PARTIAL ZERO-FORCING PRECODING FOR INTERFERENCE CHANNELS
WITH LIMITED TRANSMITTER COOPERATION

by

Siddarth Hari

A thesis submitted in conformity with the requirements
for the degree of Master of Applied Science
Graduate Department of Electrical and Computer Engineering
University of Toronto

Copyright © 2010 by Siddarth Hari
Abstract

Partial Zero-Forcing Precoding for Interference Channels with Limited Transmitter Cooperation

Siddarth Hari
Master of Applied Science
Graduate Department of Electrical and Computer Engineering
University of Toronto

2010

This thesis looks at the problem of designing a coding strategy for interference channels with rate-limited transmitter cooperation. We first consider a simple communication model in which the classic two-user Gaussian interference channel is augmented by rate-limited conferencing links between the transmitters. The main contribution is a partial zero-forcing precoding strategy based on a shared-private rate splitting scheme at the transmitter, in which each transmitter communicates part of its message to the other transmitter, and subsequently partially pre-subtracts the interfering signal using a zero-forcing precoder. We extend the proposed strategy to a class of multiuser interference channels, and outline a distributed algorithm to compute the precoder coefficients. The partial zero-forcing precoding strategy is shown to be particularly effective in certain high SNR/INR regimes, and simulation results for a multicell system highlight the cooperation gain due to the proposed strategy.
Acknowledgements

I would like to thank my advisor, Professor Wei Yu, for the encouragement provided throughout the course of this work. His remarkable intuition and insight were very helpful in the completion of this thesis work.

I would like to thank the students and staff in the Communication Group for creating a wonderful environment in which to study and do research. I would like to especially thank my colleague Hayssam Dahrouj for his help in getting me settled at the university, and for all the interesting discussions and suggestions in these last two years.

The financial support of a Research Assistantship provided by Professor Yu and a University of Toronto scholarship is gratefully acknowledged.

Finally, I would like to thank my parents for their unwavering support and encouragement through all these years.
Contents

1 Preliminaries .......................................................... 1
   1.1 Motivation ......................................................... 1
   1.2 Thesis Outline .................................................. 2
   1.3 Notation .......................................................... 3

2 Two-user interference channel ..................................... 4
   2.1 Background and Related Work .................................. 5
   2.2 System Model .................................................... 11
   2.3 Coding Strategy .................................................. 12
   2.4 Characterization of Cooperation Gain ......................... 14
      2.4.1 Achievable Rate ............................................. 14
      2.4.2 Low-Noise Regime ........................................... 15
   2.5 Asymptotic Sum Capacity in the Very Weak Interference Regime 17
      2.5.1 Sum Capacity Upper Bound .................................. 18
      2.5.2 Asymptotic Sum Capacity ................................... 20
   2.6 Concluding Remarks ............................................. 22

3 Multicell systems with constrained backhaul ..................... 23
   3.1 Background and Related Work .................................. 23
   3.2 System Model .................................................... 25
   3.3 Coding Strategy .................................................. 28
3.3.1 Shared-private rate splitting ........................................ 28
3.3.2 Zero-forcing precoder ................................................. 29
3.3.3 Power allocation ....................................................... 30
3.4 Distributed signal processing ........................................... 30
3.4.1 Precoder coefficients .................................................. 31
3.4.2 Power Allocation ....................................................... 32
3.5 Characterization of Cooperation Gain ................................. 33
3.5.1 Achievable Rate ......................................................... 33
3.5.2 Low noise regime ...................................................... 33
3.5.3 Simulations ............................................................ 36

4 Conclusions ................................................................. 47
4.1 Summary ................................................................. 47
4.2 Suggestions for further research ....................................... 48

A Deterministic Interference Channel .................................... 51

B Convergence Analysis .................................................... 58
B.1 Precoder coefficients .................................................. 59
B.2 Power allocation ....................................................... 60

Bibliography ................................................................. 64
# List of Figures

2.1 Common-private rate splitting ................................. 6
2.2 Gaussian vector broadcast channel with two-antenna transmitter and two single-antenna receivers ................................. 8
2.3 Interference channel with common information ................ 9
2.4 Discrete memoryless MAC with partially cooperating encoders ........ 10
2.5 Interference channel with limited transmitter cooperation ........ 11
2.6 Transmitter Cooperation via private-shared rate splitting .......... 13
2.7 Genie-aided channel .............................................. 18
3.1 A 19-cell system with constrained backhaul. Only the backhaul links from $BS_2$ are shown. .............................................. 27
3.2 Comparison of the improvement in average downlink rates versus number of bits shared for the shared-private rate-splitting (SP) and quantize-and-forward (QF) strategies ........................................ 39
3.3 Histogram of the average improvement in downlink rates as a fraction of the number of bits shared by each transmitter over 1000 randomly generated instances of the channel. In each instance, the number of bits shared by each transmitter is (a) 20%, (b) 40%, (c) 60% and (d) 80% of the average baseline rate. We observe that the proposed transmission strategy based on shared-private rate splitting and zero-forcing precoding achieves a gain of almost one-bit-per-shared-bit. ........................................ 42
3.4 Histogram of the average spectral radius of the iteration matrix (see Appendix B) for 1000 randomly generated instances of the channel $H$. In each instance of the channel, the average spectral radius is computed over 1000 row-permutations of $H$. A sufficient condition for convergence is $\rho(M) < 1$. The plot shows that this condition is satisfied in around 85% of the randomly generated instances of the channel.

3.5 Histogram of the spectral radius of the iteration matrix for the power allocation algorithm (see Appendix B) for 1000 randomly generated instances of the channel $H$. A necessary and sufficient condition for convergence is $\rho(M) < 1$. We observe that this condition is always satisfied.

3.6 Convergence behaviour of the distributed algorithm for the design of the precoder.

3.7 Convergence behaviour of the distributed algorithm for the power allocation.

A.1 Deterministic point-to-point channel.

A.2 Deterministic interference channel: $n_{11} = 3, n_{12} = 1, n_{21} = 2, n_{22} = 4$.

A.3 Deterministic interference channel: $n_{11} - n_{21} = 1, n_{22} - n_{12} = 3$.

A.4 Deterministic interference channel with limited transmitter cooperation.

A.5 Deterministic interference channel: $n_{11} - n_{21} = 1, n_{22} - n_{12} = 3$.

A.6 Deterministic interference channel with limited transmitter cooperation.
Chapter 1

Preliminaries

1.1 Motivation

Interference is a central phenomenon in wireless communication when multiple transmitters communicate with their respective intended receivers over a shared communication medium. Conventional wireless systems are designed with a cellular architecture in which base-stations from different cells communicate with their respective remote terminals independently. Interference management consists of fixing the frequency reuse pattern such that the communication links used in neighbouring cells are orthogonalized in the frequency domain, and do not interfere with each other. In currently deployed cellular networks, interference signals are typically too weak to be detected by out-of-cell users, and are treated as background noise. However, to increase area spectral efficiency of cellular networks, emerging cellular standards are increasingly designed with frequency reuse factor of 1. In this case, intercell interference becomes the dominant limiting factor.

There has been a lot of recent interest in exploring coding strategies for interference mitigation via cooperation between the nodes in a network. In particular channel models where the base-stations cooperate via rate-limited backhaul links have received a great deal of attention. These backhaul links enable transmitter cooperation which can be
employed to reduce the effect of interference in the downlink.

In this thesis we look at the problem of designing coding strategies for interference channels with rate-limited transmitter cooperation. This design problem consists of two coupled parts namely

- What information about a user’s message should be communicated to the other transmitters over the cooperation links?

- Using the additional information available about the messages of the interfering users, how should the transmit signals be designed for the interference channel?

1.2 Thesis Outline

The rest of this thesis is organized as follows.

In Chapter 2 a simple communication model is considered in which the classic two-user Gaussian interference channel is augmented by noiseless rate-limited digital conferencing links between the transmitters. A shared-private rate-splitting technique is introduced, which enables each transmitter to communicate part of its message to the other transmitter. The proposed coding strategy partial zero-forcing precoding is described, and its performance over this channel is characterized in certain asymptotic high SNR, INR regimes.

In Chapter 3 the idea of partial zero-forcing precoding is extended to a restricted class of multiuser interference channels with limited cooperation. The system model under consideration is motivated by an envisioned architecture for new cellular networks with local cooperation, in which each base station is connected to its six neighbouring base stations via rate-limited backhaul links. A distributed algorithm is outlined to iteratively compute the precoder coefficients locally at various transmit nodes. The cooperation gain is characterized analytically in an asymptotic low-noise regime. We present simulation
Table 1.1: Commonly used notation in the thesis

results which highlight the cooperation gain due to the proposed strategy in a multicell system.

Chapter 4 offers some conclusions and suggestions for further research.

Appendix A revisits the proposed coding strategy by interpreting it in the context of a deterministic interference channel. The simplicity of the deterministic channel model provides an intuitive understanding of the coding strategy proposed in this thesis.

Appendix B examines the convergence behaviour of the distributed algorithm proposed in Chapter 3 for computing the precoder coefficients and the corresponding power allocation.

1.3 Notation

Table 1.1 lists some common notation that is used in this thesis. We use the standard definitions for these terms, which can be found in [1].
Chapter 2

Two-user interference channel

In this chapter a communication model is considered in which the classic two-user Gaussian interference channel is augmented by noiseless rate-limited digital conferencing links between the transmitters. We propose a partial zero-forcing precoding strategy based on a shared-private rate splitting scheme at the transmitter, in which each transmitter communicates part of its message to the other transmitter, and subsequently partially pre-subtracts the interfering signal using a zero-forcing precoder. We prove an outer bound and show that the proposed strategy is asymptotically sum-capacity achieving in a very weak interference regime, where both the signal-to-noise ratio (SNR) and the interference-to-noise ratio (INR) go to infinity while their ratio in dB scale is kept fixed. In this case, every cooperation bit results in one-bit gain in sum capacity. We also consider a different asymptotic regime where the transmit power constraints and the channel gains are fixed while the noise powers go down to zero. In this case, if one compares with the achievable sum rate with interference treated as noise, one cooperation bit can in fact result in more than one-bit gain in achievable sum rate.
2.1 Background and Related Work

**Definition 1.** A discrete memoryless interference channel is denoted by a quintuple \((\mathcal{X}_1, \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1, \mathcal{Y}_2)\), where \(\mathcal{X}_1\) and \(\mathcal{X}_2\) are two finite input alphabet sets, \(\mathcal{Y}_1\) and \(\mathcal{Y}_2\) are two finite output alphabet sets, and \(p(y_1, y_2|x_1, x_2)\) is a collection of conditional probability distributions of \((y_1, y_2)\in \mathcal{Y}_1 \times \mathcal{Y}_2\) given \((x_1, x_2)\in \mathcal{X}_1 \times \mathcal{X}_2\). The conditional probability \(p_n(y_1, y_2|x_1, x_2)\) of \(y_1, y_2\in \mathcal{Y}_1^n \times \mathcal{Y}_2^n\) given \(x_1, x_2\in \mathcal{X}_1^n \times \mathcal{X}_2^n\) satisfies:

\[
p_n(y_1, y_2|x_1, x_2) = \prod_{i=1}^{n} p(y_{1i}, y_{2i}|x_{1i}, x_{2i})
\]

where \(x_j = (x_{j1}, x_{j2}, \cdots, x_{jn}) \in \mathcal{X}_j^n\) and \(y_j = (y_{j1}, y_{j2}, \cdots, y_{jn}) \in \mathcal{Y}_j^n, j \in \{1, 2\}\).

**Definition 2.** An interference channel lies in the strong interference regime if

\[
I(X_1; Y_1|X_2) \leq I(X_1; Y_2|X_2)
\]

\[
I(X_2; Y_2|X_1) \leq I(X_2; Y_1|X_1)
\]

for all product probability distributions on \(\mathcal{X}_1 \times \mathcal{X}_2\).

We use the standard definitions from Network Information Theory of codes, error probabilities, achievable rate-pairs and capacity region for the interference channel. The reader is referred to [1] for more details.

The largest known achievable rate region for the classic two-user interference channel is due to Han and Kobayashi [2], where a common-private rate splitting technique is employed to enable the receiver to partially decode and subtract the interfering signal. In this coding strategy, each user’s signal is comprised of a superposition of two signals: a common message \(U_i\) and a private message \(V_i\) (see Fig. 2.1). The common messages are meant to be decoded at both receivers, while the private message is only decoded by the intended receiver. The Han-Kobayashi scheme also includes a time-sharing factor \(Q\). The resulting rate region, known as the Han-Kobayashi achievable rate region,
Figure 2.1: Common-private rate splitting

is a intersection of the achievable rate regions \((T_i, T_j, S_i)\) of two three-user multiple access channels \(((U_i, U_j, V_i) \rightarrow Y_i, (i, j) \in \{(1, 2), (2, 1)\})\), projected on a two-dimensional subspace \((R_i, R_j)\), where \(T_i\) and \(S_i\) are the rates to user \(i\) of the common and private messages respectively, and \(R_i = T_i + S_i\).

