Dynamic Modeling and Active Vibration Control of a Planar 3-PRR Parallel Manipulator with Three Flexible Links

by

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Abstract

Given the advantages of parallel manipulators and lightweight manipulators, a 3-PRR planar parallel manipulator with three lightweight intermediate links has been developed to provide an alternative high-speed pick-and-place positioning mechanism to serial architecture manipulators in electronic manufacturing, such as X-Y tables or gantry robots. Lightweight members are more likely to exhibit structural deflection and vibrate due to the inertial forces from high speed motion, and external forces from actuators. Structural flexibility effects are much more pronounced at high operational speeds and accelerations. Therefore, this thesis presents the dynamics and vibration control of a 3-PRR parallel manipulator with three flexible links.

Firstly, a procedure for the generation of dynamic equations for a 3-PRR parallel manipulator with three flexible intermediate links is presented based on the assumed mode method. The dynamic equations of the parallel manipulator with three flexible intermediate links are developed using pinned-pinned boundary conditions. Experimental modal tests are performed using an impact hammer and an accelerometer to identify the mode shapes, frequencies, and damping ratios of flexible intermediate links. The mode shapes and frequencies, obtained from experimental modal tests, match very well the assumed mode shapes and frequencies obtained based on pinned-pinned boundary conditions, and therefore the dynamic model developed is validated.

Secondly, this thesis presents the investigation on dynamic stiffening and buckling of the flexible links of a 3-PRR parallel manipulator by including the effect of
longitudinal forces on the modal characteristics. Natural frequencies of bending vibration of the intermediate links are derived as the functions of axial force and rigid-body motion of the manipulator. Dynamic stiffening and buckling of intermediate links is investigated and configuration-dependant frequencies are analyzed. Furthermore, using Lagrange multipliers, the fully coupled equations of motions of the flexible parallel manipulator are developed by incorporating the rigid body motions with elastic motions. The mutual dependence of elastic deformations and rigid body motions are investigated from the analysis of the derived equations of motion. Open-loop simulation without joint motion controls and closed-loop simulation with joint motion controls are performed to illustrate the effect of elastic motion on rigid body motions and the coupling effect amongst flexible links. These analyses and results provide valuable insight into the design and control of the parallel manipulator with flexible intermediate links.

Thirdly, an active vibration control strategy is developed for a moving 3-PRR parallel manipulator with flexible links, each of which is equipped with multiple PZT control pairs. The active vibration controllers are designed using the modal strain rate feedback (MSRF). The amplification behavior of high modes is addressed, and the control gain selection strategy for high modes is developed through modifying the IMSC method. The filters are developed for the on-line estimation of modal coordinates and modal velocity. The second compensator is used to cut off the amplified noises and unmodeled dynamics due to the differentiation operation in the developed controller. The modal coupling behavior of intermediate links is examined with the modal analysis of vibrations measured by the PZT sensors. The error estimation of the moving platform is examined using the measurement of PZT sensors.

Finally, an active vibration control experimental system is built to implement the active vibration control of a moving 3-PRR parallel manipulator with three flexible links. The smart structures are built through mounting three PZT control pairs to each intermediate flexible link. The active vibration control system is set up using National Instruments LabVIEW Real-Time Module. Active vibration control experiments are conducted for the manipulator moving with high-speed, and experimental results demonstrate that the vibration of each link is significantly reduced.
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Nomenclatures

General Symbols

\( A_i \)  
base point of the \( i^{th} \) kinematic chain

\( B_i \)  
revolute joint point connecting the \( i^{th} \) slider and the \( i^{th} \) intermediate link

\( \overline{b}_i \)  
vector from \( B_i \) to \( C_i \)

\((b_{ix}, b_{iy})\)  
X and Y direction components of \( \overline{b}_i \)

\( b_x \)  
width of a PZT sensor

\( D \)  
damping matrix

\( C_i \)  
revolute joint point connecting the intermediate link and the moving platform

\( C_p \)  
capacitance of PZT sensors

\( d_{31} \)  
piezoelectric constant of PZT materials

\( E \)  
output matrix of active vibration control with PZT transducers

\( E^* \)  
coefficient matrix between \( G \) and \( G^* \)

\( E_i \)  
Young’s modulus of the \( i^{th} \) flexible links

\( \overline{e}_i \)  
vector from the center of the platform to the \( i^{th} \) vertex of the platform

\( E_p \)  
Young’s modulus of PZT materials

\((e_{ix}, e_{iy})\)  
X and Y direction components of \( \overline{e}_i \)

\( F_{ai} \)  
driving force on the \( i^{th} \) slider
\( F_i \) \quad \text{inertial force of the } i^{th} \text{ intermediate link}

\( F_p \) \quad \text{inertial force of the moving platform}

\( F^k_{pi} \) \quad \text{modal force vector produced by the } k^{th} \text{ actuator on the } i^{th} \text{ intermediate link}

\( F_{si} \) \quad \text{inertial force of the } i^{th} \text{ slider}

\( F_{fg} \) \quad \text{modal force from the coupling between rigid body motion and elastic motion}

\( f^0_{ij} \) \quad j^{th} \text{ modal natural frequency of the } i^{th} \text{ intermediate link without the effect of axial forces on lateral stiffness}

\( f_{ij} \) \quad j^{th} \text{ modal natural frequency of the } i^{th} \text{ intermediate link with the effect of axial forces on lateral stiffness}

\( f^i_u \) \quad \text{modal control force of the } i^{th} \text{ mode}

\( G \) \quad \text{control gain matrix of active vibration control with SRF}

\( G^* \) \quad \text{coefficient matrix between control input and state variable}

\( H \) \quad \text{constraint equations of closed-loop chains}

\( h_p \) \quad \text{height of a PZT transducer}

\( h_i \) \quad \text{1/2 height of the flexible linkage}

\( I_i \) \quad \text{second area moment of inertia for the } i^{th} \text{ intermediate link}

\( I_{ii} \) \quad \text{inertia of the } i^{th} \text{ linkage about its mass center}

\( I_p \) \quad \text{inertia of the moving platform about its center}

\( J \) \quad \text{cost function}

\( J_{Bi} \) \quad \text{concentrated rotational inertia at the revolute joint } B_i
$J_{Ci}$ concentrated rotational inertia at the revolute joint $C_i$

$J_{G1}$ derivative of $H$ w. r. t $\overline{\rho}$

$J_{G2}$ derivative of $H$ w. r. t $\overline{\beta}$

$J_{G3}$ derivative of $H$ w. r. t $\overline{X}_p$

$J_P$ Jacobian matrix relating the velocities of sliders to the velocities of the platform

$J_{\beta i}$ Jacobian matrix relating the rotation of linkages to the velocities of the platform

$K$ modal stiffness matrix

$K_f$ conventional modal stiffness matrix

$K_p$ modal stiffness matrix due to the effect of axial forces on lateral stiffness

$K_D$ derivative control gain

$K_I$ integral control gain

$K_P$ proportional control gain

$k_{ai}^k$ actuator constant of the $k^{th}$ PZT sensor on the $i^{th}$ intermediate link

$k_{di}^i$ control gain of the $i^{th}$ modal velocity feedback

$k_{di}^k$ control gain of the $k^{th}$ PZT actuator on the $i^{th}$ intermediate link

$k_{ij}^f$ $j^{th}$ modal stiffness of the $i^{th}$ intermediate link without the effect of axial forces on lateral stiffness

$k_{ij}^p$ $j^{th}$ modal stiffness of the $i^{th}$ intermediate link with the effect of axial forces on lateral stiffness
$k^i_p$  control gain of the $i^{th}$ mode feedback

$k^k_{si}$ sensor constant of the $k^{th}$ PZT sensor on the $i^{th}$ intermediate link

$L$ Lagrangian

$l_i$ length of $i^{th}$ intermediate linkages

$M$ modal mass matrix

$M^*$ mass matrix of a dynamic system

$m_{Bi}$ concentrated mass at the revolute joint $B_i$

$m_{Ci}$ concentrated mass at the revolute joint $C_i$

$m_i$ mass of the $i^{th}$ linkage

$m_{ii}$ $i^{th}$ modal mass

$m_{ij}$ $j^{th}$ modal mass of the $i^{th}$ intermediate link

$m_p$ mass of the moving platform

$m^k_{pi}$ bending moment produced by the $k^{th}$ actuator on the $i^{th}$ intermediate link

$m_{si}$ mass of the $i^{th}$ slider

$m_{sh}$ mass of shafts of flexible linkages

$P_{Bi}$ force applied at point $B_i$ of the $i^{th}$ intermediate link by $i^{th}$ the slider

$P_{xi}$ axial force in the $i^{th}$ intermediate link

$[P_x]_{cr}$ Euler’s bucking load for the $i^{th}$ intermediate link

$Q$ Generalized force vector

$q$ Generalized displacement vector
$Q$ total amount of charge generated on surfaces of a PZT sensor

$R$ projection matrix

$r$ number of the assumed modes

$T_L$ the kinetic energy of the $i^{th}$ linkage

$T_P$ kinetic energy of the moving platform

$T_S$ kinetic energy of the three sliders

$u$ control input vector

$V$ Potential energy of the manipulator system due to the deformation of links

$w_i(x)$ deformation of the $i^{th}$ intermediate link at the location $x$

$x$ distance from the arbitrary point to $B_i$ on the intermediate links

$\overline{X}_P$ position and orientation vector of the moving platform

$X_i = \{x_i, y_i, \beta_i\}$ position and orientation of the $i^{th}$ linkage

$(x_{ai}, y_{ai})$ $x$ and $y$ coordinate of the point $A_i$ in the global coordinate

$(x_{bi}, y_{bi})$ $x$ and $y$ coordinate of the point $B_i$ in the global coordinate

$(x_{ci}, y_{ci})$ $x$ and $y$ coordinate of the $i^{th}$ vertex of the moving platform

$(x'_{ci}, y'_{ci})$ $x$ and $y$ coordinate of the $i^{th}$ vertex of the moving platform in the local frame attached the moving platform

$(x_k^1, x_k^2)$ left and right end of the $k^{th}$ PZT actuator

$(x_p, y_p)$ $x$ and $y$ coordinate of the moving platform in the global coordinate

$(x_p', y_p')$ coordinates of the moving platform in the coordinate system attached to the moving platform
$V_{ai}^k$ voltage applied to the $k^{th}$ PZT actuator of the $i^{th}$ intermediate link

$V_{si}^k$ voltage produced by the $k^{th}$ PZT sensor of the $i^{th}$ intermediate link

$w_i$ width of the intermediate links

$w_p$ width of the PZT transducers

$y$ control output vector

$z$ state space vector

**Greek Symbols**

$\alpha_i$ angle between the $i^{th}$ linear guide and the X axis of the fixed frame

$\vec{\beta}$ vector composed of $\beta_i$

$\beta_i$ angle between the $i^{th}$ linkage and the $i^{th}$ linear guide

$\phi$ phase angle

$\varphi_p$ orientation angle of the moving platform

$\Phi$ constraint equations

$\Phi_m$ modal analyzer

$\Phi_q$ Jacobian matrix of $\Phi$ a w.r.t the generalized variable $q$

$\psi$ mode shape matrix

$\psi_{ij}$ $j^{th}$ mode shape function of the $i^{th}$ intermediate link

$\vec{\eta}$ vector composed of $\eta_{ij}$

$\eta_{ij}$ $j^{th}$ modal coordinate of the $i^{th}$ intermediate link

$\lambda$ the vector of Lagrangian multiplier
\[ \dot{\theta}_i \] rotation speed of the \( i^{th} \) linkage in the plane

\[ \bar{\rho} \] vector composed of \( \rho_i \)

\[ \bar{\rho}_{d} \] desired value of \( \bar{\rho} \)

\[ \rho_i \] translation of the \( i^{th} \) slider with respect to the home position

\[ \rho_{\Delta i} \] mass per unit length of the \( i^{th} \) intermediate link

\[ \omega_c \] compensator resonant frequency

\[ \omega_{i} \] \( i^{th} \) order modal frequency

\[ \omega_{jd} \] desired \( j^{th} \) natural frequency

\[ \omega_s \] structural natural frequency

\[ \xi_i \] damping ratio of the \( i^{th} \) mode

\[ \xi_i^* \] damping ratio of the \( i^{th} \) mode with active control

**Notations**

\((\bullet)^T\) transpose

\((\bullet)^{-1}\) inverse

\((\bullet)\) first order time derivative

\((\bullet)\) second order time derivative

\((\bullet)'\) first order spatial derivative

\((\bullet)''\) second order spatial derivative
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<td>ADC</td>
<td>analog/digital converter</td>
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<tr>
<td>AMM</td>
<td>assumed mode method</td>
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<td>DAC</td>
<td>digital/analog converter</td>
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<td>DC</td>
<td>direct current</td>
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<td>DOF</td>
<td>degree of freedom</td>
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<td>DSRF</td>
<td>direct strain rate feedback</td>
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<td>DSP</td>
<td>digital signal processing</td>
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<td>DAE</td>
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<td>finite element method</td>
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<td>GRFP</td>
<td>global rational fraction polynomial</td>
</tr>
<tr>
<td>GUI</td>
<td>graphical user interface</td>
</tr>
<tr>
<td>IMSC</td>
<td>independent modal space control</td>
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<tr>
<td>KED</td>
<td>kineto-elasto-dynamics</td>
</tr>
<tr>
<td>LPM</td>
<td>lumped parameter method</td>
</tr>
<tr>
<td>LTI</td>
<td>liner time-invariant</td>
</tr>
<tr>
<td>MSRF</td>
<td>modal strain rate feedback</td>
</tr>
<tr>
<td>ODE</td>
<td>ordinary differential equation</td>
</tr>
<tr>
<td>P</td>
<td>prismatic joint</td>
</tr>
<tr>
<td>PC</td>
<td>personal computer</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>--------------------------------------</td>
</tr>
<tr>
<td>PID</td>
<td>proportional and integral and derivative</td>
</tr>
<tr>
<td>PPF</td>
<td>positive position feedback</td>
</tr>
<tr>
<td>PVDF</td>
<td>polyvinylidene fluoride</td>
</tr>
<tr>
<td>PZT</td>
<td>Lead Zirconate Titanate</td>
</tr>
<tr>
<td>R</td>
<td>revolute joint</td>
</tr>
<tr>
<td>SRF</td>
<td>strain rate feedback</td>
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<tr>
<td>w.r.t</td>
<td>with respect to</td>
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Chapter 1

Introduction

In this introductory chapter, the motivation of this thesis is given and the major research topics relative to this work are reviewed and briefly discussed. The objective, outline, and contributions of the thesis are presented.

1.1 Thesis Motivation

In general, industrial robot manipulators can be classified into two types according to their configurations. The robot manipulator of the first type, as shown in Figure 1.1, is called a serial link manipulator. This type of manipulator consists of successive links, usually hinged at rotary joints which can be actuated in a coordinated fashion to position the end-effector. The other type of robot manipulator, as shown in Figure 1.2, is called a parallel link manipulator (Merlet [1], Tsai [2]). A parallel manipulator consists of two platforms, namely the base platform and the mobile platform. The base platform is fixed in space or attached to the end-effector of another robot. The mobile platform is movable with respect to the base platform. The two platforms are usually linked with six or three linear actuators. Generally speaking, serial link manipulators have the advantage of access to larger workspaces over parallel manipulators. However, parallel link manipulators provide higher strength/weight, stiffness/weight ratios and accuracy than serial manipulators. Moreover, parallel manipulators allow the actuators to be fixed to the base or to be located close to the base of the mechanisms, which minimizes the
inertia of the moving parts and which makes it possible to use more powerful actuators. Therefore, since the Stewart platform was proposed to provide rapid multi-axis motion for flight simulators by Stewart [3], parallel robot manipulators have gained growing interest for applications in various manufacturing industries, precision optics (Cash et al. [4]), nanomanipulation (Chung and Choi [5], Xu and Li [6]), and medical surgery (Xu and Li [7], Taylor and Stoianovici [8]).

On the other hand, traditional industrial robots are built to be massive in order to increase stiffness, as shown in Figure 1.3, and therefore move at speeds much lower than the fundamental natural frequency of the system due to the limitations in the drive motor output torque. The practical solution to this problem is to design and construct light weight manipulators, as shown in Figure 1.4, which are capable of moving swiftly. In contrast to the rigid manipulators, light
weight manipulators offer advantages such as higher speed, better energy efficiency, improved mobility and higher payload-to-arm weight ratio. However, at high operational speeds, inertial forces of moving components become quite large, leading to considerable deformation in the light links, generating unwanted vibration phenomena. Such manipulators are called flexible manipulators (Wang and Gao [9], Carlos et al. [10]). It is a challenging task to achieve high accuracy end-effector motion for flexible manipulators due to unwanted structural vibrations. Hence, elastic vibrations of light weight links must be considered in the design and control of the manipulators with link flexibility.

![A manipulator with rigid links](image1)

![A manipulator with flexible links](image2)

The presence of piezoelectric materials provides a promising solution to vibration suppression (Preumont [14], Clark and Saunders [15], Fuller et al. [16]). Piezoelectric materials have the potential to measure and control the vibration of distributed parameters system due, to their direct piezoelectric effect and converse piezoelectric effect. Direct piezoelectric effect means that piezoelectric materials respond to mechanical forces or pressures and generate electric charge. Conversely, an application of an electric field to piezoelectric materials can produce mechanical stress or strain, which is called the converse piezoelectric effect. Smart structures
with surface-mounted or embedded piezoelectric materials have received growing interests in vibration control of flexible structures in the various industry fields, including aerospace, car industry, machine tools, consumer products, and even civil engineering, due to the piezoelectric material properties, such as mechanical simplicity, small volume, light weight, large bandwidth, efficient conversion between energy and mechanical energy, and simple integration with various metallic and composite structures.

Considering the promising characteristics of parallel manipulators, and lightweight manipulators, parallel manipulators with light weight links are developed, to provide an alternative high-speed pick-and-place positioning mechanism to serial architecture manipulators, such as X-Y tables or gantry robots, designed for electronic manufacturing. The parallel manipulator presented in this research, as shown in Figure 1.5, is categorized as a 3-PRR type because it has three symmetric closed-loop chains, each of which consists of a prismatic joint (P), and two consecutive revolute joints (R) (Ren [11], Wang [12]). Due to its inherent stiffness and accuracy, and less inertia of moving links, this proposed mechanism can be used in high-speed and high-accuracy robotic applications as a planar positioning and orientation device (Gosselin [13]). Light weight links are used to better meet the demands of high speed and high acceleration placement. However, light weight members are more likely to deflect and vibrate due to the inertial forces and external forces such as those arising from actuators. Structural flexibility effects are much more pronounced at high operational speeds and accelerations. Active vibration control with smart structures provides a promising solution to the suppression of unwanted vibration. Smart structures are comprised of sensing, actuating, tuning, controlling and computing capabilities. These characteristics allow the smart structures to be employed to control
the static and elasto dynamic response of distributed parameter systems operation under variable service conditions in uncertain environments.

Figure 1.5 A 3-PRR parallel manipulator with flexible intermediate links

To address this concern, this thesis aims at the development of dynamic models to investigate the dynamic characteristics of these structural vibrations when the parallel robot moves with high speed and acceleration, and the development of effective vibration control methodologies to suppress these unwanted structural vibrations with the application of smart structures.

1.2 Literature Review

This thesis involves several research areas, such as parallel manipulators, dynamics of manipulators and mechanisms with link flexibility, and active vibration control with smart structures, etc. This section will review the research literature relative to these research topics.
1.2.1 Parallel Robot Manipulators

In the applications where high load carrying capacity, high-speed, and precise positioning are of paramount importance, it is desirable to have an alternative to conventional serial manipulators. In general, it is expected that manipulators have the end-effector connected to the ground via several chains having actuation in parallel, and therefore have greater rigidity and superior positioning capability. This makes the parallel manipulators attractive for certain applications and the last two decades have witnessed considerable research interest in this direction. The research efforts mainly involved inverse position kinematics, direct position kinematics, singularities, workspace and dexterity, and dynamics and control, etc. (Dasgupta and Mruthyunjaya [17]).

Significant achievements, which have driven research on parallel manipulators from its infancy into the status of a popular research topic, were reported by Earl and Rooney [18], and Hunt [19]. Work reported by Earl and Rooney [18] presented methods for the analysis and synthesis of the kinematic structures with both serial and parallel mechanisms. Hunt [19] studied the structural kinematics of parallel manipulators on the basis of screw theory. Fichter and McDowell [20] formulated the inverse kinematics equations of individual limbs of parallel manipulators and implemented the method on the Steward platform.

The direct position kinematics problem drew much attention in the research on the Stewart platform during the late 1980s and early 1990s. Lin et al. [21] formulated the closed-form solutions to the direct position kinematics using the input-output equations of spherical four-bar mechanisms. Nair and Maddocks [22] proposed a decomposition scheme which divides the direct position kinematics into two parts. One part is linear and design-dependant, the other
involves solving certain nonlinear design-independent equations. Wang and Chen [23] presented a numerical approach with nonlinear-equation-solving algorithms to obtain the direct position kinematics solution. A neural network solution was developed by Geng and Haynes [24] for the solution of the direct position kinematics of the Stewart platform manipulator.

Merlet [25] presented the extensive use of Grassman geometry to enumerate geometric singularity conditions in detail. An investigation was conducted to the force singularity of the Stewart platform in work (Gosselin and Angeles [26]). Gosselin [27] presented the important association of the conditioning of the static transformation with the stiffness of the Stewart platform. Tekeda and Funabashi [28] studied singularity and ill-conditioning in terms of a new transmission index and pressure angle, and pointed out the unusability of the ill-conditioned zones in the workspace.

A workspace analysis method of the Stewart platform is developed by Yang and Lee [29] to determine some particular sections of the positional workspace with constant orientation for very specialized structures of the manipulator. Luh et al. [30] presented a general formulation for the workspace and dexterity analysis of parallel manipulators in terms of rank-deficiency of the Jacobian of the constraints by incorporating inequality constraints of slack variables. The work reported by Masory and Wang [31] addressed the problem of determining workspace sections including the constraints of joint angle limits and leg interface.

Compared to the vast literature on the kinematics of the parallel manipulators, research reports on the dynamics and control of parallel manipulators are relatively few. Do and Yang [32] presented the inverse dynamics of the Stewart platform using the Newton-Euler approach. Liu et al. [33] developed Lagrangian equations of motion with simplified assumptions regarding the
geometry and inertia distribution of the parallel manipulator. A dynamic control strategy was proposed in work (Hatip and Ozgoren [34]) for the control for the Stewart platform assumed to be mounted on a ship and used as a motion stabilizer.

To summarize, the research on dynamics and control of parallel manipulators has accomplished much, with significant issues yet to be resolved. With the consideration of link flexibility, the dynamic formulation of parallel manipulators is more complicated and challenging.

1.2.2 Dynamic Modeling of Manipulators and Mechanisms with Link Flexibility

Flexible manipulators and mechanisms have important applications in space exploration, manufacturing automation, construction, mining, hazardous operations, and many other areas, due to their attractive advantages over the rigid-arm counterparts, such as smaller actuators and more maneuverability, higher speeds of operation, greater radio of payload to manipulator mass, lower mounting strength required and more compact link design, less material and power consumption, etc. However, lightweight members are more likely to deflect and vibrate due to the inertial and external forces. Flexibility effects are much more pronounced at high operational speeds. Therefore, the research interest in the dynamics and control of flexible manipulators and mechanisms has increased significantly in recent decades in order to fully exploit the potential offered by flexible manipulators, and to achieve less vibration and greater accuracy. Significant progresses have been made in many aspects during the past decades, as can be seen from the survey papers (Erdman and Sandor [35], Lowen and Jandrasits [36], Lowen and Chassapis [37], Shabana [38], Dwivedy and Eberhard [39], Book [40], Benosman and Vey [41]). Unfortunately, taking into account the flexibility of the arms, manipulators and mechanisms are highly nonlinear and exhibit coupled dynamics, which make the control problems of such systems difficult. Most
of reported works in this field addressed the manipulators and mechanisms with single flexible links, and much fewer experimental works were reported compared with the vast publications of theoretical formulation and numerical simulation. Compared with the dynamic model of flexible serial manipulators and four-bar mechanisms, research reports on dynamic models for parallel manipulators are rather few in number. Recently, with the consideration of link flexibility, Giovagnoni [42] presented a general approach for the dynamic analysis of flexible closed-chain manipulators using the principle of virtual work. Lee and Geng [43] developed a dynamic model of a flexible Stewart platform using Lagrange equations. Zhou et al. [44] established dynamic equations of flexible 3-Parallel-Revolute-joint-and-Spherical-joint (3PRS) manipulator for vibration analysis using the finite element method (FEM). A dynamic model was developed based on FEM for a 3-PRR planar parallel manipulator with flexible links in reported works (Piras et al. [45], Wang and Mills [46]).

Several research topics relevant to dynamic modeling are reviewed as following: the discretization of the continuous elastic deformation, dynamic stiffening and buckling, and one-pass method versus two-pass method, etc.

**Discretization of Continuous Elastic Deformation**

Manipulators and mechanisms with flexible links are inherently continuous dynamic systems with an infinite number of degrees of freedom. Their governing equations of motion are highly nonlinear, coupled, ordinary and partial differential equations, which are normally infeasible to solve. Therefore, the infinite degrees of freedom associated with such distributed parameter systems usually should be approximated by finite-dimensional models based on discretization techniques, such as the finite element method (FEM), the assumed mode method
(AMM), the lumped parameter method (LPM), and the transfer matrix methods (TMM). Among them, FEM and AMM are investigated more widely and deeply.

Finite Element Method

The dynamic model of mechanisms and manipulators with link flexibility must generally capture the mass, stiffness and damping characteristics of the links, and external loading. The finite element approach provides an easier and systematic modeling technique for complex mechanical systems and lays the groundwork for a general approach to the modeling of elastic mechanisms and manipulators.

Winfrey [47], Erdman [48], and Iman [49] were among the first investigators to apply the finite element method to elastic mechanism systems. Nath and Ghosh [50] presented a refined application of the finite element method in which the effects of the Coriolis, tangential and normal components of elastic acceleration were considered. In addition, the effects of distributed rigid body as well as elastic axial forces on the transverse vibrations of mechanisms were included. Cleghorn, Fenton and Tabrrok [51] presented a more refined application of FEM in which the need to model the mechanism as a series of instantaneous structures was eliminated. Turic and Midha [52] developed the generalized equations of motion for the dynamic analysis of elastic mechanism systems. Most recently, the dynamic equations based on FEM to model a planar parallel manipulator with three flexible links were established in our preliminary works (Piras et al. [45], Wang and Mills [46]).

In FEM, all the generalized coordinates are physically meaningful. However, the concepts of natural frequencies are not explicitly exhibited. Furthermore, all of the mathematical models
for flexible linkages involve a large number of degrees of freedom, and hence a large number of equations of motion must be solved, which leads to computational inefficiency and hence expensive computational costs. Therefore, its application to the design of controller based model is limited.

**Assumed Mode Method**

In the assumed mode method, the elastic deflection is described by an infinite number of separable harmonic modes. Since the first few modes are dominant in dynamics, the modes are truncated to a finite number of modal series, in terms of spatial mode eigenfunctions and time-varying mode amplitudes. There are several ways to choose link boundary conditions and mode eigenfunctions. Different kinds of mode shape functions can be obtained from different types of boundary conditions, such as clamped boundary conditions (Book [53]), pinned-pinned boundary conditions (Asada *et al.* [54]), and free-free boundary conditions (Baruh and Tadikonda [55]). Hastings and Book [56], and Barbieri *et al.* [57] have reported that if the beam-to-hub inertia ratio is very small (0.1 or less), the clamped condition yields better results compared with pinned boundary condition.

To obtain accurate dynamic equations of manipulators or mechanisms with link flexibility, another important issue is to determine boundary conditions, and then select the proper sets of modes for problems of elastic beams that undergo large rigid-body displacement. Bellezza *et al.* [58] presented a mathematical model for a flexible slewing beam with comparison of clamped and pinned boundary conditions at the root end. Low [59] developed experimental investigation of the boundary condition of slewing beams using a high-speed camera system, and experimental results suggested that exact natural frequencies were intermediate between the clamped and
pinned cases. Shabana [60] demonstrated that different sets of mode shapes and natural
frequencies associated with different sets of boundary conditions can be used to obtain the same
solution provided that the co-ordinate system is properly selected.

Experimental modal analysis (EMA) (Ewins [61]) is a potential way to identify boundary
conditions of manipulators and mechanism systems. EMA has been widely used in the
experimental identification of structural dynamic characteristics. An EMA formulation was
presented for a simply supported plate using piezoelectric actuators and sensors in the reported
work (Saunders et al. [62]). Wang et al. discussed the feasibility of modal testing with
piezoelectric transducers bonded to a cantilever beam (Wu et al. [63]). Natural frequencies and
damping ratios of a parallel robot were measured by Hardage and Wiens [64] using a hammer an
accelerometer when the robots is stationary at different configurations. However, experimental
identification of flexible manipulators and mechanisms has not been investigated thoroughly due
to the coupling effect between rigid body motion and elastic deformation (Midha [65]).

The AMM provides some physical insights, such as the concept of natural frequencies,
into the system. However, its generalized coordinates, i.e., the assumed harmonic modes, don’t
possess any explicit physical meanings. In addition, how to select the appropriate or best mode
eigenfunctions for a given flexible mechanism and manipulator system is not a clearly answered
problem.

*Lumped Parameter Method*

There are two kinds of lumped parameter methods which are used to describe the flexible
links. One is the lumped-mass method (Sadler and Sandaor [66]) in which each beam is
represented by a finite number of equally spaced point masses connected by massless, elastic
elements. The lumped mass is called a station, while the elastomer without mass is called field. The flexible mechanisms and manipulators are modeled by combining the stations (distributed joints and lumped masses) and fields (massless elastomers) into a composite system. Ge et al. [67] proposed a new lumped-mass model which is presented by combing both AMM and FEM. The other one of LPM is called the finite segment method which assumes that a flexible body is composed of a number of discrete rigid segments, which are connected by springs and/or dampers. The flexibility is simulated by the springs and dampers. The stiffness and damping coefficients are calculated according to the physical characteristics of the elements. These coefficients are then used in the analysis to simulate the flexibility of the links. Tosunoglu et al. [68] used this model for identification of inaccessible oscillations in n-link flexible robotic systems. Recently, Megahed and Hamza [69] presented a mathematical model for planar flexible link manipulators with rigid tip connections to revolute joints.

Compared with FEM and AMM, no special elements and mode shapes are needed in the lumped parameter method. Therefore, it is not as cumbersome as FEM and AMM. However, LPM is not suitable to the flexible links with complex geometrical shapes.

*Transfer Matrix Method*

In general, the transient response of flexible link system subjected dynamic loads can be obtained by the assumed modal superposition or the use of finite element models or lumped parameter models. In these methods, however, it is necessary to use a large number of nodes or modes resulting in a need for very large computers for their managements and regulation. In order to overcome these disadvantages, a distinctive alternative to these methods is the use of the transfer matrix method. With the mechanisms and manipulators modeled as an instantaneous
structure, transfer matrix techniques can be used to calculate the natural frequencies and corresponding mode shapes. These natural frequencies and mode shapes can then be used to with the modal superposition technique to calculate the response of the mechanism.

The transfer matrix method was used widely in structure mechanics and rotor dynamics of linear time invariant system. Dokanish [70] developed finite element-transfer matrix method to solve the problems of plate structure vibration analysis, in which the finite element technique and the transfer matrix technique were combined. Kitis [71] applied the transfer matrix method to investigate the dynamic response of elastic four-bar mechanisms by combining the lumped parameter model of flexible links. Rui and Lu [72] developed the discrete time transfer matrix method for modeling multi-flexible-body systems, in which chain multi-body systems, branched multi-body systems network multi-body systems have been discussed in detail.

Dynamic Stiffening and Buckling

Dynamic analysis of mechanical systems with link flexibility must properly take account of both gross body motion and concurrent small elastic deformation of the flexible links, and accurately include the important coupling effects existing between these two modes of dynamic behavior. One of the important coupling effects, so-called geometrical non-linearities, results from the variations in flexible body stiffness induced by inertial, internal constraint, and external loads, for the analysis of flexible links under the action of large external loads or with high-speed motion. Typical examples are buckling and dynamic stiffening (also called stress, geometric, and rotational stiffening). In particular, dynamic stiffening has been investigated for beams undergoing large rotational motion. Kane et al. [73] investigated the geometric stiffening of a rotating beam using non-linear beam theory. Yoo et al. [74] established dynamic equations of a
beam undergoing overall motions incorporating geometric stiffening. Piedboeuf and Moore [75] presented six methods of modeling a flexible rotating beam including geometric stiffening by taking into account foreshortening in the beam model. Behzad and Bastami [76] studied the effect of axial forces produced by the Poisson effect on the natural frequency of a shaft rotating about its longitudinal axis (not its transverse axis). Liu and Hong [77] put forward a criterion on inclusion of geometric stiffening term in dynamic motions of a flexible beam using an influence ratio. Since external loading and inertial forces can induce axial forces in beams, few research works include the effect of the dynamic stiffening term in modeling lateral deformation of flexible links of manipulators (Yang and Sadler [78]) and four-bar mechanisms (Cleghorn et al. [79]). However, the issue of dynamic stiffening in dynamic modeling of parallel manipulators with link flexibility has received little attention. Note that the axial forces in the links of parallel manipulators are larger compared with manipulators with an open-loop. When an axial force in a link is in compression and increases to the buckling limit, buckling occurs along the link transverse direction. The physical meaning of buckling is that the axial compressive force is greater than the elastic restoring force, and remains so with increasing deflection. At this stage, the natural frequency of the link becomes zero. Buckling of single beam has been investigated and is well understood. Bokaian [80] investigated the relative critical bulking load of a single-span beam with different boundary conditions, e.g. different combination amongst free, pinned, clamped, and sliding at both ends of the beam. However, much less works about buckling are reported in the design and dynamic analysis in four-bar mechanisms (Gong [81]), and especially in parallel manipulators.

