SET STABILIZATION FOR SYSTEMS WITH LIE GROUP SYMMETRY

by

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Abstract

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This thesis investigates the set stabilization problem for systems with Lie group symmetry. Initially, we examine left-invariant systems on Lie groups where the target set is a left or right coset of a closed subgroup. We broaden the scope to systems defined on smooth manifolds that are invariant under a Lie group action. Inspired by the solution of this problem for linear time-invariant systems, we show its equivalence to an equilibrium stabilization problem for a suitable quotient control system. We provide necessary and sufficient conditions for the existence of the quotient control system and analyze various properties of such a system. This theory is applied to the formation stabilization of three kinematic unicycles, the path stabilization of a particle in a gravitational field, and the conversion and temperature control of a continuously stirred tank reactor.
Dedication

To God and my extended family.
Acknowledgements

Foremost, I thank Professor Manfredi Maggiore. I am inspired by your knowledge and enthusiasm for this subject. I very much appreciate your dedication to this thesis.

I also thank my extended family for their support. And I cannot forget the professors, staff and students in the Systems Control Group, including Kai Hoefner. Thank you for making this experience one that I will cherish.
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Chapter 1

Introduction

A key aspect of control theory is the problem of stabilization. Whether it is the stabilization of equilibria or trajectories, the focus is for the system to exhibit and maintain desirable behaviour at steady state. Much attention has been placed on the stabilization of equilibria. While highly practical, there are problems that do not coincide with this framework. As an illustration, we consider a simple example. The CEO of the robotics company XYZ that produces laboratory equipment just claimed to market analysts on Wall Street that the company expects $10 million in revenue by the end of the fiscal year compared to $5 million achieved at the end of this year. To ensure the company will flourish, this claim needs to be met. Because of this claim, investors have moved their money into XYZ to share in this supposed profit. Hence, the company has additional resources to expend to validate the CEO’s claim. The question circling the company is how should the CEO use these resources. The main divisions of the company are:

- Production: responsible for the production of lab equipment.
- Marketing: responsible for advertising the company and its products.
- Research & Development: responsible for developing and improving products.
Chapter 1. Introduction

The CEO will ration the additional resources in these divisions to meet his claim, but doesn’t know how much to place in each division.

The CEO’s dilemma can be considered as a control problem. The states of the system are the revenues generated by each of the divisions of the company listed above. The control inputs are the fractions of the additional resources received by each division. The dynamics of the system are straightforward: increasing the resources a division receives increases the revenue it generates. The control objective is to generate $10 million in revenue by the end of the fiscal year. Hence, the CEO needs to stabilize the sum of the revenues generated by these division to $10 million. However, this is not an equilibrium stabilization problem. Assuming that it is feasible, the CEO can decide to place all the resources into any particular division, equally share the resources between the divisions or anything policy in between to achieve his claim. Hence, the CEO needs to stabilize any point of the state space that corresponds to the revenue generated by the above divisions summing to $10 million. This control problem lies within the scope of the set stabilization problem.

Other problems such as the state agreement or synchronization problem and the path following problem can be cast within the set stabilization framework. Therefore, this is our point of departure.

1.1 Set stabilization problem

This thesis examines smooth, nonlinear dynamic systems modeled by the ordinary differential equation

\[ \dot{p} = X_0(p) + \sum_{i=1}^{m} X_i(p)u_i \]  

(1.1)

where \( p \) is the state and lies on a manifold \( M \), \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \) is the control input and \( X_0, \ldots, X_m \) are smooth vector fields on \( M \).

Set Stabilization Problem. Given a controlled invariant embedded submanifold
$S^*$ of $M$, find a control law $u$ that stabilizes $S^*$ for the system given in (1.1).

While the above example illustrates a problem outside the framework of equilibria stabilization, we provide a more concrete example representative of the systems studied in this thesis.

**Example 1.1.1** (Formation Stabilization of Three Kinematic Unicycles). In this problem, we consider stabilizing the formation pattern of three independent kinematic unicycles moving on the plane. Figure 1.1 depicts a kinematic unicycle and the states that describe its configuration.

![Figure 1.1: The configuration of the kinematic unicycle](image)

The configuration of each unicycle is described by an element of $\mathbb{R}^2 \times S^1$. Hence, the state space for this system is $M = (\mathbb{R}^2 \times S^1)^3$ and the dynamics are given by

\[
\begin{align*}
\dot{x}_i &= \cos \theta_i \\
\dot{y}_i &= \sin \theta_i \\
\dot{\theta}_i &= u_i,
\end{align*}
\]

for $i = 1, 2, 3$. Inspired by the work done in [9], we consider the control objective of stabilizing the unicycles in an equilateral triangle formation of fixed length $d$. We will investigate the stabilization of the unicycles in the triangular formation moving clockwise around the circular path shown in Figure 1.2. In this configuration the unicycles
are moving tangential to the circle. Using a suitable control, each unicycle can follow this circular path at unit speed and remain in the triangular formation. Hence this configuration is controlled invariant. We would also like to assign the locations of the individual unicycles in the triangular formation. We want the order of the unicycles to be $1 \rightarrow 2 \rightarrow 3$ in the clockwise direction as shown in Figure 1.2. Since the position and angular orientation of this triangle is not fixed, this problem is not the stabilization of a point in $M$, but the stabilization of a particular subset of $M$. In Chapter 3, we investigate this example in some more detail.

The goal of this thesis does not involve the direct solution of the set stabilization problem. Rather, we consider the formation of another control system such that finding a control law that stabilizes an equilibrium of this system is equivalent to the solution of a given set stabilization problem. This places the set stabilization problem into the well studied framework of equilibrium stabilization.

To highlight this equivalence, consider the set stabilization problem for a linear time-invariant (LTI) system. The problem involves an LTI system of the form

$$\dot{x} = Ax + Bu,$$

Figure 1.2: Equilateral triangular formation of three kinematic unicycles
where $x$ is the state belonging to a vector space $\mathcal{X}$ and $u$ belongs to an admissible set of control values $\mathcal{U}$. We wish to stabilize an $(A, B)$-invariant subspace $\mathcal{P}$. As mentioned, we want to form a control system for which the stabilization of an equilibrium is equivalent to the stabilization of $\mathcal{P}$ in $\mathcal{X}$. For the LTI case, the solution is straightforward. We can define the quotient space $\mathcal{X}/\mathcal{P}$, which is also a vector space consisting of the affine subspaces of $\mathcal{X}$ modulo $\mathcal{P}$, and its induced quotient map $P : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{P}$. Using this quotient map and the controlled invariance property of $\mathcal{P}$, we can define linear maps $\bar{A} : \mathcal{X} \rightarrow \mathcal{X}$ and $\bar{B} : \mathcal{U} \rightarrow \mathcal{X}$ to derive a quotient system on $\mathcal{X}/\mathcal{P}$ given by

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u.$$ (1.3)

This system describes the motion between the affine subspaces of $\mathcal{P}$ for the system (1.2). Since $\mathcal{P}$ is $(A, B)$-invariant, it can be shown that $P(\mathcal{P})$ is an equilibrium of (1.3).

The key point of this construction is that we can stabilize $\mathcal{P}$ in $\mathcal{X}$ by constructing a control law that stabilizes the equilibrium $P(\mathcal{P})$ in $\mathcal{X}/\mathcal{P}$ and applying it to (1.2). This reduces the set stabilization problem on $\mathcal{X}$ to an equilibrium stabilization problem on $\mathcal{X}/\mathcal{P}$.

The solution of the set stabilization problem in the LTI case exploits the structure of the state space and the target set as well as the structure in the control system. This approach does not apply to problems such as the kinematic unicycle example previously given as the system in not LTI. Nevertheless, we want to be able to use this method of equivalence to solve the set stabilization problem for systems described by (1.1).

In work [16], the method applied for solving the set stabilization problem involves a feedback transformation allowing one to isolate a subsystem describing the motion system (1.1) towards and away from the target set $S^*$, with the special requirement that the subsystem in question be LTI. The dynamics of this subsystem are known as the \textbf{transversal dynamics}, and the problem in question is referred to as transverse
feedback linearization. By forming such a system, one can design a control law that stabilizes the target set for the original system. Mathematically, we wish to find a set of local coordinates \((x_1, x_2) \in \mathbb{R}^n\) on an open subset \(U\) of the manifold \(M\) and a feedback transformation \(u = \alpha(x) + \beta(x)v\) such that the target set \(S^*\) in these local coordinates is given by the set \(\{(x_1, x_2) \in \mathbb{R}^n \mid x_2 = 0\}\). In addition, the system (1.1) in these coordinates is given by

\[
\begin{align*}
\dot{x}_1 &= X_{0,1}(x_1, x_2) + \sum_{i=1}^{m} X_{i,1}(x_1, x_2)v_i \\
\dot{x}_2 &= Ax_2 + Bv,
\end{align*}
\]

where the pair \((A, B)\) is controllable. Note that in these coordinates, the set \(S^*\) is represented by the slice \(x_2 = 0\) and the dynamics of \(x_2\) form a subsystem that is independent of \(x_1\) and is LTI. Thus, in the open subset \(U\), stabilizing \(S^*\) for the system given in (1.1) is equivalent to stabilizing the origin for the LTI subsystem with state \(x_2\). While this method extends the investigation of the set stabilization problem via equivalence to a class of nonlinear systems, it is a localized approach and poses a very strong restriction on the nature of the transverse dynamics. The motivation of this thesis is to have global equivalence results that do not require the quotient dynamics to be LTI. Finding such results for general nonlinear systems is a hard open problem. In order to make the problem tractable, in this thesis we consider classes of nonlinear systems and target sets that have special structure given by Lie group symmetries. Initially, we consider left-invariant systems defined on Lie groups. Later, we make use of Lie group actions to extend the framework to systems defined on general manifolds.

1.2 Literature review

In this section, we present a brief review on some works that inspired our use of Lie groups. Some of the early research on control systems in Lie groups includes the work
by Jurdjevic and Sussmann in [7] and Brockett in [2]. These paper study controllability properties of right-invariant systems. The authors exploit the algebraic structure of the state space to relate the controllability of the system to the Lie subalgebra generated by the control vector fields. While not directly related to the work in this thesis, the study shows various attractive properties of systems defined on Lie groups.

Of direct relevance to this thesis is the work by Grizzle and Marcus in [5], where the authors investigate systems that possess symmetries induced by a Lie group action. They provide various local and global cascade decompositions of such systems in terms of lower dimensional subsystems and feedback loops. The notion of symmetry investigated by Grizzle was later generalized by Nijmeijer and van der Schaft in [17], developing a notion of partial symmetry.

In [3], Dirr and Helmke examine right-invariant systems on matrix Lie groups. The results relevant to this thesis are the formation of a quotient space from a Lie subgroup and the formation of a quotient control system on such space. The authors also provide necessary and sufficient conditions based on the original system for the quotient control system to be accessible.

Recently, Morin and Samson [12, 14, 15] gave a characterization of controllability for systems on Lie groups based on a notion of transverse function. Based on this notion, the authors were able to provide a constructive equilibrium stabilization methodology.

### 1.3 Organization and contribution of thesis

This thesis is organized as follows. In Chapter 1 we introduce the set stabilization problem with a motivating example involving three kinematic unicycles and provide the basis for solving the set stabilization problem. In Chapter 2 we provide a review of the mathematical background used in this thesis. We assume the reader has knowledge of basic differential geometry and focus the attention on Lie group theory. The majority
of the results of this thesis are presented in Chapter 3. We restrict to left-invariant systems on Lie groups and provide various equivalent necessary and sufficient conditions for solving the set stabilization problem in a manner similar to the LTI case. We also make use of feedback transformations to provide desirable structure to the problem. We apply the results of this chapter to the kinematic unicycle example previously given. In Chapter 4 we extend the results of Chapter 3 to the case of systems invariant under a Lie group action. In addition, we extend the accessibility results of [3] to include systems described in this chapter. Lastly, in Chapter 5, we apply the results of Chapter 4 to two examples involving a particle in a gravitational field and a continuously stirred tank reactor for the production of ethanol.

1.4 Statement of contributions

The following is a list of contributions made in this thesis.

1. Theorem 3.4.1, Theorem 3.6.2. Necessary and sufficient conditions for the existence of a quotient control system in the case of Lie groups. Theorem 3.4.1 is a direct consequence of a general theorem in differential geometry (Theorem 2.2.15).

2. Theorem 3.7.1. Necessary and sufficient conditions for being able to split the control vector fields into tangential and transversal vector fields in the case of Lie groups.

3. Theorem 4.3.1, Theorem 4.4.2. Necessary and sufficient conditions for the existence of a quotient control system in the case of a system invariant under a Lie group action. As in contribution 1, Theorem 4.3.1 is a direct consequence of a theorem in differential geometry (Theorem 2.2.15).

4. Theorem 4.5.1. Necessary and sufficient conditions for splitting the control vector fields into tangential and transversal vector fields in the case of a system invariant under a Lie group action.
5. Theorem 4.6.1, Theorem 4.6.4. Necessary and sufficient conditions for accessibility of the quotient control system. These theorems generalize a result by Dirr and Helmke in [3], which was developed for systems on matrix Lie groups, to the more general setting of control systems on manifolds invariant under Lie group actions.
Chapter 2

Mathematical Preliminaries

The purpose of this chapter is to address the key definitions and results that are used in this thesis. While we do address some concepts in differential geometry, the basic concepts are not presented. These concepts include the definitions and properties of the following:

• Smooth manifold and its tangent space

• Submanifolds and their tangent spaces

• Differential of a map between manifolds (the generalization of derivatives of maps on Euclidean spaces)

• Vector fields on a manifold

All of these concepts are presented in detail in such works as [10]. We also make the standing assumption that every object that has a degree of differentiability is assumed smooth. This includes manifolds, maps between manifolds, vector fields and distributions.
2.1 Notation

We provide a table with some of the notation used in this thesis.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>Set of real numbers</td>
</tr>
<tr>
<td>( \mathbb{Z} )</td>
<td>Set of integers</td>
</tr>
<tr>
<td>( M )</td>
<td>Smooth manifold</td>
</tr>
<tr>
<td>( TM )</td>
<td>Tangent bundle of ( M )</td>
</tr>
<tr>
<td>( T_p M )</td>
<td>Tangent space of ( M ) at ( p )</td>
</tr>
<tr>
<td>( F_* (p) : T_p M \to T_{F(p)} N )</td>
<td>The differential at ( p ) of the map ( F : M \to N )</td>
</tr>
<tr>
<td>( F_* X(p) )</td>
<td>If ( F : M \to N ) is smooth, it indicates the tangent vector ( F_* (p) X(p) \in T_{F(p)} N ).</td>
</tr>
<tr>
<td>( C^\infty(U) )</td>
<td>Set of real valued functions on ( U \subset M )</td>
</tr>
<tr>
<td>( \mathfrak{X}^\infty(M) )</td>
<td>Set of smooth vector fields on ( M )</td>
</tr>
<tr>
<td>( \phi^X(t, p) )</td>
<td>Flow map of the vector field ( X )</td>
</tr>
<tr>
<td>( \phi^X_0(t) )</td>
<td>The map ( t \mapsto \phi^X(t, p) )</td>
</tr>
<tr>
<td>( \phi^X_1(p) )</td>
<td>The map ( p \mapsto \phi^X(t, p) )</td>
</tr>
<tr>
<td>( I^\Delta_p )</td>
<td>The maximal integral manifold through ( p ) of a nonsingular involutive distribution ( \Delta )</td>
</tr>
</tbody>
</table>

We place inequalities as subscripts on \( \mathbb{R} \) or \( \mathbb{Z} \) to restrict the set relative to the inequality. For example, \( \mathbb{R}_{>0} \) denotes the set of positive real numbers.

2.2 Differential Geometry

In this section we present some basic definitions and results from differential geometry.

We begin with the definition of a submersion.

**Definition 2.2.1.** A map \( F : M \to N \) is a submersion if, for every \( p \in M \), the differential of \( F \) at \( p \) is onto, i.e., \( \text{rank } F_* = \text{dim } N \).

The significance of submersions lies in the fact, stated next, that their level sets are embedded submanifolds, i.e., subsets of the manifold inheriting their topology and differentiable structure from the manifold.
Theorem 2.2.2 (Preimage Theorem). If $F : M \rightarrow N$ is a submersion, then the preimage of any $q \in N$ is a closed, embedded submanifold of $M$ with dimension $\dim M - \dim N$. Moreover, for each $q \in N$, letting $S = F^{-1}(q)$ we have

$$(\forall p \in S) \ T_pS = \ker F_*(p).$$

Next, we have an important definition relating two vector fields that may exist on different manifolds.

Definition 2.2.3. Let $F : M \rightarrow N$ be a map between manifolds and let $X \in \mathfrak{X}^\infty(M)$ and $Y \in \mathfrak{X}^\infty(N)$. The vector fields $X$ and $Y$ are said to be $F$-related if $\forall p \in M$ the following diagram commutes:

\[
\begin{array}{ccc}
T_pM & \xrightarrow{F_*(p)} & T_{F(p)}N \\
p \mapsto X & & \mapsto Y \\
p \in M & \xrightarrow{F} & F(q) \in N
\end{array}
\]

In other words,

$F_*X(p) = Y(F(p)).$

The significance of the notion of $F$-relatedness lies in the fact that if $X$ and $Y$ are two $F$-related vector fields on manifolds $M$ and $N$, respectively, then $F$ maps the integral curves of $F$ on $M$ to integral curves of $Y$ on $N$. This result is stated in the next proposition, taken from [10].

Proposition 2.2.4. Suppose $F : M \rightarrow N$ is a map between manifolds and let $X \in \mathfrak{X}^\infty(M)$ and $Y \in \mathfrak{X}^\infty(N)$. The vector fields $X$ and $Y$ are $F$-related if and only if the following diagram commutes

\[
\begin{array}{ccc}
M & \xrightarrow{\phi^X_t} & M \\
\downarrow F & & \downarrow F \\
N & \xrightarrow{\phi^Y_t} & N
\end{array}
\]
In other words,

\[(\forall p \in M) \ F(\phi^X_t(p)) = \phi^Y_t(F(p)).\]

For a given vector field \(X \in \mathfrak{X}(M)\) and a map \(F : M \to N\), it is sometimes possible to define a unique vector field on \(N\) which is \(F\)-related to \(X\). When it exists, such a vector field will be denoted \(F_*X : N \to TN\) and will be referred to as the **push-forward of \(X\) by \(F\)**.

Now the definitions of Lie derivatives and Lie brackets.

**Definition 2.2.5.** If \(X \in \mathfrak{X}(M)\) is a smooth vector field on \(M\) and \(\lambda \in C^\infty(M)\) is a smooth real-valued function, the **Lie derivative of \(\lambda\) along \(X\)**, denoted \(L_X \lambda\), is the function \(M \to \mathbb{R}\) which pointwise is the directional derivative of \(\lambda\) in the direction of \(X\),

\[L_X \lambda(p) = \lim_{h \to 0} \left[ \lambda(\phi^X_h(p)) - \lambda(p) \right].\]

We see from the definition above that the Lie derivative of a real valued function is a real valued function. Using Lie derivatives, one can define a binary operator which takes two vector fields and returns a vector field, as in the next definition.

