Tracial state spaces of higher stable rank simple $C^*$-algebras

by

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Abstract

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Ten years ago, J. Villadsen constructed the first examples of simple C*-algebras with stable rank other than one or infinity. Villadsen’s examples all had a unique tracial state.

It is natural to ask whether examples can be found of simple C*-algebras with higher stable rank and more than one tracial state; by building on Villadsen’s construction, we describe such examples that admit arbitrary tracial state spaces.
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Chapter 1

Introduction

The topological stable rank for C*-algebras was introduced by Marc A. Rieffel in [13], as a notion of noncommutative dimension for C*-algebras. Given a C*-algebra $A$, the stable rank of $A$ is the least integer $n$ such that the subset $\text{Lg}_n(A)$ of $A^n$, consisting of all the $n$-tuples that generate $A$ as a left ideal, is dense in $A^n$. If no such $n$ exists, the stable rank of $A$ is defined to be infinity.

For a commutative C*-algebra $C(X)$, the stable rank is proportional to the covering dimension of the spectrum $X$ of the algebra; there are commutative C*-algebras with arbitrary stable rank values, including infinity. If we on the other hand look at the class of simple C*-algebras, the picture there was not so clear at first. While many examples of simple C*-algebras with stable rank one (or infinity) were known, it was not until ten years ago that the first examples of simple C*-algebras with stable rank other than one or infinity were found, by Jesper Villadsen ([21]).

These Villadsen algebras can appear to be quite strange at first; they are simple, unital and stably finite algebras with highly nontrivial K-theory, as indicated by holes present in their $K_0$ groups, and that came to life through an application of powerful tools from differential topology into C*-algebra theory. These tools were used originally by Villadsen ([20],[21]) and later by others to settle several open questions in the theory of nuclear
C*-algebras ([15],[14],[18]), including the construction of the first counterexamples to the strongest form of Elliott’s classification conjecture for separable amenable C*-algebras ([19]).

Andrew Toms later produced more stably finite examples of higher stable rank simple C*-algebras using Villadsen’s techniques and a generalized mapping torus construction ([17]); despite the complexity of the C*-algebras in both classes of examples, their tracial simplices are remarkably simple: these C*-algebras all admit a unique tracial state. It is therefore natural to ask whether examples can be found of simple C*-algebras with higher stable rank and more than one tracial state (as is the case with so many classes of simple C*-algebras with stable rank one).

The purpose of this thesis is to settle this question affirmatively by concretely producing examples of simple C*-algebras with stable rank other than one and more than one tracial state. In fact, our main result states that such C*-algebras can have arbitrary tracial state spaces (Theorem 3.3.2). The constructed examples constitute a generalization of Villadsen’s class of examples in [21], and the arbitrary tracial state spaces are obtained by an adaptation to our setting of the techniques used by Klaus Thomsen to prove that a certain class of AI-algebras admits arbitrary tracial state spaces ([16]).

The content of this thesis is organized as follows:

In Chapter 2, we look at the notions of stable and real ranks in a little more detail, and give a brief description of the construction of Villadsen’s algebras, as well as some of the techniques used to obtain the higher stable ranks; this is meant to provide the reader with a picture of the construction that will be generalized in the following chapter.

In Chapter 3, we proceed to generalize Villadsen’s construction in order to obtain examples with more than one tracial state. In the first section we recall Thomsen’s argument to produce his AI examples, and identify the problems in adapting his method to Villadsen algebras; the remainder of this section, and also the second section, deal with generalizing results of Thomsen to this new setting. The third section contains our
main result, Theorem 3.3.2, that shows the class of Villadsen examples can be expanded to allow for richer structure on traces. The fourth and final section deals with estimating the real rank for the $C^*$-algebras obtained, and examining some related open questions.
Chapter 2

Villadsen algebras

In this chapter we examine a class of C*-algebras constructed by Villadsen in [21], the first examples of simple unital stably finite C*-algebras with stable rank other than one, reviewing some of the technical aspects and tools necessary for their construction.

It is worth mentioning that Villadsen constructed earlier another class of simple C*-algebras in [20], that are also often referred to as Villadsen algebras. These were AH algebras with stable rank one, and lots of tracial states; their interesting feature was perforation in the $K_0$ group, which in particular showed that there are simple AH algebras that don’t have slow dimension growth. We will use the term Villadsen algebras specifically in reference to his second class of examples.

2.1 Stable and real ranks

As stated in the introduction, given a unital C*-algebra $A$ and a positive integer $n$, one says that the (left) stable rank of $A$ is less than or equal to $n$ if the subset $Lg_n(A)$ of $A^n$ consisting of all the $n$-tuples that generate $A$ algebraically as a left ideal, is dense in $A^n$; the stable rank of $A$ is then defined to be the least such $n$, and denoted $sr(A)$. If no such $n$ exists, the stable rank of $A$ is defined to be infinity.

The topological stable rank was introduced by Rieffel as a noncommutative dimension
for C*-algebras; the following theorem illustrates well the relation between the topological stable rank and dimension, for commutative C*-algebras.

**Theorem 2.1.1.** If $X$ is a compact Hausdorff space, then

$$sr(C(X)) = \lfloor \dim(X)/2 \rfloor + 1.$$  

It follows from the definition of stable rank that a unital C*-algebra has stable rank one when its set of invertible elements is dense in the algebra. Examples of simple stable rank one C*-algebras include for instance the class of simple unital AH-algebras with slow dimension growth ([7]).

Examples of simple C*-algebras with infinite stable rank include all simple unital purely infinite C*-algebras ([13]).

In what follows we deal with a number of homogeneous C*-algebras of the form $p(C(X) \otimes K)p$, for $X$ a compact metric space, $p$ a projection in $C(X) \otimes K$, and $K$ the algebra of compact operators on separable Hilbert space, and work with their stable ranks; these can be calculated using the formula in the following result, that generalizes the result above for commutative C*-algebras, and follows from Theorem 7 of [12].

**Theorem 2.1.2.** Let $A$ be a given $m$-homogeneous unital C*-algebra. Then,

$$sr(A) = \lceil \lfloor \dim(\hat{A})/2 \rfloor / m \rceil + 1,$$

where $\hat{A}$ denotes the spectrum of $A$ with its natural topology, and $\lceil \rceil$, $\lfloor \rfloor$ are the ceiling and floor functions, respectively.

In other words, for $A = p(C(X) \otimes K)p$ as above we have

$$sr(A) = \lceil \lfloor \dim(X)/2 \rfloor / \text{rank}(p) \rceil + 1.$$  

The real rank for C*-algebras was introduced by Brown and Pedersen in [6] and is, unlike the stable rank, a purely C*-algebraic invariant: given a unital C*-algebra $A$, the
real rank of $A$, denoted $\text{RR}(A)$, is the least integer $k$ such that the set of $(k+1)$-tuples of self-adjoint elements of $A$ that generate $A$ as a left ideal, is dense in $A_{sa}^{k+1}$ ($A_{sa}$ denotes the set of self-adjoint elements of $A$). If no such $k$ exists, the real rank of $A$ is defined to be infinity.

The real rank also provides a notion of noncommutative dimension for C*-algebras. For a commutative C*-algebra, say $C(X)$ for $X$ compact, the real rank is actually equal to the dimension of $X$. As shown in [6], Proposition 1.2, the real rank of a C*-algebra $A$ can be related to its stable rank by the inequality

$$\text{RR}(A) \leq 2\text{sr}(A) - 1.$$ 

The work of Villadsen in [21] seem to suggest that, in the case of a simple C*-algebra $A$, one has $\text{RR}(A) \leq \text{sr}(A)$; further evidence to this is given by Theorem 3.4.2 in the next chapter.

### 2.2 Obtaining the higher stable ranks

The Villadsen algebras are AH algebras presented as $A = \lim(A_i, \phi_i)$, where each $A_i$ is composed of a single homogeneous building block, $A_i = p_i(C(X_i)\otimes K)p_i$, for some very particular choices of compact metric spaces $X_i$, projections $p_i \in C(X_i)\otimes K$, and *-homomorphisms $\phi_i$, which we will describe below.

The first key aspect of these algebras is that they do not have slow dimension growth. Recall that an AH algebra $A$ is said to have slow dimension growth if it can be presented as an AH system $A = \lim(A_i, \phi_i)$, with $A_i = \bigoplus_{j=1}^{n_i} p_{i,j}(C(X_{i,j})\otimes K)p_{i,j}$, such that

$$\limsup_{i \to \infty} \sup \left\{ \frac{\text{dim}(X_{i,1})}{\text{rank}(p_{i,1})}, \ldots, \frac{\text{dim}(X_{i,n_i})}{\text{rank}(p_{i,n_i})} \right\} = 0.$$ 

Simple AH algebras with slow dimension growth necessarily have stable rank one ([7]), so to obtain higher stable rank examples one must consider AH systems such that the
dimensions of the base spaces $X_{i,j}$ grow relatively fast compared to the rank of the projections $p_{i,j}$ as $i$ tends to infinity.

Given compact metric spaces $X$ and $Y$, recall that a $*$-homomorphism

$$\phi : M_n(C(X)) \longrightarrow M_m(C(Y))$$

is said to be diagonal if $n$ divides $m$ and there are continuous maps $\lambda_1, \ldots, \lambda_{m/n} : Y \longrightarrow X$ (the eigenvalue maps of $\phi$) such that

$$\phi(a) = \bigoplus_{i=1}^{m/n} a \circ \lambda_i,$$

for all $a$ in $M_n(C(X))$. Many interesting classes of $C^*$-algebras can be obtained as inductive limits with diagonal $*$-homomorphisms, such as AF algebras, Goodearl algebras, and even the first class of Villadsen’s examples in [20], among others.

It has been shown recently by G. Elliott, T. Ho and A. Toms that arbitrary simple unital AH algebras that can be presented as an AH system with diagonal $*$-homomorphisms have stable rank one ([9]). The surprising aspect of their result is that no restriction is made on the dimension growth of the base spaces; for instance, you could have such an inductive limit with infinite dimensional base spaces at each step (and therefore infinite stable rank at each finite stage), but the stable rank of the limit would still drop down to one! Therefore, more general $*$-homomorphisms must be considered in constructing higher stable rank examples.

