SINGULARITY FORMATION IN NONLINEAR HEAT AND MEAN CURVATURE FLOW EQUATIONS

by

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Abstract

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In this thesis we study singularity formation in two basic nonlinear equations in $n$ dimensions: nonlinear heat equation (also known as reaction-diffusion equation) and mean curvature flow equation.

For the nonlinear heat equation, we show that for an important or natural open set of initial conditions the solution will blowup in finite time. We also characterize the blowup profile near blowup time. For the mean curvature flow we show that for an initial surface sufficiently close, in the Sobolev norm with the index greater than $\frac{n}{2} + 1$, to the standard $n$-dimensional sphere, the solution collapses in a finite time $t_*$, to a point. We also show that as $t \to t_*$, it looks like a sphere of radius $\sqrt{2n(t_* - t)}$. 
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Chapter 1

Introduction

This thesis consists of two parts. The first part deals with the nonlinear heat equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \\
u(x, 0) = u_0(x)
\end{cases}
\] (1.1)

for \( p > 1 \). Here \( u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R} \). Its main results are a proof that solutions for certain open set of initial conditions with a single absolute maximum blowup and a characterization of blowup profile.

In the second part we study the behavior of mean curvature flow (MCF) of hypersurfaces in \( \mathbb{R}^{n+1} \) with initial conditions close to spheres. Given an initial simple, closed hypersurface \( S_0 \) in \( \mathbb{R}^{n+1} \) the MCF determines a family \( \{S_t \mid t \geq 0\} \) of closed hypersurfaces in \( \mathbb{R}^{n+1} \), given by immersions \( X(\cdot, t) : \Omega \to \mathbb{R}^{n+1} \), satisfying the following evolution equation:

\[
\frac{\partial X}{\partial t} = -H(X)\nu(X),
\] (1.2)

where \( \Omega \subset \mathbb{R}^{n+1} \) is a fixed \( n \)-dimensional hypersurface, \( \nu(X) \) and \( H(X) \) are the outward unit normal vector and mean curvature at \( X \in S_t \), respectively. Here we show that for an initial surface sufficiently close, in the Sobolev norm with the index greater than \( n/2 + 1 \), to the standard \( n \)-dimensional sphere, the solution collapses in a finite time, \( t_* \), to
a point, $z_*$, approaching exponentially fast the spheres of radii $\sqrt{2n(t_*-t)}$, centered at $z(t)$.

Equation (1.1) arises in the problem of heat flow, or, more generally, in the problems involving diffusion, and is a model for a large class of nonlinear parabolic equations, which are ubiquitous in mathematics and its applications.

The local well-posedness of (1.1) is well known (see, e.g. [8] for the Sobolev spaces $\mathcal{H}^\alpha$, $0 \leq \alpha < 2$). Moreover for some data $u_0(x)$, the solutions $u(x,t)$ might blow up in finite time $t^* > 0$. In what follows a solution $u(x,t)$ is said to blow up at time $t^*$ if it exists in $L^\infty$ for $[0,t^*)$ and $\sup_x |u(x,t)| \to \infty$ as $t \to t^*$. Thus, two key problems about (1.1) are

1. Describe initial conditions for which solutions of Equation (1.1) blow up in finite time;

2. Describe the blow-up profile of such solutions.

It is expected (see e.g. [10]) that the blow-up profile is universal — it is independent of lower power perturbations of the nonlinearity and of initial conditions within certain spaces.

There is rich literature regarding the blow-up problem for Equation (1.1). We review quickly relevant results. Starting with [39], various criteria for blow-up in finite time were derived, see e.g. [39, 8, 16, 31, 65, 66, 74, 76, 82, 33, 37]. For example, if $u_0 \in \mathcal{H}^1 \cap L^{p+1}$ and $\mathcal{E}(u_0) < 0$, where $\mathcal{E}(u)$ is the energy functional for (1.1) given by

$$\mathcal{E}(u) := \int \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1},$$

(1.3)

then it is proved in [65] that $\|u(t)\|_2^2$ blows up in finite time $t^*$. By the observation

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq \|u(t)\|_\infty^{p-1} \|u(t)\|_2^2$$
we have that \( \| u(t) \|_\infty \) blows up in finite time \( t^{**} \leq t^* \) also. (In this paper, we denote the norms in the \( L^p \) spaces by \( \| \cdot \|_p \).)

Blow-up at a single point was studied as early as [86] (see also [37]). The first result on asymptotics of the blow-up for arbitrary dimension \( n \geq 1 \) was obtained in the pioneering paper [43] where the authors show that under the condition

\[
|u(x, t)(t^* - t)\frac{1}{p-1}| \text{ is bounded on } B_1 \times (0, t^*),
\]

where \( B_1 \) is the unit ball in \( \mathbb{R}^n \) centred at the origin, and either \( p \leq \frac{n+2}{n-2} \) or \( n \leq 2 \) and assuming blow-up takes place at \( x = 0 \), one has

\[
\lim_{\lambda \to 0} \lambda^{\frac{2}{p-1}} u(\lambda x, t_* + \lambda^2 (t - t_*)) = \pm \left( \frac{1}{p-1} \right)^{\frac{1}{p-1}} (t^* - t)^{\frac{1}{p-1}} \text{ or 0.}
\]

This result was further improved in several papers (see e.g. [45, 44, 49, 69, 84, 35, 36, 37, 10, 71, 68, 72]). A blow-up solution satisfying the bound (1.4) is said to be of type I. This bound was proven under various conditions in [45, 71, 68, 87, 46]. Furthermore, the limits of \( H^1 \)-blow-up solutions \( u(x, t) \) as \( t \uparrow T \), outside the blow-up sets were established in [49, 34, 69, 84, 35, 36, 37, 10, 72, 32].

For \( p > 1 \), dimension \( n = 1 \), Herrero and Velázquez [50] (see also [35]) proved that if the initial condition \( u_0 \) is continuous, nonnegative, bounded, even and has only one local maximum at 0, and if the corresponding solution blows up, then

\[
\lim_{t \uparrow t^*} (t^* - t)^{\frac{1}{p-1}} u(y((t^* - t) \ln |t^* - t|)\frac{1}{p-1}, t) = (p - 1)^{\frac{1}{p-1}} \left[ 1 + \frac{p - 1}{4p} y^2 \right]^{\frac{1}{p-1}} \quad (1.5)
\]

uniformly on sets \( |y| \leq R \) with \( R > 0 \). Further extensions of this result are achieved in [49, 84, 34, 35].

Later for dimension \( n = 1 \) Bricmont and Kupiainen [10] constructed a co-dimension 2 submanifold, of initial conditions such that (1.5) is satisfied on the whole domain. More specifically, given a small function \( g \) and a small constant \( b > 0 \), they find constants \( d_0 \) and \( d_1 \) depending on \( g \) and \( b \) such that the solution to (1.1) with the datum

\[
u_0^*(x) = (p - 1 + bx^2)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0 + d_1 x}{p - 1 + bx^2} \right)^{\frac{1}{p-1}} + g(x)
\]

(1.6)
has the convergence \((1.5)\) uniformly in \(y \in (-\infty, +\infty)\). The result of \([10]\) was generalized in \([70, 32]\) (see also \([41]\)), where it is shown that there exists a neighborhood \(U\), in the space \(\mathcal{H} := L^{p+1} \cap H^1\), of \(u_0^*\), given in \((1.6)\), such that if \(u_0 \in U\), then the solution \(u(x,t)\) blows up in a finite time \(t^*\) and satisfies \((1.5)\) for \(x \in \mathbb{R}\). They conjectured that this asymptotic behavior is generic for any blow-up solution.

In 1992, Merle \([69]\) proved that given a finite number of points \(x_1, x_2, \ldots, x_k\) in \(I = (-1, 1)\) (or any other domain \(I\) in \(\mathbb{R}\)), there is a positive solution to the nonlinear heat equation which blows up up at time \(T\) with blow-up points \(x_1, x_2, \ldots, x_k\). This theorem can be generalized to allow the sign \((+\infty\text{ or } -\infty)\) to be chosen at each blow-up point \(x_i\).

In \([20]\) precise blowup asymptotics were derived for \((1.1)\) in dimension 1 for even initial conditions. This work developed a different approach based on dynamical rescaling, method of majorants and strong linear estimates. The results of \([20]\) were extended to general initial conditions and consequently to moving blowup point in \([60]\). Our results and techniques extend those of \([20]\).

The following key properties of equation \((1.1)\) explain important features of the expressions above:

- \((1.1)\) is invariant with respect to the scaling transformation,

\[
    u(x,t) \to \lambda^\frac{2}{p-1} u(\lambda x, \lambda^2 t)
\]  

\((1.7)\)

for any constant \(\lambda > 0\), i.e. if \(u(x,t)\) is a solution, so is \(\lambda^\frac{2}{p-1} u(\lambda x, \lambda^2 t)\).

- \((1.1)\) has \(x\)-independent (homogeneous) solutions:

\[
    u_{hom} = [u_0^{-p+1} - (p-1)t]^{-\frac{1}{p-1}}.
\]  

\((1.8)\)

These solutions blow up in finite time \(t^* = ((p-1)u_0^{p-1})^{-1}\) for \(p > 1\).
Chapter 1. Introduction

The linearization of (1.1) around \( u_{\text{hom}} \) shows that the solution \( u_{\text{hom}} \) is unstable. Moreover, it is shown in [43] that if either \( n \leq 2 \) or \( p \leq (n+2)/(n-2) \), then the equation (1.1) has no other self-similar solutions of the form \( (T-t)^{-\frac{1}{p-1}}\phi \left( x/\sqrt{T-t} \right) \), \( \phi \in L^\infty \), besides \( u_{\text{hom}} \).

We consider (1.1) with initial conditions which have, modulo a small perturbation, a maximum at the origin, are slowly varying near the origin and are sufficiently small, but not necessarily vanishing, for large \( |x| \). In particular, the energy \( E(u) \) for such initial conditions might be infinite. We show that the solutions of (1.1) for such initial conditions blowup in a finite time \( t^* \) and we characterize asymptotic dynamics of these solutions.

As it turns out, the leading term is given by the expression

\[
\lambda(t) \frac{p-1}{2} \left[ \frac{c(t)}{p-1 + \lambda^2(t)(x - \zeta(t))b(t)(x - \zeta(t))} \right]^{\frac{1}{p-1}}
\]

(1.9)

where \( b(t) \) a real, symmetric \( n \times n \)-matrix \( b(t) = (b_{ij}(t)) \), \( yby := \sum_{i,j=1}^n y_i b_{ij} y_j \) for a \( n \times n \)-matrix \( b := (b_{ij}) \) and the parameters \( \lambda(t) \), \( b(t) \), \( c(t) \) and \( \zeta(t) \) obey certain dynamical equations whose solutions give

\[
\begin{align*}
\lambda(t) &= \lambda_0 (t^* - t)^{-\frac{1}{2}} (1 + o(1)) \\
b(t) &= \frac{(p-1)^2}{4 p |\ln |t^*-t||} \left( I + O\left( \frac{1}{|\ln |t^*-t||^{1/2}} \right) \right) \\
c(t) &= 1 - \frac{p-1}{2p |\ln |t^*-t||} (1 + O\left( \frac{1}{|\ln |t^*-t||} \right)) \\
\zeta(t) &= O(1).
\end{align*}
\]

(1.10)

with \( \lambda(0) = \sqrt{c_0 + \frac{2}{p-1} Tr(b(0))} \), \( c_0 > 0 \), \( b(0) > 0 \) depends on the initial datum. Here \( o(1) \) is in \( t^* - t \). Moreover, we estimate the remainder, the difference between \( u(x, t) \) and (1.9).

In our approach, as in [20], we do not fix the time-dependent scale in the self-similarity (blowup) variables but let its behavior, as well as behavior of other parameters (\( b \) and \( c \)) to be determined by the equation.

We will deal, without specifying it, with weak solutions of Equation (1.1) in the sense detailed in section 2.2. The local existence of such solutions is well known and is
presented for readers’ convenience there. These solutions can be shown to be classical for $t > 0$.

In what follows we use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $f \lesssim g$ for two positive functions $f$ and $g$, satisfying $f \leq C g$ for some universal constant $C$. The main result is the following theorem.

**Theorem 1.** Let $1 \leq c_0 \leq 4$, $b_0 := (b_{0ij}) > 0$ be a real, symmetric, positive $n \times n$-matrix with $\|b_0\| \ll 1$. Suppose the initial data $u_0 \in L^\infty(\mathbb{R}^n)$ for (1.1) satisfy the conditions

$$\|\langle x \rangle^{-m}(u_0(x) - \left(\frac{c_0}{p - 1 + x_0 x}\right)^{\frac{1}{p-1}})\|_\infty \leq \delta_m,$$

with $m = 0, 3$, $0 \leq \delta_0 \ll 1$ and $\delta_3 = C\|b_0\|^2$. Then

1. There exists a time $t^* \in (0, \infty)$ such that the solution $u(x, t)$ exists on the interval $[0, t^*)$ and blows up at $t \to t^*$.