**Definition 2.2.6.** If \(X, Y \in \mathfrak{X}(M)\) are two vector fields on \(M\), then the **Lie bracket of \(X\) and \(Y\)**, denoted \([X,Y]\), is the vector field defined implicitly by the relation

\[ (\forall \lambda \in C^\infty(M)) \ L_{[X,Y]}\lambda = L_X(L_Y\lambda) - L_Y(L_X\lambda). \]

That the relation above does indeed uniquely characterize a vector field on \(M\) is far from trivial. In the case of vector fields \(X(p)\) and \(Y(p)\) on \(\mathbb{R}^n\), one can give an explicit expression of the Lie bracket in coordinates,

\[ [X,Y](p) = \frac{\partial Y}{\partial p} X(p) - \frac{\partial X}{\partial p} Y(p), \]
where $\frac{\partial X}{\partial p}$ and $\frac{\partial Y}{\partial p}$ denote the Jacobian matrix of the vector functions $X, Y : \mathbb{R}^n \to \mathbb{R}^n$. The next theorem states that the Lie bracket operation behaves in a natural way under transformations.

**Theorem 2.2.7.** Let $F : M \to N$ be a smooth map and $X_1, X_2 \in \mathfrak{X}^\infty(M)$, $Y_1, Y_2 \in \mathfrak{X}^\infty(N)$. If $X_i$ if $F$-related to $Y_i$, $i = 1, 2$, then $[X_1, X_2]$ is $F$-related to $[Y_1, Y_2]$, i.e.,

$$(\forall p \in M) \ F_*[X_1, X_2](p) = [F_*X_1, F_*Y_2](F(p)) = [Y_1, Y_2](F(p)).$$

Next, the definition of a distribution on a manifold.

**Definition 2.2.8.** A smooth distribution (or simply a distribution) $\Delta$ on a manifold $M$ is the assignment to each $p \in M$ of a subspace $\Delta(p) \subset T_p M$ which admits in a neighborhood of $p$ a set of smooth local generators. In other words, for each $p \in M$ there exists a neighborhood $U$ of $p$ in $M$ and smooth vector fields $X_1, \ldots, X_k$ on $U$ such that

$$(\forall q \in U) \ \Delta(q) = \text{span}\{X_1(q), \ldots, X_k(q)\}.$$ 

A distribution $\Delta$ is nonsingular if $\dim(\Delta(p))$ is constant as $p$ varies over $M$. In this case the integer $\dim(\Delta(p))$ is called the dimension of $\Delta$. A vector field $X \in \mathfrak{X}^\infty(M)$ is said to belong to the distribution, denoted $X \in \Delta$, if for all $p \in M$, $X(p) \in \Delta(p)$. A distribution $\Delta$ is involutive if

$$(\forall X, Y \in \mathfrak{X}^\infty(M)) \ (X, Y \in \Delta) \implies [X, Y] \in \Delta.$$ 

The following proposition shows that we can, at least locally, write any vector field in a nonsingular distribution in terms of a set of vector fields that pointwise span the distribution, called its local generators.

**Proposition 2.2.9.** If $\Delta$ is a nonsingular distribution on $M$ of dimension $d$, then for all $p \in M$ there exist a neighborhood $U$ of $p$ in $M$ and smooth vector fields $X_1, \ldots, X_d$
on $M$ such that

$$(\forall q \in U) \Delta(q) = \text{span}\{X_1(q), \ldots, X_d(q)\},$$

and every $\tau \in \Delta$ can be expressed on $U$ as

$$\tau = \sum_{i=1}^{d} f_i X_i, \quad f_1, \ldots, f_d \in C^\infty(U).$$

We next present the notion of maximal integral manifold of a distribution, which generalizes the concept of maximal integral curve of a vector field.

**Definition 2.2.10.** A submanifold $N$ of a manifold $M$ is called an integral manifold of the distribution $\Delta$ on $M$ if for all $p \in N$, $\Delta(p) = T_pN$. The maximal integral manifold $N$ of $\Delta$ through $p$, denoted $I^\Delta_p$, is an integral manifold of $\Delta$ such that $p \in I^\Delta_p$, and any other integral manifold of $\Delta$ containing $p$ is contained in $I^\Delta_p$.

In the same way that an integral curve of a vector field has the property that the tangent line to each point of the curve is the subspace spanned by the evaluation of the vector field at that point, an integral manifold $N$ of $\Delta$ through $p$ has the property that its tangent space $T_pN$ coincides with the evaluation of the distribution $\Delta$ at $p$. It can be shown that if $\Delta$ admits an integral manifold through $p$, then there exists a unique maximal integral manifold $I^\Delta_p$. Sufficient conditions for the existence of integral manifolds are provided by Frobenius’ theorem.

**Theorem 2.2.11 (Frobenius).** If $\Delta$ is a nonsingular and involutive distribution on $M$, then through every $p \in M$ there is a maximal integral manifold of $\Delta$. Moreover, involutivity of $\Delta$ is a necessary condition for the existence of maximal integral manifolds of $\Delta$ through every $p \in M$.

The theorem above implies that the maximal integral manifolds of a nonsingular and involutive distribution $\Delta$ partition $M$. We call such a partition a foliation, and we call each maximal integral manifold a leaf of the foliation. The foliation just described
provides a well defined equivalence relation: \( p_1 \) and \( p_2 \) are equivalent if they lie on the same leaf of the foliation. The equivalence classes are the maximal integral manifolds of \( \Delta \). With this, we can define a quotient space \( M/F_\Delta \) consisting of the equivalence classes, and the induced quotient map \( \pi : M \to M/F_\Delta \) defined as \( p \mapsto I^\Delta_p \).

In general, the quotient space \( M/F_\Delta \) may not be a manifold, but if it is, then a natural question to ask is under what conditions is a vector field \( X \) on \( M \) \( \pi \)-related to a vector field on \( M/F_\Delta \), so that we can use Proposition 2.2.4 to relate the flows of \( X \) and \( \pi_*X \) in a nice manner. We begin with a definition.

**Definition 2.2.12.** Let \( M \) be a manifold with a foliation \( F_\Delta \) of a nonsingular, involutive distribution \( \Delta \) such that \( M/F_\Delta \) is a manifold, and let \( \pi : M \to M/F_\Delta \) be the associated quotient map. A vector field \( X \in \mathfrak{X}^\infty(M) \) is **projectable** if there exists a smooth vector field \( Y \in \mathfrak{X}^\infty(M/F_\Delta) \) such that \( Y = \pi_*X \), i.e., \( X \) and \( Y \) are \( \pi \)-related.

In order to characterize which vector fields are projectable, we need the notion of invariance of a distribution.

**Definition 2.2.13.** A distribution \( \Delta \) is said to be **invariant under a vector field** \( X \) if \( \forall \tau \in \Delta \) we have \( [X, \tau] \in \Delta \). We write this condition concisely as \( [X, \Delta] \subset \Delta \).

This is an infinitesimal condition for the definition of a distribution being invariant under a vector field. The next result provides an “integral” characterization of the same property. The result follows from [6, Theorem 2.1.9].

**Theorem 2.2.14.** If \( \Delta \) is a nonsingular and involutive distribution on a manifold \( M \) and \( X \) is a complete vector field\(^1\) on \( M \), then \( \Delta \) is invariant under \( X \) if and only if

\[
(\forall t \in \mathbb{R}) \ (\forall p \in M) \ \phi^X_t(I^\Delta_p) = I^\Delta_{\phi^X_t(p)}.
\]

\(^1\)A vector field \( X \) on \( M \) is complete if the domain of the flow map \( \phi(t, p) \) is \( \mathbb{R} \times M \).
The theorem above states that $\Delta$ is invariant under $X$ if and only if the flow of $X$ maps any maximal integral manifold of $\Delta$ to another integral manifold of $\Delta$. We are now ready to state conditions for a vector field to be projectable.

**Theorem 2.2.15.** Let $M$ be a manifold with a foliation $F_\Delta$ of a nonsingular, involutive distribution $\Delta$ such that $M/F_\Delta$ is a manifold. Also, let $\pi : M \to M/F_\Delta$ be the quotient map and let $X$ be a vector field on $M$. Then, $\pi_* X$ is a globally well-defined vector field on $M/F_\Delta$ if and only if $\Delta$ is invariant under $X$.

This construction, in essence, motivates the solution of the set stabilization problem in the previously mentioned LTI case. We will use this result extensively in this thesis, for it is the underlying method by which we will prove existence of quotient control systems. Related to the notion of invariance under a vector field $X$ of a distribution $\Delta$, is the notion of invariance of a family of vector fields.

**Definition 2.2.16.** If $\mathcal{F}$ is a family of vector fields on a manifold $M$, then for $p \in M$, we define the **evaluation of $\mathcal{F}$ at $p \in M$**, $\mathcal{F}(p) \subset T_p M$, as

$$\mathcal{F}(p) = \{X(p) : X \in \mathcal{F}\}$$

A family of vector fields $\mathcal{F}$ on $M$ is said to be **invariant under a vector field $X$** if

$$(\forall \tau \in \mathcal{F}) [X, \tau] \in \mathcal{F}.$$  

We remark that if $\mathcal{F}$ is a family of vector fields on $M$ such that, for all $p \in M$, its evaluation $\mathcal{F}(p)$ is a subspace of $T_p M$, then the pointwise evaluation of $\mathcal{F}$ defines a distribution on $M$. In general, the invariance under $X$ of the family $\mathcal{F}$ does not imply the invariance of the associated distribution.

We conclude this section with the definition of a Lie algebra of vector fields and Lie algebra homomorphisms.
**Definition 2.2.17.** A Lie algebra of over $\mathbb{R}$ is a vector space $\mathfrak{g}$ equipped with a binary bracket operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

(i) $[\cdot, \cdot]$ is bilinear,

(ii) $[\cdot, \cdot]$ is skew-commutative, i.e.,

$$\forall X, Y \in \mathfrak{g} \quad [X, Y] = -[Y, X],$$

(iii) $[\cdot, \cdot]$ satisfies the Jacobi identity,

$$\forall X, Y, Z \in \mathfrak{g} \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The set of smooth vector fields on $M$, $\mathfrak{X}^\infty(M)$, with the operations of pointwise sum and scalar multiplication is an $\mathbb{R}$-vector space. Equipped with the Lie bracket in Definition 2.2.6, $\mathfrak{X}^\infty(M)$ becomes a **Lie algebra of vector fields**.

**Definition 2.2.18.** Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras. A **Lie algebra homomorphism** is a linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ that preserves brackets: $[\Phi(X), \Phi(Y)] = \Phi([X, Y])$. A **Lie algebra antihomomorphism** is a linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ that negates brackets: $[\Phi(X), \Phi(Y)] = -\Phi([X, Y])$.

As for any homomorphism defined on an algebraic structure, the important property of Lie algebra homomorphisms and antihomomorphisms is that they preserve the structure induced by the Lie bracket.

### 2.3 Lie Groups

We now introduce the key background material used in this thesis. We start by defining a general algebraic group.
**Definition 2.3.1.** A **group** is a set $G$ closed under a binary operator $*: G \times G \to G$ with the following properties:

- **Associativity:** $\forall g_1, g_2, g_3 \in G \ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

- **Existence of identity:** $\exists e \in G$ such that $\forall g \in G \ g * e = e * g = g$.

- **Existence of inverses:** $\forall g \in G \ \exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

We will occasionally use the term **group multiplication** or simply **multiplication** as a reference to the operation of $*$. Also, for the remainder of this thesis, we use the juxtaposition $g_1g_2$ to denote the multiplication of $g_1, g_2 \in G$.

We stress that the binary operator need not be commutative. If it is, then the group is called **abelian**.

**Example 2.3.2.** Let's consider the set of $n \times n$ invertible real matrices. If we endow this set with matrix multiplication as a binary operator, then it is obvious that each of the properties stated above is true, making this set a group called the general linear group. It is one of the more recognizable groups and is denoted by $GL(n)$. It describes all the invertible linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$.

As with most algebraic structures, we pay attention to subsets of a group that, with induced structure, are also groups. This leads to the definition of a subgroup of a group.

**Definition 2.3.3.** A **subgroup** is a subset $H$ of a group $G$ closed under the binary operator of $G$ and, with the induced operator from $G$, is a group.

Associativity of the operator is not an issue when determining if a subset is a subgroup. However, closure under the operator and the existence of inverses need to be fulfilled. Note that closure under the operator and the existence of inverses is sufficient for the existence of identity.
Example 2.3.4. Consider the set of \( n \times n \) orthogonal matrices with determinant one as a subset of \( \text{GL}(n) \),

\[
\text{SO}(n) = \{ A \in \text{GL}(n) : \det(A) = 1, \ A^T A = I \}. \tag{2.1}
\]

By properties of determinants and the interplay between matrix multiplication and inversion with transposes, this subset satisfies the conditions for a subgroup. Each element in this subgroup describes a rotation operator on \( \mathbb{R}^n \).

Next, consider the set of matrices

\[
\text{SE}(n) = \left\{ A \in \text{GL}(n+1) : A = \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix}, R \in \text{SO}(n), v \in \mathbb{R}^n \right\}.
\]

It is readily seen that \( \text{SE}(n) \) is a subgroup of \( \text{GL}(n+1) \). Such a subgroup is naturally identified with the group of rigid motions on \( \mathbb{R}^n \), as we will see in the following.

At this point, one should notice that the notion of a subgroup of a group generalizes the notion of a subspace of a vector space. With regard to this, we construct the generalization of an affine subspace of a vector space. An affine subspace is simply a subspace translated by a fixed vector. More precisely, for a vector space \( V \) with a subspace \( U \), we can define the equivalence relation \( \sim \) on \( V \) as \( x \sim y \) if \( x - y \in U \). This partitions \( V \) into disjoint affine subspaces.

Bringing this to the viewpoint of groups, this is a special case of partitioning the group into cosets. For a group \( G \) with a subgroup \( H \), we define two equivalence relations on \( G \). We say \( g_1 \sim_L g_2 \) if \( g_1^{-1}g_2 \in H \) and \( g_1 \sim_R g_2 \) if \( g_1g_2^{-1} \in H \). In general, these equivalence relations will provide two distinct partitions of \( G \). The equivalence classes are known as left and right cosets, respectively. We provide the definition.

**Definition 2.3.5.** Suppose \( H \) is a subgroup of a group \( G \). The **left coset** of \( H \) modulo
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$G$ containing $g \in G$ is the subset

$$gH = \{gh : h \in H\} \subset G,$$

while right coset of $H$ modulo $G$ containing $g \in G$ is the subset

$$Hg = \{hg : h \in H\} \subset G.$$

If all the left and right coincide, i.e., $\forall g \in G$, $gH = Hg$, then the subgroup $H$ is called normal. Note that if the group is abelian, as is the case for a vector space, then any subgroup is normal.

Lastly, we provide the definition of a map between groups which respects their algebraic structures.

**Definition 2.3.6.** A homomorphism from a group $G$ into a group $G'$ is a map $F : G \to G'$ such that

$$F(g_1g_2) = F(g_1)F(g_2)$$

$\forall g_1, g_2 \in G$. If $F$ is invertible, then it is called an isomorphism.

A homomorphism is a generalization of a linear map between vector spaces. The group theory presented thus far is purely algebraic; there is no notion of differentiability. Our goal is to be able to define control systems on groups and hence we require further structure. This brings us to the definition of a Lie group and a Lie subgroup.

**Definition 2.3.7.** A Lie group is a group $G$ that is also a manifold with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by

$$m(g_1, g_2) = g_1g_2, \quad i(g) = g^{-1},$$

are smooth. A Lie subgroup of $G$ is a subgroup of $G$ endowed with a topology and
smooth structure making it into a Lie group and an immersed submanifold of $G$. An embedded Lie subgroup is a Lie subgroup which is also an embedded submanifold.

**Example 2.3.8.** It can be shown that $\text{GL}(n)$ is a open subset of $\mathbb{R}^{n\times n}$ and hence is a manifold. We know that $(A, B) \mapsto m(A, B) = AB$ is a polynomial function of the entries of $A, B$, and furthermore, $A \mapsto A^{-1}$ is a rational function of the entries of $A$. Hence, $m$ and $i$ are smooth making $\text{GL}(n)$ a Lie group. Following the same reasoning, all matrix groups are also Lie groups, and for this reason they are sometimes referred to as matrix Lie groups.

The following theorem is taken from [10].

**Theorem 2.3.9 (Closed Subgroup Theorem).** Suppose $G$ is a Lie group and $H \subset G$ is a subgroup that is also a closed subset of $G$. Then $H$ is an embedded Lie subgroup.

**Example 2.3.10.** Consider the subgroup $\text{SO}(n)$ contained in $\text{GL}(n)$. From the continuity of the maps $A \mapsto \det(A)$ and $A \mapsto A^T A$ on $\text{GL}(n)$, we have $\text{SO}(n)$ is a closed subset of $\text{GL}(n)$ and hence by the theorem above it is a Lie subgroup.

We end this section by adding smoothness to the definition of a homomorphism and isomorphism on a Lie group.

**Definition 2.3.11.** A Lie group homomorphism from $G$ to $G'$ is a homomorphism $F : G \rightarrow G'$ that is also smooth. If $F$ is also invertible, then it is called a Lie group isomorphism.

### 2.4 Lie Group Actions

Lie groups can provide additional structure to a manifold via a Lie group action. Actions arise in problems involving symmetry. Consider a rigid body with an axis of symmetry such as a cylinder. If one rotates the cylinder about this axis, its configuration does not
change. Hence, its configuration is invariant under a certain set of rotation transformations. This set of rotations acts on the cylinder and leaves its configuration invariant.

**Definition 2.4.1.** Suppose $G$ is a Lie group and $M$ is a manifold. A **left action** of $G$ on $M$ is a map $\xi : G \times M \to M$ that satisfies

- $(\forall p \in M) \ \xi(e, p) = p$.
- $(\forall g_1, g_2 \in G) \ (\forall p \in M) \ \xi(g_1, \xi(g_2, p)) = \xi(g_1g_2, p)$.

Analogously, a **right action** of $G$ on $M$ is a map $\xi : G \times M \to M$ that satisfies

- $(\forall p \in M) \ \xi(e, p) = p$.
- $(\forall g_1, g_2 \in G) \ (\forall p \in M) \ \xi(g_1, \xi(g_2, p)) = \xi(g_2g_1, p)$.

Given $g \in G$ and $p \in M$, we define the maps $\xi_g : M \to M$ and $\xi^p : G \to M$ by $\xi_g \triangleq \xi(g, \cdot)$ and $\xi^p \triangleq \xi(\cdot, p)$, respectively.

**Example 2.4.2.** The group $\text{SO}(2)$ acts on $\mathbb{R}^2$ by matrix multiplication: $\xi(A, x) = Ax$. The Lie group $\text{SO}(2)$ describes all rotations on $\mathbb{R}^2$ and hence the action takes a vector $x \in \mathbb{R}^2$ and simply rotates it by a certain angle depending on the matrix $A \in \text{SO}(2)$.

**Definition 2.4.3.** The maps $L : G \times G \to G$ and $R : G \times G \to G$ defined as

$$L(g, h) = gh, \quad R(g, h) = hg,$$

are called **left and right translations**. For each $g \in G$, we denote by $L_g : G \to G$ and $R_g : G \to G$ the maps

$$L_g : h \mapsto L(g, h), \quad R_g : h \mapsto R(g, h).$$

The **conjugation map** $C_g : G \to G$ is defined as $C_g(h) = ghg^{-1}$. 
It is readily seen that left and right translations are actions of $G$ on itself. We will see that these actions have the previously defined left and right cosets as their orbits.

**Definition 2.4.4.** An action $\xi : G \times M \to M$ is:

- **Transitive** if $\forall p_1, p_2 \in M, \exists g \in G$ such that $\xi(g, p_1) = p_2$.

- **Free** if the only element of $G$ that fixes every point in $M$ is the identity, i.e., if $\xi(g, p) = p \ \forall p \in M$, then $g = e$.

