over a wedge showing the decrease in Mach stem length and the transition from MR to RR. Reference point: $M_s = 1.16421$, $\theta_w = 41.75951^\circ$. 
3.2. Displaying Simulation Results with the $L$-$\alpha$ Graph

Simulation results shown in Figure 3.6. The speed of the incident shock $V_i$ and Mach stem shock $V_m$ are given for each case, as well as the resulting Mach stem length $L$. These graphs are showing only the last part of the selected position-time data taken well after the incident shock has passed the corner, and these are the same data points used to calculate $L$. This avoids any “startup” effects as described in chapter 4. Starting from the top left notice how the value of $L$ is 2.051% and the Mach stem and incident shocks are clearly at different positions at any given time. Moving toward the bottom right the Mach stem smoothly drops to near zero in the last case where the value of $L$ is 0.029%. The Mach stem and incident shocks are basically at the same position meaning that the simulation has resulted in a RR.

3.2.2 Sudden transition at high Mach numbers

For stronger shocks, when $M_s$ is greater than about 1.6, the transition becomes more sudden. This means that going from MR to RR, the length of the Mach stem may decrease gradually when in the MR region but, unlike smooth transitions at lower Mach numbers, $L$ suddenly drops to zero at the transition boundary. The results in the $L$-$\alpha$ plane then look slightly different, and one must decide how to interpret the results. Figure 3.7 shows the $L$-$\alpha$ plane for a set of Navier-Stokes results for a reference point of $M_s = 2.61563$ and $\theta_w = 50.76217^\circ$. As an example, consider the wedge results represented by the quatrefoils. Notice how the higher $\alpha$ data, which is clearly in the MR region, seems to hover around $L = 6\%$ from the far right of the figure where $\alpha = 0.04$ to about $\alpha = 0.02$, just past the middle of the figure, where the data suddenly drops to a zero Mach stem length. Also, due to the effects of sudden transition at higher Mach numbers, only two points are needed to determine the transition. In these cases, the abrupt decrease in the Mach stem length makes it clear where transition occurs.

A series of results from Navier-Stokes simulations for a reference point of $M_s = 2.61563$ and $\theta_w = 50.76217^\circ$ are shown in Figure 3.8. This is a relatively strong incident shock in this study and certainly within the sudden transition region. The top half, shows a series of constant density contours while the bottom half shows the corresponding position-time graphs. Notice how the higher $\alpha$ simulations are actually more complicated irregular MR instead of simple MR. The method used to determine the transition point doesn’t depend on the complexity of the MR, just on whether the Mach stem exists. As the simulations approach the transition boundary, the contours in the last two cases
Figure 3.7: Sample of high Mach number numerical simulation results showing the change in non-dimensional Mach stem length, $L$, along the $\alpha$ axis, as well as the calculated transition points. The inset diagrams shows the position of the reference point ($M_s = 2.61561$, $\theta_w = 50.76217^\circ$) in the $M_s$-$\theta_w$, and $M_s$-$\sin(\omega)$ planes.

$(\alpha = 0.0235$ and $\alpha = 0.0150)$ show how the double MR pattern suddenly changes to an RR pattern, and that the Mach stem length drops suddenly from 5.412 % to 0.069 %.

The position-time graphs in Figure 3.8 show only the last part of the data which is used to determine the value of $L$, just like all cases in this study. As described in the next chapter, using all the data from the corner would smooth out the sudden drop making it unclear where the transition occurs. How sudden the transition occurs seems to depend on the strength of the incident shock, but the shape of the $L$-$\alpha$ graph shown in Figure 3.7 is typical of the sudden transitions studied in this work. The $L$-$\alpha$ results of all reference points are shown in the appendix.

Although there are many approaches to determine the location of sudden transition, the method for determining the transition in this work was the same for smooth transition. This may seem like it’s not even worth mentioning, but this decision was made after considering the other approaches. The deciding factor for using this approach is related to the position-time graphs of the incident and Mach stem shocks, for computations with and without a boundary layer. The benefit of keeping the method of determining the transition consistent for both types becomes more clear when a smooth and sudden $L$-$\alpha$
3.2. Displaying Simulation Results with the $L-\alpha$ Graph

Figure 3.8: A series of constant density contours (top) and their corresponding position-time graphs (bottom) from Navier-Stokes simulations over a wedge showing the decrease in Mach stem length and the sudden transition from MR to RR. Reference point: $M_s = 2.61563$, $\theta_w = 50.76217^\circ$. 
Figure 3.9: Figures 3.5 and 3.7 plotted with the same limits on the $\alpha$-axis showing the difference between a smooth and sudden MR to RR transition.

