HIGH FREQUENCY TRADING IN A REGIME-SWITCHING MODEL

by

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Abstract

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One of the most famous problem of finding optimal weight to maximize an agent’s expected terminal utility in finance literature is Merton’s optimal portfolio problem. Classic solution to this problem is given by stochastic Hamilton-Jacobi-Bellman Equation where we briefly review it in chapter 1. Similar idea has found many applications in other finance literatures and we will focus on its application to the high-frequency trading using limit orders in this thesis. In [1], major analysis using the constant volatility arithmetic Brownian motion stock price model with exponential utility function is described. We re-analyze the solution of HJB equation in this case using different asymptotic expansion. And then, we extend the model to the regime-switching volatility model to capture the status of market more accurately.
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Chapter 1

Optimal Portfolio Selection Problem

We consider an agent who trades over a fixed time interval $[0, T]$ with initial wealth of $x$ dollars where his objective is to maximize his expected terminal utility by optimally choosing portfolio weights. We consider market with two assets available for trading. First one is riskless money market account where he makes deterministic short rate of interest $r$. We denote this asset by $B$ where its dynamics follows

$$dB_t = rB_t dt$$

Second asset is risky one where its dynamics is given by standard Black-Scholes dynamics, in other words geometric Brownian motion. For example, it could be an exchange traded stock of certain company. We denote this by $S$ where it dynamics is given by

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

Here, $W_t$ denotes standard 1-dimensional Brownian motion. Its drift rate $\alpha$ and volatility $\sigma$ are assumed to be constant. Only choice an agent can make at time $t$ is how much he should allocate his asset between riskless and risky ones. We denote $\Delta_t$ for his relative weight placed on riskless asset at time $t$. Hence, his weight allocated on risky asset at time $t$ becomes $1 - \Delta_t$. Combining all of the above, we get the dynamics of an agent’s
portfolio wealth process, which we denote by $X$, as follows

$$dX_t = [\Delta_t r + (1 - \Delta_t)\alpha]X_t dt + (1 - \Delta_t)\sigma X_t dW_t$$

Now the objective is to maximize the expected utility of its terminal wealth $X_T$. So we state the problem as

$$\max_{\Delta} E[\Phi(X_T)] \quad (1.1)$$

$$X_0 = x$$

where $\Phi$ is the utility function of given agent. This stochastic optimal control problem is simplified version of Merton’s portfolio problem without consumption constraint. We need some simplifying assumptions to make this problem mathematically well defined. We assume an agent’s portfolio is self-financing and there is no transaction cost with unlimited short selling and continuous trading is allowed. With these assumptions, this problem is solved by deriving the stochastic Hamilton-Jacobi-Bellman (HJB) equation satisfied by its optimal value function $V$. $V$ function is a function of two variables, time $t$ and initial wealth $x_0$. Value of $V$ is defined by the value of the maximum expected terminal utility when we follow the portfolio allocation given by solution of (1.1). In other words, it is simply the value of our function we are trying to maximize when our control variables are chosen to be the solution of a problem. From [5], we get the following HJB PDE for this problem

$$V_t + \sup_{u \in U} \mathcal{A}^u V = 0 \quad (1.2)$$

$$V(T, x) = \Phi(x)$$

where $\mathcal{A}^u$ denotes the infinitesimal generator of process $X$ when we use the strategy given by $u$ in set $U$. $U$ denotes set of all possible portfolio allocation strategy here. Then we solve the PDE (1.2) to obtain optimal value function $V$. In some cases of utility function, the solution is available in analytic form. Otherwise, we have to solve it numerically. Given
the optimal value function, now we can back out the optimal $\Delta_t$ to decide the allocation weight at time $t$.

