THROUGHPUT ANALYSIS OF MULTIPLE ACCESS RELAY CHANNEL UNDER COLLISION AVOIDING RELAYING SCHEMES

by

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Abstract

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Despite much research on the throughput of relaying networks under idealized interference models, many practical wireless networks rely on physical-layer protocols that preclude the concurrent reception of multiple transmissions. In this work, we develop analytical frameworks for the uplink of a multi-source single-channel relay-aided wireless system where transmissions are scheduled to avoid collisions. We study amplify-and-forward and decode-and-forward strategies, in both time-sharing and network-coded variants, and provide mathematical models to investigate their achievable rate regions. Both general and optimal power allocations are considered. We also find the cut-set outer bounds for the rate regions. Moreover, we present a comparison between these methods with the simple time sharing scheme. Our numerical results reveal that optimizing power allocation favors the time sharing scheme significantly more than it does the relaying schemes, so that time sharing under some circumstances can provide higher maximum sum rates, even if the links to the relay have strong channel gains. The proposed analysis provides a means to quantitatively evaluate the efficacy of relaying under the collision model, leading to pragmatic design guidelines.
Dedication

This thesis is dedicated to my parents, Ali Akbar and Mahnaz, and my lovely sister,

Asma who brighten up my path of success.
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Chapter 1

Introduction

The pioneering work of Cover and El Gamal [5] introduced the concept of relaying in a single-transmitter single-relay network. Although this network model was first studied in [35], [5] was the first article to address the exact capacity region of these networks under special cases. Even though the exact performance characterization of this simplest relay network remains an open problem, the capacity region under some special conditions is known. For example, [8] gives the capacity regions for degraded and reversely degraded relays, as well as an achievable rate region and an outer bound for the more general case, which are the best bounds known to-date. Thanks to the promising results of [5], much research has been devoted to understanding the effect of relays on the performance of various systems with respect to different network metrics [9,11,17–19,21,30,31]. In particular, [18] introduced an abstract representation of a relay network, termed the multiple access relay channel (MARC), which has received much further study [17,29–31]. In MARC, multiple sources communicate with a destination node with assistance from one relay. Figure 1.1 illustrates such a network with two sources.

This model is a simple extension of the relay network introduced in [5]. The broadcast nature of each node in addition to the multiple access structure of the transmissions makes this model suitable for wireless networks. Furthermore it represents a lot of commercial
Figure 1.1: Multiple access relay channel with two sources 1 and 2, relay 3, and destination 4.

wireless networks, e.g. the uplink of a relay-aided cellular network. In this chapter, we will briefly describe the need behind further investigation of MARC and provide insights on many existing opportunities regarding this network model.

1.1 Motivation and Problem Description

The former studies of relay aided networks are based on physical-layer models that allow different nodes to transmit simultaneously, while accounting for the interference at the receiver as a weighted sum of the transmissions. One particularly strong yet commonly made assumption is the full-duplex communication of relays, such that a relay can receive packets while forwarding packets in the same physical channel at the same time. To implement such a relay in practice would require multiple antennas and possibly sophisticated coding and signal processing techniques. No widely deployed wireless technology, e.g., GSM, IEEE 802.11, or IEEE 802.16, supports this mode of operation. To fill this gap between theoretical models and practical networks, half-duplex relays, which cannot transmit and receive at the same time, have been analyzed in [21, 30, 31]. Sankaranarayanan et al. [31] derived outer bounds on the capacity region of a constrained
MARC model, in which the relay operates in the receive and transmit state respectively for a fraction $\alpha$ and $1 - \alpha$ of the time. In [30], Sankar et al. specialized these bounds to the orthogonal MARC channel with additive white Gaussian links, where the relay communicates in an orthogonal channel with the destination. They developed optimal power allocation policies that maximize the sum rate. Maric et al. [21] studied a two-sender, two-receiver channel model with one relay node. They derived the rate regions of several simple relaying strategies for a half-duplex relay that receives data in one block and transmits it to the destinations in the following block. In spite of the usage of half-duplex relays, these models still allow the relay and destination to receive data from multiple sources simultaneously.

Simultaneous packet reception is not impossible to achieve, but it generally requires sophisticated transmitters or receivers. For example, there are theoretical studies and practical implementations, reported in the literature, on multiuser detection through successive interference cancelation [2, 4, 15]. However, these techniques require that the received signals differ substantially either in power or in coding [34, 36]. It is also possible to perform distributed source beamforming [3, 25], where the sources form a virtual array of antennas to transmit simultaneously toward the destination, but this technique requires strict time synchronization among the sources at the symbol modulation level, and the same message must be transmitted by all sources. More recently, in [12], the authors proposed ZigZag decoding to allow the resolution of two colliding packets at the receiver in practical wireless networks with simple physical layer implementations. However, ZigZag decoding requires repeated transmissions whenever a collision occurs, so its applicability to relaying is unclear. Furthermore, it has not yet been used in any commercial network. In this thesis, we focus on relaying wireless networks with simple transceivers, where concurrent transmissions lead to packet collisions. We study the throughput capacity of MARC under the practical constraint where transmissions from different sources and the relay are scheduled to avoid collisions. Our goal is to provide performance limitations.
of MARC with simple transmitter and receivers under more realistic assumptions. This model in a single-source single-relay network was studied in [24]. Our work considers a wider variety of relaying strategies.

1.2 Multiple Access Channels

Multiple-access channel (MAC) as illustrated in Figure 1.2 consists of a group of two (or more) senders transmitting information to a common receiver. MAC was first introduced in [33], where Shannon describes a channel with two inputs and one output. Assuming concurrent transmission of senders, MAC characterizes an interference channel in which the transmission of each user produces interference for the transmission of other users. MAC can resemble many pragmatic wireless communication systems, e.g., a satellite receiver communicating with independent ground stations, or mobile users of a cellular network communicating with a base station. Thanks to this special feature, MAC has been an interesting topic of numerous studies [20, 22, 23, 26, 28].

For a given code \(2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_m}, n\) and a set of encoding functions \(X_i : W_i \rightarrow \mathcal{X}_i, i \in \{1, 2, \ldots, m\}\) in a MAC, the rate \(m\)-tuple \((R_1, R_2, \ldots, R_m)\) is said to be achievable, if \(P_e^{(n)} \rightarrow 0\). \(P_e^{(n)}\) in this definition is the average probability of error. Given this definition of achievability, the capacity region of the multiple-access channel is the closure of the set of all achievable rate \(m\)-tuples \((R_1, R_2, \ldots, R_m)\). As illustrated in [8] the capacity of MAC can be formulated as follows:

**Theorem 1.** The capacity of a multiple-access channel is the closure of the convex hull of all \((R_1, R_2, \ldots, R_m)\) satisfying

\[
R(S) \leq I(X(S); Y|X(S^c)) \quad \text{for all } S \subseteq \{1, 2, \ldots, m\}.
\]

for some product distribution \(p_1(x_1) \ldots p_m(x_m)\) on \(X_1 \times X_2 \times \ldots \times X_m\)

This capacity region provides insights on two interesting properties of MAC. First, As stated in theorem 1 this capacity region is a convex set. We can prove the convexity of
the capacity region using the idea of time sharing between each pair of points in the set. Second, we can conclude the interference nature of MAC from these set of inequalities, since decoding the transmission of each subset of nodes depends on the full knowledge of the transmission of other sources (interference cancelation).

A key assumption in the Theorem 1 is the independence of the sources. [6, 13, 28] has extended these results to a more general class of MAC in which the sources can be correlated. [7, 10, 16, 27, 38] has also studied the effect of feedback on the capacity of MAC. In particular, [10] proved that feedback can increase the capacity of MAC.

### 1.3 Thesis Contributions

Our main technical contributions are as follows:\footnote{The results of this thesis are documented in part in [14]}:

- We develop analytical frameworks to study the achievable rate regions of amplify-and-forward (AF) and decode-and-forward (DF) in such an environment, with an AWGN channel model. For both cases, we consider time-share forwarding, where
the sources separately utilize the relay, and combined forwarding, where the relay forwards a linear combination of the source messages as a variant of network coding [1] in either the physical layer for AF or the network layer for DF.

- The time-sharing and network-coded AF and DF schemes provide inner bounds to the capacity region of this network. We further derive a cut-set outer bound (OB) for the capacity region and compare it with the above inner bounds. We demonstrate how the gaps between these bounds vary given different channel conditions.

- We study the effect of power allocation in the above transmission strategies. Optimal power allocation to maximize the sum rate is considered for each strategy, and comparison is made with equal power allocation.

- We compare the rate regions of AF and DF with that of simple time sharing (TS), where the sources take turns to transmit directly to the destination without the relay. We show that optimizing power allocation favors TS significantly more than it does the relaying schemes. Our numerical results elucidate the channel conditions under which relaying is beneficial.

Different relaying strategies and power allocation schemes require different levels of implementation complexity. Hence, the proposed analytical frameworks and the subsequent numerical results provide a first step toward quantitative design guidelines on how to efficiently utilize relaying for throughput improvement. To the best of our knowledge, this work is the first analytical study on multiple access relaying networks under the collision model.

1.4 Thesis Organization

Chapter 2 includes a thorough description of our network model with a special interest in the two source case. We will investigate traditional relaying schemes and provide
their respective rate regions under the collision model. In addition, we will introduce novel relaying strategies which are variants of network coding and provide closed form formulation of their rate regions. In order to increase the sum-throughput of the proposed relaying strategies, we formulate an optimization frame work in which our goal is to properly allocate a fix amount of total power among nodes. A brief discussion on the implication of the results will be followed at the end of the chapter.

Chapter 3 describes the generalization of our proposed relaying strategies to a more general network with $N \geq 2$ sources. We will provide a closed form expression for the inequalities governing the rate region obtained by each method. Furthermore, we extend our network coding relaying schemes to account for unequal allocation of times to the sources. This chapter also includes a discussion on a partitioning scheme which is available in the general case. The numerical results in Chapter 4 illustrate the efficacy of this partitioning scheme in increasing the achievable rate regions of our relaying strategies.

Chapter 4 contains our numerical results for various scenarios. We, first, look at some scenarios under equal and optimal power allocation schemes for the original MARC model with two sources. We provide a detailed discussion on the performance of each strategy in each scenario. Then, we look into a four source network to further investigate the effect of partitioning.

Finally, we conclude this thesis in Chapter 5 and propose potential future research works.
Chapter 2

Communication under Collision Model

This chapter contains the description of our model and the communication protocols within this model. First, we modify our system under collision model. An introduction of our relaying schemes for the special case of \(N = 2\) sources comes in the sequence. Our goal is to provide simple yet throughput efficient relaying strategies that can operate with simple transmitters and receivers. Although the proposed relaying schemes may look simple, our numerical analysis in Chapter 4 proves their efficacy under various network conditions. The proposed analysis in Section 2.2 are extended in Chapter 3 to give the corresponding rate regions in the general case.

Furthermore, we develop an optimization framework in Section 2.4 to increase the achievable rate region of our relaying strategies under general knowledge of the network. Our optimal power allocation scheme designates a limited sum of powers amongst nodes in a way that the sum-rate of the sources will increase.
2.1 System Model

We consider the MARC model as shown in Figure 1.1, in which nodes one and two are the sources, node three is the relay, and nodes four is the destination. Furthermore, we impose the constraint that only one node can transmit at a time in a broadcast manner, hence we avoid collision of any sort. We assume a common block coding scheme for message transmission [5], in which the nodes send their information in blocks of equal lengths. The links are assumed to experience additive white Gaussian noise, so that when node \(i\) transmits the \(n^{th}\) block \(X^{(n)}_i\), node \(j\) receives \(Y^{(n)}_j\) given by

\[
Y^{(n)}_j = h_{ij}X^{(n)}_i + Z^{(n)}_{ij}. \tag{2.1}
\]

where \(h_{ij}\) is the channel gain from node \(i\) to node \(j\) that bears the effects of channel fading, and \(Z^{(n)}_{ij}\) is an independent Gaussian random variable. We normalize all signal powers in the system, so that \(Z^{(n)}_{ij}\) has unit variance. We denote by \(P_i\) the upper limit on the average transmission power of node \(i\), which can be chosen arbitrarily in general and will be optimized in Section 2.4 to increase the maximum sum-throughput of the system. Then, the achievable rate, in bits per channel use, on link \(ij\) is limited by the Shannon bound [32]:

\[
R_{ij} \leq C(h_{ij}^2 P_i) = \frac{1}{2} \log_2(1 + h_{ij}^2 P_i). \tag{2.2}
\]

The relay, when active, transmits \(X^{(n)}_3\), based on previously received signals [5]:

\[
X^{(n)}_3 = f(Y^{(n-1)}_3, \ldots, Y^{(1)}_3). \tag{2.2}
\]

This encoding function can capture numerous relaying strategies, amongst which we will consider only variants of AF and DF.