Figure 3.9 includes both the previously shown Figures 3.5 and 3.7 plotted with the same limits on the $\alpha$-axis. The inset diagrams show where the reference points on the detachment criterion lie in the $M_s$-$\theta_w$ and $M_s$-$\sin(\omega)$ planes. With the same length of $\alpha$-axis, the suddenness of the higher Mach number transition is more clearly seen compared...
to a lower Mach number smooth transition case.

The sudden disappearance of the Mach stem at the transition boundary for the case of stronger shock waves is not a new phenomenon. This feature was originally discovered by the analysis of experimental data plotted in a different manner by Bleakney and Taub [6]. The sudden change from RR to MR was noticed when they plotted their experimental results in the $\omega'-\omega$ plane which was introduced in their paper. The experimental results for RR and MR were very close to the analytical transition lines for two- and three-shock theory respectively in this $\omega'-\omega$ plane but the transition between these two reflection patterns was indeed discontinuous. This sudden transition was later confirmed by Kawamura and Saito [19] and also by Henderson and Lozzi [16].
Chapter 4

Circumventing Complications of CFD Results and Grid Refinement

The accuracy of the transition boundaries between RR and MR determined by using the methods outlined in the previous chapters depends on how some complications related to the position-time graphs and the resolution of the grid are circumvented. The complications that arise due to nonlinearities in the position-time graphs from both the Euler and Navier-Stokes simulations and how these are dealt with are discussed in section 4.1, while the effect that the resolution of the grid has on the \( L-\alpha \) graph and ultimately on the calculated transition between RR and MR are discussed in section 4.2.

4.1 Avoiding Nonlinearity in Position-Time Graphs

The accurate determination of the incident shock and Mach stem velocities from the position-time graphs that are taken from the CFD simulations is a critical step toward calculating the non-dimensional Mach stem length and eventually finding the transition boundary between MR and RR. The data points on these position-time graphs, however, do not always line up so that the linear fit used to determine the velocities perfectly represents the data and thus makes it difficult to calculate \( L \) accurately. This has been seen in position-time graphs from both the Euler and Navier-Stokes computations.

For Euler computations, one would expect that since this type of unsteady shock reflection process studied in this work is self-similar, the position-time graphs for both the incident and Mach stem shocks should follow a straight line. This linear trend is observed for lower Mach number simulations that do not involve a sudden transition.
When considering higher Mach number simulations from the sudden transition regime, self-similar position-time graphs are observed only when considering points along the $\alpha$-axis far from the area where the MR transitions to RR. For points along the $\alpha$-axis close to transition when $L$ drops to zero, a simulation may begin with no Mach stem, but one can develop at later stages. This creates an “s” shape in the position-time graph, and it also means that the shock reflection is not perfectly self-similar for these points. This delayed development and formation of the Mach stem can adversely affect the calculated value of $L$, and also affect the ultimate extrapolation to a zero Mach stem length, which determines the transition from MR to RR. Figure 4.1 shows the position-time graph of a shock wave with a strength of $M_s = 2.0406$ striking a surface inclined at $\theta_w = 51.0713^\circ$, without including a boundary layer. The inset diagram at the top left roughly shows

Figure 4.1: The resulting position-time graph of an inviscid simulation when considering all the data from the corner. Stations A, B, C, and D are close-ups of the data and the linear fits. Although the fit for the Mach stem looks good on the entire diagram, it does not necessarily represent the position-time data in any of the closeups.
it difficult to determine the non-dimensional Mach stem length.

When solving the Navier-Stokes equations the most noticeable feature to be considered is the effect of the boundary layer. Even though the boundary layer develops behind the shock on the wedge surface it does change the structure of the position-time graph. Unlike the inviscid simulations which only displayed noticeable nonlinearities when sudden transition occurs, the effects of the boundary layer on the position-time graph are seen with any strength shock. Figure 4.2 shows the position-time graph of a shock wave with a strength of $M_s = 2.0411$ striking a surface inclined at $\theta_w = 47.5963^\circ$ with a boundary layer. Notice once again that the close-ups A, B and C to the right of the position-time graph show that the linear fit does not capture the trend of the data. The presence of the boundary layer delays the emergence of a Mach stem. This happens even for higher $\alpha$ simulations far from the theoretical transition that may be well within the MR region. Therefore, this “startup” time that exists before the Mach stem appears for these MR cases adds a non-self-similar aspect to the viscous computations.

