RADIATED ELECTRIC AND MAGNETIC FIELDS CAUSED BY LIGHTNING RETURN STROKES TO THE TORONTO CN TOWER

by

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Abstract

In the present PhD work, three sophisticated models based on the “Engineering” modeling approach have been utilized to conveniently describe and thoroughly analyze details of Lightning events at the CN Tower. Both the CN Tower and the Lightning Channel are represented by a number of connected in series Transmission Line sections in order to account for the variations in the shape of the tower and for plasma processes that take place within the Lightning Channel. A sum of two Heidler functions is used to describe the “uncontaminated” Return Stroke current, which is injected at the attachment point between the CN Tower and the Lightning Channel. Reflections and refractions at all points of mismatched impedances are considered until their contribution becomes less than 1% of the originally injected current wave.

In the proposed models, the problem with the current discontinuity at the Lightning Channel front, commonly taken care of by introducing a “turn-on” term when computing radiation fields, is uniquely treated by introducing reflected and transmitted components.

For the first time, variable speed of propagation of the Return Stroke current front has been considered and its influence upon the predicted current distributions along the whole Lightning Channel path and upon the radiated distant fields analyzed.
Furthermore, as another novelty, computation of the electromagnetic field is accomplished in Cartesian Coordinates. This fact permits to relax the requirement on the verticality of the Lightning Channel, normally imposed in Cylindrical Coordinates. Therefore, it becomes possible to study without difficulty the influence of a slanted Lightning Channel upon the surrounding electromagnetic field.

Since the proposed sophisticated Five-Section Model has the capability to represent very closely the structure of the CN Tower and to emulate faithfully the shape of, as well as physical processes within the Lightning Channel, it is believed to have the potential of truthfully reproducing observed fields.

The developed modeling approach can be easily adapted to study the anticipated radiated fields at tall structures even before construction.
Acknowledgments

During the last several years in my studies at the University of Toronto I have been focusing primarily on Lightning at tall structures, and Lightning at the CN Tower in particular. My research interest in this field was first ignited by one of the world’s most prominent figures in the area of Lightning, namely my supervisor Professor Emeritus Wasyl Janischewskyj. I have learned from him a great deal not only about the fascinating natural phenomenon, but also about conducting research as a professional activity. This PhD thesis work is greatly influenced by Professor Janischewskyj and would not exist without his valuable guidance and supervision. I am truly thankful to him and highly appreciate his advice and support. I am also extremely thankful to my co-supervisor Professor Reza Iravani, who in addition to teaching me two outstanding courses on “Static Power Converters” introduced me to technical writing.

Since 2005, I have been a member of the Lightning Studies Group, which is a highly dedicated focus group consisting of professors, students, and industry professionals. Professor Wasyl Janischewskyj is the Group Chair and all remaining members are mostly involved in research of the Lightning activity at the CN Tower, but also in research of Lightning phenomena in general. I have had the rare opportunity to learn from some of the best Lightning researchers nowadays, who have been very kind to share their vast knowledge and experience with me. Throughout the years, we have discussed and resolved various Lightning related issues, both theoretical and practical, and I would like to extend my gratitude to all of them. In particular, I would like to thank Professor Ali M. Hussein (Ryerson University, Toronto), Professor Jen-Shih Chang (McMaster University, Hamilton), Professor Volodymyr Shostak (Kyiv Polytechnic Institute, Ukraine), Professor Farhad Rachidi (Swiss Federal Institute of Technology), and Dr. William A. Chisholm (Kinectrics/University of Quebec at Chicoutimi). There have been a number of former and current students with whom I have been working on a daily basis, and I would like to mention Dr. E. Petrache, Dr. D. Pavanello, M. Milewski, and K. Yandulska. My sincere thanks go to all of you.

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Chapter I
Introduction to Lightning

I.1 Outline

In Chapter I after introducing basic historical background regarding Lightning and Lightning studies, typical features associated with the phenomenon are reported. Discussed are also the Global Electric Circuit and Megalightning events that have become quite popular areas of research lately.

Particular attention is paid to the producer of Lightning – the cumulonimbus cloud, its formation and structure. Moreover, the mechanism of electrification of the cloud is elaborated on.

Categorization of Lightning is then discussed and the processes involved in the development of a negative cloud-to-ground flash are fully described since this is the most common of all cloud-to-ground Lightning.

Finally, Lightning at tall structures and rocket-triggered Lightning are discussed. These geometry-initiated Lightning processes are of great interest to researchers because they allow collection of useful data regarding Lightning flash parameters that can be used in modeling considerations.

I.2 Preface

Lightning has always been intriguing, but also disturbing for mankind. It is present in many religions and myths of most of the ancient and modern civilizations. People have been trying to understand its origins and mechanisms for many years, but there is still some ambiguity and many questions remain unanswered.

Numerous examples of Lightning striking churches were reported in the Middle Ages in Europe. Back then, it was believed that ringing the church bells would disperse Lightning. Some churches were destroyed by Lightning and many bell ringers were killed performing their duties. In some cases churches were used to store gunpowder. Back in 1769, in one such particular case (St. Nazaire in Brescia, Italy) the Lightning-initiated explosion killed 3000 people and wiped out 1/6 of the city. On the other hand a number of churches never suffered any damage due to
Lightning and the reason for that appeared to be their unintentionally provided Lightning protection path consisting of metallic roofs connected to ground via metallic rain drains. Back in these times, many wooden ships having wooden masts were also damaged or destroyed by Lightning.

First scientific studies of Lightning date back to the second half of the eighteen century and were carried out by Benjamin Franklin. He proved by experiment that thunderclouds contain electrical charge and that Lightning is electrical in nature. In the experiment he performed, assisted by his son, he observed sparks jumping from a key, which was tied to the bottom of the string of probably the best known kite worldwide, to his knuckles. Franklin must have been lucky. There have been others that were killed by similar experiments. He later suggested the famous Lightning rod protection and such rods were first used for Lightning protection in 1752 in France. Even nowadays, Franklin’s invention is still used as the primary Lightning protection of structures.

In the late nineteenth century photography and spectroscopy were invented and were used in Lightning research. Leader processes and individual strokes belonging to a flash were identified. Major advances in understanding of Lightning became possible after the invention of the double-lens streak camera by Boys in 1900.

The early days of modern Lightning research are associated with C.T.R. Wilson [1], who was first to use Electric Field measurements in England to find the charge distribution within a thunderstorm and the charges involved in a Lightning flash.

Recent contributions to understanding of Lightning are primarily motivated by the necessity to provide Lightning protection from direct strikes to aircraft, spacecraft and electrical and electronic installations vulnerable to direct Lightning strikes. Such events are relatively rare and are well handled. What is more probable is the secondary effect due to Lightning strikes in the vicinity of an object or installation, namely the radiated Electromagnetic Pulse. The associated Electric and Magnetic Fields could travel hundreds of km and influence electrical and electronic equipment in such a way as to disrupt their operation or even to destroy some sensitive electronic devices.
With advancements in all technological areas, Lightning research has picked up quite significantly in the recent years. There are a large number of experiments and publications readily available in the technical literature from different authors from all over the world. Some of the “hot” spots for conducting Lightning research are usually associated with regions with high Lightning activity such as Florida (USA), Brazil, South Africa, and Japan (winter thunderstorms), but also with particular locations where high rise towers exist including CN Tower (Canada), Ostankino tower (Russia), Monte San Salvatore (Switzerland), Peissenberg Tower (Germany), and Gaisberg Tower (Austria). Such tall towers are frequently struck by Lightning and thus are very convenient means for direct measurements of the Lightning current. Knowledge of the current would be useful if one used it in a computer model to estimate the impact of radiated fields at a distance for example. Nowadays, computer capabilities are becoming better every couple of months and, consequently, computer modeling is becoming even more handy and convenient means for studying of Lightning.

In the present thesis work use is made of the contemporary “Engineering” modeling approach and of present computing capabilities to come up with a sophisticated model adequately describing Lightning events at the CN Tower in Toronto. The outcomes of the model are the radiated Electric Field components as well as the Magnetic Field components at a certain distance, provided the Lightning Return Stroke current is known. The implementation of the model is elaborated on and the achieved results are fully analyzed. Such a model could be easily adapted to produce results for any other tall structure.

The reader is first presented with the opportunity to go through a quick overview of a few of the aspects regarding Lightning, its physics and its effects. Three of the very good and thorough references on these topics, where interested readers could review in depth more information are [2], [3], and [4].

I.3 Some Summarized Facts Regarding Worldwide Lightning

Lightning could be defined as a transient, high current electric discharge over a path length in the order of kilometres [2]. The primary producer of Lightning is the cumulonimbus thundercloud.

Globally, the Lightning flash rate is somewhere in the range of tens to a hundred per second. There are some 16 million Lightning storms in the world every year [5]. Most of the Lightning
discharges are occurring between thunderclouds or inside them. The type of Lightning most commonly occurring at Earth’s ground, and for that reason most studied is downward negative Lightning. Positive Lightning discharges are approximately 10% of the global Lightning activity (cloud-to-ground flashes) and are somewhat less understood. This type of Lightning is observed in the dissipating stage of a thunderstorm, during winter thunderstorms, when shallow clouds are present, during severe thunderstorms, and over forest fires [2], [3].

The Lightning flash consists of several phases. The whole event is initiated by a corona process. This phase then transforms into a streamer phase. Please, note that these processes take place on different scales. The developed streamer later matures and moves the process into a leader phase. In the leader phase, a “Stepped Leader” traverses progressively in steps of some tens of meters lengths in general direction towards ground. At some point a connection with an upward leader from ground might take place after which a “Final Jump” occurs. Finally the Lightning flash process ends up with a single “Return Stroke” or possibly with several ”Subsequent Strokes”, each of which is initiated by the “Dart Leader”. The stroke process may be repeated until the local charge in the cloud is depleted.

The produced Electric and Magnetic Fields are transient with variations in time from ns to ms. Typical speed of the downward dart leader is $10^7 m/s$. The average peak power output of a single Lightning stroke is about a terawatt ($10^{12} W$). The diameter of the Lightning Channel is about a centimetre (0.4 in). NASA scientists have found that the radio waves created by Lightning define a safe zone in the radiation belt surrounding the earth. This zone, known as the Van Allen Belt slot, can potentially be a safe haven for satellites, offering them protection from the Sun's radiation [6],[7],[8].

Negative Lightning flashes usually contain up to five strokes but this number could be much larger (up to twenty strokes). The duration of such a Lightning flash is some hundreds of ms and the inter-stroke times are in the range of tens of ms (there have been observed inter-stroke times of few ms with more sensitive equipment). The average Return Stroke current for negative flashes characteristically exhibits a peak value of $30kA$ after a rise time of a few $\mu s$ and decays to half-peak value in about $50\mu s$. The respective positive flash current is in the order of 10 times larger as compared to the negative flash current. The associated voltage is proportional to the length of the Lightning flash. The potential gradient inside a well-developed Return Stroke
channel is in the order of hundreds of volts per metre. The lowered charge to ground is in the range of several Coulombs. The Lightning Channel temperature is near 30,000K and the channel pressure is around 10atm or higher. This compresses the surrounding air and creates a supersonic shock wave which decays to an acoustic wave that is heard as thunder [9].

### I.4 Global Atmospheric Electric Circuit

Fig. I.1 depicts one of the contemporary interpretations of the global electric circuit. The Ionospheric potential at 60-80km height is estimated to be 250kV with respect to the Earth’s surface. The total global current produced by thunderstorms is approximately 1250A and flows through the fair weather part of the circuit, Earth, and closes as point discharge currents below the thunderstorms. The fair weather regions are everywhere from the Ionosphere and Mesosphere, down through the Stratosphere to the Troposphere (including flat or mountainous regions) and comprise 99% of the Earth’s surface area. The remaining 1% of the Earth’s surface area is covered with thunderstorms at all times. Values of all resistors seen in the three branches on the right hand side of the “Equivalent Circuit” at the bottom of Fig. I.1 are 5Ω, 95Ω and 100Ω respectively. This is the assumed distributed resistance of the fair weather circuit and essentially most of the resistance is close to Earth’s surface. The resistor at the boundary layer above mountains or above Antarctica is neglected.

![Global Electric Circuit Diagram](image)

Fig. I.1 - Global Electric Circuit (adapted from Rycroft et al., JASTP, 2000, Fig. 5)
Other respective values are clearly marked in Fig. I.1, which also shows an equivalent electrical diagram. It is a well-known fact that not all types of clouds have the potential to carry large amounts of charge and consequently to produce Lightning. Thunderclouds (especially of the cumulonimbus type) are considered as the batteries, or the sources, in the global electric circuit. In general, the sources driving the global atmospheric electric circuit could be categorized in 2 major classes – DC and AC sources. DC sources are the consequences of charges collected in electrified shower clouds during thunderstorms. AC sources are caused by inter-cloud, intra-cloud and cloud-to-ground discharges of Lightning strikes.

An experiment proving that the atmospheric electric circuit looks like the approximation shown in Fig. I.1 has not been conducted yet. Therefore, this topic is still open and a subject of debate as is the case with many other issues in the field of Lightning research. For example, there are a number of Lightning related physical and chemical processes that take place during the formation and dissipation of thunderstorms. Processes of most concern to us, and commonly seen by the naked eye, including negative and positive cloud-to-ground Lightning flashes, will be investigated later on in this Chapter. Particular attention will be paid to negative cloud-to-ground Lightning. Other processes occur in the upper layers of the atmosphere. They belong to most spectacular Lightning observations and have been monitored only comparatively recently since they often require special observation equipment. These are briefly discussed below.

I.5 Upper Atmospheric Lightning (Megalightning)

I.5.1 Sprites, Blue Jets, and Elves

Lately, scientists and researchers from around the globe, working in various study areas, have become quite interested in Lightning related events that are occurring at high altitudes in the Stratosphere and the Ionosphere.

A chart showing in somewhat up to scale positive/negative cloud-to-ground Lightning discharges, Blue Jets, Sprites and Elves is seen in Fig. I.2.
Each of these atmospheric phenomena is really interesting, intriguing, and has impact on various aspects of human life. It seems that they are also directly related to each other.

Research of Megalightning events became possible in the recent years with the development of modern video and other types of recording equipment with high resolution and zooming capabilities. Sprites, Blue Jets, and Elves have been studied also because of concerns of their interaction with spacecraft in particular. In fact some of the modern space shuttles carry onboard equipment that is used to monitor and study these high altitude events.

Sprites are large scale electrical discharges which occur high above a thunderstorm cloud, giving rise to a wide range of visual shapes. They are triggered by the more powerful cloud-to-ground discharges, namely the positive Lightning (marked with +CG in Fig. I.2 above) [10]. Their name stems from the mischievous sprite (air spirit) Puck in Shakespeare's Midsummer Night's Dream.
Sprites appear reddish-orange or greenish-blue in color and have hanging tendrils below with arcing branches above their location. They can be preceded by a reddish halo [11]. Those phenomena often occur in clusters, sitting at 80-145 km above Earth's surface. Since they were first photographed by scientists from the University of Minnesota on July 6, 1989, Sprites have been observed tens of thousands of times [12].

Blue jets look like giant sparks that short-circuit the Stratosphere. They extend from the top of the cumulonimbus (12-15 km) above a thunderstorm, typically in a narrow cone, to the lowest levels of the Mesosphere (48-50 km) above Earth (see Fig. I.2). They are brighter than sprites and are blue in colour. The first recorded blue jet dates back to October 21, 1989. A video was taken from the space shuttle as it passed over Australia [13].

Elves often appear as dim, flattened, and expanded doughnut shaped glows that are around 400 km in diameter. They last for approximately one ms [14]. Elves occur in the ionosphere 97 km above the ground over thunderstorms. Their colour is now believed to be a red hue. Elves were first recorded off French Guiana on October 7, 1990 on a space shuttle mission. Their name comes from the abbreviation (Emissions of Light and Very Low Frequency Perturbations from Electromagnetic Pulse Sources) [15].

All those phenomena described above are driven by power sources, namely the thunderclouds formed during thunderstorms.

I.6 Structure and Formation of Cumulonimbus

A photograph of a typical thundercloud observed in central New Mexico is seen in Fig. I.3. The hypothetical charge distribution is clearly marked and also the temperature values at different heights are shown. Such a cloud has the shape of an anvil and is thought to have a tri-pole charge separation structure with a negative screening layer at the top [3]. Fig. I.4 is depicting the idealized gross charge structure of the typical thundercloud shown in Fig. I.3. There is a large positive charge at the top of the cloud sitting at a height of approximately 12 km, a large negative charge at about 7 km height and a small positive charge at the bottom of the cloud at about 2 km height. The approximate charges are +40 C, -40 C, and +3 C respectively.
Fig. I.3 - Typical Thundercloud Observed in New Mexico (adapted from [3], Fig 3.1)

Please, note that the presented values are applicable to thunderclouds found in New Mexico, USA and are shown for illustrative purposes. The heights vary for different regions of the world. For example the heights of the cloud charges observed during winter storms in Japan are consistently lower. Please, check Fig. I.5.

Fig. I.4 - Vertical Tri-Pole Structure of a Thundercloud (adapted from [3], Fig 3.2a)
The total Electric Field and contributing components that would be measured at ground level due to major bulk charges, produced by thunderclouds represented by a tri-pole charged structure seen in Fig. I.4, as a function of the distance from the vertical axis of the tri-pole, are shown in Fig. I.6.
Clearly, all three charges have significant influence upon the observed total Electric Field. At far distances the total Electric Field is practically very close to zero, at distances in the range 2.5-10km away from the cloud the produced field is positive and at near distances (up to approximately 2.5km away), the total Electric Field is evaluated to be negative having maximum negative peak value directly under the vertical axis of the thundercloud. The question remains how these major bulk charges are formed, separated, and moved to their respective locations to form the tri-pole structure of the thundercloud.

I.6.1 Thunderstorm Formation Requirements

So far there is no available fully satisfactory theory that explains the exact mechanisms of electrification of thunderclouds. A list of observations regarding thunderstorm formation requirements and accompanying processes was first proposed by Mason [16] and later augmented and extended by Moore and Vonnegut [17].

Those requirements could also be found summarized in [4]. Among some of the most important observations with regards to formation of thunderclouds that should be mentioned is the fact that the electrification of clouds is closely connected to processes taking place in the freezing layer inside a thundercloud. Ice and supercooled air appear to be crucial in separating of opposite charges within the cloud. Furthermore, strong electrification is observed whenever the cloud exhibits strong convective activity with rapid vertical development. An agreement exists that Lightning generally originates in the vicinity of high-precipitation regions.

In the past decades several mechanisms trying to explain the formation of thunderclouds in agreement with the requirements mentioned above have been proposed. All of these mechanisms explain to some extent some of the involved processes, but also lack argumentation regarding some of the remaining observed phenomena. A brief list of the suggested mechanisms is shown in [4] together with relevant references.

I.6.2 Thundercloud Electrification Mechanisms

Not a long time ago, there were two major vying theories trying to explain the mechanism of electrification of clouds and those are namely the Precipitation and the Convection hypothesis.
They are two very different models both trying to explain the old-fashioned dipole structure of the thundercloud. A good reference containing more details regarding this topic is [18].

![Fig. I.7 - Precipitation Mechanism (adapted from [18])](image)

The precipitation model is considering the phenomenon observed whenever a garden sprinkler is working. The dispersed water consists of larger and smaller droplets. The heavier water droplets are pulled down towards ground whereas the smaller size droplets (mist of water) remain in the air and possibly are moved or blown away by the wind. Following the same principle, there are water droplets, mist, graupel, ice crystals, and various other smaller or bigger water formations within a thundercloud. The heavy particles are descending towards ground and the lighter ones are suspended and free to move inside the cloud. While moving downwards, the heavy particles (including raindrops, hailstones, ice particles, and graupel) collide with suspended supercooled precipitation particles and it is assumed that the ice crystals transfer positive charge to the mist, while they become negatively charged. This means that the lower part of the cloud would be predominantly negatively charged and the upper part would be positively charged invoking the dipole structure representing a thundercloud.
The convection mechanism was first proposed by Grenet [19].

This model is considering an analogue to the Van de Graaff generator in which charges are sprayed onto a moving belt that later transports them to a high voltage terminal. The first question that comes to mind now is how the charges in a cloud would be supplied and where do they come from. In that theory, on one hand, charges are assumed to be supplied by the upper atmosphere and by cosmic rays that impinge on the air molecules found at the top of the cloud. Those cosmic rays ionize the air molecules and effectively separate positive and negative charges. On the other hand, charges are supplied by corona discharge of positive ions produced by strong Electric Fields found near grounded sharp objects. These positive charges are lifted upwards by strong upwinds (warm air masses rising due to convection) - analogous to the moving belt in the Van de Graaff generator. At some point these positive charges reach higher
altitudes inside the cloud and attract the negative charges formed at the top of the cloud by the cosmic rays. While entering the cloud, the negative charges attach themselves to water droplets and form the so-called “screening” negative layer. There are strong downdrafts at the outskirts of the cloud that are assumed to carry downward the negatively charged particles. This again would result in a dipole charge separation found in the thundercloud.

![Diagram of cloud electrification](image)

*Fig. I.9 - Graupel-Ice Mechanism of Cloud Electrification (adapted from [3], Fig.3.13)*

In actual fact both mechanisms are observed in all thunderclouds, but the two theories are completely different and they do not invoke each other. Furthermore, it has been found out that the thundercloud features a tripole structure, rather than a dipole structure. Several modifications of the precipitation model have been proposed throughout the years in order to present an explanation for the observed lower positive charge and also for the fact that usually the rain carries positive charges. There are many unanswered questions and the exact mechanism of electrification of thunderclouds is still under investigation.

Among all the available theories that are trying to explain that mechanism, it seems that lately most researchers have relative agreement in considering the “graupel-ice mechanism”, shown in Fig. I.9.
Essentially, this mechanism attributes the formation of electrical particles to the collisions between precipitation particles (graupel) and cloud particles (small ice crystals) in the presence of water droplets. Precipitation particles are usually larger than cloud particles and are falling at a higher speed as compared to cloud particles. The graupel particles fall through the surrounding smaller ice crystals and supercooled water droplets. Those droplets freeze and stick to ice surfaces as they touch them (rimming). Below certain temperature (reversal temperature), the falling graupel particles acquire a negative charge when colliding with the ice crystals, whereas at temperatures higher than the critical value those graupel particles become positively charged. The temperature reversal value is found to be between -10° and -20° C. This mechanism could explain the formation of the lower positive charge.

Besides the lack of thorough understanding of the exact mechanism of electrification of the thundercloud, another phase of the Lightning process is of importance, namely development of the Lightning flash, which is produced by the cumulonimbus cloud. Lightning events have been observed, documented, and analyzed for many years. Let us first start the examination of Lightning flashes by looking into their categorization.

### 1.7 Categorization of Lightning

![Fig. I.10 - Types of Lightning Discharge from Cumulonimbus Clouds (adapted from Rakov [3])](image)

Most of the Lightning discharges occur within the cloud (intra-cloud) or between clouds and are called cloud discharges. The best-studied and well-documented type is the cloud to-ground Lightning. This type is of practical interest because it is the one causing injuries and death to
people and livestock, forest fires, and also destruction and disturbances in power grids and installations and electrical and electronic equipment. In addition, there are very rare discharges from cloud to air. Fig. I.10 depicts the associated percentage of Lightning type occurrence.

According to Berger [20] there are four types of Lightning between the cloud and earth in terms of direction of motion (upward or downward) and positive or negative charge sign of the leader that initiates the Lightning stroke. Those four types are presented in Fig. I.11 below.

Fig. I.11 - Categorization of Cloud-to-Ground Lightning (adapted from [2], Fig. 1.3)

I.7.a) Category 1 Lightning accounts for about 90 percent of all cloud-to-ground Lightning. It is initiated by downwards moving negative leader and causes transfer of negative cloud charge to the Earth.

I.7.b) Category 2 Lightning is due to positively charged moving upwards leader and causes neutralizing of negative cloud charge.

I.7.c) Category 3 Lightning is initiated by a moving downwards but positive leader. This time the charge being neutralized is positive. This accounts for less than 10 percent of cloud-to-ground Lightning.
1.7.d) *Category 4* Lightning is due to negatively charged moving upwards leader and transfers positive cloud charge to the Earth.

Categories 2 and 4 are very rare events and are generally initiated by very tall and sharp objects such as mountain tops or tall towers. Category 2 will be given special attention later in the thesis. Category 4 Lightning flashes are extremely rare and usually neutralize a very large amount of charge in one stroke only which has extremely high associated current.

1.7.1 Negative Cloud-to-Ground Lightning

This is the most common type of Lightning discharge from cloud to ground and thus will be reviewed in some more detail below. Lightning of this type is striking short or flat objects and a photograph of such an event is seen in Fig. I.12. It is very typical for negative flashes to ground to exhibit downward branching, clearly visible in the figure below. In some cases more than one branch is contacting the ground at the same time (also visible in the blow-out in Fig. I.12).

![Fig. I.12 - A Photograph of a Negative Cloud-to-Ground Lightning Flash](image)
The processes involved in cloud-to-ground Lightning are explained using the idealized Lightning flash time development diagram adapted from Uman [2] and shown in Fig. I.13.

The whole process is started by the stepped leader, which initiates the first stroke of a flash by jumping in a series of discrete steps from cloud to ground. The stepped leader is initiated by a preliminary breakdown within the cloud. This breakdown process is believed to take place in the lower regions between the smaller positive charge and the negative charge. It preconditions the area for the stepped leader to take place. The stepped leader steps are usually some tens of meters long and their duration is in the order of $1\mu$s. Stops between individual steps are in the order of $50\mu$s and the average speed of stepped leader propagation is around $2 \times 10^5 \text{m/s}$.

![Fig. I.13](image-url)
The average stepped leader current is estimated to be in the range 100-1000\(A\). The associated radiated Electric and Magnetic Fields have duration of about 1\(\mu s\) and risetimes 0.1\(\mu s\) or less.

The electric potential at the bottom of the negatively charged stepped leader channel with respect to the potential of ground is in the order of more than 10\(^7\)\(V\). This high potential difference drives the local Electric Field at ground level and near the grounded objects to rise in excess of the air breakdown value. Consequently, one or even several upward moving discharges are created and the attachment process begins. As soon as one of these upwards extending branches contacts the downward moving stepped leader, at some tens of meters above ground, the final jump occurs and the leader is effectively connected to ground. This initiates the flow of the first Return Stroke current, which propagates upwards in the Lightning Channel path already ionized by the stepped leader and reaches the top of the Lightning Channel in about 100\(\mu s\).

A Lightning flash could contain just one stroke or several (up to 20) strokes. If the charge lowered to ground in the first stroke depleted the available cloud charge, there might not be any further subsequent strokes. On the other hand, if there was still additional charge available in the cloud after the first-stroke took place, and this charge was conveniently located close to the top of the already ionized Lightning Channel path, a dart leader might be formed that would propagate down the residual channel without branching and initiate a subsequent stroke. The additional charge brought to the Lightning Channel and responsible for subsequent strokes is believed to be driven by K and J processes. The charge and current associated with dart leaders are in the order of 1\(C\) and 1\(kA\) respectively and the duration of the Electric Field changes is usually about 1\(ms\). These changes are similar to first Return Stroke changes, but exhibit faster risetimes and lower overall values.

The time between subsequent strokes belonging to a flash is several \(ms\). This interstroke interval is sometimes up to tenths of a second provided a continuing current is flowing after the stroke in the channel. Such current is associated with direct charge transfer from cloud to ground and linear Electric Field change for about 0.1\(s\).

**I.7.2 Positive Cloud-to-Ground Lightning**

Positive cloud-to-ground Lightning is a relatively rare event and is observed to increase in occurrence towards the end (mature stage) of thunderstorms. This type of Lightning is believed
to be produced by the upper positively charged region of the cumulonimbus and occurs whenever the negatively charged region is depleted, thus providing less shielding for the positive charge to connect to ground. Since the positive charge is sitting at higher altitude, the associated voltage potential and charge lowered to ground are substantially higher as compared to a regular negative cloud-to-ground Lightning flash. The associated positive flash current may be in the range of 200-300kA. Positive flashes usually consist of a single stroke, followed by a continuing current. Another observation regarding positive flashes is that they predominantly occur during winter thunderstorms and are relatively uncommon during summer thunderstorms. Positive flashes are frequently observed and reported during winter storms in Japan. The structure of the thunderclouds in these Japanese winter storms is somewhat different in terms of vertical arrangement of the large positive and negative charges. Essentially, the upper positive charge is not directly on top of the negative charge, and thus often has a clear view towards ground (see Fig. I.5). In addition, the distance to ground is not as large and this makes it even easier for the positive charge to find its way to ground. More relevant information could be also found in [2-4].

### I.7.3 Tall Structures Initiated and Rocket Triggered Lightning

It is assumed that grounded objects that are rising above 500m above ground level experience only upward flashes, objects of 100-500m height experience both downward an upward flashes and structures of less than 100m height experience only downward Lightning [2].

The tall structure initiated and rocket triggered Lightning are predominantly of the upward initiated type (see Category 2 and Category 4 in Fig. I.11). The Return Stroke current neutralizes most often negative cloud charge through several strokes. Those strokes usually resemble subsequent strokes to flat ground since they feature fast risetimes and peak currents in the order of only tens of kA. It is not uncommon to observe bipolar discharges occasionally and sometimes, very rarely, only positive charge may be transferred to ground.

Lightning to tall structures is one very interesting area of research involving Category 2 and Category 4 Lightning events (see Fig. I.11). It is a relatively new field of study, since very tall towers started to become necessary for broadcasting of radio and TV programming some tens of years ago. Tall structures are usually frequently struck by Lightning and can be instrumented to directly measure Lightning current. Lightning researchers realized the convenience of such tall objects for performing Lightning studies and started taking advantage of the situation. A few
telecommunication towers around the world have been equipped permanently, others temporarily with current measuring equipment and were used to perform different studies throughout the years. For example, a Rogowski coil was mounted on the CN Tower in Toronto soon after its completion back in the 70’s. Currently, there are two permanently mounted Rogowski coils on the CN Tower and Lightning current derivative is routinely captured at that site some 50-70 times per year. Very soon, after performing the very first measurements, it was noticed that Lightning at tall structures is somewhat different from Lightning to flat ground and this is due to the propagation processes that take place inside the structure. Tall towers have considerable influence (enhancement) upon the recorded Lightning current and consequently upon the radiated Electric and Magnetic Fields. Some relevant information is introduced in [21-27]. Furthermore, some of the most comprehensive and thorough studies performed by researchers at tall towers could be reviewed in [28-40]. Essentially these studies show that Lightning current waveforms are affected by transient processes that take place along the tall structure. Currents measured close to the top of such structures are less influenced as opposed to those measured close to ground level. The current amplitudes at the bottom are significantly affected and may be much larger than the corresponding ones at the top.

Tall structures are often struck by Lightning and this is mostly because of the involved mechanism of discharge initiation. This mechanism is quite similar to the analogous one involved when “classical” rocket initiated Lightning is considered (described further down).

Essentially, at the tip of the tall structure the Electric Field is intensified and positive charges may form a positively charged leader or leaders that in turn may propagate upwards some tens to hundreds of meters. From the other end, there might be just enough negative charge, attracting even further the developing positive leaders. That charge could be easily neutralized under the existing favorable conditions (short distance to the grounded object and presence of upwards propagating positive leaders). In case one of the positive leaders found its way up to the negatively charged region of the cloud, a continuous current would start flowing and neutralizing some of the available charge. At this time, if more charge were available a number of flashes could take place.

The first successful tests with rocket-triggered Lightning were implemented back in 1960. Rockets trailing thin, grounded wires were launched from a research vessel off the west coast of
Florida [41]. Triggered Lightning experiments and results have been considered and summarized in [42] and [2].

Rocket triggered Lightning discharges are initiated in a very similar way to the tall objects Lightning discharges, e.g. grounded telecommunication towers or antennas initiated discharges. The difference between them is mainly in the fact that whenever a tall structure is considered there is an associated static structure, which may contain reinforced concrete and metallic down-conductors comprising the Lightning protection of the structure, whereas the rocket triggered Lightning makes use of a grounded wire that propagates upwards at a certain speed.

Basically in a successful triggered Lightning experiment several processes take place one after another, which are described briefly here. First, a rocket trailing a grounded wire is launched toward a charged cloud. Then, the rocket ascending at a certain speed may trigger a positively charged leader. That leader consequently vaporizes the wire and continues on its own to bridge the gap between the ground and the cloud. Upon reaching the cloud charge, the leader may become quite well branched upwards. A continuous current starts flowing in the preconditioned channels and effectively neutralizes some negative cloud charge. After that several sequences of dart leader Return Strokes may take place along the same discharge paths.

There is another way of triggering Lightning by launched rockets, called altitude triggering. In this technique the metallic wire is not grounded and a bi-directional leader process is involved. More details could be reviewed in [43] and [44].
Chapter II
Development of Expressions for Current Distribution at any Level along the CN Tower and the Lightning Channel

II.1 Outline

In Chapter II the reader will find information pertinent to modern models used to describe Lightning events. The existing four major classes of models are reviewed and the most used “Engineering” models discussed in more detail.

An original contribution shown here is the different treatment of the extension of the “Engineering” transmission based model that includes the presence of a tall structure in the Lightning Channel path.

The CN Tower was chosen to be modeled as the struck tall object, because there are data records including directly measured current, vertical component of the Electric and azimuthal component of the Magnetic Field, as well as video records, and Lightning Channel front speed records that have been accumulated by former and current researchers for more than 30 years. These data are very useful for verifying the simulated currents and fields by various proposed modeling approaches.

In the chapter, there are a few newly proposed solutions to some of the existing difficulties and inconsistencies associated with modified transmission line based models. In addition to the thorough description of the proposed modeling approach using a Single-Section Model of the CN Tower, full details are also found in Appendix I. Appendices II and III contain all relevant information for the Three- and Five-Section Model representation of the CN Tower respectively.

II.2 Modeling of Lightning (to Flat Ground and to Tall Objects)

Four major classes of Lightning Return Stroke models are summarized and discussed in great detail in [45]. In actual fact, most of the published readily available models could be classified under one or in certain cases maybe two of these classes. They are briefly introduced below.
II.2.1 Gas Dynamic (Physical) Models

These are looking into the radial evolution of a short segment of the Lightning Channel and its associated shock wave and involve solving of three hydrodynamic equations describing conservation of mass, momentum, and energy. These equations are coupled to two equations of state. A known channel current as a function of time is used as an input parameter and the outputs of such described model are the pressure, mass density, and temperature as functions of radial coordinate and time (e.g. [46], [47], and [48]).

Such models were used to describe laboratory spark discharge in air (e.g. [46-48]), but speculated to be applicable to Lightning Return Stroke modeling as well. According to some recent works (e.g. [49-57]) the physical modeling approach assumes that the plasma column is straight and cylindrically symmetrical, the algebraic sum of positive and negative charges in any volume element is “zero”, and local thermodynamic equilibrium exists at all times.

II.2.2 Electromagnetic Models

These models are usually based on a lossy, thin-wire antenna approximation of the Lightning Channel. They involve a numerical solution of Maxwell’s equations to find the current distribution at any point along the channel. Such current is then plugged in known relations for different components of Electric and Magnetic Fields to evaluate them at a certain distance away (e.g. [58-61]).

II.2.3 Distributed-circuit Models

This class of models can be considered as an approximation of the Electromagnetic models. Here, the Lightning discharge is assumed to represent a transient process on a vertical transmission line of specific resistance, inductance, and capacitance. The distributed-circuit models (sometimes referred to as R-L-C TL models) are used to determine the channel current as a function of time and height and can also be used for evaluation of Electric and Magnetic Fields at a distance (e.g. [62] and [63]).

II.2.4 “Engineering” Models
The distribution of the channel current (at any given point and time along the channel) is deduced from observed Lightning Return Stroke characteristics such as the current at the channel base, the speed of the upward-propagating Lightning Channel front, and the channel luminosity profile. These models have just a few extra parameters that need to be adjusted and predict reasonably well the electromagnetic field profiles computed at a distance (e.g. [64-65], and [45]).

In this thesis work three sophisticated “Engineering” models, representing the CN Tower and the associated Lightning Channel by variable number of transmission lines, are developed and thoroughly analyzed. The decision to use “Engineering” models was made primarily due to the fact that there are readily available data regarding the channel-base current, the Return Stroke velocity, radiated Electric and Magnetic Fields at a certain distance, and visual records pertinent to the CN Tower Lightning events. In order to refresh the reader’s knowledge and also to look at the details pertaining to the adopted modeling approach, a brief review of the different most commonly used “Engineering” models follows.

II.3 Most Used “Engineering” Models

An “Engineering” model could be defined by an equation that relates the current along the Lightning Channel $I(z', t)$ at any height $z'$ and any time $t$ to the current at the Lightning Channel base $(z'= 0)$ [45]. The “Engineering” model could be also defined in terms of line charge density $\rho_L(z', t)$ along the channel using the continuity equation [66]. Some of the “Engineering” models could be summarized and described mathematically by the following expression [67]:

$$ I(z', t) = u(t - z'/v_f)P(z')(0, t - z'/v) $$  \hspace{1cm} (II.1)

where:

- $u$ Heaviside function equal to unity for $t \geq z'/v_f$, and zero for $t < z'/v_f$

- $P(z')$ height dependent current attenuation factor [68]

- $v_f$ upward propagating Return Stroke speed
\( v \) ~ current wave propagating speed

Values for \( P(\zeta') \) and \( v \) for five “Engineering” models are summarized in Table II.1 below.

<table>
<thead>
<tr>
<th>Model</th>
<th>( P(\zeta') )</th>
<th>( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmission Line Model (TL) [69]</td>
<td>1</td>
<td>( v_f )</td>
</tr>
<tr>
<td>Modified TL Model with Linear Current Decay with Height (MTLL) [70]</td>
<td>( 1 - \zeta'/H )</td>
<td>( v_f )</td>
</tr>
<tr>
<td>Modified TL Model with Exponential Current Decay with Height (MTLE) [71]</td>
<td>( e^{\left(\frac{-\zeta'}{\psi}\right)} )</td>
<td>( v_f )</td>
</tr>
<tr>
<td>Bruce-Golde Model (BG) [72]</td>
<td>1</td>
<td>( \infty )</td>
</tr>
<tr>
<td>Traveling Current Source Model (TCS) [73]</td>
<td>1</td>
<td>( -c )</td>
</tr>
</tbody>
</table>

In Table II.1:

\( H \) ~ total channel height

\( \psi \) ~ current decay constant (assumed to be 2000 m [71])

\( c \) ~ speed of light

In the above shown models \( v_f \) is usually assumed to be a constant value.

The most used “Engineering” models could also be divided into two groups namely the TL (transmission line) based models and the TCS (traveling current source) based models. They are summarized in Table II.2 and in Table II.3 below [45].

The degree of reproduction of Lightning electromagnetic field waveforms by “Engineering” models is analyzed in [74].
Table II.2
TL based Models (Adapted from [45])

<table>
<thead>
<tr>
<th>Model</th>
<th>( I(z', t), \rho_t(z', t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TL [69]</td>
<td>[ I(z', t) = I(0, t - z'/v) ]</td>
</tr>
<tr>
<td></td>
<td>( \rho_t(z', t) = \left( \frac{I(0, t - z'/v)}{v} \right) )</td>
</tr>
<tr>
<td>MTLL [70]</td>
<td>[ I(z', t) = (1 - z'/H)I(0, t - z'/v) ]</td>
</tr>
<tr>
<td></td>
<td>( \rho_t(z', t) = (1 - z'/H) \left( \frac{I(0, t - z'/v)}{v} \right) + \frac{Q(z', t)}{H} )</td>
</tr>
<tr>
<td>MTLE [71]</td>
<td>[ I(z', t) = e^{\frac{z'}{v}} I(0, t - z'/v) ]</td>
</tr>
<tr>
<td></td>
<td>( \rho_t(z', t) = e^{\frac{z'}{v}} \left( \frac{I(0, t - z'/v)}{v} \right) + e^{\frac{z'}{v}} \frac{Q(z', t)}{\psi} )</td>
</tr>
</tbody>
</table>

\( Q(z', t) = \int_{\frac{z'}{v}}^{t} I(0, \tau - z'/v) d\tau \); \( v = \nu_f = \text{const} \); \( H = \text{const} \); \( \psi = \text{const} \).