\(^1\)The channel gains, in general, are complex valued numbers; however, We are interested in their magnitude only. Therefore, throughout this thesis \(h_{ij}\) represents the magnitude of the channel gain.
2.2 Communication Strategies and Rate Regions

In this section, we describe various communication strategies and present analysis to derive the achievable rate region for each. We denote the achievable rates of the two sources as $R_1$ and $R_2$, respectively.

2.2.1 Time Sharing

Without using the relay, the sources can time share the channel to transmit directly to the destination. Suppose the portions of time dedicated to source 1 and source 2 are $\alpha$ and $1 - \alpha$ respectively. The rate region of TS is simply given by

$$R_1 \leq \alpha C(h_{11}^2 P_1),$$
$$R_2 \leq (1 - \alpha) C(h_{21}^2 P_2),$$
$$0 \leq \alpha \leq 1.$$  (2.3)

For the purpose of illustration in the rest of this paper, we are mainly interested in the case $\alpha = \frac{1}{2}$, where the sources share equal time.

2.2.2 Amplify-and-Forward

Considering the constraint that only one node can transmit at a time, the relay has two options to assist the sources. It can either forward the two received blocks from the sources in separate time intervals, which we term Time-Sharing AF (TSAF), or it can merge the received blocks and transmit the combined signal, which we term Combined AF (CAF).

**Time-Sharing AF:** The network essentially operates as two separate single-source, single-relay AF channels. We split the transmission time into four equal parts. Each source takes one-fourth of the time to broadcast its block to the relay and the destination. The relay then takes one-fourth of the time to forward each of the two blocks.
Without loss of generality, suppose the sources transmit their blocks in every other one-fourth time interval. Whenever source $i$ transmits $X_i^{(n)}$ in one block, the relay receives $Y_3^{(n)} = h_{i3}X_i^{(n)} + Z_{i3}^{(n)}$. It then transmits $X_3^{(n+1)} = \alpha_iY_3^{(n)}$ in the following block, where $\alpha_i$ is chosen to satisfy the relay's power constraint

$$\alpha_i \leq \sqrt{\frac{P_3}{h_{i3}^2 P_1 + 1}}. \quad (2.4)$$

Thus, the received signal at the destination in four consecutive blocks is

$$Y_4^{(n)} = \begin{bmatrix}
  h_{14} & 0 & 0 & Z_{14} \\
  0 & h_{24} & 0 & Z_{24} \\
  \alpha_1 h_{13} h_{34} & \alpha_2 h_{23} h_{34} & \alpha_1 h_{34} Z_{13} + Z_{34}^{(n-2)} & \alpha_2 h_{34} Z_{23} + Z_{34}^{(n)}
\end{bmatrix}. \quad (2.5)$$

Note that here and for the rest of this paper, we keep the block index of $Z_{ij}^{(n)}$ only when it is necessary to indicate the independence between noise in different blocks, and omit it whenever there is no risk of confusion.

The above equation describes a multiple access channel (MAC) with two transmitters and one receiver. Consequently, the capacity region satisfies [34]

$$R_1 \leq \frac{1}{4} C\left((h_{14}^2 + \frac{\alpha_1^2 h_{13}^2 h_{34}^2}{\alpha_1^2 h_{34}^2 + 1})P_1\right),$$

$$R_2 \leq \frac{1}{4} C\left((h_{24}^2 + \frac{\alpha_2^2 h_{23}^2 h_{34}^2}{\alpha_2^2 h_{34}^2 + 1})P_2\right). \quad (2.6)$$

The one-fourth factors above are the natural consequence of time sharing. Note that this capacity region defines a square rate region as opposed to a pentagon, which is the known rate region for the general MAC. The pentagon region of the general MAC is due to the sum-rate constraint as a direct consequence of the simultaneous reception of both inputs at the destination, which is not allowed here.

**Combined AF:** This strategy can be considered as analog network coding [1]. We split the transmission time into three equal parts. The sources each takes one-third of
the time to broadcast their blocks, \( X_1^{(n-1)} \) and \( X_2^{(n)} \). Then, the relay transmits a linear combination of the received signals in the subsequent block

\[
X_3^{(n)} = \beta_1 Y_3^{(n-2)} + \beta_2 Y_3^{(n-1)}.
\]

such that \( \beta_1 \) and \( \beta_2 \) satisfy the relay power constraint

\[
\beta_1^2 P_1 + \beta_2^2 P_2 \leq P_3. \tag{2.7}
\]

Consequently, the channel output at the destination in three successive blocks, for a given choice of the \((\beta_1, \beta_2)\) pair, is

\[
Y_4^{(n)} = \begin{bmatrix}
    h_{14} & 0 & \beta_1 h_{13} h_{34} \\
    0 & h_{24} & \beta_2 h_{23} h_{34} \\
    \beta_1 h_{13} h_{34} & \beta_2 h_{23} h_{34} & Z_{14} + \beta_2 h_{34} Z_{23} + Z_{34}
\end{bmatrix} X_1^{(n-2)} + \begin{bmatrix}
    Z_{14} \\
    Z_{24} \\
    \beta_1 h_{34} Z_{13} + \beta_2 h_{34} Z_{23} + Z_{34}
\end{bmatrix} + \begin{bmatrix}
    h_{14}^2 \\
    h_{24}^2 \\
    \beta_1 h_{13} h_{34}^2 + \beta_2 h_{23} h_{34}^2
\end{bmatrix} P_1 + \begin{bmatrix}
    h_{24}^2 \\
    \beta_2 h_{23} h_{34}^2 + \beta_1 h_{13} h_{34}^2 + 1 \\
    (\beta_1^2 + \beta_2^2) h_{34}^2 + 1
\end{bmatrix} P_2.
\]

We can show that the resulting rate region is (see Appendix 6.1)

\[
R_1 \leq \frac{1}{3} C((h_{14}^2 + \frac{\beta_1^2 h_{13}^2 h_{34}^2}{(\beta_1^2 + \beta_2^2) h_{34}^2 + 1}) P_1),
\]

\[
R_2 \leq \frac{1}{3} C((h_{24}^2 + \frac{\beta_2^2 h_{23}^2 h_{34}^2}{(\beta_1^2 + \beta_2^2) h_{34}^2 + 1}) P_2),
\]

\[
R_1 + R_2 \leq \frac{1}{3} C((h_{14}^2 + \frac{\beta_1^2 h_{13}^2 h_{34}^2}{(\beta_1^2 + \beta_2^2) h_{34}^2 + 1}) P_1 + (h_{24}^2 + \frac{\beta_2^2 h_{23}^2 h_{34}^2}{(\beta_1^2 + \beta_2^2) h_{34}^2 + 1}) P_2
\]

\[
+ (h_{14}^2 + \frac{\beta_1^2 h_{13}^2 h_{34}^2}{(\beta_1^2 + \beta_2^2) h_{34}^2 + 1}) P_1 + \frac{\beta_2^2 h_{23}^2 h_{14}^2 h_{34}^2}{(\beta_1^2 + \beta_2^2) h_{34}^2 + 1} P_1 P_2). \tag{2.9}
\]

The overall rate region is the union of (2.9) over all possible choices of \((\beta_1, \beta_2)\) that satisfy (2.7). It is apparent that this rate region is a pentagon. In this regard, the combination of source messages at the relay has a similar effect as the simultaneous message reception in the general MAC.

### 2.2.3 Decode-and-Forward

Similar to the AF case, we consider two options for DF, termed Time-Sharing DF (TSDF) and Combined DF (CDF).
**Time-Sharing DF:** Similarly to TSAF, each source takes one-fourth of the time to broadcast its block to the relay and the destination. When source \( i \) transmits \( X_i^{(n)} \), the relay attempts to decode it and forward \( a_i X_i^{(n)} \) in the next block, where \( a_i \) satisfies

\[
a_i \leq \sqrt{\frac{P_3}{P_i}}. \tag{2.10}
\]

If relay decoding is always successful, the destination receives in four sequential blocks

\[
Y_4^{(n)} = \begin{bmatrix}
    h_{14} \\
    0 \\
    a_1 h_{34} \\
    0
\end{bmatrix} X_1^{(n-3)} + \begin{bmatrix}
    0 \\
    h_{24} \\
    0 \\
    a_2 h_{34}
\end{bmatrix} X_2^{(n-1)} + \begin{bmatrix}
    Z_{14} \\
    Z_{24} \\
    Z_{34}^{(n)} \\
    Z_{34}^{(n-2)}
\end{bmatrix}. \tag{2.11}
\]

Again, the above equation describes a MAC with rate region \([34]\)

\[
R_1 \leq \frac{1}{4} C((h_{14}^2 + a_1^2 h_{34}^2) P_1),
\]

\[
R_2 \leq \frac{1}{4} C((h_{24}^2 + a_2^2 h_{34}^2) P_2). \tag{2.12}
\]

To account for relay decoding failures, we note that the sources are independent and the transmissions occur separately. Therefore, the following rate constraints are imposed to the source-relay links separately:

\[
R_1 \leq \frac{1}{4} C((h_{13}^2) P_1),
\]

\[
R_2 \leq \frac{1}{4} C((h_{23}^2) P_2). \tag{2.13}
\]

Hence, the overall rate region is

\[
R_1 \leq \frac{1}{4} \min \{ C((h_{13}^2 P_1), C((h_{14}^2 + a_1^2 h_{34}^2) P_1) \},
\]

\[
R_2 \leq \frac{1}{4} \min \{ C((h_{23}^2 P_2), C((h_{24}^2 + a_2^2 h_{34}^2) P_2) \}. \tag{2.14}
\]

**Combined DF:** This is a variant of the linear network coding approach \([1]\). As in CAF, we partition the transmission time into three equal portions among the sources and the relay. The sources each takes one-third of the time to broadcast their blocks, \( X_1^{(n-1)} \).
and $X_2^{(n)}$. Then, if and only if both blocks are decoded successfully, the relay transmits a linear combination of the blocks in the subsequent block

$$X_3^{(n)} = b_1 X_1^{(n-2)} + b_2 X_2^{(n-1)}.$$ 

where $b_1$ and $b_2$ are chosen to satisfy the relay power constraint

$$b_1^2 P_1 + b_2^2 P_2 \leq P_3. \quad (2.15)$$

Hence, the destination receives in three sequential blocks

$$Y_4^{(n)} = \begin{bmatrix} h_{14} & 0 & 0 \\ 0 & h_{24} & 0 \\ b_1 h_{34} & b_2 h_{34} & Z_{34} \end{bmatrix} \begin{bmatrix} X_1^{(n-2)} \\ X_2^{(n-1)} \\ Z_{34} \end{bmatrix}. \quad (2.16)$$

The rate region corresponding to (2.16) can be derived similarly to that of CAF. Further considering the relay decoding conditions in (2.13), the rate region of CDF for a specific choice of $(b_1, b_2)$ can be written as

$$R_1 \leq \frac{1}{3} \min \{C(h_{14}^2 P_1), C((h_{14}^2 + b_1^2 h_{34}^2) P_1)\},$$

$$R_2 \leq \frac{1}{3} \min \{C(h_{24}^2 P_2), C((h_{24}^2 + b_2^2 h_{34}^2) P_2)\},$$

$$R_1 + R_2 \leq \frac{1}{3} \min \{C(h_{14}^2 P_1) + C(h_{24}^2 P_2), C((h_{14}^2 + b_1^2 h_{34}^2) P_1 + (h_{24}^2 + b_2^2 h_{34}^2) P_2 + (h_{14}^2 h_{24}^2 + b_1^2 b_2^2 h_{34}^2 P_1 P_2) \}. \quad (2.17)$$

The overall rate region is the union of (2.17) over all choices of $(b_1, b_2)$ that satisfy (2.15). Similar to the CAF case, this rate region generally is also a pentagon.

