Lyapunov-based Stability Analysis of a One-Pump

One-Signal Co-pumping Raman Amplifier

by

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A thesis submitted in conformity with the requirements for the degree of Masters of Applied Science
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2010

Abstract

We consider the boundary control problem to stabilize the power of a signal and a pump propagating down a Raman amplifier. This is essentially an initial-boundary value problem (IBVP) of a hyperbolic system with Lotka-Volterra type nonlinearities. We treat the system as a control problem with states in the function space and use Lyapunov-based analysis to demonstrate asymptotic stability in the $C^0$- and the $L^2$-sense. The stability conditions are derived for closed-loop systems with a proportional controller and a dynamic controller, and confirmed by simulations in MATLAB.
Acknowledgements

First and foremost, I would like to thank my supervisor, Professor Lacra Pavel. Her insightful suggestions and passion for research have greatly inspired me in numerous ways. The constant support and guidance from her throughout the course is what made this thesis possible.

Next, I wish to express my sincere gratitude to my parents and brother, who have always stood by me in times of distress. Their continuous encouragement is what motivated, and will continue to motivate me to keep going.

Finally I would like to thank the Department of Electrical and Computer Engineering for the opportunity and support of the completion of this thesis. I have gained invaluable experience from it.
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Ω  Length span of the Raman amplifier

$C^0(\Omega)$  Space of continuous functions defined on $\Omega$

$L^2(\Omega)$  Space of square integrable functions defined on $\Omega$

$H^1(\Omega)$  Space of square integrable and continuously differentiable functions

$H^1(\Omega)_{bc}$  $H^1(\Omega)$ with boundary conditions embedded

$s(t, z), p(t, z)$  Signal and pump power at time $t$ and distance $z$ along the amplifier

$\bar{s}(z), \bar{p}(z)$  Steady state signal and pump power profile

$w$  System state after change of coordinates

$y_d$  Output signal power level reference

$k_1, k_2, \varepsilon$  Controller gain parameters

$\sigma$  State variable of the dynamic controller

$V, W$  Lyapunov functional used in stability analysis

$\mu$  Tuning parameter for the weighted inner product in Lyapunov functional

$r$  Scaling parameter for the Lyapunov functional of the dynamic control

$\lambda_i, \alpha_i, c_i$  Coefficients for the non-normalized problem
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Chapter 1

Introduction

The topic of this thesis is the Lyapunov-based stability analysis of the closed-loop system of a boundary-controlled Raman amplifier. In this chapter, we will first discuss the motivation of our study of the Raman amplifier control problem. Then we will briefly review the literature on Raman amplifier control, as well as on generic systems with dynamics similar to the optical network in question. Finally, we will state the contribution and the organization of this thesis.

1.1 Motivation

As a solution for the increasing bandwidth demands of communication networks, fiber optic technology has been developed and is widely used in long-haul data transmission. However, in long-distance communication, the attenuation of signals along the fiber becomes a critical drawback for optical systems [1]. Hence it is necessary to install cascaded optical amplifiers to repeatedly amplify all signals in the network. One common choice is using Raman amplifiers.

Raman amplifier exploits the optical fiber itself as amplification medium. These fibers usually span tens to hundreds of kilometers in length. By setting up extra pump lasers that
propagate through the span of the amplifier, energy is transferred from the pumps to the signals (known as the Stimulated Raman Scattering effect) and results in signal boost. In order to provide the desired gain for signals in different channels, it is sometimes necessary to set up several pumps at different wavelengths, as depicted in Figure 1.1.

![Figure 1.1: General setup of a Raman amplifier](image)

The advantage of using a Raman amplifier is that unlike an Erbium Doped Fiber Amplifier (EDFA), it does not require specially doped optical fibers. Hence the device manufacturing cost is lower. The power of the pump lasers can also be tuned to provide optimal amplification even when the signal powers fluctuate.

However, along Raman amplifier, the dynamics of the signal and pump coupling is non-linear and involves spatial derivatives, i.e., described by nonlinear partial differential equations (PDEs). From the system control perspective, this indicates that the system state is of infinite dimensions. As a result, the stability analysis for finite-dimensional systems cannot be directly applied. Hence the motivation for our study is to develop stability analysis for a closed-loop Raman amplifier system.
1.2 Literature Review

There have been works that investigate the Raman amplifier problem in both theoretical and experimental perspectives. The common approach is to take finite difference approximation for the spatial derivative of the system, and analyze the approximated system of ordinary differential equations (ODEs) instead. As shown in [2], the solution to the ODE system converge to the PDE system. In addition, a linear feedback controller based on the signal and pump powers at the ends of the amplifier is presented in [3]. The linearization and finite difference approximation result in a system of 588th order. A 10th-order controller is designed and it is verified through simulations that step-tracking of signal and pump powers can be achieved.

While the convergence from the ODE to the PDE system is proven, there is no indication that the stabilization and regulation properties for the finite dimensional system will also hold for the actual PDE system. Furthermore, the finite difference approximation in [3] results in a high-order linear system and controller.

The experimental results on the Raman amplifier control problem demonstrate steady state tracking with simple PID control, as shown in [4] and [5]. They indicate that high-order and complex controllers are not needed to achieve optimal performance. However, these results are obtained through heuristic tuning and no rigorous analysis is provided.

On the other hand, boundary controller designs based on Lyapunov functional and $L^2$ stability analysis have been used for different hyperbolic systems [6] and reaction-diffusion systems [7], both of which are described by systems of PDEs.

In [6], Coron implements controllers at both boundaries and shows that the closed-loop system converges to a constant steady state. The system considered is the hyperbolic system of conservation laws. An analysis based on the linearization of the PDE system is developed using
a quadratic-like Lyapunov function. While the control action in the Raman amplifier problem is also implemented at the boundaries, the steady state pump and signal powers are not constant along the length of the fiber.

In [7], Rionero demonstrates stability of a given steady state solution in the \( L^2 \)-norm with respect to initial state perturbations based on quadratic-like Lyapunov functions as well. However, no control action is formulated. From [6] and [7], it seems that our Raman amplifier problem, also a system of PDEs, can be analyzed in a similar manner.

### 1.3 Contribution

Inspired by the results mentioned above, the subject of this thesis is on the boundary control design and stability analysis for the closed-loop Raman amplifier system. Specifically, we will consider a one-pump one-signal co-pumping system, i.e., the pump and the signal lasers are propagating in the same direction. This is an initial-boundary value problem (IVBP) of a system of first-order hyperbolic PDEs with Lotka-Volterra type nonlinearity coupling. Assuming that the steady state signal and pump powers are given, the pump input is used as the control signal to stabilize the system. From a systems control perspective, the signal and pump power along the amplifier will be considered as the infinite-dimensional state of the problem. We can then perform Lyapunov-based analysis to derive the stability conditions in \( C^0 \)- and \( L^2 \)-norm of the system state.

The main contribution of our study is that we will construct a Lyapunov functional for the Raman amplifier system. Since this functional is dependent on the infinite-dimensional state, the stability analysis we formulate can be directly applied on the nonlinear and infinite-dimensional system. In other words, we can treat the original Raman amplifier problem without
the need to resort to linearization and finite difference approximation.

Furthermore, we can derive stability conditions of the closed-loop system for simple boundary controllers. This not only confirms the experimental results demonstrated in [4] and [5], but also gives rise to a systematic design recipe for simple boundary controllers over heuristically tuning the parameters.

### 1.4 Organization

The thesis is organized as follows. It will start with a review of the background material and important theorems and results used in this thesis. Then Chapter 3 formulates the one-pump one-signal, normalized Raman amplifier boundary control problem. Chapter 4 presents the Lyapunov functional candidate inspired by the Lotka-Volterra ODEs [8], and summarize its important properties. In Chapter 5 we state the asymptotic and $C^0$-norm, and the global $L^2$-norm stability conditions derived based on the Lyapunov functional. Chapter 6 contains the generalization to a Raman amplifier control problem with coefficients. Chapter 7 is where we present the simulation results in MATLAB, and finally, in Chapter 8 we summarize the important findings and conclude the thesis.
Chapter 2

Background

From Chapter 1, it is apparent that the spatial dynamics, combined with the nonlinearities in the Raman amplifier problem present difficulties when performing stability analysis. Hence in this chapter, we introduce several important mathematical concepts and results. They will help us formulate the Raman amplifier problem rigorously in a fashion similar to a standard control problem, thus allowing the Lyapunov-based approach and the $C^0$- and $L^2$-norm stability analysis.

2.1 Function Spaces

In this section, we review the definitions of different function spaces. Recall in a standard control problem, the state belongs to a vector space. And given certain initial conditions, the state vector progresses according to the system dynamics with respect to time. However, the signal and pump powers are functions of time and space, and the value of the function progresses with time given initial and boundary conditions. In other words, it is necessary to define function spaces in the spatial perspective if we are to treat the power mappings along the amplifier
altogether as the system state of the control problem.

2.1.1 Hölder Spaces

Let $\mathcal{X}$ be a metric space, and consider the space $C^0(\mathcal{X})$, defined as

$$C^0(\mathcal{X}) = \{ f | f : \mathcal{X} \to \mathbb{R}, \text{f continuous} \} \quad (2.1)$$

In other words, $C^0(\mathcal{X})$ is the space of all continuous functions on $\mathcal{X}$ [9].

For example, take a real-valued function defined over the normalized Raman amplifier length span as $\Omega := [0, 1]$. Then $C^0(\Omega)$ can denote the space of all continuous power mappings along the amplifier. Equivalently, $\forall f \in C^0(\Omega)$, $f$ is a continuous function defined on a compact set $\Omega$. Hence $f$ is bounded, and we can define the norm for $C^0(\Omega)$, or the $C^0$-norm, as

$$\|f\|_{C^0(\Omega)} = \max_{z \in \Omega} |f(z)| \quad (2.2)$$

where $|\cdot|$ is simply the notation for absolute value.

Next, the concept of continuity can be extended to spatial derivatives of the functions, and this gives the notation of $C^k(\Omega)$ as the space of bounded and continuous functions $f$ such that $D^\beta f$, defined by

$$D^\beta f = \frac{d^\beta f(z)}{dz^\beta}$$

is also bounded and continuous $\forall \beta \leq k$, as given in [9], pp. 37. Equivalently,

$$C^k(\Omega) = \{ f \in C^0(\Omega) | D^\beta f \in C^0(\Omega), \ \forall |\beta| \leq k \} \quad (2.3)$$
Of course, \( C^l(\Omega) \subset C^k(\Omega) \) \( \forall l > k \). The \( C^k \)-norm is defined as

\[
\|f\|_{C^k(\Omega)} = \sum_{\beta=0}^{k} \|D^\beta f\|_{C^0(\Omega)}
\] (2.4)

For the Raman amplifier problem, we are interested in functions that are at least once continuously differentiable, i.e.,

\[
C^1(\Omega) = \left\{ f \in C^0(\Omega) \mid \frac{df}{dz} \in C^0(\Omega) \right\}
\] (2.5)

\[
\|f\|_{C^1(\Omega)} = \|f\|_{C^0(\Omega)} + \left\| \frac{df}{dz} \right\|_{C^0(\Omega)}
\] (2.6)

where \( \|\cdot\|_{C^0(\Omega)} \) is as defined in (2.2)

### 2.1.2 \( L^2 \)-Space

Another common family of function spaces is the \( L^p \) spaces. In the Raman amplifier problem, the concept of \( L^2 \)-space will be used. \( L^2 \)-space denotes the space of all square-integrable functions. For example, let \( \Omega \) denote the open set \((0, 1)\). \( f \in L^2(\Omega) \) implies that it has \( L^2 \)-norm defined by

\[
\|f\|_{L^2(\Omega)} = \left( \int_{\Omega} |f(z)|^2 \, dz \right)^{1/2} < \infty
\] (2.7)

As shown in the equation, the \( L^2 \)-norm is the square root of \( \langle f, f \rangle \), the inner product of the function \( f \) with itself.
2.1.3 Sobolev Spaces

The family of Sobolev spaces is a natural generalization of the $L^p$-spaces to include conditions on the differentiability of functions within the space, similar to the extension from $C^0$ to $C^k$ as shown in Section 2.1.1 [9]. Take the $L^2$-space, for example, we can define Sobolev spaces $W^{k,2}(\Omega)$ as

$$W^{k,2}(\Omega) = \{ f \in L^2(\Omega) | D^\beta f \in L^2(\Omega), \ \forall |\beta| \leq k \}$$

(2.8)

In other words, $L^2(\Omega)$ can be rewritten as $W^{0,2}(\Omega)$. And $W^{l,2}(\Omega) \subset W^{k,2}(\Omega) \forall l > k$ [10].

As a parallel to (2.5), let $H^1(\Omega) = W^{1,2}(\Omega)$, then we have

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) | \frac{df}{dz} \in L^2(\Omega) \right\}$$

(2.9)

For a general Sobolev space $W^{k,p}(\Omega)$, its norm is defined as

$$\|f\|_{W^{k,p}(\Omega)} = \left( \sum_{\beta=0}^{k} \|D^\beta f\|_{L^p(\Omega)}^p \right)^{1/p}$$

(2.10)

as given in [11]. Hence the $H^1$-norm is

$$\|f\|_{H^1(\Omega)} = \left( \|f\|_{L^2(\Omega)}^2 + \left\| \frac{df}{dz} \right\|_{L^2(\Omega)}^2 \right)^{1/2}$$

(2.11)

where $\|\cdot\|_{L^2(\Omega)}$ is as defined in (2.7)
2.2 Sobolev Embedding Theorem

In the previous section, we have introduced the definitions of several function spaces. Here we introduce the Sobolev Embedding Theorem from [11], pp. 208, which gives conditions for which the different function spaces can be linked together.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and let \( 1 \leq r < \infty \), and \( 0 < m < \infty \). Denote by \( W^{m,r}(\mathbb{R}^n) \) the Sobolev space consisting of all real-valued functions on \( \Omega \). Then the first part of Sobolev Embedding Theorem states that if \( j, p \) are integers such that \( 1 \leq p < \infty \) and \( 0 \leq j < m \), and

\[
\frac{1}{p} = \frac{1}{r} + \frac{j}{n} - \frac{m}{n}
\]

then \( W^{m,r}(\Omega) \subset W^{j,p}(\Omega) \).

The next part of the theorem relates the Sobolev space \( W^{m,r}(\Omega) \) with Hölder spaces and \( L^p \)-spaces. First of all, with \( m, r, \) and \( n \) as above,

\[
W^{m,r}(\Omega) \subset L^{nr/(n-mr)}(\Omega) \quad \text{for } mr < n \tag{2.12}
\]

and

\[
W^{m,r}(\Omega) \subset C^k(\Omega) \quad \text{for } 0 \leq k < m - \frac{n}{r} \tag{2.13}
\]

Specifically, when \( m = 1 \), there exists constant \( C_1 > 0 \) such that for any \( u \in W^{1,r}(\Omega) \)

\[
\|u\|_{C^0(\Omega)} \leq C_1 \|u\|_{W^{1,r}(\Omega)} \tag{2.14}
\]

for \( \infty > r > n \). Then consider the case where \( \Omega \subset \mathbb{R}^1 \) (i.e., \( n = 1 \), and the Sobolev space
\[ \mathcal{H}^1(\Omega) = \mathcal{W}^{1,2}(\Omega) \ (r = 2). \] Indeed \( \infty > r > n \) and the inequality becomes

\[ \| u \|_{C^0(\Omega)} \leq C_1 \| u \|_{\mathcal{H}^1(\Omega)} \]  

(2.15)

### 2.3 Hyperbolic Systems

Consider a first-order quasilinear system,

\[ \mathcal{A}(t, z, x) \frac{\partial x}{\partial t} + \mathcal{B}(t, z, x) \frac{\partial x}{\partial z} = \mathcal{F}(t, z, x) \]  

(2.16)

where \( x(t, z) = [x_1(t, z), \cdots, x_n(t, z)]^T \) is an unknown vector function, and \( \mathcal{A}(t, z, x), \mathcal{B}(t, z, x), \mathcal{F}(t, z, x) \) are given smooth functions of \((t, z, x)\). Then the system described by (2.16) is called hyperbolic [12] on a certain domain \( \mathcal{R} \) of \((t, z, x)\) if

- \( \det \mathcal{A} \neq 0 \)
- \( \exists n \) real generalized eigenvalues of \( \mathcal{B} \) with respect to \( \mathcal{A} \), i.e., \( \exists \lambda_1(t, z, x), \cdots, \lambda_n(t, z, x) \)
  such that
  \[ \det(\mathcal{B} - \lambda_l \mathcal{A}) = 0 \quad \forall l = 1, \cdots, n \]
- \( \mathcal{B} \) is diagonalizable with respect to \( \mathcal{A} \)
There are different types of problems associated with \((2.16)\). The first one is the initial-value problem (IVP), in which we are given that

\[
x_i(0, z) = x_{i\phi} = \phi_i(z), \quad i = 1, \ldots, n
\]

\[
\Downarrow
\]

\[
x(0, z) = x_0 = \phi(z)
\]

where \(\phi\) is a smooth function of \(z\). Also known as the Cauchy problem, this has been well-studied and an existence and uniqueness theorem has been derived for \(z \in (-\infty, \infty)\) [13], [14].