- **Proper** if the map $(g, p) \to (\xi(g, p), p)$ has the property that the preimage under this map of any compact set in $M \times M$ is a compact set in $G \times M$.

The **orbit of** $p \in M$ under the action $\xi$ is the set

$$\xi_0(p) \triangleq \{\xi_g(p) : g \in G\}.$$  

The orbits under the action of a Lie group on a manifold partition the manifold. If the action is transitive, it is easily seen that the only orbit of $\xi$ is the entire manifold $M$.

**Example 2.4.5.** The action of $\text{GL}(2)$ on $\mathbb{R}^2$ given by $(A, x) \mapsto Ax$ is transitive because for any $x, y \in \mathbb{R}^2$ there exists a nonsingular matrix $A$ such that $Ax = y$. Hence, there is only one orbit under this action.

Consider now the action of the Lie subgroup $\text{SO}(2)$ on $\mathbb{R}^2$ given, as above, by $(A, x) \mapsto Ax$. As argued earlier, the action represents rotation in $\mathbb{R}^2$, and therefore the orbit of $x \in \mathbb{R}^2$ will contain all possible rotations of $x$. Hence, its orbit is the circle of radius equal to the Euclidean norm of $x$. More generally, the orbits of the action of $\text{SO}(n)$ on $\mathbb{R}^n$ are $n$-spheres.

Finally, $\text{SE}(n)$ acts on the affine space $\{(x, 1) : x \in \mathbb{R}^n\}$ via matrix multiplication,

$$\begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + v \\ 1 \end{bmatrix}.$$
We see that the action roto-translates a point \( x \in \mathbb{R}^n \), and for this reason the group \( \text{SE}(n) \) is associated with the group of rigid motions in \( \mathbb{R}^n \).

**Example 2.4.6.** A Lie subgroup \( H \) of a Lie group \( G \) acts on \( G \) in a natural way through left or right translations. For left translations we have \( \xi : H \times G \to G, \xi(h, g) = hg \), while for right translations we have \( \xi(h, g) = gh \). The orbit of \( g \in G \) using left translations is the set \( \{hg : h \in H\} \), which is the right coset \( Hg \). Similarly, the orbit of \( g \) using right translations is the left coset \( gH \). Hence, the left and right cosets can either be defined as equivalence classes of the previously defined equivalence relations \( \sim_L \) and \( \sim_R \), or as the orbits of right and left translations restricted to \( H \times G \).

### 2.5 Quotient Operations using Lie Group Actions

We now consider the use of Lie group actions to construct quotient spaces. We begin with the special case of a subgroup \( H \) of a Lie group \( G \) acting on \( G \) through left of right translations, as in the previous example. The next result is taken from [10].

**Theorem 2.5.1.** Let \( G \) be a Lie group and let \( H \) be a closed subgroup of \( G \), and consider the left coset space \( G/H \) induced by the equivalence relation \( \sim_L \). Such left coset space has a unique smooth manifold structure such that the quotient map \( \pi : G \to G/H \) given by \( \pi(g) = gH \) is a smooth submersion. The same holds for the right coset space induced by the equivalence relation \( \sim_R \), with quotient map \( \pi(g) = Hg \).

We can further extend this result to the case of a Lie group \( G \) acting on a manifold \( M \) via \( \xi \). We wish to create an orbit (quotient) space that consists of the orbits of \( M \) modulo \( G \). To do so, we need an equivalence relation \( \sim_\xi \) on \( M \) with equivalence classes equal to the orbits of \( M \) modulo \( G \). In the case of forming a left coset space, this equivalence relation is \( \sim_L \). We can equivalently define \( \sim_L \) by defining two points to be equivalent if they lie in the same left coset. This equivalent definition is used to generalize Theorem
2.5.1 to the case of a Lie group action. Given \( p_1, p_2 \in M \), we let \( p_1 \sim_\xi p_2 \) if they lie on the same orbit of \( M \) modulo \( G \). The equivalence classes of \( \sim_\xi \) are obviously the orbits of \( M \) modulo \( G \) and hence the orbit space \( M/G \) is well defined, but it may not be a geometrically well defined object such as a manifold. The following theorem places conditions on \( \xi \) to remedy this problem.

**Theorem 2.5.2 (Quotient Manifold Theorem).** Suppose a Lie group \( G \) acts smoothly, freely and properly on a smooth manifold \( M \). Then the orbit space \( M/G \) is a topological manifold of dimension equal to \( \dim(M) - \dim(G) \), and has a unique smooth structure with the property that the quotient map \( \pi : M \to M/G \) is a smooth submersion.

Note that the action of \( H \) on \( G \) via left or right translations is free and proper and hence the conditions of the Quotient Manifold Theorem are met, giving the result in Theorem 2.5.1. Also note that, since the orbits of \( \xi \) are level sets of \( \pi \), and since \( \pi \) is a submersion, Theorem 2.2.2 implies that the orbits of \( \xi \) are closed, embedded submanifolds of \( M \).

### 2.6 Invariant Vector Fields

Now that we have defined a Lie group action, we can define vector fields on a manifold that are invariant under a Lie group action. Such notion of invariance embodies the notion of symmetries in an ordinary differential equation.

#### 2.6.1 Left-Invariant Vector Fields

We start by noting that while we restrict to left-invariant vector fields, right-invariant vector fields can be defined in a similar fashion and have equivalent properties. We give definition of a left-invariant vector field.
Definition 2.6.1. Let $G$ be a Lie group. A vector field $X$ on $G$ is called **left-invariant** if it satisfies

$$(\forall g, g' \in G) \ (L_g)_*(X(g')) = X(L_g(g')) = X(gg').$$

By comparing this definition to Definition 2.2.3, we see that a left-invariant vector field is $L_g$-related to itself for all $g \in G$. Because of the linearity of $(L_g)_*$, the set of left-invariant vector fields is a subspace of $\mathfrak{X}^\infty(M)$. Since two left-invariant vector fields $X_1, X_2$ on $G$ are $L_g$-related to themselves, Theorem 2.2.7 implies that $[X_1, X_2]$ will be $L_g$-related to itself and therefore left-invariant. Hence, the set of left-invariant vector fields on $G$ forms a Lie algebra. Interestingly, if $X$ is a left-invariant vector field on $G$, for any $g \in G$ one can determine $X(g)$ given $X(e)$, for Definition 2.6.1 implies that $X(g) = (L_g)_*X(e)$. Hence, a left-invariant vector field depends solely on the value it takes at the identity element. This implies that the dimension of $\text{Lie}(G)$ is equal to the dimension of $T_eG$. Consequently, $\text{Lie}(G)$ has the same dimension as $G$.

**Definition 2.6.2.** The **Lie algebra of left-invariant vector fields** on a Lie group $G$ is denoted $\text{Lie}(G)$. $\text{Lie}(G)$ is canonically identified with $T_eG$ through the map $v \in T_eG \mapsto X \in \text{Lie}(G), \ X(e) = (L_g)_*v$.

Suppose we have a Lie subgroup $H$ of $G$. A logical question to ask is how are $\text{Lie}(H)$ and $\text{Lie}(G)$ related. The following proposition from [10] answers this question.

**Proposition 2.6.3.** Suppose $H \subset G$ is a Lie subgroup. The subset $\mathfrak{h} \subset \text{Lie}(G)$ defined by

$$\mathfrak{h} = \{X \in \text{Lie}(G) : X(e) \in T_eH\}$$

is a Lie subalgebra of $\text{Lie}(G)$ canonically isomorphic to $\text{Lie}(H)$.

In this thesis, if $H$ is a Lie subgroup of $G$, then we identify $\text{Lie}(H)$ by the Lie subalgebra $\mathfrak{h}$ given in the proposition.
2.6.2 Left-Invariant Vector Fields on Matrix Lie Groups

The Lie algebra of left-invariant vector fields for a matrix Lie group is handled in a straightforward fashion. It can be shown that $(L_g)_*$ for a matrix Lie group is simply left multiplication by the matrix $g$. Hence, a left-invariant vector field on a matrix Lie group has the form $g \mapsto gX$ where $X \in T_eG$ is a matrix. Therefore, one can identify $\text{Lie}(G)$ by the set of matrices spanning $T_eG$ with the Lie bracket taken as the matrix commutator: $[X_1, X_2] = X_1X_2 - X_2X_1$.

2.6.3 The Exponential Map

Consider a left invariant vector field $g \mapsto gX$ on a matrix Lie group $G$, and the associated matrix differential equation

$$\dot{g} = gX.$$  

The above is a linear differential equation with solution $\phi^X_{(g_0)}(t) = g_0 \exp(tX)$, where $\exp(\cdot)$ denotes the matrix exponential. Based on the observation that $\phi^X_{(e)}(1) = \exp(X)$, the notion of matrix exponential can be generalized to the Lie group setting by defining the exponential map $\exp: \text{Lie}(G) \to G$ as

$$\exp X = \phi^X_{(e)}(1).$$

Just as in matrix differential equations, for a left-invariant vector field $X$ on a general Lie group $G$, the integral curve through $g_0 \in G$ is given by

$$t \mapsto (L_{g_0})(\exp(tX)) = g_0 \exp(tX).$$

Since this expression is defined for all $t \in \mathbb{R}$, it follows that left-invariant vector fields are complete. In order to derive the expression above, note that by Proposition 2.2.4 the integral curve through $g_0 \in G$ is simply the integral curve through $e \in G$ mapped by $L_{g_0}$,
and the integral curve through \( e \) is \( \exp(tX) \), since

\[
\frac{d}{dt}(\exp(tX)) = \frac{d}{dt}\phi^t_{(e)}(1) = \frac{d}{dt}\phi_{(e)}(t) = X(\phi^t_{(e)}(t)) = X(\phi^t_{(e)}(1)) = X(\exp(tX)).
\]

### 2.6.4 Vector Fields Invariant under a Lie Group Action

**Definition 2.6.4.** Let \( G \) be a Lie group acting on a manifold \( M \) via \( \xi \). A vector field \( X \) on \( M \) is **\( \xi \)-invariant** if

\[
(\forall p \in M)(\forall g \in G) \ (\xi_g)_*(X(p)) = X(\xi_g(p)).
\]

We see from the definition that a \( \xi \)-invariant vector field is \( \xi_g \)-related to itself for all \( g \in G \) and hence, by Proposition 2.2.4, \( \xi_g \) maps integral curves of \( X \) to integral curves of \( X \). For this reason, the notion of \( \xi \)-invariance of a vector field embodies the concept of symmetry of a differential equation.

**Example 2.6.5.** Consider the vector field in \( \mathbb{R}^2 \), \( X = \text{col}(x_2, -x_1) \). Its integral curves are concentric circles centred at the origin. Therefore, we expect this vector field to be invariant under rotations. Consider the action of SO(2) on \( \mathbb{R}^2 \) by matrix multiplication, \( (A, x) \mapsto Ax \). For any \( \theta \in \mathbb{R} \),

\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \in \text{SO}(2),
\]

we have

\[
(\xi_A)_*(X(p)) = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} x_1 \sin \theta + x_2 \cos \theta \\ -(x_1 \cos \theta - x_2 \sin \theta) \end{bmatrix} = X(Ax).
\]

Hence, \( X \) is invariant under the action of SO(2) on \( \mathbb{R}^2 \).
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Similar to left-invariant vector fields on a Lie group, the set of $\xi$-invariant vector fields on $M$ forms a Lie subalgebra of $\mathfrak{X}^\infty(M)$.

2.6.5 Using Actions to Push Vector Fields from $G$ to $M$

If $\xi: G \times M \to M$ is an action of $G$ on a manifold $M$ and $v \in \text{Lie}(G)$ is a left-invariant vector field on $G$, one can use $v$ to define a vector field on $M$ using the following construction [18]. Fix a point $p$ on the manifold $M$. Consider the integral curve of $v$ through the identity in $G$, $\exp(tX)$, and its image under the action, $\xi(\exp(tv), p)$. Using the properties of the action $\xi$, it is easy to see that the map $\mathbb{R} \times M \to M$, $(t, p) \mapsto \xi(\exp(tv), p)$ just constructed defines a flow on the manifold $M$. Its infinitesimal generator is the vector field $X \in \mathfrak{X}^\infty(M)$ defined as

$$
X(p) = \frac{d}{dt} \bigg|_{t=0} \xi(\exp(tv), p)
= (\xi^p)_*(v(e)).
$$

With this definition, we can define the map $\Phi : \text{Lie}(G) \to \mathfrak{X}^\infty(M)$ by

$$
\Phi(v)(p) = (\xi^p)_*(v(e)).
$$

(2.2)

By linearity of $(\xi^p)_*$, $\Phi$ is linear. In addition, it can be shown that $\Phi$ is a Lie algebra antihomomorphism, i.e.,

$$
(\forall v, w \in \text{Lie}(G)) \ [\Phi(v), \Phi(w)] = -\Phi([v, w]),
$$

and hence $\Phi(\text{Lie}(G))$ is a Lie subalgebra of $\mathfrak{X}^\infty(M)$ of dimension $\text{dim}(G)$. The vector field $\Phi(v)$ is called an infinitesimal generator of the action $\xi$. 
2.7 Control Systems

In this thesis, we consider control affine systems of the form

$$\dot{p} = X_0(p) + \sum_{i=1}^{m} u_i X_i(p), \quad (2.3)$$

where $p \in M$ is the state, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is the input and $X_0, \ldots, X_m \in \mathfrak{X}^\infty(M)$. We assume throughout this thesis that $\forall p \in M, X_1(p), \ldots, X_m(p)$ are linearly independent. We say this system is $\xi$-invariant if $X_0, \ldots, X_m$ are $\xi$-invariant. In particular, if $M$ is a Lie group and the vector fields $X_0, \ldots, X_m$ are left-invariant, then the system is called left-invariant.

We continue by introducing feedback equivalence of control systems and the notion of a controlled invariant distribution. We begin by defining a regular feedback transformation and feedback equivalence.

**Definition 2.7.1.** A **regular feedback transformation** (or, simply, a feedback transformation) for the system given in (2.3) is a mapping of the form $u = \alpha + \beta v$ where $\alpha : M \to \mathbb{R}^m$ and $\beta : M \to \text{GL}(m)$.

**Definition 2.7.2.** Two affine control systems on $M$ given by

$$\dot{p} = X_0(p) + \sum_{i=1}^{m} u_i X_i(p), \quad \dot{p} = \bar{X}_0(p) + \sum_{i=1}^{m} v_i \bar{X}_i(p),$$

are said to be **feedback equivalent** under a feedback transformation $u = \alpha + \beta v$ if

$$\bar{X}_0 = X_0 + \sum_{i=1}^{m} \alpha_i X_i$$

$$\bar{X}_j = \sum_{i=1}^{m} \beta_{ji} X_i \text{ for } j = 1, \ldots, m.$$ 

Lastly, we consider accessibility of a system defined by (2.3). Let $\mathcal{U}$ denote the set of measurable signals $u(t) = (u_1(t), \ldots, u_m(t)) : \mathbb{R} \to \mathbb{R}^m$. 
Definition 2.7.3. For the system given in (2.3), the **reachable set** in exactly time $T$ from $p \in M$ is the set

$$\mathcal{R}(p, T) = \{ \phi_{(p)}^u(T) : u \in \mathcal{U} \}$$

where $\phi_{(p)}^u(T)$ denotes the solution of (2.3) at time $T$ with initial condition $p$ and control signal $u(t)$. The set of reachable states in time at most $T$ is given by

$$\mathcal{R}(p, \leq T) = \bigcup_{t \in [0, T]} \mathcal{R}(p, t).$$

Definition 2.7.4. The system given in (2.3) is **accessible from** $p$ if there exists $T > 0$ such that the interior of $\mathcal{R}(p, \leq t)$ is nonempty $\forall t \in (0, T]$. We say (2.3) is **accessible** if it is accessible from each point $p \in M$.

Next we define the control Lie algebra of a system given in (2.3) and a theorem relating the control Lie algebra to the accessibility of (2.3).

Definition 2.7.5. The **control Lie algebra** for (2.3) is the Lie algebra given by

$$\mathcal{A} = \text{span}_\mathbb{R} \{ [[[X_{i_1}, X_{i_2}], X_{i_3}], \ldots, X_{i_p}] : k \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_k \in \{0, \ldots, m\} \}$$

We say (2.3) satisfies the **Lie algebra rank condition** (LARC) if $\mathcal{A}(p) = T_p M \forall p \in M$.

The next theorem taken from [8] relates accessibility to the LARC.

**Theorem 2.7.6.** If (2.3) satisfies the LARC, then it is accessible.

The LARC is sufficient for accessibility, but not necessary. If the vector fields in (2.3) are analytic, then the LARC is also necessary.

### 2.8 Set Stability

Lastly, we define what it means for a closed, invariant subset of a manifold $M$ to be stable and asymptotically stable. For this definition, we assume the system is given by
\[ \dot{p} = X(p). \]

**Definition 2.8.1.** A closed, invariant set \( S^* \subset M \) is called **locally stable** if for every open set \( U \) containing \( S^* \), there exists an open set \( V \) such that \( \phi^X_t(V) \subset U \) for all \( t \geq 0 \). It is **locally asymptotically stable** if there exists an open set \( V \) such that for every \( p_0 \in V \) and for every open set \( U \) containing \( S^* \), there exists a \( T \geq 0 \) such that \( \phi^X_t(p_0) \in U \) for all \( t \geq T \).

If \( V = M \) in either of these definitions then stability would be global. These definition reduces to the typical definition of Lyaponuv stability when \( S^* = \{ p_0 \} \) where \( p_0 \) is an equilibrium.
Chapter 3

Coset Stabilization on Lie Groups

3.1 Introduction

When considering the set stabilization problem, it is beneficial to separate the system into the dynamics that describe motion along the target set and the motion towards or away from the target set. The problem in this chapter is to define a control system that describes the latter. These dynamics will be referred to as the transverse dynamics of the system. For example, consider the problem of stabilizing a particular leaf of a foliation on a manifold. The transverse dynamics describe the motion of the system between leaves of the foliation. By creating a control system that describes this motion, we can solve the set stabilization problem by controlling the transverse motion towards the target leaf.

The methodology for constructing such a control system is inspired by the previously discussed set stabilization problem for an LTI system. This chapter extends this problem to a more general class of systems. We study systems defined on Lie groups where the target set is a closed Lie subgroup; this is a generalization of the stabilization of a subspace in a vector space. While we do not need to restrict the class of control systems, by studying left-invariant systems we are able to derive more intuitive results that, at best, could only be shown locally for a general control affine system.
In summary, this chapter studies left-invariant systems evolving on a Lie group $G$ of the form
\[ \dot{g} = X_0(g) + \sum_{i=1}^{m} u_i X_i(g), \] (3.1)
where the target set is a left or a right coset of a closed Lie subgroup $H$.

