Abstract

On the Plane Fixed Point Problem

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Several conjectured and proven generalizations of the Brouwer Fixed Point Theorem are examined, the plane fixed point problem in particular. The difficulties in proving this important conjecture are discussed. It is shown that it is true when strong additional assumptions are made. Canonical examples are produced which demonstrate the differences between this result and other generalized fixed point theorems.
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Chapter 1

Introduction and Motivation

The Brouwer Fixed Point Theorem is a classic result in topology from the 1800s. It, along with other fixed point theorems, have been indispensable in proving many theorems. As such, it was only natural to attempt to extend Brouwer’s result to a larger class of subsets of $\mathbb{R}^n$.

There has been much work in producing additional necessary and/or sufficient conditions for a subset $S$ of $\mathbb{R}^n$ to have the property that any continuous function $f : S \to S$ yields a point $x \in S$ such that $f(x) = x$. If all such functions from $S$ to $S$ have this property, then $S$ is said to have the fixed point property. The case $\mathbb{R}^2$ has been of particular interest, and has not yet been fully explored.

To this end, R.H. Bing outlined a number of open problems and conjectures in his 1969 paper *The Elusive Fixed Point Theorem* [1]. In it, Bing asked if every compact, connected set $K \in \mathbb{R}^2$ with connected complement had the fixed point property. This is the plane fixed point problem.

This problem is still open, however, progress has been made. One relatively recent theorem proved by Charles Hagopian in 1996 was that any set $C$ that is compact and path connected has the fixed point property if and only if $C$ has trivial fundamental group [2]. In particular, since any compact, connected set with connected complement has a trivial
fundamental group, the plane fixed point problem is true for all path connected sets.

We shall discuss why many promising techniques break down when attempting to prove the plane fixed point problem. This will be followed by a proof of a modified version of the plane fixed point problem, one where an additional restriction is applied to the complement of the set, ensuring that it cannot be too pathological. We shall then give examples that demonstrate the differences in the categories of sets that satisfy the hypotheses in the Brouwer Fixed Point Theorem, Hagopian’s theorem, the plane fixed point problem, and our modified plane fixed point problem.
Chapter 2

Pathological Cases

We shall first discuss some examples illustrating why the plane fixed point problem is difficult. Although we shall see that the typical methods used to prove the fixed point property on other sets do not apply directly here, they can be modified to lend insight as to the behavior of possible fixed point-free functions. As such, these basic ideas still offer strong intuitive value. The example we shall look at is a closed disc with several of spirals converging inward towards the boundary of the disc, as shown in figure 2.1. This shape resembles a spiral galaxy with \( n \) arms (we shall call it a galaxy set). Note that we can also create galaxies with infinitely many (even uncountably many) arms, but for the sake of this discussion a finite number will suffice.

Figure 2.1: Galaxy set with 2 arms
Why is this example pathological? We shall compare it to the unit disc, which is a common set to which the Brouwer Fixed Point Theorem is applied. First, it is not path connected (if the number of spiral sections is \( n \), then it has \( n + 1 \) path connected components). This means that if we try to join two points by a continuous curve, we cannot assume that this can be done with the entire curve contained in the set. Indeed, the most that we can hope for is the points \( a \) and \( b \) to be joined by line segments connecting a sequence of points, starting from \( a \) and ending at \( b \), such that all points in the sequence lie in the set, as in figure 2.2.

![Figure 2.2: An approximate curve from \( a \) to \( b \) in a galaxy set](image)

Since the set is connected, for any positive \( \epsilon \), we can find such a sequence of points from \( a \) to \( b \) such that the distance between successive points is less than \( \epsilon \). Note that as we reduce \( \epsilon \), the resultant curves can vary drastically. In the case of the galaxy, as \( \epsilon \) gets smaller, the resultant curve must wind more and more around the central disc. If we quantify the number of times such a curve winds around the center of the disc, then as \( \epsilon \) goes to 0, the number of revolutions approaches infinity. Hence, we cannot hope for some sort of convergence to a "nice" curve (see figure 2.3).