**One-pass Method versus Two-pass Method**
The accurate and complete flexible multibody dynamics not only involves the effect of the rigid body motions or nominal motions on elastic deformations, but also includes the influence of elastic deformations on rigid body motion and elastic deformations of other flexible links. Amongst the published work which addresses modeling of flexible mechanical systems, especially closed-loop mechanism systems, the majority of the investigations assume that elastic deformations of the links of mechanical systems do not have a significant effect on the rigid body motions. Some research work has included the effect of the nominal or rigid body motions on the elastic motions by introducing the Coriolis and centrifugal forces (Turcic, and Midha [82]). With this assumption, the mechanical systems are modeled first as a rigid multibody system. Rigid body methodologies are applied to calculate the inertia and joint forces, and then these forces are applied to a linear elastic problem to determine the elastic motion. The complete motion of the system is obtained by superimposing the elastic motion and rigid body motion. Hence, this approach only models the effect of rigid body motion on elastic motion. This type of method, sometimes called a two-pass method, usually requires that the rigid body motions are prescribed or known a priori. It is valid and reasonable to model mechanism systems such as a flexible four-bar crank with a large flywheel at the crank (Turcic, and Midha [82]). The flywheel, by virtue of its large inertia, ensures that the rigid body motion is maintained independent of the elastic motion. Moreover, there are limitations in the case of using two-pass method to model lightweight and high-speed mechanisms and manipulators where control systems are incorporated. For example, in a joint motion control system, feedback measured from links or joints includes effects from both rigid body motion and elastic motion. The joint motion control is implemented based on this feedback, and hence the overall motion of the system is influenced not only by the rigid motion but also the elastic motion.
Few research works presented one-pass methods to model mechanical systems with link flexibility. The one-pass method takes into full account the dynamic coupling between the rigid body and elastic motions. Amongst the research works utilizing the one-pass method, little work has been reported addressing flexible mechanisms with closed-loop chains (Gasparetto [83], Nagarajan and Turcic [84], El-Absy and Shabana [85], Yang and Sadler [86], Karkoub and Yigit [87]), especially for flexible parallel manipulators, compared with flexible manipulators with open-loop chains (Huston [88], Usuro et al. [89], Lee [90]). Note that the system equations of motion become more complicated by incorporating joint constraint equations. These algebraic equations are then combined with the differential equations of motion using the Lagrange multiplier technique. Therefore, the final equations of motion become a mixed set of differential and algebraic equations (DAEs). The solution to these DAEs is more complex than the solution of ordinary differential equations. Moreover, it is cumbersome to derive equations of motion in a closed form with the constraints eliminated. For example, Nagarajan and Turcic [84] presented the formulation of the equations of motion the elastic mechanisms with mutual dependence between rigid body and elastic motions based on FEM. Karkoub and Yigit [87] investigated the dynamic modeling and controller design for a flexible four-bar mechanism using the one-pass method based on AMM. However, only one link is considered flexible, and therefore coupling between flexible links is not revealed.

1.2.3 Vibration Control of Flexible Manipulators and Mechanisms

Concurrent to the work on dynamic modeling as reviewed above, the investigation of vibration control of flexible manipulators and mechanisms has been undertaken by many researchers for several decades. Design or control strategies have been proposed to attenuate the
unwanted vibration of the flexible links. For example, links of mechanisms and manipulators were built with composite materials having inherently superior damping characteristics and higher stiffness to weight ratios (Ghazavi et al. [91], Sung and Thompson [92]). The vibration of mechanism was dissipated by introducing additional damping materials (El-Dannah and Farghaly [93], Sisemore et al. [94]). The vibration of mechanisms was attenuated through optimizing the cross-sectional geometrics of the mechanism links (Zhang et al. [95], Cleghorn et al. [96]). These three methods are usually referred to as passive vibration control. Ulbrich and Stein [97] presented a design of an additional electromechanical actuator, which is integrated into a four-bar mechanism at the proximal end of the follower link so that the deflection of the flexible linkages is reduced by controlling the additional actuator. Another alternate strategy advocates reduction of structural vibrations through controlling joint motions or torques based on input shaping (Singhose et al. [98], Shan et al. [99]), singular perturbation techniques (Carlos et al. [10], Siciliano and Book [100]), etc. However, the successful realization of joint motion or torque control schemes may be very difficult, if not impossible, to achieve due to hardware limitations. These limitations include saturation of the motor, signal noise from the sensor, and parameter variations. These difficulties many be mitigated by adopting the use of smart materials featuring distributed actuators and sensors fixed to flexible links to control the unwanted vibration.

1.2.4 Vibration Control with Smart Structures

In recent decades, “smart” or “intelligent” structures have attracted growing research interests in the vibration control of flexible structures. A smart structure involves four key elements: actuators, sensors, control strategies, and power conditioning electronics. Many types of actuators and sensors, such as piezoelectric materials, shape memory alloys, electro-strictive
materials, magneto-strictive materials, electro-rheological fluids, magneto-rheological fluids, and fiber optics, are being considered for various applications. The state of the art of smart structures was reviewed with details in the literature (Preumont [14], Clark and Saunders [15], Fuller et al. [16], Chopra [101], Alkhatib and Golnaraghi [102]).

Some of the most widely used smart sensors and actuators are made of piezoelectric materials. The most popular piezoelectric materials used in vibration control are lead-zirconate-titanate (PZT) which is a ceramic, and polyvinylidene fluoride (PVDF), which is a polymer. PZT is extensively used both as sensors and actuators because it requires lower actuation voltages, and can be used for a wide range of frequency.

The research on active vibration control of flexible structure using piezoelectric actuators and sensors was initiated by Bailey and Hubbard [103]. Early investigation in this area mainly involved the vibration control of the space-based flexible structures and simple flexible beams. Various control strategies have been developed for the active vibration control of simple flexible beams and space-based structures. The principal methods that can be found in the literature for controlling unwanted vibrations in flexible structures include positive position feedback (PPF) control (Goh and Caughey [104], Fanson and Caughey [105], Song et al. [106], Rew et al. [107], Moheimani et al. [108]), velocity feedback control or strain rate feedback (SRF) control (Song et al. [109], Sun et al. [110], Juang and Phan [111]), and resonant control (Pota et al. [112], Halim and Moheimani [113], Tjahyadi et al. [114]). The PPF control method was proposed by Goh and Caughey [104]. The basic idea of PPF control is to introduce a second-order auxiliary compensator (filter), which is forced by the position response of the structure. The positive output of the auxiliary system is then fed back as an input to the structure. The design of a PPF
controller requires that the natural frequencies of the vibration are known exactly, and do not vary with time. When these vibration frequencies are not known exactly or change with time, the performance of the PPF control will be adversely affected. Furthermore, any structural natural modes below the compensator frequency will experience increased flexibility. Another drawback of the PPF control is that it is not suited for structures with closely spaced vibration frequencies. The resonant control method proposed by Pota et al. [112] is based on the resonant characteristics of the flexible structure. The controller applies high gain at the resonant frequency which rolls off quickly away from the resonant frequency thus avoiding spillover. A limitation to this resonant control approach is the sensitivity of the method to the vibration frequency characteristic. Another shortcoming with resonant controllers is their limited performance in terms of adding damping to the structure. Both velocity feedback and SRF control have a much wider active damping frequency region, and therefore give the designer more flexibility than the PPF control and the resonant control. The selection of a precise compensator frequency for the SRF or velocity feedback control is not as demanding as the PPF control. Provided that the compensator frequency is larger than the structural frequency, some damping can be guaranteed. This property is suitable for flexible structures and systems with close vibration frequency components. Another attractive property of velocity feedback is that the existence of spillover does not destabilize the closed-loop system although velocity feedback allows for spillover. One of the problems with velocity feedback controls is that feedback controllers require the implementation of a differentiator, which will amplify noise or unmodeled vibration signals at higher frequencies. This problem can be solved by adding a compensator or filter to the control system, providing the closed-loop system with a high frequency roll-off.
A major difficulty in the vibration control of flexible structures results from the fact that flexible structures are commonly distributed parameter systems with an infinite number of vibration modes. However, in practice, only a limited small number of modes of the structures can be or are required to be controlled. Therefore, there exist uncontrolled modes which may lead to spillover, a phenomenon in which control energy flows to the uncontrolled modes of the system. Spillover may cause the destabilization of the closed-loop control system. One effective strategy to overcome the spillover problem is to design the vibration controller in the independent modal space with collocated PZT actuators and sensors. The IMSC method (Meirovitch and Baruh [115], Baz and Poh [116], Singh et al. [117], Baz et al. [118]) was developed to prevent from the spillover problem by controlling each mode separately. In the design of the IMSC, the modal control effort for each mode is calculated with only the respective displacement and velocity feedback for the corresponding mode. Independent displacement and velocity gains are selected as modal gains for each mode. That means an independent controller can be developed for each vibration mode. The significant disadvantage of the IMSC is that each mode requires its own sensor/actuator pair. In the design of a modal feedback controller, the modal coordinates and/or modal velocities for the modes targeted for control must be known. Therefore, real-time monitoring or sensing modal coordinate is significant in the vibration control of flexible structures. Various methods have been presented to extract the modal coordinates from the outputs of the sensors. These methods include state observers (Brogan [119]), temporal filters (Hallauer et al. [120]), and modal filters (Meirovitch and Baruh [121]). It has been shown that the use of observers causes observation spillover from the residual modes, which can destabilize the residual modes. Moreover, the design of observers can represent a serious problem for large-order systems, because few guidelines exist for the selection of the
observer gains. Using temporal filters, the outputs of sensors are processed using high-pass or low-pass filters to filter out the contribution of each mode. Such a method does not work well when the modes are closely spaced. Using modal filters, the task of extracting modal coordinates for the sensor outputs is distinct from the control task, which permits the use of modal filters in conjunction with any modal filter feedback control method. Furthermore, modal filters only involve spatial integration, which is a smoothing operation that cannot lead to instability. Two different modal filtering methods have been presented for modal filters. One is a modal filter with a distributed element (Collins et al. [122]). The other is a modal filter with discrete elements (Sumali et al. [123]). It is often difficult to implement modal filtering with a distributed element because it requires one sensor for each mode. Therefore, it is reasonable to perform modal filtering with discrete sensors in real time. Sensing modal coordinates in real time involves interpolations or curve fitting. All computations must be carried out within a single sampling period because the controllers are implemented in discrete time.

1.2.5 Application of Smart Structures to Manipulators and Mechanisms

Piezoelectric transducers have been introduced for the vibration control of manipulators and mechanisms. Most of early research work only involved the active vibration control of manipulators and mechanisms with a single link with a single bonded actuator and sensor. Compared with numerical simulation, few investigations have been made toward experimental investigations, especially for parallel manipulators due to their complicated dynamics. Shan et al. [124] detailed the experimental investigation of a single-link flexible manipulator with two PZT actuators. The residual vibration was effectively reduced based on the PPF control and the velocity feedback control. Experimental work reported by Liao and Sung [125] was conducted to
suppress the elastodynamic response of a flexible four bar mechanism. Only the rocker link was flexible, with the crank and coupler link rigid. The Elastodynamic vibrations were suppressed using one piezoceramic actuator and sensor pair bonded at the midpoint of the flexible rocker link. The controller was implemented based on LQG/LTR observer. Thompson and Tao [126] undertook an experimental study on the suppression of connecting rod vibration suppression of a slider-crank mechanism. Two sets of PZT control pairs were bonded to the flexible connecting rod, and temporal filters (band-pass filters) were used in the design of the controller. Preiswerk and Venkatesh [127] presented a finite-element analysis of active vibration control of a slider-crank mechanism and a six toggle mechanism with flexible links, with only simulation results provided. Little attention has been paid to multi-mode vibration control in the field of vibration control for moving manipulators and mechanisms with flexible links. Extending these works, our preliminary work (Wang [12]) presents the experimental and theoretical investigation of active vibration of a 3-PRR parallel manipulator with only one flexible intermediate link with two PZT actuators bonded to this flexible intermediate link. This preliminary work only involved the suppression of the residual vibration with only the first mode targeted for control.

Compared with simple beams, experimental implementation of the vibration control of moving flexible manipulators and mechanisms is a more challenging problem, especially for parallel manipulators. For the proposed manipulator, rigid body motion and elastic deformation are dynamically coupled. The characteristic leads to the coupling of vibration modes, and therefore may cause spillover. It is problematic to extract independent mode coordinates using only temporal filters, which are commonly used in the vibration control strategy of simple beams with fixed bases. In practice, controller design is based on the known or modeled dynamics.
However, the dynamic responses measured by PZT sensors include the unmodeled or unknown dynamics, including for example compliance and clearance dynamics from the bearings, ball screw mechanisms, and motors. Therefore, the vibrations of the intermediate links are very complicated, and are the combination of free structural vibrations and forced vibrations, which contain many frequency components which are closely spaced. To more precisely measure modal coordinates or velocities, it is desirable to use as many PZT sensors as possible. However, the number of PZT sensors is limited by available space on the flexible links, and the real-time capability of the computing platform. A trade-off must be made between the real-time capability of the controller computing hardware and the number of sensors and actuators.

### 1.3 Thesis Objectives

The overall objective of this thesis is to investigate the dynamics and vibration control of a planar parallel manipulator with link flexibility. The detailed objectives include:

1) To establish a dynamic model and investigate the structural vibration characteristics of a 3-PRR parallel manipulators with three flexible links, including the investigation of coupling effect between rigid body motions and elastic deformations, and the effect of longitudinal loads on the lateral vibration when the manipulator is moving with high-speed.

2) To develop effective active vibration control strategies to suppress the unwanted vibration of the 3-PRR parallel manipulator using multiple PZT transducers with the multiple modes targeted for control when the manipulator is moving at high speed and high acceleration.
3) To develop and build an active vibration control experimental system of the parallel manipulator with three flexible links, each of which is equipped with multiple PZT sensors and actuators, and to implement the active vibration control experiments to verify the proposed active vibration control strategy.

1.4 Thesis Overview

This thesis presents the dynamics and vibration control of a 3-PRR parallel manipulator with three flexible links. The details involve establishing dynamic equations, performing experimental modal tests, conducting numerical simulations, developing active vibration control strategies, designing an effective vibration controller with multiple modes targeted for control, developing an experimental system, and implementing active vibration control experiments. The outline of the remainder of the thesis is as follows:

Chapter 2 introduces the prototype and basic structure of a 3-PRR parallel manipulator developed in the Laboratory for Nonlinear Systems Control at the University of Toronto, and presents the kinematics and dynamics of this parallel manipulator to provide a theoretical foundation for the research work done in this thesis.

Chapter 3 develops the structural dynamic equations of motion for a 3-PRR parallel manipulator with three flexible intermediate links using the assumed mode method based on the pinned-pinned boundary conditions. Experimental modal tests are performed to validate the assumed boundary conditions. Numerical simulation results are given and analyzed.

Chapter 4 formulates dynamic stiffening and buckling of flexible linkages of a 3-PRR parallel manipulator by including the effect of longitudinal forces on the modal characteristics.
Chapter 1

Introduction

Numerical simulations and analyses of configuration-dependent frequency properties and axial forces are performed to illustrate the dynamic stiffening and buckling.

Chapter 5 presents coupling characteristics between rigid body motions and elastic deformations of a 3-PRR manipulator with three flexible intermediate links. Using Lagrange multipliers, the coupled dynamic equations of the flexible parallel manipulator are developed by incorporating rigid body motions with elastic motions. Open-loop simulation without joint motion controls and closed-loop simulations with joint motion controls are performed to illustrate the coupling effect between rigid body motions and elastic motions.

Chapter 6 addresses the dynamic modeling and active vibration control of a 3-PRR parallel manipulator with three flexible links with bonded PZT actuators and sensors. An active vibration controller is designed based on direct strain rate feedback control, and control forces from PZT actuators are derived in the modal space and incorporated to the dynamic equations of the manipulator system. Numerical simulations are performed and the results show that the proposed active vibration control strategy is effective.

Chapter 7 addresses the experimental implementation of active vibration control of a moving 3-PRR parallel manipulator with three flexible links, each of which is equipped with multiple PZT control pairs. A Modal Strain Rate Feedback (MSRF) controller is developed and designed using an IMSC strategy. The gain amplification problem for high modes is addressed. An efficient multi-mode control strategy has been proposed through modifying the IMSC method. The active vibration control system setup is built using National Instruments LabVIEW Real-Time Module. The software and hardware developed are introduced in detail. The simplified and efficient modal filter is developed to extract modal coordinates in real time and a second order
compensator is developed to filter amplified noises and unmodeled high frequency dynamics. Experimental results demonstrate that the first and second mode vibration of each link is significantly reduced. Experiments are conducted with different numbers of PZT control pairs bonded at the different locations of links. Comparison of these cases provides insight and guidance into the selection of the number and locations of PZT transducers. The position errors of the moving platform have been examined using measured voltages of PZT sensors.

Chapter 8 presents conclusions and future work. Appendix A presents the detailed expressions of the matrices relevant to dynamic equations established in this thesis. Appendix B addresses the bonding procedure of PZT transducers to the intermediate links. Appendix C introduces the building procedure of a desktop PC as a RT target PC. Appendix D gives NI 6733 Pinout, and Appendix E gives NI 6031 Pinout.

1.5 Thesis Contributions

The contributions achieved in this thesis include:

1) Structural dynamic equations of the proposed 3-PRR parallel manipulator with three flexible intermediate links have been developed based on the assumed mode method.

2) Coupling dynamic equations of the flexible parallel manipulator have been established with one-pass method, and coupling effect of rigid body motions and elastic deformations has been investigated. The investigation of the coupling effect provides insight into the design of the joint motion controller.

3) Dynamic stiffening and buckling behavior have been investigated by including the effect of longitudinal forces on lateral stiffness in the dynamic model of the parallel manipulator.
with flexible links. This investigation provides the valuable insights into the improvement of dynamic performance though optimizing the structure and motion of the proposed parallel manipulator moving with high-speed.

4) An efficient multi-mode control strategy has been developed for the active vibration control of the moving parallel manipulator with three flexible intermediate links, each of which is equipped with multiple bonded PZT actuators and sensors. The amplification behavior of modal controllers for high modes has been addressed based on PZT transducers, and an efficient strategy for determining the control gains of high modes are developed. A MSRF controller has been designed using the IMSC strategy. Simplified and efficient modal filterers have been developed to extract modal coordinates in real time, and a second order compensator is used to filter amplified and unmodeled high frequency dynamics.

5) The active vibration control experimental setup of the proposed parallel manipulator has been built up with three smart structures. The active vibration control software system has been developed using NI LabVIEW and implemented using two PCs with NI LabVIEW Real-Time Module and Two-CPU programming techniques to guarantee the execution of the vibration controller with multiple channels in real time.

6) The unwanted vibration of the parallel manipulator has been significantly suppressed experimentally using the developed MSRF controller, and provides the solid foundation for the practical application of parallel manipulators with link flexibility, with first two modes targeted for control. The motion error estimation of the moving platform has been presented using the measurements from the PZT sensors bonded to the flexible links with
the consideration of the shortening. Active vibration control experiments are also conducted with one, or two or three PZT control pairs applied on one intermediate link. Experimental results demonstrate that at least three PZT control pairs are required to design the modal filter and accurately extract modal coordinates, and hence effectively suppress the unwanted vibration of each intermediate link.
Chapter 2

Kinematics and Dynamics of a 3-PRR Planar Parallel Manipulator Prototype

2.1 Introduction

This chapter introduces a 3-PRR planar parallel manipulator test bed, including its mechanical structure and control system. The experimental work in this thesis is conducted on this parallel manipulator test bed, which was designed and assembled by the Laboratory for Nonlinear Systems Control (LNSC) at the University of Toronto. The design and assembly, and calibration of the test bed were a group effort in The LNSC. The concept design of the parallel manipulator was conducted by Kang [128], architecture and singularity analysis was performed by Heerah [129], and mechanical design was implemented by Yeung. The motion control system of the parallel manipulator was initially developed by Zhu [130], and then was improved by Ren [11]. The kinematics and dynamics of the parallel manipulator with rigid intermediate links are presented in this chapter since they are used in the dynamic modeling work of this thesis.

2.2 Mechanical Structure and Coordinate System of the Parallel Manipulator

The 3-PRR planar parallel manipulator developed at the LNSC is shown in Figure 2.1. In order to facilitate the formulation of kinematics and dynamics, a generalized coordinate system is defined in Figure 2.2.
Figure 2.1 Prototype of the 3-PRR parallel manipulator [11]

The parallel manipulator presented is categorized a 3-PRR type because it has three symmetrical kinematic chains, each of which has one active prismatic (P) joint, followed by two consecutive passive revolute (R) joints. Each active prismatic joint is implemented by a ball screw and linear guide mechanism THK KR3306 driven at $A_i, i = 1, 2, 3$, as shown in Figure 2.2, by an Aerotech BM200 DC brushless servo motor. The ball screw converts the rotation of motors into the translational motion of the slider along the linear guide, i.e., $\rho_i, i = 1, 2, 3$. Variable $F_{ai}$ is the driving force acted on the $i^{th}$ slider. The motion range of three sliders is 400mm. The origin of $\rho_i$ is selected at the center of each linear guide. Revolute joints collocated with the sliders,
at $B_i, i=1,2,3$, connect the slider with the intermediate linkages. The other ends of the linkages, $C_i, i=1,2,3$, connect the linkages with the moving platform, which constrains the motion of the links. The moving platform, i.e., the end-effector, is of a regular triangle shape, and moves in the plane with three degrees of freedom. Three intermediate links connect the sliders and the moving platform. Each intermediate link has identical geometric parameters. The base platform is fixed to the ground.

Figure 2.2 Coordinate system of the 3-PRR parallel manipulator
The 3-PRR parallel manipulator test bed is of aluminum and steel construction. The material of the platform and intermediate links is AA7075T6 (SAE215) with the density of $2.425 \times 10^3 \text{kg/m}^3$. The material of brackets (sliders) AS4140 with density of $7.689 \times 10^3 \text{kg/m}^3$. The other parts of the manipulator are made of steel. Other parameters of the manipulator are listed in Table 2.1.

<table>
<thead>
<tr>
<th>Items</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of the platform (mm)</td>
<td>$100.0 \times 30.0 \times 4.0$</td>
</tr>
<tr>
<td>Size of the base platform (mm)</td>
<td>$876.3 \times 101.6 \times 10.0$</td>
</tr>
<tr>
<td>Size of the intermediate links (mm)</td>
<td>$200.0 \times 30.0 \times 30.0$</td>
</tr>
<tr>
<td>Lead of the ball screw (mm)</td>
<td>6.0</td>
</tr>
<tr>
<td>Mass of the moving platform (kg)</td>
<td>4.045</td>
</tr>
<tr>
<td>Motor inertia (kg·m²)</td>
<td>$1.3 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

### 2.3 Motion Control System and Software

The three-axis motion control system is a PC/DSP based hybrid control system, as shown in Figure 2.3. The hardware architecture consists of a joint motion control PC, a MCX-DSP-ISA 120Mflop/sec DSP controller, three Aerotech BM200 DC brushless motors with peak torque $3.5 \text{N} \cdot \text{m}$, and three corresponding BA20 SineDrive amplifiers, three homing sensors, and six limit switches. The DSP control board is installed in the slot of the joint motion control PC. The DC brushless motors with their built-in rotary encoders and linear guides are fixed on the base platform. The resolution of the encoders is 0.75 $\mu\text{m}$. The joint motion controller is developed based on a PC/DSP architecture. Each two of six limit switches, mounted on opposite directions
of each slider, protect the slider from moving outside of the workspace limits. Three homing sensors are used to determine the origin of the manipulator. The joint motion control PC implements the high-level control while the DSP implements the low-level control, which is responsible for the execution of motion control algorithms.

Figure 2.3 Motion control system of the parallel manipulator [11]

Figure 2.4 Motion control software interface
The control system software is developed using Visual C++6.0 to provide the user with a high level software environment. The control algorithms are then compiled and downloaded to the DSP control board which runs continuously once the controller is started up. For trajectory tracking control of the 3-PRR manipulator, the developed Windows Based software, as shown in Figure 2.4, offers user-friendly interfaces. Through inputting parameters on the tracking control interfaces, as shown in Figure 2.5, the manipulator can exert tracking motions following different trajectories repetively, such as triangles, quadrilaterals, and circles, etc. For details refer to [130].

![Figure 2.5 Trajectory tracking control interface](image_url)
2.4 Kinematics of the Parallel Manipulator with Rigid Intermediate Links

The motion control system was developed based on the inverse kinematics. The motion of sliders and intermediate links is determined based on the prescribed motion of the moving platform.

As shown in Figures 2.2 and 2.6, the origin of the fixed frame is located at \( O \), which is the position of the center point \( P \) of the moving platform when \( \alpha_i = \beta_i \). Variable \( \alpha_i \) is the angle at \( A_i \) between the X-axis of the fixed frame and the \( i^{th} \) linear guide, and \( [\alpha_1, \alpha_2, \alpha_3] = [150^\circ, 270^\circ, 30^\circ] \). Variable \( \beta_i \) is the angle at \( B_i \) between the X-axis of
the fixed frame and the $i^{th}$ intermediate link. The position and orientation of the moving platform at its mass center can be written with respect to the reference X-Y frame as a vector $\overline{X}_p = (x_p, y_p, \varphi_p)^T$, the displacement of the sliders from their origin is expressed as a vector $\overline{\rho} = (\rho_1, \rho_2, \rho_3)^T$, and the position and orientation of the $i^{th}$ link at its mass center are written as $\overline{X}_i = (x_i, y_i, \beta_i)^T$, $i = 1,2,3$.

The parallel manipulator consists of three symmetric kinematic chains. For the $i^{th}$ chain, a loop close equation can be written in forms of vectors

$$\overline{OP} + \overline{PC}_i = \overline{OA}_i + \overline{A_iB_i} + \overline{B_iC_i}$$

(2.1)

The right-hand side of equation (2.1), the coordinates of $C_i$, is given as

$$x_{ci} = x_{ai} + \rho_i \cos \alpha_i + l \cos \beta_i$$

(2.2)

$$y_{ci} = y_{ai} + \rho_i \sin \alpha_i + l \sin \beta_i$$

(2.3)

where $x_{ai}$ and $y_{ai}$ are the coordinates of the origin of the $i^{th}$ slider, and $l$ is the length of the intermediate link.

![Figure 2.7 Schematic diagram of the moving platform coordinates](image)

From the left side of equation 2.1, the coordinates of $C_i$ can be written as
Chapter 2  Kinematics and Dynamics of a 3-PRR Planar Parallel Manipulator prototype

\[ x_{ci} = x_p + x_{ci}' \cos \varphi_p - y_{ci}' \sin \varphi_p \]  
\[ y_{ci} = y_p + x_{ci}' \sin \varphi_p + y_{ci}' \cos \varphi_p \]  
(2.4)  
(2.5)

\( x_{ci}' \) and \( y_{ci}' \) are \( x \) and \( y \) coordinates of \( C_i \) respectively, measured from the mass center of the platform, \( P \), when \( \varphi_p \) is zero, as shown in Figure 2.7.

From equations (2.2-2.5), a closed-form solution is expressed as

\[ \rho_i = M_i \pm \sqrt{l_i^2 - S_i^2} \]  
(2.6)

where \( M_i = (x_{ci} - x_{ai}) \cos \alpha_i + (y_{ci} - y_{ai}) \sin \alpha_i \), and \( S_i = (x_{ci} - x_{ai}) \sin \alpha_i - (y_{ci} - y_{ai}) \cos \alpha_i \).

It is clear that there are two possible solutions for each individual chain, and the manipulator can take on a maximum of eight configurations for a set of given coordinates of the platform. However, in the design motion control system, only one solution represents a reasonable solution in terms of the prismatic joint positions at the previous sampling time. Note, only if the argument of the square root in equation (2.6) becomes zero, close equation (2.6) have a unique solution. If the argument turns out to be negative, there is no solution to satisfy given kinematic requirements.

To obtain the velocity equations, we perform the derivative of equations (2.1) with respect to time

\[ (\dot{x}_p \vec{i} + \dot{y}_p \vec{j}) + \phi_p (\vec{k} \times \vec{e}_i) = \dot{\rho}_i \vec{a}_i + \dot{\beta}_i (\vec{k} \times \vec{b}_i) \]  
(2.7)

where \( \vec{a}_i = (\cos \alpha_i, \sin \alpha_i)^T \), \( \vec{b}_i \) is the positional vector from \( B_i \) to \( C_i \), \( \vec{e}_i \) is the positional vector from \( P \) to \( C_i \), and \( \vec{k} \) is unit vector orthogonal to both \( \vec{i} \) and \( \vec{j} \).

The left-hand side of equation (2.7) can be interpreted as summation of the velocity of the platform at its mass center and a velocity term due to the rotation of the platform. The right-hand
side of equation (2.7) is the summation of the velocity of the slider and the velocity term due to rotation of the link.

To remove the velocity term related to the rotational velocity of the link, \( \dot{\beta}_i \), in equation (2.7), we apply vector dot product on both sides of equation (2.7) by \( \vec{b}_i \)

\[
(\dot{x}_p\vec{i} + \dot{y}_p\vec{j}) \cdot \vec{b}_i + \phi_p(\vec{a}_i \times \vec{b}_i) \cdot \vec{k} = \dot{\rho}_i(\vec{a}_i \cdot \vec{b}_i)
\]  

(2.8)

From equation (2.8), we have

\[
\dot{\rho}_i = \frac{1}{a_i \cdot b_i} (b_{ix} \dot{x}_p + b_{iy} \dot{y}_p + e_{ix} b_{iy} - e_{iy} b_{ix}) (\dot{x}_p \quad \dot{y}_p \quad \phi_p)^T = J_p \dot{X}_p \]  

(2.9)

where \( b_{ix} \) and \( b_{iy} \) are the X and Y direction components of the positional vector from \( B_i \) to \( C_i \), and \( e_{ix} \) and \( e_{iy} \) are the X and Y direction components of the positional vector from \( P \) to \( C_i \).

Combining equation (2.9), for \( i = 1,2,3 \), into vector-matrix form, we have

\[
\dot{\rho} = \begin{bmatrix} \dot{\rho}_1 \\ \dot{\rho}_2 \\ \dot{\rho}_3 \end{bmatrix} = \begin{bmatrix} J_{p1} & \dot{x}_p \\ J_{p2} & \dot{y}_p \\ J_{p3} & \phi_p \end{bmatrix} = J_p \dot{X}_p
\]  

(2.10)

Equation (2.10) presents the velocity relationship between the sliders and the platform with the Jacobian matrix \( J_p \), and will be used to formulate acceleration equations and dynamic equations. The velocity of sliders can be expressed by the Jacobian matrix and the velocity of the platform.

Performing vector cross product to equation (2.7) by \( \vec{b}_i \), we have

\[
(\dot{x}_p\vec{i} + \dot{y}_p\vec{j}) \times \vec{b}_i + \phi_p(\vec{k} \times \vec{a}_i) \times \vec{b}_i = \dot{\rho}_i(\vec{a}_i \times \vec{b}_i) - \dot{\beta}_i^2 |\vec{b}_i|^2 \vec{k}
\]  

(2.11)

From equations (2.9) and (2.11), the angular velocity of the link is given as
\[ \beta_i = \frac{1}{|b_i|^2} \left( (b_{ix} - b_{iy} a_{iy} + e_{ix} b_{iy}) - (b_{iy} a_{ix} - b_{ix} a_{iy}) J_{\beta_i} \right) \left[ \begin{array}{c} \hat{X}_p \\ \hat{X}_p \end{array} \right] = \left[ J_{\beta_1}, J_{\beta_2}, J_{\beta_3} \right] \left[ \begin{array}{c} \hat{X}_p \\ \hat{X}_p \end{array} \right] \] (2.12)

Since the angular velocity of the \( i^{th} \) intermediate link is known, the linear velocity at its center is given as

\[ \left( \begin{array}{c} \hat{x}_i \\ \hat{y}_i \end{array} \right) = (\hat{x}_p \hat{i} + \hat{y}_p \hat{j}) + \phi_p (\hat{k} \times \hat{e}_i) - 0.5 \hat{\beta} \hat{k} \times \hat{b}_i \] (2.13)

Augmenting all velocity terms gives the following equation

\[ \left( \begin{array}{c} \hat{x}_i \\ \hat{y}_i \\ \hat{\beta}_i \end{array} \right) = \left[ \begin{array}{ccc} 1 + 0.5 b_{iy} J_{\beta_1} & 0.5 b_{iy} J_{\beta_2} & - e_{iy} + 0.5 b_{iy} J_{\beta_3} \\ - 0.5 b_{ix} J_{\beta_1} & 1 - 0.5 b_{ix} J_{\beta_2} & e_{ix} - 0.5 b_{ix} J_{\beta_3} \\ J_{\beta_1} & J_{\beta_2} & J_{\beta_3} \end{array} \right] \left( \begin{array}{c} \hat{X}_p \\ \hat{X}_p \end{array} \right) = \left[ J_{\beta_1}, J_{\beta_2}, J_{\beta_3} \right] \left( \begin{array}{c} \hat{X}_p \\ \hat{X}_p \end{array} \right) \] (2.14)

where the Jacobian matrix \( J_{\beta} \) represents the velocity relationship between the \( i^{th} \) linkage and platform.

