Two theorems of Dye in the almost continuous category

by

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Abstract

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This thesis studies orbit equivalence in the almost continuous setting. Recently A. del Junco and A. Şahin obtained an almost continuous version of Dye’s theorem. They proved that any two ergodic measure-preserving homeomorphisms of Polish spaces are almost continuously orbit equivalent. One purpose of this thesis is to extend their result to all free actions of countable amenable groups. We also show that the cocycles associated with the constructed orbit equivalence are almost continuous.

In the second part of the thesis we obtain an analogue of Dye’s reconstruction theorem for étale equivalence relations in the almost continuous setting. We introduce topological full groups of étale equivalence relations and show that if the topological full groups are isomorphic, then the equivalence relations are almost continuously orbit equivalent.
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1.1 Orbit equivalence

In 1959 Henry Dye proved the following surprising theorem:

**Theorem 1.1.1** (Dye, 1959). Suppose that $(X, \mu)$ and $(Y, \nu)$ are Lebesgue spaces with non-atomic probability measures, $T : X \to X$ and $S : Y \to Y$ are invertible ergodic measure-preserving transformations. Then there exist invariant subsets $X_1 \subset X$ and $Y_1 \subset Y$ of full measure and a bijection $\varphi : X_1 \to Y_1$ such that $\varphi$, $\varphi^{-1}$ are measurable, $\varphi(\mu|_{X_1}) = \nu|_{Y_1}$, and for every $x \in X'$, $\varphi$ maps $T$-orbits onto $S$-orbits,

$$\varphi(\text{Orb}_T(x)) = \text{Orb}_S(\varphi(x)),$$

where $\text{Orb}_T(x) = \{T^k x\}_{k \in \mathbb{Z}}$.

Since then the subject of orbit equivalence has been a major area of study. In (Dye, 1963) H. Dye showed that a free action of an abelian group is orbit equivalent to a single transformation and conjectured that the same holds for actions of arbitrary amenable groups. This conjecture was proved by D. Ornstein and B. Weiss:
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**Theorem 1.1.2** (Ornstein & Weiss, 1980). *Suppose that $(X, \mu)$ is a Lebesgue space with a non-atomic probability measure. Then any non-singular action of an amenable group $G$ on $X$ is orbit equivalent to a single transformation.*

G. Hjorth showed that a countable group is amenable if, and only if, it induces only one equivalence relation on a standard Borel probability space considered up to orbit equivalence, see (Hjorth, 2005). In 1981 A. Connes, J. Feldman and B. Weiss proved Dye’s theorem for equivalence relations:

**Theorem 1.1.3** (Connes et al., 1981). *Suppose that $R \subset X \times X$ is a non-singular amenable countable equivalence relation. Then there exists a non-singular transformation $T: X \to X$ such that, up to a null set,*

$$R = \{(x, T^n x) : x \in X, n \in \mathbb{Z}\}.$$

Orbit equivalence was considered in various settings. W. Krieger in (Krieger, 1976) classified in measurable context all ergodic *non-singular* transformations up to orbit equivalence. Orbit equivalence in a pure topological context was considered by D. Sullivan, B. Weiss, and J. Wright. In (Sullivan et al., 1986) they showed that any two countable groups of homeomorphisms acting ergodically on perfect Polish spaces are orbit equivalent in the following sense: there exist invariant dense $G_\delta$ subsets $X' \subset X$, $Y' \subset Y$ and a homeomorphism $\varphi: X' \to Y'$ such that $\varphi$ is an orbit equivalence. It is also possible to consider topological orbit equivalence in a stricter sense: without ignoring any “negligible” subsets. The task of classifying all dynamical systems up to orbit equivalence becomes harder. See, for example, (Giordano et al., 2008, Theorem 1.6).

In the pure Borel context S. Jackson and S. Gao obtained the following result:
Theorem 1.1.4 (Gao & Jackson, preprint). Let $G$ be a countable abelian group acting in a Borel manner on a standard Borel space and let $R_G$ denote the induced equivalence relation. Then $R_G$ is Borel orbit equivalent to a $\mathbb{Z}$-action: there exist a Borel transformation $T : X \to X$ such that $\text{Orb}_G(x) = \text{Orb}_T(x)$.

It is natural to ask if in Theorem 1.1.1 the map $\varphi$ implementing the orbit equivalence can be continuous if the transformations $T$ and $S$ are continuous. The first result of this sort was obtained by T. Hamachi and M. Keane (2006). They showed that the binary and ternary odometers are almost continuously (a. c.) orbit equivalent in the following sense: there exist invariant subsets $X_1 \subset X$ and $Y_1 \subset Y$ of full measure such that the map $\varphi : X_1 \to Y_1$ implementing orbit equivalence and its inverse $\varphi^{-1}$ are continuous. This was soon extended to many other classes of $\mathbb{Z}$-actions. Recently A. del Junco and A. Şahin obtained the following very general result:

Theorem 1.1.5 (del Junco & Şahin, preprint). Suppose that $(X, \mu)$ and $(Y, \nu)$ are Polish spaces with non-atomic Borel probability measures. Suppose also that $T$ and $S$ are ergodic measure-preserving homeomorphisms of $(X, \mu)$ and $(Y, \nu)$. Then there are invariant $G_\delta$-subsets $X' \subset X$ and $Y' \subset Y$ of full measure and a homeomorphism $\varphi : X' \to Y'$ which maps $\mu|_{X'}$ to $\nu|_{Y'}$ and $T$-orbits onto $S$-orbits.

In Chapter 3 of this thesis we answer two questions from (del Junco & Şahin, preprint). Firstly, we show that Theorem 1.1.5 also holds for actions of discrete amenable groups (Theorem 3.5.2). The proof is an adaptation of the del Junco-Şahin technique to the case of amenable groups. The main difficulty here lies in using
Rohlin’s lemma\(^1\). In the case of a single transformation \(T\) one obtains a collection of pairwise disjoint subsets

\[ B, TB, \ldots, T^{n-1}B \]

that cover most of the space. Rohlin’s lemma for arbitrary amenable groups provides instead many collections, not just one, and within each collection (quasi-tower) sets have “small” intersections. The proof therefore becomes longer and more technical.

We also answer another question raised by A. del Junco and A. Şahin concerning continuity of the cocycles. Consider two groups \(G_1\) and \(G_2\) acting on \(X\) and \(Y\) respectively and an orbit equivalence between them \(\varphi : X \to Y\). Then \(G_1\) acts on \(Y\) by \(\varphi g \varphi^{-1}\), \(g \in G_1\), and this action has the same orbits as \(G_2\). Hence for each \(g \in G_1\) there is a \(G_2\)-valued function \(C_{1,g}(y)\), such that

\[ \varphi g \varphi^{-1}(y) = C_{1,g}(y) y \quad \text{for all} \quad y \in Y. \]

A. Del Junco and A. Şahin asked if it is possible to construct orbit equivalence in Theorem 3.5.2 so that the associated functions \(C_{1,g}(y)\), called cocycles, are continuous. In Section 3.5 we answer this question positively. In fact, the original del Junco-Şahin construction and our method both produce orbit equivalences with continuous cocycle maps.

Finally, we would like to mention a recent result due to A. del Junco, D. Rudolph and B. Weiss.

**Theorem 1.1.6** (del Junco *et al.*, 2009). *Suppose that \((X, \mu)\) and \((Y, \nu)\) are separable metric spaces with non-atomic Borel probability measures and \(G_1, G_2\) are countable groups of measure-preserving transformations acting ergodically on \(X\)

\(^1\)See section 3.3 for the statement of Rohlin’s lemma.
and $Y$. If $\varphi: X \to Y$ is any measurable orbit equivalence of $G_1$ and $G_2$ then there exist elements $^2\sigma \in [G_1]$ and $\tau \in [G_2]$ such that $\tau \varphi \sigma$ (which is also an orbit equivalence) is a homeomorphism between invariant subsets $X_1$ and $Y_1$ of full measure.

That is, if we do not require the subsets $X_1$ and $Y_1$ to be $G_\delta$ (to be Polish), then any orbit equivalence can be “regularized” to obtain an almost continuous orbit equivalence. It remains an open question whether $X_1$ and $Y_1$ can be always taken $G_\delta$, but in the case of almost continuous Kakutani equivalence this requirement cannot always be met, see (del Junco et al., 2009). Note that Theorem 1.1.6 also holds for non-singular equivalence relations.

### 1.2 Full groups

Any orbit equivalence $\varphi: X \to Y$ of two dynamical systems $G_1$, $G_2$ induces an algebraic isomorphism $f: [G_1] \to [G_2]$,

$$f(T) = \varphi \circ T \circ \varphi^{-1}, \quad T \in [G_1]$$

of their full groups. In 1963 H. Dye proved that every orbit equivalence arises in this way:

**Theorem 1.2.1** (Dye, 1963). *Two ergodic measure-preserving group actions* $G_1$ and $G_2$ are orbit equivalent if, and only if, there exists an algebraic isomorphism $f: [G_1] \to [G_2]$ of the full groups, and for every such isomorphism there is unique

---

$^2$See page 15 for the definition of the full group $[G]$. 
measure-preserving orbit equivalence \( \varphi : X_1 \to Y_1 \) between subsets of full measure such that \( f(T) = \varphi T \varphi^{-1} \) for all \( T \in [G_1] \).

Note that \( \varphi \) in Theorem 1.2.1 is an orbit equivalence. This theorem also holds for ergodic measure-preserving equivalence relations with only minor modifications to the plan.

A similar result was obtained recently in the pure Borel context by B. Miller and C. Rosendal, see (Miller & Rosendal, 2007). T. Giordano, I. Putnam, and C. Skau considered this in the topological setting, see (Giordano et al., 1999). They showed that two Cantor minimal systems are orbit equivalent if, and only if, their full groups are algebraically isomorphic. Recently this was improved by S. Bezuglyi and K. Medynets who showed that it is enough to have an isomorphism of commutator subgroups to reconstruct the orbit equivalence. Note that in the measure-theoretic case the full group \( [\mathcal{R}] \) of an ergodic equivalence relation is simple (see also Theorem 4.3.3).

In Chapter 4 we establish an analog of Dye’s “reconstruction theorem” 1.2.1 in the almost continuous setting. That is, we view two mappings as equal if they agree on a dense \( G_\delta \) subset of full measure. We assume that equivalence relations \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are measure-preserving, ergodic and étale. The definition of étaleness (see page 16, Definition 2.4.1) is due to J. Renault, see (Renault, 1980). Our main result is Theorem 4.1.1: if topological full groups \( [\mathcal{R}_1]_{top} \) and \( [\mathcal{R}_2]_{top} \) are isomorphic, then \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are almost continuously orbit equivalent and the isomorphism can be obtained using formula (1.1).

We want to mention here two results from (Kittrell & Tsankov, preprint). Firstly, they showed that every homomorphism \( f : [\mathcal{R}] \to G \) is automatically continuous if \( G \) is a separable topological group and \( \mathcal{R} \) is ergodic and measure-preserving,
where $[\mathcal{R}]$ is equipped with the uniform topology $d(T, S) = \mu(T \neq S)$. Secondly, they showed that for every non-trivial countable measure-preserving equivalence relation $\mathcal{R}$ its full group $[\mathcal{R}]$ is homeomorphic to the separable Hilbert space $\ell^2$. We do not know if these results can be proved in the almost continuous setting.

1.3 Organization of the thesis

This thesis is organized as follows:

In Chapter 2 we introduce necessary definitions and establish basic lemmas that are used later. Section 2.2 is dedicated to properties of zero-dimensional Polish spaces and their clopen subsets. Then in Section 2.3 we review basic definitions concerning group actions and equivalence relations. In Section 2.4 we introduce étale equivalence relations and discuss their properties. In Section 2.5 we study topological full groups. We show that if the topology on the induced equivalence relation $\mathcal{R}_G$ is induced by the group action in the usual way, then the topological full group $[\mathcal{R}_G]_{top}$ of $\mathcal{R}_G$ equals $[G]_{top}$. Hence our definition is a proper extension of topological full groups to equivalence relations.

In Chapter 3 we study continuous actions of discrete amenable groups. The main result of this chapter is Theorem 3.5.1. In the first section we introduce the basic objects that we are going to operate with, columns and arrays, and show how one can construct arrays with specific properties. In Section 3.3 we present Rohlin’s lemma for amenable groups and the necessary definitions. Section 3.4 is devoted to the proof of the main lemma. In Section 3.5 we prove our main result: any continuous measure-preserving ergodic action of an amenable group is almost continuously orbit
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equivalent to a single homeomorphism. We also show that the cocycles associated with the constructed orbit equivalence are continuous. In the last section we list several open problems.

In Chapter 4 we prove another theorem of H. Dye in the almost continuous setting. We show that if $R_1$ and $R_2$ are ergodic measure-preserving étale equivalence relations, and their topological full groups are algebraically isomorphic, then $R_1$ and $R_2$ are almost continuously orbit equivalent, and the isomorphism $f$ between their topological full groups is given by the formula

$$f(T) = \varphi T \varphi^{-1}, \quad \text{for all } T \in [R_1]_{top},$$

where $\varphi$ is the constructed almost continuous orbit equivalence. The structure of the proof is similar to the measure-theoretic case in the exposition by A. Kechris, see (Kechris, preprint). In Section 4.2 we prove that every element of the topological full group $[R]_{top}$ can be written as a product of ten involutions and as a product of five commutators. Then we obtain an analog of S. Eigen’s theorem (Eigen, 1981): the topological full group $[R]_{top}$ is simple. In Section 4.4 we show that $f$ maps functions with disjoint supports into functions whose support is also disjoint. In Section 4.5 we define a set mapping $\Phi$ and prove that it is induced by a point mapping $\varphi$. The idea of mapping everything through an intermediate symbol space that we use in Lemma 4.5.2 is originally from (Danilenko & del Junco, preprint).
Chapter 2

Background

2.1 Introduction

In this chapter we present the necessary facts about zero-dimensional Polish spaces, group actions and étale equivalence relations. Zero-dimensional Polish spaces are the most natural underlying spaces for considering almost continuous orbit equivalence (see Theorem 2.4.6). Our main goals in this chapter are (a) to learn how to construct clopen sets with certain properties; (b) to establish properties of étale equivalence relations, and (c) to define topological full groups and prove basic facts that are used in the later chapters.

In Section 2.2 we review zero-dimensional Polish spaces and basic properties of clopen sets and clopen partitions that are used in the next chapters. Then we give precise definitions of a group action, an étale equivalence relation and a topological full group. Their basic properties are established. We conclude the chapter with the definition of almost continuous orbit equivalence.
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2.2 Zero-dimensional Polish spaces

2.2.1 Polish spaces

Definition 2.2.1. A Polish space is a topological space homeomorphic to a complete separable metric space.

The following theorem is well-known, a proof can be found in (Engelking, 1989) or (Kechris, 1995, Theorem 3.11):

**Theorem 2.2.1.** Let $X$ be a Polish space.

1. (P. Alexandrov) A $G_\delta$-subset of $X$ is also Polish.

2. If a subset of $X$ is Polish, then it is $G_\delta$.

Definition 2.2.2. A topological space is called **zero-dimensional** if it has a basis consisting of clopen subsets.

Note that if a Polish space is zero-dimensional, then it has a countable clopen basis. This is a consequence of a more general statement: if a topological space $X$ has a countable basis, then every basis of $X$ contains a countable basis.

The following result is sometimes useful:

**Theorem 2.2.2** (Bogachev (2007), Theorem 9.6.3). Let $(X, \mu)$ be a Polish space with a non-atomic Borel probability measure. Then there exist a $G_\delta$ subset $X_1 \subset X$ of full measure such that $(X_1, \mu|_{X_1})$ is homeomorphic to the space of irrational numbers $(0,1) \setminus \mathbb{Q}$ with Lebesgue measure.

Definition 2.2.3. Following Danilenko & del Junco (preprint) we call a dense $G_\delta$ subset of $X$ of full measure **virtually full**. A subset $B \subset X$ is called **virtually**
open (closed) if there is a virtually full set $X_1$ such that $B \cap X_1$ is open (closed) in $X_1$. A subset $B$ is **virtually clopen** if there is a virtually full set $X_1$ such that $B \cap X_1$ is clopen in $X_1$.

**Remark 2.2.1.** Note that it is possible that a set is virtually clopen, but there is no clopen set that differs from it by a set of measure zero.

### 2.2.2 Clopen sets

It is a simple consequence of the definition of a clopen set that clopen sets form an *algebra*. Below we prove several other properties of clopen sets that will be used in the next chapters. Everywhere in this subsection $X$ is assumed to be a zero-dimensional Polish space and $\mu$ is a non-atomic Borel probability measure with full support, $\text{supp}\, \mu = X$.

**Lemma 2.2.3.** Suppose that $A$ is clopen subset of $X$ and $\{A_i\}_{i \in \mathbb{N}}$ is a partition of $A$ into clopen sets. Then every union of the atoms of the partition is clopen: for any subset $I \subset \mathbb{N}$ the union $\bigcup_{i \in I} A_i$ is clopen.

**Proof.** Clearly the unions $\bigcup_{i \in I} A_i$, $\bigcup_{i \in \mathbb{N} \setminus I} A_i$ are open. Since they are complements of each other in $X$ they are both closed, hence clopen. \qed

**Definition 2.2.4.** A partition of $X$ is called **clopen** if every set in the partition is clopen.

**Lemma 2.2.4.** Let $\{X_i\}_{i \in \mathbb{N}}$ be a clopen partition of $X$. Then for any sequence of clopen sets $E_i \subset X_i$ the union $\bigcup_{i \in \mathbb{N}} E_i$ is again clopen.
Proof. The collection \( \{ E_i, X_i \setminus E_i \}_{i \in \mathbb{N}} \) is a clopen partition of the space. The union \( \bigcup_{i \in \mathbb{N}} E_i \) is now clopen by Lemma 2.2.3. □

Lemma 2.2.5. For any clopen set \( A \subset X \) and any \( \varepsilon > 0 \) there exist a partition of \( A \) into finitely many clopen subsets \( A_1, \ldots, A_k \) with \( \mu A_i < \varepsilon \). Consequently any open set can be written as a countable disjoint union of clopen sets with measure less than \( \varepsilon \).

Proof. Fix a clopen subset \( A \subset X \). Since every point has a neighbourhood with measure less than \( \varepsilon \), the set \( \{ B \subset X : B \text{ is clopen and } \mu B < \varepsilon \} \) is a basis. Thus we can find a countable clopen basis \( \{ B_i \}_{i \in \mathbb{N}} \) such that \( \mu B_i < \varepsilon \). The clopen sets \( A_i \) defined inductively,

\[
A_1 = A \cap B_1, \quad A_n = \left( A \setminus \left( \bigcup_{k=1}^{n-1} A_k \right) \right) \cap B_n,
\]

partition \( A \) and \( \mu A_n \leq \mu B_n < \varepsilon \). If only finitely many of \( A_i \) are non-empty, the proof is finished. Otherwise, since the sum of all measures \( \mu A_n \) is \( \mu A \), there exist a number \( N \in \mathbb{N} \) such that \( \sum_{n>N} \mu A_n < \varepsilon \). Therefore we can replace all \( A_n, n > N \), with the union \( \bigcup_{n>N} A_n \), which is clopen by Lemma 2.2.3. □

Remark 2.2.2. It follows from Lemma 2.2.5 that any measurable set \( E \subset X \) can be approximated with a clopen set \( D \) so that \( \mu(E \triangle D) < \varepsilon \), since \( E \) can be approximated with an open subset \( G \), and we can make \( G \) clopen by disregarding a subset \( G_\varepsilon \subset G \) of very small measure. See also (del Junco et al., 2009, Lemma 2.8) for another proof of this fact. In fact, one can find a virtually clopen subset \( D \) such that \( \mu(E \triangle D) < \varepsilon \) and \( \mu E = \mu D \) (del Junco et al., 2009, Lemma 2.10).
The following lemma is from (del Junco & Şahin, preprint, Lemma 2, part (b)).