“Engineering” models have been validated by different authors primarily using two common approaches. One way to do that is by considering a typical channel base current and propagation speed as inputs to the chosen model, and then comparing the output of the model, namely the computed Electric and Magnetic Fields, to typically observed ones. The other approach is applicable whenever recorded information regarding the channel base current, propagation speed, and recorded radiated fields are readily available. Such records exist for rocket-triggered Lightning and at instrumented tall towers. Using those captured data, one could reproduce the “uncontaminated” current waveform and based on that compute the respective radiated field components. The computed fields could be then compared to the actual ones in order to validate the modeling approach. Since the question of what exactly happens at the physical origin of Lightning (at the end of the Lightning Channel in the cloud where the charge is located) is usually not treated, the “Engineering” models yield satisfactory results that could be trusted to some extent in the first tens of \( \mu s \) (up to usually 50\( \mu s \)). After that time, depending on the used propagation speed, reflections within the Lightning Channel and possibly other influences from the cloud charge may come into play and eventually impose additional changes in the model.
Table II.3
TCS based Models (Adapted from [45])

<table>
<thead>
<tr>
<th>Model</th>
<th>$I(z',t), \rho_L(z',t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BG [72]</td>
<td>$I(z') = I(0,t)$</td>
</tr>
<tr>
<td></td>
<td>$\rho_L(z',t) = \left( \frac{I(0,z'/v_f)}{v_f} \right)$</td>
</tr>
<tr>
<td>TCS [73]</td>
<td>$I(z') = I(0,t + z'/c)$</td>
</tr>
<tr>
<td></td>
<td>$\rho_L(z',t) = \left( -\frac{I(0,t + z'/c)}{c} \right) + \frac{I(0,z'/v^<em>)}{v^</em>}$</td>
</tr>
<tr>
<td>Diendorfer and Uman (DU) [75]</td>
<td>$I(z') = I(0,t + z'/c) - I(0,z'/v^*)e^{-\left(\frac{z'}{v_f}\right)^2/\tau_D}$</td>
</tr>
<tr>
<td></td>
<td>$\rho_L(z',t) = -\frac{I(0,t + z'/c)}{c} - \frac{I(0,z'/v^<em>)}{v_f} + \frac{\tau_D}{v^</em>} \frac{dI(0,z'/v^*)}{dt}$</td>
</tr>
</tbody>
</table>

$v^* = v_f / (1 + v_f / c); \quad v_f = \text{const}; \quad \tau_D = \text{const}$

The presence of a high tower in the Lightning Channel path has been found to exert a major influence upon the Lightning Return Stroke current as well as upon the emanated Electric and Magnetic Fields. The respective current and field waveforms recorded at tall towers are substantially different in terms of peak and overall values as compared to the corresponding ones in the case whenever the tall structure were not present. An elaborate study looking into Lightning at tall structures in which two “Engineering” models were extended in order to take into consideration the presence of a tall tower in the path of the Lightning Channel could be found in [76]. The reader may also wish to check the more recent reference [77].

All “Engineering” models that consider extended object as a part of the Lightning Channel feature a discontinuity at the Return Stroke front. This discontinuity stems from the fact that propagation along the tall object is considered to take place at the speed of light and propagation along the rest of the Lightning path (within the Lightning Channel) occurs at a different speed. That velocity is usually more than twice less than the speed of light. Basically a current wave is
assumed to be injected at the junction point between the upper portion of the Lightning Channel and the lower portion of the Lightning Channel, represented by the tall structure. Two current waves start their trip down the tall object and up the channel respectively. In such cases, the current wave that travels downward towards the ground is reflected at the bottom and after its round trip to the tip of the tall object it is transmitted into the Fully-Ionized, by the upward propagating wave, portion of the Lightning Channel. The wave reflected from the bottom of the tall object and transmitted into the Lightning Channel continues upwards at the speed of light until it catches up with the propagating at a lower speed Return Stroke channel front wave. At this point the first, faster propagating, current wave is assumed to vanish and it does not have any contribution in the region beyond the channel front. The associated discontinuity should be considered when computing Electric and Magnetic Fields. One way to do so is by introducing an additional term (“turn-on” term) [78-65-79-66] in the relations for Electric and Magnetic Fields. Fields computed using such a “turn-on” term could be found in [80].

In the present thesis work the discontinuity is treated in a different way. Instead of forcing the transmitted into the Lightning Channel components that catch up with the channel front to be zero, parts of them are allowed to be reflected at the channel front and other parts are transmitted and continue upwards together with the initial wave at the same slower speed. More details are found in the presented Single-, Three-, and Five-Section Models later in the thesis.

II.4 Modeling of Lightning Events at the CN Tower

The presence of the CN Tower should always be considered when modeling Lightning events at that site, since as already mentioned it has substantial influence upon the current and radiated distant fields. Lightning events at the CN Tower have been modeled mainly using electromagnetic modeling approach [81-83] where the structure was represented by a 3-D wire object, and also “Engineering” modeling approach [75-84-85-86], where the tower was represented by one, three [87-88] or even by five [89] transmission lines connected in series. First, before presenting the proposed in this work three models based on the “Engineering” approach, the reader is offered the opportunity to review a few facts regarding the CN Tower and the instrumentation currently used to capture Lightning events, including the derivative of the Lightning current, the radiated Electric (vertical component $E_z$) and Magnetic (azimuthal
component $H\phi$) fields at 2km away from the CN Tower, and visual records of the respective Lightning events.

II.4.1 The CN Tower

![The CN Tower in Toronto](image)

Fig. II.1 - The CN Tower in Toronto

Lightning has been observed and recorded for more than 30 years at the CN Tower in Toronto. Currently, this tower is one of the few permanently instrumented tall structures worldwide. Many
papers regarding the specifics of the site and observed Lightning parameters are readily available (e.g. [90-94]).

The CN Tower in Toronto was completed in June 1976 after 40 months of construction. It is rising 553.33\(m\) (1,815\(ft\)) and has held the title “World’s tallest free-standing building” for more than 30 years. It is still the world’s tallest free-standing structure instrumented for Lightning Research.

II.4.1.1 A Few Facts

Although the Lightning flash density for the Toronto region is only about 2\(fl/year/km^2\), Lightning strikes the CN Tower some 50 to 70 times per year. While visible copper Lightning protection strips run inside the CN Tower, due to skin effect, the bulk of the Lightning current is conducted down the tower, as designed, by the bonded concrete reinforcing steel. Both feed into the massive grounding rods below ground level and thus ensure that each Lightning strike safely finds its way to ground [95].

In accordance with information provided by the CN Tower publicity, the top of the CN Tower is 625.09\(m\) (2,050\(ft\), 10\(in\)) above sea level. The Glass Floor is 113 stories high, which is 342\(m\) (1,122\(ft\)) above ground level, the Skypod – 114 stories – 346\(m\) (1,136\(ft\)) above ground level, and the Space Deck – 147 stories or 447\(m\) (1,465\(ft\)) above ground level. At the Skypod Level the circumference is 109.1\(m\) (358\(ft\)), and at the base of the CN Tower the circumference is 208.7\(m\) (684.52\(ft\)) and the radius is 33.2\(m\) (109.2\(ft\)). Total weight of the CN Tower is 117,910 metric tonnes (130,000 tons).

II.4.1.2 Instrumentation at the CN Tower

As shown in Fig. II.2, there are two permanently installed current sensing elements on the structure at 474\(m\) and at 509\(m\) above ground level. Both elements are Rogowski coils and are used to capture directly the derivative of the Lightning current. The captured raw data are relayed via coaxial (lower coil) and fiber-optic (higher coil) cables to a digitizer where they are stored onto a PC. Later on the raw data are initially treated using LabView and finally processed using a specially developed integration routine in Matlab to produce the actual Lightning current waveform as it is captured at the respective level.
A similar routine for processing and a set-up for capturing raw data corresponding to the vertical Electric Field ($E_z$) and the azimuthal component of the Magnetic Field ($H_\phi$) radiated during Lightning events at the CN Tower are used at the University of Toronto ($2\text{km}$ north of the CN Tower). The sensing elements in this case are broadband active field sensors. Both set-ups (at the CN Tower and at the University of Toronto) have GPS time stamping and the recorded current strokes and respective Electric and Magnetic Fields belonging to a certain Lightning flash event can be well correlated. Furthermore, two DVD recorders ($16\text{ms}$ time resolution) and a High Speed Digital Camera ($2\text{ms}$ resolution) are used to capture the Lightning flash trajectory. The High Speed Camera and one of the DVD recorders are operated at the University of Toronto, while the second DVD recorder is mounted at Kinectrics ($11\text{km}$ to the West of the CN Tower). The High Speed Camera records bear computer clock time-stamps and the second DVD recording set has been recently augmented with a time-inserter that is set regularly. The main purpose of the video recording is to confirm that captured current and fields belong to a
Lightning flash landing on the CN Tower. The secondary purpose (accomplished mainly using the High Speed Camera which has higher time resolution) is to help in establishing the Lightning flash multiplicity (number of strokes in a flash). Since the two DVD recorders are now recording the Lightning flash activity at the CN Tower from two directions that are almost perpendicular with respect to each other, a third application is currently being investigated and this is namely the opportunity to reproduce the 3-D trajectory of each Lightning stroke. This would be useful in studies related to the influence of the Lightning Channel inclination and tortuosity upon the recorded currents, fields and field-current relationships. In many cases there is a difference in the number of recorded strokes by different equipment due to the particular device and set-up characteristics [96].

Plots of the current derivative of the third stroke of a Lightning event captured at the 474\text{m} above ground level and of the respective processed current waveform are shown in Fig. II.3. The corresponding radiated $E_z$ and $H_\phi$ for the same event form August 19, 2005 are shown in Fig. II.4. Fig. II.5 depicts its Lightning flash trajectory and the respective High Speed Camera record is presented in Fig. II.6.
Fig. II.4 - Recorded $E_z$ and $H\phi$ Waves

Fig. II.5 - Video Frame of the August 19, 2005 event (14:11:41)
II.4.2 Three Sophisticated “Engineering” Models

Bermudez et al. [97] showed that transmission line modeling approach is quite adequate and useful to model Lightning events at tall structures. In their experiment two scaled down models of the CN Tower were used, the first one representing the structure by one transmission line section only, and the second one - by three transmission line sections. It was observed that the three section model yielded better results, since it represented the structure in greater detail. Fig. II.7 shows the experimental setup and the three section model used. The equivalent circuit diagram is seen in Fig. II.8, and the obtained results by the three section model in Fig. II.9. Injected narrow pulse signals were used in the experiment. The presented plot in Fig. II.9 was achieved using injected voltage pulse of $2\text{ns}$ width. Figs. II.7 to II.9 are adapted from [97].
Fig. II.7 - Experimental Set-Up

Fig. II.8 - Reduced Scale Model - Equivalent Circuit Diagram
Practically, the experiment validated the transmission line representation of an elevated strike object that is struck by Lightning. This is part of the motivation behind developing of the Five-Section Model presented in this thesis.

Connecting in series a number of transmission line sections representing a tall structure, such as the CN Tower, is a convenient means to capture and reproduce in detail the tower’s structural peculiarities. In previous studies, readily available in the Technical literature [86-89] and [98], Single- and Three-Section “Engineering” models based on the TL modeling approach [69] were used. At the University of Toronto, as shown in [87] and in [89], different multi-section representations of the CN Tower were considered. In particular, the Five-Section modeling approach was developed and explored in detail for the first time in [89]. This thesis is contributing to the technical literature the sophisticated transmission line based “Engineering” Single-, Three-, and Five-Section Models of the CN Tower. Multiple sections are considered in order to represent the structure in more detail. Full derivations and considerations regarding the Single-, Three-, and Five-Section Models could be reviewed in Appendices I, II, and III respectively. In the main body, only highlights and some final expressions are shown. The Single-Section model is shown almost in full details in order to introduce the modeling approach. The models include reflections coming from the upwards-propagating Lightning Channel front and also transmitted components beyond the channel front. Another novel assumption introduced
in these models is that all contributions that are less than 1% in magnitude of the original injected current wave (Return Stroke current) are neglected.

II.4.2.1 Single-Section Model

Full details regarding the Single-Section Model, including expanding of newly introduced terms \( \sigma \) and \( \xi \) are shown in Appendix I.

To visualize the processes of reflections and refractions at different levels of the CN Tower and in the Lightning Channel, a Lattice diagram is useful. To illustrate the use of the Lattice Diagram, in view of its reduced complexity, the case of the Single-Section representation will be discussed in detail (see Fig. II.10). However, principles described are directly applicable to the Three- and Five-Section modeling approaches.

---

1 Note, that the Lightning Channel is shown for the general case, when both Not-Fully Ionized (dashed line) and Fully Ionized (thick line) portions of the Lightning Channel exist, while in the instance of time when the Return Stroke current is initiated only the Not-Fully Ionized part of the Lightning Channel is present.
Following assumptions are considered for the Single-Section Model, most of which are also valid in case of the Three- and Five-Section Models:

- The CN Tower is represented by one transmission line section of constant surge impedance, which was calculated using Chisholm’s formula for cone representation of a transmission line tower [99] \(Z_t = 110 \Omega\);

- The grounding system is considered to be a simple resistance \(Z_g = 30 \Omega\);

- Reflection Coefficients:

  - \(k_b = (Z_t - Z_g)/(Z_t + Z_g)\);
  - \(k_l = (Z_t - Z_{ch})/(Z_t + Z_{ch})\);
  - \(k_c = (Z_{ch} - Z_{tch})/(Z_{ch} + Z_{tch})\);

- The Lightning Channel is vertical and attached to the tip of the CN Tower;

- Lightning Channel length \(z_{MAX} = 8km\);

- Lightning Channel is split into two sections (Initially the Lightning Channel is represented by a single transmission line section of constant surge impedance \(Z_{tch} = 495 \Omega\) to account for the Not-Fully Ionized portion. As soon as a current wave starts its way up the Lightning Channel, the Lightning Channel is already consisting of two sections and that part, up to where the current wave has already progressed, becomes Fully Ionized, thus having a different (reduced) surge impedance \((Z_{ch} = 330 \Omega)\). Used values are \(4.5xZ_t\) and \(3xZ_t\) respectively;

- Speed of propagation in the Not-Fully Ionized portion of the Lightning Channel is constant and is assumed to be \(v = 1.9e8m/s\);

- Constant for the exponential decay of propagation within the Not-Fully Ionized portion of the Lightning Channel is \(\psi = 2000m\);

- Speed of propagation in the Fully Ionized portion of the Lightning Channel as well as in the CN Tower is constant and is \(c = 3e8m/s\) (speed of light);
- It is assumed there is no decay in propagation within the Fully Ionized portion of the Lightning Channel and in the CN Tower;

- Contributions due to reflections and refractions at different locations along the structure of the tower and in the Lightning Channel are followed and considered until they become less than 1% in amplitude of the original injected wave;

- A sum of two Heidler functions [100] is used to approximate the injected “uncontaminated” Lightning current.

The “uncontaminated” Lightning current is injected at the attachment point between the CN Tower and the Lightning Channel. Two current waves having identical waveshapes, but amplitudes inversely proportional to the surge impedances of the CN Tower and the Not-Fully Ionized portion of the Lightning Channel start their way down the tower and up the channel respectively. The wave going down the tower is travelling at the speed of light $c$, while the one propagating upwards in the Not-Fully Ionized part of the Lightning Channel is traveling at a reduced speed $v$.

![Fig. II.11 - Lattice Diagram for the Single-Section Model](image)
Let’s look at the Lattice diagram in Fig. II.11 and follow first the injected current wave that goes down the CN Tower. It reaches the bottom of the tower where it “sees” a different impedance and consequently here a reflection takes place. The reflected wave goes back to the tower top and is here reflected back down the tower, while a part of it is refracted into the Lightning Channel. The reflected portion reaches once again the bottom where it is once more reflected towards the top of the structure (where it is split again into reflected and refracted parts).

The part originally refracted into the Lightning Channel continues up the Fully Ionized portion of the channel at the speed of light and at some point catches up with the slower propagating Lightning Channel front. At this point a portion of the originally refracted wave is reflected back towards the CN Tower top while a part of it is transmitted beyond and continues to flow in the Not-Fully ionized portion of the Lightning Channel together with the initially injected current wave and at the same lower speed $v$. Same fate follows all other reflections not only from the bottom of the tower but also those reflected back into the Lightning Channel at the discontinuity between the ionized portion of the Lightning Channel and the tower top.

II.4.2.1.1 Current Distributions for Single-Section Model
II.4.2.1.1.1 Major Components (inside the CN Tower)

$$i_M = \sum_{n=0}^{\infty} \left[ k_n^m k_n^b i_0 \left( t - \frac{h - z}{c} + \frac{2nh}{c} \right) \right]$$

II.4.2.1.1.2 Additional Components: (inside the CN Tower)$^2$

$$i_{A,1} = k_b (1 + k_r) k_c (1 - k_r) i_0 \left( t - \frac{2h\xi + h - z}{c} \right)$$

$$i_{A,2} = k_r^2 (1 + k_r) k_c (1 - k_r) i_0 \left( t - \frac{2h\xi + h + z}{c} \right)$$

$^2$ There are further additional components that are neglected since their amplitudes are less than 1% of the initial injected current wave.
where:

\[ 0 \leq z \leq h \quad \sigma = \frac{v}{c-v} \quad \xi = \frac{c+v}{c-v} \]

II.4.2.1.1.3 Channel Base Current (Initial Current Propagating upwards in the Not-Fully Ionized portion of the Lightning Channel)

\[ i_{cb} = i_0 \left( t - \frac{z - h}{v} \right) e^{\left( \frac{h-z}{v} \right)} \]  

(II.4)

where: \( h \leq z \leq z_{MAX} \)

II.4.2.1.1.4 Internal Components “bouncing” within the Ionized Portion of the Lightning Channel

\[ i_{j,(2m-1)} = \sum_{n=0}^{\infty} k_b^n k_i^{m-1} (1+k_i) k_c^n (-k_i)^n \]

\[ \quad \times j_0 \left( t - \frac{2mh}{c} \xi^n - \frac{(z-h)}{c} \right) \]  

(II.5a)

\[ i_{j,(2m)} = \sum_{n=0}^{\infty} k_b^n k_i^{m-1} (1+k_i) k_c^{n+1} (-k_i)^n \]

\[ \quad \times i_0 \left( t - \frac{2mh}{c-v} \xi^n - \frac{h(1+2m\sigma\xi^n)-z}{c} \right) \]  

(II.5b)

where:

\[ h < z < h(1+2m\sigma\xi^n) \leq z_{MAX} \quad m = 1,2,3,... \]

II.4.2.1.1.5 Components Transmitted into the Not-Fully Ionized Portion of the Lightning Channel
\[ i_{Tm} = \sum_{n=0}^{x} k_{k}^{n} k_{t}^{n-1} (1 + k_{t}) k_{c}^{n} (-k_{c})^{n} (1 + k_{c}) \]
\[ i_{o} \left( t - \frac{2mh}{c - V} \zeta^{n} - z - h(1 + 2m\sigma\zeta^{n}) \right) e^{-h-z \over V} \] (II.6)

where:
\[ h(1 + 2m\sigma\zeta^{n}) < z \leq z_{MAX} \quad m = 1, 2, 3, ... \]

In order to find the current at any specified level of the tower or of the channel, corresponding contributions indicated for that level by the Lattice Diagram are summed up.

**II.4.2.2 Three-Section Model and Five-Section Model**

Following the same principles and considerations applicable to the Single-Section Model, any multi-section model of the CN Tower could be implemented. The author has considered a Three-Section (Fig. AII.1) and a Five-Section Model (Fig. AIII.1) of the CN Tower. Basically, individual sections of the CN Tower constituting the Three-Section or the Five-Section Model, and the Lightning Channel, are treated in the same way as indicated above for the case of the Single-Section Model. Table AII.1 contains the values of the lengths of different sections, their surge impedances, and respective reflection coefficients pertaining to the Three-Section Model and Table AIII.1 the same quantities corresponding to the Five-Section Model.

Full details regarding the Three- and Five-Section Model current contributions are not reproduced here, but the reader may find all of them in Appendices II and III.

It should be mentioned that one of the major salient features of the CN Tower, namely the Skypod, is considered in the Three-Section Model. It has a great influence upon the current distribution along the tower and the Lightning Channel respectively. Representing the structure more closely helps in computing of a closer current waveform to the recorded one. In actual fact, the newly introduced two additional locations of mismatched impedances influence the simulated current waveshape at 474m above ground level (one of the heights where recording is implemented) at early times in such a way to produce the otherwise missing peak in the Single-Section Model calculations (please check Figs. AI.5 and AII.33). This peak is normally always observed in the actual records, but it is not reproduced by the Single-Section Model at all.
Furthermore, the inclusion of even more structural details, such as the Space deck in the Five-Section Model, in particular makes the model more realistically representing the actual tower. This in turn helps in reproducing more accurate current waveform at the respective height of interest. The most sophisticated model (Five-Section Model) used so far yields the best results possible using the described modeling approach. The fine details that could be reproduced are shown in Fig. AIII.83 in Appendix III.

A partial Lattice Diagram of the Five-Section Model is shown in Fig AIII.2, which is restricted to $2.5\mu s$. It is seen that the complexity of such a diagram is enormous even for such a short timeframe. This was one of the reasons why it was decided to follow and use reflections only until their amplitudes are higher than 1% of the original injected current.

Caution should be exercised when certain contributions to the current within the Lightning Channel are described, since the point of reflection between the Not-Fully Ionized and Fully Ionized portions of the Lightning Channel is not static. The propagation speed of the Return Stroke channel front $v$ is affecting times and locations of reflection points and thus the current distributions in the Lightning Channel and consequently currents in the CN Tower as well.

The values of the impedances used in the Three- and Five-Section Model are chosen considering previous publications [87], [89] and based on recorded currents. They are fine tuned in order to achieve as close as possible predicted current waveforms at $474m$ or at $509m$ above ground level. The process of estimation of the impedances of different sections involved computation of the ratios between distinctive peaks found in the recorded current derivatives. These ratios are corresponding to the reflection coefficients at the borders of mismatched impedances, and thus are used to approximate the surge impedance of distinctive sections comprising the multi-section representation of the CN Tower. Theoretical expressions currently exist for transmission line towers that are considered as a single transmission line section only. Developing of relations for a multi-section representation of a transmission line tower or any high-rise structure could be a separate area of research.

The reader may have also noticed that the reflection coefficients at ground level as well as at the top of the CN Tower are slightly different in the three considered models. It should be kept in mind that these models are only approximations. However the reflection coefficient values used in the Five-Section Model arguably may be considered to be the most accurate.
Chapter III
Development of Expressions for Currents Pertinent to Variable Speed of Propagation of the Lightning Channel Front

III.1 Outline

Chapter III contains the second original contribution, which is modification of the proposed relations from Chapter II to include the possibility for varying the Lightning Channel front speed. Variable Lightning Channel front speed is treated and discussed for the first time and all developed expressions for the considered increasing and decreasing cases using the Single-Section Model are shown. The first and second “legs” of major reflections are discussed thoroughly, and then based on the produced relations further ones corresponding to subsequent “legs” are written down. Moreover, in order to somewhat reduce the computation burden, a graphical solution approach is proposed. The graphical solution approach is then extended to account for the more sophisticated representation of the CN Tower by the Three- and Five-Section Models.

In the end of the chapter several special cases involving variable and constant Lightning Channel speed are simulated and produced results showing currents at particular levels analyzed.

III.2 Variable Lightning Return Stroke Speed at Different Height

In the previous Chapter II, three sophisticated models for describing Lightning events at the CN Tower were introduced. In all of them, the speed of propagation of the Return Stroke current (or Lightning Channel front) into the Not-Fully Ionized portion of the Lightning Channel was assumed to be constant and less than the speed of light. It would be interesting to look into the possibility of using a variable instead of constant speed of propagation, which in fact would be more realistic and closer to the actually observed phenomenon.

It is commonly perceived that the speed of propagation of the Lightning Channel front somewhat decreases with height, which could be approximated, for instance, by a parabolic function. While confirming that fact, Chang et. al. [101] showed that, based on their measurements at the CN
Tower site, the speed of propagation of the Return Stroke channel is actually increasing with height in some cases. This is clearly seen in Fig. III.1 below.

Based on the data recorded and presented in Fig.III.1, one could choose to approximate the increase in speed with height by a linear function or by second, third, or any \( n^{th} \) order function. After coming up with the appropriate representation of the variation of speed with height, it could be incorporated into the corresponding Lattice diagram.

The study presented below considers a general case of variable front propagation speed. While the chosen approach may be utilized for both, the decreasing and increasing front speed, the case of increasing speed is elaborated in detail. It is utilizing the “Engineering” transmission line models from Chapter II and a linear approximation (performed using MatLab) of the increase in speed with height. The propagation speed increases up to a specified value and remains constant after that.

![Graph showing speed variation](image)

Fig. III.1 - Speed Variation of the Progressing Upwards Lightning Channel Front Recorded by ALPS (Adapted from [101], figure 8 – July 14, 04 at 4:33:56h)

Linear increase in speed with height corresponds to a non-linear increase of height with time. Such behavior is observed in Fig. III.2 up to a certain instance of time, after which point, the reached speed is assumed to remain constant and thus further increase of height in time is linear.
The equation for speed as a function of height as a linear approximation could be represented by the following relation:

\[
\frac{dz}{dt} + az = b, \quad (III.1)
\]

where:

\[
\frac{dz}{dt} = v \text{ is the speed of the Return Stroke front;}
\]

\[
z \text{ is the height;}
\]

\[
a, b \text{ - constants that are later evaluated based on Fig. III.1:}
\]

Eq. III.1 is an ODE and is solved below:

\[
e^{at} \frac{dz}{dt} + ae^{at} z = be^{at}
\]

\[
\left( e^{at} z \right)' = be^{at}
\]

\[
\int \left( e^{at} z \right) dt = \int be^{at} dt
\]

\[
e^{at} z + k = \frac{b}{a} e^{at} + k_1
\]

\[
e^{at} z = \frac{b}{a} e^{at} + C
\]

\[
z = \frac{b}{a} + Ce^{-at}
\]

Finally:

\[
z(t) = \frac{b}{a} + Ce^{-at} \quad (III.2)
\]

In particular, if one considered a specific case such as the one presented in Fig. III.1, \( a \) and \( b \) could be found directly from that Fig. III.1:
\[ a = -66000 \]
\[ b = 20000000 \]

The initial condition is: \( t = 0; z = 553 \)

\[ z(t = 0) = 553 = -303.03 + Ce^0 \]
\[ C = 856.03 \]

Finally:

\[ z(t) = -303.03 + 856.03e^{66000t} \]

The last measured speed of propagation of the Lightning Channel is at \( z = 1303m \) above ground level and using the approximation (III.1) one could calculate speed \( v = 105998000m/s \).

The corresponding time turns out to be \( t = 9.53\mu s \).

The Lattice diagram showing the first three major reflections coming from ground level and trapped in the Lightning Channel may be found below in Fig. III.2. In the Diagram critical instances of time are indicated, at which either the Lightning Channel front or the top of the CN Tower are reached. These times are measured geometrically directly from the diagram.

![Partial Lattice Diagram for Increase in Lightning Channel Speed](image)

Fig. III.2 - Partial Lattice Diagram for Increase in Lightning Channel Speed
For the Lightning Channel front propagation, relation (III.2) is used up to \( t = 9.53 \mu s \), and after that time a linear relation (III.3) is utilized, which is developed as follows.

Linear approximation for constant speed \( v = 105998000 m/s \):

\[
1303 = 105998000t + C
\]
\[
t = 9.53 \mu s \Rightarrow C = 292.8391
\]

The linear approximation then becomes:

\[
z = 105998000t + 292.8391 \tag{III.3}
\]

The approach described for the increasing front speed and shown in Fig. III.2 may be used for any speed variation, increasing, decreasing or even one described by a curve provided as a table of heights versus times. In what will follow the case of the decreasing front speed variation will be considered. A parabolic change of channel height \( z \) with time \( t \) will be adopted.

### III.3 Cases Involving Parabolic Decrease in Lightning Channel Propagation Speed

In order to pursue the case of a front speed decreasing with time, a parabolic relation is used to approximate the decrease in speed of the Return Stroke with increasing height up to the previously discussed time \( t = 9.53 \mu s \) where the speed is becoming constant and consequently described by a linear function (shown later).

\[
t = d(z + h)^2 - q \tag{III.4}
\]

### III.3.1 Relations Pertaining to First Major Reflection from Ground

Derivations of heights and times associated with first major reflection from bottom of CN Tower that penetrates into the Fully-Ionized portion of Lightning Channel are discussed below. The corresponding detailed Lattice diagram is depicted in Fig. III.3.

The geometrical location of the initial point (where the “uncontaminated” current is injected) is once again:
\[ z = h, t = 0 \]  

(III.5)

If the initial conditions (III.5) are substituted into (III.4) it follows that:

\[ 0 = d(4h^2) - q \]  

(III.6)

In order to produce a more realistic case, let us assume that \( d = 4e^{-12} \)

Then based on the chosen value of \( d \), using (III.6) one can compute the corresponding value \( q = 4.8929e^{-6} \).

Let us now examine the resulting plot shown in Fig. III.3. Since it is assumed that propagation in the tower and in the Fully Ionized portion of the Lightning Channel occurs at the speed of light and since at \( z = z_1, t = t_1 \), one could easily find out that at

\[ z_1 = h \]  

(III.7)
\[ t_1 = \frac{2h}{c} \]  \hspace{1cm} (III.8)

Since the equation describing the first reflection from ground is

\[ z = ct - h \]  \hspace{1cm} (III.9)

its intersection with the parabola of equation (III.4) (trajectory of the Return Stroke front) occurs at

\[ z_2 = \frac{(1 - 2cdh) + \sqrt{1 + 4c^2d^2q}}{2cd} \]  \hspace{1cm} (III.10)

where:

\[ t_2 = \frac{1 + \sqrt{1 + 4c^2d^2q}}{2c^2d} \]  \hspace{1cm} (III.11)

Please, note that at this point \((z_2, t_2)\) the current contribution \(i_{i,1}\) reflected at the bottom of the tower and transmitted into the Fully Ionized portion of the Lightning Channel has caught up with the Lightning Channel front and a Transmitted component \(i_{T,1}\) is produced that continues upwards the channel together with the initial injected current \(i_{ch}\) at the same slower speed. There is a corresponding reflection \(i_{i,2}\) as well, and it starts its way down the Fully Ionized portion of the Lightning Channel at the speed of light until it reaches the top of the tower at time \(t = t_3\) and the corresponding height is once again \(z_3 = h\). At this point another set of two current waves is also produced. The transmitted one starts its way inside and down the tower \(i_{T,3}\) and the reflected one \(i_{i,3}\) goes back into the Fully Ionized portion of the Lightning Channel heading upwards toward the Lightning Channel front. It is found out that:

\[ z_3 = h \]  \hspace{1cm} (III.12)

\[ t_3 = \frac{2z_2}{c} \]  \hspace{1cm} (III.13)
The next point of interest is at time $t = t_4$ when the internal component $i_{i,3}$ catches up with $i_{ch}$ and $i_{r,1}$. At this time another internal component $i_{i,4}$ is produced and starts its way down towards the top of the tower at the speed of light. A transmitted component $i_{r,3}$ is also produced and continues upwards at the slower speed of propagation of the existing components $i_{ch}$ and $i_{r,1}$. Later on the same process is repeated at different times and heights. This could be written down in the following way:

$$z = z_4, t = t_4$$  \hspace{1cm} (III.14)

Solving for $z_4$ and for $t_4$ yields:

$$z_4 = \frac{(1 - 2cdh) + \sqrt{4(1 - 4cdh) + (1 + 4c^2dq)^2 + 4\sqrt{1 + 4c^2dq}}}{2cd}$$  \hspace{1cm} (III.15)

$$t_4 = \frac{3 - 8cdh + 2\sqrt{1 + 4c^2dq} + \sqrt{4(1 - 4cdh) + (1 + 4c^2dq)^2 + 4\sqrt{1 + 4c^2dq}}}{2c^2d}$$  \hspace{1cm} (III.16)

At $z = z_5, t = t_5$

$$z_5 = h$$

Based on the geometrical relations seen in Fig. III.3 one could write down that:

$$t_5 = t_4 + \frac{z_4 - h}{c}$$  \hspace{1cm} (III.17)

$$z = z_6, t = t_6$$  \hspace{1cm} (III.18)

Also, for convenience let

$$E = (1 + 4c^2dq)$$  \hspace{1cm} (III.19)

$$F = (1 - 4cdh)$$  \hspace{1cm} (III.20)

Then it could be shown that:
\[
z_6 = \frac{(1 - 2cdh) + \sqrt{4E + 8F + 4\sqrt{E + 4\sqrt{F + E + 4\sqrt{E}}}}}{2cd} 
\]  
(III.21)

\[
t_6 = \frac{(5 - 16cdh) + 2\sqrt{E + 2\sqrt{4F + E + 4\sqrt{E + \sqrt{F + E + 4\sqrt{E + 4\sqrt{F + E + 4\sqrt{E}}}}}}}}{2c^2d} 
\]  
(III.22)

Now the components that contribute to the description of the current at any level and at any time by the Single-Section Model could be written down.

III.3.1.1 Channel-Base Current (Initial Current Propagating upwards in the Not-Fully Ionized Portion of the Lightning Channel)

\[
i_{ch} = \frac{z}{z_{ch}} i_0 \left( t - \frac{z - h}{v} \right) e^{\frac{h-z}{v}} 
\]  
(III.23)

The speed of propagation of the Lightning Channel front is found to be (derivative \( \frac{dz}{dt} \)):

\[
v = \frac{1}{2\sqrt{d} \sqrt{q + t}} 
\]  
(III.24)

so now:

\[
i_{ch} = \frac{z}{z_{ch}} i_0 \left( t - (z - h)(2\sqrt{d} \sqrt{q + t}) \right) e^{\frac{h-z}{v}} 
\]  
(III.25)

for \( h \leq z \leq z_{MAX} \)

III.3.1.2 Internal Components (Trapped in the Fully-Ionized Portion of the Lightning Channel)

\[
i_{t,1} = k_b (1 + k_l) i_0 \left( t - t_i - \frac{z - h}{c} \right) 
\]  
(III.26)
\[ i_{f,2} = k_h (1 + k_i) k_c t_0 \left( t - t_2 - \frac{z_2 - z}{c} \right) \]  
(III.27)

\[ i_{f,3} = k_h (1 + k_i) k_c (-k_i) t_0 \left( t - t_3 - \frac{z - h}{c} \right) \]  
(III.28)

\[ i_{f,4} = k_h (1 + k_i) k_c^2 (-k_i) t_0 \left( t - t_4 - \frac{z_4 - z}{c} \right) \]  
(III.29)

One can also write down the general relations:

\[ i_{f,(2n-1)} = k_h (1 + k_i) k_c^{n-1} (-k_i)^{n-1} t_0 \left( t - t_{(2n-1)} - \frac{z - h}{c} \right) \]  
(III.30)

\[ i_{f,(2n)} = k_h (1 + k_i) k_c^n (-k_i)^n t_0 \left( t - t_{2n} - \frac{z_{2n} - z}{c} \right) \]  
(III.31)

for \( n = 1,2,3,... \) and \( h \leq z \leq z_{(2n)} \), and \( t_{(2n-1)} \leq t \leq t_{(4n-1)} \)

### III.3.1.3 Additional Components

\[ i_{A,1} = k_h (1 + k_i) k_c (1 - k_i) t_0 \left( t - t_3 - \frac{h - z}{c} \right) \]  
(III.32)

\[ i_{A,2} = k_h^2 (1 + k_i) k_c (1 - k_i) t_0 \left( t - t_3 - \frac{h + z}{c} \right) \]  
(III.33)

\[ i_{A,3} = k_h^2 (1 + k_i) k_c (1 - k_i) k_i t_0 \left( t - t_3 - \frac{2h}{c} - \frac{h - z}{c} \right) \]  
(III.34)

\[ i_{A,4} = k_h^3 (1 + k_i) k_c (1 - k_i) k_i t_0 \left( t - t_3 - \frac{2h}{c} - \frac{h + z}{c} \right) \]  
(III.35)

or in general:

\[ i_{A,(2n-1)} = k_h^n (1 + k_i) k_c (1 - k_i) k_i^{n-1} t_0 \left( t - t_3 - \frac{2(n-1)h}{c} - \frac{h - z}{c} \right), \text{ for } t_{(4n-1)} \leq t \leq t_{(4n-1)} + \frac{h}{c} \]  
(III.36)
\[ i_{A(2n)} = k_n^{n+1}(1+k_i)k_c(1-k_i)k_{n-1}^n i_0 \left( t - t_3 - \frac{2(n-1)h}{c} - \frac{h+z}{c} \right) \], for \( t_{(4n-1)^+} < t \leq t_{(4n-1)} + \frac{2h}{c} \) (III.37)

for \( n = 1,2,3,... \) and \( 0 \leq z \leq h \)

### III.3.1.4 Transmitted Components

\[ i_{T,1} = k_b(1+k_i)(1+k_c)\xi_0(t-t_2) e^{\left( \frac{h-z}{c} \right)} \] (III.38)

\[ i_{T,2} = k_b(1+k_i)k_c(-k_i)(1+k_c)\xi_0(t-t_4) e^{\left( \frac{h-z}{c} \right)} \] (III.39)

\[ i_{T,3} = k_b(1+k_i)k_c^2(-k_i)^2(1+k_c)\xi_0(t-t_6) e^{\left( \frac{h-z}{c} \right)} \] (III.40)

or in general:

\[ i_{T,(n)} = k_b(1+k_i)k_c^{n-1}(-k_i)^{n-1}(1+k_c)\xi_0(t-t_{(2n)}) e^{\left( \frac{h-z}{c} \right)} \] (III.41)

for \( n = 1,2,3,... \) and \( z_{(2n)} \leq z \leq z_{\text{MAX}} \), and \( t_{(2n)} \leq t \)

### III.3.2 Relations Pertaining to Second Major Reflection from Ground

Let us now look into the derivations of heights and times for components in the Fully Ionized portion of the Lightning Channel due to the second major reflection from bottom of the CN Tower as shown in Fig. III.4. The same nomenclature regarding heights and times is used, but the values are different.