2. When $t \leq t^*$, there exist unique, positive, $C^1$ real valued, $n$-vector valued and $n \times n$-matrix valued functions $\lambda(t)$ and $c(t)$, $\zeta(t)$ and $b(t)$, respectively, with $b(t) \lesssim b(0)$, such that $u(x, t)$ can be decomposed as

$$u(x, t) = \lambda^{\frac{2}{p-1}}(t)\left(\frac{c(t)}{p - 1 + \lambda^2(t)(x - \zeta(t))b(t)(x - \zeta(t))}\right)^{\frac{1}{p-1}} + \eta(x, t)$$

with the fluctuation part, $\eta$, admitting the estimate $\|\langle \lambda(t)(x - \zeta(t))\rangle^{-m}\eta(x, t)\|_\infty \lesssim \delta_m(t)$, $m = 0, 3$, with $\delta_0(t) = \delta_0 \ll 1$ and $\delta_3(t) = \|b(t)\|^2$.

3. The functions $\lambda(t)$, $b(t)$, $c(t)$ and $\zeta(t)$ are of the form (1.10).

We note that no smoothness of initial conditions is required, also note that (1.1) is an $L^2$-gradient system $\partial_t u = -\text{grad} \mathcal{E}(u)$, with the energy defined in (1.3). Most of the works mentioned above use this fact in an essential way. It is not used in here and we expect our analysis can be extended to non-gradient systems. Specific forms of initial conditions and nonlinearity enter into only two places: orthogonal decomposition and
Lyapunov-Schmidt splitting (Sections 2.4 and 2.8). Note that our approach is closest to that of [20] (see also [89]). Our results extend those of [20] in two aspects. First we address the problem of blow-up in arbitrary dimensions. Second we consider more general initial conditions so that the blowup center is not fixed but in general moves.

The mean curvature flow is the steepest descent flow for the area functional. It arises in applications, such as models of annealing metals [73] and other problems involving phase separation and moving interfaces ([27, 57, 11]). It has been recently successfully applied by Huisken and Sinestrari to topological classification of surfaces and submanifolds ([54, 55], see also [61]). It is closely related to the Ricci and inverse mean curvature flow.

Mean curvature flow was first studied by Brakke [9]. Evans and Spruck [28] constructed a unique weak solution of the nonlinear PDE for certain functions whose zero level set evolves in time according to its mean curvature. Similar results were obtained by Chen, Giga and Goto [15] and by Ambrosio and Soner [3]. The short-time existence in Hölder spaces was proven in [9, 52, 28, 56]. Ecker and Huisken [24] established interior estimates and the short time existence for initial surface which is locally Lipschitz continuous follows from this immediately. Higher codimension mean curvature flows were studied by Mu-Tao Wang [85] (see also [61]). For more results on the existence, uniqueness and regularity of the solution one can see [9, 29, 30, 58, 17].

The question of the long time existence is, as usual, more subtle. Ecker and Huisken [25] showed longtime existence for mean curvature flow in the case of linearly growing graphs. Later they [24] proved that if the initial surface is a locally Lipschitz continuous entire graph over $\mathbb{R}^n$ then the solution will exist for all times.

However, the most interesting aspect of the mean curvature flow is formation of singularities, with two canonical examples being solutions with initial conditions being Euclidian spheres or cylinders. In the latter cases, the solution is again a family of spheres
or cylinders collapsing to their center or axis, with the radii evolving as $\sqrt{2n(t_* - t)}$ or $\sqrt{2(n - 1)(t_* - t)}$. The behaviour of solutions in the last two cases are fairly different. In the former case, the first result here was due to Gage and Hamilton \[40\], who showed that initial convex plane curves shrink to a ‘round’ point, i.e. approach asymptotically circles of radii $\sqrt{2(t_* - t)}$. Later, Grayson \[48\] showed that any embedded plane curves always shrink smoothly until they are convex, and then to points by the evolution theorem of convex curves. For higher dimensions the latter result does not hold. In a seminal work, \[52\], Huisken showed that under mean curvature flow a convex hypersurface in $\mathbb{R}^n$, $n \geq 3$, shrinks smoothly to a point, getting spherical in the limit. For more results see \[18, 19, 62, 63, 88\].

Our main result is as follows.

**Theorem 2.** Let $\Omega$ be the standard $n$-dimensional sphere and let a surface $S_0$, defined by an immersion $x_0 \in H^s(\Omega)$, for some $s > \frac{n}{2} + 1$, be close to $\Omega$, in the sense that $\|x_0 - 1\|_{H^s} \ll 1$. Then there exist $t_* < \infty$ and $z_* \in \mathbb{R}^{n+1}$, s.t. (1.2) has the unique solution, $S_t$, $t < t_*$, and this solution contracts to the point $z_*$, as $t_* \to \infty$. Moreover, $S_t$ is defined by an immersion $x(\cdot, t) \in H^s(\Omega)$, with the same $s$, of the form

$$x(\omega, t) = z(t) + R(\omega, t)\omega,$$

for some $z(t) \in \mathbb{R}^{n+1}$ and $R(\cdot, t) \in H^s(\Omega)$, satisfying $z(t) = z_* + O((t_* - t)^{\frac{1}{2n}(n+\frac{1}{2} - \frac{1}{m})})$ and

$$R(\omega, t) = \lambda(t)(\sqrt{\frac{n}{a(t)}} + \xi(\omega, t)), \quad (1.12)$$

with $\lambda(t), a(t)$ and $\xi(\cdot, t)$ which satisfy

$$\lambda(t) = \sqrt{2a_* (t_* - t)} + O((t_* - t)^{\frac{1}{2} + \frac{1}{2n}(1 - \frac{1}{m})}),$$

$$a(t) = -\lambda(t)\dot{\lambda}(t) = a_* + O((t_* - t)^{\frac{1}{2n}(1 - \frac{1}{m})})$$
and $\|\xi(\cdot, t)\|_{H^s} \lesssim (t_* - t)^{\frac{1}{2n}}$. Moreover, $|z_*| \ll 1$. 

Remark 1. If the initial condition $x_0$ is invariant under the transformation $x_i \to -x_i$ for any $i = 1, \cdots, n+1$, then $z(t) = 0$ and the proof below simplifies considerately.

The alternative scenario of formation of singularities under the mean curvature flow is neckpinching (see [2, 4, 11, 7, 23, 28, 15, 22, 53, 81, 80] and [5, 6], for the Ricci flow, and references therein). A remarkable difference between this scenario and the one described above is that, unlike spheres, the cylinders are not stable under the mean curvature flow. For instance, it follows from results of [42] that for an open set of initial conditions arbitrarily close to a cylinder which have an arbitrary shallow ‘waist’, the solution to the mean curvature flow forms a ‘neck’ which pinches in a finite time.

The form of expression (1.12) above is a reflection of a large class of symmetries of the mean curvature flow:

- (1.2) is invariant under rigid motions of the surface, i.e. $X \mapsto RX + a$, where $R \in O(n+1)$, $a \in \mathbb{R}^{n+1}$ and $X = X(u,t)$ is a parametrization of $S_t$, is a symmetry of (1.2).

- (1.2) is invariant under the scaling $X \mapsto \lambda X$ and $t \mapsto \lambda^{-2}t$ for any $\lambda > 0$.

Our approach utilizes these symmetries in an essential way. It uses the rescaling of the equation (1.2) by a parameter $\lambda(t)$ whose behaviour is determined by the equation itself and a series of differential inequalities for a Lyapunov-type functions.

This thesis is broken into two parts, with the first part dealing with NLH and the second with MCF. These parts are independent and contain their own introductions. These introductions repeat the corresponding parts of the present introduction plus description of the organizations of each chapter.
Chapter 2

On blowup in nonlinear heat equations

2.1 Introduction

In this chapter we study the blow-up problem for the $n$-dimensional nonlinear heat equation (or the reaction-diffusion equation)

$$
\begin{align*}
\partial_t u &= \Delta u + |u|^{p-1} u \\
 u(x, 0) &= u_0(x)
\end{align*}
$$

with $p > 1$. Here $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$. Equation (2.1) arises in the problem of heat flow, or, more generally, in the problems involving diffusion, and is a model for a large class of nonlinear parabolic equations, which are ubiquitous in mathematics and its applications.

The local well-posedness of (2.1) is well known (see, e.g. [8] for the Sobolev spaces $H^\alpha$, $0 \leq \alpha < 2$). Moreover for some data $u_0(x)$, the solutions $u(x,t)$ might blow up in finite time $t^* > 0$. In what follows a solution $u(x,t)$ is said to blow up at time $t^*$ if it exists in $L^\infty$ for $[0,t^*)$ and $\sup_x |u(x,t)| \to \infty$ as $t \to t^*$. Thus, two key problems about (2.1) are
1. Describe initial conditions for which solutions of Equation (2.1) blow up in finite time;

2. Describe the blow-up profile of such solutions.

It is expected (see e.g. [10]) that the blow-up profile is universal — it is independent of lower power perturbations of the nonlinearity and of initial conditions within certain spaces.

There is rich literature regarding the blow-up problem for Equation (2.1). We review quickly relevant results. Starting with [39], various criteria for blow-up in finite time were derived, see e.g. [39, 8, 16, 65, 66, 74, 76, 82, 33, 37]. For example, if \( u_0 \in H^1 \cap L^{p+1} \) and \( E(u_0) < 0 \), where \( E(u) \) is the energy functional for (2.1) given by

\[
E(u) := \int \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1},
\]

then it is proved in [65] that \( \|u(t)\|_2^2 \) blows up in finite time \( t^* \). By the observation

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq \|u(t)\|_\infty^{p-1} \|u(t)\|_2^2
\]

we have that \( \|u(t)\|_\infty \) blows up in finite time \( t^{**} \leq t^* \) also. (In this paper, we denote the norms in the \( L^p \) spaces by \( \|\cdot\|_p \).)

Blow-up at a single point was studied as early as [86] (see also [37]). The first result on asymptotics of the blow-up for arbitrary dimension \( n \geq 1 \) was obtained in the pioneering paper [43] where the authors show that under the condition

\[
|u(x,t)|(t_* - t)^{\frac{1}{p-1}} \text{ is bounded on } B_1 \times (0, t_*),
\]

where \( B_1 \) is the unit ball in \( \mathbb{R}^n \) centred at the origin, and either \( p \leq \frac{n+2}{n-2} \) or \( n \leq 2 \) and assuming blow-up takes place at \( x = 0 \), one has

\[
\lim_{\lambda \to 0} \lambda^{\frac{2}{p-1}} u(\lambda x, t_* + \lambda^2 (t - t_*)) = \pm \left( \frac{1}{p - 1} \right)^{\frac{1}{p-1}} (t_* - t)^{-\frac{1}{p-1}} \quad \text{or} \quad 0.
\]
This result was further improved in several papers (see e.g. \cite{45, 44, 19, 34, 69, 84, 35, 36, 37, 10, 71, 68, 72}). A blow-up solution satisfying the bound \eqref{eq:2.3} is said to be of type I. This bound was proven under various conditions in \cite{45, 71, 68, 87, 46}. Furthermore, the limits of $H^1$-blow-up solutions $u(x,t)$ as $t \uparrow T$, outside the blow-up sets were established in \cite{49, 34, 69, 84, 35, 36, 37, 10, 72, 32}.

For $p > 1$, dimension $n = 1$, Herrero and Velázquez \cite{50} (see also \cite{35}) proved that if the initial condition $u_0$ is continuous, nonnegative, bounded, even and has only one local maximum at 0, and if the corresponding solution blows up, then

$$
\lim_{t \uparrow t^*}(t^* - t)^{\frac{1}{p-1}} u(y((t^* - t) \ln |t^* - t|)^{\frac{1}{2}}, t) = (p - 1)^{-\frac{1}{p-1}} \left[ 1 + \frac{p - 1}{4p} y^2 \right]^{-\frac{1}{p-1}}
$$

uniformly on sets $|y| \leq R$ with $R > 0$. Further extensions of this result are achieved in \cite{49, 84, 34, 35}.

Later for dimension $n = 1$ Bricmont and Kupiainen \cite{10} constructed a co-dimension 2 submanifold, of initial conditions such that \eqref{eq:2.4} is satisfied on the whole domain. More specifically, given a small function $g$ and a small constant $b > 0$, they find constants $d_0$ and $d_1$ depending on $g$ and $b$ such that the solution to \eqref{eq:2.1} with the datum

$$
u_0^*(x) = (p - 1 + bx^2)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0 + d_1 x}{p - 1 + bx^2} \right)^{\frac{1}{p-1}} + g(x) \tag{2.5}
$$

has the convergence \eqref{eq:2.4} uniformly in $y \in (-\infty, +\infty)$. The result of \cite{10} was generalized in \cite{70, 32} (see also \cite{41}), where it is shown that there exists a neighborhood $U$, in the space $\mathcal{H} := L^{p+1} \cap \mathcal{H}^1$, of $\nu_0^*$, given in \eqref{eq:2.5}, such that if $u_0 \in U$, then the solution $u(x,t)$ blows up in a finite time $t^*$ and satisfies \eqref{eq:2.4} for $x \in \mathbb{R}$. They conjectured that this asymptotic behavior is generic for any blow-up solution.

