ESSAYS IN POLITICAL ECONOMY AND THE ECONOMICS OF ORGANISATIONS

by

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Abstract

This thesis groups three papers in applied microeconomic theory that focus on political economy and the economics of organisations.

The first chapter studies the equilibrium outcomes of a dynamic game of electoral competition between two policy-motivated parties. I model incumbent policy persistence: parties commit to implement a policy for their full tenure in office, and hence in any election only the opposition party is free to choose a new platform. The model gives rise to novel equilibrium policy dynamics: governments alternate in power; parties compromise, that is, starting from differentiated ideological positions, they gradually move towards proposing platforms which resemble one another; however, they never capitulate, that is, party labels matter and parties maintain distinct policy goals.

The second chapter studies a directed search model of competition between sellers that control the quality of buyers’ private information about goods. As better informed buyers extract more informational rents from trade, sellers may try to attract buyers by offering better information. First, I establish how the characteristics of exogenously fixed sale mechanisms determine equilibrium information provision. Information provision is higher under competition than under monopoly, yet partial information is provided for many sale mechanisms. Second, when sellers commit to both information provision and mechanisms, I identify simple conditions under which every equilibrium has full information. In these equilibria, sellers capture the efficiency gains of information provision and compete only over non-distortionary rents offered to buyers.

Retaining the option to develop a currently inactive project often requires maintaining specialised stocks of knowledge. However, standard models of experimentation treat the choice of one project over another as entailing only an implicit opportunity cost. In the third chapter, I characterise the optimal experimentation policy in a model in which undeveloped projects have explicit maintenance costs and can be irreversibly discarded. Projects which in the absence of maintenance costs would be developed only after more promising projects fail are sometimes developed first and then discarded early. Maintenance costs alter optimal project development by providing incentives to bring the option value of less promising projects forward.
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Chapter 1
Two-Party Competition with Persistent Policies

1.1 Introduction

If the parties are viewed in [a] temporal framework, one may better appraise the old saw that the parties offer the electorate only a choice between tweedledum and tweedledee. In fact, the differences between the parties vary from stage to stage in the conversion of controversy into new consensus. (Key (1958), p.247.)

Political parties are long-lived organisations that compete over sequences of elections linked through persistent political outcomes. Prime examples of persistent outcomes are the roles of incumbent and opposition, which have strategic importance and are inherited from previous elections. In this paper, I let incumbency status generate the key dynamic linkage of a model of two-party elections in which (i) governments alternate, (ii) parties compromise, that is, starting from differentiated ideological positions, they gradually move towards proposing platforms which resemble one another, yet (iii) they never become as indistinguishable as tweedledum and tweedledee; party labels matter and parties maintain distinct policy goals. These are novel and inherently dynamic insights into partisan competition which bridge standard results from static models by displaying the feature that Key refers to above: party competition leads to gradual transitions from divergent to convergent outcomes.

More precisely, I formulate a dynamic game of policy competition between two ideological parties that have ideal (single-dimensional) policies on each side of that of the median voter. Voters are myopic and support the party whose current policy yields them higher utility. Under incumbent policy persistence, parties commit to enact specific policies for their entire tenure in office, as opposed to their current term. In each election, incumbents champion (or rather defend) the policies they implemented in their previous term when facing the voters, while opposition parties are free to choose a new platform. Opposition parties are forward-looking and understand that the platforms that carry them to office will support their bids for reelection. The key insights of my model make precise how opposition parties trade off winning current elections with policies they prefer against committing to more moderate policies in order to constrain their future opponents. Parties are restricted to Markov strategies, which depend on the outcomes of previous elections only insofar as these affect the state: the identity of the incumbent party and its policy.

While the model admits a complex set of Markov perfect equilibria, its long-run policy outcomes, which are the limit points of equilibrium paths given some initial state, can be simply described. I show that all equilibria have (i) alternation in power and (ii) bounded extremism in the long-run, while robust equilibria have (iii) bounded moderation. From initial states that are sufficiently distant from the median, two-party competition always leads to some convergence. The bound on long-run extremism, which is driven by parties’ incentives to impose moderation on their future

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1See Osborne (1995) for a survey of results from static models on policy convergence and divergence.
opponents, is tight. In particular, the indefinite repetition of the median policy can occur in the long-run. However, median convergence is not a robust outcome of the model. Under a natural restriction on equilibrium strategies, I show that alternations close to the median occur in the long-run only if policy dynamics start there. That is, while convergence towards the median is dynamically robust, convergence to the median is not and ideological differentiation is persistent. The benefit of committing to more moderate policies is that future opponents commit to even more moderate policies, while its cost consists of foregone policy gains in the current election. The incentives to sustain convergence unravel as policies approach the median, since when parties champion similar policies, discounting wipes out the benefits of imposing moderation on future opponents. Lastly, the bound on robust long-run moderation is tight.

The rich dynamics of my model vanish if instead elections are modelled as a sequence of independent contests. Proposition 1.1 shows that omitting incumbent policy persistence yields a repeated game with a unique subgame perfect equilibrium, and in this equilibrium both parties commit to the median policy after all histories. Hence it is opposition parties’ greater freedom to propose significant shifts in policy that generates dynamic insights that go beyond those of static models. Incumbents are associated with current policies for a variety of reasons: renouncing previous commitments or admitting policy mistakes have large electoral costs; reelected politicians’ ideologies, which drive their policy choices, rarely change substantially between terms; voters disregard incumbents’ promises of policy change through retrospective voting. In brief, my model captures the feature that while challengers are evaluated on their promises, incumbents are evaluated on their records. The assumption of full commitment is stark but it allows a simple characterisation of equilibrium outcomes. In Section 1.5.2 I discuss my paper’s relationship with those on dynamic legislative bargaining and show how my results persist in a model in which incumbents can revise their policies with positive probability. Two features of my model are critical for my results. First, incumbents understand that they will be evaluated on their records and that choosing non-median policies puts them at a disadvantage relative to their opponents. Second, this disadvantage is larger for incumbent policies that are further away from the median. For example, similar results would obtain in a model in which parties commit to policies simultaneously but an exogenous fraction of voters are retrospective and fail to take incumbents’ new policy commitments into account when making voting decisions. My assumption of full commitment is equivalent to having all voters evaluate incumbents retrospectively.

Proposition 1.2 shows that the equilibrium policy paths of my model support two distinct patterns of power and alternation. In the first case, the initial policy is absorbing and the incumbent remains in power forever. These trivial policy dynamics arise only if a leftist (rightist) incumbent party is implementing a policy to the right (left) of the median in the initial state, sapping the competitive incentives of its opposition. Otherwise, the party system is competitive, both parties

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2 More dramatically, Duggan and Fey (2006) show that any policy path can be enforced by some subgame perfect equilibrium of the repeated two-party Downsian model with forward-looking voters.

3 This has been documented by Miller and Wattenberg (1985) and Nadeau and Lewis-Beck (2008) for presidential elections in the United States. Both find evidence that voters tend to evaluate incumbents retrospectively and challengers prospectively.

4 For dynamic legislative bargaining models, see Baron (1996), Baron et al. (2008), Bowen and Zahran (2009), Duggan and Kalandrakis (2009), Fong (2008), Kalandrakis (2004) and Kalandrakis (2007).
hold office and successive opposition parties win elections by committing to increasingly moderate policies. Such policy dynamics converge to an alternation at policies symmetric about the median, and in the long run, incumbents are defeated by opposition parties that are equally preferred by the median voter. Policies that are supported as symmetric alternations in the long-run of some equilibrium are the long-run policy outcomes of the model.

Few previous dynamic models of elections generate plausible patterns of alternation that persist in the long-run. On this note, predictable left-right alternation is not an essential feature of my results. In fact, the extension in Section 1.5.2 discussed above generates random alternation and staggered convergence paths. Models of dynamic elections with imperfect information about politicians’ preferences based on Duggan (2000) typically do not generate alternation in the long-run; successive extreme incumbents survive for one term in office until a sufficiently moderate candidate is elected and survives all challenges. A notable exception is Kalandrakis (2009), in which incumbent parties previously believed to be moderate are replaced when their preferences become extreme and they implement extreme policies. In Van Weelden (2009), while candidates’ preferences are commonly known, there is no alternation on the equilibrium path as voters use the threat of alternation to induce moderate policies.

Proposition 1.3 characterises the set of long-run policy outcomes, which consists of all sufficiently moderate policy alternations. That is, extreme policies are transient and are not observed on equilibrium paths after enough elections. In the United States, periods separating what Key (1955) has termed ‘critical elections’ have been shown to consist of a process of stabilisation in which ‘polarization gives way to conciliation. As it does, the parties move from the poles toward the center and the distance between them narrows.’ I show that a tight upper bound on the extremism of any alternating outcome reached in the long-run is given by the most moderate of the preferred alternations of each party. Discounting ensures that both parties prefer some alternation to the repetition of the median, as when alternations are sufficiently moderate their gains from enacting policies on their side of the median dominates the discounted disutility of opponents’ policies. This bound on long-run extremism makes precise the implications of the model’s key dynamic trade-off by identifying the policies that are sufficiently extreme that they provide incentives for some party to enact more moderate policies in order to rein in its future opponents. That the bound on long-run extremism is tight follows from equilibrium construction.

Of the long-run outcomes of the model, some are reached only if they occur in the initial state, whereas others can be reached from more extreme states through sequences of elections decided by increasingly moderate policies. A robust long-run policy outcome is a long-run policy outcome that can be reached from some initial state with a policy that differs from the policy outcome itself. To study robust outcomes, I require that parties’ strategies be consistent. Such Markov strategies
do not allow parties to condition on an incumbent’s exact policy when committing to policies in the interior of their set of winning policies. Since incumbents’ policies matter to opposition parties only when they constrain their policy choices, consistency strengthens the requirement that Markov strategies depend only on payoff-relevant information. Consistency is a natural restriction that rules out complex equilibrium coordination off the equilibrium path due to the existence of multiple best-responses and allows simple characterisations of payoffs and policies on equilibrium convergence paths.

Proposition 1.4 characterises the set of robust long-run policy outcomes under equilibria in consistent strategies and shows it to contain all alternating outcomes that are sufficiently extreme. This tight bound on the moderation of robust long-run outcomes is derived explicitly and is strictly away from the median. Alternating outcomes close to the median are never reached by consistent equilibrium policy dynamics that start from more extreme states; they are long-run outcomes only if policy dynamics start there. This result states that parties that start ideologically differentiated stay differentiated. On an equilibrium convergence path, moderate policy commitments are supported by opponents’ promises of further moderate commitments in future elections. That is, moderation must be self-reinforcing. The proof that robust long-run outcomes have bounded moderation shows that the incentives to commit to moderate policies unravel as convergence paths approach the median. In particular, I construct bounds on how much policy moderation each party is willing to implement at each step of a convergence path in response to an opponent’s proposed moderate move in the next election. When policy dynamics are sufficiently close to the median, parties’ ‘demands’ for moderation are incompatible. Discounting is critical to this argument. As policies approach the median, moderate moves of similar size by a party and its opponent have similar effects (in absolute value) on its payoffs, yet a party suffers the loss from its moderate move in full today while the gain from its opponent’s moderation is discounted. Hence, convergence breaks down near the median since both parties require their opponents to bear most of the cost of sustaining it. The bound on robust long-run moderation is shown to be tight through equilibrium construction.

Dynamic models of asynchronous policy competition can be traced back to Downs (1957) and were first formally presented in Kramer (1977) and Wittman (1977). They study models similar to mine in which, crucially, parties are myopic. Their models differ from each other only in their assumptions about parties’ preferences. Kramer (1977) assumes that parties are office-motivated and maximise votes, while Wittman (1977) assumes that parties are policy-motivated.

More generally, the idea that forward-looking incumbents have incentives to strategically position current policies to affect future political outcomes has had numerous applications. Closer to my paper are the infinite horizon models of dynamic legislative bargaining, spatial electoral competition and public goods provision. In dynamic legislative bargaining models, a legislator

\footnote{Related to these papers is the literature on competition between myopic adaptive parties, such as Kollman et al. (1992), Kollman et al. (1998), de Marchi (1999) and Laver (2005). Kollman et al. (1992) generate policy dynamics that moderate over time yet stay bounded away from the median in the long-run. In their framework, these dynamics result from policy experimentation by myopic parties.}

\footnote{In a well-known paper, Alesina and Tabellini (1990) show how incumbents accumulate excessive public debt in order to ‘tie the hands’ of future governments that may not share their preferences over public goods spending. For a review of this literature, consult Persson and Tabellini (2000). Bai and Lagunoff (2009) also present a useful discussion of this literature in the context of their more general model.}
is recognised each period to propose some policy which is put to a vote against the status quo. Current policies persist by becoming next period’s status quo. As opposed to my characterisation of equilibrium outcomes, papers on dynamic legislative bargaining typically study specific equilibria. The model of [Baron (1996)] is most closely related to mine. He characterises an equilibrium in which all policy paths converge to the median policy, which contrasts with the non-robustness of policy outcomes near the median in my model. His result follows from the median legislator eventually being recognised, proposing the median policy and never supporting anything other than the status quo in future periods. In Section 1.5.2, I show that it is [Baron (1996)]’s assumption of the existence of a median legislator that is critical for median convergence: in the legislative bargaining version of my model, robust convergence outcomes are still bounded away from the median.

Dynamic models of electoral competition between candidates with privately known policy preferences generate incentives to choose moderate policies to maintain a reputation for moderate preferences. In these models, candidate selection by parties is nonstrategic and candidates’ informational advantage is derived from having been drawn at random from the voting population or the party’s membership[9] while in my model parties can commit to any policy. In the absence of signalling by privately informed candidates, [Van Weelden (2009)] shows that similar intuition and dynamics can obtain. However, as noted above, these models typically do not generate alternation in the long-run, while all incumbents are replaced on most equilibrium paths of my model; moderation does not guarantee reelection, since opponents can respond by championing more moderate policies themselves.

Less closely related, [Azzimonti (2009)] embeds two-party probabilistic elections into a dynamic growth model in which two segments of the population differ in their taste for the composition of public goods spending, and studies the inefficiencies in agents’ capital accumulation decisions due to political instability[10] [Bai and Lagunoff (2009)] present a general dynamic model of ‘policy-endogenous polical power’ in which current policies determine future political power and provide an application to the provision of public goods. [Acemoglu et al. (2008)] study the long-run outcomes of a dynamic constitutional choice game. In contrast to my results, their equilibrium characterisations rely on players being arbitrarily patient.

1.2 Model

Two parties, L and R, contest an infinite sequence of elections at times \( t = 0, 1, \ldots \). Each period starts with the incumbent party \( I \in \{L, R\} \) in power, and the remaining party in opposition. An election consists of a vote over which party should form the next government, with the winning party determined by majority rule. The opposition party \(-I = \{L, R\} \setminus \{I\}\) commits to implementing a policy in the policy space \( X = [0, 1] \), if elected, and for as long as it remains in power:

---

[9] Bernhardt et al. (2009) show that drawing opponents from opposite sides of the political spectrum (i.e., from different parties) makes incumbents more willing to compromise by lowering their continuation value if they lose office. Related my assumption of incumbent policy persistence, Kalandrakis (2009) assumes that a party that a party that has recently lost an election is more likely to field a candidate of a different preference type.

this is the assumption of incumbent policy persistence. Hence, in any election, the incumbent’s policy commitment is inherited from the election that brought it to power. A party may also choose not to participate in the election.

An odd number of voters have symmetric single-peaked preferences over policies, and their ideal policies are distributed over policy space \( X \). Some policy \( M \) corresponds to the median of voters’ ideal policies. Distance preferences for all voters ensure that the median voter is decisive in single elections. Voters are myopic and in all voting subgames, I restrict attention to the equilibrium in weakly undominated strategies in which voters support the party that will enact a policy closest to their ideal policy if brought to power in this election. As the median voter is decisive, the party whose policy is closest to \( M \) wins the election. I assume for simplicity that ties are broken in favour of the opposition party. Myopic voting is a plausible assumption in large elections and has the benefit of focusing attention solely on the competition between the parties. Moreover, in Section 1.5.1 I show that all the equilibrium outcomes studied in the paper persist if voters are forward-looking.

To formalise the dynamic game, define a state \((I, x)\), with \( I \in \{L, R\} \) and \( x \in X \), which records the identity of the incumbent party along with its policy commitment. Given a state \((I, x)\), the corresponding stage game is a single-agent decision problem with the following timing:

- The opposition party – \(I\) commits to a policy \(z \in X\), or does not contest the election, written \(z = \text{Out}\).
- Elections are held. Party \(I\) wins if and only if \(|x - M| < |z - M|\).
- Parties \(L\) and \(R\) have single-peaked preferences over policies around 0 and 1 and represented by \(u_L\) and \(u_R\) respectively. Suppose, without loss of generality, that \(M \leq \frac{1}{2}\), so that party \(L\) is (weakly) favoured by the median voter. Assume that \(u_L(0) = u_R(1) = 0\), \(u_L(u_R)\) is strictly decreasing (increasing), twice continuously differentiable and strictly concave.

It is not critical that parties’ ideal policies are located at the extremes of the policy space, only that these be on opposite sides of \(M\). Concavity simplifies the results but can be relaxed. It captures two key features of parties’ payoffs: the benefits of policy compromise by a party’s opponent always more than offset its loss from its own compromise, and parties are more willing to compromise when facing extreme policies. Given state \((I, x)\), let \(W(I, x)\) be the set of winning policies for the opposition party. Note that for any \(x \in X\) and \(J \in \{L, R\}\), \(W(J, x) = [\min\{2M - x, x\}, \max\{2M - x, x\}]\) and \(W(J, x') \subset W(J, x)\) whenever \(|x' - M| < |x - M|\). Payoffs to parties and the median voter, along with the set of winning policies \(W(L, x)\) in some state \((L, x)\), are illustrated in Figure 1.1

Transitions between states are given as follows: the current period’s winning party and policy become next period’s incumbent party and incumbent policy, respectively. Formally, define the

\[\text{Transition:} \quad (I, x) \rightarrow (J, x')\]

\[\text{Definition:} \quad W(J, x') \subset W(J, x)\]

\[\text{Property:} \quad |x' - M| < |x - M|\]

\[\text{Conclusion:} \quad W(L, x) \]

\[\text{Illustrated in Figure 1.1}\]

\[\text{In fact, my results hold if I assume instead that parties commit to implement policies only for two terms. Assuming full commitment simplifies the analysis.}\]

\[\text{This is the only tie-breaking rule consistent (in the limit) with the equilibrium paths of the model, as any rule that selects the incumbent with positive probability would lead the opposition party to prefer committing to a marginally more moderate policy that wins with probability } 1.\]
Figure 1.1: Policy Space and Parties’ Preferences.

*state transition function* $\tau : (\{L, R\} \times X) \times (X \cup \{Out\}) \to \{L, R\} \times X$ by

$$
\tau((I, x), z) = \begin{cases} 
(I, x) & \text{if } |x - M| < |z - M| \text{ or } z = \text{Out}, \\
(-I, z) & \text{if } |x - M| \geq |z - M|.
\end{cases}
$$

The dynamic game proceeds as follows: given some initial state $(I, x)$, the two parties take part in an infinite sequence of elections, where the transition between stage games is given by $\tau$. A history starting from $(I, x)$ is a sequence $\{(I^i, x^i)\}_{i=1}^{\infty} \in (\{L, R\} \times X)^{\infty}$ with $N \leq \infty$ such that $(I^1, x^1) = \tau((I, x), z)$ and $(I^i, x^i) = \tau((I^{i-1}, x^{i-1}), z^i)$ for $i > 1$ for some $z^i, z^i \in X \cup \{Out\}$. The payoff to party $J$ from terminal history $\{(I^i, x^i)\}_{i=1}^{\infty}$ starting from $(I, x)$ is

$$
\sum_{i=1}^{\infty} \delta^i_J u_J(x^i),
$$

where $\delta_J < 1$ is party $J$’s discount factor.

**Definition 1.1.** A *Markov strategy* for party $J$ is a function $\sigma_J : \{L, R\} \times X \to X \cup \{Out\}$, with the restriction that $\sigma_J(J, x) = x$ for all $x \in X$.

The restriction captures the assumption of incumbent policy persistence. Let $\Sigma_J$ be the set of Markov strategies for party $J$. Henceforth, the term strategy always refers to a Markov strategy. While the restriction to pure strategies affects the set of equilibria of the game, it does not affect the set of long-run policy outcomes, as will be clear given the results of Proposition 1.2. With slight abuse of notation, the *state path* $\{(I^i, x^i)\}_{i=1}^{\infty}$ induced by profile $(\sigma_L, \sigma_R)$ starting from $(I, x)$ is defined
recursively by
\[
(I^1, x^1) = \tau((I, x), \sigma_{-1}(I, x)), \\
(I^i, x^i) = \tau((I^{i-1}, x^{i-1}), \sigma_{-i-1}(I^{i-1}, x^{i-1})).
\]
The policy path \(\{x^i\}_{i=1}^{\infty}\) induced by \((\sigma_L, \sigma_R)\) starting from \((I, x)\) is the policy sequence of the corresponding state path. Discounted payoffs to party \(J \in \{I, -I\}\) from policy path \(\{x^i\}_{i=1}^{\infty}\) induced by \((\sigma_L, \sigma_R)\) starting from \((I, x)\) are given by
\[
V_J(\sigma_L, \sigma_R; (I, x)) = \sum_{i=1}^{\infty} \delta^{i-1} u_J(x^i).
\]

**Definition 1.2.** A Markov perfect equilibrium is a strategy profile \((\sigma_L, \sigma_R)\) such that, for each state \((R, r)\),
\[
\sigma_L(R, r) \in \arg \max_{\sigma'_L \in \Sigma_L} V_L(\sigma'_L, \sigma_R; (R, r)),
\]
and for each state \((L, \ell)\)
\[
\sigma_R(L, \ell) \in \arg \max_{\sigma'_R \in \Sigma_R} V_R(\sigma_L, \sigma'_R; (L, \ell)).
\]
Henceforth, the term equilibrium always refers to Markov perfect equilibrium. The one-deviation property allows the following characterisation of Markov perfect equilibria: \((\sigma_L, \sigma_R)\) is an equilibrium if and only if, for each state \((R, r)\),
\[
\sigma_L(R, r) \in \arg \max_{z \in X} \left\{1_{z \in W(R, r)} \left[ u_L(z) + \delta_L V_L(\sigma_L, \sigma_R; (L, z)) \right] + 1_{z \notin W(R, r)} \left[ u_L(x) + \delta_L V_L(\sigma_L, \sigma_R; (R, r)) \right] \right\},
\]
along with the corresponding condition for party \(R\) in states \((L, \ell)\). The restriction to Markov strategies limits implicit equilibrium coordination by conditioning strategies only on current states and not on entire histories of play. To shed light on parties’ long-run interactions, it seems preferable to assume that challengers’ behaviour depends on incumbents’ policies only insofar as they affect available winning policies. Given that parties square off in elections that are years apart and often involve different politicians, strategies that with all else equal differentiate between events that occurred even a few elections ago would have problematic interpretations.

**1.3 Outcomes Without Incumbent Policy Persistence**

My model’s novel policy dynamics are due to incumbent policy persistence, which transfers competition from within to across elections and dampens the incentives that lead to median convergence in standard models. To illustrate this, consider instead the repeated game in which incumbent and opposition parties simultaneously commit to policies. This stage game is the standard model of electoral competition between policy-motivated parties. To make this game history independent, the tie-breaking rule cannot depend on the identity of the incumbent party, and any
random tie-breaking rule will do. As is well known, the unique Nash equilibrium of the stage game has each party commit to the median policy\footnote{See Osborne (1995).} Call the repeated simultaneous move game the \textit{model without incumbent policy persistence}. Proposition \ref{proposition:median_policy} shows that only one of the long-run equilibrium outcomes of the model with incumbent policy persistence arises in the absence of this assumption.

**Proposition 1.1.** In the unique subgame perfect equilibrium of the model without incumbent policy persistence, parties commit to the median policy after all histories\footnote{All proofs of my results are in Appendix \ref{appendix:proofs}.}

In the model without policy persistence, any party can enforce the policy path $(M, M, ...)$ after any history by committing to the median policy in all elections and hence any subgame perfect equilibrium yields party $J$ at least the payoff $\frac{1}{1-\delta_j}u_j(M)$ following all histories. Since this payoff results from the repetition of the stage game’s unique Nash equilibrium, this is in fact $J$’s lowest subgame perfect equilibrium payoff. Concavity of parties’ payoffs ensures that when policies deviate from the median, one party loses more than the other gains, which implies that any equilibrium path that differs from the median after some history cannot simultaneously guarantee payoffs of $\frac{1}{1-\delta_j}u_j(M)$ to $J$ and $\frac{1}{1-\delta_j}u_{-j}(M)$ to $-J$. It is the asynchronicity of policy choices along with discounting which allow deviations from the median under incumbent policy persistence. In that model, party $J$ can enforce policy path $(M, M, ...)$, and hence needs to be guaranteed a payoff of at least $\frac{1}{1-\delta_j}u_j(M)$ in equilibrium, \textit{only} following histories in which it can commit to new policies. In an equilibrium in which parties alternate in office, this opportunity arises every other period. While parties understand that they may be worse off relative to policy path $(M, M, ...)$ as an incumbent, they are offered sufficient payoff in their first term of office to balance this (discounted) loss. Policy paths exhibit a form of dynamic inconsistency for incumbents since if they could, they would prefer to commit to the median policy.

**1.4 Outcomes With Incumbent Policy Persistence**

This section presents my results for the model with incumbent policy persistence.

**1.4.1 Myopic Strategies**

The myopic models of Wittman (1977) and Kramer (1977) provide benchmark strategy profiles that fail to constitute Markov perfect equilibria when parties are forward-looking. Given state $(R, r)$ with $r > M$, the unique optimal choice in the stage game for opposition party $L$ is to commit to $\max\{0, 2M - r\}$, its most extreme winning policy. If $r \leq M$, committing to policy $r$, committing to a losing policy and $\text{Out}$ are all optimal. These are the optimal actions of the myopic policy-motivated parties of Wittman (1977). Define a myopically optimal strategy for party $L$, $\sigma_{L}^{my}$, as

$$
\sigma_{L}^{my}(R, r) = \begin{cases} 
\max\{0, 2M - r\} & \text{if } r \geq M, \\
\text{Out} & \text{if } r < M.
\end{cases}
$$
Party $R$’s myopically optimal strategy, $\sigma_{RM}$, is defined symmetrically. Myopic strategies generally fail to constitute an equilibrium since when faced with an incumbent whose policy is sufficiently extreme, an opposition party finds it optimal to sacrifice present payoffs and commit to a moderate policy in order to face more moderate (myopic) opponents in future elections. In fact, Proposition 1.3 establishes the precise (yet restrictive) condition under which myopic behaviour may be dynamically optimal.

On the other hand, a naive extension of static median convergence results to my model has opposition parties commit to the median policy in all states, mimicking the equilibrium strategies of the model without incumbent policy persistence. These are the optimal actions of Kramer (1977) myopic vote-maximising parties. These strategies fail to constitute an equilibrium since an opposition party expecting future opponents that always select median policies has no incentive to win the current election with the median policy: the sole cost of winning an election with non-median policies is the extremism it may generate in opponents’ future policies.

1.4.2 Equilibrium Policy Dynamics: Alternation

The restriction to Markov strategies does not eliminate equilibrium multiplicity, and the model’s set of equilibria admits no simple description. I focus instead on characterising equilibrium outcomes, and in particular those that persist in the long-run. Long-run policy outcomes are defined, naturally, as limit points of sequences of policies induced by equilibrium dynamics from some initial state.

**Definition 1.3.** Policy $y$ is a long-run policy outcome under equilibrium $(\sigma_L, \sigma_R)$ starting from $(I, x)$ if $y$ is a limit point of the policy path induced by $(\sigma_L, \sigma_R)$ starting from $(I, x)$.

A policy that is a long-run policy outcome under some equilibrium starting from some state is called simply a long-run policy outcome. Proposition 1.2 characterises equilibrium dynamics along with the properties of their limit points.

**Proposition 1.2.** Consider some equilibrium $(\sigma_L, \sigma_R)$ and some state $(I, x)$ along with the policy path $\{y^i\}$ induced by $(\sigma_L, \sigma_R)$ starting from $(I, x)$. Suppose that $(I, x) = (R, r)$.

i. If $r \leq M$, then $y^i = r$ for all $i$.

ii. If $r > M$, then a) incumbents are always defeated on the equilibrium path, unless $y^i = M$ for some $i$, b) $\{y^i\}$ has a pair of limit points $(\hat{\ell}, 2M - \hat{\ell})$ for some $\hat{\ell} \leq M$, and c) $\sigma_L(R, 2M - \hat{\ell}) = \hat{\ell}$ and $\sigma_R(L, \hat{\ell}) = 2M - \hat{\ell}$.

The case of $(I, x) = (L, \ell)$ is symmetric.

In any equilibrium, party $L$ will stay out, or commit to some losing policy, whenever $(R, r)$ is such that $r < M$, that is, when party $R$ is on the left of the political spectrum. The policy path most favourable to $L$ that can be sustained in any equilibrium from such a state is $(r, r, r, ...)$, which $L$ can attain by failing to contest any election and trapping dynamics at the initial policy. Call policy outcome $y \neq M$ trivial if it is a long-run policy outcome under $(\sigma_L, \sigma_R)$ starting from $(I, x)$ if and
only if \( y = x \) and the policy path \( \{x^i\} \) induced by \((\sigma_L, \sigma_R)\) from \((I, x)\) is such that \( x^i = y \) for all \( i \geq 1 \). From now on, the term long-run policy outcome always refers to a long-run policy outcome that is not trivial. That is, since item \( \text{[i]} \) of Proposition \( \text{[1.2]} \) shows that all policies can be reached by some equilibrium dynamics, I restrict attention to outcomes that can be reached by nontrivial dynamics.

Item \( \text{[ii]} \) of Proposition \( \text{[1.2]} \) ensures that nontrivial equilibrium dynamics entail alternation in power and convergence to symmetric pairs of policies of the form \((\ell, 2M - \ell)\) for some \( \ell \leq M \). Figure \( \text{[1.2]} \) illustrates this result, depicting a possible policy path induced by some equilibrium profile from state \((R, r)\) with \( r > M \). On the equilibrium path, no party stays out or commits to policies that either lose or are on their opponent’s side of the median. The policy path alternates around the median and has at most a pair of limit points \((\hat{\ell}, \hat{r})\) since the sequences of each party’s winning policies are monotone. The pair of long-run policies \((\hat{\ell}, \hat{r})\) need not be reached by the policy path. Furthermore, it must be that \( \hat{r} = 2M - \hat{\ell} \). The final component of item \( \text{[ii]} \) of Proposition \( \text{[1.2]} \) states that limits of alternating equilibrium dynamics are absorbing; if the dynamics start at one of the limiting policies, they stay there.

![Figure 1.2: Illustration of Equilibrium Policy Dynamics.](image)

The proofs of Proposition \( \text{[1.2]} \) and of the results to follow depend only on properties of parties’ preferences over symmetric policy alternations, which vary according to the initial policy. To clarify this, define the functions \( \{\Psi^\theta_L : [0, M] \rightarrow \mathbb{R}\}_{\theta \in \{+, -\}} \) for party \( L \) as

\[
\Psi^+_L(\ell) = u_L(\ell) + \delta_L u_L(2M - \ell), \quad \text{and} \\
\Psi^-_L(\ell) = u_L(2M - \ell) + \delta_L u_L(\ell).
\]

The discounted sum \( \frac{1}{1 - \delta_L^L} \Psi^-_L(\ell) \) is party \( L \)’s payoff from alternation at policies \((\ell, 2M - \ell)\) starting from \( \ell \), while \( \frac{1}{1 - \delta_L^L} \Psi^+_L(\ell) \) is its payoff to the same alternation when starting from \( 2M - \ell \). Functions \( \{\Psi^\theta_R : [0, M] \rightarrow \mathbb{R}\}_{\theta \in \{+, -\}} \) for party \( R \) are defined symmetrically. Strict concavity of parties’ utility functions yields a natural preference order over symmetric alternations, whose properties are collected in the following lemma.
Lemma 1.1. There exist uniquely defined policies \( \ell^* \) and \( r^* \) such that

\[
\ell^* = \arg \max_{\ell \in [0, M]} \Psi_L^+(\ell) \in [0, M], \quad \text{and}
\]

\[
r^* = 2M - \arg \max_{\ell \in [0, M]} \Psi_R^+(\ell) \in (M, 2M].
\]

\( \Psi_L^-(\Psi_R^-) \) is strictly increasing (decreasing) and both \( \Psi_L^+ \) and \( \Psi_R^- \) are strictly concave for \( J \in \{L, R\} \).

Given \( \ell \in [0, M) \), the concavity of \( u_L \) ensures that the cost to \( L \) of a moderate move away from \( \ell \) is dominated by the benefit of a moderate move away from \( 2M - \ell \). That \( \Psi_L^+ \) is single-peaked around \( \ell^* < M \), \( L \)'s favoured alternation, follows from discounting. When the payoff to \( L \) from alternating pairs are evaluated starting from \( L \)'s policy, a shift to a more moderate alternation ensures that party \( L \) suffers the full loss to moderation in its own policy, while the larger benefit of \( R \)'s moderation is discounted. For any \( \delta_L < 1 \), \( \ell^* \) is bounded away from the median as

\[
\lim_{\ell \to M} u_L(\ell) = \lim_{\ell \to M} u_L(2M - \ell).
\]

Policies \( \ell^* \) and \( r^* \) are key in the characterisation of long-run policy outcomes. When \( \ell^* > 0 \), it is uniquely determined by

\[
\frac{u_L'(\ell^*)}{u_L'(2M-r^*)} = \delta_L,
\]

is increasing in \( \delta_L \) and converges to \( M \) as \( \delta_L \) converges to 1. As party \( L \) becomes less short-sighted, the cost of \( R \)'s future policies increases and its preferred alternation comes closer to the median. Similarly, \( \ell^* \) is increasing in \( L \)'s disutility for policies away from its ideal point, captured by the concavity of \( u_L \).

Meanwhile, when the payoffs to \( L \) from alternations are evaluated starting from \( R \)'s policy, \( L \) always prefers more moderate alternations. In particular, \( L \)'s favoured alternation is that around \( M \) since \( L \)'s loss from moderating its own policy, smaller than \( L \)'s gain from \( R \) moderating its policy, is discounted.

### 1.4.3 Long-Run Policy Outcomes: Bounded Extremism

Proposition 1.3 shows that long-run policy outcomes admit a simple characterisation and display *bounded extremism*. That is, while sufficiently extreme policies can be observed on some equilibrium paths, they are transient.

**Proposition 1.3.** Policy \( \ell \leq M \) is a long-run policy outcome if and only if \( \ell \in [\max\{\ell^*, 2M-r^*\}, M] \).

Figure 1.3 illustrates Proposition 1.3 when \( \ell^* \geq 2M - r^* \). The dotted section of the policy space indicates the set of long-run policy outcomes. All symmetric policy pairs more moderate than \( (\ell^*, 2M-\ell^*) \), such as \( (\ell, r) \) and \( (\ell', r') \), are long-run policy outcomes, with \( (\ell^*, 2M-\ell^*) \) being the most extreme such pair. The bound on long-run extremism follows for the same reason, given in Section 1.4.1 that myopically optimal strategies fail to constitute an equilibrium: facing a sufficiently extreme alternation (in the long-run), some party will prefer to rein in future opponents’ policies by committing to more moderation. Proposition 1.3 makes precise how long-run deviations from the median are driven by parties’ myopic preference for their own policies. If parties are close to myopic, a wide range of alternating outcomes are supported in equilibrium.

\(^{15}\)Consider \( v_L \) obtained from \( u_L \) by applying some increasing concave transformation. Then for any \( \ell \in (0, M) \),

\[
\frac{v_L'((\ell))}{v_L'(2M-r)} < \frac{v_L'((\ell))}{v_L'(2M-r)},
\]

and hence \( \ell^* \) approaches \( M \) as parties’ utilities become more concave.
However, if only one party is arbitrarily far-sighted, policies are arbitrarily close to the median in the long-run.

There are two steps to the proof of Proposition 1.3. The first establishes the existence of the bound on extremism, given by \[ \max \{ \ell^*, 2M - r^* \} \]

This step hinges on a useful lower bound on party \( L \)'s equilibrium payoff: any equilibrium path following a commitment to some winning policy \( \ell \) yields a payoff of at least \[ \frac{1}{1 - \delta_L^+} \Psi_L^+ (\ell) \]

To see this, consider a strategy for opposition party \( L \) which sets policy \( \ell \) in the current election and responds myopically to all of \( R \)'s subsequent policies. The payoff to \( L \) from this strategy is \[ u_L (\ell) \] in this election, along with a sequence of payoffs \{ \Psi^-L (r_i) \} in the subsequent pairs of elections, for some sequence of policies \{ r_i \} such that \[ \ell \leq 2M - r_i \] for all \( i \). By Lemma 1.1, each payoff in this sequence is at least \[ \frac{1}{1 - \delta_L^+} \Psi_L^+ (\ell) \] and hence the payoff to selecting winning policy \( \ell \) must be at least \[ \frac{1}{1 - \delta_L^+} \Psi_L^+ (\ell) \]. If \( \ell < \ell^* \) were a long-run policy outcome, party \( L \) could win the election in state \( (R, 2M - \ell) \) by committing to policy \( \ell^* \) and guarantee itself a payoff of at least \[ \frac{1}{1 - \delta_L^+} \Psi_L^+ (\ell^*) \], its preferred alternation. However, \( L \)'s equilibrium payoff in state \( (R, 2M - \ell) \) is \[ \frac{1}{1 - \delta_L^+} \Psi_L^+ (\ell) \], yielding the desired contradiction.

The second step in the proof of Proposition 1.3 shows that the bound on long-run extremism is tight by constructing an equilibrium under which all policies \( \ell \in [\max \{ \ell^*, 2M - r^* \}, M] \) are long-run policy outcomes. Consider the strategy \( \sigma^\ell_L \) such that

\[
\sigma^\ell_L (R, r) = \begin{cases} 
\ell^* & \text{if } r \in [2M - \ell^*, 1], \\
2M - r & \text{if } r \in [M, 2M - \ell^*), \\
Out & \text{if } r \in [0, M). 
\end{cases}
\]

In Appendix 1.7, I show that if \( \ell^* \geq 2M - r^* \), then \( (\sigma^\ell_L, \sigma^\text{my}_R) \) is an equilibrium. If \( \ell^* < 2M - r^* \), strategy \( \sigma^\ell_R \) can be defined by reversing the roles of the two parties and then \( (\sigma^\text{my}_L, \sigma^\ell_R) \) is an equilibrium. These equilibria provide the exact condition under which parties' behaviour in Wittman (1977) can be said to be dynamically rational: myopically optimal strategies form an
equilibrium if and only if $\max\{\ell^*, 2M - r^*\} = 0$.

Figure 1.4, depicting the interval $[0, M]$, illustrates equilibrium strategies $(\sigma_L^{\ell^*}, \sigma_R^{my})$. The directed curve above (below) the interval from point $\ell$ represents the equilibrium action of party $L$ ($R$) in state $(R, 2M - \ell)$ ($(L, \ell)$). In equilibrium, from any $(L, \ell)$ with $\ell < \ell^*$ or $(R, r)$ with $r > 2M - \ell^*$ policies settle on alternation $(\ell^*, 2M - \ell^*)$ in at most two elections.

In moderate states $(L, \ell)$ for some $\ell \geq \ell^*$ and $(R, r)$ for some $r \leq 2M - r^*$, both parties respond myopically. In these states their preferences over alternations coincide; both prefer more extreme alternations when evaluated starting from their own policy. Parties’ preferences over alternations also coincide in extreme states $(L, \ell)$ for some $\ell < 2M - r^*$ and $(R, r)$ for some $r > r^*$. In these states, both parties prefer more moderate alternations when evaluated starting from their own policy. However, having both parties committing to more moderate policies cannot be an equilibrium and some party, in this case $L$, must be responsible for bringing policy dynamics towards more moderate alternations. Since party $R$ knows party $L$ will commit to $\ell^*$ in the next election against any winning policy $r \in [2M - \ell^*, 2M - \ell]$ it champions in the current election, committing to myopic policy $2M - \ell$ is optimal. For intermediate states $(L, \ell)$ for some $\ell \in [2M - r^*, \ell^*)$ and $(R, r)$ for some $r \in (2M - \ell^*, r^*)$, parties’ preferences over alternations diverge and party $L$, which prefers more moderate pairs, ensures that policy paths converge.

1.4.4 Robust Long-run Policy Outcomes: Bounded Moderation

A long-run policy outcome $y$ is the limit of equilibrium policy dynamics given some initial state. In particular, ‘steady state’ outcome $y$ need not be dynamically stable in the following sense: given an initial state with policy more extreme than $y$, equilibrium policy dynamics need not have $y$ as a limit point. For example, in the equilibrium $(\sigma_L^{\ell^*}, \sigma_R^{my})$, all policies $\ell \in (\ell^*, M]$ occur in the long-run only starting from $(L, \ell)$ or $(R, 2M - \ell)$.

**Definition 1.4.** Policy $y$ is a robust long-run policy outcome if it is a long-run policy outcome under some equilibrium $(\sigma_L, \sigma_R)$ starting from some state $(I, x)$ such that $x$ is not a long-run policy
outcome under \((\sigma_L, \sigma_R)\) starting from \((I, x)\).

Long-run policy outcomes that are not robust are poor predictions of equilibrium play since they fail to arise given any different initial state. Robustness is a weak requirement of dynamic stability as it necessitates only the existence of a single policy \(x\) that lies on an equilibrium path that has \(y\) as a limit point.\(^{16}\) Verifying robustness requires information about equilibrium convergence paths to ‘steady states’. This is a difficult task for general Markov perfect equilibria, since equilibrium paths in which parties have multiple best-responses can support complex equilibrium coordination by allowing deviations to trigger continuation play that is quite different from equilibrium play. While I restrict parties to Markov strategies precisely to eliminate such coordination, the standard definition of the state yielded by the coarsest partition of strategically equivalent histories is not sufficient to do this in my model.\(^{17}\) This suggests that a refinement of Markov strategies is required.