### 3.2 Formulating the Kinematic Unicycle Problem

We return to the problem given in Example 1.1.1 of stabilizing three kinematic unicycles in an equilateral triangle formation of fixed length $d$. We equivalently model each unicycle as a control system on $SE(2)$. Given a configuration $(x_i, y_i, \theta_i) \in \mathbb{R}^2 \times S^1$ of the $i^{th}$ unicycle, we can express this configuration as a matrix in $SE(2)$ given by
\[
\begin{bmatrix}
\cos \theta_i & -\sin \theta_i & x_i \\
\sin \theta_i & \cos \theta_i & y_i \\
0 & 0 & 1
\end{bmatrix}.
\]

With this, we can define the motion of the three unicycles as a left-invariant system on $G = (SE(2))^3$ given by
\[
\dot{g}_i = g_i X_0 + u_i g_i X_1, \quad i \in \{1, 2, 3\},
\] (3.2)
where
\[
X_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

In the triangular formation, the heading angle of unicycle 2 equals the heading angle of unicycle 1 minus $\frac{2\pi}{3}$. Furthermore, the position of unicycle 2 equals the position of unicycle 1 translated by a distance $d$. This implies that $g_2$ is simply a roto-translation
of $g_1$. Since $\text{SE}(2)$ describes all roto-translations in $\mathbb{R}^2$, there exists a matrix $M \in \text{SE}(2)$ such that $g_2 = g_1 M$. A similar argument can be made for unicycle 3, i.e., $g_3 = g_2 M$. Figure 3.1 depicts how these transformations relate the unicycles’ configurations. Hence, the set of configurations of the unicycles in the triangular formation is given by

$$S^* = \{(g_1, g_2, g_3) \in G : g_1^{-1} g_2 = M, \ g_2^{-1} g_3 = M\}.$$

The problem has become the stabilization of $S^*$. Note that the matrix $M$ will depend on the length $d$. For $d = 1$, then

$$M = \begin{bmatrix}
\cos(-\frac{2\pi}{3}) & -\sin(-\frac{2\pi}{3}) & \cos(-\frac{\pi}{3}) \\
\sin(-\frac{2\pi}{3}) & \cos(-\frac{2\pi}{3}) & \sin(-\frac{\pi}{3}) \\
0 & 0 & 1
\end{bmatrix}.$$

Figure 3.1: Configuration transformations between the unicycles

To bring this stabilization problem within the scope of this chapter, we require a closed Lie subgroup of $G$ that has a coset equal to $S^*$. We will continue the solution of this problem further into this chapter.
3.3 Problem Statement

We present the problem studied in this chapter.

**Coset Stabilization Problem**: Given the left-invariant system (3.1) and a target (left or right) coset $S^*$ of a closed subgroup $H$ of $G$, find a control law $u$ that stabilizes $S^*$ in $G$.

As previously mentioned, the primary goal of this thesis does not directly deal with the construction of such a control law. Rather, we are interested in the equivalence of this problem to one that is more viable. Specifically, we are concerned with the construction of another control system for which the stabilization of a particular equilibrium is equivalent to the above problem. Hence, the given set stabilization problem will be brought to the well studied framework of equilibrium stabilization.

We outline the methodology considered in this chapter to establish this equivalent stabilization problem. Firstly, we construct a state space on which the coset $S^*$ is reduced to a point, and secondly, we define a control affine system on this state space, known as the quotient control system that describes the motion transversal to $S^*$. The set stabilization problem will be equivalent to finding a control law that stabilizes the point $S^*$ for this newly defined system.

3.4 Methodology

3.4.1 State Space Reduction

The construction of the state space for the quotient control system is inspired by the stabilization of a subspace for an LTI system. In the LTI case, the state space for the quotient control system is the quotient of the original state space modulo the target subspace; this reduces the target set to a point in the quotient space. In addition, the quotient space is a vector space on which an LTI system that describes the transversal
motion is well defined. This reduces the set stabilization problem to an equilibrium stabilization problem on the quotient space.

In our setting, Theorem 2.5.1 extends the methodology just described to a general Lie group when the target set is a closed Lie subgroup. Given a Lie group $G$ with a closed Lie subgroup $H$, we can always define a quotient space $G/H$ consisting of the cosets (left or right) of $G$ modulo $H$. The quotient space can be given a unique manifold structure of dimension $\dim(G) - \dim(H)$ such that the quotient map $\pi : G \to G/H$ is a smooth submersion. With this, the coset $S^*$ will reduce to a point in $G/H$. The problem then becomes: under what conditions does there exist a unique control system on $G/H$ whose vector fields are $\pi$-related to $X_0, \ldots, X_m$? This is the subject of the next section.

### 3.4.2 Construction of Quotient Control System

We have established that we want the quotient control system to describe the motion of (3.1) transversal to the cosets of $G$ modulo $H$. Since the state space for the quotient system consists of the cosets of $G$ modulo $H$, any control system on this space will describe the flow from coset to coset. Therefore, to complete the construction of the quotient control system, we need to define a control system on $G/H$ that parallels (3.1). Mathematically, we need a control system such that for any piecewise continuous signal $u(t)$, the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\phi^u_t} & G \\
\downarrow^{\pi} & & \downarrow^{\pi} \\
G/H & \xrightarrow{\tilde{\phi}^u_t} & G/H
\end{array}
$$

commutes, where $\phi^u_t(g_0)$ is the solution of (3.1) with initial condition $g_0$ and control signal $u(t)$, and $\tilde{\phi}^u_t(g_0)$ is the solution of the quotient control system on $G/H$ with initial condition $g_0$ and the same control signal. The commutative diagram ensures the projection of an integral curve of (3.1) onto its motion between cosets is an integral curve of the quotient control system with corresponding initial condition. In other words, the flow of
the quotient control system is $\pi$-related to the flow of (3.1). This is the defining feature of the quotient control system as we wish to control the motion of (3.1) between cosets via quotient control system. With the above property, the motion of (3.1) from coset to coset will be as predicted by the quotient control system. Hence, by designing a controller that stabilizes $\pi(S^*)$ in $G/H$ for the quotient control system, we are guaranteed that this controller will stabilize $S^*$ in $G$ when applied to (3.1). Letting $q = \pi(g)$ we have

$$
\dot{q} = \pi_*(g)(X_0(g)) + \sum_{i=1}^{m} u_i \pi_*(g)(X_i(g))
$$

Hence, if it exists, the quotient control system is given by

$$
\dot{q} = \tilde{X}_0(q) + \sum_{i=1}^{m} u_i \tilde{X}_i(q), \quad (3.3)
$$

where $\tilde{X}_j \circ \pi = \pi_* X_j$ for $j = 0, \ldots, m$, i.e., each vector field $\tilde{X}_j$ is $\pi$-related to $X_j$. It is key to note that constructing the quotient control system in this fashion is equivalent to the above commutative diagram. Hence, this is the unique quotient control system. The problem with this construction is that the functions $g \mapsto \pi_*(g)(X_j(g))$ $j = 0, \ldots, m$ may not be written entirely in terms of the quotient state variable $q = \pi(g)$. Hence, a quotient control system may not exist.

Theorem 2.2.15 provides the conditions necessary and sufficient for the projectability of vector fields from a manifold $M$ to the quotient space $M/F_\Delta$ where $F_\Delta$ is the foliation of a nonsingular and involutive distribution $\Delta$ on $M$. To bring our problem into this framework, we define a distribution $\Delta$ whose integral manifolds are the cosets of $G$ modulo $H$. It is obvious that $\Delta(g)$ should equal the tangent space at $g$ of the coset of $g$, i.e., $\Delta(g) = T_g(gH)$ for the case of left cosets and $\Delta(g) = T_g(Hg)$ for the right coset case. Equivalently, since for each $q \in G/H$, the level set $\pi^{-1}(q)$ is a coset (left or right)
of $H$, by the preimage theorem (Theorem 2.2.2) we have

$$\Delta(g) = \ker(\pi_*(g)).$$

Since each coset has dimension $\dim(H)$, $\Delta$ is nonsingular. By construction, each coset of $H$ is an integral manifold of $\Delta$, and therefore by Frobenius’ Theorem $\Delta$ is involutive.

The next result is a straightforward consequence of Theorem 2.2.15.

**Theorem 3.4.1.** A left-invariant control system of the form given in (3.1) admits a unique quotient control system on $G/H$ given by (3.3) where $\tilde{X}_j \circ \pi = \pi_* X_j$ if and only if $\Delta$ is invariant under $X_j$ for $j = 0, \ldots, m$.

We should state that Theorem 3.4.1 does not require the control system to be left-invariant. However, we will make use of the left-invariance property for further study in this chapter.

### 3.5 Projectability using Feedback Transformations

Given a system described by (3.1), Theorem 3.4.1 provides necessary and sufficient conditions for the existence of a unique quotient control system. In this section, we investigate whether these conditions can be weakened by the use of a regular feedback transformation. In particular, if a system does not meet the conditions of Theorem 3.4.1, does there exist a feedback equivalent left-invariant system that does meet these conditions? If so, we can construct the quotient control system for this feedback equivalent system and solve the set stabilization problem, which by feedback equivalence, solves the original problem. The answer to this question, presented in Proposition 3.5.2 below, is **no** and thus Theorem 3.4.1 cannot be weakened through feedback equivalence.

We consider regular feedback transformations of the form $u = \alpha + \beta v$ where $\alpha : G \to \mathbb{R}^m$ and $\beta : G \to \text{GL}(m)$. Under a transformation of this type, (3.1) becomes a new
control system given by
\[ \dot{g} = \bar{X}_0(g) + \sum_{i=1}^{m} v_i \bar{X}_i(g) \]  
(3.4)
where \( \bar{X}_0 = X_0 + \sum_{i=1}^{m} \alpha_i X_i \) and \( \bar{X}_j = \sum_{i=1}^{m} \beta_{ji} X_i \) for \( j = 1, \ldots, m \). We investigate conditions on the original system for which the system given in (3.4) is left-invariant and admits a quotient control system.

The following proposition restricts the feedback transformations considered.

**Proposition 3.5.1.** The control system given in (3.4) is left-invariant if and only if \( \alpha \) and \( \beta \) are constant.

**Proof.** (\( \Leftarrow \)) Let \( \alpha \) and \( \beta \) be constant. Then \( \forall g, g' \in G \),

\[
(L_g)_* \bar{X}_0(g') = (L_g)_* (X_0(g') + \sum_{i=1}^{m} \alpha_i X_i(g'))
\]
\[
= X_0(L_g(g')) + \sum_{i=1}^{m} \alpha_i X_i(L_g(g'))
\]
\[
= \bar{X}_0(L_g(g')).
\]

Hence, \( \bar{X}_0 \) is left-invariant. Also, for each \( j \in \{1, \ldots, m\} \)

\[
(L_g)_* \bar{X}_j(g') = (L_g)_* (\sum_{i=1}^{m} \beta_{ji} X_i(g'))
\]
\[
= \sum_{i=1}^{m} \beta_{ji} X_i(L_g(g'))
\]
\[
= \bar{X}_j(L_g(g')).
\]

Therefore, each \( \bar{X}_j \) is left-invariant as well.

(\( \Rightarrow \)) Let (3.4) be a left-invariant control system. This implies \( \forall g, g' \in G \),

\[
(L_g)_* \bar{X}_0(g') = \bar{X}_0(L_g(g'))
\]
\[
= X_0(L_g(g')) + \sum_{i=1}^{m} X_i(L_g(g')) \alpha_i(L_g(g')).
\]
But since
\[(L_g)_* \bar{X}_0(g') = (L_g)_* X_0(g') + \sum_{i=1}^{m} (L_g)_* X_i(g') \alpha_i(g')\]
\[= X_0(L_g(g')) + \sum_{i=1}^{m} X_i(L_g(g')) \alpha_i(g')\]
and \(X_1, \ldots, X_m\) are linearly independent, we need \(\alpha(g') = \alpha(L_g(g')) \forall g, g' \in G\). This implies that \(\alpha\) must be constant. A similar argument shows that \(\beta\) must also be constant.

In order to preserve the left-invariance of the vector fields after applying a feedback transformation, such transformation must be constant. Can we apply a constant feedback transformation such that the resulting feedback equivalent system admits a quotient control system? The following result answers this question.

**Proposition 3.5.2.** Let two control systems on \(G\), \((X_0, \ldots, X_m)\) and \((\bar{X}_0, \ldots, \bar{X}_m)\), be related by a constant feedback transformation, and let \(\Delta\) be a smooth distribution on \(G\). Then, \(\Delta\) is invariant under \(X_0, \ldots, X_m\) if and only if it is invariant under \(\bar{X}_0, \ldots, \bar{X}_m\).

**Proof.** \((\Rightarrow)\) Let \(\Delta\) be invariant under \(X_0, \ldots, X_m\). Since the Lie bracket is linear in the first term and \(\alpha\) and \(\beta\) are constant, it is obvious that \(\Delta\) is invariant under \(\bar{X}_0, \ldots, \bar{X}_m\).

\((\Leftarrow)\) Let \(\Delta\) be invariant under \(\bar{X}_0, \ldots, \bar{X}_m\). Since \(\beta\) is invertible, we can write \(X_j\) as a constant linear combination of \(\bar{X}_1, \ldots, \bar{X}_m\) for \(j = 1, \ldots, m\). By linearity of the Lie bracket, \(\Delta\) is invariant under \(X_1, \ldots, X_m\). And since \(X_0 = \bar{X}_0 - \sum_{i=1}^{m} \alpha_i X_i\), \(\Delta\) is also invariant under \(X_0\).

Given two left-invariant control systems that are feedback equivalent, Proposition 3.5.2 implies that one of the systems admits a quotient control system if and only if the other does as well. Hence, the conditions required to admit a quotient control system given in Theorem 3.5.1 cannot be weakened using feedback equivalence. Given a left-invariant control system that does not admit a quotient control system, there does not
exist a constant, regular feedback transformation such that the resulting control system admits a quotient control system. Therefore, there does not exist a feedback equivalent, left-invariant control system that admits a quotient control system.

3.6 Lie Algebraic Projectability Conditions

Thus far, the question of projecting a vector field onto the quotient space has not made use of the left-invariance property of the vector fields. In this section, we derive equivalent conditions for the existence of a well defined quotient system that make use of this additional structure placed on the vector fields, namely the fact that the collection of left invariant vector fields on $G$ forms a Lie algebra.

3.6.1 Target Set is a Left Coset of $H$

We begin by investigating the case where the target set is a left coset of $H$. The result is stated as a proposition.

**Proposition 3.6.1.** The distribution $\Delta$ defined using left cosets of $H$ as $\Delta(g) = T_g(gH)$ is invariant under a left-invariant vector field $X$ if and only if $\text{Lie}(H)$ is invariant under $X$.

**Proof.** ($\Rightarrow$) Let $\Delta$ be invariant under $X$ and let $\tau \in \text{Lie}(H)$. We need to show $[X,\tau] \in \text{Lie}(H)$. We have $\tau(g) \in T_g(gH) = \Delta(g) \ \forall g \in G$. Hence $\tau \in \Delta$. Since $\Delta$ is invariant under $X$, $[X,\tau](e) \in \Delta(e) = T_eH$. Because the Lie bracket of two left-invariant vector fields is left-invariant, the fact that $[X,\tau](e) \in T_eH$ implies that $[X,\tau] \in \text{Lie}(H)$. Therefore, $\text{Lie}(H)$ is invariant under $X$.

($\Leftarrow$) Let $\text{Lie}(H)$ be invariant under $X$ and let $\tau \in \Delta$. We know $\forall g \in G$, there exists a neighbourhood $U_g$ of $g$ such that $\tau|_{U_g} \in \text{span}_{C^\infty(U_g)}(\text{Lie}(H))$. In other words, $\tau|_{U_g} = f^iX_i$
where \( f^i \in C^\infty(U_g) \) and \( X_i \in \text{Lie}(H) \). Therefore\(^1\)

\[
[X, \tau](g) = [X, f^i X_i](g) = f^i(g)[X, X_i](g) + (L_X f^i)(g)X_i(g).
\]

Since \([X, X_j], X_j \in \text{Lie}(H)\), we have \([X, X_j](g), X_j(g) \in T_g(gH)\) for each \( j \). So, \([X, \tau](g) \in T_g(gH) = \Delta(g) \forall g \in G\) which implies \([X, \tau] \in \Delta\). Therefore, \( \Delta \) is invariant under \( X \). \( \square \)

Proposition 3.6.1 provides more intuition to the problem than Theorem 3.4.1. Rather than checking the invariance property of the distribution \( \Delta \), in this case the existence of the quotient control system is depended on the family of vector fields \( \text{Lie}(H) \); this is a more insightful aspect of the problem. In addition, since \( \text{Lie}(H) \) is a finite dimensional vector space, one only needs to compute \( n \) Lie brackets where \( n = \text{dim}(H) \). With this proposition, we restate the conditions for the existence of a quotient control system.

**Theorem 3.6.2.** If \( G/H \) is defined using left cosets of \( H \), a left-invariant control system given by (3.1) admits a unique quotient control system on \( G/H \) given by (3.3) if and only if \( \text{Lie}(H) \) is invariant under \( X_j \) for \( j = 0, \ldots, m \).

**Example 3.6.3.** In this example, we make use of Proposition 3.6.1 to illustrate the projectability of vector fields for a specific case. We take \( G = \text{SL}(3) \), the group of \( 3 \times 3 \) matrices with determinant 1. We consider the stabilization of a left coset of the Heisenberg subgroup \( H \) defined as the subgroup of \( G \) consisting of \( 3 \times 3 \) upper triangular matrices with all diagonal entries equal to 1.

Since \( G \) and \( H \) are matrix Lie groups, their Lie algebras of left-invariant vector fields are simply a Lie algebra of matrices with the the matrix commutator. It can be shown that \( \text{Lie}(G) \) is the set of \( 3 \times 3 \) traceless matrices and \( \text{Lie}(H) \) is the set of \( 3 \times 3 \) upper triangular matrices with all diagonal entries equal to zero.

\(^1\)The second equality follows from the identity \([X, \lambda Y] = \lambda[X, Y] + (L_X \lambda)Y\), where \( X, Y \) are vector fields and \( \lambda \) is a smooth real-valued function.
We determine the vector fields in $\text{Lie}(G)$ that can be projected onto the quotient space $G/H$ defined using left cosets. By Proposition 3.6.1, a vector field $A \in \text{Lie}(G)$ can be projected if and only if $[A, B] \in \text{Lie}(H) \ \forall B \in \text{Lie}(H)$.

We let

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & b_{12} & b_{13} \\
0 & 0 & b_{23} \\
0 & 0 & 0
\end{bmatrix}.$$  

where $a_{11} + a_{22} + a_{33} = 0$. To determine the projectable matrices $A$, we compute $[A, B]$ for the above matrices:

$$\begin{bmatrix}
-a_{21}b_{12} - a_{31}b_{13} & a_{11}b_{12} - a_{22}b_{12} - a_{32}b_{13} & a_{11}b_{13} + a_{12}b_{23} - a_{23}b_{12} - a_{33}b_{13} \\
-a_{31}b_{23} & a_{21}b_{12} - a_{32}b_{23} & a_{21}b_{13} + a_{22}b_{23} - a_{33}b_{23} \\
0 & a_{31}b_{12} & a_{31}b_{13} + a_{32}b_{23}
\end{bmatrix}.$$  

For this matrix to be in $\text{Lie}(H)$ for all values of $b_{12}, b_{13}, b_{23}$, we must have $a_{21} = a_{31} = a_{32} = 0$. Therefore, the subset of matrices in $\text{Lie}(G)$ that project onto $G/H$ via $\pi_*$ is

$$\left\{ \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & 0 & a_{33}
\end{bmatrix} : a_{11} + a_{22} + a_{33} = 0 \right\}.$$  

### 3.6.2 Target Set is a Right Coset of $H$

Now we consider the case where the target set is a right coset of $H$. We state the result as a theorem without proof because this result is a special case of Theorem 4.4.3 in the next chapter for which a proof is given.

**Theorem 3.6.4.** If $G/H$ is defined using right cosets of $H$, then a left-invariant control system given by (3.1) always admits a unique quotient control system on $G/H$ given by (3.3).
While surprising, this result exploits the symmetry in the problem; namely, the use of left translations providing invariance to the vector fields as well as producing the cosets defining the quotient space. We use this theorem to solve the set stabilization equivalence problem for the kinematic unicycle example further into this chapter.