### 2.2.4 Outer Bound

For comparison with the constructive inner bounds of the capacity region achieved by the communication strategies above, we develop a cut-set OB using the Max-Flow Min-Cut theorem [37]. It is clear from Figure 1.1 that two important cuts are the one that
isolates the destination and the one that isolates the sources. Let \( \gamma_i \) be the portion of time dedicated to node \( i \), for \( i = 1, 2, 3 \). The sum-capacity of these cuts leads to the following cut-set bound:

\[
R_1 \leq \gamma_1 C(h_{14}^2 P_1) + \min\{\gamma_3 C(h_{34}^2 P_3), \gamma_1 C(h_{13}^2 P_1) + \gamma_2 C(h_{23}^2 P_2)\},
\]

\[
R_2 \leq \gamma_2 C(h_{24}^2 P_2) + \min\{\gamma_3 C(h_{34}^2 P_3), \gamma_1 C(h_{13}^2 P_1) + \gamma_2 C(h_{23}^2 P_2)\},
\]

\[
R_1 + R_2 \leq \gamma_1 C(h_{14}^2 P_1) + \gamma_2 C(h_{24}^2 P_2) + \min\{\gamma_3 C(h_{34}^2 P_3), \gamma_1 C(h_{13}^2 P_1) + \gamma_2 C(h_{23}^2 P_2)\}.
\]  

(2.18)

The overall OB is the union of (2.18) over all choices of \( \gamma_i \) that satisfy

\[
\gamma_i \geq 0, \ i = 1, 2, 3, \\
\sum_{i=1}^{3} \gamma_i = 1.
\]

(2.19)

Since equal-time block coding is used in AF and DF, we are also interested in the equal time-share outer bound (E-OB) where \( \gamma_i = \frac{1}{3} \) for all \( i \). Section 4 provides a comparison between OB and E-OB.

### 2.3 Discussion of the Results

In this section, we were mainly concerned on simple relaying strategies, since our goal is to prevent the need for elaborate transmitters and receivers. In spite of their simplicity, our novel modes of relaying, CAF and CDF, are promised to provide better rate regions than conventional relaying schemes, TSAF and TSDF. TSAF and TSDF suffer from surfeit time consumption imposed by independent assistance of relay to each node. In each of these methods, the relay should have the same amount of time as each of the nodes to assist their transmission in DF or AF manner; hence, we have the leading one-fourth coefficient in front of the logarithmic expressions of the inequalities governing the rate region. The time-consumption of the relay has been relaxed in the CAF and CDF by allowing one transmission of the relay for every two transmission of the sources, hence the
one-third leading coefficient. In this way, the sources have more time to transmit fresh information to the channel compared with the TSAF and TSDF. Although, CAF and CDF, make better use of time in comparison with TSAF and TSDF, they are still behind the simple-time sharing scheme. In general, no relaying strategy can beat the TS method in terms of time. The operation of relay demands some portion of the time to be allocated to the relay that can otherwise be used by the sources to transmit fresh information into the channel. The only thing that we can do is to reduce the time consumed by the relay in our relaying strategies, which is what we have done in CAF and CDF in comparison with TSAF and TSDF.

In spite of their advantage in terms of time consumption, CDF and CAF cannot provide as strong copies of each source's message as we can have using TSAF and TSDF. The relay assists the transmission of the sources individually in TSAF and TSDF; hence, it designates all its power to transmit a copy of the message of each source. However, the relay should divide its power among the messages of the sources in CAF and CDF. Therefore, CAF and CDF sacrifice the quality, in terms of the power, of the copies of the sources' messages to buy sources more time to transmit fresh information into the channel.

2.4 Power allocation

The analysis in section 2.2 assumes that $P_i$ for each source and relay node is given. Since wireless devices often have limited energy supply, in this section, we develop an optimization framework for allocating a limited sum of transmission power among the nodes, in order to maximize the achievable sum rate $R_1 + R_2$. Suppose $P$ units of total
power is available, the general form of our optimization problem is

\[
\begin{align*}
\text{Maximize} & \quad C_{R_1+R_2}, \\
\text{Subject to} & \quad P_1 + P_2 + P_3 \leq P, \\
& \quad P_i \geq 0, \quad i = 1, 2, 3, \\
& \quad g(P_1, P_2, P_3) \leq 0. 
\end{align*}
\]  

(2.20)

where \(C_{R_1+R_2}\) is the sum-rate capacity, defined as the maximum achievable sum rate, given in Section 2.2 by the right-hand side of the inequalities that describe the rate regions for different communication strategies, and \(g(P_1, P_2, P_3) \leq 0\) represents the relay power constraint, expressed in (2.4), (2.7), (2.10), and (2.15) for TSAF, CAF, TSDF, and CDF, respectively. Note that for all relaying strategies, \(g(P_1, P_2, P_3)\) is a linear function in \(P_i\). Furthermore, it is easy to show that the sum-rate constraint in (2.9) and (2.17) dominates the individual rate constraints. Although the general formulation above is comparable to the well-known water-filling problem, the additional constraint on \(g(.)\) prevents us from applying those results to our case.

Our optimization framework requires general knowledge of the network. Gathering this information is a very difficult task to do in practical systems. A simple yet common power allocation scheme is to designate powers equally among the nodes. As we will see shortly, equal designation of power among nodes in some scenarios is the optimal power allocation scheme. Moreover, in practice, nodes may interchange their roles throughout the time. A node which acts as a source in one time slot, can be used as a relay in another timeslot. A common example of such a network is a network of sensors in a field. In practice, this optimization framework may not be implementable; however, it can be used as an upper bound on the performance of the desired network in terms of sum throughput.

For TS, TSAF, and CAF, the sum-rate capacity consists of logarithmic functions. In these cases, the above optimization problem is convex. By applying Lagrangian multipli-
ers and considering the Karush-Kuhn-Tucker conditions, we obtain the following optimal power allocations. The detailed derivations are provided in Appendix 6.5. For TS, we have

\[ P_1 = \left( P + \frac{h_2^2}{2} - \frac{h_4^2}{2} \right)^+, \]
\[ P_2 = \left( P + \frac{h_4^2}{2} - \frac{h_2^2}{2} \right)^+. \] (2.21)

For TSAF, we have

\[
P_1 = \left[ \frac{(\alpha_2^2 - \alpha_1^2)(\alpha_2^2 h_2^2 + 1) + (P - \frac{\alpha_1^2 + \alpha_2^2}{2})\alpha_1^2 h_1^2}{\alpha_1^2 h_1^2 + \alpha_2^2 h_2^2 + \alpha_1^2 \alpha_2^2 h_1^2 h_2^2} \right]^+, \]
\[
P_2 = \left[ \frac{(\alpha_1^2 - \alpha_2^2)(\alpha_1^2 h_1^2 + 1) + (P - \frac{\alpha_1^2 + \alpha_2^2}{2})\alpha_2^2 h_1^2}{\alpha_1^2 h_1^2 + \alpha_2^2 h_2^2 + \alpha_1^2 \alpha_2^2 h_1^2 h_2^2} \right]^+, \]
\[
P_3 = \alpha_1^2 h_1^2 P_1 + \alpha_2^2. \] (2.22)

For CAF, we have

\[
P_1 = \left[ \frac{\frac{h_1^2 + 1}{h_1^2}}{h_1^2 h_2^2 + \frac{h_1^2 + 1}{h_1^2 + h_4^2}} + \frac{P - \beta_1^2}{2(\beta_1^2 + 1)} \right]^+, \]
\[
P_2 = \left[ \frac{\frac{h_2^2 + 1}{h_2^2}}{h_1^2 h_2^2 + \frac{h_1^2 + 1}{h_1^2 + h_4^2}} + \frac{P - \beta_2^2}{2(\beta_2^2 + 1)} \right]^+, \]
\[
P_3 = \beta_1^2 h_1^2 P_1 + \beta_2^2 h_2^2 P_2 + \beta_1^2 + \beta_2^2. \] (2.23)

where \([\cdot]^+\) denotes \(\max\{0, \cdot\}\), ensuring that only non-negative power assignments are allowed. Note that the above solutions for TSAF and CAF are functions of the amplification parameters \(\alpha_1, \alpha_2, \beta_1, \) and \(\beta_2\). Hence, to obtain the maximum sum rate, an additional step of numerical optimization over these parameters is necessary.

For TSDF and CDF, the minimum functions in the sum-rate capacity leave us with a non-convex problem. In order to simplify the analysis in these cases, we consider only the condition that the channel gain of the links connecting the sources to the relay are sufficiently high, such that the overall sum rate is dominated (limited) by the sum rate
from the sources and the relay to the destination. In this case, we can rewrite (2.14) as

\[ R_1 \leq \frac{1}{4} C((h_{14}^2 + a_1^2 h_{34}^2)P_1), \]
\[ R_2 \leq \frac{1}{4} C((h_{24}^2 + a_2^2 h_{34}^2)P_2). \] (2.24)

and (2.17) as

\[ R_1 \leq \frac{1}{3} C((h_{14}^2 + b_1^2 h_{34}^2)P_1), \]
\[ R_2 \leq \frac{1}{3} C((h_{24}^2 + b_2^2 h_{34}^2)P_2), \]
\[ R_1 + R_2 \leq \frac{1}{3} C((h_{14}^2 + b_1^2 h_{34}^2)P_1 + (h_{24}^2 + b_2^2 h_{34}^2)P_2) \]
\[ + (h_{14}^2 h_{24}^2 + b_1^2 h_{34}^2 h_{34}^2 + b_2^2 h_{14}^2 h_{34}^2)P_1 P_2). \] (2.25)

This way, the power optimization problem for TSDF and CDF can be approximated by a convex problem. It is intuitive that, in general, the relay is beneficial only if it has a strong link to the sources. Therefore, the imposed condition still allows our analysis to be applicable to most of the important scenarios for TSDF and CDF. This is also confirmed in our numerical results in Section 4.

Solving the convex versions of optimal power allocation in TSDF and CDF using a procedure similar to those for TSAF and CAF, we have for TSDF

\[ P_1 = \frac{P a_1^{-2}}{a_1^{-2} + a_2^{-2} + 1}, \]
\[ P_2 = \frac{P a_2^{-2}}{a_1^{-2} + a_2^{-2} + 1}, \]
\[ P_3 = \frac{P}{a_1^{-2} + a_2^{-2} + 1}. \] (2.26)

and for CDF

\[ P_1 = \left[ \frac{P}{2(b_1^2 + 1)} + \frac{b_1^2 + 1}{b_1^2 + 1} (h_{14}^2 + b_1^2 h_{34}^2) + (h_{24}^2 + b_2^2 h_{34}^2) \right], \]
\[ P_2 = \left[ \frac{P}{2(b_2^2 + 1)} + \frac{b_2^2 + 1}{b_2^2 + 1} (h_{24}^2 + b_2^2 h_{34}^2) - \frac{b_1^2 + 1}{b_1^2 + 1} (h_{24}^2 + b_2^2 h_{34}^2) \right], \]
\[ P_3 = b_1^2 P_1 + b_2^2 P_2. \] (2.27)
We briefly discuss some implications of the above power optimization results in the following.

First, in all of the above relaying strategies, the optimal power allocation attempts to equalize the received SNR of each source at the destination. For example, in TSDF and CDF, we observe that the assigned powers are approximately proportional to the inverse of the dedicated power coefficients at the relay. Similar arguments can be made about TSAF and CAF. Hence, the optimal power policies, despite the constraint on \( g(\cdot) \), have a water-filling [34] flavor in them.

Second, unlike in relaying, the optimal power allocation in TS provides greater power for the channel with better conditions. In particular, the following relation exists:

\[
P_1 - P_2 = h_{24}^{-2} - h_{14}^{-2},
\]

so that \( h_{14} > h_{24} \Rightarrow P_1 > P_2 \). This can be considered a form of opportunistic transmission [34].

Finally, equal power assignment in some scenarios is the optimal policy; e.g. In CDF, if we choose \( b_1 = b_2 = \frac{1}{\sqrt{2}} \) and consider a symmetric model, where \( h_{13} = h_{23} \), and \( h_{14} = h_{24} \), equation (2.27) results in equal assignment of powers, \( P_i = \frac{1}{3}, \quad i = 1, 2, 3 \). We further study the performance of equal power allocation in chapter 4.