**Lemma 2.2.6.** For any open set $B \subset X$ and a sequence $\{r_i\}_{i \in \mathbb{N}}$, $r_i > 0$, such that $\sum_{i \in \mathbb{N}} r_i = \mu B$, there are disjoint open subsets $B_i \subset B$ with $\mu B_i = r_i$.

**Lemma 2.2.7.** Let $\{E_i\}_{i=1}^n$, $E_i \subset X$, be a finite collection of disjoint measurable sets. For every $\varepsilon > 0$ there exist disjoint clopen sets $D_i$ such that $\mu(E_i \Delta D_i) < \varepsilon$.

**Proof.** Find clopen $F_i$ such that $\mu(E_i \Delta F_i) < \frac{\varepsilon}{4n}$. Then for all $i \neq j$

\[
\mu(F_i \cap F_j) \leq \mu(F_i \setminus E_i) + \mu(F_j \setminus E_j) < \frac{\varepsilon}{4n} + \frac{\varepsilon}{4n} = \frac{\varepsilon}{2n}.
\]

Let $D_i = F_i \setminus \bigcup_{j \neq i} F_j$. By construction $D_i$ are clopen and pairwise disjoint, and

\[
\mu D_i = \mu\left( F_i \setminus \bigcup_{j \neq i} F_j \right) \geq \mu F_i - (n - 1) \frac{\varepsilon}{2n} > \mu F_i - \frac{\varepsilon}{2}.
\]

Therefore

\[
\mu(E_i \Delta D_i) \leq \mu(E_i \Delta F_i) + \mu(F_i \Delta D_i) < \varepsilon/4n + \varepsilon/2 < \varepsilon.
\]

2.3 Group actions and equivalence relations

Everywhere in this section $X$ is a Polish space, $\mu$ is a non-atomic Borel probability measure on $X$, $\text{supp} \mu = X$, and $\mathbb{G}$ is a countable discrete group. By a transformation we mean an invertible mapping $T: X \rightarrow X$.

**Definition 2.3.1.** The support of a map $T: X \rightarrow X$, denoted by $\text{supp} T$, is the set of all points $x \in X$ where $T \neq \text{id}$:

\[
\text{supp}(T) \defeq \{ x \in X : T(x) \neq x \}.
\]
Definition 2.3.2. A group action of $G$ on $X$ is a Borel map $(g, x) \mapsto g \cdot x$ such that

1. $\text{id} \cdot x = x$ for all $x \in X$, and

2. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and all $x \in X$.

In this thesis we will only consider continuous group actions, that is, actions such that every map $x \mapsto gx$ is a homeomorphism.

Definition 2.3.3. A group action is called measure-preserving if every transformation $g \in G$ is measure-preserving. The action of $G$ is called ergodic if every measurable set $E \subset X$ such that $gE = E$ for all $g \in G$ has measure zero or one. The action of $G$ is called free if $gx \neq x$ for all $g \in G \setminus \{\text{id}\}$ and all $x \in X$. The action of $G$ is called non-singular if every $g \in G$ is non-singular ($\mu g$ is equivalent to $\mu$).

Definition 2.3.4. Given a group action $G$, the orbit of a point $x \in X$ is the set

$$\text{Orb}_G(x) \overset{\text{def}}{=} G \cdot x = \{g \cdot x: g \in G\}.$$ 

For a single transformation $T: X \rightarrow X$ we write $\text{Orb}_T(x) \overset{\text{def}}{=} \{T^n x\}_{n \in \mathbb{Z}}$.

Definition 2.3.5. Suppose that a countable group $G$ acts on $X$. The equivalence relation induced by the action of $G$ is

$$\mathcal{R}_G \overset{\text{def}}{=} \{(x, y): y = g \cdot x \text{ for some } g \in G\} \subset X \times X.$$ 

If $\mathcal{R} \subset X \times X$ is an equivalence relation, then the range $r(x, y)$ and the source $s(x, y)$ maps are the canonical projections $r, s: \mathcal{R} \rightarrow X$

$$r(x, y) \overset{\text{def}}{=} y, \quad s(x, y) \overset{\text{def}}{=} x.$$
The equivalence class of a point \( x \in X \) is denoted by \([x]_R\),
\[ [x]_R \overset{\text{def}}{=} \{ y \in X : (x, y) \in R \}. \]

A subset \( A \subset X \) is called \( R \)-saturated if it consists of equivalence classes of \( R \),
\[ A = \bigcup_{x \in A} [x]_R \] and for any set \( A \subset X \) we denote by \([A]_R\) its \( R \)-saturation,
\[ [A]_R \overset{\text{def}}{=} \bigcup_{x \in A} [x]_R. \]

Suppose that an equivalence relation \( R \) is a Borel subset of \( X \times X \) and \( \mu \) is a non-atomic Borel probability measure on \( X \). We identify maps that are equal a. e. on \( X \).

**Definition 2.3.6.** The **full group** \([R]\) is the collection of all transformations \( T : X \to X \) such that \( \text{Orb}_T(x) \) consists only of points from the same equivalence class of \( R \),
\[ \text{Orb}_T(x) \subset [x]_R. \]
That is, \((x, Tx) \in R\) for every \( x \in X \). The **full group** \([G]\) of a group action \( G \) is by definition the group \([R_G]\). For every \( T \in [G] \) there is a corresponding \( G \)-valued map \( C_T : X \to G \), called a **cocycle** of \( T \), such that we have
\[ T(x) = C_T(x) \cdot x \quad \text{a. e.} \]

In general there can be many different cocycles corresponding to the same \( T \in [G] \). If \( G \) acts freely then the cocycle is unique.

**Definition 2.3.7.** An equivalence relation \( R \) is called **measure-preserving** if every element of the full group is measure-preserving, \( \mu(A) = \mu(T^{-1}(A)) \) for every
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\( T \in [\mathcal{R}] \) and Borel \( A \subset X \). It is called **ergodic** if every \( \mathcal{R} \)-saturated Borel set has measure zero or one. An equivalence relation \( \mathcal{R} \) is called **non-singular** if every transformation from the full group \([\mathcal{R}]\) is non-singular.

Note that a group action \( G \) is measure-preserving (ergodic) if and only if \( \mathcal{R}_G \) is measure-preserving (ergodic), where \( \mathcal{R}_G \) is the induced equivalence relation. Also, an equivalence relation \( \mathcal{R} \) is measure-preserving if and only if every partially defined automorphism whose graph is supported by \( \mathcal{R} \) is measure-preserving, see (Kechris & Miller, 2004, proposition 2.1).

### 2.4 Étale equivalence relations

The following definition extends the notion of continuity to equivalence relations.

**Definition 2.4.1.** Suppose that \( X \) is a Polish space and \( \mathcal{R} \subset X \times X \) is a countable equivalence relation equipped with a topology \( \mathcal{T} \). The pair \((\mathcal{R}, \mathcal{T})\) is called an **étale equivalence relation** if

1. \( \mathcal{T} \) is Hausdorff and second countable;

2. the diagonal \( \Delta = \{(x, x) : x \in X\} \) is open in \( \mathcal{T} \);

3. the range and source maps are local homeomorphisms: for any point \((x, y) \in \mathcal{R}\) there is an open set \( U \subset \mathcal{R} \), \( U \in \mathcal{T} \), \((x, y) \in U \), such that \( r(U) \) and \( s(U) \) are open in \( X \) and \( r: U \to r(U) \) and \( s: U \to s(U) \) are homeomorphisms;

4. if \( U \) and \( V \) are open in \( \mathcal{T} \), then the set

   \[
   UV = \{(x, z) : (x, y) \in U, (y, z) \in V \text{ for some } y \in X\}
   \]
is also open, and

5. if $U$ is open in $T$, then the set $U^{-1} = \{(x, y) : (y, x) \in U\}$ is also open.

**Remark 2.4.1.** Definition 2.4.1 is due to J. Renault, (Renault, 1980), see also (Patterson, 1999). Usually $X$ is required to be locally compact, and $R$ to be $\sigma$-compact, but we shall have no use for these conditions. Note that if $X$ is locally compact, then $R$ is also locally compact and in this case the following are equivalent: (a) $R$ is second-countable; (b) $R$ is metrizable and $\sigma$-compact; (c) $R$ is Polish, see (Kechris, 1995, Theorem 5.3).

**Remark 2.4.2.** If $X$ is zero-dimensional we can require the set $U$ from part 3 of Definition 2.4.1 to be clopen.

**Remark 2.4.3.** When $U$ is clear from the context we will often write $r^{-1}$ and $s^{-1}$ instead of $r^{-1}|_{r(U)}$ and $s^{-1}|_{s(U)}$.

For the proof of the following theorem see (Giordano et al., 2004, proposition 2.3). It is proved there under different assumptions about $X$, but the proof only uses the fact that $X$ is a separable zero-dimensional metric space.

**Theorem 2.4.1.** Every étale equivalence relation on a zero-dimensional Polish space is generated by a countable group of homeomorphisms.

It is not known whether $R$ can be generated by a free action of some group $G$ and the same question in the ergodic, measure-preserving case was answered negatively in (Furman, 1999, Theorem D). In the Borel case an example of an equivalence relation that is not freely generated was given in (Adams, 1988), and the answer in the topological setting was given in (Hjorth & Molberg, 2006).
of homeomorphisms acts freely on $X$, then the induced equivalence relation $R_G$ is étale. An étale topology on $R_G$ is generated by the sets

$$B_{g,U} = \{(x, gx): g \in G, \ x \in U \subset X, \ U \text{ is open}\}. \quad (2.1)$$

If the action of $G$ is not free we have the following criterion (the “if” direction can be found in (Dahl, 2008)):

**Theorem 2.4.2.** Suppose that a countable group $G$ acts a zero-dimensional Polish space $X$ by homeomorphisms. Then the induced equivalence relation $R_G$ is étale in some topology if, and only if, the set

$$\text{fix}(g) = \{x \in X: gx = x\}$$

is clopen for all $g \in G$.

**Proof.** If every $\text{fix}(g)$ is clopen, it is straightforward to check that the topology generated by the sets $B_{g,U}$ is étale (we require $\text{fix}(g)$ to be clopen so that the collection $\{B_{g,U}\}$ is a base).

If $R_G$ is étale in some topology $T$, then it follows from part 2 of the definition of étaleness that the set of fixed points of each $g \in G$ is clopen: the set of fixed points is clearly closed; it is open since we can take a small neighbourhood $B \subset R$ of any fixed point, the set $r(B \cap \Delta)$ is then open and consists of fixed points. Taking the union over all fixed points we see that $\text{fix}(g)$ is open.

Another simple consequence of Definition 2.4.1 is the following theorem.

**Theorem 2.4.3.** Suppose that $R$ is an étale equivalence relation. If $X_1 \subset X$ is an $R$-saturated $G_\delta$ subset, then $R_1 = R \cap (X_1 \times X_1)$ is étale.
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Proof. Since $X_1$ is $G_δ$ it is Polish, and since $X_1$ is $\mathcal{R}$-saturated $\mathcal{R}_1$ is a sub-equivalence relation of $\mathcal{R}$ and the conditions of Definition 2.4.1 are satisfied.

Suppose that $\mu$ is a non-atomic Borel probability measure on $X$.

Definition 2.4.2. The support of a measure $\mu$ is the largest subset of $X$ for which every neighbourhood of every point of the set has positive measure,

$\text{supp}(\mu) \overset{\text{def}}{=} \{x \in X : \mu(O_x) > 0 \text{ for every open } O_x \ni x\}$.

Notice that we always have $\mu(\text{supp}(\mu)) = 1$.

For any Polish space $X$ with a non-atomic Borel probability measure $\mu$ and an étale equivalence relation $\mathcal{R}$ it is possible to find a virtually full $\mathcal{R}$-saturated subset $X_1 \subset X$ such that $\mu|_{X_1}$ has full support. This is a generalization of Theorem 2.4 from (del Junco et al., 2009). To establish it we need the following lemma.

Lemma 2.4.4. Suppose that $(X, \mu)$ is a Polish space with a non-atomic Borel probability measure, and $\mathcal{R}$ is a measure preserving (or non-singular) étale equivalence relation on $X$. If a subset $E \subset X$ has measure zero, $\mu(E) = 0$, and $E$ is $F_\sigma$, then the $\mathcal{R}$-saturation $[E]_\mathcal{R}$ is also $F_\sigma$ and has measure zero.

Proof. For every $(x, y) \in \mathcal{R}$, $x \neq y$, find a neighbourhood $U_{x,y} \subset \mathcal{R}$ such that the range and the source maps restricted to $U_{x,y}$ are homeomorphisms and

$s(U_{x,y}) \cap r(U_{x,y}) = \emptyset$.

Since $\mathcal{R}$ is second countable we can find countably many neighbourhoods $U_{x,y}$ still covering all $\mathcal{R} \setminus \Delta$. Relabel them to obtain a sequence $\{U_i\}_{i \in \mathbb{N}}$. Because $\mathcal{R}$ is non-singular, $r \circ s^{-1}(s(U_i) \cap E)$ has measure zero and we have

$[E]_\mathcal{R} = \bigcup_{i \in \mathbb{N}} r \circ s^{-1}(s(U_i) \cap E)$.

(2.2)
Since \([E]_R\) is a countable union of sets of measure zero, it also has measure zero.

To prove that \([E]_R\) is \(F_\sigma\) recall that \(X\) is metrizable, hence every open set in \(X\) is \(F_\sigma\). Since finite intersections of \(F_\sigma\) sets are again \(F_\sigma\) it follows that \(s(U_i) \cap E\) is \(F_\sigma\) and equation (2.2) shows that \([E]_R\) is also \(F_\sigma\). \(\square\)

As an immediate corollary of Lemma 2.4.4 we have

**Lemma 2.4.5.** Suppose that \((X, \mu)\) is a Polish space with a non-atomic Borel probability measure, and \(R\) is a measure preserving (or non-singular) étale equivalence relation on \(X\). If a \(G_\delta\) subset \(D \subset X\) has full measure, \(\mu(D) = 1\), then there is an \(R\)-saturated \(G_\delta\) subset \(D_1 \subset D\) of full measure.

**Theorem 2.4.6.** Suppose that \((X, \mu)\) is a Polish space with a non-atomic Borel probability measure, and \(R \subset X \times X\) is a measure preserving (or non-singular) étale equivalence relation. Then there is an \(R\)-saturated \(G_\delta\) subset of full measure \(X_1 \subset X\) such that \(X_1\) is zero-dimensional and all sets that are open in the topology of \(X_1\) have positive measure. That is, \(\text{supp}(\mu|_{X_1}) = X_1\). Consequently, the same is true for a free continuous action of a countable group \(G\).

**Proof.** First, construct a zero-dimensional \(G_\delta\) subset \(X' \subset X\) of full measure; see (Bogachev, 2007, Theorem 9.6.3) or (del Junco & Şahin, preprint) for the construction. Find a countable dense subset \(\{x_i\}_{i \in \mathbb{N}} \subset X\). If there is a neighbourhood of \(x_i\) of measure zero, find the largest open ball \(B_i\) centred at \(x_i\) of measure zero. Let \(U\) be the union of all \(B_i\). The set \(U\) is open and since \(X\) is metrizable \(U\) is \(F_\sigma\). Hence \(X' \setminus U\) is again \(G_\delta\) and by Lemma 2.4.5 there is a \(G_\delta\) set \(X_1 \subset X' \setminus U\) of full measure which is \(R\)-saturated. Now, if \(E \subset X_1\) is open and \(\mu E = 0\), then there is an open set \(O \subset X\), such that \(E = O \cap X_1\). Since \(X_1\) has
full measure we conclude that $\mu O = 0$. Let us show that in fact $E$ is empty. For any $x \in E$ there exist an open ball $B_r(x) \subset O$ of radius $r = r(x) > 0$. Find a point $x_i$ such that $d(x, x_i) < r/2$. Then the open ball $B_{r/2}(x_i) \subset B_r(x)$ contains $x$ and has measure zero. Hence $B_{r/2}(x_i) \subset B_i \subset U$. Contradiction. Therefore $E$ is empty. Thus all non-empty open subsets of $X_1$ have positive measure.

\Box

2.5 Topological full groups

Everywhere in this section $X$ is a zero-dimensional Polish space with a non-atomic probability measure $\mu$ such that $\text{supp} \mu = X$.

Recall that in the measure-theoretic category sets are equal if their symmetric difference has measure zero. Up until now everything was done in the measure-theoretic setting. We will now switch to the “almost continuous category”. From now on by $A = B$ we mean that there exists a virtually full\footnote{See Definition 2.2.3 on page 10.} subset $X_0 \subset X$ such that $A \cap X_0 = B \cap X_0$. Similarly, we say that two mappings $f, g$ are equal if there exists a virtually full subset $X_0$ such that $f|_{X_0} = g|_{X_0}$. Hence $f : X \to Y$ is understood as an equivalence class of virtually equal maps. We will call a transformation $f : X \to X$ continuous (or virtually continuous) if there is a member of the equivalence class of $f$ which is defined on a virtually full subset and continuous. Similarly, disjoint will mean virtually disjoint, et cetera.

Suppose that a group $\mathbb{G}$ acts on $X$ by non-singular (or measure-preserving) homeomorphisms.
Definition 2.5.1. The topological full group \([G]_{\text{top}}\) of a free continuous group action \(G\) is the set of all functions \(T \in [G]\) with a (virtually) continuous cocycle, 
\[
\{ T \in [G] : \text{the cocycle } C_T \text{ is (virtually) continuous} \}
\]

Notice that every \(T \in [G]_{\text{top}}\) is continuous. Continuity of the cocycle is equivalent to the following condition: \(C_T\) is defined virtually everywhere and for every \(g \in G\) the set 
\[
E_g = C_T^{-1}(g) = \{ x \in X : T(x) = g \cdot x \}
\]
is virtually clopen. The definitions are equivalent because the collection \(\{E_g\}_{g \in G}\) virtually partitions \(X\) and hence any countable union of \(E_g\) is still clopen (see Lemma 2.2.3). Therefore for any subset \(A \subset G\) the set \(C_T^{-1}(A)\) is clopen.

Notice that if \(T \in [G]\) is continuous, then \(\{E_g\}_{g \in G}\) is a partition of \(X\) and each \(E_g\) is closed, but a priori the sets \(E_g\) do not have to be open or almost open.

Suppose that a countable group \(G\) acts on \(X\) by homeomorphisms and every \(\text{fix}(g)\) is clopen. Then we can similarly define \([G]_{\text{top}}\) as the set of all mappings that have a (virtually) continuous cocycle:

Definition 2.5.2. The topological full group \([R]_{\text{top}}\) consists of all transformations \(T \in [R]\) such that there is a clopen partition \(X = \bigcup_{i \in \mathbb{N}} X_i\) and \(T|_{X_i} = g_i\) for some \(g_i \in G\).