There is an additional pathological problem which arises. Choose such a sequence of segments, \( \gamma \), and then consider \( f(\gamma) \) for some continuous function \( f \) from the set to itself. By \( f(\gamma) \) we mean the sequence of line segments joining the image under \( f \) of the original points. The maximum separation distance between these new points may be
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Figure 2.3: Approximate curves in a galaxy set, $\epsilon$ decreasing from left to right

larger than the original maximum separation distance. This means that as we look at successive images under $f$, the curves that are formed become more and more "rough", i.e. look less and less like continuous curves in the set.

Next, the disc has a continuous Jordan curve as its boundary. We can then prove that, given a continuous function $f$ from it to itself, this function has a fixed point in the following way.

Let $\gamma$ be the Jordan curve that is the boundary of the disc. Consider the curve $\gamma'$ given by $\gamma'(t) = f(\gamma(t)) - \gamma(t)$. If $f$ has a fixed point on its boundary, then we are done. If not, then since $f$ maps the boundary into the disc and the disc is convex, the curve $\gamma'$ has nonzero winding number around 0. Since the disc is simply connected, $\gamma$ can be contracted to a point. This induces a continuous deformation from $\gamma'$ to a point. Since a point has zero winding number, this means that $\gamma'$ must go through 0 at some point. This immediately implies that $f$ must have a fixed point.

In the case of the galaxy set, we can attempt the same approach, however, the boundary is not a Jordan curve (it is not even path connected). As such, it is not clear how to begin applying this technique; we could choose an approximate polygonal boundary (join points on the boundary with line segments) as above, but it is not clear which points to choose or in what order to connect them (figure 2.4). One also runs into the difficulty as above of the successive images of this approximate boundary having segments
of increasing length, as discussed previously.

Figure 2.4: Two possible approximate boundaries
Chapter 3

Main Result

We seek to prove that all sets $S$ that are compact, connected, have connected complement, and which have an additional condition on $S^c$ have the fixed point property. The precise statement of the theorem which we will prove first requires a preliminary definition.

**Definition 1.** Given two curves $\alpha, \beta \in \mathbb{R}^2$ and an $\epsilon > 0$, we say that $\alpha$ and $\beta$ are $\epsilon$-close if we can find a homotopy from $\alpha$ to $\beta$ in $\mathbb{R}^2$, $F(s,t)$, such that $|F(s,t_1) - F(s,t_2)| < \epsilon$ for all $s, t_1, t_2 \in [0, ..., 1]$.

Fundamentally, this says that $\alpha$ and $\beta$ are $\epsilon$-close if we can continuously deform $\alpha$ into $\beta$ in such a way that throughout the deformation, no point leaves the $\epsilon$-neighborhood of its initial position. We can now state the modified plane fixed point problem whose proof is our goal.

**Theorem 2.** Any set $S \in \mathbb{R}^2$ has the fixed point property if it is compact, has a connected complement, is itself connected, and has the additional property that, for every $\epsilon > 0$, there is a Jordan curve $\alpha \in S^c$ such that for any Jordan curve $\beta \in S^c \cap \text{int}(\alpha)$, $\beta$ is $\epsilon$-close to $\gamma$, where $\gamma$ is homotopic to $\alpha$ in $\text{img}(\alpha)$.

Note that $\text{int}(\alpha)$ is the interior of $\alpha$, and $\text{img}(\alpha)$ is the image of $\alpha$ (a subset of $\mathbb{R}^2$). We shall say that any set that possesses these properties is a **feasible** set. The main tool in proving this theorem is a basic result in algebraic topology:
Theorem 3. Given continuous closed curves $\alpha, \beta \in \mathbb{R}^2 - (0,0)$ such that the winding number of $\alpha$ differs from that of $\beta$, then $\alpha$ cannot be homotopic to $\beta$ in $\mathbb{R}^2 - (0,0)$.