2.5 Chapter Summary

We studied the MARC model under collision avoidance supposition. This model allows two modes of operation for the relay. In the first mode, the relay is silent and sources time-share the available amount of time amongst themselves. In the second mode, relay is active and consumes some portion of the available time to assist transmission of the sources. When relay is active, it can help the sources either individually, which leads us to the traditional AF and DF strategies, or it can assist both of the sources at once. We proposed two relaying strategies for the relay in the combined mode of assistance, CAF
and CDF, which are variants of analog and digital network coding, respectively. In CAF, 
the relay forms a linear combination of its reception from both sources in two consecutive 
time slots and transmit this combination in the next available time slot. Relay in CDF 
mode of assistance, first, decodes the transmission of the sources and transmit a linear 
combination of the decoded data in the next time slot.

We derived the achievable rate region for all of the above mentioned relaying strategies 
in Section 2.2. An interesting common property of all of the above mentioned methods 
is that the reception of receiver in consecutive time slots resembles a MAC channel; 
hence, the respective rate regions bear the properties of MAC. In comparison with the 
traditional modes of relaying, our novel relaying schemes make better use of the available 
time; however, they can’t provide the destination with better copies of transmission of 
the sources.

Exploiting the Max-flow Min-Cut theorem, we derived an outer bound on the achieve-
able rate region for MARC. This outer bound provide us with some insight on how close 
are the achievable rate regions of our relaying schemes to the capacity of MARC.

Finally, we established an optimization framework to increase the sum-rate through-
put of these strategies. Given the full knowledge of the network, our power allocation 
scheme designates a limited sum of power amongst nodes to increase the sum-rate of 
the sources. Our relaying strategies under this optimal power allocation scheme provide 
proper utilization of resources, power and time, in our network of interest. As we demon-
strate later in chapter 4, this power allocation scheme can preclude the need for the relay 
in MARC.
Chapter 3

General Multiple Access Relay Channel

So far we have provided new strategies for the MARC model which is a special case of a more general class of networks known as multiple access relay channels. In this chapter, we will extend our results in the previous chapter to this general network with more than two sources. The extended network is shown in Figure 1.1. The reader is already familiar with the general idea of each method; hence, in this chapter, we only briefly describe each relaying strategy and reception of the receiver since they closely resemble those in the previous chapter. Then, we provide the achievable rate region for each method in the extended network where the achievable rate of each source is denoted as $R_i, \ i = 1, \ldots, N$.

Deriving optimal power allocations in the extended network is not as straightforward as in MARC. As we will discuss in Section 3.4, the main obstacle is in choosing a proper objective function (in a sense that we will describe later). Once we select the objective function, the problem formulation and the result derivation follows our general optimization framework as discussed in Section 2.4.

In Section 3.3, we will discuss partitioning as a new opportunity that gives relay more dynamic in assisting the sources. With proper partitioning of the sources, the relay can
Chapter 3. General Multiple Access Relay Channel

Figure 3.1: Multiple access relay channel with $N$ sources $1, \ldots, N$, relay $N + 1$, and destination $N + 2$.

assist each group with the most suitable strategy for that group.

3.1 Communication Strategies of Relay in The Extended Network

3.1.1 Time Sharing

Similar to the two-source case, the sources time share the channel to transmit directly to the destination. Suppose the portions of time dedicated to source $i$ is $\alpha_i$. The rate region of TS is simply given by\footnote{Proofs for all the achievable rate regions discussed in this section can be found in the appendices.}

$$R_i \leq \alpha_i C(h_i^2 N^{1/2} P_i),$$

$$0 \leq \alpha_i \leq 1,$$

$$\sum_{i=1}^{N} \alpha_i = 1.$$  \hspace{1cm} (3.1)
Again, we are mainly interested in the case $\alpha_i = \frac{1}{N}$, $i = 1, \ldots, N$, where the sources share equal time.

### 3.1.2 Amplify-and-Forward

**Time-Sharing AF:** The network essentially operates as $N$ separate single-source, single-relay AF channels. We split the transmission time into $2N$ equal slots. Without loss of generality, suppose the sources transmit their blocks in every other time slot. Whenever source $i$ transmits $X_{i}^{(n)}$ in one block, the relay receives $Y_{N+1}^{n} = h_{i}^{N+1} X_{i}^{(n)} + Z_{i}^{(n)}$. It then transmits $X_{N+1}^{(n+1)} = \alpha_{i} Y_{N+1}^{(n)}$ in the following block, where $\alpha_{i}$ is chosen to satisfy the relay’s power constraint

$$\alpha_{i} \leq \sqrt{\frac{P_{N+1}}{h_{i}^{2} N+1 P_{i} + 1}}. \quad (3.2)$$

Thus, the received signal at the destination in $2N$ consecutive blocks is

$$Y_{N+2}^{(n)} = \sum_{i=1}^{N} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_{i} h_{i}^{N+2} h_{i+1}^{N+1} \\ \vdots \\ 0 \end{bmatrix} X_{i}^{(n-2N+2i-1)} + \begin{bmatrix} Z_{1}^{N+1} \\ \vdots \\ 0 \\ \alpha_{1} h_{N+1}^{N+2} Z_{1}^{N+1} + Z_{N+1}^{(n-2N+2)} \\ \vdots \\ \alpha_{N} h_{N+1}^{N+2} Z_{N}^{N+1} + Z_{N+1}^{(n)} \end{bmatrix}. \quad (3.3)$$

The above equation describes a MAC with $N$ transmitters and one receiver. Consequently, the capacity region satisfies [34] (see Appendix 6.2)

$$R_{i} \leq \frac{1}{2N} C \left( h_{i}^{2} N+2 P_{1} + \frac{(\alpha_{i} h_{N+1}^{N+2} h_{i+1}^{N+1} N+2)^{2} P_{i}}{1 + \alpha_{i}^{2} h_{N+1}^{2} N+2} \right), \quad 1 \leq i \leq N. \quad (3.4)$$

**Combined AF:** We partition the transmission time into $N + 1$ equal slots. The sources each takes one fraction of the time to broadcast their blocks, $X_{i}^{(n+N-i)}$, $i =$
1, \ldots, N. In the subsequent time slot, the relay transmits a linear combination of the previously received signals, i.e.,

$$X_{N+1}^{(n)} = \sum_{i=1}^{N} \beta_i Y_{N+1}^{(n+N-i)},$$

such that $\beta_i, i = 1, \ldots, N$, satisfy the relay power constraint

$$\sum_{i=1}^{N} \beta_i^2 (h_{iN+1}^2 P_i + 1) \leq P_{N+1}. \quad (3.5)$$

Consequently, the channel output at the destination in $N + 1$ successive blocks, for a given choice of the $(\beta_1, \ldots, \beta_N)$, is

$$Y = \begin{bmatrix}
    h_{1N+2} \\
    0 \\
    \vdots \\
    X_1 + \ldots + X_N \\
    0 \\
    [\beta_1 h_{1N+1} h_{N+1N+2}] \\
    [\beta_N h_{NN+1} h_{N+1N+2}]
\end{bmatrix}
+ \begin{bmatrix}
    Z_{1N+2} \\
    \vdots \\
    Z_{NN+2} \\
    [\sum_{i=1}^{N} \beta_i h_{N+1N+2} Z_{iN+1} + Z_{N+1N+2}]
\end{bmatrix}. \quad (3.6)$$

We can show that the resulting rate region is (see Appendix 6.3)

$$\sum_{i \in A} R_i \leq \frac{0.5}{N+1} \log \left( \prod_{i \in A} (1 + h_{iN+2}^2 P_i) \right) \left[ 1 + \sum_{i \in A} \frac{(\beta_i h_{iN+1} h_{N+1N+2}^2 P_i)}{N_{N+1N+2} (1 + h_{iN+2}^2 P_i)} \right],$$

$A \subseteq \{1, \ldots, N\}$,

$$N_{N+1N+2} = 1 + \sum_{i=1}^{N} \beta_i^2 h_{N+1N+2}^2. \quad (3.7)$$

The overall rate region is the union of (3.7) over all possible choices of $(\beta_1, \ldots, \beta_N)$ that satisfy (3.5).
3.1.3 Decode-and-Forward

**Time-Sharing DF:** Each source takes $\frac{1}{2N}$ of the time to broadcast its block to the relay and the destination. When source $i$ transmits $X_i^{(n)}$, the relay attempts to decode it and forward $a_iX_i^{(n)}$ in the next block, where $a_i$ satisfies

$$a_i \leq \sqrt{\frac{P_{N+1}}{P_i}}. \quad (3.8)$$

If relay decoding is always successful, the destination receives in $2N$ sequential blocks

$$Y_{N+2}^{[n]} = \sum_{i=1}^{N} h_{iN+2} \begin{bmatrix} a_i h_{i+1N+2} X_i^{(n-2N+2i-1)} \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} Z_{1N+2} \\ \vdots \\ Z_{N+1N+2} \end{bmatrix}. \quad (3.9)$$

Again, the above equation describes a MAC with rate region [34]

$$R_i \leq \frac{1}{2N} C(h_{iN+2}^2 P_i + (a_i h_{i+1N+2})^2 P_i), \quad 1 \leq i \leq N. \quad (3.10)$$

To account for relay decoding failures, we note that the sources are independent and the transmissions occur separately. Therefore, the following rate constraints are imposed to the source-relay links separately:

$$R_i \leq \frac{1}{2N} C(h_{iN+1}^2 P_i), \quad 1 \leq i \leq N. \quad (3.11)$$

Hence, the overall rate region is

$$R_i \leq \frac{1}{2N} \min \left\{ C(h_{iN+1}^2 P_i), C(h_{iN+2}^2 P_i + (a_i h_{i+1N+2})^2 P_i) \right\},$$

$$1 \leq i \leq N. \quad (3.12)$$
**Combined DF:** As in CAF, we partition the transmission time into \( N + 1 \) equal portions among the sources and the relay. The sources each take one time slot to broadcast their blocks, \( X_i^{(n-N+i)} \), \( i = 1, \ldots, N \). Then, if and only if all blocks are decoded successfully, the relay transmits a linear combination of the blocks in the subsequent block

\[
X_{N+1}^{(n)} = \sum_{i=1}^{N} b_i X_i^{(n-N+i-1)},
\]

where \( b_i, i = 1, \ldots, N \) are chosen to satisfy the relay power constraint

\[
\sum_{i=1}^{N} b_i^2 P_i \leq P_{N+1}.
\]

Hence, the destination receives in \( N + 1 \) sequential blocks

\[
Y = \begin{bmatrix}
  h_{1N+2} \\
  0 \\
  . \\
  . \\
  0 \\
  b_1 h_{N+1N+2}
\end{bmatrix}
X_1 + \ldots +
\begin{bmatrix}
  0 \\
  . \\
  X_N + \\
  . \\
  0 \\
  b_N h_{N+1N+2}
\end{bmatrix}
\begin{bmatrix}
  Z_{1N+2} \\
  . \\
  . \\
  . \\
  Z_{NN+2} \\
  Z_{N+1N+2}
\end{bmatrix}.
\]

The rate region corresponding to (3.14) can be derived similarly to that of CAF. Further considering the relay decoding conditions in (3.11), the rate region of CDF for a specific choice of \((b_1, \ldots, b_N)\) can be written as (see Appendix 6.4)

\[
\sum_{i \in A} R_i \leq \frac{0.5}{N+1} \min\{ \sum_{i \in A} \log(1 + h_{iN+1}^2 P_i), \log \left( \prod_{i \in A} (1 + h_{iN+2}^2 P_i) \prod_{i \in A} \left( 1 + h_{iN+2}^2 P_i \right) \right) \},
\]

\( A \subseteq \{1, \ldots, N\} \).

The overall rate region is the union of (3.15) over all choices of \((b_1, \ldots, b_N)\) that satisfy (3.13).
3.2 Relay Strategies With Unequal Time Allocation

Our proposed relaying schemes so far consider equal share of time for each node. In this section, we will generalize these strategies to allow unequal allocation of time to each node. For each of the relaying strategies, the rely has to receive sufficient blocks of information from each source before it can combine $m_i$, $i \in \{1, \ldots, N\}$, blocks of information from source $i$ in DF or AF manner and transmit the resulting block to the destination. These strategies allow one transmission of the relay for every $m_i$, $i \in \{1, \ldots, N\}$ transmission of source $i$. Obviously our previous model is the special case of $m_1 = \ldots = m_N = 1$.