The next type of problem is the boundary-value problem (BVP). In our case, with the amplifier span given by \(\Omega = [0, 1]\), a boundary value problem indicates that there are values assigned at \(z = 0\) and \(z = L\), i.e., the boundary condition.

The last type of problem associated with system (2.16) incorporates both the initial condition and the boundary condition, and is known as the initial-boundary value problem (IBVP). The Raman amplifier problem investigated in the thesis will also be in this form, since we are attempting to stabilize the signal and pump power by implementing control at the boundary of the amplifier.

### 2.4 Gateaux and Fréchet differentials

Next we review the concepts of Gateaux and Fréchet differentials based on [15], pp. 171-177. These are generalizations of the directional derivative in a finite-dimensional space.

Let \(\mathcal{X}\) be a vector space, \(\mathcal{Y}\) be a normed space, and \(T\) a transformation defined for all \(x \in D \subset \mathcal{X}\) and having range \(R \subset \mathcal{Y}\).
**Definition 1.** Let \( x \in D \subset \mathcal{X} \) and let \( h \) be arbitrary in \( \mathcal{X} \). If the limit

\[
d_G T(x; h) = \lim_{\alpha \to 0} \frac{1}{\alpha} [T(x + \alpha h) - T(x)]
\]

(2.18)

exists, it is called the Gateaux differential of \( T \) at \( x \) with increment \( h \). If such limit exists for each \( h \in \mathcal{X} \), the transformation \( T \) is said to be Gateaux differentiable at \( x \).

Specifically, let \( f \) be a functional on \( \mathcal{X} \), then the Gateaux differential of \( f \), if it exists, is

\[
d_G f(x; h) = \frac{d}{d\alpha} f(x + \alpha h) \bigg|_{\alpha=0}
\]

(2.19)

and, for each fixed \( x \in \mathcal{X} \), \( d_G f(x; h) \) is a functional with respect to the variable \( h \in \mathcal{X} \).

A stronger concept for directional derivative is the Fréchet differential, as given in [15], pp. 172.

**Definition 2.** Let \( T \) be a transformation defined on an open domain \( D \) in a normed space \( \mathcal{X} \) and having range in a normed space \( \mathcal{Y} \). If for fixed \( x \in D \) and each \( h \in \mathcal{X} \) there exists \( d_G T(x; h) \in \mathcal{Y} \) which is linear and continuous with respect to \( h \) such that

\[
\lim_{\|h\| \to 0} \frac{\|T(x + h) - T(x) - d_G T(x; h)\|}{\|h\|} = 0
\]

(2.20)

then \( T \) is said to be Fréchet differentiable at \( x \) and its Fréchet differential with increment \( h \) is equal to the Gateaux differential, \( d_G T(x; h) \), in this case.
2.5 Existence and Uniqueness of Solutions to the IBVP

In Section 2.3 and 2.4, we introduced the IBVP associated with a hyperbolic system, and established the concept of functional derivative related to its solution. However, the problem is not meaningful if such solution does not exist. Hence in this section, we will establish the concept of classical solutions to an IVBP, the existence and uniqueness of such solutions, and review the conditions derived for a hyperbolic Lotka-Volterra system in [16]. This is critical because the problem we formulate and the assumption we make must lead to valid solutions defined at least within the specified domain before we can proceed with the analysis.

There have been existence and uniqueness results proven for various hyperbolic systems’ IBVPs. For example, in [17], a Cauchy $2 \times 2$ quasilinear hyperbolic system with quadratic interaction is considered. And an abstract Cauchy problem approach is used in [18] to treat semilinear hyperbolic system with dissipative boundary conditions in $L^2$-spaces based on $C^0$-semigroup theory.

In [16], a similar method is employed on a hyperbolic system with Lotka-Volterra type nonlinearities, of the form

\[
\frac{\partial x}{\partial t}(t, z) + a(z) \frac{\partial x}{\partial z}(t, z) + b(z)x(t, z) = f(x(t, z))
\]  

(2.21)

for $t \geq 0, z \in \Omega$, with boundary condition

\[
x(t, 0) = G(x(t, 1))
\]  

(2.22)
and initial condition

\[ x(0, z) = x_0(z), \quad z \in \Omega \tag{2.23} \]

where \( x(t, z) = [x_i(t, z)] \in \mathbb{R}^n \) and the coefficients \( a(z) \) is diagonal, and \( b(z) \) takes values in the space of bounded linear operators. Furthermore, \( G \) is a smooth mapping of class \( C^2 \) that satisfies the Lipschitz property, i.e., \( G(0) = 0 \), and \( \forall x, y \in \mathbb{R}^n \),

\[ \|G(x) - G(y)\| \leq K_g \|x - y\| \tag{2.24} \]

for some \( K_g \neq 1 \), and \( \| \cdot \| \) denotes the Euclidean norm, namely \( \|v\| = \sqrt{v^T v} \). Lastly, the nonlinear reaction \( f(x) = [f_i(x)] \) is given by

\[ f_i(x) = -\sum_{j<i} x_i x_j + \sum_{j>i} x_i x_j \quad i = 1, \cdots, n \tag{2.25} \]

Based on [19] and [20], we regard \( x \) as a function of \( t \) with values in a Banach function space \( L^2(\Omega) \) over \( z \), and write \( x(t) \) for \( x(t, \cdot) \). Then (2.21)-(2.23) can be reformulated as the following abstract differential equation,

\[ \frac{dx}{dt}(t) + A x(t) = F(x(t)) \]

\[ x(t, 0) = G(x(t, 1)) \]

\[ x(0) = x_0 \tag{2.26} \]

where \( A \) is a closed linear operator, defined by

\[ A x := a \cdot \frac{dx}{dz} + b \cdot x \tag{2.27} \]
with \( \cdot \) denoting point-wise multiplication. \( F \) is a nonlinear operator where \( F(x) = [f_i(x)] \).

\( \frac{dx}{dt}(t) \) denotes the Fréchet derivative of the \( L^2(\Omega) \)-valued function \( x \).

Note that the initial condition \( x_0 \) must satisfy the compatibility condition

\[
x_0(0) = G(x_0(1))
\]

(2.28)

And \( \mathcal{H}^1_{bc}(\Omega) \) is used to denote the function space with the boundary condition embedded, i.e.,

\[
\mathcal{H}^1_{bc}(\Omega) = \{ x \in \mathcal{H}^1(\Omega) | x(0) = G(x(1)) \}
\]

(2.29)

Let \( T > 0 \) and denote, by \( C^1([0, T]; L^2(\Omega)) \), the space of continuously differentiable functions on \([0, T]\), that returns values of square integrable functions defined on \( \Omega \). We define a classical solution to the reformulated abstract problem as follows,

**Definition 3.** A function \( x : [0, T] \rightarrow L^2(\Omega) \) is a classical solution of (2.26) on \([0, T]\) if \( x \in C^1([0, T]; L^2(\Omega)) \), \( x(t) \in \mathcal{H}^1_{bc}(\Omega) \) for all \( t \in [0, T] \) and \( x(t) \) satisfies (2.26) for all \( t \in [0, T] \).

The function \( x(t) \) is a classical solution on \([0, \infty)\) if \( x(t) \) is a classical solution on \([0, T]\) for every \( T \geq 0 \).

Finally we present the main existence and uniqueness results from [16] that will be used in our Raman amplifier problem formulation:

**Proposition 1.** \( \forall x_0 \in \mathcal{H}^1_{bc}(\Omega), \exists T > 0 \) such that (2.26) has a unique classical solution, \( x(t) \), defined \( \forall t \in [0, T] \), and

\[
x \in C^0([0, T]; \mathcal{H}^1_{bc}(\Omega)) \cap C^1([0, T]; L^2(\Omega))
\]
And there exists $C_0 > 0$ such that,

$$\|x\|_{C^0} \leq C_0\|x\|_{H^1}$$  \hspace{1cm} (2.30)

**Proposition 2.** Assume that $\forall T_0 > 0$, if $x$ is the unique local classical solution of (2.26) on $[0, T]$,

$$x \in C^0([0, T]; \mathcal{H}_bc^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

for $0 < T < T_0$, and the following uniform a-priori bound holds

$$\|x\|_{C^0} \leq K(T_0)$$

$$|x(t, 1)| \leq K_1(T_0), \forall t \in [0, T]$$  \hspace{1cm} (2.31)

for some $K(T_0), K_1(T_0) > 0$ and independent of $T$. Then (2.26) admits a unique global solution, i.e.,

$$x \in C^0([0, \infty); \mathcal{H}_bc^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$$

**Remark 1.** Proposition 1 gives the local existence and uniqueness result based on an abstract Cauchy problem setup and the function spaces definitions. In other words, if we could formulate the Raman amplifier problem in the same form, we can use Proposition 1 to show local existence and uniqueness of signal and pump power functions in time. Furthermore, if a priori uniform bounds are given for the power functions, then we can use Proposition 2 to get global existence and uniqueness of solutions in time.
Chapter 3

Problem Formulation

In this chapter, we present the Raman amplifier problem formulation. We consider the system with a signal laser and a pump laser propagating in the same direction, and the coefficients are normalized. The control objective is to use the downstream signal power as feedback to design the upstream pump power, in order to stabilize the signal and pump powers along the Raman amplifier.

In essence, this is an initial-boundary value problem (IBVP) where the control design determines the boundary conditions. We propose two simple feedback controls: a pure proportional controller, and a controller with integral actions. We assume that there exist steady state solutions for the signal and pump powers of the closed-loop system. And the stabilization of the system corresponds to the signal and pump converging to their steady state powers.

Instead of a finite-dimensional vector in a standard control problem, the state in our problem will be of infinite dimensions since it is a continuous function defined over the spatial span of the amplifier. In other words, the state space will be the normed function space as defined in Chapter 2. We introduce change of coordinates with respect to the given steady state functions.
of the system, and the point of convergence is now shifted to the origin of the function space.

### 3.1 Co-pumping Raman Amplifier Problem

![Block diagram of a co-pumping Raman amplifier](image)

Figure 3.1: Block diagram of a co-pumping Raman amplifier

We consider a normalized one-pump, one-signal Raman amplifier, as depicted in Figure 3.1. The length span of the amplifier is denoted by $\Omega = [0, 1]$; $s(t, z)$ denotes the power of the signal to be amplified, at time $t$ and distance $z$ along the amplifier, whereas $p(t, z)$ is the power of the signal generated by the pump to provide the desired gain. And $s$ and $p$ are both non-negative.

We use a feedback boundary controller, where the difference between the output signal power $s(t, 1)$, and the constant reference signal power $y_d$, is measured and used to generate the control signal, which is the input pump power, $p(t, 0)$. This is an economic setup since a Raman amplifier usually spans a few kilometers, and it is difficult to take measurement of the signal or pump power along its length.

The Raman amplifier dynamics can be described by the following first-order hyperbolic
system with nonlinear reaction,

\[
\frac{\partial s}{\partial t}(t, z) = -\frac{\partial s(t, z)}{\partial z} - s(t, z) + p(t, z)s(t, z)
\]

\[
\frac{\partial p}{\partial t}(t, z) = -\frac{\partial p(t, z)}{\partial z} - p(t, z) - p(t, z)s(t, z)
\]  \quad (3.1)

for \( t \geq 0, z \in \overline{\Omega} \). \( s \) and \( p \) are non-negative and stand for the signal and pump power at time \( t \) and distance \( z \) along the amplifier. We use the notation \( \overline{\Omega} = [0, 1] \) for the closed interval of the amplifier span.

The initial conditions are given by

\[
s(0, z) = s_0(z), \quad p(0, z) = p_0(z), \quad z \in \Omega
\]  \quad (3.2)

and the boundary conditions are given by

\[
s(t, 0) = \bar{s}(0), \quad p(t, 0) = g(s(t, 1), y_d) \quad \forall t \geq 0
\]  \quad (3.3)

where \( \bar{s}(0) \) is the steady-state upstream signal power, which will be explained in Section 3.2, and \( g \) is the control action to be designed. In the case of an output tracking problem, we would like the output signal power \( s(t, 1) \), to converge to the desired output setpoint, \( y_d \).

In order to obtain a feasible problem, it is also important that the initial conditions and the boundary conditions are compatible with each other. Hence we arrive at the compatibility
conditions such that

\[ s_0(0) = \bar{s}(0) \]

\[ p_0(0) = g(s_0(0), y_d) \]  \quad (3.4)

**Remark 2.** *The problem described by (3.1)-(3.3) is in fact in the form of (2.21)-(2.23) in Section 2.5. Therefore, if we let \( x(t) = [s(t, \cdot), p(t, \cdot)]^T \), then we can use Propositions 1 and 2 to demonstrate the existence and uniqueness of \( s(t, \cdot) \) and \( p(t, \cdot) \) locally, and globally if \( \|s(t, \cdot)\|_{C^0}, \|p(t, \cdot)\|_{C^0}, s(t, 1), \) and \( p(t, 1) \) are bounded.*

### 3.2 Change of Coordinates

Assume that a set of feasible and unique steady state signal and pump power levels exist for the closed-loop system given by (3.1) and (3.3), or mathematically,

**Assumption** \( \exists (\bar{s}, \bar{p}) \) unique and positive functions such that

\[
\begin{align*}
\frac{d\bar{s}}{dz}(z) &= -\bar{s}(z) + \bar{s}(z)\bar{p}(z) \\
\frac{d\bar{p}}{dz}(z) &= -\bar{p}(z) - \bar{s}(z)\bar{p}(z), \quad z \in \Omega
\end{align*}
\]  \quad (3.5)

and \((\bar{s}, \bar{p})\) satisfy the boundary conditions, namely, \( \bar{p}(0) = g(\bar{s}(1), y_d) \), for any given controller \( g \). We can denote the steady state in vector form,

\[
\bar{w}(z) = \begin{bmatrix} \bar{w}_1(z) \\ \bar{w}_2(z) \end{bmatrix} := \begin{bmatrix} \bar{s}(z) \\ \bar{p}(z) \end{bmatrix}
\]  \quad (3.6)
The feasibility condition (3.5) is a set of ODEs, in other words, \( \forall \bar{s}(0), \bar{p}(0) > 0, \exists \bar{s}(z) = \phi_1(\bar{s}(0), \bar{p}(0), z), \bar{p}(z) = \phi_2(\bar{s}(0), \bar{p}(0), z) \), where \( \phi_1, \phi_2 \) are continuous functions that can be uniquely determined.

In this case, at steady state,

\[
\bar{s}(1) = \phi_1(\bar{s}(0), \bar{p}(0), 1)
\]

Furthermore,

\[
\bar{p}(0) = g(\bar{s}(1), y_d) = g(\phi_1(\bar{s}(0), \bar{p}(0), 1), y_d)
\]

In other words, when given \( \bar{s}(1) \) and \( y_d \), the implicit function above can be solved and to uniquely determine \( \bar{p}(0) \). As shown in [21], such a steady state selection problem can be solved using extremum seeking technique to compute the optimal input pump power \( \bar{p}(0) \).

Now, we can introduce a change of variables with respect to the steady state solution, \( \bar{w}(z) \).