Furthermore, the asynchronous structure of my model has each party solve a single-agent decision problem when in opposition so that general Markov strategies lead to choice behaviour that should be labelled as inconsistent. In particular, opposition parties should not condition on the exact policy of the incumbent when choosing policies in the interior of their set of winning policies. A party which commits to a moderate policy is unconstrained by the incumbent’s policy, and hence facing a slightly more moderate incumbent should not lead it to change its policy choice. Figure 1.5 illustrates this requirement. Suppose that party \(L\) chooses winning policy \(\ell > 2M - r\) from set of winning policies \([2M - r, r]\) for some \(r > M\). Consistency requires that Party \(L\) choose the same policy from a set of winning policies \([2M - r', r]\) for some \(r' \in [M, r)\) such that \(2M - r' < \ell\). The choice of any other policy from the smaller set of winning policies could be justified by equilibrium considerations, but not by any fundamental political constraints.

![Figure 1.5: Consistent Markov Strategies.](image)

\(^{16}\)Note that trivial long-run policy outcomes are not robust.

\(^{17}\)See Fudenberg and Tirole (1991). The definition of the state cannot be refined through the coarsest common consistent partition of histories from Maskin and Tirole (2001). Since parties never move simultaneously after any history, they need not share a common consistent partition, and the results of Maskin and Tirole (2001) do not yield more than strategic equivalence in my model.
The main result of this section is a characterisation of the set of robust long-run policy outcomes under equilibria in consistent strategies.\footnote{\textup{A class to which, notably, the equilibria $(\sigma^*_L,\sigma^*_R)$ and $(\sigma^*_L,\sigma^*_R)$ belong.}} As foreshadowed in the previous discussion, it turns out that requiring that parties choices not display the inconsistencies illustrated above eliminates the patterns of indifference that can lead to complex coordination off the equilibrium path. The definition of consistent strategies highlights the restriction’s relationship to standard requirements for choice behaviour.

**Definition 1.5.** Markov strategy $\sigma_{-1}$ is consistent if for any pairs of states $(I,x)$ and $(I,x')$, whenever

i. $\tau((I,x),\sigma_{-1}(I,x)) = \tau((I,x'),\sigma_{-1}(I,x))$, and

ii. $\sigma_{-1}(I,x) \neq \sigma_{-1}(I,x')$,

then $\tau((I,x),\sigma_{-1}(I,x')) \neq \tau((I,x'),\sigma_{-1}(I,x'))$.

A consistent Markov perfect equilibrium is a Markov perfect equilibrium in consistent Markov strategies.

Note that if $\tau((I,x),z) = \tau((I,x'),z)$ for opposition party policy $z$ that is winning in both states $(I,x)$ and $(I,x')$, then the sequences of policies induced by $z$ are the same in both states. Hence, Definition 1.5 states that if $\sigma_{-1}(I,x)$ induces identical outcomes in both states $(I,x)$ and $(I,x')$ and $\sigma_{-1}(I,x)$ is not chosen in state $(I,x')$, then $\sigma_{-1}(I,x')$ cannot induce identical outcomes in both states. Consistency goes beyond requiring that parties condition only on payoff relevant information since the histories that are considered relevant to their decisions are defined relative to their states. Consistency goes beyond requiring that parties condition only on payoff relevant information since the histories that are considered relevant to their decisions are defined relative to their states. Consistency goes beyond requiring that parties condition only on payoff relevant information since the histories that are considered relevant to their decisions are defined relative to their states. Consistency goes beyond requiring that parties condition only on payoff relevant information since the histories that are considered relevant to their decisions are defined relative to their states.

Proposition 1.4 characterises robust long-run outcomes under consistent strategies and shows that they display bounded moderation. This does not contradict the results of Section 1.4; centripetal forces are present and policy paths tend to converge toward the median. However, policies do not converge to the median. The model admits median politics as a long-run policy outcome only if the initial incumbent party champions the median, otherwise parties remain differentiated and settle into clearly defined party identities.

**Proposition 1.4.** There exists $\ell^{**} \in (\max\{\ell^*,2M-r^*\},M)$ such that policy $\ell \leq M$ is a robust long-run policy outcomes in consistent Markov strategies if and only if $\ell \in [\max\{\ell^*,2M-r^*\},\ell^{**}]$.

Figure 1.6 illustrates Proposition 1.4. The dotted line indicates the set of long-run policy outcomes, while the dashed line indicates the subset of these pairs that are robust under equilibria in consistent strategies. For example, both policies in pair $(\ell,r)$ are robust, while policies in pair $(\ell',r')$, more moderate than $(\ell^{**},2M-\ell^{**})$, are not. On equilibrium convergence paths a party’s commitment to a more moderate policy must be reciprocated in future elections by its opponent.
When converging to sufficiently moderate policy alternations, parties' value their opponents' (discounted) moderate moves so little that they are unwilling to commit to policies moderate enough to sustain convergence.

When studying the convergence outcomes of the model, it is convenient to focus on the symmetric images of party $R$'s policies with respect to the median, mapping converging dynamics into a single increasing sequence of policies. The convergence path $\{y^i\}$ to policy $\hat{\ell} \in (0, M]$ under equilibrium $(\sigma_L, \sigma_R)$ starting from $(I, x)$ is a sequence such that

i. If $(I, x) = (R, r)$ for some $r > 2M - \hat{\ell}$, then $y^i = x^i$ for $i$ odd and $y^i = 2M - x^i$ for $i$ even, where $\{x^i\}$ is the sequence of policies induced by $(\sigma_L, \sigma_R)$ starting from $(I, x)$.

ii. If $(I, x) = (L, \ell)$ for some $\ell < \hat{\ell}$, then $y^i = x^i$ for $i$ even and $y^i = 2M - x^i$ for $i$ odd.

iii. $\{y^i\} \rightarrow \hat{\ell}$.

Consistent strategies allow simple characterisations of parties' policy choices and payoffs on equilibrium convergence paths. Lemma 1.3 in Appendix 1.7 characterises strategies along convergence paths in consistent strategies and is illustrated in Figure 1.7 showing a section of some convergence path $\{y^i\}$ initiated by party $R$ committing to policy $2M - y^i$, to which $L$ responds by moderating to $y^{i+1}$. By consistent strategies, $\sigma_L(R, r) = y^{i+1}$ for all $r \in (2M - y^i, 2M - y^{i+1}]$, that is, $L$ moderates to $y^{i+1}$ when facing an incumbent $R$ championing a policy more moderate than $2M - y^i$. Furthermore, consistency implies that $\sigma_R(L, \ell) = 2M - \ell$ for all $\ell \in [y^i, y^{i+1})$, that is, $R$ responds myopically whenever $L$ stops short of moderating to $y^{i+1}$.

As noted in Section 1.4.1, it is optimal to respond myopically to an opponent that always selects the median policy. Figure 1.8 shows that consistent equilibria display this behaviour locally. That

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19Note that if policy $\hat{\ell} \leq M$ is a robust long-run policy outcome under consistent equilibria and $y^i < \hat{\ell}$ is on a convergence path to $\hat{\ell}$, then all $\ell \in (y^i, \hat{\ell})$ are also on a convergence path to $\hat{\ell}$. In this sense, convergence outcomes under consistent equilibria can be said to be 'strongly' robust since convergence to $\hat{\ell}$ occurs from all more extreme states.
Figure 1.7: Convergence Paths under Consistent Equilibria.

is, consistent equilibrium convergence paths define alternating sets of policies in which a locally myopic party meets a locally moderate party. Parties stake out non-negotiable ‘core’ issues and their opponents compromise on the corresponding policies on the other side of the median. The location of parties’ core issues may seem idiosyncratic since they compromise over neighbouring policies. However, core issues are not due to parties’ preferences for particular policies but arise endogenously as a tool to sustain policy convergence.

Section 1.4.3 noted that \( \frac{1}{1-\delta_L} \Psi_L^+(y^i) \) is a lower bound on \( L \)'s payoff in state \((R, 2M - y^i)\). Lemma 1.4 in Appendix 1.7 shows that if \( 2M - y^i \) lies on a consistent equilibrium convergence path then this payoff is also an upper bound. That is, \( L \)'s payoff at \((R, 2M - y^i)\) is computed ‘as though’ equilibrium dynamics were absorbed by an alternation at the symmetric pair of policies \((y^i, 2M - y^i)\). However, since in state \((R, 2M - y^{i+2})\) party \( L \) receives the payoff to an alternation at \((y^{i+2}, 2M - y^{i+2})\), its payoff upon gaining office on convergence paths to policies more moderate than \( \ell^* \) is strictly decreasing and after each spell in opposition, parties regret their previous moderate policies. Lemma 1.4 also shows that \( L \)'s payoff in state \((R, 2M - y^i)\) satisfies

\[
\Psi_L^+(y^i) - \Psi_L^+(y^{i+1}) = \delta_L \left[ \Psi_L^-(y^{i+2}) - \Psi_L^-(y^{i+1}) \right].
\] (1.1)

The left-hand side of (1.1) is the cost (computed in payoffs to alternations starting from \( L \)'s policy) of choosing moderate policy \( y^{i+1} \) while the right-hand side is the (discounted) benefit (computed in payoffs to alternations starting from \( R \)'s policy) of party \( R \)'s subsequent moderate move to \( 2M - y^{i+2} \). These costs and benefits are balanced by the choice of \( y^{i+1} \). Moderation is self-reinforcing: if parties anticipate an end to convergence in the future current incentives to choose moderate policies unravel. That is, if \( y^i \geq \ell^* \) then (1.1) cannot be satisfied for \( y^{i+2} = y^{i+1} \) unless \( y^{i+1} = y^i \).

Equation (1.1) also explains why party \( L \) is willing to sustain convergence paths to alternations

\[\text{In fact, this holds for all equilibria, not just those in consistent strategies. For the same reasons as above, but without relying on payoff condition (1.1), it can be shown that only the most extreme alternating outcomes, (max}\{\ell^*, 2M - r^*\}, 2M - \text{max}\{\ell^*, 2M - r^*\}\), are ever reached from a more extreme state in a finite number of elections in any equilibrium.\]
more moderate than \((\ell^*, 2M - \ell^*)\), that is, why \(\ell^{**} > \ell^*\). Around \(\ell^*\), the cost of moving to a more moderate alternation is of second-order importance, while the benefit of \(R\)'s moderation is of first-order importance. Around \(\ell^*\), \(L\) is willing to bear almost all of the cost of sustaining convergence.

The recursive relationship in (1.1), along with the corresponding relationship for party \(R\), allow the derivation of the bound on moderation \(\ell^{**} \in (\ell^*, M)\). Fix one round of moderation from \((R, 2M - y^i)\) as the moves, first by \(L\), then by \(R\), that take the state to \((R, 2M - y^{i+2})\). Then (1.1) describes the share of the total moderation \(y^{i+2} - y^i\) that \(L\) is willing to undertake. The bound \(\ell^{**}\), derived explicitly in Appendix 1.7, is the most moderate policy for which the parties' ‘supply’ of moderation is consistent with convergence in the limit as \(y^{i+2} \to y^i\). Convergence to moderate policies fails as the shares of any given round of moderation that parties are willing to undertake become too small. To see this, consider the polar case of convergence to the median. As a convergence path approaches \(M\), moderate moves of similar sizes by parties \(L\) or \(R\) have similar effects on \(L\) payoffs, yet the gain from \(R\)'s moderation is discounted. Since the same observation holds for \(R\), both parties require their opponents to make larger moderate moves than they do, which contradicts convergence.

As in section 1.4.3, the bound \(\ell^{**}\) is shown to be tight through the construction of equilibria. In that section, a single equilibrium yields all long-run policy outcomes. Here, an equilibrium under which policy \(\hat{\ell} \in (\max\{\ell^*, 2M - r^*\}, \ell^{**})\) is a robust long-run policy outcome is constructed for each such \(\hat{\ell}\). Given a policy path \(\{y^i\}\) such that \(y^0 = \ell^*, \{y^i\} \to \hat{\ell}\) and satisfying (1.1), Appendix 1.7 provides the equilibrium strategies and verifies their optimality. As in equilibrium \((\sigma^L_\ell, \sigma^R_{my})\), policies from any state more extreme than \(\ell^*\) move rapidly to \(\ell^*\), and from there a convergence path ensures they approach \(\hat{\ell}\). The key step is to show that the sequence \(\{y^i\}\) exists, which follows by iterating the recursive relationship in (1.1) forward from \(y^0 = \ell^*\) through the choice of \(y^1\) and establishing the conditions under which this operation defines a converging policy path. Given any \(\hat{\ell} \in (\ell^*, \ell^{**})\), some policy \(y^1 > \ell^*\) can be found such that \(\{y^i\} \to \hat{\ell}\). From above, when \(\hat{\ell} < \ell^{**}\) the share of moderation around \(\hat{\ell}\) that parties are willing to undertake exceeds the amount of moderation that needs to be allocated to sustain convergence. The result hinges on the concavity of \(\Psi^L_+\) and \(\Psi^-\), as this ensures that parties become less willing to compromise as policies get closer to the median and hence the share of moderations that parties are willing to undertake at all \(\ell'\) with \(\ell' < \ell < \ell^{**}\) are larger than those they are willing to undertake at \(\ell\).

1.5 Discussion and Extensions

1.5.1 Forward-looking Voters

Myopic voting guarantees that all future governments are at least as moderate as the current incumbent. However, forward-looking voters may choose to elect opposition parties with more extreme platforms than incumbents if this generates preferred continuation play. On any equilibrium convergence path of the model, the median voter has no incentive to support the incumbent since by voting against a (weakly) more moderate opposition, it is worse off in this election and faces the same choice in the next election. More generally, the median voter approves of converging equilibrium outcomes and has an incentive to vote for opposition parties that propose
moderate policies. However, equilibria with myopic voters do not persist as equilibria of a game in which voters are forward-looking. As the previous discussion suggests, the difficulties with myopic voting arise off equilibrium convergence paths.

Consider the extension of the model in which voters are forward-looking. I restrict attention to equilibria in which the median voter is decisive\(^{21}\) and consider a single representative median voter with utility function \(u_M\) and discount factor \(\delta_M\). A strategy for the voter is \(\sigma_M : (\{L, R\} \times X) \times (X \times \{Out\}) \rightarrow \{0, 1\}\), where \(\sigma_M((I, x), z) = 0\) if and only if the median voter supports incumbent \(I\) with policy \(x\) in an election opposing it to \(-I\) with policy \(z\). Assume that the median voter never abstains so that in particular \(\sigma_M((I, x), Out) = 0\) for all \((I, x)\). Denote the set of strategies for \(M\) as \(\Sigma_M\). As in Section 1.2, a profile of strategies \((\sigma_L, \sigma_R, \sigma_M)\) along with state \((I, x)\) determines discounted payoff \(V_I(\sigma_L, \sigma_R, \sigma_M; (I, x))\) for player \(J \in \{L, R, M\}\).

**Definition 1.6.** A Markov perfect equilibrium with forward-looking voters is a strategy profile \((\sigma_L, \sigma_R, \sigma_M)\) such that for each state \((I, x)\), (i) given \(\sigma_M\), \((\sigma_L, \sigma_R)\) form a Markov perfect equilibrium, and (ii) for any policy \(z\),

\[
\sigma_M((I, x), z) \in \arg \max_{\sigma'_M \in \Sigma_M} V_M(\sigma_L, \sigma_R, \sigma'_M; (I, x)).
\]

To see that equilibria in which parties use consistent strategies are not equilibria with forward-looking voters, consider a consistent equilibrium convergence path \(\{y^t\}\), a policy \(y^t\) such that \(\sigma_L(R, 2M - y^t) = y^{t+1}\), a state \((L, y')\) for some \(y' \in (y^t, y^{t+1})\) and a deviation by \(R\) to \(2M - y' + \epsilon\) for some \(\epsilon < y' - y^t\). The median voter’s myopic strategy, \(\sigma_M^{my}\), calls for a vote against \(R\). If it does so, its payoff \(V^L_M\) is given by

\[
V^L_M = u_M(y') + \delta_M u_M(2M - y') + \delta^2_M V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y')).
\]

If instead the median voter votes for \(R\), its payoff \(V^R_M\) is given by

\[
V^R_M = u_M(2M - y' + \epsilon) + \delta_M V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y')).
\]

By symmetry of \(u_M\),

\[
\lim_{\epsilon \to 0} \left( V^L_M - V^R_M \right) = \delta_M(1 - \delta_M) \left[ \frac{1}{1 - \delta_M} u_M(2M - y') - V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y')) \right] < 0,
\]

since the equilibrium path following \((R, 2M - y')\) consists of a converging path of policies all strictly closer to the median than \(2M - y'\). In the sections of the convergence path in which party \(R\) responds myopically, the median voter finds it costly to punish extreme deviations by party \(R\). To do so, it must vote for the incumbent party \(L\) and keep it in power for another term, but this delays \(R\)’s (myopic) victory by one period and the resumption of convergence by two periods.

---

\(^{21}\)In general, this entails more restrictive assumptions on voters’ preferences than those used so far. Banks and Duggan (2006) show that sufficient conditions for median decisiveness is that all voters have quadratic utilities and a common discount factor.
Voting for deviating party $R$ in this election lets a more moderate party $L$ gain office in the next election.

Proposition 1.5 shows that given any alternating consistent equilibrium convergence path, it is possible to construct voter and party strategies that enforce this path in an equilibrium with forward-looking voters. Hence, the equilibrium outcomes of this paper are not due to myopic voting.

**Proposition 1.5.** Consider consistent equilibrium $(\sigma_L, \sigma_R)$ in the game with myopic voters. Consider state $(I, x)$ such that $I = L$ and $x \leq M$ or $I = R$ and $x \geq M$, along with policy path $\{y^i\}$ induced from $(I, x)$ by $(\sigma_L, \sigma_R)$. Then there exist an equilibrium with forward-looking voters $(\sigma'_L, \sigma'_R, \sigma_M)$ such that the policy path $\{y'^i\}$ induced from $(I, x)$ by $(\sigma'_L, \sigma'_R, \sigma_M)$ is such that $y^i = y'^i$ for all $i \geq 2$.

In the equilibrium $(\sigma'_L, \sigma'_R, \sigma_M)$, the median voter sometimes votes against myopically preferred policies. In particular, in the sections of the convergence path in which party $R$ responds myopically under consistent strategies, the median voter supports more extreme policies by $R$ to ensure a quicker resumption of convergence. The equilibrium strategies are illustrated in Figure 1.8. Consider policy $y^i$ such that $\sigma_L(R, 2M - y^i) = y^{i+1}$. In Appendix 1.7 for $\ell \in [y^i, y^{i+1})$ I define function $z^{i+1}(\ell) \in [y^i, \ell)$ such that $2M - z^{i+1}(\ell)$ is the most extreme policy by $R$ supported by the median voter against $\ell$ in state $(L, \ell)$. Note that to ‘resume’ convergence, the median voter is never willing to support a policy by $R$ that is more extreme than $2M - y^i$. Suppose, for example, that the median voter supported a proposal $r \in (2M - y^i, 2M - y^{i-1})$. In the next election, $L$ does not commit to a moderate policy (in fact it commits to $z^i(r) \in [y^{i-1}, r)$), and it takes two elections to return to state $(R, 2M - y^i)$. Against this, the median voter prefers to vote against $R$ and wait two elections to arrive at the more moderate state $(L, y^{i+1})$.

![Figure 1.8: Policy Dynamics of Equilibria with Forward-looking Voters.](image-url)

When voters are myopic, in states $(L, \ell)$ with $\ell > M$ party $R$ does not participate in any elections and policy dynamics get trapped. When the median voter is forward-looking, it may vote in favour of policies by $R$ that are more extreme than $\ell$ but lead to renewed convergence. Hence the result of Proposition 1.5 applies only to alternating convergence paths, and in Appendix
I show that parties on alternating convergence paths have no incentive to commit to a policy on their opponent’s side of the median solely to have convergence eventually resume from a more extreme initial state.

1.5.2 Legislative Bargaining

This section discusses in detail the relationship between my paper and the dynamic legislative bargaining model of Baron (1996), which features a single-dimensional policy space. There are three crucial differences between my paper and Baron (1996). First, I do not assume that the median voter is represented by a party that shares its preferences over policies. Second, incumbent policy persistence generates a history-dependent proposer recognition rule. Finally, the equilibria I construct in Section 1.4.4 have the median voter strictly prefer to support opposition parties on the equilibrium path.

To view my model as a legislative bargaining model, reinterpret voters as legislators, with $M$ denoting the ideal policy of the median legislator. However, in contrast to Baron (1996), only two legislators can be recognised to propose policies; these are legislators $L$ and $R$ that have ideal policies 0 and 1. For simplicity, assume that they are recognised each period with equal probability. In the legislative bargaining model, a state $(I, x)$ consists of the current proposer along with the status quo. A proposal strategy for party $I$ is $\sigma_I : \{I\} \times X \rightarrow X$ 22 As above, I assume that the median legislator is decisive in equilibrium. Consider voting strategies $\sigma_M$ for the median legislator, where now $\sigma_M((I, x), z) = 0$ if and only if $M$ supports the status quo. An equilibrium of the legislative bargaining game is as in Definition 1.6 with the relevant reinterpretations.

A convergence path \{y_i\} in the legislative bargaining game is as defined above but its description no longer corresponds to the realised equilibrium policy path. Given a strategy for the median legislator, consistent proposal strategies for voters are as in Definition 1.5 In Appendix 1.7 I show that myopic voting is optimal for the median legislator when facing consistent strategies. Since it is without loss of generality to assume that in any equilibrium the median legislator supports proposal $y_{i+1}$ in state $(L, y_{i+1})$, consistent Markov proposal strategies along with $\sigma_M = \sigma^M_M$ imply that if $\sigma_L(L, 2M - y_i) = y_{i+1}$, then $\sigma_L(L, y_{i+1}) = y_{i+1}$. Hence under consistent proposal strategies a convergence path describes a lottery over equilibrium policy paths; policy dynamics are staggered and the status quo may remain unchanged for some time while the same legislator is recognised several periods in a row. When a new legislator is recognized, the status quo resumes its convergence.

Proposition 1.6 shows that the nonconvergence result of Proposition 1.4 is due to the median legislator never being recognised.

**Proposition 1.6.** In any equilibrium of the legislative bargaining model in consistent proposal strategies, any limit point of some convergence path from state $(I, x)$ with $x \neq M$ is bounded away from $M$.

The proof shows that the main features of the results of Section 1.4.4 in particular those concerning convergence path payoffs under consistent strategies, can be reproduced in the legislative

---

22 It is the norm in legislative bargaining models to describe the state as solely the status quo, before a new proposer is drawn. I model the state as being described after a proposer has been drawn simply to maintain consistency in notation with the earlier sections. This also explains the use of the redundant notation $\sigma_I(I, x)$ for party $I$’s strategy.
bargaining setting. I do not derive the conditions for the existence of convergence paths, but these hinge on assumptions about parties’ preferences over the staggered versions of alternating outcomes. Discounting ensures that parties have a preferred such staggered alternation that is bounded away from the median. As in my main model, convergence beyond these preferred staggered alternations requires convergence paths satisfying conditions like those of (1.1). It is also clear that convergence paths cannot approach the median for the same reasons as in my model.

Baron (1996) characterises an equilibrium in which the median voter is indifferent between supporting the status quo and the new proposal in all periods. On the convergence paths of consistent equilibria, the median voter strictly prefers to vote against the status quo. The equilibrium of Baron (1996) is in fact closely related to the equilibrium \((\sigma^L_L, \sigma^R_R)\) of Section 1.4.3. In that equilibrium, when play has reached a symmetric alternation, the median voter is indifferent between both parties’ policies. Given continuation play, it would vote for any more moderate policy, since this leads to more moderate alternations, and vote against all more extreme policies.

An iid recognition rule makes it easier to verify that myopic voting is optimal for the median legislator. Consider, for example, the problematic states for myopic voting under incumbent policy persistence. Take \(y^i\) such that \(\sigma_L(L, 2M - y^i) = y^{i+1}\), and consider state \((R, \ell)\) for \(\ell \in (y^i, y^{i+1})\). \(R\) is expected to propose \(r = 2M - \ell\). Suppose it deviates to \(r' \in (r, 2M - y^i]\). If the median legislator supports \(R\), policy \(r'\) is passed and in the next period the median legislator faces a lottery between a freezing of convergence at \(r'\) and a resumption of convergence by \(L\) proposing \(y^{i+1}\). If instead it supports the status quo, in the next period the median legislator faces a lottery between a freezing of convergence at \(\ell\) and a resumption of convergence by \(L\) proposing \(y^{i+1}\). The median legislator supports the status quo since \(|M - \ell| < |M - r'|\). Since the median legislator does not affect the lottery over future proposers by its vote, it faces no cost to punish deviations.

1.6 Conclusion

This paper has studied the policy dynamics of a game of electoral competition between two policy-motivated parties. Although incumbent policy persistence allows opposition parties to win elections with extreme policies, an incentive to commit to more moderate policies is generated by the benefits of imposing moderation on future opponents. At some opportunity cost which consists of foregone policy gains in the current election, parties can, and in equilibrium do, commit to more moderate future electoral outcomes by championing moderate policies. Furthermore, since the incentives to moderate vanish as policies approach the median, convergence toward the median is a dynamically robust phenomenon, while convergence to the median is not.

The rich policy dynamics of the model are generated by incumbent policy persistence. It is not unrealistic to suggest that incumbents and challengers are subjected to different standards by voters. In an election, incumbent politicians typically have little choice but to ‘run on their record’. Their performance in office is fresh in the minds of voters, who have had years to derive information about incumbents’ aptitudes and preferences from their decisions. Compounding this effect, opposition candidates or parties often elaborate and expound their platforms relative to the policies enacted by incumbents. Whatever the accepted evaluation of a politician’s or party’s term in office, incumbents can only have marginal success in drawing voters’ attention away from their
record. As a consequence, their ability to propose policies to voters that differ considerably from those they championed while in office is constrained. Office-holding politicians are acutely aware of this and act accordingly. In a recent example, while less than a year into his first term, Barack Obama already frames his efforts to pass a health care reform bill through its effects on a bid for reelection which is more than three years away: ‘I intend to be president for a while and once a bill passes, I own it. And if people look and say, ‘You know what? This hasn’t reduced my costs[,] … insurance companies are still jerking me around,’ I’m the one who’s going to be held responsible.’

1.7 Appendix

Proof of Proposition 1.1. As noted in the text, \( \frac{1}{1-\delta} u_J(M) \) is a subgame perfect equilibrium payoff for party \( J \) following any history. Since party \( J \) can always enforce this payoff by committing to policy \( M \) following any history, this payoff is the lowest SPE payoff for \( J \). Hence a policy path \( \{y^i\} \) is a subgame perfect equilibrium policy path only if \( \sum_{i=0}^{\infty} \delta^i u_J(y^i) \geq \frac{1}{1-\delta} u_J(M) \) for all \( J \) and all \( i \).

The first step in the proof shows that the game’s only subgame perfect equilibrium policy path following any history is the indefinite repetition of the median policy. Strict concavity is needed to ensure that if \( y \neq M \) is strictly on party \( J \)’s side of the median, then \( u_J(y) - u_J(M) < u_J(M) - u_{\neg J}(y) \). This holds since any strictly concave functions \( u_L \) and \( u_R \) defined on \([0,1]\) with \( u_L \) strictly decreasing and \( u_R \) strictly increasing can be normalised such that \( |u_L'(M)| = |u_R'(M)| \). Suppose \( y < M \). By strict concavity, for all \( \ell \in [y,M] \) we have \( |u_L'(\ell)| < |u_L'(M)| = |u_R'(M)| < |u_R'(\ell)| \), and hence \( u_L(y) - u_L(M) < u_R(M) - u_R(y) \).

Consider subgame perfect equilibrium policy path \( \{y^i\} \) following some history with \( y^0 \neq M \), and suppose that \( y^0 \) is on \( J \)'s side of the median. Define

\[
D^0_J = 0, \\
D^0_{\neg J} = \frac{u_{\neg J}(M) - u_{\neg J}(y^1)}{\delta_{\neg J}}.
\]

For any \( i \geq 1 \) and \( y^i \) (weakly) on \( J \)'s side of the median, define \( D^i_J \) and \( D^i_{\neg J} \) recursively as

\[
D^i_J = \max \left\{ 0, \frac{D^{i-1}_J + [u_J(M) - u_J(y^i)]}{\delta_J} \right\}, \\
D^i_{\neg J} = \frac{D^{i-1}_{\neg J} + [u_{\neg J}(M) - u_{\neg J}(y^i)]}{\delta_{\neg J}}.
\]

That is, interpret \( D^i_J \geq 0 \) as the payoff ‘debt’ for party \( J \) at stage \( i \) of subgame perfect equilibrium policy path \( \{y^i\} \) relative to path \( (M,M,\ldots) \). This debt collects all deviations from payoff \( u_J(M) \); if party \( J \) makes a loss with respect to \( u_J(M) \) at \( y^i \), then the equilibrium payoff from \( y^{i+1} \) needs to.

---


24 Any assumptions that yields this property are sufficient for the result of Proposition 1.1. For example, if \( u_L \) and \( u_R \) are weakly concave but strictly concave in a neighbourhood of \( M \).
yield an excess of at least $D^j_k$ over $\frac{1}{1-\delta_j} u_j(M)$. Debts grow by factor $\frac{1}{1-\delta_j}$ each period since they are incurred in the current period and reimbursed in later periods. Negative debts are never incurred since party $J$ must be guaranteed the payoff $\frac{1}{1-\delta_j} u_j(M)$ after all histories.

Since $y^0 \neq M$, debts $(D^0_L, D^0_R)$ are such that $D^j_k > 0$ for some $j$. Suppose without loss of generality that $\delta_k \leq \delta_R$. First note that for all $i > 0$, it cannot be that $D^i_k = D^i_R = 0$, since $D^0_j > 0$ and whenever $D^j_k < D^j_{k-1}$, it must be that $y^i$ is strictly on $J$’s side of the median and hence that $D^j_k > D^j_{k-1}$. Next, note that for all $j$, we have that $\liminf_{i \to \infty} D^j_i = 0$, and also that $D^j_k = 0$ infinitely often. To see this, suppose that there exists some $k$ such that $D^j_k > 0$ for all $i \geq k$. Then the equilibrium value to party $J$ from subgame perfect equilibrium policy path $\{y^i\}_{i=k}^\infty$ is strictly less than $\frac{1}{1-\delta_j} u_j(M)$, a contradiction.

Suppose now that $y^0 < M$, and hence that $D^0_L = 0 < D^0_R$. Then either

i. $D^i_L = 0$ for all $i > 0$.

ii. $D^i_L > 0$ for some $i > 0$.

In case i it must be that $y^i \leq M$ for all $i > 0$, and hence that $\lim_{i \to \infty} D^i_k \geq \lim_{i \to \infty} D^0_k = \infty$, a contradiction. We now see that assuming $y^0 < M$ is without loss of generality. First, any subgame perfect equilibrium policy path that deviates from the median policy after some history must have some subsequence that begins at stage $k$ with debt levels $D^k_L = 0 < D^k_R$. Second, assume instead that $D^0_L > 0 = D^0_R$. Then either $D^i_R = 0$ for all $i$, which leads to contradiction, or there exists $k$ such that $D^k_L = 0$, in which case we must have $D^k_R > 0$. Now consider case ii above. There must exist $n > m > 0$ with $n - m > 1$ such that $D^m_R > 0, D^m_L = D^m_R = 0$ and $D^i_L > 0$ for $i \in \{m+1, \ldots, n-1\}$. We want to show that $D^m_L < D^m_R$. Consider the sequence $\{\hat{y}^i\}_{i=m+1}^n$ that solves the following minimisation problem.

$$
\min_{\{\hat{y}^i\}_{i=m+1}^n} D^m_R \quad \text{subject to} \quad D^m_L = D^m_R = 0, \text{given } D^m_R > 0. \quad (1.2)
$$

$\{\hat{y}^i\}_{i=m+1}^n$ exists since $D^m_R$ is continuous and $X^{n-m}$ is compact. Suppose that $\{\hat{y}^i\}_{i=m+1}^n$ is such that $\hat{D}^{n-1}_L > 0$, where $\hat{D}^i$ is the debt of party $J$ under $\{\hat{y}^i\}_{i=m+1}$. Hence since $D^m_L = 0$ it must be that $\hat{y}^n < M$. Suppose that $\hat{D}^{n-2}_L + [u_R(M) - u_R(\hat{y}^{n-1})] < 0$, which implies that $\hat{D}^{n-1}_L = 0$ and that $\hat{y}^{n-1} > M$. For $\epsilon > 0$, consider $\hat{y}^{n-1} = \hat{y}^{n-1} - \epsilon$ and $\hat{y}^n = \hat{y}^n + \eta\epsilon$, where $\eta\epsilon$ is chosen such that $\hat{D}^n_L = 0$. For sufficiently small $\epsilon$, we have that $\hat{D}^{n-1}_L = \hat{D}^{n-1}_R = 0$ and $\hat{D}^n_R < \hat{D}^n_R$, a contradiction. Now suppose that $\hat{D}^{n-2}_L + [u_R(M) - u_R(\hat{y}^{n-1})] \geq 0$. $\hat{D}^n_R$ is strictly increasing in $\hat{y}^{n-1}$ if

$$
- \frac{u_R'(\hat{y}^{n-1})}{\delta_R^2} - \frac{u_R'((\hat{y}^n)\hat{y}^{n-1})}{\delta_R \hat{y}^{n-1}} > 0,
$$

where $\frac{d\hat{y}^n}{d\hat{y}^{n-1}}$ is given by

$$
\frac{u_R'((\hat{y}^{n-1})}{\delta_R} - \frac{u_R'((\hat{y}^n)\hat{y}^{n-1})}{\delta_L \hat{y}^{n-1}} = 0,
$$

or $\frac{d\hat{y}^n}{d\hat{y}^{n-1}} = -\frac{1}{\delta_L u_R'((\hat{y}^n)\hat{y}^{n-1})}$, which comes from partially differentiating the constraint $D^m_R = 0$ with
In any MPE of point i and part of point ii, consider the following claim:

\[
\frac{u'_L(\hat{y}^{n-1})}{u'_L(\hat{y}^n)} > \frac{\delta_L u'_R(\hat{y}^{n-1})}{\delta_R u'_R(\hat{y}^n)}.
\]

Say \(\hat{y}^{n-1} \geq M\). Then \(|u'_L(\hat{y}^{n-1})| > |u'_R(\hat{y}^{n-1})|\), \(\frac{\delta_L}{\delta_R} \leq 1\) and \(|u'_L(\hat{y}^n)| < |u'_R(\hat{y}^n)|\) (since \(y^n < M\)) imply that (1.3) holds, and hence that \(\{\hat{y}\}_{i=m+1}^{n}\) does not solve (1.2), a contradiction. Hence it must be that \(\hat{y}^{n-1} < M\).

This pairwise necessary condition for optimality can be used all along the sequence \(\{\hat{y}\}_{i=m+1}^{n}\) to show that a solution to (1.2) with \(\hat{y}^n < M\) must have \(\hat{y}^i < M\) for all \(i \in \{m+1, ..., n-1\}\). But consider instead sequence \(\{\tilde{y}\}_{i=m+1}^{n}\) with \(\tilde{y}^i = M\) for all \(i\). This sequence satisfies the constraints of (1.2), and is such that \(\tilde{D}_n^R = \frac{D_n^R}{\tilde{D}_R^R} < D_n^R\) for any \(\{y\}_{i=m+1}^{n}\) with \(D^{n-1} < M\). Hence, for the purported equilibrium sequence from above, we have as desired that \(D_n^R > D_n^R\). Considering the full policy sequence, we have that whenever \(D_l^i > 0\) for \(i \in \{m+1, n-1\}\), then \(D_n^R > D_n^R\). Furthermore, whenever \(D_l^i = 0\) for \(i \in \{m+1, n-1\}\), then again \(D_n^R > D_n^R\) since \(D_l^i = 0\) only if \(y^i \leq M\), and as shown above if \(D_l^R = 0\), then \(D_n^R > 0\). Hence, given the SPE path \(\{y^i\}\) following some history for which \(D_n^R > 0\), we have that \(\lim_{i \to \infty} D_R^i = \infty\), a contradiction.

The previous argument shows that the unique SPE policy path following any history is \((M, M, ...\)). It remains to be shown that both parties’ strategies must call for them to commit to the median following any history. If party J’s strategy calls for some policy \(y \neq M\) after some history, then party \(-J\) must win the election with policy \(M\). Since \(y \neq M\), party \(-J\) can win the election with a policy it prefers to \(M\), say \(y^'\). Since following any deviation, party \(-J\) payoffs revert to \(\frac{1}{1-\sigma_{-J}}(M)\), deviating to \(y^'\) is profitable for \(-J\).

\[\square\]

1.7.1 Policy Dynamics

Proof of Proposition 1.2 Consider state \((R, r)\) and policy path \(\{x^i\}\) induced by \((\sigma_L, \sigma_R)\) from \((R, r)\). First note that the policy path following state \((R, M)\) can only be \((M, M, ...\)). To prove the rest of point ]\[\text{and part of point } ii\[consider the following claim: In any MPE \((\sigma_L, \sigma_R)\), \(\sigma_L(R, r) \in X \setminus W(R, r) \cup \{Out\}\) for all \(r < M\) and \(\sigma_L(R, r) \leq M\) for all \(r > M\). The corresponding claims for party \(R\) are symmetric. To show this, consider some MPE \((\sigma_L, \sigma_R)\) with \(\sigma_L(R, r) \in [r, 2M - r]\) for some \(r < M\). Consider a one-shot deviation by \(L\) at state \((R, r)\) to \(Out\). The payoff to this deviation is

\[u_L(r) + \delta_L V_L(\sigma_L, \sigma_R; (R, r)),\]

while the payoff to \(\sigma_L(R, r)\) is \(V_L(\sigma_L, \sigma_R; (R, r))\). Hence the deviation is unprofitable if and only if

\[V_L(\sigma_L, \sigma_R; (R, r)) \geq \frac{1}{1 - \delta_L} u_L(r). \tag{1.4}\]

Since \(r < M\), the policy path following \((R, r)\) most favourable to \(L\) is \((r, r, ...)\). Hence we have that

\[V_L(\sigma_L, \sigma_R; (R, r)) \leq \frac{1}{1 - \delta_L} u_L(r). \tag{1.5}\]
(1.4) and (1.5) imply that \( V_L(\sigma_L, \sigma_R; (R, r)) = \frac{1}{1-\delta L} u_L(r) \), which holds if and only if \( \sigma_L(R, r) = r \) and \( \sigma_R(L, r) = r \). Now consider a deviation for \( R \) in state \( (L, r) \) to \( r^d \in (r, 2M - r] \). Any policy path \( \{x^i\} \) induced by \( (\sigma_L, \sigma_R) \) from \( (R, r^d) \) must be such that \( x^i > r \) for all \( i \). Hence the payoff to \( r^d \) is

\[
u_R(r^d) + \sum_{i=1}^{\infty} \delta_R^{2i-1} [u_R(x^i) + \delta_R u_R(x^{i+1})] \geq \frac{1}{1-\delta_R} u_R(r),
\]

a contradiction. For the second part of the claim, take \( (R, r) \) for some \( r > M \) such that \( \sigma_L(R, r) > M \). Consider a deviation to some \( \ell^d \in (M, \sigma_L(R, r)) \). By the first part of the claim, the payoff to \( \ell^d \) is given by

\[
\frac{1}{1-\delta_L} u_L(\ell^d) > \frac{1}{1-\delta_L} u_L(\sigma_L(R, r)) = V_L(\sigma_L, \sigma_R; (R, r)),
\]

a contradiction. In a similar manner, if \( (R, r) \) for some \( r > M \) is such that \( \sigma_L(R, r) = \text{Out} \), considering a deviation to some \( \ell^d \in (M, r) \) yields the desired contradiction.

For point (ii) of Proposition (1.2) note that by the previous claim, the sequence \( \{x^i\}_{i \text{ odd}} \) is weakly increasing and bounded by \( x^1 \) and \( M \), and hence converges to some limit \( \hat{\ell} \). The sequence \( \{x^i\}_{i \text{ even}} \) is weakly decreasing and bounded by \( M \) and \( x^2 \), and hence converges to some limit \( \hat{r} \). Furthermore, it must be that \( \hat{\ell} = 2M - \hat{r} \). Suppose instead that \( \hat{\ell} - (2M - \hat{\ell}) = \epsilon > 0 \). Consider \( n \in \mathbb{N} \) such that \( \hat{\ell} - x^i < \epsilon \) for all \( i \geq n \) odd. Then for \( j \geq n \) odd

\[
2M - \hat{\ell} < 2M - \hat{\ell} + \epsilon = \hat{r},
\]

and hence \( x^{i+1} \notin W(L, x^i) \) and there can be no \( \sigma_R(L, x^i) \) such that \( \tau((L, x^i), \sigma_R(L, x^i)) = x^{i+1} \), a contradiction. A similar argument shows that it cannot be that \( \hat{\ell} < 2M - \hat{r} \). Hence \( \hat{r} = 2M - \hat{\ell} \).