This problem illustrates the idea of using utility function to make an optimal decision to allocate the assets. Since utility function is different for each agent, the optimal weight for each of them will be also different. Hence we also incorporate each investor’s risk tolerance into the consideration when choosing the portfolio strategy. In the remainder of thesis, we extend this idea to the high-frequency stock trading world. Now the objective of an agent becomes choosing optimal bid/ask quote to maximize the expected utility of the terminal wealth. We will give a brief summary of model and problem setup in Chapter 2, and then discuss its solution and extensions in rest of the chapters.
Chapter 2

High-frequency trading model

We use basic setup from [1] as our starting point. The mid-market price, or mid-price, of the stock we are going to trade evolves according to the following SDE

\[ dS_u = \sigma dW_u \]

with initial value \( S_t = s \). Here, \( W \) is a standard 1-dimensional Brownian motion and \( \sigma \) is a constant representing volatility of the stock so the stock price is defined as arithmetic Brownian motion without drift. We have few simplifying assumptions with the model. First, we assume that money market pays no interest, i.e. \( r \) is 0. Second, we assume the agent who trades this stock in a limit order book setting has no opinion on the drift or autocorrelation structure of the stock. Last, we assume the constant volatility. This assumption will be removed in the following chapters where we extend the model to the regime-switching framework.

2.1 Limit orders

Limit order is defined as an order to buy a security at not more, or sell at not less, than a specific price. A limit order placed by the trader gets executed only when someone is willing to buy or sell his stock at a specified order price, hence a trader is not exposed
to the risk of his order getting executed in an unfavourable price as the case of market order. This gives a trader control of the price where he wants his trade to be executed. The agent in our case sets limit order to buy a unit of stock at price of $p^b$ and sell a unit of stock at a price of $p^a$. $p^b$ is called a bid price and $p^a$ is called an ask price. The difference between these two prices is so called bid/ask spread. We define

$$\delta^b = s - p^b$$

and

$$\delta^a = p^a - s$$

so these are the amount of profit an agent makes each time buy and sell orders are being executed, respectively. Intuitively, the further away from mid-price the agent sets his bid/ask price, the lower the chances his orders getting executed. Therefore, it is reasonable to model the rate which limit orders get executed as the decreasing function of the spread an agent is quoting. Following this approach, we assume sell limit orders get executed at the Poisson rate of $\lambda^a(\delta^a)$ and buy limit orders get executed at the Poisson rate of $\lambda^b(\delta^b)$.

From this, we can define a wealth process of an agent since a wealth jumps every time the limit order gets executed. Let $X_t$ be the dollar amount of wealth of an agent at time $t$. Then, $X_t$ satisfies the following relation

$$dX_t = p^a dN_t^a - p^b dN_t^b$$

where $N_t^b$ corresponds to the number of stocks bought and $N_t^a$ corresponds to the number of stocks sold. As discussed earlier, these two are counting processes having probability of jump being equal to $\lambda^a(\delta^a)dt$ and $\lambda^b(\delta^b)dt$, respectively for next time interval $dt$. We also define number of stockes in the inventory for each time $t$ as difference of these two

$$q_t = N_t^b - N_t^a$$
Now, the agent who set limit orders have control over $\delta^a$ and $\delta^b$ to maximize his expected terminal utility. We write this as a value function $u$ and our objective is to find $\delta^a$ and $\delta^b$ that will give us an optimal value function. We will assume the exponential utility function is used to make our analysis analytically tractable.

$$u(t, s, x, q) = \max_{\delta^a, \delta^b} E_t[-exp(-\gamma(X_T + qTS_T))]$$

where $s$ is the initial stock price $S_t$, $x$ is the initial wealth in dollar amount $X_t$ and $q$ is the initial number of stocks in the inventory $q_t$.

### 2.2 Optimal quotes

Key steps in solving above type of stochastic optimal problem is to derive Hamilton-Jacobi-Bellman (HJB) equation for function $u$. For our type of problem, it was studied in [7] first. [7] used dynamic programming principle to derive HJB equation for an economic agent who tries to maximize the expected terminal utility by controlling bid/ask quotes. A author of [1] uses this result to derive HJB equation satisfied by our value function $u$ where it is given by

$$u_t + \frac{1}{2}\sigma^2 u_{ss} + \max_{\delta^b} \lambda^b(\delta^b)[u(t, s, x - s + \delta^b, q + 1) - u(t, s, x, q)]$$

$$+ \max_{\delta^a} \lambda^a(\delta^a)[u(t, s, x + s + \delta^a, q - 1) - u(t, s, x, q)] = 0$$

$$u(T, s, x, q) = -exp(-\gamma(x + qs)) \quad (2.1)$$

This is highly non-linear PDE where solution $u$ depends on the variable $s, x$ and $t$ continuously and discrete on $q$. We can simplify this equation using the fact that our choice of utility function is exponential. We use the following exponential utility ansatz

$$u(t, s, x, q) = -exp(-\gamma x)exp(-\gamma \theta(t, s, q)) \quad (2.2)$$

With direct substitution of this to (2.1) gives us the following PDE for $\theta$.

