Existence of Critical Points for the Ginzburg-Landau Functional on Riemannian Manifolds

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
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Abstract

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Doctor of Philosophy
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2009

In this dissertation, we employ variational methods to obtain a new existence result for solutions of a Ginzburg-Landau type equation on a Riemannian manifold. We prove that if $N$ is a compact, orientable 3-dimensional Riemannian manifold without boundary and $\gamma$ is a simple, smooth, connected, closed geodesic in $N$ satisfying a natural nondegeneracy condition, then for every $\epsilon > 0$ sufficiently small, $\exists$ a critical point $u^\epsilon \in H^1(N; \mathbb{C})$ of the Ginzburg-Landau functional

$$E^\epsilon(u) := \frac{1}{2\pi |\ln \epsilon|} \int_N |\nabla u|^2 + \frac{(|u|^2 - 1)^2}{2\epsilon^2}$$

and these critical points have the property that $E^\epsilon(u^\epsilon) \rightarrow \text{length}(\gamma)$ as $\epsilon \rightarrow 0$.

To accomplish this, we appeal to a recent general asymptotic minmax theorem which basically says that if $E^\epsilon$ $\Gamma$-converges to $E$ (not necessarily defined on the same Banach space as $E^\epsilon$), $v$ is a saddle point of $E$ and some additional mild hypotheses are met, then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, $E^\epsilon$ possesses a critical point $u^\epsilon$ and $\lim_{\epsilon \rightarrow 0} E^\epsilon(u^\epsilon) = E(v)$. Typically, $E$ is only lower semicontinuous, therefore a suitable notion of saddle point is needed.

Using known results on $\mathbb{R}^3$, we show the Ginzburg-Landau functional $E^\epsilon$ defined above $\Gamma$-converges to a functional $E$ which can be thought of as measuring the arclength of a limiting singular set. Also, we verify using regularity theory for almost-minimal currents that $\gamma$ is a saddle point of $E$ in an appropriate sense.
Dedication

For my fiancée, Melissa.

Acknowledgements

I would like to thank my supervisor Robert Jerrard for his guidance and patience over the past few years.
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Chapter 1

Introduction

Over the past 30 years, there has been an extensive study of the equation

\[-\Delta u + \frac{1}{\epsilon^2}(|u|^2 - 1)u = 0, \quad (1.1)\]

where \( u : N \to \mathbb{R}^k, k = 1, 2, \) and \( N \) is typically an open, bounded subset of \( \mathbb{R}^{k+m}, m \geq 1, \) with smooth boundary. (1.2)

Of course, solutions of (1.1) are critical points of the energy functional

\[ E^\epsilon(u) = \frac{1}{2\pi |\ln \epsilon|} \int_N |\nabla u|^2 + \frac{(|u|^2 - 1)^2}{2\epsilon^2} dX. \quad (1.3) \]

When \( k = 1, \) (1.3) is usually referred to as the Allen-Cahn functional and the Ginzburg-Landau functional when \( k = 2. \) Many problems regarding these functionals involve making a connection between critical points of (1.3) and \( m \)-dimensional minimal surfaces in certain geometric settings. One method for establishing this connection uses the notion of \( \Gamma \)-convergence. Since being introduced in the 1970’s by De Giorgi, the concept of \( \Gamma \)-convergence of a family of functionals to a limiting functional \( E \) has proven to be a useful vehicle in describing the asymptotic behaviour of sequences of minimizers as \( \epsilon \to 0. \) The definition of \( \Gamma \)-convergence (see Section 2.3) is designed to guarantee that, if a sequence of minimizers converges, then the limit has got to be a minimizer of \( E. \) Basic
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Γ-convergence results for (1.3) are proved in [15],[14] \((k = 1)\) and [10],[1] \((k = 2)\), and these results automatically lead to a description of the limiting behaviour of a sequence \(\{u^\epsilon\}\) as \(\epsilon \to 0\), where \(u^\epsilon\) is a minimizer of \(E^\epsilon\). In particular, the energy of \(u^\epsilon\) concentrates around a minimal surface of dimension \(m\).

Γ-convergence was originally believed to only be useful for describing the asymptotic behaviour of sequences of minimizers. The first result to go beyond this was that of Kohn and Sternberg [12], which did not appear until more than 10 years after the basic definitions and examples of Γ-convergence had been developed. They show that local minimizers of \(E^\epsilon\) (with \(k = 1\)) exist provided \(E^\epsilon\) Γ-converges to \(E\) and an isolated local minimizer of \(E\) exists. A similar type of result for \(k = 2\) is proved in [16].

Recently, existence of more general critical points of (1.3) for \(k = 1\) has been proved in [13] \((N\) satisfying (1.2)) and [17] \((N\) a compact Riemannian manifold of dimension \(k + m\)) via linearization techniques that require precise control over the spectrum of \(E^\epsilon\). These techniques have not yet been extended to cover the \(k = 2\) case, as the Ginzburg-Landau energy defined for vector-valued functions has worse spectral properties than its scalar counterpart. The first existence result for general critical points of (1.3) with \(k = 2\) and \(N\) satisfying (1.2) \((m = 1)\) is established in [11] by Γ-convergence arguments. The intent of this thesis is to do the same for \(N\) a compact 3-dimensional manifold. It should be noted that these Γ-convergence arguments merely yield existence and information about critical values whereas in the \(k = 1\) case, the authors are able to obtain a precise description of the critical points near given \(m\)-dimensional minimal surfaces satisfying natural nondegeneracy conditions. Hence the results obtained for \(k = 2\) are strictly weaker than those for \(k = 1\).

The above history regarding the functional

\[
E^\epsilon(u) = \frac{1}{2\pi|\ln \epsilon|} \int_N |\nabla u|^2 + \frac{(|u|^2 - 1)^2}{2\epsilon^2},
\]

\[u : N \to \mathbb{R}^k, k = 1, 2\]

can be summarized as follows:
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\[ N \subset \mathbb{R}^{k+m} \text{ open, bounded with smooth boundary} \]

<table>
<thead>
<tr>
<th>Result</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>basic ( \Gamma )-convergence results, description of sequences of minimizers as ( \epsilon \to 0 )</td>
<td>Modica,Mortola (1977)</td>
<td>Jerrard,Soner (2002)</td>
</tr>
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<td>( m = 1 )</td>
<td>( m = 1 )</td>
<td></td>
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<tr>
<td>existence of critical points of ( E^\epsilon ) for ( \epsilon ) sufficiently small</td>
<td>Kowalczyk (2005)</td>
<td>Jerrard,Sternberg (to appear)</td>
</tr>
<tr>
<td>( m = 1 )</td>
<td>( m = 1 )</td>
<td></td>
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</tbody>
</table>

\[ N \text{ a compact (} k+m \text{-dimensional manifold} \]

<table>
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<tr>
<th>Result</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
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<tbody>
<tr>
<td>existence of critical points of ( E^\epsilon ) for ( \epsilon ) sufficiently small</td>
<td>Picard,Ritoré (2003)</td>
<td>Mesaric (2009)</td>
</tr>
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<td>( m = 1 )</td>
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In [11], Jerrard and Sternberg formulate and prove an abstract theorem (Theorem 3.1) which implies the existence of critical points of \( E^\epsilon \) assuming \( E^\epsilon \) \( \Gamma \)-converges to a limiting functional \( E \) which possesses a nondegenerate critical point and other mild hypotheses are met. Since \( E \) is typically only lower semicontinuous, one difficulty for them was to come up with a suitable notion of a nondegenerate critical point. In practice, it is not an easy task to show that a candidate for a critical point in the sense of JS (Jerrard and Sternberg) satisfies the conditions of the definition (Definition 2.1). JS use their abstract theorem to prove new existence results for the Ginzburg-Landau equation in 3 dimensions. This is accomplished by finding a reasonable candidate for a critical point of \( E \) (the \( \Gamma \)-limit of the Ginzburg-Landau functional) and then verifying that it is indeed one via \textit{ad hoc} arguments depending heavily on the specific setting. In this thesis, we use
a more robust method to identify critical points in the sense of JS of certain functionals which merely lower semicontinuous.

Let $U, V$ be Banach spaces, $E^\epsilon_U, E^\epsilon_V : U, V \to (-\infty, \infty], \epsilon \in (0, 1]$ and $V_0 = \{ v \in V : E^\epsilon_V(v) < \infty \}$. We say that $E^\epsilon_U \Gamma$-converges to $E^\epsilon_V$ as $\epsilon \to 0$ if there exists a continuous map $P^\epsilon_{UV} : U \to V$ and a map $Q^\epsilon_{UV} : V_0 \to U$ (not necessarily continuous) satisfying

**lower bound:**

if $v \in V_0$ and $\{ u_\epsilon \} \subset U$ is a sequence such that $\| P^\epsilon_{UV}(u_\epsilon) - v \|_V \to 0$ as $\epsilon \to 0$, then
$$\liminf E^\epsilon_U(u_\epsilon) \geq E^\epsilon_V(v)$$

and

**upper bound:**

for every $v \in V_0$, $\| P^\epsilon_{UV}(Q^\epsilon_{UV}(v)) - v \|_V \to 0$ and
$$\limsup E^\epsilon_U(Q^\epsilon_{UV}(v)) \leq E^\epsilon_V(v)$$ as $\epsilon \to 0$.

This definition is not exactly standard but is equivalent to other definitions.

As mentioned above, JS take $E^\epsilon_U$ to be the Ginzburg-Landau functional

$$E^\epsilon_U(u) = \frac{1}{2\pi |\ln \epsilon|} \int_{\Omega} |\nabla u|^2 + \frac{(|u|^2 - 1)^2}{2\epsilon^2} dX,$$

where $u \in U = H^1(\Omega; \mathbb{C}), \Omega$ is a bounded domain in $\mathbb{R}^3$ with smooth boundary. It is known (see [1],[10]) that $E^\epsilon_U \Gamma$-converges as $\epsilon \to 0$ to a limiting functional $E^\epsilon_V$ that can be thought of as measuring the arclength of a limiting singular set. More precisely, $V_0 = \mathcal{R}'_1(\Omega)$, which is the space consisting of elements $T$ that are unions of oriented Lipschitz curves in $\Omega$ equipped with a norm that basically gives the minimum area over all 2-d surfaces in $\Omega$ with boundary $T$ and $E^\epsilon_V(T) = M(T)$, which can be interpreted as the length of the union of curves. In the language of geometric measure theory (GMT), $V$ is the space of 1-currents $T$ that are boundaries with finite flat norm and $M(T)$ is referred to as the ‘mass of $T$’.

The abstract theorem of JS roughly says that if $E^\epsilon_U$ is a sequence of functionals that $\Gamma$-converges to a limiting functional $E^\epsilon_V$ and $v_s$ is a saddle point of $E^\epsilon_V$ (see Definition
2.1), then, under certain mild additional hypotheses, $E_{U}$ has a critical point $u^{\epsilon}$ for every $\epsilon$ sufficiently small and $E_{U}(u^{\epsilon}) \to E_{V}(v_{s})$ as $\epsilon \to 0$ (it need not be true that the critical points converge in any sense to $v_{s}$).

To use the abstract theorem, a saddle point $v_{s}$ is needed. Their candidate for a saddle point is the 1-current associated with an oriented line segment $M \subset \Omega$ joining points $x_{0}, y_{0} \in \partial \Omega$. To be consistent with our notation below, we will label this 1-current as $T_{M}$ (they label it as $T_{*}$). They fix open sets $\partial \Omega^{-}, \partial \Omega^{+}$ containing $x_{0}, y_{0}$ respectively and assume the distance function $d_{0} : \partial \Omega^{-} \times \partial \Omega^{+} \to \mathbb{R}_{+}$ defined by $d_{0}(x, y) = |x - y|$ has a non-degenerate critical point at $(x_{0}, y_{0})$, i.e.,

$$\nabla d_{0}(x_{0}, y_{0}) = 0 \quad \text{and} \quad \det D^{2}d_{0}(x_{0}, y_{0}) \neq 0.$$

The second condition implies that 0 is not an eigenvalue of $D^{2}d_{0}(x_{0}, y_{0})$. To prove that $T_{M}$ is a saddle point of $E_{V}$ in the sense of Definition 2.1, continuous maps

$$P_{WV} : V \to \mathbb{R}^{l}, Q_{VW} : W \to V_{0}$$

are constructed (here, $l$ denotes the number of negative eigenvalues of $D^{2}d_{0}(x_{0}, y_{0})$) and a number $\delta_{0} > 0$ is found to satisfy $P_{WV}(v_{s}) = 0$,

$$Q_{VW}(0) = v_{s},$$

$$P_{WV} \circ Q_{VW}(w) = w \text{ for all } w \in W,$$

$$\sup_{w \in W, \|w\| \geq a} E_{V}(Q_{VW}(w)) < E_{V}(v_{s}) \text{ for all } a \in (0, r_{1})$$

and

$$E_{V}(v_{s}) < E_{V}(v) \text{ for } v \in \{v \in V : 0 < \|v - v_{s}\|_{V} \leq \delta_{0}, P_{WV}(v) = 0\}$$

where $W$ is the open $l$-dimensional ball of radius $r_{1}$ centered at the origin for some $r_{1} > 0$ appropriately chosen. The construction of these maps relies solely on the presence of a boundary. The most difficult property to verify is (1.6); that $T_{M}$ is a strict local minimizer of $E_{V}$ in the flat norm topology among all $T \in V$ with $P_{WV}(T) = 0$. This is
accomplished by assuming
\[ M(T) \leq M(T_M) = L, \quad P_{WV}(T) = 0 \]
and the flat norm of \( T - T_M \) is less than some number \( \delta_0 > 0 \) to be chosen
\[ (1.7) \]
and then showing \( T = T_M \). To do this, they first show the existence of a ‘piece of \( T \),’ labeled \( T' \), whose support consists of a single Lipschitz curve \( \subset \Omega \) that runs from \( \partial \Omega^- \) to \( \partial \Omega^+ \) and stays in a cylinder of radius \( r < R \) about \( M \), assuming \( \delta_0 < cr^2 \) for some absolute constant \( c > 0 \). It then suffices to show \( T' = T_M \). This is done by showing the endpoints of the support of \( T' \) coincide with those of \( T_M \), taking \( r \) sufficiently small, and using the fact that \( M(T') \leq L \).

The following dissertation develops a more systematic approach of the above that works directly with the spectrum of the Jacobi operator \( J \) associated with \( M \) (\( J \) acts on normal vectorfields on \( M \) and is defined through the second variation of \( E_V \)). We take \( E_\epsilon_U \) to be
\[ E_\epsilon_U(u) = \frac{1}{2\pi |\ln \epsilon|} \int_N |\nabla u|^2 + \frac{(|u|^2 - 1)^2}{2\epsilon^2}, \]
where \( u \in U = H^1(N; \mathbb{C}), N \) is a 3-dimensional Riemannian manifold, \( V = F_1'(N) \) and \( E_V(T) = M(T) \).

In Chapter 5, we show that \( E_\epsilon_U \) \( \Gamma \)-converges to \( E_V \) along with a compactness result (see Theorem 5.1). Here, \( P_{V,U}^{\epsilon} \) is independent of \( \epsilon \) and is the 1-current associated with the jacobian of \( u \). The compactness property (2.8) and lower bound property are verified using local coordinates along with Theorem 3.2 in [11] and Theorem 5.2 in [10] respectively, and a map \( Q_{U,V} : \text{Im}Q_{VW} \to U \) is constructed satisfying the upper bound property (the abstract theorem does not require that the full \( \Gamma \)-limit hold; it suffices that \( Q_{U,V} \) be defined for every \( v \) of the form \( v = Q_{VW}(w), w \in W \)). We then verify that the additional hypotheses of the abstract theorem are satisfied to obtain our main result: there exists \( \epsilon_0 > 0 \) such that if \( \epsilon \in (0, \epsilon_0) \), \( E_\epsilon_U \) possesses a critical point \( u^\epsilon \) and \( \lim_{\epsilon \to 0} E_\epsilon(U^\epsilon) = E(v_s) \), where \( v_s \) a saddle point of \( E_V \) in the sense of Definition 2.1.

To use the abstract theorem, we need a saddle point \( v_s \). Our candidate for \( v_s \) is any
smooth, closed, oriented geodesic in $N$ that does not intersect itself with index $l$ and nullity 0 (this means that the associated Jacobi operator $J$ has $l$ negative eigenvalues $\lambda_1 \leq \cdots \leq \lambda_l$ and 0 is not an eigenvalue of $J$). This generalizes the assumptions in [11]. We label such a curve as $\gamma$ and the associated 1-current as $T_M$. The proof that $T_M$ is a saddle point in the sense of Definition 2.1 is the content of Chapter 4. To do this, we need to construct maps $P_{WV}: V \to \mathbb{R}^l, Q_{VW}: W \to V_0$ and find a number $\delta_0 > 0$ satisfying properties (1.4)-(1.6), where, as in [11], we take $W$ to be the open $l$-dimensional ball of radius $r_1$ centered at the origin for some appropriately chosen $r_1 > 0$.

The intuition behind the construction of $P_{WV}$ comes from the following: if a closed, oriented, Lipschitz curve $\beta$ can be written as the graph of a normal vectorfield $u$ over $M$ with $\|u\|_{W^{1,\infty}}$ sufficiently small, then the length of $\beta$ can be computed in terms of $u$ through a Taylor expansion. Since $\gamma$ is a geodesic, the first-order term vanishes and, letting $L := M(T_M) = \text{length}_M$,

$$\text{length}(\beta) = L + \frac{1}{2}(Ju, u)_{L^2} + \|u\|_{W^{1,\infty}}O(\|u\|_{H^1}^2).$$

Sturm-Liouville theory allows us to express $u = \sum_{i=1}^{\infty} c_i z^i$, where $\{z^i\}$ is an orthonormal (in $L^2$) basis of eigenfunctions of $J$ for the space of $L^2$ normal vectorfields on $M$, and thus, letting $n = \sum_{i=1}^l c_i z^i, p = \sum_{i=l+1}^{\infty} c_i z^i$,

$$\text{length}(\beta) \geq L - \frac{1}{2} \left( \frac{c}{2} - \lambda_1 \right) \|n\|_{H^1}^2 + \frac{c}{4} \|p\|_{H^1}^2$$

for some small positive constant $c$. In this situation, $T_\beta$ will be close to $T_M$ in the flat norm, where $T_\beta$ is the 1-current associated with $\beta$, so we would like to say that if $P_{WV}(T_\beta) = 0$, then $\text{length}(\beta) \geq L$ with equality if and only if $\beta = M$. This can be accomplished by setting $P_{WV}(T_\beta)$ to be $(c_1, ..., c_l)$ so that $n$ is identically 0. $P_{WV}(T_\beta)$ is thought of as the projection of $T_\beta$ onto the ‘unstable directions’ of the functional $E_V$ near $T_M$. The definition of $P_{WV}$ is then extended to all of $V$.

The argument above suggests that for each $w \in W$, we associate $Q_{VW}(w)$ with the curve that is the graph of the normal vectorfield $u_w = \sum_{i=1}^l w_i z^i$ over $M$. Since $\|u_w\|_{H^1}^2$, ...
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goes to 0 as \( r_1 \) goes to 0, \( Q_{VW}(0) \) will be a strict local maximum of \( M \) if \( r_1 \) is small enough (this is (1.5)). \( \{Q_{VW}(w) : w \in W\} \) represents roughly the unstable manifold of \( M \) near \( M \).

Sections 4.1 and 4.2 are spent verifying that there exists a \( \delta_0 > 0 \) so that (1.6) holds. As in [11], we assume (1.7) and show \( T = T_M \) for some \( \delta_0 > 0 \). First, we find a number \( r_0 > 0 \) so that the exponential map is well-defined on a thin tube \( K_{r_0} \) of radius \( r_0 \) about \( M \) and write \( K_{r_0} \) in coordinates through a map \( \psi \). Then, assuming \( \delta_0 \leq r_0^3 \), we show that there is a ‘piece of \( T' \), again labeled \( T' \), that consists of a single Lipschitz curve with no boundary that lies in \( K_{r_0/4} \) and is homologous to \( T_M \) in \( K_{r_0} \) (this means \( T' - T_M \) is a boundary in \( K_{r_0} \)), along with some other good properties (see Lemma 4.4). The proof of this relies strongly on the fact that \( \gamma \) is a geodesic. It then suffices to show \( T' = T_M \).

The tactics of JS can no longer be used since the support of \( T' \) has no endpoints. To do this, we introduce an auxiliary functional \( M^* = M + C_* |P_{WV}|^2 \), where \( C_* > 0 \) is to be chosen, and we show \( T' = T_M \) assuming

\[
T_M \text{ is the unique minimizer of } M^* \text{ among all } T \text{ such that } \\
\text{the support of } T \text{ lies in } K_{r_0/4} \text{ and } T \text{ is homologous to } T_M \text{ in } K_{r_0}.
\]

(1.8) is verified in section 4.2 and this is where the ideas of Brian White [20] are introduced and expanded upon. This is done by analyzing \( Q_{min} = \psi^{-1}_T T_{min} \), where \( T_{min} \) is any minimizer of \( M^* \) among all \( T \) such that the support of \( T \) lies in \( K_{r_0/4} \) and \( T \) is homologous to \( T_M \) in \( K_{r_0} \). We show that \( Q_{min} \) is \( (F, \omega, \delta) \)-minimal, where \( F \) is the pullback of \( M \) by \( \psi \). This means that for any closed \( T \) with support in a ball of radius \( \rho \in (0, \delta] \), \( F(Q_{min}) \) may be bigger than \( F(Q_{min} + T) \), but no bigger than \( F(Q_{min} + T) + \omega(\rho) M(Q_{min} + T) \), where \( \omega : (0, \delta] \to \mathbb{R}_+ \) satisfies \( \lim_{\rho \to 0^+} \omega(\rho) = 0 \). Using regularity theory, we conclude that the support of \( T_{min} \) can be written as the graph of a \( C^1 \) normal vectorfield \( u_{min} \) over \( M \) whose \( H^1 \)-norm goes to 0 as \( r_0 \) goes to 0, and then we use a Taylor expansion as described above to show \( u_{min} \) is identically 0, taking \( r_0 \) smaller if necessary.

In [20], Brian White develops a minmax characterization of minimal surfaces. Through-
out his paper, $F$ denotes a smooth, parametric, elliptic integrand (see section 2.2) on a Riemannian manifold $N$ and $M$ denotes a smooth, compact embedded submanifold (with or without boundary) of $N$. His main result states that if $M$ is a critical point of $F$ and is strictly stable for $F$ (the eigenvalues of the associated Jacobi operator are all positive), then there exists a neighbourhood of $M$ such that if $S \neq M$ is homologous to $M$ in this neighbourhood, then $F(M) < F(S)$. This is used to show that if $M$ has index $l$ and nullity 0, then $M$ is strictly minimizing in a neighbourhood of $M$ for an auxiliary functional $F^*$ essentially of the form $F^*(S) = F(S) + C_* | \int_S \tilde{f}(x)dx|^2$ for suitable $\tilde{f}, C_* > 0$. Also, he shows that there exists a family of surfaces $\{M_w\}$, indexed by $w$ in a neighbourhood of the origin in $\mathbb{R}^l$, such that $F(M_w)$ has a unique local maximum at $w = 0$. The maps $P_{WV}, Q_{VW}$ constructed in this thesis are very similar to the maps $S \mapsto \int_S \tilde{f}(x)dx, w \mapsto M_w$ respectively constructed by Brian White. The overall approach taken in section 4.2 is very similar to that taken in [20], however, it is not clear that his work can be cited to suit our needs. Moreover, his paper is written in a way that makes it difficult for readers who are not extremely well-versed in GMT regularity theory to extract all of the details. Therefore, the arguments are reconstructed in section 4.2.
Chapter 2

Definitions

Throughout this paper, $U_d^a(p)$ ($B_d^a(p)$) will denote the open (closed) $d$-dimensional ball of radius $a$ centered at $p$. We will make the convention $U_a^d := U_a^d(0)$, $B_a^d := B_a^d(0)$.