To complete the proof of Proposition (1.2), it remains to be shown that \( \sigma_L(R, \hat{r}) = \hat{\ell} \) and \( \sigma_R(L, \hat{\ell}) = \hat{r} \). Suppose first that \( x^i = \hat{\ell} \) for some \( i \) odd. Then \( x^i = \hat{\ell} \) for all \( j > i \) odd and it must be that \( \sigma_L(R, \hat{r}) = \hat{\ell} \) and \( \sigma_R(L, \hat{\ell}) = \hat{r} \). Suppose now that \( x^i \neq \hat{\ell} \) for all \( i \), and that \( \sigma_R(L, \hat{\ell}) = r < \hat{r} \). Consider \( \Delta > 0 \) such that

\[
u_L(\hat{\ell}) + \frac{\delta_L}{1-\delta_L^2} \mathbf{W}_L (2M - r) > \frac{1}{1-\delta_L^2} \mathbf{W}_L(\hat{\ell}) + \Delta.
\]

Such a \( \Delta \) exists by Lemma (1.1) since \( r < \hat{r} \). Since \( \nu_L \) is continuous and \( \{x^i\}_{i \text{ odd}} \rightarrow \hat{\ell} \), there exists \( n \in \mathbb{N} \) and \( \epsilon > 0 \) such that for all \( i \geq n \) odd, \( \hat{\ell} - x^i < \epsilon \) and \( \nu_L(x^i) - \nu_L(\hat{\ell}) < \Delta \). Now, for any
\[ j \geq n \text{ odd} \]

\[
V_L(\sigma_L, \sigma_R; (R, x^{j-1})) = u_L(x^j) + \sum_{i=1}^{\infty} \delta_L^{2i-1} [u_L(x^{j+2i-1}) + \delta_L u_L(x^{j+2i})] \\
\leq u_L(x^j) + \sum_{i=1}^{\infty} \delta_L^{2i-1} \Psi_L^-(x^{j+2i-1}) \\
\leq u_L(x^j) + \frac{\delta_L}{1 - \delta_L^2} \Psi_L^-(\hat{\ell}) \\
< \frac{1}{1 - \delta_L^2} \Psi_L^+(\hat{\ell}) + \Delta. \tag{1.7}
\]

The first inequality follows from the fact that \( x^{j+2i+1} \geq 2M - x^{j+2i} \) for all \( i \). The second inequality follows by Lemma 1.1 from the fact that \( x^{j+2i} \geq \hat{r} \) for all \( i \). In state \((R, x^{j-1})\), consider a deviating strategy by \( L, \sigma'_L \), with the properties

\[
\sigma'_L(R, x^{j-1}) = \hat{\ell} \text{ and} \\
\sigma'_L(R, r') = 2M - r' \text{ for all } r' \leq \hat{r}.
\]

Consider the policy path \( \{x^i\} \) induced by \((\sigma'_L, \sigma_R)\) from \((R, x^{j-1})\). The payoff to \( \sigma'_L \) is

\[
u_L(\hat{\ell}) + \sum_{i=1}^{\infty} \delta_L^{2i-1} \Psi_L^-(2M - x^{2i}) \geq u_L(\hat{\ell}) + \frac{\delta_L}{1 - \delta_L^2} \Psi_L^-(2M - \hat{r}) \\
> \frac{1}{1 - \delta_L^2} \Psi_L^+(\hat{\ell}) + \Delta \\
> V_L(\sigma_L, \sigma_R; (R, x^{j-1})),
\]

a contradiction. The first inequality follows from Lemma 1.1 and the fact that \( x^{2i} \leq \hat{r} \) for all \( i \), the second from (1.6) and the third from (1.7). The same proof applies to show that \( \sigma_L(R, \hat{r}) = \hat{\ell} \).

\[\square\]

### 1.7.2 Bounded Extremism

**Proof of Proposition 1.3** The following lemma provides a lower bound on equilibrium payoffs.

**Lemma 1.2.** Consider MPE \((\sigma_L, \sigma_R)\). In state \((R, r)\) with \( r > M \), the payoff to party \( L \) from policy \( \ell \in W(R, r) \) for some \( \ell \leq M \) is at least \( \frac{1}{1 - \delta_L^2} \Psi_L^+(\ell) \). The statement for party \( R \) is symmetric.

**Proof of Lemma 1.2** Given state \((R, r)\) with \( r > M \), consider the strategy \( \sigma'_L \) for \( L \) with the properties

\[
\sigma'_L(R, r) = \ell \in W(R, r) \text{ and} \\
\sigma'_L(R, r') = 2M - r' \text{ for all } r' \leq 2M - \ell.
\]
Consider the policy path \( \{x^i\} \) induced by \((\sigma_L^r, \sigma_R)\) from \((R, r)\). The payoff to \(\sigma_L^r\) is
\[
u_L(\ell) + \sum_{i=1}^{\infty} \delta_L^{i-1} \Psi_L^+(x^{2i}) \geq \frac{1}{1 - \delta_L^2} \Psi_L^+(\ell),
\]
where the inequality follows by Lemma 1.1 since \(x^{2i} \leq 2M - \ell\) for all \(i\).

The following claim establishes the bound on the extremism of long-run policy outcomes:

*If policy \(\ell\) is a long-run policy outcome, then \(\ell \geq \max\{\ell^*, 2M - r^*\}\). To show this, suppose that \(\ell^* \geq 2M - r^*\) and that \(\ell < \ell^*\) is a long-run policy outcome under \((\sigma_L, \sigma_R)\) starting from some state. By Lemma 1.2, party \(L\) can obtain a payoff of at least \(\frac{1}{1 - \delta_L^2} \Psi_L^+(\ell^*)\) by committing to \(\ell^*\) in state \((R, r)\). However, \(V_L(\sigma_L, \sigma_R; (R, r)) = \frac{1}{1 - \delta_L^2} \Psi_L^+(\ell) < \frac{1}{1 - \delta_L^2} \Psi_L^+(\ell^*)\) by Lemma 1.1 since \(\ell < \ell^*\), a contradiction.*

To complete the proof of Proposition 1.3, the following claim verifies the equilibrium construction of Section 1.4.3:

*If \(\ell^* \geq 2M - r^*\), the strategy profile \((\sigma_L^r, \sigma_R^{my})\) forms an equilibrium. If \(\ell^* < 2M - r^*\), the strategy profile \((\sigma_L^{my}, \sigma_R^{my})\) forms an equilibrium.*

To show this, suppose that \(\ell^* \geq 2M - r^*\). First verify the optimality of \(L\)'s proposed strategy. Given \((\sigma_L^r, \sigma_R^{my})\) compute
\[
V_L(\sigma_L^r, \sigma_R^{my}; (R, r)) = \begin{cases} 
\frac{1}{1 - \delta_L^2} \Psi_L^+(\ell^*) & \text{for } r \in [2M - \ell^*, 1], \\
\frac{1}{1 - \delta_L^2} \Psi_L^+(2M - r) & \text{for } r \in [M, 2M - \ell^*], \\
\frac{1}{1 - \delta_L^2} \nu_L(r) & \text{for } r \in [0, M].
\end{cases}
\]

Note that for all \(r, r'\) such that \(r > r', \sigma_L(R, r) \in W(R, r)\) and \(\sigma_L(R, r) \neq \sigma_L(R, r') \in W(R, r')\),
\[
V_L(\sigma_L^r, \sigma_R^{my}; (R, r)) > V_L(\sigma_L^{my}, \sigma_R^{my}; (R, r')).
\]

Hence, at any state \((R, r)\) such that \(\sigma_L(R, r) \in W(R, r)\), party \(L\) cannot profit from one-shot deviation \(\ell^d\) such that \(\sigma_L(R, r') = \ell\) for some \(r' \neq r\). Hence only one-shot deviations \(\ell^d \in [0, \ell^*) \cup (M, 1]\) can be profitable for \(L\) at some state.

The value of setting \(\ell^d \in [0, \ell^*)\) if winning at \((R, r)\) is
\[
u_L(\ell^d) + \delta_L u_L(2M - \ell^d) + \delta_L^2 V_L(\sigma_L^{\ell^d}, \sigma_R^{my}; (R, 2M - \ell^d))
\]
\[= \Psi_L^+(\ell^d) + \frac{\delta_L^2}{1 - \delta_L^2} \Psi_L^+(\ell^*).\]

\(\ell^d \in [0, \ell^*)\) is winning only in states \((R, r)\) with \(r \in [2M - \ell^d, 1] \cup [0, \ell^d]\). For \(r \in [2M - \ell^d, 1]\)
\[
V_L(\sigma_L^{\ell^d}, \sigma_R^{my}; (R, r)) = \frac{1}{1 - \delta_L^2} \Psi_L^+(\ell^d)
\]
\[> \Psi_L^+(\ell^d) + \frac{\delta_L^2}{1 - \delta_L^2} \Psi_L^+(\ell^*),\]
where the inequality follows from Lemma 1.1 since $\ell_d < \ell^*$. For $r \in [0, \ell^d)$

\[
V_L(\sigma^L, \sigma^R; (R, r)) = \frac{1}{1 - \delta^2_L} u_L(r) > \frac{1}{1 - \delta^2_L} u_L(\ell^d) + \frac{\delta^2_L}{1 - \delta^2_L} \Psi_L^+(\ell^d),
\]

where the inequality follows since $r \leq \ell^d$.

The value of setting $\ell^d \in (M, 1]$ if winning at $(R, r)$ is

\[
\frac{1}{1 - \delta^2_L} u_L(\ell^d).
\]

$\ell^d \in (M, 1]$ is winning only in states $(R, r)$ with $r \in [2M - \ell^d, M] \cup [\ell^d, 1]$. For $r \in [2M - \ell^d, M]$

\[
V_L(\sigma^L, \sigma^R; (R, r)) = \frac{1}{1 - \delta^2_L} u_L(r) > \frac{1}{1 - \delta^2_L} u_L(\ell^d),
\]

where the inequality follows since $r < \ell^d$. For $r \in [\ell^d, 1]$

\[
V_L(\sigma^L, \sigma^R; (R, r)) > \frac{1}{1 - \delta^2_L} u_L(M)
\]

\[
> \frac{1}{1 - \delta^2_L} u_L(\ell^d),
\]

where the first inequality follows since $r > M$, and the second since $\ell^d > M$. Hence, no profitable deviation for $L$ exists and $\sigma^L$ is optimal when facing $\sigma^R$.

Now verify the optimality of $R$’s proposed strategy. Given $(\sigma^L, \sigma^R)$ compute

\[
V_R(\sigma^L, \sigma^R; (L, \ell)) = \begin{cases} 
    u_R(2M - \ell) + \frac{\delta_R}{1 - \delta^2_R} \Psi_R^+(\ell^*) & \text{for } \ell \in [0, \ell^*), \\
    \frac{1}{1 - \delta^2_R} u_R(\ell) & \text{for } \ell \in [\ell^*, M), \\
    \frac{1}{1 - \delta^2_R} u_R(\ell) & \text{for } \ell \in [M, 1).
\end{cases}
\]

Again, note that for all $\ell < \ell'$, $\sigma_R(L, \ell) \in W(L, \ell)$ and $\sigma_R(L, \ell) \neq \sigma_R(L, \ell') \in W(L, \ell')$

\[
V_R(\sigma^L, \sigma^R; (L, \ell)) > V_R(\sigma^L, \sigma^R; (L, \ell')).
\]

Hence, at any state $(L, \ell)$ such that $\sigma_R(L, \ell) \in W(L, \ell)$, party $R$ cannot profit by deviating to any $r^d$ such that $\sigma_R(L, \ell') = r^d$ for some $\ell' \neq \ell$. Hence only one-shot deviations $r^d \in [0, M)$ can be profitable for $R$ at some state. That these cannot be profitable for $R$ follows from a verification similar to that for deviations $\ell^d \in (M, 1]$ for $L$ above. Hence, no profitable deviation for $R$ exists and $\sigma^R$ is optimal when facing $\sigma^L$. \qed
1.7.3 Consistent Markov Perfect Equilibria

The following Lemma characterises convergence paths under consistent strategies.

**Lemma 1.3.** Consider consistent Markov strategies $\sigma_L$ and $\sigma_R$.

i. If $\sigma_L(R, r) = \ell \in (\max\{2M - r, 0\}, M]$ for some $r > M$, then $\sigma_L(R, r') = \ell$ for all $r' \in [2M - \ell, r)$.

ii. Suppose $(\sigma_L, \sigma_R)$ form a consistent equilibrium. If $\sigma_L(R, r) = \ell \in (\max\{2M - r, 0\}, M]$ for some $r > M$, then $\sigma_R(L, \ell') = 2M - \ell'$ for all $\ell' \in [\max\{2M - r, 0\}, \ell)$.

Both statements for $R$ are symmetric.

**Proof of Lemma 1.3.** Part i is immediate from the definition of consistent Markov strategies. For part ii consider consistent equilibrium $(\sigma_L, \sigma_R)$, $r > M$ and $\sigma_L(R, r) = \ell > \max\{2M - r, 0\}$. Suppose for some $\ell' \in [\max\{2M - r, 0\}, \ell)$, $\sigma_R(L, \ell') = r' < 2M - \ell'$. There are two cases. First, suppose that $r' \geq 2M - \ell$. Consider the one-shot deviation by $R$ to $2M - \ell'$ in state $(L, \ell')$. The payoff to this deviation is

$$u_L(2M - \ell') + \delta_R V_L(\sigma_L, \sigma_R; (2M - \ell')) > u_L(r') + \delta_R V_L(\sigma_L, \sigma_R; (R, r'))$$

$$= V_R(\sigma_L, \sigma_R; (L, \ell')).$$

a contradiction. The inequality follows since $\sigma_L(R, r') = \ell$ for all $r' \in [2M - \ell, r]$ and $r' < 2M - \ell'$.

Second, suppose $r' < 2M - \ell$. Then by the part i of the lemma it must be that $\sigma_R(L, \ell'') = r'$ for all $\ell'' \in [\ell', 2M - r']$. By reversing the roles in the proof of the first case above, it can be seen that $L$ can profitably deviate to $2M - r'$ at $(R, r')$. \qed

The following lemma characterises payoffs on consistent equilibrium convergence paths.

**Lemma 1.4.** Consider long-run policy outcome $\hat{\ell} > \max\{\ell^*, 2M - r^*\}$, associated consistent equilibrium $(\sigma_L, \sigma_R)$ and convergence path $\{y^i\} \to \hat{\ell}$ starting from some state. Take state $(R, 2M - y^i)$ such that $\sigma_L(R, 2M - y^i) = y^{i+1}$ with $i > 1$. Then

$$V_L(\sigma_L, \sigma_R; (R, 2M - y^i)) = \frac{1}{1 - \delta_L^2} \Psi_L^+(y^i).$$

(1.8)

Furthermore,

$$\frac{1}{1 - \delta_L^2} \Psi_L^+(y^i) = u_L(y^{i+1}) + \delta_L \frac{1}{1 - \delta_L^2} \Psi_L^-(y^{i+2}).$$

(1.9)

The case of state $(L, y^i)$ such that $\sigma_R(L, y^i) = 2M - y^{i+1}$ with $i > 1$ is symmetric.

**Proof of Lemma 1.4.** Consider state $(R, 2M - y^i)$ such that $\sigma_L(R, 2M - y^i) = y^{i+1}$ with $i > 1$. Since $\hat{\ell} > \max\{\ell^*, 2M - r^*\}$, we have that $y^i < y^{i+1}$ for all $i$. Since $i > 1$, by Lemma 1.3 there exists $\epsilon > 0$ such that for all $\ell \in (y^i - \epsilon, y^{i+1}]$, $\sigma_R(L, \ell) = 2M - y^i$. For any $\hat{\epsilon} \in (0, \epsilon)$, consider one-shot
deviation by $L$ at $(R, 2M - y^i + \varepsilon)$ to $y^{i+1} = \sigma_L(R, 2M - y^i)$. The value to this deviation is given by

$$V_L(\sigma_L, \sigma_R; (R, 2M - y^i)) \leq V_L(\sigma_L, \sigma_R; (R, 2M - y^i + \varepsilon)) = u_L(y^i - \varepsilon) + \delta_L u_L(2M - y^i) + \delta_L^2 V_L(\sigma_L, \sigma_R; (R, 2M - y^i)),$$

where the inequality follows from equilibrium. This yields

$$V_L(\sigma_L, \sigma_R; (R, 2M - y^i)) \leq \frac{1}{1 - \delta_L^2} [u_L(y^i - \varepsilon) + \delta_L u_L(2M - y^i)]$$

for any $\varepsilon \in (0, \epsilon)$, and hence by the continuity of $u_L$.

$$V_L(\sigma_L, \sigma_R; (R, 2M - y^i)) \leq \frac{1}{1 - \delta_L^2} \psi_L(\sigma_L, \sigma_R; (R, 2M - y^i)).$$

Lemma 1.2 yields the opposite inequality and hence

$$V_L(\sigma_L, \sigma_R; (R, 2M - y^i)) = \frac{1}{1 - \delta_L^2} \psi_L(\sigma_L, \sigma_R; (R, 2M - y^i)).$$

The final claim of the lemma follow since

$$V_L(\sigma_L, \sigma_R; (R, 2M - y^i)) = u_L(y^{i+1}) + \delta_L u_L(2M - y^{i+2}) + \delta_L^2 V_L(\sigma_L, \sigma_R; (R, 2M - y^{i+2})).$$

\[\square\]

### 1.7.4 Bounded Moderation

To construct the bound on long-run moderation, define mappings $\alpha_L : [\max\{\ell^*, 2M - r^*\}, M] \to (0, 1]$ and $\alpha_R : [\max\{\ell^*, 2M - r^*\}, M] \to (0, 1]$ such that

$$\frac{u'_L(\ell)}{u'_L(2M - \ell)} = \frac{\delta_L}{\delta_L^2 + \alpha_L(\ell)(1 - \delta_L^2)} \quad \text{and} \quad \frac{u'_R(\ell)}{u'_R(2M - \ell)} = \frac{\delta_R}{\delta_R^2 + \alpha_R(\ell)(1 - \delta_R^2)}.$$  

(1.10)

Define $\ell^{**}$ such that $\alpha_L(\ell^{**}) + \alpha_R(\ell^{**}) = 1$. First show that $\alpha_L$, $\alpha_R$ and $\ell^{**} \in (\max\{\ell^*, 2M - r^*\}, M)$ are well-defined. To see this, note that since $u_L$ is concave $u'_L(2M - \ell)$ is strictly increasing in $\ell \in [\ell^*, M]$, with a minimum of $\delta_L$ and a maximum of 1. Now $\frac{\delta_L}{\delta_L^2 + \alpha_L(\ell)(1 - \delta_L^2)}$ is strictly decreasing in $\alpha_L \in [0, 1]$, with a minimum of $\delta_L$ and a maximum of $\frac{1}{\delta_L}$: $\alpha_L(\ell)$ is well-defined for $\ell \in [\max\{\ell^*, 2M - r^*\}, M]$, since $\frac{u'_L(\max\{\ell^*, 2M - r^*\})}{u'_L(2M - \max\{\ell^*, 2M - r^*\})} \geq \delta_L$. Also, $\alpha_L(\ell) \in (0, 1]$ for all $\ell$ since $\alpha_L(M) = \frac{\delta_L}{1 + \delta_L}$ and $\alpha_L(\ell^*) = 1$. Similarly, $\alpha_R(\ell)$ is well-defined. Furthermore, $\alpha_L(\ell) + \alpha_R(\ell)$ is
strictly decreasing in \( \ell \in [\max\{\ell^*, 2M - r^*\}, M] \), with \( a_L(M) + a_R(M) < 1 \) and \( a_L(\max\{\ell^*, 2M - r^*\}) + a_R(\max\{\ell^*, 2M - r^*\}) > 1 \). Thus \( \ell^{**} \in (\max\{\ell^*, 2M - r^*\}, M) \).

To understand the derivation of \( a_L \) and \( a_R \), consider \( y^i, y^{i+2} = y^i + \Delta \) for some \( \Delta > 0 \) and \( a_L \in [0, 1] \) such that

\[
\frac{1}{1 - \delta^2_L} \Psi^+_L(y^i) = u_L(y^i) + a_L \Delta + \frac{\delta_L}{1 - \delta^2_L} \Psi^-_L(y^i + \Delta). \tag{1.11}
\]

\( a_L \) is well-defined since evaluating (1.11) at \( a_L = 0 \) yields

\[
\frac{1}{1 - \delta^2_L} \Psi^+_L(y^i) = u_L(y^i) + \frac{\delta_L}{1 - \delta^2_L} \Psi^-_L(y^i) < u_L(y^i) + \frac{\delta_L}{1 - \delta^2_L} \Psi^-_L(y^i + \Delta),
\]

while evaluating (1.11) at \( a_L = 1 \) yields

\[
\frac{1}{1 - \delta^2_L} \Psi^+_L(y^i) > \frac{1}{1 - \delta^2_L} \Psi^+_L(y^i + \Delta) = u_L(y^i + \Delta) + \frac{\delta_L}{1 - \delta^2_L} \Psi^-_L(y^i + \Delta),
\]

where both inequalities follow from Lemma 1.1. The limit of (1.11) as \( \Delta \to 0 \) yields that \( a_L \) is determined by (1.10) evaluated at \( y^i \).

Proof of Proposition 1.4. The following claim establishes the bound on the moderation of robust long-run policy outcomes: If policy \( \ell \leq M \) is a robust long-run policy outcome under some consistent equilibrium, then \( \ell \leq \ell^{**} \). To show this, the following lemma establishes the properties of the recursive equation (1.1) that determine consistent equilibrium convergence path policies that allow us to determine possible convergence points.

Lemma 1.5. Consider robust long-run policy outcome \( \ell \) under consistent equilibrium \((\sigma_L, \sigma_R)\) and associated convergence path \( \{y^i\} \) starting from some state.

i. Suppose that

\[
\frac{u_L' (\ell)}{u_L' (2M - \ell)} < \frac{\delta_L}{\delta^2_L + a_L (1 - \delta^2_L)} \tag{1.12}
\]

for some \( a_L \in [0, 1] \) and that \( \sigma_L(R, 2M - y^{i-1}) = y^i \) for some \( i \). Then \( y^i - y^{i-1} > \frac{\alpha_L}{1 - \alpha_L} (y^{i+1} - y^i) \).

ii. Conversely, suppose that

\[
\frac{u_L' (y^i)}{u_L' (2M - y^i)} > \frac{\delta_L}{\delta^2_L + a_L (1 - \delta^2_L)} \tag{1.13}
\]

for some \( a_L \in [0, 1] \) and that \( \sigma_L(R, 2M - y^{i-1}) = y^i \). Then \( y^i - y^{i-1} < \frac{\alpha_L}{1 - \alpha_L} (y^{i+1} - y^i) \) for all \( i \geq j \).
The case for party R is symmetric.

Proof of Lemma 1.3. To prove part [3] of the lemma, first prove the following claim: Suppose that for some \( \alpha_L \in [0,1] \) and \( y, \Delta \) such that \( y - \Delta \in [\ell^*, M] \)

\[
\Psi_L^+(y - \Delta) - \Psi_L^+(y - (1 - \alpha_L)\Delta) \leq \delta_L[\Psi_L^-(y) - \Psi_L^-(y - (1 - \alpha_L)\Delta)],
\]

(1.14)

then for any \( y' \leq y \) and \( n \in \mathbb{N} \) such that \( y' - 2^n \Delta \in [\ell^*, M] \)

\[
\Psi_L^+(y' - 2^n \Delta) - \Psi_L^+(y' - 2^n(1 - \alpha_L)\Delta) \leq \delta_L[\Psi_L^-(y') - \Psi_L^-(y' - 2^n(1 - \alpha_L)\Delta)]
\]

(1.15)

with the inequality strict if \( y' \neq y \) or \( n > 0 \). Note that (1.14) implies that on an infinite convergence path for some consistent equilibrium for which \( \sigma(L, \ell) = 2M - (y - \Delta), \sigma(L, 2M - (y - \Delta)) - y \geq \alpha_L \Delta \). The claim states that if party R’s successive policy choices on some consistent equilibrium convergence path are \( 2M - (y - \Delta) \) and \( 2M - y \) and party L is (weakly) willing to moderate to \( y - (1 - \alpha_L)\Delta \) when in state \( (R, 2M - (y - \Delta)) \), then in another consistent equilibrium convergence path in which party R’s successive policies are \( 2M - (y' - \Delta') \) and \( 2M - y' \) with \( y' \leq y \), then party L is strictly willing to moderate to \( y' - (1 - \alpha_L)\Delta' \) in state \( (R, 2M - (y' - \Delta')) \), where \( \Delta' = 2^n \Delta \) for some \( n \in \mathbb{N} \).

To prove the claim, note first that, for \( y' \leq y \)

\[
\Psi_L^+(y' - \Delta) - \Psi_L^+(y' - (1 - \alpha_L)\Delta) \leq \Psi_L^+(y - \Delta) - \Psi_L^+(y - (1 - \alpha_L)\Delta)
\]

\[
\leq \delta_L[\Psi_L^-(y) - \Psi_L^-(y - (1 - \alpha_L)\Delta)]
\]

\[
\leq \delta_L[\Psi_L^-(y') - \Psi_L^-(y' - (1 - \alpha_L)\Delta)],
\]

with the first and third inequalities strict if \( y' \neq y \). The first inequality follows from the strict concavity of \( \Psi_L^+ \), the second from (1.14), and the third from the strict concavity of \( \Psi_L^- \). Given (1.14), the above shows that

\[
\Psi_L^+(y - 2\Delta) - \Psi_L^+(y - (2 - \alpha_L)\Delta) < \delta_L[\Psi_L^-(y - \Delta) - \Psi_L^-(y - (2 - \alpha_L)\Delta)],
\]

and

\[
\Psi_L^+(y - (2 - \alpha_L)\Delta) - \Psi_L^+(y - 2(1 - \alpha_L)\Delta) < \delta_L[\Psi_L^-(y - (1 - \alpha_L)\Delta) - \Psi_L^-(y - 2(1 - \alpha_L)\Delta)].
\]

(1.16)

\[\text{25That is, moderate by } \alpha_L \Delta.\]
Hence we have that

\[
\delta_L[\Psi_L^-(y) - \Psi_L^-(y - 2(1 - \alpha_L)\Delta)] = \delta_L[\Psi_L^-(y) - \Psi_L^-(y - (1 - \alpha_L)\Delta)] \\
+ \delta_L[\Psi_L^-(y - (1 - \alpha_L)\Delta) - \Psi_L^-(y - 2(1 - \alpha_L)\Delta)] \\
> \Psi_L^+(y - \Delta) - \Psi_L^+(y - (1 - \alpha_L)\Delta) \\
+ \Psi_L^+(y - (2 - \alpha_L)\Delta) - \Psi_L^+(y - 2(1 - \alpha_L)\Delta) \\
> \Psi_L^+(y - 2\Delta) - \Psi_L^+(y - \Delta(2 - \alpha_L)) \\
+ \Psi_L^+(y - (2 - \alpha_L)\Delta) - \Psi_L^+(y - 2(1 - \alpha_L)\Delta) \\
= \Psi_L^+(y - 2\Delta) - \Psi_L^+(y - 2(1 - \alpha_L)\Delta).
\]

The first inequality follows from (1.14) and (1.16), and the second inequality follows from Lemma 1.1 since \(y - (1 - \alpha_L)\Delta = y - \Delta(2 - \alpha_L) - (y - 2\Delta) = \alpha_L\Delta\). The claim follows by applying the above argument recursively.

To complete the proof of part ii of Lemma 1.5, consider (1.12). This condition guarantees that for arbitrarily small \(\Delta\), party \(L\) is willing to take up share \(\alpha_L\Delta\) of moderation \(\Delta\) from \(y - \Delta\) to \(y\). Hence, there exists some \(\hat{\Delta}\) such that for all \(\Delta < \hat{\Delta}\),

\[
\Psi_L^+(\hat{\ell} - \Delta) - \Psi_L^+(\hat{\ell} - (1 - \alpha_L)\Delta) < \delta_L[\Psi_L^-(\hat{\ell}) - \Psi_L^-(\hat{\ell} - (1 - \alpha_L)\Delta)].
\]

Thus, by the earlier claim, for all \(y < \hat{\ell}\) and \(\Delta\) such that \(y - \Delta > \ell^*\),

\[
\Psi_L^+(y - \Delta) - \Psi_L^+(y - (1 - \alpha_L)\Delta) < \delta_L[\Psi_L^-(y) - \Psi_L^-(y - (1 - \alpha_L)\Delta)].
\]

This implies that for \(y^i\) such that \(\sigma_L(R, 2M - y^{i-1}) = y^i, y^i - y^{i-1} > \frac{\alpha_L}{1 - \alpha_L}(y^{i+1} - y^i)\).

The proof of part ii of Lemma 1.5 follows along the lines of part i. While part i is backward-looking, part ii is forward-looking. That is, part i establishes that if at the limit point of a consistent equilibrium convergence path party \(L\) is willing to undertake share \(\alpha_L\) of all marginal moderations, then it was also willing to undertake share \(\alpha_L\) of all past moderate moves. In contrast, part ii shows that if at some point on a convergence path, party \(L\) would be unwilling to undertake share \(\alpha_L\) of marginal moderations, then it will undertake less than share \(\alpha_L\) of all future moderations on the convergence path. Evidently, part ii is useful to establish conditions for nonconvergence, while part i helps establish conditions for convergence. \(\square\)

Now to show that moderation is bounded by \(\ell^{**}\), consider a robust long-run policy outcome \((\hat{\ell}, 2M - \hat{\ell})\) with \(\hat{\ell} > \ell^{**}\) and associated consistent equilibrium \((\sigma_L, \sigma_R)\). Consider state \((R, r)\) with \(2M - r < \hat{\ell}\) and convergence path \(\{y^i\} \to \hat{\ell}\) given \((R, r)\) with \(\sigma_L(R, 2M - y^0) = y^1\). Fix \(n\) such that \(y^n > \ell^{**}\) and \(\sigma_L(R, 2M - y^n) = y^{n+1}\). Hence

\[
\frac{u_L'(y^n)}{u_L'(2M - y^n)} > \frac{u_L'(\ell^{**})}{u_L'(2M - \ell^{**})} = \frac{\delta_L}{\delta_L^2 + \alpha_L(\ell^{**})(1 - \delta_L^2)}.
\]
and hence by part \[5\] of Lemma \[1.5\] for all \(j \geq n\),
\[
y^{j+1} - y^j < \frac{\alpha_L(\ell^{**})}{1 - \alpha_L(\ell^{**})}(y^{j+2} - y^{j+1}).
\]

Similarly, if \(j \geq n + 1\) and \(\sigma_R(L, y^j) = 2M - y^{j+1}\) then
\[
y^{j+1} - y^j < \frac{\alpha_R(\ell^{**})}{1 - \alpha_R(\ell^{**})}(y^{j+2} - y^{j+1}).
\]

This yields that for all \(j \geq n + 1\),
\[
y^{j+1} - y^j < \frac{\alpha_L(\ell^{**})}{1 - \alpha_L(\ell^{**})} \frac{\alpha_R(\ell^{**})}{1 - \alpha_R(\ell^{**})}(y^{j+3} - y^{j+2}) < (y^{j+3} - y^{j+2}).
\]

Hence the convergence path \(\{y^j\} \to \ell\) contains a nonconverging subsequence, a contradiction.

To show that the bound on long-run moderation is tight, given a strictly increasing sequence \(\{y^j\} \to \ell\) with \(y^0 = \ell^*\) and \(y^j, y^{j+1}\) and \(y^{j+2}\) satisfying the conditions of Lemma \[1.4\] for all \(i \geq 1\), consider the following strategies

\[
\sigma_L^R(R, r) = \begin{cases} \ell^* & \text{for all } r \geq 2M - \ell^*, \\ 2M - r & \text{for all } r \in (2M - y^j, 2M - y^{j-1}) \text{ with } i > 0 \text{ odd}, \\ y^j & \text{for all } r \in [2M - y^j, 2M - y^{j-1}] \text{ with } i > 0 \text{ even}, \\ 2M - r & \text{for all } r \in [M, 2M - \ell], \\ \text{Out} & \text{for all } r < M. \end{cases}
\]

\[
\sigma_R^L(L, \ell) = \begin{cases} 2M - \ell & \text{for all } \ell < \ell^*, \\ y^j & \text{for all } \ell \in [y^{j-1}, y^j] \text{ with } i > 0 \text{ odd}, \\ 2M - \ell & \text{for all } \ell \in (y^{j-1}, y^j) \text{ with } i > 0 \text{ even}, \\ 2M - \ell & \text{for all } \ell \in [\ell, M], \\ \text{Out} & \text{for all } \ell > M. \end{cases}
\]

If instead \(\ell^* < 2M - r^*\), then for robust long-run policy outcome \((\ell, 2M - \ell)\) with \(\ell < 2M - r^*\), strategies \((\sigma_L^R, \sigma_R^L)\) can be constructed in a similar manner with the roles of the parties reversed.

The following claim verifies that these strategies form an equilibrium: Suppose that \(\ell^* \geq 2M - r^*\). Given \(\ell > \ell^*\) and a strictly increasing sequence \(\{y^j\} \to \ell\) with \(y^0 = \ell^*\) and \(y^j, y^{j+1}\) and \(y^{j+2}\) satisfying the conditions of Lemma \[1.4\] for all \(i \geq 1\), strategies \((\sigma_L^R, \sigma_R^L)\) form a form a consistent equilibrium under which \(\ell\) is a robust long-run policy outcome. The equilibrium \((\sigma_L^R, \sigma_R^L)\) in the case of \(\ell^* < 2M - r^*\) can be determined similarly. To show this, suppose \(\ell^* \geq 2M - r^*\). First verify the optimality of \(L\)'s
proposed strategy. Given $\sigma_L^2, \sigma_R^2$ and the conditions of the lemma for \( \{ y^i \} \), compute

\[
V_L(\sigma_L^2, \sigma_R^2; (R, r)) = \begin{cases} 
  u_L(\ell^*) + \frac{\delta_L}{1-\delta_L} \Psi_L^-(y^1) & \text{for } r \in [2M - \ell^*, 1], \\
  u_L(2M - r) + \delta_L u_L(2M - y') + \frac{\delta_L}{1-\delta_L} \Psi_L^+(y'^1) & \text{for } r \in (2M - y', 2M - y'^1) \text{ with } i > 0 \text{ odd}, \\
  u_L(y') + \frac{\delta_L}{1-\delta_L} \Psi_L^-(y'^1) & \text{for } r \in [2M - y', 2M - y'^1] \text{ with } i > 0 \text{ even}, \\
  \frac{1}{1-\delta_L} \Psi_L^+(2M - r) & \text{for } r \in [M, 2M - \ell^*], \\
  \frac{1}{1-\delta_L} u_L(r) & \text{for } r \in [0, M].
\end{cases}
\]

Note that for all $r, r'$ such that $r > r'$, $\sigma_L(R, r) \in W(R, r)$ and $\sigma_L(R, r) \neq \sigma_L(R, r') \in W(R, r')$, \( V_L(\sigma_L^2, \sigma_R^2; (R, r)) > V_L(\sigma_L^2, \sigma_R^2; (R, r')). \)

Hence, at any state $(R, r)$ such that $\sigma_L(R, r) \in W(R, r)$, party $L$ cannot profit by deviating to any $\ell^d$ such that $\sigma_L(R, r') = \ell$ for some $r' \neq r$. Hence only one-shot deviations $\ell^d \in [0, \ell^*) \cup (\bigcup_{j=0, i \text{ even}}[y^{i-1}, y^i)) \cup (M, 1]$ can be profitable for $L$ at some state. The value to setting $\ell^d \in [0, \ell^*)$ if winning at $(R, r)$ is

\[
\Psi_L^+(\ell^d) + \delta_L^2 V_L(\sigma_L^2, \sigma_R^2; (R, 2M - \ell^d)) = \Psi_L^+(\ell^d) + \delta_L^2 V_L(\sigma_L^2, \sigma_R^2; (R, 2M - \ell^*))
\]

$\ell^d \in [0, \ell^*)$ is winning only in states $(R, r)$ with $r \in [2M - \ell^d, 1] \cup [0, \ell^d]$. For $r \in [2M - \ell^d, 1]$

\[
V_L(\sigma_L^2, \sigma_R^2; (R, r)) > \Psi_L^+(\ell^d) + \delta_L^2 V_L(\sigma_L^2, \sigma_R^2; (R, 2M - \ell^d)) = \Psi_L^+(\ell^d) + \delta_L^2 V_L(\sigma_L^2, \sigma_R^2; (R, 2M - r)).
\]

since

\[
V_L(\sigma_L^2, \sigma_R^2; (R, r)) = u_L(\ell^*) + \frac{\delta_L}{1-\delta_L} \Psi_L^-(y^1) > \frac{1}{1-\delta_L} \Psi_L^+(\ell^*) > \frac{1}{1-\delta_L} \Psi_L^+(\ell^d).
\]

The first inequality follows from Lemma 1.1 and the fact that $y^1 > \ell^*$, and the second inequality from Lemma 1.1 and the fact that $\ell^d < \ell^*$. That a deviation to $\ell^d \in [0, \ell^*)$ in states $(R, r)$ with $r \in [0, \ell^d]$ is not profitable follows from an argument similar to that in Lemma 1.4. The value of setting $\ell^d \in [y^{i-1}, y^i)$ for $i > 0$ odd if winning at $(R, r)$ is

\[
\Psi_L^+(\ell^d) + \delta_L^2 V_L(\sigma_L^2, \sigma_R^2; (R, 2M - \ell^d)).
\]
\[ \ell^d \in [y^{i-1}, y^i) \] is winning only in states \((R, r)\) with \(r \in [2M - \ell^d, 1] \cup [0, \ell^d].\) Consider
\[
V_L(\sigma^L_\ell, \sigma^R_\ell; (R, 2M - y^i)) = \frac{1}{1 - \delta^L_\ell} \Psi^+_L(y^{i-1}) \\
= \Psi^+_L(y^{i-1}) + \delta^L_\ell V_L(\sigma^L_\ell, \sigma^R_\ell; (R, 2M - y^{i-1})) \\
\geq \Psi^+_L(\ell^d) + \delta^L_\ell V_L(\sigma^L_\ell, \sigma^R_\ell; (R, 2M - \ell^d)),
\]
where the inequality follows from Lemma 1.1 and the fact that \(\ell^* < y^{i-1} \leq \ell^d\) and the fact that \(V_L(\sigma^L_\ell, \sigma^R_\ell; (R, 2M - y^{i-1})) = V_L(\sigma^L_\ell, \sigma^R_\ell; (R, 2M - \ell^d)).\) Hence, the value to \(\ell^d\) is weakly smaller than the value following action \(y^i = \sigma_\ell(R, 2M - y^i),\) and hence for all states \((R, r)\) with \(r \in [2M - \ell^d, 1]\) deviation to \(\ell^d\) by \(L\) cannot be profitable. That a deviation to \(\ell^d \in [y^{i-1}, y^i)\) in states \((R, r)\) with \(r \in [0, \ell^d]\) is not profitable follows from an argument similar to that in the case of equilibrium \((\sigma^L_\ell^*, \sigma^R_\ell^m)，\) as does the argument that there is no profitable deviation to \(\ell^d \in (M, 1].\)

Arguments very similar to those for \(L\) above can determine \(R\)'s payoffs under \((\sigma^L_\ell^*, \sigma^R_\ell^m)\) and verify that it constitutes an equilibrium. Clearly \(\hat{\ell}\) is a robust long-run policy outcome under \((\sigma^L_\ell^*, \sigma^R_\ell^m)\) since policy dynamics have \(\hat{\ell}\) as a limit point starting from all more extreme states.

To complete the proof of Proposition 1.4, let \(Y\) be the set of increasing extended real-valued sequences.

**Definition 1.7.** Define mapping \(B : (\ell^*, M] \to Y\) such that \(B(y)^0 = \ell^*, B(y)^1 = y,\) for each \(i \geq 2\) with \(i\) even \(B(y)^i\) solves
\[
\Psi^+_L(B(y)^i-2) - \Psi^+_L(B(y)^i-1) = \delta_L[\Psi^+_L(B(y)^i) - \Psi^+_L(B(y)^i-1)],
\]
and for each \(i \geq 3\) with \(i\) odd, \(B(y)^i\) solves
\[
\Psi^+_R(B(y)^i-2) - \Psi^+_R(B(y)^i-1) = \delta_R[\Psi^+_R(B(y)^i) - \Psi^+_R(B(y)^i-1)],
\]
if solutions \(B(y)^j \leq M\) exist to (1.17) and/or (1.18). If not, set \(B(y)^i = \infty\) for all \(j \geq i.\) Define mapping \(\Gamma : (\ell^*, M] \to R \cup \{\infty\}\) such that \(\Gamma(y) = \lim_{i \to \infty} B^i(y)\).

Equations (1.17) and (1.18) restate the payoff conditions of Lemma 1.4. Suppose that \(\ell^* \geq 2M - r^*\) and that there exists a consistent equilibrium under which \(\hat{\ell} \in (\ell^*, \ell^{**})\) is a robust long-run policy outcome. In that case, there exists a convergence path \(\{y^i\} \to \hat{\ell}\) from state \((L, \ell^*).\) Suppose that in state \((L, \ell^*)\) party \(R\) selects policy \(2M - y\) for \(y \in (\ell^*, M].\) The mapping \(B\) recovers the full sequence of equilibrium convergence path policies. When no such path exists, we have \(B(y)^i = \infty\) for some \(i.\) Iteration on \(B\) yields a candidate for the sequence posited in the claim for equilibrium \((\sigma^L_\ell^*, \sigma^R_\ell^m),\) which is acceptable if the limit of of \(B(y),\) that is \(\Gamma(y),\) is contained in \((\ell^*, \ell^{**}).\) The following claim makes this precise: Mapping \(B\) is such that

1. The mapping \(\Gamma\) is well-defined, increasing, strictly increasing on \(\{y : \Gamma(y) < \infty\},\) right-continuous on \(\{y : \Gamma(y) < \ell^{**}\}\) and left-continuous on \(\{y : \Gamma(y) < \infty\}\).
2. For any \(\hat{\ell} \in (\ell^*, \ell^{**}),\) there exists \(y\) such that \(\Gamma(y) = \hat{\ell}.\)
iii. A strictly increasing sequence \( \{y^i\} \rightarrow \bar{\ell} \) with \( y^0 = \ell^* \) and \( y^i, y^{i+1} \) and \( y^{i+2} \) satisfying the conditions of Lemma\[1.4\] for all \( i \geq 1 \).

To show this, not that for \( y^1 \in (\ell^*, M], \Gamma(y^1) \) is the limit an increasing extended real-valued sequence and hence is well-defined. For the monotonicity of \( \Gamma \), consider \( y^1, \bar{y}^1 \in (\ell^*, M] \) such that \( y^1 < \bar{y}^1 \), along with induced sequences \( \{B(y^1, i)\} = \{y^i\} \) and \( \{B(\bar{y}^1, i)\} = \{\bar{y}^i\} \). First show that for \( i \geq 1 \), whenever \( \infty > \bar{y}^i - y^i > y^i \), \( \bar{y}^i - \bar{y}^{i-1} > y^i - y^{i-1} \), and \( y^{i+1}, \bar{y}^{i+1} < \infty \), it is the case that \( \bar{y}^{i+1} - \bar{y}^i > y^{i+1} - y^i \) and \( \bar{y}^{i+1} > y^{i+1} \). Suppose \( \bar{y}^{i-1} - y^{i-1} = \epsilon = y^i - \bar{y}^i \), where \( \epsilon \geq 0 \). Hence

\[
\Psi_L^+(y^i - \epsilon) - \Psi_L^+(\bar{y}^i - \epsilon) - \delta_L[\Psi_L^-(y^{i+1}) - \Psi_L^- (\bar{y}^i)]
\]

\[
> \Psi_L^+(y^{i-1}) - \Psi_L^+(y^i) - \delta_L[\Psi_L^- (y^{i+1}) - \Psi_L^-(y^i)]
\]

\[
= 0,
\]

where the inequality follows by Lemma\[1.1\] since \( y^i - \bar{y}^i > \epsilon \). Define \( \bar{y}^{i+1} \) such that

\[
\Psi_L^+(\bar{y}^{i+1} - \epsilon) - \Psi_L^+(\bar{y}^i - \epsilon) - \delta_L[\Psi_L^-(\bar{y}^{i+1} + \epsilon) - \Psi_L^-(\bar{y}^i)]
\]

\[
> 0,
\]

and hence \( \bar{y}^{i+1} > \bar{y}^i + \epsilon > y^{i+1} \) and \( \bar{y}^{i+1} - \bar{y}^i > y^{i+1} - \bar{y}^i - \epsilon > y^{i+1} - y^i \). By induction, if \( y^1, \bar{y}^1 \in \{y : \Gamma(y) < \infty\} \), this implies that for each \( i \geq 1 \), \( \bar{y}^i > y^i \), and

\[
\Gamma(y^1) = \lim_{i \to \infty} \bar{y}^i
\]

\[
> \lim_{i \to \infty} y^i
\]

\[
= \Gamma(y^1).
\]

The above argument also shows that if \( y^1 < \bar{y}^1 \), then \( y^i < \bar{y}^i \) for all \( i \) such that \( \bar{y}^i < \infty \), and hence that \( \Gamma(y^1) \leq \Gamma(\bar{y}^1) \).