$$\theta_t + \frac{1}{2}\sigma^2 \theta_{ss} - \frac{1}{2}\sigma^2 \gamma \theta^2 + \max_{\delta^b} \frac{\lambda^b(\delta^b)}{\gamma} [1 - e^{\gamma(s - \delta^b - \rho b)}] + \max_{\delta^a} \frac{\lambda^a(\delta^a)}{\gamma} [1 - e^{-\gamma(s + \delta^a - \rho a)}] = 0$$
\[ \theta(T, s, q) = qs \] 

where \( r^b \) and \( r^a \) are given by the following relations

\[ r^b(t, s, q) = \theta(t, s, q + 1) - \theta(t, s, q) \] \hspace{1cm} (2.4)

\[ r^a(t, s, q) = \theta(t, s, q) - \theta(t, s, q - 1) \] \hspace{1cm} (2.5)

Above \( r^b \) and \( r^a \) are actually definition of reservation bid and ask price of the stock when inventory is \( q \). This is also called an indifference price since it makes no difference for the agent to buy or sell a single stock at this price in terms of his expected terminal utility.

Now, from the first optimality condition in (2.3) that its first derivative must vanish, we can deduce the implicit relations for the optimal distances \( \delta^b \) and \( \delta^a \).

\[ s - r^b(t, s, q) = \delta^b - \frac{1}{\gamma} \ln(1 - \gamma \frac{\lambda^b(\delta^b)}{\partial \delta^b(\delta^b)}) \] \hspace{1cm} (2.6)

\[ r^a(t, s, q) - s = \delta^a - \frac{1}{\gamma} \ln(1 - \gamma \frac{\lambda^a(\delta^a)}{\partial \delta^a(\delta^a)}) \] \hspace{1cm} (2.7)

To summarize how an agent should calculate optimal bid and ask quote, he first needs to solve the PDE (2.3) to be able to compute \( \theta \). Then, he uses this to compute reservation bid and ask price from equation (2.4) and (2.5). Finally, he uses the implicit relation (2.6) and (2.7) to calculate optimal bid and ask spreads he is going to place on top of current mid-price of the stock. In next chapters, we will focus on the method of solving PDE (2.3) both under constant volatility and regime-switching volatility models.
Chapter 3

Asymptotic expansion in $\gamma$

For simplicity, we assume the symmetric and exponential rates of arrival for both buy and sell orders. In other words, $\lambda^b$ and $\lambda^a$ are given by

$$\lambda^b(\delta) = \lambda^a(\delta) = Ae^{-k\delta} \quad (3.1)$$

Substituting this form into (2.6) and (2.7), we get

$$\delta^b = s - r^b(t, s, q) + \frac{1}{\gamma}ln(1 + \frac{\gamma}{k}) \quad (3.2)$$
$$\delta^a = r^a(t, s, q) - s + \frac{1}{\gamma}ln(1 + \frac{\gamma}{k}) \quad (3.3)$$

Again, substituting optimal values in (3.2) and (3.3) to PDE (2.3), we get

$$\theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \frac{A}{k + \gamma}(e^{-k\delta^a} + e^{-k\delta^b}) = 0 \quad (3.4)$$

$$\theta(T, s, q) = qs$$

To solve non-linear PDE (3.4), we will expand $\theta$ function in $\gamma$, investor’s risk preference parameter. In [1], asymptotic expansion in $q$ was done but it is not ideal for two reasons. First, $q$ takes discrete values as it represents the number of stocks in the current inventory. Second, value of $q$ can go as high as any integer number which can’t guarantee that we can ignore higher order terms. We use $\gamma$ instead as its values are usually taken as 0.01.
0.1, etc. So \( \theta \) is written as

\[
\theta(t, s, q) = \theta^0(t, s, q) + \gamma \theta^1(t, s, q) + \frac{1}{2} \gamma^2 \theta^2(t, s, q) + \cdots
\]