3.2.1 Unequally Combined Decode and Forward (UCDF)

We divide the transmission time into $\sum_{i=1}^{N} m_i + 1$ time slots. In time slots $\sum_{j=1}^{i-1} m_j + 1$ till $\sum_{j=1}^{i} m_j$ the $i^{th}$ source broadcasts its blocks $X_i(j)$, $j \in \{1, \ldots, m_i\}$. The relay decodes the transmission of each source and if this decoding process is successful, it produces a linear combination consisting $m_i$ transmitted block of source $i$, $i \in \{1, \ldots, N\}$ and transmits the resulting block in the subsequent time slot. The coefficients of these linear combinations are chosen to satisfy the long term constraint on the average power of relay.

We can represent the received data at the receiver as follows:

$$
Y = \sum_{i=1}^{N} B_i + \begin{bmatrix}
Z_{1N+2}^{(1)} \\
Z_{1N+2}^{(m_1)} \\
\vdots \\
Z_{1N+2}^{(m_i)} \\
\vdots \\
Z_{N+1N+2}^{(1)} \\
\vdots \\
Z_{N+1N+2}^{(m_N)} \\
Z_{N+1N+2}
\end{bmatrix} . \quad (3.16)
$$
where

\[
B_i = \begin{bmatrix}
0 & X_i^{(1)} + \ldots + X_i^{(m_i)} \\
: & : \\
0 & 0 \\
\zeta_i^{(1)} h_{N+1+N+2} & \zeta_i^{(m_i)} h_{N+1+N+2}
\end{bmatrix}
\begin{bmatrix}
h_{iN+2} \\
0 \\
: \\
0
\end{bmatrix}
\]

Since the blocks transmitted by each source are independent of each other, we can regard them as transmissions from various nodes that are restricted to have a common rate. Therefore, Equation (3.16) represents a MAC receiver with \(\sum_{i=1}^{N} m_i\) transmitters; however, the transmission rate for \(m_i\) transmitters corresponding to source \(i\) is \(R_i\).

For the matter of simplicity and making the calculations more tractable, we make the following simplifying assumption,

\[
\zeta_i^{(j)} = \zeta_i, \quad j \in \{1, \ldots, m_i\}, \text{ and } i \in \{1, 2, \ldots, N\}. \tag{3.17}
\]

Exploiting our results in Section 3.1.3, the following formulation demonstrates the rate region of UCDF for a given choice of \((\zeta_1, \ldots, \zeta_N)\).

\[
R_i = \frac{m_i}{\sum_{i=1}^{N} m_i + 1} R_i', \\
\sum_{i=1}^{N} k_i R_i' \leq 0.5 \min \{ \sum_{i=1}^{N} k_i \log(1 + h_{iN+1}^2 P_i), \\
\log \left( \prod_{i=1}^{N} (1 + h_{iN+1}^2 P_i)^{k_i} [1 + \sum_{i=1}^{N} \frac{k_i (\zeta_i h_{iN+1+N+2})^2 P_i}{1 + h_{iN+1+N+2}^2 P_i}] \right) \},
\]

\[
0 \leq k_i \leq m_i, \text{ and } i \in \{1, \ldots, N\}. \tag{3.18}
\]
The overall rate region is the union of (3.18) over all choices of \((\zeta_1, \ldots, \zeta_N)\) that satisfy
\[
\sum_{i=1}^{N} m_i \zeta_i^2 P_i \leq P_{N+1}.
\] (3.19)

### 3.2.2 Unequally Combined Amplify and Forward (UCAF)

Similar to the UCDF source \(i\) transmit \(m_i\) blocks of information in \(m_i\) time slots dedicated to it. This time relay transmits a linear combination of its reception from the sources in the next available time slot. This linear combination consist of \(m_i\) transmitted blocks of source \(i\). For the matter of simplicity, we assume equal coefficients for the admixed blocks of one source in the linear combination. These coefficients, \(\lambda_i, i \in \{1, \ldots, N\}\) are chosen to satisfy the long term power constraint for relay.

\[
\sum_{i=1}^{N} m_i \lambda_i^2 h_{iN+1}^2 P_i + 1 \leq P_{N+1}.
\] (3.20)

We can formulate the reception of receiver in \(\sum_{i=1}^{N} m_i + 1\) blocks as

\[
Y = \sum_{i=1}^{N} B_i + \begin{bmatrix}
Z_{1N+2}^{(1)} \\
\vdots \\
Z_{1N+2}^{(m_i)} \\
\vdots \\
Z_{N+2N}^{(1)} \\
\vdots \\
Z_{N+2N}^{(m_N)} \\
\sum_{i=1}^{N} \lambda_i h_{N+1N+2} \left( \sum_{j=1}^{m_i} Z_{iN+1}^{(j)} \right) + Z_{N+1N+2}
\end{bmatrix}.
\] (3.21)
where

\[
B_i = \begin{bmatrix}
0 \\ \\
\vdots \\ \\
0 \\
X_i^{(1)} + \ldots + X_i^{(m_i)}
\end{bmatrix}
\begin{bmatrix}
h_i^{N+2} \\
0 \\
\vdots \\
0 \\
\chi_i^{(1)} h_i^{N+1} h_{N+1}^{N+2}
\end{bmatrix}
\begin{bmatrix}
h_i^{N+2} \\
0 \\
\vdots \\
0 \\
\chi_i^{(m_i)} p_i^{N+1} h_{N+1}^{N+2}
\end{bmatrix}
\]

Similar to the UCDF method, equation (3.21) represents a MAC receiver with \(\sum_{i=1}^{N} m_i\) independent transmitters. Once again, the transmission of \(m_i\) imaginary sources correspond to the transmission of source \(i\) have the common rate \(R_i\). Exploiting our results in Section 3.1.2, the achievable rate region of UCAF for a given choice of \((\lambda_1, \ldots, \lambda_N)\) is

\[
R_i = \frac{m_i}{\sum_{i=1}^{N} m_i + 1} R'_i,
\]

\[
\sum_{i=1}^{N} k_i R'_i \leq 0.5 \log \left( \prod_{i=1}^{N} (1 + h_i^{2N+2} P_i)^{k_i} \right) \left[ 1 + \sum_{i=1}^{N} k_i \frac{h_i^{N+1} h_{N+1}^{N+2}}{N_{N+1}^{N+1}(1 + h_i^{2N+2} P_i)} \right],
\]

\[
0 \leq k_i \leq m_i,
\]

\[
N_{N+1}^{N+2} = 1 + \sum_{i=1}^{N} m_i \lambda_i h_i^{2N+2}, \text{ and } i \in \{1, \ldots, N\}.
\]

The overall rate region is the union of (3.22) over all possible choices of \((\lambda_1, \ldots, \lambda_N)\) that satisfy (3.20).

### 3.2.3 Outer Bound

It is clear from Figure 3.1 that two important cuts are the one that isolates the destination and the one that isolates the sources. Let \(\gamma_i\) be the portion of time dedicated to node \(i\),
for $i = 1, \ldots, N+1$. The sum-capacity of these cuts leads to the following cut-set bound:

$$\sum_{i \in A} R_i \leq \sum_{i \in A} \gamma_i C(h_{iN+2}^2 P_i) + \min \{ \sum_{i \in A} \gamma_i C(h_{iN+2N+1}^2 P_{N+1}), \sum_{i \in A} \gamma_i C(h_{iN+1}^2 P_i) \},$$

$$A \subseteq \{1, \ldots, N\}.$$

(3.23)

The overall OB is the union of (3.23) over all choices of $\gamma_i$ that satisfy

$$\gamma_i \geq 0, \ i = 1, \ldots, N+1,$$

$$\sum_{i=1}^{N+1} \gamma_i = 1.$$

(3.24)

Since equal-time block coding is used in AF and DF, we are also interested in the equal time-share outer bound (E-OB) where $\gamma_i = \frac{1}{N+1}$ for all $i$.

### 3.3 Partitioning

The time division multiplexing nature of our system model allows us to partition the nodes into various groups. Various nodes in the network have different channel conditions and not all of them will benefit from a particular relaying scheme. Sources with very strong links to the relay may benefit from the AF relaying schemes, while the ones with poorer channel conditions might favor from DF relaying methods. With partitioning of the nodes into groups, we can pick the relaying strategies that best suited each group and achieve better rates. The above mentioned relaying strategies are just particular grouping schemes, not necessarily the best, in which we incorporate the nodes either individually or all together.

The rate region of a partition of the nodes is simply the union of the rate regions of all the groups in that partition. The rate region of each group in a partition can be easily found using the above mentioned formulations; however, these relaying schemes each require particular designation of time. Therefore, we should scale the rate region
of each group in a particular partition by the amount of time that we dedicate to that
group.

3.4 Power Allocation

In spite of the two source case, providing the optimal power allocation, in the general
case, is a difficult problem. The main obstacle is in finding a proper objective function.
The sum-rate inequality in the two-source case is the most restrictive constraint on the
sum of rates; hence, optimizing the power allocation to maximize the sum-rate inequality
seems reasonable. In general formulation of the rate region, each inequality confines the
rate region in particular subspaces. We cannot choose one of these inequalities as the
most restrictive constraint on the sum of rates. As an example, the following lemma
provides a property of the inequalities in the general formulation of the CAF case, which
illustrates the difficulty in the choice of the most restrictive constraint on the sum-rate.

Lemma 1. Let $M_A = \prod_{i \in A} (1 + h_{li}^2 P_i) \left[ 1 + \sum_{s \in A} \frac{(\alpha_i h_{siN+1} h_{N+1N+2})^2 P_s}{N_{N+1N+2}(1 + h_{li}^2 P_i)} \right]$. If $A \cap B = \emptyset$, in which
$A$ or $B \subseteq \{1, \ldots, N\}$, then $M_A \times M_B > M_{A \cup B}$.

Proof.

\[
M_A \times M_B = \prod_{i \in A} (1 + h_{li}^2 P_i) \left[ 1 + \sum_{s \in A} \frac{(\alpha_i h_{siN+1} h_{N+1N+2})^2 P_s}{N_{N+1N+2}(1 + h_{li}^2 P_i)} \right]
\]

\[
= \prod_{i \in A} (1 + h_{li}^2 P_i) \left[ 1 + \sum_{s \in A \cup B} \frac{(\alpha_i h_{siN+1} h_{N+1N+2})^2 P_s}{N_{N+1N+2}(1 + h_{li}^2 P_i)} \right]
\]

\[
+ \sum_{k \in A} \sum_{j \in B} \frac{(\alpha_k \alpha_j h_{kN+1} h_{jN+1} h_{N+1N+2})^2 P_k P_j}{N_{N+1N+2}(1 + h_{li}^2 P_i)(1 + h_{lj}^2 P_j)}
\]

\[
= M_{A \cup B} + \prod_{i \in A \cup B} (1 + h_{li}^2 P_i) \left[ \sum_{k \in A} \sum_{j \in B} \frac{(\alpha_k \alpha_j h_{kN+1} h_{jN+1} h_{N+1N+2})^2 P_k P_j}{N_{N+1N+2}(1 + h_{li}^2 P_i)(1 + h_{lj}^2 P_j)} \right].
\]
This property suggests that the inequality on the sum-rate of all \( N \) sources appears to be the more restrictive property of all. On the other hand, if \( A \cap B \neq 0 \), there are cases in which \( M_A \times M_B < M_{A \cup B} \). Therefore, it is not clear which inequalities impose the most restriction on the sum-rate. However, if the network designers are more interested in maximizing the sum-rate of particular nodes in the network, and they choose the sum-rate inequality constraint of those nodes as their objective function, equation (2.20) provides the general formulation of the optimization problem. Since the objective function in this case is a concave function, the optimized power allocation scheme can be found using the knowledge of convex optimization.

### 3.5 Chapter Summary

In this chapter, we studied the general form of multiple access relay channels with more than two sources. First, we extended our results in chapter 2 to this general case with more than two sources. Exploiting the results of Section 3.1, we modified CAF and CDF methods so that we can dedicate unequal portion of time to each source.

We introduced the concept of partitioning as a way of increasing the efficacy of the relay. Partitioning allows for more dynamic allocation of time amongst nodes. It also permits the relay to assist each partition of the nodes with the most suitable relaying strategy.

In the end, we discussed the difficulties in finding optimal power allocations. As we pointed out in Section 3.4 the main obstacle is the introduction of a proper objective function for our optimization framework. The properties of inequalities governing the rate regions for CAF and CDF methods obviate us from choosing one as the most restrictive inequality on the sum-rate; hence, we cannot pick one as the most suitable choice for the objective function.
Chapter 4

Numerical Results

We compute the rate regions and maximum sum rates, and present numerical results to illustrate the performance of different communication schemes. In all cases, we consider attenuating channels, where the channel gains are less than unity, since they are more likely to occur in practice than amplifying channels; however, the results in most cases can be extended to amplifying channels as well.