3.7 Transversal/Tangential Split of Control Vector Fields

While a feedback transformation cannot weaken the conditions in Theorem 3.4.1 for the existence of a quotient control system, they can be useful for constructing a feedback equivalent left-invariant control system with more intuitive properties for the problem at hand. We investigate feedback transformations that provide additional structure to the quotient control system; primarily, feedback transformations that split the control vector fields into two sets. The first set contains the vector fields that are everywhere tangential to the cosets that define the quotient space and the second sets contains the vector fields that are transversal to these cosets. Essentially, this allows for the decoupling of the control effort that provides motion along cosets from the control effort that provides motion between cosets.

As previously mentioned, mapping a vector field on $G$ onto $G/H$ via $\pi_*$ effectively removes the components of the vector field that are tangential to the cosets. Thus, a vector field that is everywhere tangential to cosets will be mapped to the zero vector field on $G/H$ and hence will not appear in the quotient control system. To this end, we seek a constant feedback transformation such that the resulting feedback equivalent left-invariant control system given in (3.4) has the property that for some $k \in \{0, \ldots, m\}$

(i) $\pi_*\tilde{X}_{k+1}, \ldots, \pi_*\tilde{X}_m = 0,$

(ii) $\pi_*\bar{X}_1, \ldots, \pi_*\bar{X}_k$ are linearly independent on $G.$
Property (ii) ensures that the span of the vector fields $\tilde{X}_1, \ldots, \tilde{X}_k$ does not contain any direction tangential to the cosets, thereby fully splitting the vector fields to tangential and transverse vector fields.

The following theorem provides the necessary and sufficient conditions for such a construction. We note that this result does not depend on whether the quotient space is defined using left or right cosets. The reader is reminded that $\Delta(g) = \ker(\pi_*(g))$. We also denote $\mathcal{X}(p) = \text{span}(X_1(p), \ldots, X_m(p))$.

**Theorem 3.7.1.** Given a left-invariant control system, there exists a constant, regular feedback transformation $u = \alpha + \beta v$ such that for some $k \in \{0, \ldots, m\}$ the resulting feedback equivalent, left-invariant control system satisfies properties (i) and (ii) above if and only if

$$(\forall g \in G) \quad \mathcal{X}(g) \cap \Delta(g) = (L_g)_*(\mathcal{X}(e) \cap \Delta(e))$$

**Proof.** ($\Rightarrow$) Let $u = \alpha + \beta v$ be a constant, regular feedback transformation such that $\tilde{X}_1, \ldots, \tilde{X}_m$ have the above properties. We show $\mathcal{X}(g) \cap \Delta(g) = \text{span}\{\tilde{X}_{k+1}(g), \ldots, \tilde{X}_m(g)\}$ $\forall g \in G$.

By property (ii) and since $\Delta(g) = \ker(\pi_*(g))$, we have $\text{span}\{\tilde{X}_{k+1}(g), \ldots, \tilde{X}_m(g)\} \subset \mathcal{X}(g) \cap \Delta(g)$ since $\Delta(g) = \ker(\pi_*(g))$. For converse inclusion, let $v \in \mathcal{X}(g) \cap \Delta(g)$. We can write $v = \sum_{i=1}^m a_i \tilde{X}_i(g)$. Since $v \in \ker(\pi_*(g))$, we have

$$\pi_*(g)(v) = 0$$

$$\Rightarrow \sum_{i=1}^m a_i \pi_*(g)(\tilde{X}_i(g)) = 0$$

$$\Rightarrow \sum_{i=1}^k a_i \pi_*(g)(\tilde{X}_i(g)) = 0$$

Since $\pi_*(g)(\tilde{X}_1(g)), \ldots, \pi_*(g)(\tilde{X}_k(g))$ are linearly independent, this implies that $a_1, \ldots, a_k = 0$ and hence $v \in \text{span}\{\tilde{X}_{k+1}(g), \ldots, \tilde{X}_m(g)\}$. Therefore, $\mathcal{X}(g) \cap \Delta(g) = \text{span}\{\tilde{X}_{k+1}(g), \ldots, \tilde{X}_m(g)\}$. 


We have $\forall g \in G$

$$\mathcal{X}(g) \cap \Delta(g) = \text{span}\{\bar{X}_{k+1}(g), \ldots, \bar{X}_m(g)\}$$

$$= (L_g)_*(\text{span}\{\bar{X}_{k+1}(e), \ldots, \bar{X}_m(e)\})$$

$$= (L_g)_*(\mathcal{X}(e) \cap \Delta(e))$$

$(\Leftarrow)$ Let $\mathcal{X}(g) \cap \Delta(g) = (L_g)_*(\mathcal{X}(e) \cap \Delta(e)) \forall g \in G$. Let $Y_1, \ldots, Y_m$ be a basis of $\mathcal{X}(e)$ such that for some $k \in \{0, \ldots, m\}$ $Y_{k+1}, \ldots, Y_m$ is a basis of $\mathcal{X}(e) \cap \text{ker}(\pi^*(e))$. Define the left-invariant vector fields $\bar{X}_j$ by $\bar{X}_j(g) = (L_g)_* Y_j$ for $j = 1, \ldots, m$. We know there exists $\beta \in \text{GL}(m, \mathbb{R})$ such that $\bar{X}_j = \sum_{i=1}^m \beta_{ji} X_i$ for $j = 1, \ldots, m$. Hence, the left-invariant control system given by $\dot{g} = X_0(g) + \sum_{i=1}^m v_i \bar{X}_i(g)$ is feedback equivalent to the original system under the constant, regular feedback transformation $u = \beta v$. We show that this is the feedback transformation sought in the theorem statement.

Since for each $j \in \{k+1, \ldots, m\}$ and $\forall g \in G$

$$\bar{X}_j(e) = Y_j \in \mathcal{X}(e) \cap \text{ker}(\pi_*(e))$$

$$\Rightarrow \bar{X}_j(g) \in (L_g)_*(\mathcal{X}(e) \cap \text{ker}(\pi_*(e))) = \mathcal{X}(g) \cap \text{ker}(\pi_*(g)),$$

we have $\pi_* \bar{X}_{k+1}, \ldots, \pi_* \bar{X}_m = 0$.

By property $(ii.)$, $\mathcal{X}(g) \cap \text{ker}(\pi_*(g))$ has constant dimensional $\forall g \in G$ and hence

$$\dim(\mathcal{X}(g) \cap \text{ker}(\pi_*(g))) = \dim(\mathcal{X}(e) \cap \text{ker}(\pi_*(e))) = m - k.$$

Since $\{\bar{X}_{k+1}(g), \ldots, \bar{X}_m(g)\}$ is a collection of $m - k$ linearly independent vectors that are in $\mathcal{X}(g) \cap \text{ker}(\pi_*(g))$, it form a basis for $\mathcal{X}(g) \cap \text{ker}(\pi_*(g))$.

We show $\text{span}\{\bar{X}_1(g), \ldots, \bar{X}_k(g)\} \cap \text{ker}(\pi_*(g)) = \{0\}$. Let $v \in \text{span}\{\bar{X}_1(g), \ldots, \bar{X}_k(g)\} \cap \text{ker}(\pi_*(g))$. Since $\text{span}\{\bar{X}_1(g), \ldots, \bar{X}_k(g)\} \subset \mathcal{X}(g)$, $v \in \mathcal{X}(g) \cap \text{ker}(\pi_*(g))$. Hence, $v$ can
be written as a linear combination of $\bar{X}_{k+1}(g), \ldots, \bar{X}_m(g)$ as well as a linear combination of $\bar{X}_1(g), \ldots, \bar{X}_k(g)$. Since $\bar{X}_1(g), \ldots, \bar{X}_m(g)$ are linearly independent, $v = 0$.

Lastly, we show $\pi^*(g)(\bar{X}_1(g)), \ldots, \pi^*(g)(\bar{X}_k(g))$ are linearly independent $\forall g \in G$. Let

$$\sum_{i=1}^k a_i \pi^*(g)(\bar{X}_i(g)) = 0$$

for some $a_j \in \mathbb{R}$ for $j = 1, \ldots, k$. By linearity of $\pi^*(g)$ we have $\pi^*(g)\left(\sum_{i=1}^k a_i \bar{X}_i(g)\right) = 0$, which implies that $\sum_{i=1}^k a_i \bar{X}_i(g) \in \ker(\pi^*(g))$. But since $\text{span}\{\bar{X}_1(g), \ldots, \bar{X}_k(g)\} \cap \ker(\pi^*(g)) = \{0\}$, we have $\sum_{i=1}^k a_i \bar{X}_i(g) = 0$. Since $\bar{X}_1(g), \ldots, \bar{X}_k(g)$ are linearly independent this implies that $a_j = 0$ for $j = 1, \ldots, k$. Hence $\pi^*(g)(\bar{X}_1(g)), \ldots, \pi^*(g)(\bar{X}_k(g))$ are linearly independent $\forall g \in G$. \hfill \Box

The existence of such a feedback transformation inherently requires the distribution $\mathcal{X}(g) \cap \Delta(g)$, which defines the control directions that are tangent to cosets, to have the same dimension at every point. In addition, the theorem requires this distribution to be left-invariant. This requirement ensures that it can be spanned by a finite number of left-invariant vector fields that are pointwise linearly independent.

When the target set is a left coset of $H$, the condition in Proposition 3.7.1 is always true as

$$\langle L_g \rangle_*(\mathcal{X}(e) \cap \Delta(e)) = \langle L_g \rangle_*(\mathcal{X}(e) \cap T_e H) = \mathcal{X}(g) \cap T_g (gH) = \mathcal{X}(g) \cap \Delta(g).$$

When the target set is a right coset of $H$, the condition becomes quite restrictive. Using the fact that $\Delta(g) = T_g (H g)$, we have

$$\langle \forall \ g \in G \rangle \ \mathcal{X}(e) \cap \Delta(e) = \langle L_{g^{-1}} \rangle_*(\mathcal{X}(g) \cap \Delta(g))$$
Since the vector fields $X_i$ are left invariant, we have $(L_{g^{-1}})_*X_i(g) = X_i(e)$. Moreover, 
$$\Delta(g) = T_g(Hg) = T_g(R_g(H)) = (R_g)_*T_eH = (R_g)_*(\Delta(e)).$$ Since $C_{g^{-1}}(h) = g^{-1}hg = L_{g^{-1}} \circ R_g$, we have
$$(L_{g^{-1}})_*(\Delta(g)) = (L_{g^{-1}})_*(R_g)_*(\Delta(e)) = (C_{g^{-1}})_*(\Delta(e)).$$

In conclusion, when the target is a right coset of $H$ the condition in Theorem 3.7.1 becomes
$$(\forall g \in G) \mathcal{X}(e) \cap \Delta(e) = \mathcal{X}(e) \cap (C_{g^{-1}})_*(\Delta(e)).$$

A sufficient, but not necessary condition for the above to hold is that
$$(\forall g \in G) \ T_eH = (C_g)_*T_eH.$$ This occurs, for instance, when $H$ is a normal subgroup (see page 21), since in this case $C_g(H) = H$ for all $g$ implies that $(C_g)_*T_eH = T_eH$.

It can be shown that this vector field split cannot be performed for the kinematic unicycle example.

**Example 3.7.2.** We return to Example 3.6.3. Consider the left-invariant control system on $SL(3)$ given by
$$\dot{g} = gX_0 + \sum_{i=1}^3 u_i g X_i,$$
where
$$X_0 = \begin{bmatrix} -3 & -2 & -6 \\ 0 & -1 & -1 \\ 0 & 0 & 4 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix},$$
By Example 3.6.3, all of the above vector fields can be projected onto $G/H$. Note that $X_1, X_2, X_3$ are linearly independent and are not in $\text{Lie}(H)$, implying that the quotient control system will have three control vector fields. However, the projected vector fields are not linearly independent as the intersection of $\mathcal{X}(e)$ and $\text{Lie}(H)(e)$ is nonempty. We can solve for this intersection by determining the elements in $\mathcal{X}(e)$ that have zeros along the diagonal. Doing so, we obtain

$$\mathcal{X}(e) \cap \text{Lie}(H)(e) = \{aX_1 - aX_2 - aX_3 : a \in \mathbb{R}\}$$

This is a one-dimensional subspace of $\mathcal{X}(e)$. Since $G$ is a matrix group, $(L_g)_*$ is simply left multiplication by $g$ and hence

$$(L_g)_*(\mathcal{X}(e) \cap \text{Lie}(H)(e)) = (L_g)_* \{aX_1 - aX_2 - aX_3 : a \in \mathbb{R}\} = \{agX_1 - agX_2 - agX_3 : a \in \mathbb{R}\} = \mathcal{X}(g) \cap \text{Lie}(H)(g)$$

Thus, it is possible to perform the vector field split outlined in Theorem 3.7.1; it will produce one tangential vector field. We now construct the feedback transformation $u = \beta v$ that performs this split. Recall that under this transformation we will obtain the system given by

$$\dot{g} = g\bar{X}_0 + \sum_{i=1}^{3} v_i g \bar{X}_i.$$ 

From the set $\mathcal{X}(e) \cap \text{Lie}(H)(e)$ given above, we require $\beta_{31} = -\beta_{32} = -\beta_{33}$ to ensure $\bar{X}_3$ is tangential to $\text{Lie}(H)(e)$. The remaining elements of $\beta$ are chosen so that $\beta$ is invertible.
With regard to this, we take
\[
\beta = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 1 & 1
\end{bmatrix}.
\]

Using this feedback transformation, we obtain
\[
\begin{align*}
\bar{X}_0 &= X_0, \\
\bar{X}_1 &= X_1, \\
\bar{X}_2 &= X_2, \\
\bar{X}_3 &= \begin{bmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

We have \(\bar{X}_1, \bar{X}_2 \notin \text{Lie}(H)\), but \(\bar{X}_3 \in \text{Lie}(H)\). Hence, \(\bar{X}_3\) is a tangential vector field; thus, the quotient control system for this feedback equivalent system will have two linearly independent control vector fields.

### 3.8 Equivalence of the Kinematic Unicycle Problem

We return to the problem of stabilizing three unicycles in an equilateral triangle formation and recover a result found in [9]. We have found that the set
\[
S^* = \{(g_1, g_2, g_3) \in G : g_1^{-1}g_2 = M, \ g_2^{-1}g_3 = M\},
\]
where \(M \in \text{SE}(2)\), is the target set.

Consider the subgroup \(H = \{(h, h, h) \in G : h \in \text{SE}(2)\}\). This subgroup is obviously closed in \(G\) and its right coset given by \(H(I, M, M^2)\) is equal to the set \(S^*\). Hence, we have brought this set stabilization problem into the scope of this chapter. We define \(G/H\) using right cosets of this subgroup. The right coset of a point \((g_1, g_2, g_3)\) is the set
\[
H(g_1, g_2, g_3) = \{(hg_1, hg_2, hg_3) : h \in \text{SE}(2)\},
\]
which can be rewritten as

\[ H(g_1, g_2, g_3) = \{(h_1, h_2, h_3) \in G : h_1^{-1}h_2 = g_1^{-1}g_2, \ h_2^{-1}h_3 = g_2^{-1}g_3\}. \]

Since the right coset in question is completely characterized by the pair \((g_1^{-1}g_2, g_2^{-1}g_3) \in (\text{SE}(2))^2\), we may identify \(G/H\) with \(\text{SE}(2) \times \text{SE}(2)\), and the quotient map is given by \(\pi(g_1, g_2, g_3) = (g_1^{-1}g_2, g_2^{-1}g_3)\). Note that \(\pi\) projects the configuration of the unicycles to their relative translation and rotation. The point to be stabilized in \(G/H\) to solve this set stabilization problem is \(\pi(S^*) = (M, M)\).

We remark that the choice of \(H\) above is not coincidental in that it ensures that the quotient map \(\pi\) projects a given configuration of the unicycles to their relative orientation.

By Theorem 3.6.4, since the control system is left-invariant and the target set is a right coset of \(H\), the quotient control system is well defined. In other words, the dynamics can be written entirely in terms of the relative translations and rotations. Letting \((q_1, q_2) = \pi(g_1, g_2, g_3) = (g_1^{-1}g_2, g_2^{-1}g_3)\), we have

\[
\dot{q}_1 = g_1^{-1}g_2 = -g_1^{-1}g_1^{-1}g_2 + g_1^{-1}\dot{g}_2 \\
= -g_1^{-1}(g_1X_0 + u_1g_1X_1)g_1^{-1}g_2 + g_1^{-1}(g_2X_0 + u_2g_2X_1) \\
= -X_0q_1 + q_1X_0 - u_1X_1q_1 + u_2q_1X_1.
\]

The \(q_2\) dynamics can be derived in a similar fashion to form the control system on \(G/H\) given by

\[
\dot{q}_1 = -X_0q_1 + q_1X_0 - u_1X_1q_1 + u_2q_1X_1 \\
\dot{q}_2 = -X_0q_2 + q_2X_0 - u_2X_1q_2 + u_3q_2X_1.
\]

Subsequently, stabilizing the unicycles in an equilateral triangle formation is equivalent to stabilizing the equilibrium \((M, M)\) in \(G/H\) for the quotient control system. We do
not provide a controller that performs this stabilization; with onboard sensors on each unicycle, a controller can be designed. The goal of this example was to illustrate the equivalence of the set stabilization problem to an equilibrium stabilization problem.

We remark that the quotient control system presented above has already been presented in [9] where the authors used a different, but conceptually equivalent, approach to defining the quotient space.

3.8.1 Summary

The main goal of this chapter was to show the equivalence of a coset stabilization problem for a left-invariant system on a Lie group to an equilibrium stabilization problem of a control-affine system on a manifold. We constructed an appropriate manifold that reduces the target set to a point, and used a result in differential geometry (Theorem 2.2.15) to get necessary and sufficient conditions for the existence of a unique quotient control system such that the stabilization of one of its points is equivalent to the original set stabilization problem. We also derived an equivalent Lie algebraic condition.

We investigated the use of feedback transformations. While feedback equivalence cannot weaken the conditions for the existence of a quotient control system, it can be used to split the control vector fields that are always tangent to cosets from the ones that are never tangent to cosets. We applied this theory to an academic example involving a left-invariant system on SL(2) as well as the problem of stabilization three unicycles in an equilateral triangle formation, recovering in this latter case a result in [9].

The motivation for the setup investigated in this chapter, namely a left-invariant control system on a Lie group $G$ and a target set given by a left or right coset of a Lie subgroup $H$, was the result in Theorem 2.5.1, stating that the quotient space $G/H$ is always a manifold. In the next chapter we leverage the more general Quotient Manifold Theorem (Theorem 2.5.2) to move away from the requirement that the state space of the control system be a Lie group.
Chapter 4

Orbit Stabilization on Manifolds with Lie Group Actions

4.1 Introduction

The goal of this chapter is to further the study of set stabilization to the case where the state space is a general manifold rather than a Lie group. In a similar fashion, we derive conditions for the existence of a quotient control system that describes the motion of the system transversal to the target set.

We will investigate systems that are invariant under a group action in the sense of Definition 2.6.4. While preliminary results in this chapter do not make use of this invariance, later insightful results on topics such as accessibility will depend on it. We do not provide examples of these concepts within this chapter. Rather, they will be applied through examples in Chapter 5. In summary, this chapter considers control systems of the form

\[ \dot{p} = X_0(p) + \sum_{i=1}^{m} u_i X_i(p), \]  

(4.1)
evolving on a manifold \( M \) that are invariant under a transitive and free Lie group action \( \psi : G \times M \rightarrow M \). The target set is an orbit of a Lie group action \( \hat{\theta} \) on \( M \). Note that the
two referred actions may not be equal.