Acceleration of the sliders can be formulated using a similar procedure. The acceleration of the \( i^{th} \) slider is expressed as

\[ \dot{\rho}_i = (\dot{\hat{b}}_i \cdot \hat{V}_{ci} + \dot{\beta}_i \hat{b}_i) / (\hat{a}_i \cdot \hat{b}_i) \] (2.15)

where \( \hat{V}_{ci} = (\hat{x}_p \hat{i} + \hat{y}_p \hat{j}) + \phi_p (\hat{k} \times \hat{e}_i) - \phi^2_p \hat{e}_i \).

The angular acceleration of the \( i^{th} \) intermediate link is given as

\[ \ddot{\beta}_i \hat{k} = (\dot{\hat{b}}_i \times \hat{V}_{ci} - \ddot{\rho}_i (\hat{b}_i \times \hat{a}_i)) / |\hat{b}_i|^2 \] (2.16)

The linear acceleration of the \( i^{th} \) intermediate link at its mass center is given in terms of the angular velocity and angular acceleration of the link derived as

\[ \left( \begin{array}{c} \ddot{x}_i \\ \ddot{y}_i \end{array} \right) = \ddot{\rho}_i \hat{a}_i + \ddot{\beta}_i (\hat{k} \times 0.5 \hat{b}_i) - 0.5 \hat{\beta}^2 \hat{b}_i \] (2.17)
Compared with the inverse kinematics, the forward kinematics of the parallel manipulator is much more complicated and challenging. This problem could be theoretically solved constructing and solving a high-order polynomial equation [1]. Since this thesis work is conducted based on the inverse kinematics, the details of the forward kinematics are not addressed here.

2.5 Dynamics of the Parallel Manipulator with Rigid Intermediate Links

Dynamics include inverse dynamics and forward dynamics. The inverse Dynamics of the parallel manipulator is to derive the driving forces or torques applied on the slides from the given motion of the moving platform.

Implementing the principle of virtual work to the parallel manipulator, all virtual work in the system must be zero, namely

\[
\delta \rho^T F_a + \delta \vec{X}_p^T F_p + \delta \rho^T F_s + \sum_{i=1}^{3} \delta \vec{X}_i^T F_i = 0
\]  

where \( \delta \vec{\rho} = J_p \delta \vec{X}_p \), \( \delta \vec{X}_i = J_{\beta_i} \delta \vec{X}_p \) are the virtual displacement of \( \vec{\rho} \), \( \vec{X}_p \), \( \vec{X}_i \), respectively, \( F_a = (F_{a1}, F_{a2}, F_{a3})^T \) is the actuation force acted on the slider, \( F_p = (-m_p \ddot{x}_p, -m_p \ddot{y}_p, -I_p \ddot{\omega}_p)^T \) is inertia force of the platform, \( m_p \) and \( I_p \) are the mass and the mass moment of inertia of the platform, respectively, \( F_s = (-m_{s1} \ddot{\theta}_1, -m_{s2} \ddot{\theta}_2, -m_{s3} \ddot{\theta}_3)^T \) is inertia force vector of the sliders, \( m_{s1} \) is the mass of the \( i^{th} \) slider, \( F_i = (-m_i \ddot{x}_i, -m_i \ddot{y}_i, -I_i \ddot{\omega}_i)^T \) is the inertia force vector of the \( i^{th} \) linkage, and \( m_i \) and \( I_i \) is the mass and the mass moment of inertia of the \( i^{th} \) intermediate link.

Equation (2.18) is rewritten as

\[
\delta \vec{X}_p^T (J_p^T (F_a + F_s) + F_p + \sum_{i=1}^{3} J_{\beta_i}^T F_i) = 0
\]  

(2.19)
Since $\delta X_p$ can be any virtual displacement, the outer parenthesis of equation (2.19) must be identically zero. Therefore, the actuation force vector is given as

$$F_a = -F_S - (J_T^p)^{-1}(F_p + \sum J_T^p F_i)$$  \hspace{1cm} (2.20)

Forward dynamics is to estimating motions of the parallel manipulator when actuating forces and an initial configuration of the manipulator are given. The forward dynamics of the manipulator is formulated as follows:

Equation (2.20) is rewritten as

$$\begin{bmatrix} m_p & 0 & 0 \\ 0 & m_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} \ddot{x}_p \\ \ddot{y}_p \\ \ddot{\phi}_p \end{bmatrix} + m_s J_T^p \begin{bmatrix} \ddot{\rho}_1 \\ \ddot{\rho}_2 \\ \ddot{\rho}_3 \end{bmatrix} + \sum J_T^\beta \begin{bmatrix} \ddot{\rho}_i \\ \ddot{\rho}_i \\ \ddot{\rho}_i \end{bmatrix} = J_T^p \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \end{bmatrix} = J_T^p \begin{bmatrix} F_{a1} \\ F_{a2} \\ F_{a3} \end{bmatrix}$$  \hspace{1cm} (2.21)

From equation (2.17), the acceleration vector of the sliders is expressed as

$$\begin{bmatrix} \ddot{\rho}_1 \\ \ddot{\rho}_2 \\ \ddot{\rho}_3 \end{bmatrix} = \begin{bmatrix} \frac{b_{1x}}{\alpha_1 \cdot \beta_1} & \frac{b_{1y}}{\alpha_1 \cdot \beta_1} & \frac{b_{1y} e_{1x} - b_{1x} e_{1y}}{\alpha_1 \cdot \beta_1} \\ \frac{b_{2x}}{\alpha_2 \cdot \beta_2} & \frac{b_{2y}}{\alpha_2 \cdot \beta_2} & \frac{b_{2y} e_{2x} - b_{2x} e_{2y}}{\alpha_2 \cdot \beta_2} \\ \frac{b_{3x}}{\alpha_3 \cdot \beta_3} & \frac{b_{3y}}{\alpha_3 \cdot \beta_3} & \frac{b_{3y} e_{3x} - b_{3x} e_{3y}}{\alpha_3 \cdot \beta_3} \end{bmatrix} \begin{bmatrix} \dddot{x}_p \\ \dddot{y}_p \\ \dddot{\phi}_p \end{bmatrix} + \begin{bmatrix} \beta_1^2 l^2 - \bar{\alpha}_1 \cdot \bar{\beta}_1 \phi_p^2 \\ \beta_2^2 l^2 - \bar{\alpha}_2 \cdot \bar{\beta}_2 \phi_p^2 \\ \beta_3^2 l^2 - \bar{\alpha}_3 \cdot \bar{\beta}_3 \phi_p^2 \end{bmatrix}$$  \hspace{1cm} (2.22)

From equation (2.15), we have

$$\begin{bmatrix} \dddot{x}_i \\ \dddot{y}_i \\ \dddot{\phi}_i \end{bmatrix} = \begin{bmatrix} a_{ix} & 0 & -0.5b_{iy} \\ a_{iy} & 0.5b_{ix} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dddot{\rho}_i \\ \dddot{\rho}_i \\ \dddot{\rho}_i \end{bmatrix} + \begin{bmatrix} -0.5b_{iy} \beta_i^2 \\ -0.5b_{ix} \beta_i^2 \\ 0 \end{bmatrix}$$  \hspace{1cm} (2.23)

From equations (2.15) and (2.16), we obtain
where \( d_i = \frac{b_{iy} a_{ix} - b_{ix} a_{iy}}{\bar{a}_i \cdot \bar{b}_i} \).

From equations (2.22) to (2.24), equation (2.21) can be rewritten

\[
\begin{pmatrix}
\{ \sum J^T_{B_i} \left[ \begin{array}{ccc}
  m_i & 0 & 0 \\
  0 & m_i & 0 \\
  0 & 0 & I_a
\end{array} \right] \left[ \begin{array}{ccc}
  a_{ix} & -0.5b_{iy} \\
  a_{iy} & 0.5b_{ix} \\
  0 & 1
\end{array} \right] \begin{pmatrix}
\frac{b_{ix}}{\bar{a}_i \cdot \bar{b}_i} \\
\frac{b_{iy}}{\bar{a}_i \cdot \bar{b}_i} \\
\frac{d_i b_{ix} - b_{iy}}{l^2}
\end{pmatrix}
\right] \\
\frac{b_{iy}}{\bar{a}_i \cdot \bar{b}_i} \\
\frac{b_{ix}}{\bar{a}_i \cdot \bar{b}_i} \\
\frac{d_i b_{iy} + b_{ix}}{l^2}
\end{pmatrix}
\begin{pmatrix}
\frac{b_{iy} e_{ix} - b_{ix} e_{iy}}{\bar{a}_i \cdot \bar{b}_i} \\
\frac{d_i (b_{iy} e_{ix} - b_{ix} e_{iy}) + \bar{b}_i \cdot \bar{e}_i}{l^2}
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\dot{x}_p \\
\dot{y}_p \\
\phi_p
\end{pmatrix}
\]

\[
= J^T_{P_i} \begin{pmatrix}
F_{a1} \\
F_{a2} \\
F_{a3}
\end{pmatrix} - m_i J^T_{P_i} \begin{pmatrix}
\frac{\beta^2 l^2 - \bar{e}_i \cdot \bar{b}_i \phi^2_p}{\bar{a}_i \cdot \bar{b}_i} \\
\frac{\beta^2 l^2 - \bar{e}_2 \cdot \bar{b}_2 \phi^2_p}{\bar{a}_2 \cdot \bar{b}_2} \\
\frac{\beta^2 l^2 - \bar{e}_3 \cdot \bar{b}_3 \phi^2_p}{\bar{a}_3 \cdot \bar{b}_3}
\end{pmatrix}
\]

(2.25)

Provided that the content of the parenthesis in the left-hand side of equation (2.25) is not singular, the acceleration of the platform, \( \begin{pmatrix} \ddot{x}_p & \ddot{y}_p & \ddot{\phi}_p \end{pmatrix} \), is calculated uniquely from equation
Chapter 2  Kinematics and Dynamics of a 3-PRR Planar Parallel Manipulator prototype

(2.25) at the initial time when an initial configuration of the platform and the actuation force are given, i.e., all parameters of matrix shown in equation (2.25) are known. Then, the velocity vector, \( (\dot{x}_p \quad \dot{y}_p \quad \dot{\phi}_p)^T \), and the position vector, \( (x_p \quad y_p \quad \phi_p)^T \), at the next time step, can be calculated by a numerical integral method such as the fourth order Runge-Kutta method, Adams-Moulton method (Adams-Bashfort), and Gear’s Method, etc. The acceleration of the platform, at the next time step, is calculated again from equation (2.25) based on the updated velocity vector and the position vector of the platform. The prescribed procedures are iterated until the final time results in simulation of the parallel manipulator under given actuating forces. Using the formulation of inverse kinematics, the positions, velocities, and accelerations of the slides and intermediate links can be calculated.

2.6 Summary

In this chapter, first, the mechanical structure and coordinate system of the proposed 3-PRR parallel manipulator prototype are introduced. Then, the development of the motion control system and software are addressed briefly. Finally, the kinematics and dynamics are presented for the 3-PRR parallel manipulator with rigid intermediate links, including the inverse kinematics, the inverse dynamics, and the forward dynamics.
Chapter 3

Dynamic Modeling of a 3-PRR Parallel Manipulator with Link Flexibility

3.1 Introduction

Considering the promising characteristics of parallel manipulators and light weight manipulators, a planar parallel manipulator with light weight links, as shown in Figure 3.1, is proposed to provide an alternative high-speed pick-and-place positioning mechanism to serial architecture manipulators, such as X-Y tables or gantry robots, designed for electronic manufacturing. Light weight links are used to better meet the demands of high speed and high acceleration placement. However, light weight members are more likely to deflect and vibrate due to the inertial forces and forces from actuators. Structural flexibility effects are much more pronounced at high operational speeds and accelerations of the end-effector. Therefore, we develop a dynamic model to investigate the dynamic characteristics of these structural vibrations, with the objective to develop an effective vibration control methodology that suppresses these unwanted structural vibrations.

Investigation on the dynamic modeling of manipulators and mechanisms with flexible links has been studied extensively since the 1970’s. Manipulators and mechanisms with flexible links are continuous systems with an infinite number of degrees of freedom and described by nonlinear, coupled, partial differential equations of motion. To design and implement a real-time
controller for joint motion and vibration suppression, the dynamic model formulation of flexible linkage manipulators and mechanisms has been carried out based on different discretization approaches of the flexible links. The most common approaches are the finite element method (FEM) (Piras and Cleghorn [45], Wang and Mills [46], Winfrey [47], Erdman et al. [48], Imam et al. [49], Nath and Ghosh [50], Cleghorn et al. [51], Turic and Midha [52]) and the assumed mode method (AMM) (Book [53], Asada et al. [54], Baruh and Tadikonda [55], Hustings and Book [56], Barieri and Ozguner [57], Bellezza et al. [58], Low and Lau [59], Shabana [60]). Early investigations mainly involved the modeling of flexible serial manipulators and four-bar mechanisms with detailed reviews in (Erdman and Sandor [35], Lowen and Jandrasits [36], Lowen and Chassapis [37], Shabana [38], Dwivedy and Eberhard [39], Book [40], Benosman and Vey [41]). Compared with the dynamic model of flexible serial manipulators and four-bar mechanisms, research reports on dynamic models for parallel manipulators are rather few in number. Recently, with consideration of link flexibility, Giovagnoni [42] presented a general approach for the dynamic analysis of flexible closed-chain manipulators using the principle of virtual work. Lee and Geng [43] developed a dynamic model of a flexible Stewart platform using Lagrange equations. Zhou et al. [44] established dynamic equations of flexible 3-PRS manipulator for vibration analysis using FEM. Based on AMM, Kang and Mills [131] presented a dynamic model of a 3-PRR planar parallel manipulator with flexible links. The intermediate links were modeled with pinned-free boundary conditions. However, modal identification tests (Wang and Mills [132], Zhang et al. [133]) showed that mode properties of the links match pinned-pinned boundary conditions.
Considering that the AMM is based on the dynamic behaviour of the system, i.e., natural frequencies and mode shapes, and the AMM has better computational efficiency compared with the FEM, the AMM is used to model link flexibility in this work. The flexibility of the manipulator is mainly concentrated in three intermediate links, as the rest of the structure is mechanically far stiffer. The vibration behavior is excited mainly by the driving forces from motors, the inertial forces, and the reaction due to the payload on the moving platform. It is assumed that the deformations of the intermediate links are small, relative to the length of the link, and hence can be modeled as linear elastic deformations. The intermediate links are modeled as Euler-Bernoulli beams with pinned-pinned boundary conditions. Each end of each flexible link is attached to bearing housings which have substantial rotational inertia. Here we model these structures, located at each end of each flexible links, as a concentrated rotational inertia. The effect of concentrated rotational inertia at both ends of each intermediate link is included in this model. Usually, the precise parametric modeling of a multi-link flexible robot is hard to obtain, and even if precise modeling is achieved, the model is problematic to use for real-time control as the models are very complex in nature. Many advanced control approaches are model based, implying that the dynamic parameters of the system model are incorporated into the control equations. A high order dynamic model typically implies a large number of dynamic parameters, leading to a controller which may require significant real time computation. Hence, a lower order model is preferred. Here, we neglect the axial deformation and the effect of longitudinal loads on lateral stiffness. Experimental modal tests are performed to validate the model developed in this work. The modeling errors and other structured uncertainty are considered as parameter perturbations. While not the subject of this work, adaptive and learning
control technologies may be adopted to implement the vibration control of flexible links and joint motion control, to compensate for these uncertainties.

Section 3.2 gives the coordinate system of the manipulator system and modeling of the flexibility for an individual intermediate link. The dynamic equations of motion modeling only the structural dynamics for the 3-PRR parallel manipulator with three flexible links is formulated in Section 3.3, based on the AMM, with the expressions of model coefficients detailed in the Appendix A. Results of numerical simulation and experimental modal tests are given in Section 3.4 and Section 3.5, respectively. Finally, a summary of this chapter is given in Section 3.6.

3.2 Modeling of the Structural Flexibility of an Individual Intermediate Link

The architecture and coordinate system for the parallel robot is defined Chapter 2, as shown in Figure 3.1. In this thesis, the AMM is used to model the structural flexibility of the intermediate links. To utilize the AMM, the elastic deflection of intermediate linkages, shown in Figure 3.1 by dotted lines, is described by an infinite number of separable harmonic modes. Since the first few modes dominate the dynamics, the modes are truncated to a finite number of modal series in terms of spatial mode eigenfunctions, \( \psi_j(x) \), and time-varying mode amplitudes, \( \eta_j(t) \). Therefore, according to the formulation of the AMM, flexible deformation of the \( i^{th} \) intermediate link can be expressed as,

\[
w_i(x,t) = \sum_{j=1}^{r} \eta_j(t)\psi_j(x) \quad i = 1,2,3
\]  

(3.1)

where \( \eta_j \) is the \( j^{th} \) mode coordinate of the \( i^{th} \) intermediate link, and \( \psi_j \) is the \( j^{th} \) mode shape of the \( i^{th} \) intermediate link.
Each intermediate link can be treated as an Euler-Bernoulli beam because the length of each link is much longer than its thickness. Each intermediate link has two rotary joints at two ends connecting to the platform and a slide. The platform and slides are significantly stiffer mechanical structures, and hence are assumed rigid. The pinned-pinned boundary condition is adopted for the intermediate links based on results in [132, 133], which showed that modal...
characteristics of the intermediate link match the pinned-pinned boundary condition. Therefore, position-dependant mode shape functions $\psi_j(\xi)$ are selected as in [87, 134],

$$\psi_j(x) = \sin(j \pi x / l_i) = \sin(j \pi \xi) = \psi_j(\xi) \quad j = 1, 2, \cdots, r$$

(3.2)

where $\xi = x / l_i$, $r$ is the number of selected assumed modes, and $x$ is the distance from the arbitrary point on the $i^{th}$ intermediate link to $B_i$.

Flexible generalized coordinates are defined as

$$\eta = [\eta_{11} \cdots \eta_{1r} \eta_{21} \cdots \eta_{2r} \eta_{31} \cdots \eta_{3r}]^T$$

(3.3)

where $\eta_{ij}$ is the $j^{th}$ mode coordinate of the $i^{th}$ intermediate link, $i = 1, 2, 3$, and $j = 1, 2, \cdots, r$.

Although the AMM is adopted to discretize the distributed dynamic system of the manipulator system with flexible links, the order the dynamic model obtained is generally too large to be used to implement a real-time control, due to significant computational burden. Many advanced control methods are model based, implying that the control law incorporates terms which are part of the dynamic model. If the dynamic model derived has a high order, this generally implies that there are many dynamic parameters. A model based controller, constructed from such a model, may be computationally intensive for real time control with a high control update rate. Moreover, the higher the order of the model, the more problematic it is to estimate its states. Therefore, from a practical view of point, it is indispensable to apply model order reduction techniques to reduce the order of the dynamic model developed by the AMM. There are two model order reduction methods commonly used, namely balanced truncation (Moore [135]) and modal residualization (Gangsaas and Norman [136]). The reduced order dynamic
model must be selected such that the higher frequency system dynamics omitted from the model have little energy, compared with the retained dynamics, and will not lead to a “spill-over” effect, destabilizing the system. This work focuses on dynamic modeling and validation of flexible link dynamics, hence the details of the model order reduction method used is not discussed. In the numerical simulations performed, the order of modes retained in the model are selected so that the vibration response of the flexible links from the first order mode is for example, two or three orders of magnitude larger than that of the modes omitted from the reduced order model. We assume that the modes of much higher frequency, omitted from the reduced order model, have little effect on the performance of the manipulator system, as they contain little energy.

3.3 Dynamic Equations of Motion for the Manipulator System

In this section, the general form of Lagrange’s equations is used to derive only the structural dynamic equations of motion of the 3-PRR parallel manipulator with three flexible intermediate links. The detailed formulation is provided below.

3.3.1 Kinetic Energy

The total kinetic energy of the manipulator system includes the kinetic energies of the sliders, intermediate links, and the platform.

The kinetic energy of the three sliders is written as

\[ T_s = \sum_{i=1}^{3} \frac{1}{2} m_i \dot{\rho}_i^2 \]  \hspace{1cm} (3.4)

where \( m_i \) is mass of the \( i^{th} \) slider, and \( \dot{\rho}_i \) is the linear velocity of the \( i^{th} \) slider.
With the consideration of concentrated rotational inertia at both joints (inevitable in the design, manufacture and assembly of real link structures), the kinetic energy of the three intermediate links is expressed as

\[
T_L = \sum_{i=1}^{3} \left[ \frac{1}{2} \rho_i \left( \dot{\rho}_i x_i^2 + (x_i \dot{\beta}_i + \dot{\omega}_i)^2 + 2 \rho_i (x_i \dot{\beta}_i + \dot{\omega}_i) \sin(\alpha_i - \beta_i) \right) \right] dx
+ \frac{1}{2} \left. J_{Bi} \left( \frac{d}{dt} \frac{\partial w_i}{\partial x} \right) \right|_{x=x_0}^2 + \frac{1}{2} \left. J_{Ci} \left( \frac{d}{dt} \frac{\partial w_i}{\partial x} \right) \right|_{x=x_i}^2
+ \frac{1}{2} m_{Bi} \dot{\rho}_i^2 + \frac{1}{2} m_{Ci} \left( \dot{\rho}_i^2 + (l_i \dot{\beta}_i + \dot{\omega}_i)^2 + 2 \dot{\rho}_i (l_i \dot{\beta}_i + \dot{\omega}_i) \sin(\alpha_i - \beta_i) \right)
\]  

(3.5)

where \( \rho_{Ai} \) is mass per unit length of the \( i^{th} \) link, \( \dot{\omega}_i \) is the time rate of change of bending deformation of the \( i^{th} \) intermediate link, \( \alpha_i, \beta_i \) are defined in Section 3.2, \( J_{Bi} \) and \( J_{Ci} \) are the concentrated rotational inertias at joints \( B_i \) and \( C_i \) of the intermediate links, and \( m_{Bi} \) and \( m_{Ci} \) are concentrated masses at joints \( B_i \) and \( C_i \).

In Equation (3.5), the model is linearized by noting that the amplitude of the vibrations is small compared to the length of the beam. Therefore, the total motion of the arbitrary point on the flexible intermediate links is the superposition of its rigid-body motion and its elastic motion.

The kinetic energy of the platform is expressed as

\[
T_p = \frac{1}{2} m_p (\dot{x}_p^2 + \dot{y}_p^2) + \frac{1}{2} I_p \dot{\phi}_p^2
\]  

(3.6)

where \( I_p \) is mass moment of inertia of the platform around the center point \( P \), \( m_p \) is the mass of the platform, \( \dot{x}_p \) and \( \dot{y}_p \) are the linear velocities along X-axis and Y-axis direction, respectively, and \( \dot{\phi}_p \) is angular velocity of platform around the point \( P \).
Therefore, summing the kinetic energies given in equations (3.4) to (3.6), the total kinetic energy of the system is

\[
T = \sum_{i=1}^{3} \frac{1}{2} \int \rho_i \left[ (\dot{x})^2 + \dot{\beta}_i \right] + 2 \dot{\phi}_i \sin(\alpha_i - \beta_i) \, dx + \frac{1}{2} J \left( \frac{d}{dt} \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} J c_i \left( \frac{d}{dt} \frac{\partial w}{\partial x} \right) \bigg|_{x=x_i} (3.7)
\]

\[+ \frac{1}{2} m_i \dot{\rho}_i^2 + \frac{1}{2} m_c (\dot{\beta}_i^2 + (l_i \dot{\beta}_i + \dot{w}_i)^2 + 2 \dot{\phi}_i (l_i \dot{\beta}_i + \dot{w}_i) \sin(\alpha_i - \beta_i) + \sum_{i=1}^{3} \frac{1}{2} m_{i_p} \dot{\phi}_i^2 + \frac{1}{2} m_{i_r} (\dot{x}_{i_r}^2 + \dot{y}_{i_r}^2) + \frac{1}{2} I_{i_r} \dot{\phi}_i^2
\]

### 3.3.2 Potential Energy

Since gravitational force is applied along the Z-direction, perpendicular to the X-Y plane, the potential energy due to gravitational force does not change during in-plane motions of the manipulator. Considering potential energy due to deformation of the links, the potential energy of the system is given as

\[
V = \frac{1}{2} \sum_{i=1}^{3} E_i I_{i_y} \int \left( \frac{\partial^2 w_i(x)}{\partial x^2} \right)^2 \, dx = \frac{1}{2} \sum_{i=1}^{3} E_i I_{i_y} \int \left( \frac{\partial \psi_i(\xi)}{\partial \xi} \right)^2 \, d\xi (3.8)
\]

where \(E_i\) elastic modulus of the \(i^{th}\) link, and \(I_i\) is second moment of area of the \(i^{th}\) link.

### 3.3.3 Lagrange’s Equation

This chapter focuses on the vibration characteristics of the manipulator system, and hence the rigid-motion dynamics of the manipulators system is given briefly in Chapter 2. It is assumed that the small amplitude, high-frequency structural vibrations of the manipulator have a negligible effect on its rigid-body motion, i.e., we adopt the Kineto-Elasto-Dynamics (KED)
assumptions common to much of the literature which addresses structural vibration (Lowen and Chassapis [37]). The structural deformations that the intermediate links undergo are assumed to be small, relative to the length of the links, permitting Euler-Bernoulli beam theory (Rao [134]) to be used. This assumption, validated in our experimental work, permits the deformations of the links to be modeled as linear deflections, and hence the dynamic model to describe these structural deflections is a linear elastic model. Therefore, the influence of the elastic deformation on the rigid-body motion is neglected, and the equations of motion are solved using the prescribed rigid-body motion. This assumption sometimes is referred to as the linear theory of elastic deformation. Thus, Lagrange’s equations are not formulated for rigid-motion generalized coordinates, only for flexible generalized coordinates of the manipulator system. The general form of Lagrange’s equations for flexible generalized coordinates is given as

\[
\frac{d}{dt} \frac{\partial(T-V)}{\partial \eta_j} - \frac{\partial(T-V)}{\partial \eta_j} = 0 \quad i = 1,2,3 \quad j = 1,2,\ldots,r
\]  

(3.9)

Substituting equations (3.7) and (3.8) into (3.9), we have

\[
\left( m_i \int_0^1 \psi_y^2 d\xi + \left( \frac{(j\pi)^2}{l_i^2} \left( J_{bi} \psi_y(0) \right)^2 + J_{ci} \left( \psi_y(l_i) \right)^2 \right) \right) \ddot{\eta}_y + \frac{(j\pi)^4 E}{I_i} \eta_y(t) \int_0^1 \left( \psi_y \right)^2 I_i(\xi) d\xi =

\]

\[ -m_i \ddot{\rho}_i \sin(\alpha_i - \beta_i) \int_0^1 \psi_y d\xi - m_i \dot{\beta} \int_0^1 \psi_y \dot{\xi} d\xi + m_i \ddot{\beta} \int_0^1 \psi_y \ddot{\xi} d\xi + m_i \ddot{\beta} \cos(\alpha_i - \beta_i) \int_0^1 \psi_y d\xi
\]

(3.10)

Equation (3.10) can be rewritten in matrix form as

\[
M\dddot{\eta} + K\ddot{\eta} = -M_\rho \dddot{\rho} - M_\beta \dddot{\beta} + F_{fg}
\]

(3.11)
where $m_i = \rho_{A_i} l_i$ is the mass of the $i$th intermediate link, $M$ is the modal mass matrix, $K$ is the structural modal stiffness matrix, $-M_{\rho} \ddot{\rho} - M_{\beta} \ddot{\beta}$ reflects the effect of rigid-body motion on elastic vibration of flexible links, and $F_{fg}$ is the modal force from the coupling between rigid-body motion and elastic motion. The detailed expressions for $M_{\rho}, -M_{\beta}$, and $F_{fg}$ are given in Appendix A. In equation (3.10), the effect of concentrated rotational inertia is clearly illustrated as the structural vibration frequencies of the flexible intermediate links decrease with an increase of concentrated rotational inertia at both ends, which is inevitable in the practical design, manufacture and assembly of such mechanisms.

### 3.4 Numerical Simulation and Analysis

Numerical simulations for the 3-PRR parallel manipulator with three flexible intermediate links are presented. In these simulations, a circular motion, as shown in Figure 3.1, is assigned as a desired trajectory for the mass center point of the moving platform with constant $\varphi_{\rho}$. The equations for the circle are:

- $x_p = 50 \cos(10\pi) - 50 (mm)$,
- $y_p = 50 \sin(10\pi)(mm)$,

and $\varphi_{\rho} = 45^\circ$ constant. The intermediate links are modeled as aluminum alloy with elastic modulus and mass density $E = 7.1 \times 10^9 \text{ N/m}^2$, $\rho = 2.77 \times 10^3 \text{ kg/m}^3$, respectively. Three intermediate linkages have identical geometric parameters. The length of each link is $200 mm$, cross-section width $2 mm$, and the height of cross-section $30 mm$. The concentrated rotational inertias at both ends of intermediate links are:

$$J_{Bi} = J_{Ci} = 5.815 \times 10^{-5} \text{ kg} \cdot \text{m}^2.$$

In these simulations, the first three modes are selected to model the structural flexibility of intermediate links, namely $r = 3$ in equation (3.1). Therefore, flexible generalized
coordinates in equations (3.11) are \( \eta = [\eta_{11}, \eta_{12}, \eta_{13}, \eta_{21}, \eta_{22}, \eta_{23}, \eta_{31}, \eta_{32}, \eta_{33}]^T \). The rigid-body motion of slides and intermediate linkages is derived from the given motion of the moving platform by solving inverse kinematics of the parallel manipulator. Substituting the rigid-body motion into equations (3.11), the equations become ordinary differential equations (ODEs). With the initial conditions \( \dot{\eta} = 0_{9\times1}, \ddot{\eta} = 0_{9\times1} \), the equations (3.11) are solved for the first two cycles of the moving platform using MATLAB ODEs solver *ode113* based on a variable order Adams-Bashforth-Moulton method [137].

![The deformation at the midpoint of intermediate links](image1)

**Figure 3.2** Deformation at the midpoint of intermediate links

Figure 3.2 shows the elastic vibration at the midpoint of intermediate links. The vibration amplitudes for three links are different from each other due to their different rigid-body motions. Figure 3.3 reveals that the amplitude of the first mode vibration of the first intermediate link is
two magnitudes larger than the amplitude of the second mode vibration. It is obvious that the first mode is sufficiently accurate to describe the vibration of the flexible intermediate link. Therefore, it is reasonable to reduce the number of vibration mode to small finite number, which it can be further reduced for real-time control. The effect of concentrated rotational inertia at both ends of the intermediate links is clearly illustrated in Figure 3.2 and Figure 3.3. The concentrated rotational inertia significantly decreases the vibration frequencies of flexible intermediate links. The power density of frequency spectra for the vibration response of the first intermediate link at the midpoint is shown in Figure 3.4 and Figure 3.5. It illustrates that the vibration centralizes at the frequencies of 10Hz, 20Hz, and 116.4 Hz for the case without concentrated rotational inertia, and 10 Hz, 20Hz and 70.5Hz for the case with concentrated rotational inertia. From equation (3.11), the vibration response includes free vibration and forced vibration from inertial and coupling forces. From equation (3.11), the $j^{th}$ mode vibration frequencies of the $i^{th}$ intermediate links can be written as

$$\omega_j = \left[\frac{(j \pi/l)^2 \sqrt{EI/\rho A}}{1 + \frac{(j \pi)^2}{l_i^2} \left[J_b \left(\psi_j(0)\right)^2 + J_c \left(\psi_j(l_i)\right)^2\right]/\rho_i l_i \int_0^1 \psi_j^2(\xi) d\xi}\right]^{\frac{1}{2}}.$$  

Therefore, the first three mode frequencies are calculated to be 116.4 Hz, 465.6 Hz, and 1047.5 Hz for the case without the effect of concentrated rotational inertia, and 70.5 Hz, 170.6 Hz and 280.8 Hz for the case with the effect of concentrated rotational inertia. That means the vibration with the frequency of 116.4 Hz in Figure 3.4 and the vibration with the frequency of 70.5 Hz in Figure 3.5, are free vibrations, and the vibrations with frequencies of 10 Hz and 20 Hz are forced vibrations. These are further confirmed by the investigation of the inertial and coupling forces applied on the intermediate link, as shown in Figures 3.6 to 3.9. Figure 3.6 shows the time history of the inertial and coupling forces applied on the first intermediate link expressed in the first mode space.
Frequency spectra for the inertial and coupling forces are given in Figures 3.7 to 3.9. It is illustrated that frequencies of these forces are 10, 20 and 30 Hz. The power density of 10 Hz is the largest, which is consistent with the frequency spectra of the vibration response as shown in Figure 3.4 and Figure 3.5. Figure 3.4 and Figure 3.5 also show that the first vibration frequency of the first intermediate link decreases from 116.4 Hz to 70.5 Hz due to inclusion of the effect of concentrated rotational inertia at both ends. Therefore, this rotational inertia will have a significant effect on the modal characteristics of the flexible manipulator system, and it is necessary to include concentrated rotational inertia at both ends of the intermediate links in the dynamic modeling of the flexible manipulator.