In 1992, Merle \cite{69} proved that given a finite number of points $x_1$, $x_2$, $\ldots$, $x_k$ in $I = (-1,1)$ (or any other domain $I$ in $\mathbb{R}$), there is a positive solution to the nonlinear heat equation which blows up up at time $T$ with blow-up points $x_1$, $x_2$, $\ldots$, $x_k$. This theorem can be generalized to allow the sign ($+\infty$ or $-\infty$) to be chosen at each blow-up
In [20] precise blowup asymptotics were derived for (2.1) in dimension 1 for even initial conditions. This work developed a different approach based on dynamical rescaling, method of majorants and strong linear estimates. The results of [20] were extended to general initial conditions and consequently to moving blowup point in [60]. Our results and techniques extend those of [20].

The following key properties of equation (2.1) explain important features of the expressions above:

- (2.1) is invariant with respect to the scaling transformation,

\[ u(x,t) \rightarrow \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \] (2.6)

for any constant \( \lambda > 0 \), i.e. if \( u(x,t) \) is a solution, so is \( \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \).

- (2.1) has \( x \)-independent (homogeneous) solutions:

\[ u_{hom} = [u_0^{-p+1} - (p-1)t]^{-\frac{1}{p-1}}. \] (2.7)

These solutions blow up in finite time \( t^* = (p-1)u_0^{p-1})^{-1} \) for \( p > 1 \).

The linearization of (2.1) around \( u_{hom} \) shows that the solution \( u_{hom} \) is unstable. Moreover, it is shown in [43] that if either \( n \leq 2 \) or \( p \leq (n+2)/(n-2) \), then the equation (2.1) has no other self-similar solutions of the form \((T-t)^{-\frac{1}{p-1}} \phi(x/\sqrt{T-t})\), \( \phi \in L^\infty \), besides \( u_{hom} \).

We consider (2.1) with initial conditions which have, modulo a small perturbation, a maximum at the origin, are slowly varying near the origin and are sufficiently small, but not necessarily vanishing, for large \( |x| \). In particular, the energy \( E(u) \) for such initial conditions might be infinite. We show that the solutions of (2.1) for such initial conditions blowup in a finite time \( t^* \) and we characterize asymptotic dynamics of these solutions.
As it turns out, the leading term is given by the expression

$$
\lambda(t)^{\frac{2}{p-1}} \left[ \frac{c(t)}{p-1 + \lambda^2(t)(x - \zeta(t))b(t)(x - \zeta(t))} \right]^{\frac{1}{p-1}} \tag{2.8}
$$

where $b(t)$ is a real, symmetric $n \times n$-matrix $b(t) = (b_{ij}(t))$, and $y := \sum_{i,j=1}^{n} y_i b_{ij} y_j$ for a $n \times n$-matrix $b := (b_{ij})$ and the parameters $\lambda(t), b(t), c(t)$ and $\zeta(t)$ obey certain dynamical equations whose solutions give

$$
\lambda(t) = \lambda_0(t^* - t)^{-\frac{1}{2}} (1 + o(1))
$$

$$
b(t) = \frac{(p-1)^2}{4p|\ln|t^* - t||} \left( 1 + O\left( \frac{1}{|\ln|t^* - t||^{1/2}} \right) \right)
$$

$$
c(t) = 1 - \frac{p-1}{2p|\ln|t^* - t||} \left( 1 + O\left( \frac{1}{|\ln|t^* - t||} \right) \right)
$$

$$
\zeta(t) = O(1). \tag{2.9}
$$

with $\lambda(0) = \sqrt{c_0 + \frac{2}{p-1} Tr b(0)}$, $c_0 > 0$, $b(0) > 0$ depends on the initial datum. Here $o(1)$ is in $t^* - t$. Moreover, we estimate the remainder, the difference between $u(x, t)$ and (2.8).

In our approach, as in [20], we do not fix the time-dependent scale in the self-similarity (blowup) variables but let its behavior, as well as behavior of other parameters ($b$ and $c$) to be determined by the equation.

We will deal, without specifying it, with weak solutions of Equation (2.1) in the sense detailed in the next section. The local existence of such solutions is well known and is presented for readers’ convenience in the next section. These solutions can be shown to be classical for $t > 0$.

In what follows we use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $f \lesssim g$ for two positive functions $f$ and $g$, satisfying $f \leq C g$ for some universal constant $C$. The main result of this chapter is the following theorem.

**Theorem 3.** Let $1 \leq c_0 \leq 4$, $b_0 := (b_{0ij}) > 0$ be a real, symmetric, positive $n \times n$-matrix with $\|b_0\| \ll 1$. Suppose the initial data $u_0 \in L^\infty(\mathbb{R}^n)$ for (2.1) satisfy the conditions

$$
\left\| \langle x \rangle^{-m} \left( u_0(x) - \left( \frac{c_0}{p - 1 + xb_0 x} \right)^{\frac{1}{p-1}} \right) \right\|_\infty \leq \delta_m, \tag{2.10}
$$

with $m = 0, 3$, $0 \leq \delta_0 \ll 1$ and $\delta_3 = C \|b_0\|^2$. Then
(1) There exists a time $t^* \in (0, \infty)$ such that the solution $u(x,t)$ exists on the interval $[0, t^*)$ and blows up at $t \to t^*$.

(2) When $t \leq t^*$, there exist unique, positive, $C^1$ real valued, $n$-vector valued and $n \times n$-matrix valued functions $\lambda(t)$ and $c(t)$, $\zeta(t)$ and $b(t)$, respectively, with $b(t) \lesssim b(0)$, such that $u(x,t)$ can be decomposed as

$$u(x,t) = \lambda^{\frac{2}{p-1}}(t)[(\frac{c(t)}{p-1 + \lambda^2(t)(x - \zeta(t))b(t)(x - \zeta(t))})^{\frac{1}{p-1}} + \eta(x,t)]$$

with the fluctuation part, $\eta$, admitting the estimate $\|\langle \lambda(t)(x - \zeta(t)) \rangle^{-m}\eta(x,t)\|_{\infty} \lesssim \delta_m(t)$, $m = 0, 3$, with $\delta_0(t) = \delta_0 \ll 1$ and $\delta_3(t) = \|b(t)\|^2$.

(3) The functions $\lambda(t)$, $b(t)$, $c(t)$ and $\zeta(t)$ are of the form (2.9).

We note that no smoothness of initial conditions is required, also note that (2.1) is an $L^2$-gradient system $\partial_t u = -\text{grad} \, \mathcal{E}(u)$, with the energy defined in (2.2). Most of the works mentioned above use this fact in an essential way. It is not used here and we expect our analysis can be extended to non-gradient systems. Specific forms of initial conditions and nonlinearity enter into only two places: orthogonal decomposition and Lyapunov-Schmit splitting (Sections 2.4 and 2.8). Note that our approach is closest to that of [20] (see also [89]). Our results extend those of [20] in two aspects. First we address the problem of blow-up in arbitrary dimensions. Second we consider more general initial conditions so that the blowup center is not fixed but in general moves.

This chapter is organized as follows. In Section 2.2 we give the local well-posedness of (2.1). In Sections 2.3, 2.4 we present some preliminary derivations and some motivations for our analysis. In Section 2.5 we formulate a priori bounds on solutions to (2.1) which are proven in Sections 2.9, 2.12 and 2.13. We use these bounds in Section 2.6 to prove our main result, Theorem 3. In Sections 2.8, 2.10 and 2.11 we lay the ground work for the proof of the a priori bounds of Section 2.5. In particular, in Section 2.8, using a Lyapunov-Schmidt-type argument we derive equations for the parameters $a$, $b$ and $c$ and
fluctuation $\eta$. In Section 2.10 we rescale our equations in a convenient way and in Section 2.11 we estimate the corresponding propagators. As was mentioned above, the results of Sections 2.8, 2.10 and 2.11 are used in Sections 2.9, 2.12 and 2.13 in order to prove the a priori estimates. In Appendix, we prove a convenient form of the Feynmann-Kac-type formula.

2.2 The Local Well-Posedness of (2.1)

Let $f$ be a locally Lipshitz continuous function, i.e. $\forall R > 0$ there exists $C_R > 0$ such that for all $u, v \in \mathbb{R}$ with $|u|, |v| \leq R$,

$$|f(u) - f(v)| \leq C_R |u - v|. \quad (2.11)$$

We consider the following nonlinear heat equation in $\mathbb{R}^n$

$$\partial_t u = \Delta u + f(u)$$

$$u(x, 0) = u_0(x). \quad (2.12)$$

Let $W^s := \{u \in L^\infty, (-\Delta)^{s/2} u \in L^\infty\}$, a Sobolev space. The next theorem uses the notion of mild solution to (2.12), which is given in the proof.

**Theorem 4.** Let $u_0 \in L^\infty$, and $K := 2|f(0)| + 4C_2\|u_0\|_\infty \|u_0\|_\infty$, with $C_R$ the same as in (2.11). Then there exists $t_* > \|u_0\|_\infty K^{-1}$ s.t.

- (2.12) has a unique mild solution in $C([0, t_*), L^\infty)$;
- $u$ depends continuously on the initial condition $u_0$;
- Either $t_* = \infty$ or $t_* < \infty$ and $\|u(t)\|_\infty \to \infty$ as $t \to t_*$;
- If $u_0 \in W^s$, $s > 0$, then $\|\partial_t u\|_\infty \lesssim t^{-\max(1-s,0)}$. 

Proof. Using Duhamel’s principle, Equation (2.12) can be written as the fixed point equation \( u = H(u) \), where

\[
H(u)(t) := U(t)u_0 + \int_0^t U(t-s)f(u(s))
\]  

(2.13)

where \( U(t) = e^{t\Delta} \). We say that the equation (2.12) has a **mild solution** \( u \) if \( u \in C([0, t_*], L^\infty) \) and \( u \) solves \( u = H(u) \).

Let \( R \geq 2\|u_0\|_\infty \), \( K_R := 2|f(0)| + 2C_R R \), \( T = RK_R^{-1} \) and

\[
B_R := \{ u \in C([0, T], L^\infty) \mid \sup_{0 \leq t \leq T} \| u(t) \|_\infty \leq R \}.
\]

We show that \( H \) maps \( B_R \) into itself. Indeed, by (2.11) with \( v = 0 \) and for \( u \in B_R \),

\[
|f(u)| \leq |f(0)| + C_R R =: \frac{1}{2}K_R,
\]

and by the explicit integral kernel of \( U(t) \), \( \|U(t)w\|_\infty \leq \|w\|_\infty \). Therefore, for \( u \in B_R \),

\[
\|H(u)\|_{C([0, T], L^\infty)} = \sup_{0 \leq t \leq T} \|H(u)(t)\|_\infty \leq \sup_{0 \leq t \leq T} (\|U(t)u_0\|_\infty + \int_0^t \|U(t-s)f(u(s))\|_\infty ds)
\]

\[
\leq \sup_{0 \leq t \leq T} (\|u_0\|_\infty + \frac{1}{2}K_R T) \leq R.
\]

Hence \( H : B_R \rightarrow B_R \).

Next we prove that \( H : B_R \rightarrow B_R \) is a strict contraction. We have

\[
\|H(u) - H(v)\|_{C([0, T], L^\infty)} \leq \left\| \int_0^t U(t-s)(f(u(s)) - f(v(s))) \right\|_{C([0, T], L^\infty)}
\]

\[
= \sup_{0 \leq t \leq T} \left\| \int_0^t U(t-s)(f(u(s)) - f(v(s))) \right\|_\infty
\]

\[
\leq \sup_{0 \leq t \leq T} \int_0^t \|U(t-s)(f(u) - f(v))\|_\infty ds
\]

\[
\leq \sup_{0 \leq t \leq T} \int_0^t \|f(u) - f(v)\|_\infty ds \leq \sup_{0 \leq t \leq T} T\|f(u)(t) - f(v)(t)\|_\infty
\]

\[
\leq TC_R\|u - v\|_{C([0, T], L^\infty)}.
\]

Since \( T \leq RK_R^{-1} \leq (2C_R)^{-1} \), we conclude that \( \|H(u) - H(v)\|_{C([0, T], L^\infty)} \leq \frac{1}{2}\|u - v\|_{C([0, T], L^\infty)} \). Therefore, \( H \) is a strict contraction. Hence there is a unique solution to \( H(u) = u \) in the ball \( B_R \).
To prove that the solution depends continuously on the initial condition \( u_0 \). Let \( u, v \) be the solutions with initial conditions \( u_0 \) and \( v_0 \). We estimate
\[
\|u - v\|_{C([0,T],L^\infty)} \leq \|U(t)(u_0 - v_0)\|_{C([0,T],L^\infty)} + \left\| \int_0^t U(t - s)(g(u(s)) - g(v(s))) \, ds \right\|_{C([0,T],L^\infty)}
\leq \|u_0 - v_0\|_\infty + \frac{1}{2} \|u - v\|_{C([0,T],L^\infty)}.
\]
This implies
\[
\|u - v\|_{C([0,T],L^\infty)} \leq 2\|u_0 - v_0\|_\infty,
\]
completing the proof of continuity.