These inviscid and viscous nonlinearities due to a sudden transition or the presence of a boundary layer are circumvented by skipping the early data of a position-time graph.
Figure 4.3: The resulting position-time graph of the same inviscid simulation from Figure 4.1 but considering only the last part of the data. The linear fit for the Mach stem now represents the data on the entire diagram as seen in all of the closeups.

until the Mach stem appears for a MR. The same amount of data is skipped for all of the simulations including RR even though the results are always self-similar, and thus the calculated value of the non-dimensional Mach stem length is not affected. This selection of data on the position-time graph focuses only on the duration of the simulation where the flow pattern has become virtually self-similar. Figure 4.3 is the resulting position-time graph from the same inviscid simulation from Figure 4.1, except only the last part of the data is used. The close-ups on the right show how the linear fit used to find the velocities now correctly and accurately represents the latter data.

Figure 4.4 contains the resulting position-time graph from the same viscous simulation from Figure 4.2 with only the last part of the data being used. Again, notice how skipping the nonlinear effects of the boundary layer present early in the simulation leads to an improved linear fit. It should be noted from these figures that by considering all the data from the corner simply reduces the calculated value of $L$, even though $L$ is not reduced to zero. Therefore, including nonlinearities in position-time graphs and considering all the data from the corner should not lead to errors in determining whether any given simulation results in a RR or a MR, but it will degrade the accuracy of $L$. This becomes important when using multiple simulations to estimate the transition boundary
4.1. Avoiding Nonlinearity in Position-Time Graphs

Figure 4.4: The resulting position-time graph of the viscous simulation from Figure 4.2 but considering only the last part of the data. The linear fit for the Mach stem now represents the data on the entire diagram as seen in all of the closeups.

as explained previously in chapter 3.

By considering only the later stages of a simulation, only the self-similar parts of the position-time graph are used to determine \( L \), and the sudden transition from MR to RR becomes accentuated and easier to determine a transition point. When considering all the data from the corner of the wedge, sudden transition is basically hidden within the results. By considering only the data when the Mach stem exists, all points in the transition region will have a larger \( L \). Figure 4.5 shows the \( L-\alpha \) plane for a reference point with \( M_s = 2.61563 \) and \( \theta_w = 50.76217^\circ \) and no boundary layer when all the data from the corner is used to calculated \( L \). The previously described inviscid nonlinearities occur mostly between the vertical dashed lines at \( \alpha = -0.018 \) and \( \alpha = -0.01 \). Figure 4.6 shows the \( L-\alpha \) plane for the same reference point but only uses the last parts of the position-time graphs to determine the Mach stem length. Notice how the drop to a zero Mach stem length that occurs between the vertical dashed lines is more sudden, yet the data points outside these lines seem unchanged. Also notice in Figure 4.6 that the value of \( L \) for a cone increases before it drops to zero. This peak is also seen in some Navier-Stokes simulations and would have been missed if all the position-time data from the corner had been included in the velocity calculations.
Figure 4.5: RR to MR transition for reference point with $M_s = 2.61563$ and $\theta_w = 50.76217^\circ$ and no boundary layer using all the position-time data from the corner of the wedge.

Figure 4.6: RR to MR transition for reference point with $M_s = 2.61563$ and $\theta_w = 50.76217^\circ$ and no boundary layer using only the last part of the position-time data.
4.2 Grid Refinement to Accentuate Sudden Transition