The $*$-homomorphisms considered by Villadsen are what we might call generalized diagonal; given compact metric spaces $X, Y$, a set of mutually orthogonal projections $p_1, \ldots, p_n$ in $C(Y) \otimes K$, and continuous maps $\lambda_1, \ldots, \lambda_n : Y \longrightarrow X$, a generalized diagonal $*$-homomorphism $\phi : C(X) \otimes K \longrightarrow C(Y) \otimes K$ can be constructed as follows: let $\tilde{\phi} : C(X) \longrightarrow C(Y) \otimes K$ be given by

$$\tilde{\phi}(f) = \sum_{i=1}^{n} (f \circ \lambda_i)p_i,$$
and define $\phi = (\text{id}_{C(Y)} \otimes \alpha) \circ (\tilde{\phi} \otimes \text{id}_K)$, where $\alpha : K \otimes K \to K$ is some isomorphism.

If one takes the projections $p_i$ in the above definition of $\phi$ to be trivial (that is, Murray-von Neumann equivalent to constant projections), and restricts the domain of $\phi$ to some matrix algebra over $C(X)$, a diagonal $*$-homomorphism in the usual sense is obtained; thus, the interest in these generalized diagonal maps comes from allowing for projections that are twisted in some way.

Given a projection $p$ in $C(X) \otimes K$ of constant rank, for $X$ a compact Hausdorff space, a complex vector bundle over $X$ can be obtained by associating to each $x \in X$ the range of the projection $p(x)$; similarly, each complex vector bundle of constant rank over $X$ can be seen as a projection in $C(X) \otimes K$. Under this correspondence, two complex vector bundles over $X$ are isomorphic if and only if their corresponding projections are Murray-von Neumann equivalent. It follows in particular that a projection is trivial if and only if its corresponding vector bundle is trivial. Vector bundles and projections will be therefore used interchangeably from here on.

In the next section more details are given concerning how Villadsen used nontrivial (that is, twisted) projections to obtain his higher stable rank examples. The arguments in the proof that his algebras have stable rank higher than one rely on precisely quantifying how twisted the projections are, in a certain sense; to do that, some algebraic topology is required.

The Euler class of a real vector bundle $\xi$ over a topological space $X$, denoted $e(\xi)$, is an element of the integral cohomology ring of $X$ that provides some measure on how twisted the bundle $\xi$ is; it is very useful to the present purposes in the sense that it detects whether a vector bundle has some trivial part to it. More precisely, if you add to any vector bundle $\xi$ a trivial vector bundle $\theta$, the resulting vector bundle $\xi \oplus \theta$ will have zero Euler class.
2.3 Constructing the algebras

For the convenience of the reader, in this section let us briefly review the construction of a Villadsen algebra of stable rank \( n + 1 \) for a given positive integer \( n \), as an inductive limit \( A = \lim_{\rightarrow} (A_i, \phi_i) \). All of the missing details and proofs can be found in [21].

The first algebra, \( A_1 \), is given by \( A_1 = p_1(C(\mathbb{D}^n) \otimes \mathcal{K})p_1 \) for some trivial rank one projection \( p_1 \in C(\mathbb{D}^n) \otimes \mathcal{K} \), and is thus isomorphic to the commutative algebra \( C(\mathbb{D}^n) \) of continuous, complex valued functions on the \( n \)-fold Cartesian power of the closed unit disk in the complex plane.

The disk has dimension two and therefore the stable rank of \( A_1 \) is \( n + 1 \), by the results recalled in the previous chapter. For \( i \geq 1 \), the base space for the algebra \( i + 1 \) is \( X_{i+1} = X_i \times \mathbb{CP}(a_i) \), where \( \mathbb{CP}(a_i) \) denotes the complex projective space of complex dimension \( a_i \), and \( a_i \) is some positive integer to be determined. Define generalized diagonal *-homomorphisms \( \phi_i : C(X_i) \otimes \mathcal{K} \rightarrow C(X_{i+1}) \otimes \mathcal{K} \), each obtained from two eigenvalue maps, as follows: the first is simply projection of \( X_{i+1} \) onto \( X_i \), associated with a trivial rank one projection over \( X_{i+1} \), and the second is a point evaluation to be determined, associated with a projection isomorphic to the pull-back to \( X_{i+1} \) of the universal line bundle over \( \mathbb{CP}(a_i) \).

Set \( p_{i+1} = \phi_i(p_i) \); each algebra \( A_i \) is then given by \( A_i = p_i(C(X_i)\otimes \mathcal{K})p_i \), and the connecting *-homomorphisms are obtained by restricting \( \phi_i \) to the algebra \( A_i \), for all \( i \).

The integers \( a_i \) are chosen so that \( \text{sr}(A_i) = n + 1 \) for all \( i \). This can be done by observing that the rank of the projection \( p_i \) is \( 2^{i-1} \), and using Theorem 2.1.2. Since inductive limits do not increase the stable rank ([13], Theorem 5.1), we have \( \text{sr}(A) \leq n+1 \). Finally, the point evaluations in the eigenvalue maps are chosen to ensure simplicity of the limit algebra.

For \( j = 1, \ldots, n \), let \( f_j \in C(\mathbb{D}^n) \) denote the projection onto the \( j \)-th coordinate. By construction, each projection \( p_{i+1} \) is the direct sum of a trivial rank one projection with a projection that is the pull-back to \( X_{i+1} \) of a vector bundle over \( \mathbb{CP}(a_1) \times \cdots \times \mathbb{CP}(a_i) \).
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satisfying the condition that the $n$-th power of its Euler class, in the integral cohomology ring of $\mathbb{CP}(a_1) \times \cdots \times \mathbb{CP}(a_i)$, is nonzero. Villadsen then showed that these projections have enough twist to ensure that the image of $(f_1, \ldots, f_n)$ inside $A_i^n$ via the maps $\phi_j$ lies at least distance 1 away from $\text{Lg}_n(A_i)$, for all $i$. The proof of this fact constitutes a fascinating application of differential topology in operator algebras, and shows that the limit algebra $A$ must have stable rank equal to $n + 1$.

Another surprising feature of these algebras is that their real ranks were estimated by Villadsen to be very close to their stable ranks: for a Villadsen algebra of finite stable rank $n + 1 \geq 2$ like the above, its real rank is either equal to $n + 1$ or $n$. It is not yet known which of these two possibilities actually occur; we will have more to say on this question in the next chapter.
Chapter 3

Generalizing the construction

As observed by Villadsen in [21], all C*-algebras in his second class of examples, described in the previous chapter, admit a unique tracial state.

A few years later, A. Toms described in [17] another class of higher stable rank simple C*-algebras, using a generalized mapping torus construction. These algebras can also be seen to admit a unique tracial state.

In this chapter we generalize Villadsen’s construction to produce examples of higher stable rank simple C*-algebras with more than one tracial state; an analysis of Villadsen’s algebras reveals that, in order to try and obtain more tracial states for these algebras, eigenvalue maps more general than the ones used by Villadsen must be introduced. This is by itself a nontrivial task, given that the eigenvalues affect the computations on the twist of the projections that make up the building block algebras, and these must be carefully monitored, to ensure the stable rank doesn’t drop in the limit algebra.

Other difficulties also arise, and will be discussed in the following two sections.

3.1 Order unit spaces, tracial states

Let us begin with a brief review of certain concepts that will be used later on. Given a compact, convex set $K$ in a locally convex space, consider the set $\text{Aff}(K)$ of real-valued
Chapter 3. Generalizing the construction

continuous affine functions on $K$. This set is an example of a complete order unit space ([1],[2]). By the category of complete order unit spaces we mean the category whose objects are complete order unit spaces and the morphisms of which are linear, positive, and order unit preserving maps between two such objects.

The following simple lemma will be very useful later on.

**Lemma 3.1.1.** Let $\psi : A \rightarrow B$ be a morphism in the category of complete order unit spaces. Then, $\psi$ is non-expansive, i.e., $\|\psi(a)\| \leq \|a\|$ for all $a \in A$.

**Proof.** This follows immediately from Proposition II.1.3 of [1], which states that a linear map between order unit spaces is positive if and only if it is order unit preserving and bounded with norm one.

If $K$ is a Bauer simplex, i.e., a Choquet simplex whose set of extreme points $\partial_e K$ is closed, then $\text{Aff}(K)$ is isomorphic in the category of order unit spaces to the space $C_{\mathbb{R}}(\partial_e K)$ of real valued, continuous functions on $\partial_e K$, the isomorphism being the map that sends an element of $\text{Aff}(K)$ into its restriction to the extreme boundary $\partial_e K$.

For a compact Hausdorff space $X$, the simplex $K$ of Borel probability measures on $X$ is an example of a Bauer simplex; its extreme boundary is composed of the Dirac measures (that is, the measures of the form $\mu_x$, $x \in X$, such that $\mu_x(Y)$ is equal to either 1 or zero depending on whether $x$ belongs to $Y$ or not), and is homeomorphic to $X$ (see [1], Corollary II.4.2). Thus, in this case $\text{Aff}(K)$ is isomorphic to $C_{\mathbb{R}}(X)$.

Given a unital C*-algebra $A$, we denote by $T(A)$ the space of tracial states on $A$. This space has the structure of a Choquet simplex if, for instance, $A$ is exact (by Theorem II.4.4 of [5] and the fact quasi-traces on exact C*-algebras are traces, see [11]). We can define a covariant functor from the category of unital C*-algebras and unital *-homomorphisms to the category of complete order unit spaces as follows: an object $A$ of the former is taken to $\text{Aff}(T(A))$, and a unital *-homomorphism $\phi : A \rightarrow B$ is taken to the map $\hat{\phi} : \text{Aff}(T(A)) \rightarrow \text{Aff}(T(B))$ given by $\hat{\phi}(g) = g \circ \phi^\#$, for $g \in \text{Aff}(T(A))$, where
\( \phi^\#: T(B) \rightarrow T(A) \) is the map \( \phi^\#(\tau) = \tau \circ \phi \), for \( \tau \in T(B) \).

As before, let \( K \) denote the \( C^* \)-algebra of compact operators on a separable infinite-dimensional Hilbert space \( \mathcal{H} \). Given a compact Hausdorff space \( X \) and \( p \in C(X) \otimes K \) a projection with constant rank, the simplex of tracial states on \( p(C(X) \otimes K)p \) is affinely homeomorphic to the simplex of Borel probability measures on \( X \); for a given Borel probability measure \( \mu \) on \( X \), the corresponding tracial state \( \tau \) on \( p(C(X) \otimes K)p \) is given by the equation

\[
\tau(a) = \frac{1}{\text{rank}(p)} \int_X \text{Tr}(a(x)) d\mu(x),
\]

where \( \text{Tr} \) denotes the standard unbounded trace on the \( C^* \)-algebra of bounded operators on \( \mathcal{H} \).