We state the famous Han-Kobayashi result for the sake of completeness. Let \(P^*\) be the set of probability distributions \(P^*\) that factor as:

\[
P^*(q, u_1, v_1, x_1, u_2, v_2, x_2, y_1, y_2) = P(q) \cdot P(u_1|q)P(v_1|q)P(x_1|u_1, v_1, q) \cdot P(u_2|q)P(v_2|q)P(x_2|u_2, v_2, q) \cdot P(y_1, y_2|x_1, x_2)
\]

(2.2)

Assume \(P^*\) is fixed. Let \(R_{HK}^{(1)}\) be the set of all rate pairs \((R_1, R_2)\), wherein \(R_1 = T_1 + S_1\) and \(R_2 = T_2 + S_2\) for some rate-tuples \((T_1, S_1, T_2, S_2)\), that satisfy:

\[
S_1 \leq I(V_1; Y_1|U_1, U_2, Q) \\
T_1 \leq I(U_1; Y_1|V_1, U_2, Q) \\
T_2 \leq I(U_2; Y_1|V_1, U_1, Q) \\
S_1 + T_1 \leq I(V_1, U_1; Y_1|U_2, Q) \\
S_1 + T_2 \leq I(V_1, U_2; Y_1|U_1, Q) \\
T_1 + T_2 \leq I(U_1, U_2; Y_1|V_1, Q) \\
S_1 + T_1 + T_2 \leq I(V_1, U_1, U_2; Y_1|Q)
\]

(2.3)
In the same way, we can have $\mathcal{R}_{HK}^{(2)}$ generated by the set of rate-tuples $(T_1, S_1, T_2, S_2)$ that satisfy (2.3) with indices 1 and 2 swapped everywhere. Then, the set

$$\mathcal{R}_{HK} = \bigcup_{P^* \in P^*} \mathcal{R}_{HK}^{(1)} \cap \mathcal{R}_{HK}^{(2)}$$

is an achievable rate region for the memoryless interference channel.

For the Gaussian interference channel, the capacity region in the weak interference regime remains an open problem. The Han-Kobayashi scheme allows arbitrary splits of each user’s transmit power into the common and private signals, as well as time sharing between multiple splittings. At each transmitter there is a fundamental trade-off between achieving a large rate in the direct link to its intended receiver and reducing the interference on the cross link to the non-intended receiver. While this optimization problem is not well-understood, the recent work of Etkin, Tse and Wang [3] showed that a simple power splitting strategy achieves the capacity to within one bit. The key idea in their work is to set the power of the private signal of each user such that it is received at the noise level at the non-intended receiver, so that the interference caused due to the private signal has only a small additional effect as compared to the impairments already caused by the noise.

In this chapter we study a communication model in which the classic two-user Gaussian interference channel is augmented by noiseless rate-limited digital conferencing links between the transmitters and explore interference mitigation strategies based on transmitter cooperation. This channel model is closely related to a number of well-known problems in the literature.

If the capacity of the digital link is sufficiently large for each transmitter to describe its entire message to the other transmitter, the resulting channel is a two-user Gaussian vector broadcast channel (with per-antenna power constraints). For this channel (see Fig. 2.2) Caire and Shamai showed that a dirty-paper encoding strategy achieves the sum-capacity under a sum power constraint. Moreover, in terms of throughput a simple zero-forcing strategy is asymptotically optimal for high SNR [4]. If the digital link is
unidirectional with sufficiently large capacity, so that one of the transmitters knows the message of the other completely, we have the cognitive radio channel [5].

In this chapter we focus on communication models in which the conferencing links are rate limited. In this realm, Maric et al. [6] determined the capacity region in the special case of strong-interference regime, where both receivers can decode all messages with no rate penalty. The capacity region of the memoryless interference channel, with inputs $X_1$ and $X_2$ with corresponding outputs $Y_1$ and $Y_2$, under strong interference was conjectured by Sato [7], and later established by Costa and El Gamal [8] as the union of rate pairs $(R_1, R_2)$ satisfying

$$0 \leq R_1 \leq I(X_1; Y_1 | X_2, Q)$$
$$0 \leq R_2 \leq I(X_2; Y_2 | X_1, Q)$$
$$R_1 + R_2 \leq \min\{I(X_1, X_2; Y_1 | Q), I(X_1, X_2; Y_2 | Q)\}$$

(2.4)

where $Q$ is a time sharing parameter, and the union is over all probability distributions of the form

$$p(q, x_1, x_2, y_1, y_2) = p(q)p(x_1|q)p(x_2|q)p(y_1, y_2|x_1, x_2).$$

The rates in the above channel can only increase when the transmitters can communicate
part of their information to each other. Maric et al. show that when the above channel is augmented by cooperation links of capacities $C_{12}$ (from 1 to 2) and $C_{21}$ between the transmitters, the capacity region becomes the union of rate pairs $(R_1, R_2)$ satisfying

$$0 \leq R_1 \leq I(X_1; Y_1| X_2, U) + C_{12}$$

$$0 \leq R_2 \leq I(X_2; Y_2| X_1, U) + C_{21}$$

$$R_1 + R_2 \leq \min\{I(X_1, X_2; Y_1|U), I(X_1, X_2; Y_2|U)\} + C_{12} + C_{21}$$

$$R_1 + R_2 \leq \min\{I(X_1, X_2; Y_1), I(X_1, X_2; Y_2)\}$$ (2.5)

where $U$ is an auxiliary random variable, and the union is over all probability distributions of the form

$$p(u, x_1, x_2, y_1, y_2) = p(u)p(x_1|u)p(x_2|u)p(y_1, y_2|x_1, x_2).$$

Thus, in the strong-interference regime, when transmitters 1 and 2 communicate $C_{12}$ and $C_{21}$ bits respectively of information about their signals, the rates to their respective receivers increase by $C_{12}$ and $C_{21}$ bits (for small $C_{12}$ and $C_{21}$, the first sum-rate bound in (2.5) is active).

In the weak-interference regime, Bagheri et al. [9] considered the approximate sum capacity of a symmetric interference channel for the case where the digital link between
the transmitters is unidirectional. These two works use Han-Kobayashi common-private rate splitting encoding strategy for the interference channel - in [6], for the strong interference regime the transmit signals are common messages only which can be decoded by both receivers, while in [9] the transmit signals are a superposition of common and private messages. The conferencing link is used to communicate information about common messages to the other transmitter. Note that the capacity of the backhaul link may be insufficient to completely describe even the common message alone to the other transmitter. The common message is further split into a shared-common message whose rate is equal to the capacity of the backhaul link to the other transmitter, and a secret-common message. The idea is to allow transmitters to cooperatively encode and jointly transmit the shared-common messages, thereby leading to an increase in common message rates. This coding strategy was first proposed by Willems in [10] for the multiple-access channel with conferencing encoders.

In our work we consider an alternative idea of allowing transmitters to share private messages. The idea is to mitigate interference with transmit precoding rather than receiver-based partial decoding and interference subtraction. Toward this end, we propose a shared-private rate splitting technique, which enables each transmitter to communicate
its shared message to the other transmitter, and subsequently to pre-subtract part of the interfering signal using a zero-forcing scheme, thereby leading to an increase in private message rates. The main result of this chapter is to show that a simple zero-forcing strategy can be quite effective in certain high-SNR regimes.

### 2.2 System Model

The Gaussian interference channel with a digital link between the transmitters is modelled as follows (see Fig. 2.5):

\[
\begin{align*}
Y_1 &= h_{11}X_1 + h_{21}X_2 + Z_1 \\
Y_2 &= h_{12}X_1 + h_{22}X_2 + Z_2
\end{align*}
\]  

(2.6)

where \(X_1\) and \(X_2\) are the transmit signals with power constraints \(P_1\) and \(P_2\) respectively, \(h_{ij}\) represents the channel gain from transmitter \(i\) to receiver \(j\), and \(Z_1\), \(Z_2\) are independent additive white Gaussian noises (AWGN) with power \(N\). The noiseless digital links between the transmitters have fixed capacities \(C_{12}\) and \(C_{21}\) respectively.

A general conferencing model between the transmitters allows \(K\) conferencing rounds. We use the conferencing model proposed in Willems’s work [10]. In particular, the sources 1 and 2 generate random integers \(W_1 \in \{1, 2, \ldots, 2^{nR_1}\}\) and \(W_2 \in \{1, 2, \ldots, 2^{nR_2}\}\) respectively, at the beginning of each block of \(n\) channel uses. Each encoder is com-
pletely described by an encoding function and a set of $K$ information sharing functions $\{t_{i,1}, t_{i,2}, \ldots, t_{i,K}\}, i = 1, 2$. Each function $t_{i,k}$ maps the source message $W_i$ and the sequence of previously received communications from the other encoder into the $k$th communication $T_{i,k}$, where $T_{i,k}$ ranges over a finite alphabet $\vartheta_{i,k}$, for $k = 1, 2, \ldots, K$. The amount of information that can be exchanged is bounded by the capacities of the digital links between the transmitters:

$$\sum_{k=1}^{K} \log(|\vartheta_{i,k}|) \leq nC_{ij}. \quad (2.7)$$

The encoding function maps the source message $W_i$ and $T^K_j = (T_{j,1}, T_{j,2}, \ldots, T_{j,K})$, the information received from the other transmitter, into a codeword $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$.

In the rest of this chapter we use the following definitions to simplify the notation:

$$\begin{align*}
\text{SNR}_i &= \frac{h_{ii}^2 P_i}{N} \\
\text{INR}_i &= \frac{h_{ji}^2 P_j}{N} \\
\nu_i &= 2^{2C_{ij}} - 1 \\
\gamma(x) &= \frac{1}{2} \log(1 + x) \\
h_d &= h_{11} \times h_{22} \\
h_c &= h_{12} \times h_{21}
\end{align*}$$

### 2.3 Coding Strategy

We propose the following cooperation strategy for the interference channel with partially cooperating transmitters. We use only a single conferencing round (i.e., $K = 1$). Each transmitter splits its source messages $W_i$ into a shared $W_{t,s} \in \{1, 2, \ldots, 2^{nC_{ij}}\}$ and a private $W_{t,p} \in \{1, 2, \ldots, 2^{n(R_i - C_{ij})}\}$ part. We then let each transmitter communicate its shared message to the other transmitter over the digital link. Note that a shared message is only intended to be communicated to the other transmitter—unlike the common message of the Han-Kobayashi strategy, which is decoded and subtracted at the non-intended receiver.

Since both transmitters know both shared messages, the overall channel can be viewed as a vector broadcast channel with shared messages $(U_1, U_2)$, within an interference channel with private messages $V_1$ and $V_2$ (see Fig. 2.6). In addition, we pursue a simple
zero-forcing strategy for the broadcast channel part where the transmitters first encode $(U_1, U_2)$ jointly using zero-forcing precoding, while treating $(V_1, V_2)$ as interference, then encode $V_1$ and $V_2$ individually for their respective intended receivers. In effect, the broadcast channel sees the interference channel messages as interference, while the interference channel does not see the broadcast channel messages as interference, because they either are zero-forced at the transmitter or can be decoded-and-subtracted at the intended receiver. More specifically, the transmitted signals are

$$X_1 = U_1 - \frac{h_{21}}{h_{11}} U_2 + V_1, \quad X_2 = U_2 - \frac{h_{12}}{h_{22}} U_1 + V_2$$

(2.8)

where $U_i \sim \mathcal{N}(0, P_{s,i})$ is a codeword representing the shared message $W_{i,s}$ taken from a Gaussian codebook of size $2^{nC_{ij}}$, and $V_i \sim \mathcal{N}(0, P_{p,i})$ is a codeword representing the private message $W_{i,p}$ taken from a Gaussian codebook of size $2^{n(R_i - C_{ij})}$.

Since this work focuses on the zero-forcing strategy only, we constrain the cooperating digital link rates to be below that sufficient to achieve the fully cooperative broadcast channel rate using zero-forcing. More specifically, let $\mathcal{R}^{ZF}$ denote the achievable rate region of the same two-user channel with full transmit cooperation using zero-forcing encoding and under the same per-antenna power constraints. We assume that

$$(C_{12}, C_{21}) \in \mathcal{R}^{ZF}.$$  

(2.9)
Beyond this region, the entire $\mathcal{R}^{ZF}$ is achievable.

In the rest of this chapter we focus on the high SNR/INR regime, where zero-forcing is near optimal. In the more general case, beamforming and dirty-paper coding can be used for the underlying vector broadcast channel to further improve the achievable rates.