The study of rate regions for networks with more than two sources requires plots of dimension three or more that can introduce uncalled perplexity while adding little technical insight; hence, we focus more on the MARC model in the following few sections. A network of four sources have been studied at the end of Section 4.3 to elucidate properties of networks with higher number of sources. We, first, study the case of assigning power equally to all transmitters, then present the optimal power allocation results, and finally illustrate an example scenario where the sources and destination are fixed while the relay is moved along a line between them.
Figure 4.1: Equal power allocation: impact of various channel conditions on the performance of each strategy. (a) $h_{14} = h_{24} = h_{13} = h_{23} = h_{34} = 0.5$. (b) $h_{14} = h_{24} = 0.25$ and $h_{13} = h_{23} = h_{34} = 0.5$. (c) $h_{14} = h_{24} = h_{34} = 0.5$ and $h_{13} = h_{23} = 0.25$. (d) $h_{14} = h_{24} = h_{34} = 0.25$ and $h_{13} = h_{23} = 0.5$. (e) $h_{14} = h_{24} = 0.25, h_{13} = h_{23} = 0.9$, and $h_{34} = 0.4$. (f) $h_{14} = h_{24} = h_{13} = h_{23} = 0.25$ and $h_{34} = 0.5$.

4.1 Equal Power Allocation

We assign equal powers, $P_i = 20$, $i = 1, 2, 3$, to each of the nodes\(^1\). Figure 4.1 depicts the resulting rate regions under each communication strategy for different configuration of channel gains between the sources, relay, and destination. Figure 4.1(a): the channels have equal channel gains; 4.1(b): the direct links have the worst channels and all the links connected to the relay have equal channel gains; 4.1(c): the links between the

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\(^1\)Recall that all powers are normalized to have the unit of one noise variance.
sources and the relay have the worst conditions amongst all; 4.1(d): the links between the sources and the relay have better channel gains than others; 4.1(e): the relaying links have better conditions than the direct links, and the links between the sources and the relay do not limit the rate; and 4.1(f): the channel gain of the link between the relay and the destination is dominant.

We can observe from Figures 4.1(a) and (c) that the TS rate region far exceeds the rate regions of the relaying strategies. This is an intuitive result, since in these scenarios the channel gains of the direct link is strong compared with the relay links. Furthermore, we observe that the TS rate region touches the outer bound at $R_1 = R_2$, indicating that TS is in fact the optimal communication strategy when the direct links are strong.

In Figures 4.1(b) and (e), the links that join the relay to the other nodes have better conditions than the direct links from the sources to the destination. It is apparent that in these cases the relay strategies give better performance than TS. Moreover, CDF operates close to the E-OB, which suggests that, under the requirement of service fairness (i.e., equal time sharing between sources), there may be no pragmatic need for more elaborate relaying schemes than CDF. CDF dominates all other communication strategies, using only a simple network coding approach. This exemplifies the efficacy of network-layer relaying compared to the physical-layer approaches such as TSAF and CAF. Furthermore, CAF and CDF in these cases provide much better rate regions than TSAF and TSDF respectively. This suggests the benefit of network coding if relaying is used.

Figures 4.1(d) and (f) illustrate the circumstances where either the links that connect the sources to the relay or the link that joins the relay to the destination have better states than the other ones. We see TS still has the dominant rate region in these cases. In spite of the relay having similar channel gains as direct links, all relaying strategies use a smaller portion of the total transmission time to send fresh information to the destination, which leads to degradation in the performance. Therefore, we conclude that, under the collision model, the relay is beneficial only when both incoming and outgoing links at the
Figure 4.2: Optimal power allocation: The shaded regions represent the achievable rate regions under each relaying strategy, and the boundary of the TS’ rate region is specified by the cyan color line.

4.2 Optimal Power Allocation

Next, we study the effect of optimal power allocation. We set the total available power to $P = 60$. For fair comparison between the various communication strategies, we choose channel gains the same as the scenario of Figure 4.1(e), where the relay links have better channel conditions than the direct links. We distribute the available power optimally among the relay and sources as explained in Section 2.4.

Figure 4.2 demonstrates the achievable rate regions of different communication strate-
gies, under power allocation schemes that maximizes the sum rate. For TS, equal time sharing is assumed by default. For the relaying strategies, we plot the rate regions for different $\alpha_i$, $\beta_i$, $a_i$, and $b_i$ values. The solid black lines, in each subfigure, represent the achievable rate regions for each choice of these coefficients in the linear combination of the messages at the relay; hence, the actual achievable rate region of each strategy is the convex hull of all these solid lines. An important observation here is that, under optimal powers, there is a drastic expansion of the TS rate region from the case of equal power allocation. In contrast, CDF, the best relaying strategy under equal power allocation, experiences little rate improvement.

Another interesting observation is the dominance of TSDF over other relaying methods in term of the maximum sum rate. The rate region of TSDF is larger than the rate regions of other strategies except CDF, but TSDF reaches far better sum rates than CDF. Note that the overall rate regions in Figure 4.2 may not necessarily be larger than those in Figure 4.1(e), because of the constraint of sum-rate maximization.

Furthermore, on the one hand, the fact that the sources have less time to transmit in TSDF compared with CDF and yet TSDF can achieve better sum-rate, stresses the importance of power allocation. On the other hand, the fact that the sources have more time to transmit in CDF than in TSDF, and CDF achieves better individual rates for the sources, suggests the potential for joint power, time, and coding design. Overall, comparing Figure 4.1(e) with Figure 4.2, we can conclude that, with appropriate allocation of system resources, time and power in this case, one can reduce the need for relaying.

4.3 Case Study: Moving Relay

In this section, we present a case study, where the channel gains are determined by the distance from a moving relay to the sources and the destination. We compare the sum-rate performance of all communication strategies to select the best strategy given the
Figure 4.3: Two-Source Moving relay’s configuration.

location of the relay.

4.3.1 Two Sources

Consider the symmetric structure of Figure 1.1, where the sources and the destination form an isosceles triangle, and the relay can be placed anywhere on the perpendicular bisector that passes through the destination. Let $d_1$ represent half of the distance between the two sources, and $d_2$ represent the distance of the destination to each source. We define the position of the relay in terms of its distance to the midpoint of the line joining the sources, and compute the maximum sum-rate achievable by each strategy for any position of the relay.

We consider different channel gains given by transmission path loss:

$$h_{ij} = d_{ij}^{-\frac{r}{2}}. \quad (4.1)$$

where $d_{ij}$ is the distance between the nodes $i$ and $j$, and $r$ is the path-loss exponent.
(a) Equal power, $r = 2$, $d_1 = 0.1$ and $d_2 = 3$

(c) Equal power, $r = 2$, $d_1 = 0.25$ and $d_2 = 5$

(b) Optimal power, $r = 2$, $d_1 = 0.1$ and $d_2 = 3$

(d) Optimal power, $r = 2$, $d_1 = 0.25$ and $d_2 = 5$

Figure 4.4: Moving relay, with low path loss.

Again, for equal power allocation, we assume $P_1 = P_2 = P_3 = 20$, and for optimal power allocation, we assume $P = 60$.

Figures 4.4 and 4.5 demonstrate the numerical results under equal and optimal power allocations, for low path loss and high path loss, respectively. Note that parts of the CDF and TSDF curves for the optimal power case are cutoff in some plots, for relay positions that do not satisfy the channel conditions leading to (2.24) and (2.25).

Both figures show that, for all scenarios, whenever the relay is close to the sources or the destination, TS out performs the other strategies, even though the relay has one strong link either to the sources or to the destination. This observation validates the results obtained from Figures 4.1(d) and (f). For the relay to be beneficial, both
(a) Equal power, $r = 3$, $d_1 = 0.1$ and $d_2 = 2$

(b) Optimal power, $r = 3$, $d_1 = 0.1$ and $d_2 = 2$

(c) Equal power, $r = 4$, $d_1 = 0.1$ and $d_2 = 2$

(d) Optimal power, $r = 4$, $d_1 = 0.1$ and $d_2 = 2$

Figure 4.5: Moving relay: with high path loss.

incoming and outgoing links must be stronger than the direct source-destination link. In contrast, these figures show that the relay strategies, especially CDF and TSDF, can be helpful in the intermediate positions, when the above condition is satisfied.

Another interesting observation is the improvement to TS sum rates under power optimization, such that it can dominate all relaying strategies in some cases, no matter where the relay is, as shown in Figure 4.4(b). Optimal power allocation reduces the advantage of relaying, even when the relay is in the middle between the sources and the destination, so that the channel gains for the relay is strong. In other words, TS exploits better scheduling and power management to alleviate the impact of lower channel gains. Note that, although the relaying strategies that we have investigated are not the only
possible ones, the proximity of the TS rates to the outer bound in some cases, and the overhead cost of more elaborate relaying strategies, may still obviate the need for relaying.

As we increase the channel path loss exponent, Figure 4.5 suggests that even the optimal allocation of resources is not sufficient to support TS. Under both, equal or optimal allocation of power, the DF strategies show better performance over the AF strategies and TS. Furthermore, the relative performance between TSDF and CDF depends on the location of the relay. TSDF uses a smaller amount of time to transmit fresh information to the destination in comparison with CDF; however, it can provide stronger copies of the transmitted data for the destination by dedicating more power to the individual messages of the sources. The performance advantage of TSDF for some relay positions suggests the tradeoff between time and power in different relaying strategies.

Finally, we observe that CDF always has the highest zenith point amongst all relaying strategies, so that if the relay location is free to choose by the system designer, CDF is the overall best relaying strategy. This, again, reflects the benefit of network coding based schemes. The proposed analysis additionally provides quantitative design guidelines on where to optimally place the relay for different relaying and coding strategies.

4.3.2 Four Sources

In this section, we study a four-source network in order to illustrate how beneficial partitioning can be. Consider a symmetric square pyramid structure with four sources placed on the vertices of the base and destination as the apex, Figure 4.6. The relay moves along the line perpendicular to base in its center. $d_1$ is the distance between each source and the center of the base, and $d_2$ represents the distance of the destination to each source. In addition to the previously defined strategies, the relay can also divide sources into pairs and forward a linear combination of each pair, separately, in AF or DF manner\footnote{The rate region of these methods are a scaled version of the two source case, hence we omit them for the purpose of brevity}.
Figure 4.6: Four-Source Moving relay’s configuration.

Similar to the previous subsection, the channel gains are given by transmission path loss, and we compute the maximum sum-rate achievable by each strategy for each position of the relay. We only consider equal power allocation in this study, in which each source is designated ten units of power.

Figure 4.7 demonstrates the numerical results under low and high pass loss. As we can see in the figures, our newly modified strategies has outperformed other strategies in some positions of the relay. An interesting observation is that in every one of these figures, TSDF is the dominant strategy in the positions right before the ones in which PCDF is the dominant strategy. Furthermore, CDF provides better maximum achievable rate in the positions following positions. The advantage of TSDF over CDF is that the relay can provide better copies of each source’s message for the destination; on the other hand, CDF can dedicate greater amount of time to each source to transmit fresh information into the network. TSDF is the dominant strategy in the positions in which
the relay is much closer to the sources than the destination, hence it receives source’s message better than destination and can provide better copies for the destination. On the other side, CDF is the dominant strategy in the positions where the relay is much closer to the destination than the sources; therefore, it cannot provide very strong copies of the sources’ messages and its better to use strategies that give the relay less time to transmit. PCDF, which is an intermediate strategy in terms of the amount of time that it can dedicate to each source and quality of the copies of the sources’ messages that it can provide for the destination, dominates CDF and TSD in the in between positions.

Another interesting observation is that, If we can wrap up all our DF (or AF) strategies under the DF (or AF) headline, our newly modified strategy has increased the

Figure 4.7: Four Source-Moving relay, with low and high path loss.
effective range of the DF (or AF) method.

4.4 Chapter Summary

In this chapter, we provided experimental results to compare our relaying schemes in terms of rate region and sum-rate throughput of the network. To ease the study of rate regions and avoid complex plots in dimensions of three or more, we mainly focused on the MARC model. Our simulation results under equal designation of power to all transmitters substantiated the usefulness of relaying only in the cases where the links connected to the relay have better conditions than the direct links. The numerical data in Section 4.2 proved that this advantage can be mitigated using proper allocation of power. The simple time-sharing method under optimal allocation of power not only had better maximum sum-rate than the relaying strategies, but also it provided better rate regions.