Suppose \( \Gamma \) is not right-continuous at \( y^1 \), and that \( \Gamma(y^1) < \ell^{**} \). Then there exists \( \epsilon > 0 \) such that for any \( \delta > 0 \), \( \Gamma(y^1 + \delta) - \Gamma(y^1) > \epsilon \). Take \( \bar{\epsilon} \in (0, \min\{\epsilon, \ell^{**} - \Gamma(y^1)\}) \). Hence \( \Gamma(y^1) + \bar{\epsilon} < \ell^{**} \). Consider \( \bar{y}^1 \in (y^1, y^1 + \delta) \) and associated sequence \( \{\bar{y}^i\} \). Since \( \Gamma(\bar{y}^1) + \bar{\epsilon} < \ell^{**} \), by part ii of Lemma\[1.5\] there exist \( \alpha_L \) and \( \alpha_R \) with \( \alpha_L + \alpha_R > 1 \) such that for any \( \{\bar{y}^i\} \rightarrow \Gamma(\bar{y}^1) \) with \( \Gamma(\bar{y}^1) \leq \Gamma(y^1) + \bar{\epsilon} \), \( \bar{y}^{i+1} - \bar{y}^i < \frac{\alpha_L}{1 - \alpha_L} (\bar{y}^i - \bar{y}^{i-1}) \), \( \bar{y}^i - \bar{y}^{i-1} < \frac{\alpha_R}{1 - \alpha_R} (\bar{y}^{i-1} - \bar{y}^{i-2}) \) and

\[
\lim_{i \to \infty} \bar{y}^i < \bar{y}^0 + (\bar{y}^1 - \bar{y}^0) \frac{\frac{\alpha_L}{1 - \alpha_L} (1 + \frac{\alpha_R}{1 - \alpha_R})}{1 - \frac{\alpha_L}{1 - \alpha_L} \frac{\alpha_R}{1 - \alpha_R}}.
\]
Conversely, if \( \varphi^0 + (\tilde{g}^1 - \tilde{g}^0) \frac{a_L}{1 - a_L} (1 + \frac{a_R}{1 - a_R}) \leq \Gamma(y^1) + \varepsilon \), then it must be that \( \Gamma(\tilde{g}^1) < \Gamma(y^1) + \varepsilon \). Since \( \{y^i\} \to \Gamma(y^1) \), there exists \( n \in \mathbb{N} \) such that

\[
y^i + (y^{i+1} - y^i) \frac{a_L}{1 - a_L} (1 + \frac{a_R}{1 - a_R}) < \Gamma(y^1) + \frac{\varepsilon}{2}
\]

for all \( i \geq n \). Fix \( j \geq n \). Since for all \( i \geq 1 \), \( \tilde{g}^{i+1} \) is a continuous function of \( \tilde{g}^i \) and \( \tilde{g}^{i-1}, \tilde{g}^i \) can be found such that \( \tilde{g}^i - y^i \leq \frac{\varepsilon}{4} \) and \( (\tilde{g}^{i+1} - \tilde{g}^i) - (y^{i+1} - y^i) \leq \frac{\varepsilon}{4} \frac{1 - a_L}{1 - a_L} \frac{a_R}{1 - a_R} \). Then it follows that

\[
\tilde{g}^i + (\tilde{g}^{i+1} - \tilde{g}^i) \frac{a_L}{1 - a_L} (1 + \frac{a_R}{1 - a_R}) < y^i + \frac{\varepsilon}{4} + (y^{i+1} - y^i) \frac{a_L}{1 - a_L} (1 + \frac{a_R}{1 - a_R}) + \frac{\varepsilon}{4} < \Gamma(y^1) + \frac{\varepsilon}{2}.
\]

Hence \( \Gamma(\tilde{g}^i) \) is such that \( \Gamma(\tilde{g}^i) < \Gamma(y^1) + \varepsilon \), a contradiction.

Suppose \( \Gamma \) is not left-continuous at \( y^1 \), and that \( \Gamma(y^1) < \infty \). Then there exists \( \varepsilon > 0 \) such that for all \( \delta > 0 \), \( \Gamma(y^1) - \Gamma(y^1 - \delta) > \varepsilon \). Take \( j \in \mathbb{N} \) such that \( y^j > \Gamma(y^1) - \varepsilon + \eta \) for \( \eta \in (0, \varepsilon) \). Fix \( \tilde{g}^1 \) such that \( y^j - \tilde{g}^j < \eta \). Hence \( \tilde{g}^j > y^j - \eta > \Gamma(y^1) - \varepsilon \), and hence \( \Gamma(\tilde{g}^j) > \Gamma(y^1) - \varepsilon \), since \( \tilde{g}^j \) is increasing, a contradiction.

The set \( \{ y : \Gamma(y) < \ell^{**} \} \) is nonempty since \( \lim_{y \to y^1} \Gamma(y^1) = \ell^{*} \), and hence by continuity of \( \Gamma \) on \( \{ y : \Gamma(y) < \ell^{**} \} \), for each \( \ell \) with \( \ell < \ell^{**} \), there exists \( y \) such that \( \Gamma(y) = \ell \). Finally, since \( \Gamma \) is left-continuous on \( \{ y : \Gamma(y) < \infty \} \), there exist a \( y \) such that \( \Gamma(y) = \ell^{**} \).

### 1.7.5 Forward-looking Voters

**Proof of Proposition 1.5** Consider consistent equilibrium convergence path \( \{y^i\} \) with associated consistent equilibrium strategies \((\sigma_L, \sigma_R)\). Assume for now that on convergence paths, the median voter votes according to \( \sigma_M^{my} \). To construct strategies \((\sigma'_L, \sigma'_R)\) in the game with forward-looking voters, the profile \((\sigma_L, \sigma_R)\) needs to be modified in two ways. First, consider policy \( y^i \) such that \( \sigma_L(R, 2M - y^i) = y^i + 1 \). For \( x \in [y^i, y^{i+1}] \), define \( z^{i+1}(x) \in [y^i, x] \) such that

i. If \( u_M(x) - u_M(y^i) > \delta_M \left[ V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^i)) - \frac{1}{1 - \delta_M} u_M(x) \right] \),

then \( z^{i+1}(x) \) solves

\[
u_M(x) - u_M(z^{i+1}(x)) = \delta_M \left[ V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^i)) - \frac{1}{1 - \delta_M} u_M(x) \right].
\]

ii. If \( u_M(x) - u_M(y^i) \leq \delta_M \left[ V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^i)) - \frac{1}{1 - \delta_M} u_M(x) \right] \),

then \( z^{i+1}(x) = y^i \).
That is, R commits to $2M - z^{i+1}(x)$ as ‘punishment’ for $L$ being in power with policy $x$ as opposed to $y^{i+1}$ and $z^{i+1}(x)$ is the most extreme such punishment that the median voter supports. For $y^i$ such that $\sigma_R(L, y^i) = 2M - y^{i+1}$ and for $x \in (2M - y^{i+1}, 2M - y^i), z^{i+1}(x) \in [y^i, 2M - x)$ can be defined symmetrically.

Second, given some $\sigma_M$ and $\ell > M$, let $r(\ell) > \ell$ be the most extreme commitment by $R$ in state $(L, \ell)$ that the median voter supports and that $R$ has the incentive to make. If the median voter accepts $r(\ell)$, then policy dynamics are ‘freed’ from the policy traps of equilibria with myopic voters and, after at most one period, the equilibrium path rejoins convergence path $\{y^i\}$. For $r < M$, define $\bar{\ell}(r) < r$ symmetrically. Note that, as with the functions $\{z^{i+1}(\cdot)\}, r(\cdot)$ and $\bar{\ell}(\cdot)$ are determined only by how parties and the median voter evaluate convergence paths under $(\sigma_L, \sigma_R, \sigma_M^\text{my})$. Now define strategy $\sigma_R^\ell$ as

$$
\sigma_R^\ell(R, r) = \begin{cases} 
   z^{i+1}(r) & \text{if } r \in (2M - y^{i+1}, 2M - y^i) \text{ for } y^i \text{ such that } \sigma_R(L, y^i) = 2M - y^{i+1}, \\
   \bar{\ell}(r) & \text{if } r < M \text{ and } u_L(\bar{\ell}(r)) + \delta_i V_L(\sigma_L, \sigma_R; (L, \bar{\ell}(r))) \geq \frac{1}{1 - \delta_i} u_L(r) \\
   \sigma_L(R, r) & \text{otherwise.}
\end{cases}
$$

$\sigma_R^\ell$ can be defined symmetrically. Let $\sigma_M$ be a best-response to $(\sigma_L^\ell, \sigma_R^\ell)$ in which the median voter supports the opposition party when indifferent. Given the parties’ strategies, the median voter has no incentive to vote for the incumbent on a convergence path. Hence, given convergence path policy $y^i$ such that $\sigma_L(R, 2M - y^i) = y^{i+1}$, we have that $V_K(\sigma_L^\ell, \sigma_R^\ell, \sigma_M^\text{my}; (R, 2M - y^i)) = V_K(\sigma_L, \sigma_R, \sigma_M^\text{my}; (R, 2M - y^i))$ for $K \in \{L, R, M\}$. I do not describe the median voter’s equilibrium strategy explicitly, but instead show how it responds to parties’ deviations from the convergence path $\{y^i\}$ to show that parties have no more incentive to deviate from the convergence path under $(\sigma_L^\ell, \sigma_R^\ell, \sigma_M^\text{my})$ than under $(\sigma_L, \sigma_R, \sigma_M^\text{my})$.

Consider state $(R, r)$ with $2M - r \in [y^i, y^{i+1})$ for $y^i$ such that $\sigma_R(L, y^i) = 2M - y^{i+1}$. The median voter votes against $\ell \in [y^i, z^{i+1}(r))$ since the payoff to voting in favour of $\ell$ is

$$u_M(\ell) + \delta_M V_M(\sigma_L^\ell, \sigma_R^\ell, \sigma_M; (L, y^i)) < u_M(r) + \delta_M u_M(z^{i+1}(r)) + \delta_M^2 V_M(\sigma_L^\ell, \sigma_R^\ell, \sigma_M; (L, y^i)),$$

by the definition of $z^{i+1}(r)$, where the right-hand side is the payoff to voting in favour of $r$. The median voter votes against $\ell \in [y^{i+1}, y^i)$ since the payoff to voting in favour of $\ell$ is

$$u_M(\ell) + \delta_M u_M(z^{i+1}(\ell)) + \delta_M^2 V_M(\sigma_L^\ell, \sigma_R^\ell, \sigma_M; (R, 2M - y^{i+1})) < u_M(r) + \delta_M u_M(z^{i+1}(r)) + \delta_M^2 V_M(\sigma_L^\ell, \sigma_R^\ell, \sigma_M; (L, y^i)),$$

since $|M - \ell| > |M - r|, |M - z^{i+1}(\ell)| > |M - z^{i+1}(r)|$ and $V_M(\sigma_L^\ell, \sigma_R^\ell, \sigma_M; (R, 2M - y^{i+1})) < V_M(\sigma_L^\ell, \sigma_R^\ell, \sigma_M; (L, y^i))$. Similarly, the median voter votes against $\ell \in [y^{k-1}, y^k)$ for $y^k$ such that $\sigma_L(R, 2M - y^{k-1}) = y^k$ and $k \leq i - 2$, and against $\ell \in [y^{k-1}, y^k)$ for $y^k$ such that $\sigma_R(L, y^{k-1}) = 2M - y^k$ and $k \leq i - 1$. That is, in state $(R, r)$, the median voter rejects all policies $\ell \in [0, z^{i+1}(r))$. It may or may not vote for policies $\ell \in (z^{i+1}(r), 1]$. A similar argument shows that in state $(R, r)$ with $2M - r \in [y^i, y^{i+1})$ for $y^i$ such that $\sigma_L(R, 2M - y^i) = y^{i+1}$, the median voter rejects any $\ell \in [0, r]$ and may or may not support $\ell \in (r, 1]$, but always supports $\ell = y^{i+1}$.
Now consider parties’ incentives. First, whenever a party’s equilibrium policy is being accepted, it never gains by committing to policies that are sure to be rejected, since it faces the same choice in the next election. Consider again state \((R, r)\) with \(2M - r \in [y^i, y^{i+1}]\) for \(y^i\) such that \(\sigma_R(L, y^i) = 2M - y^{i+1}\). The payoff to party \(L\) from policy \(\ell \in [z^{i+1}(r), y^{i+1}]\) that is accepted by the median voter is

\[
u_L(\ell) + \delta_L u_L(2M - y^{i+1}) + \delta_L^2 V_L(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^{i+1})),
\]

which is decreasing in \(\ell \in [y^i, y^{i+1}]\). From above, policies \(\ell \in [0, z^{i+1}(r)]\) cannot be profitably proposed since they are rejected by the median voter, while policies in \([y^{i+1}, M]\), if accepted, yield to party \(L\) at most the payoff it obtains from such deviations under \((\sigma_L, \sigma_R, \sigma_{my}^M)\). Hence committing to \(z^{i+1}(r)\) is optimal for party \(L\).

Now consider policy \(y^i\) such that \(\sigma_L(R, 2M - y^i) = y^{i+1}\) and state \((R, r)\) with \(2M - r \in [y^i, y^{i+1}]\). The payoff from \(\ell \in [2M - r, y^{i+1}]\), if accepted by the median voter, is given by

\[
u_L(\ell) + \delta_L u_L(2M - z^{i+1}(\ell)) + \delta_L^2 V_L(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^i))
\]
\[
\leq u_L(\ell) + \delta_L u_L(2M - \ell) + \delta_L^2 V_L(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^i))
\]
\[
< V_L(\sigma, \sigma_R, \sigma_{my}^M; (R, 2M - y^i)).
\]

The first inequality follows from \(z^{i+1}(\ell) \leq \ell\) and the second since \(V_L(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^i)) > \frac{1}{1 - \delta_l} \nu_L^+(\ell)\). This shows that \(y^{i+1}\) is \(L'\) preferred winning policy in \([y^i, y^{i+1}]\) given \((\sigma'_L, \sigma'_R, \sigma_M)\). As the median voter rejects any policy \(\ell \in [0, 2M - r], L\) cannot profitably deviate to such policies. Finally, deviations to any policies \(\ell \in [y^{i+1}, M]\) are never profitable since even if they are accepted by the median voter, \(L\)’s payoffs are no higher than under \((\sigma_L, \sigma_R, \sigma_{my}^M)\).

It remains to deal with states \((R, r)\) with \(r < M\). By construction, in these states \(\sigma'_L\) is optimal. It needs to be shown that in states \((R, r)\) with \(r \geq M\), party \(L\) does not want to deviate to some \(\ell^d > M\). Consider state \((R, r)\) with \(r \geq M\), and suppose party \(L\) deviates to \(\ell^d > M\) such that \(\sigma'_R(L, \ell^d) = r(\ell^d)\) and take \(\{y^i\}\) to be the convergence path from \((R, r(\ell^d))\). It must be that \(y^i \geq 2M - r(\ell^d)\). The payoff to party \(L\) from \(\ell^d\) is given by

\[
u_L(\ell^d) + \delta_L u_L(\hat{r}(\ell^d)) + \sum_{i=1}^{\infty} \delta^2 u_L(\delta_L)(2M - y^{i+1})] < u_L(\ell^d) + \frac{\delta_L}{1 - \delta_L} u_L(M)
\]
\[
< \frac{1}{1 - \delta_L} u_L(M).
\]

The first inequality follows by Lemma 1.1 and the second since \(\ell^d > M\). On the equilibrium path, \(V_L(\sigma_L, \sigma_R; (R, r)) \geq \frac{1}{1 - \delta_L} u_L(M)\), and hence deviation to \(\ell^d\) is not profitable for \(L\).

\[\square\]

### 1.7.6 Legislative Bargaining

**Proof of Proposition 1.6** Consider consistent proposal strategies \((\sigma_L, \sigma_R)\) that generate convergence path \(\{y^i\} \rightarrow \hat{\ell}\) when the median legislator is decisive and \(\sigma_M = \sigma_{my}^M\). It will be shown that \(\sigma_{my}^M\) is indeed a best response for the median legislator. It is straightforward to establish results
equivalent to Lemma [1.3] that characterises consistent proposal strategies on convergence paths.

Consider a convergence path \( \{ y^i \} \rightarrow \ell \) with policy \( y^i \) such that \( \sigma_L(L, 2M - y^i) = y^{i+1} \). Since each legislator is recognised with equal probability in each period, legislator \( L \)'s equilibrium pay-off is given by

\[
V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) = u_L(y^{i+1}) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (L, y^{i+1})) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (R, y^{i+1}))
\]

\[
= \frac{2}{2 - \delta_L} \left[ u_L(y^{i+1}) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (R, y^{i+1})) \right], \quad (1.19)
\]

where the second equality is due to consistent proposal strategies. A lower bound on \( V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) \) can be determined as in Section [1.4.3] by considering a deviation to \( y^i \) by \( L \) in state \( (L, 2M - y^i) \). Hence

\[
V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) \geq u_L(y^i) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (R, y^i)). \quad (1.20)
\]

By convergence and consistent strategies, \( \sigma_R(R, y^i) = \sigma_R(R, 2M - y^i) = 2M - y^i \), and hence, as for (1.19) above,

\[
V_L(\sigma_L, \sigma_R, \sigma_M; (R, y^i)) = \frac{2}{2 - \delta_L} \left[ u_L(2M - y^i) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) \right]. \quad (1.21)
\]

Under consistent strategies, an upper bound on \( V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) \) can be obtained as in Section [1.7.3] by considering a deviation to \( y^{i+1} \) in state \( (R, 2M - y^i + \epsilon) \) for small \( \epsilon \). That is

\[
V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) \geq u_L(y^i) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) + \frac{1}{2}\delta_L V_L(\sigma_L, \sigma_R, \sigma_M; (R, y^i)). \quad (1.22)
\]

Finally, (1.20), (1.22) and (1.21) yield

\[
V_L(\sigma_L, \sigma_R, \sigma_M; (L, 2M - y^i)) = \frac{2 - \delta_L}{2(1 - \delta_L)} \left[ u_L(y^i) + \frac{\delta_L}{2 - \delta_L} u_L(2M - y^i) \right]. \quad (1.23)
\]

This is the equivalent of (1.8) which states that \( L \)'s equilibrium pay-off at \( (R, 2M - y^i) \) is the pay-off to alternation at \( (y^i, 2M - y^i) \). Expression (1.23) incorporates the fact that the future sequence of proposers is random and that convergence is staggered. A calculation like the one in (1.21) yields
Equation (1.24) is the equivalent of (1.9), the second-order differential equation that determines consistent equilibrium convergence path policies, in the legislative bargaining model. Conditions for existence of convergence paths in this model would hinge on the properties of the payoffs of legislators $L$ and $R$ relative to (1.24). However, for the purposes of Proposition 1.6, all that is required is that (1.24) must hold along any convergence path in consistent proposal strategies.

As in 1.7.4, a bound on the moderation of convergence outcomes can be derived by constructing ‘compromise’ functions $\alpha_L$ and $\alpha_R$. An argument as in 1.7.4 shows that given some $y < M$, $\alpha_L(y)$ can be defined as

$$\frac{u_L'(y)}{u_L'(2M-y)} = \frac{\delta_L}{\alpha_L(y)(2 - \delta_L) + (1 - \alpha_L(y)) \frac{\delta_1}{2 - \delta_L}}.$$ 

In particular, $\alpha_L(M) = \frac{\delta_1}{2} < \frac{1}{2}$, and a similar argument shows that $\alpha_R(M) < \frac{1}{2}$. Hence, as in Section 1.7.4, as convergence paths approach the median, both legislators require that their opponent’s next moderate move be larger than their own current moderate move, which contradicts convergence.

I have assumed that median voter behaves myopically. In fact, it can shown that this voting strategy is optimal. Consider policy $y'$ such that $\sigma_L(L,2M - y') = y'^{i+1}$. Suppose that in state $(L,2M - y')$ legislator $L$ proposes $z \in [y^i, y'^i]$. If the median voter votes in favour of $z$ its payoff is given by

$$u_M(z) + \frac{1}{2} \delta_M V_M(\sigma_L, \sigma_R, \sigma^M_L; (L,z)) + \frac{1}{2} \delta_M V_M(\sigma_L, \sigma_R, \sigma^M_L; (R,z)) > u_M(2M - y') + \frac{1}{2} \delta_M V_M(\sigma_L, \sigma_R, \sigma^M_M; (L,2M - y')) + \frac{1}{2} \delta_M V_M(\sigma_L, \sigma_R, \sigma^M_M; (R,2M - y')),$$

where the right-hand side is the payoff to supporting the status quo. This follows since $u_M(z) > u_M(2M - y')$, $V_M(\sigma_L, \sigma_R, \sigma^M_L; (L,z)) = V_M(\sigma_L, \sigma_R, \sigma^M_L; (L,2M - y'))$ since $\sigma_L(L,z) = \sigma_L(2M - y') = y'^{i+1}$ and $V_M(\sigma_L, \sigma_R, \sigma^M_M; (R,z)) > V_M(\sigma_L, \sigma_R, \sigma^M_M; (R,2M - y'))$ since $\sigma_R(R,\ell) = 2M - \ell$ for $\ell \in [y^i, y'^i]$. Similar arguments show that the median legislator accepts any policy $z \in [y'^{i+1}, 2M - y'^i]$ and rejects any policy $z \in [y^i, 2M - y']$. Furthermore, these arguments do not depend on which legislator makes the proposal, since future periods’ draws of proposers are not affected by the identity of the legislator responsible for the status quo policy.
Chapter 2
Competing Through Information Provision

2.1 Introduction

(Christie’s and Sotheby’s) embarked on cutthroat competition to get goods for sale (... and) provide ever more luxurious services. Catalogues became ever fatter, printed in colour, on glossy art paper. (...) On the inside page of Sotheby’s catalogue of the Old Master paintings sale held in London on Dec. 13 (2001), six “specialists in charge” are listed. (...) They identify the paintings, research them, know which world specialist on this or that painter needs to be contacted, and, more mundanely, which client is most likely to be interested in what painting, etc.

Competing sellers are typically thought of as proposing prices to buyers, or more generally sale mechanisms. However, as the quality of buyers’ information about goods affects their gains from trade, sellers may try to attract buyers by offering better information. This paper considers a market in which sellers post levels of information provision that are observed by potential buyers before they choose which seller to visit. When considering how much information to reveal to buyers, sellers trade off market share against the cost of selling goods to buyers with better private information.

Privately informed buyers gain informational rents through trade. Conceptually, a buyer’s information about his valuation for a good has two elements; the private knowledge of some personal attributes, along with an understanding of how these characteristics relate to the good’s properties. Sellers cannot influence the first kind of knowledge, but controlling the information about their goods affects the second kind. As noted by Bergemann and Pesendorfer (2007), by providing less information to buyers before trading, sellers give out fewer informational rents during the exchange process. Restrictions on information come at a cost, since in the presence of more than one buyer, higher information provision increases surplus by better identifying the buyer that most prefers the good. Furthermore, and this is the novel insight of this paper, if sellers compete for buyers, the latter may shun low-information selling sites.

I show that the effect of information provision depends critically on its role in competition. If sellers choose information provision independently of sale mechanisms, competition is channelled only through the level of information, which depends on the characteristics of the sale mechanisms. Sellers prefer mechanisms that have inefficient allocations and high rents, as these soften competition. On the other hand, when sellers choose sale mechanisms and information provision jointly, they channel competition away from inefficient restrictions on information and into redistributive rent transfers to buyers. They provide full information and allocate goods efficiently based on that information.

The case of the auction houses of Christie’s and Sotheby’s, related at the beginning of the sec-

1International Herald Tribune, 12/01/2002.
tion, provides a good example of competitive information provision. In that industry, the services surrounding an auction play a critical role in allowing potential customers to better evaluate an object’s worth to them. In the early 1990’s, competition between the auction houses stiffened considerably, and expanding the services that provide information to buyers became an important competitive tool. Furthermore, later in the decade Phillips, a minor auction house, tried to break the Christie’s-Sotheby’s duopoly. It did so by providing high guarantees to sellers who consigned objects there, but it also tried to match the bigger auction houses’ superior capacity to inform buyers by luring away some of their teams of experts. However, eventually Phillips became “less willing to provide lavish guarantees and loans. It emerged that Phillips’s cash, rather than its expertise, had lured sellers of high-quality art; they returned to Christie’s and Sotheby’s.” Another example is the website of Multiple Listing Service, mls.com, which allows real-estate agents to advertise houses for sale by posting pictures and descriptions. Rival agents adopt different strategies and the quality of the information revealed in the advertisements varies widely. Some agents post a bare-bones description of the house along with a picture of the house’s exterior. Others provide pictures of some of the rooms, some even post a full slide show. Bergemann and Pesendorfer (2007) provide other examples of both monopolistic and competitive markets where information provision decisions are important.

In this paper, I present a model of directed search in which two sellers with unit supplies compete for the unit demands of two buyers by promising information. Sellers commit to information structures and sale mechanisms, after which buyers choose the seller to visit and sales take place. As in Peters and Severinov (1997), sorting occurs ex ante; buyers obtain their private information only once they choose a seller. If fully informed, buyers either have (independent) high or low valuations for either seller’s objects. However, buyers’ information is mediated by the information structures offered by sellers. Information structures, as in Bergemann and Pesendorfer (2007), map signals controlled by sellers into buyers’ inferences about their valuations for goods. Sellers cannot observe signals’ effects on buyers’ estimates of their valuations, but instead control ex post distributions of values. By providing more information, sellers release private signals that allow buyers to differentiate their private values from the public expectation, interpreted as reflecting that pool of public knowledge about the goods’ ex ante characteristics accessed by any potential buyer. As in Damiano and Li (2007), Gauza and Penalva (2007), Johnson and Myatt (2006) and Ivanov (2008), I consider information structures ordered by the precision with which they allow buyers to access their true valuations. In my model, information structures have a simple correlated structure; sellers choose the probability with which all buyers get access to their valuations for their good upon visiting their site. Ex post, all buyers visiting a particular seller are informed or uninformed.

In the subgame following the sellers’ announcements, I assume that buyers sort into sale sites according to that subgame’s unique symmetric mixed strategy equilibrium. This restriction, common in directed search, rules out equilibrium coordination among buyers and ensures smooth transitions between choices. The resulting equilibrium is unique and symmetric, and it provides insights into the effects of information structures on buyers’ decisions. The model is designed to capture the strategic interactions between buyers and sellers, and to shed light on the role of information provision in competitive markets. It also allows for the analysis of the effects of information costs and frictions on search outcomes. The model is illustrated with numerical examples, which demonstrate the strategic behavior of buyers and sellers and the impact of information structures on market outcomes. The analysis of the model provides valuable insights into the design of information structures and the role of information in competitive markets.
responses in sellers’ profits to changes in their announcements. In equilibrium, sellers face a random demand, whose distribution they affect through their choice of strategy. I consider two variants for the sellers’ strategy sets. In the first, sellers commit only to information provision, while in the second each commits both to information provision and to sale mechanisms. In the first case, information provision is determined independently of sale mechanisms, which can be set by previous competitive outcomes, industry standards or regulation. This centers attention on the effects of competition through information when it is layered onto pre-existing terms of trade. In the second case, information provision and sale mechanisms are determined jointly. In both cases, I restrict attention to symmetric equilibria of the game between the sellers.

In Section 2.3 sale mechanisms are exogenously fixed and common to both sites, and sellers can attract buyers only by promising more information. Information provision increases buyer rents across demand states (i.e., when one buyer or two buyers are present) and generates a novel version of a trade-off well known in models of directed search and competing auctioneers: higher information attracts more traffic yet decreases profits-per-head. Fixing the mechanism determines the shape of this trade-off, which, through competition, determines equilibrium information provision. Furthermore, in the presence of more than one buyer, higher information provision increases surplus as it identifies the buyer who most values the good.

I establish a number of comparative statics results for equilibria under regular mechanisms, which include common mechanisms such as auctions and prices. Under regular mechanisms a monopolist would not release information, so that any gains in informational efficiency are due solely to competition. First, equilibrium information provision is increasing in the efficiency of the sale mechanism’s allocations in informed states. That is, competition creates a complimentarity between allocative and informational efficiency. High-surplus mechanisms increase sellers’ gains from information provision and lead to traffic-stealing and more intense competition. Second, equilibrium information provision is decreasing in the rents offered to buyers, since increased rents soften competition between the sellers. Third, sellers’ equilibrium profits are always lower under mechanisms with higher allocative efficiency. Fourth, sellers’ equilibrium profits are not monotone in the rents offered to buyers by sale mechanisms. Profits always increase if a mechanism offers higher rents in the one-buyer demand state, while they may drop if a mechanism offers higher rents in the two-buyer demand state. Higher rents in the one-buyer state makes the two-buyer state relatively unattractive and stiffens the competition between the buyers, while higher rents in the two-buyer demand state reduces buyers’ aversion to meeting at a site.

In Section 2.4 sellers commit to both sale mechanisms and information provision. When the effects of information are no longer mediated by the characteristics of exogenously fixed mechanisms, sellers can disentangle their rent and information provision decisions. Under a no-exclusion assumption for informed low-valuation types, I characterise a class of symmetric equilibria in which sellers capture the efficiency benefits of increased information. In these equilibria, sellers provide full information, hold auctions and compete over the rents offered to buyers by setting appropriate reserve prices. Closely related to Coles and Eeckhout (2003), who present a two-buyer, two-seller model of directed search with sale mechanisms under perfect information,
a continuum of symmetric equilibria exist that are differentiated by the sharing of a fixed level of surplus between buyers and sellers. In all equilibria, competition drives the marginal buyer’s rents to its contribution to site surplus. The full information result exploits the ex ante nature of rent and information promises; profiles in which sellers do no offer full information are vulnerable to deviations in which sellers provide more information, adjust buyers’ rents through transfers to keep their visit decisions constant, and pocket the extra surplus.

Recent work in mechanism design, auctions and optimal pricing has found that when given some means of doing so, monopolists often substantially alter the informational attributes of their customers. In a model in which a seller designs a sale mechanism ex post, Bergemann and Persandorfer (2007) characterise optimal information structures, which take a discrete monotone partitional form. Ganuza and Penalva (2007) study information provision in second-price auctions when buyers’ ex post distributions of valuations are ordered by dispersion. They show that the seller’s incentive to limit buyers’ information vanishes as the number of buyers grows and the competition between them for the good wipes out their informational rents (on this see also Board (2009)). In contrast, when the seller designs a mechanism ex ante and hence can ‘sell’ information to buyers, Esö and Szentes (2007) show that the seller can capture all rents accruing from the information it controls and provides full information. In a model of monopoly pricing, Johnson and Myatt (2006) assume that sellers’ information provision orders buyers’ ex post distributions of valuations by sequences of rotations. In a result recalling that of Lewis and Sappington (1994), Johnson and Myatt (2006) find conditions under which a seller’s optimal choice of information provision is to either release all or none of the available signals. Bergemann and Välimäki (2006b) survey provides more references to related literature.

The question of how the incentives to provide information extend to a competitive market has received little attention to date. Damiano and Li (2007) present a model of two-seller competition with information provision and ex post price competition which generalises that of Moscarini and Ottaviani (2001). With a single buyer and price competition, information does not enhance surplus and in equilibrium sellers provide information to differentiate goods ex post and soften competition. Ivanov (2008) studies a related model with any number of sellers and continuous type distributions and shows that as the number of sellers increases there is a unique symmetric equilibrium with full information provision.

2.2 Model

Sellers: Two sellers, a and b, have a single good for sale.

Buyers: Two buyers have unit demands. An informed buyer’s valuation for either seller’s good is either $\theta_H$ or $\theta_L$, with $\theta_H > \theta_L$. The sellers’ goods are ex ante similar to buyers; the prior distribution

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6See Section 2.4 for a more detailed discussion of the two papers’ results.

7For random variables $X$ and $Y$ with distribution functions $F$ and $G$, $Y$ is said to be more dispersed than $X$ if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha < \beta < 1$. See Shaked and Shanthikumar (2007).

8See Section 2.4.1 for a discussion of my full information result and its relation to that of Esö and Szentes (2007).

9Continuous distribution function $G$ is said to be obtained from distribution $F$ by (clockwise) rotation around $z$ if $F(x) \leq G(x)$ for all $x \leq z$ and $F(x) \geq G(x)$ for all $x \geq z$. 
of buyer valuations for either good is \((p_H, p_L)\). The expected value of any good for a buyer is 
\[
\bar{\theta} = p_L \theta_L + p_H \theta_H.
\]

**Information Provision:** In the first stage of the game, sellers commit to information provision. Information structures are as follows: seller \(k\) posts a probability \(\pi_k\) with which information about the good is revealed at site \(k\) to all buyers that choose to attend it. Ex post, either all buyers at site \(k\) are informed or all are uninformed. Informed buyers’ values for goods are private and uninformed buyers have known expected value \(\bar{\theta}\) for any good.

**Remarks:** The two-seller, two-buyer setup is restrictive but counters well-known equilibrium existence and tractability issues in finite directed search and competing auctions, which explains why Peters and Severinov (1997), following McAfee (1993), focus on large economies in which a seller’s impact on market conditions vanishes. Burguet and Sákovics (1999) prove existence of a symmetric equilibrium in a 2-seller, \(n\)-buyer framework, but their characterisation is difficult to work with. See Hernando-Veciana (2005) and Virág (2009) for existence results in finite competing auctions, and Galenianos and Kircher (2009) along with Galenianos et al. (2011) for directed search equilibria in finite markets.

Having sellers choose the probability of providing information and not directly choosing some ex post distribution of types simplifies the model by reducing the ex post information states to two; informed and uninformed. However, the essential feature is that choices of \((\pi_a, \pi_b)\) differentiate the sites with respect to information ex ante. In fact, the information structures of my model can be seen to be discrete examples of those of Johnson and Myatt (2006). Consider ex post distribution of valuations \(F^\pi\) for a single buyer over valuation space \(\{\theta_L, \bar{\theta}, \theta_H\}\) generated by the information structure of my model with probability \(\pi\). \(\bar{\theta}\) is a rotation point for the family of distributions \(\{F^\pi\}\) since for \(\pi > \pi', F^\pi(x) \geq F^{\pi'}(x)\) for all \(x < \bar{\theta}\) and \(F^\pi(x) \leq F^{\pi'}(x)\) for all \(x \geq \bar{\theta}\).

**Demand and Information States:** Once sorted into selling sites, buyers either receive information about the good or not, learn the realisation of the demand state, and take part in the sale mechanism. Let \(\eta \in \{1, 2\}\) denote the demand state of a sale site\(^{10}\) and \(\tau \in \{i, u\}\) its information state, where \(i\) stands for informed and \(u\) for uninformed. The state of a sale site is given by \((\eta, \tau) \in \{1, 2\} \times \{i, u\}\).

**Sale Mechanisms:** How goods are delivered to the buyers attending site \(k\) may be exogenously fixed or committed to by the seller in tandem with \(\pi_k\). Terms of trade at site \(k\) are given by direct incentive compatible mechanisms\(^{11}\). These mechanisms specify allocations and transfers as functions of reported types for all information and demand states of the market and are constrained to be anonymous. Also, I assume that sellers cannot charge entry fees to buyers prior to the state of the site being realized. That is, all buyer participation decisions are ex post. A complete, and standard, presentation of the sale mechanisms, and corresponding payoffs for buyers and sellers, is reported to Appendix 2.6. Let \(\Gamma\) be the set of direct incentive compatible mechanisms for my model. Importantly, any mechanism \(\gamma_k \in \Gamma\) at site \(k\) induces ex ante rents for buyers in state \((\eta, \tau)\), \(R_{k, \eta, \tau}^{\gamma}\). These rents are computed before buyers learn their types and hence, in informed states, ex ante rents are the average of \(\theta_H\) and \(\theta_L\)-type rents. Denote the ex ante surplus at site \(k\) in state \((\eta, \tau)\)

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\(^{10}\)To lighten notation, I ignore the demand state in which no buyer visits a seller.

\(^{11}\)As is known from the literature on common agency (see Epstein and Peters 1999, Martimort and Stole 2002 and Peters 2001), restricting sellers to direct mechanisms is not without loss of generality.
under mechanism $\gamma_k$ as $S_k^{\eta,\tau}$. The ex ante surplus is obtained by averaging total gains from trade in state $(\eta, \tau)$ over buyer types, and it depends on mechanisms’ allocation rules. Let $S_k^{\eta,\tau}$ be the maximal available surplus at site $k$ in state $(\eta, \tau)$.

**Seller Strategies:** To focus on competition in information provision, in Section 2.3 sale mechanisms are fixed and a strategy for seller $k$ is a probability $\pi_k \in [0, 1]$. In Section 2.4 sellers compete by promising both information and mechanisms, and a strategy for seller $k$ is $(\pi_k, \gamma_k) \in [0, 1] \times \Gamma$, a probability $\pi_k$ along with a mechanism $\gamma_k$.

**Buyers’ Subgame:** Given sellers’ sale mechanisms and information offers $(\pi_a, \gamma_a, \pi_b, \gamma_b)$, buyers simultaneously choose which site to visit. A strategy for a buyer is $q : ([0, 1] \times \Gamma)^2 \rightarrow [0, 1]$, where $q$ denotes the probability with which the buyer visits seller $a$. The buyers’ subgame has a large number of equilibria; I consider symmetric mixed strategy equilibria. It has been argued, notably by Levin and Smith (1994) in the context of a single auction with entry and by Burdett et al. (2001) in a directed search model, that the equilibria with symmetric mixed strategies by buyers and random demand are more appealing than asymmetric pure strategy equilibria which generate fixed demand. Burdett et al. (2001) show that there exist many equilibria with pure actions on the equilibrium path in which sellers’ equilibrium offers are supported by buyers’ threats to revert to the mixed strategy equilibrium in the buyers’ subgame. In such equilibria coordination improves buyers’ payoffs relative to the mixed strategy equilibrium but yields behaviour that is not relevant for the questions studied here.

**Buyers’ Subgame Equilibrium:** Given strategy $(\pi_a, \gamma_a)$ for seller $a$ and a visit probability $q$ for buyers, a bidder attending auction site $a$ expects rents

$$R_a(\pi_a, \gamma_a, q) = E_\eta E_\tau R_a^{\eta,\tau}$$

$$= q \left[ \pi_a R_a^{2,j} + (1 - \pi_a) R_a^{2,u} \right] + (1 - q) \left[ \pi_a R_a^{1,j} + (1 - \pi_a) R_a^{1,u} \right].$$

(2.1)

The first expectation above is with respect to the binomial distribution with parameter $q$ of the number of opponents faced by a buyer present at site $a$, and the second with respect to the binomial distribution with parameter $\pi_a$ over information states at site $a$. Similarly, given strategy $(\pi_b, \gamma_b)$ for seller $b$ and visit probability $q$, a bidder attending auction site $b$ expects rents

$$R_b(\pi_b, \gamma_b, q) = E_\eta E_\tau R_b^{\eta,\tau}$$

$$= (1 - q) \left[ \pi_b R_b^{2,j} + (1 - \pi_b) R_b^{2,u} \right] + q \left[ \pi_b R_b^{1,j} + (1 - \pi_b) R_b^{1,u} \right].$$

Buyers’ visit decisions depend on whether or not sellers’ mechanisms generate congestion effects, that is, whether their rents at a given site decrease when the other buyer visits it more frequently. Site $k$’s mechanism generates congestion effects if $\pi_k R_k^{1,j} + (1 - \pi_k) R_k^{1,u} \geq \pi_k R_k^{2,j} + (1 - \pi_k) R_k^{2,u}$. It this is the case, in the unique symmetric mixed strategy equilibrium of the buyers’ subgame the
visit probability must satisfy \(^{12}\)
\[
q \begin{cases} 
1 & \text{if } R_a(\pi_a, \gamma_a, 1) \geq R_b(\pi_b, \gamma_b, 0), \\
0 & \text{if } R_a(\pi_a, \gamma_a, 1) \leq R_b(\pi_b, \gamma_b, 0), 
\end{cases}
\]
while if both \(R_a(\pi_a, \gamma_a, 1) < R_b(\pi_b, \gamma_b, 0)\) and \(R_a(\pi_a, \gamma_a, 0) > R_b(\pi_b, \gamma_b, 1)\), \(q \in (0, 1)\) is the unique solution to
\[
R_a(\pi_a, \gamma_a, q) = R_b(\pi_b, \gamma_b, q). \tag{2.2}
\]
Natural sales mechanisms, such as posted prices and auctions, always generate congestion effects and hence (2.2) pins down buyer behaviour uniquely for these mechanisms. All exogenous mechanisms considered in this paper, as well as the equilibrium endogenous mechanisms, will generate congestion effects. However, as in Coles and Eeckhout (2003), since off the equilibrium path sellers can commit to mechanisms that do not generate congestion effects, it is necessary to determine buyers’ behaviour for such mechanisms.