The dichotomy that either \( t_* = \infty \) or \( t_* < \infty \) and \( \|u(t)\|_\infty \to \infty \) as \( t \to t_* \) follows in a standard way from the fact the local existence time depends only on \( \|u_0\|_\infty \) (and properties of the function \( f \)).

By the explicit integral kernel of \( U(t) \), \( u(\cdot,t) \in C^\infty(\mathbb{R}^n) \forall t > 0 \).

Finally, let \( u_0 \in W^s \), \( s > 0 \). Then differentiating \( H(u)(t) \) w.r.to \( t \) and using that
\[
\partial_t U(t) = \Delta U(t) = -t^{1-\epsilon}(-\Delta)^{1-\epsilon} U(t)(-\Delta)^\epsilon
\]
and that \((-\Delta)\alpha U(t)\) is bounded in \( L^\infty \) for any \( \alpha \geq 0 \), we conclude that \( u \in C^1([0,t_*),L^\infty) \). The relation \( u \in C([0,t_*),W^s) \) is shown in the same way as the first statement of the theorem.

\[
\square
\]

2.3 Blow-Up Variables and Almost Solutions

Let \( z(t) \in \mathbb{R}^n \), \( \lambda(t) > 0 \) be differentiable functions and let \( \alpha(t) \) satisfy the equation
\[
\lambda^{-2} \dot{\alpha} - a\alpha = -\lambda^{-1} \dot{z},
\] (2.14)
with \( a(t) = \dot{\lambda}(t)/\lambda^3(t) \). We introduce the blowup variables
\[
y := \lambda(t)(x - z(t)) - \alpha(t) \text{ and } \tau := \int_0^t \lambda^2(s) \, ds
\]
and define the new function
\[
v(y,\tau) := \lambda^{-\frac{s^2}{\tau^2}}(t)u(x,t).
\] (2.15)
Plugging (2.15) into (2.1) we obtain
\[ \partial_\tau v = \left( \Delta_y - ay \cdot \nabla_y - \frac{2a}{p-1} \right) v + |v|^{p-1} v, \]  
(2.16)

where, as above, \( a(t) = \dot{\lambda}(t)/\lambda^3(t) \). The initial condition for this equation is obtained from the initial condition for (2.1) as \( v(y, 0) = \lambda_0^{-\frac{2p}{p-1}} u_0(z_0 + \frac{y + \alpha_0}{\lambda_0}) \), for some \( \lambda_0, z_0 \) and \( \alpha_0 \).

From the local well-posedness of (2.1) and using rescaling, we can conclude that there exists \( T > 0 \) s.t. (2.16) has a unique mild solution in \( C([0, T), L^\infty) \) and the solution depends continuously on the initial condition. Moreover, either \( T = \infty \) or \( T < \infty \) and \( \|v(\tau)\|_\infty \to \infty \) as \( \tau \to T \).

The equation (2.16) has the following family of homogeneous, static (i.e. \( y \) and \( \tau \)-independent) solutions: \( a \) is a constant and
\[ v_a := \left( \frac{2a}{p-1} \right)^{\frac{1}{p-1}}. \]  
(2.17)

This family of solutions corresponds to the homogeneous solution (2.7) of the nonlinear heat equation with the parabolic scaling \( \lambda^{-2} = 2a(T - t) \), where the blowup time, \( T := \left[ u_0^{p-1}(p-1) \right]^{-1} \), is dependent on the initial value, \( u_0 \) of the homogeneous solution \( u_{hom}(t) \).

If the parameter \( a \) is \( \tau \) dependent but \( |a_\tau| \) is small, then the above solutions are good approximations to the exact solutions. A richer family of approximate solutions is obtained by solving the equation \( ay \cdot \nabla_y v + \frac{2a}{p-1} v = v^p \), obtained from (2.16) by neglecting the \( \tau \) derivative and second order partial derivative in \( y \). This equation has the general solution
\[ v_{bc} := \left( \frac{c}{p-1 + yby} \right)^{\frac{1}{p-1}}, \]  
(2.18)

for \( c = 2a \) and all \( b := (b_{ij}), b_{ij} \in \mathbb{R}, \) real, symmetric \( n \times n \)-matrices. Here recall \( yby := \sum_{i,j=1}^n b_{ij}y_iy_j \). In what follows we take \( b \geq 0 \), so that \( v_{bc} \) is nonsingular. Note that \( v_{0,2a} = v_a \).
2.4 Reparametrization of Solutions

In this section we split solutions to (2.16) into the leading term - the almost solution
$$V_{ab}(y) = v_{bc} := \left(\frac{c}{p-1+gy}\right)^{\frac{1}{p-1}},$$
with $c = a + \frac{1}{2}$, and a fluctuation $\eta$ around it. More precisely, we would like to parametrize a solution by a point on the manifold $M_{as} := \{V_{ab} | a \in \mathbb{R}^+, b \in \mathbb{R}^{n \times n}\}$ of almost solutions and the fluctuation orthogonal to this manifold (large slow moving and small fast moving parts of the solution). For technical reasons, it is more convenient to require the fluctuation to be almost orthogonal to the manifold $M_{as}$. More precisely, we require $\xi$ to be orthogonal to the vectors $\phi_{a}^{(i)}(y) = e^{-\frac{a}{3} |y|^2}, 1 \leq i \leq n$, which are almost tangent vectors to the above manifold, provided $b$ is sufficiently small.

From now on, we fix the relation between the parameters $a$ and $c$ as $c = a + \frac{1}{2}$. Denote by $M_n$ the space of real, symmetric, $n \times n$ matrices and by $M_n^+$, the positive cone in this space. Let $u_{\lambda,z}(y) := \lambda^{-\frac{2}{3}} u(x)$, with $x = z + \lambda^{-1}(y + \alpha)$. We define neighborhoods
$$U_{\epsilon} := \{v \in L^\infty(\mathbb{R}^n) | \| e^{-\frac{1}{2} |y|^2} (v - V_{ab}) \|_\infty = o(\|b\|) \}$$
for some $1/4 \leq a \leq 1, 0 < b \leq \epsilon$
and
$$\tilde{U}_{\epsilon} := \{u \in L^\infty(\mathbb{R}^n) | u_{\lambda,z} \in U_{\epsilon} \}.$$ 

The following statement will be used to reparametrize the initial conditions.

**Proposition 5.** There exist an $\epsilon_0 > 0$ and a unique $C^1$ functional $g : U_{\epsilon_0} \to \mathbb{R}^+ \times M_n^+ \times \mathbb{R}^n$, such that any function $u_{\lambda,z_0} \in U_{\epsilon_0}$ can be uniquely written in the form
$$u_{\lambda,z_0} = V_{ab} + \eta,$$
with $\eta \perp \phi_{a}^{(ij)}, 0 \leq i, j \leq n$, in $L^2(\mathbb{R}^n, e^{-\frac{a|y|^2}{4}} dy), (a, b, z) = g(u_{\lambda,z_0})$. Moreover, if $\frac{1}{4} \leq a_0 \leq 1, 0 < b_0 \leq \epsilon_0$ and $\| (y)^{-m} (u_{\lambda,z_0} - V_{a_0 b_0}) \|_\infty \leq \delta_m$ with $m = 0, 3, \delta_3 = O(\|b_0\|^2)$
and $\delta_0$ small, we have
\[
|g_1(u_{\lambda,z_0}) - (a_0, b_0)| \lesssim \|b_0\|^2,
\]
(2.20)
\[
|g_2(u_{\lambda,z_0}) - z_0| \lesssim \|b_0\|,
\]
(2.21)
\[
\|\langle y \rangle^{-3}(u_{\lambda,z_0} - V_g(u_{\lambda,z_0}))\|_{\infty} \lesssim \|b_0\|^2,
\]
(2.22)
\[
\|u_{\lambda,z_0} - V_g(u_{\lambda,z_0})\|_{\infty} \lesssim \delta_0 + \|b_0\|.
\]
(2.23)

for $g(u_{\lambda,z_0}) = (g_1(u_{\lambda,z_0}), g_2(u_{\lambda,z_0}))$, where $g_1(u_{\lambda,z_0}) = (a, b)$ and $g_2(u_{\lambda,z_0}) = z$.

Proof. Let $V_{labz}(x) := \lambda^{\frac{2}{n+1}} V_{ab}(y)$, $V_\mu \equiv V_{labz} \mu = (a, b, z)$, and $\varphi^{(ij)}_{az}(x) := e^{\frac{-\lambda |y|^2}{4}} \varphi^{(ij)}_a(y)$, with $y := \lambda(x - z) - \alpha$. The orthogonality conditions on the fluctuation can be written as $G(\mu, u) = 0$, where $\mu = (a, b, z)$ and $G: \mathbb{R}^+ \times \mathbb{M}^+_{n+1} \times \mathbb{R}^n \times L^\infty(\mathbb{R}^n) \to \mathbb{M}^+_{n+1}$ is defined as
\[
G(\mu, u) := \langle V_\mu - u, \varphi^{(ij)}_{az} \rangle.
\]

Here and in what follows, all inner products are $L^2(\mathbb{R}^n)$ inner products. Whenever it is convinient we identitify $\mu$ with a $(n + 1) \times (n + 1)$-matrix: $\mu_{00} := a$, $\mu_{0i} = \mu_{i0} = z_z$, $\mu_{ij} := b_{ij}$, $1 \leq i, j \leq n$ and let $\mathbb{M}^+_{n+1} := \{\mu \in \mathbb{M}_{n+1} | a \geq 0, b \geq 0, z \in \mathbb{R}^n\}$ and $\mathbb{M}_{n+1, \varepsilon} := \{\mu \in \mathbb{M}_{n+1} | a \in [\varepsilon, 1], 0 < b \leq \varepsilon, z \in \mathbb{R}^n\}$.

Let $X := e^\frac{1}{4} |y|^2 L^\infty(\mathbb{R}^n)$ with the corresponding norm. Using the implicit function theorem we will prove that for any $\mu_0 := (a_0, b_0, z_0) \in \mathbb{M}^+_{n+1, \varepsilon}$ there exists a unique $C^1$ function $\tilde{g}: X \to \mathbb{M}_{n+1}$, defined in a neighborhood $\tilde{U}_{\mu_0} \subset X$ of $V_{\mu_0}$, such that $G(\tilde{g}(u), u) = 0$ for all $u \in \tilde{U}_{\mu_0}$. Let $B_{\varepsilon}(V_{\mu_0})$ and $B_{\delta}(\mu_0)$ be the balls in $X$ and $\mathbb{R}^{n+1}$ around $V_{\mu_0}$ and $\mu_0$ and of the radii $\varepsilon$ and $\delta$, respectively.

Note first that the mapping $G$ is $C^1$ and $G(\mu_0, V_{\mu_0}) = 0$ for all $\mu_0$. We claim that the linear map $\partial_\mu G(\mu_0, V_{\mu_0})$ is invertible.

Lemma 6. $\partial_\mu G(\mu, u)$, for $u \in \tilde{U}_{\varepsilon_0}$, is invertible.
Proof. Let the indices $\alpha$ and $\beta$ run over the pairs $(i, j)$, $0 \leq i \leq j \leq n$. We compute

$$\partial_\mu G(\mu, u) = A_1 + A_2$$  \hspace{1cm} (2.24)

where the $(\alpha, \beta)$—th entries of $A_1$ and $A_2$ are

$$A_1(\alpha, \beta) = \langle \partial_\mu V_\mu, \phi_{az}^{(\beta)} \rangle$$  \hspace{1cm} (2.25)

and

$$A_2(\alpha, \beta) = \langle V_\mu - u, \partial_\mu \phi_{az}^{(\beta)} \rangle,$$

respectively. We write $A_1$ in the block form

$$A_1 = \begin{pmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{pmatrix},$$

where $K_{11} = \langle \partial_{(a,b^{\text{diag}})} V_\mu, \phi_{az}^{(i)} \rangle$, with $0 \leq i \leq n$, $K_{22} = \langle \partial_{\text{off-diag}} V_\mu, \phi_{az}^{(ij)} \rangle$, with $1 \leq i < j \leq n$, $K_{33} = \langle \partial_z V_\mu, \phi_{az}^{(0i)} \rangle$, with $1 \leq i \leq n$ and similarly for the other entries. For $b > 0$ and small, we compute using change of variable $y = \lambda (x - z) - \alpha$, that (see Appendix B.1 for more details)

$$K_{11} = -\lambda^{-n+\frac{2}{p-1}} \frac{(2\pi)^{\frac{n}{2}} (a + \frac{1}{2})^{\frac{1}{p-1}}}{a^\frac{n}{2} (p-1)^{\frac{1}{p-1}}} \begin{pmatrix}
-\frac{1}{a+\frac{1}{2}} & -\frac{1}{a+\frac{1}{2}} & -\frac{1}{a+\frac{1}{2}} & \cdots & -\frac{1}{a+\frac{1}{2}} \\
\frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \cdots & \frac{1}{(p-1)a} \\
\frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \cdots & \frac{1}{(p-1)a} \\
\frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \cdots & \frac{1}{(p-1)a} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \frac{1}{(p-1)a} & \cdots & \frac{3}{(p-1)a}
\end{pmatrix} + O(||b||)$$  \hspace{1cm} (2.26)

is an $(n+1) \times (n+1)$ matrix,

$$K_{22} = -\lambda^{-n+\frac{2}{p-1}} \frac{(a + \frac{1}{2})^{\frac{1}{p-1}}}{a^\frac{1}{2} (p-1)^{\frac{1}{p-1}}} \frac{2}{(p-1)^2a} \left( \frac{2\pi}{a} \right)^{n/2} I_{a(n-1)} \times 2(a-1) + O(||b||)$$  \hspace{1cm} (2.27)

and

$$K_{33} = -\lambda^{-n+\frac{2}{p-1}} \frac{(a + \frac{1}{2})^{\frac{1}{p-1}}}{a^\frac{1}{2} (p-1)^{\frac{1}{p-1}}} \left( \frac{2\pi}{a} \right)^{n/2} b + o(||b||)$$  \hspace{1cm} (2.28)
is an \( n \times n \) matrix. Moreover,

\[
K_{ij} = o(\|b\|) \quad \text{for} \quad 1 \leq i \neq j \leq 3. \tag{2.29}
\]