How is this result applicable to our theorem? Suppose we have a function $f$ defined on some set $K \subset \mathbb{R}^2$. Further, suppose that we can find a closed curve $\alpha \subset K$ such that $\alpha$ is homotopic to a point in $K$, and such that the curve $\beta(s) = \alpha(s) - f(\alpha(s))$ lies in $\mathbb{R}^2 - (0,0)$ and has non-zero winding number about 0. Then we can deduce that $f$ has a fixed point - if $f$ does not have a fixed point, then the homotopy from $\alpha$ to a point $x \in K$ induces a homotopy in $\mathbb{R}^2 - (0,0)$ from $\beta$ to the curve $\gamma(s) = x - f(x)$ (a constant curve). As we have produced a homotopy from a curve with nonzero winding number to one which has winding number 0 in $\mathbb{R}^2 - (0,0)$, we have a contradiction to theorem 3, and so $f$ has a fixed point.

We state several necessary definitions and then prove a theorem generalizing this result to feasible sets. First, given a feasible set $S$, we will need a more robust notion of what a curve in $S$ is, and a more concrete definition of some of its properties:

Definition 4. An $\epsilon$-close polygonal curve in a set $S \subset \mathbb{R}^2$ is a curve formed by joining with line segments points $p_1, \ldots, p_n \in S$ with $|p_i - p_{i+1}| < \epsilon$ for $i < n$. If $p_1 = p_n$, we say that it is a closed $\epsilon$-close polygonal curve.

If such a polygonal curve $\alpha$ lay entirely in $S$, and we had a function $f$ defined on $S$, then we could consider the curve $\beta(s) = \alpha(s) - f(\alpha(s))$, as we did above. However, clearly we may have situations when the line segments do not lie completely in $S$. As such, we must also produce a new definition of this difference curve, which we accomplish by extending $f$ to the line segments via linear interpolation:

Definition 5. Given a polygonal curve $\alpha$ joining points $p_1, \ldots, p_n \in S \subset \mathbb{R}^2$, and given a continuous function $f : S \to S$, define the difference curve of $\alpha$ with respect to $f$, $\alpha_f$, as follows. Parameterize $\alpha$ so that $\alpha : [0, n] \to S$, $\alpha(i + x) = x \cdot p_{i+1} - (x - 1) \cdot p_i$ for $x \in [0, 1], i \in 0, \ldots, n - 1$. Then a parameterization of $\alpha_f$ is given by
\[ \alpha_f(i + x) = \alpha(i + x) - (x \cdot f(p_{i+1}) - (x - 1) \cdot f(p_i)) \]

again with \( x \in [0, 1] \), and with \( i \in 0, \ldots, n - 1 \).

Given a closed polygonal curve \( \alpha \), if \( \alpha_f \cap \{(0, 0)\} = \emptyset \), then we can compute the winding number of \( \alpha_f \) around \((0, 0)\) in the usual way. In general we cannot of course assume this, so we shall endeavor to prove that if there is a homotopy from \( \alpha_f \) to a constant curve that does not in some sense leave \( S \) (the target feasible set) very much, then a fixed point must exist. We require several preliminary definitions before we state this observation rigorously.

**Definition 6.** Given a feasible set \( S \) and a continuous function \( f : S \to S \), define the **minimal displacement distance** of \( f \) as

\[ \delta_f = \inf_{x \in S} |x - f(x)| \]

**Definition 7.** Given a feasible set \( S \) and a continuous function \( f : S \to S \), define the **displacement radius** \( R_f \) as

\[ R_f = \sup \{ r : 0 \leq r \leq \frac{\delta_f}{2}, \forall x \in S, \forall y \in S, |x - y| < r \Rightarrow B_{\frac{\delta_f}{2}}(x) \cap b(f(x), f(y)) = \emptyset \} \]

given \( b(f(x), f(y)) \) is the smallest closed ball that contains \( f(x) \) and \( f(y) \), and \( B_{\frac{\delta_f}{2}}(x) \) is the closed ball about \( x \) of radius \( \frac{\delta_f}{2} \).