### 2.4 Characterization of Cooperation Gain

#### 2.4.1 Achievable Rate

The overall achievable rates of the proposed partial zero-forcing scheme can be computed by summing the rates of the respective shared messages and the private messages. The shared message is conveyed by the broadcast channel, where each receiver $i$ decodes the shared message $U_i$ while treating (both) private messages as interference. The receiver can then subtract $U_i$ from its received signal, and proceed to decode its private message $W_i$ treating the private message of the other user as noise. Note that zero-forcing precoding ensures that each receiver would not see the shared message of the other user as interference in either private or shared message decoding.

The rates of the shared messages are fixed by the capacities of the conferencing links. To compute the private message rates, we first compute the shared and private message powers. The zero-forcing precoding structure of the broadcast channel gives

$$
\begin{align*}
C_{12} &\leq \gamma \left( \frac{(1 - \frac{h_c}{h_d})^2 h_{11}^2 P_{s,1}}{N + h_{11}^2 P_{p,1} + h_{21}^2 P_{p,2}} \right) \\
C_{21} &\leq \gamma \left( \frac{(1 - \frac{h_c}{h_d})^2 h_{22}^2 P_{s,2}}{N + h_{22}^2 P_{p,2} + h_{12}^2 P_{p,1}} \right). 
\end{align*}
$$

(2.10)

Further, the power constraints at the transmitters give

$$
\begin{align*}
P_{s,1} + (h_{21}^2/h_{11}^2) P_{s,2} + P_{p,1} &\leq P_1 \\
(h_{12}^2/h_{22}^2) P_{s,1} + P_{s,2} + P_{p,2} &\leq P_2.
\end{align*}
$$

(2.11)
Setting the inequality constraints as equalities, we have a system of four linear equations in four unknowns, namely \( P_{s,1}, P_{s,2}, P_{p,1} \) and \( P_{p,2} \). After some algebra, we obtain the following expressions for the power of the private messages at the transmitters
\[
P_{p,1} = \frac{1}{\Delta} \left[ (h_c^2 \nu_1 + h_d^2 \nu_2 + (h_d - h_c)^2)P_1 - (\nu_1 + \nu_2)h_{21}^2 h_{22}^2 P_2 \right. \\
- \left. N(h_{22}^2 \nu_1 + h_{21}^2 \nu_2) - N \nu_1 \nu_2 (h_{22}^2 - h_{21}^2) \left( \frac{h_d + h_c}{h_d - h_c} \right) \right] \tag{2.12}
\]
where
\[
\Delta = (h_d - h_c)^2 + (h_d^2 + h_c^2)(\nu_2 + \nu_1) + (h_d + h_c)^2 \nu_1 \nu_2
\]
and \( P_{p,2} \) is obtained by swapping the indices 1 and 2 in (2.12). The total achievable rate to receiver \( i \) is then given by
\[
R_i = \gamma \left( \frac{h_{ii}^2 P_{p,i}}{N + h_{ji}^2 P_{p,j}} \right) + C_{ij}. \tag{2.13}
\]
Note that the condition (2.9) ensures that \( P_{p,1} \) and \( P_{p,2} \) are non-negative.

### 2.4.2 Low-Noise Regime

It is instructive to investigate the rate improvement due to the proposed cooperation strategy in the asymptotic low-noise regime with \( N \to 0 \) while fixing transmit power and channel gains. Interestingly, if we compare with a baseline case where interference is always treated as noise, the improvement in sum rate due to one bit of transmitter cooperation can be more than one bit.

For an interference channel with no transmit cooperation, if we always treat interference as noise, the asymptotic achievable rates as \( N \to 0 \) are
\[
R_1 = \gamma \left( \frac{\text{SNR}_1}{\text{INR}_1} \right), \quad R_2 = \gamma \left( \frac{\text{SNR}_2}{\text{INR}_2} \right) \tag{2.14}
\]

**Theorem 1.** For the interference channel in which the transmitters partially cooperate over digital links of fixed capacities \( C_{12} \) and \( C_{21} \) satisfying (2.9), the following rate pair
is achievable asymptotically as \( N \to 0 \)

\[
R_1 = \gamma \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) + C_{12} \\
R_2 = \gamma \left( \frac{1}{\Gamma \frac{\text{SNR}_2}{\text{INR}_2}} \right) + C_{21}
\]  

(2.15)

where

\[
\Gamma = \frac{h_d^2 \nu_1 + h_c^2 \nu_2 + (h_d - h_c)^2 - (\nu_1 + \nu_2) h_{12}^2 h_{22}^2 P_2/P_1}{h_d^2 \nu_1 + h_c^2 \nu_2 + (h_d - h_c)^2 - (\nu_1 + \nu_2) h_{12}^2 h_{11}^2 P_1/P_2}.
\]

(2.16)

**Proof.** Substituting the expressions (2.12) for \( P_{p,1} \) and \( P_{p,2} \) in (2.13), and taking the limit as \( N \to 0 \), we get (2.15).

The form of (2.15) is somewhat misleading. Note that every bit shared by transmitter \( i \) enables the other transmitter to pre-subtract and reduce the interference seen at its receiver. In other words, every bit shared by one transmitter results in an improvement in the communication rate to the other receiver. Simultaneously, as the other transmitter uses more of its transmit power in zero-forcing the shared message, the interference at receiver \( i \) also reduces. Unfortunately, the complicated form of (2.16) cannot be easily manipulated to reflect how much gain is achieved to each receiver.

We can obtain an asymptotic gain of more than one-bit-per-bit if

\[
(\Gamma - 1) \frac{\text{SNR}_1}{\text{INR}_1} + \left( \frac{1}{\Gamma} - 1 \right) \frac{\text{SNR}_2}{\text{INR}_2} > 0
\]  

(2.17)

Although the left-hand side of (2.17) is a messy function of the channel parameters, there are simple cases in which the gain can be easily characterized.

Consider first the *symmetric interference channel*. Let \( \delta \) denote the improvement in sum rate obtained asymptotically over (2.14) as \( N \to 0 \), and let \( C_{12} = C_{21} = C \). We have

\[
\delta = \gamma \left( \frac{\text{SNR}}{\text{INR}} \right) + \gamma \left( \frac{1}{\Gamma \frac{\text{SNR}}{\text{INR}}} \right) + 2C - 2\gamma \left( \frac{\text{SNR}}{\text{INR}} \right)
\]

\[
= \frac{1}{2} \log \left( \frac{1 + (\Gamma + \frac{1}{\Gamma}) \frac{\text{SNR}}{\text{INR}}}{1 + 2 \frac{\text{SNR}}{\text{INR}} + \frac{\text{SNR}^2}{\text{INR}^2}} \right) + 2C
\]

\[
= 2C
\]  

(2.18)
since $\Gamma = 1$. Thus, for the symmetric channel every cooperation bit asymptotically increase the sum-rate by one bit.

The second case is the high signal-to-interference-and-noise (SINR) regime, where every cooperation bit results in one bit improvement in sum rate. We have

$$
\delta = \gamma \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) + \gamma \left( \frac{1}{\Gamma} \frac{\text{SNR}_2}{\text{INR}_2} \right) + C_{12} + C_{21} \\
- \gamma \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) - \gamma \left( \frac{\text{SNR}_2}{\text{INR}_2} \right) \\
\approx \frac{1}{2} \log \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) + \frac{1}{2} \log \left( \frac{1}{\Gamma} \frac{\text{SNR}_2}{\text{INR}_2} \right) + C_{12} + C_{21} \\
- \frac{1}{2} \log \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) - \frac{1}{2} \log \left( \frac{\text{SNR}_2}{\text{INR}_2} \right) \\
= C_{12} + C_{21} \tag{2.19}
$$

2.5 Asymptotic Sum Capacity in the Very Weak Interference Regime

The main caveat of the results in the previous section is that the rate improvement is computed against a baseline case when interference is treated as noise. Treating interference as noise is not always optimal, particularly in the low-noise regime. As $N \rightarrow 0$, considerable rate gain can be obtained with a Han-Kobayashi strategy that includes common messages.

In this section, instead of letting $N \rightarrow 0$, we investigate a different asymptotic regime, where both SNR and INR go to infinity while their ratio in dB scale is kept fixed. In a very weak interference regime, we show that every cooperation bit yields exactly one bit improvement in sum rate asymptotically.
2.5.1 Sum Capacity Upper Bound

**Theorem 2.** The achievable sum rate in the interference channel wherein the transmitters partially cooperate over digital links of fixed capacities $C_{12}$ and $C_{21}$ is bounded by

$$R_1 + R_2 \leq \min \left\{ R_{bc}, \gamma \left( \frac{\text{INR}_1}{1 + \text{INR}_2} \right) + \gamma \left( \frac{\text{INR}_2}{1 + \text{INR}_1} \right) + C_{12} + C_{21} \right\} \quad (2.20)$$

where $R_{bc}$ is the sum capacity of a Gaussian vector broadcast channel with a two-antenna transmitter and two single-antenna receivers [4, Theorem 1].

**Proof.** Consider a genie-aided channel in which signals $S_1 = h_{12}X_1 + Z_2$ and $S_2 = h_{21}X_2 + Z_1$ are available as side-information to receivers 1 and 2 respectively. We derive an upper bound for the sum capacity of this genie-aided channel (see Figure 2.7).

Let $C^1$ and $C^2$ be two $(2^{nR_1}, n), (2^{nR_2}, n)$ codes such that $P_{e,1}^{(n)}, P_{e,2}^{(n)} \to 0$, where $P_{e,i}^{(n)} = \Pr(\{\hat{W}_i \neq W_i\})$. By Fano’s inequality, for $n$ sufficiently large

$$\frac{1}{n} H(W_i|Y^n_i) \leq \frac{1}{n} + R_i P_{e,i}^{(n)} < \epsilon. \quad (2.21)$$

Since conditioning cannot increase entropy, we have

$$H(W_i|Y^n_i, S^n_i, T^K_j) < n\epsilon \quad (2.22)$$
where $S^n_i$ is the information provided to receiver $i$ by the genie, and $T^n_j$ is the information provided to transmitter $i$ from transmitter $j$ during the conference. Now,

\[
\begin{align*}
  n(R_1 + R_2) &= H(W_1) + H(W_2) \\
  &= I(W_1; Y^n_1, S^n_1, T^n_2) + H(W_1|Y^n_1, S^n_1, T^n_2) \\
  &\quad + I(W_2; Y^n_2, S^n_2, T^n_1) + H(W_2|Y^n_2, S^n_2, T^n_1) \\
  \leq& I(W_1; Y^n_1, S^n_1, T^n_2) + I(W_2; Y^n_2, S^n_2, T^n_1) + 2n\epsilon \\
  &= I(W_1; T^n_2) + I(W_1; Y^n_1, S^n_1|T^n_2) \\
  &\quad + I(W_2; T^n_1) + I(W_2; Y^n_2, S^n_2|T^n_1) + 2n\epsilon \\
  &= H(T^n_2) - H(T^n_2|W_1) + I(W_1; S^n_1|T^n_2) + I(W_1; Y^n_1|T^n_2, S^n_1) + H(T^n_1) \\
  &\quad - H(T^n_1|W_2) + I(W_2; S^n_2|T^n_1) + I(W_2; Y^n_2|T^n_1, S^n_2) + 2n\epsilon \\
  \leq& H(T^n_2) + h(S^n_1|T^n_2) - h(S^n_1|X^n_1) + h(Y^n_1|T^n_2, S^n_1) \\
  &\quad - h(Y^n_1|X^n_1, S^n_1) + H(T^n_1) + h(S^n_2|T^n_1) - h(S^n_2|X^n_2) \\
  &\quad + h(Y^n_2|T^n_1, S^n_2) - h(Y^n_2|X^n_2, S^n_2) + 2n\epsilon \\
  &= H(T^n_2) + h(S^n_1|T^n_2) - h(Z^n_2) + h(Y^n_1|T^n_2, S^n_1) - h(S^n_2) + H(T^n_1) \\
  &\quad + h(S^n_2|T^n_1) - h(Z^n_2) + h(Y^n_2|T^n_1, S^n_2) - h(S^n_1) + 2n\epsilon \\
  \leq& h(Y^n_1|S^n_1) + h(Y^n_2|S^n_2) - h(Z^n_2) + h(Z^n_1) - h(Z^n_2) + n(C_{12} + C_{21}) + 2n\epsilon \\
  \leq& n\left(\gamma \left(\frac{\text{INR}_1}{1 + \text{INR}_1} + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) + \gamma \left(\frac{\text{INR}_2}{1 + \text{INR}_2} + \frac{\text{SNR}_2}{1 + \text{INR}_2}\right) + C_{12} + C_{21} + 2\epsilon\right)
\end{align*}
\]

where

(a) follows from Fano’s inequality (2.22);

(b) is due to $H(T^n_j|W_i) \geq 0$;

(c)
(c) follows from the fact that the encoding function is a mapping from \((W_i, T_j^K)\) to \(X^n_i\), so \(S^n_i - X^n_i - (W_i, T_j^K)\) forms a Markov chain;

(d) is due to \(h(S^n_i | T^K_j) - h(S^n_i) \leq 0, h(Y^n_i | T^K_j, S^n_i) \leq h(Y^n_i | S^n_i)\), and \(H(T^K_i) \leq nC_{ij}\);

(e) follows from [3, Thm. 1]

\[\square\]

### 2.5.2 Asymptotic Sum Capacity

We now characterize the asymptotic sum capacity of a Gaussian interference channel with transmitter cooperation link capacities \(C_{12}\) and \(C_{21}\) in the limit as SNR\(_1\), INR\(_1\), SNR\(_2\), INR\(_2\) → ∞ while their log ratios

\[
\alpha_1 = \frac{\log(\text{INR}_1)}{\log(\text{SNR}_1)} \quad \alpha_2 = \frac{\log(\text{INR}_2)}{\log(\text{SNR}_2)}
\]

are kept constant. Denote the asymptotic sum capacity as \(C^\infty_{\text{sum}}(C_{12}, C_{21})\). A fundamental result in [3] is that in the very weak interference regime, defined by \(\alpha_1 \leq \frac{1}{2}\) and \(\alpha_2 \leq \frac{1}{2}\),

\[
C^\infty_{\text{sum}}(0, 0) \approx \frac{1 - \alpha_1}{2} \log(\text{SNR}_1) + \frac{1 - \alpha_2}{2} \log(\text{SNR}_2) \quad (2.33)
\]

where \(f(x) \approx g(x)\) denotes \(\lim f(x) - g(x) = 0\).