To sum up all our results under one headline, we illustrated an example in which the sources and the destination were fixed in an isosceles triangle configuration and the relay can move along the perpendicular bisector of the sources. We computed the maximum sum-rate achievable by each of our communication schemes for each position of the relay. Furthermore, we repeated this experiment with an extended network consisting of four sources. The sources and the destination were placed on the vertices of an equilateral pyramid with sources as the base. The relay can move along the line joining the apex to the center of the base. In order to evaluate our partitioning method, we introduced two other relaying strategies, PCDF and PCAF, in which the relay can assist a pair of sources in each one of its transmissions. Our simulation results corroborated the advantage of these methods for some positions of the relay. These positions were interestingly placed in between the positions where traditional relaying methods and our new relaying schemes can provide better maximum sum-rates.
Chapter 5

Conclusion

5.1 Research Summary

In this thesis, we studied a multiple access relay network under the collision model where concurrent transmissions are not allowed. We presented various relaying strategies, including time-sharing AF and DF, and variants of linear network coding in the physical layer and the network layer respectively combined with AF and DF. We developed their achievable rate regions and compared them with that of source time sharing without the relay and with cut-set outer bounds of the capacity region. We first provided a detailed explanation of each method in MARC and later extended the results to the general case of multiple access relay networks with an arbitrary number of sources. We issued the concept of partitioning as a way of increasing the rate region of our relaying scheme in the extended network. Partitioning allows the relay to provide each partition of nodes with the best relaying scheme; however, it requires us to dedicate a bigger portion of time to the relay.

We further derived optimal power allocation policies for each communication strategy under the MARC model and discussed the obstacles in choosing an appropriate objective function in the general case.
Our numerical results demonstrated that optimal power control policies under some conditions can obviate the need for relaying, which emphasizes the importance of resource management in the networks under consideration. Comparing the rate regions under equal and optimal power allocations gives us insights on the amount of performance gain that optimal power control policies can introduce, in comparison with the amount of complexity that they add to our network.

5.2 Future Work

Although there exist plenty of research papers on the study of multiple access relay channels, our definition of the collision model for this type of network is new and demands further investigation. This thesis is only the beginning of a fresh look into multiple access relay channels under the collision model. We can summarize some possible extensions of this work as follows:

- We studied simple yet effective relaying strategies that can work with simple transmitters and receivers. Our simulation results in some scenarios illustrate a noticeable gap between the achievable rate region of these strategies and the provided outer bound. Hence, one possible extension is to explore into other relaying strategies that can close this gap, or to provide a tighter outer bound.

- The UCDF and UCAF relaying strategies give us the flexibility to allocate arbitrary amount of times to each source. The large number of inequalities involved in governing the rate regions of each of these methods makes it very difficult to compare these rate regions for various time allocations amongst nodes. Hence, providing a time allocation scheme that can increase the achievable rate region or the sum-rate throughput of the network under each of these strategies would be possible future work.
• As we discussed in Chapter 3, the introduction of a proper objective function is the main barrier in finding optimal power allocations in the general network. Hence, it would be interesting to find appropriate objective functions for this optimization framework.

• Our optimal power allocation schemes, even in the simple case of MARC fail to optimize the sum-rate throughput with respect to all the parameters involved in the formulation of sum-rate. Our solutions are valid only for a given choice of coefficients, and we need an exhaustive search on the coefficients to find the maximum-achievable rate, which scales linearly with the number of users. Therefore, the next step is to find the coefficients that maximize our power allocation schemes.

• Our optimal power allocation schemes for the DF strategies are only valid in the situations where the sum-rate is determined by the second argument in the $\min$ function. This assumption is not a bad assumption to make, since the relay is only beneficial in the situations where the links connected to it have better conditions than the direct links. However, it would be more interesting, if one could find a solution that work for any possible channel condition.

• In chapter 3, we discussed partitioning as a new approach to increase the achievable rate region of our relaying strategies. However, the optimal partitioning scheme that maximizes the sum-rate throughput of the system is still unknown and requires further investigation.
Chapter 6

Appendices

6.1 Proof of CAF Rate Region

Equation (2.8) describes a multiple access channel (MAC) with two transmitters and one receiver. Hence, the rate pair \((R_1, R_2)\) should satisfy

\[
R_1 \leq \frac{1}{3} \max_{p(x_1)p(x_2)} I(X_1; Y|X_2),
\]

\[
R_2 \leq \frac{1}{3} \max_{p(x_1)p(x_2)} I(X_2; Y|X_1),
\]

\[
R_1 + R_2 \leq \frac{1}{3} \max_{p(x_1)p(x_2)} I(X_1, X_2; Y).
\]  

(6.1)

where \(p(x_i)\) is the probability function of \(X_i\), and \(I(A; B) = H(A) - H(A|B)\) where \(H(.)\) is the relative entropy function. The one-third fraction in front of each inequality is the consequence of time sharing between the nodes.
For the first inequality, we have

\[
I(X_1; Y|X_2) = H(Y|X_2) - H(Y|X_1, X_2) \\
= H(\begin{bmatrix} h_{14} \\ h_{13}h_{34} \\ Z_{14} \\ Z_{24} \end{bmatrix} + \begin{bmatrix} X_1^{n-3} \\ Z_{14} \\ Z_{24} \\ h_{34}(\beta_1 Z_{13} + \beta_2 Z_{23}) + Z_{34} \end{bmatrix}) \\
- H(\begin{bmatrix} Z_{14} \\ Z_{24} \\ h_{34}(\beta_1 Z_{13} + \beta_2 Z_{23}) + Z_{34} \end{bmatrix}).
\]

(6.2)

Under the hypothesis of an AWGN channel model, \( H(Y|X_1, X_2) = C((2\pi e)^3 \det K_Z - 1) \), where \( K \) represents the covariance matrix. Furthermore, we have \( H(Y|X_2) \leq C((2\pi e)^3 \det K_{X_1, Z} - 1) \). Hence,

\[
I(X_1; Y|X_2) \leq C((2\pi e)^3 \det K_{X_1, Z} - 1) - C((2\pi e)^3 \det K_Z - 1)
\]

\[
= \frac{1}{2} \log(\det(\begin{bmatrix} h_{14}^2 P_{1} + 1 & 0 & \beta_{13}h_{14}h_{34}P_{1} \\ 0 & 1 & 0 \\ h_{13}h_{14}h_{34}P_{1} & \beta_{13}^2 h_{34}^2 + h_{34}^2 (\beta_1^2 + \beta_2^2) + 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h_{34}^2 (\beta_1^2 + \beta_2^2) + 1 \end{bmatrix}))
\]

\[
\Rightarrow R_1 \leq \frac{1}{3} C((h_{14}^2 + \frac{\beta_{13}^2 h_{14}^2 h_{34}^2}{h_{34}^2 (\beta_1^2 + \beta_2^2) + 1})P_1).
\]

(6.3)

The other rate region boundaries can be derived similarly from the other inequalities. This completes the proof of the rate region (2.9).

### 6.2 Proof of General TSAF Rate Region

The \( 2N \) factor in the denominator is introduced, since source \( i \) will only transmit in one out of every \( 2N \) available time slot. The information of the sources are independent of
each other, and the relay assists each source individually. Therefore, the rate region can be computed for each source individually. Assume that source $i$ has transmitted its own information at the $2i - 1^{th}$ time slot, and the relay has forwarded its reception in the following time slot. We can formulate the reception of the receiver at the $2i^{th}$ time slot as follows.

$$\mathbf{Y} = \begin{bmatrix} h_{id} \\ \alpha_i h_{ir} h_{rd} \end{bmatrix} \mathbf{X}_i + \begin{bmatrix} Z_{id} \\ Z_{rd} + \alpha_i h_{rd} Z_{ir} \end{bmatrix}, \quad \alpha_i^2 (h_{ir}^2 P_i + 1) \leq P_r.$$  

Let $\Delta_i = 1 + h_{iN+2}^2 P_i$, $\delta_i = \alpha_i h_{iN+1} h_{N+1N+2} h_{iN+2} P_i$, and $N^o = 1 + \alpha_i^2 h_{N+1N+2}^2$. We can compute the constraint on the rate of the $i^{th}$ transmitter using the capacity formula of the direct link.

$$R_i \leq \frac{0.5}{2N} \log \left( \frac{\det \begin{bmatrix} \Delta_i & \delta_i \\ \delta_i & N^o + (\alpha_i h_{iN+1} h_{N+1N+2})^2 P_i \end{bmatrix}}{\det \begin{bmatrix} 1 & 0 \\ 0 & N^o \end{bmatrix}} \right)$$

$$= \frac{0.5}{2N} \log \left( 1 + h_{iN+2}^2 P_i + \frac{(\alpha_i h_{iN+1} h_{N+1N+2})^2 P_i}{1 + \alpha_i^2 h_{N+1N+2}^2} \right).$$

### 6.3 Proof of General CAF Rate Region

Equation (2.8) represents the received data at a MAC receiver. Hence, the rate region for this receiver obeys the capacity region of the MAC. The denominator of the fraction outside the $\log$ arguments represents the number of channel uses per each node, the sources and the relay. Without loss of generality, we assume that we want to find the inequality governing the sum-rate of the first $k$ sources. Let $\Delta_i = 1 + h_{iN+2}^2 P_i$, $\delta_i = \beta_i h_{iN+2} h_{iN+1} h_{N+1N+2} P_i$, and $N^o = N_{N+1N+2}$. According to the general solution of the
MAC we have the following set of equations.

\[
\sum_{i=1}^{k} R_i \leq I\left( \bigcup_{i=1}^{k} X_i; Y \bigg| \bigcup_{i=k+1}^{N} X_i \right) =
\]

\[
\max_{p(x_1) \ldots p(x_k)} \left[ H\left( Y \bigg| \bigcup_{i=1}^{N} X_i \right) - H\left( Y \bigg| \bigcup_{i=k+1}^{N} X_i \right) \right] =
\]

\[
0.5 \log \left( \det \begin{bmatrix}
\Delta_1 & 0 & \ldots & \ldots & 0 & \delta_1 \\
0 & \Delta_2 & 0 & \ldots & 0 & \delta_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & \Delta_k & \delta_k \\
\delta_1 & \ldots & \ldots & \ldots & \delta_k & \sum_{i=1}^{k} \frac{(\delta_i)^2}{P_i} + N^\alpha
\end{bmatrix} \right)
\]

\[
0.5 \log \left( \det \begin{bmatrix}
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & N^\alpha
\end{bmatrix} \right) = 0.5 \log(M_k) - 0.5 \log(N^\alpha).
\]
We are only left to find an expression for the $M_k$.

$$M_k = \Delta_k \times \det \begin{bmatrix} \Delta_1 & 0 & \ldots & \delta_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \Delta_{k-1} & \delta_{k-1} \\ \delta_1 & \ldots & \delta_{k-1} & \sum_{i=1}^{k} \frac{(\delta_i)^2}{P_i} + N^o \end{bmatrix}$$

$$+ (-1)^{2k+1}(\delta_k) \times \det \begin{bmatrix} \Delta_1 & 0 & \ldots & 0 \\ 0 & \Delta_2 & 0 & \ldots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & \Delta_{k-1} \\ \delta_1 & \ldots & \ldots & \delta_k \end{bmatrix}$$

$$\Delta_k M_{k-1} + \Delta_k \times \det \begin{bmatrix} \Delta_1 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & \Delta_{k-1} \\ \delta_1 & \ldots & \ldots & \delta_{k-1} \end{bmatrix}$$

$$(-1)^{4k+1}(\delta_k)^2 \prod_{i=1}^{k-1} \Delta_i \Rightarrow M_k = \Delta_k M_{k-1} + \prod_{i=1}^{k} \Delta_i \left( \frac{(\delta_k)^2}{P_k} - (\delta_k)^2 \prod_{i=1}^{k-1} \Delta_i \right)$$

$$= \Delta_k M_{k-1} + \frac{(\delta_k)^2}{P_k} \prod_{i=1}^{k-1} \Delta_i = \ldots = N^o \prod_{i=1}^{k} \Delta_i + \left( \sum_{i=1}^{k} \frac{(\delta_i)^2}{P_i \Delta_i} \right) \left( \prod_{j=1}^{k} \Delta_j \right).$$