The topological full group of a free group action is a common object of study in a topological setting, see, for example, (Bezuglyi & Kwiatkowski, 2000) or (Bezuglyi et al., 2005).

Definition 2.5.3. The topological full group \([R]_{\text{top}}\) of an étale equivalence relation \(R\) is the collection of all functions \(T \in [R]\) that defined on a virtually
full subset and there is a countable partition of $X$ into virtually clopen sets $X_i$ such that $T|_{X_i} = r \circ s^{-1}$, $T$ is a homeomorphism between $X_i$ and $T(X_i)$, and the range $r$ and the source maps $s$ are (virtual) homeomorphisms on the part of the graph of $T$ supported by $X_i$, $\{(x,Tx): x \in X_i\}$.

**Lemma 2.5.1.** Suppose that $\mathcal{R}$ is an étale equivalence relation on $X$. Then the set $[\mathcal{R}]_{\text{top}}$ is indeed a group.

**Proof.** It is clear that $\text{id} \in [\mathcal{R}]_{\text{top}}$. Fix a function $T \in [\mathcal{R}]_{\text{top}}$. We will show that $T^{-1}$ also belongs to $[\mathcal{R}]_{\text{top}}$. By definition there is a clopen partition $\{X_i\}_{i \in \mathbb{N}}$ of $X$ such that (a) $T$ is a homeomorphism between $X_i$ and $T(X_i)$; (b) the range and the source maps are homeomorphisms on $\{(x,Tx): x \in X_i\}$. Then $U_i = s^{-1}(X_i)$ is open in $\mathcal{R}$ and by definition of étaleness the set $U_i^{-1}$ is again open. Notice that since the range map $r: U_i \to T(X_i)$ is a homeomorphism, $s^{-1}: T(X_i) \to U_i^{-1}$ is also a homeomorphism. Similarly, $r: U_i^{-1} \to X_i$ is a homeomorphism. Hence

$$T^{-1} = r \circ s^{-1}: T(X_i) \to X_i$$

belongs to the topological full group $[\mathcal{R}]_{\text{top}}$. In a similar way, using part 4 of the definition of étaleness, one shows that $[\mathcal{R}]_{\text{top}}$ is closed under compositions. Hence $[\mathcal{R}]_{\text{top}}$ is a group. \qed

**Lemma 2.5.2.** Suppose that $(\mathcal{R}_G,T)$ is an étale equivalence relation induced by an action of a countable group $G$, and the topology $T$ is given by equation (2.1), page 18. Then $[\mathcal{R}_G]_{\text{top}} = [G]_{\text{top}}$.

**Proof.** Since the sets

$$B_{g,U} = \{(x,y): x \in U \text{ open}, y = g \cdot x\} \subset \mathcal{R}_G$$


are open, the inclusion $[\mathcal{G}]_{\text{top}} \subset [\mathcal{R}\mathcal{G}]_{\text{top}}$ is obvious.

Consider a function $T \in [\mathcal{R}\mathcal{G}]_{\text{top}}$. We will prove that $T \in [\mathcal{G}]_{\text{top}}$. Find a clopen partition $\{X_i\}$ of $X$ from the definition of the topological full group such that the range and the source maps are homeomorphism on the corresponding sets

$$U_i = \{(x, Tx): x \in X_i\} \in \mathcal{T}.$$ 

For any point $(x, Tx) \in U_i$ there is an element $B_{g,U}$ from the basis of the étale topology $\mathcal{T}$ (see equation 2.1) such that $(x, Tx) \in B_{g,U}$. Then $U_i \cap B_{g,U}$ is again clopen and by construction $T = g$ on the clopen set $s(U_i \cap B_{g,U})$ containing $x$. Hence we obtain a partition of $X$ into open sets $E_g$ such that $T = g$ on each $E_g$. To finish the proof, notice that each $E_g$ is also closed, hence clopen. 

Lemmas 2.5.1 and 2.5.2 show that $[\mathcal{G}]_{\text{top}}$ is also a group. The following lemma is a simple consequence of Definition 2.5.3. We will use it extensively to construct elements of the topological full group.

**Lemma 2.5.3.** Let $T \in [\mathcal{R}]$. If there is a clopen partition $\{X_i\}_{i \in \mathbb{N}}$ such that $T|_{X_i} = T_i$ for some $T_i \in [\mathcal{R}]_{\text{top}}$, then $T$ belongs to $[\mathcal{R}]_{\text{top}}$.

We will also frequently use the fact that if $\mathcal{R}$ is ergodic then one can map any open set $A$ onto any other open set $B$ of the same measure with an element $T$ of the topological full group $[\mathcal{R}]_{\text{top}}$. This is the content of the next lemma.

**Lemma 2.5.4.** Let $\mathcal{R}$ be an ergodic measure-preserving étale equivalence relation.

1. If $A$ and $B$ are open sets, $\mu A = \mu B > 0$, then there is an involution $T \in [\mathcal{R}]_{\text{top}}$ that maps virtually all of $A$ onto virtually all of $B$ and is supported
by $A \cup B$,

$$T(A) = B, \quad \text{supp} \, T \subset A \cup B.$$  

Equivalently, there are disjoint clopen sets $A_i \subset A$, $i \in \mathbb{N}$, and $B_i \subset B$ such that $T(A_i) = B_i$ and $\mu\left( \bigcup_{i \in \mathbb{N}} B_i \right) = \mu B$.

2. For any open set $A$ there is an involution $T \in [R]_{\text{top}}$ with $\text{supp} \, T = A$.

In case of a measure-preserving group of homeomorphisms $G$ acting freely and ergodically on $X$ we can also require that $T|_{A_i} = g_i$ for some $g_i \in G$.

**Proof.** First of all, notice that for any two disjoint sets of positive measure $A$ and $B$ there is a pair $(x, y) \in R$ such that $x \in A$ and $y \in B$. Otherwise for every point $x \in A$ its $R$-class $[x]_R$ is included in $X \setminus B$ and we have found an $R$-saturated set of measure strictly between zero and one. There exist an open set $U \subset R$ containing $(x, y)$, such that $r|_U$, $s|_U$ are homeomorphisms. Since $X$ is zero-dimensional we can require $U$ to be clopen. Therefore we found a mapping

$$r \circ s^{-1}: s(U) \cap A \to r(U) \cap B$$

of a clopen subset of $A$ onto a clopen subset of $B$. To be precise,

$$(s(U) \cap A) \cap s \circ r^{-1}(r(U) \cap B)$$

is the (open) part of $A$ which can be mapped and we can find a clopen subset of it that contains $x$. Denote by $\Lambda(A, B)$ the set of all clopen $C \subset A$ obtained in this way (and such that $r \circ s^{-1}(C) \subset B$). We will now prove the lemma using a standard exhaustion argument.

Let $A'_0 = A$, $B'_0 = B$ and let $\lambda_0 = \sup \mu(C) > 0$, where the supremum is taken over all $C \in \Lambda(A'_0, B'_0)$. Find $A_1 \in \Lambda(A'_0, B'_0)$ such that $\mu(A_1) > \frac{\lambda_0}{2}$ and let
\[ B_1 = r \circ s^{-1}(A_1), \quad A'_1 = A'_0 \setminus A_1, \quad B'_1 = B'_0 \setminus B_1. \]

Continue in the same way. After \( n \) steps let \( \lambda_{n+1} = \sup \mu(C) > 0 \), where the supremum is taken over all \( C \in \Lambda(A'_n, B'_n) \).

Find \( A_{n+1} \in \Lambda(A'_n, B'_n) \) such that \( \mu(A_{n+1}) > \frac{\lambda_{n+1}}{2} \) and let

\[ B'_{n+1} = r \circ s^{-1}(A_{n+1}), \quad A'_{n+1} = A'_n \setminus A_{n+1}, \quad B'_{n+1} = B'_n \setminus B_{n+1}. \]

There are three possibilities. If \( \lambda_N = 0 \) for some \( N \) the construction is finished. Suppose that the process is infinite and \( \lim_{n \to \infty} \lambda_n = \lambda^* > 0 \). Then we would find infinitely many disjoint subsets \( A_i \) of \( A \) with \( \mu A_i > \frac{\lambda^*}{2} > 0 \), which is impossible.

The last possibility is \( \lim_{n \to \infty} \lambda_n = 0 \). In this case let \( A^* = A \setminus \bigcup_{i \in \mathbb{N}} A_i \) and \( B^* = B \setminus \bigcup_{i \in \mathbb{N}} B_i \).

We are going to show that \( \mu A^* = \mu B^* = 0 \). To arrive at a contradiction, suppose that \( \mu A^* > 0 \). Then for almost every point \( x \in A^* \) there is a point \( y \in B^* \) and a clopen set \( U \subset \mathcal{R} \) such that \((x, y) \in U\), and \( r|_U \), \( s|_U \) are homeomorphisms, and \( s(U) \subset A \), \( r(U) \subset B \). Let \( D_U = A^* \cap s(U) \). Suppose that for some \( U \) \( \mu D_U > 0 \). Then we would have an estimate on \( \lambda_n \), \( \lambda_n \geq \mu D_U \), because \( A_i \) do not intersect \( A^* \) and \( B_i \) do not intersect \( B^* \). This would contradict \( \lim_{n \to \infty} \lambda_n = 0 \). Hence \( \mu D_U = 0 \) for all such \( U \) and almost all of \( A^* \) is covered by clopen sets \( s(U) \) that intersect \( A^* \) over a set of measure zero. Recall that \( X \) is Polish, therefore second-countable and there is a countable subcover (Munkres, 2000, Theorem 30.3). Denote the union of all sets from the subcover \( O \). Thus almost all of \( A^* \) is covered by an open set \( O \) with the property \( \mu(O \cap A^*) = 0 \). Hence \( \mu A^* = 0 \) and we arrive at a contradiction.

The involution \( T \) is then defined as follows:

\[
T(x) = \begin{cases} 
  r \circ s^{-1}(x), & \text{for } x \in A_i; \\
  s \circ r^{-1}(x), & \text{for } x \in B_i; \\
  \text{id}, & \text{otherwise.}
\end{cases}
\]
To prove the second statement, use Lemma 2.2.6 to find disjoint clopen subsets $A_1, A_2 \subset A$ such that $\mu A_1 = \mu A_2 = \frac{\mu A}{2}$. Everything now follows from part (1) applied to $A = A_1$ and $B = A_2$.

We will conclude this chapter with the definition of almost continuous orbit equivalence for étale equivalence relations. One possibility is to define it as a homeomorphism between virtually full subsets that maps orbits to orbits. It is desirable to also require the cocycles to be continuous. This works for group actions, but cocycles are not defined for equivalence relations. Recall that two equivalence relations $R_1$ and $R_2$ are orbit equivalent if, and only if, there is a mapping $\varphi$ such that $\varphi[R_1]|_{\top} \varphi^{-1} = [R_2]$.

**Definition 2.5.4.** Étale equivalence relations $R_1 \subset X \times X$ and $R_2 \subset Y \times Y$ are called **almost continuously orbit equivalent** if there exist saturated dense $G_\delta$ subsets of full measure $X_1 \subset X$ and $Y_1 \subset Y$ and a measure-preserving homeomorphism $\varphi : X_1 \to Y_1$ such that $\varphi[R_1]|_{\top} \varphi^{-1} = [R_2]|_{\top}$. Continuous group actions $G_1$ on $X$ and $G_2$ on $Y$ are almost continuously orbit equivalent if the induced equivalence relations $R_{G_1}$ and $R_{G_2}$ are étale and almost continuously orbit equivalent.

The next lemma shows that if $\varphi$ is an almost continuous orbit equivalence, then it indeed maps orbits to orbits between two saturated virtually full subsets.

**Lemma 2.5.5.** Suppose that there are saturated $G_\delta$ subsets of full measure $X_1 \subset X$ and $Y_1 \subset Y$ and a measure-preserving homeomorphism between them $\varphi : X_1 \to Y_1$ such that $\varphi[R_1]|_{\top} \varphi^{-1} = [R_2]|_{\top}$.
Then there exist saturated $G_\delta$ subsets of full measure $X^* \subset X_1$ and $Y^* \subset Y_1$ such that $\varphi(X^*) = Y^*$ and for every $x \in X^*$, $\varphi([x]_{\mathcal{R}_1} \cap X^*) = [\varphi(x)]_{\mathcal{R}_2} \cap Y^*$.

**Proof.** Without loss of generality we can assume that $X_1$ and $Y_1$ are zero-dimensional, see Theorem 2.4.6. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are étale we can find countable groups $G_1 \subset [\mathcal{R}_1]_{\text{top}}$ and $G_2 \subset [\mathcal{R}_2]_{\text{top}}$ acting on $X_1$ and $Y_1$ by homeomorphisms such that $G_i$ is orbit equivalent to $\mathcal{R}_i$, $i = 1, 2$ (these actions are not necessarily free).

By taking countable intersections find a $G_\delta$ subset of full measure $Y_2 \subset Y_1$ which is $G_2$ and $\varphi G_1 \varphi^{-1}$-invariant, and for every element $g_1 \in G_1$ $\varphi g_1 \varphi^{-1}$ equals some element of $[\mathcal{R}_2]_{\text{top}}$ everywhere on $Y_2$. Let $X_2 = \varphi^{-1}(Y_2)$ and consider $x \in X_2$, $x = \varphi^{-1}(y)$. Since $\varphi g_1 \varphi^{-1}(y) = g_2 y$ for some $g_2 \in G_2$ we have $\varphi(g_1 x) = g_2 \varphi(x)$ and $\varphi(\text{Orb}_{G_1}(x)) \subset \text{Orb}_{G_2}(\varphi(x))$.

Similarly, we can find a $G_\delta$ set of full measure $X_3 \subset X_1$ which is $G_1$ and $\varphi^{-1} G_2 \varphi$-invariant and such that $\text{Orb}_{G_2}(\varphi(x)) \subset \varphi(\text{Orb}_{G_1}(x))$. Let $X^* = X_1 \cap X_2$ and $Y^* = \varphi(X^*)$. By construction $\varphi : X^* \to Y^*$ maps orbits onto orbits and $X^*, Y^*$ are invariant $G_\delta$ subsets of full measure. \qed
Chapter 3

Dye’s theorem for amenable group actions

3.1 Introduction

The main result of this chapter is Theorem 3.5.1: any free ergodic measure-preserving action of a countable amenable group by homeomorphisms is almost continuously orbit equivalent to a single homeomorphism. Using Theorem 1 from (del Junco & Şahin, preprint) we establish that any two free ergodic measure-preserving actions of countable amenable groups by homeomorphisms are almost continuously orbit equivalent. Note that this includes the continuity of the associated cocycles.

Definition 3.1.1. A countable group $G$ is called amenable if there exists a finitely additive probability measure $\mu$ on $G$ which is left-invariant: for any $g \in G$ and $A \subset G$, $\mu(A) = \mu(gA)$. 
There are many equivalent definitions of amenability, see, for example, (Kechris & Miller, 2004, Theorem 5.14). We will not use amenability of the group directly. Its usage is hidden in Rohlin’s lemma.

The chapter is organized in the following way: first we introduce basic objects that we are going to operate with, columns and arrays, and show how one can construct arrays with specific properties. The main objective here is to construct refinements, extensions and stacking of arrays. In Section 3.3 we present Rohlin’s lemma for amenable groups and the necessary definitions. We also explain how Rohlin’s lemma is used later. Section 3.4 is devoted to the proof of the main lemma. In Section 3.5 we state and prove the main theorem. In the last section we list several open problems.

**Remark 3.1.1.** Everywhere in this chapter except for Section 3.3 \((X,\mu)\) is a Polish space with a non-atomic Borel probability measure such that \(\text{supp} \mu = X\). All equalities in this chapter are understood in the almost continuous category, see the discussion in the beginning of Section 2.5, page 21. For example, we regard two subsets \(A\) and \(B\) of \(X\) as equal if there exists a virtually full subset \(X_0 \subset X\) such that \(A \cap X_0 = B \cap X_0\). Two maps are equal if there is a virtually full subset \(X_0\) such that their restrictions on \(X_0\) are (pointwise) equal.

### 3.2 Columns, arrays and their basic properties

In this section we define columns, arrays, and develop all the machinery necessary for proving Lemma 3.4.1 and Theorem 3.5.1. Everywhere here we assume that \(X\) is a zero-dimensional Polish space with a non-atomic probability measure \(\mu\), and a countable discrete group of homeomorphisms \(\mathbb{G}\) acts on \(X\) freely and ergodically.
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The proofs, with a few exceptions, are adaptations of the same results for \( \mathbb{Z} \)-actions from (del Junco & Şahin, preprint).

**Definition 3.2.1.** A column \( \mathcal{B} \) is a finite collection of clopen sets \( \{ B_i \}_{i=0}^{h-1} \) together with elements of the group, called level maps, \( b_{ij} \in G \) such that \( b_{ij}(B_i) = B_j \) and

\[
b_{kj}b_{ik} = b_{ij}, \quad b_{ij} = b_{ji}^{-1}.
\]  

(3.1)

The set \( B_0 \) is called the base of the column and \( h \) is called the height of \( \mathcal{B} \). The width of \( \mathcal{B} \) is the measure of its base, \( \mu(B_0) \). We will also write \( |\mathcal{B}| \) for the union of all levels of \( \mathcal{B} \),

\[
|\mathcal{B}| \overset{\text{def}}{=} \bigcup_{i=0}^{h-1} B_i.
\]  

(3.2)

For a point \( x \in |\mathcal{B}| \setminus B_{h-1} \) by \( \mathcal{B}(x) \) we denote its image in the next level under the corresponding level map:

\[
\mathcal{B}(x) \overset{\text{def}}{=} b_{i,i+1}(x) = b_{0,i+1} \cdot b_{i,0}(x) \quad \text{for} \quad x \in B_i, \quad i \neq h-1.
\]  

(3.3)

The \( \mathcal{B} \)-orbit (or a \( \mathcal{B} \)-fibre) of \( x \) is the set of all images under the level maps

\[
\text{Orb}_\mathcal{B}(x) \overset{\text{def}}{=} \{ b_{0j} \cdot b_{i0}(x) \}_{j=0}^{h-1} = \{ g_{ij}x \}_{j=0}^{h-1}, \quad x \in B_i.
\]  

(3.4)

A slice of \( \mathcal{B} = \{ B_i \}_{i=0}^{h-1} \) is any column of the form \( \mathcal{S} = \{ E_i \}_{i=0}^{h-1} \) with level maps \( c_{ij} = b_{ij} \) and levels \( E_i \subseteq B_i \). For a given clopen subset \( E_i \subseteq B_i \) the slice of \( \mathcal{B} \) over \( E_i \) is the column \( \mathcal{S} = \{ b_{ij}(E_i) \}_{j=0}^{h-1} \) with the same level maps as \( \mathcal{B} \).