Since \( K_{11}, K_{22} \) and \( K_{33} \) are invertible, the matrix \( A_1 \) is also invertible. Furthermore, by the Schwarz inequality

\[
\|A_2\| \lesssim \|u - V_{\mu_0b_0}\|_X = o(\|b\|). \tag{2.30}
\]

Therefore there exist \( \varepsilon_0 \) and \( \varepsilon_1 \) such that the matrix \( \partial_{\mu} G(\mu, u) \) has an inverse for \( \mu \in \mathbb{M}_{n+1,\varepsilon} \) and \( u \in \bigcup_{\mu \in \mathbb{M}_{n+1,\varepsilon}} B_{\varepsilon_1}(V_{\mu}) \).

Moreover, from (2.24)-(2.30) we know that \( \partial_{\mu} G \) can be written as

\[
\partial_{\mu} G = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + R,
\]

where \( A_{11} = O(1) \) and has an \( O(1) \) inverse, \( A_{22} = O(\|b\|) \) and has an \( O(\|b\|^{-1}) \) inverse, \( A_{12} = o(\|b\|) \) and \( A_{21} = o(\|b\|) \). Then we have

\[
(\partial_{\mu} G)^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{2.31}
\]

where \( B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = O(1) \), \( B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} = O(\|b\|^{-1}) \), \( B_{12} = -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} = o(1) \) and \( B_{21} = -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = o(1) \).

Hence by the implicit function theorem, the equation \( G(\mu, u) = 0 \) has a unique solution \( \mu = \tilde{g}(u) \) on a neighborhood of every \( V_{\mu}, \mu \in \mathbb{M}_{n+1,\varepsilon} \), which is \( C^1 \) in \( u \). Our next goal is to determine these neighborhoods.

To determine a domain of the function \( \mu = \tilde{g}(u) \), we examine closely a proof of the implicit function theorem. Proceeding in a standard way, we expand the function \( G(\mu, u) \) in \( \mu \) around \( \mu_0 \):

\[
G(\mu, u) = G(\mu_0, u) + \partial_{\mu} G(\mu_0, u)(\mu - \mu_0) + R(\mu, u),
\]
where $R(\mu, u) = O(|\mu - \mu_0|^2)$ uniformly in $u \in X$. Here $|\mu|^2 = |a|^2 + \|b\|^2 + |z|^2$ for $\mu = (a, b, z)$. Inserting this into the equation $G(\mu, u) = 0$ and inverting the matrix $\partial_\mu G(\mu_0, u)$, we arrive at the fixed point problem $\alpha = \Phi_u(\alpha)$, where $\alpha := \mu - \mu_0$ and $\Phi_u(\alpha) := -\partial_\mu G(\mu_0, u)^{-1}[G(\mu_0, u) + R(\mu, u)]$. By the above estimates there exists an $\varepsilon_1$ such that the matrix $\partial_\mu G(\mu_0, u)^{-1}$ is bounded in $u \in B_{\varepsilon_1}(V_{\mu_0})$. Define

$$|\mu|_{b_0} = |a| + \|b\| + \|b_0\||z|$$

for $\mu = (a, b, z)$, then from (2.31) we have $|\partial_\mu G^{-1}\mu|_{b_0} \lesssim |\mu|$. It follows that

$$|\Phi_u(\alpha)|_{b_0} \lesssim |G(\mu_0, u)| + |\alpha|^2. \quad (2.32)$$

Furthermore, using that $\partial_\mu \Phi_u(\alpha) = -\partial_\mu G(\mu_0, u)^{-1}[G(\mu, u) - G(\mu_0, u) + R(\mu, u)]$, we obtain that there exist $\varepsilon \leq \varepsilon_1$ and $\delta$ such that $\|\partial_\mu \Phi_u(\alpha)\| \leq \frac{1}{2}$ for all $u \in B_{\varepsilon}(V_{\mu_0})$ and $\alpha \in B_\delta(0)$. Pick $\varepsilon$ and $\delta$ so that $\varepsilon \ll \delta \ll \|b_0\| \ll 1$. Then, for all $u \in B_{\varepsilon}(V_{\mu_0})$, $\Phi_u$ is a contraction on the ball $B_\delta(0)$ and consequently has a unique fixed point in this ball. This gives a $C^1$ function $\mu = \tilde{g}(u)$ on $B_{\varepsilon}(V_{\mu_0})$ satisfying $|\mu - \mu_0| \leq \delta$. An important point here is that since $\varepsilon \ll \|b(0)\|$ we have that $b > 0$ for all $V_{ab} \in B_{\varepsilon}(V_{\mu_0})$. Now, clearly, the balls $B_{\varepsilon}(V_{\mu_0})$ with $\mu_0 \in M_{n+1,\varepsilon_0}$ cover the neighbourhood $\tilde{U}_{\varepsilon_0}$. Hence, the map $\tilde{g}$ is defined on $\tilde{U}_{\varepsilon_0}$ and is unique, and the same is true for the map $g$, defined on as $g(u_{\lambda, z_0}) = \tilde{g}(u)$, which implies the first part of the proposition.

Now we prove the second part of the proposition. The definition of the function $G(\mu, u)$ implies $G(\mu_0, u) = \lambda^{-n+\frac{2}{p-1}} \left( \left( V_{a_0b_0} - u_{\lambda, z_0}, e^{-\frac{a_0b_0^2}{4}} \phi_{ij}(y) \right) \right)$, therefore

$$|G(\mu_0, u)| \lesssim \|e^{-\frac{1}{4}b^2}(u_{\lambda, z_0} - V_{a_0b_0})\|_{\infty}. \quad (2.33)$$

This inequality together with the estimate (2.32) and the fixed point equation $\alpha = \Phi_u(\alpha)$, where $\alpha = \mu - \mu_0$ and $\mu = g(u_{\lambda, z_0})$, implies

$$|g(u_{\lambda, z_0}) - \mu_0|_{b_0} \lesssim \|e^{-\frac{1}{4}b^2}(u_{\lambda, z_0} - V_{a_0b_0})\|_{\infty}. \quad (2.34)$$
From one of the conditions of the proposition, r.h.s. of (2.34) = $O(\|b_0\|^2)$ if $a_0 \in [\frac{1}{4}, 1]$. The last estimate implies (2.20) and (2.21). Using Equation (2.34) we obtain
\[
\|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_g(u_{\lambda, z_0}))\|_\infty \leq \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{\mu_0})\|_\infty + \|\langle y \rangle^{-3}(V_g(u_{\lambda, z_0}) - V_{\mu_0})\|_\infty
\]
\[
\lesssim \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{\mu_0})\|_\infty + g(u_{\lambda, z_0}) - \mu_0
\]
\[
\lesssim \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{\mu_0})\|_\infty,
\]
which leads to (2.22). Finally, to prove Equation (2.23), we write
\[
\|u_{\lambda, z_0} - V_g(u_{\lambda, z_0})\|_\infty \leq \|u_{\lambda, z_0} - V_{a_0, b_0}\|_\infty + \|V_g(u_{\lambda, z_0}) - V_{a_0, b_0}\|_\infty.
\]
A straightforward computation gives $\|V_{ab} - V_{a_0, b_0}\|_\infty \lesssim |a - a_0| + \frac{|b - b_0|}{\|b_0\|}$. Since by (2.20), $|a - a_0| + |b - b_0| = O(\|b_0\|^2)$, we have $\|V_{ab} - V_{a_0, b_0}\|_\infty \lesssim \|b_0\|$. This together with the fact $\|u_{\lambda, z_0} - V_{a_0, b_0}\|_\infty \leq \delta_0$ completes the proof of (2.23). \qed

Now we establish a reparametrization of solution $u(x, t)$ on small time intervals. In Section 2.6 we convert this result to a global reparametrization. In the rest of the section it is convenient to work with the original time $t$, instead of rescaled time $\tau$. We let $I_{t_0, \delta} := [t_0, t_0 + \delta]$ and define for any time $t_0$ and constant $\delta > 0$ three sets:
\[
A_{t_0, \delta} := C^1(I_{t_0, \delta}, [1/4, 1]), \quad B_{t_0, \delta, \epsilon_0} := C^1(I_{t_0, \delta}, M_+^{n, \epsilon_0}) \quad \text{and} \quad C_{t_0, \delta} := C^1(I_{t_0, \delta}, [-1, 1]^n),
\]
where, recall the constant $\epsilon_0$ from Proposition 5

Recall $u_{\lambda, z}(y, t) := \lambda(t)^{-\frac{2}{\alpha + 1}}u(x, t)$, with $x = z(t) + \lambda^{-1}(t)(y + \alpha(t))$. Suppose $u(\cdot, t)$ is a function such that for some $\lambda_0 > 0$
\[
\sup_{t \in I_{t_0, \delta}} \|b^{-1}(t)\|\|\langle y \rangle^{-3}(u_{\lambda, z}(\cdot, t) - V_{a(t), b(t)})\|_\infty \ll 1 \quad (2.35)
\]
for some $a \in A_{t_0, \delta}, b \in B_{t_0, \delta, \epsilon_0}, z \in C_{t_0, \delta}, \lambda(t)$ satisfying $\lambda(t_0) = \lambda_0$ and $\lambda^{-1}(t)\partial_t \lambda(t) = a(t)$ and $\alpha(t)$ satisfying $a(t_0) = \alpha_0$ and $\partial_t \alpha(t) - \lambda^2(t)a(t)\alpha(t) + \lambda(t)\partial_t z(t) = 0$. We define the set
\[
U_{t_0, \delta, \epsilon_0, \lambda_0, a_0} := \{u \in C^1(I_{t_0, \delta}, \langle y \rangle^3 L^\infty(\mathbb{R}^n)) \mid (2.35) \text{ holds for some } a \in A_{t_0, \delta}, b \in B_{t_0, \delta, \epsilon_0} \}
\]
Proposition 7. Suppose \( u \in \mathcal{U}_{t_0,\delta,\varepsilon_0,\lambda_0,\alpha_0} \) and \( \lambda_0^2 \delta \ll 1 \). Then there exists a unique \( C^1 \) map \( g_\# : \mathcal{U}_{t_0,\delta,\varepsilon_0,\lambda_0,\alpha_0} \to \mathcal{A}_{t_0,\delta} \times \mathcal{B}_{t_0,\delta,\varepsilon_0} \times \mathcal{C}_{t_0,\delta} \), such that for \( t \in I_{t_0,\delta} \), \( u(\cdot, t) \) can be uniquely represented in the form

\[
    u_\lambda(y, t) = V_{g_\#(u)(t)}(y) + \phi(y, t),
\]

with \( (a(t), b(t), z(t)) = g_\#(u)(t) \) and

\[
    \phi(\cdot, t) \perp \phi_{a(t)}^{(ij)} \text{ in } L^2(\mathbb{R}^n, e^{-\frac{\lambda_0}{4} |y|^2} dy),
\]

\[
    \lambda^{-3}(t) \partial_t \lambda(t) = a(t) \text{ and } \lambda(t_0) = \lambda_0,
\]

\[
    \partial_t \alpha(t) - \lambda^2(t)a(t)\alpha(t) + \lambda(t)\partial_z(t) = 0 \text{ and } \alpha(t_0) = \alpha_0.
\]

Proof. For any function \( a \in \mathcal{A}_{t_0,\delta} \), we define a function

\[
    \lambda(a, t) := (\lambda_0^{-2} - 2 \int_{t_0}^{t} a(s) ds)^{-\frac{1}{2}}.
\]

Let \( \lambda(a)(t) := \lambda(a, t) \). Next we define a function

\[
    \alpha(a, z)(t) := e^{\int_{t_0}^{t} \lambda^2(s) a(s) ds} \alpha_0 - \int_{t_0}^{t} e^{\int_{s}^{t} \lambda^2(\gamma) a(\gamma) d\gamma} \lambda(s) \partial_z z(s) ds.
\]

Define the \( C^1 \) map \( G_\# : C^1(I_{t_0,\delta}, \mathbb{R}^+) \times C^1(I_{t_0,\delta}, \mathbb{M}^+_{\mathbb{R}^n}) \times C^1(I_{t_0,\delta}, \mathbb{R}^n) \to C^1(I_{t_0,\delta}, \mathbb{R}^{(n+2)(\alpha+1)}) \) as

\[
    G_\#(\mu, u)(t) := G(\mu(t), u_{\lambda(a), z}(\cdot, t)),
\]

where \( t \in I_{t_0,\delta} \), \( \mu = (a, b, z) \) and \( G(\mu, u) \) is the same as in the proof of Proposition 5. The orthogonality conditions on the fluctuation can be written as \( G_\#(\mu, u) = 0 \).