Let

\[
w(t, z) = \begin{bmatrix} w_1(t, z) \\ w_2(t, z) \end{bmatrix} := \begin{bmatrix} s(t, z) - \bar{s}(z) \\ p(t, z) - \bar{p}(z) \end{bmatrix}
\]

Then the PDEs (3.1) can be written in the new coordinates and in vector form,

\[
\frac{\partial}{\partial t} w(t, z) + a(z) \frac{\partial}{\partial z} w(t, z) + b(z) w(t, z) = f(w(t, z))
\]

for \( t \geq 0, z \in \Omega \), where

\[
a(z) := I, \quad b(z) := \begin{bmatrix} 1 - \bar{w}_2(z) & -\bar{w}_1(z) \\ \bar{w}_2(z) & 1 + \bar{w}_1(z) \end{bmatrix}, \quad f(w) := \begin{bmatrix} w_1 \quad w_2 \\ -w_1 \quad w_2 \end{bmatrix}
\]
which is of the form (2.21). Note that since $s, p > 0$, this implies that $w_1(t, z) \in (-\bar{w}_1(z), \infty)$ and $w_2(t, z) \in (-\bar{w}_2(z), \infty)$, for $t \geq 0$ and $z \in \Omega$.

The initial condition (3.2) becomes

$$w(0, z) = \begin{bmatrix} s_0(z) - \bar{w}_1(z) \\ p_0(z) - \bar{w}_2(z) \end{bmatrix} := w_0(z), \quad z \in \Omega \quad (3.10)$$

And the boundary condition (3.3) becomes

$$w(t, 0) = \begin{bmatrix} 0 \\ G_2(w(t, 1)) \end{bmatrix} := G_2(w(t, 1)), \quad t \geq 0 \quad (3.11)$$

where $G_2(w(t, 1))$ is the control action to be designed, but now in the new coordinates.

Finally, the compatibility condition (3.4) becomes

$$w_0(0) = \begin{bmatrix} 0 \\ G_2(w(0, 1)) \end{bmatrix}, \quad z \in \Omega \quad (3.12)$$

### 3.3 Choice of Boundary Control

#### 3.3.1 Proportional Controller

We propose two simple boundary controllers for which the stability conditions of their corresponding closed-loop systems will be analyzed. The first controller is a static proportional controller, given by the following:

$$p(t, 0) = g(s(t, 1), y_d) = k_1 (s(t, 1) - y_d) + \bar{p}(0) \quad (3.13)$$
where $k_1$ is the proportional gain parameter to be tuned. After change of coordinates, this corresponds to

$$w_2(t, 0) = G_2(w(t, 1)) = k_1 w_1(t, 1)$$  \hspace{1cm} (3.14)

### 3.3.2 Controller with Integral Action

The second controller we propose is a dynamic one, with integral action

$$p(t, 0) = g(s(t, 1), y_d) = k_1 s(t, 1) + k_2 \sigma(t)$$  \hspace{1cm} (3.15)

$$\dot{\sigma}(t) = -\varepsilon \sigma(t) + s(t, 1) - y_d$$

where $\varepsilon$ is a small positive parameter, ensuring almost tracking to the desired set points and rejection of step disturbances. After change of coordinates, this corresponds to

$$w_2(t, 0) = G_2(w(t, 1)) = k_1 w_1(t, 1) + k_2 \sigma(t)$$  \hspace{1cm} (3.16)

$$\dot{\sigma}(t) = -\varepsilon \sigma(t) + w_1(t, 1)$$

### 3.4 Control Objective

In this chapter, we first set up the initial-boundary-value problem given by (3.1-3.4), in terms of the signal and pump power functions, $s(t, z)$ and $p(t, z)$. Then, with the change of coordinates with respect to the steady-state solution, the new variable $w(t, z)$ is introduced, and the problem is rewritten as (3.8-3.12). Note that (3.8) is of the same form as (2.21), which can be
reformulated as an abstract Cauchy problem in the form of (2.26), by denoting

\[ w(t) := w(t, \cdot) \]  

(3.17)

as the system state. And

\[ w_0 := w_0(\cdot) \]  

(3.18)

as the initial state of the system. In other words, the state space is now a function space, as opposed to a vector space for ODEs. Then we can apply the existence and uniqueness results from Section 2.5. In addition, the convergence of the signal and pump power to the steady-state power, is now equivalent to an origin stabilization problem in the function space. Depending on the property of the space in which \( w(t) \) resides, the convergence of the system state can be derived using different function norms. For instance, if \( w(t) \in L^2(\Omega) \), then the convergence to the origin can be represented by \( \| w(t) \|_{L^2} \rightarrow 0 \) as \( t \rightarrow \infty \).
Chapter 4

Lyapunov Functional

In the previous chapter, we have formulated the one-pump one-signal Raman amplifier stabilization problem. Since the system state is a function, we cannot apply the stability analysis methods for finite-dimensional systems directly. Hence in this chapter, we present a Lyapunov functional dependent on the signal and pump powers along the amplifier and state its properties.

The Lyapunov functional is constructed based on the Lyapunov function used for a system of ODEs with Lotka-Volterra type nonlinearity [8]. The functional is positive definite on the domain on which the state is defined. Furthermore, it is bounded by the $\mathcal{L}^2$-norm of the system state.

4.1 Choice of Lyapunov Functional

Consider a two-dimensional system described by normalized Lotka-Volterra ODEs [8]:

\[
\begin{align*}
\dot{x}_1 &= -x_1 - x_1 x_2 \\
\dot{x}_2 &= -x_2 + x_1 x_2
\end{align*}
\]  

(4.1)
where \( x_i > 0 \).

Comparing with PDEs (3.1) that describe the Raman amplifier dynamics, we see that the nonlinear actions within the two systems are very similar.

Hence we propose a Lyapunov functional based on the Lyapunov function used for a normalized Lotka-Volterra system of ODEs, as described in (4.1),

\[
\mathbf{v}(\tilde{\mathbf{x}}) = \sum_i v_i(\tilde{x}_i(z)) \tag{4.2}
\]

where

\[
\tilde{\mathbf{x}} = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix} := \begin{bmatrix}
x_1 - \bar{x}_1 \\
x_2 - \bar{x}_2
\end{bmatrix}
\]

\( \bar{x}_1, \bar{x}_2 > 0 \) denote the steady-state solution of (4.1), and

\[
v_i(x) = x - \bar{x}_i \ln \left( \frac{x}{\bar{x}_i} + 1 \right) \tag{4.3}
\]

where \( v(x) \) is defined on \( x \in (-\bar{x}, \infty) \). The function \( v(x) \) is continuously differentiable and positive definite \( \forall x \in (-\bar{x}, \infty) \).

This can be extended to functions \( w(z) \) defined over \( \Omega \). Assume that we are given some \( \bar{w} \in H^1(\Omega) \subset C^0(\overline{\Omega}) \) such that \( \bar{w}(z) \) is positive over the length span. Then let

\[
\bar{w}_m = \min_{z \in \Omega}, \quad \| \bar{w} \|_{C^0} = \max_{z \in \Omega} \bar{w}(z) \tag{4.4}
\]

Similarly, we can obtain \( w_m \) and \( \| w \|_{C^0} \) for \( w \in H^1(\Omega) \). Then consider \( w \in H^1(\Omega) \) such that \( w_m > -\bar{w}_m \). Thus \( w(z) + \bar{w}(z) > 0, \forall z \in \overline{\Omega} = [0, 1] \), so that \( v(w(z)) \) is well-defined and continuously differentiable on \( \overline{\Omega} \).
Recall \( w(t, z) (3.7) \) as the deviation from the desired steady state \( \bar{w}(z) (3.6) \), and consider the entropy like function

\[
H(t) := \int_{\Omega} v(w(t, z)) \cdot e^{-\mu z} \, dz
\] (4.5)

where \( v \) is as defined in (4.2). The term \( e^{-\mu z}, \mu > 0 \), is instrumental to obtain a strict Lyapunov function, as in [18], [6].

Notice, however, that our control problem in Chapter 3 is formulated in terms of \( w(t) \in \mathcal{L}^2(\Omega) \). Hence the Lyapunov functional must be carefully constructed so that it can be used directly on our nonlinear hyperbolic system, and demonstrates critical properties that would lead to the convergence of the function norms of \( w(t) \) to 0.

Based on \( H (4.5) \), for our stability analysis, we propose the following Lyapunov functional,

\[
V(w) := \int_{\Omega} v(w) \cdot e^{-\mu z} \, dz
\] (4.6)

for any \( w \in \mathcal{H}^1_{bc}(\Omega) \subset \mathcal{C}^0(\Omega) \), such that \( w_m > -\bar{w}_m \). Since for any \( t_0 > 0 \), \( w(t_0) \) is a function of \( z \in \Omega \), \( v(w) \) is dependent on \( z \) as well.

### 4.2 Properties of Lyapunov Function for the Lotka-Volterra ODEs

In this section, we will discuss the properties of \( v(x) (4.3) \), the component used in constructing \( v(\bar{x}) \), the Lyapunov function for the Lotka-Volterra ODEs. This will lead to the discussion of the properties of \( V(w) \) in Section 4.3.

By analyzing \( v(x) \), we obtain the following lemma:
Lemma 1. Fix $\bar{x} > 0$, and define $v(x)$ as in (4.3),

(i) There is a positive function $d$ such that

$$v(x) \leq \gamma \Rightarrow |x| \leq d(\gamma)$$

(ii) For any $\bar{a}, \bar{b} > 0$ such that $\frac{1}{\bar{b}} < \bar{x} < \frac{1}{\bar{a}}$,

$$\frac{1}{2} \bar{a} |x|^2 \leq v(x) \leq \frac{1}{2} \bar{b} |x|^2$$

$\forall x \in \mathcal{R}$ where

$$\mathcal{R} := \left\{ x \in \mathbb{R} \mid \frac{1}{\bar{b}} - \bar{x} \leq x \leq \frac{1}{\bar{a}} - \bar{x} \right\} \Box$

Proof  (i) follows from the properties and graph of Lambert W (also known as product log) function [22]. Or, by evaluating the first and second derivatives of $v(x)$, we obtain that

$$\frac{dv}{dx} = 1 - \frac{\bar{x}}{x + \bar{x}}$$

$$\frac{d^2 v}{dx^2} = \frac{\bar{x}}{(x + \bar{x})^2}$$

It can be observed that on the domain over which $v(x)$ is defined, i.e., $x \in (-\bar{x}, \infty)$, $v(x)$ attains a global minimum of 0 at $x = 0$ because $v(x)$ is convex. Hence (i) is proven.

(ii) Consider the function

$$f(x) = v(x) - \frac{1}{2} \bar{a} |x|^2$$

where $\bar{a}$ is chosen such that $\frac{1}{\bar{a}} > \bar{x}$. Notice that $f(0) = 0$ and $f(x)$ is continuous and
twice differentiable on $x \in (-\bar{x}, \infty)$, where

$$f'(x) = \frac{x}{x + \bar{x}} - \bar{a}x$$

$$f''(x) = \frac{\bar{x}}{(x + \bar{x})^2} - \bar{a}$$

Then at $x = 0$, $f'(x) = 0$. And at $x = \frac{1}{\bar{a}} - \bar{x}$, $f'(x) = 0$ and $f''(x) = \bar{a}^2 (\bar{x} - \frac{1}{\bar{a}}) < 0$.

In other words, $f(x)$ attains its local maximum at $x = \frac{1}{\bar{a}} - \bar{x}$. Beyond this point, $f(x)$ decreases. Therefore $\forall x \in [0, \frac{1}{\bar{a}} - \bar{x}]$,

$$f(x) = v(x) - \frac{1}{2} \bar{a} |x|^2 \geq 0$$

Or equivalently,

$$\frac{1}{2} \bar{a} x^2 \leq v(x) \quad \forall x \in \left[0, \frac{1}{\bar{a}} - \bar{x}\right] \quad (4.7)$$

Now, consider the function

$$g(x) = \frac{1}{2} \tilde{b} |x|^2 - v(x)$$

where $\tilde{b}$ is chosen such that $\frac{1}{\tilde{b}} < \bar{x}$. Similar to $f(x)$, $g(x)$ is continuous and twice differentiable on $x \in (-\bar{x}, \infty)$, and

$$g'(x) = \tilde{b}x - \frac{x}{x + \bar{x}}$$

$$g''(x) = \tilde{b} - \frac{\bar{x}}{(x + \bar{x})^2}$$

Then $g'(0) = 0$. And at $x = \frac{1}{\tilde{b}} - \bar{x} < 0$, $g'(x) = 0$ and $g''(x) = \tilde{b}^2 \left(\frac{1}{\tilde{b}} - \bar{x}\right) < 0$. Hence
\[ \forall x \in [\frac{1}{b} - \bar{x}, 0], \]

\[
g(x) = \frac{1}{2} \bar{b} |x|^2 - v(x) \geq 0
\]

Or equivalently,

\[ \frac{1}{2} \bar{b} x^2 \geq v(x) \quad \forall x \in \left[ \frac{1}{b} - \bar{x}, 0 \right] \tag{4.8} \]

Combining (4.7) and (4.8), we have that

\[
\frac{1}{2} \bar{a} |x|^2 \leq v(x) \leq \frac{1}{2} \bar{b} |x|^2
\]

\[ \forall x \in \mathcal{R} \text{ where} \]

\[
\mathcal{R} := \left\{ x \in \mathbb{R} \mid \frac{1}{b} - \bar{x} \leq x \leq \frac{1}{a} - \bar{x} \right\} \quad \square
\]

Lemma 1 (i) links the Lyapunov function of the Lotka-Volterra system with the Euclidean norm of the state variable. And (ii) shows that the Lyapunov function is in fact bounded by the norm. This is significant because for our system, the origin stabilization problem has to be solved based on using function norms of the system state \( w \).

In addition, notice that we can increase the range of validity for the inequality from Lemma 1 by choosing smaller \( \bar{a} \) and larger \( \bar{b} \). In fact, the range \( \to (−\bar{x}, \infty) \) in the extreme. This shows that the inequality condition is semi-global.

### 4.3 Properties of Lyapunov Functional

The Lyapunov functional we have constructed for our problem (3.8-3.12), \( V(w) \), is given in (4.6). Before we discuss its properties, first we need to derive \( \dot{V}(w) \), a functional defined at a generic point \( w \) within the function state space, \( \mathcal{H}^1(\Omega) \), analogous to the case of an ODE control.
problem, i.e., the derivative of $V$ along the trajectories of the solution to (3.8-3.12).

Based on standard functional differentiability, the Gateaux differential, as defined in (2.19), $d_G V(w; \eta)$, of $V(w)$ in the direction of $\eta \in \mathcal{H}^1(\Omega)$ is defined as

$$d_G V(w; \eta) = \int_{\Omega} v_w(w(z)) \eta(z) e^{-\mu z} dz \quad (4.9)$$

Next introduce the functional $\dot{V}$ defined by

$$\dot{V}(w) = \int_{\Omega} v_w(w(z)) \eta(z) e^{-\mu z} dz \quad (4.10)$$

where $v_w(w(z))$ is the derivative of $v$ with respect to $w$, and

$$\eta = -a(z) \frac{dw}{dz}(z) - b(z) w(z) + f(w(z))$$

which comes from (3.8). Here, $\dot{V}(w)$ is taken as the Gateaux derivative at $w$ in the direction of $\eta$ given as above.

This definition is motivated as follows. For $w(t) \in C^1(\overline{\Omega})$ let as in [23]

$$\dot{V}(w(t)) := \lim_{\delta \to 0} \frac{1}{\delta} (V(w(t + \delta)) - V(w(t))) \quad (4.11)$$

Using (4.6), after interchanging the integration with the limit, yields

$$\dot{V}(w(t)) := \int_{\Omega} \lim_{\delta \to 0} \frac{1}{\delta} \left( v(w(t + \delta, z))e^{-\mu z} - v(w(t, z)) e^{-\mu z} \right) dz$$
or

\[
\dot{V}(w(t)) = \int_{\Omega} \frac{\partial}{\partial t} (v \circ w)(t, z) e^{-\mu z} dz
\]

Since \(v\) and \(w(t, z)\) are differentiable this yields

\[
\dot{V}(w(t)) = \int_{\Omega} v_w(w(t, z)) \frac{\partial w}{\partial t}(t, z) e^{-\mu z} dz
\]

and based on (3.8)

\[
\dot{V}(w(t)) = \int_{\Omega} v_w(w(t, z)) \left[-a(z) \frac{\partial w}{\partial z}(t, z) - b(z) w(t, z) + f(w(t, z))\right] e^{-\mu z} dz
\]

which leads to the definition of \(\dot{V}\) as in (4.10).