If some sellers’ sale mechanisms does not generate congestion effects, visit probability \(q\) satisfies
\[
q \begin{cases} 
0 & \text{if } R_a(\pi_a, \gamma_a, 0) < R_b(\pi_b, \gamma_b, 0) \text{ and } R_a(\pi_a, \gamma_a, 1) < R_b(\pi_b, \gamma_b, 1), \\
1 & \text{if } R_a(\pi_a, \gamma_a, 0) > R_b(\pi_b, \gamma_b, 0) \text{ and } R_a(\pi_a, \gamma_a, 1) > R_b(\pi_b, \gamma_b, 1). 
\end{cases}
\]
However, if either \(R_a(\pi_a, \gamma_a, 0) \geq R_b(\pi_b, \gamma_b, 0)\) and \(R_a(\pi_a, \gamma_a, 1) \leq R_b(\pi_b, \gamma_b, 1)\) or \(R_a(\pi_a, \gamma_a, 0) \leq R_b(\pi_b, \gamma_b, 0)\) and \(R_a(\pi_a, \gamma_a, 1) \geq R_b(\pi_b, \gamma_b, 1)\), then both \(q = 1\) and \(q = 0\) are equilibria, along with any \(q\) satisfying (2.2). That is, when mechanisms do not generate congestion effects, buyers have an incentive to coordinate onto a common site, and the strategies allowing for this coordination are symmetric. Hence symmetry alone does not yield a unique equilibrium. I assume that in such cases the equilibrium selected is the mixed strategy equilibrium satisfying (2.2). It is possible to justify this selection by noting that in the symmetric pure strategy equilibrium one seller receives no visits and makes no profits, and hence has an incentive to offer a different mechanism at the offer stage.\(^{13}\)

**Equilibrium:** With the equilibrium in the buyers’ subgame fixed, buyer behaviour is characterised by \(q\). When interior, its responses to information provision and mechanisms is given by (2.2). In the rest of the paper, an equilibrium refers to a subgame perfect equilibrium of the full game with buyer strategies given by (2.2). Throughout the paper, I consider symmetric equilibria in the sellers’ strategies.

**Sellers’ Profits:** Profits of seller \(k\), given strategy profile \((\pi_k, \gamma_k, \pi_{-k}, \gamma_{-k})\), can be expressed as surplus less rents as
\[
P_k(\pi_k, \gamma_k, \pi_{-k}, \gamma_{-k}) = E_{\eta} E_{\tau} \left[ S_k^{\eta, \tau} - \eta R_k^{\eta, \tau} \right]. \tag{2.3}
\]

\(^{12}\) To lessen notation, the visit probability generated by \((\pi_a, \gamma_a, \pi_b, \gamma_b)\) will simply be denoted by \(q\), with its dependence on information provision and mechanisms understood.

\(^{13}\) Coles and Eeckhout (2003) give a different justification for ignoring pure strategy symmetric coordination equilibria. They note that since the mixed strategy equilibrium is always determined by (2.2), a seller that wishes to induce the mixed strategy outcome can always change his mechanism to induce congestion effects without varying rents and hence have the mixed equilibrium be the unique symmetric equilibrium of the subgame.
The first expectation is with respect to the binomial distribution with parameter \( q \) (if \( k = a \)) or \( 1 - q \) (if \( k = b \)) of demand at site \( k \), and the second with respect to the binomial distribution with parameter \( \pi_k \) over information states at site \( k \).

**A Characterisation of Incentive Compatible Mechanisms:** Note that buyers’ sorting decisions, as expressed by (2.2), depend only on information provision and expected rents \( R_\eta^\tau \). In particular, buyer decisions are not affected by how rents are shared between types conditional on being informed. This ex ante feature of rent promises allows a useful characterisation of incentive-compatible mechanisms, which simplifies sellers’ strategy sets. Crucially, as Lemma 2.6 in Appendix [2.6.3] illustrates, we can restrict \( \theta_H \)-type incentive-compatibility constraints to be binding in states \( (1, i) \) and \( (2, i) \). This is without loss of generality since any incentive-compatible mechanism at site \( k \) that achieves rents \( \{ R_\eta^\tau \} \) with non-binding \( \theta_H \)-type incentive constraints can be replaced by an incentive compatible mechanism that achieves the same levels of expected rents with the same allocations, but in which these constraints bind. Under this new mechanism, profits are unchanged and all traffic and information provision incentives are preserved. The proof is simple: given an incentive compatible mechanism in which the incentive constraint of \( \theta_H \)-types in state \( (\eta, i) \) does not bind, we can increase \( \theta_L \)-type rents and decrease \( \theta_H \)-type rents through transfers until the constraint binds, while ensuring that the expected rents in demand state \( (\eta, i) \) are unchanged.

Denote by \( \tilde{\Gamma} \) the set of incentive compatible mechanisms with binding \( \theta_H \)-type incentive compatibility constraints. My result shows that restricting sellers to offering mechanisms in \( \tilde{\Gamma} \) does not alter the set of equilibrium outcomes of the game, that is, information provision, allocations, rents and visit probabilities. Denote low-type rents under mechanism \( \gamma_k \) in state \( (\eta, \tau) \) by \( r_\eta^\tau \). These are the rents offered to \( \theta_L \)-types in informed states and to the uninformed otherwise. Let \( l_\eta^\tau \) be the expected informational rents in state \( (\eta, i) \). Lemma 2.7 in Appendix [2.6] shows that mechanisms \( \gamma_k \in \tilde{\Gamma} \) are characterised by monotone allocation probabilities, \( r_\eta^\tau \geq 0 \) for all states \( (\eta, \tau) \) and expected rents

\[
R_\eta^\tau = r_\eta^\tau \quad \text{for} \quad \eta \in \{1, 2\},
R_1^\tau = r_1^\tau + l_1^\tau,
R_2^\tau = r_2^\tau + l_2^\tau.
\]

### 2.3 Fixed Mechanisms

In this section, sale mechanisms are exogenously fixed and common to both sale sites and sellers commit solely to information provision. This centers attention on information’s impact on competition in the case that terms of trade have already been determined. Exogenous mechanisms constrain the rent offers that sellers can extend to buyers through their choice of information provision. The goal of this section is to understand how sale mechanisms affect sellers’ trade-off between traffic and profit-per-buyer, and through this effect, equilibrium information provision.
2.3.1 Second-Price Auctions

I start with an example in which sellers hold second-price auctions without reserve prices irrespective of how many buyers visit them. As Board (2009) and Ganuza and Penalva (2007) derive the optimal information structures for monopolists in a second-price auction with two buyers, this example constitutes a useful benchmark to gauge the effects of competition.

With second-price auctions, buyers obtain the good for free in the one-buyer state, and capture the full surplus $\bar{\theta}$. In the two-buyer state, to bid their best estimate of their true value is a weakly dominant strategy for buyers. When uninformed, this best estimate is $\bar{\theta}$.

A buyer that attends site $a$, given $\pi_a$ and $q$, expects rents

$$ R_a(\pi_a, q) = q\pi_a p_H p_L (\theta_H - \theta_L) + (1-q)\bar{\theta}, $$

while a bidder attending site $b$, given $\pi_b$ and $q$, expects rents

$$ R_b(\pi_b, q) = (1-q)\pi_b p_H p_L (\theta_H - \theta_L) + q\bar{\theta}. $$

In the mixed strategy equilibrium of the buyers’ subgame, the probability with which buyers visit site $a$ is given by

$$ q = \frac{\bar{\theta} - \pi_b p_H p_L (\theta_H - \theta_L)}{\bar{\theta} - \pi_a p_H p_L (\theta_H - \theta_L) + \bar{\theta} - \pi_b p_H p_L (\theta_H - \theta_L)}. \quad (2.4) $$

The profits of seller $a$, given $(\pi_a, \pi_b)$ and the resulting $q$, are given by

$$ P_a(\pi_a, \pi_b) = q^2 \left[ \pi_a \left( p_H^2 \theta_H + (1-p_H^2) \theta_L \right) + (1-\pi_a)\bar{\theta} \right] $$

$$ = q^2 \left[ \bar{\theta} - \pi_a p_H p_L (\theta_H - \theta_L) \right]. \quad (2.5) $$

The term in the brackets of (2.5) is the expected price paid by the buyer who obtains the good in the two-buyer state. This price decreases in $\pi_a$, since the seller then gives away a higher share of the surplus as informational rents. Denote this price by $w_a(\pi_a)$. Suppose a single second-price auctioneer faced a fixed set of two buyers, then its profits given information provision $\pi$ would be $w(\pi)$.

Proposition 2.1. (No Information under Monopoly) A second-price auctioneer with no reserve price facing two buyers maximises profits by setting $\pi = 0$.

This result is known from Board (2009) and Ganuza and Penalva (2007). Returning to my model, note that (2.4) can be rewritten as

$$ q = \frac{w_b(\pi_b)}{w_a(\pi_a) + w_b(\pi_b)}. \quad (2.6) $$

Since buyers get all the surplus if alone, $q$ depends only on how much profits sellers get from
demand states with two buyers. Thus (2.5) becomes

\[ P_a(\pi_a, \pi_b) = \left[ \frac{w_b(\pi_b)}{w_a(\pi_a) + w_b(\pi_b)} \right]^2 w_a(\pi_a) \]

\[ = w_b(\pi_b) \left[ \frac{w_b(\pi_b)}{w_a(\pi_a) + w_b(\pi_b)} \cdot \frac{w_a(\pi_a)}{w_a(\pi_a) + w_b(\pi_b)} \right] \]

\[ = w_b(\pi_b) q(1-q). \quad (2.7) \]

Clearly, seller a’s choice of information influences profits in (2.7) only through its effect on \( q(1-q) \), which attains a maximum when \( q = \frac{1}{2} \). Seller a can attain this maximum by setting \( \pi_a = \pi_b \). This leads to the following result.

**Proposition 2.2. (Any Level of Information under Competition)** When the sale mechanism is a second-price auction with no reserve price, \((\pi_a, \pi_b)\) is an equilibrium if and only if \( \pi_a = \pi_b \).

This surprising result states that a seller’s best-response to any information offer by an opponent is to match that promise. Rewriting the rents of a buyer attending site a yields

\[ R_a(\pi_a, q) = \bar{\theta} - q w_a(\pi_a). \quad (2.8) \]

That is, it is as though seller a gives an entering buyer an ‘attendance fee’ \( \bar{\theta} \), but imposes a ‘congestion charge’ of \( w_a(\pi_a) \) when the other buyer is also present. Rents can be rewritten in this particular form since the sale mechanism is a second-price auction with no reserve price, yet this does not depend on my assumptions about buyers’ types and sellers’ information structures. Suppose buyers’ true valuations were instead given by some continuous random variable \( Y \) with mean \( \bar{\theta} \). Denote by \( Y_{1:2} \) and \( Y_{2:2} \) the expected values of the first and second order statistics of \( Y \), then \( Y_{1:2} + Y_{2:2} = 2\bar{\theta} \). Rewriting rents as in (2.8) uses the discrete version of this identity, which in turn allows the representation of profits in (2.7). Similarly, this result is not due to my special correlated information structures. If instead \( \pi \) indexed ex post valuations \( Y^\pi \) with \( \mathbb{E} Y^\pi = \bar{\theta} \) for all \( \pi \), then \( Y^\pi_{1:2} + Y^\pi_{2:2} = 2\bar{\theta} \) for all \( \pi \), and the result of Proposition 2.2 still follows. Hence while the result of Proposition 2.2 is not due to my model’s special information structures, it does depend critically on there being only two buyers and two sellers.\(^{14}\)

### 2.3.2 Equilibrium with Regular Mechanisms

The case of second price auctions, while special, demonstrates that the set of equilibria in information provision for any exogenous incentive compatible mechanisms will be difficult to deal with in general. Here, I introduce a class of mechanisms, called *regular mechanisms*, which impose restriction on mechanisms’ allocations and rent offers to uninformed types\(^{15}\).

---

\(^{14}\)The result of Proposition 2.2 also depends critically on the other assumptions of the model, for example that information provision is costless. Say providing information required a cost of \( c \). Then seller a’s profits would be given by (2.7), less some cost term that depends on \( q \) and \( c \). Thus at symmetric profiles marginal profits are negative, so that the only symmetric equilibrium has no information provision.

\(^{15}\)All definitions that involve mechanism allocations are stated formally in Appendix 2.6.4.
Definition 2.1. (No Waste) A mechanism $\gamma_k \in \Gamma$ has no waste if and only if the good is always delivered to some buyer.

In a mechanism with no waste, the full surplus ($\bar{\theta}$) is realized in the one-buyer and uninformed states, while in state $(2, i)$ the full surplus is realized only when a $\theta_L$-type never obtains the good when a $\theta_H$-type is present.

Definition 2.2. (Regular Mechanisms) An incentive compatible mechanism $\gamma$ is regular if and only if

i. (Exploiting the uninformed) $R^{1,u} = R^{2,u} = 0$.

ii. (Congestion effects) $R^{1,i} > R^{2,i}$.

iii. $\gamma$ has no waste.

Property i states that, in uninformed states, a regular mechanism fully exploits the buyers’ lack of information. Sellers benefit from restricting buyers’ information through an easing of incentive constraints and, in regular mechanisms, sellers capture all gains from trade when buyers have no private information. Property ii states that regular mechanisms generate congestion effects and a buyer strictly prefers being alone at a selling site. Finally, that regular mechanisms have no waste is a sufficient condition for expected surplus in the two-buyer state to be increasing in information provision, that is, $S^{2,i} \geq \bar{\theta}$. Total available surplus in the two-buyer state always increases in information provision, yet the sale mechanism’s allocation rules may sufficiently restrict delivery of the good in informed states that realized surplus decreases in information provision.

Regular mechanisms combine the properties that make the study of ex ante competition through information provision interesting: sellers extract more rents from poorly informed buyers; buyers, who compete for goods, dislike the presence of other buyers and; information provision does not solely redistribute rents, but enhances total surplus. In informed states, standard mechanisms that always deliver the good to some buyer, such as auctions with reserve prices lower than $\theta_L$, or a uniform price (independent of demand state) less than $\theta_L$, can be components of regular mechanisms when combined with take-it-or-leave-it offers of $\bar{\theta}$ in uninformed states.

Under a regular mechanism $\gamma$, seller a’s profits are

$$\mathcal{P}_a(\pi_a, \gamma, \pi_b, \gamma) = q^2 \left[ \pi_a S^{2,i} + (1 - \pi_a)\bar{\theta} - 2\pi_a R^{2,i} \right] + 2q(1-q) \left[ \bar{\theta} - \pi_a R^{1,i} \right].$$

(2.9)

Seller a’s profits in the one-buyer state, $\bar{\theta} - \pi_a R^{1,i}$, are clearly decreasing in $\pi_a$. Furthermore, as shown in Appendix 2.6.4, that $\gamma$ has no waste implies that seller a’s profits in the two-buyer state, $\pi_a S^{2,i} + (1 - \pi_a)\bar{\theta} - 2\pi_a R^{2,i}$, are also decreasing in $\pi_a$. Note that this feature implies that, as in the case of second price auctions, a monopolist facing two buyers under regular mechanisms would never provide information. Hence, any information provision achieved in equilibrium with regular mechanisms is due to competition.

At symmetric profiles, the market is shared equally between the two sellers. In particular, an equally split market maximises the probability that a seller is visited by a single buyer ($2q(1-q)$),
which means that marginal shifts in information provision at symmetric profiles have no effect on this probability\footnote{This observation, often useful in the the rest of the paper, is due to the binomial distribution of demand at sale sites. That is, if $X \sim B(n, q)$ then $\frac{\partial \Pr(X = k)}{\partial q} > 0$ whenever $k > qn$, where $qn$ is the mean state of $X$. If $qn$ is an integer, then $\frac{\partial \Pr(X = qn)}{\partial q} = 0$. That is, if $q$ is increased marginally, states above the mean state become more likely and states below the mean less likely, while the probability of the mean state is unchanged.}. This simplifies the expression for marginal profits at symmetric profiles under regular mechanism $\gamma$, which is given by

$$
\frac{\partial P_a(\pi_{a\gamma}, \pi_b, \gamma)}{\partial \pi_a} \bigg|_{\pi_a = \pi_b = \pi} = \frac{\partial q}{\partial \pi_a} \bigg|_{\pi_a = \pi_b = \pi} \left[ \pi S^{2,j} + (1 - \pi) \bar{\theta} - 2\pi R^{2,j} \right] + \frac{1}{4} \left[ S^{2,j} - \bar{\theta} - 2R^{2,j} \right] - \frac{1}{2} R^{1,j}.
$$

(2.10)

The first term of the right-hand side \ref{2.10} is the increased traffic effect of an increase in information provision, which says that seller $a$ gains more rents to two-buyer state profits more often. The two last terms are the decreased profit-per-head effect, since seller $a$ now hands over more rents to all visiting buyers in each state. Since the right-hand side of \ref{2.10} can cross 0 at most once, regular mechanisms produce a unique symmetric equilibrium candidate profile.

Lemma 2.1. (Unique Candidate for Symmetric Equilibrium) In games with regular mechanisms, there is a unique candidate profile for symmetric equilibrium in information provision, given by

$$
\pi^* \equiv \begin{cases} 
\frac{-(R^{1,j} + R^{2,j})\bar{\theta}}{2R^{1,j}(S^{2,j} - \bar{\theta} + (R^{1,j} + R^{2,j}))} & \text{if } 2R^{1,j} > \bar{\theta} \text{ and } R^{1,j} + R^{2,j} > \frac{2R^{1,j}(S^{2,j} - \bar{\theta})}{2\bar{\theta} - \bar{\theta}}, \\
1 & \text{otherwise.} 
\end{cases}
$$

(2.11)

Clearly, with regular mechanisms, no equilibrium with $\pi = 0$ can exist, as uninformed buyers get no rents and any deviation by some seller from such a profile to any $\pi' > 0$ would attract all buyers. Hence relative to monopoly, competition always improves informational efficiency. Lemma 2.1 depends on the fact that the decreased profit-per-head effect is negative and does not depend on $\pi_a$ by the linearity of the information structures. Also, I show in Appendix 2.7 that $\frac{\partial \bar{q}}{\partial \pi_a} \bigg|_{\pi_a = \pi_b = \pi}$ is decreasing in $\pi$, the symmetric level of information provision. This, along with the fact that under regular mechanisms sellers’ profits in the two-buyer state are decreasing in $\pi$, implies that the increased traffic effect, though positive, is decreasing in $\pi$. That is, buyers are less sensitive to information provision when in a high-information environment, and also in such environments the profits generated by more frequent buyer visits are lower.

Since the profit function in \ref{2.9} is not concave in $\pi_a$, the solution to setting \ref{2.10} to zero is not sufficient to establish the existence of a symmetric equilibrium. In fact, the behaviour of \ref{2.9} in $\pi_a$ is complex. In Appendix 2.7 I present conditions on mechanisms’ rents that guarantee that seller $a$’s profit function is single-peaked around $\pi_a = \pi^*$ when $\pi_b = \pi^*$ and $\pi^* < 1$\footnote{See the proof of Proposition 2.3}. In the same way, it is possible to derive sufficient conditions for the existence of full-information equilibria when $\pi^* = 1$. I focus on interior symmetric equilibria in order to derive comparative statics results that describe how varying the features of regular mechanisms affect equilibrium levels of information provision and seller profits.
First, I consider shifts in the allocative efficiency of the mechanisms that leave rents unchanged. These changes can be implemented by changing mechanisms’ allocations and adjusting rents through transfers. I then consider shifts in rents that leave expected surplus unchanged. Such shifts can be implemented through changes in transfers, without affecting allocations. Let $\Psi$ be the set of regular mechanisms $\gamma$ such that: (i) The information provision game between sellers with mechanism $\gamma$ has a unique symmetric equilibrium $(\pi^*, \pi^*)$ with $\pi^* < 1$, and; (ii) There exists a neighbourhood $N$ of $\gamma$ in the space of regular mechanisms such that any $\hat{\gamma} \in N$ induces a unique symmetric equilibrium in information provision $(\hat{\pi}^*, \hat{\pi}^*)$ with $\hat{\pi}^* < 1$. The proof of Proposition 2.3 shows $\Psi$ to be nonempty.

**Proposition 2.3. (Information Provision Increases in Surplus and Decreases in Rents)** For a regular mechanism $\gamma \in \Psi$, the symmetric equilibrium information provision $\pi^*$ is such that

$$\frac{\partial \pi^*}{\partial S_{2,i}} > 0,$$

and

$$\frac{\partial \pi^*}{\partial R_{1,i}} < \frac{\partial \pi^*}{\partial R_{2,i}} \leq 0,$$

with $\frac{\partial \pi^*}{\partial R_{2,i}} = 0$ if and only if $S_{2,i} = \bar{\theta}$.

Higher surplus in the two-buyer state increases the gains to information provision, which leads to increased competition between sellers and more information provision. Competition generates a complimentarity between allocative and informational efficiency. More efficient mechanisms, by realising higher surplus, lead sellers to attempt to capture more of it by providing information.

Similarly, more generous mechanisms lead to lower equilibrium information provision, since higher rents dampen the competition between sellers by increasing the cost of attracting more buyers. However, the drop in equilibrium information provision is more pronounced when rents in the one-buyer state rather than in the two-buyer state are increased. Rewrite buyer rents from attending site $a$ in (2.1) as

$$R_a(\pi_a, \gamma, q) = \pi_a \left[ R_{1,i} - q(R_{1,i} - R_{2,i}) \right]. \quad (2.12)$$

That is, when attending site $a$ and conditional on being informed, it is as though a buyer is paid a ‘attendance fee’ of $R_{1,i}$, while he suffers a ‘congestion charge’ of $R_{1,i} - R_{2,i}$ whenever the other buyer is also present. An increase in $R_{1,i}$ affects buyer rents to attending site $a$ in two ways as both the attendance fee and the congestion charge increase. The second effect reduces buyers’ incentives to visit a deviating seller with higher probability, as this increases their chance of meeting at the same site. This buyer inertia further softens the competition between sellers. On the other hand, an increase in $R_{2,i}$ reduces the congestion charge suffered by a buyer at site $a$. By making buyers less averse to meeting their opponents at a site, this increases sellers’ incentives to deviate from symmetric profiles and hence intensifies competition between them.

**Proposition 2.4. (Profits Decrease in Surplus and Nonmonotone in Rents)** For a regular mecha-
nism $\gamma$, suppose conditions i and ii of Proposition 2.3 hold. Then
\[
\frac{\partial P_a(\pi^*, \gamma, \pi^*, \gamma)}{\partial S^{2,i}} < 0.
\]
Furthermore,
\[
\frac{\partial P_a(\pi^*, \gamma, \pi^*, \gamma)}{\partial R^{1,i}} > 0
\]
and
\[
\frac{\partial P_a(\pi^*, \gamma, \pi^*, \gamma)}{\partial R^{2,i}} \geq (>) 0 \text{ if } R^{1,i} + R^{2,i} \leq (>) S^{2,i} - \bar{\theta} \sqrt{2}.
\]

Since equilibrium profits are decreasing in two-buyer state surplus, sellers’ preferred mechanisms generate inefficiencies in allocations. This is the more negative implication of the complementarity between allocative and informational efficiency; given a choice, sellers would lower both to soften their competitive environment. If sellers could collude and commit to sales mechanisms while anticipating future competition in information they would protect themselves against its effects by selecting mechanisms with inefficient allocations.

According to Proposition 2.4, when rents in the one-buyer state increase, the drop in the equilibrium level of information provision raises profits enough to compensate for the rent increase, while this is not always the case for increases in rents in the two-buyer state. For example, note that under any mechanism in which $S^{2,i} = \bar{\theta}$, we have that \( \frac{\partial P_a(\pi^*, \gamma, \pi^*, \gamma)}{\partial R^{2,i}} < 0 \). Since changes in rents are achieved through transfers, the different mechanisms considered in the second result of Proposition 2.4 have the same allocative efficiency. However, the mechanisms preferred by sellers may foster inefficient outcomes by leading to low levels of information provision.

2.3.3 Examples

**Ex Post Optimal Mechanisms**

The results of the previous section can be used to study the situation in which sellers commit to levels of information provision but cannot commit to sale mechanisms. In that case, once buyers have chosen sale sites, sellers deliver their good through each state’s *ex post optimal mechanisms*. When buyers are uninformed, sellers optimally make take-it-or-leave-it offers of $\bar{\theta}$. When buyers are informed, the optimal mechanisms for both the one and two-buyer states depend on whether or not sellers prefer to exclude $\theta_L$-types and sell only to $\theta_H$-types. For both demand states, a seller strictly prefers to sell to $\theta_L$-types whenever $\theta_L > p_H \theta_H$. When $\theta_L$-types are excluded, sellers extract all informational rents from $\theta_H$-types. In that case, buyers expect no rents from any demand state regardless of the level of information provision. The interesting case is when $\theta_L > p_H \theta_H$ and informed $\theta_H$-types obtain rents.

**Assumption 2.1. (No Exclusion under Ex Post Optimal Mechanisms)** $\theta_L > p_H \theta_H$.

Note also that under Assumption 2.1, the ex post optimal mechanisms are regular and can be described by rent levels for low types $r^\eta, \tau = 0$ for all $\eta \in \{1, 2\}, \tau \in \{i, u\}$ and expected informational rents $I_{1,i}^j = p_H(\theta_H - \theta_L)$ and $I_{2,i}^j = \frac{1}{2} p_L p_H(\theta_H - \theta_L)$. By Lemma 2.1, there is a
unique candidate $\pi^*$ for symmetric equilibrium and since under ex post optimal mechanisms $R_{1,i} = p_H(\theta_H - \theta_L)$, it follows that

$$2R_{1,i} - \bar{\theta} = p_H\theta_H - p_L\theta_L - 2p_H\theta_L$$

$$= p_H\theta_H + p_L\theta_L - 2\theta_L$$

$$< \theta_L(p_L - 1)$$

$$< 0,$$

where the first inequality follows from $\theta_L > p_H\theta_H$. Thus, by (2.11), under optimal sale mechanisms, the only candidate for symmetric equilibrium is full information provision, which can be shown to constitute an equilibrium.

**Proposition 2.5. (Full Information with no Commitment to Mechanisms)** *Under Assumption 2.1 and ex post optimal mechanisms, the unique symmetric equilibrium has full information provision.*

To show that full information provision is indeed a symmetric equilibrium, I show that seller $a$’s profits are increasing in $\pi_a$ when $\pi_b = 1$. When buyers face the optimal mechanisms once sorted, expected rents are low. This increases the sensitivity of their sorting decisions to shifts in information provision and enhances sellers’ traffic-stealing incentives. Sellers achieve their favoured ex post outcomes, yet competition leads them to make their most costly ex ante information commitments.

**Pricing Mechanisms**

Consider *pricing mechanisms*, where $t_{\eta,\tau}$ is the price charged by the sellers in state $(\eta, \tau)$. Note that, as in Coles and Eeckhout (2003), I allow sellers to set prices that vary across demand states. When two buyers are present at the same site and both their values exceed the relevant price, each obtains the good with equal probability. Such a pricing mechanism is regular if

i. *(Exploiting the uninformed)* $t_{\eta,u} = \bar{\theta}$ for $\eta \in \{1, 2\}$.

ii. *(Congestion effects)* $\bar{\theta} - t_{1,j} > \frac{1}{2}(\bar{\theta} - t_{2,j}).$

iii. *(No waste)* $t_{1,j}, t_{2,j} \leq \theta_L$.

Thus, by Lemma 2.1, under pricing mechanisms respecting i, ii and iii there is a unique candidate $\pi^*$ for symmetric equilibrium. Furthermore, if $\pi^* < 1$, this candidate profile is indeed a symmetric equilibrium. From (2.11), $\pi^* < 1$ if

$$t_{1,j} < \frac{\bar{\theta}}{2}.$$  \hspace{1cm} (2.13)

The second condition of (2.11) is always satisfied for pricing mechanisms since $S_{2,j} = \bar{\theta}$. Thus, for $t_{1,j}$ and $t_{2,j}$ satisfying ii, iii and (2.13) the level of information provision in symmetric equilibrium

\[^{18}\text{This follows by (2.18) in Appendix 2.7.}\]
is given by
\[ \pi^* = \frac{\theta}{2(\theta - t_{1,i})} < 1, \]
which does not depend on \( t_{2,i} \) and is increasing in \( t_{1,i} \) (decreasing in \( R_{1,i} \)). Applying Proposition 2.4 to this example, it follows that
\[ \frac{\partial P_a(\pi^*, \gamma_1, \pi^*, \gamma_2)}{\partial t_{1,i}} < 0 \]
and \[ \frac{\partial P_a(\pi^*, \gamma_1, \pi^*, \gamma_2)}{\partial t_{2,i}} > 0. \]
That is, a seller’s preferred pricing mechanism charges a minimal price in state \((1, i)\) and a maximal price in state \((2, i)\), while still respecting \( ii, iii \) and \( 2.13 \). This happens when \( t_{1,i} = 0 \) and \( t_{2,i} = \theta_L \). In this pricing mechanism, sellers give away the good when one buyer is present but charge the highest price that leads to no exclusions in the two-buyer state. Equilibrium information provision is \( \pi^* = \frac{1}{2} \). The sellers’ favoured pricing mechanism has low information provision and makes buyers very averse to meeting one another at the same site by providing large rents to a buyer who is alone. While regular pricing mechanisms are equally efficient with respect to informed allocations, the mechanism most preferred by sellers is the least informationally efficient.

2.4 Endogenous Mechanisms

In this section, sellers commit jointly to information provision and sale mechanisms. Before stating my main result, the following definition provides further properties of mechanisms’ allocations.

**Definition 2.3. (Partial and Full Allocative Efficiency)** A mechanism \( \gamma_k \in \Gamma \) has partial allocative efficiency (PAE) if and only if the good is always sold to some buyer in uninformed states, and to a \( \theta_H \)-type in informed states if such a type is present.

A mechanism \( \gamma_k \in \Gamma \) has full allocative efficiency (FAE) if and only if it has partial allocative efficiency and the good is always sold to a \( \theta_L \)-type in informed states if no \( \theta_H \)-type is present.

Under FAE, the surplus in state \((2, i)\) is maximized and denoted it by \( S_{2,i} \). A mechanism with PAE may exclude \( \theta_L \)-types.\(^{19}\)

**Proposition 2.6. (Symmetric Equilibrium with Endogenous Mechanisms)** Under Assumption 2.1, \((\pi, \gamma_1, \pi, \gamma_2) \in ([0, 1] \times \Gamma)^2 \) is a symmetric equilibrium if and only if \( \pi = 1, \gamma \) has full allocative efficiency, \( R_{2,i} \leq R_{1,i} \) and \( R_{1,i} = S_{2,i}^2 \).

Proposition 2.6 characterises symmetric equilibria under Assumption 2.1. While this assumption guarantees allocative efficiency in monopoly, efficient mechanisms also lead monopolists not to provide information. This does not happen here as sellers manage to disentangle information and rent provision decisions even in the presence of competition. Sellers’ incentives to do so stem from providing large rents to a buyer who is alone.

\(^{19}\)To relate this to earlier definitions, any mechanism with FAE has no waste, but a mechanism with no waste may allocate the good to a \( \theta_L \)-buyer in the presence of a \( \theta_H \)-buyer in state \((2, i)\). A mechanism with PAE need not have no waste.
from the fact that while providing rents is solely redistributive, providing information enhances efficiency. Sellers post auctions and take advantage of their allocative efficiency by providing full information. Competition then determines non-distortionary rents.

There is a continuum of equilibria that are ranked from the most favourable to sellers (with rents \( R^{1,i} = \frac{S^{2,i}}{2} \) and \( r^{2,i} = 0 \)) to the most favourable to buyers (with rents \( R^{1,i} = \frac{S^{2,i}}{2} \) and \( R^{2,i} = R^{1,i} \)). All mechanisms have congestion effects and as seen in Section 2.4.3 the condition that \( R^{1,i} = \frac{S^{2,i}}{2} \) has the interpretation that the seller equates the rents owed the marginal buyer (\( R^{1,i} \)) to its contribution to site surplus (\( \frac{S^{2,i}}{2} \)). The equilibria differ in how the surplus is shared between buyers and sellers, yet full information, \( FAE \) and symmetric seller strategies ensure that outcomes are (constrained) efficient. Profits are not driven to zero in any equilibrium. In the one-buyer state, profits are positive since they are given by \( \bar{\theta} - \frac{S^{2,i}}{2} \) and it is the case that \( 2 \bar{\theta} > \frac{S^{2,i}}{2} \). In the two-buyer state, profits are \( \frac{S^{2,i}}{2} - 2R^{2,i} \), which is positive except in the equilibrium most favourable to buyers. That sellers do not compete away all profits in the presence of traffic effects has been noted in the literature on competing auctions. Congestion effects and mixed strategies by buyers smooth out jumps in demand induced by changes in rent offers and competition between sellers is not as fierce as in Bertrand competition.

The continuum of rent levels supported in equilibrium is closely related to Coles and Eeckhout (2003). Adjusting for the fact that with high and low-type buyers surplus levels vary across demand states and that incentive constraints imply that buyers cannot be made to expect zero rents, the rent levels pinned down by Proposition 2.6 mirror theirs. In their paper with known valuations, a mechanism consists of demand state-dependent prices which are all equally efficient. In my model, information provision, allocations and rent levels are interdependent and must be determined simultaneously. The benefits of screening between types imply that in my model auctions have an efficiency advantage. A by-product of my model’s setup is that it yields a clear interpretation of why competition fixes rents only in the one-buyer state, which is simply a consequence of equating marginal rents to marginal contributions to site surplus.

### 2.4.1 Equilibrium Information Provision

This section derives necessary conditions for full information provision in equilibrium.

**Lemma 2.2.** (Full or No Information in Equilibrium) Suppose that \((\pi_a, \gamma_a, \pi_b, \gamma_b)\) is an equilibrium, that \( E_q E_r S^{y,z}_a \) is strictly increasing (decreasing) in \( \pi_a \), and that it is not the case that \( \gamma_a \) and \( \gamma_b \) are the ex post optimal mechanisms. Then \( \pi_a = 1 \) (\( \pi_a = 0 \)).

Intuitively, as information provision increases the potential size of the surplus, it allows Pareto-improving deviations for sellers from any profile with less than full information. Fixing a profile of mechanisms, information provision also has a distributive effect through rents as it shifts probability among information states within and across demand states. However, since sellers commit

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\(^{20}\)In the presence of coordination among buyers, the efficient distribution of buyers across sale sites has one of them with each seller. In the absence of coordination, efficiency requires maximising the likelihood of having one buyer at each site, which happens when \( q = 1/2 \).

\(^{21}\)In Peters and Severinov (1997) where, as is the case here, buyers sort into sites before observing their values, the (unique) symmetric equilibrium in reserve prices of second-price auctions is bounded away from cost. This is true also in the duopolistic model of Burguet and Sákovics (1999) where, however, buyers know their values before sorting.
ex ante to both state-contingent rents and information, consider a deviation from a strategy profile with less than full information in which a seller increases information provision and offsets its effect on buyer rents through transfers. In this way, buyers’ sorting decisions are unaffected and sellers pocket the newly generated surplus. The one proviso to the above argument is that the initial mechanisms must be such that more information actually increases the expected surplus at site $k$, $E_q E_{\tau} S^\eta_k$. If some buyer types are excluded by the mechanism, this need not be the case. However, in this case, reduced information provision will generate constrained efficiency gains that the seller can capture through transfers.

The proof shows that given an equilibrium in which surplus is increasing in information provision with $\pi_a < 1$, unless it is the case that buyers’ rents are at a minimum (i.e. at the ex post optimal mechanisms), seller $a$ can always increase information provision and adjust transfers so as to keep buyer rents constant. Allocations are unchanged and so, by assumption, higher surplus is generated and seller profits increase, contradicting the fact that $\pi_a$ can be a component of an equilibrium.

Full-information symmetric equilibria could arise for some of the exogenous mechanisms of Section 2.3. However, as sellers could not commit to sale mechanisms, the rationale for their existence was quite different. There, increasing information provision was profitable only if the increase in traffic generated compensated the seller for the higher rents now offered to buyers and full-information equilibria exist under mechanisms that generate more incentives for traffic-stealing. With ex ante commitments to mechanisms, Lemma 2.2 shows that sellers can deviate to a full information profile and capture its efficiency benefits without concerning themselves with traffic effects, since they control state-contingent rents. Although information provision is also efficient when mechanisms are exogenously fixed, sellers lack the tools to exploit it.

The intuition that sellers can exploit efficiency gains through ex ante offers is very general. The result of Esö and Szentes (2007), while apparently similar, presents significant differences. When the seller controls the release of signals but does not observe their realizations, Esö and Szentes (2007) show that it can achieve the same allocation and profits as under the optimal mechanism in the case in which it directly observes the signals by suitably controlling the entry fees paid by buyers before they get access to the new information. In my paper, sellers compete for buyers and cannot charge entry fees. Given any strategy profile for sellers with $\pi_k < 1$, I need to check whether seller $k$ has a profitable deviation that involves an increase in information provision, which is the case when expected surplus is increasing in information provision. The reason why such a deviation cannot be guaranteed for any profile of mechanisms is that, as put by Bergemann and Pesendorfer (2007), buyers’ participation constraints must hold ex post.

### 2.4.2 Equilibrium Allocations

This section presents results on the efficiency of equilibrium allocations in the game with endogenous mechanisms. The first result shows that holding auctions is weakly dominant.\footnote{This is as in McAfee (1993), where, however, the focus is on large markets. In small markets, arguments must consider the effect of a change in any seller’s mechanism on market-wide rents and profits.}
Lemma 2.3. (No Exclusions of $\theta_H$ or Uninformed Types) A strategy $(\pi_k, \gamma_k)$ for seller $k$ in which $\gamma_k$ does not have partial allocative efficiency is weakly dominated.

More specifically, for any profile in which seller $k$ posts a mechanism that does not have PAE, I can find an alternative mechanism with PAE that leaves buyer rents and hence visit decisions unchanged and yields strictly higher profits to seller $k$, whenever buyers visit seller $k$ with positive probability. This result states that not only will equilibrium mechanisms have PAE, but that it is without loss of generality when searching for equilibria to consider deviations from candidate profiles that have PAE.

The proof deals with $\theta_H$-type and uninformed allocations separately, and mirrors analogous results in the monopoly framework. It shows that profits can be increased and $\theta_L$-types made less willing to mimic $\theta_H$-types if seller $k$ increases $\theta_H$-type allocations and transfers simultaneously, keeping $\theta_H$-types at the same level of rents. Similarly, a profile in which uninformed buyers are excluded with positive probability is vulnerable to a deviation where a seller increases both allocation probabilities and transfers, keeping buyers at the same level of rents.

In my model, the classic arguments from the monopoly case that determine $\theta_L$-type allocations cannot be applied directly due to their competitive effects on traffic across sale sites. In the monopoly case, Assumption 2.1 determines whether sellers excludes $\theta_L$-types in either demand state, since in any mechanism in which $\theta_L$-types are excluded with some probability, the seller can increase profits by increasing both $\theta_L$-types' allocation probabilities and transfers, keeping their rent level constant, even if this increases $\theta_H$-type rents (through the binding incentive compatibility constraint for $\theta_H$-types). This increases rents expected over informed types. The problem with this argument in my framework is that an increase in rents in any state increases traffic but may decrease the likelihood of the one-buyer state (when $q > \frac{1}{2}$), and hence its effect on total profits may depend on the relation between profits in the one-buyer and two-buyer states. The next result, unlike Lemma 2.3, presents only a necessary condition on $\theta_L$-type allocations in symmetric equilibria.

Lemma 2.4. (No Exclusion of $\theta_L$-types in Symmetric Equilibrium) Under Assumption 2.1 if $(\pi, \gamma, \pi, \gamma)$ is a symmetric equilibrium, then $\gamma$ has full allocative efficiency.

From Lemma 2.3, PAE is necessary for any equilibrium in which both sellers are visted with positive probability, and in a symmetric equilibrium $q = \frac{1}{2}$. To show that under Assumption 2.1 $\theta_L$-types always receive the good in the absence of $\theta_H$-types in a symmetric equilibrium, the proof applies the argument for the monopoly case outlined above to find a deviation from any symmetric equilibrium that violates FAE. The difficulty mentioned above is dealt with by the fact that at a symmetric profile small increases in traffic have a negligible effect on the probability of the one-buyer state. The proof of Lemma 2.4 also guarantees that profits in the two-buyer state are nonegative.

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23 In the two-buyer state, it may be the case that $\theta_L$-types receive the good even in the presence of $\theta_H$-types, and that the resource constraint binds, so that the seller cannot allocate the good more often to $\theta_H$-types without allocating it less often to $\theta_L$-types. But then the seller can simply ‘free up’ allocation probabilities by delivering the good less often to $\theta_L$-types and keep their rents constant by decreasing their transfers.
Without Assumption 2.1, a seller wants to exclude $\theta^L$-types to depress $\theta^H$-type rents. Marginally, whether this is profitable depends on whether the increased profits from $\theta^H$-types compensate the drop in traffic in the two-buyer state. This traffic-rents trade-off will also involve the level of information provision. Without Assumption 2.1, it is difficult to derive a simple necessary condition on $\theta^L$-type allocations which, as above, does not depend on information provision.

2.4.3 Equilibrium Rents

This section derives necessary conditions on equilibrium rents under Assumption 2.1.

Lemma 2.5. (Equilibrium Rents) Under Assumption 2.1 if $(\pi, \gamma, \pi, \gamma)$ is a symmetric equilibrium, then $R^2,i \leq R^1,i$ and $R^1,i = \bar{S}^L,i$. Under the regular mechanisms of Section 2.3, buyers faced congestion effects and preferred being alone at a sale site when informed. Lemma 2.5 confirms that a seller will always impose congestion effects in a symmetric equilibrium. The intuition for this is as follows. As in (2.12), rewrite a buyer’s expected rents at site $a$ from a symmetric profile with $\pi = 1$ as

$$R^{1,i} + q(R^2,i - R^1,i),$$

that is, as an ‘attendance fee’ of $R^1,i$ along with a ‘bonus’ (‘congestion charge’) of $R^2,i - R^1,i$ when another buyer attends and $R^2,i > R^1,i$ ($R^2,i \leq R^1,i$). If $R^2,i > R^1,i$, decreasing $R^2,i$ lowers the bonus, but buyers remain indifferent between attending sites $a$ and $b$ only if this bonus is handed out more often, i.e., if $q$ increases. As sellers can decrease rents while increasing traffic, profiles with $R^2,i > R^1,i$ admit a profitable deviation.

The condition $R^1,i = \bar{S}^L,i$ states that the marginal buyer attending a site is awarded his marginal contribution to site surplus. To see this, note that seller $a$’s profits at symmetric profiles with FAE are marginally increasing in $R^1,i$ (or $R^2,i$) whenever $R^1,i < \bar{S}^L,i$. A marginal buyer drawn to site $a$ by a marginal change in rents receives $R^1,i$, its ‘attendance fee’, from seller $a$. On the other hand, this marginal buyer brings its share of the surplus when another buyer is also present, $\bar{S}^L,i$, to site $a$. Since the probability of the one-buyer state is unaffected by small changes in $q$ at a symmetric profile, a marginal buyer brings nothing to that state. A seller will want to attract a marginal buyer whenever his contribution exceeds the cost of luring him. Similarly, if $\bar{S}^L,i < R^1,i$, a seller can gain by shedding a marginal buyer through a decrease in rents.