If \( f \) has no fixed points, then by the compactness of \( S \) and the continuity of \( f \), \( \delta_f > 0 \) and \( R_f > 0 \). We need one last definition before we state our theorem:

**Definition 8.** Given a closed continuous curve \( \gamma \in \mathbb{R}^2 \), we define the **interior** of \( \gamma \) as the union of the image of gamma with all bounded components of the complement of the image of \( \gamma \).
With these definitions in hand, we may now state our theorem:

**Theorem 9.** Given a feasible set $S$, and a continuous function $f : S \to S$, $f$ has a fixed point if we can find an $\epsilon$-close closed polygonal curve $\gamma$ such that $\epsilon < \frac{R_f}{20}$, $\gamma_f$ does not pass through $(0, 0)$ and has nonzero winding number, and $\text{img}(\gamma)$ is in the $\frac{R_f}{20}$ neighborhood of $S$.

We begin with a lemma which embodies the main tool to prove the above theorem:

**Lemma 1.** Given $S, f$ as usual and $\gamma$ a closed polygonal curve in $S$ such that $\gamma_f$ has nonzero winding number, if we can find a continuous function $g : \text{int}(\gamma) \to \mathbb{R}^2$ such that $\gamma_g = \gamma_f$, then $g$ must contain a fixed point.

**Proof.** We prove this by contradiction - assume that $g$ has no fixed point. We know that $\gamma_f$ has nonzero winding number, and from the definition of $\text{int}(\gamma)$, it is clear that $\gamma$ is homotopic to a point in $\text{int}(\gamma)$ (with homotopy $H$). As such, this homotopy induces a homotopy $F$ from $\gamma_f$ to a nonzero point defined by $F(s, t) = H(s, t) - f(H(s, t))$, which has winding number 0 about $(0, 0)$. Since $g$ has no fixed points, this induced homotopy lies in $\mathbb{R}^2 - \{(0, 0)\}$, and so by theorem 3 this is a contradiction.

**Proof of Theorem 9.** We shall prove the theorem by contradiction, so begin by assuming that $f$ has no fixed points. We are given the existence of a curve $\gamma$ with the aforementioned properties.

First, because $\gamma$ can have interpolated line segments that intersect, and at such an intersection point $\gamma_f$ may differ on the different interpolated segments, the following is done. We slightly modify $\gamma_f$ to produce $\omega$, a curve such that for each pair of line segments $L_1$ and $L_2$ that intersect in $\gamma$, $\omega$ agrees with $\gamma_f$ on the endpoints of each line segment, but $\omega$ is also the same at both intersection points (the point for intersection on the first line segment, and that on the second line segment). Since we can produce $\omega$ by only slightly modifying $\gamma_f$, $\omega$ and $\gamma_f$ have the same winding number.
Since $\gamma$ is a closed polygonal curve, we can divide up $\text{int}(\gamma)$ into the union of $n \in \mathbb{N}$ closed simple closed polygonal curves (here simple means that the only point of self-intersection is the initial point). Hence, let

$$\text{int}(\omega) = \bigcup_{0<i\leq n} P_i$$

Where $P_i$ is the $i$th closed simple polygonal curve. This is shown in figure 3.1.