**Theorem 3.** In the very weak interference regime where \(\alpha_1 \leq \frac{1}{2}\) and \(\alpha_2 \leq \frac{1}{2}\), every transmit cooperation bit improves the asymptotic sum capacity by exactly one bit as SNR\(_1\), INR\(_1\), SNR\(_2\), INR\(_2\) → ∞ while \(\alpha_1\) and \(\alpha_2\) are kept fixed. More precisely, assuming \(C_{12}, C_{21}\) satisfy (2.9), then

\[
C^\infty_{\text{sum}}(C_{12}, C_{21}) = C^\infty_{\text{sum}}(0, 0) + C_{12} + C_{21}.
\]

**Proof.** The achievability follows from the achievable rate computation in Section IV.
From (2.12), we have
\[
\frac{h_{11}^2 P_{p,1}}{N} = \frac{1}{\Delta} \left[ (h_c^2 \nu_1 + h_d^2 \nu_2 + (h_d - h_c)^2) \text{SNR}_1 ight. \\
\left. - (\nu_1 + \nu_2) h_d^2 \text{INR}_1 + \lambda_1 \right] 
\] (2.34)
\[
\frac{h_{21}^2 P_{p,2}}{N} = \frac{1}{\Delta} \left[ (h_c^2 \nu_2 + h_d^2 \nu_1 + (h_d - h_c)^2) \text{INR}_1 ight. \\
\left. - (\nu_1 + \nu_2) h_d^2 \text{SNR}_1 + \lambda_2 \right] 
\]
where \( \lambda_1, \lambda_2 \) and \( \Delta \) are terms that will eventually vanish in subsequent calculations as \( \text{SNR}_i, \text{INR}_i \to \infty \). Now, since \( \alpha_1 \) and \( \alpha_2 \) are fixed,
\[
\text{INR}_1 = \text{SNR}_1^{\alpha_1}, \quad \text{INR}_2 = \text{SNR}_2^{\alpha_2}. 
\] (2.35)
Substituting (2.34) and (2.35) in (2.13), we get
\[
R_1 = \gamma \left( \frac{[h_c^2 \nu_1 + h_d^2 \nu_2 + (h_d - h_c)^2] \text{SNR}_1 - (\nu_1 + \nu_2) h_d^2 \text{SNR}_1^{\alpha_1} + \lambda_1}{[h_c^2 \nu_2 + h_d^2 \nu_1 + (h_d - h_c)^2] \text{SNR}_1^{\alpha_1} - (\nu_1 + \nu_2) h_d^2 \text{SNR}_1 + \lambda_2 + \Delta} \right) + C_{12} \\
\approx \gamma (\Gamma_1 \text{SNR}_1^{1-\alpha_1}) + C_{12}, \quad \text{as} \ \text{SNR}_1 \to \infty 
\] (2.36)
where
\[
\Gamma_1 = \frac{h_c^2 \nu_1 + h_d^2 \nu_2 + (h_d - h_c)^2 - (\nu_1 + \nu_2) h_d^2 \text{SNR}_1^{\alpha_1-1}}{h_c^2 \nu_2 + h_d^2 \nu_1 + (h_d - h_c)^2 - (\nu_1 + \nu_2) h_d^2 \text{SNR}_1^{1-\alpha_1}} 
\] (2.37)
Similarly,
\[
R_2 \approx \gamma (\Gamma_2 \text{SNR}_2^{1-\alpha_2}) + C_{21}, \quad \text{as} \ \text{SNR}_2 \to \infty 
\] (2.38)
where
\[
\Gamma_2 = \frac{h_c^2 \nu_2 + h_d^2 \nu_1 + (h_d - h_c)^2 - (\nu_1 + \nu_2) h_d^2 \text{SNR}_2^{\alpha_2-1}}{h_c^2 \nu_1 + h_d^2 \nu_2 + (h_d - h_c)^2 - (\nu_1 + \nu_2) h_d^2 \text{SNR}_2^{1-\alpha_2}} 
\] (2.39)
It is easily verified that \( h_d^2 \text{SNR}_2^{1-\alpha_2} = h_d^2 \text{SNR}_1^{\alpha_1-1} \). It immediately follows that \( \Gamma_1 \Gamma_2 = 1 \).
Therefore, as \( \text{SNR}_1, \text{SNR}_2 \to \infty \), we have
\[ R_1 + R_2 \approx \gamma (\Gamma_1 \text{SNR}_1^{1-\alpha_1}) + \gamma (\Gamma_2 \text{SNR}_2^{1-\alpha_2}) + C_{12} + C_{21} \]
\[ \approx \frac{1}{2} \log (\Gamma_1 \text{SNR}_1^{1-\alpha_1}) + \frac{1}{2} \log (\Gamma_2 \text{SNR}_2^{1-\alpha_2}) + C_{12} + C_{21} \]
\[ = \frac{1 - \alpha_1}{2} \log (\text{SNR}_1) + \frac{1 - \alpha_2}{2} \log (\text{SNR}_2) + C_{12} + C_{21} \] (2.40)

The converse follows from the outer bound in Theorem 2:

\[ R_1 + R_2 \leq \gamma \left( \text{INR}_1 + \frac{\text{SNR}_1}{1 + \text{INR}_2} \right) + \gamma \left( \text{INR}_2 + \frac{\text{SNR}_2}{1 + \text{INR}_1} \right) + C_{12} + C_{21} \]
\[ < \frac{1}{2} \log \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) + \frac{1}{2} \log \left( \frac{\text{INR}_1 + \text{INR}_2 + \text{INR}_1}{\text{SNR}_1 + \text{SNR}_1 + \text{INR}_2} \right) \]
\[ + \frac{1}{2} \log \left( \frac{\text{SNR}_2}{\text{INR}_2} \right) + \frac{1}{2} \log \left( \frac{\text{INR}_2 + \text{INR}_2 + \text{INR}_2}{\text{SNR}_2 + \text{SNR}_2 + \text{INR}_1} \right) \]
\[ + C_{12} + C_{21} \]
\[ = \frac{1 - \alpha_1}{2} \log (\text{SNR}_1) + \frac{1 - \alpha_2}{2} \log (\text{SNR}_2) + C_{12} + C_{21} \]
\[ + \frac{1}{2} \log \left( \frac{\text{SNR}_1^{\alpha_1-1} + \text{SNR}_2^{\alpha_1-1} + \text{INR}_1}{\text{INR}_2} \right) \]
\[ + \frac{1}{2} \log \left( \frac{\text{SNR}_2^{\alpha_2-1} + \text{SNR}_2^{\alpha_2-1} + \text{INR}_2}{\text{INR}_1} \right) \] (2.41)

The sum of the last two terms vanishes as \( \text{SNR}_1, \text{SNR}_2 \to \infty \) for \( \alpha_1, \alpha_2 < 1/2 \). This completes the proof. \( \Box \)

### 2.6 Concluding Remarks

In this chapter we explored the use of cooperation link at the transmitters for interference mitigation in the two-user Gaussian interference channel. We proposed a zero-forcing strategy which pre-subtracts part of the interfering signal. This strategy is particularly effective in high SNR/INR regimes—it gives one bit of sum-capacity improvement for every cooperation bit in a very weak interference regime; it can yield more than one-bit-per-cooperation-bit when compared with the case where interference is treated as noise.
Chapter 3

Multicell systems with constrained backhaul

In this chapter we extend the partial zero-forcing precoding strategy to a restricted class of multiuser interference channels. The system model under consideration is motivated by an envisioned architecture for new cellular networks with local cooperation, in which each base station is connected to its six neighbouring base stations via rate-limited backhaul links. An algorithm is outlined to iteratively compute the precoder coefficients in a distributed fashion at various transmit nodes. The cooperation gain is characterized analytically in an asymptotic low-noise regime, wherein, by broadcasting one bit of information about its message to the other transmit nodes, a transmitter can improve the rate to its intended receiver by one bit. Finally, we present simulation results which highlight the cooperation gain due to the proposed strategy in a multicell system.

3.1 Background and Related Work

Conventional wireless systems are designed with a cellular architecture in which base-stations from different cells communicate with their respective remote terminals independently. Signal processing is performed on a per-cell basis with intercell interference
treated as background noise. Interference management consists of fixing the frequency reuse pattern and by fixing the maximum PSD levels of each base station. To increase area spectral efficiency of cellular networks, emerging cellular standards are increasingly designed with frequency reuse factor of 1. In this case, intercell interference becomes the dominant limiting factor.

The performance of a conventional network can be significantly improved if joint signal processing is enabled across the different base-stations to minimize or even to cancel intercell interference. There has been a lot of recent interest in the design of cellular systems wherein base stations are interconnected and can cooperate via rate-limited backhaul links [11]. Base stations can exchange control information based on their channel estimates of the direct and interfering links, and the effect of interference can be reduced by means of joint power control and coordinated beamforming.

In particular, when base stations are linked by high-capacity delay-free links, in addition to channel state information they can share the full data signals of their respective users. The various base-station antennas essentially function as a single antenna array, enabling joint transmission to improve communication rates in the downlink. Phase synchronization between the various transmit nodes is not required when we have perfect channel estimation, but frequency synchronization of the carrier signals is needed. Under this scenario, the resulting channel is a multiple-input single-output (MISO) multiuser broadcast channel. Although the information-theoretic throughput capacity of this channel has been characterized [4], efficient transmission schemes are still being studied. In particular, linear precoding schemes, which provide a good tradeoff between performance and complexity, have received a lot of interest. The most common linear precoding scheme is zero-forcing (ZF) beamforming, which performs very well in the high signal-to-noise (SNR) regime.

There has been a great deal of research on the use of zero-forcing beamforming for the MISO multiuser broadcast channel. Zero-forcing decouples the multiuser channel into
independent subchannels for the different users since the precoder inverts the channel. ZF precoding is known to closely related to the concept of generalized inequalities in linear algebra. The pseudo-inverse precoder is optimal among the set of generalized inverses for maximizing any performance measure under a total power constraint [12]. For per-antenna power constraints, the optimal precoder matrix among the set of generalized inverses depends on the specific performance criterion. The two classical criteria, fairness and throughput, are considered in [12], wherein the design problems are recast as convex optimization problems (SOCP and MAXDET respectively) which can be efficiently solved.

However, an important problem associated with multi-cell signal processing is the volume of additional backhaul traffic required between cooperating base stations. In current cellular systems, the typical per-cell data rate is of the order of 100 Mbps for a 10 MHz bandwidth. A reasonable expectation is that the data rates are doubled when full cooperation is employed. However, there is a ten-fold increase in the amount of data each BS would need to process. While the capacity of the backhaul (e.g. fibre-optics) can easily support full cooperation, considerations of decoding delays and processing complexity motivate the recent interest in alternate models where the base-stations cooperate via rate-limited backhaul links. Typically, channel state information is shared first, then, only a substream of user data or a quantized version of the transmit signals is shared among the base-stations, which allows partial interference cancellation. In this realm, quantizing, dirty-paper coding, beamforming and time-sharing strategies for transmitter cooperation are discussed in [13, 14].