2.4.4 Sufficiency

The proof of Proposition 2.6 follows from the results of the previous sections. The necessity of FAE for symmetric equilibrium has been established in Lemma 2.4. Under FAE, information provision increases the surplus available at a selling site since two buyers generate more surplus when informed than when uninformed, as $\bar{S}^2,i = \bar{S}^2,i > \theta$ and $S^1,i = \bar{\theta}$, and hence Lemma 2.2 states that $\pi = 1$ is necessary for symmetric equilibrium unless both sellers commit to the ex post

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24Interpret the marginal buyer as the mass involved in a marginal increase in $q$.
25This follows from (2.26) in Appendix 2.7.
optimal mechanisms. The necessity of full information under Assumption 2.1 for ex post optimal mechanisms follows from Proposition 2.5. Lemma 2.5 provides the conditions for equilibrium rents. Note that $R_{2,i} \leq R_{1,i} = \bar{S}_{2,i}$ implies that $2R_{2,i} \leq \bar{S}_{2,i}$ and hence that profits in the two-buyer state are nonnegative. The sufficiency argument is direct; taking a profile satisfying the conditions of the proposition, I show that no deviation can be profitable.

2.5 Conclusion

This paper has analysed the strategic interactions of sellers who compete for buyers by committing to information provision. When mechanisms are exogenously fixed and sellers compete solely through offers of information, they may prefer to compete in high-rent environments, as these lessen the intensity of competition and lead to lower information provision. Furthermore, as higher surplus mechanisms increase sellers’ competitive incentives to provide information, they prefer to compete in environments with low allocative efficiency, and hence low information provision. When sellers commit to both information provision and mechanisms, under a no-exclusion assumption, all symmetric equilibria have full information provision. However, a variety of rent levels are supported in equilibrium as a result of different equilibrium offers of mechanisms. In a sense, this result shows that sellers prefer to compete through mechanisms rather than through information provision. By doing so they maximize the available surplus, and competition determines the equilibrium share of this surplus going to buyers.

2.6 Appendix: Sale Mechanisms

2.6.1 Definitions

Let $\Psi_{\eta,\tau}$ denote the set of report profiles that can be received by the seller in state $(\eta, \tau)$. That is,

$$\Psi_{\eta,\tau} = \begin{cases} \{(\theta_m, \theta_n)\}_{(m,n) \in \{L,H\}^2} & \text{if } \eta = 2 \text{ and } \tau = i, \\ \{\theta_m\}_{m \in \{L,H\}} & \text{if } \eta = 1 \text{ and } \tau = i, \\ \emptyset & \text{if } \tau = u. \end{cases}$$

An anonymous direct mechanism for seller $k$ is a collection of functions

$$\left\{ \left\{ x_k^{\eta,\tau} : \Psi(\eta, \tau) \to [0, 1], y_k^{\eta,\tau} : \Psi(\eta, \tau) \to \mathbb{R} \right\} \right\}_{\eta \in \{1,2\}, \tau \in \{i,u\}},$$

where $x_k^{\eta,\tau}(\psi)$ and $y_k^{\eta,\tau}(\psi)$ are, respectively, the probability a buyer obtains the good and the transfer he must pay to seller $k$ when the report profile is $\psi \in \Psi_{\eta,\tau}$ in state $(\eta, \tau)$. Since no report is necessary when buyers are uninformed, I write probabilities and transfers as $x_k^{\eta,u}$ and $y_k^{\eta,u}$, respectively, for $\eta \in \{1, 2\}$. Also, since mechanisms are anonymous, define $x_k^{2,i}(\theta_m, \theta_n)$ as the probability that a buyer reporting $\theta_m$ obtains the good when the other buyer reports $\theta_n$. A similar remark
holds for the transfer function $y_{k}^{2j}(\theta_m, \theta_n)$. The allocation probabilities satisfy

$$x_{k}^{1,\tau}(\psi) \leq 1 \quad \text{for } \psi \in \Psi^{1,\tau} \text{ and } \tau \in \{i, u\},$$

$$x_{k}^{2,u} \leq \frac{1}{2},$$

$$x_{k}^{2,j}(\theta_m, \theta_n) + x_{k}^{2,j}(\theta_n, \theta_m) \leq 1 \quad \text{for } (m, n) \in \{L, H\}^2. \quad (2.15)$$

In state (2, i) at site $k$, each buyer only knows his own valuation. For $j \in \{H, L\}$, define the reduced form transfers and winning probabilities as $X_{k}^{2,j}(\theta_i) = E_{\theta,j}x_{k}^{2,j}(\theta_j, \theta_{-j})$ and $y_{k}^{2,j}(\theta_i) = E_{\theta,j}y_{k}^{2,j}(\theta_j, \theta_{-j})$. Incentive-compatible direct mechanisms respect a set of state-contingent incentive and participation constraints. When no information is released at site $k$, no incentive constraints apply. The relevant participation constraints are

$$x_{k}^{1,u} \bar{\theta} - y_{k}^{1,u} \geq 0, \quad (PC_{k}^{1,u})$$

$$x_{k}^{2,u} \bar{\theta} - y_{k}^{2,u} \geq 0. \quad (PC_{k}^{2,u})$$

In state (1, i) at site $k$, the set of constraints is given by

$$x_{k}^{1,i}(\theta_H)\theta_H - y_{k}^{1,i}(\theta_H) \geq x_{k}^{1,i}(\theta_L)\theta_H - y_{k}^{1,i}(\theta_L), \quad (IC_{k}^{1,i}(\theta_H))$$

$$x_{k}^{1,i}(\theta_L)\theta_L - y_{k}^{1,i}(\theta_L) \geq x_{k}^{1,i}(\theta_H)\theta_L - y_{k}^{1,i}(\theta_H), \quad (IC_{k}^{1,i}(\theta_L))$$

$$x_{k}^{1,i}(\theta_L)\theta_L - y_{k}^{1,i}(\theta_L) \geq 0. \quad (IC_{k}^{1,i}(\theta_L))$$

As is well known, the participation constraint of the $\theta_H$-type, $(PC_{k}^{1,i}(\theta_H))$, is satisfied whenever $(IC_{k}^{1,i}(\theta_H))$ and $(IC_{k}^{1,i}(\theta_L))$ hold.

The constraints that need to be satisfied in state (2, i) at site $k$ are given by

$$X_{k}^{2,i}(\theta_H)\theta_H - Y_{k}^{2,i}(\theta_H) \geq X_{k}^{2,i}(\theta_L)\theta_H - Y_{k}^{2,i}(\theta_L), \quad (IC_{k}^{2,i}(\theta_H))$$

$$X_{k}^{2,i}(\theta_L)\theta_L - Y_{k}^{2,i}(\theta_L) \geq X_{k}^{2,i}(\theta_H)\theta_L - Y_{k}^{2,i}(\theta_H), \quad (IC_{k}^{2,i}(\theta_L))$$

$$X_{k}^{2,i}(\theta_L)\theta_L - Y_{k}^{2,i}(\theta_L) \geq 0. \quad (PC_{k}^{2,i}(\theta_L))$$

As in the single buyer case, the participation constraint of the $\theta_H$-type, $(PC_{k}^{2,i}(\theta_H))$, is satisfied whenever $(IC_{k}^{2,i}(\theta_H))$ and $(PC_{k}^{2,i}(\theta_L))$ hold. The class of incentive compatible direct mechanisms for this problem is denoted by $\Gamma$, and a particular mechanism at site $k$ by $\gamma_{k}$.

### 2.6.2 Rents and Profits

Given mechanism $\gamma_{k}$ at site $k$, expected rents are given by

$$R_{k}^{\eta,u} = x_{k}^{\eta,u} \bar{\theta} - y_{k}^{\eta,u} \quad \text{for } \eta \in \{1, 2\},$$

$$R_{k}^{1,i} = E_{\theta} \left[ x_{k}^{1,i}(\theta)\theta - y_{k}^{1,i}(\theta) \right],$$

$$R_{k}^{2,i} = E_{\theta} \left[ X_{k}^{2,i}(\theta)\theta - Y_{k}^{2,i}(\theta) \right],$$
and the surplus is given by

\[ S_{k}^{1,u} = x_{k}^{1,u} \hat{\theta}, \]
\[ S_{k}^{2,u} = 2x_{k}^{2,u} \hat{\theta}, \]
\[ S_{k}^{1,i} = p_{H}x_{k}^{1,i}(\theta_{H})\theta_{H} + p_{L}x_{k}^{1,i}(\theta_{L})\theta_{L}, \]
\[ S_{k}^{2,i} = 2 \left[ p_{H}x_{k}^{2,i}(\theta_{H})\theta_{H} + p_{L}x_{k}^{2,i}(\theta_{L})\theta_{L} \right]. \]

Given strategy profile \((\pi_{k}, \gamma_{k}, \pi_{-k}, \gamma_{-k})\), the profits of seller \(k\) are given by

\[ \mathcal{P}_{k}(\pi_{k}, \gamma_{k}, \pi_{-k}, \gamma_{-k}) = E_{\eta} E_{\tau} E_{\psi} \left[ \eta y_{k}^{H,T}(\psi) \right]. \]

The first expectation is taken with respect to the binomial distribution with parameter \(q\) (if \(k = a\)) or \(1 - q\) (if \(k = b\)) of demand at site \(k\), and the second with respect to the binomial distribution with parameter \(\pi_{k}\) over information states at site \(k\). The final expectation is taken with respect to the distribution of truthful reports in state \((\eta, \tau)\).

### 2.6.3 A Characterisation of Incentive-Compatible Mechanisms

Lemma 2.6 shows that it is without loss of generality to restrict sellers to offering mechanisms in which \(\theta_{H}\)-type incentive-compatibility constraints are binding in states \((1,1)\) and \((2,1)\).

**Lemma 2.6. (\(\theta_{H}\)-Type Incentive-Compatibility Constraints Bind)** Given any strategy profile \((\pi_{k}, \gamma_{k}, \pi_{-k}, \gamma_{-k})\) for sellers, there exists a mechanism \(\gamma_{k} \in \Gamma\) in which \((\mathcal{IC}_{k}^{1,i}(\theta_{H}))\) and \((\mathcal{IC}_{k}^{2,i}(\theta_{H}))\) are binding, allocations are as in \(\gamma_{k}\) and such that under profile \((\pi_{k}, \gamma_{k}, \pi_{-k}, \gamma_{-k})\) buyers’ rents and sellers’ profits are the same as under profile \((\pi_{k}, \gamma_{k}, \pi_{-k}, \gamma_{-k})\).

**Proof of Lemma 2.6** Consider an incentive compatible mechanism \(\gamma_{k}\) at site \(k\) such that \((\mathcal{IC}_{k}^{1,i}(\theta_{H}))\) is slack. In particular, say

\[ x_{k}^{1,i}(\theta_{H})\theta_{H} - y_{k}^{1,i}(\theta_{H}) = x_{k}^{1,i}(\theta_{L})\theta_{H} - y_{k}^{1,i}(\theta_{L}) + C, \]

with \(C > 0\). Consider an alternative mechanism \(\tilde{\gamma}_{k}\) identical to \(\gamma_{k}\) except that

\[ \tilde{y}_{k}^{1,i}(\theta_{H}) = y_{k}^{1,i}(\theta_{H}) + p_{L}C \]
\[ \tilde{y}_{k}^{1,i}(\theta_{L}) = y_{k}^{1,i}(\theta_{L}) - p_{H}C. \]

In that case,

\[ \tilde{x}_{k}^{1,i}(\theta_{H})\theta_{H} - \tilde{y}_{k}^{1,i}(\theta_{H}) = x_{k}^{1,i}(\theta_{H})\theta_{H} - y_{k}^{1,i}(\theta_{H}) - p_{L}C \]
\[ = x_{k}^{1,i}(\theta_{L})\theta_{H} - y_{k}^{1,i}(\theta_{H}) - C + p_{H}C \]
\[ = x_{k}^{1,i}(\theta_{L})\theta_{H} - y_{k}^{1,i}(\theta_{L}) + p_{H}C \]
\[ = \tilde{x}_{k}^{1,i}(\theta_{L})\theta_{H} - y_{k}^{1,i}(\theta_{L}). \]
Thus, $\bar{IC}_k^{1,j}(\theta_H)$ binds. Since under $\bar{\gamma}_k$ the transfer of type $\theta_L$ has been decreased, $\bar{PC}_k^{1,j}(\theta_L)$ is satisfied. Since both $\bar{IC}_k^{1,j}(\theta_H)$ and $\bar{PC}_k^{1,j}(\theta_L)$ hold, then so does $\bar{PC}_k^{1,j}(\theta_H)$. Finally, under $\bar{\gamma}_k$ $\theta_H$-types are worse off and $\theta_L$-types are better off, so that $\bar{IC}_k^{1,j}(\theta_L)$ holds. Hence $\bar{\gamma}_k$ is incentive compatible.

Profits for seller $k$ in state $(1, i)$ under mechanism $\bar{\gamma}_k$ are given by

$$p_H\bar{y}_k^{1,j}(\theta_H) + p_L\bar{y}_k^{1,j}(\theta_L) = p_Hy_k^{1,j}(\theta_H) + p_Ly_k^{1,j}(\theta_L) + p_HpLC - p_LPHC$$

where the last line is profits under $\gamma_k$ in state $(1, i)$. Profits in other states are also unaffected. The proof for the case in which $IC_0^{2,j}(\theta_H)$ is slack is identical, with reduced-form mechanisms replacing the mechanisms. To that end, note that in state $(2, i)$, profits under mechanism $\gamma_k$ are given by

$$p_H^2 \left[ 2y_k^{2,j}(\theta_H, \theta_H) + 2p_Lp_H \left[ y_k^{2,j}(\theta_H, \theta_L) + y_k^{2,j}(\theta_L, \theta_H) \right] + p_L^2 [2y_k^{2,j}(\theta_L, \theta_L)] \right] = 2 \left[ p_HY_k^{1,j}(\theta_H) + p_LY_k^{1,j}(\theta_L) \right].$$

As the proof manipulates mechanisms in different demand states independently, given an original profile where the incentive compatibility constraints of $\theta_H$-types in both demand states are slack, one could find a rent and profit-equivalent mechanism with incentive constraints binding in both states by the same procedure.

Denote $\bar{\gamma}_k$ as the $IC(\theta_H)$-equivalent of $\gamma_k$. Similarly, denote by $\bar{\Gamma}$ the set of $IC(\theta_H)$-equivalent mechanisms. Given information provision $(\pi_a, \pi_b)$, a game with mechanisms $(\gamma_a, \gamma_b) \in (\Gamma \setminus \bar{\Gamma})^2$ generates the same distribution over outcomes as a game with mechanisms $(\bar{\gamma}_a, \bar{\gamma}_b)$, where $\bar{\gamma}_k$ is the $IC(\theta_H)$-equivalent mechanism of $\gamma_k$. That is, excluding mechanisms in $\Gamma \setminus \bar{\Gamma}$ does not reduce the set of equilibria in terms of information provision. On the other hand, when sellers also choose mechanisms, it is not the case that equilibrium mechanisms must belong to $\bar{\Gamma}$. However, Lemma 2.6 states that excluding mechanisms in $\Gamma \setminus \bar{\Gamma}$ does not reduce the set of equilibrium allocations, traffic levels and payoffs. In what follows, incentive compatible mechanisms refers to mechanisms in $\bar{\Gamma}$.

Given mechanism $\gamma_k$ at site $k$, we can rewrite the expected rents promised at site $k$ as

$$R_k^{\eta,\mu} = r_k^{\eta,\mu} \text{ for } \eta \in \{1, 2\},$$

$$R_k^{1,j} = r_k^{1,j} + p_Hx_k^{1,j}(\theta_L)(\theta_H - \theta_L),$$

$$R_k^{2,j} = r_k^{2,j} + p_Hx_k^{2,j}(\theta_L)(\theta_H - \theta_L).$$

Furthermore, Lemma 2.6 justifies the use of the following well-known result, whose proof is standard and omitted.

**Lemma 2.7. (Characterisation of $IC(\theta_H)$-Equivalent Mechanisms)** $\gamma_k \in \bar{\Gamma}$ if and only if $x_k^{1,j}(\theta_H) \geq x_k^{1,j}(\theta_L)$, $X_k^{1,j}(\theta_H) \geq X_k^{1,j}(\theta_L)$, $y_k^{\tau,\eta} \geq 0$ for all $\eta \in \{1, 2\}$ and $\tau \in \{i, u\}$ and $\theta_H$-type rents are given by
\( x_k^{1,j}(\theta_H - \theta_L) \) in state \((1, i)\) and \( X_k^{2,i}(\theta_H - \theta_L) \) in state \((2, i)\).

### 2.6.4 Properties of Allocations

I give the formal definitions, in terms of the underlying mechanisms, of the properties of allocations used in the text.

**Definition 2.1 (No Waste)** A mechanism \( \gamma_k \in \Gamma \) has no waste if and only if

\[

d_{1,i} = d_{1,i}^{\theta_H} = d_{1,u} = 1,
\]

\[

d_{2,u} = \frac{1}{2},
\]

\[

d_{2,i}^{\theta_H} = d_{2,i}^{\theta_L} = \frac{1}{2}
\]

As noted in Section 2.3.2, that \( \gamma \) has no waste implies that seller \( a \)'s profits in the two-buyer state decrease in \( \pi_a \), since the term in the first brackets of (2.9) is linear in \( \pi_a \) and

\[

S^{2,j} - \bar{\theta} - 2R^{2,j} = S^{2,j} - \bar{\theta} - 2 \left( r^{2,j} + X^{2,j}(\theta_L)p_H(\theta_H - \theta_L) \right)
\]

\[

\leq S^{2,j} - \bar{\theta} - 2X^{2,j}(\theta_L)p_H(\theta_H - \theta_L)
\]

\[

= \theta_H \left[ 2p_H X^{2,j}(\theta_H) - p_H \right] + \theta_L \left[ 2p_L X^{2,j}(\theta_L) - p_L \right] - 2X^{2,j}(\theta_L)p_H(\theta_H - \theta_L)
\]

\[

= (\theta_H - \theta_L) \left[ p_L - 2X^{2,j}(\theta_L) \right]
\]

\[

\leq 0. \tag{2.16}
\]

The second line follows since \( r^{2,j} \geq 0 \), the fourth since \( p_H X^{2,j}(\theta_H) + p_L X^{2,j}(\theta_L) = \frac{1}{2} \) under no waste, and the last since \( X^{2,j}(\theta_L) \geq \frac{p_L}{2} \) under no waste.

As noted in Section 2.3.2 under Assumption 2.1, the ex post optimal mechanisms are regular, and are described by allocation probabilities

\[

x_k^{1,j}(\theta_H) = x_k^{1,j}(\theta_L) = x_k^{1,u} = 1,
\]

\[

x_k^{2,u} = \frac{1}{2},
\]

\[

X_k^{2,j}(\theta_H) = \frac{p_H}{2} + p_L,
\]

\[

X_k^{2,j}(\theta_L) = \frac{p_L}{2},
\]

and rent levels for low types \( r^{h,\tau} = 0 \) for all \( h \in \{1, 2\}, \tau \in \{i, u\} \).

**Definition 2.2 (Partial and Full Allocative Efficiency)** A mechanism \( \gamma_k \in \Gamma \) has partial allocative
efficiency (PAE) if and only if

\[ x^1_k(\theta_H) = x^1_k = 1, \]
\[ x^2_k = \frac{1}{2}, \]
\[ x^2_k(\theta_H, \theta_L) = 1, \text{ and } x^2_k(\theta_H, \theta_H) = \frac{1}{2}. \]

A mechanism \( \gamma_k \in \Gamma \) has full allocative efficiency (FAE) if and only if it has partial allocative efficiency and also

\[ x^1_k(\theta_L) = 1, \]
\[ x^2_k(\theta_L, \theta_L) = \frac{1}{2}. \]

2.7 Appendix: Proofs

Proof of Lemma 2.1. Setting (2.10) equal to zero and checking the conditions for which \( \pi < 1 \), we obtain the expression for \( \pi^* \). By the argument in the text, all that needs to be shown is that \( \partial q / \partial \pi_a \bigg|_{\pi_a = \pi_b = \pi} \) is decreasing in the symmetric probability \( \pi \). By (2.2) and using the fact that \( R^{1,u} = R^{2,u} = 0 \) for regular mechanisms, we have

\[ q = \frac{\pi_a R^{1,i} - \pi_b R^{2,i}}{(R^{1,i} - R^{2,i})(\pi_a + \pi_b)^2}, \]

and it can be verified that

\[ \frac{\partial q}{\partial \pi_a} \bigg|_{\pi_a = \pi_b = \pi} = \frac{R^{1,i} + R^{2,i}}{4\pi(R^{1,i} - R^{2,i})^2}, \]

which is decreasing in \( \pi \).

Proof of Proposition 2.3. The first part of the proof is the following lemma which provides sufficient conditions for the existence of interior symmetric equilibria.

Lemma 2.8. Given a regular mechanism \( \gamma \) that generates rents such that

i. \( 2R^{1,i} > \tilde{\theta} \) and \( R^{1,i} + R^{2,i} > \frac{2R^{1,i}(S^{2,i} - \tilde{\theta})}{2R^{1,i} > \tilde{\theta}}. \)

ii. \( 2R^{1,i}(R^{2,i})^2 - 6(R^{1,i})^2 R^{2,i} + 8(R^{1,i})^3 - \tilde{\theta}(4(R^{1,i})^2 + R^{1,i} R^{2,i} - (R^{2,i})^2) \leq 0. \)

the symmetric equilibrium of the game between sellers is \( \pi^* = \frac{-(R^{1,i} + R^{2,i})\tilde{\theta}}{2R^{1,i}(S^{2,i} - \theta - (R^{1,i} + R^{2,i}))} < 1. \)

Proof of Lemma 2.8. Point i of the statement ensures that \( \pi^* < 1 \). Consider a candidate symmetric profile \((\pi, \pi)\) and a deviation by seller \( a \) to \( \pi + \lambda \) for \( \lambda \in (-\pi, 1 - \pi] \), which induces traffic level
$q^\lambda \in (0, 1]$. Then we have that

$$q^\lambda = \frac{\pi(R^{1,i} - R^{2,i}) + \lambda R^{1,i}}{(R^{1,i} - R^{2,i})(2\pi + \lambda)} = \frac{1}{2} + z,$$

with $z = \frac{\lambda(R^{1,i} + R^{2,i})}{2(R^{1,i} - R^{2,i})(2\pi + \lambda)}$, (2.17)

Also,

$$\mathcal{P}_a(\pi + \lambda, \pi) - \mathcal{P}_a(\pi, \pi) = z(z+1) \left[ \pi S^{2,i} + (1 - \pi)\bar{\theta} - 2\pi R^{2,i} \right] - 2z^2 \left[ \bar{\theta} - \pi R^{1,i} \right]
+ \left( \frac{1}{2} + z \right)^2 \left[ S^{2,i} - \bar{\theta} - 2R^{2,i} \right] - 2\lambda \left( \frac{1}{2} + z \right) (\frac{1}{2} - z) R^{1,i}
= \frac{\lambda^2}{D} \left[ 4R^{1,i}(S^{2,i} - \bar{\theta}) \left[ (R^{1,i} + R^{2,i})(R^{1,i} - R^{2,i})\bar{\theta}
- 2\lambda(R^{1,i})^2(S^{2,i} - \bar{\theta} - (R^{1,i} + R^{2,i})) \right]
+ (R^{1,i} + R^{2,i})^2 \bar{\theta} \left[ S^{2,i} - \bar{\theta} - (R^{1,i} + R^{2,i})(5R^{1,i} - R^{2,i}) \right]
+ (R^{1,i} + R^{2,i})^2 \right] \right]
\leq F \left[ (S^{2,i} - \bar{\theta})(4(R^{1,i})^2 + R^{1,i}R^{2,i} - (R^{2,i})^2)
- 2R^{2,i}(R^{1,i} + R^{2,i})(2R^{1,i} - R^{2,i}) \right] \leq (2.18)
< H \left[ 2R^{1,i}(R^{2,i})^2 - 6(R^{1,i})^2 R^{2,i} + 8(R^{1,i})^3
- \bar{\theta}(4(R^{1,i})^2 + R^{1,i}R^{2,i} - (R^{2,i})^2) \right]. \tag{2.19}
$$

Where $D, F, H > 0$ are functions of parameters. The second equality follows from setting $\pi = \pi^*$ and rearranging terms. The first inequality follows from the fact that $q^\lambda \leq 1$ when $\lambda \leq \frac{\bar{\theta}(R^{1,i} + R^{2,i})(R^{1,i} - R^{2,i})}{-2R^{1,i}R^{2,i}(S^{2,i} - \bar{\theta} - (R^{1,i} + R^{2,i}))}$. The last inequality follows since $\pi^* < 1$ when $S^{2,i} - \bar{\theta} < \frac{(R^{1,i} + R^{2,i})(2R^{1,i} - \bar{\theta})}{2R^{1,i}}$.

To show that the set of regular mechanisms $\Psi$ is nonempty, note that (2.18) implies that under any mechanism in which $S^{2,i} = \bar{\theta}$ (a pricing mechanism in the two-buyer state), deviations from the symmetric profile $(\pi^*, \pi^*)$ are strictly not profitable for seller $a$. Hence given any regular mechanism $\gamma$ with $X^{2,i}(\theta_L) = X^{2,i}(\theta_H) = \frac{1}{2}$ and rents that satisfy condition $i$ of Lemma 2.8, there is a neighbourhood $N$ of $\gamma$ in the space of regular mechanisms such that for all $\hat{\gamma} \in N \gamma$, $\hat{\gamma}$ satisfies
condition i of Lemma 2.8 and the term inside the brackets of (2.18) is negative. Thus all such \( \hat{\gamma} \) induce a unique symmetric equilibrium \((\pi^*, \pi^*)\) with \( \pi^* < 1 \).

Finally, the derivatives mentioned in the proposition can be computed directly to yield

\[
\frac{\partial \pi^*}{\partial R_{1,i}} = -\frac{2R_{1,i}(R_{1,i}^j + R_{2,i}^j) - 2\bar{\theta}R_{2,i}^j(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j))}{(-2R_{1,i}^j(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j)))^2} < 0,
\]

and

\[
\frac{\partial \pi^*}{\partial R_{2,i}} = -\frac{2\bar{\theta}R_{1,i}^j(S^2_{2,i} - \bar{\theta})}{(-2R_{1,i}^j(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j)))^2} \leq 0.
\]

From these it can be checked that

\[
\frac{\partial \pi^*}{\partial R_{1,i}} - \frac{\partial \pi^*}{\partial R_{2,i}} = \frac{-2\bar{\theta}(R_{1,i}^j + R_{2,i}^j)(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + 2R_{2,i}^j))}{(-2R_{1,i}^j(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j)))^2} < 0.
\]

\[\square\]

**Proof of Proposition 2.4.** The profits of both sellers at a symmetric equilibrium with a regular mechanism \( \gamma \) are given by

\[
\frac{1}{4} \left[ \pi^* (S^2_{2,i} - \theta - 2R_{2,i}^2) + \bar{\theta} \right] + \frac{1}{2} \left[ \bar{\theta} - \pi^* R_{1,i}^j \right].
\]

Direct computation yields

\[
\frac{\partial P_a(\pi^*, \pi^*)}{\partial R_{1,i}} = \frac{\bar{\theta}}{8(R_{1,i}^j(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j)))^2} \left[ R_{1,i}^j(S^2_{2,i} - \bar{\theta})(R_{1,i}^j + R_{2,i}^j) \right]
\]

\[
+ R_{2,i}^j(S^2_{2,i} - \bar{\theta} - 2(R_{1,i}^j + R_{2,i}^j))(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j)) \right] > 0
\]

\[
\frac{\partial P_a(\pi^*, \pi^*)}{\partial R_{2,i}} = \frac{\bar{\theta}R_{1,i}^j}{2(R_{1,i}^j(S^2_{2,i} - \bar{\theta} - (R_{1,i}^j + R_{2,i}^j)))^2} \left[ (S^2_{2,i} - \bar{\theta})^2 - 2(R_{1,i}^j + R_{2,i}^j)^2 \right].
\]

\[\square\]

**Proof of Proposition 2.5.** I show that if \( \gamma \) is the ex post optimal mechanism under Assumption 2.1.
then $\mathcal{P}_a(\pi_a, \gamma, 1, \gamma)$ is increasing in $\pi_a$. Given $\pi_a \leq 1$, $q \leq \frac{1}{2}$, and if $\pi_a$ is such that $q > 0$, then

$$
\mathcal{P}_a(\pi_a, \gamma, 1, \gamma) = \left( \frac{\pi_a - \frac{p_l}{2}}{(1 + \pi_a)(1 - \frac{p_l}{2})} \right)^2 \theta + 2 \left( \frac{\pi_a - \frac{p_l}{2}}{(1 + \pi_a)(1 - \frac{p_l}{2})^2} \right) \left( \pi_a - \frac{p_l}{2} \right) \left( \frac{(1 - \frac{p_l}{2})\left(1 - \frac{\pi_a}{2}\right)}{((1 + \pi_a)(1 - \frac{p_l}{2}))^2} \right) \left( \theta - \pi_a p_H(\theta_H - \theta_L) \right)
$$

$$=
\left( \frac{\pi_a - p_l}{((1 + \pi_a)(1 - p_l))^2} \right) \left( \theta (p_H \pi_a + 2 - \frac{p_l}{2} + 2(1 - \frac{\pi_a}{2})p_H(\theta_H - \theta_L)) \right)
\equiv A(\pi_a)(B(\pi_a) + C(\pi_a))
$$

Where $B(\pi_a)$ is clearly increasing in $\pi_a$, while it can be shown that $A(\pi_a)$ and $C(\pi_a)$ are increasing whenever $\pi_a \leq 1$ and $\pi_a \leq \frac{1}{p_l}$, respectively, which is always true.

**Proof of Lemma 2.2**

Lemma 2.2, stated in terms of $IC(\theta_H)$-equivalent mechanisms, requires that it not be the case that $r^2_a = r^1_a = r^{1,u}_a = 0$. This condition simply states that it is always possible, for at least one state, to decrease transfers in an incentive compatible way. Any mechanism $\gamma_a \in \Gamma$ that satisfies this last property would have its $IC(\theta_H)$-equivalent mechanism satisfy the property that it not be the case that $r^2_a = r^2_a = r^1_a = r^{1,u}_a = 0$ (through Lemma 2.6). The following proof then applies to all incentive compatible mechanisms that are components of some equilibrium, since a best response to a $IC(\theta_H)$-equivalent mechanism is also a best-response to the original mechanism.

Suppose that $(\pi_a, \gamma_a, \pi_b, \gamma_b)$ is an equilibrium, that $E_q E_{\tau} S^\eta_{\pi,\tau}$ is increasing in $\pi_a$, that it is not the case that $r^2_a = r^2_a = r^1_a = r^{1,u}_a = 0$ and that $\pi_a < 1$. Consider a deviation by seller $a$ to a profile in which

$$\pi_a = \pi_a + \lambda$$

$$p^\eta_{\pi,\tau} = p^\eta_{\pi,\tau} - \delta^\eta_{\tau},$$

where $\lambda \in (0, 1 - \pi_a]$ and $\delta^\eta_{\tau} < r^\eta_{\pi,\tau}$ for all $(\eta, \tau)$. For this deviant profile not to affect buyers’ visit decisions (or expected rents), we need

$$q \left[ \left( \pi_a + \lambda \right) \left[ r^2_a - \delta^2 + z^2_a \right] + \left( 1 - \pi_a - \lambda \right) \left[ r^2_a - \delta^2 \right] \right]$$

$$+ \left( 1 - q \right) \left[ \left( \pi_a + \lambda \right) \left[ r^1_a - \delta^1 + z^1_a \right] + \left( 1 - \pi_a - \lambda \right) \left[ r^1_a - \delta^1 \right] \right]$$

$$= q \left[ \pi_a \left[ r^2_a + z^2_a \right] + \left( 1 - \pi_a \right) \left[ r^2_a + z^2_a \right] \right] + \left( 1 - q \right) \left[ \pi_a \left[ r^1_a + z^1_a \right] + \left( 1 - \pi_a \right) \left[ r^1_a + z^1_a \right] \right],$$

or

$$\left( \pi_a + \lambda \right) \left[ q \delta^2 + \left( 1 - q \right) \delta^1 \right] + \left( 1 - \pi_a - \lambda \right) \left[ q \delta^2 + \left( 1 - q \right) \delta^1 \right]$$

$$= \lambda \left[ q \left[ r^2_a + z^2_a - r^2_a \right] + \left( 1 - q \right) \left[ r^1_a + z^1_a - r^1_a \right] \right],$$

(2.21)

where $z^1_a = r^1_a + p_H x^1_a(\theta_L)(\theta_H - \theta_L) \geq 0$ and $z^2_a = r^2_a + p_H x^2_a(\theta_L)(\theta_H - \theta_L) \geq 0$ are the expected informational rents given the allocations of the original mechanism. The sign of the
right-hand side (RHS) of (2.21) is given by the properties of the mechanism at site \( a \). It is positive if buyers prefer, on average, to be informed at the site, and negative if buyers prefer, on average, to be uninformed.

Suppose \( \pi_a > 0 \). Suppose \( \text{RHS}(\lambda) > 0 \). Set \( \delta^{\eta,\tau} = 0 \) for all \( (\eta, \tau) \neq (2, u) \) and \( \delta^{2, u} > 0 \). Then

\[
\lim_{\lambda \to 0} \text{LHS}((\delta^{\eta,\tau}), \lambda) > \lim_{\lambda \to 0} \text{RHS}(\delta)
\]

\[
= 0
\]

since \( \pi_a < 1 \). Also

\[
\text{LHS}((\delta^{\eta,\tau}), 1 - \pi_a) = 0
\]

\[
< \text{RHS}(1 - \pi_a).
\]

Hence there exists \( \hat{\lambda} \in (0, 1 - \pi_a) \) such that \( \text{LHS}((\delta^{\eta,\tau}), \hat{\lambda}) = \text{RHS}(\hat{\lambda}) \).

Suppose \( \text{RHS}(\lambda) < 0 \). Suppose \( \pi_a > 0 \). By assumption, there exists some \( r_{\hat{\eta}, \hat{\tau}} > 0 \). Set \( \delta^{\eta,\tau} = 0 \) for all \( (\eta, \tau) \neq (\hat{\eta}, \hat{\tau}) \) and \( \delta^{\hat{\eta}, \hat{\tau}} \) such that

\[
\text{LHS}((\delta^{\eta,\tau}), 1 - \pi_a) > \text{RHS}(1 - \pi_a) \tag{2.22}
\]

Fix \( \hat{\lambda} \in (0, 1 - \pi_a) \) such that \( \text{LHS}((\delta^{\eta,\tau}), \hat{\lambda}) = \text{RHS}(\hat{\lambda}) \). Such a \( \hat{\lambda} \) exists by (2.22) and since

\[
\lim_{\lambda \to 0} \text{RHS}(\lambda) = 0
\]

\[
> \lim_{\lambda \to 0} \text{LHS}((\delta^{\eta,\tau}, \lambda)
\]

as \( \pi_a > 0 \) and \( \delta^{\hat{\eta}, \hat{\tau}} < 0 \).

Suppose \( r_{2, u}^a = r_{1, u}^a = 0 \). Then buyers get no rents from visiting seller \( a \) in equilibrium. Either \( q = 0 \) and seller \( a \) makes no profits in equilibrium, or buyers get no rents from either site in equilibrium. The first case cannot occur in equilibrium, as any deviation for sellers that ensure positive profits and visit probabilities is profitable, and such deviations always exist. In the second case, any seller could deviate by offering marginally more rents and capturing all buyer visits, another contradiction. Hence there is some \( \hat{\eta} \) with \( r_{\hat{\eta}, \hat{\tau}}^a > 0 \). Set \( \delta^{\hat{\eta}, \tau} = 0 \) for all \( (\eta, \tau) \neq (\hat{\eta}, \hat{\tau}) \) and \( \delta^{\hat{\eta}, u} < 0 \). Set \( \hat{\lambda} \) such that \( \text{LHS}((\delta^{\eta,\tau}), \hat{\lambda}) = \text{RHS}(\hat{\lambda}) \). Such a \( \hat{\lambda} \) exists since \( \text{LHS}((\delta^{\eta,\tau}), 1) = 0 > \text{RHS}(1) \) and

\[
\lim_{\lambda \to 0} \text{RHS}(\lambda) = 0
\]

\[
< \lim_{\lambda \to 0} \text{LHS}((\delta^{\eta,\tau}, \lambda)
\]

as \( \delta^{\hat{\eta}, u} < 0 \).

Finally, if \( \text{RHS}(\lambda) = 0 \), buyers are indifferent between informed and uninformed states at site \( a \) and a seller can increase information provision without shifting traffic by setting \( \delta^{\eta,\tau} = 0 \) for all \( \eta \in \{1, 2\}, \tau \in \{i, u\} \).
In all cases, the arguments above yield a deviation for seller $a$ which keeps rent payouts unchanged and strictly increases the surplus available at site $a$. This implies that $(\pi_a, \gamma_a, \pi_b, \gamma_b)$ is not an equilibrium.

**Proof of Lemma 2.3.** My argument proceeds with mechanisms in $\bar{\Gamma}$. However, if a mechanism in $\Gamma \setminus \bar{\Gamma}$ without PAE were a component of an equilibrium, applying the following proof to its IC($\theta_H$)-equivalent (through Lemma 2.6) would yield a contradiction, since a best response to a IC($\theta_H$)-equivalent mechanism is also a best-response to the original mechanism.

Consider an incentive compatible mechanism $\gamma_k$ at site $k$ such that $x_k^{1,2}(\theta_H) < 1$. Consider an alternative mechanism $\hat{\gamma}_k$ identical to $\gamma_k$ except that

\[
\hat{x}_k^{1,2}(\theta_H) = x_k^{1,2}(\theta_H) + \epsilon
\]

\[
\hat{y}_k^{1,2}(\theta_H) = y_k^{1,2}(\theta_H) + \epsilon \theta_H,
\]

where $\epsilon \in (0, 1 - x_k^{1,2}]$. We have $x_k^{1,2}(\theta_H) > x_k^{1,2}(\theta_L) \geq \hat{x}_k^{1,2}(\theta_H) > x_k^{1,2}(\theta_L)$ and $\rho^{1,2} = \rho^{1,2} \geq 0$ since $\gamma_k \in \bar{\Gamma}$, and so $\hat{\gamma}_k \in \bar{\Gamma}$. Note that $\hat{R}^{1,2} = R^{1,2}$ and hence buyer rents and visit decisions are unaffected. However, seller $k$’s profits are higher under $\hat{\gamma}_k$ than under $\gamma_k$ if buyers sometimes visit $k$ since $\theta_H$-type transfers in the one-buyer state are higher.

As noted in the text, the proof needs to be modified in the two-buyer state if $X_k^{2,2}(\theta_H) < p_L + \frac{1}{2}p_H$ and if the constraint $x_k^{2,2}(\theta_H, \theta_L) + x_k^{2,2}(\theta_L, \theta_H) \leq 1$ from (2.15) is binding under the original mechanism $\gamma_k$. If this is not the case, then the previous proof applies to the reduced-form mechanisms. If not, it must be that $x_k^{2,2}(\theta_L, \theta_H) > 0$, that is, a $\theta_L$-type is sometimes allocated the good in the presence of a $\theta_H$-type. Consider an alternative mechanism $\hat{\gamma}_k$ identical to $\gamma_k$ except that

i. $\theta_L$-types never get preference over $\theta_H$-types, $\hat{x}_k^{2,2}(\theta_H, \theta_L) = 1$ and $\hat{x}_k^{2,2}(\theta_L, \theta_H) = 0$, so that

\[
\hat{X}^{2,2}(\theta_L) = X_k^{2,2}(\theta_L) - p_H x_k^{2,2}(\theta_L, \theta_H)
\]

\[
\hat{X}^{2,2}(\theta_H) = X_k^{2,2}(\theta_H) + p_L x_k^{2,2}(\theta_L, \theta_H).
\]

ii. Transfers are adjusted so that rents to both types are unchanged

\[
\hat{Y}^{2,2}(\theta_L) = Y_k^{2,2}(\theta_L) - \theta_L (X_k^{2,2}(\theta_L) - \hat{X}^{2,2}(\theta_L))
\]

\[
\hat{Y}^{2,2}(\theta_H) = Y_k^{2,2}(\theta_H) + \theta_H (X_k^{2,2}(\theta_H) - \hat{X}^{2,2}(\theta_H)).
\]

By condition i and since $\gamma_k \in \bar{\Gamma}$, we have that $\hat{X}^{2,2}(\theta_H) > X^{2,2}(\theta_H) \geq X^{2,2}(\theta_L) > \hat{X}^{2,2}(\theta_L)$. Along with condition ii, this implies that $\hat{\gamma}_k \in \bar{\Gamma}$.

Profits to seller $k$ in the two-buyer state under $\hat{\gamma}_k$ are given by

\[
2 \left[ p_L \hat{Y}^{2,2}(\theta_L) + p_H \hat{Y}^{2,2}(\theta_H) \right] = 2 \left[ p_L Y^{2,2}(\theta_L) + p_H Y^{2,2}(\theta_H) + p_H p_L (\theta_H - \theta_L) x^{2,2}(\theta_L, \theta_H) \right]
\]

\[
> 2 \left[ p_L Y^{2,2}(\theta_L) + p_H Y^{2,2}(\theta_H) \right],
\]
where the last expression is profits to seller $k$ in the two-buyer state under $\gamma_k$. The inequality follows since by hypothesis $x^{2,i}(\theta_L, \theta_H) > 0$. Thus seller $k$ gains by offering $\hat{\gamma}_k$ if buyers visit $k$ with positive probability since traffic and one-buyer state profits are unchanged and two-buyer state profits are higher. Furthermore, under $\hat{\gamma}_k$ it is the case that $\hat{x}^{2,i}(\theta_H) = p_L + \frac{1}{2}p_H$.