The same parabolic relation (III.4) is considered in this case:

\[ t = d(z+h)^2 - q \] (III.42)

The initial point is (similarly to III.5):
Substituting the initial conditions yields the same resultant relation as in (III.6):

$$0 = d(4h^2) - q$$  \hspace{1cm} (III.44)

Again let $d = 4e^{-12} \Rightarrow q = 4.8929e^{-6}$

The same considerations as previously described when analyzing Fig. III.3 apply in this case. The produced relations are written down:

$$z = z_1, t = t_1$$  \hspace{1cm} (III.45)
\[ z_1 = h \]  
\[ t_1 = \frac{4h}{c} \]

\[ z = z_2, t = t_2 \]

\[ z_2 = \frac{(1 - 2cdh) + \sqrt{8cdh + 4c^2 dq + 1}}{2cd} \]  
\[ t_2 = \frac{(1 + 4cdh) + \sqrt{8cdh + 4c^2 dq + 1}}{2c^2 d} \]

\[ z = z_3, t = t_3 \]

\[ z_3 = h \]

\[ t_3 = t_1 + \frac{z_2 - h}{c} \]

\[ z = z_4, t = t_4 \]

\[ z_4 = \frac{(1 - 2cdh) + \sqrt{4(1 - 2cdh) + (1 + 4c^2 dq) + 4\sqrt{8cdh + (4c^2 dq + 1)}}}{2cd} \]

let:

\[ E = 1 + 4c^2 dq \]  
\[ F = 1 - 4cdh \]  
\[ I = 1 - 2cdh \]  
\[ J = 1 + 4c^2 dq \]  
\[ G = 8cdh + J \]  

(III.46)  
(III.47)  

(III.48)  
(III.49)  

(III.50)  

(III.51)  

(III.52)  

(III.53)  

(III.54)  

(III.55)  

(III.56)  

(III.57)  

(III.58)  

(III.59)
\[ H = \sqrt{4I + J + 4\sqrt{G}} \quad (III.60) \]

so now:

\[ z_4 = \frac{I + \sqrt{4I + J + 4\sqrt{8cdh + J}}}{2cd} \quad (III.61) \]

\[ t_4 = \frac{2\left(1 - cdh + \sqrt{8cdh + J}\right) + I + \sqrt{4I + J + 4\sqrt{8cdh + J}}}{2c^2d} \quad (III.62) \]

\[ z = z_5, t = t_5 \quad (III.63) \]

\[ z_5 = h \quad (III.64) \]

\[ t_5 = t_4 + \frac{z_4 - h}{c} \quad (III.65) \]

\[ z = z_6, t = t_6 \quad (III.66) \]

\[ z_6 = \frac{I + \sqrt{4F + E + 4I + 4\sqrt{8cdh + J} + 4 \sqrt{4I + J + 4\sqrt{8cdh + J}}}}{2cd} \quad (III.67) \]

\[ t_6 = \frac{3I + 2 - 6cdh + 2\sqrt{G} + 2H + \sqrt{4F + E + 4I + 4\sqrt{G} + 4H}}{2c^2d} \quad (III.68) \]

Now the considered current contributions are written down as follows.

**III.3.2.1 Channel-Base Current (Initial Current Propagating upwards in the Not-Fully Ionized Portion of the Lightning Channel)**

\[ i_{ch} = \frac{z}{z_{ich}} i_0 \left( t - \frac{z - h}{v} \right) e^{\left( \frac{h - z}{\nu} \right)} \quad (III.69) \]
The speed of propagation of the Lightning Channel front is found to be (derivative $\frac{dz}{dt}$):

$$v = \frac{1}{2\sqrt{d} \sqrt{q + t}}$$  \hspace{1cm} (III.70)

so now:

$$i_{ch} = \frac{z}{z_{ch}} i_0 \left( t - (z - h) \left( 2\sqrt{d} \sqrt{q + t} \right) \right)^{\frac{z-h}{v}}$$  \hspace{1cm} (III.71)

for $h \leq z \leq z_{MAX}$

III.3.2.2 Internal Components (Trapped in the Fully-Ionized Portion of the Lightning Channel)

$$i_{I,1} = k_h^2 k_i (1 + k_i) i_0 \left( t - t_1 - \frac{z-h}{c} \right)$$  \hspace{1cm} (III.72)

$$i_{I,2} = k_h^2 k_i (1 + k_i) k_c i_0 \left( t - t_2 - \frac{z_2 - z}{c} \right)$$  \hspace{1cm} (III.73)

$$i_{I,3} = k_h^2 k_i (1 + k_i) k_c (-k_i) i_0 \left( t - t_3 - \frac{z-h}{c} \right)$$  \hspace{1cm} (III.74)

$$i_{I,4} = k_h^2 k_i (1 + k_i) k_c^2 (-k_i) i_0 \left( t - t_4 - \frac{z_4 - z}{c} \right)$$  \hspace{1cm} (III.75)

Or in general:

$$i_{I,(2n-1)} = k_h^2 k_i (1 + k_i) k_c^{n-1} (-k_i)^{n-1} i_0 \left( t - t_{(2n-1)} - \frac{z-h}{c} \right)$$  \hspace{1cm} (III.76)

$$i_{I,(2n)} = k_h^2 k_i (1 + k_i) k_c^n (-k_i)^{n-1} i_0 \left( t - t_{2n} - \frac{z_{2n} - z}{c} \right)$$  \hspace{1cm} (III.77)
for \( n = 1, 2, 3, \ldots \) and \( h \leq z \leq z_{(2n)} \), and \( t_{(2n-1)} \leq t \leq t_{(4n-1)} \)

### III.3.2.3 Additional Components

\[
i_{A,1} = k_b^2 k_t (1 + k_t) k_c (1 - k_t) j_0 \left( t - t_3 - \frac{h - z}{c} \right)
\]

(III.78)

\[
i_{A,2} = k_b^2 k_t (1 + k_t) k_c (1 - k_t) j_0 \left( t - t_3 - \frac{h + z}{c} \right)
\]

(III.79)

\[
i_{A,3} = k_b^3 k_t^2 (1 + k_t) k_c (1 - k_t) j_0 \left( t - t_3 - \frac{2h}{c} - \frac{h - z}{c} \right)
\]

(III.80)

\[
i_{A,4} = k_b^3 k_t^2 (1 + k_t) k_c (1 - k_t) j_0 \left( t - t_3 - \frac{2h}{c} - \frac{h + z}{c} \right)
\]

(III.81)

or in general:

\[
i_{A,(2n-1)} = k_b^{n+1} k_t^n (1 + k_t) k_c (1 - k_t) j_0 \left( t - t_3 - \frac{2(n-1)h}{c} - \frac{h - z}{c} \right), \text{ for } t_{(4n-1)} \leq t \leq t_{(4n-1)} + \frac{h}{c}
\]

(III.82)

\[
i_{A,(2n)} = k_b^{n+2} k_t^n (1 + k_t) k_c (1 - k_t) j_0 \left( t - t_3 - \frac{2(n-1)h}{c} - \frac{h + z}{c} \right), \text{ for } t_{(4n-1)} + \frac{h}{c} \leq t \leq t_{(4n-1)} + \frac{2h}{c}
\]

(III.83)

for \( n = 1, 2, 3, \ldots \) and \( 0 \leq z \leq h \)

### III.3.2.4 Transmitted Components

\[
i_{T,1} = k_b^2 k_t (1 + k_t) (1 + k_c) j_0 (t - t_4) e^{\left( \frac{h - z}{\nu} \right)}
\]

(III.84)

\[
i_{T,2} = k_b^2 k_t (1 + k_t) k_c (1 + k_t) j_0 (t - t_4) e^{\left( \frac{h - z}{\nu} \right)}
\]

(III.85)

\[
i_{T,3} = k_b^2 k_t (1 + k_t) k_c^2 (1 + k_c) j_0 (t - t_4) e^{\left( \frac{h - z}{\nu} \right)}
\]

(III.86)
or in general:

\[ i_{T(n)} = k_h^2 k_i (1 + k_i) k_c^{n-1} (- k_i)^{n-1} (1 + k_c) k_0 (t - t_{(2n)}) \left( \frac{h-z}{v} \right) \]  

(III.87)

for \( n = 1, 2, 3, \ldots \) and \( z_{(2n)} \leq z \leq z_{MAX} \), and \( t_{(2n)} \leq t \)

Similar derivations could be produced for any subsequent major reflection inside the CN Tower. One has to pay attention to the fact that at some point in time and height, the relation describing the speed of propagation of the Lightning Channel front is switching from variable to constant. At this time the components comprising the current at any height and at any time along the tower and the Lightning Channel also have to be modified accordingly in order to produce consistent results.

### III.4 Graphical Solution Approach for Cases Involving Variable Lightning Channel Propagation Speed

Derivations of heights and times for components in Fully Ionized portion of the Lightning Channel due to first major reflection from bottom of CN Tower that penetrates into the Fully Ionized portion of Lightning Channel are considered below.

Since the development of relations for \( z \) and \( t \) that need to be included in the current components, comprising the total current at any level along the Lightning Channel, requires considerable effort and attention to the details, an alternate approach to find their numerical values is proposed. The numerical evaluation of the values for \( z \) and \( t \) directly from a plot provides the exact numbers when compared to the analytically computed ones (matching could be shown in the parabolic cases), but is done instantly and without room for errors, provided the drawing is precise. Another advantage is also the fact that any type of curve related to the Lightning Channel front (Not-Fully Ionized portion of the Lightning Channel) propagation could be treated in the same way easily. The speed of propagation of the Lightning Channel front is the derivative \( \frac{dz}{dt} \), but it could be estimated at any point also by using numerical differentiation with high accuracy.
Proposed method:

- Prepare an exact drawing based on the Lattice diagram pertinent to the particular case (for variable speed of propagation of the Lightning Channel front).

- Find the intersection points (their values) between the curve associated with the Lightning Channel front propagation and the respective line representing a specific reflection leg.

- Use these values in the equations developed previously for different current components.

Let us consider now increase in speed of the Lightning Channel front as shown in [101], Fig. 8.

It is seen in Fig. III.5 that for:

\[
z_1 = h
\]  

(III.88)

\[
t_1 = \frac{2h}{c}
\]  

(III.89)
These values for height and time could be easily computed or taken directly from the plot.

In the proposed methodology, $z_2, t_2$ are taken from the plot.

At height $z$, and the associated time $t$, we have:

$$z_3 = h \quad (\text{III.90})$$
$$t_3 = t_2 + \frac{z_2 - h}{c} \quad (\text{III.91})$$

The values at height $z_4$ and the associated time $t_4$ are retrieved from the plot again.

At height $z$, and the associated time $t$, we have:

$$z_5 = h \quad (\text{III.92})$$
$$t_5 = t_4 + \frac{z_4 - h}{c} \quad (\text{III.93})$$

Once again, the values at height $z_6$ and the associated time $t_6$ are taken directly from the plot.

One could continue in the same manner until needed.

Based on the presented considerations, the different components of the current described by the Single-Section model become:

**III.4.1 Channel-Base Current (Initial Current Propagating upwards in the Not-Fully Ionized Portion of the Lightning Channel)**

$$i_{ch} = \frac{z_{ch}}{z_{arch}} \left( t - \frac{z - h}{v} \right) e^{\frac{h-z}{\nu}} \quad (\text{III.94})$$

for $h \leq z \leq z_{MAX}$
The speed of propagation of the Lightning Channel front is \( \frac{dz}{dt} \) which is found using numerical differentiation.

### III.4.2 Internal Components (Trapped in the Fully-Ionized Portion of the Lightning Channel)

\[
i_{I,(2n-1)} = k_h (1 + k_i) k_c (1 - k_i)^{n-1} i_0 \left( t - t_{(2n-1)} - \frac{z - h}{c} \right)
\]

\[
i_{I,(2n)} = k_h (1 + k_i) k_c^{n-1} i_0 \left( t - t_{2n} - \frac{z_{2n} - z}{c} \right)
\]

for \( n = 1,2,3,\ldots \) and \( h \leq z \leq z_{(2n)}, \) and \( t_{(2n-1)} \leq t \leq t_{(4n-1)} \)

### III.4.3 Additional Components

\[
i_{A,(2n-1)} = k_h^n (1 + k_i) k_c (1 - k_i) k_i^{n-1} i_0 \left( t - t_3 - \frac{2(n-1)h}{c} - \frac{h - z}{c} \right), \text{ for } t_{(4n-1)} \leq t \leq t_{(4n-1)} + \frac{h}{c}
\]

\[
i_{A,(2n)} = k_h^{n+1} (1 + k_i) k_c (1 - k_i) k_i^{n-1} i_0 \left( t - t_3 - \frac{2(n-1)h}{c} - \frac{h + z}{c} \right), \text{ for } t_{(4n-1)} + \frac{h}{c} \leq t \leq t_{(4n-1)} + \frac{2h}{c}
\]

for \( n = 1,2,3,\ldots \) and \( 0 \leq z \leq h \)

### III.4.4 Transmitted Components

\[
i_{T,(n)} = k_h (1 + k_i) k_c (1 - k_i)^{n-1} (1 + k_c) i_0 (t - t_{(2n)}) e^{\left( \frac{h - z}{\nu} \right)}
\]

for \( n = 1,2,3,\ldots \) and \( z_{(2n)} \leq z \leq z_{\text{MAX}}, \) and \( t_{(2n)} \leq t \)

### III.4.5 Extension of the Proposed Graphical Solution Approach to Account for Sophisticated Representation of the CN Tower
The graphical solution approach proposed above could be adapted and used with the Three- and Five Section Models as well. Basically, one should modify accordingly some of the equations and Lattice diagram drawings seen in Appendices II and III, and measure the appropriate times and heights of interest. A study looking into the predicted by the Single-, Three-, and Five Section Models current waves at specific heights of interest is presented further below. In order to come up with the required current components comprising the total current at the respective level, Lattice drawings for Three- and Five-Section Models found in Appendices II and III are utilized. A list of the selected considered diagrams follows.

Three-Section Model:

Current inside the CN Tower – Fig. AII.8, Fig. AII.9, Fig. AII.10, Fig. AII.15, Fig. AII.16, Fig. AII.17, Fig. AII.24, Fig. AII.25, and Fig. AII.26.

Current inside the Lightning Channel - Fig. AII.27, Fig. AII.28, Fig. AII.29, Fig. AII.30, Fig. AII.31, and Fig. AII.32.

Five-Section Model:

Current inside the CN Tower – Fig. AIII.12, Fig. AIII.34, Fig. AIII.42, Fig. AIII.52 and Fig. AIII.64.

Current inside the Lightning Channel - Fig. AIII.75, Fig. AIII.76, Fig. AIII.77, Fig. AIII.78, Fig. AIII.79, Fig. AIII.80, Fig. AIII.81, and Fig. AIII.82.

III.5 Influence of Different Speed of Propagation of the Return Stroke upon the Computed Current Waveforms at Different Levels

In order to appreciate and visualize the influence of the speed of the Lightning Channel front upon the channel progression in height, Fig. III.6 and Fig. III.7 are offered.
An attempt to use somewhat realistic values was implemented. The speed of the Return Stroke channel was set to 1/3 of the speed of light for the case involving constant speed of propagation and it is presented by a thick line in Figs. III.6 and III.7.

![Lightning Channel Front Speed at Various Heights](image)

**Fig. III.6 - Lightning Channel Front Speed at Various Heights**

The considered case with decreasing speed is shown in dashed lines in the same two figures and it is based on Eq. (III.4) up to $t = 9.53\,\mu$s, after which instance of time the speed becomes constant and is set to the attained at this time value. The respective constants used were $v = 7.6858e^7 \, m/s$, $d = 3.1595e^{-12}$, and $q = 3.8649e^{-6}$. Similarly, in the case involving increase in the speed with height, relation described by analogous to Eq. (III.2) up to $t = 9.53\,\mu$s is used, and after that the attained value of speed is adopted as a constant speed ($v = 1.347e^8 \, m/s$). This case is shown in dotted lines in Figs. III.6 and III.7.

It is anticipated that the currents at different levels along the structure and in the Lightning Channel will be influenced to some extend by the occurring at different times otherwise identical in magnitude reflected and transmitted components in the three different cases. This is also valid for the respective radiated Electric and Magnetic Fields at a distance, as will be shown later.
It is clearly seen in Figs. III.6 and III.7 that using decreasing, constant, or increasing speed of propagation of the Return Stroke front affects the height to which the channel front propagates for a certain time. For example, if one used the increasing speed case shown above, one should consider other influences such as what would happen at the upper end of the Lightning Channel (in the cloud) within the timeframe of interest. This is so, because the whole Lightning Channel length was assumed to be 8000 m, but at somewhat higher speed as is the considered case, the Return Stroke is reaching to a higher altitude within the 60 µs simulation time-frame. Consequently, some modifications must take place in the relations describing the current distribution at any height and time.

Fig. III.8 is depicting the currents calculated inside the CN Tower at 509 m above ground level (location of current sensing element) using the three different speeds of progression of the Lightning Channel front utilizing the Single-Section Model. Figs. III.9 and III.10 show the same corresponding waveforms but calculated using the Three- and Five-Section Models respectively. The calculations are limited up to a simulation time-frame of 50 µs in order to avoid dealing with reflections from top of the Lightning Channel for the case involving increasing speed of propagation of the Return Stroke channel in the range up to 9.53 µs. The plots are showing the
first 20µs of the simulation in order to better observe the differences in the waveforms achieved in the three different cases. The injected current is described by a sum of two Heidler functions [100] with input parameters seen in [22].

It could be pointed out that when the Single-Section Model is used, the influence of the different speed of propagation of the Return Stroke channel kicks in after the main reflection from ground level took place and has relatively small effect upon the produced current waveshape at this level. When the Three- and Five-Section Models are considered some influences take place at earlier times, but again it could be said that the overall waveform is not influenced greatly, although there are some observed differences mostly visible in the time-frame between 6-8µs. In the Single-Section Model computation results it is observed that the case involving decreasing speed has slightly more pronounced influence with comparison to the respective case involving increasing in the speed of propagation of the Lightning Channel, which is exactly the opposite of what is observed in the results yielded by the other two models.
Fig. III.9 - Computed Waveforms at 509m Above Ground Level Using the Three-Section Model for Three Different Speeds of Propagation of the Lightning Channel Front

Fig. III.10 - Computed Waveforms at 509m Above Ground Level Using the Five-Section Model for Three Different Speeds of Propagation of the Lightning Channel Front
The relatively minor effects observed inside the CN Tower could be attributed to the fact that there are stronger influences that are imposed by the current initially injected into the tower and also the major reflections taking place inside the structure. The influence of the Lightning Channel is somewhat diminished due to the chosen values of respective surge impedances which translate into particular values of reflection and transmission coefficients. In the actual computations the transmitted components into the tower are much smaller in amplitude than the major components bouncing inside the CN Tower and thus have small effect, however their influence is clearly seen in Figs.III.8-10.

The current inside the Lightning Channel, computed using the Single-Section Model, is shown at two different levels in Figs. III.11 and III.12 below.

![Fig. III.11 - Computed Waveforms at 900m Above Ground Level Using the Single-Section Model for Three Different Speeds of Propagation of the Lightning Channel Front](image)

The same results calculated by the Three- and Five-Section Models are depicted in Figs. III.13-III.14 and in Figs.III.15-III.16 respectively.
In contrast to the considered cases involving currents inside the CN Tower, the influence of the chosen speed of propagation has considerable effect upon the computed waveforms at the two representative levels where the current is computed inside the Lightning Channel. That influence is not only noticeable in the different times of arrival of the current wave to the respective levels, but also in the reached peak values of the currents.

Those two heights (900\(m\) and 1100\(m\) above ground level) were deliberately chosen so that the attained speed for the decreasing case for example is higher as compared to the constant and to the increasing cases at 900\(m\) above ground level, and lower respectively as compared to the constant and to the increasing cases at 1100\(m\) above ground level.

If one takes a closer look at Fig.III.11, Fig.III.13, or Fig.III.15 (900\(m\) above ground level) one will notice that the computed waveform for the decreasing speed occurs first, then the waveform corresponding to the constant speed case appears and the last one to show up is the increasing speed case waveform. If one checked out Fig. III.6, one would notice the reason for that, namely the speeds are chosen in such a way that the associated with the decreasing case speed is the
highest, the second highest is the constant speed and the slowest is the increasing case speed at this level.

Fig. III.13 - Computed Waveforms at 900m Above Ground Level Using the Three-Section Model for Three Different Speeds of Propagation of the Lightning Channel Front

Fig. III.14 - Computed Waveforms at 1100m Above Ground Level Using the Three-Section Model for Three Different Speeds of Propagation of the Lightning Channel Front
The maximum peak amplitudes seen in Fig. III.11 have different values. The highest value is achieved when the increasing case speed is considered, the lowest value is achieved when the decreasing speed case is considered, and between them the constant speed case value is found. In
contrast to that, as seen in Fig. III.12, at 1100m above ground level, the lowest and highest peak values switched places and belong to the increasing and the decreasing speed cases respectively, while the constant speed case is again in between them.

The corresponding peaks in the waveforms computed by multi-section models of the CN Tower show different behavior. The major difference from the waveforms computed by the Single-Section Model at 900m above ground level is observed in the waveform computed for the case involving increasing speed of the Lightning Channel front, which shows a noticeably lower peak value than the other two considered speed cases. An altered difference in the peaks is observed at 1100m above ground level. The Single-Section Model is predicting a peak produced using the decreasing speed case, which is higher than the constant and the increasing speed cases values. The Three- and Five Section Models, however, predict variable speed (decreasing and increasing cases) peak values that are not only lower with respect to the constant speed case peak value, but also very close to each other in amplitude. Finally, the waveform computed by the Three-Section Model at 1100m above ground level involving the decreasing speed of propagation features a slightly lower peak than the corresponding one involving increasing speed of propagation, while the respective waveforms computed by the Five-Section Model predict exactly the opposite trend.

Based on these observations and on the fact that a multi-section model is representing in better detail the structure of the CN Tower, one could draw a conclusion that the current waves computed at different levels along the Lightning Channel by the Three- or by the Five-Section Model would yield better, closer to the actual waveshapes results, than those predicted by the Single-Section Model. This would also translate into differences in the calculated radiation fields at a distance. To achieve the highest degree of accuracy, the most sophisticated Five-Section Model should be used.

Furthermore it should also be mentioned that based on the Lattice diagrams presented in this Chapter III, (e.g. that seen in Fig. III.2) and the modeling approach already described previously in Chapter II, if one wanted to consider variable speed of propagation of the Lightning Channel front, one should modify the relations describing the current distribution in time and along the Lightning Channel appropriately. The process would include assigning appropriate times of occurrence of different reflections at corresponding levels of interest. Those times may differ
from the analogous case involving constant speed considerably. When such modified current expressions are plugged into the Single-, Three-, or Five-Section Model, different waveforms would be computed at respective levels with comparison to the originally achieved ones using constant speed of propagation of the Lightning Channel front (compare Figs. III.8-III.16).
Chapter IV
Development of Expressions for Fields at a Distance

IV.1 Outline

Chapter IV consists of two essential parts. First, known theoretical expressions are shown. Second, newly proposed derivations corresponding to radiated Electric and Magnetic Field components are introduced. The chapter begins with a review of the fundamental laws of Electromagnetic Theory. After that several derivations and considerations that are useful in coming up with formulae pertinent to radiation fields and retarded potentials are presented. At this point the third original contribution of the thesis is considered, development of time-domain expressions for Electric in Magnetic fields in Cartesian Coordinates. Developed expressions for the $x$, $y$, $z$ components of the Electric and Magnetic Fields are written down, and the derivation details are found in Appendix IV. Since, these expressions look like a very powerful tool to study the impact of radiation fields due to Lightning flashes with an arbitrary shape of the Lightning Channel upon electrical and electronic equipment, the idea for prospective studies is discussed at the end of the chapter.

IV.2 Fundamental Laws – General Description

For convenience, a general overview including Maxwell’s equations is given below. More information could be found in [102-108].

IV.2.1 Gauss' Law

This law describes how electric charge can create and alter Electric Fields. In particular, Electric Fields tend to point away from positive charges, and towards negative charges. Gauss' law is the primary explanation of why opposite charges attract, and charges of the same polarity repel. The charges create certain Electric Fields, which other charges can then respond to by electric force.

Differential form:
Free Charge and Current

\[ \nabla \cdot D = \rho_f \]

Total Charge and Current

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]

Integral form:

Free Charge and Current

\[ \oint_S D \cdot dA = Q_{f,S} \]

Total Charge and Current

\[ \oint_S E \cdot dA = \frac{Q_S}{\varepsilon_0} \]

where:

\( \nabla \cdot \) The divergence operator, [/m]

\( D \) Electric Displacement Field, [C/m\(^2\)]

\( E \) Electric Field [V/m]

\( \rho_f \) Free Charge Density, [C/m\(^3\)]

\( \rho \) Total Charge Density (including both free and bound charge), [C/m\(^3\)]

\( dA \) Differential vector element of surface area A, with infinitesimally small magnitude and direction normal to surface S, [m\(^2\)]

\( Q_{f,S} \) Net unbalanced free Electric Charge enclosed by the Gaussian surface S, [C]

\[ \oint_S E \cdot dA \] The flux of the Electric Field through any closed Gaussian surface S, [Jm/C]

\( Q_S \) Net unbalanced Electric Charge enclosed by the Gaussian surface S (including both free and bound charge), [C]

\( \varepsilon_0 \) Permittivity of free space (universal constant), [F/m]
IV.2.2 Gauss' Law for Magnetism

The law states that magnetism is unlike electricity in that there are no distinct "north pole" and "south pole" particles (such particles, which exist in theory only, would be called magnetic monopoles) that attract and repel the way positive and negative charges do. Instead, north poles and south poles come in pairs (magnetic dipoles). In particular, unlike the Electric Field, which tends to point away from positive charges and towards negative charges, Magnetic Field lines always occur as closed loops (e.g. pointing away from the north pole outside of a bar magnet but towards it inside the magnet).

Differential form:

\[ \nabla \cdot \mathbf{B} = 0 \]

Integral form:

\[ \oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \]

where:

\[ B \quad \text{Magnetic Field, [Wb/m}^2\text{]} \]

\[ \oint_S \mathbf{B} \cdot d\mathbf{A} \quad \text{The flux of the Magnetic Field through any closed surface } S, \text{ [Wb]} \]

IV.2.3 Faraday's Law of Induction

It describes how a changing Magnetic Field can create an Electric Field. This is, for example, the operating principle of many electric generators: Mechanical force (e.g. the force of water falling through a hydroelectric dam) spins a huge magnet, and the changing Magnetic Field creates an Electric Field which supplies consumers with electricity through the power grid.
Differential form:

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

Total Charge and Current

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

Integral form:

Free Charge and Current

\[ \oint_{\partial S} E \cdot dl = -\frac{\partial \Phi_{B,S}}{\partial t} \]

Total Charge and Current

\[ \oint_{\partial S} E \cdot dl = -\frac{\partial \Phi_{B,S}}{\partial t} \]

where:

\[ \oint_{\partial S} E \cdot dl \] Line integral of the Electric Field along the boundary \( \partial S \) (therefore necessarily a closed curve) of the surface S, [J/C]

\[ \Phi_{B,S} = \int_{S} B \cdot dA \] Magnetic Flux through any surface S (not necessarily closed), [Wb]

dl Differential vector element of path length tangential to the path/curve, [m]

\[ \frac{\partial}{\partial t} \] Partial derivative with respect to time, [/s]

IV.2.4 Ampère's Law with Maxwell's Correction

This law states that Magnetic Fields can be generated in two ways: By electrical current (this was the original "Ampère's law") and by changing Electric Fields (this was Maxwell's correction, also called the displacement current term).

Differential form:

Free Charge and Current

Total Charge and Current
\[ \nabla \times H = J_f + \frac{\partial D}{\partial t} \quad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]

Integral form:

**Free Charge and Current**

\[ \oint H.dl = I_{f,S} + \frac{\partial \Phi_{D,S}}{\partial t} \]

**Total Charge and Current**

\[ \oint B.dl = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t} \]

where:

\( \nabla \times \) The curl operator, [T/m]

\( H \) Magnetic Field, [A/m]

\( \mu_0 \) Permeability of free space (universal constant), [H/m]

\( J_f \) Free current density (not including bound current), [A/m²]

\( J \) Total current density (including both free and bound current), [A/m²]

\[ \oint B.dl \] Line integral of the Magnetic Field over the closed boundary \( \partial S \) of the surface \( S \), [Tm]

\( I_{f,S} = \int_{S} J_f.dA \) Net free electrical current passing through the surface \( S \) (not including bound current), [A]

\( I_S = \int_{S} J.dA \) Net electrical current passing through the surface \( S \) (including both free and bound current), [A]

\( \Phi_{E,S} = \int_{S} E.dA \) Electric flux through any surface \( S \) (not necessarily closed), [Jm/C]
\( \Phi_{D,S} = \int_S D \cdot dA \) Flux of Electric displacement Field through any surface S (not necessarily closed), [C]

### IV.3 Maxwell’s Work

Maxwell's correction to Ampère's law was very important. With the included correction, the laws state that a changing Electric Field could produce a Magnetic Field, and vice-versa. It follows that it is possible to have stable, self-perpetuating waves of oscillating Electric and Magnetic Fields, with each field driving the other. (These waves are called electromagnetic radiation.) The four Maxwell's equations describe these waves quantitatively, and moreover predict that the waves should have a particular, universal speed, which can be simply calculated in terms of two easily-measurable physical constants (called the electric constant and magnetic constant).

Maxwell's equations are generally applied to macroscopic averages of the fields, which deviate considerably on a microscopic scale in the vicinity of individual atoms. The quantities permittivity and permeability of a material are normally also defined in a macroscopic sense.

The eight original Maxwell's equations can be written in modern vector notation as follows:

<table>
<thead>
<tr>
<th>No</th>
<th>Law</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>The Law of Total Currents</td>
<td>( J_{\text{tot}} = J + \frac{\partial D}{\partial t} )</td>
</tr>
<tr>
<td>2.</td>
<td>The Equation of Magnetic Force</td>
<td>( \mu H = \nabla \times A )</td>
</tr>
<tr>
<td>3.</td>
<td>Ampere’s Circuital Law</td>
<td>( \nabla \times H = J_{\text{tot}} )</td>
</tr>
<tr>
<td>4.</td>
<td>Electromotive Force created by Convection, Induction, and by Static Electricity.</td>
<td>( E = \mu \nu \times H - \frac{\partial A}{\partial t} - \nabla \Phi )</td>
</tr>
<tr>
<td></td>
<td>Equation Name</td>
<td>Equation Formula</td>
</tr>
<tr>
<td>---</td>
<td>-------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>5.</td>
<td>The Electric Elasticity Equation</td>
<td>$E = \frac{1}{\varepsilon} D$</td>
</tr>
<tr>
<td>6.</td>
<td>Ohm’s Law</td>
<td>$E = \frac{1}{\sigma} J$</td>
</tr>
<tr>
<td>7.</td>
<td>Gauss’ Law</td>
<td>$\nabla \cdot D = \rho$</td>
</tr>
<tr>
<td>8.</td>
<td>Equation of Continuity</td>
<td>$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$</td>
</tr>
</tbody>
</table>

where:

- $H$ Magnetizing Field (called by Maxwell the "Magnetic Intensity")
- $J$ Electric Current Density (with $J_{tot}$ being the total current including displacement current)
- $D$ Displacement Field (called by Maxwell the "Electric Displacement")
- $\rho$ Free Charge Density (called by Maxwell the "Quantity of Free Electricity")
- $A$ Magnetic Vector Potential (called by Maxwell the "Angular Impulse")
- $E$ "Electromotive Force" (as called by Maxwell) corresponding to Electric Field
- $\Phi$ Electric Potential (called by Maxwell also the "Electric Potential")
- $\sigma$ Electrical Conductivity (Maxwell called the inverse of Conductivity the "Specific Resistance")

When Maxwell derives the electromagnetic wave equation in 1865, he uses equation 4 to cater for electromagnetic induction rather than Faraday's law of induction, which is used in modern textbooks. (Faraday's law itself does not appear among his equations.) However, Maxwell drops the $\mu \nu \times H$ term from equation 4 when he is deriving the electromagnetic wave equation, as he considers the situation only from the rest frame.
IV.4 Constitutive Relations

In order to apply Maxwell's equations (the formulation in terms of Free Charge and Current, and $D$ and $H$), it is necessary to specify the relations between $D$ and $E$, and $H$ and $B$. These are called Constitutive Relations, and correspond physically to specifying the response of bound charge and current to the field, or equivalently, how much polarization and magnetization a material acquires in the presence of electromagnetic fields.

IV.4.1 Case without Magnetic or Dielectric Materials

In the absence of magnetic or dielectric materials, the relations are simple:

$$D = \varepsilon_0 E$$
$$H = B / \mu_0$$  \hspace{1cm} (IV.1)

where $\varepsilon_0$ and $\mu_0$ are the two universal constants, called the permittivity of free space and permeability of free space, respectively.

IV.4.2 Case of Linear Materials

In a "linear", isotropic, non-dispersive, uniform material, the relations become:

$$D = \varepsilon E$$
$$H = B / \mu$$  \hspace{1cm} (IV.2)

where $\varepsilon$ and $\mu$ are constants (which depend on the material), called the permittivity and permeability, respectively, of the material.

IV.4.3 General Case

For real-world materials, the constitutive relations can usually still be written:

$$D = \varepsilon E$$
$$H = B / \mu$$
but $\varepsilon$ and $\mu$ are not, usually, simple constants, but functions. For example, they can depend upon:

- The strength of the fields;
- The direction of the fields;
- The frequency with which the fields vary;

If further there are dependencies on:

- The position inside the material;
- The history of the fields;

then the constitutive relations take a more complicated form [109]:

\[
D(r,t) = \varepsilon_0 E(r,t) + P(r,t), \\
H(r,t) = \frac{1}{\mu_0} B(r,t) - M(r,t), \\
P(r,t) = \varepsilon_0 \int d^3r' dt' \tilde{\chi}_{\text{elec}}(r,r',t,t';E)E(r',t'), \\
M(r,t) = \frac{1}{\mu_0} \int d^3r' dt' \tilde{\chi}_{\text{magn}}(r,r',t,t';B)B(r',t')
\]

in which the permittivity and permeability functions are replaced by integrals over the more general electric and magnetic susceptibilities.

**IV.5 Maxwell's Equations in Terms of E and B for Linear Materials**

Using the constitutive relations from above, Maxwell's equations in a linear material (differential form only) become:
\[ \nabla E = \frac{\rho_f}{\varepsilon} \]
\[ \nabla B = 0 \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times B = \mu_0 I_f + \mu_\varepsilon \frac{\partial E}{\partial t} \] (IV.4)

These are formally identical to the general formulation in terms of \( E \) and \( B \) (given above when formulation in terms of total charge and current is considered), except that the permittivity of free space was replaced with the permittivity of the material, the permeability of free space was replaced with the permeability of the material, and only free charges and currents are included.

**IV.5.1 Formulation in Vacuum**

Starting with the equations appropriate in the case without dielectric or magnetic materials, and assuming that there is no current or electric charge present in the vacuum, the Maxwell equations in free space become:

\[ \nabla E = 0 \]
\[ \nabla B = 0 \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \] (IV.5)

These equations have a solution in terms of traveling sinusoidal plane waves, with the Electric and Magnetic Field directions orthogonal to one another and the direction of travel, and with the two fields in phase, traveling at the speed:

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \] (IV.6)

In fact, Maxwell's equations explain how these waves can physically propagate through space. The changing Magnetic Field creates a changing Electric Field through Faraday's law. That Electric Field, in turn, creates a changing Magnetic Field through Maxwell's correction to
Ampère's law. This perpetual cycle allows these waves, known as electromagnetic radiation, to move through space, always at velocity \( c \).

Maxwell discovered that this quantity \( c \) equals the speed of light in vacuum, and concluded that light is a form of electromagnetic radiation.

**IV.6 Potentials**

Maxwell's equations can be written in an alternative form, involving the Electric Potential (also called Scalar Potential) and Magnetic Potential (also called Vector Potential), as follows [110].

First, Gauss' law for magnetism states:

\[
\nabla \cdot B = 0 \tag{IV.7}
\]

By Helmholtz's theorem, \( B \) can be written in terms of a vector field \( A \), called the Magnetic Potential:

\[
B = \nabla \times A \tag{IV.8}
\]

Plugging Eq. IV.8 this into Faraday's law, we get:

\[
\nabla \times \left( E + \frac{\partial A}{\partial t} \right) = 0 \tag{IV.9}
\]

By Helmholtz's theorem, the quantity in parentheses can be written in terms of a scalar function \( \Phi \), called the Electric Potential:

\[
E + \frac{\partial A}{\partial t} = -\nabla \Phi \tag{IV.10}
\]

Combining Eqs. IV.8 and IV.9 with the remaining two Maxwell's equations yields the four relations:
\[ E = -\nabla \Phi - \frac{\partial A}{\partial t} \]
\[ B = \nabla \times A \]
\[ \nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot A) = -\frac{\rho}{\varepsilon_0} \]
\[ \left( \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \right) - \nabla \left( \nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 J \]

(VI.11)

These equations, taken together, are as powerful and complete as Maxwell's equations. In addition, the problem has been reduced somewhat, as the Electric and Magnetic Fields each have three components which need to be solved for (six components altogether), while the Electric and Magnetic Potentials have only four components altogether. On the other hand, these equations appear more complicated than Maxwell's equations using just the Electric and Magnetic Fields.

In fact, Eqs. VI.11 can be simplified by taking advantage of the fact that there are many different choices of \( A \) and \( \Phi \) consistent with a given \( E \) and \( B \).

IV.7 Radiation Fields

IV.7.1 Currents and Charges as Sources of Fields

Usually, the current distribution is localized in some region of space (e.g. on a wire antenna.) Electric and Magnetic Fields are generated by such a current source and these fields can propagate to far distances from the source location.

Normally it is more convenient to work with the Electric and Magnetic Potentials rather than with the Electric and Magnetic Fields. Basically, using two of Maxwell’s equations these potentials are introduced, then, making use of the other two, written in terms of these potentials, the wave-equations are found.

The potentials \( A \) and \( \Phi \) are not uniquely defined. Adding constants to them may change them. Even more freedom is possible, known as gauge invariance of Maxwell’s equations. In actual fact, for any scalar function \( f(r,t) \), the following gauge transformation leaves \( E \) and \( B \) invariant:
When radiation problems are discussed the Lorentz condition is usually imposed.

$$\nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$  \hspace{1cm} (IV.13)

Using the first two equations seen in (IV.11) (implied by Maxwell’s equations $\nabla \cdot B = 0$ and $\nabla \times E = -\frac{\partial B}{\partial t}$) and (IV.13) in the remaining two Maxwell equations:

$$\nabla \cdot E = \frac{D}{\varepsilon}$$  \hspace{1cm} (IV.14)

$$\nabla \times B = \mu J + \frac{1}{c^2} \frac{\partial E}{\partial t}$$

and after a few mathematical manipulations in which also considered is the identity $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$, one can come up with the following relations known as the wave equations:

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{\rho}{\varepsilon}$$  \hspace{1cm} (IV.15)

$$\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu J$$

As seen in Eq. (IV.15), the densities $\rho$ and $J$ could be considered as the sources that generate the potentials $\Phi$ and $A$. These potentials could be then used in the first two relations of Eq. (IV.11) to come up with the Electric and Magnetic Fields.

### IV.8 Retarded Potentials

The retarded potentials have the following form:
\[ \Phi(r,t) = \int_{V'} \frac{\rho(r',t - \frac{R}{c})}{4\pi R} d^3r' \]  
\[ A(r,t) = \int_{V'} \frac{\mu J(r',t - \frac{R}{c})}{4\pi R} d^3r' \]  

(IV.16)

In (IV.16) \( R = |r - r'| \) is the distance from the field observation point \( r \) to the source point \( r' \) (please, see Fig. AIV.1). Integration is performed over the volume \( V' \), in which the source densities \( \rho \) and \( J \) are not zero.

In case there is a change at the source point \( r' \), that change will not be felt at the observation point \( r \) for a period of time equal to \( R/c \) seconds, if propagation is assumed to take place at the speed of light (otherwise the denominator in \( R/c \) will have a different value than \( c \)).

**IV.9 General Notes Regarding Radiation from a Finite Vertical Antenna**

After this brief introduction to some of the fundamentals of Electromagnetic Theory, let us now consider the problem involving electromagnetic radiation from Lightning Return Strokes.