### 3.2 System Model

This chapter considers a multiuser interference channel with constrained backhaul, in which $M$ transmitters communicate with their respective intended receivers over a shared
Chapter 3. Multicell systems with constrained backhaul

communication medium while mutually interfering with each other. The transmit nodes are indexed by \( \{1, 2, \ldots, M\} \). Each transmit node is connected to a subset \( \mathcal{N}(i) \), called the neighbourhood of transmitter \( i \), of the other transmitters in the system via noiseless rate-limited digital backhaul links of capacities \( C_i \). (Note that in the previous chapter, the capacity of the backhaul link connecting transmitter \( i \) to transmitter \( j \) was denoted by \( C_{ij} \). Here we assume that the backhaul links connecting transmitter \( i \) to the transmitters in \( \mathcal{N}(i) \) have identical capacities. Consequently, we drop the sub-script \( j \) in the notation for the capacity of the backhaul link from transmitter \( i \) to transmitter \( j \) (\( j \in \mathcal{N}(i) \)), and denote it simply by \( C_i \).)

The multiuser Gaussian interference channel can be written as:

\[
Y_i = \sum_{j=1}^{M} h_{ij}X_j + Z_i, \quad \forall i \in \{1, 2, \ldots, M\}
\]  

(3.1)

where \( X_j \) is the signal transmitted by node \( j \) under power constraint \( P_j \), \( Y_i \) is the received signal at receiver \( i \), \( h_{ij} \) is the channel gain from transmitter \( j \) to receiver \( i \) and \( Z_i \) is the additive white Gaussian noise (AWGN) at receiver \( i \) with power \( N \).

The system model is motivated by an envisioned architecture for new cellular networks with local cooperation in which each base station is connected only to its six neighbouring base stations. Moreover the cooperation links connecting each base station to its neighbours are rate-limited, allowing only a substream of user data or a quantized version of the transmit signals to be shared. Consider the cellular system with 19 cells shown in Figure 3.1. For example, \( BS_2 \) is only connected to its six neighbours, i.e., \( \mathcal{N}(2) = \{1, 3, 7, 8, 9, 10\} \). It cooperates only locally by exchanging control (CSI) data and \( C_2 \) bits of information about its signal with these neighbouring \( BS \)s.

In this chapter we consider interference channels that satisfy the following two properties:

- \( H \) is a “weak” interference channel, wherein the direct channel from a transmitter
Figure 3.1: A 19-cell system with constrained backhaul. Only the backhaul links from BS 2 are shown.

| $h_{ij}$ | $h_{ii}$ | $\forall i \in \{1, 2, \ldots, M\}$. |

(3.2)

- The interference links from out-of-neighbourhood transmit nodes are relatively weak, i.e.

$$h_{ik}^2 < h_{ij}^2, \quad \forall k \notin \mathcal{N}(i); j \in \mathcal{N}(i)$$

(3.3)

The first assumption — regarding weak interference — is motivated by our interest in zero-forcing precoding. For such interference channels it intuitively makes sense to pre-subtract interference at the transmitter. On the other hand, if any of the interfering links is strong, then joint encoding and transmission using both direct and interfering links can be employed to communicate to the receiver.

The second assumption is again motivated by our interest in a multicell system with limited local cooperation, wherein the interference due to out-of-neighbourhood transmitters is due to base stations that are two cells away. The total interference from these base stations located two (or more) cells away is typically comparable to the noise at the receivers. These interfering links are too weak to have any significant additional effect compared to the impairments already caused by additive noise.
Definition 3. An interference channel is defined to be fully connected if each transmitter is connected to all the remaining transmitters in the system by the backhaul network, i.e.,

\[ \mathcal{N}(i) = \{1, 2, \cdots, M\} \setminus i, \quad \forall i \in \{1, 2, \cdots, M\}. \quad (3.4) \]

3.3 Coding Strategy

We extend the strategy of shared-private rate-splitting and partial zero-forcing precoding of the previous chapter to the multiuser interference channel.

3.3.1 Shared-private rate splitting

Transmitter \( i \) splits its source message \( W_i \) into two parts, namely a shared message \( W_{i,s} \in \{1, 2, \ldots, 2^{nC_i}\} \), and a private message \( W_{i,p} \in \{1, 2, \ldots, 2^{n(R_i-C_i)}\} \). Node \( i \) then broadcasts its shared message to the transmitters to which it is connected, i.e., it communicates the same \( C_i \) bits of information about its message to all the transmitters in \( \mathcal{N}(i) \).

Let \( U_i \sim \mathcal{N}(0, P_{s,i}) \) denote a codeword representing the shared message \( W_{i,s} \) taken from a Gaussian codebook of size \( 2^{nC_i} \), and \( V_i \sim \mathcal{N}(0, P_{p,i}) \) denote a codeword representing the private message \( W_{i,p} \) taken from a Gaussian codebook of size \( 2^{n(R_i-C_i)} \). The transmitted signals are of the form

\[ X_i = T_{ii}U_i - \sum_{j \in \mathcal{N}(i)} T_{ij}U_j + V_i \quad (3.5) \]

where \( T_{ij} \) are the precoding coefficients.

Quantize-and-forward

The transmitters use the shared-private rate-splitting strategy to communicate information about their signals to the other transmitters. In Section (3.5.3) we compare this strategy to a quantize-and-forward strategy, wherein each transmitter communicates only a private message to its receiver. Let \( V_i \sim \mathcal{N}(0, P_{p,i}) \) denote a codeword representing the (private) message \( W_i \) taken from a Gaussian codebook of size \( 2^{nR_i} \). Transmitter \( i \)
quantizes its private-message codeword $V_i$ with a Gaussian quantizer using $C_i$ bits and communicates the quantizer output $\hat{V}_i$ to the other transmitters. The transmitters then pre-subtract the quantized interfering signals. The transmitted signals are of the form

$$X_i = T_{ii}\hat{V}_i - \sum_{j \in N(i)} T_{ij}\hat{V}_j + (V_i - \hat{V}_i)$$

(3.6)

where $T_{ij}$ are the precoding coefficients.

### 3.3.2 Zero-forcing precoder

For a fully connected interference channel, we choose the channel inverse $H^{-1}$ as the precoding matrix.

For an interference channel that is not fully connected, recall that we are only interested in the case when the interfering links from out-of-neighbourhood transmit nodes are relatively weak, i.e.

$$h^2_{ik} \ll h^2_{ij}, \quad \forall k \notin N(i); j \in N(i)$$

Using this assumption 3.3, we approximate the channel $H$ by $\tilde{H}$ where

$$\tilde{H}_{ij} = \begin{cases} h_{ij}, & \text{if } j \in N(i) \cup \{i\} \\ 0, & \text{otherwise} \end{cases}$$

(3.7)

Receiver $i$ effectively sees interference from the other transmitters in $N(i)$, and treats the interference from out-of-neighbourhood transmitters as noise. Transmitter $i$ knows a part of the interfering signal, and pre-subtracts this known interference using zero-forcing precoding. The precoding coefficients are obtained as the solution to the following system of $M^2$ linear equations

$$(\tilde{H}T)_{ij} = I_{ij} \quad \forall j \in N(i) \cup \{i\}$$

$$T_{ij} = 0, \quad \forall j \notin N(i) \cup \{i\}$$

$$\forall i \in \{1, 2, \cdots, M\}$$

(3.8)
where $I_{ij}$ is the $(i,j)$-th element of the $M \times M$ identity matrix. Since the channel gains $h_{ij}$ are random, we can expect Eqn. (3.8) to have a unique solution. We will see in the later sections of this chapter that this precoder has two desirable properties — it can be implemented in a distributed fashion, and it provides significant improvement in throughput in the downlink.

### 3.3.3 Power allocation

Each transmitter is constrained by its total transmit-power budget. At each transmitter, we need to calculate how much of its available power needs to be allocated to the shared message, and how much power is required to be spent for zero-forcing the shared messages of its neighbouring transmitters. The remaining power is then allocated to transmission of the private message.

The rates of the shared messages are fixed by the capacities of the conferencing links. In order for the shared messages to be decoded at the respective intended receivers, the following SINR constraints must be satisfied

$$\gamma \left( \frac{(HT)_{ii}^2 P_{s,i}}{N + \sum_{j=1,j\neq i}^M (HT)_{ij}^2 P_{s,j} + \sum_{j=1}^M (h)_{ij}^2 P_{p,j}} \right) \geq C_i$$

(3.9)

Further, the power constraints at the transmitters give

$$T_{ii}^2 P_{s,i} + \sum_{j \in N(i)} T_{ij}^2 P_{s,j} + P_{p,i} \leq P_i$$

(3.10)

where $T_{ij}$ are coefficients of the zero-forcing precoder defined in (3.8).

### 3.4 Distributed signal processing

Recall that the pre-coding coefficients are obtained as the solution to a system of $M^2$ linear equations (3.8). The classical algorithms, like the Jacobi or Gauss-Siedel algorithms, for iteratively solving a system of linear equations can be easily implemented using parallel architectures with multiple computing nodes.
3.4.1 Precoder coefficients

We now look at implementation of (3.8) in a distributed fashion. We have

\[(\tilde{H} T)_{ij} = I_{ij} \quad \forall j \in \mathcal{N}(i) \cup \{i\}\]

\[T_{ij} = 0, \quad \forall j \notin \mathcal{N}(i) \cup \{i\}\]

\[\forall i \in \{1, 2, \cdots, M\}\]

We have

\[(\tilde{H} T)_{ij} = \tilde{H}_{ii} T_{ii} + \sum_{k=1, k \neq i}^{M} \tilde{H}_{ik} T_{kj}\]

\[= h_{ii} T_{ij} + \sum_{k \in \mathcal{N}(i)} h_{ik} T_{kj}\]

(3.11)

since \(\tilde{H}_{ik} = 0\) if \(k \notin \mathcal{N}(i) \cup \{i\}\).

We propose the following algorithm based on Gauss-Siedel iterations, to compute the precoder coefficients in an iterative, distributed fashion at the various transmit nodes using only local exchange of control information. The precoder is initialized as an identity matrix, corresponding to the case when each node transmits only its own shared message and private message. Every node communicates its local channel-state information (direct and cross channel gains) to its neighbouring nodes. At the \(n\)-th iteration, node \(i\) solves the following equation for every \(j \in \mathcal{N}(i)\):

\[T_{ij}^{(n)} = \frac{1}{h_{ii}} \left[ I_{ij} - \sum_{k \in \mathcal{N}(i), k < \pi(i)} h_{ik} T_{kj}^{(n)} - \sum_{k \in \mathcal{N}(i), k > \pi(i)} h_{ik} T_{kj}^{(n-1)} \right]\]

(3.12)

and communicates the updated precoder coefficients \(T_{ij}^{(n)}\) to its neighbouring nodes. \(\pi : \{1, 2, \cdots, M\} \rightarrow \{1, 2, \cdots, M\}\) is a random permutation of \(\{1, 2, \cdots, M\}\). At each iteration the precoding coefficients for the nodes are updated in a random order, and each node uses the latest available information about the coefficients of its neighbours. The algorithm is terminated after a fixed number of iterations. Note that in order to update its precoding coefficients using (3.12), node \(i\) only needs to know the local channel
conditions (namely the direct-channel gain $h_{ii}$, and the cross channel gains $h_{ij}, j \in \mathcal{N}(i)$ of the interfering links from the neighbouring nodes to its receiver) and the precoder coefficients of the neighbouring nodes.

### 3.4.2 Power Allocation

Once the precoder coefficients are determined using the algorithm outlined in previous subsection, the nodes need to determine the amount of power to be allocated to their shared and private message codebooks. Let $I$ denote the average interference from out-of-neighbourhood users. The receivers treat interference from out-of-neighbourhood users as noise. The receivers do not see any interference from the shared messages of neighbouring transmitters since these have been pre-subtracted using the zero-forcing precoder.