**Definition 3.2.2.** An array \( \mathcal{T} \) of height \( h \) is a finite or countable collection of pairwise disjoint columns \( \mathcal{T}_i \) of the same height \( h \). For a point \( x \in \mathcal{T}_i \) by \( \mathcal{T}(x) \) we denote the image of \( x \) under the corresponding column map, \( \mathcal{T}(x) \overset{\text{def}}{=} \mathcal{T}_i(x) \). The \( \mathcal{T} \)-orbit \( \text{Orb}_\mathcal{T}(x) \) of \( x \) equals \( \text{Orb}_{\mathcal{T}_i}(x) \). Similarly, \( |\mathcal{T}| \overset{\text{def}}{=} \bigcup_i |\mathcal{T}_i| \) and the base
of $T$ is the union of the bases of its columns. The width of $T$ is the measure of its base. The levels of $T$ are the levels of its columns.

Note that a column by itself is an array. The base of $T$ is not a level of $T$ if $T$ has more than one column; the total measure of $T, \mu(|T|)$ is $h\mu(B)$, where $h$ is the height of $T$ and $B$ is its base.

**Definition 3.2.3.** A subarray $T'$ of $T$ is an array such that each column of $T'$ is a slice of a column of $T$. A refinement of $T$ is a subarray $R$ such that $|T| = |R|$ virtually everywhere.

**Lemma 3.2.1.** Suppose that $T$ is an array with base $B$, and $\mu B = \sum_{i=1}^{\infty} r_i, r_i > 0$. Then there are disjoint subarrays $T_i$ of $T$ such that they fill out $T$ ( $|T|$ virtually equals $\bigcup_{i \in \mathbb{N}} |T_i|$ ) and the measure of the base of $T_i$ is $r_i$.

**Proof.** Using Lemma 2.2.6 we can find open sets $C_i \subset B$ such that $\mu C_i = r_i$ and since $X$ is zero-dimensional each $C_i$ is a union of countably many disjoint clopen
subsets $C_{i,j}$, $j \in \mathbb{N}$. Then, if $B_n$ are the bases of columns of $T$, let $T_i$ be the collection of slices of columns $B_n$ over $B_n \cap C_{i,j}$, $n, j \in \mathbb{N}$.

**Definition 3.2.4.** Suppose that $B = \{B_i\}_{i=0}^{h-1}$ and $C = \{C_j\}_{j=0}^{k-1}$ are disjoint columns with level maps $b_{ij}$ and $c_{lk}$ respectively, and $\mu B_0 = \mu C_0$. If there is an element $g \in G$ such that $g(B_0) = C_0$ we can form a concatenation of $B$ and $C$

$$D = \{B_0, \ldots, B_{h-1}, C_0, \ldots, C_{k-1}\} = \{D_i\}_{i=0}^{h+k-1}$$

with level maps $d_{ij}$ being $b_{i,j_1}$ or $c_{i,j_1}$ when $D_i, D_j$ are from the same column:

$$D_i = B_{i_1}, \ D_j = B_{j_1} \quad \text{or} \quad D_i = C_{i_1}, \ D_j = C_{j_1},$$

and

$$d_{ij} = \begin{cases} c_{0,j_1} \cdot g \cdot b_{i_1,0}, & \text{when } D_i = B_{i_1} \in B, \ D_j = C_{j_1} \in C; \\ b_{0,j_1} \cdot g^{-1} \cdot c_{i_1,0}, & \text{when } D_i = C_{i_1} \in C, \ D_j = B_{j_1} \in B, \end{cases}$$

where $g \in G$ is the element that maps $B_0$ onto $C_0$. The concatenation of several columns $C_1, \ldots, C_n$ is defined in the same way provided their bases $C_{0,1}, \ldots, C_{0,n}$ themselves form a column.

**Remark 3.2.1.** Note that $d_{ij}$ indeed satisfy the equation (3.1). It is clear from the definition that $d_{ij} = d_{ji}^{-1}$ and if, for example, $D_i = B_{i_1}$, $D_j = C_{j_1}$, and $D_k = C_{k_1}$, then

$$d_{kj} \cdot d_{ik} = c_{k_1,j_1} \cdot (c_{0,k_1} \cdot g \cdot b_{i_1,0}) = c_{0,j_1} \cdot g \cdot b_{i_1,0} = d_{ij}.$$ 

**Remark 3.2.2.** Clearly concatenation depends on the order of columns. Even in the case of two columns a concatenation of $B$ and $C$ will never be equal to a concatenation of $C$ and $B$. 


Definition 3.2.5. An array $E$ is called an extension of an array $T$ if there is a refinement $R$ of $T$ such that each column of $E$ is a concatenation of columns of $R$ and $|E| = |T|$ virtually everywhere.

Definition 3.2.6. Suppose that $T_1, \ldots, T_n$ are disjoint arrays of equal width. A stacking of $T_1, \ldots, T_n$ is an array $S$ such that $|S| = \bigcup_{i=1}^{n} |T_i|$ and there are refinements $R_1, \ldots, R_n$ of $T_1, \ldots, T_n$ such that every column $C$ of $S$ is a concatenation of columns $C_1, \ldots, C_n$, where each $C_i$ is a column of $R_i$.

Lemma 3.2.2. 1. For any finite collection $T_1, \ldots, T_n$ of arrays of equal width there is a stacking $S$ of $T_1, \ldots, T_n$.

2. For any array $T$ of height $h$ there is an extension $E$ of $T$ of height $hn$ for any $n \in \mathbb{N}$.

Proof. (1) It is enough to prove this for $n = 2$. If $A$ and $B$ are the bases of $T_1$ and $T_2$ respectively, by Lemma 2.5.4 there are disjoint clopen sets $A_i \subset A$, $i \in \mathbb{N}$ and elements of the group $g_i \in G$ such that $\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \mu A$ and the sets $B_i = g_i(A_i) \subset B$ are disjoint and fill out $B$. That is, $\mu\left(\bigcup_{i \in \mathbb{N}} B_i\right) = \mu B$. If $\{C_j\}$ and $\{D_k\}$ are the bases of the columns of $T_1$ and $T_2$, let

$$A_{ijk} = A_i \cap C_j \cap g_i^{-1}(D_k) \quad \text{and} \quad B_{ijk} = g_i(A_{ijk}).$$

Clearly the sets $A_{ijk}$ fill out $A$. Let $C_{ijk}$ and $D_{ijk}$ be the slices of $T_1$ and $T_2$ with bases $A_{ijk}$ and $B_{ijk}$ respectively. The arrays $E_1 = \{C_{ijk}\}$ and $E_2 = \{D_{ijk}\}$ are refinements of $T_1$ and $T_2$. By construction the concatenations of $C_{ijk}$ and $D_{ijk}$ exist (via $g_i$) and form a stacking of $T_1$ and $T_2$.

(2) Divide $T$ into $n$ subarrays $T_1, \ldots, T_n$ of width $\text{width}(T)/n$. By part (1) there is a stacking of $T_1, \ldots, T_n$, which gives the desired extension.  \qed
Definition 3.2.7. The diameter $\text{diam} \ T$ of an array $\ T$ is the supremum of the diameters of its levels.

Lemma 3.2.3. For any array $\ T$ and any $\varepsilon > 0$ there is a refinement $\mathcal{R}$ of $\ T$ such that $\text{diam} \ \mathcal{R} < \varepsilon$.

Proof. First, suppose that $\ T$ contains only one column $\{B_{ij}\}_{i=0}^{h-1}$ with level maps $g_{ij} \in G$. Since $X$ is separable and zero-dimensional there exist a countable clopen partition $P$ of $X$ into sets of diameter less than $\varepsilon$. For an $h$-tuple

$$p = (p_0, \ldots, p_{h-1}) \in P^h$$

consider the set $B_p = \bigcap_{i=0}^{h-1} g_{i,0}(p_i \cap B_i)$. Note that $g_{0,0}(p_0 \cap B_0) = p_0 \cap B_0$; moreover, $g_{0,i}(B_p) \subset p_i$. Hence $B_p$ is a clopen subset of $B_0$ with $\text{diam} B_p < \varepsilon$, and $\{B_p\}_{p \in P^h}$ is a clopen partition of the base $B_0$. Therefore the slices of $\ T$ over the sets $B_p$ form a refinement $\mathcal{E}$ of $\ T$ with $\text{diam} \mathcal{E} < \varepsilon$.

Consider now the general case when $\ T$ has countably many columns. One can apply the refinement procedure defined above to every column of $\ T$ so that the diameter of each refined column is less than $\varepsilon/2$. Hence the supremum of the diameters is at most $\varepsilon/2$ and less than $\varepsilon$. \qed

3.3 Rohlin’s lemma

Remark 3.3.1. Everywhere in this section $(X, \mu)$ is a measure space and all results are in the measure-theoretic category.
In this section we discuss Rohlin’s lemma (Rohlin, 1949), which plays the central role in the proof of the main Lemma 3.4.1. In case of a single transformation $T: X \to X$ it is usually stated as follows:

**Theorem 3.3.1** (V. Rohlin). Let $X$ be a Lebesgue space with a non-atomic probability measure $\mu$ and let $T: X \to X$ be an ergodic measure-preserving transformation. Then for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there is a measurable subset $B \subset X$ such that the sets

$$B, T(B), T^2(B), \ldots, T^{n-1}(B)$$

are pairwise disjoint and cover $X$ up to a set of measure less than $\varepsilon$,

$$\mu\left(X \setminus \bigcup_{i=0}^{n-1} T^i(B)\right) < \varepsilon.$$  

For the standard proof see (Petersen, 2000) or (Glasner, 2003). Theorem 3.3.1 also holds for aperiodic transformations, see (Halmos, 1956) or (Cornfeld et al., 1982); it was extended to the case of a non-invertible transformation in (Heinemann & Schmitt, 2001). (Connes et al., 1981) contains an extensive overview on the history of Rohlin’s lemma. Lemma 8 in (Connes et al., 1981) (see also Lemma 9.5 in (Kechris & Miller, 2004)) is a generalization of Rohlin’s lemma to equivalence relations.

A natural generalization of Rohlin’s lemma to group actions would be to pick a finite subset of group elements $F \subset G$ and for a given $\varepsilon > 0$ look for a measurable set $B$ such that the sets $\{f \cdot B\}_{f \in F}$ are disjoint and cover most of the space,

$$\mu(FB) = \mu\left(\bigcup_{f \in F} f \cdot B\right) > 1 - \varepsilon.$$  

**Definition 3.3.1.** A finite subset $T \subset G$ tiles $G$ if there is a set of centres $C \subset G$ such that $\{Tc\}_{c \in C}$ is a partition of $G$. If for every finite set $K \subset G$ there is a tile $T \subset G$ that contains $K$, then $G$ is called monotileable.
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Theorem 3.3.2 (Ornstein & Weiss (1980)). Suppose that the action of $G$ is free and measure-preserving.

1. If Rohlin’s lemma is valid for a finite set $F$ then $F$ tiles $G$.

2. If $G$ is amenable, and $F$ tiles $G$, then Rohlin’s lemma is valid for $F$.

In applications we use Rohlin’s lemma 3.3.1 with “large” $n \in \mathbb{N}$; that is, such that the segment $[0, n]$ is “almost invariant” under translations. Another way to look at it is to say that for a fixed $k \in \mathbb{N}$ and a large $n \in \mathbb{N}$ the $[0, k]$-border of $[0, n]$ is “small”. Hence the following definition.

Definition 3.3.2. Suppose that $K \subset G$ is finite. The $K$-boundary of $F \subset G$ contains each $g \in G$ such that $Kg$ intersects both $F$ and $G \setminus F$:

$$
\partial_K F \overset{\text{def}}{=} \{ g \in G : Kg \cap F \neq \emptyset, \ Kg \cap (G \setminus F) \neq \emptyset \}.
$$

(3.5)

The $K$-interior of $F$ is the set

$$
F \setminus \partial_K F = \{ f \in F : Kf \subset F \}.
$$

(3.6)

When $K$ is clear from the context we will denote the $K$-interior of $F$ by $\overset{\circ}{F}$.

Definition 3.3.3. Let $\delta > 0$. A finite set $F \subset G$ is called $(K-\delta)$-invariant if its $K$-interior covers more than $1-\delta$ of $F$:

$$
\left| \{ f \in F : Kf \subset F \} \right| > (1-\delta)|F|,
$$

where $|\cdot|$ denotes the counting measure.

It is not known whether for an arbitrary amenable group $(K-\delta)$-invariant tiles exist. The following theorem can be found in (Ornstein & Weiss, 1980) and (Weiss, 2001):
Theorem 3.3.3. If $G$ is a solvable group, or a finite extension of a solvable group, or an increasing union of such groups, or a residually finite group, then for any finite $K$ and $\delta > 0$ there exist a $(K - \delta)$-invariant set $F$ that tiles the group.

The following lemma states that a large subset of an almost invariant set is almost invariant:

Lemma 3.3.4. Suppose $\delta < 1/2$, $F$ is $(K - \delta)$-invariant and $F_1$ is a subset of $F$ such that $|F_1| > (1 - \delta)|F|$. Then $F_1$ is $(K - \delta_1)$-invariant with $\delta_1 = 2\delta(|K| + 1)$.

Proof. Let $\hat{F}$ and $\hat{F}_1$ be the $K$-interior of $F$ and $F_1$ respectively, and let $D = F \setminus F_1$. Then $|\hat{F} \cap F_1| > (1 - 2\delta)|F| > (1 - 2\delta)|F_1|$. Let $f \in \hat{F} \cap F_1$. If $Kf$ is not a subset of $F_1$, then $Kf \cap D \neq \emptyset$ and hence $f \in K^{-1}D$, $|K^{-1}D| \leq |K| \cdot |D| < \delta|K||F|$.

Therefore

$$|\hat{F}_1| > (1 - 2\delta)|F_1| - \delta|K||F| > (1 - 2\delta)|F_1| - 2\delta|K||F_1|.$$  

The most general version of Rohlin’s lemma (for an arbitrary amenable group) uses the notion of $\varepsilon$-quasi-tiles:

Definition 3.3.4. Finite sets $\{E_\lambda\}_{\lambda \in \Lambda}, E_\lambda \subset G$, are called $\varepsilon$-disjoint if there are sets $F_\lambda \subset E_\lambda$ such that $F_\lambda$ are pairwise disjoint and $|F_\lambda| > (1 - \varepsilon)|E_\lambda|$. Similarly, sets $\{X_\lambda\}_{\lambda \in \Lambda}, X_\lambda \subset X$, are called $\varepsilon$-disjoint if there are subsets $Y_\lambda \subset X_\lambda$ such that the $Y_\lambda$ are pairwise disjoint and $\mu Y_\lambda > (1 - \varepsilon)\mu X_\lambda$. A collection $\{T_1, \ldots, T_n\}$ of subsets of $G$ $\alpha$-covers $E \subset G$ if

$$|E \cap \left( \bigcup_{i=1}^n T_i \right)| \geq \alpha|E|.$$
Definition 3.3.5. Finite subsets \( \{T_1, \ldots, T_n\} \) of \( G \) \( \varepsilon \)-quasi-tile \( G \) if

\[ e \in T_1 \subset T_2 \subset \ldots \subset T_n \]

and for any finite \( D \subset G \) there are sets of centres \( C_i \subset G, \ 1 \leq i \leq n \), such that

1. for a fixed \( i \) the sets \( \{T_i c\}_{c \in C_i} \) are \( \varepsilon \)-disjoint;

2. for \( i \neq j \) \( T_i C_i \cap T_j C_j = \emptyset \);

3. the sets \( \{T_i C_i\}_{i=1}^{n} \) \((1 - \varepsilon)\)-cover \( D \).

It is well-known that \( \varepsilon \)-quasi-tiles exist, see (Ornstein & Weiss, 1980), (Ornstein & Weiss, 1987):

Theorem 3.3.5. Given \( \varepsilon > 0 \), there is an \( N = N(\varepsilon) \in \mathbb{N} \) such that in any amenable group \( G \) for any finite \( K \subset G \), \( \delta > 0 \), and \( \delta_i, \ 1 \leq i \leq N \), there exist subsets \( \{T_1, \ldots, T_N\}, \ T_i \subset G \), such that

1. each \( T_i \) is \( (K - \delta) \)-invariant;

2. \( T_{i+1} \) is \( (T_i T_i^{-1} - \delta_i) \)-invariant;

3. the sets \( T_i \) \( \varepsilon \)-quasi-tile the group \( G \).

We are going to use a version of Rohlin’s lemma proved in (Begun & del Junco, 2007), first stated in (Ornstein & Weiss, 1980).

Theorem 3.3.6 (Rohlin lemma for amenable groups). Given \( \delta > 0 \) there are \( n \in \mathbb{N} \) and \( \eta > 0 \) such that for any discrete amenable group \( G \), any sequence \( \{T_i\}_{i=1}^{n} \) that \( \eta \)-quasi-tiles \( G \) and any free measure-preserving action of \( G \) on \( X \), there are subsets \( L_i \subset T_i \) and subsets \( \{E_i\}_{i=1}^{n}, \ E_i \subset X \), such that
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1. $|L_i| > (1 - \delta)|T_i|;$

2. $L_iE_i$ and $L_jE_j$ are disjoint for $i \neq j;$

3. the sets $\{lE_i\}_{l \in L_i}$ form a $\delta$-quasi-tower: they are $\delta$-disjoint for a fixed $i$;

4. $\mu\left(\bigcup_{i} L_i E_i\right) > 1 - \delta$.

This is the version of Rohlin’s lemma that will be used in section 3.4. Let us review its statement in detail. Collections $\{E_i\}_{l \in L_i}$ are called “quasi-towers”; each level $lE_i$ is a translate of the base $E_i$, but the levels are now not disjoint. Nonetheless, it is guaranteed that the intersections are relatively small. Part 1 simply states that $L_i$ are “large” subsets of “almost invariant” sets. Finally, by part 2 the quasi-towers are pairwise disjoint and by part 4 they cover most of the space. Theorem 3.3.5 guarantees that $\eta$ can be made as small as we please; accordingly we will assume later that $\eta$ is at least as small as $\delta$.

3.4 Main lemma

The following lemma is the main ingredient in the proof of Theorem 3.5.2. Roughly it says that for any finite set $F \subset G$ and any array $\tau$ there is an extension $\sigma$ such that for most $x \in X$ their $\sigma$-orbit (“fibre”) is $F$-closed. In the case of a single transformation this fact is usually proved by using induced transformations. If $A \subset X$ is clopen, then the induced transformation $T_A$ is continuous (see, for example, Lemma 4.3 from del Junco et al. (2009)) and it is possible to prove Lemma 7 from (del Junco & Şahin, preprint) in the standard way (see Katznelson & Weiss (1991)).
Lemma 3.4.1. Suppose that $\tau$ is an array partitioning $X$, $F \subset \mathbb{G}$ is finite and $\varepsilon > 0$ is fixed. Then there is an extension $\sigma$ of $\tau$, and a clopen set $X^*$ such that $\mu(X^*) > 1 - \varepsilon$ and $fx \in \text{Orb}_\sigma(x)$ for every $x \in X^*$ and $f \in F$.

Proof. 1 It is enough to prove the lemma for $F \subset \mathbb{G}$ such that $e \in F$ and $F = F^{-1}$.