Time domain solutions to the electromagnetic radiation from a finite, straight vertical antenna above a perfectly conducting plane have been derived in Spherical Coordinates by Uman et. al and shown in [111]. Moreover, time domain solutions to the same problem have been developed in Cylindrical Coordinate system by Master and Uman [112]. In this thesis work the same problem, as described above, is examined in Cartesian Coordinates. Significantly, this approach allows to relax the requirement of the formulation in Cylindrical Coordinates for verticality of the antenna. Now the antenna may be in an arbitrary position, including that inclined with respect to ground. Considered are the time-dependent forms of Maxwell’s equations and their solutions in terms of retarded Scalar and Vector Potentials. Expressions for the \( x \), \( y \), and \( z \) components of the total Electric and Magnetic Fields in time are given in differential form. The derived expression for the \( z \) component of the Electric Field is actually the well known from Cylindrical Coordinates vertical component of the Electric Field (\( Ez \)) while derived expression
for the $x$ component of the Magnetic Field is the azimuthal ($H\phi$) component of Cylindrical Coordinates respectively. Experimental data of these field components have been and continue to be routinely captured (currently both $Ez$ and $H\phi$ are monitored) 2km away from the CN Tower in Toronto, Canada. Computed fields will be compared to actually recorded ones in Chapter V, where results of simulations will be fully analyzed and discussed as well.

**IV.10 Derivation Considerations**

Herewith below, geometrical considerations, derived expressions for vertical component of the Electric Field and for azimuthal component of the Magnetic Field in Cylindrical Coordinates, as well as derived expressions for $x$, $y$, and $z$ components of the Electric and Magnetic Fields in Cartesian Coordinates are shown. Details regarding deriving the newly proposed relations in Cartesian Coordinates are presented in Appendix IV.

In Fig. IV.1, both the tall object and the Lightning Return Stroke Channel are assumed to be vertical antennas above a perfectly conducting ground, where the observation point $P$ lies at a certain distance $d$ from the bottom of the tall object. The azimuthal component of the Magnetic Field and the vertical component of the Electric Field, due to a vertical dipole of infinitesimal length $dz'$ along the $z$ axis at height $z'$ observed at point $P$ are given in Cylindrical Coordinates in [22].

![Fig. IV.1 - Geometrical Relations Pertinent to Radiating Vertical Antenna above Perfectly Conducting Ground and Observation Point P](image-url)
The reader may wish to also consider references [102-110] for more information regarding electromagnetic radiation.

IV.10.1 Expressions for Electric and Magnetic Field Components in Cylindrical Coordinates

IV.10.1.1 Azimuthal Component of Magnetic Field ($H_\phi$)

\[
dH_\phi (\rho, \phi, z, z', t) = \frac{dz'}{2\pi} \left[ \frac{\rho}{R^3} i\left(z', t - \frac{R}{c}\right) + \frac{\rho}{cR^2} \frac{\partial i\left(z', t - \frac{R}{c}\right)}{\partial t} \right]
\]  

(IV.17)

IV.10.1.2 Vertical Component of Electric Field ($E_z$)

\[
dE_z (\rho, \phi, z, z', t) = \frac{dz'}{2\pi\varepsilon_0} \left[ \frac{2(z-z')^2 - \rho^2}{R^5} \int_{R/c}^t i(z', t - R/c) d\tau \right. \\
\left. + \frac{2(z-z')^2 - \rho^2}{cR^4} i(z', t - R/c) - \frac{\rho^2}{c^2 R^3} \frac{\partial i(z', t - R/c)}{\partial t} \right]
\]  

(IV.18)

where $\rho, \phi, z$ are the coordinates of point $P$, $c$ is the speed of light, $R$ is the distance between the dipole and the observation point $R = \sqrt{\rho^2 + (z-z')^2}$, $i(z' t)$ is the dipole current, $\varepsilon_0$ is the permittivity of free space, and $t - R/c$ is the retarded time.

In Eqs. (IV.17) and (IV.18) above the terms in brackets proportional to the current and to its
derivative are known as the Induction and Radiation Terms respectively. In Eq. (IV.18), an additional term, which is proportional to the integral of current is seen and it is known as the Electrostatic Term. At relatively close distances all terms have significant contribution to the total Electric Field, however, since the Induction and Electrostatic Terms in Eq. (IV.18) are inversely proportional to $R^4$ and $R^5$ respectively they can be neglected at far distances.

Eqs. (IV.17) and (IV.18) are applicable for the simplified case, when vertical antennas are assumed to represent both the tall object and the Lightning Channel. In reality, the Lightning Channel is most often inclined and has some tortuosity. In order to compute the field components emanating from non-vertical antennas above perfectly conducting ground, it comes advantageous to use relations derived in Cartesian Coordinates. Fig. IV.2 could be used to set up the problem and to come up with general solutions to such topologies.

Again, similar assumptions are adopted as in the case considered using Cylindrical Coordinates, but this time, instead of the vertical linear dipole of infinitesimal length, the infinitesimal element is part of an antenna in a general 3-D position in free space, and the associated dipole current is decomposed along $x$, $y$, and $z$ axes.

Fig. IV.2 - Geometry of the Problem in Cartesian Coordinates
IV.10.2 Expressions for Electric and Magnetic Field Components in Cartesian Coordinates

IV.10.2.1 x, y, and z Components of the Magnetic Field

The \( x, y \), and \( z \) components of the Magnetic Field due to the element \( dV' \) with associated current \( i(x', y', z', t) \) at the observation point \( P(x, y, z) \) are given in Cartesian Coordinates:

\[
dH_x(x, x', y, y', z, z', t) = \frac{1}{4\pi} \left[ \frac{Y}{R^2} \left( \frac{1}{R} \frac{\partial i_x}{\partial t} + \frac{1}{c} \frac{\partial i_z}{\partial t} \right) - \frac{Z}{R^2} \left( \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{1}{c} \frac{\partial i_z}{\partial t} \right) \right]
\]

\[
dH_y(x, x', y, y', z, z', t) = \frac{1}{4\pi} \left[ \frac{Z}{R^2} \left( \frac{1}{R} \frac{\partial i_x}{\partial t} + \frac{1}{c} \frac{\partial i_y}{\partial t} \right) - \frac{X}{R^2} \left( \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{1}{c} \frac{\partial i_z}{\partial t} \right) \right]
\]

\[
dH_z(x, x', y, y', z, z', t) = \frac{1}{4\pi} \left[ \frac{X}{R^2} \left( \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{1}{c} \frac{\partial i_z}{\partial t} \right) - \frac{Y}{R^2} \left( \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{1}{c} \frac{\partial i_z}{\partial t} \right) \right]
\]

IV.10.2.2 x, y, and z components of the Electric Field

Using the same geometrical considerations as for the Magnetic Field components, the \( x, y \), and \( z \) components of the Electric Field due to the element \( dV' \) with associated current \( i(x', y', z', t) \) at the observation point \( P(x, y, z) \) are given in Cartesian Coordinates:

\[
dE_x(x, x', y, y', z, z', t) = \frac{1}{4\pi \varepsilon_0} \left[ \frac{3X^2}{cR^4} \frac{1}{R} \frac{\partial i_x}{\partial t} + \left( \frac{X^2}{c^2 R^3} - \frac{1}{R} \right) \frac{\partial i_x}{\partial t} + \left( \frac{3X^2}{R^5} - \frac{1}{R^3} \right) \frac{\partial i_x}{\partial t} \right]_{i_x} d\tau
\]

\[
+ \left( \frac{3XY}{cR^4} \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{XY}{c^2 R^3} \frac{\partial i_y}{\partial t} + \left( \frac{3XY}{R^5} \right) \frac{\partial i_y}{\partial t} \right)_{i_y} d\tau
\]

\[
+ \left( \frac{3XZ}{cR^4} \frac{1}{R} \frac{\partial i_z}{\partial t} + \frac{XZ}{c^2 R^3} \frac{\partial i_z}{\partial t} + \left( \frac{3XZ}{R^5} \right) \frac{\partial i_z}{\partial t} \right)_{i_z} d\tau
\]

(IV.22)
\[ dE_x(x, x', y, y', z, z', t) = \]
\[ = \frac{1}{4\pi \varepsilon_0} \left[ \left( \frac{3XY}{cR^4} \right) \dot{i}_x + \left( \frac{XY}{c^2R^3} \right) \frac{\partial i_x}{\partial t} + \left( \frac{3XY}{R^5} \right) \int_0^t i_x \, d\tau \right] \]
\[ + \left( \frac{3Y^2}{cR^4} \right) i_y + \left( \frac{Y^2}{c^2R^3} \right) \frac{\partial i_y}{\partial t} + \left( \frac{3Y^2}{R^5} \right) \int_0^t i_y \, d\tau \]
\[ + \left( \frac{3YZ}{cR^4} \right) i_z + \left( \frac{YZ}{c^2R^3} \right) \frac{\partial i_z}{\partial t} + \left( \frac{3YZ}{R^5} \right) \int_0^t i_z \, d\tau \]
\[ \text{(IV.23)} \]

\[ dE_z(x, x', y, y', z, z', t) = \]
\[ = \frac{1}{4\pi \varepsilon_0} \left[ \left( \frac{3ZX}{cR^4} \right) \dot{i}_x + \left( \frac{XZ}{c^2R^3} \right) \frac{\partial i_x}{\partial t} + \left( \frac{3XZ}{R^5} \right) \int_0^t i_x \, d\tau \right] \]
\[ + \left( \frac{3YZ}{cR^4} \right) i_y + \left( \frac{YZ}{c^2R^3} \right) \frac{\partial i_y}{\partial t} + \left( \frac{3YZ}{R^5} \right) \int_0^t i_y \, d\tau \]
\[ + \left( \frac{3Z^2}{cR^4} \right) \dot{i}_z + \left( \frac{Z^2}{c^2R^3} \right) \frac{\partial i_z}{\partial t} + \left( \frac{3Z^2}{R^5} \right) \int_0^t i_z \, d\tau \]
\[ \text{(IV.24)} \]

In the relations of Eqs. (IV.19-IV.24) \( R \) is the distance from the infinitesimal element \( dV'(x', y', z') \) to the point \( P(x, y, z) \), where the field is calculated, \( c \) is the speed of light, \( t - R/c \) is the retarded time, \( X = (x - x') \), \( Y = (y - y') \), \( Z = (z - z') \), \( i_x = i(x', t - R/c) \, dx \), \( i_y = i(y', t - R/c) \, dy \), and \( i_z = i(z', t - R/c) \, dz \).

The derived above equations for \( x, y, z \) components of \( dH \) and of \( dE \) hold for the general case when all components have contribution to the total Magnetic or Electric Field. In the case, when Lightning current path is vertical, perfectly conducting ground is assumed, and the observation point \( P \) lies on the \( y \) axis (see Fig. IV.1), the \( y \) and \( z \) components of the Magnetic Field, as well as the \( x \) and \( y \) components of the Electric Field are equal to “zero” due to the absence of \( i_x \) and \( i_y \) currents and some geometrical considerations. In such a case, the numerical result for \( H_x \) (Cartesian Coordinates) coincides with \( H_\phi \) (Cylindrical Coordinates) and \( E_z \) (Cartesian Coordinates) with \( E_\theta \) (Cylindrical Coordinates) respectively. On the other hand, the derived expressions for Magnetic and Electric fields in Cartesian Coordinates appear to be very useful and powerful tools for calculating the radiated electromagnetic fields when the condition for
verticality of the Lightning Channel is relaxed. In fact, now the opportunity to study the influence of the Lightning Channel arbitrarily inclined or of Lightning Channel tortuosity upon the radiated distant fields becomes quite tractable. Please, note that for the case of the CN Tower, when a perfectly conducting ground and an image are assumed Eqs. IV.19-IV.24 should be multiplied by a coefficient equal to two.

IV.11 Possibilities to Study the Influence of Lightning Channel Inclination Upon Radiated Electric and Magnetic Fields

Although, it has been shown (e.g. Rachidi et. al [22]) that the contribution of the Lightning Channel to the total radiated electromagnetic fields at 2km away usually is in the order of 2-3 times smaller than the contribution of the CN Tower, it is still quite considerable and has significant influence upon magnitudes and waveshapes of the observed fields. Furthermore, as will be shown in Chapter V of this thesis, contribution of any inclined Lightning Channel may be considerably larger depending on the specificity of each considered case. Let us now consider an arbitrarily inclined Lightning Channel as shown in Fig. IV.3, in order to appreciate the potential for the influence of the Lightning Channel inclination. As an input current, a Heidler function [100], could be used. If such current was injected at the tip of the CN Tower and provided the current at any level along the CN Tower and the Lightning Channel at any given time was appropriately described (e.g. Single-Section Model that is considered in [113]), one could use Eqs. (IV.19-IV.24) to come up with the associated, radiated at a certain distance, field components. When working in Cartesian Coordinates different components of the current \(i_x,i_y,i_z\) are derived from the total current \(i\) using geometrical relations applicable to the particular case study. In some convenient cases when the Lightning Channel is assumed to lie in the \(y-z\) plane (angle \(\phi\) equals 0°, 180°, or 360°, see Fig. IV.4) and to be inclined at a specific angle \(\gamma\) with respect to the \(z\) axis (see Fig. IV.3), there are no \(x\) components contributing to the total current \(i\). In other special cases (angle \(\phi\) equals 90° or 270°, see again Figs. IV.3 and IV.4) there are no \(y\) components.
In general all $x$, $y$, and $z$ current components exist and these $i_x$, $i_y$, and $i_z$, together with the geometrical relations pertinent to the case study, are precisely what we need in order to compute different field components using the proposed relations from Eqs. (IV.19-IV.24). Appendix V contains details regarding geometrical relations and current components needed to assess the electromagnetic radiation due to inclined Lightning Channel.

It looks like it is important to perform a thorough analysis in order to find out whether the Lightning Channel contribution would be significantly higher or smaller in some particular real world cases when the Lightning Channel has some tortuosity and some inclination and the observation point is also located at a specific arbitrary position in 3-D space. The expressions in Cartesian Coordinates presented in this chapter provide a possibility to compute all the radiated components of the Electric and Magnetic Fields for such geometries and eventually compare some of them to corresponding existing records. Such a detailed study has not been performed until this moment but could now be undertaken.
Fig. IV.4 - Angle $\phi$ in the $x$-$y$ Plane
Chapter V
Studies, Analyses, and Experimental Results

V.1 Outline

This chapter is devoted to analysis and examination of the developed sophisticated modeling of Lightning events at the CN Tower. The newly introduced relations for computing radiated distant fields due to Lightning Return Strokes to the CN Tower, together with current described using the developed multi-section models, are taken advantage of and employed in studies presented in this Chapter.

First, a general analysis, using a typically observed current is performed. The typical “uncontaminated” Lightning current is used as input to the three developed in the thesis “Engineering” models. Plots of computed current derivative and current at a specific location on the CN Tower (where one of the current sensing elements is mounted), as well as another plot showing the current at several different levels of CN Tower and along the Lightning Channel are provided.

Next, radiated fields are treated. Plots of the Electric and the Magnetic Field at 2km from the CN Tower are shown, in which the respective Induction, Radiation, and Static components are clearly marked.

Furthermore, a full sensitivity study looking into the influence of different channel inclinations upon the radiated Electric and Magnetic Fields is performed. Appendix V contains details regarding the used geometries and corresponding currents. For the first time, different speeds of propagation of the Return Stroke in the Not-Fully Ionized portion of the Lightning Channel is studied. This includes non-linear variation in the speed of the Return Stroke front.

Later in the chapter, four discrete cases of channel inclination, considered by another author [114], are analyzed by the techniques developed in this thesis.

Next, a recorded event from August 19, 2005 is considered and computationally modelled. It was chosen because for that particular event, a full data set exists including captured current, vertical component of Electric Field, azimuthal component of Magnetic Field, and video records.
Lastly, the potential for computing and analyzing components of the Electric and Magnetic Fields ($E_x$, $E_y$ and $H_y$, $H_z$) for practical cases of interest that have not been considered in detail by other authors is introduced and briefly discussed.

**V.2 Computation of Current Waveforms**

Let us now consider an arbitrary event in order to compare the outputs of the three multi-section models described in Chapter II.

**V.2.1 Channel Base Current**

The analytical relation adopted to represent the channel base current, whose specific waveshape and amplitude are intended to emulate the actually existing Return Stroke current, is the one proposed by Heidler and it is frequently referred to as the Heidler function [100].

\[
 i(h,t) = \frac{I_0}{\eta} \frac{(t/\tau_1)^n}{1 + (t/\tau_1)^n} e^{\frac{-t}{\tau_2}} 
\]  

(V.1)

where:

- $h$  height at which the Return Stroke current is initiated
- $t$  instance of time
- $I_0$  amplitude of the channel-base current
- $\tau_1$  front time constant
- $\tau_2$  decay time constant
- $n$  exponent having values in the range from 2-10
- $\eta$  the amplitude correction factor given by
In studies that follow, the channel base current is represented by a sum of two Heidler functions and it has the following analytical representation:

$$i_0(h,t) = \frac{I_{01}}{\eta_1} \left( \frac{t}{\tau_{11}} \right)^{\eta_1} e^{-\frac{t}{\tau_{11}}} + \frac{I_{02}}{\eta_2} \left( \frac{t}{\tau_{12}} \right)^{\eta_2} e^{-\frac{t}{\tau_{12}}}$$  \hspace{1cm} (V.3)

This equation holds for the instance of time $t$. The reader should bear in mind that different current contributions, forming the total current at a respective level above ground level, have respective time delays (e.g. see equations pertinent to Single-Section Model - Eqs. A1.1 - A1.6).

The input to all considered models is a Heidler function with parameters utilized in previous publications (e.g. [22]).

For convenience, the used values are shown below and the corresponding plot of the produced waveform is depicted in Fig. V.1.

![Fig. V.1 - Heidler Current Used in Simulations](image)
\[ I_{o1} = 10.7 \text{kA}; \tau_{11} = 0.25 \mu\text{s}; \tau_{21} = 2.5 \mu\text{s}; n_1 = 2 \]
\[ I_{o2} = 6.5 \text{kA}; \tau_{12} = 2 \mu\text{s}; \tau_{22} = 230 \mu\text{s}; n_2 = 2 \]

V.2.2 Effects of Chosen Models

Assuming \( v = 1.9e8 \text{ m/s} \) as constant propagation velocity within the Not-Fully Ionized portion of the Lightning Channel, waveforms of the Return Stroke current derivative and of the current, both computed at the 474 m level, using a variable number of CN Tower sections, are shown in Fig. V.2 and Fig. V.3 respectively.

As is seen in these two figures, waveforms of the current derivative and of the current itself are more elaborately represented when a more detailed model of the CN Tower is considered. Respective changes in the waveshapes are spaced in time according to reflection points considered in different models. While only two reflections (from bottom and top of the CN Tower) are seen for the Single-Section Model, additional two reflections are caused by the top and bottom of the Skypod for the Three- and the Five-Section Models. Further two reflections, from top and bottom of the Space Deck, are associated only with waveforms computed by the Five-Section Model. Since these last two reflections are spaced very closely in time, they are not easily discernable on the original timescale of Fig. V.2, and therefore an enlargement plot for the region \( 0.35-0.50 \mu\text{s} \) and \( 38-42 \text{kA/\mu s} \), showing clearly the influence of these two additional reflections, is provided. In order to capture the effect of these reflections upon the computed current derivative waveform, one should use a high resolution time step, because reflections are coming from points of mismatched impedances which are only 9m apart (see Fig. AIII.1). This is corresponding to a travel time of 60ns for the round trip of 18m. Consequently, a time step of 1ns was needed in calculations to produce the current derivatives for the Five-Section Model shown in Fig. V.2. The computations pertinent to Single and Three-Section Models could be carried out using a time step of 100ns, which was found to be sufficient in terms of precision in the produced current derivative and current waveforms.
Fig. V.2 - Injected and Calculated Current Derivatives at 474m Above Ground Level using the Single-, Three- and Five-Section CN Tower Model for the Arbitrary Lightning Event (the enlargement of the area in the circle ($t = 0.35-0.50\mu s$ and $di = 38-42kA/\mu s$) is shown on top)
Fig. V.3 - Injected and Calculated Current Waveshapes at 474m Above Ground Level using Single-, Three- and Five-Section CN Tower Model for the Arbitrary Lightning Event

Another aspect of the Five-Section Model concerns reflections from the Space Deck, which travel up the Tower and are partially reflected and partially refracted into the Lightning Channel. For the Five-Section Model these occur at a much earlier time than those coming from the Skypod for the Three-Section Model and from the ground for the Single-Section Model. In case of the current wave (see Fig. V.3), for the Five-Section Model, the top part of the front is seen to be somewhat increased by this First Set of Space Deck Reflections.

There are other changes in the computed waveforms that are due to reflections coming from interactions among different discontinuities of the CN Tower and from reflections within the Lightning Channel. These arrive at the observation point at various times and either add or subtract to form the final waveshapes.

If one looks closely at Fig. V.2, presenting the current derivative at 474m above ground level computed by the Five-Section Model, one can notice distinctive peaks occurring at particular instances of time. Since these peak values correspond to maximum steepnesses of the current wave, one can associate these peaks with maximum front steepnesses of the current wave reflected at discontinuities of the structure. To better visualize these reflection points within the
CN Tower structure for the Five-Section Model, and to correlate them in time, Fig. V.4 is offered.

In Fig. V.4 the following abbreviations are used:

- **Ht**: height of the tower
- **HRC**: height of mounting of the lower Rogowski coil with respect to ground
- **H4**: top level of the Space deck with respect to ground
- **H3**: bottom level of the Space deck with respect to ground
- **H2**: top level of the Skypod with respect to ground
- **H1**: bottom level of the Skypod with respect to ground

Fig. V.4 - Travel Times inside the CN Tower for the Five-Section Model
It is well worth noting that, due to the peculiarities of the CN Tower structure, some of the considered reflections arrive from different reflection points at the 474m recording location at the same instances of time. This is the case for example for the current waves traveling the path HRC-H2-HRC and HRC-H3-HRC-Ht-HRC, each taking 0.72\(\mu s\).

In actual fact many different contributions add up to each other or cancel off in order to produce the final computed waveform at the respective level and time of interest. Similar considerations are used to come up with the current distributions along the Lightning Channel as well.

While Fig. V.3 gives the waveshapes of the current at the 474m above ground level, computed using different number of CN Tower sections, Fig. V.5 provides information about current waveforms at different levels of the CN Tower and of the Lightning Channel, all determined using the Five-Section representation of the CN Tower. Please, note the delay in the starting instances of current waveshapes at different levels above ground indicating the time needed for the incident current wave to travel down the tower at the speed of light \(c\), and its counterpart to progress up the Not-Fully Ionized portion of the Lightning Channel at the reduced propagation velocity \(v=1.9e8m/s\).

![Fig. V.5 - Injected Current and Computed Current Waveforms, using Five-Section Model of CN Tower, at Various Heights along the CN Tower and the Lightning Channel](image)

Computed peak amplitudes are affected by the numerous reflections and refractions, making the
current wave at the bottom of the CN Tower to have the maximum amplitude of all. One could also notice in the current wave computed at 4000\textit{m} above ground, which starts around 18\textmu s, that there is a strong attenuation experienced during travel within the Lightning Channel.

As already shown in Chapter IV, detailed waveshapes of the Lightning current at all levels of the CN Tower and within the Lightning Channel constitute the required set of input quantities needed to compute Electric and Magnetic Fields at a distance.

V.3 Computation of Electric and Magnetic Fields at a Distance

The relations used to compute different components of the Electric and Magnetic Fields are discussed in Chapter IV, and shown in Appendix IV.

V.3.1 Base Case

The base case considers the situation where the Lightning Channel is vertical. As an input current, the Heidler function introduced in the preceding Section is used.

![Graph] Fig. V.6 - Computed $E_z$ and $H\phi$ at 2km away from the CN Tower (Single-Section Model)
Eqs. (IV.19-IV.24) are evaluated using current produced by the Single-, Three-, and Five-Section Model and computing Electric and Magnetic Field components $E_z$ and $H_\phi$ at a distance $2km$ away from the CN Tower. Resultant graphs are shown in Figs. V.6-V.8, where the individual contribution terms are clearly marked.

It is observed in Figs. V.6-V.8 that the waveforms of both $E_z$ and $H_\phi$ have initial peaks which look very similar to each other. This is mainly due to the Radiation Term component which is proportional to the derivative of the injected current. The contribution of the Induction Term, which is proportional to the injected current, also has similar influence upon the total calculated $E_z$ and $H_\phi$ field components. After the peak, the $E_z$ waveform ramps up and this is because of the contribution of the Static Term, which is proportional to the integral of the injected current (or the amount of transferred charge). Since there is no Static Term in the $H_\phi$ field, the tail of the curve is not ramping up, but similar trends in the instances of appearance of distinctive “bumps” and “dips” are observed in both calculated fields.

![Computed $E_z$ and $H_\phi$ at 2km away from the CN Tower (Three-Section Model)](image)

Fig. V.7 - Computed $E_z$ and $H_\phi$ at 2km away from the CN Tower (Three-Section Model)
Fig. V.8 - Computed $E_z$ and $H\phi$ at 2km away from the CN Tower (Five-Section Model)

Although at first glance, the overall waveforms in the presented plots of Figs. V.6-V.8 might look the same, under closer examination one could notice that there are some important differences which are mainly observed in the finer waveform detail and in the peak values. In order to better appreciate these deviations Fig. V.9 is prepared.

Fig. V.9 - Computed $E_z$ and $H\phi$ at 2km away from the CN Tower
It is clearly visible in Fig. V.9 that there is a difference in the peak values yielded by the three employed models. The peak value is most influenced by the Radiation Term. It is apparent in Fig. V.2 that, when the Five-Section Model is used, the first peak is reaching the highest value, because there is a positive reflection coming from the top of the Space Deck. The Three-Section Model does not include reflections from the Space Deck and thus yields a slightly lower first peak. There is a considerable contribution due to the reflection coming from the Skypod, which is considered in both the Three- and Five-Section Models and consequently these two models predict higher peaks than the Single-Section Model as expected.

It is also observed in Fig. V.9 that the curve computed using the Single-Section Model is smoother when compared to the corresponding ones calculated by the Three- and Five-Section Models. In the case of the Single-Section Model only reflections at the extremities of the tower are considered and the curve is basically influenced only by these. In the waveforms described by the Three- and Five-Section Models reflections from the top and bottom of the additionally introduced Skypod and Space Deck contribute to the computed waveshapes and produce a more elaborated profile. Nevertheless, the waveforms predicted by the three models at later times (when the reflections have been somewhat suppressed) are very close to each other and even the simple Single-Section Model yields reasonably matching results to ones computed using the most sophisticated Five-Section Model.

It is worth pointing out that although the first influences due to the reflections at the Space Deck when the Five-Section Model is considered are comparably small, they have relatively high contribution since they occur at early times both, within the Tower itself and inside the Fully Ionized part of the Lightning Channel, where they travel only a small distance until they reach the top of the propagating upwards Lightning Channel front. At that point parts of them are reflected and parts of them continue beyond the Lightning Channel front together with the initial injected wave at the slower speed of propagation \( v \). These components influence the total currents and in turn the radiated fields at very early times.

V.3.2 Sensitivity Studies

Relations pertinent to cases in which an inclination of the Lightning Channel is considered are shown in Appendix V. Taking advantage of such expressions (V.A1-V.A24), one could perform
unlimited number of case studies, involving various inclined or tortuous channel trajectories. In this Thesis, based on developments shown in Chapter IV, sensitivity analyses were carried out, using the previously introduced Single-, Three-, and Five-Section Models, for straight Lightning Channel that is inclined at different angles $\gamma$ with respect to the vertical Lightning Channel, and lies at various angles $\phi$ (please see Fig. IV.3) in the $x$-$y$ plane. Moreover, a sensitivity study looking into the influence of variable Lightning Channel front speed was also implemented.

V.3.2.1 Sensitivity Study of Lightning Channel Inclination

Plots showing computed waveforms of $E_z$, $H_x$, for straight vertical Lightning Channel (base case) could be seen in Figs. V.6-V.8. Similar plots were prepared for a number of study case scenarios in which the inclination angle $\gamma$ was set to specific fixed angles ($\gamma = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ$). For each fixed inclination angle $\gamma$, calculations were implemented for rotation angles $\phi = 0-360^\circ$. The values of the first peaks were then extracted and recorded. Corresponding plots are shown in Figs. V.10-V.11 for the Single-Section Model, in Figs. V.12-V.13 for the Three-Section Model, and in Figs. V.14-V.15 for the Five-Section Model.

Fig. V.10 - Computed $E_z$ as a function of $\phi$ for different $\gamma$ (Single-Section Model)
Fig. V.11 - Computed $H_x$ as a function of $\phi$ for different $\gamma$ (Single-Section Model)

It is observed in Figs. V.10 and V.11 that in general, for a given angle $\gamma$, peak amplitudes of both $E_z$ and $H_x$ start at a maximum value for $\phi = 0^\circ$, reach a minimum at $\phi = 180^\circ$ and climb back to the maximum at $\phi = 360^\circ$. Furthermore, please observe that with increasing values of Lightning Channel inclination angle $\gamma$, maximum values of the Electric as well as of the Magnetic Field do not necessarily increase monotonically, while their minima continuously diminish with $\gamma$.

Fig. V.12 - Computed $E_z$ as a function of $\phi$ for different $\gamma$ (Three-Section Model)
Fig. V.13 - Computed $H_x$ as a function of $\phi$ for different $\gamma$ (Three-Section Model)

Fig. V.14 - Computed $E_z$ as a function of $\phi$ for different $\gamma$ (Five-Section Model)
When considering the results yielded by the Five-Section Model (Figs. V.14-V.15) one could notice that the peak values of \( E_z \) are affected by the inclined Lightning Channel from \(-10.95\%\) (for the minima) up to \(+12.48\%\) (for the maxima). Corresponding changes for \( H_x \) are \(-11.01\%\) and \(+9.66\%\).

Fig. V.15 - Computed \( H_x \) as a function of \( \phi \) for different \( \gamma \) (Five-Section Model)

Fig. V.16 - Peak Values of \( H_x \) as a Function of \( \gamma \)
In the performed sensitivity study, it was found that the three considered models of the CN Tower indicate similar tendencies regarding peak values of $E_z$ and $H_x$. However, the more sophisticated multi-section models predict results that are in closer agreement with each other. This tendency is depicted for the case of the radial Magnetic Field component in Fig. V.16.

When examining Fig. V.16 in detail one can notice that the Three and Five-Section Models yield quite similar results, which differ noticeably from the ones achieved using the Single-Section Model. It is clearly seen that the Single-Section Model predicts in all considered cases lower $H_x$ peak values and this is due to the fact that major points of reflections were not considered (Space Deck and Skypod of the CN Tower).

Use of the newly introduced Cartesian Coordinates presents the opportunity to consider different trajectories of the Lightning Channel in computation of $E_z$ and $H_x$. Each introduced more sophisticated “Engineering” model (Single, Three and Five Sections representing the CN Tower and two sections representing the Lightning Channel) reproduces closer and closer the Lightning current waveshape along the CN Tower and in the Lightning Channel. Although it has been shown [22] that the Lightning Channel has 2-3 times smaller contribution to producing the total radiated fields than the CN Tower itself, the performed sensitivity study shows that different channel inclinations may influence the waveforms of radiated fields by more than 10%.

V.3.2.2 Sensitivity Study of Different Lightning Channel Front Speeds

In this study, the employed relations describing the current at any height and at any time along the CN Tower and the Lightning Channel are presented in Chapter III. Corresponding expressions for the vertical component of the Electric as well as the azimuthal component of the Magnetic Field are shown in Chapter IV. Assumed is a vertical Lightning Channel and the simulations are limited up to 60$\mu$s. However, the produced plots shown in the following Figs. V.17-V.22 are limited up to 20$\mu$s in order to better visualize the deviations in the waveforms at early times. Considered speeds of propagation of the Lightning Channel front are seen in Fig. III.6.
Fig. V.17 - $E_z$ Computed Using the Single-Section Model

Fig. V.18 - $H_x$ Computed Using the Single-Section Model
It is observed in the waveforms shown in Figs. V.17-V.18 that, in addition to reflection details, the peak values of $E_z$ and of $H_x$ computed by the use of the Single-Section Model at 2$km$ away from CN Tower, are also affected by the different speed of propagation of the Lightning Channel front.

Fig. V.19 - $E_z$ Computed Using the Three-Section Model

Fig. V.20 - $H_x$ Computed Using the Three-Section Model
In addition to the observed deviations in the predicted peaks in Figs. V.19-V.22, it is also seen that the calculated resultant radiation fields calculated with different front propagation velocities
feature different decay steepness after the major peaks. Moreover, the fine profile observed in both $E_z$ and $H_x$ waveforms is containing some considerable deviations mostly in the region around and after $6\mu s$. Potentially, it may be possible that in some particular cases these deviations may become larger in values and thus have some more pronounced influence on affected devices susceptible to electromagnetic radiation.

In summary, it could be said that the developed formulae regarding $E_z$ and $H_x$ components in Cartesian Coordinates coupled with the proposed new approach to study the influence of variable Lightning Channel propagation speed upon the radiated fields at a particular distance present the opportunity to conduct unlimited different scenario cases involving inclined Lightning Channel.

V.3.3 Validation of Newly Proposed Formulae

V.3.3.1 Comparison to Another Author’s Study Results

In order to validate the relations for Electric and Magnetic Fields derived in Cartesian Coordinates (in particular Eq. IV.19 and Eq. IV.24 shown in Chapter IV), let us now consider the same cases analyzed in [114]. In that study, the current used was represented by a Double-exponential function:

$$I_s(t) = \frac{I_1}{\eta_1} \left[ \exp(-a_1 T(t)) - \exp(-a_2 (T(t))^2) \right]$$

(V.1)

In Eq. V.1, the parameters used were:

$I_1 = 15kA; \eta_1 = 0.0973; a_1 = 0.005e6; a_2 = 30.64e12;$

$T(t) = \frac{a_1}{a_2} - t;$

Different geometries for all considered cases in [114] are shown in Fig. V.23. This time the computation of $E_z$ and $H_x$ components was carried out using the newly proposed equations in Cartesian Coordinates (Eqs. IV.24 and IV.19). Predicted $E_z$ and $H_x$ waveforms at $2km$ away from the CN Tower for the simulated cases seen in Fig. V.23 are presented in Fig. V.24 and in Fig. V.25. The appearance (scales and waveforms) of these plots is matched to the corresponding ones in [114] for easier comparison.
Fig. V.23 - Simulated Cases of Lightning Return Stroke to the Tower with Different Channel Inclinations (Adapted from [114])

The details pertaining to the simulated cases are as follows:

- Case A – base case when the Lightning Channel is vertical and thus $\phi = 0^\circ$ and $\gamma = 0^\circ$;

- Case B1 (B2) – $\phi = 0^\circ$ and $\gamma = 22.5^\circ (45^\circ)$;

- Case C1 (C2) – $\phi = 90^\circ$ and $\gamma = 22.5^\circ (45^\circ)$;

- Case D1 (D2) – $\phi = 180^\circ$ and $\gamma = 22.5^\circ (45^\circ)$;

- Case E – $\phi = 90^\circ$ and $\gamma_1 = 67.5^\circ$; $\gamma_2 = 22.5^\circ (l_1=100m, l_2=2600m)$. 

While there are some minor differences, the curves shown in Figs. V.24-V.25 are stacked in exactly the same order as the corresponding ones in [114]. For $\phi = 0^\circ$ (cases B1 and B2)
amplitudes of both $E_z$ and $H_x$ exceed those of the base case ($\phi = 0^\circ$ and $\gamma = 0^\circ$), for $\phi = 90^\circ$ (cases C1 and C2) they lie below the base case, finally for $\phi = 180^\circ$ (cases D1 and D2) their amplitudes are smallest of all. The reader should be alerted in particular to the curves representing case E, where the change in the inclination of the Lightning Channel is clearly reproduced at around 1.2 $\mu$s. On the basis of these observations it may be stated that the used Eqs. IV.19 and IV.24 have been successfully validated.

V.3.3.2 Comparison to a Real Event

It was already mentioned that $H\phi$ and $E_z$ are routinely captured at the University of Toronto. Using Eqs. IV.19 and IV.24 one could compute, in Cartesian Coordinate system, theoretical predictions of the waveforms of the fields. An actual event, whose current was recorded at the 474m level and respective fields at 2km away from CN Tower on August 19, 2005, shown in Figs. II.3-II.6 will be considered for the purpose of a comparison.

In this prediction the Five-Section Model of the CN Tower was used. As the first step, the appropriate injection current was derived through comparison of the observed and computed current at the 474m level. Fig. V.26 shows the considered current derivative and current records and the simulated ones after the final adjustments.

![Fig. V.26 - Recorded and Calculated Current and Current Derivative at 474m Above Ground](image-url)
It is worth noting that the speed of propagation of the Lightning Channel front also plays quite an important role in simulating the current waveshape at concrete levels. If a different speed of propagation were assumed, certain reflections would take place at corresponding specific instances of time and thus influence the final waveshape in a different manner. The most pronounced influence comes from the major reflection from the top of the Fully-Ionized Lightning Channel.

Fig. V.27 is offered in order to appreciate the impact of the chosen speed of propagation in the Not-Fully-Ionized part of the Lightning Channel. In Fig. V.27 several major points of reflections are clearly marked by arrows and the recorded waveform is plotted in thick line. The calculated wave using speed of propagation of the Lightning Channel front equal to $1.9\times10^8\,\text{m/s}$ is shown in dashed-dotted line and the dotted line is corresponding to speed of propagation equal to $1.0\times10^8\,\text{m/s}$.

Fig. V.27 - Recorded and Calculated Current at 474\,m Above Ground

It is clearly seen in Fig. V.27 that when the higher speed of propagation is adopted the major reflection from the top of the Fully-Ionized Lightning Channel occurs at later time. Since the current wave calculated by the lower speed ($1.0\times10^8\,\text{m/s}$) is a better match to the recorded waveform, it was selected as the appropriate one together with the following set of Heidler...
parameters:

\[ I_{o1} = 1.5kA; \tau_{11} = 0.15\mu s; \tau_{21} = 7\mu s; n_1 = 2 \]

\[ I_{o2} = 3.5kA; \tau_{12} = 0.25\mu s; \tau_{22} = 90\mu s; n_2 = 2 \]

Such described current and Lightning Channel propagation speed were then plugged in the relations for distant fields in order to carry out the computations and produce the respective plots depicting the vertical component of the Electric Field and the azimuthal component of the Magnetic Field for which corresponding records exist.

The calculation of radiated fields due to the CN Tower and the Lightning Channel at a distance of 2\( km \) away from the tower included reconstruction of the Lightning Channel in 3-D space based on the visual records from Figs II.5 and II.6. The assumed Lightning Channel trajectory is shown in Fig V.28 and in Fig. V.29, and the corresponding heights and angles are as follows:

\[ h_1=553m; h_2=587m; h_3=621m; h_4=689m; h_5=807m; \]

\[ 1=45^\circ; 2=45^\circ; 3=45^\circ; 4=30^\circ; 5=30^\circ; 6=45^\circ; 7=45^\circ. \]
Fig. V.29 - Reconstructed Lightning Channel Trajectory in 3-D Space (Drawing Reproduction)

Fig. V.30 - Recorded and Calculated $E_z$ and $H_x$ at 2 km away from CN Tower
Fig V.30 shows the recorded and computed at 2km away from the CN Tower (in Pratt building of University of Toronto) waveforms of $E_z$ and $H_x$ respectively.

As is seen in Figs V.26-V.27, the observed and the computed current derivative and current were successfully closely matched, since there was the possibility to adjust the injected current and speed of Return Stroke front propagation. Radiated fields shown in Fig V.30, while displaying correct shape of waveforms, do not have their amplitudes even closely matched. There are a few reasons for that. It is shown in another study [80] that due to omission in modeling of existing metallic frames and pipes, found in buildings sitting in the propagation path between the radiating tall structure and the recording equipment, the predicted fields are underestimating the actual fields and usually the deviation is more than 25%. Furthermore, there is some enhancement due to the metallic edge of the roof where the sensors are mounted and this may be the reason that the Electric Field is even further apart from that measured than the Magnetic Field.

Also, please note that the Lightning Channel is reconstructed in 3-D space based on observations in 2-D, which will bring some additional error. Currently, there is a possibility to capture the Lightning Channel trajectory in 3-D and it is planned in future studies to develop techniques for precise determination of the 3-D shape of the Lightning Channel to model the radiated fields. It is well observed in Figs. V.10-V.16, and in Figs. V.23-V.25 that the Lightning Channel has some considerable influence upon the total radiated fields and thus its contribution should be carefully evaluated in each considered case.