Figure 3.3 First three vibration modes of the 1st intermediate link
Chapter 3  Dynamic Modeling of a 3-PRR Parallel Manipulator with Link Flexibility

Figure 3.4 Frequency spectra of vibration for the 1st link

Figure 3.5 Frequency spectra of vibration for the 1st link

Figure 3.6 Inertial and coupling forces of the 1st mode for the 1st link

Figure 3.7 Frequency spectra of F1
3.5 Experimental Modal Test Validation

To validate the model developed in this work, experimental modal tests are performed using an impact hammer and an accelerometer to identify the mode shapes, frequencies, and damping ratios of flexible intermediate links. During these tests, the three drive motors are locked at their home positions, and the moving platform is at the center of the workspace. The impact hammer is used to hit link 1 at point 1, and the accelerometer is attached to the link sequentially from point 1 to point 9, as shown in Figure 3.10. Using an HP 35670A dynamic response analyzer with inputs from the hammer and accelerometer, the Frequency Response Function (FRF) for each test location is measured and recorded. For each accelerometer location, the test is performed 15 times, taking the average result. The FRFs measured at point 2 to point 5 are given in Figure 3.11. There are no vibrations that are measured at point 1 and point 9, which are the locations of the rotation joints $B_1$ and $C_1$, and the vibration at the midpoint 5 due to the second mode is almost zero as shown in Figure 3.10. These results coincide with the assumption of pinned-pinned boundary conditions used in the model presented this work. To further validate the dynamic model presented in this work, the measured FRFs are analyzed using
Experimental Modal Analysis (EMA) [18]. The first two mode shapes, frequencies and ratios are derived using the Global Rational Fraction Polynomial (GRFP) method, a standard frequency domain modal analysis approach. The first two mode shapes, shown in Figure 3.12 and Figure 3.13, illustrate that the mode shapes from experimental modal tests match very well the mode shapes using in the modeling presented this work. The measured frequencies are 76.6 Hz for the first mode, and 231.2 Hz for the second mode. The damping ratios are 0.057 for the first mode and 0.017 for the second mode. The first frequency measured is very close to the first frequency of the theoretical model given in equation (3.11). The difference between the theoretically predicted frequency of the second mode frequency is larger compared with the difference in the first mode frequency, but these modeling errors can be overcome by adopting vibration control methods, such as adaptive control and iterative learning control.

![Figure 3.10 Location of an accelerometer along the flexible link](image-url)
Figure 3.11 Magnitudes of frequency response functions with logarithmic scale

Figure 3.12 First mode shape
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3.6 Summary

In this chapter, a procedure for the generation of dynamic equations for a 3-PRR parallel manipulator with three flexible intermediate links is presented based on the assumed mode method. It is assumed that the deformations of the intermediate links are small, relative to the length of the links. This assumption permits the deformation of the links to be modeled as linear deflections, and hence the dynamic model to describe these structural deflections is a linear elastic model. The mode shape functions are selected by modeling intermediate links as Euler-Bernoulli beams with pinned-pinned boundary conditions. Considering the structure of the intermediate links, i.e., each end of each link is attached to bearing housings with substantial rotational inertia, the effect of concentrated rotational inertia, located at each end of each flexible link, is included in the model developed.

Figure 3.13 Second mode shape
Model reduction techniques are applied to the AMM developed in this work. Two reasons exist for such model order reduction. As discussed, many advanced control approaches are model based, and therefore, we must apply model order reduction techniques to reduce the order of the dynamic model developed with the AMM, to permit model based controllers to be used for real-time control. Secondly, reduced order models are often used for simulation purposes, reducing computational burden. The reduced order dynamic model must be selected such that the higher frequency system dynamics omitted from the model have little energy, compared with the retained dynamics, and will not lead to a “spill-over” effect, destabilizing the system. In numerical simulations given in this work, the first three modes are selected. We assume that the higher frequency modes, omitted from the reduced order model, have little effect on the dynamic behavior of the manipulator system, as they contain little energy.

Numerical simulations and experimental modal tests are performed to illustrate the effectiveness of the developed model through the comparison and analysis of mode characteristics of the manipulator system. Numerical simulations of vibration responses, coupling forces and inertial forces are presented. The corresponding frequency spectra analysis is performed using the Fast Fourier Transform (FFT). Experimental modal tests are performed using an impact hammer and an accelerometer to identify the mode shapes, frequencies, and damping ratios of flexible intermediate links. The mode shapes and frequencies, obtained from experimental modal tests, match very well the assumed mode shapes and frequencies obtained based on pinned-pinned boundary conditions obtained with the dynamic model developed this chapter.
Chapter 4

Dynamic Stiffening and Buckling Behavior

4.1 Introduction

Dynamic analysis of mechanical systems with link flexibility requires that both gross body motion and concurrent small elastic deformation of flexible links be accounted for, but also includes important coupling effects which exists between rigid body motion and flexible mode behavior. An important coupling effect results from the variations in flexible body stiffness induced by inertial forces, internal constraint forces, and external loads. This phenomenon arises under the action of large external loads or during high-speed motion. Typical examples of this behavior are buckling and dynamic stiffening (also called stress, geometric, and rotational stiffening). In particular, dynamic stiffening has been investigated on beams undergoing large rotational motion (Kane et al. [73], Yoo et al. [74], Pieboeuf and Moore [75], Behzad and Bastam [76], Liu and Hong [77]). However, fewer research reports on this subject have been published for four-bar mechanisms (Song [81]), and especially for parallel manipulators.

This chapter is mainly concerned with dynamic stiffening and buckling, providing a thorough investigation of the dynamic modeling of a 3-PRR parallel manipulator with three flexible intermediate links. The dynamic equations of the flexible parallel manipulator are developed based the assumed mode method. The effect of longitudinal loads on lateral stiffness is investigated considering the high-speed motion of the manipulator. Natural frequencies of bending vibration of intermediate links are derived as the functions of axial force and rigid-body
motion of the manipulator. Dynamic stiffening and buckling of intermediate links are investigated and configuration-dependent frequencies are analyzed. Simulation results further verify the theoretical derivations and analyses.

Section 4.2 formulates the dynamic equations of motion for the parallel manipulator with three flexible links including the effect of the axial forces on lateral vibrations. In Section 4.3, the dynamic stiffening and buckling behavior is investigated through the frequency analysis of the manipulator system. Numerical simulations and analyses are performed in Section 4.4. Section 4.5 offers the summary of this chapter.

4.2 Dynamic Equations with the Effect of Axial Forces on Lateral Stiffness

In this section, with the consideration of the effect of longitudinal forces on lateral stiffness, the structural dynamic equations of motion of the 3-PRR parallel manipulator with three flexible intermediate links are derived using the general form of Lagrange’s equations.

The total kinetic energy of the manipulator system was formulated as in Chapter 3, and therefore is not presented in this chapter.

4.2.1 Potential Energy Including the Effect of Axial Forces on Later Stiffness

The potential energy of the flexible manipulator system arises from two sources: the elastic deformation of flexible links and gravity. However, since gravitational force is applied along the Z-direction, perpendicular to the X-Y plane, the potential energy due to gravity is not included in this work.
This work investigates geometric stiffening and buckling of intermediate links, specifically, how the longitudinal loads acting on the intermediate links affect the lateral stiffness of the links. The total potential energy includes two terms. The first term, $V_1$, is the flexural strain energy. The second term, $V_2$, is the potential energy due to longitudinal loads along the flexible intermediate links undergoing bending deformation. The total potential energy of the system is given as

$$V = \frac{1}{2} \sum_{i=1}^{3} E_i I_i (w_i')^2 dx + \frac{1}{2} \sum_{i=1}^{3} P_{xi}(x)(w_i')^2 dx = V_1 + V_2$$

(4.1)

where $E_i$ elastic modulus of the $i^{th}$ link, $I_i$ second area moment of the $i^{th}$ link, $P_{xi}$ is the longitudinal loads in the $i^{th}$ link, $w_i'$, and $w_i''$. Variable $P_{xi}$ is positive when the longitudinal load is in tension, and $P_{xi}$ is negative when the longitudinal load is in compression (Rao [134]). $V_1$ is obtained in Chapter 3, and the detailed formulation is presented in Chapter 3.

To derive the potential energy $V_2$, the longitudinal force $P_{xi}$ in the $i^{th}$ link must be determined. The $i^{th}$ intermediate link and slider are separated from the manipulator system as shown in Figure 4.1. The left diagram represents the force analysis of the $i^{th}$ slider, and the right diagram corresponds to the force analysis of the $i^{th}$ intermediate link. $F_{ai}$ is the driving force applied on the slider $i^{th}$ link, and $P_{Bi}$ is the force applied at the point $B_i$ of the $i^{th}$ link by the $i^{th}$ slider.

The force balance equations of the $i^{th}$ slide and intermediate link can be written as
\[ F_{ai} + P_{Bi} = m_i \ddot{ \rho}_i \] (4.2)

\[ P_{xi} = P_{Bi} \cos(\beta_i - \alpha_i) + \rho_{Ai} [ \dot{ \rho}_i \cos(\beta_i - \alpha_i) x - \frac{\dot{ \beta}_i^2}{2} x^2 ] \] (4.3)

Then, the axial load applied on the \( i^{th} \) intermediate link at the location \( x \) is written as

\[ P_{xi} = (m_i \ddot{ \rho}_i - F_{ai}) \cos(\beta_i - \alpha_i) + \rho_{Ai} [ \dot{ \rho}_i \cos(\beta_i - \alpha_i) x - \frac{\dot{ \beta}_i^2}{2} x^2 ] \] (4.4)

Figure 4.1 Force analysis of the \( i^{th} \) slider and intermediate link

Equation (4.4) illustrates that the axial force on the \( i^{th} \) intermediate link changes with the configuration of the manipulator. This force may be compressive at some configurations, and extensive for others. The analysis of dynamic stiffening and buckling will be further discussed in Section 4.3.

Therefore, the potential energy \( V_2 \) in equation (4.1), is given as
Chapter 4  Dynamic Stiffening and Buckling Behavior

\[
V_2 = \frac{1}{2} \sum_{i=1}^{3} \int_0^l (m_i \ddot{r}_i - F_{ai}) \cos(\beta_i - \alpha_i) + \rho_{ai} \dot{r}_i \cos(\beta_i - \alpha_i) \dot{x} - \rho_{ai} \ddot{r}_i \frac{\beta_i^2}{x^2} \left( \frac{\partial w_i(x)}{\partial x} \right)^2 dx
\]

\[
= \frac{1}{2} \sum_{i=1}^{3} \frac{1}{l_i} (m_i \ddot{r}_i - F_{ai}) \cos(\beta_i - \alpha_i) \int_0^l \sum_{j=1}^{r} \eta_2 (t)(\eta_2' (\xi))^2 d\xi
\]

\[
+ \frac{1}{2} \sum_{i=1}^{3} \rho_{ai} \dot{r}_i \cos(\beta_i - \alpha_i) \int_0^l \sum_{j=1}^{r} \eta_2 (t)(\eta_2' (\xi))^2 d\xi
\]

\[
- \frac{1}{4} \sum_{i=1}^{3} \rho_{ai} l_i \beta_i^2 \int_0^l \sum_{j=1}^{r} \eta_2 (t)(\eta_2' (\xi))^2 d\xi
\]

(4.5)

The flexural stain energy \( V_1 \) in equation (4.1) depends on geometric parameters and material properties of intermediate links, but is not configuration-dependent. In contrast, equation (4.5) illustrates that the potential energy \( V_2 \) in equation (4.1) not only depends on geometric parameters and the material properties of intermediate links, but is also configuration-dependent.

Substituting equations (4.5) into equation (4.1), the total potential energy of the parallel manipulator with flexible intermediate links can be calculated. Combining the kinetic energy derived in chapter 3 and the potential energy, the dynamic equations of the flexible parallel manipulator system can be derived using Lagrange's equation.

4.2.2 Lagrange’s Equation

The generalized coordinates for the manipulator with flexible intermediate links include rigid-body motion generalized coordinates and flexible generalized coordinates. However, this chapter focuses on the effect of dynamic stiffening on the vibration characteristics of the manipulator system, and hence the rigid-motion dynamics of the manipulators system is neither derived nor discussed here. It is assumed that the small amplitude, high-frequency structural vibrations of the manipulator have a negligible effect on its rigid-body motion, i.e., we adopt the Kineto-Elasto-Dynamics (KED) assumptions, common to much of the literature which addresses
structural vibration (Lowen et al. [37]). Therefore, the influence of the elastic deformation on the rigid-body motion is neglected, and the equations of motion are solved using the prescribed rigid-body motion. Thus, Lagrange’s equations are not formulated for rigid-motion generalized coordinates, only for flexible generalized coordinates of the manipulator system. Therefore, the formulation of Lagrange’s equations for the flexible generalized coordinates is provided in detail below,

\[
\frac{d}{dt} \left( \frac{\partial (T-V)}{\partial \ddot{\eta}_j} \right) - \frac{\partial (T-V)}{\partial \dot{\eta}_j} = 0 \quad i = 1,2,3 \quad j = 1,2,\ldots,r
\]  

(4.6)

Substituting the calculated kinetic and potential energy into equation (4.6), we have

\[
\begin{align*}
&\left( m_i \int_0^1 \psi_g^2 d\xi \right) \ddot{\eta}_j + \left( \frac{E_i}{l_i} (\psi_g')^2 l_i (\xi) d\xi \right) \eta_j + \left( \frac{1}{l_i} (m_{\alpha_i} \ddot{\rho}_i - F_{\alpha_i}) \cos(\beta_i - \alpha_i) \right) \int_0^1 (\psi_g')^2 d\xi \\
&+ \rho_{\alpha_i} \ddot{\rho}_i \cos(\beta_i - \alpha_i) \int_0^1 \xi (\psi_g')^2 d\xi - \frac{1}{2} \rho_{\alpha_i} l_i \beta_i^2 \int_0^1 \xi^2 (\psi_g')^2 d\xi \right) \eta_j = -m_i \ddot{\rho}_i \sin(\alpha_i - \beta_i) \int_0^1 \psi_g d\xi \\
&- m_i l_i \beta_i \int_0^1 \psi_g \xi d\xi + m_i \rho_{\alpha_i} \beta_i \cos(\alpha_i - \beta_i) \int_0^1 \psi_g d\xi 
\end{align*}
\]  

(4.7)

\( i = 1,2,3 \quad j = 1,2,\ldots,r \)

Equation (4.7) can be rewritten in matrix form as

\[
M \dddot{\eta} + (K_f + K_p) \dddot{\eta} = -M_{\rho} \dddot{\rho} - M_{\beta} \dddot{\beta} + F_{fs}
\]  

(4.8)

where \( M \) is the modal mass matrix of the parallel manipulator system, \( K_f \) is the conventional modal stiffness matrix, \( K_p \) is the modal stiffness matrix due to the effect of axial forces on lateral stiffness, \(-M_{\rho} \dddot{\rho} - M_{\beta} \dddot{\beta}\) is the modal force vector caused by the effect of rigid-body motion on elastic vibration of the flexible links, \( F_{fs} \) is the modal force vector from the coupling
between rigid-body motion and elastic motion, and \( m_i = \rho_i l_i \) is the mass of the \( i^{th} \) intermediate link. Detailed expressions for \( M, M_\rho, M_\beta \), and \( F_{fs} \) are given in Appendix A.

### 4.3 Dynamic Stiffening and Buckling of Intermediate Links

The axial forces on the intermediate links of parallel manipulators become significant due to high-speed motion and high payload. Therefore, the effect of axial forces on lateral stiffness is not negligible. In this section, the analysis of dynamic stiffening and buckling of intermediate links is conducted based on properties of the natural frequencies of the parallel manipulator.

#### 4.3.1 Link Mode Vibration Frequency

From equation (4.8), it is clear that the solution to flexible modal coordinates \( \vec{\eta} \) includes two parts: free vibration and forced vibration. The generalized force, i.e., the right hand side of equation (4.8), has no impact on the modal properties of the elastic deformation of the manipulator, and only affects the vibration amplitude. Therefore, evaluation of the modal characteristics only involves the homogeneous part of equation (4.8), given as

\[
M \ddot{\vec{\eta}} + (K_f + K_p)\vec{\eta} = 0
\]

(4.9)

where \( M, K_f \), and \( K_p \) are diagonal matrices. The equation of motion of the \( j^{th} \) mode for the \( i^{th} \) intermediate link can be expressed,

\[
m_i \ddot{\eta}_i + (k_f + k_p)\eta_i = 0
\]

(4.10)

where
Chapter 4  Dynamic Stiffening and Buckling Behavior

\[ m_{ij} = \rho_{Ai} l_i \int \psi_{ij}^2 d\xi \]  
(4.11)

\[ k_{ij}^f = \frac{E_i}{l_i^3} \int l_i (\xi) \psi_{ij}^{\kappa^2} d\xi \]  
(4.12)

\[ k_{ij}^p = \frac{1}{l_i} (m_n \ddot{\psi}_{ij} - F_{\text{long}}) \int \psi_{ij}^2 d\xi + \rho_{Ai} \ddot{\psi}_{ij} c_i \frac{1}{2} m_i \dot{\psi}_{ij}^2 \int \psi_{ij}^2 \xi d\xi - \frac{1}{2} m_{ij} \beta^2 \frac{1}{2} \int P_{\text{long}} \psi_{ij}^{\kappa^2} (x) dx \]  
(4.13)

Then, the natural frequency of the \( j^{th} \) mode for the \( i^{th} \) intermediate link, \( f_{ij} \), is given,

\[ f_{ij} = \frac{1}{2\pi} \sqrt{\frac{k_{ij}^f + k_{ij}^p}{m_{ij}}} \]  
(4.14)

4.3.2 Dynamic Stiffening

The conventional formulation for the natural frequency of the \( j^{th} \) mode for the \( i^{th} \) intermediate link, \( f_{ij}^0 \), is given by neglecting the effect of longitudinal loads acting on the links, and assuming the cross-section of the \( i^{th} \) intermediate link is constant, hence we have

\[ f_{ij}^0 = \frac{1}{2\pi} \sqrt{\frac{k_{ij}^f}{m_{ij}}} = \frac{\pi (l_i/j_i)^2}{2} \frac{E_i l_i}{\rho_{Ai}} \]  
(4.15)

Equation (4.15) shows that the conventional natural frequency \( f_{ij}^0 \) is a function of geometric parameters \( l_i \) and \( J_i \), and material parameters \( E_i \) and \( \rho_{Ai} \). Hence, the natural frequency is independent the rigid-body motion of the manipulator system, and therefore doesn’t change with the configuration of the manipulator system. The effect of geometric stiffening is clearly shown in equation (4.14) as a result of the potential energy due to longitudinal loads along the
flexible intermediate links undergoing bending deformation. Equations (4.13) and (4.14) reveal that the frequency, \( f_{ij} \), is not only a function of \( l_i, I_i, E_i, \) and \( \rho_{Ai} \), but also a function of rigid-body motion, \( \rho_i, \beta_i \), and driving forces acting on the sliders, \( F_{ai} \), and axial loads acting on intermediate flexible links, \( P_{si} \). Therefore, the actual frequency, \( f_{ij} \), is configuration-dependent due to inclusion of the effect of longitudinal forces. From equation (4.14), we also find that the effect of longitudinal forces on \( f_{ij} \) increases with the speed and acceleration of the rigid-body motion of the manipulator system. Variable \( f_{ij} \) increases (stiffness increases) and leads to stiffening when \( P_{si} \) is positive, and decreases (stiffness decreases) and may cause buckling when \( P_{si} \) is negative. The axial forces of intermediate links changes with configuration. They may be in tension at some configurations, and in compression at other configurations.

These results provide a mechanism to optimize or control the motion of the manipulator to increase the stiffness of intermediate links and decrease unwanted vibration. For example, for the end-effector motion, the configurations of the manipulator may be optimized so that the axial forces in the flexible links are in extension to increase the stiffness. It can be concluded that in the modeling and control of the manipulator with flexible links, the effect of longitudinal loads on lateral stiffness should be included when the manipulator moves with high-speed or undertakes large payloads.

### 4.3.3 Buckling Analysis

With the general analysis of the elastic stability analysis, buckling occurs when axial forces are greater than the elastic restoring forces of the intermediate links. Specifically, the \( i^{th} \) intermediate link will buckle if the axial force is compressive and of such magnitude that the
natural frequency of the first mode shape, calculated from equation (4.14), is purely imaginary. It is known that for a pinned-pinned link or beam, e.g., the $i^{th}$ intermediate link, the critical axial load or Euler’s buckling load is given (Gong [81]),

$$[P_x]_{cr}^i = \frac{\pi^2 E_i I_i}{l_i^2}$$  \hspace{1cm} (4.16)

To prevent the $i^{th}$ intermediate link from buckling, its maximum compressive axial forces must be

$$|P_{xi}|_{\text{max}} < [P_x]_{cr}^i$$  \hspace{1cm} (4.17)

Therefore, geometric parameters of intermediate links, payload, and motion velocity and acceleration should be determined according to equation (4.17) in the design and dynamic modeling of the parallel manipulator with flexible intermediate links.

### 4.4 Numerical Simulations and Analysis

Numerical simulations for the 3-PRR parallel manipulator with three flexible intermediate links are presented. In these simulations, a circular motion is used as a desired trajectory for the mass center of the moving platform with constant orientation $\varphi_\rho$. The equations for the trajectory are $x_\rho = 50\cos 2\pi ft - 50(mm)$, $y_\rho = 50\sin 2\pi ft (mm)$, and $\varphi_\rho = 45^\circ$ constant. Here, $f$ is defined as the frequency of the mass center moving along the described circular trajectory. The intermediate links are modeled with an aluminum alloy having Young’s modulus and mass density $E = 7.1 \times 10^{10} \text{ N/m}^2$ and $\rho = 2.77 \times 10^3 \text{ kg/m}^3$, respectively. The three intermediate
linkages have identical geometric parameters. The length of each link is 200 mm, cross-section width 2 mm, and the height of cross-section 30 mm.

This chapter focuses on the study of dynamic stiffening and buckling by including the effect of longitudinal loads on lateral vibration frequency properties of the intermediate links. It is assumed that the small amplitude, high-frequency structural vibrations of the manipulator have a negligible effect on the robot rigid-body motion, i.e., we adopt the KED assumptions common to much of the literature which addresses structural vibration. Therefore, the influence of the elastic deformation on the rigid-body motion is neglected to simplify our analysis. Flexible generalized coordinates $\vec{\eta}$ are obtained by solving equation (4.8) using the prescribed rigid-body motion. In the numerical simulations performed, the order of modes retained in the model is selected so that the vibration response of the flexible links from the first order mode is for example, two or three orders of magnitude larger than that of the modes omitted from the reduced order model. We assume that the modes of much higher frequency, omitted from the reduced order model, have little effect on the dynamic behavior of the manipulator system, as they contain little energy. In these simulations, the first three modes are selected to model the structural flexibility of the intermediate links. Therefore, flexible generalized coordinates in equation (4.8) are $\vec{\eta} = [\eta_{11} \eta_{12} \eta_{13} \eta_{21} \eta_{22} \eta_{23} \eta_{31} \eta_{32} \eta_{33}]^T$. The rigid-body motion of sliders and intermediate linkages is derived from the given motion of the moving platform by solving the inverse kinematics of the parallel manipulator. Substituting the rigid-body motion into equation (4.8), the equations become ordinary differential equations (ODEs). With the initial conditions $\vec{\eta} = 0_{9\times1}$, $\dot{\vec{\eta}} = 0_{9\times1}$, equation (4.8) is solved by MATLAB ODEs solver ode113 based on a variable order Adams-Bashforth-Moulton method [137].
Figures 4.2 and 4.3 show the bending deformation at the midpoints of intermediate links with $f = 10 \text{ Hz}$ and $20 \text{ Hz}$, respectively. The effect of axial forces on bending deformation is illustrated in these figures. The effect is not pronounced when $f = 10 \text{ Hz}$. However, when $f$ increases to $20 \text{ Hz}$, the effect is significant.

Figure 4.4 reveals that the amplitude of the first mode vibration of the first intermediate link is two magnitudes larger than the amplitude of the second mode vibration. It can be seen that the first mode is sufficiently accurate to describe the vibration of the flexible intermediate link. Therefore, it is reasonable to reduce the number of vibration modes to small finite number. From Figure 4.4, it can also be seen that the effect of axial forces on bending deformation decreases with the increase of the number of modes modeled. Figure 4.6 shows the driving force acting on the first slider required, illustrating the effect of intermediate link flexibility and also the effect of neglecting intermediate link flexibility, respectively. This result illustrates that the driving force corresponding to the two cases is almost the same. The difference is less than 2\% as shown in Figure 4.7. It is seen that the intermediate link flexibility has little effect on the rigid-body motion of the manipulator system. Therefore, the KED simplification for simulation in this work is justified.

The solution to equation (4.8) includes two parts: free vibration and forced vibration. Without including the effect of longitudinal forces, the $j^{th}$ mode frequency of free vibration for the $i^{th}$ intermediate link can be written as

$$f_{ij}^0 = \frac{\pi}{2} \left( \frac{l_i}{l_j} \right)^2 \sqrt{\frac{E_i}{\rho_{Ai}}}.$$  

Therefore, the first three natural frequencies are calculated to be 116.4 Hz, 465.6Hz, and 1047.5Hz. The amplitudes and frequencies of forced vibration for the intermediate links are determined by inertial and coupling
forces $F_1 = -M_{\rho}$, $F_2 = -M_{\beta}$, and $F_3 = F_{fg}$. These inertial and coupling forces are given in Figure 4.5, corresponding to the first mode of the first intermediate link with $f = 20 \text{ Hz}$.

To investigate the dynamic stiffening and buckling, equation (4.14) is used to calculate the natural lateral vibration frequencies of the intermediate links. Figures 4.8 to 4.10 compare the first three mode frequencies with the axial force in the first intermediate link. These three figures illustrate clearly the effect of the longitudinal forces on the natural lateral vibration frequencies of the intermediate links. The natural lateral vibration frequencies of the intermediate links exhibit configuration dependency. When the link is in tension, the lateral stiffness increases, and therefore the frequency increases. When the axial force is in compression, the lateral stiffness decreases, and therefore the vibration frequency decreases.

These results provide insight into the control of the motion of the manipulator to increase the stiffness of intermediate links and decrease undesired vibration. For example, for the described end-effector motion, the configurations of the manipulator can be optimized so that the axial forces in the flexible links are in extension to increase the stiffness.
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Figure 4.2 Deformation of intermediate links when $f=10\text{Hz}$

Figure 4.3 Deformation of intermediate links when $f=20\text{Hz}$
Table 4.1 Effect of longitudinal force on natural frequencies of the first intermediate link

<table>
<thead>
<tr>
<th>With the effect of longitudinal force</th>
<th>Without the effect of longitudinal force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>Maximum</td>
</tr>
<tr>
<td>The first mode frequency</td>
<td>116.4 Hz</td>
</tr>
<tr>
<td>The second mode frequency</td>
<td>465.6 Hz</td>
</tr>
<tr>
<td>The third mode frequency</td>
<td>1047.5 Hz</td>
</tr>
<tr>
<td>90.2 Hz</td>
<td>441.7 Hz</td>
</tr>
<tr>
<td>1024 Hz</td>
<td>1075 Hz</td>
</tr>
<tr>
<td>44%</td>
<td>11%</td>
</tr>
<tr>
<td>5%</td>
<td></td>
</tr>
</tbody>
</table>

The effect of longitudinal forces on natural frequencies of the first intermediate links is further summarized in Table 4.1. It can be seen that longitudinal forces have a larger effect on flexural vibration for the lower mode frequencies than for the higher mode frequencies. These results are consistent with the theoretical formulation of modal characteristics in Section 4.3.

Figure 4.4 First three mode amplitudes when $f = 20$Hz
The inertial and coupling force applied on the first intermediate link

Figure 4.5 Inertial and coupling forces when $f = 20$Hz

Figure 4.6 Driving forces required when $f = 20$Hz

Figure 4.7 Difference of driving forces when $f = 20$Hz

Figure 4.8 Axial forces and the 1st modal frequency when $f = 20$Hz
Chapter 4  Dynamic Stiffening and Buckling Behavior

The effect of longitudinal load on the frequency of the second mode

The effect of longitudinal load on the frequency of the third mode

Figure 4.9 Axial forces and the 2nd modal frequency when $f = 20\text{Hz}$

Figure 4.10 Axial forces and the 3rd modal frequency when $f = 20\text{Hz}$

Figure 4.11 Axial forces for different $f$

Figure 4.12 1st mode frequencies change for different $f$

Figure 4.13 Axial forces for $f = f_{cr}$

Figure 4.14 Frequency for $f = f_{cr}$
The effect of longitudinal forces on natural vibration frequencies of the intermediate links becomes paramount with the increase of the speed of rigid-body motion. This is further verified by Figure 4.12, which shows the first natural mode frequency change for the first intermediate link corresponding to $f = 10 \text{ Hz}, 20 \text{ Hz},$ and $30 \text{ Hz}$. The results clearly illustrate that the linkage natural vibration frequency changes with the configuration of the parallel manipulator, and the vibration frequency change increases with the speed of the moving platform. The reason for this dynamic behavior is that the axial forces increase with the speed of the manipulator, which is shown in Figure 4.11.

According to the analysis of buckling in Section 4.3, the Euler’s buckling load of the first intermediate link is calculated as
\[
[P_x]_{cr} = \frac{\pi^2 E I_1}{l_1^2} = 360.2 \text{ N}.
\]
Based on equation (4.4), the axial force of the first intermediate link is obtained as shown in Figure 17 when the motion speed of the moving platform increases to $f_{cr} = 31.66 \text{ Hz}$. The maximum compressive force in Figure 4.13 reaches 360.2 N. The first mode frequency of the first intermediate link is also given in Figure 4.14. It clearly illustrates that the lowest frequency reaches 0 Hz. That means that buckling occurs in the intermediate link when $f = 31.66 \text{ Hz}$, and simulation results are consistent with the theoretical analysis in Section 4.3. The above simulation results illustrate that buckling should be considered, and geometric parameters of intermediate links, payload, and motion velocity and acceleration should be determined according to equation (4.17) in the design and dynamic modeling of the parallel manipulator with flexible intermediate links.

To summarize, the dynamic stiffening and buckling should be considered by including the effect of longitudinal forces on lateral stiffness in the design and dynamic simulation when
the parallel manipulator moves with high-speed. The natural frequencies of flexural vibration for the intermediate links are configuration-dependent and increase with axial forces when axial forces are positive, and decrease when axial forces are negative. Axial forces have a more significant influence on the lower order modal characteristics than on the higher order modal characteristics. The influence increases with increasing speed of rigid-body motion of the manipulator system.

### 4.5 Summary

In this chapter, the dynamic stiffening and buckling have been investigated by examining the effect of longitudinal forces on lateral vibration modal characteristics of a 3-PRR parallel manipulator with three flexible intermediate links. First, a procedure for the formulation of dynamic equations of the parallel manipulator has been presented based on the assumed mode method. The mode shape functions are selected by modeling intermediate links as Euler-Bernoulli beams with pinned-pinned boundary conditions verified by modal experimental tests in Chapter 3. The effect of longitudinal forces on lateral stiffness is included in this dynamic model. Then, the natural frequencies of flexural vibration of intermediate links are derived as the functions of axial force and rigid-body motion during high speed motion. The mode frequencies indicate both stiffening and buckling by analyzing the effect of longitudinal forces on lateral stiffness and configuration-dependent frequency property. Finally, numerical simulations validate the theoretical analysis and derivation on the modal characteristics and the dynamic stiffening and buckling of intermediate links.