2.1 Basic Geometric Measure Theory

Let $\Omega$ denote an open set of a Riemannian manifold $N$ of dimension $n$. Assume also that $\partial \Omega$ is smooth.

Define

$$D_k(\Omega) := \{\text{continuous, linear functionals on } D^k(\Omega)\}$$

where $D^k(\Omega)$ denotes the space of $C^\infty$ $k$-forms with compact support in $\Omega$. If $T \in D_k(\Omega)$, then $T$ is referred to as a $k$-current in $\Omega$.

The boundary of a $k$-current $T$, denoted $\partial T$, is the $(k - 1)$-current defined by

$$\partial T(\eta) := T(d\eta), \quad \eta \in D^{k-1}(\Omega).$$

The mass and flat norm of a $k$-current $T \in D_k(\Omega)$ are given by

$$M(T) := \sup_{\eta \in D^k(\Omega) : \|\eta\|_{L^\infty(\Omega)} \leq 1} |T(\eta)|$$

(2.1)
and
\[ F(T) := \inf \{ M(S) : S \in D_{k+1}(\Omega), \partial S = T \} \]  
(2.2)
respectively. We set \( F(T) = +\infty \) if there does not exist \( S \in D_{k+1}(\Omega) \) with finite mass such that \( \partial S = T \).

Let \( R_k(\Omega) \) denote the space of rectifiable, integer multiplicity \( k \)-currents in \( \Omega \). If \( T \in R_k(\Omega) \), we will write \( T = \tau(\Gamma, m, \vec{T}) \) to indicate that for \( \eta \in D^k(\Omega) \),
\[ T(\eta) = \int_{\Gamma} \langle \eta(x), \vec{T}(x) \rangle m(x) dH^k(x) = \int_{\Gamma} \langle \eta(x), \vec{T}(x) \rangle d\|T\|(x), \]

where \( \Gamma = \text{spt}T \) is a \( k \)-rectifiable set (a union of countably many Lipschitz \( k \)-submanifolds of \( \Omega \) and a set of \( H^k \)-measure zero), \( \vec{T}(x) \) is, at \( H^k \) a.e. \( x \), a simple \( k \)-vector of unit length orienting the approximate tangent space \( apT_x \Gamma \) and \( m \) is a non-negative integer-valued \( H^k \)-integrable function called the multiplicity of the current.

Set
\[ R_k'(\Omega) := \{ T \in R_k(\Omega) : M(T) < \infty, T = \partial S \text{ for some } S \in R_{k+1}(\Omega) \}, \]
\[ F_k'(\Omega) := \{ T \in D_k(\Omega) : T = \partial S \text{ for some } S \in D_{k+1}(\Omega), F(T) < \infty \}. \]

The restriction of \( T = \tau(\Gamma, m, \vec{T}) \in R_k(\Omega) \) to a set \( A \subset \Omega \), denoted \( T\lfloor A \), is the \( k \)-current defined by
\[ (T\lfloor A)(\eta) := \int_{\Gamma \cap A} \langle \eta(x), \vec{T}(x) \rangle m(x) dH^k(x), \quad \eta \in D^k(\Omega). \]

For general \( T \in D_k(\Omega) \) with finite mass, define \( (T\lfloor A)(\eta) = T(\chi_A \eta) \). In this case, the right-hand side can be understood by representing \( T \) by a measure \( \nu \) and restricting \( \nu \) to \( A \).

Suppose \( \Omega = \phi(O) \) for some diffeomorphism \( \phi \) and open set \( O \subseteq \mathbb{R}^n \). Then any \( \eta \in D^1(\Omega) \) can be expressed as
\[ \eta(x) = \sum_{i=1}^n \eta_i(x) dX_i, \]
where $X = \phi^{-1}(x)$. If $f : N \to \mathbb{R}$ is Lipschitz, then for a.e. $x$,

$$df(x) = \sum_{i=1}^{n} \frac{d}{dX_i}(f \circ \phi)|_{\phi^{-1}(x)}dX_i.$$ 

Let $f : \Omega_1 \to \Omega_2$ be a Lipschitz function, where $\Omega_i$ is an open subset of a Riemannian manifold $N_i$ of dimension $n_i$, $i = 1, 2$. For $T \in D_k(\Omega_1)$, define $f^* T \in D_k(\Omega_2)$ by

$$f^* T(\eta) = T(f^* \eta), \quad \eta \in D^k(\Omega_2).$$

Here, $f^* \eta$ is the pullback of $\eta$ by $f$. If $\Omega_i$ can be written in coordinates through a map $\phi_i$ and $\eta = \sum_{i=1}^{n_2} \eta^i(x) dX_i \in D^1(\Omega_2)$, $X = \phi_2^{-1}(x)$, then

$$f^* \eta(x) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \eta^j(f(x)) \frac{d}{dX_j} h_i(\tilde{X}) d\tilde{X}_i$$

where $h = \phi_2^{-1} \circ f \circ \phi_1$ and $\tilde{X} = \phi_1^{-1}(\tilde{x})$.

If $I \subset \mathbb{R}$ is an interval and $\beta : I \to \Omega$ is a Lipschitz curve, we say that a 1-current $T$ corresponds to integration over $\beta$ if for any $\eta \in D^1(\Omega)$,

$$T(\eta) = \int_I \eta^i(\beta(t)) \alpha'_i(t) dt$$

where $\alpha(t) = \phi^{-1}(\beta(t))$.

Let

$$\mathcal{I}_k(\Omega) = \{ T \in \mathcal{R}_k(\Omega) : M(\partial T) < \infty \}.$$ 

If $T \in \mathcal{I}_k(\Omega)$, then we say that $T$ is an integral $k$-current in $\Omega$.

Any integral 1-current can be written as a finite or countable sum

$$T = \sum_i T_i,$$

where each $T_i$ corresponds to integration over a Lipschitz curve $\beta_i$,

$$M(T) = \sum_i M(T_i) = \sum_i \mathcal{H}^1(\beta_i)$$
and
\[ M(\partial T) = \sum_i M(\partial T_i). \]
This can be seen by isometrically embedding \( N \) into \( \mathbb{R}^{n+m} \) for some \( m \geq 0 \) (see [11], p.6).
Note that if \( \partial T = 0 \), then \( \partial T_i = 0 \) for every \( i \).

If \( T \) is a \( k \)-current in \( \Omega \) such that \( M(T) + M(\partial T) < \infty \) and \( f : \Omega \to \mathbb{R} \) is Lipschitz, then for a.e. \( s \) define
\[ \langle T, f, s \rangle := (\partial T) \cup \{ x : f(x) \leq s \} - \partial (T \cup \{ x : f(x) \leq s \}). \]

The \((k-1)\)-currents \( \langle T, f, s \rangle \) are referred to as ‘the slices of \( T \) by level sets of \( f \)’.

If \( T \) is a 1-current corresponding to integration over a Lipschitz curve \( \beta : I \to \Omega \), where \( I \) is an interval, then there is an explicit formula for \( \langle T, f, s \rangle \), namely,
\[ \langle T, f, s \rangle = \sum_{t \in I; \beta(t) \in f^{-1}(s)} \text{sign} \left( \frac{d}{dt} f(\beta(t)) \right) \delta_{\beta(t)}. \tag{2.3} \]
Here, we use the convention \( \text{sign}(0) = 0 \). This is a special case of Theorem 4.3.8(2) in [7].

Suppose now that \( N \) is 3-dimensional. For \( u = u_1 + iu_2 \in H^1(\Omega) \), let \( J(u) \) denote the 2-form \( du_1 \wedge du_2 \) and \( j(u) \) denote the 1-form \( \frac{1}{2i}(\bar{u}du - ud\bar{u}) \). We can identify \( J(u) \) with a 1-current, denoted \( J(u) \), defined through its action on 1-forms \( \eta \in D^1(\Omega) \) by
\[ J(u)(\eta) := \int \eta \wedge J(u). \]
Similarly, \( j(u) \) can be associated with a 2-current, denoted \( j(u) \), that acts on 2-forms \( \eta \in D^2(\Omega) \) via
\[ J(u)(\eta) := \int \eta \wedge j(u). \]
One can check through integration by parts that \( J(u) = \frac{1}{2} \partial(\star j(u)) \). Thus,
\[ F(\star J(u)) \leq \frac{1}{2} M(\star j(u)) = \frac{1}{2} ||j(u)||_{L^1(\Omega)}. \tag{2.4} \]

If \( \phi \) is a diffeomorphism that takes some open subset \( O \) of \( \mathbb{R}^3 \) onto an open subset \( \Omega \) of \( N \), then
\[ \phi_*(\star J(u)) = \star J(u \circ \phi^{-1}). \tag{2.5} \]
2.2 Regularity

In this section, assume $N = \mathbb{R}^n$ and $\Omega$ is an open subset of $\mathbb{R}^n$. This terminology is used in Sections 4.1, 4.2 and Chapter 6. We encourage the reader to bypass this section for now until needed.

A parametric integrand of degree $k$ on an open set $\Omega$ is a continuous real-valued function $\Psi = \Psi(X, \xi)$ defined for $X \in \Omega$ and $\xi \in \Lambda_k(\mathbb{R}^n)$ which is homogeneous of degree 1 in the second variable, i.e.,

$$\Psi(X, c\xi) = c\Psi(X, \xi) \quad \text{for } c > 0.$$  

We will assume that $\Psi$ is non-negative.

For $T = \tau(\Gamma, m, \vec{T}) \in \mathcal{R}_k(\Omega)$ and $X \in \Omega$, define

$$\Psi(T) := \int_{\Gamma} \Psi(X, \vec{T}(X))m(X)dX$$

and

$$\Psi_X(T) := \int_{\Gamma} \Psi(X, \vec{T}(X))m(\vec{X})d\vec{X}.$$  

Note that if $\Psi(X, \xi) = |\xi|$, then $\Psi(T)$ is just the mass of $T$.

For the rest of this section, $\Psi$ denotes a parametric integrand of degree 1.

Let $(\epsilon_1, ..., \epsilon_n)$ be an orthonormal reference frame in $\mathbb{R}^n$ centered at $X_0 \in \Omega$. We write $c = (c_1, ..., c_n) = (\vec{c}, c_n)$ for the associated coordinates of a point $X \in \Omega$, i.e., $X = X_0 + \sum_{i=1}^{n} c_i \epsilon_i = X_0 + Ac^T$ for some $n \times n$ matrix $A$ satisfying $AA^T = A^TA = I$.

Given $\Psi, X_0$, the reference frame $(\epsilon_1, ..., \epsilon_n)$ and $p = (p_1, ..., p_{n-1})$, let

$$\Psi^\parallel(\vec{c}, c_n, p) := \Psi(X, \sum_{i=1}^{n-1} p_i \epsilon_i + \epsilon_n).$$

$\Psi^\parallel$ is called the non-parametric integrand associated with $\Psi$ and the frame $(\epsilon_1, ..., \epsilon_n)$.

$\Psi$ is said to be $\lambda$-elliptic in $\Omega$ if for every flat $T_1 = \tau(\Gamma, m_1, \vec{T}_1)$ (this means $\vec{T}_1$ is constant on $\Gamma_1$) and $X \in \Omega$,

$$\lambda^{-1}(M(T_2) - M(T_1)) \leq \Psi_X(T_2) - \Psi_X(T_1).$$
for all rectifiable \( T_2 \) with compact support and \( \partial T_2 = \partial T_1 \).

Suppose \( K \) is a compact subset of \( \Omega \), \( Q \) is a rectifiable 1-current with compact support in \( K \) and \( \omega \) is a positive function defined for \( \rho \in (0, \delta] \) such that \( \lim_{\rho \to 0^+} \omega(\rho) = 0 \). We say that \( Q \) is \((\Psi, \omega, \delta)\)-minimal if

\[
\Psi(Q \downarrow K) \leq \Psi(Q \downarrow K + T) + \omega(\rho)M(Q \downarrow K + T)
\]

for all rectifiable \( T \) with compact support in \( K, \partial T = 0 \) and \( \text{diam}(\text{spt} T) \leq \rho \leq \delta \).

For \( y \in \mathbb{R}^{n-1}, t \in \mathbb{R} \) and \( X = (y, t) \), let \( p(X) := t \). Define

\[
E(T, t, a) := \frac{M(T \downarrow C(t, a)) - M(p_t(T \downarrow C(t, a)))}{a},
\]

where \( C(t, a) := \mathbb{R}^{n-1} \times B_a^1(t), a > 0 \). \( E(T, t, a) \) is referred to as the excess of \( T \) in \( C(t, a) \).

Let \( C_a := C(0, a) \).

### 2.3 \( \Gamma \)-Convergence and Saddle Point

Suppose \( U, V \) are Banach spaces, \( E_U^\epsilon, E_V : U, V \to (-\infty, \infty], \epsilon \in (0, 1] \) and \( V_0 = \{v \in V : E_V(v) < \infty\} \). We say that \( E_U^\epsilon \) \( \Gamma \)-converges to \( E_V \) as \( \epsilon \to 0 \) if for all \( \epsilon \in (0, 1] \), there exists a continuous map \( P_{VU}^\epsilon : U \to V \) and a map \( Q_{UV}^\epsilon : V_0 \to U \) (not necessarily continuous) satisfying

- **lower bound:**
  
  if \( v \in V_0 \) and \( \{u_\epsilon\} \subset U \) is a sequence such that \( \|P_{VU}^\epsilon(u_\epsilon) - v\|_V \to 0 \) as \( \epsilon \to 0 \),
  
  then \( \liminf E_U^\epsilon(u_\epsilon) \geq E_V(v) \) \hspace{1cm} (2.6)

  and

- **upper bound:**

  for every \( v \in V_0, E_U^\epsilon(Q_{UV}^\epsilon(v)) \to E_V(v) \) and \( \|P_{VU}^\epsilon(Q_{UV}^\epsilon(v)) - v\|_V \to 0 \) as \( \epsilon \to 0 \). \hspace{1cm} (2.7)
We will only be interested in Γ-limits for which the following compactness condition is satisfied:

\[
\text{if } \sup_{\epsilon \in (0,1]} E_U^\epsilon(u_\epsilon) < \infty, \text{ then } \{P_{VV}^\epsilon(u_\epsilon)\}_{\epsilon \in (0,1]} \text{ is precompact in } V. \tag{2.8}
\]

Given a $C^1$ functional $E : U \to \mathbb{R}$, a sequence $\{u_j\}_{j=1}^\infty$ is said to be a Palais-Smale sequence if

\[
\|\nabla E(u_j)\|_{U^*} \to 0 \text{ as } j \to \infty \quad \text{and} \quad \{E(u_j)\}_{j=1}^\infty \text{ is bounded.}
\]

The functional $E$ is said to satisfy the Palais-Smale condition if every Palais-Smale sequence is precompact in $U$.

**Definition 2.1.** We say that $E_V$ has a saddle point at $v_s \in V_0$ if there exists an integer $j \geq 0$, a number $\delta_0 > 0$, a neighbourhood $W \subset \mathbb{R}^j$ of 0, a continuous map $P_{WV} : V \to \mathbb{R}^j$ such that $P_{WV}(v_s) = 0$ and a continuous map $Q_{VW} : W \to V_0$ satisfying the conditions

\[
E_V(v_s) < E_V(v) \text{ for } v \in \{v \in V : 0 < \|v - v_s\|_V \leq \delta_0, P_{WV}(v) = 0\}, \tag{2.9}
\]

\[
Q_{VW}(0) = v_s, \tag{2.10}
\]

\[
P_{WV} \circ Q_{VW}(w) = w \text{ for all } w \in W \tag{2.11}
\]

and

\[
\text{for every } a > 0, \sup_{w \in W : |w| \geq a} E_V(Q_{VW}(w)) < E_V(v_s). \tag{2.12}
\]
Chapter 3

Some Known Results

Recall that we are assuming that $N$ is a Riemannian manifold of dimension $n$ and $\Omega$ is an open subset of $N$ with smooth boundary.

The following Sturm-Liouville type existence result can be deduced from the theory of bounded, linear, compact, symmetric operators (see Appendix D, Theorem 7 in [6]).

**Lemma 3.1.** For $y \in C^1([0,L];\mathbb{R}^{n-1})$, consider the boundary value problem

$$S(y) := -y'' + Ay = \lambda y, \quad y(L) = B^T y(0), \quad \frac{d}{dt}y(L) = B^T \frac{d}{dt}y(0)$$

where $y^T = (y_1, \ldots, y_{n-1})$, $A : [0,L] \to M_{n-1,n-1}(\mathbb{R})$ is a smooth function such that $(A(t))^T = A(t)$ for all $t \in [0,L]$ and $B \in M_{n-1,n-1}(\mathbb{R})$ with $BB^T = B^T B = I$. There exists a non-decreasing, unbounded sequence $\{\lambda_i\} \subset \mathbb{R}$ and $\{Y_i\} \subset C^1([0,L];\mathbb{R}^{n-1})$ such that $\{Y_i\}$ forms an orthonormal basis for $L^2([0,L];\mathbb{R}^{n-1})$ and for each $i$, $Y_i$ satisfies

$$S(Y_i) = \lambda_i Y_i, \quad Y_i(L) = B^T Y_i(0), \quad \frac{d}{dt}Y_i(L) = B^T \frac{d}{dt}Y_i(0).$$

A useful inequality related to slices is the following (see [18], p.158). Refer to Section 2.1 for the definition of $\langle T, f, s \rangle$, the slice of $T$ by $f^{-1}(s)$.

**Lemma 3.2.** Suppose $T \in D_k(\Omega)$, $M(T) + M(\partial T) < \infty$ and $f : \overline{\Omega} \to \mathbb{R}$ is Lipschitz. Then

$$\int M(\langle T, f, s \rangle) ds \leq \sup_{x \in \Omega} |\nabla f(x)| M(T).$$
The following is a consequence of the convex hull property (see [20], p.207), which says that the support of a minimizer of a frozen elliptic integrand with boundary constraint $B$ must lie in the convex hull of $\text{spt}B$.

**Lemma 3.3.** Let $\Psi$ be an elliptic, parametric integrand of degree $k$ on $\Omega \subseteq \mathbb{R}^n$, $K$ be a compact, convex subset of $\Omega$ and $X \in \Omega$. Suppose $Q$ is a rectifiable $k$-current supported in $K$ and $T$ minimizes $\Psi_X$ among all rectifiable $k$-currents with boundary equal to $\partial Q$. Then $\text{spt}T \subset K$.

Below is an isoperimetric type inequality.

**Lemma 3.4.** Suppose $N$ is compact and $T \in \mathcal{R}'_1(N)$. There exist constants $c_N, C_N > 0$ such that if $M(T) < c_N$, then we can write $T = \partial S$ for some rectifiable 2-current $S$ with

$$M(S) \leq C_N (M(T))^2.$$ 

Lemma 3.4 can be deduced as follows: first, cover $N$ by finitely many open sets $\{\Omega_j\}_{j=1}^m$ where each $\Omega_j$ is diffeomorphic to a ball $U^n_{R_j}$. Let $c_N$ be minimum of $\text{diam}(\Omega_j \cap \Omega_k)$ over all $j, k$ such that $\Omega_j \cap \Omega_k \neq \emptyset$. Writing $T$ as a sum of indecomposables $T = \sum_i T_i$, we must have for each $i$ that $\text{spt}T_i \subset \Omega_j$ for some $j$. Applying 4.2.10 in [7] to the pullback of $T_i$ and using Lemma 3.3, we can find $S_i \in \mathcal{R}_2(\Omega_j)$ such that $\partial S_i = T_i$ and $M(S_i) \leq C_j M(T_i)^2$. Setting $S = \sum_i S_i$ gives the desired result.

Next, we state a cone type inequality for integral 1-currents on $\mathbb{R}^n$ (see Proposition 3.4 in [19]). Basically, for a given closed curve, this result is obtained by considering the ‘cone’ swept out by contracting the curve to a suitable point.

**Lemma 3.5.** If $T \in \mathcal{R}'_1(\mathbb{R}^n)$, then we can write $T = \partial S$ for some $S \in \mathcal{I}_2(\mathbb{R}^n)$ satisfying

$$M(S) \leq 2 \text{diam}(sptT)M(T).$$
The following is the important homotopy formula for 1-currents; see [18], p.139 for a proof. Given 2 non-self-intersecting Lipschitz curves \( f_1, f_2 \) supported in an open cylinder \( \Omega = U_R^{n-1} \times (a, b) \) that run from \( U_R^{n-1} \times \{ t = a \} \) to \( U_R^{n-1} \times \{ t = b \} \), we use this to write the difference of the curves as a boundary in \( \Omega \) and also to get an upper bound for the mass of this surface in terms of \( f_1, f_2 \). In fact, Lemma 3.5 can probably be deduced from Lemma 3.6 below.

**Lemma 3.6.** Let \( f_1, f_2 : I \to \Omega \) be Lipschitz functions, where \( I \subset \mathbb{R} \) is an open interval and \( \Omega \) is an open, convex subset of \( \mathbb{R}^n \). Define \( h : [0, 1] \times I \to \Omega \) by

\[
h(s, t) = sf_2(t) + (1 - s)f_1(t).
\]

If \( T \in D_1(I) \) and \( h|_{[0,1] \times \text{sp} T} \) is proper, then \( h_\#(E^1 \downarrow [0, 1] \times T) \in D_2(\Omega) \) and

\[
(f_2)_\#T - (f_1)_\#T = \partial h_\#(E^1 \downarrow [0, 1] \times T) + h_\#(E^1 \downarrow [0, 1] \times \partial T).
\]

Moreover,

\[
\mathcal{M}(h_\#(E^1 \downarrow [0, 1] \times T)) \leq \sup_{\text{sp} T} \sup_{\text{sp} T} |f_1 - f_2| \sup_{\text{sp} T} (|f'_1| + |f'_2|) \mathcal{M}(T).
\]

Here, \( E^1 \) is the standard 1-current obtained by integration of 1-forms over \( \mathbb{R} \).

Finally, we state the aforementioned abstract theorem due to Jerrard and Sternberg [11], which is the backbone to proving our main theorem (Theorem 5.2).

**Theorem 3.1.** Suppose that \( U, V \) are Banach spaces and that \( \{ E^i_U \}_{i \in (0, 1]} \) is a family of \( C^1 \) functionals mapping \( U \) to \( \mathbb{R} \) that \( \Gamma \)-converge to a limiting functional \( E_V : V_0 \to \mathbb{R} \) via maps \( P_{UV}^\epsilon : U \to V \) and \( Q_{UV}^\epsilon : V_0 \to U \). Assume also that the compactness condition (2.8) holds.