Now we expand every terms involving \( \gamma \) and \( \theta \) in this way in equation (3.2), (3.3) and (3.4). Collecting the terms with same order, we end up with the following 0th order PDE satisfied by \( \theta^0 \)

\[
\begin{align*}
\theta^0_t + \frac{1}{2}\sigma^2 \theta^0_{ss} + \frac{A}{ek}(e^{k(\Delta^* \theta^0 - s)} + e^{k(s - \Delta^* \theta^0)}) &= 0 \\
\theta^0(T, s, q) &= qs
\end{align*}
\]  
(3.5)

where following notations were used

\[
\Delta^* \theta = \theta(t, s, q + 1) - \theta(t, s, q)
\]
\[
\Delta_s \theta = \theta(t, s, q) - \theta(t, s, q - 1)
\]

Solution to this 0th order equation is easily found to be

\[
\theta^0(t, s, q) = qs + \frac{2A}{ek}(T - t)
\]

Substituting this solution to the 1st order term gives us the following PDE satisfied by \( \theta^1 \)

\[
\begin{align*}
\theta^1_t + \frac{1}{2}\sigma^2 \theta^1_{ss} + \frac{A}{e}(\Delta^* \theta^1 - \Delta_s \theta^1) &= \frac{1}{2}\sigma^2 q^2 + \frac{A}{2k^2e} \\
\theta^1(T, s, q) &= 0
\end{align*}
\]  
(3.6)

In order to solve this PDE, we first observe that the solution \( \theta^1 \) does not depend on \( s \) as there is no term involving \( s \) other than second order derivative term in \( s \) which vanishes anyway. Hence, solution to the below equation is also a solution of (3.6).

\[
\begin{align*}
\theta^1_t + \frac{A}{e}(\Delta^* \theta^1 - \Delta_s \theta^1) &= \frac{1}{2}\sigma^2 q^2 + \frac{A}{2k^2e} \\
\theta^1(T, s, q) &= 0
\end{align*}
\]  
(3.7)
Solution of this equation can be obtained by using Feynman-Kac theorem. Let $N_t$ and $M_t$ be independent Poisson processes with intensity $\frac{A}{e}$. Then our equation exactly corresponds to the generator if its difference, hence the solution has a stochastic representation which can be solved explicitly as follows

\[
\theta^1(t, s, q) = E\left[\int_t^T \frac{1}{2} \sigma^2 (q + N_s - M_s)^2 + \frac{A}{2k^2e} ds\right] = \int_t^T E\left[\frac{1}{2} \sigma^2 (q + N_s - M_s)^2 + \frac{A}{2k^2e}\right] ds = (T - t)\left[-\frac{1}{2} \sigma^2 q^2 + \frac{A}{2k^2e} - \frac{1}{2} \sigma^2 A e (T - t)\right] (3.8)
\]

Note that we assumed we can interchange integral and expectation sign to obtain the solution. This gives us solution to the (3.4) up to the 1st order expansion

\[
\theta(t, s, q) \approx qs + \frac{2A}{ek}(T - t) + \gamma(T - t)\left[-\frac{1}{2} \sigma^2 q^2 + \frac{A}{2k^2e} - \frac{1}{2} \sigma^2 A e (T - t)\right] (3.9)
\]

Combining this result with (3.2) and (3.3), we find that

\[
\delta^b = \gamma(T - t)\frac{1}{2}(2q + 1)\sigma^2 + \frac{1}{\gamma}ln(1 + \frac{\gamma}{k}) (3.10)
\]

\[
\delta^a = -\gamma(T - t)\frac{1}{2}(2q - 1)\sigma^2 + \frac{1}{\gamma}ln(1 + \frac{\gamma}{k})
\]

We then find a both reservation price and bid/ask spread from this as well.