Theoretical derivation and simulation results provide a valuable insight into the design and control of parallel manipulator with flexible intermediate links. It can be concluded that in
the modeling and control of parallel manipulators with flexible intermediate links, the effect of longitudinal loads on lateral stiffness should be included when the manipulator moves with high-speed or experiences large payloads. The motion of parallel manipulators can be optimized and controlled to increase the stiffness of intermediate links and decrease undesired vibration. For example, for the end-effector motion examined in this work, the configuration of the manipulator may be optimized so that the axial forces in the flexible links are in extension to increase the stiffness. Geometric parameters of intermediate links, payload, and motion velocity and acceleration should be determined according to buckling conditions in the design and dynamic modeling of the parallel manipulator with flexible intermediate links.
Chapter 5

Coupling Characteristics of Rigid Body Motion and Elastic Deformation

5.1 Introduction

Dynamic modeling of mechanisms and manipulators with multiple flexible links is a challenging task. The accurate flexible multibody dynamics not only involves the effect of the rigid body motions or nominal motions on elastic deformations, but also includes the influence of elastic deformations on rigid body motion and elastic deformations of other flexible links. Amongst the published research reports which addressed modeling of flexible mechanical systems, the majority of the investigations presented solutions to the dynamic equations utilizing the two-pass method. With the two-pass method, the mechanical systems are modeled first as a rigid multibody system. Rigid body methodologies are applied to calculate the inertia and joint forces, and then these forces are applied to a linear elastic problem to determine the elastic motion. Hence, this approach only models the effect of rigid body motion on elastic motion. It is valid and reasonable to model mechanism systems such as a flexible four-bar crank with a large flywheel at the crank (Turcic and Midha [82]) using two-pass method. The flywheel, by virtue of its large inertia, ensures that the rigid body motion is maintained independent of the elastic motion. Few research reports presented one-pass methods to model mechanical systems with link flexibility. The one-pass method takes into full account the dynamic coupling between the rigid
This chapter is mainly concerned with the effect of the elastic motions on rigid body motion, and the coupling effect among elastic motions of a 3-PRR planar parallel manipulator with three flexible intermediate links. The fully coupled equations of motions of the flexible parallel manipulator are developed by incorporating the rigid body motions with elastic motions in modeling and using Lagrange multipliers. The equations clearly reveal the mutual dependence of elastic deformations and rigid body motions. This has potential applications in suppressing vibrations through controlling joint motions. For simplicity, PID feedback is selected for the gross rigid body motion control (joint motion), since this work focuses on the effect of elastic motions on rigid body motions. Open-loop simulations without joint motion controls and closed-loop simulations with joint motion controls are performed to investigate the effect of elastic motions on rigid body motions and the coupling effect between flexible links.

Section 5.2 formulates the dynamic equations of the flexible parallel manipulator with the proposed one-pass method. The reduction and analysis of the established dynamic equations are addressed in Section 5.3. Section 5.4 presents numerical simulations and results. Section 5.5 gives the summary of this chapter.

5.2 Dynamic Equations with One-pass Method

In this section, the fully coupled equations of motion of the 3-PRR parallel manipulator with three flexible intermediate links are presented by incorporating the rigid body motions with
Chapter 5    Coupling Characteristics of Rigid Body Motion and Elastic Deformation

elastic motions as a single set of dynamic equations, using Lagrange multipliers. The detailed formulation is given below.

The parallel manipulator with three flexible links and its coordinate system are presented in Chapter 3. The generalized coordinates for the manipulator with flexible intermediate links include rigid body motion generalized coordinates and flexible generalized coordinates. Rigid body motion coordinates are selected as \( \rho = [\rho_1 \ \rho_2 \ \rho_3] \), \( \beta = [\beta_1 \ \beta_2 \ \beta_3] \), and \( X_p = [x_p \ \ x_p \ \ \phi_p] \), and therefore the generalized coordinates of the flexible parallel manipulator system are \( X = [\rho \ \ \beta \ \ X_p \ \ \eta] \in R^{9+3r} \).

The rigid body motion coordinates: \( \rho, \beta, \) and \( X_p \) are not independent, and satisfy six constraint equations. From the geometry of three closed-loop chains as shown in Figure 2.6, these constraint equations are given as

\[
A_i P + PC_i - A_i B_i - B_i C_i = 0 \quad i = 1,2,3 \quad (5.1)
\]

Writing equations (5.1) into an X-axis component and a Y-axis component, the six constraint equations are written

\[
H_{2i-1} = x_{ai} + \rho_i \cos \alpha_i + l_i \cos \beta_i - x_p - x_{ci} \cos \phi_p + y_{ci} \sin \phi_p = 0 \quad (5.2)
\]

\[
H_{2i} = y_{ai} + \rho_i \sin \alpha_i + l_i \sin \beta_i - y_p - y_{ci} \sin \phi_p - x_{ci} \cos \phi_p = 0 \quad (5.3)
\]

where \( x_{ci} \) and \( y_{ci} \) are the x and y coordinates of \( C_i \), respectively, measured from the mass center of the platform, \( P \), when \( \phi_p \) is zero.
In general, the constraint equations involve elastic deformations. However, the longitudinal deformations are neglected and the transverse deformations are assumed small in this work. Therefore, the distance $\overline{B_iC_i}$ between adjacent joints of the intermediate links is constant. We model these links under pinned-pinned boundary conditions. This allows us to express the end effector trajectory in terms of the rigid body coordinates only, and therefore, constraint equation (5.2) and equation (5.3) do not depend on the elastic deformation variable $\eta$ (Ledesma and Bayo [138]).

Using the Lagrange multiplier method, the dynamic equations of the manipulator with three flexible intermediate links is given as

$$\frac{d}{dt} \left( \frac{\partial(T-V)}{\partial \dot{X}_i} \right) - \frac{\partial(T-V)}{\partial X_i} = Q_i + \sum_{k=1}^{6} \lambda_k \frac{\partial G_k}{\partial X_i}$$  \hspace{1cm} (5.4)

where $Q_i$ is the generalized force, $\lambda_k \equiv k^{th}$ Lagrange multiplier, and $i = 1, 3, \ldots, 9 + 3r$.

Substituting the kinetic and potential energies calculated in Chapter 3, and equations (5.2) and (5.3) into equation (5.4), then the dynamic equations of the flexible parallel manipulator are written as the following differential-algebraic equation (DAE),

$$H_{2i-1} \equiv x_{ai} + \rho_i \cos \alpha_i + l_i \cos \beta_i - x_i - x_{ci} \cos \varphi_p + y_{ci} \sin \varphi_p = 0$$

$$H_{2i} \equiv y_{ai} + \rho_i \sin \alpha_i + l_i \sin \beta_i - y_i - y_{ci} \sin \varphi_p - y_{ci} \cos \varphi_p = 0$$  \hspace{1cm} (5.5)
where \( F_{ici} \) are inertial and coupling forces, \( J_{G1}, \ J_{G2}, \) and \( J_{G3} \) are the derivatives of \( H \) with respect to \( \bar{\rho}, \bar{\beta}, \) and \( \bar{X}_p \), superscript \( T \) represents the transpose of a matrix, and \((F_{\rho}, F_{\beta}, F_{X_p}, F_{\eta})^T\) is the external force vector applied on generalized coordinates \( \bar{\rho}, \bar{\beta}, \bar{X}_p, \eta \).

Detailed expressions for the matrices in equation (5.5) are given in Appendix A.

Equation (5.5) clearly shows the dynamic coupling between the rigid body motions and the elastic motions of the manipulator, although the constraint equations do not depend on elastic deformations.

### 5.3 Reduction and Analysis of Dynamic Equations

For convenience of formulation, equation (5.5) is rewritten in a general form

\[
M^* \ddot{q} + \Phi_q^T \lambda = Q \tag{5.6}
\]

\[
\Phi(q,t) = 0 \tag{5.7}
\]

where the matrix \( M^* \) is a \( n \times n \) mass matrix of a dynamic system, the vector \( q \) represents a set of \( n \) unknown dependent coordinates, the matrix \( \Phi_q \) is the \( m \times n \) Jacobian matrix of a set of \( m \) nonlinear constraint equations in (5.7), the vector \( \lambda \) represents \( m \) Lagrange multipliers, and the vector \( Q \) represents a set of \( n \) generalized forces.

The resulting equations are DAEs. These equations are differential equations with respect to the generalized coordinates and algebraic equations with respect to the Lagrange multipliers, as the derivatives of the Lagrange multipliers do not appear. After the equations describing the motions are derived, these equations can be utilized in their differential-algebraic form (Karkoub and Yigit [87]), or the equations may be reduced to an unconstrained differential form [139-143].
There are numerical integration methods for mixed systems of differential equations that permit adding algebraic equations. However, they are not the most numerically efficient and are not free from stability problems in the simulation of mechanical systems. Therefore, several methods have been developed to avoid the direct integration of DAEs. These methods can be classified into two formulations: the dependent coordinate formulation, and the independent formulation.

### 5.3.1 Dependent Coordinate Formulation

There are two methods for formulating different equations of motion in terms of dependent coordinates. In order to avoid direct integration of DAEs, one method (Jalon and Bayo [140]) uses the acceleration kinematic equations obtained by differentiating the constraint equation (5.7) twice with respect to time:

\[
\Phi_q \ddot{q} = -\dot{\Phi}_t - \Phi_q \dot{q} = c \tag{5.8}
\]

Combining Equations (5.6) and (5.8), we have:

\[
\begin{bmatrix}
M^* & \Phi_q^T \\
\Phi_q & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
Q \\
c
\end{bmatrix} \tag{5.9}
\]

Equation (5.9) represents a system of \( n + m \) equations with \( n + m \) unknown, the coefficient matrix is symmetrical and, in general, non-positive definite, and also very sparse in many practical cases. In this formulation, it is necessary to eliminate the dependent constraint equations which were previously developed and corresponding Lagrange multipliers, in order to obtain a nonsingular coefficient matrix in equation (5.9) and make possible its resolution.
A second formulation for differential equations of motion with dependent coordinates is based on a projection matrix $R$ (Jalon and Bayo [140]), with dimensions $n \times (n - m)$. The $n - m$ columns of the matrix $R$ represents a basis of the nullspace of the Jacobian $\Phi_q$; that is, a basis of the subspace of possible motions. The simplest approach to calculate the projection matrix $R$ is to select the independent coordinates as a subset of the dependent ones. The matrix $R$ verifies the following relationship for holonomic systems: $\Phi_q R = 0$. Premultiplying equation (5.6) by the matrix $R^T$, we have

$$R^T M \ddot{\mathbf{q}} + R^T \Phi_q^T \lambda = R^T \mathbf{Q}$$

(5.10)

By virtue of $\Phi_q R = 0$, the term containing the Lagrange multipliers is cancelled, and thus equation (5.10) is rewritten

$$R^T M \ddot{\mathbf{q}} = R^T \mathbf{Q}$$

(5.11)

Combining equations (5.8) and (10), the equations of motion are given as:

$$\begin{bmatrix} \Phi_q \\ R^T M^T \end{bmatrix} \ddot{\mathbf{q}} = \begin{bmatrix} c \\ R^T \mathbf{Q} \end{bmatrix}$$

(5.12)

Equation (5.12) represents a system of $n$ equations with $n$ unknown. The upper part of equation (5.12), corresponding to the Jacobian $\Phi_q$, has been previously factored in order to calculate the matrix $R$. Therefore, the system of equations can be solved with very little additional effort. The simplest approach to calculate the projection matrix $R$ is to select the independent coordinates as a subset of the dependent ones.
The above two methods formulate the differential equations of motion in terms of the dependent accelerations vector \( \ddot{q} \). The drawback of the two methods is the violation of the constraint equations achieved during the numerical integration process. In these two formulations, we calculate the accelerations \( \ddot{q} \) of the mechanism or manipulator at \( t \), and then numerically integrate \( \ddot{q} \) to give \( \dot{q} \) and \( q \) at \( t + \Delta t \) but with limited precision. This error will accumulate during the numerical integration process, leading to an unacceptable violation of constraint Equation (5.7). To overcome this problem, Baumgarte [141] proposed a stabilization method. This method is to replace equation (5.8) with the following system

\[
\ddot{\Phi} + 2\alpha \dot{\Phi} + \beta^2 \Phi = 0
\]  

(5.13)

where \( \alpha \) and \( \beta \) are positive constants, usually chosen to be equal, with values between 1 and 20. The behavior of the method does not significantly depend on these values.

By using equation (5.13) instead of equation (5.8), the differential equations of motion, namely equations (5.9) and (5.13), are modified accordingly to:

\[
\begin{bmatrix}
M^* & \Phi_q^T \\
\Phi_q & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
Q \\
-\Phi_{\dot{q}} - \dot{\Phi}_q \dot{q} - 2\alpha(\Phi_q \dot{q} + \Phi_{\dot{q}} - \beta^2 \Phi)
\end{bmatrix}
\]

(5.14)

\[
\begin{bmatrix}
\Phi_q \\
R^T M^*
\end{bmatrix}
\ddot{q} =
\begin{bmatrix}
-\Phi_{\dot{q}} - \dot{\Phi}_q \dot{q} - 2\alpha(\Phi_q \dot{q} + \Phi_{\dot{q}} - \beta^2 \Phi) \\
R^T Q
\end{bmatrix}
\]

(5.15)

Another method to solve this constraint violation problem is a penalty formulation proposed by Bayo et al. [142]. In essence, this method directly incorporates the constraint equations as a dynamical system, penalized by a large factor, into the equations of motions. The larger the
penalty factor the better the constraints will be achieved at the cost of introducing some numerical ill-conditioning.

5.3.2 Independent Coordinate Formulation

The independent velocity vector \( \dot{p} \) with \( f = n - m \) elements (\( f \) is the number of degree of freedom for the system) normally is given by the projection of the dependent velocities \( \dot{q} \) on certain vectors defined by the rows of a matrix \( S \), given as: \( \dot{p} = S \dot{q} \), where \( S \) is the matrix defining the linear combination of \( \dot{q} \). It can be proved that \( \dot{q} = R \dot{p} \) (Jalon and Bayo [140]). The need to appropriately select the independent coordinates can be done based on the mechanical point of view. Differentiating \( \dot{q} = R \dot{p} \) with respect to time, we have

\[
\ddot{q} = R \ddot{p} + \dot{R} \dot{p}
\]  

(5.16)

By introducing this equation into equation (5.11), and expressing the dependent accelerations in terms of the independent accelerations and velocities, the resulting system of differential equations of motion is given as

\[
R^T M^* \dot{R} \ddot{p} = R^T Q - R^T M^* \dot{R} \dot{p}
\]  

(5.17)

This equation describes the dynamic system in terms of independent accelerations. Essentially, equation (5.17) represents a general matrix transformation from the vector space of dependent acceleration and forces to the vector space of independent accelerations and forces. In this formulation, the numerical integration routine needs the evaluation of matrix \( R \) and the term \( \dot{R} \). Therefore, it is necessary to calculate the corresponding dependent positions and velocities. This involves the solution of the nonlinear constraint equations, namely equation (5.7), to obtain the
dependent position in terms of the independent coordinates, which can be done based on the
kinematic analysis of the mechanism or manipulator system. As the constraint equations are
automatically fulfilled, in this formulation, there does not exist any constraint violation problem.
Other authors (Wehage and Haug [143]) proposed the coordinate partitioning method or
extraction method to address the independent coordinate formulation. This method integrates
some of the vector’s elements in equations (5.9) and (5.12), namely, those corresponding to the
independent coordinates. Essentially, the two methods are the same.

5.3.3 Analysis of the Established Equations of Motion

In this work, the independent coordinate formulation is used to solve the differential
equations of motion previously established, namely equation (5.5). The number of degrees of
freedom is \(3 + 3r\) in this parallel manipulator with three flexible links, i.e., 3 degrees of rigid
body motion degree and \(3r\) degrees of elastic motion. Based on the kinematic analysis of this
manipulator (Heerah et al. [128]), the accelerations and velocities of dependent rigid body
motion coordinates can be expressed in terms of independent rigid body coordinates
\(\vec{\rho} = [\rho_1, \rho_2, \rho_3]^T\) or \(\vec{x}_p = [x_p, x_p, \varphi_p]^T\). Equation (5.5) is comprised of \(9 + 3r\) differential
equations, \(15 + 3r\) variables, and 6 algebraic constraint equations. For convenience of joint motion
controller design and analysis of coupling between rigid body motions and elastic motions, it is
desirable to reduce equation (5.5) to \(3 + 3r\) differential equations with \(3 + 3r\) independent
coordinates, in a closed form with the Lagrange multipliers eliminated. The \(3 + 3r\) independent
coordinates are selected as \(\vec{\eta} = [\eta_{11}, \ldots, \eta_{1r}, \eta_{21}, \ldots, \eta_{2r}, \eta_{31}, \ldots, \eta_{3r}]^T\) and
\(\vec{\rho} = [\rho_1, \rho_2, \rho_3]^T\).
Equation (5.5) is a set of linear algebraic equations with respect to the Lagrange multipliers \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^T \). Therefore, the Lagrange multipliers can be expressed using six linear independent equations of equation (5.5). Partitioning equation (5.5), we have

\[
\begin{bmatrix}
M_{11} & M_{12} & 0 & M_{14} \\
M_{41} & M_{42} & 0 & M_{44}
\end{bmatrix}
\begin{bmatrix}
\ddot{\rho} \\
\ddot{\overline{X}}_P \\
\ddot{\eta}
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & K(\eta)
\end{bmatrix} = \begin{bmatrix}
F_{ic1} \\
F_{ic4}
\end{bmatrix} + \begin{bmatrix}
F_{\rho} \\
F_{\eta}
\end{bmatrix} + \begin{bmatrix}
J_{G1}^T & \lambda_1 \\
J_{G1} & \lambda_2 \\
J_{G2} & \lambda_3 \\
J_{G2}^T & \lambda_4 \\
J_{G3} & \lambda_5 \\
J_{G3}^T & \lambda_6
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\lambda_6
\end{bmatrix}
\]  
(5.18)

\[
\begin{bmatrix}
M_{21} & M_{22} & 0 & M_{24} \\
0 & 0 & M_{33} & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\rho} \\
\ddot{\overline{X}}_P \\
\ddot{\eta}
\end{bmatrix} = \begin{bmatrix}
F_{ie2} \\
F_{ie4}
\end{bmatrix} + \begin{bmatrix}
F_{\rho} \\
F_{\eta}
\end{bmatrix} + \begin{bmatrix}
J_{G2}^T & \lambda_1 \\
J_{G2} & \lambda_2 \\
J_{G3} & \lambda_3 \\
J_{G3}^T & \lambda_4 \\
J_{G2}^T & \lambda_5 \\
J_{G3}^T & \lambda_6
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\lambda_6
\end{bmatrix}
\]  
(5.19)

From equation (5.19), we have

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\lambda_6
\end{bmatrix} = \begin{bmatrix}
J_{G2}^T \\
J_{G3}^T
\end{bmatrix}^{-1} \begin{bmatrix}
M_{21} & M_{22} & 0 & M_{24} \\
0 & 0 & M_{33} & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\rho} \\
\ddot{\overline{X}}_P \\
\ddot{\eta}
\end{bmatrix} - \begin{bmatrix}
J_{G2}^T & \lambda_1 \\
J_{G2} & \lambda_2 \\
J_{G3} & \lambda_3 \\
J_{G3}^T & \lambda_4 \\
J_{G2}^T & \lambda_5 \\
J_{G3}^T & \lambda_6
\end{bmatrix}^{-1} \begin{bmatrix}
F_{ie2} \\
F_{ie4} \\
F_{\rho} \\
F_{\eta} \\
F_{X_p}
\end{bmatrix}
\]  
(5.20)

Substituting equation (5.20) into equation (5.18), we have
\[
\begin{bmatrix}
M_{11} & M_{12} & 0 & M_{14} \\
M_{41} & M_{42} & 0 & M_{44}
\end{bmatrix}
- \begin{bmatrix}
J_{G1}^T & J_{G2}^T & 0 & M_{21} & M_{22} & 0 & M_{24}
0 & J_{G3}^T & 0 & M_{A3}
\end{bmatrix}
\begin{bmatrix}
\ddot{\rho} \\
\ddot{\beta} \\
\dddot{X}_p \\
\dddot{\eta}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
K_d \\
\eta
\end{bmatrix}
\]

(5.21)

The procedure from equation (5.18) to equation (5.21) eliminates the Lagrange multipliers based on the projection matrix method discussed in Section 5.3.2. The projection matrix \( R \) is a basis of the nullspace of the Jacobian matrix of the constraint equations.

Now, we can use the kinematic analysis of the parallel manipulator, and express the dependent coordinates in terms of independent coordinates. The relationships amongst \( \ddot{\rho}, \ddot{\beta}, \) and \( \dddot{X}_p \) are given as

\[
\ddot{\beta} = J_\beta \dot{\rho} \\
\dddot{X}_p = J_p \ddot{\rho}
\]

(5.22)

(5.23)

Differentiating equations (5.22) and (5.23) with respect to time, we have

\[
\dddot{\beta} = J_{\beta\beta} \ddot{\rho} + J_\beta \dot{\rho} \\
\dddot{X}_p = J_p \dddot{\rho} + J_p \ddot{\rho}
\]

(5.24)

(5.25)

where the expressions of \( J_\beta \) and \( J_p \) are provided in [32] and Chapter 2. Note that quantities \( J_\beta \) and \( J_p \) are not only functions of the independent coordinates \( \rho \), but also functions
of dependent coordinates $\ddot{\beta}$ and $\ddot{X}_p$. Therefore, the acceleration vector of dependent coordinates may be expressed in terms of independent coordinates as

$$
\begin{pmatrix}
\ddot{\beta} \\
\ddot{\eta} \\
\ddot{X}_p
\end{pmatrix} =
\begin{pmatrix}
I_1 & 0 & 0 \\
J_\beta & 0 & 0 \\
0 & J_\rho & 0
\end{pmatrix}
\begin{pmatrix}
\ddot{\beta} \\
\ddot{\eta} \\
\ddot{X}_p
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & 0 \\
J_\beta & 0 & 0 \\
0 & J_\rho & 0
\end{pmatrix}
\begin{pmatrix}
\ddot{\rho} \\
\ddot{\eta} \\
\ddot{X}_p
\end{pmatrix}
$$

(5.26)

Substituting equations (5.26) into equation (5.21), the equations of motion with independent coordinates, in closed form are given as

$$
\begin{pmatrix}
\ddot{\rho} \\
\ddot{\eta}
\end{pmatrix} + K \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\ddot{\rho} \\
\ddot{\eta}
\end{pmatrix} + \begin{pmatrix}
F_{s1} \\
F_{s2}
\end{pmatrix} = \begin{pmatrix}
F_\rho \\
F_\eta
\end{pmatrix}
$$

(5.27)

$$
[M_s] =
\begin{pmatrix}
M_{11} & M_{12} & 0 & M_{14} \\
M_{41} & M_{42} & 0 & M_{44}
\end{pmatrix}
- \begin{pmatrix}
J^{T}_{G1} & J^{T}_{G2} & J^{T}_{G3}
\end{pmatrix}^{-1}
\begin{pmatrix}
M_{21} & M_{22} & 0 & M_{24} \\
0 & 0 & M_{33} & 0 \\
0 & I_2 & 0
\end{pmatrix}
\begin{pmatrix}
I_1 & 0 \\
J_\beta & 0 \\
J_\rho & 0 \\
0 & I_2
\end{pmatrix}
$$

(5.28)

$$
\begin{pmatrix}
F_{s1} \\
F_{s2}
\end{pmatrix} =
\begin{pmatrix}
F_{ic1} \\
F_{ic4}
\end{pmatrix}
+ \begin{pmatrix}
J^{T}_{G1} & J^{T}_{G2} & J^{T}_{G3}
\end{pmatrix}^{-1}
\begin{pmatrix}
F_{ic2} + F_\beta \\
F_\rho
\end{pmatrix}
+ 
\begin{pmatrix}
M_{11} & M_{12} & 0 & M_{14} \\
M_{41} & M_{42} & 0 & M_{44}
\end{pmatrix}
- \begin{pmatrix}
J^{T}_{G1} & J^{T}_{G2} & J^{T}_{G3}
\end{pmatrix}^{-1}
\begin{pmatrix}
M_{21} & M_{22} & 0 & M_{24} \\
0 & 0 & M_{33} & 0 \\
0 & I_2 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
J_\beta \ddot{\rho} \\
J_\rho \ddot{\rho}
\end{pmatrix}
$$

(5.29)

where $I_1$ is a $3 \times 3$ unit matrix, and $I_2$ is a $3r \times 3r$ unit matrix. This procedure presented is based on the independent coordinate formulation in Section 5.3.2. Comparing equation (5.26)
with equation (5.16), we find the projection matrix \( R \) is given by
\[
\begin{bmatrix}
I_1 & 0 \\
J_p & 0 \\
0 & I_2
\end{bmatrix}
\]
In this procedure, the dynamic analysis of the flexible manipulator system, i.e., the numerical integration of equation (5.27), involves the calculation of matrix \( R \) and matrix \( \dot{R} \). Hence, it is necessary to calculate the corresponding dependent positions and velocities. This involves the solution of the constraint equations, namely equations (5.7), (5.22) and (5.23), which can be achieved based on the kinematic analysis of the manipulator system. Therefore, the constraint equations are satisfied utilizing this procedure.

Equation of motions (5.27) for the parallel manipulator system becomes \( 3 + 3r \) differential equations with constraint equations eliminated. In equation (5.28), the mass matrix \( M_\varepsilon \) is non-diagonal, and therefore, coupling exists between generalized rigid body motion coordinates \( \vec{\rho} \) and the generalized elastic motion coordinates \( \vec{\eta} \) although the rigid kinematics is used in equation 5.5. The coupling also appears in the vector \(( F_{s1} \ F_{s2} )^T \). It is obvious that the elastic deformation is seen in the dynamic equations of rigid body motion, and can be suppressed through controlling joint motions. This fact provides a potential solution to suppress unwanted vibration by controlling the joint driving forces, which may be implemented through the design of joint motion controllers using input shaping (Singhose et al. [98]) or singular perturbation approaches (Siciliano and Book [100]).

To summarize, the formulation presented in this work leads to the differential-algebraic equations of motion, namely equation (5.5), as follows: first a set of independent coordinates (\( \vec{\rho} \) and \( \vec{\eta} \)) are chosen, and then the Lagrange multipliers and acceleration terms of dependent
coordinates \( \ddot{\beta} \) and \( \ddot{X}_p \) are eliminated from equation (5.5). Based on the formulation from equation (5.6) to equation (5.27), the reduced order differential equations of motion, namely equation (5.27), are obtained. This equation contains only acceleration terms of independent coordinates, but it is a function of displacement and velocity of both independent and dependent coordinates. By using initial values of the independent coordinates, we may solve the constraint equations for \( \ddot{\beta}, \ddot{X}_p, \dddot{\beta}, \) and \( \dddot{X}_p \). These values are used to numerically integrate the differential equation, solving for the independent coordinates. This cycle is repeated for each interval of time. After each interval we have to check whether independent coordinates are chosen properly so that constraint equations can be solved for the dependent coordinates. If that is not appropriate, a new set of independent coordinates has to be chosen.

5.4 Numerical Simulation and Analysis

Numerical simulations for the 3-PRR parallel manipulator with three flexible intermediate links are presented. In these simulations, a 50 milli-second circular motion is assigned as a desired trajectory for the mass center point of the moving platform with constant orientation \( \varphi_p \). This motion serves to excite structural vibrations in the flexible linkages. The robot is simulated under closed-loop control during the first 52.5 milli-second rigid body motion, then the simulation continues under open-loop or closed-loop control until \( t = 1800 \text{ms} \). The simulation is terminated at \( t = 1800 \text{ms} \) because the manipulator approaches a singular configuration at that time. The equations for the trajectory are \( x_p = 30 \cos 2\pi f \ t - 30 \text{(mm)} \), \( y_p = 30 \sin 2\pi f \ t \text{(mm)} \), and \( \varphi_p = 45^\circ \) (constant) with \( f = 20 \text{Hz} \) for \( t \leq 50 \text{ms} \), and the input trajectories are \( x_p = 0, y_p = 0 \) for \( t > 50 \text{ms} \). The intermediate links are
modeled as aluminum alloy with Young’s modulus $E = 7.1 \times 10^{10} \text{ N/m}^2$ and mass density $\rho = 2.77 \times 10^3 \text{ kg/m}^3$. The three intermediate linkages have identical geometric parameters. The length of each link is 200 mm, cross-section width 2 mm, and the height of cross-section 30 mm.

For simplicity, three joint motions are controlled separately using a proportional and integral and derivative (PID) controller with gains $k_p = 1000$, $k_i = 150000$, and $k_d = 500$. Thus, the driving forces applied to three sliders are given as

$$F_a = k_p (\ddot{\rho}_d - \ddot{\rho}) + k_i \int (\dot{\rho}_d - \dot{\rho}) dt + k_d (\dot{\rho}_d - \dot{\rho})$$  \hspace{1cm} (5.30)

where $\ddot{\rho}_d$ and $\dot{\rho}_d$ are desired values of the three sliders, calculated for the desired motion of the platform using inverse kinematics.

With the initial values of generalized elastic deformation $\ddot{\rho}_0 = 0$, velocity $\dot{\rho}_0 = 0$, and the initial position $\rho_0$ and velocity $\dot{\rho}_0$ of sliders calculated from given motion of the platform, equation (5.27) is solved with MATLAB ODE solver ode113 based on a variable order Adams-Bashforth-Moulton method [137]. In this work, two simulation strategies are adopted to examine the effect of elastic deformations on rigid body motions and flexibility coupling between the intermediate links. In the first strategy, joint motion PID control is applied only when $t \leq 52.5 ms$. The purpose of extending control time 2.5 ms is to prevent the manipulator moving near singular configurations due to inertial motion. At $t > 52.5 ms$, no joint motion control and driving forces are applied to the three sliders. We investigate the coupling...
characteristics of elastic deformations and rigid body motions when \( t > 52.5 ms \). In the second strategy, joint motion PID control is applied at all times.

Figures 5.1 to 5.4 illustrate the simulation response under strategy 1, with PID control, equation (5.30) applied for \( 0 \leq t \leq 52.5 ms \), and open-loop control for \( t > 52.5 ms \). Figure 5.1 presents the elastic deformations of the three flexible intermediate links at their midpoints. It is interesting to find that there exists obvious exchange of elastic strain potentials between the intermediate links from 200ms-1800ms, which means flexibility coupling occurs between the elastic deformations of the intermediate links. The linear and angular displacements of the platform, calculated based on one-pass method and two-pass method, are given in Figures 5.2 through 5.4. These figures illustrate that the difference of rigid body motion of the moving platform, as determined with the simulations of the one-pass method and those of the two-pass method, increases with time when \( t > 50 ms \). The reason is that the energy from the elastic deformation of the intermediate links affects the rigid body motion of the manipulator system in the simulation using the one-pass method. Therefore, elastic motions of the intermediate links and the coupling amongst elastic deformations of intermediate links have a significant effect on rigid body motions of the manipulator systems. In the open-loop simulations with the two-pass method, motion errors of the system are generated due to neglecting the effect of elastic motion on rigid body motion and coupling amongst elastic deformations of the intermediate links.
Figure 5.1 Midpoint deformation of the three flexible intermediate links

Figure 5.2 Position of the platform along X axis
Chapter 5    Coupling Characteristics of Rigid Body Motion and Elastic Deformation

Figure 5.3 Position of the platform along Y axis

Figure 5.4 Angular position of the platform
The equations of constraint, given by equation (5.5), are calculated for the duration of the simulations. Therefore, the constraint equations are not violated due to numerical drift during the simulations. Figure 5.5 shows the time history of the parallel manipulator energy. This figure illustrates that the total energy of the manipulator system is conserved after 52.5 ms. It further shows that the constraints are not violated in the simulation and the simulations are not ill conditioned. From this fact, we can conclude the divergence between the results of one-pass method and that of two-pass method is not caused by numerical drift during the simulation.

Figures 5.6 through 5.12 show simulations performed with the closed-loop simulation strategy. The linear and angular displacements of the platform, calculated based on the one-
pass method and the two-pass method, are given in Figures 5.6 through 5.8. These results are substantially different from the case of the open-loop simulation in that elastic deformations have negligible effect on rigid body motion of the manipulator system in the level of displacement in the case of closed-loop simulation. This is because introduction of the joint motion controller as a variable coefficient torsional spring and damper, has the potential to absorb the elastic strain energy. However, the effect of elastic motions leads to apparent changes of velocity and acceleration of rigid body motion, which is illustrated in Figures 5.9 and 5.10. These changes will further impact on the dynamics behaviors of the manipulator system.

![Graph showing position of the platform along X axis](image)

**Figure 5.6 Position of the platform along X axis**
Chapter 5  Coupling Characteristics of Rigid Body Motion and Elastic Deformation

Figure 5.7 Position of the platform along Y axis

Figure 5.8 Angular position of the platform
Chapter 5    Coupling Characteristics of Rigid Body Motion and Elastic Deformation

Figure 5.9 Angular velocity of the platform

Figure 5.10 Angular acceleration of the platform
Figure 5.11 Deformation of the first intermediate link at the midpoint with joint motion PID control

Figure 5.12 Deformation of the intermediate links at the midpoint
To further show the coupling effect between the elastic motions and the rigid body motions of the manipulator system, the midpoint deformation of the first intermediate link is given in Figure 5.11. It is illustrated that flexible deformation of the intermediate link is seen in the joint motion controller based on the one-pass method. The vibration of the links is suppressed by PID joint motion controller due to the coupling nature of rigid body motions and elastic motions in the equations developed with the one-pass method. Therefore, it is feasible to suppress the unwanted vibration through joint motion control using feedback technologies based on the one-pass method. Figure 5.11 also reveals that the effect of elastic motions on rigid body motion has significant impact on the elastic deformation of the intermediate links, although the effect on the rigid body displacement of the manipulator system is negligible in the closed-loop simulation. However, the vibration is not suppressed by joint motion control based on the two-pass method. This fact is further confirmed in Figure 5.12, which shows that the vibrations of the flexible intermediate links are suppressed due to the PID joint motion controllers. Notice the control gains of the PID joint motion controllers are adjusted using trial and error based on the requirement of the smallest trajectory error and best vibration suppression. While not the subject of this work, the control gains might be optimized to best suppress the vibration, and the deformation or stain of the intermediate links could be included in the joint motion controller as well.