Consider a $P_i$ for any valid $i$. Let the vertices of $P_i$ be $a_1, ..., a_m$. Since the interior of $P_i$ is in the $\frac{R_f}{20}$-neighborhood of $S$, we can choose a finite number of points $b_1, ..., b_k \in S$ such that $b_i \in \text{int}(P_i)$, and for every ball $B$ with center in $\text{int}(P_i)$ with radius $\frac{R_f}{20}$, $B \cap \{b_1, ..., b_k, a_1, ..., a_m\} \neq \emptyset$. In essence, the $b_i$’s "fill in" the interior of $P_i$, so that any ball of radius $\frac{R_f}{20}$ contains at least one of these points, or one of the $a_i$’s. Because of this, we can find a triangulation of $\text{int}(P_i)$, $Q$, such that the vertices of $Q$ are a subset of $a_1, ..., a_m, b_1, ..., b_k$, $P_i$ is contained in $Q$, the maximal edge length is at most $\frac{R_f}{5}$ and any triangle in the triangulation can be contained in some ball of radius $\frac{R_f}{2}$. In figure 3.2, we see an example triangulation.

Now, define a function $g_i$ on $\text{int}(P_i)$ as follows: $g_i = f$ on $b_1, ..., b_k$. For each $a_i$, if $a_i$ is one of the vertices of $\gamma$, let $g_i(a_i) = f(a_i)$. If $a_i$ is the intersection of two interpolated edges, then let $g_i(a_i)$ be such that $a_i - g_i(a_i)$ agrees with $\omega$ at the point of intersection.
We extend $g_i$ to the edges of $Q$ by linear interpolation, and we then extend $g_i$ to the interiors of all of the triangles that make up $Q$ also by linear interpolation between the edges. Since each such triangle is contained in a ball of radius $\frac{R_f}{2}$, and since $f$ has no fixed points by assumption, $g_i$ has no fixed points.

We can find similar functions $g_i$ for all $i$, $0 < i \leq n$. If $P_i$ and $P_j$, $i \neq j$ share a common edge, then $g_i$ and $g_j$ agree on this edge. Hence, we can define $g$ on all of $\text{int}(\gamma)$ as the union of all $g_i$’s. By the construction of $g$, $\gamma_g = \omega$, and so $\gamma_g$ has a nonzero winding number. By Lemma 1, $g$ must have a fixed point, which means that $g_i$ must have a fixed point for some $i$, which is a contradiction. Hence, $f$ has a fixed point.

Let us recap where we stand. We want to prove Theorem 2, which says that any feasible set $S$ has the fixed point property. We have shown that, given a function $f : S \rightarrow S$, if we can find a closed polygonal curve $\gamma$ in $S$ with small joining segments such that the winding number of $\gamma$ is nonzero, and such that the interior of $\gamma$ is close to $S$, then we will have shown that $f$ has a fixed point. We now introduce the definition and existence of polygonal approximate boundaries.

**Definition 10.** An $\epsilon$-polygonal $\delta$-approximate boundary $\alpha$ of a feasible set $S$ is a closed $\epsilon$-close polygonal curve in $S$ such that there is a Jordan curve $\beta \subseteq S^c$ such that $S \subseteq \text{int}(\beta)$ and $\beta$ is $\delta$-close to $\alpha$. 
Lemma 2. For a feasible set $S$, for any $\epsilon, \delta > 0$, there exists an $\epsilon$-polygonal $\delta$-approximate boundary $\alpha$ of $S$.

Proof. Without loss of generality, assume that $\epsilon \leq \delta$. If $\epsilon > \delta$ then the following proof holds, except with $\epsilon$ replaced with $\delta$. Because $S$ is connected, we can find a Jordan curve $\alpha \in S^c$ such that $S \subset \text{int}(\alpha)$, and $\alpha$ lies in the $\frac{\epsilon}{4}$-neighborhood of $S$. Since $\alpha$ is in the $\frac{\epsilon}{4}$-neighborhood of $S$, we can find an $\epsilon$-close polygonal curve $\beta$ in $S$ such that $\alpha$ and $\beta$ are $\epsilon$-close, as desired (since $\delta \geq \epsilon$). To find $\beta$, proceed as follows. Begin with a point $x_1 \in \alpha$, and choose a point $y_1 \in S$ with $|x_1 - y_1| < \frac{\epsilon}{4}$. Next, move along $\alpha$ until we reach a point on $\alpha$, $x_2$, such that $|x_1 - x_2| = \frac{\epsilon}{4}$. Similarly to before, find a point $y_2 \in S$ with $|x_2 - y_2| < \frac{\epsilon}{4}$. Proceed in this way all the way around $\alpha$ until we reach a final point $x_n \in \alpha$ such that moving along $\alpha$, all points between $x_n$ and $x_1$ are within $\frac{\epsilon}{4}$ of $x_1$. By choosing $y_1, ..., y_n, y_1$ as our points to construct our polygonal curve $\beta$, we see that $\alpha$ and $\beta$ have the desired properties.