V.3.4 Prospects for Computation of the Ex, Ey Components of the Electric Field and Hy, Hz Components of the Magnetic Field

$E_x$, $H_y$, and $H_z$ components have not been considered much in technical literature until now due to several reasons including the lack of formulae to compute them and the fact that they have not been routinely measured. $E_x$, $H_y$, and $H_z$ are also not considered since they do not have any contribution when the Lightning Channel is considered to be vertical as is assumed in most “Engineering” models. An expression describing the radial component ($E_y$) of the Electric Field in Cylindrical Coordinates is found in [112]. For observation point on the $y$ axis and for vertical Lightning Channel, which are commonly assumed, the $E_y$ is also equal to zero, though.
Unfortunately, this component has not been monitored at the CN Tower site and no actual records are available. However, currently observations are being performed at the Austrian Gaisberg Tower as reported in [115-116]. The motivation behind these observations stems from the fact that the radial (horizontal) component has some influence for the more realistic cases when the observation point is not on the $y$ axis, but is elevated at some height. It is reported in [115] that at farther distances there is a difference of several orders of magnitude between the amplitudes of the $E_z$ and $E_y$ components, but nevertheless the $E_y$ component exists and has some noticeable contribution. Furthermore, in reality the Lightning Channel is never vertical, thus all the remaining components ($E_x$, $H_y$, and $H_z$) also have some contribution. It follows that in certain cases all of the above or at least some of the considered components should not be neglected as they might have potentially some considerable influence in the radiated Electric and Magnetic Fields and in turn unaccounted effects upon some sensitive electronic or electrical equipment. The newly proposed expressions (Eq. IV.19-IV.24) are now available and could be used to estimate the impact of various Electric and Magnetic Field components if an inclined and tortuous Lightning Channel were assumed for observation points at any height and distance.
Chapter VI
Concluding Remarks

The objective of this work, namely to develop a sophisticated model describing Lightning events at the CN Tower in Toronto, has been implemented successfully. In contrast to presently popular frequency-domain approaches, computational algorithms have been developed in this thesis in the time domain. Furthermore, the new formulae that have been derived bring new insights that should improve the understanding of the electromagnetic radiation process associated with Lightning at tall structures and enhance modeling of its impact upon the surroundings.

In this thesis, three transmission-line-based “Engineering” models, each representing in greater detail the structural peculiarities of the CN Tower, have been considered. In those models, the tower is represented by one, three, and five transmission line sections of respective lengths and impedances. In each case, the Lightning Channel is modeled by two transmission line sections; their respective impedances are constant, while their lengths are variable. The author has presented to the technical literature the Five-Section Model of the CN Tower for the first time.

A novelty is introduced at the border between the two sections of the Lightning Channel. In most other works, the authors assume that there is a discontinuity in the current components that reach the Return Stroke front and beyond this point they cease to exist. This is physically inconsistent and requires additional measures to rectify the situation. In the presented work this is not an issue, since the discontinuity is fully taken into account and the effects of reflected and transmitted current components are truthfully reproduced. The transmitted components are joining the originally injected current wave, add up to it and propagate at the same lower speed upwards within the Not-Fully Ionized portion of the Lightning Channel, while the reflected components head back inside the channel’s Fully Ionized portion towards the tip of the tower at the speed of light.

Another novelty and major contribution provides the possibility to take into account variable speed $v$ of Return Stroke front propagation. Until now only constant speed $v$ has been considered in the technical literature. At the same time observations at the CN Tower had indicated that this speed may be either constant, or decreasing, or even increasing as the Return Stroke front progresses up the Lightning Channel. In the thesis it is shown that the chosen speed pattern
exerts considerable effect upon the current waveshapes at different levels of the tower and of the Lightning Channel as well as upon the radiated fields at a distance.

Third major contribution is the introduction of newly developed expressions describing different components of the radiated fields in Cartesian Coordinates. These equations constitute very powerful tools that present the opportunity to study the influence of the Lightning Channel trajectory, which could assume any general shape in the 3-D space. It has been shown in the thesis that, in different cases involving the inclined channel, there is considerable variation, not only in the resultant peak values, but also in the shapes of associated waveforms as well. This means that the theoretical predictions, in some particular cases whenever the Lightning Channel is assumed to be vertical rather than resembling more closely its actual trajectory, will yield estimated radiated fields that may deviate substantially from the actual ones, and one should keep that fact in mind.

The developed theoretical expressions might be used to study the impact of field components that have not been considered until now, namely $E_x$, $H_y$, and $H_z$. As already mentioned, these components are normally neglected since for the commonly assumed vertical channel, they do not have any contribution. However, as briefly discussed in Chapter V, these components do exist for all cases involving inclined Lightning Channel for observation points at some height above ground level and might have some unforeseen effects upon the radiated fields and in turn upon the devices susceptible to electromagnetic interference. One could take advantage of the developed theoretical expressions in this thesis work, adapt them to their own study case and estimate the potential impact. Furthermore, sensitivity analysis involving inclined Lightning Channel could be performed for the $E_y$ component, as well. There is a readily available in the technical literature expression for this component, but it is in Cylindrical Coordinates and assumed vertical Lightning Channel only. Now the flexibility to relax the condition regarding the verticality of the Lightning Channel exists and could be taken advantage of.

Another quantity that has been considered by some authors, but not studied extensively, is the radial component of the Magnetic Field $H_y$. At the CN Tower this component has been monitored for various periods of time, so that some records do exist in the archives. It might be a good sub-project area to look into these records and perform some analysis using the analytical expressions developed in this thesis.
A separate research area could be a study looking into the influence of the CN Tower height upon the current waveshapes as well as upon the respective radiated fields at a distance. Along these lines, a study may be initiated, in which analysis of the influence of the CN Tower shape could be undertaken, i.e. to explore, what would be the impact if some of the major points of reflections, such as the Space Deck or the Skypod were located at different heights. On a completely general basis, adapting the proposed modeling approach, studies could be implemented during the design stage of new tall towers to establish the impact of proposed structural shapes upon Lightning-caused radiated fields.
References


Assembly of International Union of Radio Science (URSI), New Delhi, India, October 23-29, 2005.


Appendix I

Current Distribution for Single-Section Model of the CN Tower

Fig. II.10 found in Chapter II is depicting the adopted Single-Section Model. All assumptions pertinent to that model are listed in Chapter II as well. In this Appendix, only the developed expressions for currents at any time and height along the structure of the tower and along the Lightning Channel are fully shown together with more detailed respective Lattice diagrams. Furthermore, expansion of $\sigma$ and $\xi$ terms is also included. Finally, input parameters of the Heidler function primarily used in the thesis are listed under AI.3 at the end of this Appendix.

Al.1 Current Distributions for Single-Section Model

\[
\sum_{n=0}^{\infty} + k_t^n i_0 \left( t - \frac{h-z}{c} - \frac{2nh}{c} \right) + k_b^n i_0 \left( t - \frac{h+z}{c} - \frac{2nh}{c} \right) = 0
\]  

where \( 0 \leq z \leq h \)

Fig. AI.1 - Detailed Lattice Diagram for Single-Section Model (inside the CN Tower)

Al.1.1 Major Components (inside the CN Tower)

\[
i_M = \sum_{n=0}^{\infty} \left( k_t^n i_0 \left( t - \frac{h-z}{c} - \frac{2nh}{c} \right) + k_b^n i_0 \left( t - \frac{h+z}{c} - \frac{2nh}{c} \right) \right) = 0
\]  

(AI.1)
AI.1.2 Additional Contributions (inside the CN Tower)

\[ i_{A1} = k_b (1 + k_r) k_c (1 - k_r) i_0 \left( t - \frac{2 h \xi + h - z}{c} \right) \]

\[ i_{A2} = k_b^2 (1 + k_r) k_c (1 - k_r) i_0 \left( t - \frac{2 h \xi + h + z}{c} \right) \]

\[ i_{A3} = k_b^2 (1 + k_r) k_c (1 - k_r) i_0 \left( t - \frac{2 h \xi + 3 h - z}{c} \right) \]

\[ i_{A4} = k_b^2 k_r (1 + k_r) k_c (1 - k_r) i_0 \left( t - \frac{4 h \xi + h - z}{c} \right) \]  

(AI.2)

where \( 0 \leq z \leq h \) \quad \sigma = \frac{v}{c-v} \quad \xi = \frac{c+v}{c-v}

AI.1.3 Channel Base Current (initial current propagating upwards in the not-fully ionized portion of the Lightning Channel)

\[ i_{ch} = \frac{z}{z_{ch}} i_0 \left( t - \frac{z - h}{v} \right) e^{\left( \frac{h-z}{v} \right)} \]  

(AI.3)

for \( h \leq z \leq z_{MAX} \)

Fig. AI.2 - Detailed Lattice Diagram for Single-Section Model (inside the Lightning Channel)
Fig. AI.2 shows a Lattice diagram containing more details pertinent to the relations for Internal and Transmitted components. Numbers 2, 4, 6, ... 2m represent the corresponding “First”, “Second”, “Third”, ... and “mth” reflections from ground respectively.

**Al.1.4 Internal Components “bouncing” within the Ionized Portion of the Lightning Channel (contributions due to the “mth” reflection coming from ground)**

\[
i_{I,(2m-1)} = \sum_{n=0}^{\infty} k_b^n k_i^{m-1} (1 + k_i) k_c^n (- k_i)^n \left( t - \frac{2m h}{c} \right) n \left( z - \frac{h}{c} \right)
\]

\[
i_{I,(2m)} = \sum_{n=0}^{\infty} k_b^n k_i^{m-1} (1 + k_i) k_c^n (- k_i)^n \left( t - \frac{2m h}{c - v} \right) n \left( h + 2m \sigma \xi^n \right) - \frac{z}{c}
\]

where: \( h < z < h \left( 1 + 2m \sigma \xi^n \right) \leq z_{MAX} \quad m = 1,2,3,...

**Al.1.5 Components Transmitted into the not-fully Ionized Portion of the Lightning Channel**

The Transmitted components are shown in Fig. AI.2 by arrows (at heights \( h, z_2, z_4, ... \) and at times \( t_0, t_2, t_4, ... \) respectively), which are parallel to the dashed line representing the Return Stroke wave propagating in the upward direction (to the right). Their respective initial times and heights of occurrence are marked clearly and are found to be the following expressions:

\[
z_2 = h \left( 1 + 2m \sigma \xi^0 \right) \quad t_2 = \frac{2mh}{c - v} \xi^0
\]

\[
z_4 = h \left( 1 + 2m \sigma \xi^1 \right) \quad t_4 = \frac{2mh}{c - v} \xi^1
\]

\[
z_6 = h \left( 1 + 2m \sigma \xi^2 \right) \quad t_6 = \frac{2mh}{c - v} \xi^2
\]

\[
...\]

Now, one could derive a general relation for the Transmitted contributions due to the “mth” reflection coming from ground:
\[ i_{tm} = \sum_{n=0}^{\infty} k_{b}^n k_{i}^{m-1} (1 + k_{i}) k_{c}^n (-k_{i}) (1 + k_{c}) I_{0} \left( t - \frac{2m h}{c-v} \xi^n - z - h \left( 1 + 2m \sigma^n \xi^n \right) \right) \left( \frac{h-z}{v} \right) \] (AI.6)

where

\[ h \left( 1 + 2m \sigma^n \xi^n \right) < z \leq z_{\text{MAX}} \quad m = 1, 2, 3, \ldots \]

**Al.2 Expanding of \( \sigma \) and \( \xi \) terms**

Fig. Al.3 - Geometric Relations for Deriving Expressions for \( \sigma \) and \( \xi \) Terms

First, let us consider “line A” ( \( z = ct + A \) )

\[ t_1 = \frac{2L_1}{c}; \quad z_1 = h; \]

at \( t_1 \rightarrow ct_1 + A = h \rightarrow A = h - ct_1 \rightarrow A = h - 2L_1 \rightarrow z = ct + h - 2L_1 \)

Let us now look at the intersection of “line A” and \( z = vt + h \)

at \( t_2 \rightarrow ct_2 + h - 2L_1 = vt_2 + h \rightarrow t_2(c - v) = 2L_1 \)

\[ t_2 = \frac{2L_1}{c-v}; \quad z_2 = vt_2 + h = ct_2 + h - 2L_1 \rightarrow z_2 = h + 2L_1 \sigma \]
\[ t_3 = 2t_2 - t_1 = \frac{4L}{c - v} - \frac{2L}{c} = \frac{4L(c - 2L(c - v))}{c(c - v)} = \frac{2L(c + v)}{c(c - v)} \]

\[ t_3 = \frac{2L}{c} \xi ; \quad z_3 = h \]

where \( \sigma = \frac{v}{c - v} \), and \( \xi = \frac{(c + v)}{(c - v)} \)

Now, let us consider “line B” \( (z = ct + B) \)

at \( t_3 \to ct_3 + B = h \to B = h - ct_3 \to B = h - 2L_1 \xi \to z = ct + h - 2L_1 \xi \)

Let us look at the intersection of “line B” and \( z = vt + h \)

at \( t_4 \to ct_4 + h - 2L_1 \xi = vt_4 + h \to t_4(c - v) = 2L_1 \xi \)

\[ t_4 = \frac{2L}{c - v} \xi ; \quad z_4 = vt_4 + h = ct_4 + h - 2L_1 \xi \to z_4 = h + 2L_1 \sigma \xi \]

in a similar manner it could be shown that:

\[ t_5 = \frac{2L}{c} \xi^2 ; \quad z_5 = h \]

\[ t_6 = \frac{2L}{c - v} \xi^2 ; \quad z_6 = h + 2L_1 \sigma \xi^2 \]

or in general:

\[ t_{2n+1} = \frac{2L}{c} \xi^n ; \quad z_{2n+1} = h \]

\[ t_{2n+2} = \frac{2L}{c - v} \xi^n ; \quad z_{2n+2} = h + 2L_1 \sigma \xi^n \]

for \( n = 0, 1, 2, \ldots \)
AI.3 Input Parameters of the Heidler Function

The injected “uncontaminated” current wave, used as input to the model, is approximated by a sum of two Heidler functions. All respective parameters used are presented in Table AI.1, and the produced current derivative and current waveshapes using Eq. V.3 are depicted in Fig. AI.4. Typical corresponding current waveforms that would be computed using the Single-Section Model at 474m and at 509m above ground level are shown in Fig. AI.5. These particular heights are of interest since there are permanently mounted and operating current sensing elements at 474m and at 509m above ground level levels. A large pool of recorded data exists and the calculated resultant waveforms could be easily compared to existing captured current and current derivative waveshapes.

<table>
<thead>
<tr>
<th>$I_{o1}$ [kA]</th>
<th>$\tau_{11}$ [µs]</th>
<th>$\tau_{21}$ [µs]</th>
<th>$n_1$ [-]</th>
<th>$I_{o2}$ [kA]</th>
<th>$\tau_{12}$ [µs]</th>
<th>$\tau_{22}$ [µs]</th>
<th>$n_2$ [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.7</td>
<td>0.25</td>
<td>2.5</td>
<td>2</td>
<td>6.5</td>
<td>2</td>
<td>230</td>
<td>2</td>
</tr>
</tbody>
</table>

Fig. AI.4 - Injected (“uncontaminated” by reflections) Heidler Current Derivative and Current
In the timeframe up to 5\(\mu s\), shown in Fig. A1.5 above, the reflection coming from the ground at about 3.5\(\mu s\) (for the 509\(m\) above ground computed current and also current derivative) is clearly seen. This reflection is positive, due to the positive reflection coefficient at ground level and adds up to the current to produce the maximum current peak value at the respective time.
Appendix II

Current Distribution for Three-Section Model of the CN Tower

Fig. AII.1 - Three-Sections Model

All.1 Assumptions

All assumptions previously described in “Appendix I” hold, except for the impedances and lengths of respective CN Tower sections. Let us re-emphasize that current contributions with peak values less than 1% of the input wave peak are neglected. The values of impedances used

\[ k_{b} = 0.65 \]

\[ k_{23} = -0.24 \]

\[ k_{12} = 0.3 \]

\[ k_{c} = -0.20 \]

\[ k_{1} = -0.30 \]

\[ Z_{1} \]

\[ Z_{2} \]

\[ Z_{3} \]

\[ Z_{g} \]

\[ H(553m) \]

\[ H_{1}(335m) \]

\[ H_{2}(365m) \]

\[ \text{GROUND LEVEL 0m} \]

\[ L_{1} \]

\[ L_{2} \]

\[ L_{3} \]

\[ \text{Lightning Channel} \]

\[ \text{Fully Ionized Lightning Channel} \]

Note, that the Lightning Channel is shown for the general case, when both Not-Fully Ionized (dashed line) and Fully Ionized (thick line) portions of the Lightning Channel exist, while in the instance of time when the Return-Stroke current is initiated only the Not-Fully Ionized part of the Lightning Channel is present.
in the Three-Section Model are listed in Table AII.1 below. The reflection coefficients are computed using the relations:

- \( k_c = \frac{(Z_{ch} - Z_{tch})}{(Z_{ch} + Z_{tch})}; \)
- \( k_t = \frac{(Z_1 - Z_{ch})}{(Z_1 + Z_{ch})}; \)
- \( k_{12} = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)}; \)
- \( k_{23} = \frac{(Z_2 - Z_3)}{(Z_2 + Z_3)}; \)
- \( k_b = \frac{(Z_3 - Z_g)}{(Z_3 + Z_g)}; \)

<table>
<thead>
<tr>
<th>Section #</th>
<th>Length [m]</th>
<th>Reflection coefficient</th>
<th>Impedance [Ω]</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC - not ionized</td>
<td>8000-z</td>
<td>-0.20</td>
<td>Z_{tch} 495</td>
</tr>
<tr>
<td>LC – fully ionized</td>
<td>z (variable)</td>
<td>-0.35</td>
<td>Z_{ch} 330</td>
</tr>
<tr>
<td>L_1</td>
<td>188</td>
<td>0.30</td>
<td>Z_1 160</td>
</tr>
<tr>
<td>L_2</td>
<td>30</td>
<td>-0.24</td>
<td>Z_2 86</td>
</tr>
<tr>
<td>L_3</td>
<td>335</td>
<td>0.65</td>
<td>Z_3 140</td>
</tr>
<tr>
<td>Z_g</td>
<td>-</td>
<td>Z_g 30</td>
<td></td>
</tr>
</tbody>
</table>

Table AII.1
Surge Impedances and Lengths for Three-Section Model

All.2 Current Distributions for Three-Section Model

Fig. AII.2 - Partial Lattice Diagram for the Three-Section Model of the CN Tower
Not all components seen in Fig. AII.2 above are used to come up with the respective current waveshapes at particular heights of interest. This is so, because many of them have very little influence upon the final computed current waveform and only slow down the computation without changing the result much. Only those contributions due to reflections and refractions at different locations along the structure of the tower and in the Lightning Channel that are higher than 1% in amplitude of the original injected wave are followed. All these considered contributions used in the Three-Section Model are shown by means of their corresponding graphical representations and analytical expressions below:

**AII.2.1 Current inside L1 (i = \sum_{k=1}^{14} i_k, for 365 < z \leq 553 m)**

\[ i_1 = \sum_{n=0}^{\infty} k_{12}^n k_{1}^n i_0 \left( t - \frac{h - z}{c} - \frac{2L_{11}}{c} n \right) \]  \hspace{1cm} (AII.1)

\[ i_2 = \sum_{n=0}^{\infty} k_{12}^{n+1} k_{1}^{n} i_0 \left( t - \frac{2L_{11} - h + z}{c} - \frac{2L_{13}}{c} n \right) \]  \hspace{1cm} (AII.2)

\[ i_3 = \sum_{n=0}^{\infty} k_{1}^{n} (1 + k_{12})^{n+1} k_{23}^{\frac{n+1}{1}} (1 - k_{12})^{n+1} i_0 \left( t - \frac{2L_{12} - h + z}{c} - \frac{2L_{12}}{c} n \right) \]  \hspace{1cm} (AII.3)

\[ i_4 = \sum_{n=0}^{\infty} k_{1}^{n+1} (1 + k_{12})^{n+1} k_{23}^{\frac{n+1}{1}} (1 - k_{12})^{n+1} i_0 \left( t - \frac{2L_{12} + h - z}{c} - \frac{2L_{12}}{c} n \right) \]  \hspace{1cm} (AII.4)
Fig. AII.5 - Current contributions $i_5$ & $i_6$

\[
i_5 = \sum_{n=0}^{\infty} k_5^n (1+k_{12})^{n+1} (1+k_{23})^{n+1} k_{12}^{n+1} (1-k_{23})^{n+1} (1-k_{12})^{n+1} i_0 \left( t - \frac{h+z}{c} - \frac{2h}{c} n \right) \]  
\text{(AII.5)}

\[
i_6 = \sum_{n=0}^{\infty} k_6^n (1+k_{12})^{n+1} (1+k_{23})^{n+1} k_{12}^{n+1} (1-k_{23})^{n+1} (1-k_{12})^{n+1} i_0 \left( t - \frac{3h-z}{c} - \frac{2h}{c} n \right) \]  
\text{(AII.6)}

Fig. AII.6 - Current contributions $i_7$ & $i_8$

\[
i_7 = 2 \sum_{n=0}^{\infty} k_{12}^{n+1} k_{12}^{n+2} (1-k_{12}) k_{23} (1+k_{12}) i_0 \left( t - \frac{2(L_{12} + L_1) - h + z}{c} - \frac{2L_1}{c} n \right) \]  
\text{(AII.7)}

\[
i_8 = 2 \sum_{n=0}^{\infty} k_{12}^{n+1} k_{12}^{n+2} (1-k_{12}) k_{23} (1+k_{12}) i_0 \left( t - \frac{2(L_{12} + L_1) + h - z}{c} - \frac{2L_1}{c} n \right) \]  
\text{(AII.8)}

Fig. AII.7 - Current contributions $i_9$ & $i_{10}$

\[
i_9 = 2 \sum_{n=0}^{\infty} k_{12}^{n+1} k_{12}^{n+1} (1+k_{12}) (1+k_{23}) (1-k_{12}) (1-k_{23}) i_0 \left( t - \frac{2(h+L_1) - h + z}{c} - \frac{2L_1}{c} n \right) \]  
\text{(AII.9)}

\[
i_{10} = 2 \sum_{n=0}^{\infty} k_{12}^{n+1} k_{12}^{n+2} (1+k_{12}) (1+k_{23}) (1-k_{12}) (1-k_{23}) i_0 \left( t - \frac{2(h+L_1) + h - z}{c} - \frac{2L_1}{c} n \right) \]  
\text{(AII.10)}
Fig. AII.8 - Current contribution $i_{11}$

\[
i_{11} = k_{12}(1+k_i)k_c(1-k_i)i_0(T - \frac{2L_1\xi + h - z}{c}) \tag{AII.11}
\]

Fig. AII.9 - Current contribution $i_{12}$

\[
i_{12} = (1+k_{12})k_{23}(1-k_{12})(1+k_i)k_c(1-k_i)i_0(T - \frac{2L_{12}\xi + h - z}{c}) \tag{AII.12}
\]

Fig. AII.10 - Current contributions $i_{13}$ & $i_{14}$

\[
i_{13} = (1+k_{12})(1+k_{23})k_h(1-k_{23})(1-k_{12})(1+k_i)k_c(1-k_i)i_0(T - \frac{2h\xi + h - z}{c}) \tag{AII.13}
\]

\[
i_{14} = (1+k_{12})(1+k_{23})k_h(1-k_{23})(1-k_{12})(1+k_i)k_c(1-k_i)k_{12}i_0(T - \frac{2h\xi + z - h + 2L_1}{c}) \tag{AII.14}
\]
All.2.2 Current inside L2 \( (i = \sum_{k=1}^{k} i_k + i_{15} + i_{16} + i_{17}, \text{ for } 335 < z \leq 365m) \)

\[
i_1 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23}) k^n_b (1 - k_{23}) \left(1 - k_{12}\right)^n k^n_i i_0 \left(t - \frac{h - z}{c} - \frac{2h}{c} n\right) \quad \text{(AII.15)}
\]

\[
i_2 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} k^m_b (1 - k_{23})^{m+1} (1 - k_{12})^{m+1} k^m_i i_0 \left(t - \frac{h + z}{c} - \frac{2h}{c} n\right) \quad \text{(AII.16)}
\]

\[
i_3 = \sum_{n=0}^{\infty} k_{12} k_1 (1 + k_{12})^{n+1} (1 + k_{23}) k^n_b (1 - k_{23}) \left(1 - k_{12}\right)^n k^n_i i_0 \left(t - \frac{2L_1 + h - z}{c} - \frac{2h}{c} n\right) \quad \text{(AII.17)}
\]

\[
i_4 = \sum_{n=0}^{\infty} k_{12} k_1 (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} k^m_b (1 - k_{23})^{m+1} (1 - k_{12})^{m+1} k^m_i i_0 \left(t - \frac{2L_1 + h + z}{c} - \frac{2h}{c} n\right) \quad \text{(AII.18)}
\]

\[
i_5 = \sum_{n=0}^{\infty} (1 + k_{12}) k^{n+1}_{23} (-k_{12})^n i_0 \left(t - \frac{h + z - 2L_3}{c} - \frac{2L_2}{c} n\right) \quad \text{(AII.19)}
\]
\[ i_6 = \sum_{n=0}^{\infty} (1+k_{12})k_{23}^{n+1}(-k_{12})^{n+1}i_0\left(t - \frac{h + 2L_2 - z}{c} = \frac{2L_2}{c}n\right) \]  

(AII.20)

\[ i_7 = \sum_{n=0}^{\infty} (1+k_{12})(1+k_{23})k_2(1-k_{23})(-k_{12})^{n+1}k_{23}^{n}i_0\left(t - \frac{h + 2L_{23} - z}{c} = \frac{2L_2}{c}n\right) \]  

(AII.21)

\[ i_8 = \sum_{n=0}^{\infty} (1+k_{12})(1+k_{23})k_2(1-k_{23})(-k_{12})^{n+1}k_{23}^{n+1}i_0\left(t - \frac{h + 2L_2 + z}{c} = \frac{2L_2}{c}n\right) \]  

(AII.22)

\[ i_{15} = k_{12}(1+k_i)k_c(1-k_i)(1+k_{12})i_0\left(t - \frac{2L_i \xi + h - z}{c}\right) \]  

(AII.23)

\[ i_{16} = (1+k_{12})^2k_{23}(1-k_{12})(1+k_i)k_c(1-k_i)i_0\left(t - \frac{2L_{12} \xi + h - z}{c}\right) \]  

(AII.24)
Fig. AII.17 - Current contributions $i_{17}$

$$i_{17} = (1 + k_{12})^2 (1 + k_{32}) k_b (1 - k_{23}) (1 - k_{12}) k_c (1 - k_1) i_0 \left( t - \frac{2h \xi + h - z}{c} \right)$$ (AII.25)

All.2.3 Current inside L3 ($i = \sum_{k=1}^{12} i_k + \sum_{m=1}^{23} i_m$, for $0 < z \leq 335m$)

Fig. AII.18 - Current contributions $i_1$ & $i_2$

$$i_1 = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{32}) k_b^n (-k_{23})^n i_0 \left( t - \frac{h - z}{c} - \frac{2L_3}{c} n \right)$$ (AII.26)

$$i_2 = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{32}) k_b^{n+1} (-k_{23})^{n+1} i_0 \left( t - \frac{h + z}{c} - \frac{2L_3}{c} n \right)$$ (AII.27)

Fig. AII.19 - Current contributions $i_3$ & $i_4$

$$i_3 = \sum_{n=0}^{\infty} (1 + k_{12}) k_{23} (-k_{12})^{n+1} (1 + k_{23}) k_b^n i_0 \left( t - \frac{h + 2L_2 - z}{c} - \frac{2L_3}{c} n \right)$$ (AII.28)
\[ i_4 = \sum_{n=0}^{\infty} (1+k_{12}) k_{23} (-k_{12})^{n+1} (1+k_{23}) k_{b}^{n+1} i_0 \left( t - \frac{h + 2L_2 + z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.29)

Fig. AII.20 - Current contributions \( i_5 \) & \( i_6 \)

\[ i_5 = \sum_{n=0}^{\infty} k_{12} k_r (1+k_{12}) (1+k_{23}) k_{b}^{\nu} (-k_{23})^{\nu} i_0 \left( t - \frac{h + 2L_1 - z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.30)

\[ i_6 = \sum_{n=0}^{\infty} k_{12} k_r (1+k_{12}) (1+k_{23}) k_{b}^{\nu+1} (-k_{23})^{\nu} i_0 \left( t - \frac{h + 2L_1 + z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.31)

Fig. AII.21 - Current contributions \( i_7 \) & \( i_8 \)

\[ i_7 = \sum_{n=0}^{\infty} (1+k_{12})^2 k_{23} (1-k_{12}) k_r (1+k_{23}) k_{b}^{\nu} (-k_{23})^{\nu} i_0 \left( t - \frac{h + 2L_1 - z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.32)

\[ i_8 = \sum_{n=0}^{\infty} (1+k_{12})^2 k_{23} (1-k_{12}) k_r (1+k_{23}) k_{b}^{\nu+1} (-k_{23})^{\nu} i_0 \left( t - \frac{h + 2L_1 + z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.33)

Fig. AII.22 - Current contributions \( i_9 \) & \( i_{10} \)

\[ i_9 = \sum_{n=0}^{\infty} k_{12}^2 k_r (1+k_{12}) (1+k_{23}) k_{b}^{\nu} (-k_{23})^{\nu} i_0 \left( t - \frac{h + 4L_1 - z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.34)

\[ i_{10} = \sum_{n=0}^{\infty} k_{12}^2 k_r (1+k_{12}) (1+k_{23}) k_{b}^{\nu+1} (-k_{23})^{\nu} i_0 \left( t - \frac{h + 4L_1 + z}{c} - \frac{2L_3}{c} n \right) \]  

(AII.35)
Fig. AII.23 - Current contributions $i_{11} & i_{12}$

$$i_{11} = \sum_{n=0}^{\infty} (1 + k_{12})^{n+2} (1 + k_{23})^{n+2} k_n^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_i^{n+1} i_0 \left( t - \frac{3h - z - 2h}{c}n \right) \quad (AII.36)$$

$$i_{12} = \sum_{n=0}^{\infty} (1 + k_{12})^{n+2} (1 + k_{23})^{n+2} k_n^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_i^{n+1} i_0 \left( t - \frac{3h + z - 2h}{c}n \right) \quad (AII.37)$$

Fig. AII.24 - Current contributions $i_{18} & i_{19}$

$$i_{18} = k_{12} (1 + k_i) k_c (1 - k_i) (1 + k_{12}) (1 + k_{23}) i_0 \left( t - \frac{2L_{15} + h - z}{c} \right) \quad (AII.38)$$

$$i_{19} = k_{12} (1 + k_i) k_c (1 - k_i) (1 + k_{12}) (1 + k_{23}) k_h i_0 \left( t - \frac{2L_{15} + h + z}{c} \right) \quad (AII.39)$$

Fig. AII.25 - Current contributions $i_{20} & i_{21}$
\[ i_{20} = (1 + k_{12})^2 k_{23} (1 - k_{12})(1 + k_i)k_c (1 - k_i) (1 + k_{23})i_0 \left( t - \frac{2L_{12} \xi + h - z}{c} \right) \]  \hspace{1cm} (AII.40)

\[ i_{21} = (1 + k_{12})^2 k_{23} (1 - k_{12})(1 + k_i)k_c (1 - k_i) (1 + k_{23})k_0i_0 \left( t - \frac{2L_{12} \xi + h + z}{c} \right) \]  \hspace{1cm} (AII.41)

\[ i_{22} = (1 + k_{12})^2 (1 + k_{23})^2 k_h (1 - k_{23})(1 - k_{12})(1 + k_i)k_c (1 - k_i)k_0i_0 \left( t - \frac{2h \xi + h - z}{c} \right) \]  \hspace{1cm} (AII.42)

\[ i_{23} = (1 + k_{12})^2 (1 + k_{23})^2 k_h^2 (1 - k_{23})(1 - k_{12})(1 + k_i)k_c (1 - k_i)k_0i_0 \left( t - \frac{2h \xi + h + z}{c} \right) \]  \hspace{1cm} (AII.43)

### AII.3 Derivation of Relations for the Reflections Taking Place within the Lightning Channel

#### AII.3.1 Channel Base Current Propagating upwards & Transmitted Components in the Lightning Channel

\[ \text{Fig. AII.26} - \text{Current contributions } i_{22} \text{ & } i_{23} \]

\[ \text{Fig. AII.27} - \text{Current contributions } i_{ch}, i_{T1}, i_{T2}, i_{T3}, i_{T3}, \text{ & } i_{T5} \]
\[ i_{ch} = \frac{Z_1}{Z_{th}} i_0 \left( t - \frac{z-h}{v} \right) e^{\frac{h-z}{v}} \]  
(AII.44)

for \( z \geq 553 \text{ m} \)

\[ i_{r1} = k_{12} (1 + k_i) (1 + k_c) i_0 \left( t - \frac{2L_i}{c-v} - \frac{z-(h+2L_i\sigma)}{v} \right) e^{\frac{h-z}{v}} \]  
(AII.45)

for \( h + 2L_i\sigma < z < \infty \)

\[ i_{r2} = \left(1 + k_{12}\right) k_{23} (1-k_{12})(1+k_i)(1+k_c) i_0 \left( t - \frac{2L_{12}}{c-v} - \frac{z-(h+2L_{12}\sigma)}{v} \right) e^{\frac{h-z}{v}} \]  
(AII.46)

for \( h + 2L_{12}\sigma < z < \infty \)

\[ i_{r3} = \left(1 + k_{12}\right) (1+k_{23}) k_h (1-k_{12})(1-k_{23})(1+k_i)(1+k_c) 
\quad \quad i_0 \left( t - \frac{2h}{c-v} - \frac{z-(h+2h\sigma)}{v} \right) e^{\frac{h-z}{v}} \]  
(AII.47)

for \( h + 2h\sigma < z < \infty \)

\[ i_{r4} = \left(1 + k_{12}\right) k_{23} (1-k_{12}) k_h k_{12} (1+k_i)(1+k_c) 
\quad \quad i_0 \left( t - \frac{2(L_{12}+L_i)}{c-v} - \frac{z-(h+2(L_{12}+L_i)\sigma)}{v} \right) e^{\frac{h-z}{v}} \]  
(AII.48)

for \( h + 2(L_{12}+L_i)\sigma < z < \infty \)

\[ i_{r5} = 2 \left(1 + k_{12}\right) (1+k_{23}) k_h (1-k_{23})(1-k_{12}) k_h k_{12} (1+k_i)(1+k_c) 
\quad \quad i_0 \left( t - \frac{2(h+L_i)}{c-v} - \frac{z-(h+2(h+L_i)\sigma)}{v} \right) e^{\frac{h-z}{v}} \]  
(AII.49)

for \( h + 2(h+L_i)\sigma < z < \infty \)

All.3.2 Internal Components in the LC
Fig. AII.28 - Current contributions $i_1$ & $i_2$

\[ i_1 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_h (1 - k_{23})(1 - k_{12})(1 + k_i)k_c^n (-k_i)^n \]

\[ i_0 \left( t - \frac{2h}{c} \xi^n - \frac{z - h}{c} \right) \]  \hspace{1cm} (AII.50)

\[ i_2 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_h (1 - k_{23})(1 - k_{12})(1 + k_i)k_c^{n+1} (-k_i)^n \]

\[ i_0 \left( t - \frac{2h}{c - v} \xi^n - \frac{h(1 + 2\sigma\xi^n) - z}{c} \right) \]  \hspace{1cm} (AII.51)

for $h < z < h(1 + 2\sigma\xi^n)$,

Fig. AII.29 - Current contributions $i_3$ & $i_4$

\[ i_3 = \sum_{n=0}^{\infty} (1 + k_{12})k_{23} (1 - k_{12})(1 + k_i)k_c^n (-k_i)^n \]

\[ i_0 \left( t - \frac{2L_{12}}{c} \xi^n - \frac{z - h}{c} \right) \]  \hspace{1cm} (AII.52)

\[ i_4 = \sum_{n=0}^{\infty} (1 + k_{12})k_{23} (1 - k_{12})(1 + k_i)k_c^{n+1} (-k_i)^n \]

\[ i_0 \left( t - \frac{2L_{12}}{c - v} \xi^n - \frac{h + 2L_{12}\sigma\xi^n - z}{c} \right) \]  \hspace{1cm} (AII.53)
for $h < z < h + 2L_{12} \sigma_\xi^n$, 

$$i_5 = \sum_{n=0}^{\infty} k_{12} (1 + k_i) k_c^n (-k_i)^n i_0 \left( t - \frac{2L_1}{c} \xi^n = \frac{z - h}{c} \right)$$ (AII.54)

$$i_6 = \sum_{n=0}^{\infty} k_{12} (1 + k_i) k_c^{n+1} (-k_i)^n i_0 \left( t - \frac{2L_1}{c - v} \xi^n = \frac{h + 2L_1 \sigma_\xi^n - z}{c} \right)$$ (AII.55)

for $h < z < h + 2L_1 \sigma_\xi^n$, 

$$i_7 = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23}) k_h (1 - k_{23}) (1 - k_{12}) k_i k_{12} (1 + k_i) k_c^n (-k_i)^n i_0 \left( t - \frac{2(h + L_1)}{c} \xi^n = \frac{z - h}{c} \right)$$ (AII.56)
\[ i_n = \sum_{n=0}^{\infty} (1 + k_{i2}) k_{12} k_{i2} (1 - k_{i2}) k_{1} k_{c} (-k_{r})^n \]  

\[ j_0 \left( t - \frac{2(h + L_{1})}{c - v} \xi^n - \frac{h + 2(h + L_{1}) \sigma \xi^n - z}{c} \right) \]  

(EII.57)

for \( h < z < h + 2(h + L_{1}) \sigma \xi^n \),

\[ i_9 = \sum_{n=0}^{\infty} (1 + k_{i2}) k_{12} (1 - k_{i2}) k_{i2} (1 + k_{1}) k_{c} \xi^n (-k_{r})^n \]  

\[ j_0 \left( t - \frac{2(L_{12} + L_{1})}{c} \xi^n - \frac{z - h}{c} \right) \]  

(EII.58)

\[ i_{10} = \sum_{n=0}^{\infty} (1 + k_{i2}) k_{12} (1 - k_{i2}) k_{1} k_{c} \xi^n (-k_{r})^n \]  

\[ j_0 \left( t - \frac{2(L_{12} + L_{1})}{c - v} \xi^n - \frac{h + 2(L_{12} + L_{1}) \sigma \xi^n - z}{c} \right) \]  

(EII.59)

for \( h < z < h + 2(L_{12} + L_{1}) \sigma \xi^n \),

Using the Three-Section Model and the same injection current shown in Appendix A in Fig. A1.4, the following resultant waveforms at respective levels of interest are calculated:
It is worth noting that the waveshapes are influenced by the Skypod. In Fig. AII.33 the first most pronounced influence is well noticed in the current derivative curve, representing the computed signal at 474 m above ground level (thick line), at 1 µs. At that time, there is no change in the corresponding curve shown in Fig. AI.5, because the presence of the Skypod in the Single-Section Model is neglected.
Appendix III

Current Distribution for Five-Section Model of the CN Tower

Fig. AIII.1 - Five-Sections Model

AIII.1 Assumptions

Similar assumptions already shown in Appendices I and II are adopted. Note again that current contributions less than 1% of the incoming current wave are neglected. The respective values of the impedances together with corresponding section lengths and calculated values for the