5.5 Summary

This chapter has presented the coupling characteristics of elastic motions and rigid body motions for a 3-PRR planar parallel manipulator with three flexible intermediate links. The
coupled equations of motion of the flexible parallel manipulator have been developed utilizing the one-pass method, namely, incorporation of the rigid body motions with elastic motions as non-separated quantities based on the use of Lagrange multipliers, and then the subsequent reduction in order of the dynamic equations in a closed form which is suitable for the design of a controller. The equations clearly reveal coupling of elastic motion and rigid body motion, which provides potential solution for suppressing vibrations through controlling joint motions. Open-loop simulations and closed-loop simulations are performed. Open-loop simulation results show that the elastic motions of the intermediate links have a significant effect on rigid body motions of the manipulator system, and energy exchanges between flexible intermediate links. Closed-loop simulation results illustrate that the effect of elastic motions on rigid body motion is negligible in the level of linear and angular displacement of the manipulator system due to the implementation of joint motion controllers. Closed-loop simulation results also show that it is feasible to suppress the unwanted vibration through the design of joint motion controllers.
Chapter 6

Active Vibration Control Based on Strain Rate Feedback

6.1 Introduction

Concurrent to the work on dynamic modeling as presented in the previous chapters, increasing efforts have been invested to the investigation of vibration control of flexible manipulators and mechanisms. A variety of design or control strategies have been presented to suppress unwanted structural vibration of flexible links (Ghazavi et al. [91], Sung and Thompson [92], El-Dannah and Farghaly [93], Sisemore et al. [94], Zhang et al. [95], Cleghorn et al. [96], Ulbrich and Stein [97], Singhose et al. [98], Shan et al. [99], Siciliano and Book [100]). A promising method is through the use of smart material actuators and sensors bonded to flexible links to control unwanted vibration (Shan et al. [124], Liao and Sung [125], Thompson and Tao [126], Preiswerk and Venkatesh [127]).

Piezoelectric materials have received growing attention in vibration control of flexible structures in recent years, due to the advantageous properties of piezoelectric actuators and sensors. These properties include mechanical simplicity, small volume, light weight, large bandwidth, efficient conversion between electrical energy and mechanical energy, and simple integration with various metallic and composite structures. In this work, Lead Zirconate Titanate (PZT) sensors and actuators are employed to suppress the unwanted vibrations of the flexible
parallel manipulator. The control algorithms can be categorized into three basic families according to the controller architecture: feedforward, feedback, and hybrid structure. In the feedback control design, the controller input is determined and based on the output of vibrations sensors in the feedback path. A number of feedback control techniques have been developed for vibration control. Positive position feedback (PPF) (Goh and Caughey [104], Fanson and Caughey [105], Song et al. [106], Rew et al. [107], Moheimani et al. [108]), velocity feedback or strain rate feedback (SRF) (Song et al. [109], Sun et al. [110], Juang and Phan [111]), and resonant control (Pota et al. [112], Halim and Moheimani [113], Tjahyadi et al. [114]) are three typical examples of such control techniques. Strain rate feedback has a wider active damping frequency range, and hence can stabilize more than one mode simultaneously, given sufficient bandwidth. Moreover, the strain rate is readily available in practice through differentiation of voltages measured from PZT sensors. Since the 1990s, piezoelectric actuators and sensors have been introduced for vibration damping of flexible manipulators and mechanisms (Shan et al. [124], Liao and Sung [125], Thompson and Tao [126], Preiswerk and Venkatesh [127]). Most of these works investigated the residual vibration control of manipulators and mechanisms with a single flexible link bonded with single actuator and sensor. This chapter addresses the vibration control of a 3-PRR parallel manipulator with three flexible intermediate links with multiple PZT actuators and sensors bonded to these links when the manipulator moves with high speed and acceleration. The centralized bending moments from the PZT actuators bonded on flexible intermediate links, are transformed to the force vectors in modal space and are incorporated into the dynamic equation of the flexible manipulator. The resultant dynamic equations provide essential insight into the active vibration control of smart structures featuring PZT actuators and sensors. A state-place model is developed to formulate the control input and output voltages
applied to PZT actuators, and generated from PZT sensors. The development of an optimal active vibration controller is presented for the parallel manipulator with flexible links with multiple bonded PZT transducers based on SRF. Finally, numerical simulations are performed to verify that the proposed active control strategy is effective.

Section 6.2 formulates the dynamic of PZT sensors and actuators based on the SRF. In Section 6.3, Dynamic equations are presented for the parallel manipulator with three flexible intermediate links with multiple bonded PZT transducers. Section 6.4 addresses optimal control design for the SRF controller. Numerical simulations and analysis are performed in Section 6.5. Section 6.6 offers the summary of this chapter.

### 6.2 Sensor and Actuator Dynamics Based on Strain Rate Feedback Control

PZT materials have the potential to both damp vibrations and measure the vibration characteristics of distributed parameter systems, due to their direct and converse piezoelectric effect. A smart beam is constructed by bonding PZT patches to the two opposite sides of the intermediate link as shown in Figure 6.1. The PZT patches on the one side of the beam plays the role of actuators, while the PZT patches on the opposite side plays the role of sensors. One actuator and one sensor constitute a PZT control pair, and are located at the same location along the beam. Hence, this configuration can be regarded as a smart beam with collocated actuators and sensors.
6.2.1 Sensor Dynamics

Strains are generated in the PZT sensors bonded on the intermediate links due to the lateral vibration of the flexible links. According to the direct piezoelectric effect, the strains in a PZT sensor along the \( x \) direction lead to a charge signal on the upper and lower surfaces of the PZT sensor. Integration of the charge signal over the sensor area gives the total amount of charge generated. Considering the fact that PZT sensor dimension along the \( x \) direction is very short compared to the length of the beam, the axial strain over a sensor is approximated to be equal to that of the midpoint of the PZT sensor. Using an impedance converter, a charge signal is converted to a voltage signal proportional to the strain at the midpoint of a sensor. Therefore, the voltage produced by the sensor located at \( x_k \) on the \( i^{th} \) intermediate link can be expressed as

\[
V_{si}^k = k^k_s \left( \frac{\partial w_i^2(x_k,t)}{\partial x^2} \right) = k^k_s \sum_{j=1}^{r} \eta_j(t) \psi_j^i(x_k) = \frac{E_p d_{31} w_p}{C_p} (h_{p} + h_i) \sum_{j=1}^{r} \eta_j(t) \psi_j^i(x_k) \tag{6.1}
\]
where \( k^s_i \) is the sensor constant (\( V^{-1} \)), \( E_p \) Young’s modulus of PZT materials (\( N/m^2 \)), \( d_{31} \) is the piezoelectric constant (\( m \cdot V \)), \( w_p \) the width of the PZT transducer (\( m \)), \( h_i \) the half height of the link, \( h_p \) the height of the PZT transducer (\( m \)), and \( C_p \) is the capacitance of the sensor (Farad).

### 6.2.2 Actuator Dynamics

According to the converse piezoelectric effect, applying a voltage on a PZT actuator, a strain is produced in the PZT actuator. When a stiff boundary layer is used to bond the actuator to the host structure (one of the intermediate links), the shear strain is transferred between the actuator and the beam over an infinitesimal distance near the ends of the actuator (Crawley and Luis [144]). Therefore, the action resulting from the converse piezoelectric effect is represented by two concentrated bending moments applied at the both ends of the actuator. The expression of the bending moment generated by the \( k^{th} \) actuator of the \( i^{th} \) intermediate links is given as

\[
M^k_{p_i}(x_{k1}) = -M^k_{p_i}(x_{k2}) = -d_{31} \frac{E_i E_p h_i^2 w_i w_p}{E_i h_i w_i + E_p h_p w_p} \times \frac{V^k}{V^k_{ai}} = -k^s_i V^k_{ai}
\]

where \( E_i \) is the Young’s modulus of the \( i^{th} \) link, \( w_i \) is the width of the link, and \( V^k_{ai} \) is the voltage applied on the PZT actuator, \( x_{k1} \) is the coordinate of the left end of the actuator, and \( x_{k2} \) is the coordinate of the right end of the actuator.

Using modal theory, the bending moment (expressed in physical space) can be transformed to the force vector in modal space (Liao and Sung [125]). The modal force vector is given as
\[ F_{pi}^k = [\psi^i_{1i}(x_{k2}) - \psi^i_{1i}(x_{k1}), \ldots, \psi^i_{ij}(x_{k2}) - \psi^i_{ij}(x_{ki}), \ldots, \psi^i_{jr}(x_{k2}) - \psi^i_{jr}(x_{k1})] \top M_{p_{ij}}^k(x_k) \]  

(6.3)

where (\') denotes the derivative with respect to \( x \).

### 6.2.3 Strain Rate Feedback Control

In this work, strain rate feedback control (SRF) is used to damp the lateral vibrations of the flexible intermediate links. Using SRF, the derivative of a voltage from a PZT sensor is multiplied by a negative control gain \(-k_{di}^k\) and fed back to the corresponding actuator. Therefore, the voltage applied on the actuator is expressed as

\[ V_{ai}^k = -k_{di}^k \frac{\psi_{ji}(x_k)}{k_{si}^k} = -k_{di}^k \sum_{j=1}^r \eta_j(x_k) \psi_{ji}^k(x_k) \]  

(6.4)

From equation (6.3), a control force vector applied on the \( i^{th} \) link by \( m \) PZT pairs is written as

\[ F_{pi} = -\sum_{k=1}^m k_{ai}^k k_{di}^k \left[ (\psi^i_{1i}(x_{k2}) - \psi^i_{1i}(x_{k1}))\psi^i_{1i}(x_k) \ldots (\psi^i_{ji}(x_{k2}) - \psi^i_{ji}(x_{ki}))\psi^i_{ji}(x_k) \ldots (\psi^i_{jr}(x_{k2}) - \psi^i_{jr}(x_{k1}))\psi^i_{jr}(x_k) \right] [\eta_{i1}^k \ldots \eta_{ir}^k] \]  

(6.5)

In equation (6.5), we do not need to know the values of \( k_{ai}^k \) in the practical design of the controller because \( k_{ai}^k k_{di}^k \) can be regarded as a new control gain \( k_{Di}^k \), which will be selected to damp the vibrations of the flexible link.

### 6.3 Dynamic Equations with Active Vibration Control

In this chapter, the general form of Lagrange’s equations is used to derive the structural dynamic equations of motion of the 3-PRR parallel manipulator with three flexible intermediate links bonded with PZT sensors and actuators for vibration control. Since our interest in this
chapter focuses on the structural vibration control of the manipulator system, the rigid-motion dynamics of the manipulator system is not discussed here. Specifically, we adopt the Kineto-Elasto-Dynamics (KED) assumptions common to much of the literature which address structural vibration (Liao and Sung [125]). Therefore, the influence of the elastic deformation on the rigid-body motion is neglected, and the equations of motion are solved using the prescribed rigid-body motion. Thus, Lagrange’s equations are formulated only with flexible generalized coordinates of the manipulator system.

To derive Lagrange’s equation of the flexible parallel manipulator, we must calculate the total kinetic energy and potential energy of the manipulator system which are presented in Chapter 3. We will analyze and incorporate generalized forces acted on flexible intermediate linkages from PZT actuators control actions.

From equation (6.5), the generalized forces corresponding to the generalized coordinate \( \eta_{ij} \) is expressed as

\[
F_{\eta}^{ij} = \sum_{k=1}^{m} -k_{Di}^{k} \left( \psi_{ij}^{r} \left( x_{k} \right) \sum_{j=1}^{r} \left( \psi_{ij}^{r} \left( x_{k2} \right) - \psi_{ij}^{r} \left( x_{k1} \right) \right) \right) \hat{\eta}_{ij} \tag{6.6}
\]

The general form of Lagrange’s equations for flexible generalized coordinates is given as

\[
\frac{d}{dt} \left( \frac{\partial (T-V)}{\partial \dot{\eta}_{ij}} \right) - \frac{\partial (T-V)}{\partial \eta_{ij}} = F_{\eta}^{ij} \quad i=1,2,3 \quad j=1,2,\cdots,r \tag{6.7}
\]

Substituting the calculated kinetic and potential energies, and the generalized forces in equation (6.6) into (6.7), we have
\begin{align}
  m_i \ddot{\eta}_j + \int_0^1 \psi_{ij}^2 d\xi + \frac{E}{I_j} \eta_j(t) \left( \psi_j'' \right)^2 I_j(\xi) d\xi &= F_{\eta}^{ij} \\
  -m_i \ddot{\rho}_i \sin(\alpha_i - \beta_i) \int_0^1 \psi_{ij} d\xi - m_i \ddot{l}_i \int_0^1 \psi_{ij} \xi d\xi + m_i \ddot{\rho}_i \cos(\alpha_i - \beta_i) \int_0^1 \psi_{ij} d\xi
\end{align}

Equation (6.8) can be rewritten in matrix form as

\begin{equation}
  M \ddot{\eta} + K\eta = -M \ddot{\rho} - M \ddot{\beta} + F_{fg} + F_{\eta}
\end{equation}

\begin{equation}
  F_{\eta} = -C \ddot{\eta} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}
\end{equation}

\begin{equation}
  C_i = \sum_{k=1}^{m} k_{ai}^i k_{bi}^j \begin{bmatrix} (\psi_{i1}'(x_{k1}) - \psi_{i1}'(x_{k1}))\psi_{i1}(x_k) & \cdots & (\psi_{i1}'(x_{k1}) - \psi_{i1}'(x_{k1}))\psi_{i1}'(x_k) \\ \vdots & \vdots & \vdots \\ (\psi_{ir}'(x_{k1}) - \psi_{ir}'(x_{k1}))\psi_{ir}'(x_k) & \cdots & (\psi_{ir}'(x_{k1}) - \psi_{ir}'(x_{k1}))\psi_{ir}'(x_k) \end{bmatrix}
\end{equation}

where \( m_i = \rho_i l_i \) is the modal mass of the \( i^{th} \) intermediate link, \( K \) is the structural modal stiffness matrix, \(-M \ddot{\rho} - M \ddot{\beta}\) reflects the effect of rigid-body motion on elastic vibration of flexible links, and \( F_{fg} \) is the modal force from the coupling between rigid-body motion and elastic motion. The detailed expressions for \( M, -M \ddot{\rho}, -M \ddot{\beta}, \) and \( F_{fg} \) are given in the Appendix A. Parameter \( F_{\eta} \) is the modal control force vector transformed from bending moment created by PZT actuators.

Equation (17) is rewritten as

\begin{equation}
  M \ddot{\eta} + C \ddot{\eta} + K\eta = -M \ddot{\rho} - M \ddot{\beta} + F_{fg}
\end{equation}
It is clearly shown in Equation (6.12) that the essential contribution of SRF control is to introduce the damping term $\eta \dot{\theta}$ into the dynamic system, and hence suppress the vibration of the flexible links.

In control applications, the second-order system dynamic equations are generally expressed in state-variable or first-order format. The state-space representation of the dynamic equations of the parallel manipulator with flexible links may be formulated as,

$$\begin{align*}
\dot{z} &= Az + Bu + d \\
y &= Ez \\
u &= -G\dot{y}
\end{align*}$$

(6.13)

where $z = (\vec{\eta}, \dot{\vec{\eta}})^T$, $y$ is a measured voltage vector from PZT sensors, $u$ is a control voltage vector applied on PZT actuators, and $G$ is a control gain matrix.

Comparing equation (6.13) and equation (6.14), A, B, d and E are given as follows:

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ M^{-1}(M^{-1}\ddot{\beta} + M^{-1}\ddot{\beta} - F_{fr}) \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 & 0 & 0 \\ 0 & \bar{B}_2 & 0 \\ 0 & 0 & \bar{B}_3 \end{bmatrix}.$$  

$$\bar{B}_i = \begin{bmatrix} k_{i1}(\psi_{i1}'(x_{k2}) - \psi_{i1}'(x_{k1})) & \cdots & k_{i1}(\psi_{i1}'(x_{k2}) - \psi_{i1}'(x_{k1})) \\ \vdots & \vdots & \vdots \\ k_{i1}(\psi_{ir}'(x_{k2}) - \psi_{ir}'(x_{k1})) & \cdots & k_{i1}(\psi_{ir}'(x_{k2}) - \psi_{ir}'(x_{k1})) \end{bmatrix}, \quad i = 1, 2, 3,$$

$$E = \begin{bmatrix} \bar{E}_1 & 0 & 0 \\ 0 & \bar{E}_2 & 0 \\ 0 & 0 & \bar{E}_3 \end{bmatrix}, \quad \bar{E}_i = \begin{bmatrix} k_{i1}(\psi_{i1}'(x_1)) & \cdots & k_{i1}(\psi_{i1}'(x_1)) \\ \vdots & \vdots & \vdots \\ k_{i1}(\psi_{ir}'(x_m)) & \cdots & k_{i1}(\psi_{ir}'(x_m)) \end{bmatrix}, \quad i = 1, 2, 3.$$
The control input voltage applied on PZT sensors $u$ is expressed in terms of state vector as
\[
\begin{align*}
\dot{u} &= -G\dot{y} = -G\dot{E}\ddot{z} = -G[\ddot{E} \ 0] \ddot{z} = -G[0 \ \dddot{E}]z = -G^z z \\
\end{align*}
\]
(6.14)

where $E^* = [0 \ \dddot{E}]$, and $G^* = GE^*$ (note that $\dddot{E} = \dddot{E}$ since $\dddot{E}$ is not time-dependent).

The ultimate aim of state feedback vibration control is to reduce the link structural vibration to the greatest possible extent. One of the commonly used methods of modern control theory is optimal control. The method calculates the feedback gain by the minimization of a cost function or performance index, which is proportional to a measure of the system’s response. In other words, the control input $u(t)$ is designed to minimize a cost function or performance index, denoted by $J = J(z,u,t)$. Given the initial conditions $z(t_0)$ and $\dot{z}(t_0)$ subject to the constraint that
\[
\begin{align*}
\dot{z} &= Az + Bu + d \\
y &= Ez \\
\end{align*}
\]
(6.15)

For active vibration control design, the cost function is selected as
\[
J = \frac{1}{2} \int_0^\infty [z^T Q z + u^T R u] dt \\
\]
(6.17)

where weighting matrices $Q$ and $R$ indicate a tradeoff between the structural vibration control efficiency and the required control input energy.

According to the optimal control theory, the control input is given as
where $P$ is found by solving the Algebraic Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \tag{6.19}$$

Comparing equation (6.18) with (6.14), we have

$$G^* = GE^* = R^{-1}B^TP \tag{6.20}$$

Then, we can obtain the optimized control gain matrix for control voltages of PZT actuators,

$$G = R^{-1}B^TP(E^*)^T[E^*(E^*)^T]^{-1} \tag{6.21}$$

### 6.4 Numerical Simulation and Analysis

Numerical simulations for the 3-PRR parallel manipulator with three flexible intermediate links are presented. In these simulations, a circular motion is assigned as a desired trajectory for the mass center point of the moving platform with constant $\rho_p$. The equations for the circle are $x_p = 50\cos(20\pi) - 50(mm)$, $y_p = 50\sin(20\pi)(mm)$, and $\varphi_p = 45^\circ$. Three intermediate links are modeled as aluminum alloy and have identical geometric parameters listed in Table 6.1. PZT actuators and sensors are selected as material BM 532 manufactured by Sensor Technology. The material parameters and dimensions of PZT actuators are listed in Table 6.2. The first three order mode is targeted to be controlled in this simulation. Considering that the midpoints of intermediate links are the nodes of the second order mode, and to decrease the control voltages required to be applied to each PZT actuator, three PZT actuators and sensors are bonded to each of three flexible links. Three actuators and three sensors are bonded at the
midpoint, quarter point, and three quarters point for each flexible link. SRF control strategy is used to design the active vibration controller in modal space. The rule for the selection of control gain $k_{Di}$ is that required control voltages of PZT sensors is less than 50V, and the oscillatory component of the transverse deformation of the intermediate links is suppressed to be close zero in less than 0.1 second. According to this rule, the control gain $k_{Di}$ is tried and selected as $1.5 \frac{N \cdot m}{V^2}$ in this simulation. In the strategy applied in this work, no rigorous attempt to calculate the optimal number and locations has been made by the authors. The ideal number and location of sensors and actuators are the complex problems for multiple sensor-actuator pairs and multiple modal vibration controls. These problems can be solved by genetic algorithms or other optimization techniques [37, 38]. While not the subject of this work, the control gain should be adjusted according to the collocation of PZT transducers and tuned to meet the requirement of uncertain and unmodelled dynamics by advanced control approaches, such as adaptive control, learning control, etc.

<table>
<thead>
<tr>
<th>Density</th>
<th>2.77×10^3 kg/m^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus</td>
<td>7.1×10^10 N/m^2</td>
</tr>
<tr>
<td>Dimension</td>
<td>200mm×30mm×2mm</td>
</tr>
</tbody>
</table>

Table 6.2 Parameters of PZT actuators

<table>
<thead>
<tr>
<th>Piezoelectric constant $d_{31}$</th>
<th>-270×10^{-12} C/N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus</td>
<td>6.3×10^10 N/m^2</td>
</tr>
<tr>
<td>Actuator Dimension</td>
<td>25.4mm×25.4mm×0.254mm</td>
</tr>
</tbody>
</table>
Figure 6.2 Deformation at the midpoint of the 1st intermediate link

Figure 6.3 First three vibration modes of the 1st intermediate link
In the simulations presented here, the first three modes are selected to model the structural flexibility of intermediate links. Therefore, flexible generalized coordinates in equations (6.12) are \( \eta = [\eta_1, \eta_2, \eta_3, \eta_{21}, \eta_{22}, \eta_{23}, \eta_{31}, \eta_{32}, \eta_{33}]^T \). The rigid-body motion of slides and intermediate linkages is derived from the given motion of the moving platform by solving inverse kinematics of the parallel manipulator. Substituting the rigid-body motion into equations (6.12), the equations become ordinary differential equations (ODEs). With the initial conditions \( \vec{\eta} = 0_{9 \times 1}, \ddot{\vec{\eta}} = 0_{9 \times 1} \), equations (6.12) are solved for the first two periods (50 ms for a period) of the moving platform using MATLAB ODEs solver \textit{ode113} based on a variable order Adams-Bashforth-Moulton method [137].

![Figure 6.4 Frequency spectra without control](image1)

![Figure 6.5 Frequency spectra with control](image2)

Figure 6.4 Frequency spectra without control  
Figure 6.5 Frequency spectra with control

Figure 6.2 presents elastodynamic response of the midpoint of the first intermediate link with and without control action. The elastodynamic response is the superposition of structural free vibration and substantially lower frequency deflection caused by inertial forces and coupling forces. It is clearly illustrated, in this figure, that the magnitude of the midpoint deflection due to free vibration was substantially suppressed by employing PZT actuators. Figure 6.3 further
reveals that the first three modes of vibrations are effectively reduced, and the second and third mode of vibrations are reduced substantially more than the first mode of vibration. It is interesting to observe from these figures that the deflection caused by structural free vibration was completely eliminated by the imposed control voltages, which are shown in Figure 6.7, while the deflection caused by the inertial forces and coupling forces remained. This is the difference from the situation of controlling residual vibrations. This phenomenon is confirmed by frequency analysis of flexible intermediate links and the spectral analysis of the elastodynamic response, with and without PZT actuator control inputs, given in Figure 6.4 and Figure 6.5. The power spectral density for the vibration response illustrates in Figure 6.4 that the vibration energy is greatest at the frequencies of 10Hz, 20Hz, and 116.4 Hz. From equation (6.12), the vibration response includes free vibration and forced vibration from inertial and coupling forces. The $j^{th}$ mode vibration frequencies of the $i^{th}$ intermediate links can be written as

$$\omega_j = (j\pi/l_i)^2 \sqrt{EI_i/\rho} \omega_i.$$ 

Therefore, the first three mode frequencies are calculated to be 116.4 Hz, 465.6 Hz, and 1047.5 Hz. This implies that in Figure 5, the vibration with the frequency of 116.4 Hz is free vibration, and the vibrations with frequencies of 10 Hz and 20 Hz are forced vibrations. Figure 7 shows the time history of the inertial and coupling forces applied on the first intermediate link expressed in the first mode space, where $F_1$ corresponds to $-M_\rho \ddot{\theta}$, $F_2$ corresponds to $-M_S \ddot{\Phi}$, and $F_3$ corresponds to $F_{f_k}$. Frequency spectral analyses for the inertial and coupling forces are performed using FFT. Analysis results show that frequencies of these forces are 10, 20 and 30 Hz, and the power density of 10 Hz exhibits the largest peak, which is consistent with the frequency spectra of the vibration response, as shown in Figure 6.4. Comparing Figure 6.4 and Figure 6.5, it is clearly shown that the vibration with frequency of
116.4 Hz was completely suppressed, but the vibrations with frequencies of 10 and 20 Hz were not reduced. These results are consistent with the results given in Figure 6.2 and Figure 6.3.

Figure 6.6 Inertial and coupling forces of the 1st mode

Figure 6.7 Voltages on the PZT actuators of the 1st link

Figure 6.3 reveals that the amplitude of the first mode vibration of the first intermediate link is two orders of magnitudes larger than the amplitude of the second mode vibration. Therefore, the dynamic responses contributed by those neglected modes are rather small. It is obvious that the first mode is sufficiently accurate to describe the vibration of the flexible intermediate link. Therefore, it is reasonable to reduce the number of modeled vibration modes to a small finite number, which can be further reduced for real-time control.

6.5 Summary

This chapter has presented simulation results of the active vibration control of elastodynamic response of the moving 3-PRR parallel manipulators with three flexible intermediate links bonded with multiple PZT actuators and sensors. An active PZT actuator and sensor controller has been designed in modal space based on strain rate feedback control.
Incorporated with control forces from PZT actuators, a procedure for the generation of dynamic equations for a 3-PRR parallel manipulator with three flexible intermediate links has been addressed, based on the assumed mode method. Then, the design of an optimal active vibration controller is presented for the parallel manipulator with flexible links with bonded PZT transducers based on SRF. Finally, numerical simulations are performed, and simulation results show that the deflections caused by structural free vibration were almost completely eliminated by the imposed control voltages, but the deflections caused by the inertial forces and coupling forces were not reduced. The unreduced deformations may be suppressed using other control strategies such as controlling the joint torque, introducing additional microprocessor controlled actuators to the original mechanism.
Chapter 7

Vibration Control Experimental System and Implementation

7.1 Introduction

A major difficulty in the vibration control of flexible structures results from the fact that flexible structures are commonly distributed parameter systems with an infinite number of vibration modes. However, in practice, only a limited small number of modes of the structures can be or are required to be controlled. Therefore, there exist uncontrolled modes which may lead to spillover, a phenomenon in which control energy flows to the uncontrolled modes of the system. Spillover may result in instability and degradation of control performance (Preumont [14]). Therefore, the IMSC method (Meirovitch and Baruh [115], Baz and Poh [116], Singh et al. [117], Baz et al. [118]) was developed to prevent from the spillover problem by controlling each mode separately. In the design of a modal feedback controller, the modal coordinates and/or modal velocities for the modes targeted for control must be known. Therefore, real-time monitoring or sensing modal coordinate is significant in the vibration control of flexible structures. Various methods have been presented to extract the modal coordinates from the outputs of the sensors. These methods include state observers (Brogan [119]), temporal filters (Hallauer et al. [120]), and modal filters (Meirovitch and Baruh, [121]). It has been shown that the use of observers causes observation spillover from the residual modes, which can destabilize the residual modes. Using temporal filters, the outputs of sensors are processed using high-pass
or low-pass filters to filter out the contribution of each mode. Such a method does not work well when the modes are closely spaced. Using modal filters, the task of extracting modal coordinates from the sensor outputs is distinct from the control task, which permits the use of modal filters in conjunction with any modal feedback control method. Furthermore, modal filters only involve spatial integration, which is a smoothing operation that cannot lead to instability. Two different modal filtering methods have been presented for modal filters. One is a modal filter with a distributed element (Collins et al. [122]). The other is a modal filter with discrete elements (Sumali et al. [123]). It is often difficult to implement modal filtering with a distributed element because it requires one sensor for each mode. Therefore, it is reasonable to perform modal filtering with discrete sensors in real time. Sensing modal coordinates in real time involves interpolations or curve fitting. All computations must be carried out within a single sampling period because the controllers are implemented in discrete time.

There are many simulations and experiment demonstrations of active vibration control in the space-based flexible structures and simple flexible beams (Goh and Caughey [104], Fanson and Caughey [105], Song et al. [106], Rew et al. [107], Moheimani et al. [108], Alhazza et al. [146]). A few researchers introduced PZT transducers for the vibration control of manipulators and mechanisms (Shan et al. [124], Liao and Sung [125], Thompson and Tao [126], Preiswerk, and Venkatesh [127]). However, most of the early research work involved the active vibration control of manipulators and mechanisms with a single link with a single bonded actuator and sensor. Compared with numerical simulation, few investigations have been conducted towards experimental investigations, especially for parallel manipulators with multiple flexible links, due to their complicated dynamics. Wang [12] conducted residual vibration control experiments of
the planar parallel manipulator. Note that the active vibration control experiments were only implemented on one single flexible link, with one mode targeted for control.

Compared with simple beams, experimental implementation of the vibration control of moving flexible manipulators and mechanisms is a more challenging problem, especially for parallel manipulators. For the proposed manipulator, rigid body motion and elastic deformation are dynamically coupled. This characteristic leads to the coupling of vibration modes, and may cause spillover. It is problematic to extract independent mode coordinates using only temporal filters which are commonly used in the vibration control strategy of simple beams with fixed bases. In practice, controller design is based on the known or modeled dynamics. However, the dynamic responses measured by PZT sensors include the unmodeled or unknown dynamics, including for example, compliance and clearance dynamics from the bearings, ball screw mechanisms, and motors. Therefore, the vibrations of the intermediate links are very complicated, and are the combination of free structural vibrations and forced vibrations, which contain many frequency components which are closely spaced. To more precisely measure modal coordinates or velocities, it is desirable to use as many PZT sensors as possible. However, the number of PZT sensors is limited by available space on the flexible links, and the real-time capability of the computing platform. A trade-off must be made between the real-time capability of the controller computing hardware and the number of sensors.

To address the above problems and challenges, this chapter presents an experimental investigation of the active vibration control of a 3-PRR parallel manipulator with three flexible links, each of which is bonded with three PZT actuators and sensors, when the manipulator is moves along a prescribed trajectory. The control experiments are implemented in real time using
LabVIEW Real-Time and Two-CPU programming technologies. The active vibration control system consists of a host PC on which active vibration control algorithms are developed and data are collected, and a second PC used for real-time control, named the target PC. The target PC executes control algorithms downloaded from the host PC. The controller is designed in the modal space based on strain rate feedback control, and the simplified modal filters are used to measure the modal coordinates and velocities. To prevent probable destabilization resulted from the differentiation of the SRF controller, a compensator is used to cut off the amplified noises and unmodeled dynamic signals with high frequencies. The position error of the moving platform has been investigated using the measured voltages of PZT sensors with the consideration of the shortening of the intermediate links. The gain amplification problem for high modes has been addressed, and provides guidelines for the control gain selection in the design of the modal controller. An efficient multi-mode control strategy has been proposed based on modifying the independent modal space control method. Experiments are first conducted with either one, or two or three PZT control pairs applied to one intermediate link, to provide insight and guidance into the selection of an acceptable number and location of PZT control pairs.

Section 7.2 presents the development of the active vibration control experimental system for the parallel manipulator with three flexible intermediate links with multiple bonded PZT transducers. In Section 7.3 addresses Smart structures based on SRF. Sections 7.4, 7.5, and 7.6 present the direct output feedback control (DOFC), independent modal space control (IMSC), and the implementation of IMSC. The design of second order compensator (filter) is presented in Section 7.7. Mode characteristic analysis is performed in Section 7.8. An efficient control gain selection strategy is proposed for multi-mode control in Section 7.9. Section 7.10 presents an approach to estimate the position error of the moving platform with the consideration of the
shortening of the intermediate links. The experimental results are given in Section 7.11. Section 7.12 provides the summary of this chapter.

### 7.2 Active Vibration Control Experiment System Setup

![Figure 7.1 Photograph of the active vibration control system setup](image)

Figure 7.1 Photograph of the active vibration control system setup
An experimental setup, as shown in Figures 7.1 and 7.2, has been developed and built to conduct active vibration control investigations. This experimental system consists of a 3-PRR planar parallel manipulator, a three-axis motion control system, and an active vibration control system based on LabVIEW Real-Time. The 3-PRR parallel manipulator and the three axis motion control system are presented in Chapter 2. This section focuses on the active vibration control system.

Figure 7.2 Schematic diagram of the control architecture
The hardware architecture of the active vibration control system consists of PZT actuators (BM532), PZT sensors (BM532), impedance converters Kistler 558, an actuating amplifier SS08 (Sensor Technology), a signal conditioner (NI SCXI 1531), a low pass filter, two digital to analog conversion (D/A) boards (NI PCI6733, 16 bits) with eight channels and an analog to digital conversion (A/D) board (NI PCI6031E, 16 bits) with 32 channels. The A/D and D/A boards are inserted into one slot of the target PC. A signal measured by a PZT sensor is a charge signal with high impedance. Therefore, the charge signal is converted to a high-level voltage signal with low impedance by an impedance converter. Then, the voltage signal is filtered and amplified by SCXI 1531, acquired by the A/D board and input the controller. A digital output signal is converted to an analog signal by the D/A board, and filtered by the low pass filter. The filtered output signal is amplified by SS08 amplifier, and applied on a PZT actuator.