Let us recall an argument of K. Thomsen that shows that AI algebras of the form \( A = \lim_{\rightarrow} (A_i, \phi_i) \), where each \( A_i \) is a single full matrix algebra over the unit interval, admit arbitrary tracial state spaces ([16]). Given a Choquet simplex \( S \), Thomsen uses an approximate intertwining argument to show that \( \text{Aff}(S) \) can be written in the category of complete order unit spaces as an inductive limit of copies of \( C_\mathbb{R}([0,1]) \), with maps that can be lifted to diagonal \( * \)-homomorphisms between appropriately sized matrix algebras over the interval:

\[
\begin{align*}
M_{n_1}(C(I)) & \longrightarrow M_{n_2}(C(I)) \longrightarrow M_{n_3}(C(I)) \longrightarrow \cdots \longrightarrow A \\
C_\mathbb{R}(I) & \longrightarrow C_\mathbb{R}(I) \longrightarrow C_\mathbb{R}(I) \longrightarrow \cdots \longrightarrow \text{Aff}(S)
\end{align*}
\]

The inductive limit \( A \) of these matrix algebras and diagonal \( * \)-homomorphisms is then an AI algebra of the desired form with tracial state space affinely homeomorphic to \( S \).

This argument will now be generalized to algebras like the ones considered by Villadsen, as follows: given a Choquet simplex \( S \), an approximate intertwining argument is carried out to write \( \text{Aff}(S) \) as an inductive limit of spaces \( C_\mathbb{R}(X_i) \), for certain compact spaces \( X_i \), with maps that can be lifted to unital generalized diagonal \( * \)-homomorphisms.
between homogeneous C*-algebras \( p_i(C(X_i) \otimes K)p_i \) with appropriate dimensions and twist in the projections involved:

\[
p_1(C(X_1) \otimes K)p_1 \xrightarrow{\phi_1} p_2(C(X_2) \otimes K)p_2 \xrightarrow{\phi_2} p_3(C(X_3) \otimes K)p_3 \rightarrow \cdots \rightarrow A
\]

\[
C_\mathbb{R}(X_1) \xrightarrow{\psi_1} C_\mathbb{R}(X_2) \xrightarrow{\psi_2} C_\mathbb{R}(X_3) \rightarrow \cdots \rightarrow \text{Aff}(S)
\]

The generalization to allow for more general base spaces, and the lifting of maps to generalized diagonal *-homomorphisms are relatively straightforward, and are covered in the next section, in particular in Lemmas 3.2.1 and 3.2.3. Let us consider more closely some of the other potential problems that arise with this argument.

First, unlike what happens in Thomsen’s case, one must keep track of the dimension of the base spaces \( X_i \) and the rank of the projections \( p_i \), given that the stable rank of each building block algebra depends on those two quantities, by Theorem 2.1.2. Since at the \( i \)-th step a codomain base space \( X_{i+1} \) has to be fixed beforehand in order to define a map \( \psi_i : C_\mathbb{R}(X_i) \rightarrow C_\mathbb{R}(X_{i+1}) \) that will be later lifted and, after lifting, the rank of the projection \( p_{i+1} = \phi_i(p_i) \) may not give the correct stable rank, correction measures on the base spaces must be taken at each step to ensure that the stable rank ends up right.

Also, the projections \( p_{i+1} \) and, more importantly, their twists, depend on the liftings \( \phi_i \); it is therefore necessary to show it is possible to find liftings that will yield projections with the required twists. This will be done in the proof of Theorem 3.3.2.

Let us consider, in the remainder of this section, a definition and two theorems that will be used to carry out the approximate intertwining argument in the proof of Theorem 3.3.2. Given a diagram in the category of complete order unit spaces of the form

\[
A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots
\]

\[
B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \cdots
\]

let us say that the diagram is \textit{approximately commutative} if, for all \( i \), given any element \( x \) in either \( A_i \) or \( B_i \), the norm of the difference of the images of \( x \) along two distinct paths in
the diagram starting at the space $x$ is at and ending at the same place, converges to zero as the number of steps for which the two paths coincide tends to infinity (this is a minor modification of the definition for $C^*$-algebras given in [8]). The following two results are versions for complete order unit spaces of two theorems from [8] (Theorem 2.2, and the result preceding it, both results concerning $C^*$-algebras and *-homomorphisms), and are similarly proven as will be seen below. For details on the existence and construction of inductive limits of complete order unit spaces, see [16].

**Theorem 3.1.2.** Suppose that a diagram in the category of complete order unit spaces,

$$
\begin{array}{ccccccccc}
A_1 & \phi_1 & A_2 & \phi_2 & A_3 & \phi_3 & \cdots \\
\downarrow r_1 & \downarrow s_1 & \downarrow r_2 & \downarrow s_2 & \downarrow r_3 & \downarrow s_3 & \\
B_1 & \psi_1 & B_2 & \psi_2 & B_3 & \psi_3 & \cdots
\end{array}
$$

is given. For each $i$, let $a_i$ and $b_i$ be dense sequences in $A_i$ and $B_i$, respectively. Consider the finite subset $S_i$ of $A_i$ consisting of the images in $A_i$ of the first $i$ terms of the sequences $a_1, a_2, \ldots, a_{i-1}$ and $b_1, b_2, \ldots, b_i$ along all possible paths in the given diagram, and similarly the finite subset $T_i$ of $B_i$ consisting of the images in $B_i$ of the first $i$ terms of the sequences $a_1, a_2, \ldots, a_{i-1}$ and $b_1, b_2, \ldots, b_{i-1}$ along all possible paths in the given diagram. Suppose that for each $i$,

$$
\|\phi_i(a) - r_{i+1} \circ s_i(a)\| < 2^{-i}
$$

for all $a \in S_i$ and, similarly,

$$
\|\psi_i(b) - s_i \circ r_i(b)\| < 2^{-i}
$$

for all $b \in T_i$. Then, the given diagram is approximately commutative.

**Proof.** Fix an element $a \in S_i$ for some $i$, and consider two paths starting at $A_i$, coinciding until the $j$-th stage for some $j \geq i$, and ending at the same place, say at stage $k$ (that is, at $A_k$ or $B_k$). A straightforward computation, making repeated use of the triangle
inequality and Lemma 3.1.1, shows that the images of $a$ along the two paths differ by at most the sum of the prescribed differences at each triangle between stages $j$ and $k$, and this number goes to zero as $j$ tends to infinity, given that the series $\sum_{i=1}^{\infty} 2^{-i}$ is summable. For instance, if we consider $a \in A_1$ and the paths $\phi_2 \circ r_2 \circ s_1$ and $r_3 \circ s_2 \circ \phi_1$ starting at $A_1$ and ending at $A_3$, we would have (noting that $\phi_1(a) \in S_2$)

$$
\|\phi_2 \circ r_2 \circ s_1(a) - r_3 \circ s_2 \circ \phi_1(a)\| \leq \|\phi_2(r_2 \circ s_1(a) - \phi_1(a))\| + \|\phi_2(\phi_1(a)) - r_3 \circ s_2(\phi_1(a))\|
$$

$$
< \|r_2 \circ s_1(a) - \phi_1(a)\| + 2^{-2}
$$

$$
< 2^{-1} + 2^{-2}.
$$

The same argument holds for elements in $T_i$, and the density of the sequences $a_j, b_j$ ensures the desired convergence holds for arbitrary elements of $A_i$ and $B_i$, completing the proof.

**Theorem 3.1.3.** Suppose that an approximately commutative diagram in the category of complete order unit spaces,

\[
A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots
\]

\[
B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} \cdots
\]

is given. It follows that the inductive limits $A = \varinjlim(A_i, \phi_i)$ and $B = \varinjlim(B_i, \psi_i)$ are isomorphic.

**Proof.** Let $\mu_i : A_i \to A$ and $\eta_i : B_i \to B$ denote the maps into the inductive limits. For a given fixed $i$, consider the sequence of maps $\alpha_{i,j} : A_i \to B$, for $j > i$, defined by

$$
\alpha_{i,j} = \eta_{j+1} \circ s_j \circ \phi_{j,i},
$$

where $\phi_{j,i} : A_i \to A_j$ denotes the composition $\phi_{j-1} \circ \cdots \circ \phi_{i+1} \circ \phi_i$.

The sequence $\{\alpha_{i,j}\}_j$ converges pointwise; indeed, given a fixed $a \in A_i$, for $k > j$ and
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using Lemma 3.1.1 we have the following estimate:

\[ \|\alpha_{i,k}(a) - \alpha_{i,j}(a)\| = \|\eta_{k+1} \circ s_k \circ \phi_{k,i}(a) - \eta_{j+1} \circ s_j \circ \phi_{j,i}(a)\| \]
\[ = \|\eta_{k+1}(s_k \circ \phi_{k,i}(a) - \psi_{k+1,j+1} \circ s_j \circ \phi_{j,i}(a))\| \]
\[ \leq \|s_k \circ \phi_{k,i}(a) - \psi_{k+1,j+1} \circ s_j \circ \phi_{j,i}(a)\|. \]

The right-hand side is the difference in the images of \(a\) along two paths starting at \(A_i\), coinciding until the \(j\)-th stage, and ending at \(B_{k+1}\), which converges to zero as \(j\) tends to infinity by hypothesis, so the sequence \(\{\alpha_{i,j}(a)\}\) is Cauchy, and therefore converges as \(B\) is complete. Let \(\alpha_i : A_i \rightarrow B\) denote the function given by \(\alpha_i(a) = \lim_{j \rightarrow \infty} \alpha_{i,j}(a)\) for all \(a \in A_i\).

The maps \(\alpha_i, i \geq 1\), are compatible. It follows that they give rise to a map \(\alpha : A \rightarrow B\) such that \(\alpha \circ \mu_i = \alpha_i\) for all \(i\), which is linear, positive and order unit preserving. If we consider for a given \(i\) a sequence of maps \(\beta_{i,j} : B_i \rightarrow A\) by

\[ \beta_{i,j} = \mu_j \circ r_j \circ \psi_{j,i}, \]

then, in a similar way, with \(\beta_i : B_i \rightarrow A\) given by \(\beta_i(b) = \lim_{j \rightarrow \infty} \beta_{i,j}(b)\) for all \(b \in B_i\), one obtains a morphism \(\beta : B \rightarrow A\) such that \(\beta \circ \eta_i = \beta_i\) for all \(i\).