From Lemma 1 and (4.10), we arrive at the following result for properties of \(V\) and \(\dot{V}\):

**Lemma 2.** Given \(\bar{w} \in H^1(\Omega)\), such that \(\bar{w}_1(z), \bar{w}_2(z) > 0, \forall z \in \Omega\) and \(\|\bar{w}_2\|_{C^0} < 1\), consider \(V\), (4.6), and \(\dot{V}\), (4.10) defined for any \(w \in C^1(\Omega)\) that satisfies (3.12).

(i) Let any \(a_1, b_0 > 0\) such that \(\frac{1}{b_0} < \bar{w}_2(0), \frac{1}{a_1} > \bar{w}_1(1)\), and for any \(\mu > 0\) let \(\tilde{a}_1 = a_1 e^{-\mu}\).

For every \(w\) such that \(-\bar{w}_1(1) < w_1(1) \leq \frac{1}{a_1} - \bar{w}_1(1)\) and \(G_2(w(1)) \geq -\bar{w}_2(0) + \frac{1}{b_0}\),

\[
\dot{V}(w) \leq -\mu V(w) - \frac{1}{2} \tilde{a}_1 |w_1(1)|^2 + \frac{1}{2} b_0 |G_2(w(1))|^2 \tag{4.12}
\]

where \(G_2\) is the control action to be designed.

(ii) Let any \(\bar{a}, \bar{b} > 0\) such that \(\frac{1}{\bar{b}} < \bar{w}_m, \frac{1}{\bar{a}} > \|\bar{w}\|_{C^0}\), and for any \(\mu > 0\) let \(\alpha = \bar{a} e^{-\mu}, \beta = \bar{b}\).

Then for every \(w \in C^0(\Omega)\) such that \(\|w\|_{C^0} \leq d\), where \(d = \min\{\frac{1}{\bar{a}} - \|\bar{w}\|_{C^0}, \bar{w}_m - \frac{1}{\bar{b}}\}\),

the following holds
\[
\frac{1}{2} \alpha \|w\|_{L^2}^2 \leq V(w) \leq \frac{1}{2} \beta \|w\|_{L^2}^2
\] (4.13)

**Proof**  
(i) Evaluating \(\dot{V}(w)\), we obtain

\[
\dot{V}(w) = \int_\Omega \nabla_w(w(z)) \left( -\frac{dw}{dz}(z) + f(w(z)) - b(z) w(z) \right) e^{-\mu z} dz
\]
(4.14)

Note that

\[
\frac{d\nabla(w(z))}{dz} = \nabla_w(w(z)) \frac{dw}{dz}(z) + \nabla_{\bar{w}}(w(z)) \frac{d\bar{w}}{dz}(z)
\]

Using the foregoing to substitute for \(\nabla_w(w(z)) \frac{dw}{dz}(z)\) into (4.14) yields

\[
\dot{V}(w) = -\int_\Omega \frac{d\nabla(w(z))}{dz} e^{-\mu z} dz + \int_\Omega \nabla_{\bar{w}}(z) \frac{d\bar{w}}{dz}(z) e^{-\mu z} dz
\]

+ \int_\Omega \nabla_w(w(z))(f(w(z)) - b(z)w(z)) e^{-\mu z} dz

Hence

\[
\dot{V}(w) = -\int_\Omega \frac{d\nabla(w(z))}{dz} e^{-\mu z} dz + \int_\Omega \mathcal{G}(z) e^{-\mu z} dz
\]
(4.15)

where

\[
\mathcal{G}(z) := \nabla_{\bar{w}}(w(z)) \frac{d\bar{w}}{dz}(z) + \nabla_w(w(z)) \left( f(w(z)) - b(z)w(z) \right), \quad z \in \Omega
\]
(4.16)

and \(\nabla(w(z)) = \sum_i v_i(w_i(z))\). Integrating by parts in (4.15) yields

\[
\dot{V}(w) \leq - \nabla(w(z)) e^{-\mu z} \left. \right|_{z=0}^{z=1} - \mu V(w) + \int_\Omega \mathcal{G}(z) e^{-\mu z} dz
\]
(4.17)

To evaluate \(\mathcal{G}(z)\) (4.16), first note that with notation \(\bar{w}(z), b(z)\) as in (3.6,3.9), the steady-
The state equations (3.5) are rewritten as

\[
\frac{d\bar{w}_1}{dz}(z) = -\bar{w}_1(z) b_{11}(z), \quad \frac{d\bar{w}_2}{dz}(z) = -\bar{w}_2(z) b_{22}(z)
\]

where \( b_{11}(z) = (1 - \bar{w}_2(z)) \) and \( b_{22}(z) = (1 + \bar{w}_1(z)) \). Then the first term in \( G(z) \) (4.16) is written as

\[
\mathbf{v}_\bar{w}(w(z)) \frac{d\bar{w}}{dz}(z) = -\sum_i [\mathbf{v}_\bar{w}(w(z))]_i \bar{w}_i(z) b_{ii}(z)
\]

From the definition of \( \mathbf{v} \) and (4.3),

\[
[\mathbf{v}_\bar{w}(w(z))]_i = \frac{w_i(z)}{w_i(z) + \bar{w}_i(z)} \ln \left( \frac{w_i(z)}{\bar{w}_i(z)} + 1 \right), \quad [\mathbf{v}_w(w(z))]_i = \frac{w_i(z)}{w_i(z) + \bar{w}_i(z)}, \quad \forall i = 1, 2
\]

Using this together with (4.3) into foregoing yields

\[
\mathbf{v}_\bar{w}(w(z)) \frac{d\bar{w}}{dz}(z) = \sum_i \left[ \frac{w_i^2(z)}{w_i(z) + \bar{w}_i(z)} - v_i(w_i(z)) \right] b_{ii}(z)
\]

Using (3.9) the second term in \( G(z) \) (4.16) can be written as

\[
\mathbf{v}_w(w(z)) (f(u(z)) - b(z)w(z)) = -\sum_i \frac{w_i^2(z)}{w_i(z) + \bar{w}_i(z)} b_{ii}(z)
\]

Thus using the previous two expressions,

\[
G(z) = -b_{11}(z) v_1(w_1(z)) - b_{22}(z) v_2(w_2(z))
\]

Therefore, since \( v_1(w_1), v_2(w_2) \geq 0 \), and by assumption \( b_{11}(z) = 1 - \bar{w}_2(z) > 0, \forall z \in \overline{\Omega} \)
(since $||\bar{w}_2||_\infty < 1$), indeed $\mathcal{G}(z) \leq 0$, $\forall z \in \Omega$. Hence

$$\dot{V}(w) \leq -\mu V(w) - v(w(1)) e^{-\mu} + v(w(0))$$  \hspace{2cm} (4.19)$$

Using compatibility condition (3.12), (4.19) can be rewritten as

$$\dot{V}(w) \leq -\mu V(w) - v(w_1(1)) e^{-\mu} + v(G_2(w(1)))$$  \hspace{2cm} (4.20)$$

Finally, using Lemma 1 (ii), since $\frac{1}{a_1} > \bar{w}_1(1)$ and $\frac{1}{b_0} < \bar{w}_2(0)$, then $\forall w$ such that

$$-\bar{w}_1(1) < w_1(1) \leq \frac{1}{a_1} - \bar{w}_1(1),$$

$$G_2(w(1)) \geq \bar{w}_2(0) + \frac{1}{b_0}$$

we have

$$v(w_1(1)) \geq \frac{1}{2} a_1 |w_1(1)|^2$$

$$v(G_2(w(1))) \leq \frac{1}{2} b_0 |G_2(w(1))|^2$$

Therefore,

$$\dot{V}(w) \leq -\mu V(w) - \frac{1}{2} a_1 |w_1(1)|^2 e^{-\mu} + \frac{1}{2} b_0 |G_2(w(1))|^2$$

$$= -\mu V(w) - \frac{1}{2} \tilde{a}_1 |w_1(1)|^2 + \frac{1}{2} b_0 |G_2(w(1))|^2$$  \hspace{2cm} (4.21)$$
(ii) Given the choice of $d$, $\forall \|w\|_{C^0} \leq d$,

\[
\frac{1}{b} - \bar{w}_i(z) \leq w_i(z) \leq \frac{1}{a} - \bar{w}_i(z)
\]

$\forall z \in \Omega$. Using Lemma 1 (i), this indicates that

\[
\frac{1}{2} a |w_i(z)|^2 \leq v_i(w_i(z)) \leq \frac{1}{2} b |w_i(z)|^2
\]

$\forall z \in \Omega$, where $| \cdot |$ is the absolute value notation. Hence

\[
\frac{1}{2} a \sum_i |w_i(z)|^2 \leq v_i(w(z)) \leq \frac{1}{2} b \sum_i |w_i(z)|^2
\]

Therefore, given the choice of $\alpha$,

\[
\frac{1}{2} \alpha \|w\|_{L_2}^2 = \frac{1}{2} a e^{-\mu} \int_{\Omega} \sum_i |w_i(z)|^2 \, dz \\
\leq \int_{\Omega} \frac{1}{2} a \sum_i |w_i(z)|^2 e^{-\mu z} \, dz \\
\leq \int_{\Omega} \sum_i v_i(w_i(z)) e^{-\mu z} \, dz \\
\leq V(w)
\]

Similarly, it can be shown that given the expression of $\beta$, $V(w) \leq \frac{1}{2} \beta \|w\|_{L_2}^2$, and the inequality (4.13) is proven. □

Lemma 2 (i) shows that $\dot{V}$, the derivative of the Lyapunov functional along the system trajectory, has an upper bound dependent on $V$, the Lyapunov functional itself, and other terms that incorporate $w$ evaluated at $\partial \Omega$ only (i.e., $z = 0$ and $z = 1$).
This means that with the right boundary controller, we should be able to prove at least local asymptotic stability of the origin using Lyapunov-based analysis.

Lemma 2 (ii) links the Lyapunov functional with the $L^2$-norm of the system state $w$. Note, however, that the domain on which the inequality holds is bounded by the $C^0$-norm instead. Hence space embedding might be necessary when performing stability analysis.
Chapter 5

Stability Analysis

In this chapter, we will present the results of our stability analysis when the static proportional controller (3.14) and the controller with integral action (3.16) are used. We will use the constructed Lyapunov functional candidate (4.6) with its derived properties as stated in Lemma 1 and 2 in Chapter 4. The analysis is analogous to Lyapunov’s stability theorem for a control problem described by ODEs [24].

If we examine the proof for Lyapunov’s stability theorem as presented in [24], we can see that to demonstrate the asymptotic stability of the origin, i.e., the system state $x(t) \to 0$ as $t \to \infty$, it is neccessary to relate the level sets of the Lyapunov function to sets within the state space in which the Euclidean norm of the system state is bounded. On the other hand, for any $T_0 > 0$, our system state $w(T_0)$ is a function defined on $\Omega$. Therefore we must discuss the convergence of $w(t)$ in terms of certain function norms. Specifically, we have chosen to consider the $C^0$- and $L^2$-norms, defined in (2.2) and (2.7) respectively. From Chapter 4, it is observed that the properties of $V(w)$ derived are mostly related to the $C^0$-norm and the $L^2$-norm of $w(t)$. As a consequence, the analysis also results in stability in terms of these two norms.
5.1 Stability Results for Proportional Control

In this section we consider the case with a static proportional controller, as given in (3.14), \( G_2(w) = k_1 \, w_1(t, 1) \). Using this expression and Lemma 2, we can arrive at the following stability result:

**Theorem 1.** Consider the closed-loop system (3.8-3.12), (3.14), and assume that \( \| \bar{w}_2 \|_{C^0} < 1 \).

For any \( a_1, b_0, \mu > 0 \) such that \( \frac{1}{a_1} > \bar{w}_1(1) > 0, \frac{1}{b_0} < \bar{w}_2(0) \), let \( \tilde{a}_1 = a_1 e^{-\mu} \), select any \( 0 < d_1 < \min\{ \bar{w}_m, \frac{1}{a_1} - \bar{w}_1(1) \} \) where \( \bar{w}_m \) is defined in Lemma 2, and choose \( k_1 \) such that

\[
\begin{align*}
 k_1^2 &< \frac{\tilde{a}_1}{b_0} \quad \text{and} \quad -\frac{\bar{w}_2(0) - 1/b_0}{1/a_1 - \bar{w}_1(1)} < k_1 < \frac{\bar{w}_2(0) - 1/b_0}{\bar{w}_1(1)} \\
&
\end{align*}
\]

(5.1)

(i) For any initial condition \( w_0 \in \mathcal{H}_{bc}^1(\Omega) \) such that \( \| w_0 \|_{\mathcal{H}^1} \leq d_1/C_0 \), where \( C_0 \) is the constant in Proposition 1 from Chapter 2,

\[
V(w(t)) \leq e^{-\mu t} V(w_0), \quad \forall t \in [0, T]
\]

as long as \([0, T] \) is, as given in Proposition 1, the time span for which the solution \( w(t) \) exists, and \( \|w(t)\|_{C^0} \leq d_1 \).

Moreover, there exists \( \delta_{01} > 0 \) such that, for any initial condition \( w_0 \in \mathcal{H}_{bc}^1(\Omega) \) with \( \| w_0 \|_{\mathcal{H}^1} \leq \delta_{01} \leq d_1/C_0 \), the classical solution exists globally over \([0, \infty) \) and the origin is asymptotically stable in the \( C^0 \)-norm.

(ii) There exists \( 0 < \delta_0 \leq \delta_{01} \) such that, for any initial condition \( w_0 \in \mathcal{H}_{bc}^1(\Omega) \) with
\[ \|w_0\|_{H^1} \leq \delta_0, \text{ the } L^2\text{-norm of } w(t) \text{ satisfies } \forall t \in [0, \infty) \]

\[ \|w(t)\|_{L^2} \leq C e^{-\mu t}\|w(0)\|_{L^2}, \quad \forall t \geq 0 \]

for some constant \( C > 0 \), hence the origin is exponentially stable in the \( L^2\)-norm.

**Remark 3.** To select the proportional gain \( k_1 \) in practice, we can first choose the parameters \( a_1, b_0, \) and \( \mu \) based on the steady state boundary values \( \bar{w}_1(1) \) and \( \bar{w}_2(0) \). Then an explicit range can be derived for \( k_1 \) using the expression given in (5.1).

The proofs of both (i) and (ii) follow the format of the proof of Lyapunov’s Stability Theorem (Theorem 4.1, pp. 114-116) in [24]. We first establish a set of \( w \) on which its function norm is bounded, and relate it to level sets of the Lyapunov functional. Then we show that with the stated assumptions, for any initial condition \( w_0 \), as defined in 3.18, within a norm-bounded space, the corresponding solution \( w(t) \) stays within a certain level set of the Lyapunov functional. Finally, the Lyapunov functional is linked back to the function norm of \( w(t) \) to demonstrate that the norm of the solution indeed remains bounded.

**Proof** (i) From Proposition 1, we know that for any \( w_0 \in H_{bc}^1(\Omega) \) there exists \( T > 0 \) such that a unique classical solution \( w(t) \in H_{bc}^1(\Omega) \) exists for \( \forall t \in [0, T] \). Now, let

\[ D_1 := \{ u \in H_{bc}^1(\Omega) \left\| u \right\|_{C^0} \leq d_1 \} \quad (5.2) \]

where \( d_1 \) is defined in the statement of the theorem. Then, from the boundary condition (3.11), the form of the proportional controller (3.14), and (i) of Lemma 2, for any \( a_1, b_0 > \)
Given that \( w \in D_1, \|w\|_{\mathcal{C}^0} \leq d_1, \) and the first inequality is immediately satisfied. And since \( \frac{1}{b_0} < \bar{\omega}_2(0), -\bar{\omega}_2(0) + \frac{1}{b_0} > 0. \) With the choice of \( k_1 \) given in (5.1), the second inequality is satisfied as well.