With one more lemma, we can progress to the proof of the theorem:

Lemma 3. Given $S$ and $f$ as above with $f$ fixed point free, if we have closed $\epsilon$-close polygonal curves $\alpha, \beta$ in $S$ with $\epsilon < \frac{R_f}{4}$ such that $\alpha$ is $\frac{R_f}{4}$-close to $\beta$, then the winding numbers of $\alpha_f$ and of $\beta_f$ about $(0, 0)$ are well-defined and equal.

Proof. The first part of this statement to prove is that $\alpha_f$ and $\beta_f$ have well-defined winding numbers about $(0, 0)$. To prove this, we must show that $\forall s \in [0, 1], \alpha(s) \neq f(\alpha(s)), \beta(s) \neq f(\alpha(s))$, where $f$ is interpolated along the line segments in $\alpha$ and in $\beta$. Consider $\alpha$. Since each pair of successive points $p_i, p_{i+1}$ are within $\frac{R_f}{4}$ of each other, this implies that $p_i$ and $p_{i+1}$ are mapped into a ball $B_1$ that is disjoint from a ball $B_2$ containing $p_i$ and $p_{i+1}$. Hence, since we interpolate $f$ along the line joining $p_i$ to $p_{i+1}$, and since the closed ball is convex, this implies that $f(\alpha) \cap \alpha = \emptyset$ between $p_i$ and $p_{i+1}$. 
Since $p_i$ and $p_{i+1}$ where chosen as any successive points, we are done. The proof for $\beta$ is identical.

To show that $\alpha$ and $\beta$ have the same winding number, recall that since $\alpha$ is $\frac{R_f}{4}$-close to $\beta$, we have a homotopy from $\alpha$ to $\beta$ which has the property that the curve is deformed in such a way so that any point never leaves the $\frac{R_f}{4}$-neighborhood of its initial position. How does this relate to the winding number of $\alpha_f$ and of $\beta_f$? This homotopy $F$ induces a homotopy $F'$ from $\alpha_f$ to $\beta_f$ given by

$$F'(s, t) = t \cdot (F(s, 1) - f(F(s, 1))) + (1 - t) \cdot (F(s, 0) - f(F(s, 0)))$$

If we can prove that this homotopy is in $\mathbb{R}^2 - \{(0, 0)\}$ (i.e. never goes through $(0, 0)$), we will be done as per theorem 3. Given $s \in [0, 1]$, let $a = F(0, s)$ and $b = F(1, s)$. We can find successive vertices $a_1, a_2 \in \alpha$ and $b_1, b_2 \in \beta$ such that $a$ lies on the line segment that joins $a_1$ to $a_2$, and $b$ lies on the line segment that joins $b_1, b_2$. Since $|a - b| < \frac{R_f}{4}$, and since $|a_i - a| < \frac{R_f}{4}$ and $|b_i - b| < \frac{R_f}{4}$ for $i \in 1, 2$, all of $a, a_1, a_2, b, b_1, b_2$ are contained in a ball $B_1$ around $a$ of radius $\frac{3R_f}{4}$. Hence, all $a_1, a_2, b_1$ and $b_2$ are mapped to another ball $B_2$ disjoint from the first. Since the balls are convex, and since $\alpha_f$ and $\beta_f$ are formed by linear interpolation, as is $F'$, this demonstrates that $F'(s, t) \neq (0, 0)$ for any $s, t$. This completes the proof.