Let \( v_s \in V \) be a saddle point in the sense of Definition 2.1. Assume also that

\[
P_{WV} \text{ is uniformly continuous in } \{ v \in V : \| v - v_s \|_V \leq 2\delta_0 \},
\]

(3.1)

\[
Q_{UV}^\epsilon := Q_{UV}^\epsilon \circ Q_{VW}^\epsilon : W \to U \text{ is continuous for all } \epsilon,
\]

(3.2)
Chapter 3. Some Known Results

\[ \| P_{UV} \circ Q_{UV}(w) - Q_{VW}(w) \|_V \to 0 \] uniformly in \( w \in W \) as \( \epsilon \to 0 \), \hspace{1cm} (3.3)

and

\[ E_U^\epsilon(Q_{UV}(w)) \to E_V(Q_{VW}(w)) \] uniformly in \( w \in W \) as \( \epsilon \to 0 \). \hspace{1cm} (3.4)

Then given \( \delta > 0 \), there exists \( \bar{\epsilon} > 0 \) and a Palais-Smale sequence \( \{ u_j^\epsilon \} \) for every \( \epsilon \in (0, \bar{\epsilon}) \) such that

\[ \sup_j | E_U^\epsilon(u_j^\epsilon) - E_V(v_s) | \leq \delta. \]

In particular, if \( E_U^\epsilon \) satisfies the Palais-Smale condition for every \( \epsilon \), then there exists \( \epsilon_0 > 0 \) and a critical point \( u^\epsilon \) of \( E_U^\epsilon \) for every \( \epsilon \in (0, \epsilon_0) \) such that \( \lim_{\epsilon \to 0} E_U^\epsilon(u^\epsilon) = E_V(v_s) \).

Remark 3.1. Jerrard and Sternberg remark in their paper [11] that an inspection of the proof shows that we do not need the full \( \Gamma \)-limit to hold. In particular, \( Q_{UV} \) need only be defined on \( \text{Im} Q_{VW} \).
Chapter 4

Saddle Points of Mass

Throughout this section, \((N, g)\) will be a compact, orientable, \(n\)-dimensional Riemannian manifold with \(\partial N = \emptyset\) that is isometrically embedded in \(\mathbb{R}^{n+m}\), \(n \geq 2, m \geq 1\). For \(x \in N\), we identify \(T_x N\) as an \(n\)-dimensional subspace of \(\mathbb{R}^{n+m}\) in the natural way. \(M\) will denote a 1-dimensional, connected, smooth submanifold of \(N\) without boundary. We can write \(M = \{\gamma(t)\}\) for some \(\gamma \in C^\infty(\mathbb{R}/L\mathbb{Z}; N)\). Suppose also that \(\gamma\) is a geodesic, i.e.,

\[
\frac{D}{dt} (\gamma'(t)) = 0 \text{ for all } t \in \mathbb{R}/L\mathbb{Z}.
\] (4.1)

Here, \(\frac{D}{dt}\) denotes the covariant derivative. Since \(\gamma\) is a geodesic, \(|\gamma'|\) is constant. We shall take this constant to be 1 so that the length of \(\gamma\) is \(L\). Let \(T_M\) denote the multiplicity 1 current corresponding to integration over \(\gamma\). Assume \(T_M \in \mathcal{R}_1(N)\).

For \(p \in M\), we write \(T_p^\perp M\) to denote that normal space at \(M\), characterized by \(T_p M \oplus T_p^\perp M = T_p N\).

Choose \(u^i \in C^\infty(\mathbb{R}/L\mathbb{Z})\) so that \(\{u^1(t), ..., u^{n-1}(t)\}\) forms an orthonormal basis for \(T_{\gamma(t)}^\perp M\) for all \(t \in \mathbb{R}/L\mathbb{Z}\). Therefore, for any \(t \in \mathbb{R}/L\mathbb{Z}\),

\[
\{\gamma'(t), u^1(t), ..., u^{n-1}(t)\} \text{ is an orthonormal basis for } T_{\gamma(t)} N.
\] (4.2)
For \( x_1, x_2 \in N \), define
\[
d(x_1, x_2) := \inf \{ \text{length} \beta : \beta : [0, 1] \to N \text{ is piecewise differentiable,} \\
\beta(0) = x_1, \beta(1) = x_2 \}.
\]
Set \( d(x) := \text{dist}(x, M) = \inf \{ d(x, \tilde{x}) : \tilde{x} \in M \} \) and
\[
K_r := \{ x \in N : d(x) < r \}, \quad r \in (0, r_0), 0 < r_0 < 1.
\]
For \( r_0 \) sufficiently small, define \( \psi : B^{n-1}_{r_0} \times \mathbb{R}/L\mathbb{Z} \to \overline{K_{r_0}} \) by
\[
\psi(y, t) := \exp \left( \gamma(t), \sum_{i=1}^{n-1} y_i u^i(t) \right).
\]
Here, \( \exp : \{(x, v) : x \in M, v \in T^\perp_x M, |v| < r_0 \} \to K_{r_0} \) is a smooth map given by \( \exp(x, v) := \alpha(1, x, v) \), where \( s \mapsto \alpha(s, x, v) \) is the unique geodesic of \( N \) which, at the instant \( s = 0 \), passes through \( x \) with velocity \( v \). Let \([t] \in \mathbb{R}/L\mathbb{Z}\) denote the conjugacy class of \( t \in \mathbb{R} \). From now on, when thinking of \( \psi \) as a map defined on \( B^{n-1}_{r_0} \times \mathbb{R} \), we will write \( \psi(y, t) \) to mean \( \psi(y, [t]) \).

Note that for any \( t \in \mathbb{R}/L\mathbb{Z} \) and \( i = 1, ..., n-1 \),
\[
\frac{\partial}{\partial t} \psi(y, t) \bigg|_{(0, t)} = \gamma'(t) \tag{4.3}
\]
and
\[
\frac{\partial}{\partial y_i} \psi(y, t) \bigg|_{(0, t)} = d\psi_{(0, t)}(e_i) = d(\exp_{\gamma(t)})(u^i(t)) = u^i(t). \tag{4.4}
\]

Therefore, using (4.2), it follows from the inverse function theorem that for every \( t \in \mathbb{R}/L\mathbb{Z} \), there exists a neighbourhood \( \Omega_t \) of \((0, t)\) such that \( \psi|_{\Omega_t} \) is a smooth diffeomorphism. Covering \( \{0\} \times \mathbb{R}/L\mathbb{Z} \) by \( \bigcup_{i=1}^m \Omega_{t_i} \), for some \( \{t_i\} \subset \mathbb{R}/L\mathbb{Z} \) and selecting \( r_0 \) smaller if necessary so that \( B^{n-1}_{r_0} \times \mathbb{R}/L\mathbb{Z} \subset \bigcup_{i=1}^m \Omega_{t_i} \), we conclude that \( \psi : B^{n-1}_{r_0} \times \mathbb{R}/L\mathbb{Z} \to \overline{K_{r_0}} \) is a smooth diffeomorphism.

For \( x \in K_{r_0} \), let
\[
p(x) := \text{the unique point of } M \text{ closest to } x \text{ in } N
= \gamma(\psi_n^{-1}(x))
\]
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and

\[ v(x) := \text{the unique vector } \in T^\perp_{p(x)} \text{ such that } \exp_{p(x)}(v(x)) = x \]
\[ = \sum_{i=1}^{n-1} \psi_i^{-1}(x) u^i(\psi_n^{-1}(x)). \]

Set

\[ \mathcal{A} := \{ u \in C^1([0, L]) : u(t) \in T^\perp_{\gamma(t)} M \text{ for every } t \in [0, L], \]
\[ u(L) = u(0) \text{ and } \frac{D}{dt} u(L) = \frac{D}{dt} u(0) \}. \]

For \( u \in \mathcal{A}, \) there exists a positive number \( s_u \) such that \( |su(t)| < r_0 \) for all \( t \in [0, L] \)
and \( s \in (-s_u, s_u). \) Consider the variation \( h_u : (-s_u, s_u) \times [0, L] \to K_{r_0} \) defined by

\[ h_u(s, t) := \exp(\gamma(t), su(t)). \]

Let \( T_{u,s} \) denote the multiplicity 1 current corresponding to integration over \( h_u(s, \cdot). \) Note that \( T_M = T_{u,0} \) for any \( u \in \mathcal{A}. \)

The Jacobi operator \( J \) is a linear, second-order, differential operator defined through the relation

\[ \left. \frac{d^2}{ds^2} M(T_{u,s}) \right|_{s=0} = (Ju, u)_{L^2(0,L)}. \]

Explicitly,

\[ Ju = -(u'' + R(\gamma', u)\gamma'), \]

where \( R \) is the curvature of \( N \) (see chapter 9 in [5]). This relies on the assumptions that \( \gamma \) is a geodesic and an arclength parametrization.

Let \( \{u_1^p(t), \ldots, u_{n-1}^p(t)\} \) be an orthonormal basis for \( T^\perp_{\gamma(t)} M \) such that, for each \( i = 1, \ldots, n - 1, \)

\[ \frac{D}{dt} u_i^p(t) = 0 \text{ for all } t \in [0, L]. \]

Here, we use the subscript ‘p’ because each \( u_i^p \) is a ‘parallel’ vectorfield on \( M. \) Since \( \{u_1^p(0), \ldots, u_{n-1}^p(0)\} \) and \( \{u_1^p(L), \ldots, u_{n-1}^p(L)\} \) are orthonormal bases for \( T^\perp_{\gamma(0)} M = T^\perp_{\gamma(L)} M, \)
there exists a matrix \( B = (b_{ij}) \) satisfying

\[ u_i^p = \sum_{j=1}^{n-1} b_{ij} u_j^p \quad \text{and} \quad BB^T = B^T B = I. \]
If \( u \in A \) and \( u = \sum_{i=1}^{n-1} y_i u_i \), it follows that \( y(L) = B^T y(0), y'(L) = B^T y'(0) \) and

\[
Ju = \sum_{i=1}^{n-1} \left( -y_i'' + \sum_{j=1}^{n-1} a_{ij} y_j \right) u_i,
\]

where \( y^T = (y_1, \ldots, y_{n-1}) \) and \( a_{ij} = -(R(\gamma', u_i^p), \gamma^p_j) \). Note that \( a_{ij} = a_{ji} \). Let \( A = (a_{ij}) \) and \( S(y) = -y'' + Ay \). Using Lemma 3.1, there exists a non-decreasing, unbounded sequence \( \{\lambda_i\} \subset \mathbb{R} \) and \( \{Y^i\} \subset C^1([0, L]; \mathbb{R}^{n-1}) \) such that \( \{Y^i\} \) forms an orthonormal basis for \( L^2([0, L]; \mathbb{R}^{n-1}) \) and for each \( i \), \( Y^i \) satisfies

\[
S(Y^i) = \lambda_i Y^i, \quad Y^i(L) = B^T Y^i(0), \quad \frac{d}{dt} Y^i(L) = B^T \frac{d}{dt} Y^i(0).
\]

Setting \( z^i = \sum_{j=1}^{n-1} Y_j^i u_j^0 \), it follows that

\[
Jz^i = \lambda_i z^i, \quad z^i(L) = z^i(0), \quad \frac{d}{dt} z^i(L) = \frac{d}{dt} z^i(0)
\]

and \( \{z^i\} \) forms an orthonormal basis for

\[
U := \{u \in L^2(0, L) : u(t) \in T_{\gamma(t)}^1 M \text{ for a.e. } t \in [0, L]\}.
\]

**Lemma 4.1.** For any \( u \in U \cap H^1(0, L) \), we can write \( u = \sum_{i=1}^{\infty} c_i z^i \). Let \( n = \sum_{i=1}^{l} c_i z^i \) and \( p = \sum_{i=l+1}^{\infty} c_i z^i \). Then

\[
(Jn, n)_{L^2(0, L)} \geq \lambda_1 \|n\|^2_{H^1(0, L)}.
\]

Moreover, there exists a positive constant \( c \), independent of \( u \), such that

\[
\|n\|^2_{L^2(0, L)} \geq c \|n\|^2_{H^1(0, L)}
\]

and

\[
(Jp, p)_{L^2(0, L)} \geq c \|p\|^2_{H^1(0, L)}.
\]

**Proof.** The first claim is vacuous unless \( \lambda_1 < 0 \). If \( \lambda_1 < 0 \), we compute

\[
(Jn, n)_{L^2(0, L)} = \sum_{i,j=1}^{l} \lambda_i c_i c_j \delta_{ij} = \sum_{i=1}^{l} \lambda_i c_i^2 \geq \lambda_1 \|n\|^2_{L^2(0, L)} \geq \lambda_1 \|n\|^2_{H^1(0, L)}.
\]
For the second claim, note

\[ \|n\|_{L^2(0,L)}^2 \leq C \sum_{i,j=1}^l |c_i||c_j| \leq C/2 \sum_{i,j=1}^l (c_i^2 + c_j^2) = lC\|n\|_{L^2(0,L)}^2, \]

where \( C > 0 \) depends on \( z^1, \ldots, z^l \). Thus,

\[ \|n\|_{L^2(0,L)}^2 \geq (1 + lC)^{-1}\|n\|_{H^1(0,L)}^2. \]

Finally, to prove the third claim, first note

\[ \int_0^L (|p'(t)|^2 + (a(t), p(t)))dt = (Jp, p)_{L^2(0,L)} = \sum_{i,j=l+1}^\infty \lambda_i c_i c_j \delta_{ij} \geq \lambda_{l+1}\|p\|_{L^2(0,L)}^2 \]

where \( a(t) = -R(\gamma'(t), p(t))\gamma'(t) \). Now, for \( \theta \in (0, 1) \),

\[ \theta\|p'\|_{L^2(0,L)}^2 \leq \theta\|p'\|_{L^2(0,L)}^2 + (1 - \theta)((Jp, p)_{L^2(0,L)} - \lambda_{l+1}\|p\|_{L^2(0,L)}^2) \]

\[ = (Jp, p)_{L^2(0,L)} - \int_0^L (\theta(a(t), p(t)) + (1 - \theta)\lambda_{l+1}|p(t)|^2)dt. \]

Since \( |a| \leq \tilde{C}|p| \) for some constant \( \tilde{C} \) depending on \( \gamma \) and the Christoffel symbols of the connection on \( M \), we can select \( \theta \) small enough so that

\[ \theta(a, p) + (1 - \theta)\lambda_{l+1}|p|^2 \geq (\lambda_{l+1} - \theta(\tilde{C} + \lambda_{l+1}))|p|^2 \geq 0. \]

Therefore, for this value of \( \theta \),

\[ \theta\|p'\|_{L^2(0,L)}^2 \leq (Jp, p)_{L^2(0,L)} \]

and

\[ (Jp, p)_{L^2(0,L)} \geq \frac{\theta\lambda_{l+1}}{\theta + \lambda_{l+1}}\|p\|_{H^1(0,L)}^2. \]

Define \( V := \mathcal{F}_1(N) \) and \( E_V : V \to [0, \infty] \) by

\[ E_V(T) := \begin{cases} 
M(T) & \text{if } T \in V_0 := \mathcal{R}'_1(N) \\
+\infty & \text{if not.}
\end{cases} \quad (4.5) \]
**Theorem 4.1.** Suppose the Jacobi operator $J$ associated with $M$ has finite index and 0 nullity. Then $T_M$ is a saddle point of $E_V$ in the sense of Definition 2.1.

Note that our assumptions imply

$$\lambda_1 \leq \cdots \leq \lambda_l < 0 < \lambda_{l+1} \leq \cdots$$

if $l := \text{index of } J > 0$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

if $l = 0$.

To prove Theorem 4.1, we need to construct maps $P_{WV} : V \rightarrow \mathbb{R}^j, Q_{VW} : W \rightarrow V_0$ satisfying the conditions of Definition 2.1 for some nonnegative integer $j$ and neighbourhood $W$ of 0 in $\mathbb{R}^j$. We claim that this is possible with $j = l$.

If $l = 0$, we adopt the convention $\mathbb{R}^0 = \{0\}$ and set $P_{WV}(v) = 0$ for all $v \in V$. Our arguments will show $T_M$ is a local minimizer of mass in the flat norm topology.

Now we define $P_{WV}$ for $l > 0$. Keeping in mind that we need to verify (2.9), we would like to define $P_{WV}$ so that, if $T$ corresponds to integration over a Lipschitz curve and $\text{spt} T$ can be written as the graph of a normal vectorfield $u$ over $M$ with $\|u\|_{W^{1,\infty}}$ sufficiently small, then, writing $u = \sum_{i=1}^{\infty} c_i z^i$, we have $P_{WV}(T) = (c_1, \ldots, c_l)$. The reason for this is that such a current $T$ as described above will be close to $T_M$ in the flat norm and thus, if $P_{WV}(T) = 0$, we want to be able to say $M(T) \geq L$ with equality if and only if $T = T_M$. To see this, first note that we can take $s_u = 2$. Then, since $\gamma$ is a geodesic,

$$M(T) = M(T_{u,1}) = L + \frac{1}{2} (Ju, u)_{L^2} + \|u\|_{W^{1,\infty}} O(\|u\|_{H^1}^2)$$

$$= L + \frac{1}{2} (Jn, n)_{L^2} + \frac{1}{2} (Jp, p)_{L^2} + \|u\|_{W^{1,\infty}} O(\|u\|_{H^1}^2)$$

where $n = \sum_{i=1}^l c_i z^i, p = \sum_{i=l+1}^{\infty} c_i z^i$. If $c > 0$ is the constant from Lemma 4.1 and $\|u\|_{W^{1,\infty}}$ is small enough so that $\|u\|_{W^{1,\infty}} O(\|u\|_{H^1}^2) \leq \frac{c}{8} \|u\|_{H^1}^2$, we have

$$M(T) \geq L - \frac{1}{2} \left(\frac{c}{2} - \lambda_1\right) \|n\|_{H^1}^2 + \frac{c}{4} \|p\|_{H^1}^2.$$
Thus, if $P_{WV}(T) = 0$, $n$ is identically 0 and $M(T) \geq L$ with equality if and only if $T = T_M$. In this case, we can think of $P_{WV}$ as being the projection of $T$ onto the negative eigenspace associated with $J$. Before defining $P_{WV}$, we first need to make some definitions. Let $Z(x)$ be the $l$-vector $(z^1(\tau(x)) \cdot v(x), ..., z^l(\tau(x)) \cdot v(x))$ and define $\Phi : N \to (D^1(N))^l$ by
\[
\Phi(x) := \begin{cases} 
\chi(|v(x)|)Z(x)d\tau(x) & \text{for } x \in K_{r_0} \\
0 & \text{for } x \in N \setminus K_{r_0}, \end{cases}
\]
where
\[
\tau(x) := \gamma^{-1}(p(x)) = \psi^{-1}_n(x)
\]
and $\chi \in C^\infty_c([0, r_0); [0, 1])$ with $\chi(s) = 1$ for every $s \in [0, r_0/2]$. Note that $\|d\Phi\|_{L^\infty} \leq C/r_0$.

Now, for $T \in V$, set $P_{WV}(T) := T(\Phi) \in \mathbb{R}^l$. The desired property of $P_{WV}$ described above is verified in Lemma 4.3. First, we show that $P_{WV}$ is continuous with respect to the $F$-norm.

**Lemma 4.2.** $P_{WV}$ is uniformly continuous in the flat norm topology.

**Proof.** Given $T_1, T_2 \in V$, we can find $S \in D_2(N)$ such that $T_1 - T_2 = \partial S$ and $M(S) \leq 2F(T_1 - T_2)$. Then
\[
|P_{WV}(T_1) - P_{WV}(T_2)| = |(T_1 - T_2)(\Phi)|
\]
\[
= |\partial S(\Phi)|
\]
\[
= |S(d\Phi)|
\]
\[
\leq \|d\Phi\|_{L^\infty} M(S)
\]
\[
\leq 2\|d\Phi\|_{L^\infty} F(T_1 - T_2),
\]
which implies the above assertion. \qed

**Lemma 4.3.** For $u \in U$ and $t \in [0, L)$, let $\beta(t) = \exp(\gamma(t), u(t))$. Suppose $\beta$ is Lipschitz, $\text{Im}\beta \subset K_{r_0/2}$ and $T$ is the multiplicity 1 current corresponding to integration over $\beta$. 
Then, writing \( u = \sum_{i=1}^{\infty} c_i z^i \), we have
\[
P_{WV}(T) = (c_1, \ldots, c_l).
\]

**Proof.** Set \( \alpha(t) = \psi^{-1}(\beta(t)) \). Since \( T \) corresponds to integration over \( \beta \) and \( \text{Im} \beta \subset K_{r_0} \),
\[
P_{WV}(T) = T(\Phi) = \int_0^L \chi(|v(\beta(t))|)Z(\beta(t)) \frac{d}{dt}(\tau \circ \psi)_{\alpha(t)} \alpha'(t) dt
\]
By definition of \( p, v \) and \( \tau \), we have \( p(\beta(t)) = \gamma(t), v(\beta(t)) = u(t) \) and \( \tau(\beta(t)) = t \). Therefore, since \( |u(t)| < r_0/2 \) for every \( t \in [0, L] \) and \( \{z^i\}_{i=1}^{\infty} \) are orthonormal in \( L^2(0, L) \),
\[
P_{WV}(T) = \int_0^L (z^1(t) \cdot u(t), \ldots, z^l(t) \cdot u(t)) dt
\]
\[
= \left( \int_0^L z^1(t) \cdot u(t) dt, \ldots, \int_0^L z^l(t) \cdot u(t) dt \right)
\]
\[
= (c_1, \ldots, c_l).
\]
\( \square \)

Note that if \( T = T_M \) above, then \( u(t) = 0 \) for every \( t \in [0, L] \) and \( P_{WV}(T_M) = 0 \).

### 4.1 Flat Local Minimizers of Mass in \( P_{WV}^{-1}(0) \)

The goal of this section is to prove (2.9), i.e., to show that \( T_M \) is a strict local minimizer of mass in the flat norm topology among currents \( T \in \mathcal{R}_1(N) \) with \( P_{WV}(T) = 0 \). The conclusion of the following proposition is equivalent to (2.9).

**Proposition 4.1.** There exists \( \delta_0 > 0 \) such that if \( T \in \mathcal{R}_1'(N) \),
\[
F(T - T_M) < \delta_0, \tag{4.6}
\]
\[
P_{WV}(T) = P_{WV}(T_M) = 0 \tag{4.7}
\]
and
\[
M(T) \leq M(T_M) = L, \tag{4.8}
\]
then

\[ T = T_M. \]

First we would like to show that if \( T \) is close to \( T_M \) in the flat norm, then there exists a ‘piece’ of \( T \) that is uniformly close to \( T_M \). This is a partial result of the following Lemma, which relies heavily on the fact that \( T \) is 1-dimensional. Basically, this is due to the fact that if a Lipschitz curve stretches between two sets \( A \) and \( B \), then the length of the curve has to be at least the distance between \( A \) and \( B \), whereas if an \( n \)-dimensional surface, \( n \geq 2 \), stretches between two sets, the \( n \)-dimensional surface area can be arbitrarily small.