In this case,

\[
\dot{V}(w) \leq -\mu V(w) - \frac{1}{2} (\bar{a}_1 - b_0 k_1^2) w_1^2(1)
\]

Since \( k_1^2 < \frac{a_1}{b_0}, \forall w \in D_1, \) we know that

\[
\dot{V}(w) \leq -\mu V(w) \tag{5.3}
\]

Next, pick any initial condition \( w(0) \in \mathcal{H}_{bc}^1(\Omega) \) with \( \|w_0\|_{\mathcal{H}_c^1} \leq \frac{d_1}{C_0} \), where \( C_0 \) is given in Proposition 1. Then by Sobolev embedding theorem (2.14), \( \|w_0\|_{\mathcal{C}^0} \leq d_1 \) and \( w_0 \in D_1 \).

Then, from (5.3),

\[
\dot{V}(w_0) \leq -\mu V(w_0)
\]

Furthermore, as long as \( w(t) \) remains in \( D_1 \), (5.3) holds.

Next it is shown that for sufficiently small initial conditions, there exists \( K > 0 \) finite, independent of \( T \) such that the bound (2.31) holds, so that by Proposition 2 the solution
exists globally in time over $[0, \infty)$. Specifically, such a bound is shown to hold for any initial condition $w_0$ inside a sub-level set of $v$ (4.3).

Consider $h : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^+$ defined for each $(t, z)$ by

$$h(t, z) := (v \circ w)(t, z) = v(w(t, z))$$

Then using (2.26), the definition of the operator $L$ (2.27) and $a$ (3.9),

$$\frac{\partial h}{\partial t} = v_u(w(t, z)) \left( -\frac{\partial w}{\partial z}(t, z) + f(w(t, z)) - b(z) w(t, z) \right)$$

Note that the right-hand side is similar to the integrand term on the right-hand side of (4.14). Then after some manipulations as in the proof of (i), Lemma 2, it can be shown that

$$\frac{\partial h}{\partial t}(t, z) \leq -\frac{\partial h}{\partial z}(t, z) + g(t, z), \quad t \in [0, T], \ z \in \overline{\Omega} \quad (5.4)$$

where

$$g(t, z) := v_\bar{w}(w(t, z)) \frac{d\bar{w}}{dz}(z) + v_w(w(t, z)) \left( f(w(t, z)) - b(z) w(t, z) \right)$$

Recalling (4.16), note that $g(t, z)$ is the same as $G(z)$ evaluated for $w(z) = w(t, z)$. Then using (4.18),

$$g(t, z) = -b_{11}(z) v_1(w_1(t, z)) - b_{22}(z) v_2(w_2(t, z)), \quad t \in [0, T], \ z \in \overline{\Omega}$$

Thus by $b_{ii}(z) > 0$, $g(t, z) \leq 0$, $\forall t \in [0, T], \ z \in \overline{\Omega}$ and, from (5.4), $h(t, z)$ satisfies the
scalar linear differential inequality

\[
\frac{\partial h}{\partial t}(t, z) + \frac{\partial h}{\partial z}(t, z) \leq 0
\]  

(5.5)

with initial conditions \( h(0, z) = v(w_0(z)) \). Moreover, from (3.14), it follows that \( h(t, 0) = v_2(w_2(t, 0)) = v_2(k_1w_1(t, 1)) \). Using the estimates for \( v \) in (ii), Lemma 1, with \( k_1^2 < \tilde{a}_1/b_0 \), it follows that \( h(t, 0) = v_2(k_1w_1(t, 1)) \leq b_0k_1^2w_1^2(t, 1)/2 < \tilde{a}_1w_1^2(t, 1)/2 \leq v_1(w_1(t, 1)) \leq v(w(t, 1)) = h(t, 1) \). Thus on \( D_1 \), \( h(t, 0) < h(t, 1) \).

Recall (5.5) and consider the solution \( q(t, z) \) to the scalar linear equation

\[
\frac{\partial q}{\partial t}(t, z) + \frac{\partial q}{\partial z}(t, z) = 0
\]  

(5.6)

with initial conditions \( q(0, z) \) and boundary conditions \( q(t, 0) - q(t, 1) = 0 \). Note that (5.6) has a travelling wave solution, \( q(t, z) = q(0, z - t) \). Consider \( q(0, z) = \gamma \), \( \forall z \in \bar{\Omega} \), hence the constant solution of (5.6) \( q(t, z) = \gamma \), \( \forall t \in [0, T] \), \( \forall z \in \bar{\Omega} \). Now, from (5.5, 5.6), \( h(t, z) \) satisfies

\[
\frac{\partial h}{\partial t}(t, z) + \frac{\partial h}{\partial z}(t, z) \leq \frac{\partial q}{\partial t}(t, z) + \frac{\partial q}{\partial z}(t, z)
\]

and \( h(t, 0) - h(t, 1) < q(t, 0) - q(t, 1) \). By a scalar comparison lemma for differential inequalities (Theorem 1.2-4, p. 19, [25]), it follows that, if \( v(w_0(z)) = h(0, z) \leq q(0, z) = \gamma \), \( \forall z \in \bar{\Omega} \), then \( v(w(t, z)) = h(t, z) \leq q(t, z) = \gamma \), \( \forall z \in \bar{\Omega}, t \in [0, T] \), on \( D_1 \). Using the definition of \( v \) it follows that if \( v(w_0(z)) = h(0, z) \leq q(0, z) = \gamma \), \( \forall z \in \bar{\Omega} \), then \( \forall i \), \( v_i(w_i(t, z)) \leq \gamma \), \( \forall z \in \bar{\Omega}, t \in [0, T] \). Hence, by applying (i) in Lemma 1, it follows that
\forall i, |w_i(t, z)| \leq d(\gamma), \forall z \in \Omega, t \in [0, T], \text{ or } \|w(t)\|_\infty \leq d(\gamma), \forall t \in [0, T], \text{ as long as the solution remains in } D_1.

It remains to show how \( w_0 \) can be picked such that the solution \( w(t) \) stays in \( D_1 \). Recall from Lemma 1 (i), \( \forall \gamma > 0, \exists d(\gamma) > 0 \) finite such that

\[ v(x) \leq \gamma \Rightarrow |x| \leq d(\gamma) \]

Pick \( \gamma, d_0 > 0 \) such that \( d(\gamma) < d_1 \), and \( 0 < d_0 \leq d_1 \), where for any

\[ w_0 \in D_0 := \{w_0 \in H^1_{bc}(\Omega) | \|w_0\|_{C^0} < d_0\}, \quad \|v(w_0)\|_{C^0} \leq \gamma \tag{5.7} \]

Now, let \( \delta_{01} = \frac{d_0}{C_0} \leq \frac{d_1}{C_0} \) for some \( C_0 > 0 \). Then, for any \( w_0 \in H^1_{bc}(\Omega) \) with \( \|w_0\|_{H^1} \leq \delta_{01} \), by Sobolev embedding theorem, \( \|w_0\|_{C^0} \leq d_0 \), so that \( w_0 \in D_0 \) and \( \|v(w_0)\|_{C^0} \leq \gamma \).

For \( \gamma \) chosen as above it follows that \( h(t, z) = v(w(t, z)) \leq \gamma, \forall t \in [0, T], \forall z \in \Omega \), and

\[ \|w(t)\|_{C^0} \leq d(\gamma) < d_1 \quad \forall t \in [0, T] \tag{5.8} \]

Therefore,

\[ (\forall w_0 \in D_0) \Rightarrow w(t) \in D_1 \tag{5.9} \]

Since we have shown that the solution exists globally in time, we can now analyze its asymptotic properties.

Recall \( g(t, z) \) defined in (5.4). Since \( b_{11}(z) > 0, b_{22}(z) > 0 \), for \( \forall z \in \Omega = [0, 1] \), there
exists $\varepsilon > 0$ such that $b_{11}(z) \geq \varepsilon, b_{22}(z) \geq \varepsilon$, hence,

$$g(t, z) \leq -\varepsilon \left[ v_1(w_1(t, z)) + v_2(w_2(t, z)) \right] = -\varepsilon h(t, z), \quad \forall z \in \overline{\Omega}$$

Proceeding similarly as in (5.4, 5.5, 5.6) consider $q(t, z)$ the solution to

$$\frac{\partial q}{\partial t}(t, z) + \frac{\partial q}{\partial z}(t, z) + \varepsilon q(t, z) = 0 \quad (5.10)$$

for $\varepsilon > 0, z \in \overline{\Omega}$. Note that (5.10) is a scalar linear hyperbolic equation with dissipation term and boundary dissipation [26], p. 136. Its classical solution exists globally and

$$\|q(t, \cdot)\|_{C^0} \leq e^{-\varepsilon t}\|q_0\|_{C^0}, \quad \forall t \geq 0.$$ 

Then, by the same scalar comparison lemma [25] for $h$ and this $q$ above, if $v(w_0(z)) = h(0, z) \leq q_0(z), \forall z \in \overline{\Omega}$, then $v(w(t, z)) = h(t, z) \leq q(t, z) \leq e^{-\varepsilon t}\|q_0\|_{C^0}$, for every $z \in \overline{\Omega}, \forall t \geq 0$. Since $v$ is continuous and positive definite this implies that $\lim_{t \to \infty} v(w(t, z)) = 0$ point-wise for every $z \in \overline{\Omega}$, and $v(\lim_{t \to \infty} w(t, z)) = 0$ point-wise for every $z \in \overline{\Omega}$. Using $v(0) = 0$ it follows that $\lim_{t \to \infty} |w(t, z)|_{C^0} = 0$ point-wise for every $z \in \overline{\Omega} = [0, 1]$, hence $\lim_{t \to \infty} \|w(t, \cdot)\|_{C^0} = 0$ and asymptotic stability in the $C^0([0, 1])$-norm is proven.

(ii) For (ii), the first half of the proof is similar to (i). Given $d_1$ and $D_1$ as defined in (5.2), we can show that $\forall w \in D_1$,

$$\dot{V}(w) \leq -\mu V(w)$$

Next, recall Lemma 2 (ii), let

$$\mathcal{D} := \{ w \in H^1_{bc} ||w||_{C^0} \leq d \} \quad (5.11)$$
and let \( d_1 = \min\{d_1, d\} \), and correspondingly the set \( D_1 \), then \( D_1 \subset D_1 \cap D \). Next, recall from Lemma 1 (i), for a given \( d_1 \), pick any \( \gamma > 0 \) such that \( d(\gamma) < d_1 \).

Then, pick \( 0 < d_0 \leq d_1 \) such that the condition (5.7) is met, i.e., \( \forall w_0 \in D_0, \|v(w_0)\|_c^0 \leq \gamma \) on \( D_1 \subset D_1 \). Now, let an arbitrary \( \epsilon > 0 \) and pick any \( 0 < \delta_0 < \min \left\{ \sqrt{\frac{\alpha}{\beta}} \epsilon, \frac{d_0}{C_0} \right\} \), where \( \alpha, \beta > 0 \) are as chosen to satisfy Lemma 2 (ii). Then, by Sobolev Embedding Theorem, for any \( w_0 \in H_{bc}^1 \) with \( \|w_0\|_{H^1} \leq \delta_0 \), we have

\[
\|w_0\|_{L^2} \leq \delta_0 \leq \sqrt{\frac{\alpha}{\beta}} \epsilon, \quad \text{and}
\]

\[
\|w_0\|_c^0 \leq C_0 \|w_0\|_{H^1} \leq C_0 \delta_0 \leq \delta_0
\]

so that \( w_0 \in D_0 \). Then we can see that (5.9) also holds for (ii).

Furthermore, since \( D_0 \subset D \), by using RHS of (4.12) from Lemma 2 (ii),

\[
V(w_0) \leq \frac{\beta}{2} \|w_0\|_{L^2}^2 \leq \frac{\beta}{2} \delta_0^2 \leq C
\]

where \( C = \frac{\alpha \epsilon^2}{2} \). We can apply (5.9) on \( D_1 \) to show that \( w(t) \in D_1 \subset D_1, \forall t \geq 0 \).

Thus by (5.3), \( \forall t \geq 0 \),

\[
V(w(t)) \leq V(w(0)) \leq C
\]

Since \( w(t) \in D_1 \subset D_1 \), using LHS of (4.12) for \( C = \frac{\alpha \epsilon^2}{2} \) yields

\[
\|w(t)\|_{L^2} \leq \epsilon \quad \forall t \geq 0
\]
For any such \( w_0 \), since \( w(t) \in \mathcal{D}_1 \subset \mathcal{D}_1 \), from (5.3), we obtain

\[
V(w(t)) \leq e^{-\mu t} V(w_0) \quad \forall t \geq 0
\]

Since \( w_0 \in \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_1 \), and \( w(t) \) remains in \( \mathcal{D}_1 \subset \mathcal{D}_1 \), \( \forall t \geq 0 \), we can use (4.12) on both sides so that

\[
\frac{1}{2} \alpha \| w(t) \|^2_{L^2} \leq V(w(t)) \leq e^{-\mu t} V(w_0) \\
\leq \frac{1}{2} \beta e^{-\mu t} \| w_0 \|^2_{L^2}
\]

Thus \( \| w(t) \|^2_{L^2} \leq \sqrt{\frac{\beta}{\alpha}} e^{-\mu t} \| w_0 \|^2_{L^2} \). Hence exponential asymptotic stability in \( L^2 \)-norm is proven. □

Notice that the nature of the system and the Lyapunov functional \( V(w) \) invoke bounds based on the \( L^2 \)-norm of \( w(t) \), but the bounds are valid on domains constrained by its \( C^0 \)-norm.

As a result, in (i) of Theorem 1, even though we have obtained the inequality \( \dot{V} \leq -\mu V \), we are only able to demonstrate asymptotic stability of the origin in terms of the \( C^0 \)-norm.

**Remark 4.** In (ii), using the Sobolev embedding theorem, we are able to construct space intersections on which the \( C^0 \)-norm and the \( L^2 \)-norm of \( w(t) \) are both defined. And with \( V(w) \) being bounded by the \( L^2 \)-norm of \( w \), we are in fact able to show that \( \| w(t) \|^2_{L^2} \) is attenuated with time, as given by (5.12). Hence exponential asymptotic stability of the origin can be derived in terms of the \( L^2 \)-norm.

Lastly, (5.1) gives bounds on the proportional gain \( k_1 \) which depend on parameters \( a_1, b_0, \) and \( \mu \). If we choose these parameters such that the bounds on \( k_1 \) are relaxed, it indirectly
determines $\delta_0$, and hence shrinks the size of the domain in which the stable initial condition $w_0$ resides. Conversely, tightening the bounds for $k_1$ increases the domain. In fact, when $k_1 \to 0$, the stability domain is global.

5.2 Stability Results for Control with Integral Action

In this section we consider a controller with integral action, as given in (3.16):

\[
\begin{align*}
    w_2(t, 0) &= G_2(w(t, 1)) = k_1 w_1(t, 1) + k_2 \sigma(t) \\
    \dot{\sigma}(t) &= -\varepsilon \sigma(t) + w_1(t, 1)
\end{align*}
\]

However, to take into consideration of the extra state $\sigma$, the Lyapunov functional is also augmented to incorporate a $\sigma^2$ term, i.e.,

\[
    W(w, \sigma) = V(w) + r \frac{\sigma^2}{2} \quad (5.13)
\]

The stability result is as follows:

**Theorem 2.** Consider the system (3.8-3.12), (3.16), and assume all parameters $a_1, b_0, \mu, k_1$ are selected as in Theorem 1. For any $\varepsilon > \frac{\mu}{2}$, set $\bar{\varepsilon} = \varepsilon - \frac{\mu}{2} > 0$, and select $0 < r < 2 \bar{\varepsilon} (\bar{a}_1 - b_0 k_1^2)$. Finally, choose $k_2$ such that $K_2 < k_2 < K_2$, where $K_2, \bar{K}_2$ are the roots of the following quadratic expression,

\[
    \phi(k_2) = -\bar{a}_1 b_0 k_2^2 - 2 r b_0 k_1 k_2 + 2 r \bar{\varepsilon} (\bar{a}_1 - b_0 k_1^2) - r^2 \quad (5.14)
\]
Note that the coefficients of $\phi(k_2)$ can be predetermined based on the choice of $\tilde{a}_1, b_0, k_1,$ and $r$.

Then, $\exists \delta_0 > 0$ such that, for any initial condition $w_0 \in \mathcal{H}^1_{bc}(\Omega)$ with $\|w_0\|_{\mathcal{H}^1} \leq \delta_0$, the classical solution exists globally over $[0, \infty)$ and the origin is exponentially stable in the $L^2$-norm.