---

Note, that the Lightning Channel is shown for the general case, when both Not-Fully Ionized (dashed line) and Fully Ionized (thick line) portions of the Lightning Channel exist, while in the instance of time when the Return-Stroke current is initiated only the Not-Fully Ionized part of the Lightning Channel is present.
reflection coefficients pertinent to the Five-Section Model are listed in Table AIII.1 below. The reflection coefficients are visualized in Fig. AIII.1, where their “direction” is assigned. They are computed using the relations:

\[
\begin{align*}
&k_c = \frac{(Z_{ch} - Z_{tch})}{(Z_{ch} + Z_{tch})}; \\
&k_l = \frac{(Z_1 - Z_{ch})}{(Z_1 + Z_{ch})}; \\
&k_{12} = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)}; \\
&k_{23} = \frac{(Z_2 - Z_3)}{(Z_2 + Z_3)}; \\
&k_{34} = \frac{(Z_3 - Z_4)}{(Z_3 + Z_4)}; \\
&k_{45} = \frac{(Z_4 - Z_5)}{(Z_4 + Z_5)}; \\
&k_b = \frac{(Z_5 - Z_g)}{(Z_5 + Z_g)};
\end{align*}
\]

Table AIII.1

<table>
<thead>
<tr>
<th>Section #</th>
<th>Length [m]</th>
<th>Reflection coefficient</th>
<th>Impedance [Ω]</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC - not ionized</td>
<td>8000-z</td>
<td>(k_c) -0.20</td>
<td>(Z_{ch}) 495</td>
</tr>
<tr>
<td>LC – fully ionized</td>
<td>z (variable)</td>
<td>(k_l) -0.30</td>
<td>(Z_{ch}) 330</td>
</tr>
<tr>
<td>L_1</td>
<td>100</td>
<td>(k_{12}) 0.21</td>
<td>(Z_1) 160</td>
</tr>
<tr>
<td>L_2</td>
<td>9</td>
<td>(k_{23}) -0.14</td>
<td>(Z_2) 105</td>
</tr>
<tr>
<td>L_3</td>
<td>79</td>
<td>(k_{34}) 0.24</td>
<td>(Z_3) 140</td>
</tr>
<tr>
<td>L_4</td>
<td>30</td>
<td>(k_{45}) -0.17</td>
<td>(Z_4) 86</td>
</tr>
<tr>
<td>L_5</td>
<td>335</td>
<td>(k_b) 0.60</td>
<td>(Z_g) 30</td>
</tr>
<tr>
<td>Z_g</td>
<td>-</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

AIII.2 Current Distributions for Five-Section Model

Once again, in the Lattice diagram seen below all components are shown, however, in the following figures and corresponding expressions only the ones having greater than 1% of the originally injected current wave are considered.
AIII.2.1 Current inside L1 \((453 < z \leq 553m)\)

\[ i_1 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} (1 + k_{45})^{n+1} k_b^{n+1} (1 - k_{45})^{n+1} \]
\[ \cdot (1 - k_{34})^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_{12}^{n+1} t_0 \left( t - \frac{h + z - 2h_n}{c} \right) \]  

\[ i_2 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} (1 + k_{45})^{n+1} k_b^{n+1} (1 - k_{45})^{n+1} \]
\[ \cdot (1 - k_{34})^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_1^{n+1} t_0 \left( t - \frac{3h - z - 2h_n}{c} \right) \]  

Fig. AIII.2 - Partial Lattice Diagram for the Five-Section Model of the CN Tower

Fig. AIII.3 - Current contributions \(i_1\) & \(i_2\)
Fig. AIII.4 - Current contributions $i_3$ & $i_4$

\[ i_3 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} k_{34}^{n+1} (1 - k_{23})^{n+1} (1 - k_{34})^{n+1} \left( (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_t^{n+1} \right) \left( t - \frac{2L_{123} - h + z}{c} - \frac{2L_{123}}{c} n \right) \]  

(AIII.3)

\[ i_4 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} k_{45}^{n+1} (1 - k_{34})^{n+1} \left( (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_t^{n+1} \right) \left( t - \frac{2L_{123} + h - z}{c} - \frac{2L_{123}}{c} n \right) \]  

(AIII.4)

Fig. AIII.5 - Current contributions $i_5$ & $i_6$

\[ i_5 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} k_{34}^{n+1} (1 - k_{23})^{n+1} \left( (1 - k_{12})^{n+1} k_t^{n+1} \right) \left( t - \frac{2L_{123} - h + z}{c} - \frac{2L_{123}}{c} n \right) \]  

(AIII.5)

\[ i_6 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} k_{34}^{n+1} (1 - k_{23})^{n+1} \left( (1 - k_{12})^{n+1} k_t^{n+1} \right) \left( t - \frac{2L_{123} + h - z}{c} - \frac{2L_{123}}{c} n \right) \]  

(AIII.6)
Fig. AIII.6 - Current contributions $i_7$ & $i_8$

\[ i_7 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} k_{23}^{n+1} (1 - k_{12})^{n+1} k_i^{n+1} i_0 \left( t - \frac{2L_{12} - h + z}{c} - \frac{2L_{12}}{c} n \right) \]  
\[ (AIII.7) \]

\[ i_8 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} k_{23}^{n+1} (1 - k_{12})^{n+1} k_i^{n+1} i_0 \left( t - \frac{2L_{12} + h - z}{c} - \frac{2L_{12}}{c} n \right) \]  
\[ (AIII.8) \]

Fig. AIII.7 - Current contributions $i_9$ & $i_{10}$

\[ i_9 = \sum_{n=0}^{\infty} k_{12} k_i^{n} i_0 \left( t - \frac{h - z}{c} - \frac{2L_1}{c} n \right) \]  
\[ (AIII.9) \]

\[ i_{10} = \sum_{n=0}^{\infty} k_{12}^{n+1} k_i^{n+1} i_0 \left( t - \frac{2L_1 - h + z}{c} - \frac{2L_1}{c} n \right) \]  
\[ (AIII.10) \]

Fig. AIII.8 - Current contributions $i_{11}$ & $i_{12}$

\[ i_{11} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b(1 - k_{45})(1 - k_{34}) \] 
\[ (1 - k_{23})(1 - k_{12})k_i^{n+1} k_{12}^{n+1} i_0 \left( t - \frac{2(h + L_1) - h + z}{c} - \frac{2L_1}{c} n \right) \]  
\[ (AIII.11) \]
\[ i_{12} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45}) k_S (1 - k_{45})(1 - k_{34}) \]
\[ \cdot (1 - k_{23})(1 - k_{12}) k_{i_2}^{n+2} k_{i_2}^{n+1} i_0 \left( t - \frac{2(h + L_1) + h - z}{c} - \frac{2L_1}{c} n \right) \]  
(AIII.12)

**Fig. AIII.9** - Current contributions \( i_{13} \) & \( i_{14} \)

\[ i_{13} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34}) k_{45} (1 - k_{34})(1 - k_{23}) \]
\[ \cdot (1 - k_{12}) k_{i_3}^{n+1} k_{i_2}^{n+1} i_0 \left( t - \frac{2(L_{1234} + L_1) - h + z}{c} - \frac{2L_1}{c} n \right) \]  
(AIII.13)

\[ i_{14} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34}) k_{45} (1 - k_{34})(1 - k_{23}) \]
\[ \cdot (1 - k_{12}) k_{i_4}^{n+2} k_{i_2}^{n+1} i_0 \left( t - \frac{2(L_{1234} + L_1) + h - z}{c} - \frac{2L_1}{c} n \right) \]  
(AIII.14)

**Fig. AIII.10** - Current contributions \( i_{15} \) & \( i_{16} \)

\[ i_{15} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23}) k_{34} (1 - k_{23})(1 - k_{12}) k_{i_5}^{n+1} k_{i_2}^{n+1} \]
\[ i_0 \left( t - \frac{2(L_{123} + L_1) - h + z}{c} - \frac{2L_1}{c} n \right) \]  
(AIII.15)

\[ i_{16} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23}) k_{34} (1 - k_{23})(1 - k_{12}) k_{i_6}^{n+2} k_{i_2}^{n+1} \]
\[ i_0 \left( t - \frac{2(L_{123} + L_1) + h - z}{c} - \frac{2L_1}{c} n \right) \]  
(AIII.16)
Fig. AIII.11 - Current contributions \( i_{17} \) & \( i_{18} \)

\[
i_{17} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) k_{23} (1 - k_{12}) k_i^{n+1} k_{12}^{n+1}
\]

\[
\alpha_{0} \left( t - \frac{2(L_{12} + L_{1}) - h + z}{c} - \frac{2L_{1}}{c} n \right)
\]

(AIII.17)

\[
i_{18} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) k_{23} (1 - k_{12}) k_i^{n+2} k_{12}^{n+1}
\]

\[
\alpha_{0} \left( t - \frac{2(L_{12} + L_{1}) + h - z}{c} - \frac{2L_{1}}{c} n \right)
\]

(AIII.18)

Fig. AIII.12 - Current contributions \( i_{r1} \), \( i_{r2} \), \( i_{r3} \), \( i_{r4} \), & \( i_{r5} \)

\[
i_{r1} = k_{12} (1 + k_{r}) k_{c} (1 - k_{r}) \alpha_{0} \left( t - \frac{2L_{1} \xi + h - z}{c} \right)
\]

(AIII.19)

\[
i_{r2} = (1 + k_{12}) k_{23} (1 - k_{12}) (1 + k_{r}) k_{c} (1 - k_{r}) \alpha_{0} \left( t - \frac{2L_{123} \xi + h - z}{c} \right)
\]

(AIII.20)

\[
i_{r3} = (1 + k_{12}) (1 + k_{23}) k_{34} (1 - k_{23}) (1 - k_{12}) (1 + k_{r}) k_{c} (1 - k_{r}) \alpha_{0} \left( t - \frac{2L_{123} \xi + h - z}{c} \right)
\]

(AIII.21)

\[
i_{r4} = (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) k_{45} (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) (1 + k_{r}) k_{c} (1 - k_{r}) \alpha_{0} \left( t - \frac{2L_{1234} \xi + h - z}{c} \right)
\]

(AIII.22)
\[ i_{r5} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})(1 - k_{12})(1 - k_{23})(1 - k_{34})(1 - k_{45}) \]
\[ .(1 - k_{12})(1 + k_{1})(1 - k_{1})j_0 \left( t - \frac{2h\xi + h - z}{c} \right) \]  
\[ \text{(AIII.23)} \]

### AIII.2.1.1 Additional Currents

**Fig. AIII.13 - Current contribution \( i_{A1} \)**

\[ i_{A1} = (1 + k_{12})^2(1 + k_{23})^2 k_{34}^2(1 - k_{23})^2(1 - k_{12})^2 k_{1} j_0 \left( t - \frac{4L_{213} - h + z}{c} \right) \]  
\[ \text{(AIII.24)} \]

**Fig. AIII.14 - Current contribution \( i_{A2} \)**

\[ i_{A2} = (1 + k_{12})(1 + k_{23})^2 k_{34}^2(1 - k_{23})^2(-k_{12})(1 - k_{12})j_0 \left( t - \frac{2L_{123} + 2L_{23} - h + z}{c} \right) \]  
\[ \text{(AIII.25)} \]

**Fig. AIII.15 - Current contribution \( i_{A3} \)**

\[ i_{A3} = (1 + k_{12})^2(1 + k_{23})k_{34}(1 - k_{23})^2k_{1}k_{23}j_0 \left( t - \frac{2L_{123} + 2L_{12} - h + z}{c} \right) \]  
\[ \text{(AIII.26)} \]
Fig. AIII.16 - Current contributions $i_{A4}$, $i_{A5}$, & $i_{A6}$

$$i_{A4} = (1 + k_{12})^2 (1 + k_{23})^2 k_{34}(1 - k_{23})^2 (1 - k_{12})^2 k_{45}(1 + k_{34})$$

$$
(1 + k_{45})k_{b}(1 - k_{45})(1 - k_{34})k_{b}i_{0} \left( t - \frac{2L_{123} + h + z}{c} \right) \quad \text{(AIII.27)}
$$

$$i_{A5} = (1 + k_{12})^2 (1 + k_{23})^2 k_{34}(1 - k_{23})^2 (1 - k_{12})^2 k_{45}(1 + k_{34})$$

$$
(1 + k_{45})k_{b}(1 - k_{45})(1 - k_{34})k_{b}i_{0} \left( t - \frac{2L_{123} + 3h - z}{c} \right) \quad \text{(AIII.28)}
$$

$$i_{A6} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34})^2 k_{45}(1 - k_{34})^2 (1 - k_{23})^2$$

$$
(1 - k_{12})^2 k_{1}(1 + k_{45})k_{b}(1 - k_{45})i_{0} \left( t - \frac{2L_{123} + h + z}{c} \right) \quad \text{(AIII.29)}
$$

Fig. AIII.17 - Current contribution $i_{A7}$

$$i_{A7} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_{b}(1 - k_{45})$$

$$
(1 - k_{34})(1 - k_{23})(1 - k_{12})k_{23}(1 - k_{12})k_{0} \left( t - \frac{h + 2L_{2} + z}{c} \right) \quad \text{(AIII.30)}
$$

Fig. AIII.18 - Current contribution $i_{A8}$
\[ i_{A8} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45}) k_b (1 - k_{45}) \]
\[ (1 - k_{34})(- k_{23}) k_b (1 - k_{23})(1 - k_{12}) k_0 \left( t = \frac{h + 2L_3 + z}{c} \right) \]  
(AIII.31)

**Fig. AIII.19 - Current contributions \( i_{A9} \) & \( i_{A10} \)**

\[ i_{A9} = (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 (1 + k_{45})^2 k_b^2 (1 - k_{45})^2 \]
\[ (1 - k_{34})^2 (1 - k_{23})(1 - k_{23})(1 - k_{12}) k_0 \left( t = \frac{h + 2L_{345} + z}{c} \right) \]  
(AIII.32)

\[ i_{A10} = (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 (1 + k_{45})^2 k_b^2 (1 - k_{45})^2 \]
\[ (1 - k_{34})^2 (1 - k_{23})(1 - k_{23})(1 - k_{12}) k_0 \left( t = \frac{3h + 2L_{345} - z}{c} \right) \]  
(AIII.33)

**Fig. AIII.20 - Current contribution \( i_{A11} \)**

\[ i_{A11} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45}) k_b (1 - k_{45})(- k_{34}) \]
\[ k_{45}(1 - k_{34})(1 - k_{23})(1 - k_{12}) k_0 \left( t = \frac{h + 2L_4 + z}{c} \right) \]  
(AIII.34)

**Fig. AIII.21 - Current contributions \( i_{A12} \) & \( i_{A13} \)**
\[ i_{a12} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})^2 k_b^2 (1 - k_{45})^2 \]
\[ \cdot (-k_{34})(1 - k_{34})(1 - k_{23})(1 - k_{12})i_0 \left( t - \frac{h + 2L_{45} + z}{c} \right) \] (AIII.35)

\[ i_{a13} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})^2 k_b^2 (1 - k_{45})^2 \]
\[ \cdot (-k_{34})(1 - k_{34})(1 - k_{23})(1 - k_{12})i_0 \left( t - \frac{3h + 2L_{45} - z}{c} \right) \] (AIII.36)

Fig. AIII.22 - Current contributions \( i_{A14}, \) & \( i_{A15} \)

\[ i_{a14} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b^2 (-k_{45}) \]
\[ \cdot (1 - k_{45})(1 - k_{34})(1 - k_{23})(1 - k_{12})i_0 \left( t - \frac{h + 2L_5 + z}{c} \right) \] (AIII.37)

\[ i_{a15} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b^2 (-k_{45}) \]
\[ \cdot (1 - k_{45})(1 - k_{34})(1 - k_{23})(1 - k_{12})i_0 \left( t - \frac{3h + 2L_5 - z}{c} \right) \] (AIII.38)

Fig. AIII.23 - Current contribution \( i_{A16} \)

\[ i_{a16} = (1 + k_{12})(1 + k_{23})^2 (1 + k_{34})(1 + k_{45})k_b (1 - k_{45}) \]
\[ \cdot (1 - k_{34})(1 - k_{23})^2 (-k_{12})k_{34}(1 - k_{12})i_0 \left( t - \frac{h + 2L_{23} + z}{c} \right) \] (AIII.39)
Fig. AIII.24 - Current contribution $i_{A17}$

\[ i_{A17} = (1 + k_{12})(1 + k_{23})^2 (1 + k_{34})^2 (1 + k_{45})^2 k_b^2 (1 - k_{45})^2 
\times (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) (1 - k_{12}) k_0 \left( t - \frac{h + 2L_{2345} + z}{c} \right) \]  

\[ (AIII.40) \]

Fig. AIII.25 - Current contributions $i_{A18, A19}$

\[ i_{A18} = (1 + k_{12})(1 + k_{23})^2 (1 + k_{34})^2 (1 + k_{45})^2 k_b^2 (1 - k_{45})^2 
\times (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) (1 - k_{12}) k_0 \left( t - \frac{h + 2L_{2345} + z}{c} \right) \]  

\[ (AIII.41) \]

\[ i_{A19} = (1 + k_{12})(1 + k_{23})^2 (1 + k_{34})^2 (1 + k_{45})^2 k_b^2 (1 - k_{45})^2 
\times (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) (1 - k_{12}) k_0 \left( t - \frac{3h + 2L_{2345} - z}{c} \right) \]  

\[ (AIII.42) \]

AIII.2.2 Current inside L2 ($444 < z \leq 453m$)

Fig. AIII.26 - Current contributions $i_1$ & $i_2$
\[ i_1 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{34})^n (1 + k_{45})^n k_b^n (1 - k_{45})^n \]
\[ (1 - k_{34})^n (1 - k_{23})^n (1 - k_{12})^n k_i^n i_0 \left( t - \frac{h - z}{c} - \frac{2h}{c} \right) \]

\[ i_2 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} (1 + k_{45})^{n+1} k_b^{n+1} (1 - k_{45})^{n+1} \]
\[ (1 - k_{34})^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_i^{n+1} i_0 \left( t - \frac{h + z}{c} - \frac{2h}{c} \right) \]

Fig. AIII.27 - Current contributions \( i_3 \) \& \( i_4 \)

\[ i_3 = \sum_{n=0}^{\infty} (1 + k_{12})^n k_{23}^{n+1} (-k_{12})^n \left( t - \frac{h + z - 2L_{45}}{c} - \frac{2L_z}{c} \right) \]

\[ i_4 = \sum_{n=0}^{\infty} (1 + k_{12})^n k_{23}^{n+1} (-k_{12})^{n+1} \left( t - \frac{h + 2L_z - z}{c} - \frac{2L_z}{c} \right) \]

Fig. AIII.28 - Current contributions \( i_5 \) \& \( i_6 \)

\[ i_5 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_{34} (1 - k_{23})(-k_{12})^n k_{23}^n i_0 \left( t - \frac{h + z - 2L_{45}}{c} - \frac{2L_z}{c} \right) \]

\[ i_6 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_{34} (1 - k_{23})(-k_{12})^{n+1} k_{23}^{n+1} i_0 \left( t - \frac{h + 2L_{23} - z}{c} - \frac{2L_z}{c} \right) \]
Fig. AIII.29 - Current contributions $i_7$ & $i_8$

\[
i_7 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})k_{45}(1 - k_{34})(1 - k_{23})(-k_{12})^n k_{23}^n
\]

\[
J_0\left(t = \frac{h + z - 2L_5}{c} - \frac{2L_2}{c} n\right)
\]

(AIII.49)

\[
i_8 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})k_{45}(1 - k_{34})(1 - k_{23})(-k_{12})^{n+1} k_{23}^n
\]

\[
J_0\left(t = \frac{h + 2L_{2345} - z}{c} - \frac{2L_2}{c} n\right)
\]

(AIII.50)

Fig. AIII.30 - Current contributions $i_9$ & $i_{10}$

\[
i_9 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_h(1 - k_{45})(1 - k_{34}) k_{23} \left(1 - k_{23}\right)^{n+1} i_0 \left(t = \frac{h + 2L_{2345} - z}{c} - \frac{2L_2}{c} n\right)
\]

(AIII.51)

\[
i_{10} = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_h(1 - k_{45})(1 - k_{34}) k_{23} \left(1 - k_{23}\right)^{n+1} i_0 \left(t = \frac{h + z - 2L_2}{c} - \frac{2L_2}{c} n\right)
\]

(AIII.52)
Fig. AIII.31 - Current contributions $i_{11}$ & $i_{12}$

\[
i_{11} = \sum_{n=0}^{\infty} (1 + k_{12})^{n+2} k_{23}^{n+1} (1 - k_{12})^{n+1} k_{i}^{n+1} i_{0} \left( t - \frac{h + 2L_{12} - z}{c} - \frac{2L_{12}}{c} n \right) \]  \hspace{2cm} (AIII.53)

\[
i_{12} = \sum_{n=0}^{\infty} (1 + k_{12})^{n+2} k_{23}^{n+2} (1 - k_{12})^{n+1} k_{i}^{n+1} i_{0} \left( t - \frac{3L_{12} + z - L_{345}}{c} - \frac{2L_{12}}{c} n \right) \]  \hspace{2cm} (AIII.54)

Fig. AIII.32 - Current contributions $i_{13}$ & $i_{14}$

\[
i_{13} = \sum_{n=0}^{\infty} (1 + k_{12})^{n+2} (1 + k_{23})^{n+1} k_{34}^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_{i}^{n+1} i_{0} \left( t - \frac{2L_{123} - z + h}{c} - \frac{2L_{123}}{c} n \right) \]  \hspace{2cm} (AIII.55)

\[
i_{14} = \sum_{n=0}^{\infty} (1 + k_{12})^{n+2} (1 + k_{23})^{n+2} k_{34}^{n+2} (1 - k_{23})^{n+2} (1 - k_{12})^{n+1} k_{i}^{n+1} i_{0} \left( t - \frac{3L_{123} + z - L_{45}}{c} - \frac{2L_{123}}{c} n \right) \]  \hspace{2cm} (AIII.56)

Fig. AIII.33 - Current contributions $i_{15}$ & $i_{16}$
\[ i_{15} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^{n+2} \left(1 + k_{23}\right)^{n+1} \left(1 + k_{34}\right)^{n+1} k_{45} \left(1 - k_{34}\right)^{n+1} \left(1 - k_{23}\right)^{n+1} \left(1 - k_{12}\right)^{n+1} \]
\[ \cdot \left(1 - k_{12}\right)^{n+1} k_{10} \left(t - \frac{2L_{1234} - z + h}{c} - \frac{2L_{1234}}{c} n\right) \]

\[ (AIII.57) \]

\[ i_{16} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^{n+2} \left(1 + k_{23}\right)^{n+2} \left(1 + k_{34}\right)^{n+2} \left(1 + k_{45}\right)^{n+2} \left(1 - k_{23}\right)^{n+2} \left(1 - k_{12}\right)^{n+2} \]
\[ \cdot \left(1 - k_{12}\right)^{n+1} k_{10} \left(t - \frac{3L_{1234} + z - L_{5}}{c} - \frac{2L_{1234}}{c} n\right) \]

\[ (AIII.58) \]

Fig. AIII.34 - Current contributions \( i_{r1}, i_{r2}, i_{r3}, i_{r4}, \) & \( i_{r5} \)

\[ i_{r1} = k_{12} (1 + k_{12}) k_{c} (1 - k_{c}) (1 + k_{12}) j_{0} \left(t - \frac{2L_{1234} + h - z}{c}\right) \]

\[ (AIII.59) \]

\[ i_{r2} = (1 + k_{12})^{2} k_{23} (1 - k_{12}) (1 + k_{c}) (1 - k_{c}) j_{0} \left(t - \frac{2L_{1234} + h - z}{c}\right) \]

\[ (AIII.60) \]

\[ i_{r3} = (1 + k_{12})^{2} (1 + k_{23}) k_{34} (1 - k_{23}) (1 - k_{12}) (1 + k_{c}) (1 - k_{c}) \]
\[ j_{0} \left(t - \frac{2L_{1234} + h - z}{c}\right) \]

\[ (AIII.61) \]

\[ i_{r4} = (1 + k_{12})^{2} (1 + k_{23}) (1 + k_{34}) k_{45} (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) (1 + k_{c}) (1 - k_{c}) \]
\[ j_{0} \left(t - \frac{2L_{1234} + h - z}{c}\right) \]

\[ (AIII.62) \]

\[ i_{r5} = (1 + k_{12})^{2} (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_{5} (1 - k_{45}) (1 - k_{34}) \]
\[ j_{0} \left(t - \frac{2L_{1234} + h - z}{c}\right) \]

\[ (AIII.63) \]
AIII.2.2.1 Additional Currents

Fig. AIII.35 - Current contributions $i_{A7}$, & $i_{A8}$

$$ i_{A7} = k_{12} k_i (1 + k_{12}) i_0 \left( t - \frac{h - z + 2L_1}{c} \right) $$  \hfill (AIII.64)

$$ i_{A8} = k_{12} k_i (1 + k_{12}) k_{23} i_0 \left( t - \frac{2L_1 + L_{12} + z - L_{345}}{c} \right) $$  \hfill (AIII.65)

Fig. AIII.36 - Current contributions $i_{A12}$, & $i_{A13}$

$$ i_{A12} = (1 + k_{12})(1 + k_{23})(1 + k_{34}) k_{45} (1 - k_{34}) (1 - k_{23}) (-k_{12}) i_0 \left( t - \frac{h - z + 2L_{234}}{c} \right) $$  \hfill (AIII.66)

$$ i_{A13} = (1 + k_{12})^2 (1 + k_{23})(1 + k_{34}) k_{45} (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) k_i (1 + k_{23}) k_{34} (1 - k_{23}) i_0 \left( t - \frac{2L_{1234} + L_{123} + z - L_{45}}{c} \right) $$  \hfill (AIII.67)

Fig. AIII.37 - Current contributions $i_{A18}$, $i_{A19}$, & $i_{A19d}$
\[ i_{A18} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_p (1 - k_{45}) \]
\[ (-k_{34})k_{45}(1 - k_{34})(1 - k_{23})i_0 \left( t - \frac{h + 2L_4 + z}{c} \right) \]  
(AIII.68)

\[ i_{A19} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_p^2 (-k_{45}) \]
\[ (1 - k_{45})(1 - k_{34})(1 - k_{23})i_0 \left( t - \frac{h + 2L_4 + z}{c} \right) \]  
(AIII.69)

\[ i_{A19d} = (1 + k_{12})^2(1 + k_{23})(1 + k_{34})(1 + k_{45})k_p^2 (-k_{45})(1 - k_{45}) \]
\[ (1 - k_{34})(1 - k_{23})k_i (1 - k_{12})i_0 \left( t - \frac{3h + 2L_5 - z}{c} \right) \]  
(AIII.70)

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\[ i_{A20} = (1 + k_{12})(1 + k_{23})(1 + k_{34})^2(1 + k_{45})^2 k_p^2 (1 - k_{45})^2 \]
\[ (-k_{34})^2(-k_{23})(1 - k_{23})i_0 \left( t - \frac{h + 2L_{345} + z}{c} \right) \]  
(AIII.71)

\[ i_{A21} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})^2(1 + k_{45})^2 k_p^2 (1 - k_{45})^2 \]
\[ (1 - k_{34})^2(1 - k_{23})^2(-k_{12})i_0 \left( t - \frac{h + 2L_{2345} + z}{c} \right) \]  
(AIII.72)

\[ i_{A21d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})^2(1 + k_{45})^2 k_p^2 (1 - k_{45})^2 \]
\[ (1 - k_{34})^2(1 - k_{23})^2(-k_{12})(1 - k_{12})k_i i_0 \left( t - \frac{3h + 2L_{2345} - z}{c} \right) \]  
(AIII.73)

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Fig. AIII.38 - Current contributions \( i_{A20}, i_{A21}, \) & \( i_{A21d} \)

Fig. AIII.39 - Current contributions \( i_{A22}, i_{A23}, \) & \( i_{A22d} \)
\[ i_{A22} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34}) (1 + k_{45}) k_b (1 - k_{45}) \]
\[ (1 - k_{34})(1 - k_{23})^2 (1 - k_{12}) k_{45} i_0 \left( t - \frac{2h + L_{123} + z - L_{45}}{c} \right) \]  
(AIII.74)

\[ i_{A22d} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34}) (1 + k_{45}) k_b (1 - k_{45}) \]
\[ (1 - k_{34})(1 - k_{23})^2 (1 - k_{12})^2 k_{45}^2 i_0 \left( t - \frac{3h + 2L_{123} - z}{c} \right) \]  
(AIII.75)

\[ i_{A23} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34}) (1 + k_{45}) k_h (1 - k_{45}) \]
\[ (1 - k_{34})(1 - k_{23})^2 (1 - k_{12}) k_{45} i_0 \left( t - \frac{2h + L_{123} + z - L_{5}}{c} \right) \]  
(AIII.76)

\[ i_{A24} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})^2 k_h^2 (1 - k_{45})^2 \]
\[ (-k_{34})(1 - k_{34})(1 - k_{23}) i_0 \left( t - \frac{h + 2L_{45} + z}{c} \right) \]  
(AIII.77)

\[ i_{A24d} = (1 + k_{12})^2 (1 + k_{23})(1 + k_{34})(1 + k_{45})^2 k_h^2 (1 - k_{45})^2 \]
\[ (-k_{34})(1 - k_{34})(1 - k_{23})(1 - k_{12}) k_{45} i_0 \left( t - \frac{3h - z + 2L_{45}}{c} \right) \]  
(AIII.78)

**AIII.2.3 Current inside L3 \( 365 < z \leq 444 m \)**

Fig. AIII.40 - Current contributions \( i_{A24}, \& i_{A24d} \)

Fig. AIII.41 - Current contributions \( i_1 \& i_2 \)
\[ i_1 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n} (1 + k_{45})^{n} k_{n}^{2} (1 - k_{45})^{n} \]
\[ \cdot (1 - k_{34})^{n} (1 - k_{23})^{n} (1 - k_{12})^{n} k_{r}^{n} i_{0} \left( t - \frac{h - z}{c} - \frac{2h}{c} n \right) \]  
\[ (AIII.79) \]

\[ i_2 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} (1 + k_{45})^{n+1} k_{b}^{n+1} (1 - k_{45})^{n+1} \]
\[ \cdot (1 - k_{34})^{n+1} (1 - k_{23})^{n+1} (1 - k_{12})^{n+1} k_{r}^{n+1} i_{0} \left( t - \frac{h + z}{c} - \frac{2h}{c} n \right) \]  
\[ (AIII.80) \]

Fig. AIII.42 - Current contributions \( i_{r1}, i_{r2}, i_{r3}, i_{r4}, i_{r5} \)

\[ i_{r1} = k_{12} (1 + k_{r}) k_{r} (1 - k_{r}) (1 + k_{12}) (1 + k_{23}) k_{0} \left( t - \frac{2L_{12} \xi + h - z}{c} \right) \]  
\[ (AIII.81) \]

\[ i_{r2} = (1 + k_{12})^{2} k_{23} (1 - k_{12}) (1 + k_{r}) k_{c} (1 - k_{r}) (1 + k_{23}) k_{0} \left( t - \frac{2L_{123} \xi + h - z}{c} \right) \]  
\[ (AIII.82) \]

\[ i_{r3} = (1 + k_{12})^{2} (1 + k_{23})^{2} k_{34} (1 - k_{23}) (1 - k_{12}) (1 + k_{r}) k_{c} (1 - k_{r}) i_{0} \left( t - \frac{2L_{123} \xi + h - z}{c} \right) \]  
\[ (AIII.83) \]

\[ i_{r4} = (1 + k_{12})^{2} (1 + k_{23})^{2} (1 + k_{34}) k_{45} (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) \]
\[ \cdot \left( 1 + k_{r} \right) k_{c} (1 - k_{r}) i_{0} \left( t - \frac{2L_{1234} \xi + h - z}{c} \right) \]  
\[ (AIII.84) \]

\[ i_{r5} = (1 + k_{12})^{2} (1 + k_{23})^{2} (1 + k_{34}) (1 + k_{45}) k_{b} (1 - k_{45}) (1 - k_{34}) \]
\[ \cdot (1 - k_{23}) (1 - k_{12}) (1 + k_{r}) k_{c} (1 - k_{r}) i_{0} \left( t - \frac{2h \xi + h - z}{c} \right) \]  
\[ (AIII.85) \]
AIII.2.3.1 Additional Currents

\[ i_{A3} = (1 + k_{12})(1 + k_{23})k_{34}i_0 \left( t - \frac{L_{123} + z - L_{45}}{c} \right) \]  
(AIII.86)

\[ i_{A3d} = (1 + k_{12})^2(1 + k_{23})^2k_{34}(1 - k_{23})(1 - k_{12})k_i i_0 \left( t - \frac{2L_{123} + h - z}{c} \right) \]  
(AIII.87)

\[ i_{A3u} = (1 + k_{12})^2(1 + k_{23})^2k_{34}^2(1 - k_{23})(1 - k_{12})k_i i_0 \left( t - \frac{3L_{123} + z - L_{45}}{c} \right) \]  
(AIII.88)

\[ i_{A3dd} = (1 + k_{12})^2(1 + k_{23})^2k_{34}^2(1 - k_{23})(1 - k_{12})k_i i_0 \left( t - \frac{3L_{123} + z + L_4 - L_5}{c} \right) \]  
(AIII.89)

\[ i_{A3ddd} = (1 + k_{12})^2(1 + k_{23})^2k_{34}(1 - k_{23})(1 - k_{12})k_i (1 + k_{34})k_i \left( t - \frac{2L_{123} + h + z}{c} \right) \]  
(AIII.90)

\[ i_{A4} = (1 + k_{12})(1 + k_{23})k_{34}(-k_{23})i_0 \left( t - \frac{h - z + 2L_3}{c} \right) \]  
(AIII.91)
Fig. AIII.45 - Current contributions $i_5$, $i_{A5d}$, $i_{A5u}$, & $i_{A5dd}$

\[
i_{A5} = (1 + k_{12})(1 + k_{23})(1 + k_{34})k_{45}(1 - k_{34})j_0 \left( t - \frac{L_{1234} + z - L_{5}}{c} \right)
\]

\[
i_{A5d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})k_{45}(1 - k_{34})(1 - k_{23})(1 - k_{12})j_i
\]

\[
i_{A5u} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})k_{45}(1 - k_{34})(1 - k_{23})(1 - k_{12})j_i k_{34}
\]

\[
i_{A5dd} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})^2(1 + k_{45})k_{4}(1 - k_{45})(1 - k_{34})
\]

\[
k_{45}(1 - k_{34})(1 - k_{23})(1 - k_{12})j_i t_0 \left( t - \frac{2L_{1234} + h + z}{c} \right)
\]

Fig. AIII.46 - Current contribution $i_6$

\[
i_{A6} = (1 + k_{12})(1 + k_{23})(1 + k_{34})k_{45}(1 - k_{34})(-k_{23})j_0 \left( t - \frac{h - z + 2L_{34}}{c} \right)
\]
Fig. AIII.47 - Current contributions $i_{A7}$, & $i_{A7d}$

\[
i_{A7} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b z L h t (1 - k_{45})(1 - k_{34})
\]
\[
i_{A7d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})k_b z L h t (1 - k_{45})(1 - k_{34})
\]

\[
i_{A7d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})k_b z L h t (1 - k_{45})(1 - k_{34})
\]

\[
i_{A7d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})k_b z L h t (1 - k_{45})(1 - k_{34})
\]

\[
i_{A7d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})k_b z L h t (1 - k_{45})(1 - k_{34})
\]

Fig. AIII.48 - Current contributions $i_{A8}$, & $i_{A8d}$

\[
i_{A8} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})^2 k_b z L h t (1 - k_{45})(1 - k_{34})
\]
\[
i_{A8d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})^2 k_b z L h t (1 - k_{45})(1 - k_{34})
\]

\[
i_{A8d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})^2 k_b z L h t (1 - k_{45})(1 - k_{34})
\]

\[
i_{A8d} = (1 + k_{12})^2(1 + k_{23})^2(1 + k_{34})(1 + k_{45})^2 k_b z L h t (1 - k_{45})(1 - k_{34})
\]
AIII.2.4 Current inside L4 \((335 < z \leq 365m)\)

\[ i_1 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_n^h \left(1 - k_{45}\right)^n \]
\[ (1 - k_{34})^n (1 - k_{23})^n (1 - k_{12})^n k_i^i L_0 \left( t - \frac{h - z}{c} - \frac{2h}{c} n \right) \]  
\[ (AIII.101) \]

\[ i_2 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_n^h \left(1 - k_{45}\right)^{n+1} \]
\[ (1 - k_{34})^n (1 - k_{23})^n (1 - k_{12})^n k_i^i L_0 \left( t - \frac{h + z}{c} - \frac{2h}{c} n \right) \]  
\[ (AIII.102) \]

\[ i_5 = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34})^{n+1} k_{45}^n \]
\[ J_0 \left( t - \frac{h + 2L_{45} - z}{c} - \frac{2L_4}{c} n \right) \]  
\[ (AIII.103) \]

\[ i_6 = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34})^{n+1} k_{45}^{n+1} \]
\[ J_0 \left( t - \frac{h + 2L_4 + z}{c} - \frac{2L_4}{c} n \right) \]  
\[ (AIII.104) \]
Fig. AIII.51 - Current contributions $i_7$ & $i_8$

\[
\begin{align*}
    i_7 &= \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^{n+2} \left(1 + k_{23}\right)^{n+2} \left(1 + k_{34}\right) k_{45} \left(1 - k_{34}\right)^{n+1} \\
    &= \left(1 - k_{23}\right)^{n+1} \left(1 - k_{12}\right)^{n+1} k_{7} k_{6} l_{0} \left( t - \frac{2L_{1234} + h - z}{c} - \frac{2L_{1234}}{c} n \right) \
\end{align*}
\]

(AIII.105)

\[
\begin{align*}
    i_8 &= \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^{n+2} \left(1 + k_{23}\right)^{n+2} \left(1 + k_{34}\right) k_{45} \left(1 - k_{34}\right)^{n+1} \\
    &= \left(1 - k_{23}\right)^{n+1} \left(1 - k_{12}\right)^{n+1} k_{7} k_{6} l_{0} \left( t - \frac{3L_{1234} + z - L_{5}}{c} - \frac{2L_{1234}}{c} n \right) \
\end{align*}
\]

(AIII.106)

---

Fig. AIII.52 - Current contributions $i_{r1}$, $i_{r2}$, $i_{r3}$, $i_{r4}$, & $i_{r5}$

\[
\begin{align*}
    i_{r1} &= k_{12} \left(1 + k_{r}\right) k_{c} \left(1 - k_{r}\right) \left(1 + k_{12}\right) \left(1 + k_{23}\right) \left(1 + k_{34}\right) k_{0} \left( t - \frac{2L_{12} \xi + h - z}{c} \right) \
\end{align*}
\]

(AIII.107)

\[
\begin{align*}
    i_{r2} &= \left(1 + k_{12}\right)^{2} k_{23} \left(1 - k_{12}\right) \left(1 + k_{r}\right) k_{c} \left(1 - k_{r}\right) \left(1 + k_{23}\right) \left(1 + k_{34}\right) k_{0} \left( t - \frac{2L_{12} \xi + h - z}{c} \right) \
\end{align*}
\]

(AIII.108)

\[
\begin{align*}
    i_{r3} &= \left(1 + k_{12}\right)^{2} \left(1 + k_{23}\right)^{2} k_{34} \left(1 - k_{23}\right) \left(1 - k_{12}\right) \left(1 + k_{r}\right) k_{c} \left(1 - k_{r}\right) \left(1 + k_{34}\right) k_{0} \left( t - \frac{2L_{123} \xi + h - z}{c} \right) \
\end{align*}
\]

(AIII.109)

\[
\begin{align*}
    i_{r4} &= \left(1 + k_{12}\right)^{2} \left(1 + k_{23}\right)^{2} \left(1 + k_{34}\right)^{2} k_{45} \left(1 - k_{34}\right) \left(1 - k_{23}\right) \left(1 - k_{12}\right) \left(1 + k_{r}\right) k_{c} \left(1 - k_{r}\right) \left(1 + k_{45}\right) k_{0} \left( t - \frac{2L_{1234} \xi + h - z}{c} \right) \
\end{align*}
\]