**Lemma 4.4.** Suppose \( T \in \mathcal{R}'_1(N) \) satisfies (4.6) and (4.8). If \( \delta_0 \leq r_0^3 \) and \( r_0 \) is taken sufficiently small, then there exists a 1-current \( T' \in \mathcal{R}_1(N) \) such that \( \Gamma' := \text{spt} T' \) consists of a single Lipschitz curve with no boundary,

\[ \Gamma' \subset K_{r_0/4} \]  

and

\[ \Gamma' \cap \tau^{-1}(t) \neq \emptyset \text{ for all } t \in \mathbb{R}/L\mathbb{Z}. \]  

In addition,

\[ M(T - T') = M(T) - M(T') \]  

and

\[ T' - T_M = \partial S' \text{ for some } 2\text{-current } S' \text{ with } \text{spt} S' \subset K_{2r_0/3} \text{ and } M(S') < \infty. \]  

**Proof.** 1. Assumption (4.6) implies there exists \( S_1 \in \mathcal{D}_2(N) \) such that

\[ \partial S_1 = T - T_M \text{ in } N \quad \text{and} \quad M(S_1) < \delta_0. \]

We would like to replace \( T \) by a current \( \tilde{T} \) with support in \( K_{r_0} \) satisfying (4.6) and a slightly weaker form of (4.8). This can be accomplished by finding a slice of \( S_1 \) by \( d \) with sufficiently small mass.
Therefore, letting $\text{geodesics locally minimize arclength (see Proposition 3.6 in [5])},$

$$d(\psi(y, t)) = \int_0^1 |\alpha'_{y,t}(s)| ds = |\alpha_{y,t}(0)| = \sum_{i=1}^{n-1} y_i u^i(t).$$

Since $d(\psi(y, t))$ must be attained by a path connecting $\psi(y, t)$ and $\tilde{\psi}$ which, at the instant $s = 0$, passes through $\gamma(t)$ with velocity $\sum_{i=1}^{n-1} y_i u^i(t)$. Therefore, letting $Y(x) = (\tilde{\psi}_1^{-1}(x), ..., \tilde{\psi}_{n-1}^{-1}(x), 0)$, we have

$$\nabla d(x) = \sum_{i,j=1}^n g^{ij}(\tilde{\psi}_1^{-1}(x)) \frac{d}{dX_j} (d \circ \psi)|_{\tilde{\psi}_1^{-1}(x)} \psi_{X_i}(\tilde{\psi}_1^{-1}(x)) = \sum_{i,j=1}^n g^{ij}(\tilde{\psi}_1^{-1}(x)) \frac{\tilde{\psi}_1^{-1}(x)}{|Y(x)|} \psi_{X_i}(\tilde{\psi}_1^{-1}(x))$$

and

$$|\nabla d(x)|^2 = \frac{Y(x)G^{-1}(\tilde{\psi}_1^{-1}(x))(Y(x))^T}{|Y(x)|^2}.$$ 

Using Lemma 3.2, it follows that

$$\int_{r_0/2}^{r_0} M\langle S_1, d, r \rangle dr \leq \left( \sup_{x \in K_{r_0} \setminus K_{r_0/2}} |\nabla d(x)| \right) M(S_1 \cup K_{r_0} \setminus K_{r_0/2}) \leq C_1 \delta_0,$$

which implies the existence of a number $\tilde{r} \in (r_0/2, r_0)$ such that

$$M\langle S_1, d, \tilde{r} \rangle \leq 2C_1 \delta_0 / r_0 \leq 2C_1 r_0^2.$$

Let $\tilde{S}_1 = S_1 \cup K_{\tilde{r}}$ and $\tilde{T} = T \cup K_{\tilde{r}} + \langle S_1, d, \tilde{r} \rangle$. Then

$$\partial \tilde{S}_1 = \tilde{T} - T_M, \quad M(\tilde{S}_1) \leq M(S_1) < \delta_0$$

and

$$M(\tilde{T}) \leq L + 2C_1 r_0^2. \quad (4.13)$$

2. Now we would like to estimate $|\nabla \tau|$ in $K_{r_0}$. First note that

$$\frac{\partial}{\partial y_i} g_{nn} = 2 \left( \psi_{t}, \frac{D}{dy_i} \psi_t \right) = 2 \left( \psi_{t}, \frac{D}{dt} \psi_{y_i} \right) = 2 \frac{d}{dt} (\psi_t, \psi_{y_i}) - 2 \left( \frac{D}{dt} \psi_t, \psi_{y_i} \right),$$
which implies
\[
\frac{\partial}{\partial y_i} g_{nn}(0, t) = 2 \frac{d}{dt} (\gamma'(t), u'(t)) - 2 \left( \frac{D}{dt} \gamma'(t), u'(t) \right) = 0
\]
using (4.1) and (4.2). Since
\[
\tau(\psi(y, t)) = t \text{ for all } (y, t) \in U_{r_0}^{n-1} \times \mathbb{R}/\mathbb{L} \mathbb{Z},
\]
we have \( \nabla \tau(x) = \sum_{i=1}^{n} g^{in}(\tilde{\psi}^{-1}(x)) \psi_X(\tilde{\psi}^{-1}(x)) \) and
\[
|\nabla \tau(\psi(X))| = \sqrt{g^{nn}(X)}
\]
\[
= g^{nn}(0, t) + \frac{1}{2\sqrt{g^{nn}(0, t)}} (\sum_{i=1}^{n} g^{in}(0, t)) \cdot y + O(|y|^2)
\]
\[
= 1 - \frac{1}{2} (\sum_{i=1}^{n} g^{in}(0, t)) \cdot y + O(|y|^2)
\]
\[
= 1 + O(|y|^2).
\]

3. Define
\[
\Sigma_0 := \{ t \in (0, L) : M(\langle \tilde{T}, \tau, t \rangle \perp K_{r_0/8}) = 0 \},
\]
\[
\Sigma_1 := \{ t \in (0, L) : M(\langle \tilde{T}, \tau, t \rangle) = M(\langle T, \tau, t \rangle \perp K_{r_0/8}) = 1 \}
\]
and
\[
\Sigma_2 := \{ t \in (0, L) : M(\langle \tilde{T}, \tau, t \rangle) \geq 2 \}.
\]
Clearly,
\[
|\Sigma_0| + |\Sigma_1| + |\Sigma_2| \geq L. \tag{4.14}
\]
Also, using Lemma 3.2,
\[
\int_0^L M(\langle \tilde{T}, \tau, t \rangle) dt \leq \left( \sup_{x \in K_{r_0}} |\nabla \tau(x)| \right) M(\tilde{T})
\]
\[
\leq (1 + C_2 r_0^2) (L + 2C_1 r_0^2)
\]
\[
\overset{(*)}{\leq} 4L, \tag{4.15}
\]
which implies \( M(\langle \tilde{T}, \tau, t \rangle) < \infty \) for a.e. \( t \in (0, L) \). Here, \( (*) \) indicates that we are taking \( r_0 \) sufficiently small. We claim that, most of the time, \( \tilde{T} \) stays in \( K_{r_0/8} \) and intersects
level sets of $\tau$ only once, i.e., $|\Sigma_1|$ is big. This will be verified in Steps 4 and 5 by showing $|\Sigma_0|$ and $|\Sigma_2|$ are small.

4. Note that for a.e. $t \in (0, L)$,

$$\partial(\tilde{S}_1, \tau, t) = \langle \partial \tilde{S}_1, \tau, t \rangle = \langle \tilde{T}, \tau, t \rangle - \langle T_M, \tau, t \rangle = \langle \tilde{T}, \tau, t \rangle - \delta_{\gamma(t)}.$$ 

Hence, for a.e. $t \in \Sigma_0$, $\partial(\tilde{S}_1, \tau, t) \cdot K_{r_0/8} = -\delta_{\gamma(t)}$. Using this, it follows that

$$C_1 \mathbf{M}((\tilde{S}_1, \tau, t)) \geq \frac{r_0}{8} \quad \text{for a.e. } t \in \Sigma_0. \quad (4.16)$$

To see this, let $f_\epsilon(x) = \phi_\epsilon(d(x))$ where $\{\phi_\epsilon\}_{\epsilon \in (0, r_0/16)} \subset C^\infty_c([0, r_0/8); [0, r_0/8])$ satisfies $\phi_\epsilon(0) \geq r_0/8 - \epsilon, \phi'_\epsilon(0) = 0$ and $\|\phi'_\epsilon\|_\infty \leq 1$. Then $\|df_\epsilon\|_\infty = \|\nabla f_\epsilon\|_\infty \leq C_1$ and for a.e. $t \in \Sigma_0$,

$$\frac{r_0}{8} - \epsilon \leq \phi_\epsilon(0) = f_\epsilon(\gamma(t)) = -\partial(\tilde{S}_1, \tau, t)(f_\epsilon) = -\langle \tilde{S}_1, \tau, t \rangle(df_\epsilon) \leq C_1 \mathbf{M}((\tilde{S}_1, \tau, t)).$$

Let $\epsilon \to 0$ to obtain (4.16).

Now, again using Lemma 3.2,

$$\frac{r_0}{8} |\Sigma_0| \leq C_1 \int_{\Sigma_0} \mathbf{M}((\tilde{S}_1, \tau, t))dt \leq C_1 \int_0^L \mathbf{M}((\tilde{S}_1, \tau, t))dt \leq C_1(1 + C_2 r_0^2)\delta_0 \overset{(\ast)}{\leq} 2C_1 \delta_0,$$

which gives

$$|\Sigma_0| \leq 16C_1 \delta_0/r_0 \leq 16C_1 r_0^2.$$ \quad (4.17)

5. Similar to (4.15), we compute

$$\mathbf{M}(\tilde{T}) \geq \frac{1}{1 + C_2 r_0^2} \int_0^L \mathbf{M}((\tilde{T}, \tau, t))dt \geq (1 - C_2 r_0^2)(|\Sigma_1| + 2|\Sigma_2|).$$
Combining this with (4.13), (4.14) and (4.17), we see that
\[ |\Sigma_2| \leq 3(6C_1 + LC_2)r_0^2, \] which implies
\[ |\Sigma_1| \geq L - (36C_1 + 3LC_2)r_0^2. \] (4.18)

6. Writing \( \tilde{T} \) as a sum of indecomposable currents \( \tilde{T} = \sum_{i=1}^{\infty} T_i \), where each \( T_i \) corresponds to integration over a Lipschitz curve \( \tilde{\gamma}_i \), we claim that

if for some \( i \), one has \( \text{spt } \tilde{T}_i \cap K_{r_0/8} \neq \emptyset \), then \( \text{spt } \tilde{T}_i \subset K_{r_0/4} \). (4.19)

To verify this, first note that, once again using Lemma 3.2 and recalling the definition of \( \Sigma_1 \),

\[
M(\tilde{T}_\perp K_{r_0/8}) \geq \frac{1}{1 + C_2r_0^2} \int_{\Sigma_1} M(\langle \tilde{T}, \tau, t \rangle \setminus K_{r_0/8}) dt \\
\geq (1 - C_2r_0^2)|\Sigma_1| \\
\geq L - 2(17C_1 + 2LC_2)r_0^2.
\]

Thus, if the support of \( \tilde{\gamma}_i \) intersects both \( K_{r_0/8} \) and \( N \setminus K_{r_0/4} \), then \( \tilde{\gamma}_i \) must have arclength at least \( r_0/8 \) and

\[
r_0/8 \leq M(\tilde{T}_\perp (N \setminus K_{r_0/8})) = M(\tilde{T}) - M(\tilde{T}_\perp K_{r_0/8}) \leq 4(9C_1 + LC_2)r_0^2. \] (4.13)

Taking \( r_0 \) sufficiently small leads to a contradiction. Therefore, (4.19) holds.

7. For any positive integer \( j \), let

\[
\Sigma_{1,j} := \left\{ t \in \Sigma_1 : \sum_i \mathcal{H}^0(\tilde{\gamma}_i \cap \tau^{-1}(t)) = j \right\}.
\]

We claim that
\[ |\Sigma_{1,1}| \geq L - 10(7C_1 + LC_2)r_0^2. \] (4.20)

The explicit formula (2.3) for the slice of an indecomposable 1-current implies

\[
M(\langle \tilde{T}, \tau, t \rangle) \leq \sum_i M(\langle \tilde{T}_i, \tau, t \rangle) = \sum_i \mathcal{H}^0(\tilde{\gamma}_i \cap \tau^{-1}(t))
\]
for a.e. \( t \). It follows that \( \sum_i H^0(\tilde{\gamma}_i \cap \tau^{-1}(t)) \geq 1 \) for a.e. \( t \in \Sigma_1 \) and hence that

\[
\sum_{j=1}^{\infty} |\Sigma_{1,j}| = |\Sigma_1|.
\] (4.21)

Since \( |J\tau|(x) = |\nabla \tau(x)| \leq 1 + C_2 r_0^2 \), it follows from the coarea formula that

\[
\int_0^L H^0(\tilde{\gamma}_i \cap \tau^{-1}(t))dt = \int_{\tilde{\gamma}_i} |J\tau|(x)dH^1(x) \leq (1 + C_2 r_0^2)H^1(\tilde{\gamma}_i).
\]

Using this, we have

\[
L + 2C_1 r_0^2 \overset{(4.13)}{\geq} M(\tilde{T}) = \sum_i M(\tilde{T}_i) = \sum_i H^1(\tilde{\gamma}_i) \geq \frac{1}{1 + C_2 r_0^2} \int_{\Sigma_1} \sum_i H^0(\tilde{\gamma}_i \cap \tau^{-1}(t))dt = \left(1 - \frac{C_2 r_0^2}{1 + C_2 r_0^2}\right) \sum_{j=1}^{\infty} j|\Sigma_{1,j}| \geq (1 - C_2 r_0^2) \left(|\Sigma_{1,1}| + 2 \sum_{j=2}^{\infty} |\Sigma_{1,j}|\right) \overset{(4.21)}{=} (1 - C_2 r_0^2)(2|\Sigma_1| - |\Sigma_{1,1}|).
\]

This together with (4.18) implies (4.20).

8. Taking \( r_0 \) sufficiently small, we have \( |\Sigma_{1,1}| > L/2 > 0 \). Fix \( t^* \in \Sigma_{1,1} \) at which the Lebesgue density is 1. Let \( \tilde{\gamma}_{i^*} \) be the unique closed curve that intersects \( \tau^{-1}(t^*) \). The point of intersection must be in \( K_{r_0/8} \) since \( t^* \in \Sigma_1 \). Using (4.19), we conclude \( \tilde{\gamma}_{i^*} \) is entirely contained in \( K_{r_0/4} \).

As the Lebesgue density at \( t^* \in \Sigma_{1,1} \) is 1, we can find points in \( \Sigma_{1,1} \) arbitrarily close to \( t^* \). Since \( \tilde{\gamma}_{i^*} \) is a closed, Lipschitz curve, we must have

\[
\tilde{\gamma}_{i^*} \cap \tau^{-1}(t) \neq \emptyset \text{ for all } t \in \mathbb{R}/L\mathbb{Z},
\]

which verifies (4.10).
9. Now set $T'$ to be the current corresponding to integration over the curve $\tilde{\gamma}_i^*$ chosen in the step above. Writing $T$ as a sum $T = \sum T_i$ of indecomposable currents, we must have $T' = T_k$ for some $k$ since $\tilde{T}_\perp \mathbb{K}_{r_0/4} = T'_\perp \mathbb{K}_{r_0/4}$. Thus,

$$M(T - T') = M\left(\sum_{i \neq k} T_i\right) = \sum_{i \neq k} M(T_i) = \sum_i M(T_i) - M(T_k) = M(T) - M(T'),$$

which is (4.11).

10. Finally, we need to show that $T'$ is homologous to $T_M$ in $\overline{K_{2r_0/3}}$. To do this, parametrize $\tilde{\gamma}_i^*$ with respect to $t \in [t^*, t^* + L]$ and let $\alpha(t) = \psi^{-1}(\tilde{\gamma}_i^*(t))$. Using Lemma (3.6) with the convex set $\Omega = U_{r_0/2}^{n-1} \times (t^*, t^* + L)$ and $f_1(t) = \alpha(t), f_2(t) = (0, t)$ for $t \in (t^*, t^* + L)$, we can find a 2-current $S$ supported in $U_{r_0/2}^{n-1} \times (t^*, t^* + L)$ such that

$$\psi^{-1}_2 T' - T_{\{0\} \times (t^*, t^* + L)} = \partial S - T_{\alpha(t^* + L), (0, t^* + L)} + T_{\alpha(t^*), (0, t^*)}$$

with

$$M(S) \leq L(\|\alpha\|_{L^\infty} + L)(\|\alpha'\|_{L^\infty} + 1) < \infty.$$ 

Here, $T_{X_1, X_2}$ denotes the multiplicity 1 current corresponding to integration over the line segment $l_{X_1, X_2}$, where $l_{X_1, X_2} : [0, 1] \to \mathbb{R}^n$ is defined by $l_{X_1, X_2}(s) := (1 - s)X_1 + sX_2$. Therefore,

$$T' - T_M = \partial \psi S$$

with $\text{spt} \psi S \subset \overline{K_{2r_0/3}}$ and $M(\psi S) \leq C_\psi M(S) < \infty$, as required. Note that $C_\psi = 1 + O(r_0)$ since $|J\psi|(0, t) = 1$ for every $t \in \mathbb{R}/L\mathbb{Z}$. 

Our goal now is to show that $T' = T_M$. To do this, we first introduce a functional $F = \psi S$, the pullback of $S$ induced by $\psi$, then state and verify some good properties satisfied by $F$.

For $X \in U_{r_0}^{n-1} \times \mathbb{R}$ and $\xi \in \mathbb{R}^n$, let

$$F(X, \xi) := |D\psi(X)\xi|,$$
which is a parametric integrand of degree 1 (see Section 2.2). Here, we take \(D\psi(X)\xi\) to represent \(\sum_{i=1}^{n} \xi_i \psi_{X_i}(X) \in T_{\psi(X)}N\).

From this, we can define a functional on \(R_1(U_{r_0}^{n-1} \times \mathbb{R})\) by

\[
F(T) := \int_{\Gamma} F(X, \vec{T}(X)) m(X) dH^1(X),
\]

where \(T = \tau(\Gamma, m, \vec{T})\). Also, for \(X \in U_{r_0}^{n-1} \times \mathbb{R}\) and \(T = \tau(\Gamma, m, \vec{T}) \in \mathcal{R}_1(\mathbb{R}^n)\), define the ‘frozen’ integrand \(F_X\) by

\[
F_X(T) := \int_{\Gamma} F(X, \vec{T}(\hat{X})) m(\hat{X}) dH^1(\hat{X}).
\]

One can easily verify

\[
F(T) = M(\psi_2 T) \quad (4.22)
\]

and the existence of a constant \(C_3 = C_3(\psi) > 0\) such that if \(\text{spt} T \subset B^p_{\rho}(X) \subset U_{r_0}^{n-1} \times \mathbb{R}\), then

\[
|F_X(T) - F(T)| \leq C_3 \rho M(T). \quad (4.23)
\]

**Lemma 4.5.** For any \(X \in U_{r_0}^{n-1} \times \mathbb{R}\) and \(\xi \in \mathbb{R}^n\),

\[
(1 - C_3 r_0) |\xi| \leq F(X, \xi) \leq (1 + C_3 r_0) |\xi|. \quad (4.24)
\]

**Proof.** Using (4.3),(4.4) and recalling \(\{\gamma'(t), u^1(t), ..., u^{n-1}(t)\}\) is an orthonormal basis for \(T_{\gamma(t)}N\), we have

\[
|D\psi(0,t)\xi| = |\xi| \quad \text{for any } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n.
\]

Therefore, letting \(X = (y, t)\),

\[
|F(X, \xi) - |\xi|| = ||D\psi(y, t)\xi| - |D\psi(0,t)\xi||
\leq ||(D\psi(y, t) - D\psi(0,t))\xi||
\leq C_3 r_0 |\xi|.
\]

\(\square\)
From now on, we will assume that \( r_0 \) is small enough so that

\[
|\xi|/2 \leq F(X, \xi) \leq 3|\xi|/2 \tag{4.25}
\]

for all \( X \in U_{r_0}^{n-1} \times \mathbb{R} \) and \( \xi \in \mathbb{R}^n \).

**Lemma 4.6.** If \( T = \tau(\Gamma, m, \tilde{T}) \in \mathcal{R}_1(U_{r_0}^{n-1} \times \mathbb{R}) \), then

\[
(1 - C_3 r_0) M(T) \leq F(T) \leq (1 + C_3 r_0) M(T). \tag{4.26}
\]

In addition, for every \( X \in U_{r_0}^{n-1} \times \mathbb{R} \),

\[
(1 - C_3 r_0) M(T) \leq F_X(T) \leq (1 + C_3 r_0) M(T). \tag{4.27}
\]

**Proof.** Using (4.24) with \( \xi = \tilde{T}(\tilde{X}), \tilde{X} \in \Gamma \subset U_{r_0}^{n-1} \times \mathbb{R} \), we have

\[
1 - C_3 r_0 \leq F(X, \tilde{T}(\tilde{X})) \leq 1 + C_3 r_0 \tag{4.28}
\]

for every \( X \in U_{r_0}^{n-1} \times \mathbb{R} \). Multiplying through by \( m(\tilde{X}) \) and integrating over \( \Gamma \) gives (4.27). Setting \( X = \tilde{X} \) in (4.28), multiplying through by \( m(\tilde{X}) \) and integrating over \( \Gamma \) gives (4.26).

We now show that \( M(T - T') \) is small. Set

\[
\delta_1 := L - M(T) \overset{(4.11)}{=} L - M(T) - M(T - T') \overset{(4.8)}{\geq} 0
\]

and \( Q' := \psi^{-1}_t T' \). From Step 10 in the proof of Lemma 4.4, we have

\[
M(Q') = \mathcal{H}^1(\alpha) \geq L.
\]

Hence,

\[
L \leq M(Q') \overset{(4.26)}{\leq} \frac{1}{1 - C_3 r_0} F(Q') \overset{(4.22)}{=} \left(1 + \frac{C_3 r_0}{1 - C_3 r_0}\right) M(T'),
\]

which implies

\[
\delta_1 \leq \frac{C_3 L r_0}{1 - C_3 r_0}. \tag{4.29}
\]
This means, recalling (*) indicates that we are taking $r_0$ sufficiently small,

$$
M(T - T') \stackrel{(4.11)}{=} M(T) - M(T') \\
\leq \stackrel{(4.8)}{L - M(T')} \\
= \stackrel{(4.29),(*)}{\delta_1} \\
\leq \stackrel{(4.30)}{2C_3 L r_0}.
$$

Using Lemma 3.4 and taking $r_0$ sufficiently small, there exists a 2-current $S_2$ such that $\partial S_2 = T - T'$ and $M(S_2) \leq C_N(M(T - T'))^2$. Thus,

$$
M(S_2) \leq C_N(M(T - T'))^2 \stackrel{(4.11)}{=} C_N(M(T) - M(T'))^2 \leq C_N \delta_1^2.
$$

For $r \in (0,2r_0/3)$, let

$$
H_r := \{ T \in R_1(N) : \text{spt} T \subset \overline{K_r} \text{ and } T - T_M = \partial S \text{ for some} \\
S \in D_2(N) \text{ with support in } \overline{K_{2r_0/3}} \text{ and } M(S) < \infty\}.
$$

Recalling (4.9) and (4.12), $T' \in H_{r_0/4}$. Also note that $0 \notin H_r$ for any $r \in (0,2r_0/3)$.