It remains to show that \(\alpha\) and \(\beta\) are inverses of each other. For a given \(i\) and \(a \in A_i\),

\[ \beta \circ \alpha_i(\mu_i(a)) = \lim_{j \rightarrow \infty} \beta \circ \alpha_{i,j}(a) = \lim_{j \rightarrow \infty} \beta \circ \eta_{j+1} \circ s_j \circ \phi_{j,i}(a) \]
\[ = \lim_{j \rightarrow \infty} \beta_{j+1} \circ s_j \circ \phi_{j,i}(a) \]
\[ = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \mu_k \circ r_k \circ \psi_{k,j+1} \circ s_j \circ \phi_{j,i}(a) \]
\[ = \mu_k \circ \phi_{k,i}(a) \]
\[ = \mu_i(a), \]

since \(r_k \circ \psi_{k,j+1} \circ s_j \circ \phi_{j,i}\) and \(\phi_{k,i}\) are two paths starting at \(A_i\), coinciding until the \(j\)-th stage, and ending at \(A_k\), and thus the difference of the images of \(a\) along them goes
to zero as \( j \) (and therefore \( k \)) tends to infinity. It follows that \( \beta \circ \alpha = \text{id}_A \); similarly, \( \alpha \circ \beta = \text{id}_B \), whence \( A \) and \( B \) are isomorphic.

\[ \square \]

### 3.2 Generalized diagonal \(*\)-homomorphisms, Markov operators

In this section we deal with the problem of lifting maps at the level of affine functions on traces to generalized diagonal \(*\)-homomorphisms at the level of \( C^* \)-algebras. We must of course determine what special kind of morphism is to be lifted, but first we should perhaps see what the image \( \hat{\phi} \) of a generalized diagonal \(*\)-homomorphism \( \phi \) via the functor described in the previous section looks like; that’s done with the following lemma, that generalizes a similar result of Thomsen for diagonal maps between matrix algebras (Lemma 3.5 of [16]).

**Lemma 3.2.1.** Let compact Hausdorff spaces \( X, Y \), a projection \( p \) in \( C(X) \otimes \mathcal{K} \) with constant rank, and a unital generalized diagonal \(*\)-homomorphism \( \phi : (C(X) \otimes \mathcal{K})_p \rightarrow (C(Y) \otimes \mathcal{K})_{\phi(p)} \)

corresponding to a \( d \)-tuple \((\lambda_i, q_i)_{i=1}^d\), where the projections \( q_i \) have constant rank 1, be given. Then, after making the identifications of \( \text{Aff}(T((C(X) \otimes \mathcal{K})_p)) \) with \( C_{\mathbb{R}}(X) \) and \( \text{Aff}(T((C(Y) \otimes \mathcal{K})_{\phi(p)})) \) with \( C_{\mathbb{R}}(Y) \), the induced map \( \hat{\phi} : C_{\mathbb{R}}(X) \rightarrow C_{\mathbb{R}}(Y) \) is given by the formula

\[ \hat{\phi}(g) = \frac{1}{d} \sum_{i=1}^{d} g \circ \lambda_i. \]

**Proof.** Let us begin by observing that if \( x, y \) are arbitrary points in \( X, Y \) respectively, then the tracial states over \( (C(X) \otimes \mathcal{K})_p \) and \( (C(Y) \otimes \mathcal{K})_{\phi(p)} \) corresponding to the Dirac measures \( \mu_x \) and \( \mu_y \) are given respectively by the formulas \( \tau_x(a) = \frac{1}{\text{rank}(p)} \text{Tr}(a(x)), \tau_y(b) = \frac{1}{\text{rank}(p)} \text{Tr}(a(y)) \).
If \( b = \phi(a) \), then since \( \phi \) is essentially sending \( a \) to \( d \) orthogonal twisted copies of \( a \), namely \( a \circ \lambda_i \), \( 1 \leq i \leq d \), we have that

\[
\tau_y(\phi(a)) = \frac{1}{\text{rank}(\phi(p))} \text{Tr}(\phi(a)(y))
\]

\[
= \frac{1}{\text{rank}(\phi(p))} \sum_{i=1}^{d} \text{Tr}(a(\lambda_i(y)))
\]

\[
= \frac{\text{rank}(p)}{\text{rank}(\phi(p))} \sum_{i=1}^{d} \tau_{\lambda_i(y)}(a).
\]

Observing that \( \frac{\text{rank}(\phi(p))}{\text{rank}(p)} = d \), we therefore conclude that \( \phi^\#(\tau_y) = \frac{1}{d} \sum_{i=1}^{d} \tau_{\lambda_i(y)} \), and the result follows.

A linear map \( \omega : C(X) \to C(Y) \) is called a Markov operator if it is unital and positive. Markov operators are bounded, and they constitute a convex subset of the space of bounded operators from \( C(X) \) to \( C(Y) \), whose extreme boundary is given by \( \text{Hom}_1(C(X), C(Y)) \), the set of unital *-homomorphisms from \( C(X) \) to \( C(Y) \) (see [16]). The following theorem is a Krein-Milman type theorem for Markov operators due to Thomsen ([16], Theorem 2.1), included here for convenience.

**Theorem 3.2.2.** Assume that \( X \) is path-connected. Then the closed convex hull of the unital *-homomorphisms from \( C(X) \) to \( C(Y) \), in the strong operator topology, is the set of Markov operators.

The rational convex hull of \( \text{Hom}_1(C(X), C(Y)) \), denoted by \( \text{co}_Q \text{Hom}_1(C(X), C(Y)) \), is the set of Markov operators of the form

\[
\omega = \sum_{j=1}^{N} \frac{\alpha_j}{d} \omega_j,
\]

where \( N, d, \alpha_1, \ldots, \alpha_N \) are natural numbers satisfying \( \sum_{j=1}^{N} \alpha_j = d \) and with \( \omega_j \in \text{Hom}_1(C(X), C(Y)) \), \( 1 \leq j \leq N \). It follows from the above theorem and the fact that \( \mathbb{Q} \) is dense in \( \mathbb{R} \) that \( \text{co}_Q \text{Hom}_1(C(X), C(Y)) \) is dense, in the strong operator topology, in the set of Markov operators. It is the elements in this rational convex hull that can be lifted
to generalized diagonal *-homomorphisms, as the following lemma shows; it generalizes Lemma 3.6 of [16].

**Lemma 3.2.3.** Let compact Hausdorff spaces $X, Y$ and $\omega \in \text{co}_Q \text{Hom}_1(C(X), C(Y))$ be given. If $\omega = \sum_{j=1}^{N} \frac{\alpha_j}{d} \omega_j$ with $N, d, \alpha_j, \omega_j$ as above, then there are continuous maps $\lambda_i : Y \to X$, $1 \leq i \leq d$, satisfying the following: if $q_1, \ldots, q_d$ are any $d$ mutually orthogonal projections in $C(Y) \otimes K$ with constant rank equal to 1, and

$$\phi : C(X) \otimes K \to C(Y) \otimes K$$

is a generalized diagonal *-homomorphism corresponding to the $d$-tuple $(\lambda_i, q_i)_{i=1}^d$, then for any nonzero projection $p$ in $C(X) \otimes K$ with constant rank, the restricted (unital) *-homomorphism $\phi_p : (C(X) \otimes K)_p \to (C(Y) \otimes K)_{\phi(p)}$ induced by $\phi$ satisfies $\hat{\phi}_p = \omega$ (after making the proper identifications).

**Proof.** For $j = 1, \ldots, N$, let $h_j : Y \to X$ denote the continuous function such that the induced unital *-homomorphism $h_j^* : C(X) \to C(Y)$ equals $\omega_j$ (that is, for all $f \in C(X)$, we have $\omega_j(f) = f \circ h_i$).

Fix $d$ mutually orthogonal projections $q_1, \ldots, q_d$ in $C(Y) \otimes K$ with constant rank equal to 1, and define $\lambda_1, \ldots, \lambda_d : Y \to X$ in order by $\alpha_1$ copies of $h_1$, $\alpha_2$ copies of $h_2$, and so on; let $\phi : C(X) \otimes K \to C(Y) \otimes K$ be a generalized diagonal *-homomorphism corresponding to the $d$-tuple $(\lambda_i, q_i)_{i=1}^d$. It now follows from Lemma 3.2.1 that, for a given $g \in C_R(X)$, the restricted *-homomorphism $\phi_p$ satisfies

$$\hat{\phi}_p(g) = \frac{1}{d} \sum_{j=1}^{d} g \circ \lambda_j$$

$$= \frac{1}{d} \sum_{j=1}^{N} \alpha_j g \circ h_j$$

$$= \sum_{j=1}^{N} \frac{\alpha_j}{d} \omega_j(g)$$

$$= \omega(g),$$

completing the proof. \qed
3.3 The main theorem

Suppose that $M$ is a compact, connected and orientable differentiable manifold, and let $\mathbb{D}^n$ denote the $n$-fold Cartesian power of the closed unit disk in the complex plane. Let $\pi_1 : \mathbb{D}^n \times M \rightarrow \mathbb{D}^n$, $\pi_2 : \mathbb{D}^n \times M \rightarrow M$ denote the coordinate projections, let $a_j \in C(\mathbb{D}^n)$, $1 \leq j \leq n$ denote the projection onto the $j$-th coordinate, let $p$ be a trivial rank one projection in $C(\mathbb{D}^n \times M) \otimes \mathcal{K}$, and let $q$ be a projection in $C(\mathbb{D}^n \times M) \otimes \mathcal{K}$ orthogonal to $p$ of the form $q = \pi_2^* (\eta)$, where $\eta$ is a complex vector bundle over $M$ such that $e(\eta)^n$, the $n$-fold (cup) product of the Euler class $e(\eta)$ of $\eta$ by itself, is nonzero. Set $B = (C(\mathbb{D}^n \times M) \otimes \mathcal{K})_{p+q}$. The following result is due to Villadsen, and will be essential to the proof of Theorem 3.3.2:

**Theorem 3.3.1** (Villadsen ([21], Theorem 7)). Suppose that $b_1, \ldots, b_n \in B$ are such that $p b_j p = (a_j \circ \pi_1) p$ for each $j$. Then the distance from $(b_1, \ldots, b_n)$ to $L_{g_n}(B)$ is at least 1.

We are now ready for our main theorem.

**Theorem 3.3.2.** Let $m \in \{2, 3, \ldots\} \cup \{\infty\}$ and a non-empty metrizable Choquet simplex $S$ be given. Then, there exists a simple, separable, unital AH-algebra $A$ with stable rank $m$ and tracial state space affinely homeomorphic to $S$.