### 4.2 Problem Statement

As mentioned, we are extending the problem from the previous chapter to include the case of arbitrary Lie group actions on a manifold. To do so, we must analyze the previous problem statement in detail. Let us consider the problem of stabilizing a right coset of a subgroup $H$ in a Lie group $G$. While right cosets are related to the action of left translations $L$ on $G$, which is free and transitive, they are not the orbits of $L$. Rather, they are the orbits of $L$ restricted to the action of $H$ on $M$. Extending this to the case of an arbitrary action, we require a transitive and free Lie group action $\theta : G \times M \to M$ such that the target set is an orbit of $\hat{\theta} \triangleq \theta |_{H \times M}$, where $H$ is a closed subgroup $G$. This is the basis for the set stabilization problem in this chapter. We point out that, for simplicity, we denote an orbit of $\hat{\theta}$ through a point $p$ by $\theta_H(p)$ rather than $\hat{\theta}_H(p)$.

We now present the problem studied in this chapter.

**Orbit Stabilization Problem:** Let $\theta : G \times M \to M$ and $\psi : G \times M \to M$ be two transitive and free Lie group actions on $M$ and suppose that the control system (4.1) is invariant under $\psi$. Let $\hat{\theta} \triangleq \theta |_{H \times G}$ and assume that $\hat{\theta}$ is proper (it is also free because $\theta$ is free). Given a controlled invariant orbit $S^*$ of $\hat{\theta}$, find a control law stabilizing $S^*$ in $M$.

As with the previous chapter, our approach is to look for a quotient control system such describing the dynamics transversal to the orbit $S^*$.

### 4.3 Quotient Control System

By the quotient manifold theorem (Theorem 2.5.2), the orbit space $M/H$ is a manifold and the quotient map $\pi : M \to M/H$ is a smooth submersion. As in the previous chapter, we want to define a system on $M/H$ that describes the motion of (4.1) transversal to the
orbits of $\hat{\theta}$. As we argued previously, if the quotient control system exists, it is given by

$$\dot{q} = \tilde{X}_0(q) + \sum_{i=1}^{m} u_i \tilde{X}_i(q), \quad (4.2)$$

where $\tilde{X}_j \circ \pi = \pi_* X_j$ for $j = 0, \ldots, m$. In order to find conditions under which a vector field on $M$ can be projected onto $M/H$ via $\pi_*$ we need to use Theorem 2.2.15. To this end, we need a nonsingular, involutive distribution $\Delta$ whose integral manifolds are the orbits of $\hat{\theta}$. This distribution is obviously

$$\Delta(p) = T_p(\theta_H(p)) = \ker(\pi_*)(p).$$

We thus have the following.

**Theorem 4.3.1.** The $\psi$-invariant control system (4.1) admits a unique quotient control system on $M/H$ given by (4.2) if and only if $\Delta$ is invariant under $X_j$ for $j = 0, \ldots, m$.

While this theorem does not require $\psi$-invariance, we make use of this property for further study of this problem. Lastly, we note that, similar to the Lie group case, we cannot use a regular feedback transformation to weaken the conditions of Theorem 4.3.1 since $\psi$-invariance is only maintained through constant feedback transformations.

### 4.4 Lie Algebraic Projectability Conditions

In this section, we derive equivalent conditions for the existence of a quotient control system similar in spirit to those in Theorem 3.6.2. Specifically, we aspire to get a condition equivalent to Theorem 4.3.1, but involving a family of vector fields rather than the distribution $\Delta$.

Let $\Phi : \text{Lie}(G) \to \mathfrak{X}^\infty(M)$ be the Lie algebra antihomomorphism induced by the action $\theta : G \times M \to M$ defined as in Section 2.6.5. We have the following result.
Lemma 4.4.1. If $v \in \text{Lie}(H)$, then $\Phi(v) \in \Delta$.

Proof. Let $p \in M$. By the definition of $\Phi$, the flow of $\Phi(v)$ with initial condition $p$ is given by $t \mapsto \theta(\exp(tv), p)$. Since $v \in \text{Lie}(H)$, $\exp(tv) \in H \ \forall t \in \mathbb{R}$. This implies that $\theta(\exp(tv), p) \in \theta_H(p) \ \forall t \in \mathbb{R}$. Therefore, $\theta_H(p)$ is invariant under $\Phi(v)$. Since $\theta_H(p)$ is a closed embedded submanifold of $M$, the set $\theta_H(p)$ is invariant under $\Phi(v)$ if and only if $\Phi(v)$ is tangent to $\theta_H(p)$, i.e., $\Phi(v)(p) \in T_p(\theta_H(p)) = \Delta(p)$.

Theorem 4.4.2. A $\psi$-invariant control system given by (4.1) admits a unique quotient control system on $M/H$ given by (4.2) if and only if $[X_j, \Phi(\text{Lie}(H))] \subset \Delta$ for $j = 0, \ldots, m$.

Proof. ($\Rightarrow$) Obvious given Theorem 4.3.1 and the fact that $\Phi(\text{Lie}(H)) \subset \Delta$.

($\Leftarrow$) Let $[X_j, \Phi(\text{Lie}(H))] \subset \Delta$ for each vector field $X_j$ in (4.1). Let $\tau$ be a vector field in $\Delta$ and $p \in M$. Since the dimension $\Phi(\text{Lie}(H))$ is equal to the dimension of $\Delta$, the vector fields in $\Phi(\text{Lie}(H))$ generate $\Delta$. This implies that in a neighbourhood $U_p$ of $p$, $\tau = f^i(Y_i|_{U_p})$ where $Y_1, \ldots, Y_k \in \Phi(\text{Lie}(H))$ and $f^1, \ldots, f^k \in C^\infty(U_p)$. Therefore

$$[X_j, \tau](p) = [X_j, f^i Y_i](p) = f^i(p)[X_j, Y_i](p) + (L_{X_j} f^i)(p)Y_i(p).$$

Since $[X_j, Y_i], Y_i \in \Delta$, we have $[X_j, \tau](p) \in \Delta(p)$ for each $j$. Therefore by Theorem 4.3.1, the quotient control system given by (4.2) exists.

We were not able to remove $\Delta$ entirely from the projectability conditions. It is obvious that if $[X_j, \Phi(\text{Lie}(H))] \subset \Phi(\text{Lie}(H))$ for each $j$, then the quotient control system exists. However, this condition is not necessary since $[X_j, \Phi(\text{Lie}(H))]$ may not be a vector field in $\Phi(\text{Lie}(G))$.

\footnote{See Footnote on page 44.}
Next, we generalize Theorem 3.6.4 and make use of the invariance property of the control system to guarantee the existence of the quotient control system.

**Theorem 4.4.3.** If \( \psi = \theta \), then the quotient control system given by (4.2) exists.

**Proof.** Since \( \psi = \theta \), the vector fields \( X_0, \ldots, X_m \) are \( \theta \)-invariant. Therefore, to prove this theorem it suffices to show that any \( \theta \)-invariant vector field can be projected onto \( M/H \) via \( \pi_* \). Let \( X \) be a \( \theta \)-invariant vector field. Then, \( \forall t \in \mathbb{R}, \theta_H(\phi^X_t(p)) = \phi^X_t(\theta_H(p)) \). Since the orbits of \( \hat{\theta} \) are integral manifolds of \( \Delta \), we have \( \hat{\theta}_H(\phi^X_t(p)) = \theta_H(\phi^X_t(p)) = I^\Delta_{\phi^X_t(p)} \) and \( \phi^X_t(\theta_H(p)) = \phi^X_t(\hat{\theta}_H(p)) = \phi^X_t(I^\Delta_{\theta_H(p)}) \). Therefore, for all \( p \in M \) and all \( t \in \mathbb{R}, I^\Delta_{\phi^X_t(p)} = \phi^X_t(I^\Delta_{\theta_H(p)}) \). By Theorem 2.2.14, \( \Delta \) is invariant under \( X \).

In the case of stabilizing a right coset for a left-invariant system on a Lie group \( G \), we have \( \theta = \psi = L \). Hence, Theorem 3.6.4 is a special case of Theorem 4.4.3.

### 4.5 Tangential/Transversal Split of the Control Vector Fields

We continue by providing a result analogous to Theorem 3.7.1 for the case of a \( \psi \)-invariant control system. In this case, we wish to separate the control vector fields that are tangential to orbits from the ones that are transversal using a constant, regular feedback transformation. The following theorem provides the necessary and sufficient conditions for such a construction. The reader is reminded that \( \Delta(p) = \ker(\pi_*(p)) \) and \( \mathcal{X}(p) = \text{span}(X_1(p), \ldots, X_m(p)) \). Also, \( \bar{X}_j \) for \( j = 1, \ldots, m \) are the control vector fields of the \( \psi \)-invariant control system created after the application of the feedback transformation.

**Theorem 4.5.1.** Let \( p_0 \) be any point on \( M \). Given a \( \psi \)-invariant control system, there exists a constant, regular feedback transformation \( u = \alpha + \beta v \) such that for some \( k \in \{0, \ldots, m\} \) the resulting feedback equivalent, \( \psi \)-invariant control system satisfies:
(i) \( \pi_* \bar{X}_{k+1}, \ldots, \pi_* \bar{X}_m = 0 \)

(ii) \( \pi_* \bar{X}_1, \ldots, \pi_* \bar{X}_k \) are linearly independent on \( M \),

if and only if

\[
(\forall g \in G) \ X(\psi_g(p_0)) \cap \Delta(\psi_g(p_0)) = (\psi_g)_*(X(p_0) \cap \Delta(p_0)).
\]

The proof of this theorem is omitted due to its similarity to the proof of Theorem 3.7.1.

### 4.6 Accessibility of the Quotient Control System

Accessibility of a control system is a key concept since it is linked to controllability. In fact, it is a necessary condition for controllability. Controllability ensures the existence of a control that transfers the state of the system between two points in finite time. Let us consider this concept in the kinematic unicycle example given in Chapter 4; in particular, the problem of transferring the unicycles from an equilateral triangle formation to a straight line formation in finite time. In each of these formations, the relative translation and rotation between the unicycles is constant. This implies these formations are represented by distinct points on the quotient space. Therefore, finding a control that performs this transfer is tied to the controllability of the quotient control system. This is our topic of interest.

While the system in (4.1) admits invariance properties, the quotient control system, in most cases, will not share in these properties; it will be a general control affine system. Controllability results for control affine systems are limited to various sufficient conditions applicable to special cases. For this reason, we investigate the more manageable problem of accessibility, presenting conditions for the system given in (4.1) that result in the accessibility of the quotient control system given in (4.2). This will provide necessary conditions for the quotient control system to be controllable.
To study this problem of accessibility, we desire conditions for which the quotient control system satisfies the Lie algebra rank condition (LARC) defined in Section 2.7. As noted in Theorem 2.7.6, the LARC is a sufficient condition for accessibility of a smooth control system. It is also necessary if the control system is analytic. Consequently, the results in this section are at least sufficient for accessibility.

We begin by studying the control Lie algebra $\tilde{\mathcal{A}}$ of the quotient control system. By definition, $\tilde{\mathcal{A}}$ is the set of all $\mathbb{R}$-linear combinations of the family of vector fields

$$\left\{ [[\tilde{X}_{i_1}, \tilde{X}_{i_2}], \tilde{X}_{i_3}], \ldots, \tilde{X}_{i_k}] : k \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_k \in \{0, \ldots, m\} \right\}.$$ 

We can use the $\pi$-relatedness of the vector fields $\tilde{X}_0, \ldots, \tilde{X}_m$ to $X_0, \ldots, X_m$ and properties of the Lie bracket to relate $\tilde{\mathcal{A}}$ to the control Lie algebra of (4.1), denoted by $\mathcal{A}$. We have

$$\tilde{\mathcal{A}} = \text{l.c. of } \left\{ [[\pi_* X_{i_1}, \pi_* X_{i_2}], \pi_* X_{i_3}], \ldots, \pi_* X_{i_k}] : k \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_k \in \{0, \ldots, m\} \right\}.$$ 

But since, by Theorem 2.2.7, $[\pi_* X_{i_j}, \pi_* X_{i_k}] = \pi_* [X_{i_j}, X_{i_k}]$, we have

$$\tilde{\mathcal{A}} = \text{l.c. of } \left\{ \pi_* [[X_{i_1}, X_{i_2}], X_{i_3}], \ldots, X_{i_k}] : k \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_k \in \{0, \ldots, m\} \right\} = \pi_* \mathcal{A}.$$ 

Hence, the vector fields in $\tilde{\mathcal{A}}$ are simply the vector fields in $\mathcal{A}$ projected onto $M/H$ via $\pi_*$. We use this fact to produce a result relating accessibility of the quotient control system to the control Lie algebra $\mathcal{A}$. Note that $\mathcal{A}$ can be defined as a distribution on $M$ given by $\mathcal{A}(p) = \{ X(p) : X \in \mathcal{A} \}$.

**Theorem 4.6.1.** The following are equivalent:

(a) The quotient control system (4.2) satisfies the LARC.

(b) $\mathcal{A}(p) + \Delta(p) = T_p M \ \forall \ p \in M.$
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Proof. (a) ⇒ (b) Let (4.2) be accessible. By the LARC, we have ∀q ∈ M/H (π∗A)(q) = \( \tilde{A}(q)T_q(M/H) \). Since \( \pi \) is a submersion, \( T_q(M/H) = \pi_∗(T_pM) \) where \( p \in \pi^{-1}(q) \). Since \( (\pi∗A)(q) = \pi_∗(A(p)) \) and by the linearity of \( \pi_∗(p) \), we have that ∀\( p \in \pi^{-1}(q) \)

\[
(\pi∗A)(q) = T_q(M/H) \\
\Rightarrow \pi_∗(A(p)) = \pi_∗(T_pM) \\
\Rightarrow A(p) + \ker(\pi_∗(p)) = T_pM \\
\Rightarrow A(p) + \Delta(p) = T_pM. 
\]

(b) ⇐ (a) We have ∀p ∈ M

\[
A(p) + \Delta(p) = T_pM \\
\Rightarrow A(p) + \ker(\pi_∗(p)) = T_pM. 
\]

Applying \( \pi_∗(p) \) to both sides of the above equation gives

\[
\pi_∗(A(p)) = \pi_∗(T_pM) \\
\Rightarrow (\pi∗A)(q) = T_q(M/H). 
\]

Hence, the LARC for (4.2) is satisfied.

This result is intuitive. For (4.2) to be accessible, its control Lie algebra must contain all possible directions of motion on \( M/H \). These are the directions transversal to the orbits of \( \hat{\theta} \). Hence, for accessibility of (4.2), the control Lie algebra of (4.1) must contain all of the directions that are transversal to these orbits. If the sum of the control Lie algebra of (4.1) and the directions tangential to orbits equals all possible directions, then we have accessibility of (4.2). If not, this implies that the control Lie algebra of (4.1) does not contain all transversal direction and hence we do not have accessibility of (4.2).
Next, we develop a condition for the accessibility of the quotient control system that was inspired by the work in [3]. This paper restricts the problem to a right-invariant system on a matrix Lie group $G$ where the quotient control system is formed using left cosets of $G$ modulo a closed subgroup $H$. The following accessibility result is taken from [3].

**Theorem 4.6.2.** The following are equivalent:

(a) The quotient control system satisfies the LARC.

(b) $(\forall g \in G) \ A + (C_g)_*(\text{Lie}(H)) = \text{Lie}(G)$.

where Lie($H$) and Lie($G$) are Lie algebras of right-invariant vector fields. This results provides a necessary and sufficient condition for accessibility of the quotient control system that is based on families of vector fields.

We now generalize this result to the setup of this chapter. Theorem 4.6.2 requires the control system to be right-invariant as well as the orbits, in this case left cosets, to be formed using right translations. In our setting, this corresponds to the case where $\psi = \theta$. This will be also assumed for our extension of this theorem. To extend this theorem to the case of general Lie group actions, we assume the existence a transitive, free action $\bar{\theta}$ that commutes with $\theta$ in the sense that

$$
(\forall g_1, g_2 \in G) \ \theta_{g_1} \circ \bar{\theta}_{g_2} = \bar{\theta}_{g_2} \circ \theta_{g_1}.
$$

(4.3)

If such an action exists, then

$$
(\forall g \in G) \ (\forall p \in M) \ \bar{\theta}_g(\theta_H(p)) = \theta_H(\bar{\theta}_g(p)),
$$

and so $\bar{\theta}_g$ maps orbits of $\hat{\theta}$ to orbits of $\hat{\theta}$. As mentioned earlier, the action of a Lie group on itself by left or right translation enjoys property (4.3).
Theorem 4.6.2 relates accessibility to a condition involving families of vector fields on $G$. To extend this result, we need to define two families of vector fields that take the place of $\text{Lie}(H)$ and $\text{Lie}(G)$ in Theorem 4.6.2. Since $\text{Lie}(G)$ is the Lie algebra of left-invariant vector fields on $G$, it is natural to replace it with $\Theta(M) \triangleq \Phi(\text{Lie}(G))$, which is the Lie algebra of $\theta$-invariant vector field on $M$. To replace $\text{Lie}(H)$, we require a family of $\theta$-invariant vector fields that involve the distribution $\Delta$. Fixing an arbitrary $p_0 \in M$, we will see that the family of $\theta$-invariant vector fields $X$ such that $X(p_0) \in \Delta(p_0) = T_{p_0}(\theta_H(p_0))$ will make a good candidate. We denote this family by $\Theta_{p_0}(M)$. Note that this family is not necessarily closed under Lie brackets and hence may not be a Lie algebra. Finally, the conjugation map $C_g : G \rightarrow G$ in Theorem 4.6.2 will be replaced by the map $C_g : M \rightarrow M$ defined as

$$C_g(p) \triangleq \theta_{g^{-1}} \circ \bar{\theta}_g(p),$$

where $\bar{g}$ is chosen such that $\theta_g(p_0) = \bar{\theta}_g(p_0)$. Note that, for any $g \in G$ and $p \in M$, $\bar{g}$ exists since $\bar{\theta}$ is transitive. Since for each $p \in M$ different choices of $\bar{g}$ such that $\theta_g(p_0) = \bar{\theta}_g(p_0)$ yield the same value of $C_g(p)$, the map $C_g$ is well-defined. Moreover, $C_g$ is a diffeomorphism on $M$ with inverse $C_{g^{-1}} = \bar{\theta}_{g^{-1}} \circ \theta_g$. The following lemma is required for the extension of Theorem 4.6.2.

**Lemma 4.6.3.** If $X$ is a $\theta$-invariant vector field on $M$, then for all $g \in G$, $(C_g)_*X$ is also a $\theta$-invariant vector field on $M$.

**Proof.** Let $X$ be a $\theta$-invariant vector field on $M$ and let $g \in G$. The vector field $(C_g)_*X$ is given by $p \mapsto (C_g)_*X(C_{g^{-1}}(p))$. We show $(C_g)_*X$ is $\theta$-invariant by showing that

$$(\forall h \in G) (\theta_h)_* \circ (C_g)_*X(C_{g^{-1}}(p)) = (C_g)_*X(C_{g^{-1}}(\theta_h(p))).$$
We have
\[
(\theta_h)_* (C_g)_* X(C_g^{-1}(p)) = (\theta_h)_* (\theta_{g^{-1}})_* (\bar{\theta}_g)_* X(\bar{\theta}_{g^{-1}}(\theta_g(p)))
\]
\[
= (\bar{\theta}_g)_* (\theta_h)_* X(\bar{\theta}_{g^{-1}}(p))
\]
\[
= (\bar{\theta}_g)_* X(\theta_h(\bar{\theta}_{g^{-1}}(p)))
\]
\[
= (\bar{\theta}_g)_* (\theta_{g^{-1}})_* (\theta_g)_* X(\bar{\theta}_{g^{-1}}(\theta_h(p)))
\]
\[
= (\theta_{g^{-1}})_* (\bar{\theta}_g)_* X(\bar{\theta}_{g^{-1}}(\theta_h(p)))
\]
\[
= (C_g)_* X(C_g^{-1}(\theta_h(p))).
\]

Therefore, \( \forall g \in G, (C_g)_* X \) is \( \theta \)-invariant if \( X \) is \( \theta \)-invariant. This implies \( \forall g \in G, (C_g)_* (\Theta_{p_0}(M)) \) is a family of \( \theta \)-invariant vector fields on \( M \). We can now state the extension of Theorem 4.6.2.