**Remark 5.** Since the dynamic controller is dependent on the parameters $k_1, k_2$, and $\varepsilon$, to select them properly according to the conditions stated in the theorem, we can first choose $a_1, b_0, \mu,$ and $k_1$ as in Theorem 1. Then we select $\varepsilon$ based on the choice of $\mu$, $r$ based on the choice of $a_1$, $b_0$, $k_1$, and $\varepsilon$. And finally, we will be able to compute the range of the integral gain $k_2$ based on the value of these previously chosen parameters, since the bounds of $k_2$ are simply the roots of the quadratic expression (5.14)

**Proof** The proof of Theorem 2 will be similar to that of Theorem 1 (ii). However, because the Lyapunov functional has been augmented, we must evaluate $\dot{W}(w)$. Using Lemma 2 and (3.16), we obtain that

$$\dot{W}(w) \leq -\mu W(w) - \frac{1}{2} \hat{a}_1 w_1^2(1) + \frac{1}{2} b_0 k_1^2 w_1^2(1) + b_0 k_1 k_2 w_1(1) \sigma$$

$$+ \frac{1}{2} b_0 k_2^2 \sigma^2 - r \varepsilon \sigma^2 + r w_1(1) \sigma$$

$$= -\mu W(w) - \frac{1}{2} (\hat{a}_1 - b_0 k_1^2) w_1^2(1) + (b_0 k_1 k_2 + r) w_1(1) \sigma - (r \tilde{\varepsilon} \sigma) + \frac{1}{2} b_0 k_3^2 \sigma^2$$

$$= -\mu W(w) - \frac{1}{2} \begin{bmatrix} w_1(1) & \sigma \end{bmatrix} M \begin{bmatrix} w_1(1) \\ \sigma \end{bmatrix}$$

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where

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} = \begin{bmatrix}
\tilde{a}_1 - b_0 k_1^2 & -r - b_0 k_1 k_2 \\
-r - b_0 k_1 k_2 & 2 r \tilde{\varepsilon} - b_0 k_2^2
\end{bmatrix}
\]

For the \(k_1\) chosen as in Theorem 1, \(M_{11} > 0\). Evaluating the \(\det(M)\) will yield that \(\det(M) = \phi(k_2)\). Given the expression of \(\phi(k_2)\), we can compute its discriminant to be

\[
dis(\phi) = 4 r b_0 (\tilde{a}_1 - b_0 k_1^2)(2 \tilde{\varepsilon} b_0 \tilde{a}_1 - r)
\]

Since \(k_1^2 < \frac{\tilde{a}_1}{b_0}, \tilde{a}_1 - b_0 k_1^2 > 0\). And with the choice of \(r\) as in the statement of the theorem, \(r > 0\) and

\[2 \tilde{\varepsilon} b_0 \tilde{a}_1 - r > 2 \tilde{\varepsilon} b_0 \tilde{a}_1 - 2 \tilde{\varepsilon} (\tilde{a}_1 - b_0 k_1^2) > 0\]

Hence \(\phi(k_2)\) has two distinct real roots, denoted by \(K_2^{+}\) and \(K_2^{-}\). Furthermore, \(-\tilde{a}_1 b_0 < 0\), therefore \(\phi(k_2) > 0\) for \(K_2^{-} < k_2 < K_2^{+}\).

Using the quadratic formula to evaluate the roots, we can show that given \(K_2^{-} < k_2 < K_2^{+}\),

\[2 r \tilde{\varepsilon} - b_0 k_2^2 > 0\]. In otherwords, \(\det(M) > 0\), and \(M_{11}, M_{22} > 0\), which means that \(M\) is positive definite. Hence we have

\[
\dot{W}(w) \leq -\mu W(w)
\]

Then, by choosing the same \(\delta_0\) as in Theorem 1 (ii) and following similar steps in its proof, we know that \(\|w_0\|_{\mathcal{L}^2} \leq \delta_0, \exists \epsilon > 0\) where \(\|w(t)\|_{\mathcal{L}^2} \leq \epsilon\) and more importantly, by using (4.12) from Lemma 2, we can conclude that \(\|w(t)\|_{\mathcal{L}^2}^2 \leq \frac{\beta}{\alpha} e^{-\mu t} \|w_0\|_{\mathcal{L}^2}^2\). Hence exponential asymptotic stability in \(\mathcal{L}^2\)-norm is proven. □
Remark 6. For Theorem 2, the Lyapunov functional has to be modified to incorporate the controller state variable \( \sigma \). In other words, the new Lyapunov functional is dependent on the augmented closed-loop system state, \((w, \sigma)\). This led to the different approach used in the proof comparing to Theorem 1. Here we formulated the square term

\[
-\frac{1}{2} \begin{bmatrix} w_1(1) \\ \sigma \end{bmatrix} M \begin{bmatrix} w_1(1) \\ \sigma \end{bmatrix}
\]

and demonstrated that \( M \) is positive definite. Although this resulted in more complex bounds for the integral gain \( k_2 \), it also provides an insight into how systems of a larger scale (i.e., with more signals/pumps) can be treated.
Chapter 6

Generalization to Systems with Coefficients

From Chapter 3 to Chapter 5, we set up the normalized one-pump one-signal Raman amplifier problem, proposed a Lyapunov functional, and performed stability analysis on the closed-loop system to derived conditions for asymptotic stability in the $C^0$-norm, and exponential stability in the $L^2$-norm. In this chapter, we extend the Lyapunov-based analysis to a system with coefficients. We follow the same steps as in the previous chapters to derive stability conditions for the proportional controller and the controller with integral actions. This will provide a more practical recipe for boundary control design for the Raman amplifier power stabilization problem.
6.1 Problem Setup

Consider the following $2 \times 2$ first-order semilinear hyperbolic system with nonlinear reaction

\[
\begin{align*}
\frac{\partial s}{\partial t}(t, z) &= -\lambda_1 \frac{\partial s(t, z)}{\partial z} - \alpha_1 s(t, z) + c_1 p(t, z)s(t, z) \\
\frac{\partial p}{\partial t}(t, z) &= -\lambda_2 \frac{\partial p(t, z)}{\partial z} - \alpha_2 p(t, z) - c_2 p(t, z)s(t, z)
\end{align*}
\]  

(6.1)

for $t \geq 0$, $z \in \Omega$. $\lambda_i$, $\alpha_i$, and $c_i$ are coefficients dependent on the wavelengths of the pump and the signal. And we can assume that $0 < \lambda_1 < \lambda_2$ without loss of generality.

Next, the initial condition of the system is given by

\[
\begin{align*}
s(0, z) &= s_0(z), \quad p(0, z) = p_0(z), \quad z \in \Omega
\end{align*}
\]  

(6.2)

Similar to the normalized case, we assume the boundary conditions are given by

\[
\begin{align*}
s(t, 0) &= 0, \quad p(t, 0) = f(s(t, 1), y_d) \quad \forall t \geq 0
\end{align*}
\]  

(6.3)

where $f(s(t, 1), y_d)$ is the control action to be designed based on the boundary measurement $s(t, 1)$, and $y_d$ is the desired signal power level. In order to obtain a feasible problem, the compatibility conditions would require that

\[
\begin{align*}
s_0(0) &= \bar{s}(0) \\
p_0(0) &= f(s_0(1), y_d)
\end{align*}
\]  

(6.4)
Next, let $\bar{w}(z) := (\bar{s}(z), \bar{p}(z))$ denote the steady-state solution, which satisfies

\[
\begin{align*}
\lambda_1 \frac{d\bar{s}}{dz}(z) &= -\alpha_1 \bar{s}(z) + c_1 \bar{s}(z)\bar{p}(z) \\
\lambda_2 \frac{d\bar{p}}{dz}(z) &= -\alpha_2 \bar{p}(z) - c_2 \bar{s}(z)\bar{p}(z), \quad z \in \Omega
\end{align*}
\]  

(6.5)

Similar to Chapter 3, we introduce change of coordinates with respect to $\bar{w}(z)$, and arrive at an initial-boundary-value problem, as outlined below,

\[
\frac{\partial}{\partial t} w(t, z) + a(z) \frac{\partial}{\partial z} w(t, z) + b(z) w(t, z) = f(w(t, z))
\]

(6.6)

for $t \geq 0, z \in \Omega$, where $a(z) = \text{diag}(\lambda_1, \lambda_2), 0 < \lambda_1 < \lambda_2$,

\[
b(z) := \begin{bmatrix} \alpha_1 - c_1 \bar{w}(z) & -c_1 \bar{w}_1(z) \\ c_2 \bar{w}_2(z) & \alpha_2 + c_2 \bar{w}_1(z) \end{bmatrix}, \quad f(w) := \begin{bmatrix} c_1 w_1 w_2 \\ -c_2 w_1 w_2 \end{bmatrix}
\]

(6.7)

with initial condition

\[
w(0, z) = \begin{bmatrix} s_0(z) - \bar{s}(z) \\ p_0(z) - \bar{p}(z) \end{bmatrix} := w_0(z), \quad z \in \Omega
\]

(6.8)

And boundary condition

\[
w(t, 0) = \begin{bmatrix} 0 \\ G_2(w(t, 1)) \end{bmatrix} := G(w(t, 1)), \quad t \geq 0
\]

(6.9)

where $G_2(w(t, 1))$ is the control action to be designed, but now in the new coordinates.
Finally, the compatibility conditions (3.4) become

\[ w_0(0) = \begin{bmatrix} 0 \\ G_2(w(0, 1)) \end{bmatrix}, \quad z \in \Omega \]  

(6.10)

Similar to Chapter 3, our stability analysis is performed on the closed-loop systems when two different boundary controllers are applied. First is the static proportional controller, as given in (3.14), and the second one is the controller with integral action, as given in (3.16).

### 6.2 Lyapunov Functional and Its Properties

The Lyapunov functional is modified to accommodate for the coefficients, and is given by

\[ V(w) := \int_{\Omega} v(w(z)) \cdot e^{-\mu z} \, dz \]  

(6.11)

where

\[ v(w(t, z)) = \sum_i \frac{1}{c_i} v_i(w_i(t, z)) \]  

(6.12)

and \( v \) is defined in (4.3). Then Lemma 1 from Section 4.2 regarding the properties of \( v(x) \) still holds, and \( \dot{V} \) is as derived in (4.10). Using the above expressions, we can arrive at the following lemma that describes the properties of the Lyapunov functional (a parallel of Lemma 2).

**Lemma 3.** Given \( \tilde{w} \in \mathcal{H}^1(\Omega) \), such that \( \tilde{w}_1(z) > 0, \tilde{w}_2(z) > 0, \forall z \in [0, 1] \) and \( \|\tilde{w}_2\|_{C^0} < 1 \), let \( \tilde{w}_m = \min_{z \in \Omega} \tilde{w}(z) > 0 \) and \( \|\tilde{w}\|_{C^0} = \max_{z \in \Omega} \tilde{w}(z) \) and consider \( V \), (6.11), and \( \dot{V} \), (4.10).

1. Let any \( a_1, b_0 > 0 \) such that \( \frac{1}{b_0} < \tilde{w}_2(0), \frac{1}{a_1} > \tilde{w}_1(1) \), and for any \( \mu > 0 \) let \( \tilde{a}_1 = \frac{\lambda_1}{c_1} a_1 e^{-\mu}, \) and \( \tilde{b}_0 = \frac{\lambda_2}{c_2} b_0 \). For every \( w \) such that \( -\tilde{w}_1(1) < w_1(1) \leq \frac{1}{a_1} - \tilde{w}_1(1) \) and
\[ G_2(w(1)) \geq -\bar{w}_2(0) + \frac{1}{\bar{b}_0}, \]

\[ \dot{V}(w) \leq -\mu \lambda_1 V(w) - \frac{1}{2} \bar{a}_1 |w_1(1)|^2 + \frac{1}{2} \tilde{b}_0 |G_2(w(1))|^2 \quad (6.13) \]

where \( G_2 \) is the control action to be designed.

(ii) For any \( \mu > 0 \), \( \exists \alpha, \beta, d > 0 \) such that \( \forall w \) with \( \|w\|_{\mathcal{C}^0} \leq d \),

\[ \frac{1}{2} \alpha \|w\|_{L^2}^2 \leq V(w) \leq \frac{1}{2} \beta \|w\|_{L^2}^2 \quad (6.14) \]

Proof (i) The proof of Lemma 3 is very similar to that of Lemma 2. First we evaluate \( \dot{V}(w) \) to obtain

\[
\dot{V}(w) = - \int_{\Omega} v_w(w) a(z) \frac{dw}{dz} e^{-\mu z} \, dz + \int_{\Omega} v_w(w) \cdot (f(w(z)) - b(z) w(z)) e^{-\mu z} \, dz \quad (6.15)
\]

where \( a(z), b(z), \) and \( f(w(z)) \) are defined in (6.7) Next, note that

\[
\frac{dV(w)}{dz} = V_w(w) \frac{dw}{dz} + V_{\dot{w}}(w) \frac{d\dot{w}}{dz} \quad (6.16)
\]

This yields, after manipulation,

\[
\dot{V}(w) = - \int_{\Omega} \left[ \frac{d\nu_{\dot{w}}(w)}{dz} - G(z) \right] e^{-\mu z} \, dz \quad (6.17)
\]
where \( v_0(w(z)) = \sum_i \frac{\lambda_i}{c_i} v(w_i(z)) \), and

\[
G(z) := v_w(w) \left[ \text{diag}\{\lambda_i\} \right] \frac{d\bar{w}}{dz} + v_w(w) (f(w(z)) - b(z) w(z)) \tag{6.18}
\]

Integrating by parts gives

\[
\dot{V}(w) = \left[ -v_0(w(z)) e^{-\mu z} \right]_{z=0}^{z=1} - \mu \int_\Omega v_0(w(z)) e^{-\mu z} \, dz + \int_\Omega G(z) e^{-\mu z} \, dz
\leq \left[ -v_0(w(z)) e^{-\mu z} \right]_{z=0}^{z=1} - \mu \lambda_1 V(w) + \int_\Omega G(z) e^{-\mu z} \, dz \tag{6.19}
\]

since \( 0 < \lambda_1 < \lambda_2 \). To evaluate (6.18), first note that

\[
[v_{\bar{w}}(w)]_i = \frac{w_i}{w_i + \bar{w}_i} - \ln \left( \frac{w_i}{\bar{w}_i} + 1 \right)
\]

And

\[
[v_w(w)]_i = \frac{w_i}{w_i + \bar{w}_i} \quad \forall i = 1, 2
\]

Furthermore, we have the expression for \( \frac{d\bar{w}}{dz} \) from (6.5). Using these expressions, we obtain

\[
G(z) = -\frac{b_{11}(z)}{c_1} v(w_1(z)) - \frac{b_{22}(z)}{c_2} v(w_2(z)) \tag{6.20}
\]

where \( b_{11}(z) = (\alpha_1 - c_1 \bar{w}_2(z)) \), \( b_{22}(z) = (\alpha_2 + c_2 \bar{w}_1(z)) \). Now, by the assumption that \( \|\bar{w}_2\|_{C^0} < \frac{\alpha_1}{c_1}, b_{11}(z) > 0 \ \forall z \in \Omega \). In other words, \( G(z) \leq 0 \ \forall z \in \Omega \). Hence

\[
\dot{V}(w) \leq -\mu V(w) - v(w(1)) e^{-\mu} + v(w(0)) \tag{6.21}
\]
Using compatibility condition (3.12), (4.19) can be rewritten as

\[ \dot{V}(w) \leq -\mu \lambda_1 V(w) - \frac{\lambda_1}{c_1} v(w_1(1)) e^{-\mu} + \frac{\lambda_2}{c_2} v(G_2(w(1))) \] (6.22)

Finally, using Lemma 1 (ii), since \( \frac{1}{a_1} > \bar{w}_1(1) \) and \( \frac{1}{a_0} < \bar{w}_2(0) \), then \( \forall w \) such that

\[ -\bar{w}_1(1) < w_1(1) \leq \frac{1}{a_1} - \bar{w}_1(1) , \]

\[ G_2(w(1)) \geq \bar{w}_2(0) + \frac{1}{b_0} \]

we have

\[ v(w_1(1)) \geq \frac{1}{2} a_1 |w_1(1)|^2 \]

\[ v(G_2(w(1))) \leq \frac{1}{2} b_0 |G_2(w(1))|^2 \]