**Proof.** By [10], Proposition 4.2, any non-empty metrizable Choquet simplex is affinely homeomorphic to the projective limit of a sequence of finite dimensional simplices and continuous affine maps; it follows that the space $\text{Aff}(S)$ is isomorphic to the inductive limit of a sequence

$$
\mathbb{R}^{n_1} \xrightarrow{\theta_1} \mathbb{R}^{n_2} \xrightarrow{\theta_2} \mathbb{R}^{n_3} \xrightarrow{\theta_3} \cdots
$$

in the category of complete order unit spaces ($\mathbb{R}^n$ is a complete order unit space with the usual ordering and order unit given by the element $(1, 1, \ldots, 1)$).

Let us start with the case of finite $m$. Write $n = m - 1$, and consider the spaces $X_1 = \mathbb{D}^n$ and $\tilde{X}_2 = X_1 \times \mathbb{C}P(n)$, where $\mathbb{C}P(n)$ is the complex projective space of complex
dimension \( n \). Choose \( n_2 \) distinct points in \( \tilde{X}_2 \), say \( \tilde{b}_1, \ldots, \tilde{b}_{n_2} \), and a partition of unity on 
\( C_{\mathbb{R}}(\tilde{X}_2), g_1, \ldots, g_{n_2} \), such that \( g_i(\tilde{b}_j) = \delta^i_j \), \( 1 \leq i, j \leq n_2 \). Define maps 
\( \tilde{s}_2 : \mathbb{R}^{n_2} \rightarrow C_{\mathbb{R}}(\tilde{X}_2) \) 
and \( \tilde{r}_2 : C_{\mathbb{R}}(\tilde{X}_2) \rightarrow \mathbb{R}^{n_2} \) by 
\[
\tilde{s}_2(a_1, \ldots, a_{n_2}) = \sum_{j=1}^{n_2} a_j g_j \\
\tilde{r}_2(f) = (f(\tilde{b}_1), \ldots, f(\tilde{b}_{n_2}));
\]

it can be easily verified that these maps are linear, positive, and order unit preserving 
and, moreover, that \( \tilde{s}_2 \) is injective with left inverse given by \( \tilde{r}_2 \).

By choosing \( n_1 \) distinct points in \( X_1 \) and proceeding analogously as above, define 
maps \( s_1 : \mathbb{R}^{n_1} \rightarrow C_{\mathbb{R}}(X_1) \) and \( r_1 : C_{\mathbb{R}}(X_1) \rightarrow \mathbb{R}^{n_1} \) with \( r_1 \) as left inverse to \( s_1 \). Denote 
the composition map \( \tilde{s}_2 \circ \theta_1 \circ r_1 : C_{\mathbb{R}}(X_1) \rightarrow C_{\mathbb{R}}(\tilde{X}_2) \) by \( \tilde{\psi}_1 \). We obtain by construction 
the following commutative diagram:

Choose a dense sequence in the closed unit ball of \( C_{\mathbb{R}}(X_1) \) (with the supremum norm), 
say \( f_{1,1}, f_{1,2}, f_{1,3}, \ldots \). The map \( \tilde{\psi}_1 \) gives rise to a Markov operator 
\( \tilde{\psi}_1' : C(X_1) \rightarrow C(\tilde{X}_2) \) via 
\( \tilde{\psi}_1'(f) = \tilde{\psi}_1(\Re(f)) + i\tilde{\psi}_1(\Im(f)); \) we can then choose \( \gamma_1 \in \text{co}_q \Hom_1(C(X_1), C(\tilde{X}_2)) \) 
such that 
\[
\|\tilde{\psi}_1(f_{1,1}) - \gamma_1(f_{1,1})\| = \|\tilde{\psi}_1'(f_{1,1}) - \gamma_1(f_{1,1})\| < 2^{-1}.
\]
The map \( \gamma_1 \) is of the form 
\[
\gamma_1 = \sum_{j=1}^{d_1} \alpha_1^j \gamma_1^j,
\]
where \( N_1, d_1, \alpha_1^1, \ldots, \alpha_1^{N_1} \in \mathbb{N}, \sum_{j=1}^{N_1} \alpha_1^j = d_1, \gamma_1^j \in \Hom_1(C(X_1), C(\tilde{X}_2)), \) and we can 
assume that \( d_1 \geq 2^3 \).
Let $X_2 = X_1 \times \mathbb{C}P(n(d_1 - 1))$, and denote by $\pi$ the continuous projection of $\mathbb{C}P(n(d_1 - 1))$ onto $\mathbb{C}P(n)$ induced by the projection of $\mathbb{C}^n(d_1 - 1) + 1$ onto $\mathbb{C}^{n+1}$ obtained by restricting to the first $n + 1$ coordinates; the map $\eta_2 := \text{id}_{X_1} \times \pi : X_2 \rightarrow \tilde{X}_2$ is continuous and surjective, so it gives rise to a unital injective $\ast$-homomorphism $\eta_2^\ast : C(\tilde{X}_2) \rightarrow C(X_2)$. Choose $n_2$ distinct points $b_1, \ldots, b_{n_2}$ in $X_2$ such that $\eta_2(b_i) = \tilde{b}_i$, and consider the maps $r_2 : C_\mathbb{R}(X_2) \rightarrow \mathbb{R}^{n_2}$, $\psi_1 : C(X_1) \rightarrow C(X_2)$ given by

$$r_2(f) = (f(b_1), \ldots, f(b_{n_2}))$$

$$\psi_1 = \eta_2^\ast \circ \gamma_1 = \sum_{j=1}^{N_1} \frac{\alpha_j}{d_1} \eta_2^\ast \circ \gamma_1^j.$$

We obtain a diagram

$$\begin{array}{ccc}
\mathbb{R}^{n_1} & \xrightarrow{\theta_1} & \mathbb{R}^{n_2} \\
\downarrow{r_1} & \nearrow{\eta_2^\ast \circ \delta_2 \circ \theta_1} & \uparrow{r_2} \\
C_\mathbb{R}(X_1) & \xrightarrow{\psi_1} & C_\mathbb{R}(X_2)
\end{array}$$

such that the upper triangle in the diagram is commutative, and the lower triangle approximately commutes within $2^{-1}$ on the set $S_1 = \{f_{1,1}\} \subseteq C_\mathbb{R}(X_1)$. Indeed, commutativity of the upper triangle follows from the fact $r_2 \circ \eta_2^\ast = \tilde{r}_2$ by construction and, for the lower triangle, we have

$$\|\eta_2^\ast \circ \tilde{s}_2 \circ \theta_1 \circ r_1(f_{1,1}) - \psi_1(f_{1,1})\| = \|\eta_2^\ast \left(\tilde{\psi}_1(f_{1,1}) - \gamma_1(f_{1,1})\right)\| < 2^{-1},$$

using the fact $\eta_2^\ast$ is contractive.

For the next step, let $I$ denote the closed unit interval $[0,1]$. Using the spaces $\mathbb{R}^{n_3}$ and $I$, do the same construction as before to produce maps $\tilde{s}_3 : \mathbb{R}^{n_3} \rightarrow C_\mathbb{R}(I)$ and $\tilde{r}_3 : C_\mathbb{R}(I) \rightarrow \mathbb{R}^{n_3}$, with $\tilde{r}_3$ as left inverse to $\tilde{s}_3$, and define $\psi_2^1 : C_\mathbb{R}(X_2) \rightarrow C_\mathbb{R}(I)$ by the composition $\tilde{s}_3 \circ \theta_2 \circ r_2$ ($r_2$ is the same map we found in the last step), obtaining a commutative diagram just as before.

Choose a dense sequence in the closed unit ball of $C_\mathbb{R}(X_2)$, say $f_{2,1}, f_{2,2}, f_{2,3}, \ldots$, and consider the set $S_2 \subseteq C_\mathbb{R}(X_2)$ consisting of the elements $f_{2,1}, f_{2,2}, \psi_1(f_{1,1}), \eta_2^\ast \circ \tilde{\psi}_1(f_{1,1})$ (in
other words, the first two elements of the chosen dense sequence in the closed unit ball of $C_\mathbb{R}(X_2)$, and the images of $f_{1,1}$ in $C_\mathbb{R}(X_2)$ along the two possible paths in the diagram obtained in the last step). There is then $\gamma_2^1 \in \text{co}_Q \text{Hom}_1(C(X_2), C(I))$ such that $\gamma_2^1$ and $\psi_2^1$ are close within $2^{-3}$ on the set $S_2$; write

$$\gamma_2^1 = \sum_{j=1}^{N_2} \frac{\alpha_j^1}{d_2^j} \epsilon_2^j,$$

where $\sum_{j=1}^{N_2} \alpha_j^1 = d_2^1, \epsilon_2^j \in \text{Hom}_1(C(X_2), C(I))$, and assume that $d_2^1 \geq 2\sqrt{3} = 2^2$.

Let $\tilde{X}_3 = X_2 \times \mathbb{C}P(n)$; by using $n_3$ distinct points in $\tilde{X}_3$, produce maps $\tilde{s}_3 : \mathbb{R}^{n_3} \rightarrow C_\mathbb{R}(\tilde{X}_3)$ and $\tilde{r}_3 : C_\mathbb{R}(\tilde{X}_3) \rightarrow \mathbb{R}^{n_3}$ with $\tilde{r}_3$ as left inverse to $\tilde{s}_3$, just as before, and denote the composition $\tilde{s}_3 \circ \tilde{r}_3 : C_\mathbb{R}(I) \rightarrow C_\mathbb{R}(\tilde{X}_3)$ by $\psi_2^2$.

Let $S'_2$ be the subset of $C_\mathbb{R}(I)$ given by $\gamma_2^1(S_2)$. Choose $\gamma_2^2 \in \text{co}_Q \text{Hom}_1(C(I), C(\tilde{X}_3))$ such that $\gamma_2^2$ and $\psi_2^2$ are close within $2^{-3}$ on the set $S'_2$. Write

$$\gamma_2^2 = \sum_{j=1}^{M_2} \frac{\beta_j^2}{d_2^j} \delta_2^j,$$

where $\sum_{j=1}^{M_2} \beta_j^2 = d_2^2, \delta_2^j \in \text{Hom}_1(C(I), C(\tilde{X}_3))$, assuming that $d_2^2 \geq 2^2$.