Using the implicit function theorem we will first prove that for any \( \mu_0 := (a_0, b_0, z_0) \in \mathcal{A}_{t_0,\delta} \times \mathcal{B}_{t_0,\delta,\varepsilon_0} \times \mathcal{C}_{t_0,\delta} \) there exists a neighborhood \( \mathcal{U}_{\mu_0} \) of \( V_{\mu_0} \) and a unique \( C^1 \) map \( g_\# : \mathcal{U}_{\mu_0} \to \mathcal{A}_{t_0,\delta} \times \mathcal{B}_{t_0,\delta,\varepsilon_0} \times \mathcal{C}_{t_0,\delta} \) such that \( G_\#(g_\#(v), v) = 0 \) for all \( v \in \mathcal{U}_{\mu_0} \).

We claim that \( \partial_\mu G_\#(\mu, u) \) is invertible, provided \( u_{\lambda(a), z} \) is close to \( V_\mu \). We compute

\[
    \partial_\mu G_\#(\mu, u)(t) = \partial_\mu G(\mu(t), u_{\lambda(a), z}(\cdot, t)) = A(t) + B(t),
\]

where

\[
    A(t) := \partial_\mu G(\mu, v)|_{v=u_{\lambda(a), z}}, \quad B(t) := \partial_v G(\mu, v)|_{v=u_{\lambda(a), z}} \partial_\mu u_{\lambda(a), z}.
\]
Note that in (2.39) $\partial_{\mu} G(\mu, v)\bigr|_{v = u_{\lambda(a), z}}$ is acting on $\partial_{\mu} u_{\lambda(a), z}$ as an integral with respect to $y$ and let $B(t)(y)$ be the integral kernel of this operator. We have shown in Lemma 6 that the first term on the r.h.s. is invertible, provided $u_{\lambda(a), z}$ is close to $V_\mu$.

Now we show that for $\delta > 0$ sufficiently small the second term on the r.h.s. is small. Let $v := u_{\lambda(a), z}$. Assuming for the moment that $v$ is differentiable, we compute

$\partial_{\alpha} v = -\partial_{\alpha}(\lambda^{-1})\left[\frac{2}{p-1}\lambda v - (y + \alpha)\nabla_y v\right] + \lambda^{-1}\partial_{\alpha} \alpha \nabla_y v$. Combining the last two equations together with Equation (2.39) we obtain

$$[B(t)\rho](t) = \int B(t)(y)\left[-\frac{2}{p-1}\lambda v + (y + \alpha)\nabla_y v)(\partial_{\alpha} \lambda^{-1} \rho) + \lambda^{-1} \nabla_y v(\partial_{\alpha} \rho)\right] dy.$$ Integrating by parts the second term in parenthesis gives

$$[B(t)\rho](t) = -\int \left[\left(-\frac{2}{p-1}\lambda v + (y + \alpha)\nabla_y \cdot (y + \alpha)\right)\partial_{\alpha} \lambda^{-1} \rho + \lambda^{-1} \nabla_y \cdot (\partial_{\alpha} \rho)\right] B(t)(y) dy. \quad (2.40)$$

Furthermore, $\partial_{\alpha}(\lambda^{-1}) \rho = \lambda(t) \int_{t_0}^t \rho(s) ds$ and

$$\partial_{\alpha} \rho = e^{\int_{t_0}^t \lambda(s) \alpha(s) ds} \partial_{\alpha} \rho = \int_{t_0}^t \lambda(s) \partial_{\alpha} \lambda^2(s) \rho(s) ds$$

$$- \lambda(s) \partial_{\alpha} \lambda^2(s) \rho(s) ds$$

where

$$\left[\partial_{\alpha} \rho\right] = \int_{t_0}^t \lambda(s) \partial_{\alpha} \lambda^2(s) \rho(s) ds$$

Now, using a density argument, we remove the assumption of the differentiability on $v$ and conclude that (2.40) holds without this assumption. Using this expression and the inequality $\lambda(t) \leq \sqrt{2}\lambda_0$, provided $\delta \leq (4 \sup \lambda)^{-1} \lambda_0^{-2} \leq 1/4\lambda_0^{-2}$, we estimate

$$\|B(t)\rho\|_{L^\infty([t_0, t_0 + \delta])} \lesssim \delta \lambda_0^2 \|v\|_{L^\infty([t_0, t_0 + \delta])}. \quad (2.41)$$

So $B(t)$ is small, if $\delta \lesssim (\lambda_0^2 \|v\|_{L^\infty})^{-1}$, as claimed. This shows that $\partial_{\mu} G_\#(\mu, u)$ is invertible, provided $u_{\lambda(a), z}$ is close to $V_\mu$. Proceeding as in the proof of Proposition 3 we conclude the proof of Proposition 7.

2.5 A priori Estimates

Let $u(x, t), 0 \leq t \leq T$ be a solution to (2.1) with initial condition $u_0 \in U_{\epsilon_0}$ and $v(y, \tau) = \lambda^{-\frac{1}{p-1}}(t) u(x, t)$, where $y = \lambda(x - z) - \alpha$ and $\tau(t) := \int_0^t \lambda^2(s) ds$. We assume that there

\[\]
exist $C^1$ functions $a(\tau)$ and $b(\tau)$ such that $v(y, \tau)$ can be represented as

$$v(y, \tau) = \left( \frac{c(\tau)}{p - 1 + yb(\tau)y} \right)^{\frac{1}{p-1}} + e^{\frac{a(\tau)}{4}y^2} \xi(y, \tau),$$

(2.42)

where $\xi(\cdot, \tau) \perp \phi_{a(\tau)}^{(ij)}$ (see (2.19)), $\lambda^{-3}(t) \partial_t \lambda(t) = a(\tau(t))$, $c = a + \frac{1}{2}$. Since $u_0 \in U_{\epsilon_0}$, by Proposition 5,

$$\left\| \langle y \rangle^{-3} e^{-\frac{a}{4}y^2} \xi(y, 0) \right\|_\infty \lesssim \left\| b(0) \right\|^2.$$

(2.43)

In this section we formulate a priori bounds on the fluctuation $\xi$ which are proved in later sections.

Let the function $\tilde{\beta}(\tau)$, the constant $\kappa$ be defined as

$$\tilde{\beta}(\tau) := (b(0)^{-1} + \frac{4p}{(p - 1)^2} I)^{-1} \text{ and } \kappa := \min\left\{ \frac{1}{2}, \frac{p - 1}{2} \right\},$$

(2.44)

and let $\beta(\tau)$ is the largest eigenvalue of $\tilde{\beta}(\tau)$. For the functions $\xi(\tau)$, $b(\tau)$ and $a(\tau)$ we introduce the following estimating functions (families of semi-norms)

$$M_1(T) := \max_{\tau \leq T} \beta^{-2}(\tau) \left\| \langle y \rangle^{-3} e^{-\frac{a}{4}y^2} \xi(\tau) \right\|_\infty,$$

$$M_2(T) := \max_{\tau \leq T} \left\| e^{-\frac{a}{4}y^2} \xi(\tau) \right\|_\infty,$$

$$A(T) := \max_{\tau \leq T} \beta^{-2}(\tau) \left| a(\tau) - \frac{1}{2} + \frac{2Tb(\tau)}{p-1} \right|,$$

$$B(T) := \max_{\tau \leq T} \beta^{-(1+\kappa)}(\tau) \left\| b(\tau) - \tilde{\beta}(\tau) \right\|.$$

(2.45)

**Proposition 8.** Let $\xi$ to be defined in (2.42) and we assume $M_1(0), A(0), B(0) \lesssim 1$, $M_2(0) \ll 0$. Assume there exists an interval $[0, T]$ such that for $\tau \in [0, T]$,

$$M_1(\tau), A(\tau), B(\tau) \leq \beta^{-\kappa/2}(\tau).$$

Then in the same time interval the parameters $a, b$ and the function $\xi$ satisfy the following estimates

$$\left\| \frac{\partial}{\partial \tau} b(\tau) + \frac{4p}{(p - 1)^2} b^2(\tau) \right\| \lesssim \beta^3(\tau) + \beta^3(\tau) M_1(\tau)(1 + A(\tau)) + \beta^4(\tau) M_2^2(\tau) + \beta^{2p} M_1^{2p}(\tau),$$

(2.46)

and

$$B(\tau) \lesssim 1 + M_1(\tau)(1 + A(\tau)) + M_1^2(\tau) + M_1^p(\tau),$$

(2.47)
\[ A(\tau) \lesssim A(0) + 1 + \beta(0)M_1(\tau)(1 + A(\tau)) + \beta(0)M_1^2(\tau) + \beta^{2p-2}(0)M_1^p(\tau), \quad (2.48) \]

\[ M_1(\tau) \lesssim M_1(0) + \beta\tilde{\varepsilon}(0)[1 + M_1(\tau)A(\tau) + M_1^2(\tau) + M_1^p(\tau)] \\
+ [M_2(\tau)M_1(\tau) + M_1(\tau)M_2^{p-1}(\tau)], \quad (2.49) \]

\[ M_2(\tau) \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + \beta^{\frac{\kappa}{2}}(0)M_1^{\frac{2}{p}}(T)M_2^{\frac{1}{p}}(T) + M_2^2(\tau) + M_2^p(\tau) \\
+ \beta\tilde{\varepsilon}(0)[1 + M_2(\tau) + M_1(\tau)A(\tau) + M_1^2(\tau) + M_1^p(\tau)]. \quad (2.50) \]

Equations (2.46)-(2.48), (2.49) and (2.50) will be proved in Sections 2.9, 2.12 and 2.13 respectively.

**Corollary 9.** Let \( \xi \) to be defined in (2.42) and assume \( M_1(0), A(0), B(0) \lesssim 1, M_2(0) \ll 1 \). Assume there exists an interval \([0, T]\) such that for \( \tau \in [0, T] \),

\[ M_1(\tau), A(\tau), B(\tau) \leq \beta^{-\kappa/2}(0). \]

Then in the same time interval the parameters \( a, b \) and the function \( \xi \) satisfy the following estimates

\[ M_1(\tau), A(\tau), B(\tau) \lesssim 1, M_2(\tau) \ll 1. \quad (2.51) \]

(In fact, \( M_i(\tau) \lesssim M_i(0) + \beta\tilde{\varepsilon}(0), \ i = 1, 2. \))

**Proof.** Since \( \beta(\tau) \leq \beta(0) \ll 1 \), we have, by the continuity (or by Proposition 5), that for a sufficiently small time interval

\[ M_1(\tau), B(\tau), A(\tau) \leq \beta^{-\frac{\kappa}{2}}(0) \leq \beta^{-\frac{\kappa}{2}}(\tau), \quad (2.52) \]

where, recall, the definitions of \( \beta(\tau) \) and \( \kappa \) are given in (2.44). Then Equations (2.47)-(2.50) imply that for the same time interval

\[ M_1(\tau), B(\tau), A(\tau) \lesssim 1, M_2(\tau) \ll 1. \quad (2.53) \]

Thus the conditions of the proposition above are satisfied. Since \( M_1(\tau) \leq \beta^{-\frac{\kappa}{2}}(0) \), we can solve (2.48) for \( A(\tau) \). We substitute the result into Equations (2.49) - (2.50) to obtain inequalities involving only the estimating functions \( M_1(\tau) \) and \( M_2(\tau) \). Consider
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the resulting inequality for $M_2(\tau)$. The only terms on the r.h.s., which do not contain $\beta(0)$ to a power at least $\kappa/2$ as a factor, are $M_2^2(\tau)$ and $M_2^p(\tau)$. Hence for $M_2(0) \ll 1$ this inequality implies that $M_2(\tau) \lesssim M_2(0) + \beta^2(0)$. Substituting this result into the inequality for $M_1(\tau)$ we obtain that $M_1(\tau) \lesssim M_1(0) + \beta^2(0)$ as well. The last two inequalities together with (2.47) and (2.48) imply the desired estimates on $A(\tau)$ and $B(\tau)$. 