The simulations solving the Euler equations used 8 levels of adaptive mesh refinement (AMR) as mentioned in section 2.3 of chapter 2. The viscous calculations used 10 levels in order to more clearly resolve the boundary layer. Different levels of AMR, however, where used throughout this work in order to see if and how the transition results changed, and to develop increased confidence in the quality of the final results. For the lower Mach number cases, the finer grids don’t significantly change the results. The largest differences in the results occurred with the higher Mach number viscous computations. Figure 4.7 shows how the simulation results contained in the $L$-$\alpha$ graph for a reference point with $M_s = 2.04071$ and $\theta_w = 50.62809^\circ$ can change with different levels of refinement. Notice how the transition becomes more sudden when using the highest resolution, as can be seen by the development of steeper drops and sharper corners. Also, it should be noticed that as the mesh is refined from 9 to 10 levels, the results do not change much, compared to when the mesh is refined from 6 to 7 levels. This increases one’s confidence in the quality of the final results in terms of realizing that the refinement of at least 9 AMR levels is sufficient. It is also interesting how the shape of the $L$-$\alpha$ graph does change for
the higher Mach number cases, but seems to only affect the simulations that are within the transition area. Thus, the value of $L$ does not change as much when very large values of $\alpha$ are being considered.
Chapter 5

Results, Discussion and Conclusions

5.1 Numerical Transition Boundaries for Wedges and Cones

The results for the transition boundaries between RR and MR for wedges and cones plotted in the familiar $M_s$-$\theta_w$ plane are shown in Figures 5.1 and 5.2. The solid and dashed lines represent the detachment criterion (solid), sonic criterion (short dash) and mechanical equilibrium (long dash) for the analytical transition boundaries that separate the RR, MR and dual reflection regions. Experimental results for the planar flow scenario are also plotted on these graphs as filled circles, and they help to define the experimental transition boundary for the case when a boundary layer is present. The numerically determined transition points are shown by the unfilled circles for planar flows and the quatrefoils for axisymmetric flows and these points define the numerical transition boundaries for the only two self-similar reflection geometries with and without a boundary layer. Also shown is the planar flow result from the bifurcated shock tube experiment by Barbosa and Skews [1] in which the interacting plane has no boundary layer. This point is represented by a square and occurs at a shock strength of $M_s = 1.15$ and a wedge incline of $\theta = 40^\circ$. The numerical transition results showing the values of $M_s$ and $\theta_w$ are also given in tabular form in Tables 5.1 and 5.2. Some of these results are presented with more significant digits of accuracy that can be justified.
Figure 5.1: Transition boundary results determined by the numerical computations of planar shocks interacting with wedges and cones and no boundary layer.

Figure 5.2: Transition boundary results determined by the numerical computations of planar shocks interacting with wedges and cones with a viscous and thermal boundary layer.
5.1. Numerical Transition Boundaries for Wedges and Cones

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Table 5.1: Inviscid transition results for both the wedge and the cone.

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Table 5.2: Viscous transition results for both the wedge and the cone.
Chapter 5. Results, Discussion and Conclusions

The boundaries defined by the inviscid numerical transition results for both wedges and cones closely follows the trend of the analytical detachment criterion boundary. For low Mach numbers the numerical transition results lie below the analytical boundary. At a Mach number of about 1.6, which is near the region where sudden transition begins as well as the dual region, the numerical results cross the analytical detachment and sonic boundaries and remain above there for stronger shocks.

When considering the two different reflection geometries, the planar and axisymmetric transition results are very close together but exhibit measurable differences that are more clearly seen in Table 5.1.

The inviscid numerical transition results closest to the experimental result by Barbosa and Skews have a shock strength of \( M_s = 1.1764 \) and angle of \( \theta_w = 40.0351^\circ \) for the planar case and \( M_s = 1.1727 \) and \( \theta_w = 40.5740^\circ \) for the axisymmetric case. The inviscid transition boundary formed by the numerical results presented in this work is extremely close to the Barbosa and Skews result, which falls between the the numerical and analytical detachment and sonic boundaries. In fact, looking at Figure 5.1, one could certainly argue that the bifurcated shock tube result shows that the inviscid numerical boundary is at least equally good as the analytical boundary.