Fix $\delta > 0$ (to be specified later) and find finitely many columns $\{\tau_j\}_{j=1}^k$ of $\tau$ that cover most of $X$: $\mu\left(X \setminus \bigcup_{j=1}^k \tau_j \right) < \delta^2$. We will call these columns good. Let $J$ be the (finite!) set of all level maps of the good columns. Find $\delta_0 < \delta$ such that

$$2\delta_0(1 + |F \cup J|) < \delta$$

and fix $\eta < \delta_0$. By Theorem 3.3.5 there exist $T_1, \ldots, T_m$ that are $(F \cup J, \eta)$-invariant and $\eta$-quasi-tile $\mathbb{G}$. Later in the proof we are only going to use the fact that every $T_i$ is $(F \cup J, \delta_0)$-invariant. Using Rohlin’s lemma (Theorem 3.3.6) we can find sets $L_i \subset T_i$, $|L_i| > (1 - \delta_0)|T_i|$ and $E_i \subset X$ such that $\{lE_i\}_{l \in L_i}$ is a $\delta_0$-quasi-tower, sets $L_iE_i$ are pairwise disjoint and $\mu\left(\bigcup_{i=1}^m L_iE_i\right) > 1 - \delta_0$.

Claim 1. The total measure of quasi-towers $L_iE_i$ that are not $(1 - \delta)$-covered by the good columns $\tau_j$ is less than $\delta$.

\[\blacktriangleleft\] Consider the part of $X$ that is covered by such quasi-towers $L_iE_i$. Say the measure of this part is $b > 0$. Then for each of those quasi-towers at least $\delta$ part of it is not covered by the good columns $\tau_j$. Thus we have found a set of measure $\delta b$ which is not covered by the good columns. Hence $\delta b < \delta^2$ and $b < \delta$. \[\blacktriangleright\]

Without loss of generality we may assume that the first $n$ quasi-towers $L_iE_i$ are $(1 - \delta)$-covered by the good columns $\{\tau_j\}$.

\[1\text{Symbols } \blacktriangleleft \text{ and } \blacktriangleright \text{ denote the proofs of the claims.}\]
Since the sets $lE_i$, $l \in L_i$, are $\delta$-disjoint there are subsets $F_{il} \subset lE_i$ such that

$$\mu F_{il} > (1-\delta)\mu E_i$$

and $F_{il}$ are pairwise disjoint. By Lemma 2.2.7 there exist disjoint clopen sets $E_{il}$, where $1 \leq i \leq n$, $l \in L_i$, such that

$$\mu(E_{il} \triangle F_{il}) < \gamma$$

$(\gamma \ll \delta \mu E_i/|L_i|)$ to be specified later, and hence

$$\mu E_{il} \geq \mu(E_{il} \cap F_{il}) > \mu F_{il} - \gamma > (1-\delta)\mu(lE_i) - \gamma > (1-2\delta)\mu E_i.$$  \hspace{1cm} (3.7)

**Claim 2.** For a sufficiently small $\gamma$ the collection $\{E_{il}\}$ covers most of the space:

$$\mu\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} E_{il}\right) > 1 - 4\delta.$$ \hspace{1cm} (3.8)

Notice that

$$\mu\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} E_{il}\right) \geq \mu\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} F_{il}\right) - \mu\left(\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} E_{il}\right) \triangle \left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} F_{il}\right)\right),$$

and therefore

$$\mu\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} E_{il}\right) \geq \mu\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} F_{il}\right) - \gamma \sum_{i=1}^{n} |L_i|,$$

and by construction

$$\mu\left(\bigcup_{i=1}^{n} \bigcup_{l \in L_i} F_{il}\right) \geq (1-\delta)\mu\left(\bigcup_{i=1}^{n} L_i E_i\right)$$

$$\geq (1-\delta)\mu\left(\bigcup_{i=1}^{n} L_i E_i\right) - \delta$$

$$> (1-\delta)^2 - \delta > 1 - 3\delta.$$  

Hence inequality (3.8) holds when $\gamma \sum_{i=1}^{n} |L_i| < \delta$. \hspace{1cm} ▶
Claim 3. Similarly to Claim 1, each “approximate-quasi-tower” \( \{E_{il}\}_{l \in L_i} \) is well covered by the good columns \( \tau_j \):

\[
\mu\left( \left( \bigcup_{l \in L_i} E_{il} \right) \cap \left( \bigcup_{j=1}^{k} |\tau_j| \right) \right) > (1 - C\delta) \mu\left( \bigcup_{l \in L_i} E_{il} \right),
\]

where \( C = 9 \).

\[\blacktriangleleft\text{ The quasi-towers } L_i E_i, 1 \leq i \leq n, \text{ are well covered by } \{\tau_j\}_{j=1}^k \text{, and } E_{il} \text{ approximate } l E_i \text{, therefore the claim follows (for some constant } C \). \text{ For the detailed computation of } C \text{ see Appendix A, page 75. } \blacktriangleright\]

The next step is to slice the approximate quasi-towers \( \{E_{il}\}_{l \in L_i} \) up into actual columns. Fix \( i \), \( 1 \leq i \leq n \), and denote the set of all levels of \( \tau \) by \( \mathcal{L}(\tau) \). We would like to single out an element of the approximate quasi-tower and treat it as if it were the “base” of the tower. Since \( L_i \) is a subset of \( T_i \) and we cannot assume that \( e \in L_i \), choose an element \( g_0 \in L_i \) and call \( B_i = g_0^{-1} E_{i,g_0} \) the “base” of \( \{E_{il}\}_{l \in L_i} \). Define \( \alpha : B_i \rightarrow \left( \mathcal{L}(\tau) \cup \{\varnothing\} \right)^{L_i} \) as

\[
\alpha(x)_g = \begin{cases} 
  L, & \text{if } gx \in L \cap E_{ig}, \ L \in \mathcal{L}(\tau); \\
  \varnothing, & \text{if } gx \notin E_{ig}.
\end{cases}
\]

For a \( \beta \in \left( \mathcal{L}(\tau) \cup \{\varnothing\} \right)^{L_i} \) set

\[ P_\beta = \{x \in B_i : \alpha(x) = \beta \}. \]

Let \( \beta' \) be the “support” of \( \beta \):

\[ \beta' = \{g \in L_i : \beta_g \neq \varnothing \}. \]
With this notation \( \beta' P_\beta = \bigcup_{g \in \beta'} g P_\beta \) becomes the “slice” of \( \{ E_{il} \}_{l \in L_i} \) over \( P_\beta \). It follows from the explicit formula for \( P_\beta \) that every set \( P_\beta \) is clopen:

\[
P_\beta = B_i \cap \left( \bigcap_{l \in \beta'} l^{-1} (\beta_l \cap E_{il}) \right) \cap \left( \bigcap_{l \in L_i \setminus \beta'} l^{-1} (X \setminus E_{il}) \right).
\]

The union of all \( P_\beta \) is the base \( B_i \), because every point \( x \in B_i \) has a certain “name” \( \alpha(x) \in \left( \mathcal{L}(\tau) \cup \{ \emptyset \} \right)^{L_i} \). Notice that \( E_{il} \) are not perfect translates of each over, hence \( \bigcup_{g \in \beta'} g P_\beta \) is only a large subset of \( \bigcup_{l \in L_i} E_{il} \). To elaborate,

\[
\mu(lB_i \triangle E_{il}) \leq \mu(lB_i \triangle lE_i) + \mu(lE_i \triangle E_{il})
\]

\[
< \mu(g_0^{-1} E_{i,g_0} \triangle E_i) + 2 \delta \mu E_i \quad \text{(see Equation (A.3), page 75)}
\]

\[
= \mu(E_{i,g_0} \triangle g_0 E_i) + 2 \delta \mu E_i
\]

\[
< 4 \delta \mu E_i.
\]

Hence

\[
\mu(lB_i \cap E_{il}) > \mu E_{il} - 4 \delta \mu E_i
\]

\[
> \mu E_{il} - 4 \delta(1 + 3 \delta) \mu E_{il} \quad \text{(see Equations (3.7) and (A.3))}
\]
\[ > (1 - 8\delta) \mu E_{il} \]

and we can conclude that

\[ \mu \left( \bigcup \beta' P_{\beta} \right) > (1 - 8\delta) \mu \left( \bigcup_{l \in L_i} E_{il} \right). \tag{3.11} \]

**Claim 4.** The base \( B_i \) is \((1 - \delta_1)\)-covered by \( P_{\beta} \) such that \( \beta' \) is relatively large:

\[ |\beta'| > (1 - \delta_1)|L_i|, \] where \( \delta_1 = \sqrt{12\delta} \).

\[ \blacktriangleright \] First of all, notice that

\[ \mu \left( \bigcup \beta' P_{\beta} \right) > (1 - 8\delta) \mu \left( \bigcup_{l \in L_i} E_{il} \right) \]

\[ > (1 - 8\delta)(1 - 2\delta)|L_i|\mu E_{i} \]

\[ > (1 - 8\delta)(1 - 2\delta)|L_i|(1 - 2\delta)\mu B_{i} \]

\[ > (1 - 12\delta)|L_i|\mu B_{i}. \tag{3.12} \]

Suppose that \( B_i \) is \( x \)-covered by relatively sparse \( P_{\beta} \) with \( |\beta'| < (1 - \delta_1)|L_i| \). That is, \( \mu(\bigcup P_{\beta}) > x \mu B_{i} \), where the union is taken over all “sparse” \( \beta \). Then

\[ \mu \left( \bigcup_{\beta} \beta' P_{\beta} \right) < |L_i|\mu B_{i} - |L_i|\delta_1 x \mu B_{i}. \]

Combine this with inequality (3.12):

\[ (1 - 12\delta)|L_i|\mu B_{i} < |L_i|\mu B_{i} - |L_i|\delta_1 x \mu B_{i}. \]

Hence \( \delta_1 x < 12\delta \) and the claim follows. \( \blacktriangleright \)

We can strengthen Claim 4 in the following way: most \( P_{\beta} \) have large \( \beta' \) and most levels \( \beta_l \) come from the good columns \( \tau_j \).
Claim 5. The base $B_i$ is $(1-\delta_2)$-covered by $P_\beta$ such that

$$\left|\{l \in \beta': \beta_l \in \bigcup_{j=1}^k |\tau_j|\}\right| > (1-\delta_2)|L_i|,$$

where $\delta_2 = \sqrt{29\delta}$.

We know that the good columns $\tau_j$ cover at least $1 - 9\delta$ of the approximate quasi-towers $\bigcup_{l \in L_i} E_{il}$ (see Claim 3). Together with (3.11) this gives us

$$\mu\left(\left(\bigcup_{l \in L_i} \beta' P_\beta\right) \cap \left(\bigcup_{j=1}^k |\tau_j|\right)\right) > (1-9\delta-8\delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right).$$

If $B_i$ is $b$-covered by $P_\beta$ such that more than $\delta_2|L_i|$ levels are missing or do not belong to the good columns $\tau_j$, we conclude that

$$(1-17\delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right) < \mu\left(\left(\bigcup_{l \in L_i} \beta' P_\beta\right) \cap \left(\bigcup_{j=1}^k |\tau_j|\right)\right) \leq |L_i|\mu B_i - \delta_2|L_i|b\mu B_i. \quad (3.13)$$

It follows from (3.12) that

$$(1-17\delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right) > (1-17\delta)\mu\left(\bigcup_{l \in L_i} \beta' P_\beta\right)$$

$$> (1-17\delta)(1-12\delta)|L_i|\mu B_i$$

$$> (1-29\delta)|L_i|\mu B_i.$$ 

Combine this inequality with (3.13):

$$(1-29\delta)|L_i|\mu B_i < |L_i|\mu B_i - \delta_2|L_i|b\mu B_i.$$ 

Therefore $\delta_2 b < 29\delta$ and the claim follows.

Fix a $\beta$ that satisfies Claim 5 (we call such $\beta$ “good”). Denote by $\hat{L}_i$ the $(F \cup J)$-interior of $L_i$. By Lemma 3.3.4

$$|\hat{L}_i| > \left(1-2\delta_0(1+|F \cup J|)|L_i| > (1-\delta)|L_i|.$$
For each \( g \in L_i \) let \( S_{\beta,g} \) be the slice of \( \tau \) over \( gP_{\beta} \). We call a slice \textbf{good} if \( g \in \hat{L}_i \cap \beta' \) and \( \beta_g \) belongs to one of the good columns \( \tau_j \). All levels of \( S_{\beta,g} \) are of the form \( j(gP_{\beta}) \) with \( j \in J \), hence \( jg \in L_i \) and the good slices consist of levels \( lP_{\beta}, l \in L_i \).

\textbf{Claim 6.} For a good \( \beta \) any two good slices \( S_{\beta,g_1} \) and \( S_{\beta,g_2} \) are either disjoint or equal.

\begin{itemize}
\item [\( \triangleright \)] Proof. If they do intersect then they share a level \( lP_{\beta} \) for some \( l \in L_i \) and they both equal the slice of \( \tau \) over \( lP_{\beta} \).
\end{itemize}

We will call a good slice \textbf{complete} if it consists of levels \( lP_{\beta} \) with \( l \in \beta' \). Since \( \beta \) is good (it satisfies Claim 5), it is guaranteed that

\[ \left| \left\{ l \in \beta' \cap \hat{L}_i : \beta_l \subset \bigcup_{j=1}^{k} |\tau_j| \right\} \right| > (1-2\delta_2-\delta)|L_i|. \tag{3.14} \]

Any element \( l \) from the the left-hand side of (3.14) can potentially give rise to a complete good slice. If the slice over \( lP_{\beta} \) is not complete it contains a level outside of \( \beta' \). By Dirichlet’s principle the total number of slices that are not complete is bounded by \((2\delta_2+\delta)|L_i|\) and they cover no more than \((2\delta_2+\delta)h|L_i|\) levels of \( L_i \).

Let \( \mathcal{S}_\beta \) denote the set of all complete good slices \( S_{\beta,g} \). The slices in \( \mathcal{S}_\beta \) cover at least \( 1-\delta_3 \) of \( L_i \), where \( \delta_3 = h(2\delta_2+\delta) \). We are now going to construct longer columns using the slices in \( \mathcal{S}_\beta \). Find the smallest \( r_i \in \mathbb{N} \) such that \( r_i h \geq (1-\delta_3)|L_i| \).

There are at least \( r_i \) slices in \( \mathcal{S}_\beta \) (otherwise we would have elements of \( L_i \) outside of those \( \delta_3|L_i| \) discussed above and not used in any of the complete good slices). Concatenate them in any order to get a column \( C_{\beta} \) of height \( r_i h \).

\textbf{Claim 7.} Columns \( C_{\beta} \) cover at least \( 1-\delta_4 \) of \( \bigcup_{l \in L_i} E_l \), where \( \delta_4 = 8\delta+\delta_2+\delta_3 \).
Notice that
\[
\mu\left(\bigcup_{\text{good } \beta} \mathcal{E}_\beta\right) > r_i h (1 - \delta_2) \mu B_i
\]
\[
\geq (1 - \delta_3)|L_i|(1 - \delta_2) \mu B_i
\]
\[
> (1 - \delta_3 - \delta_2)|L_i|\mu B_i
\]
\[
> (1 - \delta_3 - \delta_2)\mu \left(\bigcup_{\text{all } \beta} \beta' P_\beta\right),
\]
and it follows from inequality (3.11) that
\[
(1 - \delta_3 - \delta_2)\mu \left(\bigcup_{\text{all } \beta} \beta' P_\beta\right) > (1 - \delta_3 - \delta_2)(1 - 8\delta)\mu \left(\bigcup_{l \in L_i} E_{il}\right)
\]
\[
> (1 - 8\delta - \delta_2 - \delta_3)\mu \left(\bigcup_{l \in L_i} E_{il}\right).
\]
The claim follows.

There could be countably many different columns \(\mathcal{E}_\beta\). Choose finitely many of them so that they still cover \(1 - 2\delta_4\) of \(\bigcup_{l \in L_i} E_{il}\). These columns form a clopen array \(\sigma_i\).

Let \(w_i > 0\) be the width of \(\sigma_i\). The height of \(\sigma_i\) is \(r_i h\). Our goal is to construct an array using \(\sigma_i\) as building blocks, even though they have different heights. For any \(N \in \mathbb{N}\) we can write \(w_i = k_i/N + \gamma_i\), where \(k_i \in \mathbb{N} \cup \{0\}\), \(0 \leq \gamma_i < 1/N\). Take \(N \in \mathbb{N}\) so large that \(\frac{1}{N} \sum_{i=1}^{n} w_i (r_i h) < \delta\). By Lemma 3.2.1 there exist \(k_i\) subarrays of \(\sigma_i\) of width \(1/N\). By Lemma 3.2.2 there is a stacking \(\mathcal{S}\) of these \(k_1 + \cdots + k_n\) arrays. Finally, again by Lemma 3.2.2 there exist an extension of the part of \(\tau\) supported by \(X \setminus |\mathcal{S}|\) of the same height as \(\mathcal{S}\). By adjoining this extension to \(\mathcal{S}\) we obtain an array \(\sigma\). Let
\[
X^*_i = \left(\bigcup_{\mathcal{E}_\beta \subseteq \mathcal{S}} \left(\bigcap_{f \in F} f^{-1}|\mathcal{E}_\beta|\right)\right) \cap \left(\bigcup_{l \in L_i} E_{il}\right), \quad X^* = \bigcup_{i=1}^{n} X^*_i.
\]
**Claim 8.** The measure of $X^*$ is greater than $1 - \delta_5$, where $\delta_5(\delta) \to 0$ as $\delta \to 0$.

Notice that every column $C_\beta = \{lP_\beta\}_{l \in \Lambda_\beta}$ is indexed by some set $\Lambda_\beta$ with $|\Lambda_\beta|$ greater than $(1 - \delta_5)|L_i|$. Hence by Lemma 3.3.4 $\Lambda_\beta$ is $(F, \rho)$-invariant, where $\rho = 2\delta_3(\delta + 1)$. Therefore

$$\left|\{l \in \Lambda_\beta : Fl \subset \Lambda_\beta\}\right| > (1 - \rho)|\Lambda_\beta|.$$  

This inequality implies that for any $f \in F$ the intersection $\Lambda_\beta \cap f\Lambda_\beta$ contains at least $(1 - \rho)|\Lambda_\beta|$ elements. Hence

$$\left|\bigcap_{f \in F} f^{-1}\Lambda_\beta\right| > (1 - \rho|F|)|\Lambda_\beta|.$$  

By construction columns $C_\beta$ included in $\mathcal{C}$ cover at least $(1 - 2\delta_4 - \delta)$ of $\bigcup_{l \in L_i} E_{il}$, $1 \leq i \leq n$. Moreover, since $|\hat{L}_i| > (1 - \delta)|L_i|$, the second union in the definition of $X_i$, $\bigcup_{l \in L_i} E_{il}$ covers at least $1 - \delta$ of $\bigcup_{l \in L_i} E_{il}$. Hence

$$\mu X_i^* > \left((1 - 2\delta_4 - \delta)(1 - \rho|F|) - \delta\right)\mu\left(\bigcup_{l \in L_i} E_{il}\right), \quad 1 \leq i \leq n.$$  

The claim now follows and we assume that $\delta$ was chosen so that $\delta_5 < \varepsilon$.  