Now that $d_2^1$ and $d_2^2$ are fixed, let $d_2 = d_2^1 d_2^2$, and set $X_3 = X_2 \times \mathbb{C}P(nd_1(d_2 - 1))$. By projecting $\mathbb{C}P(nd_1(d_2 - 1))$ onto $\mathbb{C}P(n)$ we can produce as before a continuous map $\eta_3$ from $X_3$ onto $\tilde{X}_3$ and $r_3 : C_\mathbb{R}(X_3) \rightarrow \mathbb{R}^{n_3}$ such that $r_3 \circ \eta_3 = \tilde{r}_3$, and obtain the following diagram:

The upper triangles on this diagram commute, the first lower triangle approximately commutes within $2^{-3}$ on the set $S_2$, and the second lower triangle approximately commutes within $2^{-3}$ on the set $S'_2$. If we now set $\tilde{\psi}_2 = \eta_3^* \circ \psi_2^2 \circ \tilde{s}_3 \circ \theta_2, \psi_2 = \eta_3^* \circ \gamma_2^2 \circ \gamma_1^1$, a
straightforward calculation (using Lemma 3.1.1 and the fact \( \eta_3^* \) is non-expansive) shows that the diagram

\[
\begin{array}{ccc}
\mathbb{R}^{n_2} & \xrightarrow{\theta_2} & \mathbb{R}^{n_3} \\
 r_2 & \searrow & r_3 \\
 C_R(X_2) & \xrightarrow{\psi_2} & C_R(X_3)
\end{array}
\]

is commutative at the upper triangle, and approximately commutative within \( 2^{-2} \) on the set \( S_2 \subseteq C_R(X_2) \) at the lower triangle. The fact that \( \psi_2 \) factors through \( C_R(I) \) will be very important later on.

The \( i \)-th step, \( i \geq 3 \), is analogous to the second step; more precisely, use the spaces \( \mathbb{R}^{n_{i+1}} \) and \( I \) and do the by now familiar construction to produce maps \( \hat{s}_{i+1} : \mathbb{R}^{n_{i+1}} \to C_R(I) \) and \( \hat{r}_{i+1} : C_R(I) \to \mathbb{R}^{n_{i+1}} \), with \( \hat{r}_{i+1} \) as left inverse to \( \hat{s}_{i+1} \), and define \( \psi^1_i : C_R(X_i) \to C_R(I) \) by the composition \( \hat{s}_{i+1} \circ \theta_i \circ r_i \), to obtain a commutative diagram.

After that, fix a dense sequence in the closed unit ball of \( C_R(X_i) \), and set \( S_i \) to be the subset of \( C_R(X_i) \) consisting of the first \( i \) elements of this sequence, together with the images in \( C_R(X_i) \) of the first \( i \) elements of all the dense sequences chosen in the previous steps along all possible paths in the diagram

\[
\begin{array}{ccccccc}
\mathbb{R}^{n_1} & \xrightarrow{\theta_1} & \mathbb{R}^{n_2} & \xrightarrow{\theta_2} & \mathbb{R}^{n_3} & \cdots & \xrightarrow{\theta_{i-1}} & \mathbb{R}^{n_i} \\
 r_1 & \searrow & r_2 & \searrow & r_3 & \cdots & \searrow & r_i \\
 C_R(X_1) & \xrightarrow{\psi_1} & C_R(X_2) & \xrightarrow{\psi_2} & C_R(X_3) & \cdots & \xrightarrow{\psi_{i-1}} & C_R(X_i)
\end{array}
\]

Choose \( \gamma^1_i \in \text{co}_Q \text{Hom}_1(C(X_i), C(I)) \) such that \( \gamma^1_i \) and \( \psi^1_i \) are close within \( 2^{-(i+1)} \) on the set \( S_i \), writing

\[
\gamma^1_i = \sum_{j=1}^{N_i} \alpha^j_i d^j_i e^j_i,
\]

where \( \sum_{j=1}^{N_i} \alpha^j_i = d^1_i \), \( e^j_i \in \text{Hom}_1(C(X_i), C(I)) \), and such that \( d^1_i \geq 2^{\sqrt{i+2}} \).

Set \( \tilde{X}_{i+1} = X_i \times \mathbb{CP}(n) \), construct morphisms \( \tilde{s}_{i+1} : \mathbb{R}^{n_{i+1}} \to C_R(\tilde{X}_{i+1}) \) and \( \tilde{r}_{i+1} : C_R(\tilde{X}_{i+1}) \to \mathbb{R}^{n_{i+1}} \) with \( \tilde{r}_{i+1} \) as left inverse to \( \tilde{s}_{i+1} \), and let \( \psi^2_i : C_R(I) \to C_R(\tilde{X}_{i+1}) \)
denote the composition $\tilde{s}_{i+1} \circ \tilde{r}_{i+1}$. Choose $\gamma_i^2 \in \text{co}_Q \text{Hom}_1(C(I), C(\tilde{X}_{i+1}))$ such that $\gamma_i^2$ and $\psi_i^2$ are close within $2^{-(i+1)}$ on the set $S'_i = \gamma_i^1(S_i) \subseteq C_R(I)$, writing

$$\gamma_i^2 = \sum_{j=1}^{M_i} \frac{\beta_j^i}{d_i^2} \delta_j^i,$$

where $\sum_{j=1}^{M_i} \beta_j^i = d_i^2$ and $d_i^2 \geq 2^{\sqrt{r_i^2}}$.

Set $d_i = d_i^1 d_i^2$, $D_k = \prod_{j=1}^{k} d_j$, and $X_{i+1} = X_i \times \mathbb{C} \mathbb{P}(nD_{i-1}(d_i - 1))$. Just as before, there exists a continuous map $\eta_{i+1}$ from $X_{i+1}$ onto $\tilde{X}_{i+1}$, and morphisms $r_{i+1} : C_R(X_{i+1}) \to \mathbb{R}^{n_{i+1}}$, $s_{i+1} : \mathbb{R}^{n_{i+1}} \to C_R(X_{i+1})$ such that $r_{i+1}$ is a left inverse to $s_{i+1}$ and $r_{i+1} \circ \eta_{i+1}^* = \tilde{r}_{i+1}$. By putting $\tilde{\psi}_i = \eta_{i+1}^* \circ \psi_i^2 \circ \tilde{s}_{i+1} \circ \theta_i$, $\psi_i = \eta_{i+1}^* \circ \gamma_i^2 \circ \gamma_i^1$, we obtain a diagram

$$\begin{array}{ccc}
\mathbb{R}^{n_i} & \xrightarrow{\theta_i} & \mathbb{R}^{n_{i+1}} \\
\downarrow r_i & & \downarrow \tilde{r}_{i+1} \\
C_R(X_i) & \xrightarrow{\psi_i} & C_R(X_{i+1})
\end{array}$$

such that its upper triangle commutes, and its lower triangle approximately commutes to within $2^{-i}$ on $S_i$.

Repeating this for all $i$ we obtain an infinite diagram in the category of complete order unit spaces,

$$\begin{array}{ccc}
\mathbb{R}^{n_1} & \xrightarrow{\theta_1} & \mathbb{R}^{n_2} \xrightarrow{\theta_2} \mathbb{R}^{n_3} \xrightarrow{\theta_3} \cdots \\
r_1 \downarrow \tilde{\psi}_1 & & r_2 \downarrow \tilde{\psi}_2 & & r_3 \downarrow \tilde{\psi}_3 \\
C_R(X_1) & \xrightarrow{\psi_1} C_R(X_2) & \xrightarrow{\psi_2} C_R(X_3) & \xrightarrow{\psi_3} \cdots
\end{array}$$

which was constructed to be approximately commutative, by Theorem 3.1.2. It then follows from Theorem 3.1.3 that the inductive limits $\lim\to \mathbb{R}^{n_i}(\theta_i)$ and $\lim\to (C_R(X_i), \psi_i)$, in the category of complete order unit spaces, are isomorphic, whence $\text{Aff}(S)$ is isomorphic to $\lim\to (C_R(X_i), \psi_i)$.

By definition, $\psi_i = \eta_{i+1}^* \circ \gamma_i^2 \circ \gamma_i^1$ for $i \geq 2$, and $\eta_{i+1}^* \circ \gamma_i^2 \in \text{co}_Q \text{Hom}_1(C(I), C(X_{i+1}))$, $\gamma_i^1 \in \text{co}_Q \text{Hom}_1(C(X_i), C(I))$. From the formula given for $\gamma_i^1$ above, and choosing $d_i^1$
orthogonal (trivial) rank one projections in $C(I) \otimes \mathcal{K}$, say $u_1^i, \ldots, u_{d_i}^i$, we now apply Lemma 3.2.3 to the map $\gamma_i^1$ to obtain continuous maps $\mu_j^i : I \rightarrow X_i$ and a generalized diagonal $\ast$-homomorphism

$$\chi_i^1 : C(X_i) \otimes \mathcal{K} \rightarrow C(I) \otimes \mathcal{K}$$

associated to the $d_i^1$-tuple $(\mu_j^i, u_j^i)$ satisfying the conclusions of the lemma.

Recall that $X_{i+1} = X_i \times \mathbb{CP}(nc_i)$, where $c_i = D_{i-1}(d_i - 1)$ (with the convention $D_0 = 1$); for $i \geq 1$, denote by $\pi_{i+1}^1$ and $\pi_{i+1}^2$ the projections of $X_{i+1}$ onto $X_i$ and $\mathbb{CP}(nc_i)$ respectively, and by $\Gamma_{nc_i}$ the universal line bundle over $\mathbb{CP}(nc_i)$. Now, consider the formula given for $\gamma_i^2$, and choose $d_i^2$ orthogonal projections in $C(X_{i+1}) \otimes \mathcal{K}$, say $v_1^{i+1}, \ldots, v_{d_i^2}^{i+1}$, such that $v_1^{i+1}$ is a trivial rank one projection and, for $j \geq 2$, each $v_j^{i+1}$, when seen as a vector bundle, is isomorphic to the bundle $\pi_{i+1}^2(\Gamma_{nc_i})$. Apply Lemma 3.2.3 to $\eta_i^2 \circ \gamma_i^2$ to get continuous maps $\lambda_j^{i+1} : X_{i+1} \rightarrow I$ and a generalized diagonal $\ast$-homomorphism

$$\chi_i^2 : C(I) \otimes \mathcal{K} \rightarrow C(X_{i+1}) \otimes \mathcal{K}$$

associated to the $d_i^2$-tuple $(\lambda_j^{i+1}, v_j^{i+1})$ satisfying the conclusions of the lemma.