Therefore,

\[ \dot{V}(w) \leq -\mu \lambda_1 V(w) - \frac{1}{2} \frac{\lambda_1}{c_1} a_1 |w_1(1)|^2 e^{-\mu} + \frac{1}{2} \frac{\lambda_2}{c_2} b_0 |G_2(w(1))|^2 \]

\[ = -\mu V(w) - \frac{1}{2} \tilde{a}_1 |w_1(1)|^2 + \frac{1}{2} \tilde{b}_0 |G_2(w(1))|^2 \] (6.23)

(ii) From (6.11),

\[ V(w) = \int_\Omega v(w(z)) e^{-\mu z} \, dz \]

\[ = \int_\Omega \sum_i \frac{1}{c_i} v_i(w_i(z)) e^{-\mu z} \, dz , \quad i = 1, 2 \]
Let $\bar{c} = \max(c_1, c_2)$, $\underline{c} = \min(c_1, c_2)$, and choose $\alpha, \beta > 0$ such that

$$\frac{2 e^{-\mu}}{\alpha \bar{c}} > \max_{\Omega} (\bar{w}_1(z), \bar{w}_2(z))$$

$$\frac{2}{\beta \underline{c}} < \min_{\Omega} (\bar{w}_1(z), \bar{w}_2(z))$$

Then from Lemma 1 (ii), $\forall w(z) \in C^1(\overline{\Omega})$ such that

$$\frac{2}{\beta \underline{c}} - \bar{w}_1(z) \leq w_i(z) \leq \frac{2 e^{-\mu}}{\alpha \bar{c}} - \bar{w}_1(z), \quad (6.24)$$

$$\frac{1}{4} \frac{\alpha}{e^{-\mu}} |w_i(z)|^2 \leq v_i(w_i(z)) \leq \frac{1}{4} \frac{1}{\underline{c}} |w_i(z)|^2 \quad \forall z \in \Omega$$

If we choose $d > 0$ as

$$d = \min_{\Omega} \left( \left| \frac{2}{\beta} - \bar{w}_1(z) \right|, \left| \frac{2 e^{-\mu}}{\alpha} - \bar{w}_1(z) \right| \right)$$

then it’s clear that $\forall w$ such that $\|w\|_{C^0} \leq d$, $w$ also satisfies (6.24). Therefore,

$$\frac{1}{2} \alpha \frac{\bar{c}}{e^{-\mu}} \sum_i |w_i(z)|^2 \leq \sum_i v_i(w_i(z)) \leq \frac{1}{2} \beta \underline{c} \sum_i |w_i(z)|^2$$

Recall the definition of $\| \cdot \|_{L^2}$ given in (2.7), then we get that $\forall w \in C^1(\overline{\Omega})$ such that
\[ \|w\|_{C^0} \leq d \]

\[
\frac{1}{2} \alpha \|w\|_{L^2}^2 \leq \frac{e^{-\mu}}{\epsilon} \int_{\Omega} \frac{1}{2} \epsilon - \mu \sum_i |w_i(z)|^2 \, dz 
\leq \frac{e^{-\mu}}{\epsilon} \int_{\Omega} \sum_i v_i(w_i(z)) \, dz 
\leq \int_{\Omega} \sum_i \frac{1}{c_i} v_i(w_i(z)) e^{-\mu z} \, dz 
\leq V(w) \leq \int_{\Omega} \sum_i v_i(w_i(z)) e^{-\mu z} \, dz 
\leq \frac{1}{2} \beta \|w\|_{L^2}^2. \quad \square
\]

**Remark 7.** Lemma 2 (i) states that assuming \( \|\bar{w}_2\|_{C^0} < 1 \), the derivative of the Lyapunov functional along the solution to the normalized system, is bounded by

\[ \dot{V}(w) \leq -\mu V(w) - \frac{1}{2} \tilde{a}_1 |w_1(1)|^2 + \frac{1}{2} b_0 |G_2(w(1))|^2 \]

For the system with coefficients, with a modified Lyapunov functional \( V(w) \) (6.11), Lemma 3 (ii) gives

\[ \dot{V}(w) \leq -\mu \lambda_1 V(w) - \frac{1}{2} \tilde{a}_1 |w_1(1)|^2 + \frac{1}{2} \tilde{b}_0 |G_2(w(1))|^2 \]

assuming that \( \|\bar{w}_2\|_{C^0} < \frac{\alpha_1}{c_1} \). The two are very similar except for the fact that the inequality itself now directly involves the coefficient \( \lambda_1 \), and indirectly, \( \lambda_1, \lambda_2, c_1, \) and \( c_2 \), based on the choice of \( \tilde{a}_1 \) and \( \tilde{b}_0 \). The values of \( \alpha_1, \alpha_2 \) only show up in the bound for \( w \) within which the inequality is valid.
6.3 Stability Analysis

Next, the stability analysis performed on closed-loop system (6.6-6.10) with the proportional controller (3.14) leads to the following theorem that shows the asymptotic stability of the origin in $C^0$-norm and the exponential stability in $L^2$-norm.

**Theorem 3.** Consider the closed-loop system (6.6-6.10), (3.14), and assume that $\|\bar{w}_2\|_{C^0} < \alpha_1 / c_1$.

For any $a_1, b_0, \mu > 0$ such that $\frac{1}{a_1} > \bar{w}_1(1) > 0$, $\frac{1}{b_0} < \bar{w}_2(0)$, let $\tilde{a}_1 = \frac{\lambda_1}{c_1} a_1 e^{-\mu}$, and $\bar{b}_0 = \frac{\lambda_2}{c_2} b_0$, select any $0 < d_1 < \min}\{\bar{w}_m, \frac{1}{a_1} - \bar{w}_1(1)\}$ and choose $k_1$ such that

$$k_1^2 < \frac{\tilde{a}_1}{\bar{b}_0} \quad \text{and} \quad -\frac{\bar{w}_2(0) - 1/b_0}{1/a_1 - \bar{w}_1(1)} < k_1 < \frac{\bar{w}_2(0) - 1/b_0}{\bar{w}_1(1)}$$ (6.25)

(i) For any initial condition $w_0 \in H^1_{bc}(\Omega)$ such that $\|w_0\|_{H^1} \leq d_1 / C_0$, then

$$V(w(t)) \leq e^{-\mu \lambda_1 t} V(w_0), \quad \forall t \in [0, T]$$

as long as the solution $w(t)$ exists and $\|w(t)\|_{C^0} \leq d_1$.

Moreover, there exists $\delta_{01} > 0$ such that, for any initial condition $w_0 \in H^1_{bc}(\Omega)$ with $\|w_0\|_{H^1} \leq \delta_{01} \leq d_1 / C_0$, the classical solution exists globally over $[0, \infty)$ and the origin is asymptotically stable in the $C^0$-norm.

(ii) There exists $0 < \delta_0 \leq \delta_{01}$ such that, for any initial condition $w_0 \in H^1_{bc}(\Omega)$ with $\|w_0\|_{H^1} \leq \delta_0$, the $L^2$-norm of $w(t)$ satisfies $\forall t \in [0, \infty)$

$$\|w(t)\|_{L^2} \leq C e^{-\mu \lambda_1 t} \|w(0)\|_{L^2}, \quad \forall t \geq 0$$
for some constant $C > 0$, hence the origin is exponentially stable in the $L^2$-norm.

The proof for Theorem 3 (i) is nearly identical to that of the proof for Theorem 1 (i) due to the observations noted in Remark 7. First, the coefficients in (6.6) resulted in different choices of $\tilde{a}_1$, $\tilde{b}_0$, as demonstrated in Lemma 3 (i). This indirectly determines $d_1$ and $\delta_{01}$. Finally, the rate of which the $C^0$-norm of $w$ decays is now dependent on $\lambda_1$, since our inequality states that

$$V(w(t)) \leq e^{-\mu \lambda_1 t} V(w_0), \quad \forall t \in [0, T]$$

The proof for Theorem 3 (ii) will be similar to Theorem 1 (ii). The choice of the new parameter, $\delta_0$, is determined by $\alpha$, $\beta$, and $d$, which are coefficient-dependent as shown in Lemma 3 (ii). Furthermore, the exponential rate at which the $L^2$-norm of $w$ decays to 0 will also be coefficient-dependent.

Finally, for the closed-loop system (6.6-6.10) with the controller with integral action (3.16), the condition for exponential stability in $\|w(t)\|_{L^2}$ is stated below.

**Theorem 4.** Consider the system (6.6-6.10), (3.16), and assume all parameters $a_1, b_0, \mu, k_1$ are selected as in Theorem 3. For any $\varepsilon > \frac{\mu}{2}$, set $\bar{\varepsilon} = \varepsilon - \frac{\mu}{2} > 0$, and select $0 < r < 2 \bar{\varepsilon} (\tilde{a}_1 - \tilde{b}_0 k_1^2)$. Finally, choose $k_2$ such that $K_2 < k_2 < K_2$, where $K_2, K_2$ are the roots of the following quadratic expression,

$$\phi(k_2) = -r^2 \tilde{a}_1 \tilde{b}_0 k_2^2 - 2 r \tilde{b}_0 k_1 k_2 + 2 r \bar{\varepsilon} (\tilde{a}_1 - \tilde{b}_0 k_1^2) - 1 \quad (6.26)$$

Note that the coefficients of $\phi(k_2)$ can be predetermined based on the choice of $\tilde{a}_1, \tilde{b}_0, k_1, \text{and } r$. 63
Then, \( \exists \delta_0 > 0 \) such that, for any initial condition \( w_0 \in \mathcal{H}^1_{bc}(\Omega) \) with \( \|w_0\|_{\mathcal{H}^1} \leq \delta_0 \), the classical solution exists globally over \([0, \infty)\) and the origin is exponentially stable in the \( L^2 \)-norm.

Again, the proof for Theorem 4 will be similar to Theorem 2.

**Remark 8.** In the closed-loop systems with both controllers, it is important to note that as the analysis extends to system with coefficients, the bounds for choosing \( \tilde{a}_1 \) and \( \tilde{b}_0 \) are modified due to the inequality expression in Lemma 3. However, the controller gains \( k_1 \) and \( k_2 \) still have the same dependency on \( \tilde{a}_1 \) and \( \tilde{b}_0 \) and other parameters. This shows that the Lyapunov-based approach indeed provides a design method that is more systematic and can be generalized easily.
Chapter 7

Simulations

In this chapter, we present the simulation results of various Raman amplifier systems with normalized coefficients defined on a normalized length span ($\Omega := [0, 1]$). The simulations are executed in MATLAB using a first-order hyperbolic PDEs solver [27]. The software developed by Shampine allows PDEs of the form

$$\frac{\partial x}{\partial t}(t, z) = f \left( t, z, x(t, z), \frac{\partial x}{\partial z}(t, z) \right)$$

as well as general boundary conditions. The Raman amplifier models are constructed using (3.1), with the original set of variables, i.e., $s(t, z)$ and $p(t, z)$ that stand for the signal and the pump power mappings respectively. Then the boundary conditions are set up depending on the controller to be applied (either the proportional (3.13), the one with integral action (3.15), or simply open-loop). The parameters are chosen based on the stability results derived in Theorems 1 and 2 from Chapter 5. Then the associated initial-boundary value problem can be solved using central finite difference method with a time step of $10^{-4}$ and a spatial discretization step of $2 \cdot 10^{-3}$. For given initial conditions and a given period of time $[0, T_0]$, the solver returns
the corresponding solution \((s_d(t, z), p_d(t, z))\), quantized by the time and the spatial steps. (See Appendix Section 9.1 for more details regarding the hyperbolic PDEs solver)

**Remark 9.** It is important to note that the computations performed by the PDE solver actually returns solutions discretized by the time and spatial steps. For simplicity, we use the same notations in this chapter to denote the discretized variables \(t\), \(z\), and the power mappings \(s\), \(p\), \(\bar{s}\), and \(\bar{p}\).

As suggested in Chapter 3, the steady state solution of the control problem, \((\bar{s}, \bar{p})\), is previously known. Hence before we set up and investigate any closed-loop system, the open-loop system is first simulated for a long period of time until \(s(t, z)\) and \(p(t, z)\) settle to determine \((\bar{s}(z), \bar{p}(z))\) on \(z \in \Omega\). In a output tracking problem, we would also need to specify the reference output signal power \(y_d\); otherwise when we are only interested in the stabilization problem, we assume that \(y_d = \bar{s}(1)\).

Since the set of PDEs and boundary conditions are set up in terms of \(s\) and \(p\) rather than the shifted \(w\) as in the analysis, the determined \((\bar{s}(z), \bar{p}(z))\) will serve as reference when we discuss the stability property and the tracking performance of the simulation results. Keeping in mind the coordinate transformation in (3.7) and the system state representation (3.17), we can see that the convergence of \(w(t)\) to the origin (either in the \(C^0\)- or \(L^2\)-norm) is equivalent to the convergence of \(s(t, z)\) and \(p(t, z)\) to \(\bar{s}(z)\) and \(\bar{p}(z)\), \(\forall z \in \Omega\), as \(t \to \infty\).

### 7.1 Initial State Deviations

We start by investigating the system performance under different initial conditions. In other words, we generate two arrays that represent the discretized signal and pump power mappings
at $t = 0$, i.e., $s_0(z)$ and $p_0(z)$. By the compatibility conditions given in (3.4), we require that $s_0(0) = \bar{s}(0)$, and $p_0(0) = f(s_0(1), y_d)$, where $f(\cdot, \cdot)$ is the boundary controller, and $y_d$ is the output signal power reference. Without the tracking requirement, we simply have $y_d = \bar{s}(1)$.

And the boundary conditions are given by (3.3).

Figure 7.1: $s(t, z)$ and $p(t, z)$ under initial state deviations with proportional control

With this setup, it is easy to see that the analysis from Chapter 4 and 5 can be directly applied on the system. Specifically, when the proportional controller or the controller with integral actions has gain parameters that satisfy (5.1) and (5.14), if the $H^1$-norm of the initial state is bounded and sufficiently close to the steady state (as given in Theorem 1 and 2), then the solution $s(t, z)$ and $p(t, z)$ converge to the steady state, $\bar{s}(z)$ and $\bar{p}(z)$.

Our simulation indeed agrees with the analysis. First we present the simulation results with a pure proportional control of $k_1 = 0.5$. The signal and pump powers are plotted against both time and distance along the amplifier in Figure 7.1. To get a better view of the surface, not all points from the arrays are plotted. But it is still apparent that the solution stabilizes very
quickly. Furthermore, in Figure 7.2, the time evolutions of the signal powers at various points along the amplifier show that the solution indeed converges to the pre-determined steady state.

Next we present the results when a controller with integral action is applied, where $k_1 = 0.5$

Figure 7.2: $s(t, z_0)$ for $z_0 = 0.25, 0.5, 0.75$ under initial state deviations with proportional control

Figure 7.3: $s(t, z)$ and $p(t, z)$ under initial state deviations with dynamic control
and $k_2 = 0.1$. Figure 7.3 shows the signal and pump power mappings and Figure 7.4 plots the signal power evolutions with respect to time at various points along the amplifier. Similar to the case with proportional control, stabilization and convergence to the given steady state can be observed. In addition, there is very little difference in performance comparing to the proportional control.

This makes sense since the integral gain improves output tracking and disturbance rejection performances. In the absence of both elements, the system will converge to the known steady state regardless of the type of the boundary controller used.

![Graphs showing signal power evolutions](image)

Figure 7.4: $s(t, z_0)$ for $z_0 = 0.25, 0.5, 0.75$ under initial state deviations with dynamic control

### 7.2 Output Tracking Performance

The next case we investigate is tracking of the output signal power, $y_d$. Again, the open-loop system is first simulated until $s$ and $p$ converge to certain steady state $\bar{s}$ and $\bar{p}$. Then, at $t = 0$,
the value of \( y_d \) is altered from the original \( \bar{s}(1) \). Therefore, the initial condition is given by

\[
\begin{align*}
    s_0(z) &= \bar{s}(z) \\
    p_0(z) &= \bar{p}(z) \quad \forall z \neq 0 \\
    p_0(0) &= f(s_0(1), y_d) \quad (7.1)
\end{align*}
\]

where the boundary condition stays the same.