(AIII.110)
\[ i_{r5} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34})^2 (1 + k_{45}) k_6 (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) \]

\[ (1 - k_{12}) (1 + k_i) k_i (1 - k_i) i_0 \left( t - \frac{2h z^c + h - z}{c} \right) \]  
(AIII.111)

### AIII.2.4.1 Additional Currents

**Fig. AIII.53 - Current contributions \( i_{A3} \) & \( i_{A4} \)**

\[ i_{A3} = (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) k_{45} i_0 \left( t - \frac{L_{1234} + z - L_5}{c} \right) \]  
(AIII.112)

\[ i_{A4} = (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) k_{45} (-k_{34}) i_0 \left( t - \frac{h + 2L_4 - z}{c} \right) \]  
(AIII.113)

**Fig. AIII.54 - Current contributions \( i_{A9} \) & \( i_{A10} \)**

\[ i_{A9} = (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_6^2 (-k_{45}) (1 - k_{45}) i_0 \left( t - \frac{h + 2L_5 + z}{c} \right) \]  
(AIII.114)

\[ i_{A10} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34})^2 (1 + k_{45})^2 k_6^2 (-k_{45}) (1 - k_{45}) \]

\[ (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) k_i i_0 \left( t - \frac{3h + 2L_5 - z}{c} \right) \]  
(AIII.115)

**Fig. AIII.55 - Current contributions \( i_{A11} \) & \( i_{A12} \)**
\[ i_{A11} = k_{12} k_t (1 + k_{12})(1 + k_{23})(1 + k_{34}) k_0 \left( t - \frac{2L_1 + h - z}{c} \right) \]  
\[ i_{A12} = k_{12} k_t (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45}) k_0 (1 - k_{45}) k_0 \left( t - \frac{h + 2L_1 + z}{c} \right) \]

Fig. AIII.56 - Current contributions \( i_{A13} \) & \( i_{A14} \)

\[ i_{A13} = (1 + k_{12})^2 k_{23} (1 - k_{12}) k_t (1 + k_{23})(1 + k_{34}) k_0 \left( t - \frac{2L_{12} + h - z}{c} \right) \]  
\[ i_{A14} = (1 + k_{12})^2 k_{23} (1 - k_{12}) k_t (1 + k_{23})(1 + k_{34})(1 + k_{45}) k_0 (1 - k_{45}) \left( t - \frac{h + 2L_{12} + z}{c} \right) \]

Fig. AIII.57 - Current contributions \( i_{A15}, i_{A16} \) & \( i_{A17} \)

\[ i_{A15} = (1 + k_{12})^2 (1 + k_{23})^2 k_{34} (1 - k_{12})(1 - k_{23}) k_t (1 + k_{34}) k_0 \left( t - \frac{2L_{123} + h - z}{c} \right) \]  
\[ i_{A16} = (1 + k_{12})^2 (1 + k_{23})^2 k_{34} (1 - k_{12})(1 - k_{23}) k_t (1 + k_{34}) k_{45} \left( t - \frac{3L_{123} + L_4 + z - L_5}{c} \right) \]  
\[ i_{A17} = (1 + k_{12})^2 (1 + k_{23})^2 k_{34} (1 - k_{12})(1 - k_{23}) k_t (1 + k_{34})(1 + k_{45}) k_0 \left( t - \frac{h + 2L_{123} + z}{c} \right) \]
Fig. AIII.58 - Current contribution $i_{A18}$

$$i_{A18} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34})^2 k_{45} (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) k_5 k_6 (1 - k_{45}) k_0 \left( t - \frac{3L_{1234} + L_5 + z}{c} \right)$$  \hspace{1cm} (AIII.123)

Fig. AIII.59 - Current contribution $i_{A19}$

$$i_{A19} = (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 k_{45} (1 - k_{34})(- k_{23}) k_0 \left( t - \frac{2L_{34} + h - z}{c} \right)$$  \hspace{1cm} (AIII.124)

Fig. AIII.60 - Current contribution $i_{A20}$

$$i_{A20} = (1 + k_{12})(1 + k_{23})^2 (1 + k_{34})^2 k_{45} (1 - k_{34})(1 - k_{23})(- k_{12})$$
$$i_0 \left( t - \frac{2L_{234} + h - z}{c} \right)$$  \hspace{1cm} (AIII.125)
Fig. AIII.61 - Current contribution $i_{a21}$

$$i_{a21} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})(-k_{34})(1 + k_{45}) k_b (1 - k_{45})$$

$$i_0 \left( t - \frac{h + 2L_4 + z}{c} \right)$$  \hspace{2cm} (AIII.126)

Fig. AIII.62 - Current contribution $i_{a22}$

$$i_{a22} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45}) k_b^2 (1 - k_{45})^2 (-k_{34})$$

$$i_0 \left( t - \frac{h + 2L_{45} + z}{c} \right)$$  \hspace{2cm} (AIII.127)

AIII.2.5 Current inside L5 \(( z \leq 335m )\)

Fig. AIII.63 - Current contributions $i_1$ & $i_2$

$$i_1 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} (1 + k_{45})^{n+1} k_b^n (1 - k_{45})^n$$

$$(1 - k_{34})^n (1 - k_{23})^n (1 - k_{12})^n k_i^n i_0 \left( t - \frac{h - z}{c} - \frac{2h}{c} n \right)$$  \hspace{2cm} (AIII.128)
\[ i_2 = \sum_{n=0}^{\infty} (1 + k_{12})^{n+1} (1 + k_{23})^{n+1} (1 + k_{34})^{n+1} (1 + k_{45})^{n+1} k_n^{n+1} (1 - k_{45})^n \]

\[ (1 - k_{45})^n (1 - k_{23})^n (1 - k_{12})^n k_n^{n} i_0 \left( t - \frac{h + z}{c} - \frac{2h}{n} \right) \]  
(AIII.129)

Fig. AIII.64 - Current contributions \( i_{r1}, i_{r2}, i_{r3}, i_{r4}, \) & \( i_{r5} \)

\[ i_{r1} = k_{12} (1 + k_c) (1 - k_c) (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) i_0 \left( t - \frac{2L_{12} \xi + h - z}{c} \right) \]  
(AIII.130)

\[ i_{r2} = (1 + k_{12})^2 k_{23} (1 - k_{12}) (1 + k_c) (1 - k_c) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) \]

\[ i_0 \left( t - \frac{2L_{123} \xi + h - z}{c} \right) \]  
(AIII.131)

\[ i_{r3} = (1 + k_{12})^2 (1 + k_{23})^2 k_{34} (1 - k_{23}) (1 - k_{12}) (1 + k_c) (1 - k_c) (1 + k_{34}) (1 + k_{45}) i_0 \left( t - \frac{2L_{1234} \xi + h - z}{c} \right) \]  
(AIII.132)

\[ i_{r4} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34})^2 k_{45} (1 - k_{34}) (1 - k_{23}) \]

\[ (1 - k_{12}) (1 + k_c) (1 - k_c) (1 + k_{45}) i_0 \left( t - \frac{2L_{1234} \xi + h - z}{c} \right) \]  
(AIII.133)

\[ i_{r5} = (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{34})^2 (1 + k_{45})^2 k_b (1 - k_{45})(1 - k_{34}) \]

\[ (1 - k_{23}) (1 - k_{12}) (1 + k_c) (1 - k_c) i_0 \left( t - \frac{2h \xi + h - z}{c} \right) \]  
(AIII.134)

**AIII.2.5.1 Additional Currents**

Fig. AIII.65 - Current contributions \( i_{a3} \) & \( i_{a4} \)
\[ i_{43} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b(-k_{45})k_b(-k_{45}) \left( t - \frac{h + 2L_5 - z}{c} \right) \]  \hspace{1cm} (AIII.135)

\[ i_{44} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b(-k_{45})k_b(-k_{45}) \left( t - \frac{h + 2L_5 + z}{c} \right) \]  \hspace{1cm} (AIII.136)

Fig. AIII.66 - Current contributions \( i_{A5}, i_{A6} \& i_{A7} \)

\[ i_{45} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b(-k_{45})\left(1 - k_{34}\right)k_b(-k_{45})k_b(-k_{45}) \left( t - \frac{h + 2L_{45} - z}{c} \right) \]  \hspace{1cm} (AIII.137)

\[ i_{46} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b(-k_{45})\left(1 - k_{34}\right)k_b(-k_{45})k_b(-k_{45}) \left( t - \frac{h + 2L_{45} + z}{c} \right) \]  \hspace{1cm} (AIII.138)

\[ i_{47} = (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_b(-k_{45})\left(1 - k_{34}\right)k_b(-k_{45})\left(1 - k_{34}\right)^2 \left( t - \frac{h + 4L_{45} - z}{c} \right) \]  \hspace{1cm} (AIII.139)

Fig. AIII.67 - Current contributions \( i_{A8} \& i_{A9} \)

\[ i_{48} = (1 + k_{12})(1 + k_{23})(1 + k_{34})\left(1 + k_{45}\right)^2k_b\left(1 - k_{45}\right)(1 - k_{34}) \left( t - \frac{h + 2L_{345} - z}{c} \right) \]  \hspace{1cm} (AIII.140)

\[ i_{49} = (1 + k_{12})(1 + k_{23})(1 + k_{34})\left(1 + k_{45}\right)^2k_b\left(1 - k_{45}\right)(1 - k_{34}) \left( t - \frac{h + 2L_{345} + z}{c} \right) \]  \hspace{1cm} (AIII.141)
Fig. AIII.68 - Current contributions $i_{A10}$ & $i_{A11}$

\[ i_{A10} = \left(1 + k_{12}\right)\left(1 + k_{23}\right)^2\left(1 + k_{34}\right)\left(1 + k_{45}\right)k_b\left(1 - k_{45}\right)\left(1 - k_{34}\right) \]
\[ \cdot \left(1 - k_{23}\right)\left(- k_{12}\right)k_0 \left( t - \frac{h + 2L_{2345} - z}{c} \right) \]  
(AIII.142)

\[ i_{A11} = \left(1 + k_{12}\right)\left(1 + k_{23}\right)^2\left(1 + k_{34}\right)\left(1 + k_{45}\right)^2k_b\left(1 - k_{45}\right)\left(1 - k_{34}\right) \]
\[ \cdot \left(1 - k_{23}\right)\left(- k_{12}\right)k_0 \left( t - \frac{h + 2L_{2345} + z}{c} \right) \]  
(AIII.143)

Fig. AIII.69 - Current contributions $i_{A12}$, $i_{A13}$, $i_{A14}$, & $i_{A15}$

\[ i_{A12} = k_{12}k_1\left(1 + k_{12}\right)\left(1 + k_{23}\right)^2\left(1 + k_{34}\right)\left(1 + k_{45}\right)k_b\left(1 - k_{45}\right)\left(1 - k_{34}\right) \]
\[ \cdot \left(1 - k_{23}\right)\left(- k_{12}\right)k_0 \left( t - \frac{2L_1 + h - z}{c} \right) \]  
(AIII.144)

\[ i_{A13} = k_{12}k_1\left(1 + k_{12}\right)\left(1 + k_{23}\right)^2\left(1 + k_{34}\right)\left(1 + k_{45}\right)k_0 \left( t - \frac{2L_1 + h + z}{c} \right) \]  
(AIII.145)

\[ i_{A14} = \left(1 + k_{12}\right)^2k_{23}\left(1 - k_{12}\right)k_1\left(1 + k_{23}\right)^2\left(1 + k_{34}\right)\left(1 + k_{45}\right)k_0 \left( t - \frac{2L_{12} + h - z}{c} \right) \]  
(AIII.146)

\[ i_{A15} = \left(1 + k_{12}\right)^2k_{23}\left(1 - k_{12}\right)k_1\left(1 + k_{23}\right)^2\left(1 + k_{34}\right)\left(1 + k_{45}\right)k_b\left(1 - k_{45}\right)\left(1 - k_{34}\right) \]
\[ \cdot \left(1 - k_{23}\right)\left(- k_{12}\right)k_0 \left( t - \frac{2L_{12} + h + z}{c} \right) \]  
(AIII.147)
Fig. AIII.70 - Current contributions \( i_{A16} \) & \( i_{A17} \)

\[
i_{A16} = (1+k_{12})^2(1+k_{23})^2k_{34}(1-k_{23})(1-k_{12})k_r(1+k_{34})(1+k_{45})
\]

\[
i_0 \left( t - \frac{2L_{123} + h - z}{c} \right)
\]

(AIII.148)

\[
i_{A17} = (1+k_{12})^2(1+k_{23})^2k_{34}(1-k_{23})(1-k_{12})k_r(1+k_{34})(1+k_{45})k_p
\]

\[
i_0 \left( t - \frac{2L_{123} + h + z}{c} \right)
\]

(AIII.149)

Fig. AIII.71 - Current contributions \( i_{A18} \) & \( i_{A19} \)

\[
i_{A18} = (1+k_{12})^2(1+k_{23})^2(1+k_{34})^2k_{45}(1-k_{34})(1-k_{23})(1-k_{12})
\]

\[
k_r(1+k_{45})k_0 \left( t - \frac{3L_{1234} + L_5 - z}{c} \right)
\]

(AIII.150)

\[
i_{A19} = (1+k_{12})^2(1+k_{23})^2(1+k_{34})^2k_{45}(1-k_{34})(1-k_{23})(1-k_{12})
\]

\[
k_r(1+k_{45})k_3k_0 \left( t - \frac{3L_{1234} + L_5 + z}{c} \right)
\]

(AIII.151)

Fig. AIII.72 - Current contributions \( i_{A20} \) & \( i_{A21} \)

\[
i_{A20} = (1+k_{12})(1+k_{23})k_{34}(-k_{23})(1+k_{34})(1+k_{45})k_0 \left( t - \frac{h + 2L_3 - z}{c} \right)
\]

(AIII.152)
\[ i_{A21} = (1 + k_{12})(1 + k_{23})k_{34}(-k_{23})(1 + k_{34})(1 + k_{45})k_0i_0 \left( t - \frac{h + 2L_3 + z}{c} \right) \] (AIII.153)

**Fig. AIII.73 - Current contributions \( i_{A22}, i_{A23}, i_{A24}, \) & \( i_{A25} \)**

\[ i_{A22} = (1 + k_{12})(1 + k_{23})^2(1 + k_{34})^2k_{45}(1 - k_{23})(-k_{12}) \]
\[ \cdot (1 + k_{45})k_0i_0 \left( t - \frac{h + 2L_{234} - z}{c} \right) \] (AIII.154)

\[ i_{A23} = (1 + k_{12})(1 + k_{23})^2(1 + k_{34})^2k_{45}(1 - k_{23})(-k_{12})(1 + k_{45}) \]
\[ k_0i_0 \left( t - \frac{h + 2L_{234} + z}{c} \right) \] (AIII.155)

\[ i_{A24} = (1 + k_{12})(1 + k_{23})^2k_{34}(1 - k_{23})(-k_{12})(1 + k_{34})(1 + k_{45}) \]
\[ i_0 \left( t - \frac{h + 2L_{23} - z}{c} \right) \] (AIII.156)

\[ i_{A25} = (1 + k_{12})(1 + k_{23})^2k_{34}(1 - k_{23})(-k_{12})(1 + k_{34})(1 + k_{45}) \]
\[ k_0i_0 \left( t - \frac{h + 2L_{23} + z}{c} \right) \] (AIII.157)

**Fig. AIII.74 - Current contribution \( i_{A26} \)**

\[ i_{A26} = (1 + k_{12})(1 + k_{23})(1 + k_{34})^2k_{45}(1 - k_{34})(-k_{23})(1 + k_{45}) \]
\[ i_0 \left( t - \frac{h + 2L_{34} - z}{c} \right) \] (AIII.158)
AIII.3 Derivation of Relations for the Processes Taking Place within the Lightning Channel

AIII.3.1 Channel Base Current propagating upwards & Transmitted Components in the Lightning Channel

\[ i_{zh} = \frac{Z_1}{Z_{tch}} i_0 \left( t - \frac{z - h}{v} \right) e^{\frac{h-z}{\psi}} \]  \hspace{1cm} (AIII.159)

for \( z \geq 553 \) m

\[ i_{z1} = k_{12} (1 + k_i) (1 + k_c) i_0 \left( t - \frac{2L_1}{c-v} - \frac{z - (h + 2L_1 \sigma)}{v} \right) e^{\frac{h-z}{\psi}} \]  \hspace{1cm} (AIII.160)

for \( h + 2L_1 \sigma < z < \infty \)

\[ i_{z2} = (1 + k_{12}) k_{23} (1 - k_{12}) (1 + k_i) (1 + k_c) i_0 \left( t - \frac{2L_{12}}{c-v} - \frac{z - (h + 2L_{12} \sigma)}{v} \right) e^{\frac{h-z}{\psi}} \]  \hspace{1cm} (AIII.161)

for \( h + 2L_{12} \sigma < z < \infty \)

\[ i_{z3} = (1 + k_{12}) (1 + k_{23}) k_{34} (1 - k_{12}) (1 - k_{23}) (1 + k_i) (1 + k_c) \]

\[ i_0 \left( t - \frac{2L_{123}}{c-v} - \frac{z - (h + 2L_{123} \sigma)}{v} \right) e^{\frac{h-z}{\psi}} \]  \hspace{1cm} (AIII.162)

for \( h + 2L_{123} \sigma < z < \infty \)
\[ i_{z4} = (1 + k_{i2}) (1 + k_{i3}) (1 + k_{i4}) k_{45} (1 - k_{34}) (1 - k_{23}) \]

\[ (1 - k_{i2}) (1 + k_{i3}) (1 + k_{i4}) \psi_g (t - \frac{2L_{1234}}{c - v} - \frac{z - (h + 2L_{1234})}{v}) e^{\frac{h-z}{v}} \]  

(III.163)

for \( h + 2L_{1234} \sigma < z < \infty \)

\[ i_{z5} = (1 + k_{i2}) (1 + k_{i3}) (1 + k_{i4}) (1 + k_{45}) k_{5b} (1 - k_{45}) (1 - k_{34}) \]

\[ (1 - k_{23}) (1 - k_{12}) (1 + k_{i3}) (1 + k_{i4}) \psi_g (t - \frac{2h}{c - v} - \frac{z - (h + 2h\sigma)}{v}) e^{\frac{h-z}{v}} \]  

(III.164)

for \( h + 2h\sigma < z < \infty \)

\[ i_{z6} = (1 + k_{i2}) (1 + k_{i3}) (1 + k_{i4}) (1 + k_{45}) k_{5b} (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) \]

\[ k_{13} (1 + k_{i3}) (1 + k_{i4}) \psi_g (t - \frac{4L_{1233}}{c - v} - \frac{z - (h + 4L_{1233} \sigma)}{v}) e^{\frac{h-z}{v}} \]  

(III.165)

for \( h + 4L_{123} \sigma < z < \infty \)

\[ i_{z7} = (1 + k_{i2}) (1 + k_{i3}) (1 + k_{i4}) (1 + k_{45}) k_{5b} (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) \]

\[ k_{k_{34}} (1 + k_{i3}) (1 + k_{i4}) \psi_g (t - \frac{2(L_{1234} + L_{123})}{c - v} - \frac{z - (h + 2(L_{1234} + L_{123}) \sigma)}{v}) e^{\frac{h-z}{v}} \]  

(III.166)

for \( h + 2(L_{1234} + L_{123}) \sigma < z < \infty \)

\[ i_{z8} = (1 + k_{i2}) (1 + k_{i3}) (1 + k_{i4}) (1 + k_{45}) k_{5b} (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) (1 - k_{12}) \]

\[ k_{k_{34}} (1 + k_{i3}) (1 + k_{i4}) \psi_g (t - \frac{2(h + L_{123})}{c - v} - \frac{z - (h + 2(h + L_{123}) \sigma)}{v}) e^{\frac{h-z}{v}} \]  

(III.167)

for \( h + 2(h + L_{123}) \sigma < z < \infty \)

**AIII.3.2 Internal Components in the Lightning Channel**

![Fig. AIII.76 - Current contributions](image-url)
\[ i_1 = \sum_{n=0}^{\infty} k_{12}(1 + k_i)k^n_c(-k_i)^n i_0 \left( t - \frac{2L_1}{c} \xi - \frac{z - h}{c} \right) \quad \text{(AIII.168)} \]

\[ i_2 = \sum_{n=0}^{\infty} k_{12}(1 + k_i)k^{n+1}_c(-k_i)^n i_0 \left( t - \frac{2L_1}{c - v} \xi - \frac{h + 2L_1 \sigma \xi^n}{c} - z \right) \quad \text{(AIII.169)} \]

for \( h < z < h + 2L_1 \sigma \xi^n \)

\[ i_3 = \sum_{n=0}^{\infty} (1 + k_{12})k^{23}_c(1 - k_{12})(1 + k_i)k^n_c(-k_i)^n i_0 \left( t - \frac{2L_1}{c} \xi - \frac{z - h}{c} \right) \quad \text{(AIII.170)} \]

\[ i_4 = \sum_{n=0}^{\infty} (1 + k_{12})k^{23}_c(1 - k_{12})(1 + k_i)k^{n+1}_c(-k_i)^n i_0 \left( t - \frac{2L_{12}}{c - v} \xi - \frac{h + 2L_1 \sigma \xi^n}{c} - z \right) \quad \text{(AIII.171)} \]

for \( h < z < h + 2L_{12} \sigma \xi^n \)

\[ i_5 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_{34}^{23}_c(1 - k_{23})(1 - k_{12})(1 + k_i)k^n_c(-k_i)^n \]

\[ i_6 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_{34}^{23}_c(1 - k_{23})(1 - k_{12})(1 + k_i)k^{n+1}_c(-k_i)^n \quad \text{(AIII.172)} \]

\[ \text{for } h < z < h + 2L_{123} \sigma \xi^n \]

\[ i_7 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})k_{45}^{34}_c(1 - k_{34})(1 - k_{23}) \]

\[ .(1 - k_{12})(1 + k_i)k^n_c(-k_i)^n i_0 \left( t - \frac{2L_{1234}}{c} \xi - \frac{z - h}{c} \right) \quad \text{(AIII.173)} \]

\[ i_8 = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})k_{45}^{34}_c(1 - k_{34})(1 - k_{23}) \]

\[ .(1 - k_{12})(1 + k_i)k^{n+1}_c(-k_i)^n i_0 \left( t - \frac{2L_{1234}}{c - v} \xi - \frac{h + 2L_{1234} \sigma \xi^n}{c} - z \right) \quad \text{(AIII.174)} \]

for \( h < z < h + 2L_{1234} \sigma \xi^n \)
\[ i_9 = \sum_{n=0}^{\infty} (1 + k_{23})(1 + k_{34})(1 + k_{45})k_n (1 - k_{45})(1 - k_{34}) \]
\[ .(1 - k_{23})(1 - k_{12})(1 + k_i)k^n_c (-k_i)^n i_0 \left\{ t - \frac{2h}{c} \xi^n - \frac{z - h}{c} \right\} \]
\[ i_{10} = \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_n (1 - k_{45})(1 - k_{34}) \]
\[ .(1 - k_{23})(1 - k_{12})(1 + k_i)k^{n+1}_c (-k_i)^n i_0 \left\{ t - \frac{2h}{c} \xi^n - \frac{h + 2h \sigma \xi^n - z}{c} \right\} \]

for \( h < z < h + 2h \sigma \xi^n \)

\[ \text{Fig. AIII.77 - Current contributions } i_{11}, i_{12}, i_{13}, \text{ & } i_{14} \]

\[ i_{11} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_{34}(1 - k_{23})(1 - k_{12})k_n k_{12} \]
\[ .(1 + k_i)k^n_c (-k_i)^n i_0 \left\{ t - \frac{2(L_{123} + L_1)}{c} \xi^n - \frac{z - h}{c} \right\} \]
\[ i_{12} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})k_{34}(1 - k_{23})(1 - k_{12})k_n k_{12} \]
\[ .(1 + k_i)k^{n+1}_c (-k_i)^n i_0 \left\{ t - \frac{2(L_{123} + L_1)}{c} \xi^n - \frac{h + 2(L_{123} + L_1) \sigma \xi^n - z}{c} \right\} \]

for \( h < z < h + 2(L_{123} + L_1) \sigma \xi^n \)

\[ i_{13} = 2 \sum_{n=0}^{\infty} (1 + k_{12})k_{23}(1 - k_{12})k_{12}(1 + k_i)k^n_c \]
\[ .(-k_i)^n i_0 \left\{ t - \frac{2(L_{12} + L_1)}{c} \xi^n - \frac{z - h}{c} \right\} \]

\( \text{(AIII.176)} \)

\( \text{(AIII.177)} \)

\( \text{(AIII.178)} \)

\( \text{(AIII.179)} \)

\( \text{(AIII.180)} \)
\[ i_{14} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) k_{23} (1 - k_{12}) k_r k_{c} (1 + k_r) k_{c}^{n+1} (\zeta - k_r)^n i_0 \left( t - \frac{2(L_{12} + L_1)}{c - v} \zeta^n - h + 2(L_{12} + L_1) \sigma_{\zeta}^n - z \right) \]  
\[ (AIII.181) \]

for \( h < z < h + 2(L_{12} + L_1) \sigma_{\zeta}^n \)

\[ i_{15} = \sum_{n=0}^{\infty} (1 + k_{12})^2 k_{23}^2 (1 - k_{12})^2 k_r (1 + k_r) k_{c}^{n+1} (\zeta - k_r)^n i_0 \left( t - \frac{4L_{12}}{c} \zeta^n - \frac{z - h}{c} \right) \]  
\[ (AIII.182) \]

\[ i_{16} = \sum_{n=0}^{\infty} (1 + k_{12})^2 k_{23}^2 (1 - k_{12})^2 k_r (1 + k_r) k_{c}^{n+1} (\zeta - k_r)^n i_0 \left( t - \frac{4L_{12}}{c} \zeta^n - \frac{h + 4L_{12} \sigma_{\zeta}^n - z}{c} \right) \]  
\[ (AIII.183) \]

for \( h < z < h + 4L_{12} \sigma_{\zeta}^n \)

\[ i_{17} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23}) k_{34} (1 - k_{23}) (1 - k_{12})^2 k_r k_{23} (1 + k_r) \]  
\[ \times k_{c}^{n+1} (\zeta - k_r)^n i_0 \left( t - \frac{2(L_{123} + L_{12})}{c} \zeta^n - \frac{z - h}{c} \right) \]  
\[ (AIII.184) \]

\[ i_{18} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23}) k_{34} (1 - k_{23}) (1 - k_{12})^2 k_r k_{23} (1 + k_r) \]  
\[ \times k_{c}^{n+1} (\zeta - k_r)^n i_0 \left( t - \frac{2(L_{123} + L_{12})}{c - v} \zeta^n - \frac{h + 2(L_{123} + L_{12}) \sigma_{\zeta}^n - z}{c} \right) \]  
\[ (AIII.185) \]

for \( h < z < h + 2(L_{123} + L_{12}) \sigma_{\zeta}^n \)
Fig. AIII.79 - Current contributions \(i_{19}, i_{20}, i_{21}, i_{22}, i_{23} \& i_{24}\)

\[i_{19} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^2 \left(1 + k_{23}\right)^2 k_{34}^2 \left(1 - k_{23}\right)^2 \left(1 - k_{12}\right)^2 k_i \left(1 + k_i\right)\]

\[k_c^n (- k_i)^n i_0 \left(t - \frac{4L_{123}}{c} \xi^n - \frac{z - h}{c}\right)\]  \hspace{1cm} (AIII.186)

\[i_{20} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^2 \left(1 + k_{23}\right)^2 k_{34}^2 \left(1 - k_{23}\right)^2 \left(1 - k_{12}\right)^2 k_i \left(1 + k_i\right)\]

\[k_c^n (- k_i)^n i_0 \left(t - \frac{4L_{123}}{c} \xi^n - \frac{h + 4L_{123}G_2^n}{c} - z\right)\]  \hspace{1cm} (AIII.187)

for \(h < z < h + 4L_{123}G_2^n\)

\[i_{21} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^2 \left(1 + k_{23}\right)^2 \left(1 + k_{34}\right)^2 k_{45} \left(1 - k_{34}\right)^2 \left(1 - k_{23}\right)^2 \]

\[(1 - k_{12})^2 k_i \left(1 + k_i\right) k_c^n (- k_i)^n i_0 \left(t - \frac{4L_{1234}}{c} \xi^n - \frac{z - h}{c}\right)\]  \hspace{1cm} (AIII.188)

\[i_{22} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^2 \left(1 + k_{23}\right)^2 \left(1 + k_{34}\right)^2 k_{45}^2 \left(1 - k_{34}\right)^2 \left(1 - k_{23}\right)^2 \left(1 - k_{12}\right)^2 \]

\[k_i \left(1 + k_i\right) k_c^{n+1} (- k_i)^n i_0 \left(t - \frac{4L_{1234}}{c} \xi^n - \frac{h + 4L_{1234}G_2^n}{c} - z\right)\]  \hspace{1cm} (AIII.189)

for \(h < z < h + 4L_{1234}G_2^n\)

\[i_{23} = \sum_{n=0}^{\infty} \left(1 + k_{12}\right)^2 \left(1 + k_{23}\right)^2 \left(1 + k_{34}\right)^2 \left(1 + k_{45}\right)^2 k_i \left(1 - k_{45}\right)^2 \left(1 - k_{34}\right)^2 \]

\[(1 - k_{23})^2 (1 - k_{12})^2 k_i \left(1 + k_i\right) k_c^n (- k_i)^n i_0 \left(t - \frac{4h}{c} \xi^n - \frac{z - h}{c}\right)\]  \hspace{1cm} (AIII.190)
\[ i_{24} = \sum_{n=0}^{\infty} \left( (1 + k_{12})^2 (1 + k_{23})^2 (1 + k_{45})^2 \right) k_{e}^2 (1 - k_{45})^2 (1 - k_{34})^2 (1 - k_{23})^2 \\
\quad \times (1 - k_{12})^2 k_{i} (1 + k_{i}) k_{e}^{n+1} (-k_{i})^n i_0 \left( t - \frac{4h}{c-v} \xi^n - h + 4h\sigma\xi^n - z \right) \] (AIII.191)

for \( h < z < h + 4h\sigma\xi^n \)

\[ i_{25} = \sum_{n=0}^{\infty} \left( (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_{e}^2 (-k_{45}) (1 - k_{45}) (1 - k_{34}) \right) \\
\quad \times (1 - k_{23}) (1 - k_{12}) (1 + k_{i}) k_{e}^n (-k_{i})^n i_0 \left( t - \frac{2(h + L_{j})}{c-v} \xi^n - \frac{z - h}{c} \right) \] (AIII.192)

for \( h < z < h + 2(h + L_{j})\sigma\xi^n \)

\[ i_{26} = \sum_{n=0}^{\infty} \left( (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_{e}^2 (-k_{45}) (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) \right) \\
\quad \times (1 - k_{12}) (1 + k_{i}) k_{e}^{n+1} (-k_{i})^n i_0 \left( t - \frac{2(h + L_{j})}{c-v} \xi^n - h + 2(h + L_{j})\sigma\xi^n - z \right) \] (AIII.193)

for \( h < z < h + 2(h + L_{j})\sigma\xi^n \)

\[ i_{27} = \sum_{n=0}^{\infty} \left( (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_{e}^{2} (1 - k_{45})^2 (-k_{34}) (1 - k_{34}) \right) \\
\quad \times (1 - k_{23}) (1 - k_{12}) (1 + k_{i}) k_{e}^n (-k_{i})^n i_0 \left( t - \frac{2(h + L_{45})}{c-v} \xi^n - \frac{z - h}{c} \right) \] (AIII.194)

for \( h < z < h + 2(h + L_{45})\sigma\xi^n \)

\[ i_{28} = \sum_{n=0}^{\infty} \left( (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_{e}^2 (1 - k_{45})^2 (-k_{34}) (1 - k_{34}) (1 - k_{23}) \right) \\
\quad \times (1 - k_{12}) (1 + k_{i}) k_{e}^{n+1} (-k_{i})^n i_0 \left( t - \frac{2(h + L_{45})}{c-v} \xi^n - h + 2(h + L_{45})\sigma\xi^n - z \right) \] (AIII.195)

for \( h < z < h + 2(h + L_{45})\sigma\xi^n \)
\[ i_{29} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_{b}(1 - k_{45})(1 - k_{34})(1 - k_{23}) \]
\[ (1 - k_{34})(1 - k_{23})(1 - k_{12})(1 + k_{t})(- k_{t})^{n}i_{0} \left( t - \frac{2(h + L_{4})}{c} \xi^{n} - z - h \right) \]  
(AIII.196)

\[ i_{30} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_{b}(1 - k_{45})(1 - k_{34})(1 - k_{23})(1 - k_{12}) \]
\[ (1 - k_{12})(1 + k_{t})k_{c}^{n+1}(- k_{t})^{n}i_{0} \left( t - \frac{2(h + L_{4})}{c} \xi^{n} - h + 2(h + L_{4}) \sigma \xi^{n} - z \right) \]  
(AIII.197)

for \( h < z < h + 2(h + L_{4}) \sigma \xi^{n} \)

![Diagram](image.png)

Fig. AIII.81 - Current contributions \( i_{31}, i_{32}, i_{33}, i_{34}, i_{35}, i_{36}, i_{37} \) & \( i_{38} \)

\[ i_{31} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_{b}(1 - k_{45})(1 - k_{34})(1 - k_{23}) \]
\[ (- k_{12})k_{23}(1 - k_{12})(1 + k_{t})k_{c}^{n}(- k_{t})^{n}i_{0} \left( t - \frac{2(h + L_{2})}{c} \xi^{n} - z - h \right) \]  
(AIII.198)

\[ i_{32} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45})k_{b}(1 - k_{45})(1 - k_{34})(1 - k_{23})(1 - k_{12}) \]
\[ k_{23}(1 - k_{12})(1 + k_{t})k_{c}^{n+1}(- k_{t})^{n}i_{0} \left( t - \frac{2(h + L_{2})}{c} \xi^{n} - h + 2(h + L_{2}) \sigma \xi^{n} - z \right) \]  
(AIII.199)

for \( h < z < h + 2(h + L_{2}) \sigma \xi^{n} \)

\[ i_{33} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})^{2}(1 + k_{34})(1 + k_{45})k_{a}(1 - k_{45})(1 - k_{34})(1 - k_{23})^{2} \]
\[ (- k_{12})k_{34}(1 - k_{12})(1 + k_{t})k_{c}^{n}(- k_{t})^{n}i_{0} \left( t - \frac{2(h + L_{23})}{c} \xi^{n} - z - h \right) \]  
(AIII.200)

\[ i_{34} = 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})^{2}(1 + k_{34})(1 + k_{45})k_{a}(1 - k_{45})(1 - k_{34})(1 - k_{23})^{2}(- k_{12}) \]
\[ k_{34}(1 - k_{12})(1 + k_{t})k_{c}^{n+1}(- k_{t})^{n}i_{0} \left( t - \frac{2(h + L_{23})}{c} \xi^{n} - h + 2(h + L_{23}) \sigma \xi^{n} - z \right) \]  
(AIII.201)
for \( h < z < h + 2(h + L_{23})\sigma \xi^n \)

\[
i_{35} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23})^3 (1 + k_{34})^2 (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) k_{45} (1 - k_{34}) (1 + k_{23}) (1 - k_{45}) (1 - k_{34})^2 (1 - k_{23})^2 \left( -k_{12} \right) k_{45} (1 - k_{12}) (1 + k_{i}) k_c^n \left( -k_{i} \right) i_0 \left( t - \frac{2(h + L_{234})}{c} \xi^n - \frac{z - h}{c} \right)
\]  

(AIII.202)

\[
i_{36} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23})^3 (1 + k_{34})^2 (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) k_{45} (1 - k_{12}) (1 + k_{i}) k_c^{n+1} \left( -k_{i} \right) i_0 \left( t - \frac{2(h + L_{234})}{c} \xi^n - \frac{z - h}{c} \right)
\]  

(AIII.203)

\[
i_{37} = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23})^3 (1 + k_{34})^2 (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) k_{45} (1 - k_{12}) (1 + k_{i}) k_c^n \left( -k_{i} \right) i_0 \left( t - \frac{2(h + L_{234})}{c} \xi^n - \frac{z - h}{c} \right)
\]  

(AIII.204)

\[
i_{38} = \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23})^3 (1 + k_{34})^2 (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34})^2 (1 - k_{23})^2 (-k_{12}) (1 - k_{23}) (1 + k_{i}) k_c^{n+1} \left( -k_{i} \right) i_0 \left( t - \frac{2(h + L_{234})}{c} \xi^n - \frac{z - h}{c} \right)
\]  

(AIII.205)

for \( h < z < h + 2(h + L_{234})\sigma \xi^n \)

\[
i_{39} = 2 \sum_{n=0}^{\infty} (1 + k_{12}) (1 + k_{23}) (1 + k_{34}) (1 + k_{45}) k_b (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) k_{34} \left( -k_{12} \right) k_{45} (1 - k_{34}) (1 + k_{23}) (1 - k_{45}) (1 - k_{34}) (1 - k_{23}) (-k_{12}) (1 + k_{i}) k_c^n \left( -k_{i} \right) i_0 \left( t - \frac{2(h + L_{234})}{c} \xi^n - \frac{z - h}{c} \right)
\]  

(AIII.206)

Fig. AIII.82 - Current contributions \( i_{39}, i_{40}, i_{41}, i_{42}, i_{43}, \) & \( i_{44} \)
\[
\begin{align*}
  i_{40} &= 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})(1 + k_{45}) k_{p} (1 - k_{45})(1 - k_{34})(- k_{23}) k_{34} (1 - k_{23}) \\
  &\quad \cdot (1 - k_{12}) (1 + k_{r}) k_{c}^{n+1} (- k_{r})^n i_{0} \left( t - \frac{2(h + L_{3})}{c - v} \xi^n - h + 2(h + L_{3}) \sigma_{\xi^{n}} - z \right) \\
  \text{for } h < z < h + 2(h + L_{3}) \sigma_{\xi^{n}} \\
  \\
  i_{41} &= 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 (1 + k_{45}) k_{p} (1 - k_{45})(1 - k_{34})^2 (- k_{23}) \\
  &\quad \cdot k_{45} (1 - k_{23}) (1 - k_{12}) (1 + k_{r}) k_{c}^{n} (- k_{r})^n i_{0} \left( t - \frac{2(h + L_{34})}{c - v} \xi^n - \frac{z - h}{c} \right) \\
  \text{for } h < z < h + 2(h + L_{34}) \sigma_{\xi^{n}} \\
  \\
  i_{42} &= 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 (1 + k_{45}) k_{p} (1 - k_{45})(1 - k_{34})^2 (- k_{23}) k_{45} (1 - k_{23}) \\
  &\quad \cdot (1 - k_{12}) (1 + k_{r}) k_{c}^{n+1} (- k_{r})^n i_{0} \left( t - \frac{2(h + L_{34})}{c - v} \xi^n - h + 2(h + L_{34}) \sigma_{\xi^{n}} - z \right) \\
  \text{for } h < z < h + 2(h + L_{34}) \sigma_{\xi^{n}} \\
  \\
  i_{43} &= 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 (1 + k_{45})^2 k_{p}^2 (1 - k_{45})^2 (1 - k_{34})^2 (- k_{23}) \\
  &\quad \cdot (1 - k_{23}) (1 - k_{12}) (1 + k_{r}) k_{c}^{n} (- k_{r})^n i_{0} \left( t - \frac{2(h + L_{345})}{c - v} \xi^n - \frac{z - h}{c} \right) \\
  \text{for } h < z < h + 2(h + L_{345}) \sigma_{\xi^{n}} \\
  \\
  i_{44} &= 2 \sum_{n=0}^{\infty} (1 + k_{12})(1 + k_{23})(1 + k_{34})^2 (1 + k_{45})^2 k_{p}^2 (1 - k_{45})^2 (1 - k_{34})^2 (- k_{23}) (1 - k_{23}) \\
  &\quad \cdot (1 - k_{12}) (1 + k_{r}) k_{c}^{n+1} (- k_{r})^n i_{0} \left( t - \frac{2(h + L_{345})}{c - v} \xi^n - h + 2(h + L_{345}) \sigma_{\xi^{n}} - z \right) \\
  \text{for } h < z < h + 2(h + L_{345}) \sigma_{\xi^{n}}
\end{align*}
\]

Using the Five-Section Model and the same injection current shown in Appendix A1 in Fig. A1.4, the following resultant waveforms at respective levels of interest are calculated:
This time, the waveshapes are influenced not only by the Skypod, but by the Space-deck as well. The influence of the Space-deck is better seen when examining the current derivative computed for the 509\textit{m} level above ground. Under closer look, one will notice that there is a small “hump” at about 0.5\textit{µs}, which is precisely due to the modeling of the Space-deck. Furthermore, the influence from the Skypod is again noticeable at 1\textit{µs} (in the curve of the current derivative computed at 474\textit{m} above ground level).
Appendix IV
Derivation of Expressions for Distant Electric and Magnetic Fields in Cartesian Coordinates

Let us start with Ampere’s circuital law (with Maxwell’s correction):

\[ \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]  

(AIV.1)

where:

- \( \nabla \times \) The curl operator
- \( B \) Magnetic flux density
- \( \mu_0 \) Permeability of free space
- \( J \) Total current density (including both free and bound current)
- \( \varepsilon_0 \) Permittivity of free space
- \( \frac{\partial}{\partial t} \) Partial derivative with respect to time
- \( E \) Electric Field intensity

When the MAGNETIC case is considered the term \( \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \) is equal to “zero”, (no displacement current along medium 1 (conductor), where the CN Tower or the associated Lightning Channel is considered as the conductor). In this case conduction and impressed current exist only. Therefore, now Eq. (AIV.1) can be rewritten:

\[ \nabla \times B = \mu_0 J \]  

(AIV.2)

According to the Equation of Magnetic Force:
\( \mu H = \nabla \times A \)  \hspace{1cm} (AIV.3)

Constitutive Relation is:

\( \mu H = B \)

Eq. (AIV.3) is then rewritten:

\( B = \nabla \times A \)  \hspace{1cm} (AIV.4)

Substituting Eq. (AIV.4) into Eq. (AIV.2) yields the following:

\( \nabla \times \nabla \times A = \mu_0 J \)  \hspace{1cm} (AIV.5)

but \( (\nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A) \) and if we used Coulomb gauge \( (\nabla \cdot A = 0) \) then Eq. (AIV.5) becomes:

\( -\nabla^2 A = \mu_0 J \)  \hspace{1cm} (AIV.6)

or:

\[
\begin{align*}
- \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) &= \mu_0 J_x \\
- \left( \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right) &= \mu_0 J_y \\
- \left( \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) &= \mu_0 J_z
\end{align*}
\]  \hspace{1cm} (AIV.6)

Each component of \( A \) obeys the Poisson equation.