**Theorem 4.6.4.** Fix \( p_0 \in M \) and let

\[
\Theta(M) = \Phi(\text{Lie}(G)), \quad \Theta_{p_0}(M) = \{ X \in \Theta(M) : X(p_0) \in \Delta(p_0) = T_{p_0}(\theta_H(p_0)) \}.
\]

Assume that there is a transitive, free action \( \bar{\theta} : G \to G \) such that (4.3) holds. Then, the following are equivalent:

(a) The quotient control system given by (4.2) satisfies the LARC.

(b) \( (\forall g \in G) \ A + (C_g)_* (\Theta_{p_0}(M)) = \Theta(M) \).

**Proof.** (a) \( \Rightarrow \) (b) Let \( g \in G \) and set \( p = \theta_g(p_0) \). Since the family vector fields given in (b) are all \( \theta \)-invariant, it suffices to prove the equality at the previously defined point
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$p_0 \in M$. By Theorem 4.6.1, accessibility of the quotient control system implies

$$\mathcal{A}(p) + \Delta(p) = T_p M$$

$$\Rightarrow \mathcal{A}(p) + T_p(\theta_H(p)) = T_p M$$

$$\Rightarrow \mathcal{A}(p) + T_p(\theta_H(p)) = \Theta(M)(p).$$

Since $\theta, \bar{\theta}$ are transitive, there exist $g, \bar{g} \in G$ such that $p = \theta_g(p_0) = \bar{\theta}_\bar{g}(p_0)$. Also, by property (4.3), $\bar{\theta}_g$ maps $\theta_H(p_0)$ onto $\theta_H(\bar{\theta}_\bar{g}(p_0)) = \theta_H(p)$. This implies that $(\bar{\theta}_g)_*$ maps $T_{p_0}(\theta_H(p_0))$ onto $T_p(\theta_H(p))$. Therefore,

$$\mathcal{A}(\theta_g(p_0)) + (\bar{\theta}_g)_* T_{p_0}(\theta_H(p_0)) = \Theta(M)(\theta_g(p_0))$$

$$\Rightarrow (\theta_g)_* \mathcal{A}(p_0) + (\bar{\theta}_g)_* T_{p_0}(\theta_H(p_0)) = (\theta_g)_* \Theta(M)(p_0).$$

Lastly, we know $\Theta_{p_0}(M)(p_0) = T_{p_0}(\theta_H(p_0))$. Therefore

$$\mathcal{A}(p_0) + (\theta_g^{-1})_* \circ (\bar{\theta}_g)_* (\Theta_{p_0}(M)(p_0)) = \Theta(M)(p_0)$$

$$\Rightarrow \mathcal{A}(p_0) + (C_g)_* (\Theta_{p_0}(M)(p_0)) = \Theta(M)(p_0).$$

Thus, the equality given in (b) to true at $p_0$.

(b) $\Rightarrow$ (a) At $p_0$, for all $g \in G$ we have

$$\mathcal{A}(p_0) + (C_g)_* (\Theta_{p_0}(M)(p_0)) = \Theta(M)(p_0)$$

$$\Rightarrow (\theta_g)_* \mathcal{A}(p_0) + (\theta_g)_* (C_g)_* (T_{p_0}(\theta_H(p_0))) = (\theta_g)_* (T_{p_0} M)$$

$$\Rightarrow \mathcal{A}(\theta_g(p_0)) + (\bar{\theta}_g)_* (T_{p_0}(\theta_H(p_0))) = T_{\theta_g(p_0)} M.$$

Since $\bar{\theta}_g$ maps $\theta_H(p_0)$ to $\theta_H(\bar{\theta}_g(p_0))$, $(\bar{\theta}_g)_*$ will map the tangent space of $\theta_H(p_0)$ to the
tangent space of $\theta_H(\bar{\theta}_g(p_0))$. Hence, we have

$$A(\theta_g(p_0)) + T_{\theta_g(p_0)}(\theta_H(\bar{\theta}_g(p_0))) = T_{\theta_g(p_0)}M$$

$$\Rightarrow A(\theta_g(p_0)) + T_{\theta_g(p_0)}(\theta_H(\theta_g(p_0))) = T_{\theta_g(p_0)}M.$$

Since $\theta$ is transitive we have $\forall p \in M$,

$$A(p) + T_p(\theta_H(p)) = T_pM$$

$$\Rightarrow A(p) + \Delta(p) = T_pM.$$

Therefore, by Theorem 4.6.1, (4.2) satisfies the LARC.

\[\square\]

### 4.6.1 Summary

In this chapter, we continued our treatment of the set stabilization problem by extending the results in Chapter 4 to the case of a general manifold. We considered the stabilization of an orbit of a Lie group action for a system invariant under another Lie group action. We paralleled the results of Chapter 4 by deriving various conditions for the existence of a unique quotient control system such that the stability of one of its points is equivalent to the stabilization of the target set for the original problem. In addition, we considered the use of feedback equivalence to ensure the control vector fields for the quotient control system are linearly independent. The new results of this chapter deal with the accessibility of the quotient control system. We produced conditions that are equivalent to the LARC and hence are sufficient for the quotient control system to be accessible. In the next chapter, we apply these results in examples to illustrate this approach to solving a set stabilization problem.
Chapter 5

Examples

5.1 Controlling a Particle in a Gravitational Field

The first example, inspired by the work in [5], is the motion control of a particle of unit mass in a gravitational field. The particle lies on the plane $\mathbb{R}^2$ with the force of the gravitational field radially constant and centered about the origin. To avoid the singularity in the gravitational field, we remove the origin from the plane. The control action is a thruster on the particle that will help guide its motion.

We define the position $p = (p_1, p_2)$ and the velocity $v = (v_1, v_2)$ of the particle, letting $x = [p \ v]^T$. The state space of this system is $M = (\mathbb{R}^2 \setminus (0, 0)) \times \mathbb{R}^2$.

Control Objective: stabilize the particle traveling counterclockwise around the unit circle in $\mathbb{R}^2$ with unit speed as seen in Figure 5.1.

5.1.1 Development of the Target Set $S^*$

The aforementioned control objective is the stabilization of a subset of $M$ that contains all the states corresponding to the particle traveling with unit speed around the unit circle. Hence, this problem becomes the stabilization of this subset; we derive equations that describe this target set $S^*$. 
When the particle is on the unit circle, it is obvious that $p_1^2 + p_2^2 = 1$. We can also derive equations that ensure the particle’s velocity is tangent to the unit circle with magnitude one. The component of $v$ in the direction of $p = (p_1, p_2)$ measures the velocity of the particle transversal to the unit circle, while the component in the direction of the vector $(-p_2, p_1)$, which is orthogonal to $p$, measures its counterclockwise tangential velocity. These components are given by $p_1v_1 + p_2v_2$ and $p_1v_2 - p_2v_1$, respectively. To meet our control objective, we required the tangential component to be one and the transversal one to be zero. Hence, we can characterize the target set as

$$S^* = \{(p_1, p_2, q_1, q_2) : p_1^2 + p_2^2 = 1, \ p_1v_2 - p_2v_1 = 1, \ p_1v_1 + p_2v_2 = 0\}.$$ 

### 5.1.2 Equations of Motion and Invariance

This is an example of a simple mechanical system. The equations of motion are derived from Newton’s second law of motion, which in this case, states that $\ddot{v}$ equals the sum of
the forces acting on the particle. The equations of motion without control are given by

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{v}_1 \\
\dot{v}_2
\end{bmatrix} =
\begin{bmatrix}
v_1 \\
v_2 \\
-\frac{p_1}{(p_1^2+p_2^2)^{3/2}} \\
-\frac{p_2}{(p_1^2+p_2^2)^{3/2}}
\end{bmatrix} \triangleq \bar{X}_0(p_1, p_2, v_1, v_2).
\]

The vector field \( \bar{X}_0 \) is invariant under the action \( \psi : S0(2) \times M \to M \) given by

\[
\psi(A, x) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x,
\]

reflecting the fact that the system has a rotational symmetry. Indeed, if we let

\[
A = \begin{bmatrix}
\cos(s) & \sin(s) \\
-\sin(s) & \cos(s)
\end{bmatrix},
\]

where \( s \in \mathbb{R} \), then we get

\[
X_0(\psi_A(x)) = X_0 \left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x \right) = \begin{bmatrix}
v_1 \cos(s) + v_2 \sin(s) \\
- v_1 \sin(s) + v_2 \cos(s) \\
- \frac{p_1 \cos(s) + p_2 \sin(s)}{(p_1^2+p_2^2)^{3/2}} \\
- \frac{p_1 \sin(s) + p_2 \cos(s)}{(p_1^2+p_2^2)^{3/2}}
\end{bmatrix} = (\psi_A)_* X_0(x).
\]

We now consider the orbits of \( \psi \). Given a position and velocity of the particle, a matrix \( A \in SO(2) \), under the action of \( \psi \), rotates its position and velocity vectors by the
same angle. This implies that the positions of $x$ and $\psi_A(x)$ lie on the same unit circle and their respective transversal and tangential velocities are equal. This is shown in Figure 5.2. For $x^* = (1, 0, 0, 1)$, the particle is on the unit circle and it is moving tangential to the unit circle with unit speed. Hence, our target set $S^*$ is the orbit of $x^*$ under the action $\psi$. For this reason, we pick $\theta = \psi$ in this example.

Next, we derive the equations of motion with the addition of control. As mentioned, the control is a thruster on the particle that applies an additional force $(\bar{u}_1, \bar{u}_2)$. Hence, the new equations of motion are given by

$$
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{v}_1 \\
\dot{v}_2
\end{bmatrix} =
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0
\end{bmatrix} \bar{u}_1 + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} \bar{u}_2.
$$

It can be shown that the control vector fields in this equation are not invariant under $\psi$. 

Figure 5.2: An example of an orbit of $\psi$
But, with the feedback transformation given by [5]

\[
\begin{bmatrix}
\bar{u}_1 \\
\bar{u}_2
\end{bmatrix} = \frac{1}{(p_1^2 + p_2^2)^{1/2}} \begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix} \begin{bmatrix} u_1 \\
\end{bmatrix}
\]

the system is feedback equivalent to the \(\psi\)-invariant system given by

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{v}_1 \\
\dot{v}_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\frac{p_1}{(p_1^2 + p_2^2)^{1/2}} \\
\frac{p_2}{(p_1^2 + p_2^2)^{1/2}}
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2 \\
X_0(x) \\
X_1(x)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\frac{p_1}{(p_1^2 + p_2^2)^{1/2}} \\
\frac{p_2}{(p_1^2 + p_2^2)^{1/2}}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \end{bmatrix}
\]

This is an example where the system is invariant under \(\psi\) and the target set is an orbit of \(\psi\). However, it is obvious that \(\psi\) is not transitive and hence this problem does not entirely fit into the framework of Chapter 4. Nevertheless, we can still define the appropriate quotient control system as previously discussed. Indeed, as the vector fields in (5.1) are \(\psi\)-invariant and hence will map orbits of \(\psi\) onto orbits of \(\psi\). By the proof of Theorem 4.4.3, this implies that they can be projected onto the quotient space \(M/G\).

### 5.1.3 Derivation of the Quotient Control System

In this section, we derive the quotient control system, starting with the quotient space \(M/G\). As mentioned, along an orbit of \(\psi\), the particle lies on a circle of \(\mathbb{R}^2\) with constant tangential and transversal velocities relative to this circle; each orbit is defined by the radius of this circle and a velocity vector. Therefore, the quotient space \(M/G\) can be identified with \(\mathbb{R}_{>0} \times \mathbb{R}^2\).

Next, we compute the quotient map \(\pi\). A point \(x = (p_1, p_2, v_1, v_2) \in M\), will be mapped by \(\pi\) to its corresponding orbit. Therefore, \(\pi\) maps \(x\) to the circle containing
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\((p_1, p_2)\) with, at each point, equal transversal and tangential velocities to \((v_1, v_2)\) relative to this circle. Remembering that the transversal and tangential velocities of \(x\) relative to the circle containing \((p_1, p_2)\) are, respectively, given by \(p_1 v_1 + p_2 v_2\) and \(p_1 v_2 - p_2 v_1\), the quotient map is given by

\[
\pi(x) = (p_1^2 + p_2^2, p_1 v_1 + p_2 v_2, p_1 v_2 - p_2 v_1) \triangleq (q_1, q_2, q_3).
\]

Hence, \(\pi\) reduces the set \(S^*\) to the point \((1, 0, 1)\) in \(M/G\).

We use \(\pi\) to determine the quotient control system. Unlike the unicycle example, we compute \(\pi_*\) in coordinates, which is simply its Jacobian matrix, to project each of the vector fields in (5.1). We have

\[
\pi_*(x) = \begin{bmatrix}
2p_1 & 2p_2 & 0 & 0 \\
v_1 & v_2 & p_1 & p_2 \\
v_2 & -v_1 & -p_2 & p_1 \\
\end{bmatrix}.
\]

Using this, we obtain the quotient control system given by

\[
\dot{q} = \begin{bmatrix}
2q_2 \\
\frac{q_2^2 + q_3^2}{q_1} - \frac{1}{\sqrt{q_1}} \\
0 \\
\end{bmatrix} \begin{bmatrix}
\dot{x}_0(q) \\
\dot{x}_1(q) \\
\dot{x}_2(q) \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\sqrt{q_1} \\
\end{bmatrix} u_1 + \begin{bmatrix}
0 \\
\sqrt{q_1} \\
\end{bmatrix} u_2 \tag{5.2}
\]

Without control, it can be seen that the point \((1, 0, 1)\) \(\in M/G\) is an equilibrium of (5.2). The stabilization of this equilibrium is equivalent to the stabilization of \(S^*\) in (5.1).

5.1.4 Vector Field Split & Accessibility

We investigate the concepts of splitting the control vector fields and accessibility of the quotient control system. Firstly, we observe that the feasibility of splitting the control
vector fields cannot be determined using Theorem 4.5.1 since it requires \( \psi \) to be transitive. Nevertheless, it can be seen in (5.2) that, since \( q_1 \) is never zero, the control vector fields are pointwise linearly independent. This implies that \( X_1, X_2 \) in (5.1) are never tangent to the orbits of \( \psi \). Therefore, these control vector fields are already split in the sense of Theorem 4.5.1.

Accessibility of the quotient control system is also trivial since (5.1) is accessible. Indeed, as

\[
[X_0, X_1](x) = -\frac{1}{(p_1^2 + p_2^2)^{1/2}} \begin{bmatrix} p_1 \\ p_2 \\ \ast_1 \\ \ast_2 \end{bmatrix}, \quad [X_0, X_2](x) = \frac{1}{(p_1^2 + p_2^2)^{1/2}} \begin{bmatrix} -p_2 \\ p_1 \\ \ast_3 \\ \ast_4 \end{bmatrix}
\]

where \( \ast_1, \ldots, \ast_4 \in \mathbb{R} \). For every \( x \in M \), \( X_1(x), X_2(x), [X_0, X_1](x), [X_0, X_2](x) \) are linearly independent. This implies that this system satisfies the LARC and is accessible. Therefore, Theorem 4.6.1 is trivially true and the quotient control system is accessible.

### 5.1.5 Stabilization of \( S^* \)

We design a controller that fulfills our objective of stabilizing the particle moving tangentially around the unit circle with unit speed. We have reduced this set stabilization problem to the stabilization of the equilibrium \((1, 0, 1)\) for the quotient control system given in (5.2). We design a controller for this problem. By defining the deviation variables

\[
\tilde{q}_1 = q_1 - 1 \\
\tilde{q}_2 = q_2 \\
\tilde{q}_3 = q_3 - 1
\]
and redefining the quotient control system in these variables, which is given by

\[
\begin{bmatrix}
2\bar{q}_2 \\
\bar{q}_2 + (\bar{q}_1 + 1)^2 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\sqrt{\bar{q}_1 + 1} \\
0
\end{bmatrix}
u_1 + \begin{bmatrix}
0 \\
0 \\
\sqrt{\bar{q}_1 + 1}
\end{bmatrix}u_2, \quad (5.3)
\]

the problem becomes the stabilization of the origin of (5.3).

We design a state feedback controller to perform this stabilization. We cancel the nonlinear terms in the drift vector field to make the system linear time-invariant and add a linear state feedback controller to stabilize the system. This state feedback controller is given by

\[
\begin{align*}
u_1 &= \frac{1}{\sqrt{\bar{q}_1 + 1}} \left[ - \left( \frac{\bar{q}_2^2 + (\bar{q}_3 + 1)^2}{\bar{q}_1 + 1} - \frac{1}{\sqrt{\bar{q}_1 + 1}} \right) - \frac{1}{2} \bar{q}_1 - 2\bar{q}_2 \right] \\
u_2 &= -\frac{1}{\sqrt{\bar{q}_1 + 1}} \bar{q}_3.
\end{align*}
\]

We note that this controller is based on linear control theory and given that the state space of the quotient control system is not Euclidean, this controller will not work globally. It may allow \(q_1\) to take values less than 0 and hence not lie in \(M/G\).

Nevertheless, the controller does perform the stabilization problem for certain initial conditions. It was applied in simulation to the system describing the particle given in (5.1). Figures 5.3 and 5.4 depict the position of the particle in \(\mathbb{R}^2\) as well as its speed with respect to time. As seen, the controller is able to stabilize the particle moving counterclockwise around the unit circle with unit speed.

### 5.1.6 Summary

Using the results of Chapter 5, we were able to solve this set stabilization problem by solving an equivalent equilibrium stabilization problem. Essentially, this solution exploited the radial invariance of the control system. According to [5], we can rewrite
the system given in (5.2) in polar coordinates given by

\[ r = \sqrt{p_1^2 + p_2^2} \]
\[ \sigma = \arctan\left(\frac{p_2}{p_1}\right) \]

to obtain the system

\[
\begin{bmatrix}
\dot{r} \\
\dot{\sigma} \\
\dot{v}_r \\
\dot{v}_\sigma
\end{bmatrix} =
\begin{bmatrix}
v_r \\
v_\sigma \\
v_r^2 \frac{1}{r^2} - \frac{1}{r^2} + u_1 \\
u_2
\end{bmatrix}
\]

The dynamics for \( r \) do not depend on \( \sigma \). Hence, when stabilizing a set that does not depend on \( \sigma \), such as a circle, we can essentially ignore \( \sigma \) from the problem. It is important to note that this is due to the invariance of the control system; without this invariance, the problem could not be solved in this fashion.
## 5.2 Control of a Continuously Stirred Tank Reactor

The second example involves the hydrolysis of chloroethane to produce ethanol using a continuously stirred tank reactor (CSTR) with a heater exchanger. The reaction considered is given by

\[
\begin{align*}
\text{C}_2\text{H}_5\text{Cl} & \quad + \quad \text{H}_2\text{O} \quad \rightarrow \quad \text{C}_2\text{H}_5\text{OH} \quad + \quad \text{HCl} \\
\text{Chloroethane} & \quad + \quad \text{Water} \quad \rightarrow \quad \text{Ethanol} \quad + \quad \text{Hydrogen chloride}
\end{align*}
\]

Since water is one of the reactants, it will also be used as the solvent for this reaction. The diagram of a CSTR, ignoring the heat exchanger, is shown in Figure 5.5. The reactants flow into the reactor at a flowrate \( F_0 \) with concentrations \( C_{0i} \) and the products and unreacted reactants exit the reactor at a flowrate \( F \) with concentrations \( C_i \). The following are typical assumptions for a CSTR:

- The volume of the liquid in the reactor is constant.
- The concentrations of the reactants and products in the reactor are spatially constant and equal to their exit concentrations.
- The temperature in the reactor is spatially constant and equal to the temperature of the outlet flow.
The density and heat capacity of the reacting liquid is constant.