The $\ast$-homomorphism given by the composition

$$\chi_i = \chi_i^2 \circ \chi_i^1 : C(X_i) \otimes \mathcal{K} \rightarrow C(X_{i+1}) \otimes \mathcal{K}$$

is also a generalized diagonal $\ast$-homomorphism, associated to a $d_i^1 \cdot d_i^2$-tuple (that is, a $d_i$-tuple, if we recall $d_i = d_i^1 \cdot d_i^2$) given by $(\mu_j^i \circ \lambda_k^{i+1}, q_{j,k}^{i+1})_{j,k}$, where each $q_{j,k}^{i+1}$ is a projection that, when seen as a vector bundle, is isomorphic to $(u_j^i \circ \lambda_k^{i+1}) \otimes v_k^{i+1}$, which in turn is itself isomorphic to $v_k^{i+1}$, since $u_j^i$ is a trivial rank one projection. For convenience, we will write this $d_i$ tuple as $(\tilde{\sigma}_j^{i+1}, q_j^{i+1})$; thus, for each $j$, $\tilde{\sigma}_j^{i+1} = \mu_j^i \circ \lambda_k^{i+1}$ for some $k$, $l$, $q_1^{i+1}$ is a trivial rank one projection and, for each $j \geq 2$, $q_j^{i+1}$ is isomorphic to $\pi_{i+1}^2(\Gamma_{nc_i})$ (as vector bundles).
The case \( i = 1 \) is easier; apply Lemma 3.2.3 to \( \psi_1 \) directly to obtain \( \chi_1 \) associated to the \( d_1 \)-tuple \( (\tilde{a}^2_j, q^2_j) \), where \( q^2_1 \) is a trivial rank one projection and \( q^2_j \) is isomorphic to \( \pi^2_2(\Gamma_{nc}) \) if \( j \geq 2 \).

Let \( \tilde{p}_1 \) be a trivial rank one projection in \( C(X_1) \otimes K \) and set \( \tilde{p}_{i+1} = \chi_i(\tilde{p}_i), i \geq 1 \). Set \( \tilde{A}_i = (C(X_i) \otimes K)_{\tilde{p}_i} \); if we consider the restricted (unital) \(*\)-homomorphisms \( \chi_i : \tilde{A}_i \rightarrow \tilde{A}_{i+1} \), it then follows by functoriality and Lemma 3.2.3 that \( \hat{\chi}_i = \psi_i \), since

\[
\hat{\chi}_i = \chi^2_i \circ \chi^1_i = \chi^2_i \circ \chi^1_i = \eta^*_i \circ \gamma^2_i \circ \gamma^1_i = \psi_i,
\]

and \( \hat{\chi}_1 = \psi_1 \) directly from the lemma. If \( \tilde{A} \) is the \( \mathrm{C}^* \)-algebra given by the inductive limit \( \lim_{\rightarrow}(\tilde{A}_i, \chi_i) \), then the above implies that \( \operatorname{Aff}(T(\tilde{A})) \) is isomorphic to \( \operatorname{Aff}(S) \).

The algebra \( \tilde{A} \) is not necessarily simple; furthermore, by construction it is an AI algebra and therefore has stable rank one. The last step of the proof consists in modifying the algebra \( \tilde{A} \) to obtain a new \( \mathrm{C}^* \)-algebra \( A \) that is simple, has stable rank \( n + 1 \), and such that \( \operatorname{Aff}(T(A)) \) is isomorphic to \( \operatorname{Aff}(T(\tilde{A})) \). We do this as follows: for each \( i \), choose a dense sequence in \( X_i \), say \( x_{i,1}, x_{i,2}, x_{i,3}, \ldots \). Let \( y_1, y_2, y_3, \ldots \) be the enumeration of the union of all these sequences given by the arrows in the diagram below (so that \( y_1 = x_{1,1}, y_2 = x_{1,2}, y_3 = x_{2,1} \) and so on):

\[
\begin{align*}
x_{1,1} &\rightarrow x_{1,2} & x_{1,3} &\rightarrow x_{1,4} \\
x_{2,1} &\rightarrow x_{2,2} & x_{2,3} &\\x_{3,1} &\rightarrow x_{3,2} & &\\x_{4,1} &\rightarrow x_{4,2} & &\\x_{5,1} & & &
\end{align*}
\]

Observe that this enumeration is such that, for any given natural number \( k \geq 2 \), the element \( y_k \) belongs to a space \( X_{l_k} \) with the property that \( l_k < k \). Now, for each \( k \), choose
an element \( z_k \in X_k \) such that \( \pi_{l_k+1} \circ \cdots \circ \pi_{l_2} \circ \pi_{l_1}(z_k) = y_k \).

For each \( i \), consider the generalized diagonal \( * \)-homomorphism
\[ \phi_i : C(X_i) \otimes K \rightarrow C(X_{i+1}) \otimes K \]
corresponding to \( (\sigma_j^{i+1}, q_j^{i+1})_{j=1}^{d_i} \), obtained from \( \chi_i \) by simply changing the eigenvalue maps as follows: set \( \sigma_1^{i+1} = \pi_{i+1} \), let \( \sigma_2^{i+1} \) denote the constant map that takes all points in \( X_{i+1} \) into \( z_i \), and set \( \sigma_j^{i+1} = \tilde{\sigma}_j^{i+1} \) if \( j = 3, \ldots, d_i \). Now, set \( p_1 = \tilde{p}_1 \), \( p_{i+1} = \phi_i(p_i) \) for \( i \geq 1 \), and set \( A_i = (C(X_i) \otimes K)_{p_i} \). The \( C^* \)-algebra \( A \) we are looking for is equal to the inductive limit \( \lim_{\rightarrow} (A_i, \phi_i) \).

Clearly, \( A \) is a unital AH-algebra. The point evaluations \( z_k \) were chosen to ensure that, for any \( a \in A_i \), there is an index \( j \geq i \) such that all the images of \( a \) along the inductive sequence after the \( j \)-th stage are everywhere nonzero. This will in turn imply in the simplicity of \( A \) by pretty much the same argument as in the proof of Proposition 2.1 of [7], and also used by Villadsen in [21].

Let us now show that \( \text{Aff}(T(A)) \) is isomorphic to \( \text{Aff}(T(\tilde{A})) \): observing that, given any \( i \) and \( g \in C_{\mathbb{R}}(X_i) \) with norm less than or equal to 1, and recalling that Lemma 3.2.1 gives us
\[ \psi_i = \hat{\chi}_i = \frac{1}{d_i} \sum_{j=1}^{d_i} g \circ \bar{\sigma}_j^{i+1}, \quad \hat{\phi}_i = \frac{1}{d_i} \sum_{j=1}^{d_i} g \circ \sigma_j^{i+1}, \]
where \( d_1 \geq 2^3 \) and \( d_i = d_1^i d_i^2 \geq 2^{i+2}2\sqrt{i+2} = 2^{i+2} \) if \( i \geq 2 \), we get
\[ \| \psi_i(g) - \hat{\phi}_i(g) \| = \frac{1}{d_i} \| g \circ \bar{\sigma}_1^{i+1} + g \circ \bar{\sigma}_2^{i+1} - g \circ \sigma_1^{i+1} - g \circ \sigma_2^{i+1} \| \leq \frac{4}{d_i} \leq 2^{-i}. \]
From this, it follows that the \( i \)-th square in the diagram below approximately commutes to within \( 2^{-i} \) on \( S_i \),
\[
\begin{array}{c}
C_{\mathbb{R}}(X_1) \xrightarrow{\hat{\phi}_1} C_{\mathbb{R}}(X_2) \xrightarrow{\hat{\phi}_2} C_{\mathbb{R}}(X_3) \xrightarrow{\hat{\phi}_3} \cdots \\
\| \quad \| \quad \| \\
C_{\mathbb{R}}(X_1) \xrightarrow{\psi_1} C_{\mathbb{R}}(X_2) \xrightarrow{\psi_2} C_{\mathbb{R}}(X_3) \xrightarrow{\psi_3} \cdots
\end{array}
\]
and therefore the whole diagram is approximately commutative in the category of complete order unit spaces, whence their inductive limits, namely $\text{Aff}(T(A))$ and $\text{Aff}(T(\tilde{A}))$, are isomorphic (again by Theorems 3.1.2 and 3.1.3). Together with the fact that $\text{Aff}(T(\tilde{A}))$ is isomorphic to $\text{Aff}(S)$, this will in turn imply that $T(A)$ is affinely homeomorphic to $S$, as desired.

It only remains to show that the stable rank of $A$ is $n + 1$. Note that, by construction, $\dim(X_1) = 2n$, $\rank(p_1) = 1$, and $\dim(X_i) = 2n D_{i-1}$, $\rank(p_i) = D_{i-1}$ for $i \geq 2$. Then, since $A_i$ is a rank($p_i$)-homogeneous unital C*-algebra, by Theorem 2.1.2 its stable rank is given by

$$sr(A_i) = \left\lceil \frac{\dim(\hat{A}_i)}{2} \right\rceil / \rank(p_i) + 1$$

$$= \left\lceil \frac{\dim(X_i)}{2} \right\rceil / \rank(p_i) + 1$$

$$= n + 1.$$

We can therefore conclude that $sr(A) \leq n + 1$, and to complete the proof it is now sufficient to show that $sr(A) \geq n + 1$ (the proof, as we shall see, is similar to the proof of Theorem 8 of [21]).