This case is slightly different from the previous one because the system is required to converge to a different point within the state space, i.e., we would like \( s(t, 1) \rightarrow y_d \neq \bar{s}(1) \) as \( t \rightarrow \infty \). With the boundary controller in place, the closed-loop system may very well reach an entirely different steady state, denoted by \( \bar{s}_n(z) \) and \( \bar{p}_n(z) \). However, based on the new steady state, we can still apply the coordinate transformation to obtain \( w(t) \), and the stability analysis will follow just the same. The rules for designing the controller gains are still given by (5.1) and (5.14). With the initial condition (7.1) already bounded, the requirement for \( (s_0, p_0) \) to be sufficiently close to \( (\bar{s}_n, \bar{p}_n) \) indirectly translates to a bound on the specified new output reference \( y_d \).

Consider a closed-loop system with a static proportional controller, i.e.,

\[
p(t, 0) = k_1(s(t, 1) - y_d) + \bar{p}(0)
\]

Applying \( y_d = \bar{s}(1) \), we get that

\[
p(t, 0) = k_1(s(t, 1) - \bar{s}(1)) + \bar{p}(0) \quad (7.2)
\]
It is trivial to see that (7.2) is satisfied if \((s, p) \rightarrow (\bar{s}, \bar{p})\), and \((s(t, 1) - \bar{s}(1)) \rightarrow 0\) as \(t \rightarrow \infty\).

However, assume that under a modified tracking condition, the system now converges to a different steady state \((\bar{s}_n, \bar{p}_n)\), which gives, as \(t \rightarrow \infty\),

\[
\bar{p}_n(0) = k_1(\bar{s}_n(1) - y_d) + \bar{p}(0)
\]

\[
\bar{p}_n(0) - \bar{p}(0) = k_1(\bar{s}_n(1) - y_d)
\]

(7.3)

In the case where \(\bar{p}_n(0) \neq \bar{p}(0)\), the new steady state signal power will not match the output reference, and tracking error is present. Since \((\bar{s}_n, \bar{p}_n)\) is not previously determined, this greatly compromises the tracking capability of the proportional controller.

Using a proportional gain of \(k_1 = 0.4\), the simulation results indeed demonstrate a 3.75% offset in the output signal power from the desired reference, shown in Figure 7.5. On the other hand, Figure 7.6 shows that the system still stabilizes to a new steady state, as our analysis from Chapter 4 and 5 indicates.
The controller with integral action is given by (3.15),

\[ p(t, 0) = k_1 s(t, 1) + k_2 \sigma(t) \]
\[ \dot{\sigma}(t) = -\varepsilon \sigma(t) + s(t, 1) - y_d \]

where \( \varepsilon \ll 1 \). This is very significant because at steady state, \( \dot{\sigma}(t) = 0 \). Hence

\[ s_n(1) - y_d = \varepsilon \bar{\sigma}(t) \]  

(7.4)

If \( \varepsilon \) can be made arbitrarily small, \( s_n(1) \to y_d \), which satisfies the tracking condition regardless of the value of \( y_d \). On the other hand, \( \varepsilon \) affects the speed of the controller. The smaller \( \varepsilon \) is, the longer it takes for the system to reach steady state. Nevertheless, it allows us to control and improve upon the tracking performance if needed.
For the simulations, we used a controller of proportional gain $k_1 = 0.4$, and integral gain $k_2 = 0.1$. $\varepsilon$ is chosen to be 0.01. As demonstrated in Figure 7.7, the steady state signal output approaches $y_d$ with a 1.02% error, much better than the case with a proportional controller. However, the settling time is also much longer. If we examine the signal and pump power mappings in Figure 7.8, we can see that due to the controller state $\sigma$ and the small parameter $\varepsilon$, the solution indeed takes longer to stabilize to a new steady state.

![Figure 7.7: $s(t, 1)$: output tracking with dynamic control](image)

Figure 7.7: $s(t, 1)$: output tracking with dynamic control
7.3 Performance under External Disturbance

The next case we investigate is the presence of an extra signal. In an optical network, a change in network structure can sometimes lead to the adding or dropping of channels onto a path. Here we examine the simple case of one extra channel being added to the existing one-pump one-signal Raman amplifier. The added signal, power denoted by $s_2(t, z)$, will serve as the external disturbance. In other words, the implemented controller does not change and is still based on the specifications of the existing signal $s(t, 1)$ and $y_d$.

Here we consider a disturbance signal of a lower wavelength, which, from the existing signal’s perspective, serves as an extra pump along the amplifier. This modifies the hyperbolic
This in turn gives a different steady state, \((\bar{s}, \bar{p}, \bar{s}_2)\). However, if we focus only on the first two differential equations in (7.5), and compare them with the original system (3.1), we can see that the external disturbance only serves to introduce noise terms \(s_2 s\) and \(p s_2\) in the equations. In other words, from the perspective of the original state \((s, p)\), the new problem is equivalent to one concerning a changed steady state (as in Section 7.2) and as well as the robustness of the boundary control when noise terms are present, as depicted in Figure 7.9.

![Figure 7.9: Block diagram of a co-pumping Raman amplifier with a disturbance channel](image)

Here we use a proportional controller of gain \(k_1 = 0.3\). Similar to the results in Section 7.2, the system stabilizes to a new steady state, as shown in Figure 7.10. In addition, from Figure 7.11, the steady state signal output is significantly different from the original one, due to controller’s poor tracking ability. The output appears to be higher since the chosen disturbance...
signal is of a lower wavelength, and its presence helped boost the original signal $s(t, z)$ along the amplifier due to the Raman Scattering Effect.

Figure 7.11: $s(t, 1)$: external disturbance with proportional control

The controller with integral action has gain parameters $k_1 = 0.4$ and $k_2 = 0.1$. Similar
stabilization behaviour is observed in Figure 7.12, and Figure 7.13 shows that the implemented controller is still able to maintain the same signal output under the presence of the external disturbance. Hence the controller with integral action is certainly the more robust one.

Figure 7.13: $s(t, 1)$: external disturbance with dynamic control
Chapter 8

Conclusion

In this thesis, we investigate the Raman amplifier stabilization problem under the Lyapunov-based approach. Specifically, the control objective is to stabilize the power functions of a signal and a pump propagating along a Raman amplifier, by applying boundary control. We study the cases with a pure proportional controller and a controller with integral actions.

In Chapter 3, we first formulate the normalized problem, and identify it as an initial-boundary value problem associated with a hyperbolic system with Lotka-Volterra nonlinearity. Then, we are able to use results from [16] to ensure existence and uniqueness of solutions to the problem. Furthermore, when we consider the system state as $\mathcal{H}^1(\Omega)$ functions, the abstract Cauchy form looks similar to the standard control problem of finite-dimensional states. Hence we introduce the change of variables with respect to the given steady state solutions, and construct Lyapunov functional in Chapter 4 inspired by the Lotka-Volterra ODEs.

Then, in Chapter 5, we are able to follow similar procedures in the proof of the Lyapunov Theorem [24] to derive conditions under which the origin of the closed-loop system is asymptotically stable in the $C^0$-norm, and exponentially stable in the $L^2$-norm. The analysis relies
mostly on the form of the Lyapunov functional, as well as the function spaces embedding that allows us to impose bounds on the $C^0$- and the $L^2$-norms. Consequently, the results are easily extended to the non-normalized system in Chapter 6, where the stability conditions are modified by the coefficients.

In Chapter 7, the MATLAB simulation results presented indeed confirm with the analysis and demonstrate stabilization of the signal and pump powers. We are also able to test the output tracking and disturbance rejection performances of the closed-loop system. As expected, the controller with integral actions outperforms the proportional one in both cases.

This thesis has taken a rigorous approach to the Raman amplifier control problem that involves spatial dynamics. We were able to derive stability conditions for two very simple boundary controllers. This in turn can provide a more systematic approach to control design over heuristically tuning the parameters.

The next logical step is to consider the case where the signal and pump propagate in opposite directions, which is also a very common setup in optical networks with Raman amplifiers. After that, one direction of future work is expanding into higher dimensions. When proving Theorem 2, we append the vector $(w, \sigma)^T$ to incorporate the additional system state added by the controller, and analyzed the Lyapunov functional expression in matrix form. This is a probable approach that can be taken to perform stability analysis for systems with more pumps and signals.

Another direction is to investigate the one-pump one-signal setup, but with different control objectives such as disturbance rejection, for which we demonstrated the simulation results in Chapter 7. In an optical network, adding and dropping of channels are often the cause of system instability. If we could also formulate the problem rigorously and derive conditions for
which the disturbance is rejected, this will greatly improve the robustness and feasibility of the boundary control design.
Bibliography


Chapter 9

Appendix

9.1 PDE Solver

This section contains descriptions of the methods used in the MATLAB PDE solver developed by Shampine [27], and the type of systems of PDEs applicable. In short, the solver can be used for IVBPs for first-ordered systems of hyperbolic PDEs in one spatial variable (denoted by $z$ in our case) and time variable $t$. The PDE can be of either of the three following forms:

\[
\frac{\partial x}{\partial t} = f(z, t, x, \frac{\partial x}{\partial z}) \tag{9.1}
\]

\[
\frac{\partial x}{\partial t} = \frac{\partial f(z, t, x)}{\partial z} + g(z, t, x) \tag{9.2}
\]

\[
\frac{\partial x}{\partial t} = \frac{\partial f(x)}{\partial z} \tag{9.3}
\]

and general boundary conditions are allowed. Our Raman amplifier problem is of the form described in Equation (9.1).

The solver utilizes four different central finite difference methods commonly used for hy-
perbolic PDEs, namely, the Lax-Friedrichs method [28], [29], the Richtmyer’s two-step variant for the Lax-Wendroff method [30], the Lax-Wendroff method with a nonlinear filter [31], and finally, the Nessyahu-Tadmor (NT) method [32].

The computation of the solution is first initialized by the following line of code:

```matlab
sol = setup(form,pdefun,t,z,x,method,bcfun)
```

where 'form' denotes the form of PDE equation used; 'pdefun' is the function handle of the actual equation; 't', 'z' are the spatial and time variable array; 'x' and 'bcfun' contain the initial and boundary conditions; finally, 'method' selects the central finite difference method to be used when treating the problem.

After the initial setup, the system of PDEs is solved by calling

```matlab
sol = hpde(sol,howfar,dt);
```

where 'howfar' and 'dt' are the time interval and time step size for which the solution will be computed and stored in a two-dimensional array.
global k sst pst xo

tscale = [-7 -6 -5 -4 -2 0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5];

n = length(tscale);
x = linspace(0,1,500);

k = 0;
[opos_ic_ol_s(1:6,:),opos_ic_ol_p(1:6,:),sst,pst] = testopos;
opos_ic_k_s = opos_ic_ol_s;
opos_ic_k_p = opos_ic_ol_p;

k = 0.5;
[opos_ic_k_s(6:n,:),opos_ic_k_p(6:n,:),sst,pst] = testoposi;

k = 0;
[opos_ic_ol_s(6:n,:),opos_ic_ol_p(6:n,:),sst,pst] = testoposi;

%---------------------------------------------------
% Controller with Integral Actions

opos_ic_pi_s = opos_ic.ol_s;

opos_ic_pi_p = opos_ic.ol_p;

[opos_ic_pi_s(6:n,:), opos_ic_pi_p(6:n,:),
 sst, pst] = testopospi1;
9.3 Simulation Code for Output Tracking

```matlab
global k sst pst xo

tscale = [-7 -6 -5 -4 -2 0 0.5 1 1.5 2 2.5
3 3.5 4 4.5 5 6 8 10 12 14];
n = length(tscale);
x = linspace(0,1,500);

%------------------------------------
% Proportional Control
opos_srefb_k_s = zeros(n,500);
opos_srefb_k_p = zeros(n,500);
opos_srefb_k_s(1:16,:) = opos_ic_ol_s;
opos_srefb_k_p(1:16,:) = opos_ic_ol_p;

opos_srefb_ol_s = zeros(n,500);
opos_srefb_ol_p = zeros(n,500);
opos_srefb_ol_s(1:16,:) = opos_ic_ol_s;
opos_srefb_ol_p(1:16,:) = opos_ic_ol_p;

k = 0.4;
[opos_srefb_k_s(6:n,:),opos_srefb_k_p(6:n,:)] =
opos_srefb_k_1;

k = 0;
```

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[opos_srefb_ol_s(6:n,:), opos_srefb_ol_p(6:n,:)] =

opos_srefb_k_1;

%------------------------------------
% Controller with Integral Actions

opos_srefb_pi_s = zeros(n,500);

opos_srefb_pi_p = zeros(n,500);

opos_srefb_pi_s(1:16,:) = opos_ic_ol_s;

opos_srefb_pi_p(1:16,:) = opos_ic_ol_p;

[opos_srefb_pi_s(6:n,:), opos_srefb_pi_p(6:n,:)] =

opos_srefb_pi_1;
9.4 Simulation Code for External Disturbance

global k ss1 ss2 ps xo
tscale = [-7 -6 -5 -4 -2 0 0.5 1 1.5 2 2.5
3 3.5 4 4.5 5 6 8 10 12 14 16 18 20 22 24 26 28 30];
n = length(tscale);
x = linspace(0,1,500);

% sst = opos_ic.ol.s(5,:);
% pst = opos.ic.ol.p(5,:);

%------------------------------------

% Proportional Control
opts_s1d_k_s1 = [z1; zeros(n-6,500)];
opts_s1d_k_s2 = [z2; zeros(n-6,500)];
opts_s1d_k_p = [z3; zeros(n-6,500)];
opts_s1d_ol_s1 = [z1; zeros(n-6,500)];
opts_s1d_ol_s2 = [z2; zeros(n-6,500)];
opts_s1d_ol_p = [z3; zeros(n-6,500)];

opts_s1d_k_s1(5,:) = ss1;
opts_s1d_k_s2(5,:) = ss2;
opts_s1d_k_p(5,:) = ps;

opts_s1d_ol_s1(5,:) = ss1;
opts_s1d_ol_s2(5,:) = ss2;
opts_s1d_ol_p(5,:) = ps;
k = 0;
[opts_s1d.ol_s1(6:n,:), opts_s1d.ol_s2(6:n,:),
 opts_s1d.ol_p(6:n,:)] = opts_d_k_1;

k = 0.4;
[opts_s1d.k_s1(6:n,:), opts_s1d.k_s2(6:n,:),
 opts_s1d.k_p(6:n,:)] = opts_d_k_1;

%------------------------------------
% Controller with Integral Actions
%------------------------------------

% Controller with Integral Actions
opts_s1d.pi_s1 = [z1; zeros(n-6,500)];
opts_s1d.pi_s2 = [z2; zeros(n-6,500)];
opts_s1d.pi_p = [z3; zeros(n-6,500)];
opts_s1d.pi_s1(5,:) = ss1;
opts_s1d.pi_s2(5,:) = ss2;
opts_s1d.pi_p(5,:) = ps;
[opts_s1d.pi_s1(6:n,:), opts_s1d.pi_s2(6:n,:),
 opts_s1d.pi_p(6:n,:)] = opts_d_pi_1;
9.5 Sample Code for Plot Generation

```matlab
figure;
subplot(1,2,1);
surf(x(1:25:500),tscale(5:n),opts_s1d_k_s2(5:n,1:25:500));
colormap gray
title('1P1S Ext. Dist. Proportional - S(t,z)');
xlabel('Distance');
ylabel('Time');

subplot(1,2,2);
surf(x(1:25:500),tscale(5:n),opts_s1d_k_p(5:n,1:25:500));
colormap gray
title('1P1S Ext. Dist. Proportional - P(t,z)');
xlabel('Distance');
ylabel('Time');

figure;
subplot(1,2,1);
surf(x(1:25:500),tscale(5:n),opts_s1d_pi_s2(5:n,1:25:500));
colormap gray
title('1P1S Ext. Dist. Dynamic - S(t,z)');
xlabel('Distance');
ylabel('Time');
```

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subplot(1,2,2);

surf(x(1:25:500),tscale(5:n),opts_sld_pi_p(5:n,1:25:500));
colormap gray
title('1P1S Ext. Dist. Dynamic - P(t,z)');
xlabel('Distance');
ylabel('Time');
zlabel('Power');