This is true because we can find $\eta \in D^1(N)$ with $\eta(x) = d\tau(x)$ for $x \in K_{r_0}$, which implies $T_M(\eta) = T_M(d\tau) = L > 0$ and $\partial S(\eta) = S(d^2\tau) = 0$ for any $S \in D_2(N)$ with spt$S \subset \overline{K_{2r_0/3}}$.

For $T \in V$, define $M^*(T) := M(T) + C_* |P_WV(T)|^2$, where $C_* > 0$ is to be chosen. We claim that

there exists a $C_* > 0$ such that $T_M$ is the unique minimizer

$$
M^* \text{ in } H_r \text{ for } r \in (0,r_0/2) \text{ and } r_0 \text{ sufficiently small.}
$$

Let’s assume this for now and complete the proof of Proposition 4.1. Since $T_M$ minimizes $M^*$ in $H_{r_0/4}$, $M(T_M) = L$ and $P_WV(T_M) = 0$, we have

$$
L = M^*(T_M) \leq M^*(T') = M(T') + C_* |P_WV(T')|^2 = L - \delta_1 + C_* |P_WV(T')|^2,
$$

which implies

$$
\delta_1 \leq C_* |P_WV(T')|^2.
$$
Using this, we have
\[
\begin{align*}
\delta_1 & \leq C_s |P_{WV}(T')|^2 \\
& \overset{(4.7)}{=} C_s |P_{WV}(T') - P_{WV}(T)|^2 \\
& = C_s |(T' - T)(\Phi)|^2 \\
& = C_s |\partial S_2(\Phi)|^2 \\
& = C_s |S_2(\Phi)|^2 \\
& \leq C_s \|d\Phi\|_{\infty}^2 (M(S_2))^2 \\
& \overset{(4.31)}{\leq} C_s \|d\Phi\|_{\infty}^2 C_N^2 \delta_1^4 \\
& \overset{(4.29), (\ast)}{\leq} \delta_1/2.
\end{align*}
\]
Since \(\delta_1 \geq 0\), this implies \(\delta_1 = 0\). Looking at the above string of inequalities, \(\delta_1 = 0\) implies \(P_{WV}(T') = 0\). Thus, \(M^*(T') = L = M^*(T_M)\). Since we are assuming (4.32), we have \(T' = T_M\). Finally, recalling (4.30), \(\delta_1 = 0\) implies \(T = T' = T_M\).

### 4.2 Regularity

This section consists of lemmas which lead to (4.32).

**Lemma 4.7.** For \(r \in (0, 2r_0/3)\), there exists a minimizer of \(M^*\) in \(H_r\).

**Proof.** First note that \(T_M \in H_r\) and

\[
m_r := \inf_{T \in H_r} M^*(T) \leq M^*(T_M) = L < \infty.
\]

Select a minimizing sequence \(\{T_i\}_{i=1}^{\infty} \subset H_r\). Then

\[
M^*(T_i) \rightarrow m_r \quad \text{as } i \rightarrow \infty.
\]

Define the norm \(F_{r_0}\) by \(F_{r_0}(T) = \inf\{M(S) : S \in D_2(N), \text{spt} S \subset \overline{K_{3r_0/2}}, \partial S = T\}\).

Since \(\text{spt} T_i \subset \overline{K_r}, M(T_i) \leq C\) and \(\partial T_i = 0\) for all \(i\), the Federer-Fleming Compactness Theorem implies that there exists a subsequence \(\{i_j\}\) and \(T_r \in R_1(N)\) such that the
support of $T_r$ lies in $\overline{K_r}$, $M(T_r) \leq C, \partial T_r = 0$ and

$$F_{r_0}(T_{ij} - T_r) \to 0 \text{ as } j \to \infty.$$ (4.33)

Using (4.33), there exists a $j$ such that $T_{ij} - T_r = \partial S_{r,j}$ for some $S_{r,j} \in D_2(N)$ with $\text{spt} S_{r,j} \subset \overline{K_{3r_0/2}}$ and $M(S_{r,j}) < 1$. Since $T_{ij} \in H_r$, $T_{ij} - T_M = \partial S_j$ for some $S_j \in D_2(N)$ with $\text{spt} S_j \subset \overline{K_{3r_0/2}}$ and $M(S_j) < \infty$. For this $j$, set $S_r = S_{r,j} + S_j \in D_2(N)$. Then $T_r - T_M = \partial S_r, \text{spt} S_r \subset \overline{K_{3r_0/2}}$ and $M(S_r) < \infty$. Therefore, $T_r \in H_r$.

Since $M$ is weakly lower semicontinuous, $P_{WV}$ is weakly continuous and (4.33) implies $T_{ij} \rightharpoonup T_r$, we have

$$M^*(T_r) \leq \liminf_{j \to \infty} M^*(T_{ij}) = m_r.$$

It follows that

$$M^*(T_r) = m_r = \min_{T \in H_r} M^*(T).$$

Lemma 4.8. Suppose $T \in H_r, r \in (0, 2r_0/3)$, and $T$ has finite mass. Then

$$M(\langle T_r, \tau, t \rangle) \geq 1 \text{ for a.e. } t \in (0, L).$$ (4.34)

In particular, $T_r$ satisfies (4.34).

Proof. As $T \in H_r$, we can write

$$T - T_M = \partial S$$

for some 2-current $S$ with $M(S) < \infty$. Suppose $M(\langle T, \tau, t \rangle) = 0$ on a set of positive measure $A \subset (0, L)$. Since

$$\partial \langle S, \tau, t \rangle = \langle T, \tau, t \rangle - \delta_{\gamma(t)} \text{ for a.e. } t \in (0, L),$$

we would have $\partial \langle S, \tau, s \rangle = -\delta_{\gamma(s)}$ for some $s \in A$. If $\eta$ is any smooth function with compact support in $N$ such that $\eta = 1$ in a neighbourhood of $\tau^{-1}(s)$, then

$$1 = \eta(\gamma(s)) = \delta_{\gamma(s)}(\eta) = -\partial \langle S, \tau, s \rangle(\eta) = -\langle S, \tau, s \rangle(d\eta) = 0,$$
which is nonsense. Therefore, (4.34) holds.

We now show that $T_r$ satisfies some good properties.

First, since

$$M(T_r) \leq M'(T_r) \leq M(T_M) = L \quad (4.35)$$

and $\partial T_r = 0$, we can write $T_r$ as a sum of indecomposable currents $T_r = \sum_i T_{r,i}$, where each $T_{r,i}$ corresponds to integration over a closed, Lipschitz curve $\gamma_{r,i}$, $M(T_r) = \sum_i M(T_{r,i})$ and $M(\partial T_r) = \sum_i M(\partial T_{r,i})$.

Similar to the definitions in the proof of Lemma 4.4, define

$$\Sigma_0^r := \{ t \in (0, L) : M(\langle T_r, \tau, t \rangle) = 0 \},$$

$$\Sigma_1^r := \{ t \in (0, L) : M(\langle T_r, \tau, t \rangle) = 1 \}$$

and

$$\Sigma_2^r := \{ t \in (0, L) : M(\langle T_r, \tau, t \rangle) \geq 2 \}.$$

Note that these sets are disjoint and (4.34) implies $|\Sigma_0^r| = 0$. Also, using Lemma 3.2 and the estimate for $|\nabla \tau|$ derived in Step 2 in the proof of Lemma 4.4,

$$|\Sigma_1^r| + 2|\Sigma_2^r| \leq \int_{\Sigma_1^r} M(\langle T_r, \tau, t \rangle)dt + \int_{\Sigma_2^r} M(\langle T_r, \tau, t \rangle)dt$$

$$= \int_0^L M(\langle T_r, \tau, t \rangle)dt$$

$$\leq (1 + C_2 r_0^2) M(T_r)$$

$$\leq (4.35) \quad L + C_2 Lr_0^2$$

$$= |\Sigma_1^r| + |\Sigma_2^r| + C_2 Lr_0^2,$$

which implies $|\Sigma_2^r| \leq C_2 Lr_0^2$ and $|\Sigma_1^r| = L - |\Sigma_2^r| \geq L - C_2 Lr_0^2$. Now, letting

$$\Sigma_{1,j}^r := \left\{ t \in \Sigma_1^r : \sum_i \mathcal{H}^0(\gamma_{r,i} \cap \tau^{-1}(t)) = j \right\}$$

and using an argument similar to the one in Step 7 in the proof of Lemma 4.4, we have

$$|\Sigma_{1,1}^r| \geq 2|\Sigma_1^r| - L - 2C_2 Lr_0^2,$$
which implies $|\Sigma_{r,1}^i| \geq L - 4C_2Lr_0^2 > L/2 > 0$. From now on, $t_r^*$ will denote a point in $\Sigma_{r,1}^i$ at which the Lebesgue density is 1. Let $\gamma_{r,i^*}$ be the unique closed curve that intersects $\tau^{-1}(t_r^*)$. It follows that

$$\gamma_{r,i^*} \cap \tau^{-1}(t) \neq \emptyset \text{ for all } t \in \mathbb{R}/L\mathbb{Z}.$$ 

For $r \in (0, 2r_0/3)$, set

$$Q_r \text{ to be the } 1\text{-current compactly supported in } B_{r}^{n-1} \times [t_r^* - 2L, t_r^* + 2L]$$

defined on cylinders of height $L$ by $\psi^{-1}_r T_r \in \mathcal{R}_1(U_{r_0}^{n-1} \times \mathbb{R}/L\mathbb{Z})$.

Note that $M(\partial Q_r) = 2$ for every $r \in (0, 2r_0/3)$. Let

$$K = B_{3r_0/4}^{n-1} \times [-2L, 3L]$$

so that $\text{spt}Q_r \subset K$ for all $r \in (0, 2r_0/3)$.

Since $Q_r$ is an integral current in $\mathbb{R}^n$, we can write $Q_r$ as a sum of indecomposable currents $Q_r = \sum_i Q_{r,i}$, where each $Q_{r,i}$ corresponds to integration over a closed $(in U_{r_0}^{n-1} \times (t_r^* - 2L, t_r^* + 2L))$ Lipschitz curve $\alpha_{r,i} : [0, L_{r,i}] \to B_{r}^{n-1} \times [t_r^* - 2L, t_r^* + 2L]$. Note that the above discussion implies that we must have $4T_{r,i^*} = \psi^*_r Q_{r,j}$ for some $j$, where $\alpha_{r,j}$ is the only curve among $\{\alpha_{r,i}\}$ satisfying $\alpha_{r,i}(L_{r,i}) = \alpha_{r,i}(0) \pm (0, 4L)$. The curves $\{\gamma_{r,i} : i \neq j\}$ correspond to closed loops in $B_{r}^{n-1} \times [t_r^* - 2L, t_r^* + 2L]$. Thus, recalling that $p(X) = t$ for $X = (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\text{spt}Q_r \cap p^{-1}(t) \neq \emptyset \text{ for all } t \in (t_r^* - 2L, t_r^* + 2L). \quad (4.36)$$

Also,

$$M(Q_r \downarrow C(t, L/2)) = M(Q_r \downarrow C(t_r^* + L/2, L/2)) \geq \mathcal{H}^1(\alpha_{r,j}|_{B_r^{n-1} \times [t_r^*, t_r^* + L]}) \geq L \quad (4.37)$$

and, using (4.22),

$$F(Q_r \downarrow C(t, L/2)) = M(T_r) \leq L \quad (4.38)$$
for any $t \in (t^*_r - 3L/2, t^*_r + 3L/2)$.

Our goal now is to show that $Q_r$ is an almost minimizer for $F$; this means that small perturbations of $Q_r$ by closed, rectifiable 1-currents $T$ supported in a tiny ball of radius $\rho$ may make $F(Q_r + T) \leq F(Q_r)$, but no smaller than $F(Q_r) - \omega(\rho) M(Q_r + T)$ for some positive function $\omega(\rho) \to 0$. Our proof will require the use of Lemma 3.3, therefore, we must first verify that $F$ satisfies the ellipticity condition described in Section 2.2.

**Lemma 4.9.** There exists $\lambda > 0$ such that $F$ is $\lambda$-elliptic in $U_{r_0}^{n-1} \times \mathbb{R}$.

**Proof.** Set $\tilde{F}_{X,c}(\xi) := F_X(\xi) - c|\xi|, X \in B_{r_0}^{n-1} \times \mathbb{R}, \xi \in \mathbb{R}^n, c > 0$.

First, we will show that if there exists a $c > 0$ such that $\tilde{F}_{X,c}$ is convex for all $X \in U_{r_0}^{n-1} \times \mathbb{R}$, then $F$ is $c$-elliptic in $U_{r_0}^{n-1} \times \mathbb{R}$. Suppose $T_i = \tau(\Gamma_i, m_i, \vec{T}_i) \in \mathcal{R}_1(\mathbb{R}^n)$ has compact support, $i = 1, 2, \partial T_1 = \partial T_2$ and $T_1$ is flat. We can find a number $a > 0$ such that $T_2 - T_1 = \partial S$ for some 2-current $S$ with $\text{spt} S \subset U_{a}^{n}$.

Note that $\int \vec{T}_2 d\|T_2\| = \int \vec{T}_1 d\|T_1\| = M(T_1) \vec{T}_1$. To see this, let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a continuous linear function and define $\phi \in D^1(\mathbb{R}^n)$ so that $\langle \phi(\vec{x}), \xi \rangle = \chi(\xi)$ for all $\vec{x} \in U_{a}^{n}$. This implies $d\phi = 0$ on $U_{a}^{n}$. Hence,

$$\chi \left( \int \vec{T}_2 d\|T_2\| - \int \vec{T}_1 d\|T_1\| \right) = \int \chi(\vec{T}_2) d\|T_2\| - \int \chi(\vec{T}_1) d\|T_1\| = (T_2 - T_1)(\phi) = \partial S(\phi) = S(d\phi) = 0.$$

Since $\chi$ is arbitrary, we have $\int \vec{T}_2 d\|T_2\| = \int \vec{T}_1 d\|T_1\|$.

Now,

$$F_X(T_1) - cM(T_1) = \tilde{F}_{X,c}(M(T_1) \vec{T}_1) = \tilde{F}_{X,c} \left( \int \vec{T}_2 d\|T_2\| \right).$$

Using Jensen's Inequality and the fact that $F_{X,c}$ is homogeneous of degree 1, we have

$$F_X(T_1) - cM(T_1) \leq \int \tilde{F}_{X,c}(\vec{T}_2) d\|T_2\| = F_X(T_2) - cM(T_2),$$
Lemma 4.10. Let \( \beta \) be here. He uses a different definition of almost-minimality, his argument does not exactly apply. In [20], Brian White proves a result similar to Lemma 4.10 below, however, since in Section 2.2 is that of Bombieri [4] since his paper gives the regularity results that we require. In [20], Brian White proves a result similar to Lemma 4.10 below, however, since he uses a different definition of almost-minimality, his argument does not exactly apply here.

\[
\begin{align*}
\beta^T D^2 \tilde{F}_{X,c}(\xi) \beta &= \beta_i \beta_k g_{ik} l_X(\xi) - \xi_i g_{ij} l_X(\xi) \sum_k g_{ik} l_X(\xi) - c \beta^T D^2 f(\xi) \beta \\
&= \frac{|l_X(\xi)|^2 |l_X(\beta)|^2 - (l_X(\xi) \cdot l_X(\beta))^2}{|l_X(\xi)|^3} - \frac{|\xi \wedge \beta|^2}{|\xi|^3} \\
&\geq \left( \frac{|l_X|^{-3} |l_X^{-1}|^{-4}}{c} \right) \frac{|\xi \wedge \beta|^2}{|\xi|^3}.
\end{align*}
\]

Here, \( l_X^{-1} \) is a map that takes the vector \( \sum_{i=1}^n \xi X_i \in T_{(X)}N \) to \( \xi \in \mathbb{R}^n \). Let
\[
c = \frac{1}{2} \min_{X \in B_{r_0}^{n-1} \times \mathbb{R}/LZ} \|l_X\|^{-3} \|l_X^{-1}\|^{-4}.
\]

Note \( c > 0 \) for \( r_0 \) sufficiently small since \( \|l_{(0,1)}\| = \|l_{(0,0)}^{-1}\| = 1 \) for every \( t \in \mathbb{R} \) and \( \|l_X\|, \|l_X^{-1}\| \) are continuous functions of \( X \). Then \( \tilde{F}_{X,c} \) is convex for all \( X \in U_{r_0}^{n-1} \times \mathbb{R} \) and \( F \) is \( \lambda \)-elliptic in \( U_{r_0}^{n-1} \times \mathbb{R} \) with \( \lambda = c^{-1} \).

Before stating Lemma 4.10, we note that the definition of \((F, \omega, \delta)\)-minimality given in Section 2.2 is that of Bombieri [4] since his paper gives the regularity results that we require. In [20], Brian White proves a result similar to Lemma 4.10 below, however, since he uses a different definition of almost-minimality, his argument does not exactly apply here.

Lemma 4.10. Let \( \omega(\rho) = \sqrt{\rho} \). There exists a positive constant \( \delta \), independent of \( r \), such that \( Q_r \) is \((F, \omega, \delta)\)-minimal for all \( r \in (0, 2r_0/3) \) and \( r_0 \) sufficiently small.

Proof. Set \( \delta := r_0^{1/4} L/2 \).

For \( X \in U_{r_0}^{n-1} \times \mathbb{R} \) and \( \rho \in (0, \delta) \), let
\[
S := \{ T \in \mathcal{R}_1(\mathbb{R}^n) : \text{spt} T \text{ is compact in } K, \text{spt} T \subset B_{\rho}^n(X) \text{ and } \partial T = 0 \}.
\]
To prove the lemma, it suffices to show
\[ F(Q_R B^a_\rho(X)) \leq (1 + C\rho) F_X(Q_R B^a_\rho(X) + \tilde{T}) \] (4.39)
for some absolute constant \( C > 0 \), where \( \tilde{T} \) minimizes \( F_X(Q_R B^a_\rho(X) + \cdot) \) in \( S \). Indeed, if the above holds, then for any \( T \in S \),
\[
F(Q_R B^a_\rho(X)) \leq (1 + C\rho) F_X(Q_R B^a_\rho(X) + \tilde{T}) \\
\leq (1 + C\rho) F_X(Q_R B^a_\rho(X) + T) \\
\leq (1 + C\rho)(F(Q_R B^a_\rho(X) + T) + C_3\rho M(Q_R B^a_\rho(X) + T)) \\
\leq F(Q_R B^a_\rho(X) + T) + \left(C_3(1 + C\rho)^{1/4} + \frac{C}{1 - C_3\rho}\right) \rho M(Q_R B^a_\rho(X) + T) \\
\leq F(Q_R B^a_\rho(X) + T) + \sqrt{\rho} M(Q_R + T).
\]
Adding \( F(Q_R K \setminus B^a_\rho(X)) \) to both sides gives the desired result. We assume that \( \tilde{T} \neq 0 \) since otherwise (4.39) follows easily from (4.23) and (4.27). Also, we assume that \( \mathcal{H}^1(\text{spt}\tilde{T} \cap \text{spt} Q_R) > 0 \) otherwise (4.39) again follows easily since \( F_X(Q_R B^a_\rho(X) + \tilde{T}) = F_X(Q_R B^a_\rho(X) + \tilde{T}) + F_X(\tilde{T}) \).

Since
\[
F_X(Q_R B^a_\rho(X) + \tilde{T}) = F_X(Q_R \text{spt} \tilde{T} + \tilde{T}) + F_X(Q_R B^a_\rho(X) \setminus \text{spt} \tilde{T})
\]
and \( \tilde{T} \) minimizes \( F_X(Q_R B^a_\rho(X) + \cdot) \) in \( S \), it follows that
\[
\tilde{T} = Q_R \text{spt} \tilde{T} + \tilde{T} \text{ minimizes } F_X \text{ among all } T \in \mathcal{R}_1(\mathbb{R}^n)
\] (4.40)
with compact support in \( K \), \( \text{spt} T \subset B^a_\rho(X) \) and \( \partial T = \partial(\text{spt} \tilde{T}) \).

To see this, let \( A = B^a_\rho(X) \setminus \text{spt} \tilde{T} \) and suppose \( T \in \mathcal{R}_1(\mathbb{R}^n) \) has compact support in \( K \), \( \text{spt} T \subset B^a_\rho(X) \) and \( \partial T = \partial(\text{spt} \tilde{T}) \). Then \( \partial(Q_R A + T - Q_R B^a_\rho(X)) = 0 \) and
\[
F_X(\tilde{T}) + F_X(Q_R A) = F_X(Q_R B^a_\rho(X) + \tilde{T}) \\
\leq F_X(Q_R A + T) \\
\leq F_X(Q_R A) + F_X(T),
\]
which implies $F_X(\hat{T}) \leq F_X(T)$, as claimed.