Note that since $V$ is constant, the inlet and outlet flowrates are equal, i.e., $F = F_0$. The reaction is endothermic, which means that it requires energy from the heat exchanger. One of the key factors in designing a reactor is the conversion of the reactants. The conversion of a reactant is the ratio of its outlet to inlet concentrations.

**Control Objective:** maintain a constant conversion rate of 60% of chloroethane while maintaining the reactor temperature at 313 K.

### 5.2.1 Modeling the CSTR

The methodology for modeling a CSTR can be reviewed from textbooks such as [4]. Firstly, we provide tables outlining the variables and parameters that describe the dynamics of the CSTR.

**Table 5.1: Variables for the CSTR**

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chloroethane inlet concentration</td>
<td>$x_1$</td>
<td>g.L$^{-1}$</td>
</tr>
<tr>
<td>Chloroethane outlet concentration</td>
<td>$x_2$</td>
<td>g.L$^{-1}$</td>
</tr>
<tr>
<td>Reactor temperature</td>
<td>$T$</td>
<td>K</td>
</tr>
<tr>
<td>Inlet fluid temperature</td>
<td>$T_{in}$</td>
<td>K</td>
</tr>
<tr>
<td>Heat exchanger rate</td>
<td>$H_{exc}$</td>
<td>J.s$^{-1}$</td>
</tr>
<tr>
<td>Flowrate through Reactor</td>
<td>$F$</td>
<td>L.s$^{-1}$</td>
</tr>
<tr>
<td>Reaction rate</td>
<td>$R_{rxn}$</td>
<td>g.L$^{-1}$.s$^{-1}$</td>
</tr>
</tbody>
</table>

**Table 5.2: Parameters for the CSTR**

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>g.L$^{-1}$</td>
</tr>
<tr>
<td>Volume</td>
<td>$V$</td>
<td>L</td>
</tr>
<tr>
<td>Heat Capacity</td>
<td>$C_p$</td>
<td>J.g$^{-1}$.K$^{-1}$</td>
</tr>
</tbody>
</table>

The heat exchange rate $H_{exc}$ is the rate of energy transferred from the heat exchanger to the reactor. The reaction rate $R_{rxn}$ measures the rate of change in the concentration of chloroethane due to the reaction. Since the reactor contains mostly water, the reacting
liquid has density and heat capacity similar to water. Hence, $\rho = 1000 \text{ g.L}^{-1}$ and $C_p = 4.18 \text{ J.g}^{-1}.\text{K}^{-1}$. The control objective is to achieve $\frac{\Delta x}{\Delta t} = 0.6$ and $T = 313$.

The inlet concentration of chloroethane $x_1$ is taken as a user-defined parameter that can be altered with respect to time. This allows the user to modify the outlet concentration of chloroethane by varying the dynamics of its inlet concentration. This is important for the separation processes that occur downstream to the reactor as their operation depends on the outlet concentration of the reactants. To this end, the dynamics for $x_1$ are given by

$$\frac{dx_1}{dt} = \bar{u}_1.$$ 

In short, the control $\bar{u}_1$ will not be used to meet the control objective, but to allow for variations in the inlet chloroethane concentration at the user’s discretion.

The basis for modeling a CSTR is the conservation of mass and energy:

$$\text{Accumulation} = \text{Input} - \text{Output} + \text{Generation} - \text{Consumption}.$$ 

We start by performing a mass balance on the reactant chloroethane. The mass of chloroethane in the reactor is equal to the product of its concentration in the reactor $x_2$ by the volume of the reactor $V$. The input (output) of chloroethane is given by the product of its corresponding inlet (outlet) concentration $x_1$ ($x_2$) by the flowrate through the reactor $F$. Lastly, its consumption is the rate of the reaction $R_{rxn}$ multiplied by $V$; we will discuss $R_{rxn}$ later in detail. Using the general mass balance equation, the dynamics for $x_2$ are given by

$$\frac{d(Vx_2)}{dt} = Fx_1 - Fx_2 - R_{rxn}V.$$
Since $V$ is constant we have

$$\frac{dx_2}{dt} = \frac{F}{V}(x_1 - x_2) - R_{rxn}. $$

Next, we perform the energy balance for the CSTR. It is not possible to determine the ‘energy’ of the liquid in the reactor, but we can determine the energy relative to some reference state. For the CSTR, we consider an energy balance relative to some reference temperature $T_{ref}$. The heat capacity $C_p$ of the reacting fluid determines how its energy changes per unit mass as its temperature varies from $T_{ref}$. Hence, the relative energy inside the reactor is the product of the mass of the reacting liquid, its heat capacity and the temperature difference from $T_{ref}$. The relative energy of the inlet and outlet flows are computed in a similar fashion. The energy consumed by the reaction depends on the rate of the reaction and amount of energy consumed per reaction. The latter value is the heat of the reaction $\Delta H$. Lastly, we incorporate the energy transfer from the heat exchanger. For simplicity, the dynamics of the heat exchanger are not included in the model. Rather, the amount of energy supplied by the heat exchanger will be an input to the system. Using this information, we can determine the energy balance for the CSTR.

$$\frac{d(\rho V C_p (T - T_{ref}))}{dt} = \rho F C_p (T_{in} - T_{ref}) + H_{exc}(t) - \rho F C_p (T - T_{ref}) - V \Delta H R_{rxn}. $$

Since it is assume that $\rho$, $V$, $C_p$ and $T_{ref}$ are constant, we obtain

$$\frac{dT}{dt} = \frac{F}{V} (T_{in} - T) - \frac{\Delta H}{\rho C_p} R_{rxn} + \frac{H_{exc}(t)}{\rho C_p V}. $$
5.2.2 Reaction Kinetics

In this section, we investigate the rate of the reaction $R_{rxn}$. For a general reaction involving $n$ reactants, the reaction rate is given by

$$R_{rxn} = kC_1^{\alpha_1}C_2^{\alpha_2} \ldots C_n^{\alpha_n}$$

where $k$ is a temperature dependent parameter and $\alpha_i \geq 0$. The hydrolysis of chloroethane involves two reactants: water and chloroethane. Since water is also the solvent, its concentration is relatively constant and hence can be incorporated in the parameter $k$. Therefore, the reaction rate is given by

$$R_{rxn} = kx_2$$

The parameter $k$ depends on temperature according to the Arrhenius equation given by

$$k = A \exp \left( \frac{-E_a}{RT} \right)$$

where $R$ is the universal gas constant, $A$ is known as the pre-exponential factor and $E_a$ is the activation energy of the reaction. Table 5.2.2 shows the values of these constants, as well as the heat of reaction, for the hydrolysis of chloroethane as taken from [20].

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas Constant</td>
<td>$R$</td>
<td>8.31</td>
<td>J.mol$^{-1}$.K$^{-1}$</td>
</tr>
<tr>
<td>Pre-exponential factor</td>
<td>$A$</td>
<td>$22.8 \times 10^{12}$</td>
<td>s$^{-1}$</td>
</tr>
<tr>
<td>Activation energy</td>
<td>$E_a$</td>
<td>$11.1 \times 10^4$</td>
<td>J.mol$^{-1}$</td>
</tr>
<tr>
<td>Heat of Reaction</td>
<td>$\Delta H$</td>
<td>750</td>
<td>J.g$^{-1}$</td>
</tr>
</tbody>
</table>
5.2.3 Control System for the CSTR

Using the previous section, we fully develop the control system for the CSTR. The states are the inlet and outlet concentrations of chloroethane as well as the temperature inside the reactor. If we assume that \( x_1 \) and \( x_2 \) are never zero, then the state space is \( \bar{M} = (\mathbb{R}_{>0})^3 \). As mentioned, the user can vary the dynamics of \( x_1 \) via \( \bar{u}_1 \). We apply the feedback transformation \( u_1 = \frac{\bar{u}_1}{x_1} \); the reason for this will become apparent.

To control this system, we use (up to constant parameters) the ratio of the flowrate through the reactor, \( u_2 = \frac{F}{V} \), as well as the heat exchanger rate, \( u_3 = \frac{H_{\text{exc}}}{\rho C_p V} \), as controls. Hence, the model becomes

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{T}
\end{bmatrix} = 
\begin{bmatrix}
0 \\
-A \exp\left(-\frac{E_a}{RT}\right) x_2 \\
c_0 \exp\left(-\frac{E_a}{RT}\right) x_2
\end{bmatrix} x_2 + 
\begin{bmatrix}
x_1 \\
0 \\
0
\end{bmatrix} u_1 + 
\begin{bmatrix}
0 \\
x_1 - x_2 \\
T_{\text{in}} - T
\end{bmatrix} u_2 + 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u_3
\]

(5.4)

where \( c_0 = \frac{A \Delta H}{\rho C_p} \). We take \( \bar{x} = (x_1, x_2, T) \). Note that without control this system does not have any equilibria.

As mentioned, the control objective is to achieve 60% conversion of chloroethane and maintain a reactor temperature of 313 K. This is a set stabilization problem with target set

\[
\bar{S}^* = \left\{ (x_1, x_2, T) : \frac{x_1}{x_2} = 0.6, \; T = 313 K \right\}.
\]

5.2.4 Invariance of the Control System

Consider the action \( \bar{\psi} : G \times \bar{M} \rightarrow \bar{M} \), where \( G \) is the multiplicative group \( \mathbb{R}_{>0} \), given by

\[
\bar{\psi}(g, \bar{x}) = (gx_1, gx_2, T).
\]
Note that along an orbit of \( \bar{\psi} \), \( x_2/x_1 \) and \( T \) are constant. Therefore, we wish to stabilize the orbit of \( \bar{\psi} \) corresponding to 60% conversion of chloroethane and reaction temperature of 313 K, which is \( \bar{\psi}_G(1, 0.6, 313) \). If the control system (5.4) were invariant under \( \bar{\psi} \), then we would set \( \theta = \bar{\psi} \) and, in light of Theorem 4.4.3, the quotient system would be guaranteed to exist. However, it is easily seen that the control system given in (5.4) is not invariant under \( \bar{\psi} \). To remedy this, let the constant \( c_0 \) become a state of the system with dynamics \( \dot{c}_0 = 0 \). We have the prolonged control system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{T} \\
\dot{c}_0
\end{bmatrix} =
\begin{bmatrix}
0 & -A \exp \left( -\frac{E_a}{RT} \right) x_2 \\
-A \exp \left( -\frac{E_a}{RT} \right) x_2 & 0 \\
-c \exp \left( -\frac{E_a}{RT} \right) x_2 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
T \\
c_0
\end{bmatrix} +
\begin{bmatrix}
0 \\
u_1 \\
u_2 \\
u_3
\end{bmatrix} = \begin{bmatrix}
0 \\
u_1 \\
u_2 \\
u_3
\end{bmatrix}.
\]

with state \( x = [x_1, x_2, T, c_0]^\top \) and state space \( M = \bar{M} \times \mathbb{R}_{>0} \). Now consider the action \( \psi : G \times M \to M \),

\[
\psi(g, x) = (gx_1, gx_2, T, c_0/g).
\]

It is easy to verify that \((\psi_g)_*X_j = X_j(\psi_g), j = 0, \ldots, 3\), and thus the system is \( \psi \)-invariant. The target set for the prolonged system becomes \( S^* = \{(x_1, x_2, T, c_0) \in M : x_1/x_2 = 0.6, T = 313K\} \).

### 5.2.5 Derivation of the Quotient Control System

In this section, we derive the quotient control system. It is obvious that the quotient space \( M/G = (\mathbb{R}_{>0})^4 \setminus \mathbb{R}_{>0} \) is simply \((\mathbb{R}_{>0})^3\). We compute the quotient map \( \pi \). The orbit of a point \( x_0 = (x_{10}, x_{20}, T_0, c_0) \in M \) is the set

\[
\left\{ (x_1, x_2, T, c) \in M : \frac{x_2}{x_1} = \frac{x_{20}}{x_{10}}, T = T_0, x_2c = x_{20}c_0 \right\}.
\]
Therefore, we have
\[ \pi(x) = \left( \frac{x_2}{x_1}, T, x_2c \right) \triangleq (q_1, q_2, q_3). \]

Notice that the target set for the quotient system is not a point, but rather the one-dimensional subspace
\[ \pi(S^*) = \{(q_1, q_2, q_3) \in M/G : q_1 = 0.6, \ q_2 = 313\}. \]

Nevertheless, this is essentially the stabilization of a point as we want to stabilize \( q_1, q_2 \) to certain values while ignoring the value of \( q_3 \). Though, we will not be able to make use of the results on accessibility or vector field splitting.

We use \( \pi \) to determine the quotient control system. We have
\[
\pi_\ast(x) = \begin{bmatrix}
-\frac{x_2}{x_1} & \frac{1}{x_2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & c & 0 & x_2
\end{bmatrix}
\]

Using this, we obtain the quotient control system given by
\[
\dot{q} = \begin{bmatrix}
-A \exp \left(-\frac{E_a}{Rq_2} \right) q_1 \\
\exp \left(-\frac{E_a}{Rq_2} \right) q_3 \\
0
\end{bmatrix} + \begin{bmatrix}
-q_1 \\
0 \\
0
\end{bmatrix} u_1 + \begin{bmatrix}
1 - q_1 \\
T_{in} - q_2 \\
\frac{q_2}{q_1} - q_3
\end{bmatrix} u_2 + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} u_3 (5.6)
\]

### 5.2.6 Stabilization of \( S^* \)

We design a controller that stabilizes (5.6) to the set \( \pi(S^*) \). We use the same methodology as the particle in a gravitational field example; we use the controls \( u_2, u_3 \) to make the system linear time-invariant and design a linear state feedback controller to stabilize the system. As previously mentioned, the control \( u_1 \) is not used to stabilize the system, but
to allow the user to change the inlet concentration of chloroethane. Hence, it is assumed that $u_1$ is known.

Using the deviation variables

\[
\begin{align*}
\bar{q}_1 &= q_1 - 0.6 \\
\bar{q}_2 &= q_2 - 313 \\
\bar{q}_3 &= q_3
\end{align*}
\]

the controller is given by

\[
\begin{align*}
u_2 &= \frac{1}{1 - \bar{q}_1} \left[ A \exp \left( \frac{-E_a}{R \bar{q}_2} \right) \bar{q}_1 - k_1 \bar{q}_1 + \bar{q}_1 u_1 \right] \\
u_3 &= -\frac{T_{in} - \bar{q}_2}{1 - \bar{q}_1} \left[ A \exp \left( \frac{-E_a}{R \bar{q}_2} \right) \bar{q}_1 - k_1 \bar{q}_1 + \bar{q}_1 u_1 \right] - \exp \left( \frac{-E_a}{R \bar{q}_2} \right) \bar{q}_3 - k_2 \bar{q}_2
\end{align*}
\]

where $k_1, k_2$ are positive design parameters. Since this controller uses linear control theory, it does not respect the fact that both $u_2, u_3$ should be positive. To ensure this, we choose $k_1 = k_2 = 0.05$; these are relatively small values to decrease the gain of the system and prevent oscillatory behaviour as the system reaches its desired state.

We tested this controller in simulation. For the simulation, we took

\[
u_1(t) = \frac{-1}{(t + 1)(t + 2)}
\]

to make $x_1$ the signal shown in Figure 5.6.

Figures 5.7 and 5.8 show the reactor temperature and the conversion of chloroethane. At steady state, the reactor is operating isothermally at 313K and the outlet chloroethane concentration reaches a constant steady state such that the conversion stabilizes at 60\%.
5.2.7 Summary

We conclude by discussing the intuition behind the solution of this problem. The solution used the coordinate change transformation given by

\[(x_1, x_2, T) \mapsto \left( \frac{x_2}{x_1}, T, cx_2 \right) \triangleq (q_1, q_2, q_3),\]

redefined the system in terms of these coordinates and stabilized only the first two coordinates to certain predefined values. We could not remove the coordinate \(q_3\) from the stabilization problem as one cannot remove the dependence of the dynamics for \(q_1\) and \(q_2\)
on $q_3$. Therefore, the set stabilization problem could not be reduced to the stabilization of a point in $M/G$. However, the invariance of the system to the action $\theta$ did allow the system dynamics to be written in terms of $\frac{x_2}{x_1}$, which was important to meet the control objective.

It should be mentioned that since $c$ is constant, essentially the coordinate change was given by

$$(x_1, x_2, T) \mapsto \left( \frac{x_2}{x_1}, T, x_2 \right)$$

This transformation would remove the need for dynamic prolongation and simplify the problem. At any rate, if $c$ were not constant and could be considered as a control input, then the use of dynamic prolongation along with the addition of an action on the control to make the system invariant under a Lie group action provides a method to solve the set stabilization problem without ignoring $c$ as a control.
Chapter 6

Conclusions

The set stabilization problem is an aspect of control theory that has been overlooked in comparison to other stabilization problems such as the stabilization of trajectories or equilibria, but indeed has its practicality. There are problems in control theory that can be simplified when brought into this framework. Such examples are the synchronization problem and the path following problem.

It is natural to solve the set stabilization problem as an equilibrium stabilization problem. In general, if one is attempting to achieve some goal and has some type of measure of the ‘distance’ from this goal, then to fulfill this goal one must reduce this distance to zero. In the case of set stabilization, if one can measure the distance from a target set and form a system that describes how this distance changes, then one simply designs a controller that stabilizes this distance to zero, which is an equilibrium stabilization problem.

The solution to the set stabilization problem for LTI and general control affine systems make use of this idea, but each are confining in some regard. The results in the LTI case are global, but are restrictive in the class of systems considered. The solution for a general control affine system is attractive since it does not restrict the class of systems, but the results are local. In our approach, we restrict to systems invariant under a Lie
group action, but achieve global results.

Initially, we restrict to left-invariant systems on Lie groups where the target set is a coset of a Lie subgroup. We determine several equivalent necessary and sufficient conditions for the existence of a quotient control system that describes the motion transversal to the target set. We continue by giving necessary and sufficient conditions for the existence of a feedback equivalent, left-invariant system that exhibits a quotient control system with linearly independent control vector fields. We apply these results in two examples including the formation stabilization of three kinematic unicycles.

We expand the problem to include systems on manifolds that are invariant under a Lie group action where the target set is an orbit of another Lie group action. We broaden the previous results to include this class of systems. We consider the accessibility of the quotient control system in relation to the original system. We produce conditions on the original system that result in the of the quotient control system satisfying the Lie algebra rank condition.

We apply the results to two examples: the motion control of a particle in a gravitational field and the conversion and temperature control of a continuously stirred tank reactor. In these examples, controllers are developed to stabilize the target set. These examples validate the presented theory while showing the applicability of the set stabilization problem.
Bibliography