The active vibration control system is built based on National Instruments LabVIEW with the LabVIEW Real-Time Module [145]. The system consists of two PCs: a host PC and a target PC. Using LabVIEW graphical programming, a LabVIEW Real-Time embedded control application is developed on the host PC, and then the program is downloaded to, and run on the target PC. The two PCs communicate over Ethernet using TCP/IP, an industry standard communication protocol. Two individual loops, namely the deterministic loop (the time-critical loop or control loop), and the nondeterministic loop (the communication loop) are designed and built during programming an application with the LabVIEW Real-Time Module. These loops run in separate threads at different priorities. The highest execution priority is set to the deterministic loop which contains deterministic control codes. The normal priority is set to the communication loop, which contains the network communication codes and logging data codes. Therefore, the real-time property can be guaranteed by prioritizing control tasks so that the most critical task
always take control of the processor when needed. The active vibration controller is developed with the identical architecture for each intermediate link with bonded PZT transducers.

### 7.3 Smart Structures Based on Strain Rate Feedback Control

The link flexibility, sensor and actuator dynamics are presented in previous chapters, and hence the details are not presented in this chapter. The “smart” structure aspects of the system proposed here refers to the use of PZT sensor and actuator system and corresponding active control system. A smart structure was built by bonding PZT sensors/actuators to the two sides of the intermediate link as shown in Figure 7.3. The PZT patches on one side of the link act as sensors, while the PZT patches on the opposite link face act as actuators. One sensor and one actuator constitute a PZT control pair based on the strain rate feedback control strategy, and are located at the same location along the length of each intermediate link. Sensor-actuator collocation increases the robustness of control laws designed for active vibration control, and results in a minimum phase system due to pole-zero interlacing (Preumont [14]).

![Figure 7.3 A smart link structure with PZT actuator and sensor patches (top view)](image-url)
Using the assumed mode method based on pinned-pinned boundary conditions, the dynamic equation of an intermediate link can be given in modal space as

$$\ddot{\eta} + D \dot{\eta} + \Omega \eta = f_w + f_u$$

(7.1)

where $\eta = [\eta_1 \cdots \eta_n]^T$ is the modal coordinate vector of an intermediate link, $D$ is the modal damping matrix, $\Omega = \text{diag}(\omega_i^2)$ is the structural modal stiffness matrix, $\omega_i$ is the $i$th order modal frequency, and $f_u$ reflects the modal force from the effect of the rigid-body motion on the elastic vibration of the flexible links, and the coupling between the rigid-body motions and the elastic motions. $f_u$ is the modal control force vector transformed from bending moment created by PZT actuators. Based on the assumption of Rayleigh damping, the damping matrix $D$ is diagonal and expressed by $D = \text{diag}(2\xi_i m_i \omega_i)$.

### 7.4 Direct Output Feedback Control

One active vibration control strategy is direct output feedback control (DOFC) (strain or/and strain rate). This control approach does not demand modal state estimation. In this method, the sensors are collocated with the actuators, and a given actuator control force is a function of the sensor output at the same location along the intermediate link. Thus, one PZT actuator and one PZT sensor constitute a PZT control pair based on the strain and strain rate feedback control strategy. Sensor-actuator collocation increases the robustness of control laws designed for active vibration control, and makes the control system be minimum phase due to pole-zero interlacing.

Using strain and strain rate feedback, the control voltage applied a PZT actuator (center point of the PZT actuator) located at $x_k$ along the intermediate link is given as,
\[ V_a(t) = -k_p V_a(t) - k_d V_a(t) = -K_x \left\{ k_p \sum_{i=1}^{n} \left( \varphi_i^r(x_k) \eta_i(t) \right) + k_d \sum_{i=1}^{n} \left( \varphi_i^r(x_k) \dot{\eta}_i(t) \right) \right\} \quad (7.2) \]

where \( k_p \) and \( k_d \) are the feedback gains in terms of voltages and voltage rates. The modal force vector produced by the PZT actuator is expressed as

\[ F_a = -K_x K_a \left[ \begin{array}{c} (\varphi_1'(x_k^2) - \varphi_1'(x_k^1)) \varphi_1^r(x_k) \\ \vdots \\ (\varphi_n'(x_k^2) - \varphi_n'(x_k^1)) \varphi_n^r(x_k) \end{array} \right] \left[ \begin{array}{c} \eta_1 \\ \vdots \\ \eta_n \end{array} \right] + k_p \left[ \begin{array}{c} \eta_1 \\ \vdots \\ \eta_n \end{array} \right] + k_d \left[ \begin{array}{c} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_n \end{array} \right] \quad (7.3) \]

where \( x_k^2 \) is the coordinate at the right end of the PZT actuator, and \( x_k^1 \) is the coordinate at the left end of the PZT actuator. Note that we assume that all PZT sensors are identical, and all PZT actuators are identical in this work. In practice, the controller is designed with only the limited low order mode targeted for control (for example, \( c \) modes). Equation (7.3) can be partitioned and written as

\[ F_a = \begin{bmatrix} F_{c} \\ F_{n-c} \end{bmatrix} = \begin{bmatrix} \psi_{c,c} & \psi_{c,n-c} \\ \psi_{n-c,c} & \psi_{n-c,n-c} \end{bmatrix} \begin{bmatrix} \eta_c \\ \eta_{c+1} \\ \vdots \\ \eta_n \end{bmatrix} + k_p \begin{bmatrix} \eta_c \\ \eta_{c+1} \\ \vdots \\ \eta_n \end{bmatrix} + k_d \begin{bmatrix} \dot{\eta}_c \\ \dot{\eta}_{c+1} \\ \vdots \\ \dot{\eta}_n \end{bmatrix} \quad (7.4) \]

In equation (7.4), \( F_c \) is the modal control force for the targeted modes. \( F_{n-c} \) is the modal control force flow into the uncontrolled modes, and it could result the control spillover. The term \( \psi_{c,n-c}(k_p \ddot{n}_{n-c} + k_d \dot{n}_{n-c}) \) in \( F_c \) is caused by the excited uncontrolled modes flowing to the modes.
targeted for control. The interactive action between controlled modes and uncontrolled modes finally could lead to the instability of the control system. Furthermore, it is clear that the control force is coupled among the controlled modes since the matrix

\[
\begin{bmatrix}
\psi_{c,x} & \psi_{c,n-c} \\
\psi_{n-c,x} & \psi_{n-c,n-c}
\end{bmatrix}
\]

in equation (7.4) is not a diagonal matrix, and hence the controlled \( c \) modes are dependent of each other. This fact leads to the problem of determining control gains when multiple PZT actuators are applied. The reason for that is that pole allocation or optimal control most likely will require gain matrices with entries independent of each other while DOFC implies that the entries of the control gain matrix are not independent for multi-actuators.

### 7.5 Independent Modal Space Controller

Independent Modal Space Control (IMSC), the use of modal coordinates in feedback control can prevent control spillover, a phenomenon that results in degradation of performance or instability. Using modal coordinates, a feedback control problem of continuous structures can be transformed to the problem of controlling several single-degree-of-freedom (SDOF) systems in parallel, with no interaction among the systems. Therefore, feedback controllers can be designed in the independent modal space so that each targeted mode is controlled by one independent modal controller associated with only its own modal displacement and velocity. With IMSC, the modal control force \( f_u^i \) for the \( i^{th} \) mode only depends on \( \eta_i \) and \( \dot{\eta}_i \). \( f_u^i \) is given as

\[
f_u^i = -k_i^i \eta_i - k_i^\dot{\eta}_i
\]

Equation (7.5) illustrates that the coupling of modal equations due to feedback is avoided with IMSC. Combining equations (7.1) and (7.3), the closed-loop modal equations can be expressed with the independent modes as
\[ \ddot{\eta}_i + (2\xi_i \omega_i + k_i^d) \dot{\eta}_i + (\omega_i^2 + k_i^u) \eta_i = f_{w_i} \quad i = 1, \cdots, n \] (7.6)

Many approaches, such as pole assignment or optimal control, exist to determine the control gain in equation (7.5). In the optimal control approach the control gains are determined through optimizing a prescribed performance index. In the IMSC method, a quadratic function, the combination of the modal potential energy \( \omega_i^2 \eta_i^2 \), modal kinetic energy \( \dot{\eta}_i^2 \), and the required modal control force \( f_u^i \), is typically chosen to be the performance index given as

\[ J = \int_0^\infty \left[ (\omega_i^2 \eta_i^2 + \dot{\eta}_i^2) + \beta (f_u^i)^2 \right] dt \] (7.7)

In equation (7.7), \( \beta \) is modal weight used to signify the importance of vibration energy suppression and the required control effort. The closed-loop solution to equation (7.7) can be obtained by solving a matrix Riccati equation, and is given as

\[
\begin{align*}
-k_i^d &= -2\xi_i \omega_i + [4\xi_i^2 \omega_i^2 + \beta^{-1} - 2\omega_i^2 + 2\omega_i (\omega_i^2 + \beta^{-1})^{1/2}]^{1/2} \\
k_i^u &= -\omega_i^2 + \omega_i (\omega_i^2 + \beta^{-1})^{1/2}
\end{align*}
\] (7.8)

### 7.6 Implementation of IMSC

In practical experiments, the output voltage of a PZT sensor corresponds to the physical link physical deformation coordinates, not modal coordinates. Therefore, to implement IMSC control, the modal coordinates must be extracted from the output voltages of discrete PZT sensors in real time. The modal coordinates, extracted in real time, are provided to the modal feedback controller. Using the modal synthesizer, the modal control voltages are transformed back to the control voltages in physical space, and are used as inputs to the PZT actuators.
The implementation of IMSC is illustrated in Figure 7.4. Note that we assume all PZT sensors are identical, and PZT actuators are identical as well. Since the length of PZT actuators is much smaller than the length of intermediate links, \( \varphi_i(x_i^R) - \varphi_i(x_i^I) \) is approximated to be \( \varphi_i(x_i) l_a \), where \( l_a \) is the length of a PZT actuator.

![Diagram of IMSC implementation based on PZT transducers](image)

The modal filter expression for each smart link is expressed as

**Figure 7.4 Schematic of IMSC implementation based on PZT transducers**

The modal filter expression for each smart link is expressed as
Chapter 7  
Active Vibration Control Experimental System and Implementation

\[ \eta(t) = \Phi_m \bar{V}(t) \]  

(7.9)

where \( \eta(t) = (\eta_1(t) \quad \eta_2(t) \quad \cdots \quad \eta_n(t))^T \) is modal coordinate vector of the intermediate link, \( \bar{V}(t) = (V_{s1}(t) \quad V_{s2}(t) \quad \cdots \quad V_{sm}(t))^T \) is the output vector of the \( m \) sensors bonded to the intermediate link, and \( \Phi_m \) is a \( n \times m \) modal coordinate transformation matrix or modal analyzer. \( \Phi_m \) is given as

\[ \Phi_m = \frac{1}{K_s} (\psi^T \psi)^{-1} \psi^T \]  

(7.10)

The matrix \( \psi \) is calculated with the values of the mode shape function at the discrete sensors

\[ \psi = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(x_m) & \varphi_2(x_m) & \cdots & \varphi_n(x_m) \end{bmatrix} \]  

(7.11)

The modal coordinates, extracted in real time, are provided to the modal feedback controller. Using the modal synthesizer, the modal control voltages are transformed to the control voltages in physical space, and are used as inputs to the PZT actuators. The implementation of the derivative algorithm tends to amplify noise or unmodeled vibration signal at higher frequencies, due to its derivative operation. In practice, it is desirable to add a compensator (shown in Figure 7.4) so that the SRF controller exhibits guaranteed stability and a roll-off at high frequency. The design of the compensator based on SRF is presented utilizing vibration theories in the next section.
For the intermediate links of the manipulator, the mode shape function of the $i^{th}$ mode is expressed as $\varphi_i(x) = \sin\left(\frac{i\pi}{l}x\right)$ based on pinned-pinned boundary conditions, which were experimentally validated in this work. The second order partial derivative of the $i^{th}$ order mode shape function is given as

$$\varphi''_i(x) = -\left(\frac{i\pi}{l}\right)^2 \sin\left(\frac{i\pi}{l}x\right) = -\left(\frac{\pi}{l}\right)^2 i^2 \varphi_i(x)$$

(7.12)

Note that the transformation from the input control voltages of PZT actuators to modal control forces, as shown in Figure 1, employs the transformation matrix consisting of $\varphi_i'(x)$, and therefore, the sign of equation (7.12) in the controller design may be ignored. It is obvious that the modal coordinate vector from the modal filter is $\begin{pmatrix} \eta_1(t) \\ 2^2 \eta_2(t) \\ \cdots \\ n^2 \eta_n(t) \end{pmatrix}$, not $\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \cdots \\ \eta_n(t) \end{pmatrix}^T$. That means the modal coordinates of the $i^{th}$ mode is amplified by $i^2$ before entering the modal controller, and then the $i^{th}$ order modal control force is further amplified by $i^2$ due to PZT actuators. This amplification behavior for higher order modes, arising from the use of PZT transducers, means the control gain of the $i^{th}$ mode is increased by a factor of $i^4$. This phenomenon has not been addressed in our previous publications since only a single mode was controlled, but it must be considered in the implementation of a multi-mode vibration controller. The modal control gain of the $i^{th}$ mode should be reduced by a factor of $i^4$ when the controller is implemented experimentally.
7.7 Compensator Design Based on SRF

The implementation of the derivative algorithm tends to amplify noise or the unmodeled vibration signal at higher frequencies due to its derivative operation. In practice, it is desirable to add a compensator so that the SRF controller exhibits guaranteed stability and a larger roll-off at high frequency [105, 109]. The basic idea is to pass the strain rate feedback signal through a second order filter with substantial damping, and generate a feedback proportional to the output of the filter. To better illustrate the operation principle of SRF controller with a compensator, it is assumed that a structure may be simplified to be a single degree of freedom, single-input-single-output vibration system using one collocated one PZT sensor and actuator pair. The block diagram which describes the control strategy is shown in Figure 7.5.

\[
\dot{\eta}_s + 2\zeta_s \omega_s \dot{\eta}_s + \omega_s^2 \eta_s = -gc_1 \omega_s^2 \eta_c
\] (7.13)

\[
\dot{\eta}_c + 2\zeta_c \omega_c \dot{\eta}_c + \omega_c^2 \eta_c = c_2 \omega_c^2 \dot{\eta}_s
\] (7.14)

Figure 7.5 Block diagram for SRF controller with compensator
where $\eta_s$ is a modal coordinate of the structure, $\zeta_s$ is the damping ratio of the structure, $\omega_s$ is the natural frequency of the structure, $\eta_c$ is a modal coordinate of the compensator, $\zeta_c$ is the damping ratio of the compensator, $\omega_c$ is the resonant frequency of the compensator, $g$ is a strain rate back feedback gain, and $c_1$, $c_2$ are constants which are determined by the sensitivity of the actuator and sensor, respectively. It should be noted that equation (7.13) and equation (7.14) are coupled by the physical interaction between control signals, through coordinate $\eta_s$, and the structure.

Assume that the vibration solution for the structure with a single degree of freedom is given in the form as

$$\eta_s = \mu e^{i\omega t}$$  \hspace{1cm} (7.15)

The steady vibration solution of the compensator is given as

$$\eta_c = \lambda e^{i(\omega t + \pi/2 - \phi)}$$  \hspace{1cm} (7.16)

where $\lambda = c_2 \mu \omega_c / \sqrt{(1 - \omega_s^2 / \omega_c^2)^2 + (2 \zeta_c \omega_s / \omega_c)^2}$ and $\phi = \tan^{-1}\left[(2 \zeta_c \omega_s / \omega_c)/(1 - \omega_s^2 / \omega_c^2)\right]$.

From equation (7.15) and equation (7.16), we have

$$\eta_c = \lambda e^{i(\omega t)} e^{i(\pi/2 - \phi)} = \frac{c_2 \omega_c}{\sqrt{(1 - \omega_s^2 / \omega_c^2)^2 + (2 \zeta_c \omega_s / \omega_c)^2}} \eta_s$$  \hspace{1cm} (7.17)

Substituting equation (7.17) into equation (7.13), the structural vibration equation with SRF controller is expressed as three different forms with the relationship between the structural
natural frequency, \( \omega_s \), and the compensator resonant frequency, \( \omega_c \). When \( \omega_s \ll \omega_c \), the phase angle \( \phi \) approaches zero. Substituting equation (7.17) into (7.13), equation (7.13) can be formulated as

\[
\ddot{\eta}_s + (2\xi_s + gc_1\lambda)\omega_s \dot{\eta}_s + \omega_s^2 \eta_s = 0
\]  
(7.18)

Equation (7.18) clearly shows that the essential role of the SRF compensator is to increase the damping ratio of the structure. When \( \omega_s = \omega_c \), the phase angle \( \phi \) is equal to \( \pi/2 \). Substituting equation (7.17) into (7.13), equation (7.10) can be expressed as

\[
\ddot{\eta}_s + 2\xi_s \omega_s \dot{\eta}_s + (\omega_s^2 + gc_1\lambda\omega_s^2)\eta_s = 0
\]  
(7.19)

Equation (7.19) shows that the SRF compensator leads to the increase of the stiffness of the structure. When \( \omega_s \gg \omega_c \), the phase angle \( \phi \) approaches \( \pi \). Substituting equation (7.17) into equation (7.13), equation (7.13) can be presented as

\[
\ddot{\eta}_s + (2\xi_s - gc_1\lambda)\omega_s \dot{\eta}_s + \omega_s^2 \eta_s = 0
\]  
(7.20)

Equation (7.20) demonstrates that the compensator tends to decrease the damping ratio of the structure in this case. Therefore, in the design of SRF control with the compensator (filter), the compensator should be designed to make sure that the targeted frequencies are below the compensator frequencies. The above strategy can be extended to control multiple modes by introducing multiple compensators in parallel (Rew et al. [107]). One compensator is designed for each targeted mode.
7.8 Mode Characteristic Analysis

Using the modal filters and the design of IMSC, the dynamic equations of intermediate links with bonded PZT control patches can be decoupled and decomposed to single-degree-of-freedom systems. These single DOF systems can expressed as second order ordinary differential equations in terms of modal coordinates. Therefore, the \( i^{th} \) mode dynamic equation of each intermediate link is given as

\[
\ddot{\eta}_i + 2\xi_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = f_{i}^\eta + \sum_{j=1}^{n} f_{ij}^w \cos(\omega_j t)
\]  

(7.21)

In equation (7.21), the second term of the right hand side represents the modal forces arising from inertia, coupling between rigid body motions and elastic deformations, constraint forces, forces resulting from motor dynamics, clearances, ball screw mechanisms and bearings. This term makes the modal coordinates complicated signals with multiple frequency components \( \omega_j \).

The modal coordinate for one certain order mode might contain the frequency components from the other modes. These modes are dynamically coupled, which leads to vibration modes that cannot be decoupled completely. The complete modal response which includes the free modal vibration and the forced vibration is expressed as

\[
\eta_i(t) = \eta_{i0} e^{-\xi_i \omega_i t} \cos(\sqrt{1-(\xi_i \omega_i)^2} \omega_i t - \phi_{i0}) + \sum_{j=1}^{n} \frac{f_{ij}^w}{\omega_j} \cos(\omega_j t - \phi_j) \left( 1 - \left( \frac{\omega_j}{\omega_i} \right)^2 \right)^{1/2} + \left[ 2\xi_i \frac{\omega_j}{\omega_i} \right]^{1/2} \right) \}
\]  

(7.22)
It is shown in equation (7.22) that the essential contribution of SRF control is to increase modal damping ratio from $\xi_i$ to $\xi^*_i$, and therefore reduce the free modal vibration and the forced vibration.

### 7.9 Efficient Control Gain Selection Strategy for Multi-mode Control

The control gains of higher modes are larger than the control gains of lower modes if the control gains of all modes are determined based on IMSC, as shown in equation (7.8). On the other hand, the vibration amplitudes of higher modes are substantially less than those of the lower modes, and hence the suppression of higher-mode vibration can be achieved with lower damping applied. Further, a mode with higher frequency can be implemented with active damping for more number of cycles in the same interval of control time, and hence less damping can be provided to that mode at each control update. Therefore, the controlled forces on PZT actuators may achieve unnecessarily high values when the modal feedback control gains of all modes are chosen based on IMSC, possible resulting in instability of the control system when multi-mode vibrations are controlled simultaneously. To overcome this problem, the IMSC is modified and an efficient control gain selection strategy for multi-mode vibration control is proposed to obtain weighted control gains of higher modes. In this strategy, the control gain of the first mode is calculated using IMSC, and the control gains of the other modes are weighed in terms of modal energy and frequency with respect to the first mode. Thus, the weighing factor for the control gains of the $i^{th}$ mode, is given as

$$
\alpha_i = \frac{\omega_i^2 \eta_i^2 + \eta_1^2}{\omega_1^2 \eta_1^2 + \eta_i^2} \times \frac{\omega_1}{\omega_i}
$$

(7.23)
Finally, the control gains of the $i^{th}$ mode are $k_p^i = \alpha_k k_p^1$ and $k_d^i = \alpha_k k_d^1$.

### 7.10 Position Error Estimation of the Moving Platform

This section proposes an approach to estimate the position error of the moving platform using the measured output voltages of the PZT sensors bonded to intermediate links so that the change of the position error can be examined when the active vibration control is performed. To simplify the estimation, we assume that the axial deformation of each intermediate link is negligible since the axial stiffness is much higher than the lateral bending stiffness. Therefore, the position error is estimated based on the shortening of the intermediate link due to lateral deformations of the intermediate links.

![Figure 7.6 Schematic of the shortening of an intermediate link](image-url)
The shortening of an intermediate link is illustrated in Figure 7.6. The dotted line represents the deformed intermediate link. Due to the effect of shortening, the intermediate link is shortened from $B_i C_i$ to $B_i C_i^*$, which causes the position error of $C_i$ on the moving platform. The error is equal to $C_i^* C_i$ (defined by $\sigma$).

The rotation of the intermediate link is approximated by

$$\theta(x,t) = -\frac{\partial w(x,t)}{\partial x} = -w'(x,t)$$ \hspace{1cm} (7.24)

where $w(x,t)$ is deformation at the location $x$ along the intermediate link. Then the projection $d\sigma(x,t)$ of a differential element $ds$ onto the x-axis is given as

$$d\sigma(x,t) = ds(1 - \cos \theta(x,t))$$ \hspace{1cm} (7.25)

Using a Taylor series expansion of $\cos \theta(x,t) = 1 - \frac{1}{2} \theta^2(x,t) + h.o.t$, the equation (7.25) is approximated [147] by

$$d\sigma(x,t) = \frac{1}{2} \theta^2(x,t)ds = \frac{1}{2}(w'(x,t))^2 ds$$ \hspace{1cm} (7.26)

Considering that $\sigma$ is much small compared to $s$ and $x$, the position error can be derived by the integration of equation (7.26) and is approximated by [25]

$$\sigma(l,t) = \int_0^l \left(\frac{1}{2}(w'(x,t))^2\right) dx = \int_0^l \left(\frac{1}{2} \sum_{i=1}^n \phi_i'(x) \eta_i(t)\right)^2 dx$$ \hspace{1cm} (7.27)
In equation (7.27), $\dot{\varphi}_i$ is the derivative of the mode shape function, which is validated in our previous work. To obtain the position error, the modal coordinate should be calculated. These can be calculated using the experimental measurement of PZT sensors using modal filter as shown in Figure 7.4. The modal coordinate are finally expressed with displacement units transformed from PZT voltage values. This transformation is carried out as follows:

The voltage of PZT sensor can be expressed as

$$V_s = K_s \sum_{j=1}^{n} \eta_j(t) \varphi_i^*(x)$$  \hspace{1cm} (7.28)

where is $K_s$ PZT sensor constant.

Substituting equation (7.12) into equation (7.28), we have

$$V_s = K_s \left(\frac{l}{\pi}\right)^{2} \sum_{j=1}^{n} i^2 \varphi_i(x) \eta_j(t)$$  \hspace{1cm} (7.29)

Using equations (7.27) and (7.29), and modal filter as shown in Figure 7.4, we obtain the position error $\sigma(l,t)$ in terms of displacement value.

### 7.11 Experimental Results and Analyses

In this work, three PZT sensors and actuators are bonded to the third intermediate link at its quarter point, midpoint and three-quarter point, as shown in Figures 7.1 and 7.3. Three actuators and sensors are selected as BM 532, manufactured by Sensor Technology. The piezoelectric constant $d_{31}$ is $-270 \times 10^{-12} C/N$, and Young's modulus is $6.3 \times 10^{10} N/m^2$. The
dimensions of each PZT actuator are $25.4\,mm \times 25.4\,mm \times 0.254\,mm$ and the dimensions of each PZT sensor are $6.35\,mm \times 6.35\,mm \times 0.254\,mm$. As presented in Section 2, the active vibration control system is set up using LabVIEW Real-Time. The sampling rate for each channel of the A/D and D/A is configured to be 1000 Hz, and the input and output voltage range of each channel of the A/D and D/A is set to be $\pm 10$ volts. The voltage gain for each channel of the Sensor Technology SS08 amplifier is set to be 30.

For all experiments conducted in this work, a circular motion was assigned as a desired trajectory for the mass center point of the moving platform with constant $\rho = 45^\circ$. The radius of the circular trajectory was set to be 30mm. The motion control of the manipulator was implemented based on the inverse kinematics model of manipulator. The positions of the three axes were obtained through the inverse kinematics with the desired position of the end-effector of the manipulator. To excite the structural vibration of the flexible intermediate links as much as possible, the maximum velocity and acceleration of three sliders were experimentally set to be $0.1\,m/s$ and $50\,m/s^2$, respectively.

The theoretically designed gains may not be exactly optimal due to the uncertainty of the dynamic model and some other unconsidered factors, such as the phase transfer properties for low pass filters between D/As to the high-voltage PZT amplifiers. Therefore, using trial and error to adjust the gains, the poles of the flexible link were allocated to a desired location where best vibration attenuation results were achieved.
7.11.1 Vibration Control Applied on One Intermediate Link

Using modal filters as presented in Section 7.6, the vibration of the intermediate link was examined in the modal space. Figures 7.7 shows the first modal coordinate when the end-effector moves twice in a circular motion. Figures 7.8 to 7.11 present the power spectral density plots of the first two modal coordinates. Figure 7.8 illustrates that the first modal vibration contains not only natural frequency components at a frequency of 71.63 Hz, but also other frequency components at 23.38 Hz, 47.25 Hz and 94 Hz. The forced vibration components, as addressed in Equation (7.22), make the vibration characteristics of the intermediate link more complicated. These non-natural frequency components likely results from the excitation of rigid body motion of the link, motor dynamics, and dynamics due to the compliance of revolute joint bearings and the ball screw mechanisms.

Figure 7.9 shows the PSD plot of the second modal vibration. The second modal vibration contains the major frequency components of 235.3 Hz (which is the second order natural frequency), and forced vibration components. The forced vibration not only includes frequency components with 23.4 Hz, 47.3 Hz and 94.0 Hz, which are paramount in the first modal vibrations, but also contains other forced vibration components with 117.6 Hz, 141.3 Hz, 164.0 Hz, 188.0 Hz, and 211.6 Hz. More interesting is that the second modal vibration contains the first modal natural frequencies 71.6 Hz. This situation likely results from the coupling among the first modal vibration, rigid body motion, motor dynamics and dynamics from the compliance of the bearings and the ball screw. These figures clearly show that the first modal vibration energy level is much higher than the second modal vibration. Therefore, the first order mode vibration is targeted for control to simplify the controller in the vibration control experiments this work.
During the experiments conducted, we found that the use of higher gains achieved high vibration reduction. However, experiments revealed that larger gains are more likely to cause instability. This is due to the fact that modes are dynamically coupled and cannot be decoupled completely even though modal filters and temporal filters are used. To trade off vibration reduction with stability, the control gains for the PZT actuators bonded to the intermediate links are finally chosen, based on trial and error, to be 2 sec, 3 sec, and 2 sec for the actuators at the quarter point, midpoint, and three-quarter point, respectively. To better investigate how multiple PZT control pairs perform vibration suppression, the system response with a number of different cases with different numbers of PZT actuator, sensor pairs were investigated for a single flexible link. Note that experiments show that the use of only one or two PZT sensors was insufficient to effectively extract the modal coordinates, hence the DSRF are used for active vibration control with one or two PZT control pairs. Experiments show that at least three PZT control pairs are required to design the modal filter and precisely extract modal coordinates. To have a benchmark for comparison, the control gain of each PZT actuator remains the same for the cases above. For the case with three PZT control pairs applied simultaneously, the MSRF controller is used. A fast Fourier transform is performed to provide a power spectral density (PSD) plot of the midpoint vibration (strain) of each intermediate link. The PSD provides a measure of signal energy level at different frequencies. A comparison of the modal energy level between the case without control and the case with control is used to better demonstrate the effectiveness of the SRF active vibration control using PZT transducers.

As shown in Table 7.1, Case A shows the results of vibration control with three PZT pairs applied to a single flexible link (the link 3). Cases B, D and E correspond to the vibration control with two PZT pairs at the quarter point, midpoint, and three-quarter point, respectively. Cases C,
F and G are for the experiments with vibration control with one PZT pair applied at the quarter point, midpoint, and three quarter point, respectively.

Table 7.1 Experiment results of the active vibration control

<table>
<thead>
<tr>
<th>Case</th>
<th>PZT actuator at the quarter point</th>
<th>PZT actuator at the midpoint</th>
<th>PZT actuator at the three-quarter point</th>
<th>PSD at the quarter point</th>
<th>PSD at the midpoint</th>
<th>PSD at the three-quarter point</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>With control</td>
<td>With control</td>
<td>With control</td>
<td>149.9 (-65.9%)</td>
<td>115.2 (-72.7%)</td>
<td>169.1 (-64.1%)</td>
</tr>
<tr>
<td>B</td>
<td>With control</td>
<td>Without control</td>
<td>With control</td>
<td>174.8 (-60.2%)</td>
<td>873.2 (+106.9%)</td>
<td>171.7 (-64.1%)</td>
</tr>
<tr>
<td>C</td>
<td>Without control</td>
<td>With control</td>
<td>Without control</td>
<td>438.8 (-0.2%)</td>
<td>41.6 (-90.2%)</td>
<td>494.6 (+5%)</td>
</tr>
<tr>
<td>D</td>
<td>With control</td>
<td>With control</td>
<td>Without control</td>
<td>123.1 (-72.8%)</td>
<td>93.7 (-77.8%)</td>
<td>540.9 (+14.8%)</td>
</tr>
<tr>
<td>E</td>
<td>Without control</td>
<td>With control</td>
<td>With control</td>
<td>535.5 (+21.8%)</td>
<td>57.9 (-86.3%)</td>
<td>300.9 (-36.1%)</td>
</tr>
<tr>
<td>F</td>
<td>With control</td>
<td>Without control</td>
<td>Without control</td>
<td>148.5 (-66.2%)</td>
<td>1291 (+205.9%)</td>
<td>553.2 (+17.5%)</td>
</tr>
<tr>
<td>G</td>
<td>Without control</td>
<td>Without control</td>
<td>With control</td>
<td>562.5 (+27.9%)</td>
<td>502.5 (+19.1%)</td>
<td>215.9 (-54.2%)</td>
</tr>
<tr>
<td>No control</td>
<td>Without control</td>
<td>Without control</td>
<td>Without control</td>
<td>439.7</td>
<td>422</td>
<td>471</td>
</tr>
</tbody>
</table>

Figure 7.7 Mode 1 for case A
Figure 7.8 PSD of mode 1 without control

Figure 7.9 PSD of mode 1 for Case A
Figure 7.10 PSD of mode 2 without control

Figure 7.11 PSD of mode 2 for Case A
Figure 7.12 Vibration at the quarter point for Case A

Figure 7.13 Vibration at the midpoint for case A
The experimental results of Case A are given in Figures 7.7 to 7.20. As shown in Figures 7.7, 7.10 and 7.11, the first order modal vibration is significantly reduced, and the second order modal vibration remains almost unchanged since only the first mode is targeted for control. The PSD plots of the first two order modal coordinates, given in Figures 7.8 to 7.11, further show that the first modal vibration of the intermediate link is significantly suppressed with the proposed control strategy in Case A, and the second modal vibration energy has no significant change. The vibrations and the corresponding PSDs at the quarter point, midpoint and three-quarter point in physical space are given in Figures 7.12 to 7.20. The PSD at frequency 70.5 Hz is reduced by 65.9% for the quarter point, by 72.7% for the midpoint, and by 64.1% for the three-quarter point. The forced vibration component with frequency 23.4 Hz is increased. This situation likely arises due to the coupling between deformation and the rigid body motions. However, the general
vibration amplitudes at all the three points are suppressed significantly as shown in Figures 7.12 to 7.14.