The SINR and transmit power constraints outlined in Section (3.3.3) simplify to:

$$\gamma \left( \frac{P_{s,i}}{N + I + h_{ii}^2 P_{p,i} + \sum_{j \in \mathcal{N}(i)} h_{ij}^2 P_{p,j}} \right) \geq C_i$$

$$T_{ii}^2 P_{s,i} + \sum_{j \in \mathcal{N}(i)} T_{ij}^2 P_{s,j} + P_{p,i} \leq P_i, \; \forall i. \quad (3.13)$$

Converting the inequality constraints into equalities (this is not necessarily optimal), using (3.13), node $i$ can compute its shared and private message powers, $P_{s,i}$ and $P_{p,i}$ respectively, in an iterative fashion using only local information about the shared and private message powers of those transmit nodes in its neighbourhood $\mathcal{N}(i)$. The update equations are

$$P_{s,i}^{(n)} - \gamma_i h_{ii}^2 P_{p,i}^{(n)} = \left[ \gamma_i (N + I + \sum_{j \in \mathcal{N}(i)} h_{ij}^2 P_{p,j}^{(n-1)}) \right]$$

$$P_{p,i}^{(n)} + T_{ii}^2 P_{s,i}^{(n)} = P_i - \sum_{j \in \mathcal{N}(i)} T_{ij}^2 P_{s,j}^{(n-1)} \quad (3.14)$$

At the $n$-th iteration each node $i$ updates its shared message power $P_{s,i}$ and private message power $P_{p,i}$ using (3.14), and communicates these to its neighbouring transmit nodes. The algorithm is terminated after a fixed number of iterations.
In this section we proposed a distributed algorithm for computing the precoder coefficients and the corresponding power allocation locally at the various transmit nodes, using only a local exchange of control information. The convergence behaviour of this distributed algorithm is examined in Appendix B.

3.5 Characterization of Cooperation Gain

3.5.1 Achievable Rate

The overall achievable rates of the proposed partial zero-forcing scheme can be computed by summing the rates of the respective shared messages and the private messages. The receivers treat interference from out-of-neighbourhood users as noise. Receiver $i$ first decodes the shared message $U_i$ while treating all private messages as interference. The receiver can then subtract $U_i$ from its received signal, and proceed to decode its private message $W_i$ treating the private message of the other users as noise. Note that zero-forcing precoding ensures that each receiver would not see the shared messages of the other (neighbourhood) users as interference in either private or shared message decoding.

The rates of the shared messages are fixed by the capacities of the conferencing links. To compute the private message rates, we first compute the shared and private message powers. Let $I$ denote the average interference from out-of-neighbourhood users. The total achievable rate to receiver $i$ is then given by

$$R_i = \gamma \left( \frac{h_{ii}^2 P_{p,i}}{N + I + \sum_{j \in N(i)} h_{ij}^2 P_{p,j}} \right) + C_i.$$  

(3.15)

3.5.2 Low noise regime

It is instructive to investigate the rate improvement due to the proposed cooperation strategy in the asymptotic low-noise regime with $N \to 0$ while fixing transmit power and channel gains. For a fully-connected multiuser Gaussian interference channel, compared
with a baseline case where interference is always treated as noise, by broadcasting one bit of information about its message to the other transmitters, a transmitter can improve the rate to its intended receiver by one bit.

**Theorem 4.** For a fully connected multiuser interference channel with \( M \) transmitter-receiver pairs, the following rate is achievable to receiver \( i \) asymptotically as \( N \to 0 \)

\[
R_i = \gamma \left( \frac{h_{ii}^2 P_i}{\sum_{j=1, j \neq i}^M h_{ij}^2 P_j} \right) + C_i
\]

if transmitter \( i \) broadcasts \( C_i \) bits of information about its transmit signal to the other transmitters.

**Proof.** Transmitter \( i \) generates codewords \( U_i \sim \mathcal{N}(0, P_u) \) and \( V_i \sim \mathcal{N}(0, \beta P_i) \), representing the shared message \( W_{i,s} \) and the private message \( W_{i,p} \) and taken from Gaussian codebooks of size \( 2^{nC_i} \) and \( 2^{n(R_i - C_i)} \) respectively. Note that this power allocation strategy is different from that proposed in Sections (3.2.3) and (2.4.1). All transmitters allocate identical amount of power \( P_u \) to their shared messages and a fixed fraction \( \beta \) of its total available transmit power for the transmission of their private messages.

Using the zero-forcing precoder \( H^{-1} \), the transmit signals are given by

\[
X_i = \sum_{j=1}^M (H^{-1})_{ij} U_j + V_i
\]

The received signal at user \( i \) is given by

\[
Y_i = U_i + \sum_{j=1}^M H_{ij} V_j + Z_i
\]

Each receiver first decodes its shared message, treating its own private message as well as private messages of the other users as interference. Note that it does not see any interference due to the shared messages of the other users, since these have been pre-subtracted using the zero-forcing precoder \( H^{-1} \) at the transmitter. Successful decoding
is possible if the following SINR constraints are satisfied:

\[
C_i \leq \gamma \left( \frac{P_{s,i}}{N + \sum_{j=1}^{M} h_{ij}^2 P_{p,j}} \right) \\
= \gamma \left( \frac{P_u}{N + \beta \sum_{j=1}^{M} h_{ij}^2 P_j} \right)
\]

\[\forall i \in \{1, 2, \cdots, M\} \tag{3.19}\]

Converting the inequality constraints to equalities, Eqn.(3.19) contains \(M\) linear equations in the single unknown parameter \(\beta\). Let \(\beta_i\) denote the value of \(\beta\) obtained by solving the \(i\)-th equation \((i = 1, 2, \cdots, M)\). Choose

\[
\beta = \frac{1}{\Gamma} \min\{\beta_1, \beta_2, \cdots, \beta_M\} \tag{3.20}
\]

where the scaling factor \(\Gamma\) is chosen such that the transmit power constraints

\[
\sum_{j=1}^{M} (H^{-1})_{ij}^2 P_u + \beta P_i \leq P_i \quad \forall i \in \{1, 2, \cdots, M\} \tag{3.21}
\]

are satisfied. Note that this choice of \(\beta\) satisfies the SINR constraints for the successful decoding of shared messages at all the receivers.

The receivers subtract the shared message from their received signal and then decode their private messages. The private rates are given by

\[
R_{i,p} = \gamma \left( \frac{h_{ii}^2 P_{p,i}}{N + \sum_{j=1, j\neq i}^{M} h_{ij}^2 P_{p,j}} \right) \\
= \gamma \left( \frac{\beta h_{ii}^2 P_i}{N + \beta \sum_{j=1, j\neq i}^{M} h_{ij}^2 P_j} \right) \tag{3.22}
\]

The total rate is obtained by summing the rates of the shared and private messages.

\[
R_i = C_i + \gamma \left( \frac{\beta h_{ii}^2 P_i}{N + \beta \sum_{j=1, j\neq i}^{M} h_{ij}^2 P_j} \right) \tag{3.23}
\]

Taking the limit as \(N \to 0\), we get (3.16). This completes the proof.
**Corollary 1.** For the two-user interference channel in which the transmitters cooperate over rate-limited links of finite capacities $C_{12}$ and $C_{21}$ the following rate pair is asymptotically achievable as $N \to 0$

\[
R_1 = \gamma \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) + C_{12} \\
R_2 = \gamma \left( \frac{\text{SNR}_2}{\text{INR}_2} \right) + C_{21}
\] (3.24)

It is interesting to compare the statement of Corollary 1 with Theorem 1 from the previous chapter. Corollary 1 states that for every bit that transmitter $i$ communicates to the other transmitter, the rate to its own receiver increases by one bit, whereas we saw that in the case of Theorem 1, for every bit that transmitter $i$ communicates to the other transmitter, there is an increase in the rates to both receivers. This is quite surprising since both are based on identical coding strategies, and only differ in the power allocation. (The discussion in the appendix sheds some light on these two results.) Also, it must be noted that these are achievability results. Even with the partial zero-forcing strategy itself, it may be possible to achieve an asymptotic gain of more than one-bit-per-shared-bit using a different power-allocation strategy.

Theorem 4 needs to be clearly distinguished from the sum-capacity results of Section 2.5. Here we are comparing with a baseline case in which interference is treated as noise. As $N \to 0$, unbounded rates can be achieved in the interference channel even with no cooperation, using a simple time-sharing strategy. In each time-slot only one transmitter is active, during which it has a perfect channel to its receiver.

### 3.5.3 Simulations

We consider a wireless cellular network comprised of 19 cells with wrap-around. Each base station is connected to its six neighbouring base stations via digital backhaul links of finite capacities. The BS-to-BS distance is set to 2.8 kilometers. The thermal noise is -174dBm/Hz. The average out-of- neighbourhood interference from base stations two cells
away is -172dBm/Hz. Taking into account a receiver noise figure of 7dB, the effective noise becomes -162dB/Hz.

We focus on downlink transmission with single user per cell and frequency reuse factor 1. We assume a single antenna at transmitter/receiver with 15dBi antenna gain. The transmitter power budget is -27 dBm/Hz. In addition, we assume an SNR gap of 6 dB, which accounts for the effect of practical channel coding and modulation.

We consider a path-loss model in which the loss due to distance is modeled using the following formula in unit dB:

\[
L = 128.1 + 37.6 \log_{10}(d) \tag{3.25}
\]

where \( d \) is the distance from base station to mobile user in unit of kilometer. Throughout the simulation we assume perfect channel estimation. Although we use real numbers for the channel gains, the ideas of previous sections are valid even when complex channels are considered, and the results of the simulation will be essentially the same.

The system configuration is summarized in Table 3.1.

In the baseline system, there is no cooperation between the transmitters with out-of-cell interference treated as noise. Figure 3.2 compares the performance (average improvement in downlink rates versus the number of bits shared by each transmitter) of the proposed precoder (3.8) implemented using the distributed algorithm described in Section 3.4, with a channel-inverse precoder.

The shared-private rate-splitting strategy is compared with a quantize-and-forward strategy, in which each transmitter communicates a quantized version of its transmit signal to the other transmitters. The transmitters pre-subtract the quantized signal from their transmit signal. The simulation results show that from the perspective of communicating information about the transmit signals over the backhaul links for the purpose of interference pre-subtraction, the shared-private rate-splitting strategy clearly outperforms quantize-and-forward.
### Table 3.1: System Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cellular Layout</td>
<td>Hexagonal Grid, 19 cell sites with wrap-around</td>
</tr>
<tr>
<td>BS-to-BS distance</td>
<td>2.8 kilometer</td>
</tr>
<tr>
<td>Frequency Reuse</td>
<td>1</td>
</tr>
<tr>
<td>BS Tx Power Budget</td>
<td>-27 dBm/Hz</td>
</tr>
<tr>
<td>Antenna Gain</td>
<td>15 dBi</td>
</tr>
<tr>
<td>SNR Gap</td>
<td>6 dB</td>
</tr>
<tr>
<td>Background Noise</td>
<td>-169 dBm/Hz</td>
</tr>
<tr>
<td>Noise Figure</td>
<td>7 dB</td>
</tr>
<tr>
<td>No. of BS Tx Antennas</td>
<td>1</td>
</tr>
<tr>
<td>No. of MS Rx Antennas</td>
<td>1</td>
</tr>
<tr>
<td>Distance-dependent path loss</td>
<td>$L(dB) = 128.1 + 37.6 \log_{10}(d), d \text{ in km}$</td>
</tr>
<tr>
<td>Channel Estimation</td>
<td>Perfect (at Tx)</td>
</tr>
</tbody>
</table>
Figure 3.2: Comparison of the improvement in average downlink rates versus number of bits shared for the shared-private rate-splitting (SP) and quantize-and-forward (QF) strategies.
The initial gain due to the partial zero-forcing precoding strategy is around one-bit-per-bit. For the sake of simplicity, in this simulation every transmitter shares the same number of bits. In Figure 3.2 as the number of shared bits increases, at the point A one of the transmitters uses up all its available power to transmit its shared message and pre-subtract the shared messages of its neighbours, and has no power left to allocate to transmitting a private message to its receiver. Consequently, any further increase in shared-message rates is not feasible. It is possible to design better schemes, at the expense of greater complexity, in which not all transmitters are required to share the same number of bits.

Figure 3.5.3 plot the histogram of the average improvement in downlink rates as a fraction of the number of bits shared by each transmitter over 1000 randomly generated instances of the channel $H$. The four sub-plots correspond to cases when the number of bits shared by each transmitter is 20%, 40%, 60% and 80% of the average baseline rate respectively. Note that as the number of shared bits increases, the median of the gain decreases. Also, the number of infeasible cases (corresponding to when one or more transmitters do not have sufficient power available to transmit their shared message and pre-subtract the shared messages of their neighbours) increases. In each of these plots, the proposed transmission strategy based on shared-private rate splitting and zero-forcing precoding achieves a gain of almost one-bit-per-shared-bit.

The algorithm to compute the power allocations forces each transmitter to use up all of its available power for its shared and private messages and for zero-forcing the shared messages of its neighbours. As we have already noted previously, using the full available power is not necessarily optimal.