When comparing the viscous numerical transition results with the analytical boundaries, notice how the former is always below the latter, meaning that RR occurs in the region labelled “MR” in Figure 5.2. Furthermore, compared to the experimental results, the numerical transition results are in very close agreement up to a Mach number of about 2.4. Beyond this high Mach number, high-temperature gas effects likely become an important part of the reflection process and may affect the transition between RR and MR. See the calculations of Lee and Glass [20] who studied these effects on the transition boundary. The viscous numerical transition results presented here therefore show that the persistence of RR into the MR region seen in experiments can be partially or perhaps fully explained by the presence of the boundary layer. Also notice that although the viscous transition results for different reflection geometries are again very close, the differences between wedge and cone transitions are slightly larger than the inviscid results and can even be seen in Figure 5.2.
5.2 Conclusion

The study has used CFD and presented numerically determined transition boundaries between RR and MR, with and without a viscous and thermal boundary layer, for the only two self-similar reflecting geometries. In addition, this work has also studied analytical and experimental transition boundaries between RR and MR that have been well known for some time. It is clear that, even though some of these boundaries may overlap for some range of $M_s$, only the numerically determined viscous transition boundary for planar geometries presented in this work is in close agreement with experimental results. This means that the discrepancy between von Neumann theory and experimental results can be explained by the presence (or absence) of a boundary layer on the wedge surface. If one is to consider the persistence of RR into the MR region as a von Neumann paradox, then the comprehensive results presented here help to resolve this paradox. Finally, this work has also shown that there is a measurable difference in transition boundaries when considering different reflecting inclined surfaces (wedge versus the cone).
References


Appendix A

Numerical Transition Results from Euler Equations

Numerical results for the non-dimensional Mach stem length $L$ as a function of distance from the theoretical boundary along the $\alpha$-axis for flows over wedges and cones without a boundary layer. These are done by solving the Euler equations and are all collected in this appendix.

Table A summarizes the transition results along with the ratio of $\alpha$ for the cone to $\alpha$ for the wedge.
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Table A.1: Numerical transition results from inviscid calculations along with the $\alpha_C/\alpha_W$ ratio.
Figure A.1: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.01323$ and $\theta_w = 17.5574^\circ$ for flows without a boundary layer.

Figure A.2: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.02704$ and $\theta_w = 23.7657^\circ$ for flows without a boundary layer.
Appendix A. Numerical Transition Results from Euler Equations

Figure A.3: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.05014$ and $\theta_w = 29.9489^\circ$ for flows without a boundary layer.

Figure A.4: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.09022$ and $\theta_w = 36.0334^\circ$ for flows without a boundary layer.
Figure A.5: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.16421$ and $\theta_w = 41.7595^\circ$ for flows without a boundary layer.

Figure A.6: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.29502$ and $\theta_w = 46.2409^\circ$ for flows without a boundary layer.
Appendix A. Numerical Transition Results from Euler Equations

Figure A.7: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.47003$ and $\theta_w = 48.7111^\circ$ for flows without a boundary layer.

Figure A.8: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.65828$ and $\theta_w = 49.85222^\circ$ for flows without a boundary layer.
Figure A.9: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 1.84921$ and $\theta_w = 50.38209^\circ$ for flows without a boundary layer.

Figure A.10: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 2.04071$ and $\theta_w = 50.62809^\circ$ for flows without a boundary layer.
Figure A.11: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 2.23233$ and $\theta_w = 50.73375^\circ$ for flows without a boundary layer.

Figure A.12: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 2.42401$ and $\theta_w = 50.76798^\circ$ for flows without a boundary layer.
Figure A.13: \( L-\alpha \) graph and numerical transition boundary results for reference point with \( M_s = 2.61563 \) and \( \theta_w = 50.76217^\circ \) for flows without a boundary layer.

Figure A.14: \( L-\alpha \) graph and numerical transition boundary results for reference point with \( M_s = 2.80725 \) and \( \theta_w = 50.73762^\circ \) for flows without a boundary layer.
Figure A.15: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 2.99894$ and $\theta_w = 50.70339^\circ$ for flows without a boundary layer.
Appendix B

Numerical Transition Results from Navier-Stokes Equations

Numerical results for the non-dimensional Mach stem length $L$ as a function of distance from the theoretical boundary along the $\alpha$-axis for flows over wedges and cones with a boundary layer. These are done by solving the Navier-Stokes equations and are all collected in this appendix.

Table B summarizes the transition results along with the ratio of $\alpha$ for the cone to $\alpha$ for the wedge.
### Table B.1: Numerical transition results from viscous calculations along with the $\alpha_C/\alpha_w$ ratio.

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Figure B.15: $L$-$\alpha$ graph and numerical transition boundary results for reference point with $M_s = 2.99894$ and $\theta_w = 50.70339^\circ$ for flows with a boundary layer.
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