We now progress to the proof of Theorem 2.

**Proof of Theorem 2** Given a feasible $S$ and $f : S \to S$, we wish to prove that $f$ has a fixed point. Assume that it does not; we shall produce a contradiction.

By the feasibility of $S$, there is a Jordan curve $\gamma$ such that, for any Jordan curve $\sigma \in S^c \cap \text{int}(\gamma)$, $\sigma$ is $\frac{R_f}{40}$-close to some curve, $\gamma'$, such that $\gamma'$ is homotopic to $\gamma$ in $\text{img}(\gamma)$. Given this $\gamma$, we find an $\epsilon$-polygonal $\epsilon$-approximate boundary $\alpha$ of $S$ with corresponding Jordan curve $\beta$ with the following properties:

(i) $\epsilon < \frac{R_f}{40}$
(ii) $\epsilon$ is so small that $int(\beta)$ is in the $\frac{R_f}{40}$-neighborhood of $S$

(iii) $\epsilon$ is so small so that $f(\alpha) \subset int(\gamma)$

This is possible because of the following reasons. First, we may insist that $\epsilon < \frac{R}{40}$ since we know that such boundaries exist by Lemma 2. Second, property (ii) is reasonable since, for any $\rho > 0$, if we choose $\epsilon$ to be small enough, the Jordan curve $\kappa$ corresponding to an $\epsilon$-polygonal $\epsilon$-approximate boundary of $S$ will have the property that $int(\kappa)$ is in the $\rho$-neighborhood of $S$. This follows from the fact that the complement of $S$ is connected. The last condition makes sense since $S$ is mapped into $S$, so since $S$ is compact, if all of the line segments joining vertices in $\alpha$ are small enough, then the interpolated image of $\alpha$ will lie entirely inside of the interior of $\gamma$. An example of this setup is in figure 3.3.

![Figure 3.3: The set S with bounding curves shown; the innermost polygonal curve is \( \alpha \), followed by the Jordan curve \( \beta \), lastly followed by the Jordan curve \( \gamma \)](image)

By $\gamma'$ we now refer to the Jordan curve that is $\frac{R}{40}$-close to $\beta$ with $\gamma'$ being homotopic to $\gamma$ in $img(\gamma)$. We want to show that $\alpha_f$ has nonzero winding number. Since $\alpha$ is $\frac{R_f}{40}$-close to $\beta$ and $\beta$ is $\frac{R_f}{40}$-close to $\gamma'$, $\alpha$ is $\frac{R_f}{20}$-close to $\gamma'$. This homotopy from $\alpha$ to $\gamma'$, $F$, produces a curve $\gamma'_f$ given by

$$\gamma'_f(s) = \gamma'(s) - f(F(s,0))$$

Since $f(\alpha) \subset int(\gamma')$ (as $int(\gamma) = int(\gamma')$), and since in the homotopy $F$, points don’t leave the $\frac{R_f}{20}$-neighborhoods of their initial positions, $\gamma'_f(s) \cap (0,0) = \emptyset$ and $\gamma'_f$ has
nonzero winding number. By Lemma 3, this implies that the winding number of $\alpha_f$ is also nonzero. Since $int(\beta)$ is $\frac{R_f}{40}$-close to $S$, and since $\alpha$ and $\beta$ are $\frac{R_f}{40}$-close, $int(\alpha)$ is in the $\frac{R_f}{20}$-neighborhood of $S$. By Theorem 9, this implies that $f$ has a fixed point. This is a contradiction, so $f$ does indeed have a fixed point.
Chapter 4

Canonical Examples

For the three theorems that we have discussed, that is, Hagopian’s theorem, the plane fixed point problem, and our modified plane fixed point problem, a natural question arises: are there sets in $\mathbb{R}^2$ that satisfy the hypotheses for some of these theorems but not others? Let the collections of sets that satisfy each of the collections of hypotheses be $A$, $B$, and $C$.