2.6 Proof of Main Theorem 3

We start with an auxiliary statement which eases the induction step. Recall the notation $I_{t_0, \delta} := [t_0, t_0 + \delta]$. We say that $\lambda(t)$ is admissible on $I_{t_0, \delta}$ if $\lambda \in C^1(I_{t_0, \delta}, \mathbb{R}^+)$ and $\lambda^{-3} \partial_t \lambda \in [1/4, 1]$. Recall that $t_*$ is the maximal existence time defined in Section 2.2.

Lemma 10. Assume $u \in C^1([0, t_*), \langle x \rangle^3 L^\infty)$, $t_0 \in [0, t_*)$ and $u_{\lambda_0}(\cdot, t_0) \in U_{\epsilon_0/2}$ for some $\lambda_0$ and for $\epsilon_0$ given in Proposition 5. Then there are $\delta = \delta(\lambda_0, u) > 0$ and $\lambda(t)$, admissible on $I_{t_0, \delta}$, s.t. (2.36) and (2.37) hold on $I_{t_0, \delta}$.

Proof. The conditions $u \in C^1([0, t_*), \langle x \rangle^3 L^\infty)$ and $u_{\lambda_0}(t_0) \in U_{\epsilon_0/2}$ imply that there is a $\delta = \delta(\lambda_0, u)$ s.t. $u \in U_{t_0, \delta, t_0, \lambda_0}$. By Proposition 7, the latter inclusion implies that there is $\lambda(t)$, admissible on $I_{t_0, \delta}$, $\lambda(t_0) = \lambda_0$, s.t. (2.36) and (2.37) hold on $I_{t_0, \delta}$. \hfill \Box

Choose $b_0$ so that $C\|b_0\|^2 \leq \frac{1}{2}\epsilon_0$ with $C$ the same as in (2.10) and with $\epsilon_0$ given in Proposition 5. Let $v_0(y) := \lambda_0^{-\frac{\beta}{2\kappa}} u_0(z_0 + \lambda_0^{-1} y)$. Then $v_0 \in U_{\epsilon_0/2}$, by the condition (2.10) with $m = 3$, on the initial conditions. Hence Proposition 5 holds for $v_0$ and we have the splitting (2.19). Denote $g(v_0) := (a(0), b(0), z(0))$.

Furthermore, by Lemma 10 there are $\delta_1 > 0$ and $\lambda_1(t)$, admissible on $[0, \delta_1]$, s.t. $\lambda_1(0) = \lambda_0$ and Equations (2.36) and (2.37) hold on the interval $[0, \delta]$. Hence, in particular, the estimating functions $M_1(\tau)$, $M_2(\tau)$, $A(\tau)$ and $B(\tau)$ of Section 5 are defined on
the interval \([0, \delta_1]\). We will write these functions in the original time \(t\), i.e. we will write \(M_i(t)\) for \(M_i(\tau(t))\) where \(\tau(t) = \int_0^t \lambda^2(s)ds\).

Recall the definitions of \(\beta(\tau)\) and \(\kappa\) are given in (2.44). Since \(\beta(0)\) is the largest eigenvalue of \(b(0)\), by Equation (2.10) and Proposition 5, \(A(0), M_1(0) \lesssim 1\) and \(M_2(0) \ll 1\), while \(B(0) \ll 1\), by the definition. We have, by the continuity, that

\[ M_1(t), A(t), B(t) \leq \beta^{-\frac{3}{2}}(0), \]  

for a sufficiently small time interval, which we can take to be \([0, \delta_1]\). Then by Corollary 9 we have that for the same time interval

\[ M_1(t), A(t), B(t) \lesssim 1, M_2(t) \ll 1. \]  

Equation (2.55) implies that \(u_{\lambda_1}(\cdot, \delta_1) \in \tilde{U}_{\epsilon_0/2}\) (indeed, by the definitions of \(M_1(t)\) and \(M_2(t)\) we have \(\|\langle y \rangle^{-3}(u_{\lambda_1}(\cdot, t) - V_{a(t)}, b(t))\| \leq M_1(t)\|b(t)\|^2\) and \(\|u(t)\|_\infty \lesssim \lambda_t^{\frac{2}{p-1}}(t)[1 + M_1(t) + M_2(t)]\)). Now we can apply Lemma 10 again and find \(\delta_2 > 0\) and \(\lambda_2(t)\), admissible on \([0, \delta_1 + \delta_2]\), s.t. \(\lambda_2(t) = \lambda_1(t)\) for \(t \in [0, \delta_1]\) and Equations (2.36) and (2.37) hold on the interval \([0, \delta_1 + \delta_2]\).

We iterate the procedure above to show that there is a maximal time \(t^* \leq t_*\) (\(t_*\) is the maximal existence time), and a function \(\lambda(t)\), admissible on \([0, t^*]\), s.t. (2.36) and (2.37) and (2.55) hold on \([0, t^*]\). We claim that \(t^* = t_*\) and \(t^* < \infty\) and \(\lambda(t^*) = \infty\). Indeed, if \(t^* < t_*\) and \(\lambda(t^*) < \infty\), then by the a priori estimate (2.55) \(u_\lambda(t) \in \tilde{U}_{\epsilon_0/2}\) for any \(t \leq t^*\). By Lemma 10 this implies that there is \(\delta > 0\) and \(\lambda_\#(t)\), admissible on \([0, t^* + \delta]\), s.t. (2.36) and (2.37) hold on \([0, t^* + \delta]\] and \(\lambda_\#(t) = \lambda(t)\) on \([0, t^*]\), which would contradict the assumption that the time \(t^*\) is maximal. Hence

\[ \text{either } t^* = t_* \text{ or } t^* < t_* \text{ and } \lambda(t^*) = \infty. \]  

The second case in (2.56) is ruled out as follows. Using the relation between the functions \(u(x, t)\) and \(v(y, \tau)\) we obtain the following a priori estimate on the (non-rescaled) solution
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$u(x,t)$ of equation (2.1):

$$\|u(t)\|_\infty \lesssim \lambda(t)^{\frac{2}{p-1}}[1 + M_1(t) + M_2(t)],$$  \hspace{1cm} (2.57)

where we used the fact $\|e^{\frac{a\tau^2}{2}} \xi(\cdot, \tau(t))\|_\infty \lesssim M_1(t) + M_2(t)$. By the estimate (2.55) above the majorants $M_j(t)$ are uniformly bounded and therefore

$$\|u(t)\|_\infty \lesssim \lambda(t)^{\frac{2}{p-1}} \text{ for } t < t^*.$$  \hspace{1cm} (2.58)

Moreover by (2.42) and the fact $\|\langle y \rangle^{-3} e^{\frac{a\tau^2}{2}} \xi\|_\infty \lesssim \|b(t)\|^2$, implied by $M_1 \lesssim 1$, give

$$|u(0,t)| \geq \lambda(t)^{\frac{2}{p-1}} \left[ \left( \frac{c(t)}{p-1} \right)^{\frac{1}{p-1}} - C\|b(t)\|^2 \right] \rightarrow \infty,$$  \hspace{1cm} (2.59)

as $t \uparrow t^*$, which implies that $t^* \geq t_*$ and therefore $t_* = t^*$.

Now we consider the first case in (2.56). In this case we must have either $t^* = t_* = \infty$ or $t^* = t_* < \infty$ and $\lambda(t^*) = \infty$, since otherwise we would have existence of the solution on an interval greater than $[0,t_*]$. Finally, the case $t^* = t_* = \infty$ is ruled out in the next paragraph. This proves the claim which can reformulated as: there is a function $\lambda(t)$, admissible on $[0,t_*]$ s.t. (2.36) and (2.37) and (2.55) hold on $[0,t_*)$ and $\lambda(t) \rightarrow \infty$ as $t \rightarrow t_*$. This gives the statements [1] and [2] of Theorem 3.

By the definitions of $A(t)$ and $B(t)$ in (2.45) and the facts that $A(t), B(t) \lesssim 1$ proved above, we have that

$$a(t) - \frac{1}{2} = -\frac{2}{p-1} Tr b(\tau) + O\left(\beta^2(\tau)\right), \quad b(t) = \tilde{\beta}(\tau) + O\left(\beta^{1+\kappa/2}(\tau)\right),$$  \hspace{1cm} (2.60)

where, recall, $\tau = \tau(t) = \int_0^t \lambda^2(s)ds$. Hence $a(t) - \frac{1}{2} = O(\beta(\tau))$. Recall that $a = \lambda^{-3} \partial_\tau \lambda$, which can be rewritten as $\lambda^{-2}(t) = \lambda_0^{-2} - 2 \int_0^t a(s)ds$ or $\lambda(t) = [\lambda_0^{-2} - 2 \int_0^t a(s)ds]^{-\frac{1}{2}}$.

Assume $t^* = \infty$. Since $|a(t) - \frac{1}{2}| = O(\beta(\tau))$, there exists a time $t^{**} < \infty$ such that $\lambda_0^{-2} = 2 \int_0^{t^{**}} a(s)ds$, i.e. $\lambda(t) \rightarrow \infty$ as $t \rightarrow t^{**}$. This contradicts the assumption that $\lambda(t)$ is defined on $[0,t^* = \infty)$. Hence $t^* < \infty$. This completes the proof of Statements [1] and [2] of Theorem 3.
Now we prove the statement [3] of Theorem 3. Equation (2.60) implies $b(t) \to 0$ and $a(t) \to \frac{1}{2}$ as $t \to t^*$. By the analysis above and the definitions of $a$, $\tau$ and $\tilde{\beta}$ (see (2.44)) we have

$$\lambda(t) = (t^* - t)^{-\frac{1}{2}}(1 + o(1)), \quad \tau(t) = -\ln |t^* - t|(1 + o(1)),$$

and

$$\tilde{\beta} (\tau(t)) = -\frac{(p - 1)^2}{4p \ln |t^* - t|} (I + o(1)).$$

This gives the first equation in (2.9). By (2.60) and the relation $c = a + \frac{1}{2}$ we have the second and third equations in (2.9). Finally, let $\zeta(t) = z(t) + \alpha(t)/\lambda(t)$. By (2.14) and (2.21) we obtain the last equation in (2.9). This completes the proof of Theorem 3.

\[ \square \]

2.7 "Gauge" Transform

Observe that the operator $\Delta_y - ay \cdot \nabla_y - \frac{2a}{p-1}$ in (2.16) is not self-adjoint. In order to convert it into a more tractable self-adjoint operator we perform a gauge transform. Let

$$w(y, \tau) := e^{-a|y|^2/4} v(y, \tau). \quad (2.61)$$

Then by (2.16), the function $w$ satisfies the equation

$$\tau_w = \left( \Delta_y - \frac{a^2 + a_r |y|^2}{4} - \frac{2a}{p - 1} + \frac{na}{2} \right) w + e^{\frac{a}{2}(p-1)|y|^2} |w|^{p-1} w \quad (2.62)$$

The approximate solutions $v_{ab}$ to (2.16) transform to $W_{abc}$, where $W_{abc}(y) := v_{bc}(y)e^{-a|y|^2/4}$. Explicitly

$$W_{abc}(y) := \left( \frac{c}{p - 1 + yby} \right)^{\frac{1}{p-1}} e^{-a|y|^2/4}. \quad (2.63)$$
2.8 Lyapunov-Schmidt Splitting (Effective Equations)

According to Lemma 10 the solution $w(y, \tau)$ of (2.62) can be decomposed as (2.42), with the parameters $a$, $b$ and $c$ and the fluctuation $\xi$ depending on time $\tau$:

$$w = W_{abc} + \xi, \quad \xi \perp \phi^{(ij)}_a, \quad 0 \leq i, j \leq n,$$

(2.64)

in the sense of $L^2(\mathbb{R}^n)$, where $W_{abc} := \left( \frac{c}{p-1+yby} \right)^{\frac{1}{p-1}} e^{-\frac{a}{4}|y|^2}, \quad c = a + \frac{1}{2}$ and $\phi_a^{(ij)}$ are defined in the beginning of Section 2.4. Plugging the decomposition (2.64) into (2.62) gives the equation (see Appendix B.2)

$$\xi_\tau = -L_{abc} \xi + N(\xi, b, c) + F(a, b, c)$$

(2.65)

where

$$L_{abc} = -\Delta y + \frac{1}{4} \left( a^2 + a_\tau \right) |y|^2 - \frac{na}{2} + \frac{2a}{p-1} - \frac{pc}{p-1+yby},$$

(2.66)

$$N(\xi, b, c) = \left[ |\xi + W_{abc}|^{p-1} (\xi + W_{abc}) - W_{abc}^p - pW_{abc}^{p-1} \xi \right] e^{\frac{a}{4}(p-1)|y|^2},$$

(2.67)

$$F(a, b, c) = \frac{1}{p-1} \left[ \Gamma_0 + \sum_{j,k} \Gamma_{jk} \left( \frac{p-1}{p-1+yby} \right) W_{abc}, \right]$$

(2.68)

with the functions $\Gamma_{jk}$ ($1 \leq j, k \leq n$) given as

$$\Gamma_0 := -\frac{cr}{c} + (c-2a) - \frac{2}{p-1} Tr b,$$  

(2.69)

$$\Gamma_{jk} := \frac{1}{a(p-1)} \left( \frac{\partial}{\partial y} b_{jk} - (c-2a) b_{jk} + \frac{2b_{jk}}{p-1} Tr b + \frac{4p}{(p-1)^2} \sum_{i=1}^n b_{ij} b_{ik} \right),$$  