Notice that by construction every $X_i$ is clopen. To finish the proof, consider a point $x \in X^*$. Then $x \in C_\beta \subset |\sigma_i|$ for some $i$. It follows from the definition of $X_i$ that $x \in \bigcap_{f \in F} f^{-1}C_\beta$. Hence $x \in f^{-1}C_\beta$ and $f x \in C_\beta$. Therefore $f x \in C_\beta$ for all $f \in F$. In other words, $Fx \subset |C_\beta|$. Since also $x \in \bigcup_{g \in L_i} E_{ig}$, we conclude that $x \in E_{ig}$ for some $g \in \hat{L}_i$ and $Fg \subset L_i$. This guarantees that $f x$ belongs to $E_{i(fg)}$ level of the approximate Rohlin quasi-tower and hence to the same fibre of $\sigma_i$. Therefore $x$ and $f x$ are in the same orbit of $\sigma$.  

$\square$
3.5 Proof of the main theorem

We are now ready to prove that any continuous ergodic measure-preserving action of a discrete amenable group is almost continuously orbit equivalent to a single transformation. Combining this result with (del Junco & Şahin, preprint, Theorem 1) we show that any two such actions are almost continuously orbit equivalent.

Theorem 3.5.1. Suppose that \((X, \mu)\) is a zero-dimensional Polish space with a non-atomic Borel probability measure \(\mu\) and an action of a discrete amenable group \(G\) on \(X\) by homeomorphisms is free, ergodic and measure-preserving. Then there is a \(G\)-invariant \(G_\delta\) subset \(X' \subset X\) of full measure and a measure-preserving homeomorphism \(S: X' \to X'\), \(S \in [G]\text{top}\), such that \(\text{Orb}_G(x) = \text{Orb}_S(x)\) for all \(x \in X'\).

Proof. Take \(\varepsilon_n = 2^{-n}\) and find finite \(F_n \subset G\) such that

\[ e \in F_n, \quad F_n^{-1} = F_n, \quad F_n \subset F_{n+1}, \quad \bigcup_{n \in \mathbb{N}} F_n = G. \]

By Lemma 3.4.1 there exist clopen sets \(X^*_n\) and a refining sequence of arrays \(\tau_n\) such that \(\mu(X^*_n) > 1 - \varepsilon_n\) and \(fx \in \text{Orb}_{\tau_n}(x)\) for every \(f \in F_n\) and \(x \in X^*_n\). Lemma 3.2.3 allows us to require \(\text{diam}(\tau_n) < \varepsilon_n\). Another condition we would like to impose is that \(\text{width}(\tau_n) < 2^{-n}\). This can be done by Lemma 3.2.2. Let \(\tau^*_n\) be the array \(\tau_n\) except for the first \(n\) levels from the bottom and the last \(n\) levels on the top. Recall how \(\tau_n(x)\) is defined: it is the image of \(x\) under the column map from the level containing \(x\) to the next level (it is not defined on the very top level). Since the order of levels between \(\tau_n\) and \(\tau_{n+1}\) is preserved, we can define

\[ Sx = \lim_{n \to \infty} \tau_n(x). \]
Consider the set
\[ X_0 = \bigcap_{N \in \mathbb{N}} \bigcup_{i > N} |\tau_i^*|. \]
As an intersection of open sets, \( X_0 \) is \( G_\delta \). Since the measure of \( |\tau_n| \) is at least \( 1 - (2n)2^{-n} \) we conclude that \( \mu X_0 = 1 \). If \( x \in X_0 \) then it belongs to the union \( \bigcup_{i > N} |\tau_i^*| \) for all \( N \) and hence \( S^n x \) is defined for all \( n \in \mathbb{Z} \). Let
\[ X^* = \left( \bigcap_{n \in \mathbb{N}} \bigcup_{i > n} X_i^* \right) \cap X_0 \]
and \( X' = \bigcap_{g \in G} g X^* \). By construction \( X' \) is a \( G \)-invariant \( G_\delta \)-subset of full measure. Furthermore, the transformation \( S \) is defined everywhere on \( X' \) and by definition \( \text{Orb}_S(x) \subset \text{Orb}_G(x) \). On the other hand, since \( X_0 \) is \( G \)-invariant, for every \( x \in X_0 \) and \( f \in F \) there is an \( n \in \mathbb{N} \) such that \( fx \in \text{Orb}_{\tau_n}(x) \), so that \( x \) and \( fx \) belong to the same \( S \)-orbit, \( fx \in \text{Orb}_S(x) \).

The levels of \( \tau_n^* \) form a basis for topology of \( X' \) because each \( x \in X_0 \) belongs to a level of \( \tau_n \) with diameter less then \( \varepsilon_n \to 0 \). The transformation \( S \) (\( S^{-1} \)) shifts levels from the basis one level up (down) and is therefore continuous.

Let us now show that the associated cocycles are continuous on \( X' \), which implies that \( g \in [S]_{\text{top}} \) for every \( g \in G \) and \( S \in [G]_{\text{top}} \). Suppose that \( gx = S^n g(x) \) and \( Sx = C_S(x) \cdot x \), where \( n_g : X' \to \mathbb{Z} \) and \( C_S : X' \to G \) are the cocycles.

Since the ranges of the cocycles are discrete it is enough to show that \( n_g^{-1}(k) \) and \( C_S^{-1}(g) \) are open for every \( k \in \mathbb{Z} \) and \( g \in G \). Let
\[ E_g = C_S^{-1}(g) = \{ x \in X' : Sx = gx \}. \]

Recall how \( S \) was constructed. It is the limit of array maps \( \tau_n \), where \( \tau_n(x) \) is the image of \( x \) under the level map \( g_{i,i+1} \) of the column \( C \) that contains \( x \) in the
i-th level. Denote by $\mathcal{L}_{n,g}$ the set of all levels $L_j$ of $\tau_n$ such that the level map to the next level equals $g$, $g_{j,j+1} = g$. Then $E_g$ is simply the union of all levels from $\mathcal{L}_{n,g}$:

$$E_g = \bigcup_{n=1}^{+\infty} \bigcup_{L \in \mathcal{L}_{n,g}} L.$$ 

Hence $E_g$ is open in $X'$, and $C_S$ is continuous on $X'$.

Let us show that $n_g$ is also continuous. By construction for every $x \in X'$ the point $gx$ belongs to the $\text{Orb}_{\tau_n}(x)$ for large enough $n$. In other words, $x$ and $gx$ belong to the same fibre of some column of $\tau_n$. Since by definition the column maps are elements of $G$, $g$ actually maps a whole level $L$ of $\tau_n$ to some other level in the same column. What is $n_g(x)$ then? For points $x \in L$ it is simply the number of levels that separate $L$ and $gL$, positive if $gL$ is higher than $L$ and negative otherwise. Let

$$E_k = n_g^{-1}(k) = \{x \in X': gx = S_k^k x\}.$$ 

Then $E_k$ is the union over all $n$ of the levels $L_j$ of $\tau_n$ such that the level map $g_{j,j+k}$ from $L_j$ to the level $k$ floors higher (or lower, when $k$ is negative) is $g$. Since all levels of $\tau_n$ are clopen $E_k$ is an open set and $n_g$ is continuous.

**Remark 3.5.1.** In the proof of the main theorem in (del Junco & Şahin, preprint) two refining sequences of arrays on $X$ and $Y$ are constructed. Just like in the proof of Theorem 3.5.1 these arrays give rise to measure-preserving homeomorphisms $T_1$ and $S_1$ orbit equivalent to $T$ and $S$ respectively and such that $\varphi : X' \to Y'$ is an isomorphism between $T_1$ and $S_1$. Repeating the same steps as in the proof of Theorem 3.5.1 we can show that the associated cocycles $n$ and $m$ are continuous,
where \( T_x = T_1^{n(x)}x \) and \( S_y = S_1^{m(y)}y \). Note that since \( \varphi^{-1}S_1\varphi = T_1 \) we have

\[
\varphi^{-1} \circ S_1^m \circ \varphi(x) = T_1^{m(\varphi(x))}x.
\]

Thus \( \varphi^{-1}h\varphi \) belongs to the topological full group \([T_1]_{top} = [T]_{top}\) for all \( h \in [S]_{top} \).

This remark together with Theorem 1.1.5 gives us the following theorem.

**Theorem 3.5.2.** Let \((X, \mu)\) and \((Y, \nu)\) be Polish spaces with non-atomic Borel probability measures. Suppose that countable discrete amenable groups \( G_1 \) and \( G_2 \) act freely and ergodically on \( X \) and \( Y \) respectively by measure-preserving homeomorphisms. Then \( G_1 \) and \( G_2 \) are almost continuously orbit equivalent.

**Remark 3.5.2.** It is possible that the same method can handle the case of non-free actions, see (Ornstein & Weiss, 1980, page 163).

### 3.6 Open problems

We conclude this chapter with a list of open problems.

1. Is it possible to extend Theorem 3.5.2 to non-singular étale ergodic amenable equivalence relations?

2. If not, can one extend Theorem 3.5.2 to non-singular transformations or to the case of infinite measures?

3. All orbit equivalences that we have constructed have continuous cocycles. Is there a continuous orbit equivalence \( \varphi \) such that the cocycles are not virtually continuous? In other words, there are no virtually full subsets \( X_1 \) and \( Y_1 \) such that all cocycles associated with \( \varphi: X_1 \to Y_1 \) are continuous.
Chapter 4

The reconstruction theorem

4.1 Introduction

In this chapter we prove another theorem of H. Dye in the almost continuous setting. In 1963 H. Dye proved that the full groups of ergodic group actions are isomorphic if, and only if, the groups are orbit equivalent and every such isomorphism $f$ of full groups arises in the following way:

$$f(T) = \varphi \circ T \circ \varphi^{-1} \quad \text{for all } T \in [G_1],$$

(4.1)

where $\varphi$ is an orbit equivalence between $G_1$ and $G_2$. Note that this theorem also holds for equivalence relations (with only minor changes in the proof). We are going to prove an analogue of this result for étale equivalence relations:

**Theorem 4.1.1.** Suppose that $(X, \mu)$ and $(Y, \nu)$ are Polish spaces with non-atomic Borel probability measures, $\mathcal{R}_1$ and $\mathcal{R}_2$ are measure-preserving ergodic étale equivalence relations on $X$ and $Y$ respectively. If the topological full groups $[\mathcal{R}_1]_{\text{top}}$ and
Chapter 4. The reconstruction theorem

\[ \mathcal{R}_2 \text{ are algebraically isomorphic, then } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ are almost continuously orbit equivalent. Moreover, for any algebraic isomorphism of the topological full groups } f: [\mathcal{R}_1]_{\text{top}} \to [\mathcal{R}_2]_{\text{top}} \text{ there is a unique almost continuous orbit equivalence } \varphi \text{ such that } f(T) = \varphi T \varphi^{-1} \text{ for any } T \in [\mathcal{R}_1]_{\text{top}}. \]

The structure of the proof follows the pattern of the measure-theoretic case in the exposition by A. Kechris, see (Kechris, preprint). In Section 4.2 we prove that every element of the topological full group can be written as a product of ten involutions and as a product of five commutators. Then we obtain an analog of Eigen's theorem: we show that the topological full group \([\mathcal{R}_1]_{\text{top}}\) is simple.

Section 4.4 studies in detail properties of an algebraic isomorphism \(f\) between the topological full groups. In Section 4.5 we define a set mapping \(\Phi\) and prove that it is induced by a point mapping \(\varphi\). Lemma 4.5.3 shows that \(f\) equals the action of \(\varphi\) by conjugation.

Remark 4.1.1. Everywhere in this chapter \((X, \mu)\) and \((Y, \nu)\) are zero-dimensional Polish spaces with non-atomic Borel probability measures such that \(\text{supp } \mu = X\), \(\text{supp } \nu = Y\). \(\mathcal{R}, \mathcal{R}_1\) and \(\mathcal{R}_2\) are étale ergodic measure-preserving equivalence relations. Similarly to Chapter 3 we work in the almost continuous category, see remark 3.1.1 on page 30.

4.2 Generators of the full group

In this section we show that the topological full group is generated by involutions, and also by commutators (elements of the form \(h = [f, g]\) for some \(f, g \in [\mathcal{R}]_{\text{top}}\)). This is done by considering periodic transformations first. In Lemma 4.2.3 we show
that any periodic $T \in \mathcal{R}_{\text{top}}$ can be written as a product of two involutions or as a commutator. We then show that any aperiodic transformation $T \in \mathcal{R}_{\text{top}}$ can be factored as $T = P \cdot S$, where $P \in \mathcal{R}_{\text{top}}$ is periodic and $S$ has small support. Theorem 4.2.6 wraps it all up by showing that any function $T \in \mathcal{R}_{\text{top}}$ is a product of five commutators (or ten involutions).

**Lemma 4.2.1.** For any map $T \in \mathcal{R}_{\text{top}}$ and number $n \in \mathbb{N}$ the set

$$P_n = \{ x : \text{Orb}_T(x) = n \}$$

is (virtually) clopen.

**Proof.** We start by showing that $P_1$ is clopen. By the definition of the topological full group $\mathcal{R}_{\text{top}}$ there is a clopen partition $\{X_i\}_{i \in \mathbb{N}}$ of $X$ such that $T$ equals $r \circ s^{-1}$ on each $X_i$, and $T|_{X_i}$ is a homeomorphism. Recall that by definition of étaleness the diagonal $\Delta = \{(x, x)\}$ is clopen. Then the intersection $P_1 \cap X_i$ equals $r(s^{-1}(X_i) \cap \Delta)$, hence it is clopen. By Lemma 2.2.4 the union $\bigcup_{i \in \mathbb{N}} (P_1 \cap X_i) = P_1$ is again clopen.

Suppose now that every set $P_i$, $i < n$, is clopen. Since we already know that the set of fixed points of any transformation from $\mathcal{R}_{\text{top}}$ is clopen, the set

$$P_1(T^n) = P_n^* = \{ x \in \text{supp} T : T^n(x) = x \}$$

is clopen. It follows then from the formula

$$P_n = P_n^* \cap \left( \bigcup_{i=1}^{n-1} P_i \right)^c$$

that $P_n$ is also clopen. \qed
Lemma 4.2.2. Suppose that $H$ is a finite group of homeomorphisms such that for every $h \in H$ the fixed point subset $\text{fix}(h) \subset X$ is clopen. Then there exists a clopen set $E$ intersecting each $H$-orbit in exactly one point.

Proof. There exists a Borel set $D$ that intersects every $H$-orbit in exactly one point, see (Kechris & Miller, 2004, Proposition 6.4). Construct a sequence of disjoint clopen sets $E_k$ in the following way: approximate the base $D$ with a clopen set $F_1$ and disjointify the images: let $E_1 = F_1 \setminus \bigcup_{h \in H} h(\text{supp}(h) \cap F_1)$. Then $E_1$ has disjoint images that cover most of $X$. If the part of $X$ not covered by the $\bigcup_{h \in H} hE_1$ has positive measure, continue in the same way. If $E_n$ has been constructed approximate the part of $D$ which is not covered by $\bigcup_{h \in H} h\left( \bigcup_{i \leq n} E_i \right)$ with a clopen set $F_{n+1}$ disjoint from every $E_i$. Disjointify the translates of $F_{n+1}$ to obtain $E_{n+1}$. Let $E = \bigcup_{i \in \mathbb{N}} E_i$. By construction $E$ is open and intersects each orbit exactly once. Since we can guarantee that the part of $X$ not covered after the nth step has measure less than $2^{-n}$ the translates of $E$ cover $X$. $\Box$

Remark 4.2.1. See also (Dahl, 2008, Lemma 3.2) and (Krieger, 1979/80).

It follows from Lemma 4.2.2 that if $T \in [\mathcal{R}]_{\text{top}}$ has period $n$ for every $x \in X$, then there exists a partition of $X$ into $n$ (virtually) clopen sets $X_i$, $0 \leq i \leq n-1$, such that $T(X_i) = X_{i+1}$, $i < n-1$, and $T(X_{n-1}) = X_0$. For example, if $T \in [\mathcal{R}]_{\text{top}}$ is an involution with full support, then there exist two disjoint clopen sets $X_1$ and $X_2$ such that $T(X_1) = X_2$, $T(X_2) = X_1$ and $X_1 \cup X_2 = X$.

Lemma 4.2.3. Suppose that $T \in [\mathcal{R}]_{\text{top}}$, every $T$-orbit is finite, and there is a number $N \in \mathbb{N}$ such that for all $x \in X$ $|\text{Orb}_T(x)| \leq N$. Then $T$ is a commutator in $[\mathcal{R}]_{\text{top}}$ and a product of two involutions from $[\mathcal{R}]_{\text{top}}$. 
Proof. By Lemma 4.2.1 sets $P_n = \{ x : |\text{Orb}_T(x)| = n \}$ are clopen, and there are only finitely many non-empty $P_n$. For simplicity we can assume that $T$ is periodic with period $n$ for all $x$.

Suppose that $A$ is a clopen set intersecting every $T$-orbit once (such a set $A$ exists by Lemma 4.2.2). Split $A$ into two (virtually) clopen sets of equal measure, $A = A_1 \cup A_2$. Let

$$T_1 = \begin{cases} T, & \text{on } B_1 = \bigcup_{i=0}^{n-1} T^i(A_1); \\ id, & \text{on } B_2 = \bigcup_{i=0}^{n-1} T^i(A_2) \end{cases}$$

and

$$T_2 = \begin{cases} id, & \text{on } B_1; \\ T, & \text{on } B_2. \end{cases}$$

Suppose that we have found an involution $Q \in [\mathcal{R}]_{\text{top}}$ that maps $B_1$ onto $B_2$ and such that $T_1 = QT_2^{-1}Q$. Then $T = T_1T_2 = (QT_2^{-1}Q^{-1})T_2 = [Q, T_2^{-1}]$, and we have shown that $T$ is a commutator. On the other hand, $T$ is a product of two involutions $Q$ and $T_2^{-1}Q^{-1}T_2$, and the lemma follows.

Let us now show that such an involution $Q$ exists. By Lemma 2.5.4 there is an involution $Q_0$ mapping $A_1$ onto $T^{n-1}(A_2)$. Let

$$Q(x) = \begin{cases} (T^{-k}Q_0T^{-k})x, & \text{for } x \in T^k(A_1); \\ (T^kQ_0T^k)x, & \text{for } x \in T^{-k}(T^{n-1}(A_2)), \end{cases} \quad (4.2)$$

so that $Q$ maps $T^kA_1$ onto $T^{-k}(T^{n-1}A_2)$. By construction $Q$ is an involution from $[\mathcal{R}]_{\text{top}}$. Then for any $x \in T^k(A_1)$ we have

$$QT_2^{-1}Qx = QT_2^{-1}(T^{-k}Q_0T^{-k}x)$$

$$= Q(T^{-k-1}Q_0T^{-k}x)$$

$$= T^{k+1}Q_0T^{k+1}(T^{-k-1}Q_0T^{-k}x) \quad \text{(by equation (4.2))}$$

$$= Tx.$$
and hence $QT_1Q = T_2^{-1}$.

\[ \text{Remark 4.2.2.} \] It follows from Lemma 2.2.4 that Lemma 4.2.3 also holds for any transformation $T \in [\mathcal{R}]_{\text{top}}$ with finite orbits.

\[ \text{Lemma 4.2.4.} \] For any $T \in [\mathcal{R}]_{\text{top}}$ and $\delta > 0$ there are $S_1, T_1 \in [\mathcal{R}]_{\text{top}}$ such that $T = S_1T_1$, $S_1$ is periodic, and $\mu(\text{supp} T_1) < \delta$.

\[ \text{Proof.} \] Let $P_n = \{ x : \text{Orb}_T(x) = n \}$, by Lemma 4.2.1 every $P_n$ is clopen. If finite orbits cover most of the space, $\sum_{n \geq 2} \mu P_n > \mu(\text{supp} T) - \delta$, then there is a number $N \in \mathbb{N}$ such that $\sum_{n=2}^{N} \mu P_n > \mu(\text{supp} T) - \delta$. We can then apply Lemma 4.2.3 on $T|\bigcup_{n=2}^{N} P_n$ to construct $S_1$ and let $T_1$ be $T$ on the rest of the space.