Using Lemma 3.3, we conclude $\text{spt} \hat{T} \subset B_{r}^{n-1} \times [-2L, 3L] \cap B_{\rho}^{n}(X)$ as $\text{spt}(Q_{rL} \text{spt} \hat{T})$ is contained in the convex set $B_{r}^{n-1} \times [-2L, 3L] \cap B_{\rho}^{n}(X)$. Thus,

$$\text{spt} \hat{T} \subset B_{r}^{n-1} \times [-2L, 3L] \cap B_{\rho}^{n}(X). \quad (4.41)$$

Let $\tilde{S}$ minimize $M$ among all 2-currents with boundary equal to $\hat{T}$. Using (4.41), Lemma 3.3 implies $\text{spt} \tilde{S} \subset B_{r}^{n-1} \times [-2L, 3L] \cap B_{\rho}^{n}(X)$. Now we would like to show $M(\tilde{S}) < \infty$. First, as $\text{spt} \hat{T} \subset B_{\rho}^{n}(X)$, Lemma 3.5 implies that there exists a 2-current $S$ such that $\partial S = \hat{T}$ and

$$M(S) \leq 4\rho M(\hat{T}). \quad (4.42)$$

Next, note that

$$\psi_2 Q_r = 4T_r \quad (4.43)$$

and

$$M(\tilde{T}) \overset{(*)}{\leq} 24L. \quad (4.44)$$

To verify (4.44), we estimate

$$M(\tilde{T}) \overset{(4.27)}{\leq} (1 - C_3 r_0)^{-1} F_X(\hat{T}) \overset{(4.40)}{\leq} (1 - C_3 r_0)^{-1} F_X(Q_{rL} \text{spt} \hat{T}) \overset{(4.27)}{\leq} \frac{1 + C_3 r_0}{1 - C_3 r_0} M(Q_{rL} \text{spt} \hat{T}),$$

which implies

$$M(\tilde{T}) \leq M(\hat{T}) + M(Q_{rL} \text{spt} \hat{T}) \overset{(4.26)}{\leq} \left( \frac{1 + C_3 r_0}{1 - C_3 r_0} + 1 \right) M(Q_{rL} \text{spt} \hat{T}) \overset{(4.22),(4.43)}{=} 24M(T_r) \overset{(4.35)}{\leq} 24L.$$
Therefore,
\[
M(\tilde{T}) \leq M(S) \leq 4\delta M(\tilde{T}) < 96L. \quad (4.45)
\]

Since \(\tilde{T}\) is homologous to 0 in \(B_{r}^{n-1} \times [-2L, 3L] \cap B_{\rho}^{n}(X)\) and \(T_{r}\) is homologous to \(T_{M}\) in \(K_{2r_{0}/3}\), it follows that \(T_{r} + \psi_{r} \tilde{T}\) is homologous to \(T_{M}\) in \(K_{2r_{0}/3}\). This, (4.41) and (4.45) imply \(T_{r} + \psi_{r} \tilde{T} \in H_{r}\). Thus, using the fact that \(T_{r}\) minimizes \(M^{*}\) in \(H_{r}\),

\[
M(T_{r}) = M^{*}(T_{r}) - C_{s}|P_{WV}(T_{r})|^{2} \\
\leq M^{*}(T_{r} + \psi_{r} \tilde{T}) - C_{s}|P_{WV}(T_{r})|^{2} \\
= M(T_{r} + \psi_{r} \tilde{T}) + C_{s}(P_{WV}(T_{r} + \psi_{r} \tilde{T}) + P_{WV}(T_{r})). \\
(4.22) \\
\leq M(T_{r} + \psi_{r} \tilde{T}) + C_{s}(2M(T_{r}) + F(\tilde{T}))\|\Phi\|_{\infty}\psi_{r} \tilde{T}(\Phi)).
\]

Note that, using (4.35), (4.26) and the above estimate for \(M(\tilde{T})\),

\[
2M(T_{r}) + F(\tilde{T}) \leq 50L.
\]

Recalling (4.42),

\[
|\psi_{r} \tilde{T}(\Phi)| = |\partial \psi_{r} S(\Phi)| \\
= |\psi_{r} S(d\Phi)| \\
\leq \|d\Phi\|_{\infty}M(\psi_{r} S) \\
\leq \|d\Phi\|_{\infty}C_{\psi}M(S) \\
\leq 4C_{\psi}\|d\Phi\|_{\infty}\rho M(\tilde{T}) \\
\leq 8C_{\psi}\|d\Phi\|_{\infty}\rho F(\tilde{T}). \quad (4.26),(*)
\]

Combining the above 2 estimates and letting \(C_{4} = 400LC_{s}C_{\psi}\|\Phi\|_{\infty}\|d\Phi\|_{\infty}\), we have

\[
F(Q_{r}, C(s, L/2)) \leq F(Q_{r}, C(s, L/2) + \tilde{T}) + C_{4}\rho F(\tilde{T})
\]

for any \(s \in [t_{r}^{*} - 3L/2, t_{r}^{*} + 3L/2]\).
Since \( \rho \leq \delta < \frac{L}{2} \), \( \text{spt} \tilde{T} \subset B^n_\rho(X) \cap C(s, L/2) \) for some \( s \in [t^*_r - 3L/2, t^*_r + 3L/2] \), and thus

\[
F(Q_r \cup B^n_\rho(X)) \leq F(Q_r \cup B^n_\rho(X) + \tilde{T}) + C_4 \rho F(\tilde{T}) \\
\leq F(Q_r \cup B^n_\rho(X) + \tilde{T}) + C_4 \rho (F(Q_r \cup B^n_\rho(X) + \tilde{T}) + F(Q_r \cup B^n_\rho(X))).
\]

This implies

\[
F(Q_r \cup B^n_\rho(X)) \leq \frac{1 + C_4 \rho}{1 - C_4 \rho} F(Q_r \cup B^n_\rho(X) + \tilde{T}) \\
\leq (1 + 4C_4 \rho) F(Q_r \cup B^n_\rho(X) + \tilde{T}) \\
\leq (1 + 4C_4 \rho) (F_X(Q_r \cup B^n_\rho(X) + \tilde{T}) + C_3 \rho M(Q_r \cup B^n_\rho(X) + \tilde{T})) \\
\leq (1 + C_4 \rho) F_X(Q_r \cup B^n_\rho(X) + \tilde{T}),
\]

which is (4.39). \( \square \)

We are now in a position to show that \( Q_r \) can be written as the graph of a \( C^1 \) function with \( H^1 \) norm on the order of \( r^\alpha_0 \) for some \( \alpha \in (0, 1) \).

**Lemma 4.11.** For \( r \in (0, 2r_0/3) \) and \( r_0 \) sufficiently small,

\[
\text{spt} T_r = \{ \exp(\gamma(t), u_r(t)), t \in [0, L] \},
\]

for some \( C^1 \) function \( u_r \) such that

\[
u_r(t) \in T^\perp_{\gamma(t)} M \text{ for all } t \in [0, L], u_r(0) = u_r(L), \frac{D}{dt} u_r(0) = \frac{D}{dt} u_r(L)\]

and

\[\|u_r\|_{W^{1, \infty}([0, L])} \to 0 \text{ as } r_0 \to 0^+.\]

**Proof.** We would like to use Lemmas 6.1 and 6.2.

To start, we show that the excess of \( Q_r \) over thin cylinders is small (refer to Section 2.2 for the definition of excess). Take \( r_0 \) smaller if necessary so that \( (2\sqrt{r_0})^{-1} (L/2 - \sqrt{r_0}) \) is an integer \( I = I(r_0) \). Fix \( \tilde{t} \in (0, L) \) and let \( t_i = \tilde{t} + 2i \sqrt{r_0}, i = -I, ..., -1, 0, 1, ..., I.\)
Since the excess is always non-negative,

\[
E(Q_r, \tilde{r}, \sqrt{r_0}) \leq \sum_{i=-I}^{I} E(Q_r, t_i, \sqrt{r_0})
\]

\[
= r_0^{-1/2} (M(Q_r \mathcal{L}C(\tilde{r}, L/2)) - L)
\]

\[
\leq r_0^{-1/2} \left( \frac{F(Q_r \mathcal{L}C(\tilde{r}, L/2))}{1 - C_3 r_0} - L \right)
\]

\[
\leq r_0^{-1/2} \left( \frac{L}{1 - C_3 r_0} - L \right)
\]

\[
= C_3 L \sqrt{r_0} \left( 1 - C_3 r_0 \right). \quad (4.26)
\]

\[
\leq r_0^{-1/2} \left( \frac{L}{1 - C_3 r_0} - L \right)
\]

\[
= C_3 L \sqrt{r_0} \left( 1 - C_3 r_0 \right) \quad (4.38)
\]

\[
\leq C_3 L \sqrt{r_0} \left( 1 - C_3 r_0 \right) \quad (\ast)
\]

Recalling (4.36) and assuming \( \tilde{X} = 0 \) for some \( \tilde{X} \in p^{-1}(\tilde{t}) \) (after a translation), it suffices to show the assumptions made in the Appendix hold with

\[
T = Q_r, \Psi = F
\]

\[
\Omega = U_{r_0}^{n-1} \times \mathbb{R}, K = B_3^{3r_0/4} \times [-2L, 3L]
\]

\[
\omega(\rho) = \sqrt{\rho}, 0 < \rho \leq \delta
\]

\[
R = \sqrt{r_0} < L.
\]

First, (T1) holds due to Lemma 4.10. Next, (T2) and (T4) follow from the discussion preceding Lemma 4.10. Recall that \( \delta \) from Lemma 4.10 was chosen to be \( r_0^{1/4} \) and thus

\[
\text{diam}(\text{spt}Q_r \mathcal{L}C_R) \leq 2\sqrt{r^2 + r_0} \leq \sqrt{13} \sqrt{r_0} \leq \delta/4,
\]

which is (T3). If we take \( r_0 \) sufficiently small, then (T5) and (6.2) will hold. Property (P2) follows from (4.25). Since \( F \) is smooth on \( \Omega \times \mathbb{R}^n \setminus \{0\} \), we can find a positive constant so that (P3)-(P6) are satisfied with \( \phi_0 = \nabla_\xi F(X_0, \xi_0) \) and \( \nu(\rho) = \sqrt{\rho} \). Finally, (P1) is a result of Lemma 4.9 and we will assume that the constant \( \lambda \) obtained in Lemma 4.9 is larger than the constants found to satisfy (P2)-(P6).

Therefore, using (6.4), we have

\[
\text{spt}(Q_r \mathcal{L}C(\tilde{r}, R/2)) = \{(y_r(s), s), s \in B^1_{R/2}(\tilde{t})\}
\]

for some \( C^1 \) function \( y_r \) with \( \|y'_r\|_{L^\infty} \leq C_5 r_0^{1/8} \).
Since \( \tilde{t} \in (0, L) \) was chosen arbitrarily, it follows that \( \text{spt}(Q_{r \perp} C(L/2, L/2)) \) coincides with the graph of a function \( y_r \in C^1([0, L]; B_{r}^{n-1}) \) satisfying \( y_r(0) = y_r(L), y'_r(0) = y'_r(L) \) and \( \|y_r\|_{W^{1, \infty}} \leq (1 + C_5) r_0^{1/8} \).

Setting
\[
u_r(t) := \sum_{i=1}^{n-1} y_{r,i}(t) u^i(t),
\]
we have
\[
\text{spt} T_r = \{ \exp(\gamma(t), u_r(t)), t \in [0, L] \}
\]
with \( u_r(0) = u_r(L), \frac{D}{dt} u_r(0) = \frac{D}{dt} u_r(L) \) and
\[
\|u_r\|_{W^{1, \infty}} \leq C_6 r_0^{1/8}.
\]

(4.46)

Finally, we complete the proof of (4.32).

**Lemma 4.12.** There exists a positive constant \( C_* \) such that if \( T_r \) is a minimizer of \( M^* \) in \( H_r, r \in (0, r_0/2) \), and \( r_0 \) is sufficiently small, then \( T_r = T_M \).

**Proof.** Write \( u_r = n_r + p_r, n_r = \sum_{i=1}^{l} c_{r,i} z^i \). Lemma (4.3) implies
\[
|P(T_r)|^2 = \sum_{i=1}^{l} c_{r,i}^2 = \|n_r\|_{L^2}^2.
\]

(4.47)

Assume \( s_r = \|u_r\|_{C([0, L])} = \|y_r\|_{C([0, L])} > 0 \), otherwise there is nothing to prove. Consider the variation \( h_r : (-2s_r, 2s_r) \times [0, L) \to K_{r_0} \) defined by \( h_r(s, t) := \psi(s y_r(t), t) \) and let \( T_{r,s} \) denote the multiplicity 1 current corresponding to integration over the Lipschitz curve \( h_r(s, \cdot) \).

Since \( |(sy_r'(t), 1)| \geq 1, \)
\[
\left| \frac{\partial}{\partial t} \psi(s y_r(t), t) \right| \geq 1 - C_3 r_0 \geq 1/2.
\]

(4.24)

Expanding \( M(T_{r,s}) \) in a Taylor series about \( s = 0 \), we have
\[
M(T_{r,s}) = L + \frac{1}{2s^2} (Ju_r, u_r) s^2 + R_{r,2}(s).
\]
Note that
\[ \frac{d}{ds} M(T_{r,s}) \bigg|_{s=0} = 0 \]
since \( \gamma \) is a geodesic.

Also note that \( |R_{r,2}(s)| \leq o(||u_r||^2_{H^1})|s|^3 \). Indeed, viewing \( F \) as a function of \( Y = (Y_1, ..., Y_{2n}) \) where

\[ Y = Y_r(s, t) = (ss_r^{-1}y_r(t), t, ss_r^{-1}y'_r(t), 1), \]

we have
\[ \frac{d^3}{ds^3} M(T_{r,s}) = \int_0^L Y^T_s \left( \frac{d}{ds} D^2 F(Y) \right) Y_s dt, \]

which implies
\[ |R_{r,2}(s)| \leq C r^{1/8} s^{-3} ||u_r||^2_{H^1} |s|^3 \quad (4.48) \]

using Taylor’s Theorem, (4.46) and the fact that \( u_r \cdot u^j = y_{r,j} \).

Therefore,
\[ M(T_r) = M(T_{r,s_r}) = L + \frac{1}{2} (Ju_r, u_r) + R_{r,2}(s_r) \quad (4.49) \]

and, using Lemma 4.1,

\[ L \begin{aligned} &\geq M^*(T_{M}) \\ &\geq M^*(T_r) \\ &\stackrel{(4.49),(4.47)}{=} L + \frac{1}{2} (Ju_r, u_r) + R_{r,2}(s_r) + C_s ||n_r||^2_{L^2} \\ &\geq L + \frac{1}{2} (Ju_r, u_r) + \frac{1}{2} (Jp_r, p_r) + C_s ||n_r||^2_{L^2} + R_{r,2}(s_r) \\ &\geq L + \left( \frac{\lambda_1}{2} + C_s c \right) ||n_r||^2_{H^1} + \frac{c}{2} ||p_r||^2_{H^1} + R_{r,2}(s_r) \\ &\geq L + \left( \frac{\lambda_1}{2} + c(C_s - \frac{c}{4}) \right) ||n_r||^2_{H^1} + \frac{c}{4} ||p_r||^2_{H^1}. \end{aligned} \]

If we choose \( C_s > \frac{c-2\lambda_1}{4c} \), then \( n_r \) and \( p_r \) are identically 0, which implies \( T_r = T_M \). \( \square \)
4.3 Construction of $Q_{VW}$

Let $W = U^l_{r_1}$ where $r_1 > 0$ is to be chosen. For $w \in B^l_{r_1}$, define $z_w(t) := \sum_{i=1}^{t} w_i z^i(t)$.
Assume $r_1$ is small enough so that

$$|z_w(t)| < r_0/2 \text{ for all } t \in [0, L] \text{ and } w \in B^l_{r_1}. \quad (4.50)$$

Define $\gamma_w : [0, L] \to K_{r_0/2}$ by

$$\gamma_w(t) := \exp(\gamma(t), z_w(t))$$

and $Q_{VW}(w)$ to be the multiplicity 1 current corresponding to integration over the Lipschitz curve $\gamma_w$.

**Lemma 4.13.** $Q_{VW}$ is continuous.

**Proof.** For $X_1, X_2 \in \mathbb{R}^n$, let $T_{X_1, X_2}$ denote the multiplicity 1 current corresponding to integration over $l_{X_1, X_2}$, where $l_{X_1, X_2} : [0, 1] \to \mathbb{R}^n$ is defined by $l_{X_1, X_2}(s) := (1-s)X_1 + sX_2$.

Fix $w^1, w^2 \in U^l_{r_1}$. Using Lemma 3.6 with $f_i(t) = \psi^{-1}(\gamma^i_w(t))$ for $i = 1, 2$, $t \in (0, L)$, we can find $S_{w^1, w^2} \in D_2(U^n_{r_0} \times (0, L))$ such that

$$\partial S_{w^1, w^2} = \psi^{-1}_1 Q_{VW}(w^2) - \psi^{-1}_2 Q_{VW}(w^1) + T_{f_1(0), f_2(0)} - T_{f_1(L), f_2(L)}$$

and

$$M(S_{w^1, w^2}) \leq L \|f_1 - f_2\|_{L^\infty} \|f'_1 + f'_2\|_{L^\infty} \leq C_8 |w^1 - w^2|.$$ 

Therefore,

$$\partial \psi^2 S_{w^1, w^2} = Q_{VW}(w^2) - Q_{VW}(w^1)$$

and

$$F(Q_{VW}(w^2) - Q_{VW}(w^1)) \leq M(\psi^2 S_{w^1, w^2}) \leq C_\psi C_8 |w^2 - w^1|,$$

which implies $Q_{VW}$ is continuous. \qed
To complete the proof of Theorem 4.1, we need to verify (2.10)-(2.12).

- Verification of (2.10):
  Clearly, $z_0$ is identically 0 and $\gamma_0 = \gamma$, which implies $Q_{VW}(0) = T_M$.

- Verification of (2.11):
  This follows directly from (4.50) and Lemma 4.3.

- Verification of (2.12):
  For $a \in (0, r_1)$ and $w \in W$ with $|w| \geq a$, $s_w = \|z_w\|_{C([0,L])} > 0$. Consider the variation $h_w : (-2s_w, 2s_w) \times [0, L] \to K_{r_0}$ defined by $h_w(s, t) = \exp(\gamma(t), s_w z_w(t))$. Let $T_{w,s}$ denote the multiplicity 1 current corresponding to integration over the Lipschitz curve $h_w(s, \cdot)$. Then, arguing as in the proof of Lemma 4.12,
  
  $$M(Q_{VW}(w)) = M(T_{w,s_w}) = L + (J_{z_w}, z_w) + o(|w|^2).$$

  Note that
  
  $$\langle J_{z_w}, z_w \rangle = \sum_{i=1}^l \lambda_i w_i^2 \leq \lambda_l |w|^2.$$

  Taking $r_1$ small enough so that $\frac{o(|w|^2)}{|w|^2} \leq -\lambda_l / 2$, we have

  $$M(Q_{VW}(w)) \leq L + \lambda_l |w|^2 / 2.$$

  Thus,

  $$\sup_{w \in U_{r_1}, |w| \geq a} M(Q_{VW}(w)) \leq L + \lambda_la^2 / 2 < L.$$
Chapter 5

Applications

Throughout this chapter, \((N, g)\) will be a 3-dimensional compact, orientable Riemannian manifold without boundary, \(U = H^1(N; \mathbb{C})\), \(V = \mathcal{F}_1(N)\) and \(E_V\) will be given by (4.5).

Define the Ginzburg-Landau energy \(E_{\epsilon}^U : U \to \mathbb{R}\) by

\[
E_{\epsilon}^U(u) := \frac{1}{2\pi |\ln \epsilon|} \int_N |\nabla u|^2 + \frac{(|u|^2 - 1)^2}{2\epsilon^2}.
\]

(5.1)

5.1 \(\Gamma\)-convergence of \(E_{\epsilon}^U\) on \(N\)

Theorem 5.1. (2.6) and (2.8) are satisfied with \(P_{V\epsilon} = P_{V\epsilon}^U(u) = \star Ju/\pi\). Furthermore, there exists a family of maps \(Q_{UV}^\epsilon : \text{Im}Q_{VW} \to U\) such that (2.7) is satisfied for every \(v \in \text{Im}Q_{VW}\).

We will verify the claims made in Theorem 5.1 through Lemmas 5.1, 5.3, 5.4 and 5.5.

Lemma 5.1. If \(\{u_{\epsilon}\} \subset U\) is a sequence of functions such that \(\sup_{\epsilon} E_{\epsilon}^U(u_{\epsilon}) < \infty\), then \(\{\star Ju_{\epsilon}/\pi\}\) is precompact in \(V\).

Proof. For every \(x \in N\), there exists an open set \(\Omega_x \subset N\), a positive number \(R_x\) and a diffeomorphism \(\varphi_x : B_{R_x} \to \overline{\Omega_x}\) such that \(\varphi_x(0) = x\) and \(\{\frac{\partial}{\partial x_1} \varphi_x(0), \frac{\partial}{\partial x_2} \varphi_x(0), \frac{\partial}{\partial x_3} \varphi_x(0)\}\) forms an orthonormal basis for \(T_xN\). Letting \(G_x\) be the \(3 \times 3\) matrix with entries
\( \frac{\partial}{\partial X_i} \varphi, \frac{\partial}{\partial X_j} \varphi \), we have \( G_x^{-1}(X) = I + O_x(|X|) \). Take \( R_x \) smaller if necessary so that \( |O_x(|X|)| \leq 1/2 \) for all \( X \in B_{R_x} \).

Since \( N \) is compact, \( N = \bigcup_{i=1}^{m} \Omega_i \) for some \( \{x_i\} \subset N \).

Let \( \Omega_i = \Omega_{x_i}, \varphi_i = \varphi_{x_i}, R_i = R_{x_i} \) and \( G_i = G_{x_i} \).

Set \( v^i_\epsilon = u_\epsilon \circ \varphi_i \). Note that

\[
E_{H^1(U_{R_i}, \mathbb{C})}(v^i_\epsilon) = \frac{1}{2\pi|\ln \epsilon|} \int_{U_{R_i}} |\nabla v^i_\epsilon(X)|^2 + \frac{1}{2\epsilon^2} (|v^i_\epsilon(X)|^2 - 1)^2 dX
\]

\[
\leq \frac{1}{\pi|\ln \epsilon|} \int_{\Omega_i} (\nabla v^i_\epsilon|_{\varphi_i^{-1}(x)} G_i^{-1}(\varphi_i^{-1}(x)) \nabla v^i_\epsilon|_{\varphi_i^{-1}(x)})^* + \frac{1}{2\epsilon^2} (|u_{\epsilon}(x)|^2 - 1)^2 \int_{\Omega_i} |J_{\varphi_i^{-1}}(x)|
\]

\[
\leq C E_{U_i}^\epsilon(u_\epsilon).
\]

In the remainder of the proof, we will denote (2.2) by \( F_{\Omega} \). Also, define \( \tilde{F}_{\Omega} \) by \( \tilde{F}_{\Omega}(T) = \inf\{M(R) + M(S) : T = R + \partial S, R \in D_1(\Omega), S \in D_2(\Omega)\} \).