Clearly, $C \subset B$. However, as we shall see from our examples, $C \not\subset A$, $A \not\subset C$, $A \not\subset B$, and $B \not\subset A$.

We shall produce examples $a$, $b$, $c$, each one which lie in the corresponding $A$, $B$, $C$.

$a$ is the topologist’s sine curve wound back on itself, as in figure 4.1:

**Theorem 11.** $a \in A$, but $a \notin B$.

**Proof.** $a \in A$ since the set is closed, bounded, and connected, and so is compact. Furthermore, it is clearly path connected. Since the complement has 2 connected components (outside the loop and inside the loop), it fails to be in $B$ (or in $C$, since $C \subset B$).

$b$ is a ”galaxy set” (figure 4.2). This set is a solid disc with $n$ ”arms” extending in a spiral pattern. Note that the arms wind infinitely many times around the central disc, and that the figure is not path connected - it has $n + 1$ path connected components.
Theorem 12. $b \in B$, but $b \notin A$.

Proof. $b \in B$ since $b$ is compact, connected, and has a connected complement. One way to verify that its complement is connected is to observe that, although the arms wind infinitely many times around the central disc, any point between two arms can be reached via a homeomorphism of $[0,...,1]$ in the complement of the set. As such, the complement is clearly path connected, and so is connected as well. $b$ is not in $A$ since it has more than one path connected component.

The last example is $c$, which is two sine curves that oscillate towards each other (figure...
4.3. A line segment is added between them to make the result compact. This is clearly connected and compact, and has a connected complement (as in the galaxy set, we can show that the complement is path connected, and so is connected).

\[ \text{Figure 4.3: Two sine curves oscillating towards each other} \]

**Theorem 13.** \( c \in C \), but \( c \notin A \).

**Proof.** \( c \notin A \) because \( c \) is not path connected. From the argument above, all that we must check is that \( c \)'s complement satisfies the additional feasibility condition to prove that \( c \in C \). Recall that we need to show that, for any \( \epsilon > 0 \), there exists a Jordan curve \( \alpha \in \mathbb{R}^2 - c \) such that for any other Jordan curve \( \beta \) in the intersection of the interior of \( \alpha \) and the complement of \( c \), \( \beta \) is \( \epsilon \)-close to \( \alpha \). An example \( \alpha \) can be seen in figure 4.4.

\[ \text{Figure 4.4: Example of an appropriate Jordan curve; the smaller the \( \epsilon \), the larger the number of oscillations in the Jordan curve} \]
The basic idea is to, given $\epsilon > 0$, choose $\alpha$ that follows $c$ until the oscillations become very dense - until the distance between successive oscillations is less than $\epsilon$. Such an $\alpha$ has the desired property because any Jordan curve $\beta$ that lies in the interior of $\alpha$ but is still in the complement of $c$ looks like $\alpha$ except with possible oscillations that are within $\epsilon$ of each other, and with possible backtracking. In the case of the oscillations, $\beta$ can follow $c$ closer towards the line $x = 0$, oscillating with greater frequency. In this case, choose $\alpha'$ to be $\alpha$ but on the last oscillation of $\alpha$, double back and repeat this segment several times. Since all of the innermost oscillations are within $\epsilon$ of this last oscillation, we can move these additional oscillations inward to $\beta$. The other problem is if $\beta$ moves along $c$, then doubles back and moves back along $c$ in the other direction. To deal with these changing directions, we simply double back and repeat the relevant segments of $\alpha$ to form $\alpha'$. These additional segments can be moved onto $\beta$. 
Chapter 5

Conclusion

The plane fixed point problem is a deceptively simple problem; despite being stated easily, it has defied attempts to prove it. When the problem is closely examined, many pathological cases quickly appear, and it also becomes apparent why many promising techniques fail to make progress on the problem. We have succeeded in proving a special case of this problem, and have shown that this special case is not immediately covered by existing sufficient criteria theorems.
Bibliography