### 6.4 Proof of General CDF Rate Region

The first terms in the $\min$ argument accounts for the direct transfer of information between the sources and the relay. With reference to our prior discussion, equation (3.15) represents the received data at a MAC receiver. Hence, the rate region for this receiver obeys the capacity region of the MAC, which has been reflected in the second term in the $\min$ argument. The denominator of the fraction outside the $\min$ arguments represents the number of channel uses per each node, the sources and the relay. All in all, we are only left to show how to derive the second term in the $\min$ argument from
the general solution of the MAC. Without loss of generality, we assume that we want to find the inequality governing the sum-rate of the first $k$ sources. Let $\Delta_i = 1 + h_i^2 N_{+2} P_i$, and $\delta_i = b_i h_i N_{+2} h_{N+1} N_{+2} P_i$. According to the general solution of the MAC we have the following set of equations.

\[
\sum_{i=1}^{k} R_i \leq I \left( \bigcup_{i=1}^{k} X_i; Y \bigg| \bigcup_{i=k+1}^{N} X_i \right)
= \max_{p(x_1) \ldots p(x_k)} \left[ H \left( Y \bigg| \bigcup_{i=k+1}^{N} X_i \right) - H \left( Y \bigg| \bigcup_{i=1}^{N} X_i \right) \right] =
0.5 \log \left( \det \begin{bmatrix}
\Delta_1 & 0 & \ldots & \ldots & 0 & \delta_1 \\
0 & \Delta_2 & 0 & \ldots & 0 & \delta_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & \Delta_k & \delta_k \\
\delta_1 & \ldots & \ldots & \delta_k & \sum_{i=1}^{k} \frac{\delta_i^2}{P_i} + 1 & \end{bmatrix} \right) -
0.5 \log \left( \det \begin{bmatrix}
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 \\
\end{bmatrix} \right) = 0.5 \log(M_k).
\]
We are only left to find an expression for the $M_k$.

$$M_k = \Delta_k \times \det \begin{bmatrix}
\Delta_1 & 0 & \ldots & \ldots & 0 & \delta_1 \\
0 & \Delta_2 & 0 & \ldots & 0 & \delta_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \Delta_{k-1} & \delta_{k-1} \\
\delta_1 & \ldots & \ldots & \delta_{k-1} & \sum_{i=1}^{k} \frac{(\delta_i)^2}{P_i} + 1
\end{bmatrix} +$$

$$(-1)^{2k+1} \delta_k \times \det \begin{bmatrix}
\Delta_1 & 0 & \ldots & \ldots & 0 \\
0 & \Delta_2 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \Delta_{k-1} & 0 \\
\delta_1 & \ldots & \ldots & \delta_{k} & \delta_k
\end{bmatrix} =$$

$$\Delta_k M_{k-1} + \Delta_k \times \det \begin{bmatrix}
\Delta_1 & 0 & \ldots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \Delta_{k-1} & 0 \\
\delta_1 & \ldots & \ldots & \delta_{k-1} & \Delta_k
\end{bmatrix} + (-1)^{4k+1}(\delta_k)^2 \prod_{i=1}^{k-1} \Delta_i \Rightarrow$$

$$M_k = \Delta_k M_{k-1} + \prod_{i=1}^{k} \Delta_i \left( \frac{(\delta_k)^2}{P_k} - (\delta_k)^2 \prod_{i=1}^{k-1} \Delta_i \right) = \Delta_k M_{k-1} + \left( \frac{(\delta_k)^2}{P_k} \prod_{i=1}^{k-1} \Delta_i \right) = \ldots$$

$$= \prod_{i=1}^{k} \Delta_i + \left( \sum_{i=1}^{k} \frac{\delta_i^2}{P_i \Delta_i} \right) \left( \prod_{j=1}^{k} \Delta_j \right).$$

### 6.5 Derivation of Optimal Power Allocations

We show the derivation of optimal power allocation for CAF. The derivation for TS, TSAF, and modified versions of TSDF and CDF are similar and are omitted due to page limitations.

Denote by $\mathbf{P}$ the vector of power allocations, and $\lambda$ the vector of Lagrangian multi-
pliers. The Lagrangian function of our problem is

\[
L(P, \lambda) = -C_{R_1 + R_2} - \sum_{i=1}^{3} \lambda_i P_i + \lambda_4(P_1 + P_2 + P_3 - P)
+ \lambda_5(\beta^2 h_{13}^2 P_1 + \beta^2 h_{23}^2 P_2 + \beta^2_1 + \beta^2_2 - P_3).
\] (6.4)

Note that the last term above corresponds to the relay power constraint specific to CAF, as shown in (2.7). To ensure \( P_i > 0, i = 1, 2, 3 \), the complementary slackness imposes \( \lambda_i = 0, i = 1, 2, 3 \). Applying the KKT conditions to (6.4) will result in,

\[
\frac{\partial L}{\partial P_3} = 0 \Rightarrow \lambda_4 = \lambda_5
\]

\[
\frac{\partial L}{\partial P_1} = \frac{\partial L}{\partial P_2} = 0 \Rightarrow \lambda_4 = \lambda_5 \quad \frac{1}{\beta^2 h_{13}^2 + 1} \left( k p_1 \right) - \frac{1}{\beta^2 h_{13}^2 + 1} \left( k p_2 \right)
= \frac{1}{\beta^2 h_{13}^2 + 1} \left( h_{14}^2 + \frac{\beta^2 h_{34}^2}{\beta^2_1 + \beta^2_2} \left( \beta^2_1 + \beta^2_2 + 1 \right) \right)
- \frac{1}{\beta^2 h_{23}^2 + 1} \left( h_{24}^2 + \frac{\beta^2 h_{34}^2}{\beta^2_1 + \beta^2_2} \left( \beta^2_1 + \beta^2_2 + 1 \right) \right).
\] (6.5)

where \( k = h_{14}^2 h_{24}^2 + \frac{\beta^2 h_{14}^2 h_{24}^2}{\beta^2_1 + \beta^2_2 + 1} + \frac{\beta^2 h_{34}^2 h_{34}^2}{\beta^2_1 + \beta^2_2 + 1} \). The choice of \( \lambda_4 = \lambda_5 = 0 \) is not acceptable, since it imposes \( P_i = 0, i = 1, 2, 3 \). Therefore, \( \lambda_i > 0 \), for \( i = 4, 5 \), and from complementary slackness

\[
P = P_1 + P_2 + P_3,
\]

\[
P_3 = \beta^2 h_{13}^2 P_1 + \beta^2 h_{23}^2 P_2 + \beta^2_1 + \beta^2_2.
\] (6.6)

Equations (6.5) and (6.6) describe a system of linear equations. The solution of this system determines the optimal power allocation.

### 6.6 Derivation of optimal power allocation for CDF

The Lagrangian function for this case is
\[ L(P, \lambda) = -C_{R_1+R_2} - \sum_{i=1}^{3} \lambda_i P_i + \lambda_4 (P_1 + P_2 + P_3 - P) \]
\[ + \lambda_5 (b_1^2 P_1 + b_2^2 P_2 - P_3). \]  

(6.7)

where the last expression is due to our special choice of \( b_1 \) and \( b_2 \) and the power limitations. If we apply the KKT conditions to (6.7) the following conclusions can be made.

\[ P_i > 0 \Rightarrow \lambda_i = 0, \ i = 1, 2, 3, \]

\[ \frac{\partial L}{\partial P_3} = 0 \Rightarrow \lambda_4 = \lambda_5, \]

\[ \frac{\partial L}{\partial P_1} = \frac{\partial L}{\partial P_2} = 0 \Rightarrow \frac{1}{b_2^2 + 1} (kp_1) - \frac{1}{b_1^2 + 1} (kp_2), \]

\[ = \frac{1}{b_1^2 + 1} (h_{14}^2 + h_{34}^2) - \frac{1}{b_2^2 + 1} (h_{24}^2 + h_{34}^2). \]  

(6.8)

where \( k = h_{14}^2 h_{24}^2 + b_2^2 h_{14}^2 h_{34}^2 + b_1^2 h_{24}^2 h_{34}^2 \). The choice of \( \lambda_4 = \lambda_5 = 0 \) is not acceptable, since it imposes \( P_i = 0, \ i = 1, 2, 3 \). Therefore, \( \lambda_t > 0, \ t = 4, 5 \) and from complementary slackness

\[ P = P_1 + P_2 + P_3, \]

\[ P_3 = b_1^2 P_1 + b_2^2 P_2. \]  

(6.9)

Equations (6.8) and (6.9) describe a system of linear equations. The solution of this system determines our optimal power allocation.

### 6.7 Derivation of optimal power allocation for TSAF

The lagrangian function for this scenario is

\[ L(P, \lambda) = -C_{R_1+R_2} - \sum_{i=1}^{3} \lambda_i P_i + \lambda_4 (P_1 + P_2 + P_3 - P), \]

\[ + \lambda_5 (\alpha_1^2 h_{13}^2 P_1 + \alpha_1^2 - P_3) + \lambda_6 (\alpha_2^2 h_{23}^2 P_2 + \alpha_2^2 - P_3). \]  

(6.10)
where the last expressions are imposed by the power constraints in the relay. If we apply the KKT conditions to (6.10) the following conclusions can be made.

\[ P_i > 0 \Rightarrow \lambda_i = 0, \ i = 1, 2, 3, \]

\[ \frac{\partial L}{\partial P_3} = 0 \Rightarrow \lambda_4 = \lambda_5 + \lambda_6. \]

(6.11)

The choice of \( \lambda_6 = \lambda_5 = 0 \) is not acceptable, since it imposes \( P_i = 0, \ i = 1, 2, 3 \). The cases \( \lambda_5 = 0, \lambda_6 > 0 \) and \( \lambda_5 > 0, \lambda_6 = 0 \) are also realized under special circumstances, which can be obtained in the limit of the more general case \( \lambda_i > 0, \ t = 5, 6 \). Hence, we choose \( \lambda_i > 0, \ t = 5, 6 \) and the complementary slackness condition imposes

\[ P = P_1 + P_2 + P_3, \]

\[ P_3 = \alpha_1^2 h_{13}^2 P_1 + \alpha_1^2 = \alpha_2 h_{23}^2 P_2 + \alpha_2. \]  

(6.12)

The solution of this linear system determines our optimal power allocation.

### 6.8 Derivation of optimal power allocation for TSDF

The lagrangian function for this scheme is

\[ L(\mathbf{P}, \lambda) = - C_{R_1 + R_2} - \sum_{i=1}^{3} \lambda_i P_i + \lambda_4 (P_1 + P_2 + P_3 - P), \]

\[ + \lambda_5 (\alpha_1^2 P_1 - P_3) + \lambda_6 (\alpha_2^2 P_2 - P_3). \]

(6.13)

where the last expressions are imposed by the power constraints in the relay. If we apply the KKT conditions to (6.10) the following conclusions can be made.

\[ P_i > 0 \Rightarrow \lambda_i = 0, \ i = 1, 2, 3, \]

\[ \frac{\partial L}{\partial P_3} = 0 \Rightarrow \lambda_4 = \lambda_5 + \lambda_6. \]

(6.14)
Following the same arguments as we did for the TSAFE case, our choice for $\lambda$s will be $\lambda_i > 0, \ t = 5, 6$. Hence,

$$P = P_1 + P_2 + P_3,$$

$$P_3 = \alpha_1^2 h_{13}^2 P_1 + \alpha_1^2 = \alpha_2^2 h_{23}^2 P_2 + \alpha_2^2. \quad (6.15)$$

The solution of this linear system determines our optimal power allocation.

### 6.9 Derivation of optimal power allocation for TS

The lagrangian function in this case is

$$L(P, \lambda) = -C_{R_1+R_2} - \lambda_1 P_1 - \lambda_2 P_2 + \lambda_3 (P_1 + P_2 - P). \quad (6.16)$$

The outcome of applying the KKT conditions to (6.16) is

$$P_i > 0 \Rightarrow \lambda_i = 0, \ i = 1, 2,$$

$$\frac{\partial L}{\partial P_1} = \frac{\partial L}{\partial P_2} = 0 \Rightarrow P_1 - P_2 = h_{24}^2 - h_{14}^2. \quad (6.17)$$

$C_{R_1+R_2}$, in this case, is the summation of two logarithmic functions, and logarithm is monotonically increasing. Hence, the maximum of $C_{R_1+R_2}$ is achieved, if $P_1 + P_2 = P$. This equation together with 6.17 describes a linear system of equations. The solution of this linear system determines our optimal power allocation.
Bibliography