Observe that, since $\phi_i$ comes from $(\sigma_j^{i+1}, q_j^{i+1})_{j=1}^{d_i}$, as vector bundles the following isomorphism holds:

$$p_{i+1} = \phi_i(p_i) \cong \bigoplus_{j=1}^{d_i} \sigma_j^{i+1*}(p_i) \otimes q_j^{i+1}.$$  

Now, if $i \geq 2$, recall that each $\sigma^{i+1}_j$ for $j \geq 3$ factors through the interval $I$; that is, there are $k, l$ such that $\sigma^{i+1}_j = \mu^i_k \circ \lambda^{i+1}_j$ with maps $\lambda^{i+1}_j : X_{i+1} \to I$, $\mu^i_j : I \to X_i$ defined before. Therefore, $\sigma^{i+1*}_j(p_i) = \lambda^{i+1*}_j(\mu^{i*}_i(p_i))$ and, since the space $I$ is compact and contractible, the vector bundle $\mu^{i*}_i(p_i)$ is trivial, whence $\sigma^{i+1*}_j(p_i)$ is a trivial vector bundle over $X_{i+1}$ with rank equalling the rank of $p_i$, namely $D_{i-1}$. In the case $j = 2$, $\sigma^{i+1*}_2(p_i)$ is also trivial since $\sigma^{i+1}_2$ is a point evaluation. From this, we can conclude that,
for \( j \geq 2, \)

\[
\sigma_j^{i+1} (p_i) \otimes q_j^{i+1} \cong \text{rank}(p_i) q_j^{i+1} \cong D_{i-1} \pi_{i+1}^2 (\Gamma_{nc_i}),
\]

where by \( k\xi \) we mean the \( k \)-fold direct sum of copies of the bundle \( \xi \) and, if we recall that \( \sigma_1^{i+1} = \pi_{i+1}^1 \), it follows that

\[
p_i + 1 \cong \pi_{i+1}^1 (p_i) \oplus (d_i - 1) D_{i-1} \pi_{i+1}^2 (\Gamma_{nc_i}) \]

\[
= \pi_{i+1}^1 (p_i) \oplus c_i \pi_{i+1}^2 (\Gamma_{nc_i}).
\]

Thus, by recursion we can write

\[
p_i + 1 \cong \xi_1 \times c_1 \Gamma_{nc_1} \times \cdots \times c_i \Gamma_{nc_i} = \xi_1 \times \zeta_i,
\]

where \( \xi_1 \) represents a trivial line bundle over \( \mathbb{D}^n \), and \( \zeta_i = c_1 \Gamma_{nc_1} \times \cdots \times c_i \Gamma_{nc_i} \) is a vector bundle over \( M_i = \mathbb{C}P(nc_1) \times \cdots \times \mathbb{C}P(nc_i) \) such that \( e(\zeta_i)^n \) is nonzero in the integral cohomology ring of \( M_i \). To see this recall that, for any positive integer \( k \), \( H^*(\mathbb{C}P(k)) \), the cohomology ring of \( \mathbb{C}P(k) \), is generated by \( e(\Gamma_k) \), with the single relation \( e(\Gamma_k)^{k+1} = 0 \).

Thus, \( e(\Gamma_{nc_j})^{nc_j} \) is nonzero for all \( j = 1, \ldots, i \). That \( e(\zeta_i)^n \) is nonzero will then follow from the Künneth formula for singular cohomology; see [20] for instance.

Now, write \( \phi_i = \phi_i^1 \oplus \phi_i^2 \), where \( \phi_i^1 \) comes from \( (\pi_{i+1}^1, q_j^{i+1}) \), and \( \phi_i^1 \) comes from \( (\sigma_j^{i+1}, q_j^{i+1}) \), \( j \geq 2 \). If we denote by \( \phi_{i,1} \) the composition \( \phi_{i-1} \circ \phi_{i-2} \circ \cdots \circ \phi_1 \) (with \( \phi_{1,1} = \phi_1 \)), then \( p_i - \phi_{i,1}^1 (p_1) \) corresponds to the pull-back to \( X_i \) of the bundle \( \zeta_{i-1} \), since \( \phi_{i,1}^1 (p_1) \) is the subprojection of \( p_i \) that, when seen as a vector bundle, is trivial with rank one. For \( j = 1, \ldots, n \), let \( a_j \in C(\mathbb{D}^n) \) be the projection onto the \( j \)-th coordinate. We have the identity

\[
\phi_{i,1}^1 (p_1) \phi_{i,1}^1 (a_j) \phi_{i,1}^1 (p_1) = \phi_{i,1}^1 (a_j) = (a_j \circ \pi_i) \phi_{i,1}^1 (p_1),
\]

where \( \pi_i \) denotes the projection of \( X_i \) onto \( X_1 = \mathbb{D}^n \). Since \( e(\zeta_{i-1})^n \neq 0 \), it follows from Theorem 3.3.1 that the distance from \( (\phi_{i,1} (a_1), \ldots, \phi_{i,1} (a_n)) \) to \( \text{Lg}_n (A_i) \) is at least one. As a consequence, \( \text{sr}(A) \geq n + 1 \), whence \( \text{sr}(A) = n + 1 = m \), as desired.
In the case $m$ is infinite, we obtain the desired algebra by doing almost the same construction as above: we replace the spaces $X_i$, using instead $X_1 = \mathbb{D}$ and, for $i \geq 1$,

$$X_{i+1} = \mathbb{D}^{(c_i)^2} \times \mathbb{C}P(c_1) \times \mathbb{C}P(2c_2) \times \cdots \times \mathbb{C}P(ic_i),$$

where $c_i = D_{i-1}(d_i - 1)$, just as before. In particular,

$$X_{i+1} = \mathbb{D}^\alpha \times X_i \times \mathbb{C}P((i + 1)c_{i+1})$$

for some non negative integer $\alpha$, so that we can define projections $\pi^{1}_{i+1}$ and $\pi^{2}_{i+1}$ of $X_{i+1}$ onto $X_i$ and $\mathbb{C}P((i + 1)c_{i+1})$ respectively, and the rest of the construction is the same. That the algebra obtained this way has infinite stable rank follows from the proof of Theorem 12 of [21].

The $C^*$-algebras constructed in 3.3.2 are structurally very similar to the algebras considered by Villadsen, and could thus rightfully be also called *Villadsen algebras*.

### 3.4 The real rank, and further questions

Let us consider the question of the real rank of the $C^*$-algebras constructed in Theorem 3.3.2. First, the real rank for the $C^*$-algebras with infinite stable rank constructed in Theorem 3.3.2 is also infinite; this follows from Theorem 13 of [21].

As mentioned in Chapter 2, Villadsen obtained an estimate for the real ranks of his $C^*$-algebras with finite stable rank. The same estimates can be shown to hold for the class of $C^*$-algebras constructed in Theorem 3.3.2; we need nothing more than what Villadsen used for his estimates and, in fact, the proof follows analogously. For completeness, this is presented below.

Recall that the real rank of an $m$-homogeneous $C^*$-algebra $A$ is given by the following formula ([3],[12]):

$$RR(A) = \left\lceil \frac{\dim(\hat{A})}{(2m - 1)} \right\rceil.$$
We need the following theorem, that uses the same notation as Theorem 3.3.1; that is, suppose that $M$ is a compact, connected and orientable differentiable manifold. Let $\pi_1: \mathbb{D}^n \times M \rightarrow \mathbb{D}^n$, $\pi_2: \mathbb{D}^n \times M \rightarrow M$ denote the coordinate projections, let $a_j \in C(\mathbb{D}^n)$, $1 \leq j \leq n$ denote the projection onto the $j$-th coordinate, let $p$ denote a trivial rank one projection in $C(\mathbb{D}^n \times M) \otimes K$, and let $q$ denote a projection in $C(\mathbb{D}^n \times M) \otimes K$ orthogonal to $p$ of the form $q = \pi_2^*(\eta)$, where $\eta$ is a complex vector bundle over $M$ such that $e(\eta)^n$ is nonzero. Set $B = (C(\mathbb{D}^n \times M) \otimes K)p + q$.

**Theorem 3.4.1** (Villadsen ([21], Theorem 9)). Suppose that $b_1, \ldots, b_n \in B_{sa}$ are such that $pb_j p = (\text{Re}(a_j) \circ \pi_1) p$ for each $j$. Then the distance from $(b_1, \ldots, b_n)$ to $L_g^n(B) \cap B^n_{sa}$ is at least 1.

**Theorem 3.4.2.** Let $A$ be one of the $C^*$-algebras constructed in Theorem 3.3.2, and suppose $A$ has finite stable rank $n \geq 2$. Then, its real rank is either equal to $n$ or $n - 1$.

**Proof.** Following the notation of the proof of Theorem 3.3.2, $A$ was written as an inductive limit of homogeneous $C^*$-algebras $A_i = p_i(C(X_i) \otimes K)p_i$, such that $\dim(X_i) = 2(n - 1)D_{i-1}$, $\text{rank}(p_i) = D_{i-1}$ for $i \geq 2$. The real rank formula above then gives

$$\text{RR}(A_i) = \lceil \dim(\hat{A})/(2m - 1) \rceil$$

$$= \lceil \dim(X_i)/(2\text{rank}(p_i) - 1) \rceil$$

$$= \lceil 2D_{i-1}(n - 1)/(2D_{i-1} - 1) \rceil,$$

which can be seen to be equal to $(n - 1) + 1 = n$ for large enough $i$, and so we have $\text{RR}(A) \leq n$.

The estimate $\text{RR}(A) \geq n - 1$ is obtained by the same argument as the estimate $\text{sr}(A) \geq n$ in Theorem 3.3.2, only we use the real parts of the coordinate projections $a_j$ and Theorem 3.4.1 instead of Theorem 3.3.1.

This estimate for the real rank brings up several questions, particularly if one considers the dichotomy now established of Villadsen algebras with one, or more than one, tracial state.
For instance, AI algebras have stable rank one, and therefore real rank either zero or one; the condition of real rank zero for an AI algebra is equivalent to the algebra admitting a unique tracial state ([16]). Therefore, for an AI algebra the real rank can be seen to be zero or one depending on whether the algebra admits a unique tracial state, or more than one.

A similar result holds for simple unital AH algebras with slow dimension growth (which, as we discussed before, also have stable rank one): if such an algebra admits a unique tracial state, then it must have real rank zero (a result that follows from [4]).

It is worth noticing that the results mentioned above for stable rank one $C^*$-algebras use a particular characterization of real rank zero; namely, a $C^*$-algebra has real rank zero if and only if its self-adjoint elements can be approximated arbitrarily well by self-adjoint elements with finite spectra ([6]). Still, one can ask whether similar results hold for Villadsen algebras. Given a Villadsen algebra with stable rank $n$ and with a unique (respectively, with more than one) tracial state, must its real rank be $n - 1$ (respectively, $n$)? What about the converse statements?

These questions have resisted all attempts at solving them so far. One complication arises from the fact that the real rank (also the stable rank, despite what the term suggests) is far from being stable. For instance, as mentioned in the previous chapter, one can have an inductive sequence of $C^*$-algebras, each with infinite real and stable ranks, but whose inductive limit has stable rank one and real rank zero. Moreover, even the slightest changes in the *-homomorphisms in the sequence can greatly affect the stable and real ranks; a good example of this is seen already in the $C^*$-algebras of Theorem 3.3.2: if one considers one such $C^*$-algebra $A$ with stable rank $n \geq 2$, then by changing just a single eigenvalue map at each stage in the construction of $A$ (a change that becomes increasingly minor as one goes further along the sequence) one obtains an AI algebra as a result. Despite this instability, it doesn’t seem unlikely at this point that one or more of the statements above could turn out to be true.
Bibliography