\[
r(t, s, q) = \frac{r^a + r^b}{2} = s - q\gamma\sigma^2(T - t) (3.11)
\]

\[
\delta^a + \delta^b = \gamma(T - t)\sigma^2 + \frac{2}{\gamma}ln(1 + \frac{\gamma}{k}) (3.12)
\]

Note that all these values are equal to the result in [1] using the asymptotic expansion in $q$ even though we used different variable to expand. From the solution (3.9) we can observe this would be the case since only $q$ dependent terms are what is important as we are taking difference in $q$ variable. And indeed if we limit the solution to the terms involving $q$ variable, they are the same.
Chapter 4

Regime-switching Model

We now consider model where volatility of stock price itself is driven by a continuous time Markov chain. This will introduce an extra variable to our problem, the state variable. Let’s denote $E = 1, 2, ..., n$ to be space of all possible states or regimes and $\sigma(i), i \in E$ denote the volatility when the stock is in the state $i$. Hence our stock price follows

$$dS_t = \sigma(I_t)dW_t$$

where $I_t$ is a continuous time Markov chain that takes values in $E$. Typical example of this type of model has $n = 2$ meaning there are 2 possible states of economy representing normal regime and volatile regime. Volatile regime could correspond to any sort of economic event in both good and bad way, such as release of more than expected earnings news or economic crisis.

With this dynamics, our value function becomes

$$u(t, s, x, i, q) = \max_{\delta^a, \delta^b} E_{t,i}[\exp(-\gamma(X_T + q_T S_T))]$$

where $I_t = i$. Using the result obtained in [2], we can derive the HJB equation for new $u$ function

$$u_t + \frac{1}{2} \sigma(i)^2 u_{ss} + \max_{\delta^b} \lambda^b(\delta^b)[u(t, s, x - s + \delta^b, i, q + 1) - u(t, s, x, i, q)]$$  \hspace{1cm} (4.1)
\[ + \max_{\delta^a} \lambda^a(\delta^a)[u(t, s, x + s + \delta^a, i, q - 1) - u(t, s, x, i, q)] \]
\[ + \sum_{j \in E} q_{ij}[u(t, s, x, j, q) - u(t, s, x, i, q)] = 0 \]
\[ u(T, s, x, i, q) = -\exp(-\gamma(x + qs)) \]

where \( q_{ij} \) is \( ij \)-th entry of the generator matrix of underlying continuous time Markov chain. With the new exponential utility ansatz and same Poisson rates of order arrival as in chapter 3, we get the following PDE for \( \theta \).

\[ u(t, s, x, i, q) = -\exp(-\gamma x)\exp(-\gamma \theta(t, s, i, q)) \quad (4.2) \]
\[ \theta_t + \frac{1}{2} \sigma(i)^2 \theta_{ss} - \frac{1}{2} \sigma(i)^2 \gamma \theta_s^2 \]
\[ + \sum_{j \in E} q_{ij}[\theta(t, s, j, q) - \theta(t, s, i, q)] + \frac{A}{k + \gamma}(e^{-k\delta^a} + e^{-k\delta^b}) = 0 \]
\[ \theta(T, s, i, q) = qs \]

From here on, all \( \theta \) function also depends on the state variable \( i \). Applying the same expansion in \( \gamma \) as in the previous chapter, we obtain 0th and 1st order equation. We have additional Markov chain generator term and also volatility \( \sigma \) depends on the state variable. 0th order equation is given by

\[ \theta_t^0 + \frac{1}{2} \sigma(i)^2 \theta_{ss}^0 + \frac{A}{e^k}(e^{k(\Delta^* \theta^0 - s)} + e^{k(s - \Delta^* \theta^0)}) \]
\[ + \sum_{j \in E} q_{ij}[\theta^0(t, s, j, q) - \theta^0(t, s, i, q)] = 0 \]
\[ \theta^0(T, s, i, q) = qs \]

We easily observe that our previous solution with constant volatility is also a solution to above equation as there is no explicit dependence in the state variable. So the solution to (4.4) is also given by

\[ \theta^0(t, s, i, q) = qs + \frac{2A}{ek}(T - t) \]