Suppose now that periodic points do not cover enough of the support,

$$\sum_{n \geq 2} \mu P_n \leq \mu(\text{supp} T) - \delta.$$ 

Fix $\varepsilon > 0$, $\varepsilon < \delta/5$, and find a large number $N \in \mathbb{N}$ such that $\mu\left(\bigcup_{n \geq N} P_n\right) < \varepsilon$ and $1/N < \varepsilon$. Let $P_{\infty}$ be the set (not necessarily clopen) of all infinite orbits,

$$P_{\infty} = \{ x : \text{Orb}_T(x) = \infty \}, \quad \mu P_{\infty} > 0.$$ 

Since $T|_{P_{\infty}}$ is aperiodic by Rohlin’s lemma there is a set $A \subset P_{\infty}$ such that the translates $T^kA$, $0 \leq k \leq N$, are disjoint and $(1-\varepsilon)$-cover $P_{\infty}$. Approximate $A$ with a clopen set $B_0$ such that $\mu(A \triangle B_0) < \varepsilon/(1+N)^2$ and let $B = B_0 \setminus \bigcup_{i=1}^{N} T^i(B_0)$.

By construction $\mu B > \mu B_0 - \varepsilon/(N+1)$ and $B$, $T(B)$, $\ldots$, $T^N(B)$ are all disjoint. Hence

$$\mu\left(\bigcup_{i=0}^{N} T^i(B)\right) > (N+1)\mu B_0 - \varepsilon > (N+1)\mu A - 2\varepsilon > \mu P_{\infty} - 3\varepsilon.$$
The constructed Rohlin tower over \( B \) is disjoint from all \( P_k, \ k \leq N \). Now let

\[
S_1 = \begin{cases} 
  T, & \text{on } \left( \bigcup_{k=1}^{N} P_k \right) \cup \left( \bigcup_{i=0}^{N-1} T^i(B) \right) \\
  T^{-N}, & \text{on } T^N(B), \\
  \text{id}, & \text{on the rest of the space.}
\end{cases}
\]

Define \( T_1 = S_1^{-1}T \). It follows from the definition of \( S_1 \) that

\[
\text{supp } T_1 \subset \left( \left( \bigcup_{n \geq N} P_n \right) \bigcup P_{\infty} \right) \setminus \bigcup_{i=0}^{N} T^i(B) \bigcup T^N(B)
\]

and \( \mu(\text{supp } T_1) < 5\varepsilon < \delta \). By construction \( S_1 \) and \( T_1 \) belong to \([R]_{\text{top}}\) and for all \( x \) we have \( |\text{Orb}_{S_1}(x)| \leq N + 1 \). Since \( S_1 \) is periodic by Lemma 4.2.3 it is a commutator and a product of two involutions.

**Lemma 4.2.5.** Let \( A \) and \( B \) be disjoint clopen subsets of positive measure. If \( T \in [R]_{\text{top}} \) and \( \text{supp } T \subset A \) then there exist \( S_1, T_1 \in [R]_{\text{top}} \) such that \( T = S_1T_1 \), where \( S_1 \) is a product of two commutators (four involutions) and

\[
\text{supp } S_1 \subset A \cup B, \quad \text{supp } T_1 \subset B.
\]

**Proof.** Apply Lemma 4.2.4 to factor \( T \) as \( T = PQ \), \( \mu(\text{supp } Q) < \mu B \), where \( P \) is periodic and hence is a commutator and a product of two involutions. Find an involution \( U \in [R]_{\text{top}} \) with \( \text{supp } U \subset A \cup B \) such that \( U(\text{supp } Q) \subset B \). Let \( T_1 = UQU \), \( \text{supp}(T_1) \subset B \). Then

\[
T = PUT_1U = P(U(T_1U^{-1}T_1^{-1}))T_1 = P[U,T_1]T_1 = S_1T_1,
\]

where \( S_1 = P[U,T_1] \) is a product of four involutions (two commutators). \( \square \)
Chapter 4. The reconstruction theorem

**Theorem 4.2.6.** Every element of $[\mathcal{R}]_{\text{top}}$ is a product of five commutators (ten involutions) from $[\mathcal{R}]_{\text{top}}$.

**Proof.** Fix an element of the full group $T \in [\mathcal{R}]_{\text{top}}$. By Lemma 4.2.4 there exist transformations $S_1, T_1 \in [\mathcal{R}]_{\text{top}}$ such that $T = S_1 T_1$, where $S_1$ is a commutator and the product of two involutions, and $\mu(\text{supp } T_1) < 1/2$. Find a sequence of disjoint (virtually) clopen sets $X_i \subset X$ with $\mu(X_i) = 2^{-i}$ and such that $X_1$ contains $\text{supp } T_1$. By Lemma 4.2.5 there are $T_2$ and $S_2 \in [\mathcal{R}]_{\text{top}}$ such that $T_1 = S_2 T_2$, where $S_2$ is the product of two commutators (four involutions) supported by $X_1 \cup X_2$ and $T_2$ is supported by $X_2$. Continue in the same way, so that on the $n$-th step we construct $S_{n+1}$ and $T_{n+1} \in [\mathcal{R}]_{\text{top}}$ such that $T_n = S_{n+1} T_{n+1}$, where $S_{n+1}$ is the product of two commutators (four involutions) supported by $X_n \cup X_{n+1}$ and $T_{n+1}$ is supported by $X_{n+1}$. Let $P_1 = \prod_{\text{even } i} S_i$ and $P_2 = \prod_{i > 1, \text{odd } i} S_i$. Then

$$P_1 = S_2 S_4 S_6 \ldots = (T_1 T_2^{-1}) (T_3 T_4^{-1}) (T_5 T_6^{-1}) \ldots = (T_1 T_3 T_5 \ldots) (T_2^{-1} T_4^{-1} T_6^{-1} \ldots),$$

$$P_2 = S_3 S_5 S_7 \ldots = (T_2 T_3^{-1}) (T_4 T_5^{-1}) (T_6 T_7^{-1}) \ldots = (T_2 T_4 T_6 \ldots) (T_3^{-1} T_5^{-1} T_7^{-1} \ldots),$$

hence $T_1 = P_1 P_2$ and $T = S_1 P_1 P_2$. Since each $P_i$ is the product of $S_j$ with disjoint support Lemma 2.5.3 guarantees that $P_1, P_2 \in [\mathcal{R}]_{\text{top}}$ and since each $S_j$ is the product of two commutators (four involutions), the same is true for $P_i$, $i = 1, 2$. Hence $T$ is the product of five commutators (ten involutions).

**Remark 4.2.3.** In the measure-theoretic setting and in the topological setting it is possible to show that every transformation $T$ from the full group is the product of three involutions, see (Ryzhikov, 1993) and (Bezuglyi & Medynets, 2008, Theorem 4.14).
4.3 Simplicity of the topological full group

Definition 4.3.1. Suppose that $H$ is a subset of a group $G$. The commutator $[H,H]$ of $H$ is the subgroup generated by the commutators $[h_1,h_2]=h_1h_2h_1^{-1}h_2^{-1}$. Every element of $[H,H]$ is a finite product of commutators.

Recall that a subgroup $N \subset G$ is called normal if for any $n \in N$ and $g \in G$ the element $gng^{-1}$ belongs to $N$.

Lemma 4.3.1. Suppose that $G$ is a group and $N$ is a normal subgroup, $N \triangleleft G$. If a subset $H \subset G$ generates $G$ and $[H,H] \subset N$, then also $[G,G] \subset N$.

Proof. Let $\pi : G \to G/N$ be the canonical projection. The inclusion $[H,H] \subset N$ means that $ab \in baN$ for all $a, b \in H$. Hence $\pi(a)\pi(b) = \pi(b)\pi(a)$ and $\pi(H)$ is abelian. It follows that $\pi(G)$ is also abelian since $H$ generates $G$. Therefore for any $a, b \in G$ we have $\pi([a,b]) = \text{id}$ and $[G,G] \subset N$. \qed

Lemma 4.3.2. For every $\varepsilon > 0$ the set

$$\{ T \in [\mathcal{R}]_{\text{top}} : \mu(\text{supp } T) < \varepsilon \}$$

generates $[\mathcal{R}]_{\text{top}}$.

Proof. Recall that $[\mathcal{R}]_{\text{top}}$ is generated by involutions. Any involution can be broken down into a composition of involutions with small disjoint support. Hence this lemma holds even with the additional requirement that $T^2 = \text{id}$. \qed

We are now ready to prove that $[\mathcal{R}]_{\text{top}}$ is simple. The measure-theoretic analogue of this theorem first appeared in (Eigen, 1981), where both measure-preserving and non-singular cases were considered. In the topological setting the commutator of the topological full group is simple, see (Bezuglyi & Medynets, 2008).
Theorem 4.3.3. The group \([-R]_{\text{top}}\) is simple.

Proof. Suppose that \(N \triangleleft \([-R]_{\text{top}}\), \(N \neq \{\text{id}\}\). We will show that \(N = \([-R]_{\text{top}}\). Fix a non-trivial transformation \(Q \in N\). Find a clopen set \(A\) of positive measure such that \(Q(A) \cap A = \emptyset\). Let \(\varepsilon = \frac{1}{2} \mu A\) and

\[
H = \{ S \in \([-R]_{\text{top}}: \mu(\text{supp } S) < \varepsilon \}.
\]

By Lemma 4.3.2 \(H\) generates \([-R]_{\text{top}}\). Fix any two mappings \(S, T \in H\). Find an involution \(U \in \([-R]_{\text{top}}\) that maps \(\text{supp } T \cup \text{supp } S\) into \(A\) and denote by \(T_U\) and \(S_U\) the conjugates of \(T\) and \(S\) by \(U\),

\[
T_U = UTU, \quad S_U = UTU.
\]

Then \(\text{supp } T_U \subset A\) and \(\text{supp } S_U \subset A\). Notice that \(S_U^{-1}\) and \(Q(T_U^{-1})Q^{-1}\) have disjoint supports and hence commute. The commutator \([Q, T_U]\) belongs to \(N\), since \(Q(T_U Q^{-1} T_U^{-1}) \in N\). Denote by \(\pi\) the canonical projection \(G \to G/N\). We have

\[
\pi(S_U T_U) = \pi(S_U (Q T_U Q^{-1})) = \pi((Q T_U Q^{-1}) S_U) = \pi(T_U S_U)
\]

and \([S_U, T_U] \in N\). Therefore the commutator \([S, T]\) also belongs to \(N\), since \([S, T]\) is the conjugate of \([S_U, T_U]\) by \(U\). Hence by Lemma 4.3.1 the commutator \([S, T] \in N\) for any two elements \(S, T \in \([-R]_{\text{top}}\). The group \([-R]_{\text{top}}\) is generated by commutators, thus \(N = \([-R]_{\text{top}}\). \qed

4.4 Properties of \(f: \([-R_1]_{\text{top}} \to \([-R_2]_{\text{top}}\)

In this section we shall consider an algebraic isomorphism \(f\) of the topological full groups \([-R_1]_{\text{top}}\) and \([-R_2]_{\text{top}}\). The main goal will be to show that if \(T, S \in \([-R_1]_{\text{top}}\)
have disjoint supports, then the supports of \( f(T) \) and \( f(S) \) are also disjoint. We will start with some preparatory lemmas.

Suppose that a topological group \( G \) acts on \( X \) and the action \( G \times X \rightarrow X \) is a continuous map. It is well-known the factor space \( X/G \) does not exhibit any pathological properties if the group \( G \) is compact (Munkres, 2000, page 199): it is still Hausdorff (regular, or normal, or second-countable) if \( X \) is. Since \( X/G \) is Hausdorff, regular and second-countable, by Urysohn’s lemma it is separable and metrizable. We prove that if a subgroup \( H \) of the topological full group is finite (compact in the discrete topology), then \( R/H \) is again étale, where

\[
R/H = \left\{ ([x]_H, [y]_H) \in X/H \times X/H : (x, y) \in R \right\}
\]

and the topology on \( R/H \) is transferred from \( R \) via the projection \( \pi : R \rightarrow R/H \).

We also show that if \( S \in [R]_{top} \) commutes with every element of \( H \), then it maps \( H \)-orbits to \( H \)-orbits and induces a mapping \( S^* \) on the factor space that also belongs to the topological full group of the corresponding equivalence relation.

**Lemma 4.4.1.** Suppose that \( R \) is a measure-preserving étale equivalence relation and \( H \subset [R]_{top} \) is a finite subgroup of the topological full group. Then \( R/H \) is again étale and measure-preserving as an equivalence relation on \( \left(X/H, \pi(\mu)\right) \).

If \( R \) is ergodic, then \( R/H \) is \( \pi(\mu) \)-ergodic, where \( \pi : X \rightarrow X/H \) is the natural projection on the factor space. Any element \( S \in [R]_{top} \) of the normalizer of \( H \) induces a function \( S^* \in \left[R/H\right]_{top} \).

**Proof.** By Lemma 4.2.2 there is a clopen set \( E \) intersecting every \( H \)-orbit once; the factor space \( X/H \) is (virtually) homeomorphic to \( E \). By Theorem 2.4.3 \( R|_E \) is étale, \( R/H \), which is isomorphic to \( R|_E \), is also étale. If \( R \) is ergodic, but \( R/H \)
is not, then there is an $\mathcal{R}/H$-saturated subset $A \subseteq E$, and $\bigcup_{h \in H} hA$ is a non-trivial $\mathcal{R}$-saturated subset, contradiction.

Consider any $S \in [\mathcal{R}]_{\text{top}}$ from the normalizer of $H$. We have $SH = HS$. Hence

$$S(\text{Orb}_H(x)) = \text{Orb}_H(S(x)).$$

Since $S$ maps $H$-orbits onto $H$-orbits we can define a function $S^*: X/H \to X/H$ : if $\pi: X \to X/H$ is the natural projection which sends $x$ to its equivalence class, then $S^*(\pi(x)) = \pi(S(x))$. On every level of $P_n = \{x: |\text{Orb}_H(x)| = n\}$ the projection $\pi$ equals some $h \in H$ and $S^*$ can be viewed as the composition of $S$ with $h$, hence $S^* \in [X/H]_{\text{top}}$.

**Lemma 4.4.2.** If $T \in [\mathcal{R}]_{\text{top}}$ is an involution, then $[T]_{\text{top}}$ is the largest abelian normal subgroup of the centralizer $C_T$ of $T$ : if $N \triangleleft C_T$ and $N$ is abelian, then $N \subseteq [T]_{\text{top}}$.

**Proof.** Since for each $x$ the orbit $\text{Orb}_T(x)$ is either $\{x\}$ or $\{x, Tx\}$ for every $T_1 \in [T]_{\text{top}}$ there is a set $E$ such that $T_1 = T$ on $E$ and $T_1 = \text{id}$ on $X \setminus E$. Hence $[T]_{\text{top}}$ is abelian. Also, for every $S \in C_T$ we have

$$S([T]_{\text{top}})S^{-1} = [STS^{-1}]_{\text{top}} = [T]_{\text{top}},$$

thus $[T]_{\text{top}}$ is a normal subgroup of $C_T$.

Suppose now that $N$ is a normal abelian subgroup of $C_T$. Consider the factor space $X/T$ with the measure $\nu = \pi(\mu)$, where $\pi: X \to X/T$ is the projection mapping. By Lemma 4.4.1 $\mathcal{R}/T$ is an ergodic étale equivalence relation on $X/T$ and every $S \in C_T$ induces a map $S^*: X/T \to X/T$ such that $S^*(\pi(x)) = \pi(S(x))$.
and $S^* \in [\mathcal{R}/T]_{\text{top}}$. Since $\text{supp} T$ is invariant under $S$ the mapping $S^*|_{(\text{supp} T)/T}$ belongs to the topological full group

$$G_0 = \left[ \mathcal{R}/T \right]_{\text{top}}/\text{supp} T.$$ 

By Theorem 4.3.3 this group is simple. Consider the (surjective) homomorphism $\rho: S \mapsto S^*|_{\text{supp} T/T}$. Since $\rho: C_T \to G_0$ is onto, $\rho(N)$ has to be a normal abelian subgroup of $G_0$ (normality is preserved under surjective homomorphisms). Thus $\rho(N) = \{\text{id}\}$. This means that

$$N \subset \ker(\rho) = \{S \in C_T: S|_{\text{supp} T} \in [T]_{\text{top}}\}.$$ 

Consider also the projection $\rho_2: S \mapsto S|_{X \setminus \text{supp} T}$. Since $\rho_2(N)$ is a normal abelian subgroup of $\rho_2(C_T) = [\mathcal{R}|_{X \setminus \text{supp} T}]_{\text{top}}$ it has to be trivial too and $N = [T]_{\text{top}}$.

Suppose that $f$ is an algebraic isomorphism between the topological full groups of two measure-preserving ergodic étale equivalence relations:

$$f: [\mathcal{R}_1]_{\text{top}} \to [\mathcal{R}_2]_{\text{top}}.$$ 

**Lemma 4.4.3.** If $T \in [\mathcal{R}_1]_{\text{top}}$ is an involution, then

$$f([T]_{\text{top}}) = [f(T)]_{\text{top}} \quad \text{and} \quad f(C_{[T]_{\text{top}}}) = C_{[f(T)]_{\text{top}}},$$

where $C_{[T]_{\text{top}}}$ is the centralizer of $[T]_{\text{top}}$ in $[\mathcal{R}_1]_{\text{top}}$.

**Proof.** Since $f$ is an isomorphism $f(C_T)$ equals $C_{f(T)}$, and $f$ maps the largest normal abelian subgroup of $C_T$ onto the largest normal abelian subgroup of $C_{f(T)}$. Hence $f([T]_{\text{top}}) = [f(T)]_{\text{top}}$. The second identity follows from the first part and the fact that $f$ is an isomorphism.

$\Box$
Lemma 4.4.4. If $T \in [\mathcal{R}]_{\text{top}}$ is an involution with full support, then $C[T]_{\text{top}} = [T]_{\text{top}}$.

Proof. We only need to prove the inclusion $C[T]_{\text{top}} \subset [T]_{\text{top}}$, since $[T]_{\text{top}}$ is abelian and the opposite inclusion is obvious. Suppose there is a mapping $S \in C[T]_{\text{top}}$ such that $S \notin [T]_{\text{top}}$. Assume there is a clopen set $A$ such that both $A$ and $S(A)$ are $T$-invariant and $A \cap S(A) = \emptyset$. Let

$$U = \begin{cases} 
  \text{id} & \text{on } S(A)^c; \\
  T & \text{on } S(A).
\end{cases}$$

By construction $U \in [T]_{\text{top}}$. For any point $x \in A$ we have $US(x) = T(S(x))$ and $SU(x) = S(x)$. By assumption $\text{supp} T = X$, hence $T(S(x)) \neq S(x)$ and $US(x) \neq SU(x)$. Since $S$ and the constructed map $U$ do not commute we arrive at a contradiction. The fact that such a clopen set $A$ exists is best illustrated with a picture.