For the channel generated by the system parameters listed in Table3.1, the convergence of the proposed distributed algorithm is not guaranteed. For the design of the precoder coefficients, Figure 3.4 plots the histogram of the “average spectral radius $\rho(\mathcal{M})$ of the iteration matrix” (see Appendix B) for 1000 randomly generated instances
of the channel $H$. A sufficient condition for convergence is $\rho(M) < 1$. The plot shows that this condition is satisfied in around 85% of the randomly generated instances of the channel. For the algorithm for distributed implementation of the power allocation, Figure 3.5 shows the histogram of the spectral radius $\rho(M)$ of the iteration matrix for 1000 randomly generated instances of the channel. A necessary and sufficient condition for convergence is $\rho(M) < 1$. We observe that this condition is always satisfied. Finally, Figures 3.6 and 3.7 show the convergence behaviour of the distributed algorithm for computing the precoder coefficients and power allocations respectively.
Figure 3.3: Histogram of the average improvement in downlink rates as a fraction of the number of bits shared by each transmitter over 1000 randomly generated instances of the channel. In each instance, the number of bits shared by each transmitter is (a) 20%, (b) 40%, (c) 60% and (d) 80% of the average baseline rate. We observe that the proposed transmission strategy based on shared-private rate splitting and zero-forcing precoding achieves a gain of almost one-bit-per-shared-bit.
Figure 3.4: Histogram of the average spectral radius of the iteration matrix (see Appendix B) for 1000 randomly generated instances of the channel $H$. In each instance of the channel, the average spectral radius is computed over 1000 row-permutations of $H$. A sufficient condition for convergence is $\rho(M) < 1$. The plot shows that this condition is satisfied in around 85% of the randomly generated instances of the channel.
A necessary and sufficient condition for convergence is $\rho(M) < 1$. We observe that this condition is always satisfied.
Figure 3.6: Convergence behaviour of the distributed algorithm for the design of the precoder
Figure 3.7: Convergence behaviour of the distributed algorithm for the power allocation
Chapter 4

Conclusions

4.1 Summary

In this thesis we looked at the problem of designing coding strategies for interference channels with rate-limited transmitter cooperation. This design problem consists of two coupled parts namely

• What information about a user’s message should be communicated to the other transmitters over the cooperation links?

• Using the additional information available about the messages of the interfering users, how should the transmit signals be designed for the interference channel?

The characterization of the optimal cooperation/transmission strategy using an information-theoretic or an optimization framework is quite difficult. In this work we proposed and characterized a simple partial zero-forcing precoding strategy based on a shared-private rate splitting scheme at the transmitter, in which each transmitter communicates part of its message to the other transmitter, and subsequently partially pre-subtracts the interfering signal using a zero-forcing precoder.

We first considered a simple communication model in which the classic two-user Gaussian interference channel is augmented by noiseless rate-limited digital conferencing links
between the transmitters, for which the proposed strategy was shown to be asymptotically sum-capacity achieving in a very weak interference regime, where both the signal-to-noise ratio (SNR) and the interference-to-noise ratio (INR) go to infinity while their ratio in dB scale is kept fixed. In this case, every cooperation bit results in one-bit gain in sum capacity. We also considered a different asymptotic regime where the transmit power constraints and the channel gains are fixed while the noise powers go down to zero. In this low-noise regime, if one compares with the achievable sum rate with interference treated as noise, one cooperation bit can in fact result in more than one-bit gain in achievable sum rate.

The extension of the partial zero-forcing precoding strategy to a restricted class of multiuser interference channels with limited cooperation was described and seen to yield significant performance gains. We provide a distributed algorithm to iteratively compute the precoding coefficients in a distributed manner at the various transmit nodes. Simulation results for a multicell system with rate-limited local cooperation show that the proposed transmission strategy based on shared-private rate splitting and zero-forcing precoding achieves a gain of almost one-bit-per-shared-bit.

### 4.2 Suggestions for further research

The general problem of designing coding strategies for interference channels with rate-limited transmitter cooperation is much broader than the scope of the ideas presented in this thesis. We have focussed on interference pre-subtraction using transmit precoding. There are many other ideas that are already known, e.g. cooperative encoding of source messages and joint transmission, receiver-based partial decoding and interference subtraction, or interference alignment, which can be incorporated into a general coding strategy. However, even within the framework of interference pre-subtraction using transmit precoding there are many avenues to be explored. Some of these are listed below.
\textbf{Parallel interference channels}

Consider a situation, in which two transmitters communicate with their respective intended receivers using $N$ orthogonal sub-carriers (as in an OFDM system) over a shared communication medium, while mutually interfering with each other on each of these sub-channels. The resulting channel can be modelled as a collection of $N$ parallel interference channels. Suppose these transmitters can cooperate using rate-limited backhaul links. An interesting question that arises is if it would be better for transmitters to communicate an equal number of bits about each of their $N$ signals and partially pre-subtract the interference in all the $N$ interference channels, or to select a few sub-channels on which they fully cooperate.

\textbf{Multicell systems with multiple users per cell}

Consider a TDMA multicell system with rate-limited cooperation, in which multiple users are simultaneously active in each cell. Inside a cell, each user is allocated a time-slot in which it is serviced by the base station. Suppose the proposed partial zero-forcing precoding strategy is employed for (intercell) interference mitigation. One drawback of the proposed strategy is that it does not work well when the interference channel is not diagonaly dominant, which may often be the case in practical systems due to a variety of reasons (e.g, shadowing). Can the users be scheduled so that in each time-slot, the resulting interference channel is diagonally dominant?

\textbf{Linear beamforming schemes}

The zero-forcing precoder is the most common linear precoding scheme. Even within the framework of shared-private rate-splitting, the throughput gain can be further increased if we can find better linear beamformers. This is an area for further exploration.

\textbf{“Shared-private” splitting for signal constellations}

An interesting avenue for future work is to examine the case when the interfering signal is not drawn from a Gaussian codebook, but rather is a point in a signal constellation (such as 256-QAM). Do there exist effective set-partitioning functions (wherein the index
of the partition can be communicated to the other transmitter) which enable us to pre-subtract some of the interference?
Appendix A

Deterministic Interference Channel

An intuitive understanding of the proposed coding strategy can be obtained by interpreting it in the context of the deterministic interference channel [15]. The idea of analyzing Gaussian networks using equivalent deterministic channel models stems from the work of Avestimehr, Diggavi and Tse [16]. These deterministic channels approximate the behaviour of the corresponding Gaussian networks in the high SNR limit, when the effect of additive noise at the receivers is negligible compared to the interactions among different signals in the shared communication medium.

Consider first the deterministic channel model for the point-to-point AWGN channel shown in Figure A.1. The real-valued channel input $x$ is written in base 2:

$$x_b = 0.b_1b_2b_3b_4\cdots$$

where we use the subscript $b$ to distinguish the resulting representation of the signal — a vector of bits — from the representation as a real number. The binary representation $x_b$ is interpreted as occupying a succession of levels on a signal scale, wherein the most significant bit coincides with the highest level. A level corresponds to a unit of power in the Gaussian channel, measured in a log scale. The effect of noise in the Gaussian channel is modelled in the deterministic channel by truncation: bits occupying levels
below the noise level are lost. The channel can be written as

$$y_b = \lfloor 2^n x_b \rfloor$$  \hspace{1cm} \text{(A.2)}$$

with the correspondence $n = \lfloor \log(\text{SNR}) \rfloor$.

For the channel shown in Figure A.1, we have $n = \lfloor \log(\text{SNR}) \rfloor = 3$:

\begin{align*}
x_b &= 0.b_1b_2b_3b_4\cdots \\
y_b &= \lfloor 2^3 x_b \rfloor = \lfloor b_1b_2b_3b_4\cdots \rfloor \\
&= b_1b_2b_3 \\
&= b_1b_2b_3 \hspace{1cm} \text{(A.3)}
\end{align*}

At the receiver bits $b_1, b_2, b_3$ are recovered while bits $b_4, b_5, \cdots$ are lost since they occupy levels below the noise.

The Gaussian interference channel (2.6) is parameterized by the four power-to-noise ratios $\text{SNR}_1$, $\text{SNR}_2$, $\text{INR}_1$ and $\text{INR}_2$. The equivalent deterministic interference channel (see Fig. A.2) is parameterized by the four integers

\begin{align*}
n_{11} &= \lfloor \log(\text{SNR}_1) \rfloor \quad n_{12} = \lfloor \log(\text{INR}_2) \rfloor \\
n_{22} &= \lfloor \log(\text{SNR}_2) \rfloor \quad n_{21} = \lfloor \log(\text{INR}_1) \rfloor \\
&= \left(\begin{array}{cccc}
n_{11} & n_{12} & 0 & 0 \\
0 & n_{22} & n_{21} & 0 \\
0 & 0 & 0 & n_{22} \\
0 & 0 & 0 & 0 \\
\end{array}\right) \hspace{1cm} \text{(A.4)}
\end{align*}

wherein the signal $x_{ib}$ from transmitter $i$, as observed at receiver $j$, is scaled by $2^{n_{ij}}$. The
Appendix A. Deterministic Interference Channel

The deterministic interference channel can be written as:

\[
y_1 = \lfloor 2^{n_{11}} x_{1b} \rfloor \oplus \lfloor 2^{n_{21}} x_{2b} \rfloor
\]

\[
y_2 = \lfloor 2^{n_{12}} x_{1b} \rfloor \oplus \lfloor 2^{n_{22}} x_{2b} \rfloor
\]

(A.5)

where the addition is performed on each bit (modulo 2) and \( \lfloor \cdot \rfloor \) is the integer-part function. For the deterministic interference channel shown in Figure A.2, \( n_{11} = 3, n_{12} = 1, n_{21} = 2, n_{22} = 4 \). The signals at transmitters 1 and 2 are \( x_{1b} = 0.a_1a_2a_3a_4 \cdots \) and \( x_{2b} = 0.b_1b_2b_3b_4 \cdots \) respectively. The signal at the receiver 1 is

\[
y_1 = \lfloor 2^3 x_{1b} \rfloor \oplus \lfloor 2^2 x_{2b} \rfloor = \lfloor a_1a_2a_3a_4 \cdots \rfloor \oplus \lfloor b_1b_2b_3 \cdots \rfloor
\]

\[
= a_1a_2a_3 \oplus 0b_1b_2
\]

(A.6)

We illustrate the improvement with transmitter cooperation using an example. For a Gaussian interference channel with no transmit cooperation, if the receivers treat interference as noise, the asymptotic achievable rates as \( N \to 0 \) are

\[
R_1 = \gamma \left( \frac{\text{SNR}_1}{\text{INR}_1} \right), \quad R_2 = \gamma \left( \frac{\text{SNR}_2}{\text{INR}_2} \right)
\]

(A.7)
For the equivalent deterministic interference channel, note that as $N \to 0$ the parameters $n_{ij}$ are not meaningful, but the differences
\begin{align*}
   n_{11} - n_{21} &= \lfloor \log(\text{SNR}_1) \rfloor - \lfloor \log(\text{INR}_1) \rfloor \\&\approx \lfloor \log \left( \frac{\text{SNR}_1}{\text{INR}_1} \right) \rfloor \\
n_{22} - n_{12} &= \lfloor \log(\text{SNR}_2) \rfloor - \lfloor \log(\text{INR}_2) \rfloor \\&\approx \lfloor \log \left( \frac{\text{SNR}_2}{\text{INR}_2} \right) \rfloor
\end{align*}
are useful parameters. Since the receivers treat interference as noise, we can model the effect of interference by truncation — bits occupying levels below that of interference are lost. The channel from each transmitter to its own receiver can be written as an equivalent deterministic point-to-point channel
\[ y_{ib} = \lfloor 2^{(n_{ii} - n_{ij})} x_{ib} \rfloor \quad (A.9) \]

The following two cases provide some insight into the different results seen in Theorem 1 of Chapter 2, and Corollary 1 of Chapter 3, even though the same coding strategy of shared-private splitting and zero-forcing precoding is employed in both cases.

Case 1: For the deterministic interference channel in Figure A.3, transmitter 1 can communicate 1 bit ($a_1$) to its receiver (bits $a_2, a_3, \cdots$ are lost due to interference), while
transmitter 2 can communicate 3 bits \((b_1, b_2, b_3)\) to its receiver (bits \(b_4, b_5, \cdots\) are lost due to interference).

Now suppose transmitter 2 communicates one bit of its transmit signal (the MSB \(b_1\)) to transmitter 1. Then transmitter 1 can “pre-subtract” a part of the interfering signal, thereby reducing the interference level at its receiver (see Figure A.4). Transmitter 1 can now communicate 2 bits \((a_1, a_2)\) to its receiver (bits \(a_3, a_4, \cdots\) are lost due to interference), while transmitter 2 can communicate 3 bits \((b_1, b_2, b_3)\) to its receiver (bits \(b_4, b_5, \cdots\) are lost due to interference). By communicating 1 bit of information, transmitter 2 enables transmitter 1 to communicate 1 additional bit to its receiver.