Now, 1st order equation is where the dependence on state variable becomes explicit. It is given by

\[ \theta_t^1 + \frac{1}{2} \sigma(i)^2 \theta_{ss}^1 + \frac{A}{e}(\Delta^* \theta^1 - \Delta_s \theta^1) \quad (4.5) \]
\[+ \sum_{j \in E} q_{ij}[\theta^1(t, s, j, q) - \theta^1(t, s, i, q)] = \frac{1}{2}\sigma(i)^2 q^2 + \frac{A}{2k^2c}\]
\[\theta^1(T, s, i, q) = 0\]

Potential analytic approach of solving this type of equation is to utilize z-transforms combined with Fourier transforms. However, this is out of this thesis’ scope and we will proceed with the finite difference method with explicit scheme to solve it.

### 4.1 Finite Difference Method

First, we again observe the solution does not depend on \(s\) as previously hence the solution to the below equation is equally a solution to the original problem.

\[\theta^1_t + \frac{A}{c}(\Delta^s \theta^1 - \Delta_i \theta^1)\]
\[+ \sum_{j \in E} q_{ij}[\theta^1(t, s, j, q) - \theta^1(t, s, i, q)] = \frac{1}{2}\sigma(i)^2 q^2 + \frac{A}{2k^2c}\]
\[\theta^1(T, s, i, q) = 0\]

Now we will use explicit finite difference scheme to approximate the \(t\)-derivative term and then solve the equation backward starting from \(T\). We use the approximation

\[\theta^1_t \approx \frac{\theta^1_{tn} - \theta^1_{tn-1}}{h}\]

where \(h\) is the length of the time slice \((h = t_n - t_{n-1})\). This give us the discretization of (4.6)

\[\theta^1_{tn-1}(s, i, q) = \theta^1_{tn}(s, i, q) - h[\frac{1}{2}\sigma(i)^2 q^2 + \frac{A}{2k^2c}]\]
\[+ \frac{A}{c}[(\theta^1_{tn}(s, i, q + 1) - \theta^1_{tn}(s, i, q)) - (\theta^1_{tn}(s, i, q) - \theta^1_{tn}(s, i, q - 1))]\]
\[+ \sum_{j \in E} q_{ij}(\theta^1_{tn}(s, j, q) - \theta^1_{tn}(s, i, q))\]

From the boundary condition, we know that \(\theta^1_T(s, i, q) = 0\) for all \(s, i, q\) hence we can solve it backward all the way back to time \(t\). Note that \(q\) can only increase by 1 or decrease by 1 each time slice. Hence if we have \(N\) time slices, then we only need to worry about value of \(q\) between initial \(q + N\) and initial \(q - N\).
Chapter 5

Simulations and Conclusions

Based on the numerical scheme developed in section 4.1, we now turn our interest to actual simulation. For fair comparison purpose, we use the same parameters used in [1], that is $s = 100$, $T = 1$, $\sigma = 2$, $dt = 0.005$, $q = 0$, $\gamma = 0.1$, $k = 1.5$ and $A = 140$. For simplicity, we will use continuous time Markov chain with 2-state where the generator is given by the following matrix

$$
\begin{pmatrix}
-0.05 & 0.05 \\
0.8 & -0.8
\end{pmatrix}
$$

There are two volatilities corresponding to each regime and we will pick $\sigma(1) = 1.8$ for the normal regime and calculate $\sigma(2)$ such that its invariant distribution will be same as $\sigma$ in the constant volatility case whch was chosen to be 2 in [1]. Invariant distribution of this chain is given by $\pi_1 = \frac{16}{17}$ and $\pi_2 = \frac{1}{17}$. Then, simple calculation gives us the value of $\sigma(2) = 4.02$.