In differential form Gauss’ law says that:

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_0}
\]

where:
\( \rho \) Total charge density (including both free and bound charge)

The scalar potential \( \Phi \) can be determined from (AIV.7):

\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}
\]  
(AIV.8)

The solution to Poisson’s equation (AIV.8) has the following form:

\[
\Phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}', t - R/c) \frac{dV'}{|\vec{r} - \vec{r}'|}
\]  
(AIV.9)

Similarly, the solution to Eq. (AIV.6) is (Magnetic Vector Potential):

\[
A(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{J(\vec{r}', t - R/c) \frac{dV'}{|\vec{r} - \vec{r}'|}}
\]  
(AIV.10)

Fig. AIV.1 - Geometrical Relations Pertinent to Deriving Expressions in Cartesian Coordinates
Components “x”, “y”, and “z” of Magnetic Field

In terms of derivatives, Eq. (AIV.4) is rewritten:

\[
\mathbf{dA} = \nabla \times \mathbf{A} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
dA_x & dA_y & dA_z
\end{vmatrix}
\]

(AIV.11)

\[
\mathbf{dA} = \hat{i} \left( \frac{\partial dA_z}{\partial y} - \frac{\partial dA_y}{\partial z} \right) + \\
\hat{j} \left( \frac{\partial dA_x}{\partial z} - \frac{\partial dA_z}{\partial x} \right) + \\
\hat{k} \left( \frac{\partial dA_y}{\partial x} - \frac{\partial dA_x}{\partial y} \right)
\]

(AIV.11)

x, y, and z components of Magnetic Vector Potential \(dA\) (Eq. (AIV.10)):

\[
dA_x = \frac{\mu_0}{4\pi} \frac{i(x', t - R/c)}{R} \, dx
\]
\[
dA_y = \frac{\mu_0}{4\pi} \frac{i(y', t - R/c)}{R} \, dy
\]
\[
dA_z = \frac{\mu_0}{4\pi} \frac{i(z', t - R/c)}{R} \, dz
\]

Now, one could come up with the relations for Magnetic Flux Densities and respectively with the Magnetic Field components along x, y, and z as shown hereafter.

In the following derivations some abbreviations are adopted, which are introduced below:

\[
i_x = i(x', t - R/c) \, dx
\]
\[
i_y = i(y', t - R/c) \, dy
\]
\[
i_z = i(z', t - R/c) \, dz
\]
\[ X \equiv (x - x') \]
\[ Y \equiv (y - y') \]
\[ Z \equiv (z - z') \]

"x" component:

\[
\frac{dB_x}{dt}(x,x',y,y',z,z',t) = \frac{\partial dA_y}{\partial y} - \frac{\partial dA_z}{\partial z} = \frac{\mu_0}{4\pi} \left[ \frac{\partial i_z}{\partial y} R - \frac{\partial i_y}{\partial z} R \right]
\]

(AIV.12)

\[
\frac{dB_x}{dt}(x,x',y,y',z,z',t) = \frac{\mu_0}{4\pi} \left[ \frac{1}{R} \frac{\partial i_z}{\partial y} + i_z \frac{\partial }{\partial y} \frac{1}{R} - \frac{1}{R} \frac{\partial i_y}{\partial z} - i_y \frac{\partial }{\partial z} \frac{1}{R} \right]
\]

(AIV.12)

\[
\frac{dB_x}{dt}(x,x',y,y',z,z',t) = \frac{\mu_0}{4\pi} \left[ \frac{Y}{R^2c} \frac{\partial i_z}{\partial t} + \frac{Y}{R^2} i_z - \frac{Z}{R^2c} \frac{\partial i_y}{\partial t} - \frac{Z}{R^2} i_y \right]
\]

(AIV.13)

"y" component:

\[
\frac{dB_y}{dt}(x,x',y,y',z,z',t) = \frac{\partial dA_x}{\partial x} - \frac{\partial dA_z}{\partial z} = \frac{\mu_0}{4\pi} \left[ \frac{\partial i_x}{\partial t} R - \frac{\partial i_z}{\partial x} R \right]
\]

(AIV.14)

\[
\frac{dB_y}{dt}(x,x',y,y',z,z',t) = \frac{\mu_0}{4\pi} \left[ \frac{1}{R} \frac{\partial i_x}{\partial z} + i_x \frac{\partial }{\partial z} \frac{1}{R} - \frac{1}{R} \frac{\partial i_z}{\partial x} - i_z \frac{\partial }{\partial x} \frac{1}{R} \right]
\]

(AIV.14)

\[
\frac{dB_y}{dt}(x,x',y,y',z,z',t) = \frac{\mu_0}{4\pi} \left[ \frac{Z}{R^2c} \frac{\partial i_x}{\partial t} + \frac{Z}{R^2} i_x - \frac{X}{R^2c} \frac{\partial i_z}{\partial t} - \frac{X}{R^2} i_z \right]
\]
\[
dH_z(x, x', y, y', z, z', t) = \\
= \frac{1}{4\pi} \left[ Z \left( \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{1}{R} \frac{\partial i_z}{\partial t} \right) - X \left( \frac{1}{R} \frac{\partial i_x}{\partial t} + \frac{1}{R} \frac{\partial i_z}{\partial t} \right) \right] \\
\text{(AIV.15)}
\]

"z" component:

\[
dB_z(x, x', y, y', z, z', t) = \\
= \frac{\partial dA_y}{\partial x} - \frac{\partial dA_x}{\partial y} = \mu_0 \frac{\partial}{\partial x} \left[ \frac{i_y}{R} - \frac{i_x}{R} \right] \\
\text{(AIV.16)}
\]

\[
dB_z(x, x', y, y', z, z', t) = \\
= \mu_0 \frac{1}{4\pi} \left[ \frac{\partial i_y}{\partial x} + \frac{\partial}{\partial x} \left( \frac{1}{R} \frac{\partial i_x}{\partial t} \right) - \frac{1}{R} \frac{\partial i_x}{\partial y} - \frac{1}{R} \frac{\partial i_x}{\partial y} \right] \\
\text{(AIV.16)}
\]

\[
dB_z(x, x', y, y', z, z', t) = \\
= \mu_0 \left[ \frac{X}{R^2} \frac{\partial i_y}{\partial t} + \frac{X}{R^2} \frac{i_y}{\partial t} - \frac{Y}{R^2} \frac{\partial i_x}{\partial t} - \frac{Y}{R^2} \frac{i_x}{\partial t} \right] \\
\text{(AIV.16)}
\]

\[
dH_z(x, x', y, y', z, z', t) = \\
= \frac{1}{4\pi} \left[ X \left( \frac{1}{R} \frac{\partial i_y}{\partial t} + \frac{1}{R} \frac{\partial i_z}{\partial t} \right) - Y \left( \frac{1}{R} \frac{\partial i_x}{\partial t} + \frac{1}{R} \frac{\partial i_z}{\partial t} \right) \right] \\
\text{(AIV.17)}
\]

Components "x", "y", and "z" of Electric Field

When the ELECTRIC case is considered the term \( \mu_0 J \) in Eq. (AIV.1) is equal to "zero" (no conduction and impressed current flowing in medium 2 (air)). Therefore, now Eq. (AIV.1) can be rewritten:

\[
\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \\
\text{(AIV.18)}
\]

Constitutive relation is:

\[
B = \mu_0 H
\]
Eq. (AIV.18) is rewritten using derivatives of $B(H)$ and $E$:

$$\nabla \times dH = \varepsilon_0 \frac{\partial dE}{\partial t} \quad \text{(AIV.19)}$$

$$\nabla \times dH = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ dH_x & dH_y & dH_z \end{vmatrix} \quad \text{(AIV.20)}$$

$$\nabla \times dH = \hat{x} \left( \frac{\partial dH_z}{\partial y} - \frac{\partial dH_y}{\partial z} \right) + \hat{y} \left( \frac{\partial dH_z}{\partial x} - \frac{\partial dH_x}{\partial z} \right) + \hat{z} \left( \frac{\partial dH_y}{\partial x} - \frac{\partial dH_x}{\partial y} \right) \quad \text{(AIV.20)}$$

Using Eq. (AIV.19) and Eq. (AIV.20) and also previously derived Eqs. (AIV.13, AIV.15 and AIV.17) - Magnetic Flux Densities (derivatives) - the relations for Electric Fields (derivatives) are derived as follows:

"$x$" component:

$$\varepsilon_0 \frac{\partial dE_x}{\partial t} = \frac{\partial dH_x}{\partial y} - \frac{\partial dH_y}{\partial z} \quad \text{(AIV.21)}$$

$$\frac{\partial dE_x(x, x', y, y', z, z', t)}{\partial t} = \frac{1}{4\pi\varepsilon_0} \left[ i_x \frac{\partial}{\partial y} \left( \frac{Y}{R^3} \right) + \frac{Y}{R^3} \frac{\partial i_y}{\partial y} + \frac{Y}{R^2c} \frac{\partial i_z}{\partial y} + \frac{Y}{R^2c} \frac{\partial i_z}{\partial t} \right] + \frac{Y}{R^3c} \frac{\partial}{\partial y} \left( \frac{\partial i_x}{\partial t} \right)$$

$$- \frac{Y}{R^3c} \frac{\partial}{\partial y} \left( \frac{X}{R} \right) - \frac{X}{R} \frac{\partial i_x}{\partial y} + \frac{Y}{R^2c} \frac{\partial i_y}{\partial t} - \frac{X}{R^2c} \frac{\partial i_z}{\partial y} - \frac{X}{R^2c} \frac{\partial i_z}{\partial t}$$

$$- \frac{X}{R} \frac{\partial i_y}{\partial z} - \frac{X}{R} \frac{\partial i_z}{\partial y} + \frac{X}{R^2c} \frac{\partial i_y}{\partial t} + \frac{X}{R^2c} \frac{\partial i_z}{\partial y} + \frac{X}{R^2c} \frac{\partial i_z}{\partial t}$$

$$+ \frac{Z}{R^3} \frac{\partial i_x}{\partial z} + \frac{Z}{R^3} \frac{\partial i_x}{\partial z} + \frac{Z}{R^2c} \frac{\partial i_z}{\partial t} + \frac{Z}{R^2c} \frac{\partial i_z}{\partial t}$$

(AIV.22)
Use the following relations in (AIV.22):

\[
\begin{align*}
    i_x \frac{\partial}{\partial y} \left( \frac{Y_i R}{R^3} \right) &= R^2 - 3Y^2 \frac{i_x}{R^3} \frac{1}{\partial y} ;
    \frac{Y_i}{R^3} \frac{\partial i_x}{\partial y} = Y^2 \frac{\partial i_x}{cR^4 \partial t} ;
    \frac{\partial}{\partial y} \left( \frac{Y_i}{cR^2} \right) \frac{i_x}{\partial t} = R^2 - 4Y^2 \frac{\partial i_x}{cR^4 \partial t} ;
    \\
    \frac{Y}{cR^2} \frac{\partial (\frac{\partial i_x}{\partial t})}{\partial t} &= \frac{Y^2}{c^2 R^3} \frac{\partial^2 i_x}{\partial t^2} .
\end{align*}
\]

\[
\begin{align*}
    i_y \frac{\partial}{\partial y} \left( \frac{X_i R}{R^3} \right) &= -3XY \frac{i_y}{R^3} \frac{1}{\partial y} ;
    \frac{X_i}{R^3} \frac{\partial i_y}{\partial y} = -XY \frac{\partial i_y}{cR^4 \partial t} ;
    \frac{\partial}{\partial y} \left( \frac{X_i}{cR^2} \right) \frac{i_y}{\partial t} = -2XY \frac{\partial i_y}{cR^4 \partial t} ;
    \\
    \frac{X}{cR^2} \frac{\partial (\frac{\partial i_y}{\partial t})}{\partial t} &= -XY \frac{\partial^2 i_y}{c^2 R^3 \partial t^2} .
\end{align*}
\]

\[
\begin{align*}
    i_z \frac{\partial}{\partial z} \left( \frac{X_i R}{R^3} \right) &= -3XZ \frac{i_z}{R^3} \frac{1}{\partial z} ;
    \frac{X_i}{R^3} \frac{\partial i_z}{\partial z} = -XZ \frac{\partial i_z}{cR^4 \partial t} ;
    \frac{\partial}{\partial z} \left( \frac{X_i}{cR^2} \right) \frac{i_z}{\partial t} = -Z^2 \frac{\partial i_z}{cR^4 \partial t} ;
    \\
    \frac{X}{cR^2} \frac{\partial (\frac{\partial i_z}{\partial t})}{\partial t} &= -XZ \frac{\partial^2 i_z}{c^2 R^3 \partial t^2} .
\end{align*}
\]

\[
\begin{align*}
    i_x \frac{\partial}{\partial z} \left( \frac{Z_i R}{R^3} \right) &= R^2 - 3Z^2 \frac{i_x}{R^3} \frac{1}{\partial z} ;
    \frac{Z_i}{R^3} \frac{\partial i_x}{\partial z} = Z^2 \frac{\partial i_x}{cR^4 \partial t} ;
    \frac{\partial}{\partial z} \left( \frac{Z_i}{cR^2} \right) \frac{i_x}{\partial t} = R^2 - 4Z^2 \frac{\partial i_x}{cR^4 \partial t} ;
    \\
    \frac{Z}{cR^2} \frac{\partial (\frac{\partial i_x}{\partial t})}{\partial t} &= -Z^2 \frac{\partial^2 i_x}{c^2 R^3 \partial t^2} .
\end{align*}
\]

To get:

\[
\begin{align*}
    \partial dE_s(x, x', y, y', z, z', t) =
    \frac{1}{4\pi \varepsilon_0} \left[ \frac{R^2 - 3Y^2 + R^2 - 3Z^2}{R^3} \right] i_x \\
    + \left( \frac{Y^2 + R^2 - 4Y^2 + Z^2 + R^2 - 4Z^2}{cR^4} \right) \frac{i_x}{\partial t} \\
    + \left( \frac{-Y^2 - Z^2}{c^2 R^3} \right) \frac{\partial^2 i_x}{\partial t^2} \\
    + \left( \frac{3XY}{R^5} \right) i_y + \left( \frac{3XY}{cR^4} \right) \frac{i_y}{\partial t} + \left( \frac{XY}{c^2 R^3} \right) \frac{\partial^2 i_y}{\partial t^2} \\
    + \left( \frac{3XZ}{R^5} \right) i_z + \left( \frac{3XZ}{cR^4} \right) \frac{i_z}{\partial t} + \left( \frac{XZ}{c^2 R^3} \right) \frac{\partial^2 i_z}{\partial t^2} \\
    \right]
\end{align*}
\]
where:

\[
\left( \frac{R^2 - 3Y^2 + R^2 - 3Z^2}{R^3} \right) = \left( \frac{2R^2 - 3X^2 - 3Y^2 - 3Z^2 + 3X^2}{R^5} \right) = \left( \frac{2R^2 - 3R^2 + 3X^2}{R^5} \right) = \left( \frac{3X^2}{R^5} - \frac{1}{R^3} \right)
\]

\[
\left( \frac{Y^2 + R^2 - 4Y^2 + Z^2 + R^2 - 4Z^2}{cR^4} \right) = \left( \frac{2R^2 - 3X^2 - 3Y^2 - 3Z^2 + 3X^2}{cR^4} \right) = \left( \frac{3X^2}{cR^4} - \frac{1}{cR^2} \right)
\]

\[
\left( -\frac{Y^2 - Z^2}{c^2 R^3} \right) = \left( -\frac{X^2 - Y^2 - Z^2 + X^2}{c^2 R^3} \right) = \left( \frac{X^2}{c^2 R^3} - \frac{1}{c^2 R} \right)
\]

So that finally after integration of Eq. (AIV.23) we can write down Eq. (IV.22).

“\( y \)” component:

\[
\varepsilon_0 \frac{\partial dE_y}{\partial t} = \frac{\partial dH_z}{\partial z} - \frac{\partial dH_x}{\partial x} \tag{AIV.24}
\]

\[
\frac{\partial dE_y(x',y',y,z',z,t)}{\partial t} = \frac{1}{4\pi\varepsilon_0} \left[ \int_y \frac{\partial}{\partial z} \left( \frac{Z}{R^3} \right) + \frac{Z}{R^3} \frac{\partial i_y}{\partial z} + \frac{Z}{R^2 c} \frac{\partial i_y}{\partial t} + \frac{Z}{R^2 c} \frac{\partial i_y}{\partial z} \right] - \frac{i_y}{R^3} \frac{\partial}{\partial z} \left( \frac{Y}{R^3} \right) - \frac{Y}{R^3} \frac{\partial i_z}{\partial z} - \frac{Y}{R^2 c} \frac{\partial i_z}{\partial t} - \frac{Y}{R^2 c} \frac{\partial i_z}{\partial z} + \frac{i_x}{R^3} \frac{\partial}{\partial z} \left( \frac{Y}{R^3} \right) - \frac{Y}{R^3} \frac{\partial i_x}{\partial z} - \frac{Y}{R^2 c} \frac{\partial i_x}{\partial t} - \frac{Y}{R^2 c} \frac{\partial i_x}{\partial z} + \frac{i_x}{R^3} \frac{\partial}{\partial z} \left( \frac{X}{R^3} \right) + \frac{X}{R^3} \frac{\partial i_x}{\partial z} + \frac{X}{R^2 c} \frac{\partial i_x}{\partial t} + \frac{X}{R^2 c} \frac{\partial i_x}{\partial z} \right] \tag{AIV.25}
\]

Use the following relations in (AIV.25):
\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{Z}{R^3} \right) &= \frac{R^2-3Z^2}{R^5} j_y; \\
\frac{\partial}{\partial z} \left( \frac{Z}{R^3} \right) &= \frac{Z^2}{R^3} \frac{\partial i_y}{\partial t}; \\
\frac{\partial}{\partial z} \left( \frac{Z}{cR^2} \right) \frac{\partial i_y}{\partial t} &= \frac{R^2-4Z^2}{cR^4} \frac{\partial i_y}{\partial t}; \\
\left( \frac{Z}{cR^2} \right) \frac{\partial^2 i_y}{\partial z \partial t} &= -\frac{Z^2}{c^2 R^5} \frac{\partial^2 i_y}{\partial t^2}. \\
\frac{\partial}{\partial x} \left( \frac{Y}{R^3} \right) &= -\frac{3YZ}{R^3} i_z; \\
\frac{\partial}{\partial z} \left( \frac{Y}{R^3} \right) &= -\frac{YZ}{cR^4} \frac{\partial i_z}{\partial t}; \\
\frac{\partial}{\partial z} \left( \frac{Y}{cR^2} \right) \frac{\partial i_z}{\partial t} &= -\frac{2YZ}{cR^4} \frac{\partial i_z}{\partial t}; \\
\left( \frac{Y}{cR^2} \right) \frac{\partial^2 i_z}{\partial x \partial t} &= -\frac{YZ}{c^2 R^5} \frac{\partial^2 i_z}{\partial t^2}. \\
\frac{\partial}{\partial x} \left( \frac{X}{R^3} \right) &= \frac{R^2-3X^2}{R^5} i_y; \\
\frac{\partial}{\partial x} \left( \frac{X}{R^3} \right) &= \frac{X^2}{R^3} \frac{\partial i_y}{\partial t}; \\
\frac{\partial}{\partial x} \left( \frac{X}{cR^2} \right) \frac{\partial i_y}{\partial t} &= \frac{R^2-4X^2}{cR^4} \frac{\partial i_y}{\partial t}; \\
\left( \frac{X}{cR^2} \right) \frac{\partial^2 i_y}{\partial x \partial t} &= -\frac{X^2}{c^2 R^5} \frac{\partial^2 i_y}{\partial t^2}.
\end{align*}
\]

To get:

\[
\frac{\partial dE_y(x', y', z', t)}{\partial t} = \frac{1}{4\pi\varepsilon_0} \left[ \left( 3XY \right) i_x + \left( 3XY \right) \frac{\partial i_x}{\partial t} + \left( \frac{XY}{c^2 R^3} \right) \frac{\partial^2 i_x}{\partial t^2} \right] \\
+ \left( \frac{2r^2-3Z^2-3X^2}{R^5} \right) j_y + \left( \frac{2r^2-3Z^2-3X^2}{cR^4} \right) \frac{\partial i_y}{\partial t} \\
+ \left( \frac{-Z^2-X^2}{c^2 R^5} \right) \frac{\partial^2 i_y}{\partial t^2} \\
+ \left( \frac{3YZ}{R^5} \right) j_z + \left( \frac{3YZ}{cR^4} \right) \frac{\partial i_z}{\partial t} + \left( \frac{YZ}{c^2 R^3} \right) \frac{\partial^2 i_z}{\partial t^2} \right]
\]  

(AIV.26)

where:
\[
\left( \frac{2R^2 - 3Z^2 - 3X^2}{R^5} \right) = \left( \frac{2R^2 - 3X^2 - 3Y^2 - 3Z^2 + 3Y^2}{R^5} \right) = \\
\left( \frac{2R^2 - 3R^2 + 3Y^2}{R^5} \right) = \left( \frac{3Y^2}{R^5} - \frac{1}{R^3} \right) \\
\left( \frac{-2R^2 - 3Z^2 - 3X^2}{cR^4} \right) = \left( \frac{2R^2 - 3X^2 - 3Y^2 - 3Z^2 + 3Y^2}{cR^4} \right) = \left( \frac{3Y^2}{cR^5} - \frac{1}{cR^3} \right) \\
\left( \frac{-Z^2 - X^2}{c^2R^3} \right) = \left( \frac{-X^2 - Y^2 - Z^2 + Y^2}{c^2R^3} \right) = \left( \frac{Y^2}{c^2R^3} - \frac{1}{c^2R} \right)
\]

Eq. (AIV.26) is integrated in order to come up with Eq. (IV.23).

“z” component:

\[
\varepsilon_0 \frac{\partial dE_z}{\partial t} = \frac{\partial dH_y}{\partial y} - \frac{\partial dH_z}{\partial z}
\]

\[
\frac{\partial dE_z(x',y',y',z',z',t)}{\partial t} = \frac{1}{4\pi\varepsilon_0} \left[ i_z \frac{\partial}{\partial x} \left( \frac{X}{R^3} \right) + X \frac{\partial i_z}{\partial x} + \frac{X}{R^3} c \frac{\partial i_z}{\partial x} \right] + X \frac{\partial}{\partial x} \left( \frac{X}{R^3} \right) \\
- i_z \frac{\partial}{\partial x} \left( \frac{Z}{R^3} \right) - \frac{Z}{R^3} \frac{\partial i_z}{\partial x} - \frac{Z}{R^3} c \frac{\partial i_z}{\partial x} \right] + X \frac{\partial}{\partial x} \left( \frac{i_z}{R^3} \right)
\]

(AIV.28)

Use the following relations in (AIV.28):

\[
\frac{i_z}{R^3} \frac{\partial}{\partial x} \left( \frac{X}{R^3} \right) = \frac{R^2 - 3X^2}{R^5} i_z; \quad X \frac{\partial i_z}{\partial c} = \frac{X^2}{cR^4} \frac{\partial i_z}{\partial t}; \quad \frac{\partial}{\partial x} \left( \frac{X}{R^3} \right) \frac{\partial i_z}{\partial t} = \frac{R^2 - 4X^2}{cR^4} \frac{\partial i_z}{\partial t};
\]

\[
\frac{X}{R^2 c} \frac{\partial}{\partial x} \left( \frac{\partial i_z}{\partial t} \right) = -\frac{X^2}{c^2R^4} \frac{\partial^2 i_z}{\partial t^2}.
\]

\[
i_z \frac{\partial}{\partial x} \left( \frac{Z}{R^3} \right) = -i_z \frac{3XZ}{R^5}; \quad Z \frac{\partial i_z}{\partial x} = -\frac{XZ}{R^4 c} \frac{\partial i_z}{\partial t}; \quad \frac{\partial}{\partial x} \left( \frac{Z}{R^3} \right) \frac{\partial i_z}{\partial t} = -\frac{2XZ}{cR^4} \frac{\partial i_z}{\partial t};
\]

\[
\frac{Z}{R^2 c} \frac{\partial}{\partial x} \left( \frac{\partial i_z}{\partial t} \right) = -\frac{Z}{c^2R^4} \frac{\partial^2 i_z}{\partial t^2}.
\]
\[ i_y \frac{\partial}{\partial y} \left( \frac{Z}{R^3} \right) = -i_y \frac{3YZ}{R^3 R^3} \frac{\partial i_y}{\partial y} \frac{Z}{R^4} \frac{\partial i_y}{\partial t} \frac{Z}{R^2} \frac{\partial i_y}{\partial t} = -2YZ \frac{\partial i_y}{\partial t} \]

\[ Z \frac{\partial}{\partial y} \left( i_y \frac{\partial i_y}{\partial t} \right) = -Z R c \frac{\partial^2 i_y}{\partial t^2} \]

\[ i_z \frac{\partial}{\partial y} \left( \frac{Y}{R^3} \right) = R^2 - 3Y^2 \frac{\partial i_z}{\partial t} \frac{Y}{R^4} \frac{\partial i_z}{\partial t} \frac{Y}{R^2} \frac{\partial i_z}{\partial t} = R^2 - 4Y^2 \frac{\partial i_z}{\partial t} \]

\[ Y \frac{\partial}{\partial y} \left( i_i \frac{\partial i_z}{\partial t} \right) = -Y^2 \frac{\partial^2 i_z}{\partial t^2} \]

To get:

\[ \frac{\partial dE_z}{\partial t} = \frac{1}{4\pi\varepsilon_0} \left[ i_y \left( \frac{3YZ}{R^3} \right) + \frac{\partial i_y}{\partial t} \frac{3YZ}{R^4} + \frac{\partial^2 i_y}{\partial t^2} \frac{Z}{R^2} \frac{Z}{c^2 R} \right] \]

\[ + i_y \left( \frac{3YZ}{R^5} \right) + \frac{\partial i_y}{\partial t} \frac{3YZ}{R^6} + \frac{\partial^2 i_y}{\partial t^2} \frac{Z}{R^3} \frac{1}{c^2 R} \]

\[ + i_z \left( \frac{3Z^2 - 1}{R^5} \right) + \frac{\partial i_z}{\partial t} \left( \frac{3Z^2}{R^5} - \frac{1}{R^5} \right) + \frac{\partial^2 i_z}{\partial t^2} \left( \frac{Z^2}{c^2 R^3} - \frac{1}{c^2 R} \right) \]

where:

\[ \left( \frac{R^2 - 3X^2 + R^2 - 3Y^2}{R^3} \right) = \left( \frac{2R^2 - 3X^2 - 3Y^2 - 3Z^2 + 3Z^2}{R^5} \right) = \left( \frac{2R^2 - 3R^2 + 3Z^2}{R^5} \right) = \left( \frac{3Z^2 - 1}{R^5} \right) \]

\[ \left( \frac{X^2 + R^2 - 4X^2 + Y^2 + R^2 - 4Y^2}{cR^4} \right) = \left( \frac{2R^2 - 3X^2 - 3Y^2 - 3Z^2 + 3Z^2}{cR^4} \right) = \left( \frac{3Z^2}{cR^4} - \frac{1}{cR^2} \right) \]

\[ \left( -\frac{X^2 - Y^2}{c^2 R^3} \right) = \left( -\frac{X^2 - Y^2 - Z^2 + Z^2}{c^2 R^3} \right) = \left( \frac{Z^2}{c^2 R^3} - \frac{1}{c^2 R} \right) \]

Eq. (IV.24) is produced after Integration of Eq. (AIV.29).
Appendix V
Geometrical Relations and Current Components Used in Calculations Involving Inclined Lightning Channel

In Fig. IV.4 of Chapter IV, angle $\phi$ varies from 0-360° in x-y plane, and angle $\gamma$ varies from 0-90° as seen in Fig. IV.3 respectively. In order to perform a sensitivity study looking into the influence of the Lightning Channel position in 3-D space upon the radiated fields at a certain distance, the x-y plane is subdivided into four quadrants and the corresponding geometrical relations and current components are shown below. The total $E_z$ or $H_x$ field at the observation point $P$ is comprised of the difference of the two contributions with equal magnitudes and opposite signs, which ultimately translates into multiplying the real solution by a factor of “two”. Nevertheless, all pertinent relations are written down.

First Quadrant (-x, +y, +z)

Fig. AV.1 - Geometrical Relations in First Quadrant
In Fig. AV.2 above, $i_x, i_y, i_z$ are the current components along $x, y,$ and $z$ axes of the total dipole current $i$ which flows through the discrete length $z'_1-z'_0$ in the direction of the growing Lightning Channel.

**Solutions for Real Part**

\[
i_x = -i \sin(\gamma) \cos(\alpha) \\
i_y = i \sin(\gamma) \sin(\alpha) \\
i_z = i \cos(\gamma)
\]  

(V.A1)

\[
R_n = \sqrt{(d - n \sin(\gamma) \sin(\alpha))^2 + (n \sin(\gamma) \cos(\alpha))^2 + (h + n \cos(\gamma))^2}
\]  

(V.A2)

\[
x = 0 \quad x = \sin(\gamma) \cos(\alpha) \quad x = 2 \sin(\gamma) \cos(\alpha) \\
z_0' \rightarrow y = d \quad z_1' \rightarrow y = d - \sin(\gamma) \sin(\alpha) \quad z_2' \rightarrow y = d - 2 \sin(\gamma) \sin(\alpha) \\
z = -h \quad z = -h - \cos(\gamma) \quad z = -h - 2 \cos(\gamma)
\]  

(V.A3)

**Solutions for Imaginary Part**

\[
i_x = -i \sin(\gamma) \cos(\alpha) \\
i_y = i \sin(\gamma) \sin(\alpha) \\
i_z = -i \cos(\gamma)
\]  

(V.A4)
\[ R_n = \sqrt{(d - n \sin(\gamma) \sin(\alpha))^2 + (n \sin(\gamma) \cos(\alpha))^2 + (h + n \cos(\gamma))^2} \] (V.A5)

\[
\begin{align*}
x &= 0 & x &= \sin(\gamma) \cos(\alpha) & x &= 2 \sin(\gamma) \cos(\alpha) \\
z_0' &\rightarrow y = d & z_1' &\rightarrow y = d - \sin(\gamma) \sin(\alpha) & z_2' &\rightarrow y = d - 2 \sin(\gamma) \sin(\alpha) \\
z &= h & z &= h + \cos(\gamma) & z &= h + 2 \cos(\gamma)
\end{align*}
\] (V.A6)

Second Quadrant (-x, -y, +z)

---

Fig. AV.3 - Geometrical Relations in Second Quadrant

Fig. AV.4 - Currents in Second Quadrant
Solutions for Real Part

\[ i_x = -i \sin(\gamma) \cos(\alpha) \]
\[ i_y = -i \sin(\gamma) \sin(\alpha) \]
\[ i_z = i \cos(\gamma) \] (V.A7)

\[ R_n = \sqrt{(d + n \sin(\gamma) \sin(\alpha))^2 + (n \sin(\gamma) \cos(\alpha))^2 + (h + n \cos(\gamma))^2} \] (V.A8)

\[ x = 0 \quad x = \sin(\gamma) \cos(\alpha) \quad x = 2 \sin(\gamma) \cos(\alpha) \]
\[ z_0' \rightarrow y = d \quad z_1' \rightarrow y = d + \sin(\gamma) \sin(\alpha) \quad z_2' \rightarrow y = d + 2 \sin(\gamma) \sin(\alpha) \] (V.A9)
\[ z = -h \quad z = -h - \cos(\gamma) \quad z = -h - 2 \cos(\gamma) \]

Solutions for Imaginary Part

\[ i_x = -i \sin(\gamma) \cos(\alpha) \]
\[ i_y = -i \sin(\gamma) \sin(\alpha) \]
\[ i_z = -i \cos(\gamma) \] (V.A10)

\[ R_n = \sqrt{(d + n \sin(\gamma) \sin(\alpha))^2 + (n \sin(\gamma) \cos(\alpha))^2 + (h + n \cos(\gamma))^2} \] (V.A11)

\[ x = 0 \quad x = \sin(\gamma) \cos(\alpha) \quad x = 2 \sin(\gamma) \cos(\alpha) \]
\[ z_0' \rightarrow y = d \quad z_1' \rightarrow y = d + \sin(\gamma) \sin(\alpha) \quad z_2' \rightarrow y = d + 2 \sin(\gamma) \sin(\alpha) \] (V.A12)
\[ z = h \quad z = h + \cos(\gamma) \quad z = h + 2 \cos(\gamma) \]
Third Quadrant (+x, -y, +z)

Fig. AV.5 - Geometrical Relations in Third Quadrant

Fig. AV.6 - Currents in Third Quadrant

Solutions for Real Part

\[
\begin{align*}
ix &= i \sin(\gamma) \cos(\alpha) \\
\iy &= -i \sin(\gamma) \sin(\alpha) \\
\iz &= i \cos(\gamma)
\end{align*}
\]  \hspace{1cm} (V.A13)

\[
R_n = \sqrt{(d + n \cos(\gamma))^2 + (n \sin(\gamma) \cos(\alpha))^2 + (d + n \sin(\gamma) \sin(\alpha))^2}
\]  \hspace{1cm} (V.A14)

\[
\begin{align*}
x = 0 & \quad x = -\sin(\gamma) \cos(\alpha) & \quad x = -2 \sin(\gamma) \cos(\alpha) \\
z_0' \rightarrow y = d & \quad z_1' \rightarrow y = d + \sin(\gamma) \sin(\alpha) & \quad z_2' \rightarrow y = d + 2 \sin(\gamma) \sin(\alpha) \\
z = -h & \quad z = -h - \cos(\gamma) & \quad z = -h - 2 \cos(\gamma)
\end{align*}
\]  \hspace{1cm} (V.A15)
Solutions for Imaginary Part

\[
\begin{align*}
  i_x &= i \sin(\gamma) \cos(\alpha) \\
  i_y &= -i \sin(\gamma) \sin(\alpha) \\
  i_z &= -i \cos(\gamma)
\end{align*}
\]  

(V.A16)

\[ R_n = \sqrt{(d + n \cos(\gamma))^2 + (n \sin(\gamma) \cos(\alpha))^2 + (d + n \sin(\gamma) \sin(\alpha))^2} \]  

(V.A17)

\[
\begin{align*}
  x &= 0 \\
  x &= -\sin(\gamma) \cos(\alpha) \\
  x &= -2 \sin(\gamma) \cos(\alpha) \\
  z_0' \rightarrow y &= d \\
  z_1' \rightarrow y &= d + \sin(\gamma) \sin(\alpha) \\
  z_2' \rightarrow y &= d + 2 \sin(\gamma) \sin(\alpha) \\
  z &= h \\
  z &= h + \cos(\gamma) \\
  z &= h + 2 \cos(\gamma)
\end{align*}
\]

(V.A18)

Fourth Quadrant (+x, +y, +z)

Fig. AV.7 - Geometrical Relations in Fourth Quadrant

Fig. AV.8 - Currents in Fourth Quadrant
Solutions for Real Part

\[
i_x = i \sin(\gamma)\cos(\alpha) \\
i_y = i \sin(\gamma)\sin(\alpha) \\
i_z = i \cos(\gamma)
\]

(V.A19)

\[
R_n = \sqrt{(n \sin(\gamma)\cos(\alpha))^2 + (d - n \sin(\gamma)\sin(\alpha))^2 + (h + n \cos(\gamma))^2}
\]

(V.A20)

\[
x = 0 \quad x = -\sin(\gamma)\cos(\alpha) \quad x = -2 \sin(\gamma)\cos(\alpha) \\
z_1' \rightarrow y = d \quad z_1' \rightarrow y = d - \sin(\gamma)\sin(\alpha) \quad z_2' \rightarrow y = d - 2 \sin(\gamma)\sin(\alpha) \\
z = -h \quad z = -h - \cos(\gamma) \quad z = -h - 2 \cos(\gamma)
\]

(V.A21)

Solutions for Imaginary Part

\[
i_x = i \sin(\gamma)\cos(\alpha) \\
i_y = i \sin(\gamma)\sin(\alpha) \\
i_z = -i \cos(\gamma)
\]

(V.A22)

\[
R_n = \sqrt{(n \sin(\gamma)\cos(\alpha))^2 + (d - n \sin(\gamma)\sin(\alpha))^2 + (h + n \cos(\gamma))^2}
\]

(V.A23)

\[
x = 0 \quad x = -\sin(\gamma)\cos(\alpha) \quad x = -2 \sin(\gamma)\cos(\alpha) \\
z_0' \rightarrow y = d \quad z_1' \rightarrow y = d - \sin(\gamma)\sin(\alpha) \quad z_2' \rightarrow y = d - 2 \sin(\gamma)\sin(\alpha) \\
z = h \quad z = h + \cos(\gamma) \quad z = h + 2 \cos(\gamma)
\]

(V.A24)