Similarly, for uninformed allocations, consider an incentive compatible mechanism $\gamma_k$ at site $k$ such that $x^{\eta,i}_k < 1$ for some $\eta \in \{1, 2\}$. Consider an alternative mechanism $\hat{\gamma}_k$, identical to $\gamma_k$ except that in state $(\eta, u)$

\[
\hat{x}^{\eta,i}_k = x^{\eta,i}_k + \epsilon
\]
\[
\hat{y}^{\eta,i}_k = y^{\eta,i}_k + \epsilon \bar{\theta},
\]

where $\epsilon \in (0, 1 - x^{\eta,i}_k]$. Thus buyer rents are the same under both mechanisms but seller $k$’s profits are higher in state $(\eta, u)$ if buyers visit seller $k$ with positive probability since the good is sold more often at higher prices. \hfill \qed

Proof of Lemma 2.4. Consider an incentive compatible mechanism $\gamma_k$ at site $k$ such that $x^{1,i}_k(\theta_L) < 1$ and the level of rents provided to type $\theta_L$ is given by $r^{1,i} \geq 0$. Then

\[
y^{1,i}_k(\theta_L) = \theta_L x^{1,i}_k(\theta_L) - r^{1,i}, \tag{2.23}
\]

and, by Lemmas 2.6 and 2.3

\[
y^{1,i}_k(\theta_H) = \theta_H - x^{1,i}_k(\theta_L)(\theta_H - \theta_L) - r^{1,i}. \tag{2.24}
\]

By (2.23) and (2.24), write seller $k$’s profits conditional on $\text{IC}^{1,i}_k(\theta_H)$ binding and type $\theta_L$ receiving rents $r^{1,i}$ as

\[
x^{1,i}_k(\theta_L)(\theta_L - p_H \theta_H) + p_H \theta_H - r^{1,i}. \tag{2.25}
\]

These are increasing in $x^{1,i}_k(\theta_L)$ whenever $\theta_L > p_H \theta_H$. Since $x^{1,i}(\theta_H) = 1$ by Lemma 2.3, an increase in $x^{1,i}(\theta_L)$ maintains incentives compatibility so seller $k$ can increase profits in state $(1, i)$ by doing so. This increases traffic to site $k$ (since rents to $\theta_H$-types increase). But at a symmetric equilibrium $q = \frac{1}{2}$ and marginal changes in traffic have negligible effects on the probability of the one-buyer state ($2q(1-q)$), so that profits of seller $k$ increase with marginal changes in $x^{1,i}(\theta_L)$ if profits in the two-buyer state are assumed to be nonnegative. However, note that this argument ensures that profits in the two-buyer state must be nonnegative in a symmetric equilibrium. If not, a seller could marginally increase transfers in the two-buyer state without affecting traffic significantly in the one-buyer state, while both traffic and losses per buyer would decrease in the two-buyer state. \hfill \qed

Proof of Lemma 2.5. To show that $R^{2,i} \leq R^{1,i}$, consider a symmetric equilibrium with $\pi = 1, FAE$
and a mechanism \( \gamma \) such that \( R_{1,j}^{1,j} < R_{2,j}^{2,j} \). This last fact implies that \( r_{2,i}^{2,j} > 0 \). Consider a mechanism \( \hat{\gamma}_k \) for seller \( k \) identical to \( \gamma \) except that \( \hat{r}_{2,i}^{2,j} = r_{2,i}^{2,j} - \Delta \). By (2.2) and the argument in the text for \( \Delta \approx 0 \), \( \hat{\gamma}_k \) leads to an infinitesimal increase in the number of buyers visiting site \( k \). Locally, moving away from a symmetric profile does not change the probability of the one-buyer state, while it increases that of the two-buyer state, where rents are now lower. This deviation is thus profitable given that profits in the two-buyer state are nonnegative (see proof of Lemma 2.4).

To show that \( R_{1,j}^{1,j} = \frac{\Delta}{2} \), consider marginal variations in \( R_{1,j}^{1,j} \) and \( R_{2,j}^{2,j} \) that leave \( \pi = 1 \) and allocative efficiency unchanged. Assume for now that \( r_{1,i}^{1,j} > 0 \) and \( r_{2,i}^{2,j} > 0 \) to ensure that it is always possible to effect such marginal changes through transfers. Profits for seller \( a \) are given by

\[
P_a(\pi_a, \gamma_a, \pi_b, \gamma_b) = q^2 [S_{2,i}^{2,i} - 2R_{a}^{2,i}] + 2q(1-q)[\hat{\theta} - R_{a}^{1,i}].
\]

At a symmetric profile, the marginal changes in the term \( q(1 - q) \) can be ignored and thus

\[
\frac{\partial P_a(\pi_a, \gamma_a, \pi_b, \gamma_b)}{\partial R_{a}^{1,i}} = 2q \left[ \frac{\partial q}{\partial R_{a}^{1,i}} (S_{2,i}^{2,i} - 2R_{a}^{2,i}) - (1-q) \right],
\]

where, at a symmetric profile with \( \pi = 1 \) we have \( q = \frac{1}{2} \) and \( \frac{\partial q}{\partial R_{a}^{1,i}} = \frac{1}{4(R_{a}^{1,i} - R_{a}^{2,i})} \). Thus

\[
\frac{\partial P_a(\pi_a, \gamma_a, \pi_b, \gamma_b)}{\partial R_{a}^{1,i}} = \left( \frac{1}{4} \right) \frac{S_{2,i}^{2,i} - 2R_{a}^{2,i}}{R_{a}^{1,i} - R_{a}^{2,i}} - \frac{1}{2} = 0 \quad \text{only when } R_{a}^{1,i} = \frac{S_{2,i}^{2,i}}{2}.
\]

In the same way, it can be computed that \( \frac{\partial P_a(\pi_a, \gamma_a, \pi_b, \gamma_b)}{\partial \pi_a^{1,i}} = 0 \) only when \( R_{a}^{1,i} = \frac{S_{2,i}^{2,i}}{2} \). That is, the same condition holds for marginal changes in expected rents in both one-buyer and two-buyer states. Since \( \frac{\partial P_a(\pi_a, \gamma_a, \pi_b, \gamma_b)}{\partial R_{a}^{2,i}} = 0 \) and \( \frac{\partial P_a(\pi_a, \gamma_a, \pi_b, \gamma_b)}{\partial \pi_a^{2,i}} = 0 \) yield the same condition, we need to worry about the existence of derivatives only when \( r_{1,i}^{1,j} = r_{2,i}^{2,j} = 0 \). But then an argument considering deviations \( R_{a}^{1,i} + \Delta \) or \( R_{a}^{2,i} + \Delta \) yields the result.

**Proof of Proposition 2.6** Fixing some profile that satisfies the assumptions of the proposition, I will first show that with \( \pi = 1 \) and FAE, no deviation consisting of either individual or joint shifts (not necessarily local) in \( R_{1,i}^{1,i} \) and \( R_{2,i}^{2,i} \) can achieve higher profits. Since the candidate profile has full information and FAE, considering changes in rents where surplus in both states is maximized gives an upper bound on the profitability of deviations that involve the same changes in rents but that include a decrease in information provision and/or allocative efficiency.

Consider some profile with \( \pi = 1 \) and associated rents \( R_{1,i}^{1,i} \geq R_{2,i}^{2,i} \). Consider a deviation profile for seller \( a \) in which

\[
\hat{R}_{a}^{1,i} = R_{a}^{1,i} + \Delta^1,
\]

\[
\hat{R}_{a}^{2,i} = R_{a}^{2,i} + \Delta^2,
\]

\footnote{I need only consider mechanisms in \( \Gamma \), by the remark in the proof of Lemma 2.3.}
where $\Delta^\eta$ for $\eta \in \{1, 2\}$ need not be positive. Clearly, seller $a$ cannot profitably deviate to any mechanism for which $\hat{q} = 0$. Also, the most profitable deviation to some mechanism such that $\hat{q} = 1$ is such that any less generous mechanism leads to $\hat{q} < 1$. Hence we can restrict attention to pairs $(\Delta^1, \Delta^2)$ such that the level of traffic $\hat{q} \in (0, 1]$ is given by (2.2). Hence $\hat{q}$ is given by

$$
\hat{q} = \frac{(R^{1,i} - R^{2,i}) + \Delta^1}{2((R^{1,i} - R^{2,i}) + \Delta^1 - \Delta^2)}
= \frac{1}{2} + z
$$

with $z = \left(\frac{1}{2}\right) \frac{\Delta^1 + \Delta^2}{2((R^{1,i} - R^{2,i}) + \Delta^1 - \Delta^2)}$.

The difference in profits can be written as

$$
P_a(1, \hat{\gamma}_a, 1, \gamma_b) - P_a(1, \gamma_a, 1, \gamma_b) = \left[S^{2,i} - 2R^{2,i}\right] (x(x + 1)) - 2 \left[m - R^{1,i}\right] x^2
- 2\Delta^2 \left(\frac{1}{2} + x\right)^2 - 2\Delta^1 \left(\frac{1}{2} + x\right) \left(\frac{1}{2} - x\right)
= C \left[S^{2,i} - 2R^{2,i}\right] \left(4((R^{1,i} - R^{2,i}) + 3\Delta^1 - \Delta^2) \left(\Delta^1 + \Delta^2\right)
- 2 \left[\bar{\theta} - R^{1,i}\right] \left(\Delta^1 + \Delta^2\right)^2
- 8 \left((R^{1,i} - R^{2,i})\right) \left((R^{1,i} - R^{2,i}) + \Delta^1\right) \left(\Delta^1 + \Delta^2\right)\right],
$$

where $C = \left(\frac{1}{4}\right) \left[\frac{1}{2((R^{1,i} - R^{2,i}) + \Delta^1 - \Delta^2)}\right]^2 > 0$. Set the original candidate profile as

$$
R^{1,i} = \frac{S^{2,i}}{2},
R^{2,i} = \frac{S^{2,i}}{2} - \epsilon, \text{ for } \epsilon \geq 0.
$$

simplifying the profit difference yields

$$
P_a(1, \hat{\gamma}_a, 1, \gamma_b) - P_a(1, \gamma_a, 1, \gamma_b) = C \left[(\Delta^1 + \Delta^2)^2 (-2\epsilon - (2\bar{\theta} - S^{2,i}))\right]
< 0 \text{ for any } (\Delta^1, \Delta^2), \text{ since } \epsilon > 0 \text{ and } 2\bar{\theta} > S^{2,i}.
$$

Thus no deviations are profitable. \qed
Chapter 3
Experimentation with Costly Project Maintenance

3.1 Introduction

When experimentation is costly, decision-makers face choices about which alternatives to investigate. For example, firms that develop new technologies focus on a few projects at a time and do not fund all competing ideas equally. Investing in multiple technologies simultaneously is costly, so that firms choose a limited set of technologies to develop initially and then decide whether to transfer their resources to other projects when experimentation results are not sufficiently encouraging. Professional sports teams can be thought of as managing their rosters in a similar way. Since the rules of the sport specify that a single player can play at a given position at a time, coaches/managers can infer information about players’ abilities only by allowing them to enter the game and replace a teammate.

In standard models of experimentation, the choice of gathering information about one alternative as opposed to another entails only an implicit opportunity cost, the foregone opportunity of learning about the inactive alternative. However, retaining the option to investigate a currently shelved alternative often involves explicit maintenance costs. For example, a research and development firm must maintain specialised stocks of knowledge simply to keep open the option of developing the technology that is not currently a priority. This consists of the upkeep of specialised equipment or the salaries of skilled workers or scientists that can be lost to other firms. In professional sports, the option to develop players of unknown quality is kept open by filling roster spots with ‘bench’ players, who may seldom get the opportunity to play but command non-negligible salaries.

In this paper, I present a standard model of experimentation, but allow alternatives that are not being investigated to have explicit maintenance costs and to be irreversibly discarded if deemed unpromising. Two risky alternatives can be good or bad and only good alternatives eventually succeed if investigated. An experimenter searches for a success among the alternatives by choosing both (i) which one to investigate now and (ii) whether or not to maintain the inactive one for (potential) future research. An alternative’s current state is characterised by the experimenter’s belief that it is good, and repeated failures make the experimenter more pessimistic about the alternative.

Without the option to discard alternatives, the optimal experimentation policy has a ‘stay-with-the-winner’ property: the alternative that is more likely to be good is investigated first. I fully characterise the optimal experimentation policy with maintenance costs and show that these lead to significant departures from this standard result. In particular, alternatives that are less likely to succeed are sometimes investigated first. In such cases, ‘losing’ alternatives are granted a ‘last chance’ to succeed, after which they are permanently discarded in favour of more promising
alternatives.

Maintenance costs alter the optimal order in which alternatives are investigated by providing incentives to bring the option value of less promising alternatives forward. To do so, the experimenter needs to investigate the losing alternative first while paying to maintain the winning alternative. The benefit is that the decision to discard the losing alternative following repeated failures is better informed, and the experimenter can then investigate the winning alternative without having to maintain the losing one.

I model experimentation as a multi-armed bandit problem. This is a well-known class of models in statistical decision theory in which an experimenter (e.g. a firm) learns about the distributions generating the payouts of various arms (e.g. technologies) by ‘pulling’ (e.g. developing) them sequentially over time. In general, these are complex multi-dimensional dynamic optimisation problems, but in the standard discounted multi-armed bandit problem with independent arms the optimal experimentation policy can be represented by a well-known (Gittins) index policy. To each arm is assigned a number (index) that depends only on the ex ante characteristics and accumulated observations of that arm. The optimal experimentation policy consists of always selecting an arm among those with maximal indices.

The index representation of the optimal experimentation policy is not robust to perturbations in standard assumptions such as geometric discounting and independent arms. Closer to my paper, Banks and Sundaram (1994) have shown that index policies are not optimal in the presence of switching costs between arms. Intuitively, the reason is that switching costs link the experimenter’s decision to move away from an active arm, that is, to incur a switching cost, with the characteristics of outside arms in a way that cannot be captured by a single index that depends only on the characteristics of the active arm. General characterisations of optimal experimentation policies with switching costs have proven difficult to obtain.

Although my results are obtained in a special framework, maintenance costs appear to be more tractable than switching costs. Maintenance costs differ from switching costs in that the latter are attributed to an inactive arm only when experimentation transitions to it and are always accompanied by an observation from that arm. Maintenance costs need to be paid whenever an inactive arm is not pulled and never generate observations from that arm. It is not clear how to even define an index policy in the presence of maintenance costs since experimentation policies need to specify both which arm is pulled and which arms are maintained. Nevertheless, even if this difficulty could be surmounted the bandit problem with maintenance costs would fail to admit a Gittins index representation for the same reason Banks and Sundaram (1994) identified for bandit models with switching costs: the index of a given maintained arm would have to be a function of the maintenance cost, and this relationship would depend nontrivially on the characteristics of outside arms.

The bandit model I consider is a continuous-time exponential bandit due to Keller et al. (2005). The exponential bandit model has proved useful in applications due to its tractability. In par-

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1See Berry and Fristedt (1985), Bergemann and Välimäki (2006a) survey the bandits literature with an eye to applications in economics.

2For details, see Jun (2004). An exception is Bergemann and Välimäki (2001), who exploit results of Banks and Sundaram (1992b) on bandits with a countable numbers of ex ante identical arms to show that an experimenter never switches back to an arm it switched away from earlier.
ticular, the experimenter’s optimal payoff functions of the two-armed bandit (optimal stopping) problem are simple to compute, which is not the case for bandits with different probability structures. Keller et al. (2005), following Bolton and Harris (1999), study strategic experimentation and the free-riding incentives of multiple agents facing a single risky arm. Keller and Rady (2009) generalise the model to ‘poisson’ bandits that allow for arms of the bad type to also generate successes. Klein and Rady (2008) allow for each of two experimenters to have perfectly negatively correlated versions of the same risky arm, and hence the state can still be described by a single-dimensional belief. Strulovici (2009) applies the model in a voting framework. Bergemann and Hege (1998) have introduced a discrete-time version of the model to study the moral hazard problem arising between bankers (principal) and venture capitalists (experimenters). In this vein, recent papers by Bonatti and Hörner (2009) and Hörner and Samuelson (2009) focus on the provision of incentives to experimenting agents.

I present the model in section 3.2. I address how to define strategies (which depend on time in the absence of a success) and Markov strategies (which depend on the state, the belief of the experimenter that each arm is good) and clarify the relationships between them. While Markov strategies along with dynamic programming methods allow for simple expressions for optimal payoffs, many of the arguments regarding when and why maintained arms should be discarded are naturally established by considering time paths of play. In Section 3.3 I use the simple structure of the continuous-time exponential bandit problem to derive expressions for the experimenter’s optimal payoffs. In Section 3.4 I characterise optimal experimentation in the benchmark model in which inactive arms have maintenance costs but the experimenter cannot discard risky arms. This is equivalent to a standard three-armed bandit with two risky arms, and hence the solution is known by the Gittins index theorem. I show that the optimal experimentation policy involves the stay-with-the-winner rule. In Section 3.5 I present the main results of the paper for the model in which inactive arms have maintenance costs and the experimenter can discard arms. First, I show that if the optimal policy ever ‘goes-with-the-loser’, it will do so in a very specific way. The losing arm will be chosen continuously for a short period, after which, in the absence of a success, it will be discarded. Losing arms are used before winning arms only if they are being granted a ‘last chance’ to succeed, else they are maintained and not used or simply discarded. Second, I give a complete characterisation of the optimal policy and show that whenever it is not the case that maintenance costs are high enough that the ‘losing’ arm is always discarded immediately, there exist regions of initial beliefs for which the experimenter starts with the losing arm.

3.2 Model

Consider a continuous time three-armed bandit problem with two risky arms, $A$ and $B$, and a safe arm $S$. An experiment consists of pulling a risky arm for some time interval $[t, t + dt]$. The probabilistic structure of the risky arms is as in Keller et al. (2005). Experiments yield either successes or failures. The type of a risky arm is $\theta \in \{ \text{Good}, \text{Bad} \}$. A risky arm of type $\theta$ that is pulled continuously in time interval $[t, t + dt]$ succeeds with probability $G dt$ for some $G > 0$ if $\theta = \text{Good}$, while it fails for sure if $\theta = \text{Bad}$. The types of risky arms $A$ and $B$ are drawn independently. Let $p_J(0)$ be the ex ante probability that arm $J$ is of type $\text{Good}$. A safe arm $S$ yields a flow payoff of
0. A success on either risky arm yields a lump-sum payment of 1 and ends the experimentation process.

Pulling risky arm \( J \) continuously in time interval \([t, t + dt]\) entails experimentation costs \( \dd{t} \). As my only departure from standard multi-armed bandit problems, I introduce explicit costs to maintaining inactive risky arms. That is, a risky arm that is maintained but not pulled in time interval \([t, t + dt]\) entails a cost of \( \dd{t} \). The experimenter can irreversibly discard risky arms without cost. That is, it can avoid paying for the maintenance of inactive arms but only at the cost permanently abandoning some of its options. There are no costs to the safe arm, which can be interpreted as an option to quit the experimentation process. The experimenter discounts future payoffs at rate \( r \).

Since experimentation ends after the first success, the only histories after which the experimenter selects an arm to pull are intervals of time in which only failures have been observed. Strategies should properly be defined on histories, however, any such strategy can be redefined to depend solely on time in the absence of a success. A \textit{strategy} is a collection \((\alpha, \phi_A, \phi_B)\) for some function \( \alpha : \mathbb{R}_+ \to [0, 1] \cup \{S\} \) and decreasing functions \( \phi_J : \mathbb{R}_+ \to \{0, 1\} \) for \( J \in \{A, B\} \). The function \( \alpha \) is an \textit{assignment rule} and \( \int_t^{t+dt} \alpha(t) \) specifies the fraction of time devoted to pulling arms \( A \) in time interval \([t, t + dt]\) if the experimenter pulls only risky arms in that interval, while \( \alpha(t) = S \) if the experimenter pulls the safe arm at time \( t \). The principal is allowed to share the responsibility for the project between the agents in any interval of time. The assumption that the experimenter cannot share the assignment between all three arms and must decide first whether or not to pull risky arms and then in what ratio is made to simplify the exposition and is in fact without loss of generality for optimal experimentation. Functions \( \phi_A \) and \( \phi_B \) specify \textit{maintenance rules}, with \( \phi_J(t) = 1 \) if and only if \( J \) is maintained at time \( t \). Strategy \((\alpha, \phi_A, \phi_B)\) is \textit{admissible} if each component is right-continuous and piecewise Lipschitz continuous. Let \( t_J \in [0, \infty) = \sup\{t : \phi_J(t) = 1\} \). Given any initial conditions \((p_A(0), p_B(0)) \in [0, 1]^2\), admissible strategy \((\alpha, \phi_A, \phi_B)\) induces uniquely defined and continuously differentiable laws of motion for the beliefs \((p_A(t), p_B(t))\) that arms \( A \) and \( B \) are of type \textit{Good} at time \( t^J \) These laws of motion are given by

\[
\begin{align*}
\dot{p}_A(t) &= \begin{cases} 
-\alpha(t)Gp_A(t)(1 - p_A(t)) & \text{for } t \in [0, t_A), \\
0 & \text{for } t \geq t_A.
\end{cases} \\
\dot{p}_B(t) &= \begin{cases} 
-(1 - \alpha(t))Gp_B(t)(1 - p_B(t)) & \text{for } t \in [0, t_B), \\
0 & \text{for } t \geq t_B.
\end{cases}
\end{align*}
\]

These laws of motion are derived in a straightforward way by requiring that the evolution of beliefs be consistent with \( \alpha \) and Bayes’ rule, and follows \cite{keller2005}. For much of the paper, it will be more convenient to work with Markov strategies. These are strategies that are conditioned on the state variable, which is the current beliefs along with the set of maintained arms. More formally, a \textit{state} consists of \((p_A, p_B, I_A, I_B) \in [0, 1]^2 \times \{0, 1\}^2\). A \textit{Markov assignment} is a function \( \beta : [0, 1]^2 \times \{0, 1\}^2 \to [0, 1] \cup \{S\} \). \textit{Markov maintenance rules} are functions

\footnote{In turn, this ensures that the optimal control problem of finding a payoff-maximising strategy is well-defined.}
φ_j: [0,1]^2 \times [0,1]^2 \to [0,1] for J \in \{A, B\} such that \( \varphi_j(p_A, p_B, I_A, I_B) = 0 \) whenever \( I_j = 0 \).

Imposing admissibility requirements directly on Markov strategies can be cumbersome. A further source of difficulty in this framework is to determine how the monotonicity (irreversibility) requirements on maintenance rules carry over to restrictions on Markov maintenance rules. To get around these issues, I rely on the admissibility requirement already stated for strategies. Markov strategy \((\beta, \varphi_A, \varphi_B)\) will be said to be admissible if given any state \((p_A, p_B, I_A, I_B)\) and initial beliefs \((p_A(0), p_B(0)) = (p_A, p_B)\), there exists a corresponding admissible strategy \((\alpha, \varphi_A, \varphi_B)\) such that for all \(t\)

\[
\alpha(t) = \beta(p_A(t), p_B(t), \varphi_A(t), \varphi_B(t)),
\]

\[
\varphi_A(t) = \varphi_A(p_A(t), p_B(t), \varphi_A(t), \varphi_B(t)),
\]

\[
\varphi_B(t) = \varphi_B(p_A(t), p_B(t), \varphi_A(t), \varphi_B(t)).
\]

Henceforth I will not explicitly restrict the experimenter to using admissible Markov strategies, or rather I will assume that the Markov strategies evoked yield well-defined solutions to the differential equations for the evolution of beliefs. However, I will verify that the optimal Markov strategies I derive, as well as the deviating strategies that support various proofs, are admissible.

A Markov strategy \((\beta, \varphi)\) is symmetric if

\[
\beta(p_B, p_A, I_B, I_A) = \begin{cases} 
1 - \beta(p_A, p_B, I_A, I_B) & \text{if } \beta(p_A, p_B, I_A, I_B) \neq S, \\
S & \text{if } \beta(p_A, p_B, I_A, I_B) = S,
\end{cases}
\]

\[
\varphi_j(p_B, p_A, I_B, I_A) = \varphi_j(p_A, p_B, I_A, I_B) \quad \text{for } J \in \{A, B\}.
\]

Given any optimal strategy \((\beta^*, \varphi^*)\), there exists an optimal symmetric strategy that achieves the same payoffs. Hence, restricting to symmetric strategies is without loss of generality for the experimenter’s payoffs. Given the restriction to symmetric strategies, it is without loss of generality to assume that \(p_A \geq p_B\). Henceforth, arm \(A\) will always be the ‘winning’ arm, with arm \(B\) the ‘losing’ arm.

Let \(W(\alpha, \phi; t, \tau)\) be the experimenter’s payoff at time \(t\) to strategy \((\alpha, \phi)\). If a success arrives at time \(\tau < \min\{t_A, t_B\}\)

\[
W(\alpha, \phi; t, \tau) = e^{-rt} - \int_t^\tau e^{-rs}(k + k)ds,
\]

while if a success arrives at \(\tau \in [\min\{t_A, t_B\}, \max\{t_A, t_B\}]\)

\[
W(\alpha, \phi; t, \tau) = e^{-rt} - \int_t^{\min\{t_A, t_B\}} e^{-rs}(k + k)ds - \int_{\min\{t_A, t_B\}}^\tau e^{-rs}kds,
\]

and finally if a success never arrives

\[
W(\alpha, \phi; t, \tau) = -\int_t^{\min\{t_A, t_B\}} e^{-rs}(k + k)ds - \int_{\min\{t_A, t_B\}}^{\max\{t_A, t_B\}} e^{-rs}kds.
\]

---

\(^4\)See Fleming and Rishel (1975), Theorem 6.1.
The expected payoff to strategy \((\alpha, \phi)\) given belief \((p_A(0), p_B(0))\) is

\[
V(\alpha, \phi; t) \equiv \mathbb{E}_\tau W(\alpha, \phi; t, \tau),
\]

where the expectation is taken over the distribution of stopping times \(\tau\) determined by \((\alpha, \phi)\) and \((p_A(s), p_B(s))\). Consider an admissible Markov strategy \((\beta, \varphi)\) and its corresponding strategy \((\alpha, \phi)\) for some state \((p, I)\). The expected payoff to \((\beta, \varphi)\) in state \((p, I)\) is given by

\[
v(\beta, \varphi; p, I) \equiv V(\alpha, \phi; 0)
\]

The objective of the experimenter is to find a payoff-maximising strategy. To this end, let \(U(t) = \max_{(\alpha, \phi)} V(\alpha, \phi; t)\). Similarly, let \(u(p, I) = \max_{(\beta, \varphi)} v(\beta, \varphi; p, I)\).

### 3.3 Preliminaries: Optimal Payoff Functions

The principal advantage of the continuous-time exponential bandit framework is its tractability, and the fact that simple expressions for optimal value functions exist for the standard two-armed bandit (optimal stopping) problem. In this section, I derive the expressions satisfied by the optimal payoff \(u\) that will support the characterisations of Sections 3.4 and 3.5. To simplify notation, let the number of beliefs listed in a state implicitly denote the set of maintained arms. Hence \((p_A, p_B)\) can stand for state \((p_A, p_B, 1, 1)\), \((p_A)\) for state \((p_A, p_B, 1, 0)\) given any \(p_B\), and so on.

In any open region of the state space in which both arms are maintained, \(u\) must satisfy the following Bellman equation

\[
u(p_A, p_B) = \max \left\{ e^{-rdt}u(p_A, p_B), u_A(p_A), u_B(p_B), \max_{\beta \in [0, 1]} \left\{ \beta p_A G + (1 - \beta) p_B G - (k + k) dt + e^{-rdt} \mathbb{E}[u(p_A + dp_A, p_B + dp_B)|p_A, p_B] \right\} \right\}.
\]

(3.1)

The first term in the brackets of (3.1) corresponds to the option of employing the safe arm in a time interval of length \(dt\). The second and third terms correspond to the options of discarding arms \(A\) and \(B\) respectively, where \(u_J\) corresponds to the optimal payoff to the two-armed bandit problem with risky arm \(J\) and the safe arm. The final term corresponds to the payoffs from maintaining both arms and allocating the experimentation effort optimally. When a risky arm has been discarded, the experimenter faces a standard optimal stopping problem with the remaining risky arm, and payoff \(u_J\) solves

\[
u_J(p_J) = \max \left\{ e^{-rdt}u_J(p_J), [p_J G - \bar{\mathbb{E}}] dt + e^{-rdt} \mathbb{E}[u_J(p_J + dp_J)|p_J] \right\}.
\]

The probability of a success in an interval of length \(dt\) is \(p_J G dt\), and the payoff to a success is 1. The probability of failure is \(1 - p_J G dt\). In case of failure, the payoff to the experimenter is
\[ u_j(p_j) + u'_j(p_j)dp_j, \] which is equal to \( u(p_j) - u'_j(p_j)p_j(1 - p_j)Gdt. \) By rewriting and cancelling dominated terms

\[ ru_j(p_j) = \max \left\{ 0, p_jG - \bar{k} - u'_j(p_j)p_j(1 - p_j) - u_j(p_j)Gp_j \right\}. \]

Hence, in an open region of beliefs in which arm \( J \) is used, \( u_j \) satisfies the differential equation

\[ u_j(p_j)(r + Gp_j) = p_jG - \bar{k} - u'_j(p_j)Gp_j(1 - p_j), \] (3.2)

which can be solved to yield

\[ u_j(p_j) = \tilde{C}_j \left( \frac{1 - p_j}{p_j} \right)^{\frac{\bar{k}}{r}} (1 - p_j) + p_j \frac{G - \bar{k}}{r + G} - (1 - p_j) \frac{\bar{k}}{r}, \] (3.3)

with the constant of integration

\[ \tilde{C}_j = \left( \frac{\bar{k}}{G - \bar{k}} \right)^{\frac{\bar{k}}{r}} \frac{G}{r + G} \]

and the stopping belief \( p_j^* = \frac{\bar{k}}{G} \) determined by value-matching and smooth-pasting conditions

\[ u_j(p_j^*) = 0, \] and
\[ u'_j(p_j^*) = 0. \]

The setup here is slightly different than in Keller et al. (2005), but the expression (3.3) admits the same interpretation. The term \( p_j \frac{G - \bar{k}}{r + G} - (1 - p_j) \frac{\bar{k}}{r} \) is the payoff to risky arm \( J \) in the absence of the ability to quit experimentation, while the term \( \tilde{C}_j \left( \frac{1 - p_j}{p_j} \right)^{\frac{\bar{k}}{r}} (1 - p_j) \) captures the option value of the safe arm \( S \), the quitting option.

Note that the part of value function (3.1) in which both arms are maintained is linear in \( \beta \). Hence, in an open region in which both arms are maintained, the optimal value is attained for \( \beta \in \{0, 1\} \), and (3.1) can be rewritten as

\[ ru(p_A, p_B) = \max \left\{ p_A G - (\bar{k} + k) - \frac{\partial u(p_A, p_B)}{\partial p_A} Gp_A(1 - p_A) - u(p_A, p_B)Gp_A, \right. \]
\[ \left. p_B G - (\bar{k} + k) - \frac{\partial u(p_A, p_B)}{\partial p_B} Gp_B(1 - p_B) - u(p_A, p_B)Gp_B \right\}. \] (3.4)

Contrary to (3.2), partial differential equation (3.4) does not have a simple solution, since such a solution must include an optimal allocation rule.

I approach the solution to (3.4) by abstracting from allocations in order to reduce the two-dimensional problem (3.4) to suitably defined single-dimensional problems. First consider an open region of the state space in which arm \( A \) is used but both arms are maintained. Then since the optimal Markov strategy is admissible there exists \( t' > 0 \) and a parametrised path \( (p_A(t), p_B) \) such that \( U(t) = u(p_A(t), p_B) \) for \( t \in (0, t') \). An argument similar to that establishing (3.2) shows
that \( U(t) \) satisfies

\[
U(t)[r + p_A(t)G] - U'(t) = p_A(t)G - (\bar{k} + \bar{k}).
\]  

(3.5)

For path \((p_A(t), p_B)\), define \( u_A(p_A(t); p_B) \equiv U(t) \). Then \( U'(t) = -u'_A(p_A(t); p_B)Gp_A(t)(1 - p_A(t)) \), which uses the law of motion for \( p_A \). Condition (3.5) can be rewritten, eliminating the dependence on time, as

\[
u_A(p_A; p_B)[r + p_A G] + u'_A(p_A; p_B)Gp_A(1 - p_A) = p_AG - (\bar{k} + \bar{k}).
\]  

(3.6)

As for (3.2), (3.6) can be solved to yield

\[
u_A(p_A; p_B) = C_A(p_B)\left(1 - p_A\right)^{\frac{\bar{c}}{r}} \left(1 - p_A\right) + p_A \frac{G - (\bar{k} + \bar{k})}{r + G} - (1 - p_A)\frac{\bar{k} + \bar{k}}{r}.
\]  

(3.7)

In (3.7), the constant of integration will in general depend on \( p_B \), since \( p_B \) may affect the payoffs when exiting the \( A \)-region. If the parametrised path \((p_A(t), p_B)\) exits the \( A \)-region in state \((p_A^*, p_B)\), then \( p_A^* \) and \( C_A(p_B) \) satisfy the value-matching and smooth-pasting properties

\[
u_A(p_A^*; C_A(p_B)) = u(p_A^*, p_B), \text{ and}
\]

\[
\frac{\partial}{\partial p_A} \nu_A(p_A^*; C_A(p_B)) = \frac{\partial}{\partial p_A} u(p_A^*, p_B).
\]

In general, \( u(p_A^*, p_B) \) is endogenous and depends on the experimentation policy once exit from the \( A \)-region occurs. If, for example, experimentation exits the \( A \)-region into the quitting region at \( p_A^* \), then \( u(p_A^*, p_B) = 0 \) and \( \frac{\partial}{\partial p_A} u(p_A^*, p_B) = 0 \), which yields that \( p_A^* = \frac{\bar{k} + \bar{k}}{G} \).

Equation (3.7) shows that when arm \( A \) is used in the optimal solution, payoffs evolve essentially as though the experimenter was facing a two-armed bandit problem with cost \( \bar{k} + \bar{k} \) for the risky arm. However, the effect of \( p_B \) on \( u(p_A, p_B) \) is captured by the constant of integration \( C_A(p_B) \), which pins down the option value of arm \( B \). It will be useful in the sequel to distinguish a payoff of the form (3.7) from the optimal payoff \( u(p_A, p_B) \) in an \( A \)-assignment region. Given \((p_A, p_B)\) and some function \( C_A(p_B) \), define the righthand side of (3.7) as \( \nu_A(p_A; C_A(p_B)) \).

3.4 Optimal Experimentation Without Discarding Arms

In this section, I present the optimal experimentation policy for the benchmark case in which risky arms cannot be discarded individually. That is, the experimenter is restricted to Markov strategies with \( \phi(p, I) = (1, 1) \) for all states \((p, I)\) such that \( \beta(p, I) \neq S \). The experimenter can quit the experimentation by moving to the safe arm and discarding both risky arms. This problem is equivalent to the standard three-armed bandit problem with direct costs to experimentation \((\bar{k} + \bar{k})\).
3.4.1 Stay-with-the-Winner

The next lemma, which is also useful in later sections, deals with the allocation of trials between risky arms conditional on continuing the experimentation. It states that the experimenter should always use the arm with the highest belief, that is, ‘stay-with-the-winner’. When beliefs \((p_A, p_B)\) are such that \(p_A > p_B\), this means using arm \(A\). When beliefs are such that \(p_A = p_B\), then both arms have the highest belief. In the discrete time version of the model, beliefs would jump down following a failure and this would (generically) ensure that there would always exist a ‘best’ arm, allowing the application of the ‘stay-with-the-winner’ rule. In continuous time, staying with the winner means sharing experimentation effort equally between both arms when beliefs are on the 45-degree line.

**Lemma 3.1.** Consider \((p_A(0), p_B(0))\) and the belief path \((p_A(t), p_B(t))\) under optimal experimentation. If \(p_A(0) > p_B(0)\), then \(\beta^*(p_A(t), p_B(t)) \in \{1, S\}\) for almost all \(t \in [0, \hat{t})\), where \(\hat{t}\) is such that \(\hat{t} = \min\{\inf\{t : p_A(t) = p_B(t)\}\,\infty\}\). If instead \(p_A(0) = p_B(0)\), then \(\beta^*(p_A(t), p_B(t)) \in \{\frac{1}{2}, S\}\) for almost all \(t\).

Lemma 3.1 mimics the Gittins index representation of the optimal experimentation policy, in which an arm’s belief is taken to be the index. In fact, its proof is essentially a simplified version of the original ‘interchange argument’ in Gittins and Jones (1974) and Gittins (1979) that establishes the optimality of the Gittins’ index for standard bandit problems. By starting with an assignment in which an arm with a non-maximal Gittins index is chosen before the arm with the maximal index, the argument shows that interchanging the order in which both arms are pulled, keeping continuations following these periods of experimentation fixed, increases the experimenter’s payoffs. A special feature of the exponential bandit problems is that the arm with the highest Gittins index is also the arm with the highest belief. Hence, the myopically optimal allocation is also dynamically optimal. Exponential bandits are continuous time versions of Bernoulli bandits in discrete time, for which a result that myopic play is optimal is shown by Berry and Fristedt (1985), who coined the term ‘stay-with-the-winner’. Their result was generalised in Banks and Sundaram (1992a), who show that dynamically optimal play is myopic for a class of two-type symmetric bandits.

3.4.2 Shared Experimentation

Before completing the characterisation of optimal experimentation without discarding arms, I describe the experimenter’s payoffs when \(p_A = p_B\). According to Lemma 3.1, if there is no transition to the safe arm, then experimentation is shared and beliefs move down the 45-degree line. The key to obtaining an expression for optimal payoffs is that the partial differential equation (3.4) can be represented as a differential equation under the assumption that \(p_A = p_B = p\) and that \(\beta(p, p) = \frac{1}{2}\) for all beliefs \(p\) greater than some quitting belief \(p^*\). In that case, the payoff \(u\) must satisfy \(\frac{\partial}{\partial p_A}u = \frac{\partial}{\partial p_B}u\). Define \(u_{AB}(p) \equiv u(p, p)\), then it follows that \(u'_{AB}(p) = 2\frac{\partial}{\partial p_A}u(p, p)\) and \(u_{AB}\)

---

5 All proofs are in Appendix 3.7.
6 For a concise presentation of the original proof, see Frostig and Weiss (1999).
solves

\[ u_{AB}(r + pG) = pG - (\bar{k} + k) - \frac{1}{2}Gp(1 - p)u'_{AB} \]

The differential equation (3.8) has solution

\[ u_{AB}(p) = \tilde{C}_{AB} \left( \frac{1-p}{p} \right)^{\frac{2}{5}} (1-p)^2 + p^2 \frac{G - (\bar{k} + k)}{r + G} + 2p(1-p) \frac{G - (\bar{k} + k)}{r + G} - (1-p)^2 \frac{\bar{k} + k}{r}. \]  

If optimal experimentation leads from shared experimentation to the safe arm at belief \( p^* \), the constant of integration \( \tilde{C}_{AB} \) and cutoff belief \( p^* = \frac{\bar{k} + k}{G} \) are determined by value-matching and smooth-pasting conditions.

\[ u_{AB}(p^*) = 0, \text{ and} \]
\[ u'_{AB}(p^*) = 0. \]

(3.8) has the same interpretation as (3.3), where \( p^2 \frac{G - (\bar{k} + k)}{r + G} + 2p(1-p) \frac{G - (\bar{k} + k)}{r + G} - (1-p)^2 \frac{\bar{k} + k}{r} \) is the payoff to shared experimentation with belief \( p \) for both arms in the absence of the ability to quit experimentation, while \( \tilde{C}_{AB} \left( \frac{1-p}{p} \right)^{\frac{2}{5}} (1-p)^2 \) is the option value of the safe arm. Note also that since \( p^* = \frac{\bar{k} + k}{G} \), the optimal quitting value is the same in the single-arm problem with experimentation cost \( \bar{k} + k \) as in the two-arm problem with equal beliefs.

### 3.4.3 Optimal Policy

Lemma 3.1 and the previous discussion characterise the optimal experimentation policy without discarding arms. Lemma 3.1 ensures that off the 45-degree line, the ‘best’ arm is used, while if on the 45-degree line, shared experimentation must continue until all experimentation ceases. Hence, when off the 45-degree line, two alternatives are available: experiment with arm A until some quitting belief, or experiment with arm A until 45-degree line is reached and then share experimentation. The discussion around (3.7) and (3.8) establishes the boundaries between the quitting and experimentation regions.

**Proposition 3.1.** When arms cannot be discarded, the following admissible Markov strategy is optimal

\[
\beta^*(p_A, p_B) = \begin{cases} 
1 & \text{if } p_A > p_B \text{ and } p_A > \frac{\bar{k} + k}{G}, \\
\frac{1}{2} & \text{if } p_A = p_B > \frac{\bar{k} + k}{G}, \\
S & \text{if } p_A \leq \frac{\bar{k} + k}{G}.
\end{cases}
\]

\[
\phi^*(p_A, p_B) = \begin{cases} 
(1,1) & \text{if } p_A > \frac{\bar{k} + k}{G}, \\
(0,0) & \text{if } p_A \leq \frac{\bar{k} + k}{G}.
\end{cases}
\]

Figure 3.1 illustrates belief paths consistent with the optimal experimentation policy when arms cannot be discarded. From belief \((p_A, p_B)\) with \( p_A > p_B > \frac{\bar{k} + k}{G} \), the use of arm A followed by
shared experimentation until belief \((k^G + \Delta k, k^G)\) is optimal. From belief \((p'_A, p'_B)\) with \(p'_A > \frac{k^G}{G} > p'_B\), only arm \(A\) will ever be used, until belief \((\frac{k^G + \Delta k}{G}, p_B)\).

![Graph](image)

**Figure 3.1: Optimal Experimentation Without Discarding of Arms.**

### 3.5 Optimal Experimentation with Discarding of Arms

This section returns to the problem in which risky arms can be discarded and characterises optimal experimentation. When arms can be discarded, the experimenter can avoid accumulating maintenance costs on inactive arms, yet may hesitate to irreversibly abandon the option of exploiting them. The experimenter has an incentive to bring the option value of inactive arms forward. How this incentive is resolved in the optimal experimentation policy is the subject of the following sections.

#### 3.5.1 When to ‘Go-with-the-Loser’

I first provide necessary conditions which deal with which arms can be discarded, and when. These establish that the patterns of optimal experimentation when arms can be discarded are very simple. The first result is quite natural and states that if an arm is ever discarded it is the losing arm \(B\), and it is discarded as soon as the experimenter no longer intends to use it. That is, there is no value in ‘stringing’ \(B\) along without pulling it, only to discard it later.