Using the above estimate for \( E_{H^1(U_{R_i}, \mathbb{C})}(v^i_\epsilon) \) and Theorem 3.2 in [11], there exists \( Q_1 \in F_1(U_{R_1}) \) and a subsequence \( \{\epsilon_j^1\} \subset (0, 1] \) such that

\[
F_{U_{R_1}}(\ast J v^1_\epsilon - \pi Q_1) \to 0 \quad \text{as} \quad j \to \infty.
\]

Since \( \sup_{\epsilon} E_{H^1(U_{R_i}, \mathbb{C})}(v^i_\epsilon) < \infty \) for all \( i = 1, \ldots, m \), we can use Theorem 3.2 in [11] \( m - 1 \) more times to conclude that for \( i = 2, \ldots, m \), there exists \( Q_i \in F_1(U_{R_i}) \) and a subsequence \( \{\epsilon_j^i\} \subset \{\epsilon_j^{i-1}\} \) such that

\[
F_{U_{R_i}}(\ast J v^i_\epsilon - \pi Q_i) \to 0 \quad \text{as} \quad j \to \infty.
\]

Setting \( \epsilon_j = \epsilon_j^m \), we have

\[
F_{U_{R_i}}(\ast J v^i_\epsilon - \pi Q_i) \to 0 \quad \text{as} \quad j \to \infty \quad \text{for all} \quad i = 1, \ldots, m.
\]
Thus, using (2.5),
\[
\mathbf{F}_{\Omega_i}(\ast Ju_{\epsilon_j} - \pi (\varphi_i)^{-1}Q_i) = \mathbf{F}_{\Omega_i}(\ast J(v_t^i \circ \varphi_i^{-1}) - \pi (\varphi_i)^{-1}Q_i) \\
= \mathbf{F}_{\Omega_i}(\varphi_i^{i} \ast Ju_{\epsilon_j} - \pi Q_i) \\
\leq C\mathbf{F}_{U_{R_i}}(\ast Ju_{\epsilon_j} - \pi Q_i) \\
\rightarrow 0 \quad \text{as } j \to \infty \text{ for all } i = 1, \ldots, m. \tag{5.2}
\]

Let \( T_j^i = \ast Ju_{\epsilon_j} - \pi (\varphi_i)^{-1}Q_i. \) Since \( T_j^i \) is a boundary in \( \Omega_i, \) (5.2) implies
\[
\tilde{\mathbf{F}}_{\Omega_i}(T_j^i) \to 0 \quad \text{as } j \to \infty \text{ for all } i = 1, \ldots, m. \tag{5.3}
\]

Let \( \{\chi_i\} \) be a partition of unity subordinate to the cover \( \{\Omega_i\}, \) \( T = \sum_{i=1}^{m}(\varphi_i)^{-1}Q_i \chi_i \) and \( T_j = \sum_{i=1}^{m} T_j^i \chi_i = \ast Ju_{\epsilon_j} - \pi T. \) It follows from (5.3) that
\[
\tilde{F}_N(T_j) \leq \sum_{i=1}^{m} \tilde{F}_N(T_j^i \chi_i) \\
\leq C \sum_{i=1}^{m} \tilde{F}_{\Omega_i}(T_j^i) \\
\rightarrow 0 \quad \text{as } j \to \infty. \tag{5.4}
\]
for some constant \( C > 0 \) depending on \( \{\chi_1, \ldots, \chi_m\}. \) In particular, \( T_j \) converges weakly to 0 as \( j \to \infty. \) From this, we can conclude \( T \) is a boundary in \( N \) using Theorem 7 in Section 5.3.2 of [8]. Thus, \( T \in \mathcal{F}_1(N). \)

Now we need to show that \( \mathbf{F}_N(T_j) \to 0 \) as \( j \to \infty. \) First, choose \( R_j, S_j \) such that \( T_j = R_j + \partial S_j \) and \( \mathbf{M}(R_j) + \mathbf{M}(S_j) \leq 2\tilde{\mathbf{F}}_N(T_j). \) Owing to (5.4), we can find \( J \in \mathbb{Z}_+ \) such that \( j \geq J \) implies \( \mathbf{F}_N(T_j) \leq c_N/2, \) where \( c_N > 0 \) is the constant mentioned in the assumptions of Lemma 3.4. Now, since \( \partial R_j = \partial T_j = 0 \ \forall j, \) we can find \( S'_j \in \mathcal{D}_2(N) \) such that \( R_j = \partial S'_j \) and \( \mathbf{M}(S'_j) \leq C_N(\mathbf{M}(R_j))^2 \) for \( j \geq J. \) Therefore, if \( j \geq J, \)
\[
\mathbf{F}_N(T_j) \leq \mathbf{F}_N(T_j - R_j) + \mathbf{F}_N(R_j) \\
\leq \mathbf{M}(S_j) + \mathbf{M}(S'_j) \\
\leq 2\tilde{\mathbf{F}}_N(T_j) + 4C_N(\tilde{\mathbf{F}}_N(T_j))^2,
\]
which implies, using (5.4), \( \mathbf{F}_N(\ast Ju_{\epsilon_j} - \pi T) \to 0 \) as \( j \to \infty, \) as required. \( \Box \)
We now state and prove a lemma that we will need in order to verify the lower bound property of $\Gamma$-convergence.

**Lemma 5.2.** If $|\nu|$ is a Radon measure on $\mathbb{R}^3$ supported on a 1-dimensional curve and $\alpha pT_{X_0}|\nu|$ exists, then

$$\lim_{r \to 0} \frac{|\nu|(\partial B^3_r(X_0))}{|\nu|(B^3_r(X_0))} = 0.$$ 

By definition, the above hypotheses mean there exists a line $l \subset \mathbb{R}^3$ passing through $X_0$ and $\theta > 0$ such that $|\nu|_r \rightharpoonup \theta H^1 l = |\nu|_0$ as $r \to 0$ where $|\nu|_r(A) := \frac{1}{r}|\nu|(X_0 + rA)$.

**Proof.** Without loss of generality, assume $X_0 = 0$. Since $|\nu|_0(\partial(\partial B^3_1)) = |\nu|_0(\partial B^3_1) = 0$ and $|\nu|_r \rightharpoonup |\nu|_0$, we have

$$\lim_{r \to 0} |\nu|_r(\partial B^3_1) = |\nu|_0(\partial B^3_1) = 0 \quad \text{and} \quad \lim_{r \to 0} |\nu|_r(B^3_1) = |\nu|_0(B^3_1) > 0.$$ 

Therefore,

$$\lim_{r \to 0} \frac{|\nu|(\partial B^3_r)}{|\nu|(B^3_r)} = \lim_{r \to 0} \frac{|\nu|_r(\partial B^3_1)}{|\nu|_r(B^3_1)} = \frac{|\nu|_0(\partial B^3_1)}{|\nu|_0(B^3_1)} = 0.$$

\hfill \Box

With this, we proceed to show (2.6) is satisfied with $P_{VU}^\epsilon = P_{VU}(u) = \ast Ju/\pi$. The idea of the proof is this: first, we use local coordinates $\phi$ and consider $Q = \pi \phi^{-1}_z T$ on a ball in $\mathbb{R}^3$. Then, since $T$ is rectifiable, we can identify a set of ‘good points’ $X$ of full $\|T\|$ measure on which we can approximate $Q \cap B_r(X)$ in a certain sense by a straight line $l$ for $r$ sufficiently small. By slicing $Q \cap B_r(X)$ orthogonal to $l$ and using known 2-d results in [10], we are able to deduce a local version of the required result. To complete the proof, we apply the local result to suitable disjoint open sets $\{\Omega_i\}$ which cover spt$T$ except for a small set of $\|T\|$-measure.

**Lemma 5.3.** If $T \in \mathcal{R}'_1(N)$ and $\{u_\epsilon\} \subset U$ is a sequence of functions such that

$$\|\ast Ju_\epsilon - \pi T\|_V \to 0 \quad \text{as} \quad \epsilon \to 0,$$ 

then $\lim \inf E_V^\epsilon(u_\epsilon) \geq M(T)$. 

\hfill \Box
Proof. If $\liminf_\epsilon E_U^\epsilon(u_\epsilon) = +\infty$, then there is nothing to prove. Therefore, assume
\[
\liminf_\epsilon E_U^\epsilon(u_\epsilon) < \infty.
\]

Let $x_0 \in \text{spt} T$ and $\Omega$ be an open subset of $N$ such that $x_0 \in \Omega$ and $\overline{\Omega}$ is diffeomorphic to $B^3_R$ for some $R > 0$ and diffeomorphism $\phi$. Assume $\phi(0) = x_0$ and \{\(\frac{\partial}{\partial X_1}\phi(0), \frac{\partial}{\partial X_2}\phi(0), \frac{\partial}{\partial X_3}\phi(0)\)\} forms an orthonormal basis for $T_{x_0}N$. Let $G = (g_{ij})$ be the $3 \times 3$ matrix with entries $g_{ij} = (\phi_{X_i}, \phi_{X_j})$, $G^{-1} = (g^{ij})$ and \{\(\epsilon_j\)\} be a subsequence such that
\[
\lim_{j \to \infty} E_{U^\epsilon_j}^\epsilon(u_\epsilon_j) = m_\Omega := \liminf_\epsilon E_{U^\epsilon}^\epsilon(u_\epsilon) < \infty. \tag{5.6}
\]

Set
\[
|v(X)|_G := \sqrt{v(X)^T G(X)v(X)},
\]
where $v$ is any vectorfield defined on $B^3_R$ taking values in $\mathbb{R}^3$. Taking $R$ small enough, we have
\[
\frac{1}{2}|v(X)|_e \leq |v(X)|_G \leq \frac{3}{2}|v(X)|_e \tag{5.7}
\]
where $|\cdot|_e$ denotes length with respect to the Euclidean inner product.

If $T = \tau(\Gamma, m, \overline{T})$ and $\eta = \eta^i dX_i \in D^1(U^3_R)$ where $\{dX_1, dX_2, dX_3\} \subset \Lambda^1(\mathbb{R}^3)$ denotes the standard orthonormal basis of covectors on $\mathbb{R}^3$, then
\[
\phi^{-1}_* T(\eta) = \int \langle \eta^i(X) d\phi^{-1}_*(\phi(X)), \overline{T}(\phi(X)) \rangle m(\phi(X)) \sqrt{\det G(X)} dX.
\]
Letting $\overline{T} \circ \phi = \sigma, \phi_{X_i}, \sigma = D\phi^{-1}(\overline{T} \circ \phi) = (\sigma_1, \sigma_2, \sigma_3), |\nu| = H^1 \pi m \circ \phi \sqrt{G}$ and $\nu = |\nu| \ll \sigma$, we have
\[
\pi \phi^{-1}_* T(\eta) = \int \eta^i d\nu_i.
\]
Clearly,
\[
|\nu|(A) = \pi M(T_\perp \phi(A)) \tag{5.8}
\]
for any $A \subset B^3_R$. As $T$ is rectifiable, it follows that for $|\nu|$-a.e. $X_0$,
\[
|\sigma(X_0)|_G = 1, \tag{5.9}
\]
\[ apT_{X_0}|\nu| \text{ exists} \] (5.10)

and

\[ |\nu|(B_r(X_0)) > 0 \text{ for all } r > 0. \] (5.11)

Also, since \( T \) has finite mass and (5.7) holds, \( \sigma \in L^1(U^3_R; d|\nu|) \), which implies

\[
\lim_{r \to 0} \frac{1}{|\nu|(B^3_r(X_0))} \int_{B^3_r(X_0)} |\sigma(X) - \sigma(X)|_e d|\nu| = 0 \text{ for } |\nu|-\text{a.e. } X_0. \] (5.12)

If \( X_0 \) satisfies (5.9)-(5.12), then we say that \( X_0 \) is a good point.

Define \( \mu^\epsilon \) for \( A \subset B^3_R \) by

\[
\mu^\epsilon(A) := \frac{1}{2\pi |\ln \epsilon|} \int_A \left( |\nabla_{X_1} v_\epsilon(X)|^2 + \frac{1}{2\epsilon^2} (|v_\epsilon(X)|^2 - 1)^2 \right) \sqrt{\det G(X)} dX,
\]

where \( v_\epsilon = u_\epsilon \circ \phi \). Recalling (5.6), there exists a subsequence of \( \{\epsilon_j\} \), still denoted \( \{\epsilon_j\} \), and a nonnegative Radon measure \( \mu \) such that \( \mu^\epsilon_j \) converges to \( \mu \) weakly as measures.

Assume, without loss of generality, that \( X_0 = 0 \) is a good point and \( \sigma(X_0) = e_3 \) (otherwise we can analyze the pullback of \( \pi \phi^{-1}_t T \) by an isometry \( \mathbb{R}^3 \to \mathbb{R}^3 \) that takes \( (0,0,0) \) to \( X_0 \) and the \( z \)-axis onto the line through \( X_0 \) in the direction of \( \sigma(X_0) \)). We claim that

\[
\frac{d|\nu|}{d\mu}(X_0) = \lim_{r \to 0} \frac{|\nu|(B^3_r(X_0))}{\mu(B^3_r(X_0))} \leq \pi \text{ for } \mu\text{-a.e. good point } X_0. \] (5.13)

From this point on, assume \( r \in (0,R) \). Since \( B^3_r(X_0) \) denotes a closed ball and \( G(X_0) = I \),

\[
\mu(B^3_r(X_0)) \geq \lim \inf \mu^\epsilon_j(B^3_r(X_0)) \geq \pi^{-1}(1 - Cr) \lim \inf \mu^\epsilon_j(B^3_r(X_0)) \] (5.14)

where, for \( A \subset B^3_R \),

\[
\mu^\epsilon(A) := \frac{1}{2|\ln \epsilon|} \int_A \left( |\nabla_{X_1} v_\epsilon(X)|^2 + \frac{1}{2\epsilon^2} (|v_\epsilon(X)|^2 - 1)^2 \right) dX.
\]

Define \( \mu^{\epsilon,t} \) by

\[
\mu^{\epsilon,t}(S) := \frac{1}{2|\ln \epsilon|} \int_S \left( |\nabla_{X_1} v_\epsilon(X_1, X_2, t)|^2 + \frac{1}{2\epsilon^2} (|v_\epsilon(X_1, X_2, t)|^2 - 1)^2 \right) dX_1 dX_2.
\]
where \( S \subset B^3_{R} \cap p^{-1}(t) \).

Since we are assuming (5.5), Theorem 5.2 in [10] implies \( \nu_3 \) can be represented by slices \( \nu^t_3 \). This means in particular that for any open ball \( U \subset B^3_{R} \),

\[
\nu_3(U) = \int \nu^t_3(U \cap p^{-1}(t)) \, dt \tag{5.15}
\]

where \( p(X) = p(X_1, X_2, t) = t \). For almost every \( t \), \( \nu^t_3 \) has the form \( \pi \sum d_i(t) \delta_{a_i(t)} \) for some integers \( d_i(t) \) and points \( a_i(t) \in U \cap p^{-1}(t) \). The theorem also implies

\[
\liminf \mu^{e_j,t}(U \cap p^{-1}(t)) \geq \nu^t_3(U \cap p^{-1}(t)) \tag{5.16}
\]

for any open ball \( U \subset B^3_{R} \). Therefore,

\[
\pi \mu(B^3_r(X_0)) \geq (1 - Cr) \liminf \mu^{e_j,t}(U^3_r(X_0)) \geq (1 - Cr) \int \mu^{e_j,t}(U^3_r(X_0) \cap p^{-1}(t)) \, dt \geq (1 - Cr) \int \nu^t_3(U^3_r(X_0) \cap p^{-1}(t)) \, dt \geq (1 - Cr) \int U^3_r(X_0) \, d\nu_3 \tag{5.14}
\]

Recalling that \( \sigma(X_0) = e_3 \) and \( G(X_0) = I \),

\[
\int U^3_r(X_0) \, d\nu_3 = \int e_3 \cdot \sigma(X) \, d|\nu| \geq |\nu|(U^3_r(X_0)) - \int_{B^3_r(X_0)} |\sigma(X) - \sigma(X)| \, d|\nu|.
\]

Combining the above 2 estimates and using Lemma 5.2 as well as (5.12), we conclude

\[
\pi \mu(B^3_r(X_0)) \geq (1 - o(1))|\nu|(B^3_r(X_0)). \tag{5.17}
\]

Here, \( o(1) \) is a quantity which \( \to 0 \) as \( r \to 0^+ \).

Since (5.11) holds, we have for \( r \) sufficiently small,

\[
\frac{|\nu|(B^3_r(X_0))}{\mu(B^3_r(X_0))} \leq \pi(1 + o(1)). \tag{5.18}
\]
Basic theorems on differentiation of measures guarantee \( \lim_{r \to 0} \frac{|\nu|(B_r^3(X_0))}{\mu(B_r^3(X_0))} \) exists \( \mu \)-a.e.

Hence, letting \( r \to 0^+ \) in (5.18) we obtain (5.13).

Now we wish to show \( M(T \cup \Omega) \leq m_\Omega \). From (5.17), we can deduce \( |\nu| \ll \mu \). Thus, using (5.8),

\[
M(T \cup \Omega) = \frac{1}{\pi} |\nu|(U_R^3) = \frac{1}{\pi} |\nu|\{|X \in U_R^3 : X \text{ is a good point}\}) = \frac{1}{\pi} \int_{\{X \in U_R^3 : X \text{ is a good point}\}} d|\nu|(X)d\mu \leq \mu(U_R^3),
\]

(5.13)

Since \( U_R^3 \) denotes an open ball,

\[
M(T \cup \Omega) \leq \mu(U_R^3) \leq \liminf_j \mu_j^\varepsilon(U_R^3) = \liminf_j E_{\Omega_j}^\varepsilon(u_\varepsilon) = m_\Omega.
\]

Finally, we need to show \( \liminf_\varepsilon E_{U}^\varepsilon(u_\varepsilon) \geq M(T) \). Fix \( \delta > 0 \). We can cover \( \text{spt} T \) by disjoint open sets \( \{\Omega_i = \psi_i(U_R^3)\} \), where \( \phi_i \) is a diffeomorphism as \( \phi \) above, and \( \sum M(T \cup \Omega_i) \geq M(T) - \delta \). Using the above result,

\[
\liminf_\varepsilon E_{U}^\varepsilon(u_\varepsilon) \geq \sum_i \liminf_\varepsilon E_{\Omega_i}^\varepsilon(u_\varepsilon) \geq \sum_i M(T \cup \Omega_i) \geq M(T) - \delta.
\]

Let \( \delta \to 0^+ \) to obtain

\[
\liminf_\varepsilon E_{U}^\varepsilon(u_\varepsilon) \geq M(T),
\]

as required. \( \square \)

Now we need to construct a map \( Q_{UW}^t = Q_{UV}^t \circ Q_{VW}^t : W \to U \) satisfying (2.7). We will denote \( Q_{UW}^t \) by \( u_t^\varepsilon \). First we need to recall and make some definitions.

First recall that \( W = U_{r_1}^t \), where \( r_1 > 0 \) is the number chosen in Section 4.3 and \( Q_{VW}(w) \) is the multiplicity 1 current corresponding to integration over \( \gamma_w \), where \( \gamma_w(t) = \exp(\gamma(t), z_w(t)), z_w(t) = \sum_{i=1}^l w_i z_i(t), t \in [0, L] \).

Let \( u_{1\varepsilon}(t) = u_1(t, w), i = 1, 2, \) be smooth functions on \( \mathbb{R}/LZ \times B_{r_1}^l \) such that, for every \( t \in \mathbb{R}/LZ \), \( \{u_{1\varepsilon}(t), u_{2\varepsilon}(t), \gamma_{1\varepsilon}(t)\} \) forms an orthonormal basis for \( T_{\gamma_w(t)}N \) and \( u_{0\varepsilon}(t) = u(t) \).
Let
\[ K_{w,r} := \left\{ \exp \left( \gamma_w(t), \sum_{i=1}^2 y_i u_i^w(t) \right) : y \in U_r^2, t \in [0, L] \right\} \]
for \( r \in (0, r_0/2] \) and
\[ M_w := \{ \gamma_w(t) : t \in [0, L] \}. \]

Recall that for \( X = (y, t) \in B_r^2 \times \mathbb{R}/L\mathbb{Z}, \psi(X) = \exp(\gamma(t), \sum_{i=1}^2 y_i u_i(t)). \) Set \( y_i(x) := \psi_i^{-1}(x) \) for \( i = 1, 2 \) and \( y(x) = y_1(x) + iy_2(x). \)

Define \( O_w : N \rightarrow N \) by
\[
O_w(x) := \begin{cases} 
\exp \left( \exp(\gamma(\tau(x)), \chi(|v(x)|)z_w(\tau(x))), \sum_{i=1}^2 y_i(x) v_i^w(x) \right) & \text{for } x \in K_{r_0}, \\
x & \text{for } x \in N \setminus K_{r_0}
\end{cases}
\]
where
\[
v_i^w(x) = u_i^w(\tau(x)) + \chi(|v(x)|) (u_i^w(\tau(x)) - u_i^w(x))
\]
and \( \chi \in C^\infty_c([0, r_0); [0, 1]) \) with \( \chi(s) = 1 \) for every \( s \in [0, r_0/2] \). Note that if \( r_1 \) is small enough, then \( O_w \) is a smooth diffeomorphism that takes \( M \) onto \( M_w \), \( K_\epsilon \) onto \( K_{w,\epsilon} \) and \( K_{r_0} \setminus K_\epsilon \) onto \( K_{r_0} \setminus K_{w,\epsilon} \) for every \( \epsilon \in (0, r_0/2) \).

Let
\[
v^0(x) := \begin{cases} 
y(x)/|y(x)| & \text{for } x \in K_{r_0}, \\
y^0(x) & \text{for } x \in N \setminus K_{r_0}
\end{cases}
\]
where \( y^0 \) is any smooth function taking values in \( S^1 \) such that \( y^0(x) = y(x)/|y(x)| \) in a neighbourhood of \( \partial K_{r_0} \). We refer the reader to the Appendix for the justification of the existence of such a function \( y^0 \).

Let \( \psi_w = O_w \circ \psi \) and \( u^0_w = v^0 \circ O^{-1}_w \). We claim that
\[
\star J u^0_w = \pi Q_{VW}(w). \tag{5.19}
\]
To see this, first note that \( \star J u^0_w : N \setminus K_{w,r_0/2} = 0 \) since \( u^0_w \) is smooth and takes values in \( S^1 \) away from \( M_w \). Hence it suffices to show
\[
\star J u^0_w : K_{w,r_0/2} = \pi Q_{VW}(w). \tag{5.20}
\]
Using (2.5) and setting \( u(X) = u(y, t) = \frac{X}{|y|} \) for \( X \in \mathbb{R}^2 \times \mathbb{R}/LZ \), we have

\[
(\psi^{-1}_w)_* (\star J u_{w}^0 K_{w,r_0/2}) = \star J u_{\psi^{-1} T_{\{0\} \times \mathbb{R}/LZ}} \psi^{-1} T_{\{0\} \times \mathbb{R}/LZ}.
\]

It is well-known that \( \star J u = \pi T_{\{0\} \times \mathbb{R}/LZ} \) (see Example 4 in [9]). Thus,

\[
(\psi^{-1}_w)_* (\star J u_{w}^0 K_{w,r_0/2}) = \pi T_{\{0\} \times \mathbb{R}/LZ} = \pi \psi^{-1} T_{\{0\} \times \mathbb{R}/LZ} = \pi \psi^{-1} (O^{-1}_w) \pi Q_{VW}(w) = (\psi^{-1}_w)_* (\pi Q_{VW}(w)),
\]

which implies (5.19).

For \( \epsilon \in (0, r_0/4) \), define

\[
v^\epsilon(x) := v^0(x) \begin{cases} \frac{|y(x)|}{\epsilon} & \text{for } x \in K_\epsilon, \\ 1 & \text{for } x \in N \setminus K_\epsilon \end{cases}
\]

and \( u^\epsilon_w := v^\epsilon \circ O^{-1}_w. \)

**Lemma 5.4.** As \( \epsilon \to 0 \), \( \|P_{VU}^\epsilon(u^\epsilon_w) - Q_{VW}(w)\|_V \to 0 \) uniformly on \( W \).