**Case 2**: Now suppose that both transmitters reduce their transmit power by one level on the signal scale (see Figure A.5), i.e

\[
x_{1b} = 0.0a_1a_2a_3\cdots
\]

\[
x_{2b} = 0.0b_1b_2b_3\cdots
\]  

(A.10)

Again we are interested in the asymptotic behaviour of the Gaussian interference channel in the limit as \(N \to 0\), wherein if both transmitters scale down their transmit power by
an identical factor (recall that the signal levels correspond to a log scale), the SINR at both receivers remains unchanged. The same behaviour is reflected in the deterministic interference channel, where the interference is reduced by one level on the signal scale, and the same rate pair (1 bit, 3 bits) is achievable as before.

Now suppose transmitter 2 decides to transmit an additional bit \( b_0 \) to its receiver. It can do so by slotting \( b_0 \) into the position of the MSB which is vacant, i.e.,

\[
x_{2b} = 0.b_0b_1b_2b_3 \cdots
\]

Transmitter 2 can now communicate 4 bits to its receiver. However, the transmission of \( b_0 \) increases the interference level at receiver 1. Consequently, all the bits sent by transmitter 1 are lost and the throughput remains unchanged. Now suppose transmitter 2 communicates \( b_0 \) to transmitter 1. Transmitter 1 can again “pre-subtract” a part of the interfering signal (by sending \( a_1 \oplus b_0 \)), reducing the interference at its receiver, which now sees a clean copy of bit \( a_1 \). The rate pair (1 bit, 4 bits) is achievable. By communicating 1 bit of information to transmitter 1, thereby enabling it to pre-subtract the additional interference, transmitter 2 can communicate 1 additional bit to its receiver without affecting the communication to receiver 1.
Figure A.5: Deterministic interference channel: $n_{11} - n_{21} = 1, n_{22} - n_{12} = 3$.

Figure A.6: Deterministic interference channel with limited transmitter cooperation
Appendix B

Convergence Analysis

This appendix examines the convergence behaviour of the distributed algorithm proposed in Chapter 3 for computing the precoder coefficients and the corresponding power allocation. The techniques and mathematical tools used for the convergence analysis are only briefly summarized here. The reader is referred to [17] for a detailed treatment.

Consider a system of linear equations $Ax = b$, where $A$ is an $n \times n$ invertible matrix. Suppose that this system has a unique solution, denoted by $x^*$. Consider an iterative algorithm to solve this system of linear equations wherein the update equation is given by

$$x^{(n+1)} = Mx^{(n)} + Gb$$  \hspace{1cm} (B.1)

where $M$ (called the “iteration matrix”) and $G$ are suitable matrices determined by $A$ and the particular algorithm (Jacobi iterations, Gauss-Siedel algorithm) being used. Let $e^{(n)} = x^{(n)} - x^*$. Then $e^{(n+1)} = Me^{(n)} = M^{n+1}e(0)$. The iterative algorithm converges to the solution $x^*$ for all choices of $x^{(0)}$ if and only if $e^{(n)}$ converges to zero for all choices of $e^{(0)}$. This happens if and only if all the eigenvalues of $M$ have a magnitude smaller than 1. In other words, the iterative algorithm converges to the solution $x^*$ if and only if the spectral radius $\rho(M)$ is less than 1.
Appendix B. Convergence Analysis

B.1 Precoder coefficients

Recall that the distributed algorithm to compute the precoder coefficients (Section 3.4.1) was based on the Gauss-Siedel iterative algorithm for solving a system of linear equations. To derive the update equation of the Gauss-Siedel algorithm, we decompose $A$ as $A = L + D + U$, where $L$ is strictly lower-triangular, $D$ is diagonal, and $U$ is strictly upper-triangular. Then we can write

$$x^{(n+1)} = [- (L + D)^{-1}U]x^{(n)} + (L + D)^{-1}b$$  \(B.2\)

The Gauss-Siedel algorithm converges to the solution $x^*$ if and only if

$$\rho\left(- (L + D)^{-1}U\right) < 1$$  \(B.3\)

Let $\pi : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ denote a random permutation of $\{1, 2, \ldots, n\}$, and let $\{\pi_1, \pi_2, \ldots, \pi_N\}$ ($N = n!$) denote the range of the function $\pi$. We denote by $\pi_j(Q)$ the resulting matrix obtained by rearranging the $n$ rows of a matrix $Q$ according to the permutation $\pi_j$.

Consider again the update equation (B.1). Since $e^{(n+1)} = Me^{(n)}$, we have

$$\frac{||e^{(n+1)}||}{||e^{(n)}||} \leq |\lambda_{max}| = \rho(M)$$  \(B.4\)

where $\lambda_{max}$ denotes the eigenvalue of $M$ with the largest magnitude.

We are now ready to state a sufficient condition for the convergence of our distributed algorithm to compute the precoder coefficients. Recall that at each iteration the precoding coefficients for the nodes are updated in a random order, and each node uses the latest available information about the coefficients of its neighbours, which corresponds to a random permutation of the rows of $A, x$ and $b$. The update equation can be written as

$$\pi_j(x^{(n+1)}) = M_n\pi_j(x^{(n)}) + G_n\pi_j(b)$$

$$= [- (L_n + D_n)^{-1}U_n]\pi_j(x^{(n)}) + (L_n + D_n)^{-1}\pi_j(b)$$  \(B.5\)
where $L_n$, $D_n$ and $U_n$ are strictly lower-triangular, diagonal, and strictly upper-triangular matrices obtained by the decomposition of $\pi_j(A)$, where $\pi_j$ is randomly chosen from the set $\{\pi_1, \pi_2, \ldots, \pi_N\}$.

**Definition 4.** Let $Q = \{Q_1, Q_2, \ldots, Q_M\}$ be a set of $M \times n$ matrices. The average spectral radius $\rho(Q)$ of $Q$ is defined as

$$\log_{10} (\rho(Q)) = \frac{1}{M} \sum_{j=1}^{M} \log_{10}(\rho(Q_j))$$  \hspace{1cm} (B.6)

Let $\mathcal{M}$ be the set of $N$ “iteration matrices” obtained using (B.5) from the $N$ different permutations of $A$ according to $\{\pi_1, \pi_2, \ldots, \pi_N\}$. We have

$$\frac{\|e^{(n+1)}\|}{\|e^{(n)}\|} \leq \rho(M_n)$$  \hspace{1cm} (B.7)

where $M_n$ is randomly chosen from $\mathcal{M}$. We have,

$$\|e^{(n+1)}\| \leq \rho(M_n)\|e^{(n)}\|$$

$$\leq \prod_{j=0}^{n} \rho(M_n)\|e^{(0)}\|$$

$$= \rho(\mathcal{M})^n\|e^{(0)}\|, \quad \text{for large } n.$$  \hspace{1cm} (B.8)

Therefore $\|e^{(n)}\| \to 0$ as $n \to \infty$ if $\rho(\mathcal{M}) < 1$. This provides a sufficient condition for the convergence of our distributed algorithm.

**B.2 Power allocation**

In this section we derive sufficient conditions for the convergence of the distributed algorithm for calculating the power allocations based on block Jacobi iterations, wherein two variables $P_{s,i}$ and $P_{p,i}$ are updated simultaneously (Section 3.4.2). The convergence analysis for block Jacobi iterations is in general non-trivial, and it may be possible to come up with tighter conditions.
Convergence Analysis using Maximum Norms

The maximum norm of a square matrix $A$ is defined as follows

$$||A||_\infty = \max_i \sum_j |a_{ij}|$$

(B.9)

where $a_{ij}$ are the entries of $A$.

We use the following important property of the spectral radius

$$\rho(A) \leq ||A||_\infty$$

(B.10)

We use (B.10) to derive the sufficient conditions. We do so by deriving the conditions under which the maximum norm of the iteration matrix is less than 1. This approach is commonly adopted in convergence analysis since the eigenvalues of $M$ are rarely known exactly.

The distributed algorithm for the power allocation outlined in Section 3.4.2 converges if

$$1 + \gamma_i h_i^2(T_i^2 - \sum_{j \in N(i)} T_j^2 - 1) + \gamma_i(h_i^2 - \sum_{j \in N(i)} h_j^2) > 0$$

B.11

Consider (3.14). Converting the inequalities in (3.13) into equality constraints and rewriting the resulting system of equations in the form $Ax = b$,

$$
\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,2M} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,2M} \\
\vdots & \ddots & \ddots & \vdots \\
a_{2M,1} & \cdots & a_{2M,2M}
\end{bmatrix}
\begin{bmatrix}
P_{s,1} \\
P_{p,1} \\
\vdots \\
P_{s,M} \\
P_{p,M}
\end{bmatrix}
= 
\begin{bmatrix}
P_1 \\
\gamma_1 N \\
\vdots \\
\gamma_M N
\end{bmatrix}
$$

(B.12)

we note that the matrix $A$ has the following structure

$$
\begin{bmatrix}
a_{2i-1,1} & a_{2i-1,2} & \cdots & a_{2i-1,2M} \\
a_{2i,1} & a_{2i,2} & \cdots & a_{2i,2M}
\end{bmatrix}
= 
\begin{bmatrix}
B_{i,1} & B_{i,2} & \cdots & B_{i,M}
\end{bmatrix}
$$

(B.13)
where
\[ B_{i,j} = \begin{bmatrix} T_{ii}^2 & 1 \\ 1 & -\gamma_i h_{ii}^2 \end{bmatrix}, \quad \text{if } j = i, \]
\[ = \begin{bmatrix} T_{ij}^2 & 0 \\ 0 & -\gamma_i h_{ij}^2 \end{bmatrix}, \quad \text{if } j \in \mathcal{N}(i), \]
\[ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{otherwise}. \] (B.14)

Rewriting the update equations (3.14) in the matrix form (B.1), using (B.13) we can infer the following structure for the iteration matrix \( M \)
\[ \begin{bmatrix} m_{2i-1,1} & m_{2i-1,2} & \cdots & m_{2i-1,2M} \\ m_{2i,1} & m_{2i,2} & \cdots & m_{2i,2M} \end{bmatrix} = -B_{i,i}^{-1} \begin{bmatrix} B_{i,1} & \cdots & B_{i,i-1} & O & B_{i,i+1} & \cdots & B_{i,M} \end{bmatrix} \]
\[ = \begin{bmatrix} -B_{i,i}^{-1} B_{i,1} & \cdots & O & \cdots & -B_{i,i}^{-1} B_{i,M} \end{bmatrix} \] (B.15)

We have, \( \forall j \in \mathcal{N}(i) \)
\[-B_{i,i}^{-1} B_{i,j} = \frac{-1}{1 + \gamma_i h_{ii}^2 T_{ii}^2} \begin{bmatrix} \gamma_i h_{ii}^2 T_{ij}^2 & -\gamma_i h_{ij}^2 \\ -\gamma_i h_{ij}^2 T_{ij}^2 & \gamma_i h_{ij}^2 T_{ii}^2 \end{bmatrix} \] (B.16)
\[ = \frac{-1}{1 + \gamma_i h_{ii}^2 T_{ii}^2} \begin{bmatrix} \gamma_i h_{ii}^2 T_{ij}^2 & -\gamma_i h_{ij}^2 \\ T_{ij}^2 & \gamma_i h_{ij}^2 T_{ii}^2 \end{bmatrix} \] (B.17)

We can now write down the maximum norm of the iteration matrix using (B.17) as follows
\[ \| M \|_\infty = \max_i \left\{ \sum_{j=1}^{M} \left| m_{2i-1,2j-1} \right| + \left| m_{2i-1,2j} \right|, \sum_{j=1}^{M} \left| m_{2i,2j-1} \right| + \left| m_{2i,2j} \right| \right\} \]
\[ = \max_i \left\{ \frac{\sum_{j \in \mathcal{N}(i)} \gamma_i h_{ii}^2 T_{ij}^2 + \gamma_i h_{ij}^2}{1 + \gamma_i h_{ii}^2 T_{ii}^2}, \frac{\sum_{j \in \mathcal{N}(i)} T_{ij}^2 + \gamma_i h_{ij}^2 T_{ii}^2}{1 + \gamma_i h_{ii}^2 T_{ii}^2} \right\} \] (B.18)

We have \( \| M \|_\infty < 1 \) if
\[ \sum_{j \in \mathcal{N}(i)} T_{ij}^2 + \gamma_i h_{ij}^2 T_{ii}^2 < 1 + \gamma_i h_{ii}^2 T_{ii}^2 \quad \text{and} \]
\[ \sum_{j \in \mathcal{N}(i)} \gamma_i h_{ii}^2 T_{ij}^2 + \gamma_i h_{ij}^2 < 1 + \gamma_i h_{ii}^2 T_{ii}^2 \] (B.19)
Some basic algebraic manipulation of (B.19) yields (B.11).
Bibliography