**Lemma 3.2.** Consider \((p_A(0), p_B(0))\) and the belief path \((p_A(t), p_B(t))\), under optimal experimentation.

1. Suppose there exist \(\hat{t}\) and \(\epsilon > 0\) such that \(\varphi^*(p_A(t), p_B(t)) \neq (1,1)\) for all \(t \in [\hat{t}, \hat{t} + \epsilon)\) and \(\beta^*(p_A(t), p_B(t)) \neq S\) for almost all \(t \in [\hat{t}, \hat{t} + \epsilon).\) Then, without loss of generality, \(\varphi^*(p_A(t), p_B(t)) = (1,0)\) for all \(t \in [\hat{t}, \hat{t} + \epsilon).\)
Suppose there exists \( \hat{t} \) such that \( \beta^*(p_A(t), p_B(t)) = \beta^*(p_A(t), p_B(t)) = 1 \) for all \( t \in [0, \hat{t}] \) and that there exists \( t' < \hat{t} \) such that \( \varphi^*(p_A(t), p_B(t)) = (1, 1) \) for almost all \( t \in [0, t') \). Then \( \varphi^*(p_A(t), p_B(t)) = (1, 1) \) for all \( t \in [t', \hat{t}] \).

The proof of Lemma 3.2 is simple. If the better arm \( A \) were discarded before \( B \), and \( B \) was used after having discarded \( A \), then inverting the roles of arms \( A \) and \( B \) would increase the experimenter’s payoff. If, on the other hand, arm \( B \) were maintained but never used again, were arm \( B \) to be discarded immediately, discoveries would occur with the same probability and maintenance costs would be avoided for a random time of positive expected length.

The next result derives the precise conditions under which the losing arm \( B \) can be used under optimal experimentation. It shows that whenever arm \( A \) is strictly better than arm \( B \) but is not used, then it must be that arm \( B \) is used exclusively for some time and then discarded in the event of failure.

**Lemma 3.3.** Suppose that \( p_A(0) > p_B(0) \) and consider the belief path \((p_A(t), p_B(t))\) under optimal experimentation. Suppose that there exists \( \hat{t} > 0 \) such that \( \beta^*(p_A(t), p_B(t)) \neq 1 \) for almost all \( t \in [0, \hat{t}] \) and \( \varphi^*(p_A(t), p_B(t)) = (1, 1) \) for all \( t \in [0, \hat{t}] \). Then there exists \( t^* \) such that \( \hat{t} \leq t^* \), \( \beta^*(p_A(t), p_B(t)) = 0 \) for almost all \( t \in [0, t^*), \varphi^*(p_A(t), p_B(t)) = (1, 1) \) for all \( t \in [0, t^*] \) and \( \varphi^*(p_A(t^*), p_B(t^*)) = (1, 0) \).

Lemma 3.3 answers the question at the head of the section regarding how the experimenter manages to bring the option value of inactive arms forward to avoid maintenance costs. Lemma 3.3 relies on Lemma 3.1 which shows that if both arms are maintained, optimal experimentation requires that the better arm be used. Hence, any period of experimentation in which arm \( B \) is used and arm \( A \) is maintained must end by arm \( B \) being discarded. That is, whenever arm \( B \) is used, it is given a ‘last chance’ before it is culled. By Lemma 3.1 the losing arm \( B \) ‘should’ be the inactive arm. To avoid both maintaining arm \( B \) while using arm \( A \) and sacrificing its option value by discarding it, the experimenter must inefficiently use arm \( B \) for a short period while maintaining the better arm \( A \). However, this is done with an eye to discarding arm \( B \) quickly in the absence of success. Hence to bring the option value of arm \( B \) forward, the experimenter faces a trade-off between inefficient assignment and the maintenance cost savings of experimenting with the better arm \( A \) in the absence of arm \( B \).

### 3.5.2 Discarding Boundary

Together, Lemmas 3.1, 3.2 and 3.3 imply that given \( p_A(0) > p_B(0) \), either \( i \) arm \( B \) is discarded immediately, \( ii \) arm \( B \) is used until it is discarded in favour of arm \( A \), \( iii \) arm \( A \) is used for some time, a switch to \( B \) occurs and \( B \) is used until it is discarded, or \( iv \) arm \( A \) is used until the 45-degree line is reached, followed by shared experimentation. From shared experimentation, either one or both arms are discarded, or arm \( B \) is used exclusively until it is discarded.

This section focuses on the experimenter’s discarding decision. To this end, suppose that \( p_A > p_B \) and that \((p_A, p_B)\) lies in an open region of beliefs in which arm \( B \) is used. Then, by Lemma 3.3 arm \( B \) will be used until it is discarded in the event of failure at some belief \( p_B^* \), and as was shown
in Section 3.3, the experimenter’s payoff satisfies
\[ u(p_A, p_B) = v_B(p_B; C_B(p_A)), \]
for some constant of integration \( C_B(p_A) \). The experimenter’s payoff at belief \( (p_A, p_B^*) \) once arm \( B \) has been discarded is given by \( u_A(p_A) \) and is independent of \( p_B \). Recall that \( u_A(p_A) \) is the optimal payoff to a single risky arm with cost \( k \) and belief \( p_A \). Hence value-matching and smooth-pasting conditions at the discarding belief \( (p_A, p_B^*) \) yield
\[ v_B(p_B^*; C_B(p_A)) = u_A(p_A), \quad \text{and} \]
\[ \frac{\partial}{\partial p_B} v_B(p_B^*; C_B(p_A)) = \frac{\partial}{\partial p_B} u_A(p_A) \]
\[ = 0. \tag{3.10} \]
Rearranging (3.10) yields
\[ C_B(p_A) \left( 1 - \frac{p_B^*}{p_B^*} \right) \frac{\dot{\tau}}{G} = \frac{G(r + k + \bar{k})}{r(r + G)} \left( \frac{p_B^*G}{p_B^*G + r} \right). \tag{3.11} \]
(3.11), along with (3.9), yield that \( p_B^* \) solves
\[ u_A(p_A) = \frac{p_B^*G - (\bar{k} + k)}{p_B^*G + r} \tag{3.12} \]
Note that the right-hand side in (3.12) is the payoff to an arm that is known to be of type Good but has a success rate \( p_B^*G \) and associated experimentation cost \( \bar{k} + k \), since the flow payoff of such an arm is \( p_B^*G - (\bar{k} + k) \), while the expected wait until a success is \( p_B^*G \) and hence the effective discount is \( p_B^*G + r \). Hence (3.12) states that at a cutoff belief \( (p_A, p_B^*) \) at which arm \( B \) is discarded, the experimenter is indifferent between its payoff to arm \( A \) in the absence of arm \( B \) and a riskless arm with a payoff equal to arm \( B \)’s flow payoff at the belief \( p_B^* \) at which it is discarded. Note that (3.12) also implies that given arm \( A \) with belief \( p_A \), there is a unique candidate cutoff state \( (p_A, p_B^*) \) at which arm \( B \) is discarded.
The preceding discussion does not say whether arm \( B \) is actually ever used when \( p_A > p_B^* \), just when it should be discarded were it to be used. Define mapping \( p_B^* : [0, 1] \to [0, 1] \) such that \( p_B^*(p_A) \) is the solution to (3.12) if it exists, and is equal to \( p_A \) otherwise. Clearly, a necessary condition for arm \( B \) to be used before arm \( A \) is that there exists belief \( p_A \) such that \( p_B^*(p_A) < p_A \). This occurs whenever, for fixed \( p_A \), there exists \( p_B \) such that \( u_A(p_A) < \frac{p_B^*G - (\bar{k} + k)}{p_B^*G + r} \). To this end, consider the mapping \( p_B \mapsto p_B^G - (\bar{k} + k) \). It is straightforward to verify that this mapping is increasing and concave. Hence, for fixed \( p_A \), the inequality \( u_A(p_A) \leq \frac{p_B^*G - (\bar{k} + k)}{p_B^*G + r} \) is easiest to satisfy.
for \( p_B = p_A \). Note that

\[
\lim_{{p_A \to 1}} \left[ u_A(p_A) - \frac{p_AG - (\bar{k} + k)}{p_AG + r}\right] = \frac{G - \bar{k}}{G + r} - \frac{G - (\bar{k} + k)}{G + r} > 0.
\] (3.13)

That is, as the probability that arm \( A \) is of type \( \text{Good} \) approaches 1, the payoff to a single risky arm with cost \( \bar{k} \) approaches the payoff to an arm known to be of type \( \text{Good} \) with cost \( \bar{k} \) and success rate \( G \). This dominates the payoff to an arm known to be of type \( \text{Good} \) with cost \( \bar{k} + \bar{k} \) and success rate \( G \). Furthermore,

\[
\lim_{{p_A \not\to \frac{\bar{k} + k}{G}}} \left[ u_A(p_A) - \frac{p_AG - (\bar{k} + k)}{p_AG + r}\right] = v_A(\frac{\bar{k} + k}{G}; \bar{C}_A)
\] > 0.
\] (3.14)

That is, as \( p_A \) approaches the quitting belief \( \frac{\bar{k} + k}{G} \) (for a risky arm with cost \( \bar{k} + \bar{k} \)), the payoff to an arm known to be of type \( \text{Good} \) with cost \( \bar{k} + \bar{k} \) and success rate \( p_AG \) approaches 0, while the payoff to a risky arm with cost \( \bar{k} \) is strictly positive, since its own quitting belief is \( \frac{\bar{k} + k}{G} \). This implies that when arms can be discarded, contrary to the results of Proposition 3.1, the experimenter will never reach the quitting belief \( \frac{\bar{k} + k}{G} \) with both arms maintained. \( (3.13) \) and \( (3.14) \) show that discarding arm \( B \) is always best either when arm \( A \) is almost sure to be good or when the belief in arm \( A \) approaches the quitting belief for the case in which arms cannot be discarded. Thus, if arm \( B \) is ever used, beliefs must lie ‘between’ \( (\frac{\bar{k} + k}{G}, \frac{\bar{k} + k}{G}) \) and \( (1, 1) \).

The following lemma collects some of the previous discussion to give a necessary and sufficient condition for the existence of a set of beliefs with positive Lebesgue measure in which arm \( B \) is used before arm \( A \).

**Lemma 3.4.** One of the two following cases must obtain. Either

i. \( u_A(p_A) > \frac{p_AG - (\bar{k} + k)}{p_AG + r} > 0 \) for all \( p_A \), and for almost all \((p_A, p_B)\), \( \varphi^*(p_A, p_B) = (1, 0) \), or

ii. there exist \( \bar{p}_A > p_A \) such that \( u_A(p_A) \leq \frac{p_AG - (\bar{k} + k)}{p_AG + r} \) if and only if \( p_A \in [p_A', \bar{p}_A] \). Then for almost all \((p_A, p_B)\) with \( \varphi^*(p_A, p_B) = (1, 1) \), \( \beta^*(p_A, p_B) = 0 \) only if \( p_A \in [p_A', \bar{p}_A] \) and \( p_B \in [p_B^*(p_A), p_A] \). Furthermore, the set \( \{(p_A, p_B) : \varphi^*(p_A, p_B) = (1, 1) \text{ and } \beta^*(p_A, p_B) = 0\} \) has positive Lebesgue measure.

Figure 3.2 illustrates the discarding boundary when the condition of part ii of Lemma 3.4 obtains. Define

\[
\mathcal{P}_M = \{(p_A, p_B) : p_A \geq p_B, p_B \geq p_B^*(p_A)\},
\]

which is the set of beliefs which is inside the discarding boundary. That is, \( \mathcal{P}_M \) is the maintenance region, the set of beliefs inside which arm \( B \) is never discarded. Further define

\[
\mathcal{P}_D = \{(p_A, p_B) : p_A \geq p_B\} \setminus \mathcal{P}_M,
\]
which is the set of beliefs outside the discarding boundary. This is the discarding region, in which arm \( B \) can be discarded immediately or maintained but never used. It is easily verified that the boundary separating \( \mathcal{P}_M \) from \( \mathcal{P}_D \) is concave. From state \((p_A, p_B)\), if it is optimal to use arm \( B \), then \( B \) must be used until \((p_A, p_B^*(p_A))\), after which \( B \) is discarded and \( A \) must be used until \( p_A^* = \frac{\bar{k}}{G} \), the quitting belief with a single risky arm.

![Discarding Boundary](image)

Figure 3.2: Discarding Boundary.

Part \( ii \) of Lemma 3.4 states that the set of beliefs for which arm \( B \) is used before arm \( A \) is nonempty whenever arm \( B \) is not always immediately discarded. The set \( \mathcal{P}_B \) in Figure 3.2 illustrates the beliefs for which the argument in the proof applies, which are those beliefs close to \((p_A', p_A)\) and \((\bar{p}_A, \bar{p}_A)\). For any beliefs \((p_A, p_B)\), the payoff to using arm \( A \) (or to shared experimentation) and maintaining \( B \) is at most the payoff to using an arm known to be of type \emph{Good} with success rate \( p_A G \) and experimentation cost \( \bar{k} + k \). However, near \((\bar{p}_A, \bar{p}_A)\), discarding arm \( B \) yields a payoff close to the payoff to an arm known to be of type \emph{Good} with success rate \( \bar{p}_A G \) but reduced experimentation cost \( \bar{k} \). Hence near \((\bar{p}_A, \bar{p}_A)\), discarding arm \( B \) yields strictly higher payoffs than either using arm \( A \) (or shared experimentation). Yet, for beliefs inside \( \mathcal{P}_M \), using arm \( B \) until the discarding boundary yields strictly higher payoffs than discarding it. By continuity, using arm \( B \) yields strictly higher payoffs than using arm \( A \) (or shared experimentation). The same argument applies around \((p_A', p_A)\). Intuitively, around \((\bar{p}_A, \bar{p}_A)\) and \((p_A', p_A)\), the experimenter has already decided that it no longer wishes to maintain both arms in the long term. Yet arm \( B \) may still be considered to be of some value, which the experimenter may want to exploit before discarding it.

### 3.5.3 Optimal Policy

The previous sections derived the possible patterns of optimal experimentation when arms can be discarded. In this final section, I complete the characterisation of optimal experimentation by
showing when it entails these various patterns. A useful starting point is to focus on beliefs on the 45-degree line. Lemma 3.3 states that if \( p_A = p_B \), then either there is shared experimentation or arm B is used until it is discarded. By Lemma 3.4, shared experimentation always ends with the selection of arm B (in order to discard it). The next lemma addresses the question of how many exit points from shared experimentation to arm B there can be on the 45-degree line, and shows that only a single belief \((p, p)\) can satisfy both the value-matching and smooth-pasting conditions associated to such an exit.

**Lemma 3.5.** Suppose there exists \( p' \prec p'' \) such that \( \beta^*(p, p) = \frac{1}{2} \) for almost all \( p \in [p', p''] \). Then there exists \( p \) and \( p \) such that \( p \leq p' \), \( p'' \leq p \) and \( \beta^*(p, p) = \frac{1}{2} \) for almost all \( p \in P \) if and only if \( P \subset \{p, p\} \).

Lemma 3.5 implies that if the belief \( p \), derived explicitly in Appendix 3.7, is such that \( p \in (P_A, P_B) \) and if \( v_{AB}(p; C_{AB}(p)) > v_B(p; C_B(p)) \) for a set of beliefs \((p, p)\) such that \( p \in (p, p + \epsilon) \) for some \( \epsilon > 0 \), then there exists \( p \in (P_A, P_B) \) such that \( \beta(p, p) = \frac{1}{2} \) for almost all \( p \in (p, p) \) and \( \beta(p, p) = 0 \) for almost all \( p \in (P_A^1, P_B^1) \). That is, optimal experimentation calls for shared experimentation only for those beliefs \((p, p)\) with \( p \in (P, P) \).

To complete the characterization of the optimal experimentation policy, I will define two sets of beliefs, \( P_B \subset P_M \) and \( P_A \) via a backwards induction argument. Intuitively, these sets will correspond to the regions of the state space in which arms A and B are used under optimal experimentation. In the following, assume that the conditions of Lemma 3.5 are met and that there exists a (unique) portion of the 45-degree line \((p, P)\) for which shared experimentation is optimal. The arguments that follow apply in a straightforward way when this is not the case.

First, let
\[
P_B^1 = \left\{ (p_A, p_B) \in P_M : p_A \in [P_A, P] \cup [P, P_A] \right\}.
\]
By Lemma 3.3, it must be that given \( p \in [P_A, P] \cup [P, P_A] \), \( \beta^*(p, p_B) = 0 \) for almost all \( p_B \in (p, p_B(p)) \). That is, if arm B is used from the 45-degree line, it cannot be that a transition to arm A occurs before arm B is discarded.

Second, consider
\[
P_M \setminus P_B^1 = \left\{ (p_A, p_B) \in P_M : p_A \in [P, P], p_B \in [P_B^1(p, P), P_B^1(p, P)] \right\},
\]
the set of beliefs in the maintenance region that have not been attributed to \( P_B^1 \). By Lemma 3.3 from such beliefs, an optimal policy will either use arm B immediately until it is discarded, or use arm A either until beliefs reach the 45-degree or until a switch to arm B occurs. Define \( v_A(p_A; C_A(p_B; p'_A)) \) to be the payoff to the experimenter in state \((p_A, p_B) \in P_M \setminus P_B^1\) were it to pull arm A until belief \( p'_A \in [\max\{p_B, P\}, P_A] \), and then switch to arm B until discarding belief \( p_B(p') \). Hence the constant of integration \( C_A(p_B; p'_A) \) depends on the belief \( p_B \) and on the switching belief \( p'_A \), but not on \( p_A \). Similarly, if \( p_B > P \), define \( v_A(p_A; C_A^{P5}(p_B)) \) to be the payoff to the experimenter in state \((p_A, p_B) \) were it to pull arm A until it reaches the 45-degree line, after which it shares experimentation until joint belief \( p \). If \( p_B \leq P \), then define \( v_A(p_A; C_A^{P5}(p_B)) = v_A(p_A; C_A(p_B; p)) \). Note that \( v_A(p_A; C_A(p_B; p'_A)) \geq v_A(p_A; C_A(p_B; p'_A)) \) if and only if \( C_A(p_B; p'_A) \geq C_A(p_B; p'_A) \) and \( v_A(p_A; C_A(p_B; p'_A)) \geq v_A(p_A; C_A^{P5}(p_B)) \) if and only if
When arms can be discarded, the following admissible Markov strategy is optimal. Conjecture 3.2. Markov assignment. Let 

\[ P_2^A = \left\{ (p_A, p_B) \in P_M \setminus P_1^B : \max \left\{ \max_{p' \in [\max\{p_B, p_A\}],p_A]} C_A(p_B; p') \right\} \leq C_A(p_B; p_A) \right\}, \]

and let \( P_B = P_1^B \cup P_2^B \). Finally, let \( P_A = P_M \setminus P_B \). Hence, all the beliefs in \( P_M \) have been attributed either to \( P_B \) or to \( P_A \).

Third, consider the beliefs in \( P_D \), those outside the discarding boundary. By Lemma 3.2, it must be that \( \phi^*(p_A, p_B) = (1, 0) \) for all \( (p_A, p_B) \in P_D \) that are not in the set \( \{(p_A, p_B) : \beta^*(p_A, p_B(p_A)) = 1\} \). That is, if arm \( B \) is maintained, it must be that it will not be discarded once beliefs reach the discarding boundary. Let

\[ Q = \left\{ (p_A, p_B) \in P_D : (p_A, p_B(p_A)) \in P_1^A \right\}. \]

That is, \( Q \) is the set of beliefs in the discarding region such that were \( A \) to be used and \( B \) maintained until the discarding bound, \( B \) would also be maintained when beliefs cross into \( P_M \). For any \( (p_A, p_B) \in Q \), define \( p_A^*(p_B) = \sup\{p_A : (p_A, p_B) \in P_A^1\} \). Furthermore, define

\[ P_A^2 = \left\{ (p_A, p_B) \in Q : v_A(p_A; C_A(p_B; p_A^*(p_B))) > v_A(p_A; C_A) \right\}. \]

Finally, let \( P_A = P_A^1 \cup P_A^2 \). The next proposition collects these various results into an optimal Markov assignment.

**Proposition 3.2.** When arms can be discarded, the following admissible Markov strategy is optimal.

\[
\beta^*(p_A, p_B) = \begin{cases} 0 & \text{if } (p_A, p_B) \in P_B, \\ 1 & \text{if } (p_A, p_B) \in P_A, \\ \frac{1}{2} & \text{if } p_A = p_B \text{ and } p \in (p, \bar{p}), \\ S & \text{otherwise.} \end{cases}
\]

\[
\beta^*(p_A) = \begin{cases} 0 & \text{if } p_B \geq \frac{k}{\tau}, \\ S & \text{otherwise.} \end{cases}
\]

\[
\phi^*(p_A, p_B) = \begin{cases} (1, 1) & \text{if } (p_A, p_B) \in P_A \cup P_B, \\ (1, 0) & \text{otherwise.} \end{cases}
\]

\[
\phi^*(p_A) = \begin{cases} 1 & \text{if } p_B \geq \frac{k}{\tau}, \\ 0 & \text{otherwise.} \end{cases}
\]

Figure 3.3 provides an illustration of the sets \( P_A \) and \( P_B \). The figure as drawn assumes that \( P_A \) is convex, which need not necessarily be the case. However, note that the boundary between sets
$\mathcal{P}_A$ and $\mathcal{P}_B$ must always be downward-sloping, else this would violate Lemma 3.3.

When shared experimentation occurs on the 45-degree line, which by continuity implies that $\mathcal{P}_A$ is nonempty, the reversal of the ‘go-with-the-winner’ property exhibits a noteworthy non-monotonicity. For fixed $p_B$, arm B is used first in state $(p_A, p_B)$ only for intermediate values of $p_A$. Suppose that arm A is believed to be of type Good with very high probability, and hence to succeed quickly. In that case, it is best for the experimenter to discard project B immediately and exploit project A. Suppose now that $p_A$ is substantially higher than $p_B$, yet arm B is still believed to be of type Good with relatively high probability. The optimal experimentation rule in the absence of maintenance costs would use arm A until its belief dropped to $p_B$, after which experimentation would be shared. However, since arm A is thought fairly likely to succeed, maintaining the option value of arm B is very costly, since this value can be realised only after arm A has failed for a long time. In this case, the optimal policy with maintenance costs starts by using arm B, discards it after a short period of failure, and moves to the clearly more promising arm A. If instead beliefs $p_A$ and $p_B$ are close to each other, it may still be optimal to develop projects as in the optimal policy without maintenance costs. In this case, discarding arm B immediately or giving it its ‘last chance’ is more costly, since the realisation of its option value is not so far away and arm A is not the clear-cut superior arm.

3.6 Conclusion

The standard approach to experimentation has been to assume that when currently occupied by other projects, keeping the option of researching various alternatives at later dates is costless. That keeping options open can involve maintenance costs is natural in many settings. This paper shows
that such costs generate new trade-offs for experimenters by giving them incentives to manage the timing of the realisation of inactive alternatives’ option values and have important implications for optimal experimentation policies.

While I have focused on the simple and tractable exponential bandit problem, it is not unreasonable to expect that my main arguments extend to more general bandit settings. If they do, this would show that maintenance costs are indeed much more tractable than switching costs. Note that more generally, the arguments used in the paper are based on finite backwards induction, where the recursion is on the set of maintained arms. At each step of the recursion, the arguments rely on maintained arms’ Gittins indices. This is made clear by the common structure of Lemmas 3.1 and 3.3 and the original ‘interchange argument’ of Gittins (1979) that establishes the optimality of index policies in standard bandit problems.

3.7 Appendix

Proof of Lemma 3.1. Suppose that \( p_A(0) > p_B(0) \), and consider the belief path under optimal experimentation \((p_A(t), p_B(t))\). The first step is to show that if there exists \( \hat{t} > 0 \) and \( \hat{T} \subset [0, \hat{t}) \) such that \( \hat{T} \) has positive Lebesgue measure and \( \beta^*(p_A(t), p_B(t)) \neq 1 \) for all \( t \in \hat{T} \), then \( \beta^*(p_A(t), p_B(t)) = 0 \) for almost all \( t \in \hat{T} \). Suppose instead that \( \beta^*(p_A(t), p_B(t)) = \alpha(t) \in (0, 1) \) for all \( t \in \hat{T} \). Let \( T_A = \int_0^\hat{t} \alpha(t) dt \). By assumption, \( T_A \in (0, \hat{t}) \).

Given allocation \( \alpha(t) \) for \( t \in [0, \hat{t}) \), and initial beliefs \((p_A(0), p_B(0))\), solving the differential equation for the evolution of beliefs yields that

\[
\begin{align*}
    p_A(t) &= \frac{1}{1 + \frac{1-p_A(0)}{p_A(0)} e^{\int_0^t \alpha(s) ds}}, \quad \text{and} \\
    p_B(t) &= \frac{1}{1 + \frac{1-p_B(0)}{p_B(0)} e^{\int_0^t (1-\alpha(s)) ds}}.
\end{align*}
\]

Belief \( p_A(t) \) depends only on the cumulative experimentation on arm \( A \) up to time \( t \), \( \int_0^t \alpha(s) ds \), and not on when this experimentation occurred within the interval \([0, t]\).

Consider an alternative admissible Markov assignment \( \hat{\beta} \) such that

\[
\hat{\beta}(p_A(t), p_B(t)) = \begin{cases} 
    1 & \text{for all } t \in (0, \hat{t} - T_A], \\
    0 & \text{for all } t \in (\hat{t} - T_A, \hat{t}),
\end{cases}
\]

with \( \hat{\beta} = \beta^* \) otherwise. Then \((\hat{p}_A(\hat{t}), \hat{p}_B(\hat{t})) = (p_A(\hat{t}), p_B(\hat{t}))\), where \((\hat{p}_A(t), \hat{p}_B(t))\) is the belief path associated with \( \hat{\beta} \). Hence, the payoffs following \( \hat{t} \) are the same under both assignments. That is, \( v(\hat{\beta}, \beta^*; p_A(\hat{t}), p_B(\hat{t})) = u(p_A(\hat{t}), p_B(\hat{t})) \). Furthermore, conditional on \((p_A(0), p_B(0))\), the probability that no success occurs until \( \hat{t} \) is the same under \( \beta \) and \( \hat{\beta} \).

Let \( \tau_{\beta^*} \) (respectively \( \tau_{\hat{\beta}} \)) be the random arrival time of a success under assignment \( \beta^* \) (respectively \( \hat{\beta} \)) in time interval \([0, \hat{t}]\). Then \( Pr[\tau_{\hat{\beta}} \leq t | p_A(0), p_B(0)] > Pr[\tau_{\beta^*} \leq t | p_A(0), p_B(0)] \) for all \( t \in (0, \hat{t}) \), that is, \( \tau_{\hat{\beta}} \) is higher than \( \tau_{\beta^*} \) in the sense of first order stochastic dominance. By discount-
ing, the experimenter’s payoff is decreasing in the arrival time of a success, and hence \( \hat{\beta} \) yields a strictly higher expected payoff than \( \beta^* \) in \([0, \hat{t}]\), or

\[
\int_0^\hat{t} [1 - (k + k)]e^{-r\hat{t}}\hat{\mu}(d\tau_{\hat{\beta}}) > \int_0^\hat{t} [1 - (k + k)]e^{-r\hat{t}}\mu^*(d\tau_{\hat{\beta^*}}),
\]

where \( \hat{\mu} \) and \( \mu^* \) are the distributions of \( \tau_{\hat{\beta}} \) and \( \tau_{\hat{\beta^*}} \), respectively. Hence,

\[
v(\hat{\beta}; p_A(\hat{t}), p_B(\hat{t})) = \int_0^\hat{t} [1 - (k + k)]e^{-r\hat{t}}\hat{\mu}(d\tau_{\hat{\beta}}) + Pr[\tau_{\hat{\beta}} > \hat{t} | p_A(0), p_B(0)]u(p_A(\hat{t}), p_B(\hat{t})]
\]

\[
\geq \int_0^\hat{t} [1 - (k + k)]e^{-r\hat{t}}\mu^*(d\tau_{\hat{\beta^*}}) + Pr[\tau_{\hat{\beta^*}} > \hat{t} | p_A(0), p_B(0)]u(p_A(\hat{t}), p_B(\hat{t})]
\]

\[
= u(p_A(0), p_B(0)),
\]

a contradiction. Hence it must be that \( \alpha(t) = 0 \) for almost all \( t \in [0, \hat{t}] \).

That is, the previous argument shows that if arm \( A \) is not used, it must be that arm \( B \) is used exclusively. Since in that case \( p_A(t) > p_B(t) \) for all \( t > 0 \), the previous argument also ensures that arm \( B \) is used until experimentation ceases, which must occur at time \( t^* \) such that \( p_B(t^*) = \frac{k + k}{u} \).

By mimicking this strategy with arm \( A \) instead of \( B \), that is, using arm \( A \) until belief \( p_A^* = \frac{k + k}{u} \) and then moving permanently to \( S \), the experimenter’s payoff at time 0 would be higher. That is, consider alternative strategy \( \hat{\beta} \) such that

\[
\hat{\beta}(p_A, p_B) = \begin{cases} 
1 & \text{if } p_A > \frac{k + k}{u} \\
0 & \text{otherwise.}
\end{cases}
\]

Then

\[
v(\beta; p_A(0), p_B(0)) = v_A(p_A(0); C_A)
\]

\[
\geq v_B(p_B(0, C_A))
\]

\[
= u(p_A(0), p_B(0)),
\]

a contradiction. \( C_A \) is the constant of integration for the optimal stopping problem with a single risky arm and direct cost \( k + k \). Hence, it must be that \( \alpha(t) = 1 \) for all \( t \) such that \( p_A(t) > p_B(t) \).

The same argument can be applied if \( p_A(0) = p_B(0) \) to show that experimentation is shared until it ceases, i.e., \( \beta^*(p_A(t), p_B(t)) = \frac{1}{2} \) for all \( t \) such that \( \varphi(p_A(t), p_B(t)) \).

\[ \square \]

**Proof of Proposition 3.1** Most of the result was proved in the text. All that remains is to show that the assignment \( \beta^* \) is admissible. Clearly, given any \( t^*, t^{**} \) and \( t' \) such that \( 0 \leq t^* \leq t^{**} \) and \( t' \geq 0 \),
any strategy \((a, \phi)\) of the form

\[
a(t) = \begin{cases} 
1 & \text{if } t < t^*, \\
\frac{1}{2} & \text{if } t \in [t^*, t^{**}), \\
S & \text{if } t \geq t^{**},
\end{cases}
\]

\[
\phi(t) = \begin{cases} 
(1, 1) & \text{if } t < t', \\
(0, 0) & \text{if } t \geq t',
\end{cases}
\]

(3.15)

is admissible. Furthermore, given any \((p_A, p_B)\), there exist \(t^*, t^{**}\) and \(t'\) such that \(0 \leq t^* \leq t^{**}\) and \(t' \geq 0\) such that a strategy \((a, \phi)\) defined as in (3.15) is such that

\[
a(t) = \beta^*(p_A(t), p_B(t), \phi_A(t), \phi_B(t)),
\]

\[
\phi_A(t) = \varphi_A(p_A(t), p_B(t), \phi_A(t), \phi_B(t)),
\]

\[
\phi_B(t) = \varphi_B(p_A(t), p_B(t), \phi_A(t), \phi_B(t)),
\]

and hence Markov strategy \((\beta^*, \varphi^*)\) is admissible.

\[\square\]

Proof of Lemma 3.2 For part \(i\), suppose there exists \(\hat{t}\) and \(\epsilon > 0\) such that \(\varphi^*(p_A(t), p_B(t)) \neq (1, 1)\) and \(\beta^*(p_A(t), p_B(t)) \neq S\) for almost all \(t \in [\hat{t}, \hat{t} + \epsilon)\). Then one arm is discarded on the equilibrium path. Let \(t^* = \inf\{t < \hat{t} : \varphi^*(p_A(t), p_B(t)) \neq (1, 1)\}\). If \(\varphi^*(p_A(t^*), p_B(t^*)) = (0, 1)\), then since \(\beta^*(p_A(t^*), p_B(t^*)) \neq S\) for almost all \(t \in [\hat{t}, \hat{t} + \epsilon)\), it must be that \(\beta^*(p_A(t^*), p_B(t^*)) = 0\) for almost all \(t \in [\hat{t}, \hat{t} + \epsilon)\). Consider a Markov strategy \((\beta', \varphi')\) such that

\[
\varphi'(p_A, p_B) = (1, 0)
\]

for all \((p_A, p_B)\) such that \(\varphi^*(p_A, p_B) = (0, 1)\),

\[
\beta'(p_A(t), p_B(t)) = 1
\]

for all \(t > t^*\) for which \(\beta^*(p_A(t), p_B(t)) = 0\),

with \((\beta', \varphi') = (\beta^*, \varphi^*)\) otherwise. Under \((\beta', \varphi')\), \(p_A'(t) \geq p_B'(t)\) for all \(t > t^*\) by the assumption of symmetric strategies, and hence for all \(t > t^*\) such that \(\beta^*(p_A(t), p_B(t)) = 0\),

\[
v(\beta', \varphi'; p_A(t), p_B(t)) = v_A(p_A(t); \hat{C}_A) \\
\geq v_B(p_B(t); \hat{C}_A) \\
= v_B(p_B(t); \hat{C}_B) \\
= u(p_A(t), p_B(t)).
\]

If the inequality is strict, this yields the required contradiction, while if it holds with equality, it is without loss of generality to discard arm \(B\) instead of arm \(A\).

For part \(ii\), suppose that there exists \(\hat{t}\) such that \(\beta^*(p_A(t), p_B(t)) = 1\) for almost all \(t \in [0, \hat{t})\) and that there exists \(t' < \hat{t}\) such that \(\varphi^*(p_A(t), p_B(t)) = (1, 1)\) for almost all \(t \in [0, t')\), but that \(\varphi^*(p_A(t''), p_B(t'')) \neq (1, 1)\) for some \(t'' \in (t', \hat{t})\). By part \(i\), \(\varphi^*(p_A(t''), p_B(t'')) = (1, 0)\). Consider Markov strategy \((\beta', \varphi')\) such that \(\varphi'(p_A(0), p_B(0)) = (1, 0)\), with \((\beta', \varphi') = (\beta^*, \varphi^*)\) otherwise.
Then we can write
\[ v(\beta', \varphi'; p_A(0), p_B(0)) = v_A(p_A(0); C'_A) + p_A(0) \frac{k}{r + G + \frac{k}{r}}, \]
and
\[ u(p_A(t), p_B(t)) = v_A(p_A(0); C_A), \]
where the constants of integration \( C_A \) and \( C'_A \) are determined at beliefs \((p_A(t''), p_B(t''))\) at which
\[ v_A(p_A(t''); C_A) = v_A(p_A(t''); C'_A) = u(p_A(t''), p_B(t'')). \]
Hence
\[ C'_A = C_A - \frac{p_A(t'') \frac{k}{r + G + \frac{k}{r}} + (1 - p_A(t'')) \frac{k}{r}}{(1 - p_A(t''))^{\frac{k}{r}} (1 - p_A(t''))}, \]
and
\[ v(\beta', \varphi'; p_A(0), p_B(0)) = v_A(p_A(0); C_A) \]
\[ - \frac{(1 - p_A(0))^{\frac{k}{r}} (1 - p_A(0))}{(1 - p_A(t''))^{\frac{k}{r}} (1 - p_A(t''))} \left[ p_A(t'') \frac{k}{r + G + \frac{k}{r}} + (1 - p_A(t'')) \frac{k}{r} \right] \]
\[ + p_A(0) \frac{k}{r + G + \frac{k}{r}} + (1 - p_A(0)) \frac{k}{r} > v_A(p_A(0); C_A). \]
The inequality follows since \( p_A(t'') < p_A(0) \).

\[ \square \]

**Proof of Lemma 3.3** Lemma 3.1 and part i of Lemma 3.2 imply that if there exists \( \hat{t} > 0 \) such that \( \beta^*(p_A(t), p_B(t)) \neq 1 \) for almost all \( t \in [0, \hat{t}] \) and \( \varphi^*(p_A(t), p_B(t)) = (1, 1) \) for all \( t \in [0, \hat{t}] \), then it must be that \( \beta^*(p_A(t), p_B(t)) = 0 \) for almost all \( t \in [0, \hat{t}] \) and that if there exists \( t^* > \hat{t} \) such that \( \beta^*(p_A(t^*), p_B(t^*)) > 0 \), then by part i of Lemma 3.2 it must be that \( \varphi^*(p_A(t^*), p_B(t^*)) \in \{(1, 0), (0, 0)\} \). That is, if arm \( A \) is not pulled it must be that arm \( B \) is, and the experimenter cannot go back to arm \( A \) without discarding arm \( B \). Since the experimenter must eventually discard \( B \) if \( p_B \) gets close to 0, it only remains to be shown that \( \varphi^*(p_A(t^*), p_B(t^*)) = (1, 0) \), that is, that the experimenter will discard \( B \) in favour of \( A \) at \( t^* \). This follows by part ii of Lemma 3.2.

\[ \square \]

**Proof of Lemma 3.4** First, \( u_A(p_A) \) is increasing and convex in \( p_A \). Also, since the mapping \( p_A \mapsto \frac{p_A G - (\frac{k}{r} + k)}{p_A G + r} \) is increasing and concave, then by (3.13) and (3.14) either \( u_A(p_A) > \frac{p_A G - (\frac{k}{r} + k)}{p_A G + r} > 0 \) for all \( p_A \) or there exist \( \bar{p}_A > p_A \) such that \( u_A(p_A) \leq \frac{p_A G - (\frac{k}{r} + k)}{p_A G + r} \) if and only if \( p_A \in [\frac{p_A G - (\frac{k}{r} + k)}{p_A G + r}, \bar{p}_A] \), where \( p_A \) and \( \bar{p}_A \) are the only two solutions to \( u_A(p_A) = \frac{p_A G - (\frac{k}{r} + k)}{p_A G + r} \).

Now suppose that the conditions of part ii obtain. A first claim is that at \((\bar{p}_A, \bar{p}_A)\), discarding
arm $B$ is strictly preferred to shared experimentation. By Lemma 3.3, if $B$ is not discarded then $\beta^*(\bar{p}_A, \bar{p}_A) = \frac{1}{2}$ and the beliefs go down the 45-degree line until some belief $(p^*, p^*)$, and hence the experimenter’s payoffs satisfy $u(p_A, p_B) = v_{AB}(\bar{p}_A; C_{AB}(p^*))$. $v_{AB}$ itself satisfies

$$v_{AB}(\bar{p}_A; C_{AB}(p^*)) = \frac{\bar{p}_A G - (\bar{k} + \bar{k})}{(r + \bar{p}_A G)} \frac{G\bar{p}_A(1 - \bar{p}_A)}{2(r + \bar{p}_A G)} v'_{AB}(\bar{p}_A; C_{AB}(p^*))$$

$$< \frac{\bar{p}_A G - (\bar{k} + \bar{k})}{(r + \bar{p}_A G)} = v_A(\bar{p}_A; \bar{C}_A).$$

Hence, by continuity, for states $(p, p)$ with $p < \bar{p}_A$ sufficiently close to $\bar{p}_A$, discarding $B$ is strictly preferred to shared experimentation. A very similar argument shows that discarding $B$ is strictly preferred to using arm $A$ for an open set of states of positive Lebesgue measure $(p_A, p_B)$ with $p_A > p_B$ sufficiently close to $(\bar{p}_A, \bar{p}_A)$. However, within the discarding boundary using arm $B$ (until the boundary) is preferred to discarding it and hence there exists an open region of positive Lebesgue measure around $(\bar{p}_A, \bar{p}_A)$ in which using arm $B$ is optimal. A very similar argument demonstrates the same result for a region around $(\bar{p}_A, \bar{p}_A)$.

\[\square\]

**Proof of Lemma 3.5** By Lemma 3.3 once the experimenter quits shared experimentation, arm $B$ is used, then discarded and replaced with arm $A$. Also, by Lemma 3.4 there exists a belief $\hat{p} > \bar{p}_A$ such that $\beta^*(p, p) = 0$ for almost all $p \in [\bar{p}_A, \hat{p}]$. Suppose there exists $p' > p''$ such that $\beta^*(p, p) = \frac{1}{2}$ for almost all $p \in [p', p'']$, and that the experimenter switches from shared experimentation to arm $B$ at belief $p^* < p''$. Hence the smooth-pasting condition at belief $p^*$ is

$$\frac{\partial}{\partial p_B} u(p^*, p^*) = \frac{1}{2} \frac{\partial}{\partial p} v_{AB}(p^*; C_{AB}(p^*))$$

$$= \frac{\partial}{\partial p_B} v_B(p^*, C_B(p^*)),$$

which, with manipulations, yields that

$$C_{AB} = \frac{C_B(p)}{\left(\frac{1-p}{p}\right)^{\frac{5}{2}} (1 - p)}$$

$$+ p \left[ G_{r - (\bar{k} + \bar{k})} - \frac{G_{r - (\bar{k} + \bar{k})}}{r + G} \right] + (1 - p) \left[ \frac{G_{r - (\bar{k} + \bar{k})}}{r + G} - \frac{G_{r - (\bar{k} + \bar{k})}}{r} - \frac{G_{r + (\bar{k} + \bar{k})}}{r(r + G)} \right]$$

$$(1 - p)^{\frac{5}{2}} p^{\frac{5}{2}} \left[ G_{r + (\bar{k} + \bar{k})} \right].$$

Meanwhile, the value matching condition is

$$v_{AB}(p^*; C_{AB}(p^*)) = v_B(p^*; C_B(p^*)),$$
which, with manipulations, yields that

$$C_{AB} = \frac{C_B(p)}{(1-p)^{\frac{\beta}{(1-p)}}} - \frac{G^2(r + \bar{k} + k)}{(1-p)^{\frac{\beta+1}{(r+G)(r+G+\bar{k})}}}.$$  

Together, these yield that

$$p^* = \frac{2(\bar{k} + k)(r + G)(r + G + \bar{k})}{2(\bar{k} + k)(r + G)(r + G + \bar{k}) + G^2(\bar{k} + k + r)}.$$  \hfill (3.16)

Clearly, $p^* \in [0, 1]$ is unique. Define $p$ to be the unique solution to (3.16).

Proof of Proposition 3.2 Most of the result was proved in the text. All that remains is to show that the assignment $(\beta^*, \varphi^*)$ is admissible, which follows from an argument very similar to that for Proposition 3.1.
References


Key, V. (1958). Politics, parties, and pressure groups.