![Figure 4.1: A $T$-invariant clopen set $A$ such that $A$ and $S(A)$ are disjoint.](image)

Definition 4.4.1. The set of transformations from $[\mathcal{R}]_{\text{top}}$ with support disjoint from the support of $T \in [\mathcal{R}]_{\text{top}}$ will be denoted by $\perp T$:

$$\perp T \overset{\text{def}}{=} \{ S \in [\mathcal{R}]_{\text{top}} : \text{supp } T \cap \text{supp } S = \emptyset \}.$$
Proof. Notice that any mapping $S \in C[T]_{top}$ can be written as a composition of two transformations $S_1 \in [T]_{top}$ and $S_2 \in \perp T$ with disjoint supports,

$$\text{supp } S_1 = \text{supp } T \quad \text{and} \quad \text{supp } S_2 = X \setminus \text{supp } T.$$ 

Clearly $[T]_{top}$ is abelian, hence the commutator of any two elements from the centralizer $C[T]_{top}$ belongs to $\perp T$. Since $\perp T$ is isomorphic to $[R_1|_{X \setminus \text{supp } T}]_{top}$, it is generated by commutators and we have the following formula:

$$\perp T = [C[T]_{top}, C[T]_{top}].$$

Therefore

$$f(\perp T) = [C[f(T)]_{top}, C[f(T)]_{top}] = \perp f(T).$$

4.5 The reconstruction mapping

Consider now a clopen set $A$. By Lemma 2.5.4 there is an involution $T \in [R_1]_{top}$ with $\text{supp } T = A$. Let

$$\Phi(A) \overset{\text{def}}{=} \text{supp } f(T).$$

This defines a set mapping that maps clopen subsets of $X$ onto clopen subsets of $Y$. It is easy to check that the collection of virtually clopen sets form an algebra.

Lemma 4.5.1. $\Phi$ is a measure-preserving isomorphism of algebras of (virtually) clopen sets.

Proof. Since $f^{-1}$ induces $\Phi^{-1}$ it follows that $\Phi$ is a bijection. To show that $\Phi$ is a homomorphism we need to prove that for any two clopen sets $A$ and $B$

$$\Phi(X \setminus A) = Y \setminus \Phi(A) \quad \text{and} \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B).$$
If two involutions $T$ and $S$ from $[R_1]_{\text{top}}$ are such that their supports are disjoint and cover the whole space, then the same is true for $f(T)$ and $f(S)$. Otherwise we would have a non-trivial mapping $R \in (\bot T \cap \bot S)$, which is impossible. Hence $\Phi(X \setminus A) = Y \setminus \Phi(A)$.

Let us now prove the second identity. For a clopen set $A$ denote by $G_A$ the set of all mappings $R$ from the topological full group such that $\text{supp} R \subset A$. For a fixed clopen set $A$ and an involution $T \in [R_1]_{\text{top}}$ with $\text{supp} T = X \setminus A$

$$f(G_A) = f(\bot T)$$

$$= \bot f(T)$$

(by Lemma 4.4.5)

$$= \{S \in [R_2]_{\text{top}} : \text{supp} S \subset (\text{supp} f(T))^c\}$$

(supp $f(T) = \Phi(X \setminus A)$)

$$= G_{\Phi(A)}.$$  

(since $Y \setminus \Phi(A^c) = \Phi(A)$)

Hence $f(G_A) = G_{\Phi(A)}$. The identity $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ now follows:

$$G_{\Phi(A \cap B)} = f(G_{A \cap B}) = f(G_A \cap G_B) = f(G_A) \cap f(G_B) = G_{\Phi(A)} \cap G_{\Phi(B)} = G_{\Phi(A) \cap \Phi(B)}.$$

To prove that $\Phi$ is measure-preserving it is enough to prove that $\Phi$ preserves measures of the form $\frac{1}{2^n}$. This is because a clopen set $A$ of positive measure is a disjoint union of clopen sets $A_i$ such that $\mu A_i = \frac{1}{2^{n_i}}$.

Suppose that $\mu A = 1/2$. Since the measure of $B = X \setminus A$ is also $1/2$, there is an involution $T$ such that $T(A) = B$. Therefore $TG_A T = G_B$. Apply $f$ to both sides of this equation:

$$f(T) G_{\Phi(A)} f(T) = G_{\Phi(B)}.$$

Hence the involution (a measure-preserving involution!) $f(T) \in [R_2]_{\text{top}}$ swaps $\Phi(A)$ and $\Phi(B)$. Since $\nu(\Phi(A) \cup \Phi(B)) = 1$ each must have measure $1/2$. The same argument shows that $\Phi$ is measure-preserving on sets of measure $1/4$ and so on. \(\square\)
For any virtually clopen set $E \subset X$ we have defined $\Phi(E) \subset Y$ which is virtually clopen and $\Phi$ is measure-preserving; $\Phi$ maps disjoint sets to disjoint sets; if $A \subset B$, then $\nu(\Phi(A) \setminus \Phi(B)) = 0$; $\Phi^{-1}$ also satisfies these properties.

We will now prove that $\Phi$ is induced by a point map defined on a virtually full subset.

**Remark 4.5.1.** If we do not require the subset to be $G_\delta$, this follows from von Neumann theorem (every homomorphism of measure algebras arises from a point homomorphism mod 0), see (Petersen, 2000, Theorem 4.7) or (Bogachev, 2007, Theorem 9.5.1). Note that we could apply Sikorski’s theorem (Kechris, 1995, theorem 15.9) since the collection of meagre sets that can be covered by an $F_\sigma$ set of measure zero is a $\sigma$-ideal.

**Lemma 4.5.2.** $\Phi$ is induced by a point map: there are virtually full subsets $X^* \subset X$ and $Y^* \subset Y$ and a mapping $\varphi : X^* \to Y^*$ such that $\Phi(A) = \varphi(A)$ for any virtually clopen $A \subset X$.

**Proof.** Let $\varepsilon_n = 2^{-n}$. We are going to construct two refining sequences of countable partitions $P_i = \{P_{ij}\}$ and $Q_i = \{Q_{ij}\}$ that virtually partition $X$ and $Y$ respectively, with every atom $P_{ij}$ ($Q_{ij}$) being virtually clopen.

Let $P_1 = \{P_{1j}\}$ be any collection of clopen sets that virtually partition $X$ with diameters $\text{diam}(P_{1j}) < \varepsilon_1$. Then the collection $Q_1 = \{\Phi(P_{1j})\}$ virtually partitions $Y$ with each $\Phi(P_{1j})$ being virtually clopen. Next find a clopen partition $Q_2 = \{Q_{2j}\}$ with $\text{diam} Q_2 < \varepsilon_2$ (in the $Y$-metric) refining $Q_1$. Now $P_2 = \{\Phi^{-1}(Q_{2j})\}$ is a clopen partition of $X$ that refines $P_1$. Continue in the same way.

If a clopen partition $P_n = \{P_{nj}\}$ with $\text{diam} P_n < \varepsilon_n$ is already constructed, then the collection $Q_n = \{\Phi(P_{nj})\}$ virtually partitions $Y$ and each atom is virtually
clopen. Find a refining partition $Q_{n+1}$ such that $\text{diam } Q_{n+1} < \varepsilon_{n+1}$ (in the $Y$-metric). Then $P_{n+1} = \{ \Phi^{-1}(Q_{n+1,j}) \}$ virtually partitions $X$ and refines $P_n$.

Find $X_0 \subset X$ and $Y_0 \subset Y$ dense $G_\delta$ subsets of full measure of $X$ and $Y$ respectively such that $P_n$ and $Q_n$ are clopen partitions of $X_0$ and $Y_0$ for all $n$. For every sequence of atoms $p_i \in P_i$ such that $p_{i+1} \subset p_i$ consider the intersection $\bigcap_{i \in \mathbb{N}} p_i$. Since the diameters of atoms converge to zero as $i$ approaches infinity, $\bigcap_{i \in \mathbb{N}} p_i$ is either empty or contains a single point.

There is a naturally defined projection $\pi$, mapping an atom of $P_i$ to the atom of $P_{i-1}$ that contains it:

$$P_1 \xleftarrow{\pi} P_2 \xleftarrow{\pi} P_3 \xleftarrow{\pi} P_4 \xleftarrow{\pi} \cdots.$$  

Each $P_i$ can be viewed as a discrete topological space with points $P_{ij}$ and a probability measure $\mu_i$, such that $\mu_i(P_{ij}) = \mu(P_{ij})$. We can consider the projective limit space

$$Z = \lim_{\leftarrow} P_i = \{(p_i)_{i \in \mathbb{N}} : p_i \in P_i \quad \text{and} \quad \pi(p_{i+1}) = p_i \}$$

with the measure $\mu^* = \lim_{\leftarrow} \mu_i$ and the usual projective limit topology. This topology coincides with the topology induced from the product space, see (Bourbaki, 1998). The product space of all possible sequences $\{p_i\}_{i \in \mathbb{N}}$ is Polish as a countable product of Polish spaces, see (Stromberg, 1994, theorem 7.9). Since $Z$ is a $G_\delta$ subset of the product space, by Alexandrov’s theorem 2.2.1 $Z$ is also Polish. There is a naturally defined mapping

$$\alpha : X_0 \to Z, \quad \alpha(x) = (p_i)_{i \in \mathbb{N}},$$

where $p_i$ is the atom of $P_i$ that contains $x$. Since $\alpha$ is a homeomorphism between $X_0$ and $\alpha(X_0)$, $\alpha(X_0)$ is also Polish and by Theorem 2.2.1 it is $G_\delta$. The
collection of sets
\[ \{(p_i)_{i \in \mathbb{N}} : p_n = P_{nm}\}, \quad \text{with} \quad P_{nm} \in \mathcal{P}_n \]
is a basis for the topology of \( Z \) and every such set intersects \( \alpha(X_0) \). Hence \( \alpha(X_0) \) is dense in \( Z \). By the definition of \( \mu^* \) the image \( \alpha(X_0) \) has full measure.

There is a similarly defined projective limit space \( \varprojlim Q_i \) which can be identified with \( Z \), and \( \nu^* = \varprojlim \nu_i = \mu^* \). There is a mapping \( \beta : Y_0 \to Z \) with \( \beta(y) = (q_i)_{i \in \mathbb{N}} \) such that \( q_i \) is an atom of \( Q_i \) and \( y \in q_i \). The image \( \beta(Y_0) \) is also a dense \( G_\delta \) subset of \( Z \). Note that \( Z \) is a complete metric space and hence it is a Baire space. Therefore the intersection \( Z^* = \alpha(X_0) \cap \beta(Y_0) \) is dense in \( Z \). We can now form the composition
\[ \varphi = \beta^{-1} \circ \alpha : X^* \to Y^*, \]
where \( X^* = \alpha^{-1}(Z_0) \) and \( Y^* = \beta^{-1}(Z_0) \). The image \( \varphi(X^*) = \beta^{-1}(Z^*) \) is dense \( G_\delta \) and \( \varphi : X^* \to Y^* \) is a homeomorphism. Both \( \alpha(X_0) \) and \( \beta(Y_0) \) are subsets of full measure in \( Z \) with respect to \( \mu^* \), hence \( \mu^*(Z_0) = 1 \). By construction for any \( P_{in} \in \mathcal{P}_i \) we have \( \varphi(P_{in}) = Q_{in} \in \mathcal{Q}_i \) and hence \( \varphi \) induces \( \Phi \).

The following two lemmas finish the proof of the main theorem.

**Lemma 4.5.3.** For every \( T \in [\mathcal{R}_1]_{\text{top}} \)
\[ f(T)(y) = \varphi T \varphi^{-1}(y) \quad \text{for all} \quad y \in Y_T \]
for some \( G_\delta \) subset of full measure \( Y_T \subset Y \) that depends on \( T \).

**Proof.** By Theorem 4.2.6 the full group \( [\mathcal{R}_1]_{\text{top}} \) is generated by involutions, so it is enough to prove this for involutions.
Suppose that $T \in [\mathcal{R}_1]_{top}$ is an involution, then

$$\text{supp}(\varphi T \varphi^{-1}) = \varphi(\text{supp } T) = \Phi(\text{supp } T) = \text{supp } f(T).$$

Suppose, on the contrary, that the virtually open set

$$B = \{ y \in \text{supp}(f(T)) : \varphi T \varphi^{-1}(y) \neq f(T)(y) \}$$

has positive measure. We can find a clopen set $C \subset B$ of positive measure such that

1. $\varphi T \varphi^{-1}(C) \cap C = \emptyset$;
2. $f(T)(C) \cap C = \emptyset$;
3. $\varphi T \varphi^{-1}(C) \cap f(T)(C) = \emptyset$.

Let $D = C \cup f(T)(C)$. By construction $D$ is $f(T)$-invariant, but not $\varphi T \varphi^{-1}$-invariant. Since $D$ is $f(T)$-invariant we have $f(T) = U_1 U_2$, where

$$U_1 = \begin{cases} f(T) & \text{on } D; \\ \text{id} & \text{on } X \setminus D \end{cases} \quad \text{and} \quad U_2 = \begin{cases} \text{id} & \text{on } D; \\ f(T) & \text{on } X \setminus D. \end{cases}$$

Both involutions $U_1$ and $U_2$ belong to the full group $[\mathcal{R}_2]_{top}$. Hence $T$ equals $f^{-1}(U_1) f^{-1}(U_2)$ and

$$\varphi T \varphi^{-1} = (\varphi f^{-1}(U_1) \varphi^{-1}) (\varphi f^{-1}(U_2) \varphi^{-1}). \quad (4.3)$$

Then

$$\text{supp}(\varphi f^{-1}(U_1) \varphi^{-1}) = \varphi(\text{supp } f^{-1}(U_1)) = \text{supp } (f f^{-1}(U_1)) = \text{supp } U_1$$

and $\text{supp}(\varphi f^{-1}(U_2) \varphi^{-1}) = \text{supp } U_2$. It follows now from equation (4.3) that $D$ is $\varphi T \varphi^{-1}$-invariant. Contradiction. $\square$
Lemma 4.5.4. There is only one mapping $\varphi$ satisfying Lemma 4.5.3: if for some $\varphi_1, \varphi_2 : X \to Y$ and every $S \in [R_1]_{top}$, $\varphi_1 S \varphi_1^{-1} = \varphi_2 S \varphi_2^{-1}$, then $\varphi_1 = \varphi_2$.

Proof. It is easy to see that $Q = \varphi_2^{-1} \varphi_1$ commutes with every element of the topological full group $[R_1]_{top}$:

$$
\varphi_2^{-1} \varphi_1 S = \varphi_2^{-1} (\varphi_1 S \varphi_1^{-1}) \varphi_1 = \varphi_2^{-1} (\varphi_2 S \varphi_2^{-1}) \varphi_1 = S \varphi_2^{-1} \varphi_1.
$$

For every clopen set $A \subset X$ there is an involution $T$ with $\text{supp} T = A$. Then

$$
A = \text{supp} T = \text{supp} (QTQ^{-1}) = Q(\text{supp} T) = Q(A)
$$

and hence $Q = id$. \qed
Appendix A

Proof of Claim 3

Proof of Claim 3 from page 43:

\[
\mu\left(\left(\bigcup_{l \in L_i} E_{il}\right) \cap \left(\bigcup_{j=1}^{k} |\tau_j|\right)\right) \geq \mu\left(\left(\bigcup_{l \in L_i} E_{il}\right) \cap L_i E_{il} \cap \left(\bigcup_{j=1}^{k} |\tau_j|\right)\right) \\
\geq \mu\left(L_i E_{il} \cap \left(\bigcup_{j=1}^{k} |\tau_j|\right)\right) - \mu\left(\bigcup_{l \in L_i} E_{il} \triangle L_i E_{il}\right). (A.1)
\]

Since we are only considering quasi-columns that are \((1 - \delta)\)-covered by the good columns \(\tau_j\) (see Claim 1 from page 41), we know that

\[
\mu\left(L_i E_{il} \cap \left(\bigcup_{j=1}^{k} |\tau_j|\right)\right) > (1 - \delta)\mu(L_i E_{il}). (A.2)
\]

Now, since the levels \(lE_i\) of the quasi-columns are well approximated by the \(E_{il}s\) :

\[
\mu(E_{il} \triangle lE_i) = \mu(E_{il} \setminus lE_i) + \mu(lE_i \setminus E_{il}) \\
\leq \mu(E_{il} \setminus F_{il}) + \mu(lE_i \setminus (E_{il} \cap F_{il})) \\
< \gamma + (\delta \mu E_i + \gamma) \\
= 2\gamma + \delta \mu E_i < 2\delta \mu E_i, (A.3)
\]
we can estimate the other term:

\[
\mu\left(\left(\bigcup_{l \in L_i} E_{il}\right) \triangle L_i E_i\right) \leq \mu\left(\bigcup_{l \in L_i} (E_{il} \triangle l E_i)\right)
\]

\[
\leq \sum_{l \in L_i} \mu(E_{il} \triangle l E_i)
\]

\[
< 2\delta |L_i| \mu E_i.
\]

Combining (A.4) and (A.2) with inequality (A.1) we obtain

\[
\mu\left(\left(\bigcup_{l \in L_i} E_{il}\right) \cap \left(\bigcup_{j=1}^k |\tau_j|\right)\right) > (1 - \delta) \mu(L_i E_i) - 2\delta |L_i| \mu E_i.
\]

(A.5)

It follows from inequality (3.7) on page 42 that

\[
\mu\left(\bigcup_{l \in L_i} E_{il}\right) > (1 - 2\delta) |L_i| \mu E_i.
\]

Without loss of generality we can assume that \( \delta < 1/6 \) and hence

\[
\frac{1}{1 - 2\delta} < 1 + 3\delta.
\]

(A.6)

Therefore

\[
(1 + 3\delta) \mu\left(\bigcup_{l \in L_i} E_{il}\right) > |L_i| \mu E_i.
\]

Also, it follows from (A.4) that

\[
\mu(L_i E_i) > \mu\left(\bigcup_{l \in L_i} E_{il}\right) - 2\delta |L_i| \mu E_i.
\]

Continue now inequality (A.5):

\[
\mu\left(\left(\bigcup_{l \in L_i} E_{il}\right) \cap \left(\bigcup_{j=1}^k |\tau_j|\right)\right) > (1 - \delta) \mu(L_i E_i) - 2\delta |L_i| \mu E_i
\]

\[
> (1 - \delta) \mu\left(\bigcup_{l \in L_i} E_{il}\right) - (1 - \delta) 2\delta |L_i| \mu E_i - 2\delta |L_i| \mu E_i
\]
\[(1 - \delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right) - 4\delta |L_i| \mu E_i > (1 - \delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right) - 4\delta (1 + 3\delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right) > (1 - \delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right) - 8\delta \mu\left(\bigcup_{l \in L_i} E_{il}\right) = (1 - 9\delta)\mu\left(\bigcup_{l \in L_i} E_{il}\right). \]
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