**Proof.** Let \( G_w \) be the 3 \times 3 matrix with entries \((\frac{d}{dX_i} \psi_w, \frac{d}{dX_j} \psi_w), G^{-1}_w = (g^{ij}_w) \) and

\[
a^\epsilon_{w,i}(x) = u^\epsilon_w(x) \frac{d}{dX_i} \left( u^\epsilon_w \circ \psi_w \right) \left( \psi^{-1}_w(x) \right) - u^0_w(x) \frac{d}{dX_i} \left( u^0_w \circ \psi_w \right) \left( \psi^{-1}_w(x) \right)
\]

where \( \times \) denotes the complex cross product. Using (5.19),

\[
\|P_{VU}^\epsilon(u^\epsilon_w) - Q_{VW}(w)\|_V
\]

\[
= \frac{1}{2\pi} \left| |j(u^\epsilon_w) - j(u^0_w)| \right|_{L^1(N)}
\]

\[
\leq \frac{1}{2\pi} \int_{K_w} \sqrt{g^{ij}_{w,\epsilon}(x) a^\epsilon_{w,i}(x) a^\epsilon_{w,j}(x)}
\]

\[
= \frac{1}{2\pi} \int_{L_w} \int_{U_2} \sqrt{g^{ij}_{w}(X) a^\epsilon_{w,i}(\psi_w(X)) a^\epsilon_{w,j}(\psi_w(X))} |J \psi_w(X)| dy dt
\]
Note that since $\psi_{w}$ is a smooth function in both $X \in B_{r_0}^2 \times \mathbb{R}/LZ$ and $w \in B_{r_1}^l$, there exists an absolute constant $C_9 > 0$ such that

$$|G_{w}^{-1}(X)| + |J\psi_{w}|(X) \leq C_9$$

for all $X \in B_{r_0}^2 \times \mathbb{R}/LZ$ and $w \in B_{r_1}^l$. Also, $a_{\epsilon, 3}(\psi_{w}(X)) = 0$ and $a_{w, i}(\psi_{w}(X)) \leq \frac{3}{|y|}$ for all $i = 1, 2, X \in U_\epsilon^2 \times \mathbb{R}/LZ, w \in W$ and $\epsilon \in (0, r_0/4)$.

Therefore,

$$\|P_{\epsilon}U(u_{w}^\epsilon) - Q_{W}(w)\|_V \leq \frac{6C_9^2}{\pi} \int_0^L \int_{U_\epsilon^2} \frac{1}{|y|} dy dt = 12C_9^2 L\epsilon$$

which implies $\|P_{\epsilon}U(u_{w}^\epsilon) - Q_{W}(w)\|_V \xrightarrow{\epsilon \to 0} 0$ uniformly on $W$.

**Lemma 5.5.** As $\epsilon \to 0$, $E_{U}(u_{w}^\epsilon) \to M(Q_{W}(w))$ uniformly on $W$.

**Proof.** First we break $E_{U}(u_{w}^\epsilon)$ into pieces and then analyze each piece:

$$E_{U}(u_{w}^\epsilon) = \frac{1}{2\pi|\ln \epsilon|} \left( \int_{K_{w, \epsilon}} |\nabla u_{w}^\epsilon|^2 + \int_{K_{\epsilon} \setminus K_{w, \epsilon}} |\nabla u_{w}^\epsilon|^2 + \int_{N \setminus K_{\epsilon}} |\nabla u_{w}^\epsilon|^2 + \int_{K_{w, \epsilon}} \frac{(|u_{w}^\epsilon|^2 - 1)^2}{2\epsilon^2} \right)$$

$$= \frac{1}{2\pi|\ln \epsilon|} (E_1 + E_2 + E_3 + E_4).$$

- **Estimate of $E_1$:** Note that for $X \in U_\epsilon^2 \times \mathbb{R}/LZ$, $u_{w}^\epsilon(\psi_{w}(X)) = \frac{y_1 + iy_2}{\epsilon}$. Using this,

$$E_1 = \int_{K_{w, \epsilon}} \nabla (u_{w}^\epsilon \circ \psi_{w})(X) G_{w}^{-1}(\psi_{w}^{-1}(X)) (\nabla (u_{w}^\epsilon \circ \psi_{w})(X))^* J\psi_{w}(X) dy dt$$

$$\leq C_9 \int_0^L \int_{U_\epsilon^2} |\nabla (u_{w}^\epsilon \circ \psi_{w})(X)|^2 dy dt$$

$$= C_9 \int_0^L \int_{U_\epsilon^2} |(1, i, 0)^T/\epsilon|^2 dy dt$$

$$= 2\pi L C_0.$$
Estimate of $E_2$: Note that for any $t \in \mathbb{R}/L\mathbb{Z}$, $\frac{d}{dt}\psi_w(X)|_{(0,t)} = u_w^i(t)$ for $i = 1, 2$ and $\frac{d}{dt}\psi_w(X)|_{(0,t)} = \gamma'_w(t)$, which implies

$$G_w(0, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |\gamma'_w(t)|^2 \end{pmatrix},$$

and $|J\psi_w|(0, t) = \sqrt{\det G_w(0, t)} = |\gamma'_w(t)|$. Also, for $X \in U_{r_0}^2 \setminus U^2_{\epsilon} \times \mathbb{R}/L\mathbb{Z}$,

$$\nabla(u_0^w \circ \psi_w)(X) = \frac{1}{|y|^3}(y_2^2 - iy_1y_2, -y_1y_2 + iy_2^2, 0)^T.$$

Thus,

$$E_2 = \int_0^L \int_{U_{r_0}^2 \setminus U^2_{\epsilon}} \nabla(u_0^w \circ \psi_w)(X)G_w^{-1}(X)(\nabla(u_0^w \circ \psi_w)(X))^*|J\psi_w|(X)dydt$$

$$= \int_0^L \int_{U_{r_0}^2 \setminus U^2_{\epsilon}} \nabla(u_0^w \circ \psi_w)(X)(I + O(|y|))(\nabla(u_0^w \circ \psi_w)(X))^*|\gamma'_w(t)| + O(|y|))dydt$$

$$= \int_0^L |\gamma'_w(t)| \int_{U_{r_0}^2 \setminus U^2_{\epsilon}} \frac{1}{|y|^2} dydt + \int_0^L \int_{U_{r_0}^2 \setminus U^2_{\epsilon}} O(|y|^{-1})dydt$$

$$= 2\pi |\ln \epsilon| \int_0^L |\gamma'_w(t)| dt + O(1).$$

Estimate of $E_3$

$$E_3 = \int_{N \setminus K_0} |\nabla y^0|^2 < \infty \quad \text{since } N \text{ is compact.}$$

Estimate of $E_4$

$$E_4 = \int_0^L \int_{U^2_{\epsilon}} \frac{(|u^0_w(\psi_w(X))|^2 - 1)^2}{2\epsilon^2} |J\psi_w|(X)dydt$$

$$\leq \pi LC_9 \int_0^\epsilon \frac{(r^2/\epsilon^2 - 1)^2}{\epsilon^2} rdr$$

$$= \pi LC_9 \frac{(r^2/\epsilon^2 - 1)^3}{0}$$

$$= \frac{\pi LC_9}{6}.$$

Combining these estimates we have

$$E'_U(u^*_w) = \int_0^L |\gamma'_w(t)| dt + O(|\ln \epsilon|^{-1})$$
with $|\ln \epsilon|O(|\ln \epsilon^{-1}|)$ bounded above by a constant independent of $w$. Therefore,

$$E^\epsilon_U(u^\epsilon_w) \overset{\epsilon \to 0}{\longrightarrow} \int_0^L |\gamma'_w(t)| dt = M(Q_{VW}(w))$$

uniformly on $w$. \hfill \Box

### 5.2 Existence of critical points for $E^\epsilon_U$ on $N$

Before we state our main result, we need to show the Ginzburg-Landau energy $E^\epsilon_U$ satisfies the Palais-Smale condition in the $H^1$ topology for every $\epsilon \in (0, 1]$.

**Proposition 5.1.** If $\{u_j\} \subset U$ is a Palais-Smale sequence associated with the functional $E^\epsilon_U$, then $\{u_j\}$ is precompact in $U$.

**Proof.** By assumption, $\{u_j\}$ satisfies

$$\sup_j E^\epsilon_U(u_j) < \infty, \|\delta E^\epsilon_U(u_j)\|_{U^*} \to 0 \text{ as } j \to \infty.$$  

The energy bound immediately implies $\|u_j\|_{H^1(N)} \leq C$. Hence there is a subsequence $\{j_k\}$ and a function $u \in H^1(N)$ such that

$$u_{j_k} \to u \text{ in } H^1(N) \quad (5.21)$$

and

$$u_{j_k} \to u \text{ in } L^p(N), 1 \leq p < 6. \quad (5.22)$$

This follows from Rellich’s Compactness Theorem and the fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence. Since

$$\delta E^\epsilon_U(u_{j_k})(v) = \frac{1}{\pi |\ln \epsilon|} \int_N \nabla u_{j_k} \cdot \nabla v + \frac{1}{\epsilon^2} (|u_{j_k}|^2 - 1) u_{j_k} \cdot v,$$

$$\delta E^\epsilon_U(u_{j_k})(u) \overset{k \to \infty}{\longrightarrow} 0 \quad (5.23)$$

and

$$|\delta E^\epsilon_U(u_{j_k})(u)| \leq C\|\delta E^\epsilon_U(u_{j_k})\|_{U^*} \overset{k \to \infty}{\longrightarrow} 0, \quad (5.24)$$
we have
\[
\lim_{k \to \infty} \int_N |\nabla u_{jk}|^2 = \frac{1}{\epsilon^2} \lim_{k \to \infty} \int_N (1 - |u_{jk}|^2)|u_{jk}|^2 = \frac{1}{\epsilon^2} \lim_{k \to \infty} \int_N (1 - |u_{jk}|^2)u_{jk} \cdot u
\]
\[
= \lim_{k \to \infty} \int_N \nabla u_{jk} \cdot \nabla u = \lim_{k \to \infty} \left( \langle u_{jk}, u \rangle_{H^1(N)} - \langle u_{jk}, u \rangle_{L^2(N)} \right)
\]
\[
\int_N |\nabla u|^2.
\]
Thus, \( u_{jk} \to u \) in \( H^1 \) and this completes the proof.

\[\square\]

**Theorem 5.2.** Suppose \( N \) is a 3-dimensional compact, orientable Riemannian manifold with \( \partial N = \emptyset, M = \{\gamma(t)\} \) is a 1-dimensional, connected, smooth submanifold of \( N \) without boundary, \( \gamma \) is a geodesic and the Jacobi operator associated with \( M \) has finite index and 0 nullity. Let \( T_M \in \mathcal{R}_1(N) \) denote the multiplicity 1 current corresponding to integration over \( \gamma \). Then for \( E_\epsilon^U, E_\epsilon^V \) defined by (5.1),(4.5) respectively, there exists \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \), \( E_\epsilon^U \) possesses a critical point \( u_\epsilon \) and \( E_\epsilon^U(u_\epsilon) \to E_\epsilon^V(T_M) \) as \( \epsilon \to 0 \).

**Proof.** Recalling Theorems 3.1,4.1,5.1 and Proposition 5.1, this will follow if we can verify (3.1)-(3.4). Since \( O_w^{-1} \) is a smooth function in \( w \in W \), (3.2) holds and (3.1),(3.3),(3.4) hold due to Lemmas 4.2,5.4,5.5 respectively. \( \square \)
Chapter 6

Appendix

Throughout this section, $\Omega$ will be an open subset of $\mathbb{R}^n$, $K$ will be a compact subset of $\Omega$ and $T \in \mathcal{I}_1(\mathbb{R}^n)$ with $M(T) < \infty$. Also, $X = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ will denote a point of $\Omega$ and $p(X) := t$.

Assumptions on $T$:

(T1) $T$ is $(\Psi, \omega, \delta)$-minimal in $\Omega$ with $\frac{\omega(\rho)}{\rho}$ decreasing in $\rho$

(T2) $\text{spt}T$ is compact in $K$, $\text{spt}T \cap C_R$ is compact $\subset C_R$ and $\text{spt}\partial(T \cap C_R) \subset \partial C_R$ (note that this implies $p_#T = \kappa E^1$ for some integer $\kappa$, where $E^1$ is the standard 1-current obtained by integration of 1-forms over $\mathbb{R}$; we will assume that $\kappa = \pm 1$)

(T3) $\text{diam}(\text{spt}T \cap C_R) \leq \delta/4$

(T4) $\Theta(\|T\|, X) \geq 1 \|T\|$-a.e.

(T5) $E(T, R) < \omega_1 = 2$

Assumptions on $\Psi$:

(P1) $\Psi$ is a non-negative, $\lambda$-elliptic parametric integrand of degree 1 on $\Omega \times \mathbb{R}^n$ that is $C^1$ on $\Omega \times S^{n-1}$

(P2) $\lambda^{-1}|\xi| \leq \Psi(X, \xi) \leq \lambda|\xi|$ for all $X \in \Omega, \xi \in \mathbb{R}^n$

(P3) $|\Psi(X, \xi) - \Psi(Y, \xi)| \leq \lambda|X - Y|$ for all $X, Y \in K, \xi \in S^{n-1}$

(P4) for every $(X_0, \xi_0) \in K \times S^{n-1}$, there is a certain 1-covector $\phi_0 = \phi_0(X_0, \xi_0)$ such
that $|\phi_0| \leq \lambda$ and

$$|\Psi(X, \xi) - \Psi(X_0, \xi_0) - \langle \phi_0, \xi - \xi_0 \rangle| \leq \lambda(|X - X_0| + |\xi - \xi_0|^2)$$

for all $(X, \xi) \in K \times S^{n-1}$

For (P5) and (P6), $X_0 \in K$ and the reference frame $(\epsilon_1, \ldots, \epsilon_n)$ centered at $X_0$ are given.

(P5) for all $(\tilde{c}, c_n) \in K$ and $|p| \leq 1$, $\lambda$ is a common bound for $\Psi^\delta, \frac{\partial \Psi^\delta}{\partial p_1}$ and their first derivatives

(P6) for all $(\tilde{c}, c_n) \in K$, all $|p| \leq 1$ and every $p^0$ with $|p^0| \leq 1$, we have

$$\left| \frac{\partial \Psi^\delta}{\partial p_i}(\tilde{c}, c_n, p) - \frac{\partial \Psi^\delta}{\partial p_i}(0, 0, p^0) - \sum_j \frac{\partial^2 \Psi^\delta}{\partial p_j \partial p_i}(0, 0, p^0)(p_j - p^0_j) \right|$$

$$\leq \lambda(|\tilde{c}| + |c_n| + \nu(|\tilde{c}| + |c_n| + |p - p^0|)|p - p^0|)$$

for some positive function $\nu$ defined on $\mathbb{R}_+$ with $\frac{\omega(\rho)}{\rho}$ is decreasing in $\rho$

For $\eta = \eta(t)dt \in D^1(U_1^1)$, define $S_j(\eta) := T(y_j \eta), j = 1, \ldots, n - 1$. We can write

$$S_j(\eta) = \int \eta(t)f_j(t)dt$$

for some $f_j \in BV(U_1^1)$. To see this, first note that the map $\eta \mapsto S_j(\eta)$ defines a bounded, linear functional on $C_c(U_1^1)$ and thus $S_j$ can be represented by a Radon measure $\nu_j$. Since

$$\int \eta'(t)d\nu_j = T(y_j \eta'dt)$$

$$= T(d(y_j \eta) - \eta dy_j)$$

$$= T(d(y_j \eta)) - T(\eta dy_j)$$

$$= \frac{\partial T(y_j \eta)}{\partial y_j} - T(\eta dy_j)$$

$$\overset{(T2)}{=} -T(\eta dy_j)$$

$$\leq M(T)\|\eta\|_{C(U_1^1)}$$

for any $\eta \in C_c^1(U_1^1)$, we can mollify $\nu_j$ to obtain measures $\nu^\epsilon_j$ which we can identify with $BV$ functions $f^\epsilon_j$ satisfying $\|f^\epsilon_j\|_{BV} \leq CM(T)$. General compactness results guarantee
that there exists \( f_j \in BV \) such that \( f_j^\epsilon \to f_j \) in \( L^1 \) as \( \epsilon \to 0 \). Therefore, \( \nu_j \) can be identified with the \( BV \) function \( f_j \) since \( \nu_j^\epsilon \) converges to \( \nu_j \) weakly as measures.

The following results can be found in [4]. Lemma 6.1 is Lemma 2 in [4] and Lemma 6.2 is a combination of Lemmas 17,18 and estimates in the proof of Lemma 18 in [4].

**Lemma 6.1.** If \( f = (f_1, \ldots, f_{n-1}) \), where \( f_j \) is the \( BV \) function above, then

\[
|Df|(U^1_R) \leq 3\sqrt{E(T,R)} R \tag{6.1}
\]

**Lemma 6.2.** There exists a positive constant \( E_0 = E_0(\lambda, \omega(\cdot), \delta, \nu(\cdot)) \) such that if

\[
R + E(T,R) \leq E_0 \quad \text{and} \quad \int_0^R \frac{\omega(\rho)}{\rho} d\rho \leq E_0, \tag{6.2}
\]

then \( f \in C^1(B^1_{R'/2}) \) and \( T \cdot C_{R'/2} = G(t) = (f(t), t) \), \( t \in B^1_{R/2} \).

Moreover, for any \( t \in B^1_{R/2} \),

\[
|f'(t) - f'(0)| \leq C \left( \Phi(R) + \int_0^R \frac{\Phi(\rho)}{\rho} d\rho \right) \tag{6.3}
\]

where

\[
\Phi(\rho) = \left( c \frac{\rho}{R} E(T, R) + E_0^{-1}(\rho + \omega(12\rho)) \right)^{1/2}.
\]

Assume \( \omega(\rho) = \rho^\alpha, \alpha \in (0, 1), 0 < \rho \leq \delta < 1 \). Combining (6.1) and (6.3), we have for any \( t \in B^1_{R/2} \),

\[
|f'(t)| \leq |f'(t) - f'(0)| + \frac{1}{R} \int_{U^1_{R/2}} |f'(0)| ds \\
\leq 2 \sup_{s \in B_{R/2}} |f'(s) - f'(0)| + \frac{1}{R} |Df|(U^1_R) \\
\leq 2C \left( 1 + \frac{2}{\alpha} \right) \left( \sqrt{\frac{cE(T,R)}{R}} + \sqrt{(1 + 12^\alpha E_0^{-1})R^{\alpha/2}} + 3\sqrt{E(T,R)} \right) \tag{6.4}
\]
Justification of existence of $y_0$:

First represent $T_M$ by a measure-valued 2-form $\mu_M$ via

$$T_M(\star \eta) = \langle \mu_M, \eta \rangle = \int (\mu_M, \eta), \quad \eta \in D^2(\eta).$$

Assume $r_0$ is sufficiently small so that $\theta = \arg(y(x))$ is defined in $K_{3r_0}$. Note that

$$d\theta(x) = -\frac{y_2(x)}{|y(x)|^2} d\psi_1^{-1} + \frac{y_1(x)}{|y(x)|^2} d\psi_2^{-1}$$

and for any $\eta \in D_2(K_{3r_0}),$

$$\langle d^2\theta, \eta \rangle = \int (d\theta, d^\star \eta) = \int \star d^\star \eta \wedge d\theta.$$

Letting $u(x) = \frac{y(x)}{|y(x)|}$ for $x \in K_{3r_0}$, we have $j(u) = d\theta$, and, recalling (5.20) with $w = 0,$

$$\langle d^2\theta, \eta \rangle = \int \star d^\star \eta \wedge j(u) = j(u)(\star \eta) = \partial \star j(u)(\star \eta) = 2 \star J(u)(\star \eta) = 2\pi T_M(\star \eta) = \langle 2\pi \mu_M, \eta \rangle.$$

Thus, $d^2\theta = 2\pi \mu_M$ in $K_{3r_0}.$

Recall that we have assumed that $T_M \in R_1'(N).$ In particular, this means $T_M = \partial S_M$ for some $S_M \in D_2(N).$ If $\eta$ is any co-closed 2-form, then

$$\langle \mu_M, \eta \rangle = T_M(\star \eta) = \partial S_M(\star \eta) = S_M(d \star \eta) = S_M(\star d^\star \eta) = 0.$$ 

Therefore, there exists a unique 2-form $\eta$ with coefficients in $W^{1,q}(N)$ for $q < 3/2$ that is orthogonal to all harmonic 2-forms and satisfies

$$-\Delta \eta = 2\pi \mu_M.$$

This is classical if the right-hand side is in $L^2$ (see [8], Chapter 5.2.5, Theorem 3). The extension to the case where the right-hand side is a measure is proved using the duality argument in [3]; see [2] for a more detailed account of a related result. Moreover, $d\mu_M = 0$ since $\partial T_M = 0$, which implies $d\eta$ is harmonic. As $\langle d\eta, \omega \rangle = 0$ for every harmonic 3-form $\omega$, it follows that $d\eta = 0.$
Chapter 6. Appendix

Fix \( x_0 \in N \setminus K_{r_0} \) and for \( x \in N \setminus K_{r_0} \), set \( \phi^0(x) = \int_{\gamma_{x_0,x}} (d^* \eta)_T \), where \( \gamma_{x_0,x} \) is any smooth curve supported in \( N \setminus K_{r_0} \) connecting \( x_0 \) and \( x \). We claim that \( \phi^0 \) is well-defined modulo \( 2\pi \). To see this, it suffices to check that if \( \beta \) is a smooth closed loop in \( N \setminus K_{r_0} \), then \( \int_{\beta} (d^* \eta)_T = 2\pi d \) for some \( d \in \mathbb{Z} \). Using Stokes’ Theorem and that fact that \( dd^* \eta = 0 \) in \( N \setminus K_{r_0} \), we deduce that there exist \( t \in \mathbb{R} / \mathbb{L} \mathbb{Z} \) and \( d \in \mathbb{Z} \) such that

\[
\int_{\beta} (d^* \eta)_T = d \int_C (d^* \eta)_T
\]

for some path \( C = C(\theta) = \psi(r_0 \cos \theta, r_0 \sin \theta, t), 0 \leq \theta < 2\pi \). Since \( dd^* \eta = d^2 \theta = 2\pi \mu_M \) in \( K_{3r_0} \),

\[
\int_C (d^* \eta)_T = \int_C (d\theta)_T = \int_C d\theta.
\]

The change of variables \( y_1(x) = r_0 \cos \theta, y_2(x) = r_0 \sin \theta \) gives

\[
\int_C d\theta = \int_0^{2\pi} \frac{-r_0 \sin \theta}{r_0^2} \ d(r_0 \cos \theta) + \frac{r_0 \cos \theta}{r_0^2} \ d(r_0 \sin \theta) = 2\pi.
\]

Define \( e^{i\zeta} \) in \( K_{3r_0} \setminus K_{r_0} \) by

\[
e^{i\zeta(x)} = e^{-i\phi^0(x)} \frac{y(x)}{|y(x)|}.
\]

Note that \( d\phi^0 = d^* \eta \) by construction and thus \( d\zeta = -d^* \eta + d\theta \). We can recover \( \zeta \) by integrating over paths (up to an additive constant). Since \( d^2 \zeta = 0 \), Stokes’ Theorem implies that \( \zeta \) is well-defined. Also, as \( \eta \) and \( \theta \) are smooth in \( K_{3r_0} \setminus K_{r_0} \), so is \( \zeta \). Now,

\[
y^0(x) = e^{i\phi^0(x)} \begin{cases} e^{i\chi(|y(x)|/3)\zeta(x)} & \text{in } K_{3r_0} \setminus K_{r_0}, \\ 1 & \text{in } N \setminus K_{3r_0} \end{cases}
\]

satisfies the required properties.
Bibliography