Figure 7.15 PSD of the vibration at the quarter point without control

Figure 7.16 PSD of the vibration at the quarter point for case A
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Figure 7.17 PSD of the vibration at the midpoint without control

Figure 7.18 PSD of the vibration at the midpoint for case A
Figure 7.19 PSD of the vibration at the midpoint without control

Figure 7.20 PSD of the vibration at the midpoint for case A
Figure 7.21 Vibration at the quarter point for case B

Figure 7.22 Vibration at the midpoint for case B
Figure 7.23 Vibration at the three-quarter point for case B

Figure 7.24 Vibration at the quarter point for case C
Figure 7.25 Vibration at the midpoint for case C

Figure 7.26 Vibration at the three-quarter point for case C
The experimental results for Case B are given in Figures 7.21 to 7.23. These figures show that the vibrations at the quarter point and three-quarter point are reduced effectively. However, the vibration at the midpoint is increased.

The vibrations at the quarter point, midpoint and three-quarter point are illustrated in Figures 7.24 to 7.26 for Case C. Figure 7.25 shows that the vibration at midpoint is reduced significantly, and Figure 7.24 and Figure 7.26 show that the vibrations at the quarter point and three-quarter point remain almost unchanged.

The other cases are listed in Table 7.1. The PSD in Table 7.1 is referred to the PSD value at the first order vibration frequency around 70.0 Hz. The first order structural vibration frequency changes slightly with the configuration of the manipulator. Table 7.1 shows that the structural vibrations at the quarter point and midpoint are suppressed, and but the vibration at three-quarter point is increased in Case D. The structural vibrations at the midpoint and three-quarter point are reduced while the structural vibration at the quarter point is increased in Case E. The structural vibration at the quarter point is reduced while the structural vibration at the midpoint and three-quarter point are increased in Case F. The structural vibrations at the quarter point and midpoint are increased while the structural vibration at the three-quarter point is reduced in Case G.

With the comparison of the Cases A to G, we conclude that Case A, namely applying three PZT pairs at the quart point, midpoint and three-quarter point simultaneously, is the most effective strategy to suppress the vibration of the intermediate link of the manipulator. Case C is an acceptable strategy for the first mode targeted for control. For the first mode, the vibration is also reduced more effectively in Case A than in Cases C. There exists spillover for all other cases except for Case A. Therefore, to conduct the active vibration control of the intermediate links of
the presented manipulator with complicated dynamics, it is critical to select the appropriate number, and location of PZT sensors and actuators.

7.11.2 Vibration Control Applied on Three Intermediate Links

It is a challenging and time consuming work to choose the appropriate feedback gains using the DOFC due to coupling between PZT pairs at the same intermediate link and flexibility coupling among the intermediate links. The control gain for each PZT pair is determined using trial and error with all the other PZT control pairs powered off. Then, individual PZT pair control gains are adjusted for each intermediate link when three PZT controls work simultaneously but only one link is controlled during experiments. Due to the dynamic coupling effect among the intermediate links, the control gains are finally adjusted again and determined for experiments where each intermediate links are controlled simultaneously with three PZT control pairs. Using DOFC, it is difficult to determine the proper control gains, if not impossible, to achieve the effective vibration suppression of three intermediate links. Therefore, the IMSC strategy as presented in Section 7.6 is used to design the active vibration suppression of the three flexible intermediate links.

The control gains for the controlled modes are determined using the proposed strategy in Section 7.9. Since the main purpose is to dampen the vibration, only modal strain rate feedback is applied in the experiments. Using trial and error strategy, the control effort weighing is chosen as $\beta = 6$ in equation (7.7). Before the control suppression is implemented, the response of the uncontrolled system is analyzed using modal filters to obtain modal displacement and velocities. With the identified natural frequencies (the natural frequency is $75 \text{ Hz}$ for the first mode), the weight factor of the control gain of the second mode can be calculated using equation (7.8). The control gain for the first mode is selected to be $3.0 \text{ N} \cdot \text{s/m}$.
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Figure 7.27 PSD of mode1 for link 1 without control

Figure 7.28 PSD of mode1 for link 1 with control
Figure 7.29 PSD of mode1 for link2 without control

Figure 7.30 PSD of mode1 for link2 with control
Figure 7.31 PSD of mode1 for link3 without control

Figure 7.32 PSD of mode1 for link3 with control
Figure 3.33 Vibration at the midpoint for link 2

Figure 7.34 PSD for the configurations A-A
The experimental results are shown in Figures 7.27 to 7.35. In all the experimental results presented here, only the first vibration mode is targeted for control. The power spectral density (PSD) of the first vibration mode is effectively suppressed by 38% for the first intermediate link as shown in Figures 7.27 and 7.28, by 41% for the second link as shown in Figures 7.29 and 7.30, and by 41% for the third link as given in Figures 7.31 and 7.32. In the experiments, we find that the vibration of the second link is suppressed significantly at some configuration, but not at the other configuration, as shown in Figure 7.33. This phenomenon can be illustrated by the vibration analysis of the second link at the configurations A-A and BB. The PSD analysis at the configurations A-A and BB is performed and given in Figures 7.34 and 7.35. These two figures clearly show that structural vibration components are much smaller than the vibration components (at the frequencies around 95 Hz) likely resulting from the excitation of rigid body motions, rotor dynamics, and the dynamics from the clearance and compliance of the bearings and ball screws. These two figures illustrate that structural vibration (with frequency around
70Hz) can be reduced effectively, and the other vibration components can not reduced effectively with the proposed control strategy. Experiments also demonstrate that the second order modal vibration remains almost unchanged since only the first mode is targeted for control.

Theoretically, with the implementation of feedback controller in the modal space, there should be no spillover. Several reasons lead to spillover in the vibration control of the presented parallel manipulator. First, the deformation of the intermediate link is dynamically coupled with the rigid body motions, the dynamics of the motors, and the dynamics resulted from the compliance and clearance of the revolute joints and ball screw mechanisms. All modes are dynamically coupled with the other mode, and hence cannot be separated completely with modal filters and temporal filters. Second, the modal filter is designed with the ideal mode shape functions. The practical mode shape functions are not exactly the same as the theoretical mode shape functions although the measured mode shapes via modal tests match the assumed mode shapes very well. Third, the parameters of the PZT sensors, for example the capacitance, are not exactly the same for each PZT sensor/actuator pair. The difference can cause real time estimation errors of the modal coordinates. To better suppress the vibration, the vibration controller should be designed with the consideration of the robustness of the controllers to dynamic uncertainty, for example, using the robust control, learning control or adaptive control.

7.11.3 Vibration Control Applied on the First Two Modes

In this Section, the first two flexible link modes are targeted for control. The control gains for the controlled modes are determined using the proposed strategy in Section 7.9. Since the main purpose is to damp the vibration, only modal strain rate feedback is applied in the experiments. The control effort weighing is chosen as $\beta = 6$ in equation (7.7). Before the control
suppression is implemented, the response of the uncontrolled system is analyzed using modal filters to obtain modal displacement and velocities. With the identified natural frequencies (the natural frequency is $75 \, Hz$ for the first mode, and $235 \, Hz$ for the second mode), the weight factor of the control gain of the second mode can be calculated using equation (7.8). The control gain for the first mode is selected to be $3.0 \, N \cdot s/m$, and the control gain for the second mode is selected to be $0.6 \, N \cdot s/m$. With these control gains, the first two modes are significantly suppressed for the proposed flexible manipulator, as shown in Figures 7.36 to 7.42. Figures 7.36 and 7.37 show that the first mode vibration of the third intermediate link was suppressed 70%, and Figures 7.38 and 7.39 demonstrate the second mode vibration is decreased by 50%. The vibration response measured at the quarter point, midpoint, and three-quarter point are significantly reduced, as shown in Figures 7.40-7.42.

Using the proposed computation method given by equation (7.22), the position error of the moving platform is calculated with the measured voltages of PZT sensors amounted to the intermediate links. The position errors in X-direction and Y direction are given in Figures 7.44 and 7.45. These two figures clearly show that the position error of the moving platform is
significantly reduced due to the implementation of the active vibration control strategy proposed in this work.

The above results show that the vibrations at $70\,\text{Hz}$ for the first mode and $235\,\text{Hz}$ for the second mode are significantly suppressed for the first mode, but the vibrations at the other frequencies have less change. This fact can be explained by Figure 7.43, which illustrates the relationship between the frequency response and the damping ratios for the first order mode vibration. The natural frequency of flexible links is $75\,\text{Hz}$ as the identified in this work. Figure 11 shows that the increase of the damping ratio (for example, from $c = 0.05$ to $c = 0.1$) leads to the significant reduction of the forced vibration close to the natural frequency. However, the effect of the damping rate change becomes much smaller for the forced vibrations with the frequencies being far from the natural frequency. In the experiments, we also found that the weighting factor $\beta$ is limited because increasing the control input effort could lead to the destabilization of the controller. To further improve the suppression of the vibration, our future work will consider feedforward control, nonlinear control, etc.
Figure 7.40 Vibration response measure by the PZT sensor at the quarter point

Figure 7.41 Vibration response measure by the PZT sensor at the midpoint

Figure 7.42 Vibration response measure by the PZT sensor at the three-quarter point
Chapter 7    Active Vibration Control Experimental System and Implementation

Figure 7.43 Forced vibration response of a flexible link (adapted from [19])

Figure 7.44 Position error of the moving platform in X direction

Figure 7.45 Position error of the moving platform in Y direction
7.12 Summary

An active vibration control strategy is proposed for a moving 3-PRR parallel manipulator with flexible links with three PZT control pairs bonded, using modal filters and second compensators based on the IMSC strategy. The modal filters are simplified and designed with assumed mode shapes which are validated by experimental modal tests. The proposed modal filters are used for the on-line estimation of modal coordinates and modal velocity. The second compensator is used to cut off the amplified noises and unmodeled dynamics due to the differentiation operation in the proposed controller. The modal coupling behavior of intermediate links is examined with the modal analysis of vibrations measured by the PZT sensors. The gain amplification problem for higher order modes has been addressed, and provides guidelines for control gain selection in the design of modal controller. An efficient multi-mode vibration control strategy has been proposed through modifying the IMSC strategy. The motion effort the moving platform is examined with the consideration of the shortening of the flexible intermediate links. A real-time experimental system has been developed with LabVIEW Real-Time. The active vibration control experiments are performed first with the proposed control strategy applied on the third intermediate link for a number of cases: single PZT control pair applied, two PZT control pair applied, and three PZT control pair applied simultaneously. The experimental results show that the vibration of the intermediate link is reduced effectively based on the proposed control strategy using three PZT control pairs simultaneously. Finally, active vibration control experiments are successfully implemented to three flexible links, each of which are equipped with three PZT control pairs at its quarter point, midpoint, and three quarter point. The experimental results show that the vibration of the flexible manipulator is suppressed effectively.
To justify the use of three PZT control pairs for the vibration suppression, the experiments are first conducted using one or two PZT control pairs applied at the different locations along the same single flexible link. The experimental results demonstrate that the vibration cannot be effectively suppressed if only one or two PZT control pairs are used. Note that since the use of only one or two PZT sensors was insufficient to effectively extract the modal coordinates, the DSRF are used for active vibration control with one or two PZT control pairs. Experiments show that at least three PZT control pairs are required to design the modal filter and accurately extract modal coordinates.

The vibration control experiments are implemented for the first two modes targeted for control in this work. The effectiveness of the vibration control needs improvement for the following reasons: the estimated assumed modal shapes are not exactly the mode shapes of the intermediate links, which leads to the errors of the modal coordinates and modal velocities; the control gains are limited due to the modal couplings. Large gains cause the destabilization of the control system. The experiments show the vibration of the intermediate link is reduced significantly at some configurations while not at other configurations. The potential solution to these issues is to consider the robustness to dynamic uncertainty to design the vibration controllers. Our continuing efforts also include extending the proposed strategy to target the higher modes for control. These challenges include overcoming the coupling among modes, and guaranteeing the real-time implementation of controllers.
Chapter 8

Conclusions and Discussion

8.1 Thesis Summary

This thesis is mainly concerned with dynamic modeling and active vibration control of a complex dynamic system, a 3-PRR planar parallel manipulator with three flexible links moving at high speed. The methodologies of dynamic modeling of a complex multibody dynamic system have been presented with the assumed mode method. The significant dynamic characteristics of the flexible parallel manipulator have been investigated, including stiffening and buckling behavior, and coupling characteristics between elastic deformations and rigid body motions. The active vibration control with smart structures are introduced and addressed to the parallel manipulator with three flexible links, each of which is bonded with multiple PZT sensors and actuators. Using modal control theory, a complex coupled dynamic control problem has been linearized and decoupled into multiple single-input-single-output problems. An active vibration control experimental system has been designed and built, and the corresponding real-time control system has been developed with NI LabVIEW Real Time Module. The developed active vibration control strategy has been experimentally implemented on the established active vibration control system of a 3 P-RRR parallel manipulator with flexible links. The developed methods and strategies of dynamic modeling and vibration control can be extended to other types of parallel manipulators or other multibody dynamic systems with flexible components. In the following, the work done in this thesis is summarized, and some significant contributions of this thesis work are restated:
To achieve high speed and high acceleration, light weight links are used to reduce inertia forces. However, light weight links are easy to deform and vibrate when the manipulator is moving at high speed. Therefore, link flexibility should be considered in the dynamic model. With the assumption of small and linear elastic deformation, a methodology has been presented based on the assumed mode method for the generation of dynamic equations for a 3-PRR parallel manipulator with three flexible intermediate links. The mode shape functions are selected by modeling intermediate links as Euler-Bernoulli beams with pinned-pinned boundary conditions, which have been validated with experimental modal tests in this work. Considering the specific structure of the intermediate links, i.e., both ends of each link are attached with bearing housings having substantial rotational inertia, the effect of concentrated rotational and translational inertia is included to make the model developed more reasonable and accurate.

The axial forces on the intermediate links of parallel manipulators are significant when the manipulator is moving at high speed and with high payload. Through including the effect of axial forces on lateral deformation, the stiffening and bucking behavior of the flexible manipulator have been investigated. The natural mode frequency equation is formulated in terms of axial forces and rigid body motions. With the frequency equation, stiffening and buckling dynamic characteristics of the flexible parallel manipulator are examined, which provides valuable insights into the design and control of parallel manipulators with flexible intermediate links. For example, joint motions of parallel manipulators can be optimized and controlled so that axial forces in the flexible links are in extension to increase the stiffness. With the frequency equation derived, geometric parameters of intermediate links and payload can be determined and selected.
to prevent buckling in the design of parallel manipulators with flexible intermediate links.

- The accurate and complete dynamic equations of flexible multibody systems not only includes the effect of rigid body motions on elastic deformations, but also considers the influence of elastic deformations on rigid body motions, and coupling effect among elastic deformations of flexible bodies. In this thesis, the fully coupled dynamic equations of the 3-PRR parallel manipulator with three flexible links have been established using a one-pass method which incorporates rigid body motions with elastic deformations based on the assumed mode method using Lagrange multipliers. The equations clearly reveal the mutual dependence of rigid body motions and elastic deformations. The numerical simulations and theoretical analyses have potential applications in suppressing the unwanted vibration through controlling joint motions.

- “Smart” structures have been designed and built for the vibration suppression of flexible intermediate links of the parallel manipulator. A smart structure consists of PZT sensor and actuator system, corresponding active controller, and filtering and amplifying circuits. Three PZT sensors/actuator control pairs are bonded to the two sides of the intermediate links. The PZT patches on one side of the link act as sensors, while the PZT patches on the opposite link face act as actuators. One sensor and one actuator constitute a PZT control pair, and are located at the same position along the length of each intermediate link. Sensor-actuator collocation increases the robustness of control laws designed for active vibration control, and makes the control system be minimum phase due to pole-zero interlacing. The active vibration control system is developed
using NI Labview Real-Time Module. The control system consists of two PCs: a host PC and Target PC. Using LabVIEW graphical programming, a LabVIEW Real-Time embedded control application is developed on the host PC, and then downloaded to, and run on the target PC. The two PCs communicate over Ethernet using TCP/IP. The two-CPU programming technique is used to further improve the real-time execution of the controller by assigning one CPU for control and calculation, and the other CPU for data logging. The active vibration controller is developed with identical architecture for each smart link. With the hardware and software developed, the real-time active vibration control can be guaranteed to be executed at the rate 1K Hz with real-time saving and logging data for nine control channels. This active vibration control system provides a reliable experimental platform for this thesis work and future research.

- An efficient modal strain rate feedback controller has been developed and implemented using the independent modal space method for the active vibration control of the moving parallel manipulator with three flexible intermediate links, each of which was equipped with multiple PZT actuators and sensors. A simplified and efficient modal filter has been proposed to monitor modal displacement and velocity for each modal controller. A second order auxiliary compensator has been developed to reduce high frequency noise arising from the use of differentiated strain rate feedback. The design of the controller with the compensator has been addressed utilizing vibration theories. The amplification behavior of the modal controller for the high modes has been addressed based on PZT sensors and actuators. An effective strategy of determining the control gains of high modes has been developed through modifying the independent modal space control strategy. The error estimation of the moving platform has been conducted.
using the measurements from PZT sensors mounted on the flexible intermediate links. Finally, active vibration control experiments have been successfully implemented for the parallel manipulator with three flexible links, each of which are equipped with three PZT control pairs at its quarter point, midpoint, and three quarter point. The experimental results show that the first two vibration modes of the flexible manipulator have been suppressed effectively with proposed active vibration control strategy and developed experimental system, and the motion precision of the moving platform has been improved. To justify the use of three PZT control pairs applied on each intermediate link, active vibration control experiments were conducted with either one PZT control pair, or two PZT control pairs or three control pairs applied on one intermediate link. The experimental results reveal that the vibration can not be effectively suppressed if only one or two PZT control pairs are used. Experiments further show that at least three PZT control pairs are required to design the modal filter and precisely extract modal coordinates. Comparison of these cases provides insight and guidance into the selection of the number and the location of PZT transducers.

8.2 Recommendations for Future Work

Dynamical modeling and vibration control of parallel manipulators with multiple flexible links is a challenging task. Many interesting and unsolved problems deserve researchers to invest significant efforts in the future to address, and hence promote the practical application of lightweight parallel manipulators. Reviewing the work done in this thesis, the following research efforts for future work could include:
- In the dynamic equations developed in this thesis, the dynamics of motors and ball screw mechanisms are neglected. To establish more accurate dynamics equations of the parallel manipulator with link flexibility, motor dynamics and dynamics from ball screw mechanisms should be included, and the effect of PZT sensors and actuators on the system dynamics should be considered and examined as well.

- In the experimental modal tests presented in this work, the modal identification tests were conducted on one flexible link since the three flexible intermediate links have identical structural parameters and materials, and hence are assumed to have the identical mode shapes. The experimental modal tests are performed when the manipulator is stationary at the home position of the parallel manipulator. Since the sliders and the platform of the test bed are much heavier compared with intermediate links, coupling effects among three flexible links are neglected, and hence simplifying the experimental modal test process. To obtain more accurate modal test results, experimental modal tests should be implemented on three flexible intermediate links at different configurations when the manipulator is moving considering the effect of nonlinear dynamics on modal shapes and frequencies.

- As addressed in this thesis, several basic assumptions were made to experimentally investigate the vibration control of the complex dynamic system of the parallel manipulator. For example, couplings among intermediate links and/or modes were neglected, and the interactions between the PZT actuators and DC servo motors were not examined in the control design. Therefore, the control problem presented in this work was linearized and decoupled to three single input single output (SISO) control problem to simplify the controller design for each link. However, in practice, when the
controllers are applied simultaneously, the dynamics of the three flexible links and
different control pairs somewhat couple each other, hence the control gains need to be
adjusted with the consideration of vibration suppression performance. Further, the
performance of the proposed controller changes with configurations. These problems
could be potentially solved with the consideration of the robustness of controllers to
dynamic uncertainty, for example, using the robust control, learning control or adaptive
control, etc., in future work. The current controller design is only targeted for the first
and second order modes. Future research effort could include extending the proposed
strategy to higher vibration modes to be targeted for control.

- For the unwanted vibration, this thesis only involves the vibration suppression with PZT
  actuators. The other strategies could be used for suppression the unwanted vibration, for
  example, controlling the joint motors with input shapes etc.

- In the current parallel manipulator test bed, three sliders and platform are much more
  massive than the intermediate links. To further improve the motion speed of the
  manipulator to reveal the configuration-dependent dynamic behavior, the mass of the
  sliders and platform should be decreased.
References


References


Appendix A

Coefficient Matrices in Dynamic Equations

\[
M_{11} = \begin{bmatrix}
m_{s1} + m_1 & 0 & 0 \\
0 & m_{s2} + m_2 & 0 \\
0 & 0 & m_{s3} + m_3
\end{bmatrix}
\]

\[
M_{12} = \begin{bmatrix}
\frac{m_{l1}}{2} & s_1 & 0 \\
0 & \frac{m_{l2}}{2} & s_2 \\
0 & 0 & \frac{m_{l3}}{2} s_3
\end{bmatrix}
\]

\[
M_{33} = \begin{bmatrix}
m_p & 0 & 0 \\
0 & m_p & 0 \\
0 & 0 & I_p
\end{bmatrix}
\]

\[
M_{22} = \begin{bmatrix}
\frac{1}{3} m_{l1}^2 + \frac{1}{2} m_1 \sum_{j=1}^{r} \eta_{1j} \psi_{1j} \xi^2 d\xi & 0 \\
0 & \frac{1}{3} m_{l2}^2 + \frac{1}{2} m_2 \sum_{j=1}^{r} \eta_{2j} \psi_{2j} \xi^2 d\xi & 0 \\
0 & 0 & \frac{1}{3} m_{l3}^2 + \frac{1}{2} m_3 \sum_{j=1}^{r} \eta_{3j} \psi_{3j} \xi^2 d\xi
\end{bmatrix}
\]

\[
F_{ic1} = \begin{bmatrix}
0.5 m_{l1} c_1 \dot{\beta}_1^2 + \sum_{j=1}^{r} m_j \ddot{\eta}_{1j} \dot{c}_1 \psi_{1j} d\xi \\
0.5 m_{l2} c_2 \dot{\beta}_2^2 + \sum_{j=1}^{r} m_j \ddot{\eta}_{2j} \dot{c}_2 \psi_{2j} d\xi \\
0.5 m_{l3} c_3 \dot{\beta}_3^2 + \sum_{j=1}^{r} m_j \ddot{\eta}_{3j} \dot{c}_3 \psi_{3j} d\xi
\end{bmatrix}
\]

\[
F_{ic2} = \begin{bmatrix}
\sum_{j=1}^{r} \ddot{\eta}_{1j} \dot{c}_1 \psi_{1j} d\xi \\
\sum_{j=1}^{r} \ddot{\eta}_{2j} \dot{c}_2 \psi_{2j} d\xi \\
\sum_{j=1}^{r} \ddot{\eta}_{3j} \dot{c}_3 \psi_{3j} d\xi
\end{bmatrix}
\]

\[
J_{G1}^T = \begin{bmatrix}
\cos \alpha_1 & \sin \alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \alpha_2 & \sin \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \alpha_3 & \sin \alpha_3
\end{bmatrix} \in \mathbb{R}^{3 \times 6}
\]
Appendix A: Coefficient Matrices in Dynamic Equations

\[ J_{G2}^T = \begin{bmatrix} s_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_3 & c_3 \end{bmatrix} \in \mathbb{R}^{3 \times 6} \]

where: \( s_2 = -l \sin \beta \) and \( c_2 = l \cos \beta \)

\[ J_{G3}^T = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ s_3 & -c_3 & s_3 & -c_3 & s_3 & -c_3 \end{bmatrix} \in \mathbb{R}^{3 \times 6} \]

where: \( s_3 = \epsilon_{ix}, \ c_3 = \epsilon_{iy} \)

\[ M = M_{44} = \begin{bmatrix} \hat{M}_1 & 0 & 0 \\ 0 & \hat{M}_2 & 0 \\ 0 & 0 & \hat{M}_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3r} \]

\[ \hat{M}_i = m_1 \begin{bmatrix} \int \psi_{ii}^2 d\xi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \int \psi_{ii}^2 d\xi \end{bmatrix} \in \mathbb{R}^{r \times r} \]

\[ K = K_f = \begin{bmatrix} \hat{K}_1^f & 0 & 0 \\ 0 & \hat{K}_2^f & 0 \\ 0 & 0 & \hat{K}_3^f \end{bmatrix} \in \mathbb{R}^{3 \times 3r} \]

\[ \hat{K}_i^f = \frac{E_i}{l_i^2} \begin{bmatrix} \int I_i(\xi)\psi_{ii}^2 d\xi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \int I_i(\xi)\psi_{ii}^2 d\xi \end{bmatrix} \in \mathbb{R}^{r \times r} \]

\[ K_p = \begin{bmatrix} \hat{K}_1^p & 0 & 0 \\ 0 & \hat{K}_2^p & 0 \\ 0 & 0 & \hat{K}_3^p \end{bmatrix} \in \mathbb{R}^{3 \times 3r} \]

\[ \hat{K}_i^p = \frac{1}{m_i} \begin{bmatrix} \hat{k}_{n1}^p & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{k}_{n1}^p \end{bmatrix} \in \mathbb{R}^{r \times r} \]

\[ \hat{k}_{n1}^p = \frac{1}{l_i} (m_n \ddot{\rho}_i - F_{ai}) c_i \int \psi_{ii}^2 d\xi + \rho_n \ddot{\rho}_i c_i \int \psi_{ii}^2 \xi d\xi - \frac{1}{2} \frac{m_i}{\beta^2} \int \xi^2 \psi_{ii}^2 d\xi \]
Appendix A: Coefficient Matrices in Dynamic Equations

\[
M_\rho = M_{41} = \begin{bmatrix}
  m_1 s_1 \int \psi_{11} d\xi & 0 & 0 \\
  \vdots & \ddots & \ddots \\
  m_1 s_1 \int \psi_{1i} d\xi & 0 & 0 \\
  0 & m_2 s_2 \int \psi_{21} d\xi & 0 \\
  \vdots & \ddots & \ddots \\
  0 & m_2 s_2 \int \psi_{2i} d\xi & 0 \\
  0 & 0 & m_3 s_3 \int \psi_{31} d\xi \\
  \vdots & \ddots & \ddots \\
  0 & 0 & m_3 s_3 \int \psi_{3i} d\xi 
\end{bmatrix}
\]

\[
M_\beta = M_{42} = \begin{bmatrix}
  m_1 l_1 \int \psi_{11} d\xi & 0 & 0 \\
  \vdots & \ddots & \ddots \\
  m_1 l_1 \int \psi_{1i} d\xi & 0 & 0 \\
  0 & m_2 l_2 \int \psi_{21} d\xi & 0 \\
  \vdots & \ddots & \ddots \\
  0 & m_2 l_2 \int \psi_{2i} d\xi & 0 \\
  0 & 0 & m_3 l_3 \int \psi_{31} d\xi \\
  \vdots & \ddots & \ddots \\
  0 & 0 & m_3 l_3 \int \psi_{3i} d\xi 
\end{bmatrix}
\]

\[
F_{r} = F_{\kappa 4} = \begin{bmatrix}
  m_1 c_i \rho_i \dot{\beta}_i \int \psi_i d\xi \\
  m_1 c_i \rho_i \dot{\beta}_i \int \psi_i d\xi \\
  m_2 c_2 \rho_2 \dot{\beta}_2 \int \psi_i d\xi \\
  \vdots \\
  m_2 c_2 \rho_2 \dot{\beta}_2 \int \psi_i d\xi \\
  m_3 c_3 \rho_3 \dot{\beta}_3 \int \psi_i d\xi \\
  \vdots \\
  m_3 c_3 \rho_3 \dot{\beta}_3 \int \psi_i d\xi 
\end{bmatrix}
\]

where \( s_i = \sin(\alpha_i - \beta_i) \) and \( c_i = \cos(\alpha_i - \beta_i) \).
Appendix B

Bonding Procedure of PZT Sensors and Actuators

To bond PZT transducers to the intermediate links, a kind of mixed glue is made from Eccobond 45 and Catalyst 15 manufactured by Emerson & Cuming Inc. The detailed steps to bond the PZT transducers are as follows:

1) Clean the bonding surface with isopropyl alcohol.

2) Draw the rectangles on each intermediate with pencil or knife to mark the location of PZT transducers according to the number and size of PZT transducers. The current PZT transducers are bonded to the quarter point, middle point, and three quarter point. Three PZT sensors are bonded on one side of a link, and three PZT actuators are bonded on the opposite side of the link. One sensor and one actuator constitute a PZT control pair, and are located at the same position along the length of each intermediate links.

3) Stick double tapes on the surface of the intermediate links according to the locations marked above, and then cut of a hole of the size of the PZT transducer on the tape.

4) Take Eccobond 45 and Catalyst 15 separately with two cleaned syringes. The volume proportion of Eccobond 45 to catalyst is set to be 1:0.815.
5) Put the above the two liquids together into a metal container, and mix round them fully using a wood stick till the mixture turns to be a kind of even glue without air in it.

6) Move several droplets of the mixed glue into the hole of each tape on the face of intermediate links, and then spread them to a quite thin and even layer within the working life of the glue, namely 120 minutes.

7) Pick the PZT transducers and place them into the holes of the tapes, and squeeze the glues so that the excessive glue flows out.

8) Using wrap films to cover the bonding PZT transducers, apply a pressure by weights, and keep the weights on the PZT transducers for about 12 hours.

9) Remove the wrap films and weights and clean the surface of the PZT transducers and linkages carefully to make the following soldering and wiring easier.
Appendix C

Configuring Procedure of a Desktop PC as a RT Target PC

To complete the following steps to configure a desktop PC as an RT target:

1) Check Desktop PC Requirements. To configure a desktop PC as an RT target, ensure that the PC meets the following requirements: Processor based on the x86 architecture; Supported Ethernet chipset, the Ethernet device from the LabVIEW Real-Time Deployment License Bundle for Standard PCs, or a supported Ethernet card; Formatted hard drive or partition on the desktop PC with the FAT32 file system. Because Windows Vista requires the NTFS file system, you cannot install RT Module software on the same partition as Windows Vista; 3.5 inch floppy drive or bootable USB port on the desktop PC. The detailed requirements for a desktop PC to be used as a LabVIEW Real-Time are detailed in the ni.com document KnowledgeBase 39NDI8PK: Requirements for a Desktop PC as a LabVIEW Real-Time Target.

2) Check compatibility with LabVIEW Real-Time on the desktop PC. Download the Real-Time PC Validator from ni.com (it may be loaded on either a floppy disk or USB jump drive) and run the Real-Time Desktop Validator start-up executable. When a system boots up, the executable will run automatically and print something similar to the two images shown below. The Real-Time Desktop Validator will run a diagnostic to learn about the following components of the desktop PC: CPU, BIOS, memory, and Ethernet
drivers. If the test passes, the desktop PC can be booted directly and run LabVIEW Real-Time applications. Information will be provided as Figure C-1.

3) Create a desktop PC boot disk using NI Measurement & Automation Explorer (MAX) on the host PC. Select Tools»RT Disk Utilities»Create Desktop PC Boot Disk in MAX to create the desktop PC boot disk, as shown in Figure C-2.

4) Turn on the desktop PC and access the BIOS configuration utility.

5) Set the boot configuration to use the floppy drive as the first boot device.

6) Disable legacy USB support.

7) Disable any unnecessary integrated peripherals that use an interrupt request line (IRQ).

   For example, disable unused serial ports or integrated sound on the desktop PC.

8) Save the configuration changes and exit the BIOS configuration utility.

9) Insert the desktop PC boot disk in the floppy drive, reboot the desktop PC and the desktop PC is booted into the real-time operating system.
Appendix C: Configuring Procedure of a Desktop PC as an RT Target PC

Searching hard drive for boot program... found.
Using boot program on hard drive.

LabVIEW Real-Time SMP Kernel: CPU cores found: 2
MAX system identification name: target1
Initializing network...
Device 1 - MAC address: 00:02:B3:EB:AA:C3 - 169.254.0.2 (primary)
Welcome to LabVIEW Real-Time 8.5

Figure C-1 Real-Time Desktop Validator Passed Test

Figure C-2 Interface of Creating Desktop PC Boot Disk
Appendix D

NI 6733 Pinout

The digital to analog conversion board NI PCI-6733, as shown in Figure D-1, is installed inside one PCI slot of the target PC. The National Instruments PCI-6733 delivers high-performance, reliable high-speed analog outputs to meet a wide range of application requirements. It provides eight analog output channels with up to 1 MS/s per channel, 16-bit resolution, and digital triggering. The board uses the RTSI bus to synchronize with additional data acquisition, motion and vision products, so users can create custom measurement solutions to test innovative designs. The pinout is shown in Figure D-2.

Figure D-1 D/A Board NI PCI 6733
Figure D-2 Pinout Connector of NI PCI 6733
Appendix E

NI 6031 Pinout

The analog to digital conversion board 6031E, as shown in Figure E-1, is installed inside one PCI slot of the target PC. The National Instruments PCI-6031E uses E Series technology to deliver high-performance, reliable data acquisition capabilities to meet a wide range of application requirements. The maximum sampling rate can get up to 100 kS/s, 16-bit resolution with 64 single-ended analog inputs. Depending on your type of hard drive, this board can stream data to disk at rates up to 100 kS/s. The pinout is shown in Figure E-2.

Figure E-1 A/D Board 6031E
Appendix E: NI 6031 Pinout

Figure E-2 NI 6031E Pinout