The simulation will be done in multiple steps. First, we simulate 1,000 sample pathes of Markov Chain for each of time $t = 0, 0.05, 0.1, \cdots$. From this, we then simulate 1,000 sample pathes of mid-price of the stock starting from 100 by adding a random increment $\pm \sigma(i) \sqrt{dt}$ where we pick $\sigma(i)$ from already simulated Markov chain’s status. And then we move on to the simulation of our strategy. For each time step, with probability
\[ \lambda^a(\delta^a)dt, \text{ the inventory variable decreases by one and the wealth increases by } s + \delta^a. \]

With probability \[ \lambda^b(\delta^b)dt, \] the inventory variable increases by one and the wealth decreases by \[ s - \delta^b. \] This gives us 1,000 sample paths of mid-price and bid/ask prices we quoted at each time. The main advantage of our approach is bid and ask quote at each time is going to be different based on the current inventory level as our goal is to maximize terminal expected utility. If we have too many stocks unsold at the inventory, we are likely to put narrower spread to clear the position as it’ll be subject to the huge terminal uncertainty if remained till the end. In [1], constant volatility model strategy is compared to the symmetric strategy where an agent places equal amount of bid and ask spreads no matter what the inventory variable is. We do the same analysis using regime-switching volatility model. Figure 5.2 is the typical sample path from the regime-switching volatility model. Below table shows the various result using Regime-switching model compared to the others. We see that regime-switching model strategy produces slightly lower profit than constant Vol but it also has lower standard deviation. This means regime-switching strategy generates more stable profit in variable market conditions than constant volatility strategy which is what we expect to see as regime-switching model is definitely tracking the market closer than constant volatility model. In other words, constant volatility model only cares about the average volatility over time hori-
Figure 5.2: Sample path of mid-price and bid/ask quote using Regime-Switching Strategy

Figure 5.3: Comparison of Bid spread

zont from \( t \) to \( T \) which will result in volatile profit profile if market itself was volatile for certain periods of time. However, regime-switching model will be able to properly react to such a possible volatile market condition and adjust its optimal bid/ask quote accordingly and it is shown as lower standard deviation of profit.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Profit</th>
<th>Std(Profit)</th>
<th>Final q</th>
<th>Std(Final q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regime-switching Vol</td>
<td>58.6</td>
<td>5.9</td>
<td>-0.053</td>
<td>2.9</td>
</tr>
<tr>
<td>Constant Vol</td>
<td>64.3</td>
<td>6.7</td>
<td>-0.143</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Figure 5.3 provides comparison of optimal bid spread each strategy would place over the time horizon from \( t = 0 \) to \( T = 1 \) where we have no stocks in the inventory, in other
Figure 5.4: Histogram of final q over 1,000 simulations using Constant Vol Strategy

Figure 5.5: Histogram of final q over 1,000 simulations using Regime-Switching Strategy
words where \( q = 0 \). As very much expected, constant volatility strategy would place its spread in somewhere between where two regimes would place. Top line corresponds to volatile regime and we see it would place much larger spread as we expect the probability of limit order being executed is much higher in such a regime. Likely, in a normal regime we would place smaller bid spread than what constant vol would place as our view in the market volatility is lower this case.

Figure 5.5 shows the histogram of final inventory over 1,000 simulations. We see that more than 80 percent of cases, final inventory ends up in \([-3, 3]\) range. And none of the final inventory goes over 10 or under -10 stocks while the most extreme scenario is 200 as our time step is 200. In general, we see that an agent would prefer to have less number of stocks left in the inventory as much as possible at the end since they don’t want to get exposed to the uncertainty of final stock price. His strategy is to generate profit in any kind of market condition by placing optimal bid/ask quote according to the market, hence the optimal strategy will tend to avoid any kind of risk coming from market uncertainties. According to the histogram, it seems like with our model parameters, having more than 5 or under 5 stocks in the final inventory would not be an optimal strategy in most of cases. It is more likely to end up with less number of stocks in the inventory in the regime-switching framework since an agent would place smaller spread in the normal regime so that more we would expect to see more orders being executed compared to the constant volatility model.

In conclusion, we have observed that regime-switching volatility model is more conservative and closely tracks the market compared to constant volatility model which results in slightly lower profit with smaller standard deviation of its profit distribution. It would have been even more stable if we divided the market into multiple regimes, more than 2. So it would be recommended for risk averse investors who would like to avoid any big
losses due to the volatile market conditions to use regime-switching volatility model.
Bibliography