(2.70)

$$G_1 := -\frac{4p(yby)(\sum_{i=1}^n (\sum_{j=1}^n b_{ij} y_j)^2)}{(p-1)^2(p-1+yby)^2}. $$

(2.71)

Proposition 11. If $A(\tau), \quad B(\tau) \leq \beta^{\frac{2}{3}}(\tau)$ and $1/4 \leq c(0) \leq 1$, then

$$\| \langle y \rangle^{-3} e^{\frac{a}{4}|y|^2} \mathcal{F} \|_\infty = O \left( |\Gamma_0| + \sum_{j,k} |\Gamma_{jk}| + \beta^{\frac{2}{3}} \right)$$

(2.72)

and

$$\| e^{\frac{a}{4}|y|^2} \mathcal{F} \|_\infty = O \left( |\Gamma_0| + \frac{1}{\beta} \sum_{j,k} |\Gamma_{jk}| + \beta \right).$$

(2.73)
where, recall, \( \langle y \rangle := (1 + y_1^2 + \cdots + y_n^2)^{\frac{1}{2}} \). Furthermore we have for \( \mathcal{N} = \mathcal{N}(\xi, b, c) \)

\[
|\mathcal{N}| \lesssim e^{\frac{a|y|^2}{4}} |\xi|^2 + e^{(p-1)\frac{a|y|^2}{4}} |\xi|^p. \tag{2.74}
\]

**Proof.** We estimate \( \|\langle y \rangle^{-3} e^{\frac{a|y|^2}{4}} \mathcal{F}\|_\infty \) using the expression of \( \mathcal{F} \) and the estimates

\[
\|e^{\frac{a|y|^2}{4}} W_{abc}\|_\infty, \quad \|\langle y \rangle^{-3} y_j y_k\|_\infty \lesssim 1.
\]

The result is

\[
\|\langle y \rangle^{-3} e^{\frac{a|y|^2}{4}} \mathcal{F}\|_\infty \lesssim |\Gamma_0| + \sum_{j,k} |\Gamma_{jk}| + \|b\|^{\frac{3}{2}}. \tag{2.75}
\]

Now we estimate \( \|e^{\frac{a|y|^2}{4}} \mathcal{F}\| \). Recall the expression of \( \mathcal{F} \) in Equation (2.68). We use the estimates

\[
\|e^{\frac{a|y|^2}{4}} W_{abc}\|_\infty, \quad \|e^{\frac{a|y|^2}{4}} \frac{b_j y_j y_k}{(p-1 + yby)^2} W_{abc}\|_\infty \lesssim 1
\]

to obtain that

\[
\|e^{\frac{a|y|^2}{4}} \mathcal{F}\|_\infty \lesssim |\Gamma_0| + \sum_{j,k} \frac{1}{\|b\|} |\Gamma_{jk}| + \|b\|. \tag{2.76}
\]

To complete the proof we estimate \( b \) in terms of \( \beta \) and \( B \) of the first bound. The assumption that \( B \leq \beta^{-\frac{p}{2}} \) implies that \( \|b\| = \beta + O\left(\beta^{1+\frac{p}{2}}\right) \), which together with estimates (2.75) and (2.76), implies estimates (2.72) and (2.73).

For (2.74) we observe that if \( W_{abc} \leq 2|\xi| \) then \( |\mathcal{N}| \leq (3^p + 2^p + p2^{p-1})e^{(p-1)\frac{a|y|^2}{4}} |\xi|^p. \) If \( W_{abc} \geq 2|\xi| \), then we use the formula \( \mathcal{N} = e^{(p-1)\frac{a|y|^2}{4}} p \int_0^1 [(W_{abc} + s\xi)^{p-1} - W_{abc}^{p-1}] \xi ds \) and consider the cases \( 1 < p \leq 2 \) and \( p > 2 \) separately to obtain (2.74). \( \square \)

Recall that \( \phi^{(ij)}_a = (\sqrt{a}y_i)^{1-\delta_a} (\sqrt{a}y_j)^{1-\delta_a} e^{-\frac{a|y|^2}{4}}, \) \( i, j = 0, \ldots, n. \)

**Proposition 12.** Suppose that \( A(\tau), M_1(\tau) \leq \beta^{-\frac{p}{2}}, B(\tau) \leq \beta^{-\frac{p}{2}} \) and \( 1/2 \leq c(0) \leq 2 \) for \( 0 \leq \tau \leq T. \) Let \( w = W_{abc} + \xi \) be a solution to (2.62) with \( \xi \perp \phi^{(ij)}_a \) in \( L^2(\mathbb{R}^n). \) Over times
\[ 0 \leq \tau \leq T, \text{ the parameters } b \text{ and } c \text{ satisfy the equations} \]

\[
\begin{align*}
\frac{\partial}{\partial \tau} b &= -\frac{4p}{(p-1)^2} b^2 - \frac{2b}{p-1} \text{Tr} b + (c - 2a)b + R_b(\xi, b, c), \\
\frac{c}{c} &= (c - 2a) - \frac{2}{p-1} \text{Tr} b + R_c(\xi, b, c),
\end{align*}
\]

(2.77)

(2.78)

where the remainders \( R_b \) and \( R_c \) are of the order \( O\left(\beta^3 + \beta^3 M_1 (1 + A) + \beta^4 M_1^2 + \beta^{2p} M_1^p\right) \) and satisfy \( R_b(0, b, c), R_c(0, b, c) = O(\beta^3) \).

**Proof.** We take inner product of the equation (2.65) with \( \phi^{(ij)}_a \) to get

\[ \langle \xi, \phi^{(ij)}_a \rangle = \langle -L_{abc} \xi + N(\xi, b, c) + F(a, b, c), \phi^{(ij)}_a \rangle. \]

We use the orthogonality conditions \( \phi^{(ij)}_a \perp \xi \) to derive (2.77) and (2.78). We start with analyzing the \( F \) term. The inner product of \( F \) with \( \phi^{(ij)}_a \) gives the expressions

\[
(p - 1) \left( \mathcal{F}, \phi^{(ij)}_a \right) = \Gamma_0 \left( W_{abc}, \phi^{(ij)}_a \right) + \left( \mathcal{G}_1 W_{abc}, \phi^{(ij)}_a \right) + \sum_{k,l} \left( \Gamma_{kl} \left( W_{abc}, a y_k y_l \phi^{(ij)}_a \right) - \Gamma_{kl} \left( \frac{a y_k y_l b}{p-1 + y y b} W_{abc}, \phi^{(ij)}_a \right) \right). \]

(2.79)

By rescaling the variable of integration so that the exponential term does not contain the parameter \( a \), expanding \( W_{abc} \) in \( b \) and we obtain the estimates

\[
\begin{align*}
\left( \mathcal{F}, \phi^{(ij)}_a \right) &= \left( \frac{c}{p-1} \right)^{\frac{1}{p-1}} \left( \frac{2\pi}{a} \right)^{\frac{p}{2}} \delta_{ij} + O\left( \|b\| \right), \\
\left( W_{abc}, \phi^{(ij)}_a \right) &= \left( \frac{a y_k y_l b}{p-1 + y y b} W_{abc}, \phi^{(ij)}_a \right) = O\left( \|b\| \right), \\
\left( \mathcal{G}_1 W_{abc}, \phi^{(ij)}_a \right) &= \left( \frac{a y_k y_l b}{p-1 + y y b} W_{abc}, \phi^{(ij)}_a \right) = O\left( \|b\|^3 \right),
\end{align*}
\]

Where, recall \( \mathcal{G}_1 := -\frac{4p(y y b)(\sum_{i=1}^n (\sum_{j=1}^n b_{ij} y_j))^2}{(p-1)^2(p-1+y y b)^2} \). Substituting these estimates into Equation
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(2.79) gives

\[ (p - 1) \langle \mathcal{F}, \phi_a^{(00)} \rangle = \left( \frac{c}{p - 1} \right)^{\frac{1}{p - 1}} \left( \frac{2\pi}{a} \right)^{\frac{\beta}{2}} (\Gamma_0 + \sum_k \Gamma_{kk}) + R_1, \]

(2.80)

\[ (p - 1) \langle \mathcal{F}, \phi_a^{(ii)} \rangle = \left( \frac{c}{p - 1} \right)^{\frac{1}{p - 1}} \left( \frac{2\pi}{a} \right)^{\frac{\beta}{2}} (\Gamma_0 + \sum_k \Gamma_{kk} + 2\Gamma_{ii}) + R_2 \text{ for } 1 \leq i \leq n, \]

(2.81)

\[ (p - 1) \langle \mathcal{F}, \phi_a^{(ij)} \rangle = 2 \left( \frac{c}{p - 1} \right)^{\frac{1}{p - 1}} \left( \frac{2\pi}{a} \right)^{\frac{\beta}{2}} \Gamma_{ij} + R_3 \text{ for } 1 \leq i < j \leq n, \]

(2.82)

where both remainders \( R_1, R_2 \) and \( R_3 \) are bounded by \( O\left( \|b\| (|\Gamma_0| + \sum_{i,j} |\Gamma_{ij}|) + \|b\|^3 \right) \).

To estimate the projection of \( \partial_\tau \xi \) onto \( \phi_a^{(ij)} \), we differentiate the orthogonality conditions \( \langle \xi, \phi_a^{(ij)} \rangle = 0 \), obtaining the relations \( \langle \xi_\tau, \phi_a^{(ij)} \rangle = -\langle \xi, \partial_\tau \phi_a^{(ij)} \rangle \). When simplified using the orthogonality conditions on \( \xi \), these relations give

\[ \langle \xi_\tau, \phi_a^{(00)} \rangle = 0 \text{ and } \left| \langle \xi_\tau, \phi_a^{(ij)} \rangle \right| \leq \left| \frac{1}{4} a^{-1} \alpha \left( y^{-3} e^{-\frac{3}{2} |y|^2} \xi, a^2 y^3 y_i y_j |y|^2 e^{-\frac{3}{2} |y|^2} \right) \right| \cdot \]

Estimating the right hand side of the second inequality by Hölder’s inequality and using the definition of \( M_1(\tau) \) gives that over times \( 0 \leq \tau \leq T \)

\[ \langle \xi_\tau, \phi_a^{(ij)} \rangle = O (|\alpha| \beta^2 M_1). \]

Next we estimate \( \alpha_\tau \). Since \( a = c - \frac{1}{2} \), we have \( \alpha_\tau = c_\tau \). From (2.69) we find

\[ c_\tau = O (\Gamma_0 + \beta^2 A) \]

for times \( 0 \leq \tau \leq T \). Substituting these estimates into the expression for \( \alpha_\tau \) gives that

\[ \alpha_\tau = O (|\Gamma_0| + \beta^2 A) \]

and hence

\[ \langle \xi_\tau, \phi_a^{(ij)} \rangle = O (\beta^2 M_1 (|\Gamma_0| + \beta^2 A)). \]

(2.83)

We now estimate the terms involving the linear operator \( \mathcal{L}_{abc} \). Write the operator \( \mathcal{L}_{abc} \) as

\[ \mathcal{L}_{abc} = \mathcal{L} + \frac{1}{4} \alpha_\tau |y|^2 - \frac{pc}{p - 1 + yby}, \]
where $L_* := -\Delta_y + \frac{1}{4}a^2|y|^2 - \frac{na}{2} + \frac{2a}{p-1}$ is self-adjoint and satisfies $L_*\phi_a^{(00)} = \frac{2a}{p-1}\phi_a^{(00)}$ and $L_*\phi_a^{(ij)} = \frac{2ap}{p-1}\phi_a^{(ij)}$ for $1 \leq i, j \leq n$. Projecting $L_{abc}\xi$ onto the eigenvectors $\phi_a^{(00)}$ and $\phi_a^{(ij)}$ of $L_*$ gives the equations

$$
\langle L_{abc}\xi, \phi_a^{(00)} \rangle = \frac{1}{4}a_\tau \left\langle \xi, \frac{pc}{p-1 + yby}e^{-\frac{2}{3}|y|^2} - \frac{\xi, \frac{pc}{p-1 + yby}e^{-\frac{2}{3}|y|^2}}{\xi, \frac{pc}{p-1 + yby}e^{-\frac{2}{3}|y|^2}} \right\rangle,
$$

$$
\langle L_{abc}\xi, \phi_a^{(ij)} \rangle = \frac{1}{4}a_\tau \left\langle \xi, a_{ij}y_j|y|^2e^{-\frac{2}{3}|y|^2} - \frac{\xi, a_{ij}y_j|y|^2e^{-\frac{2}{3}|y|^2}}{\xi, a_{ij}y_j|y|^2e^{-\frac{2}{3}|y|^2}} \right\rangle + \frac{pc}{p-1 + yby}e^{-\frac{2}{3}|y|^2}.
$$

Estimating with Hölder’s inequality gives the inequalities

$$
|\langle L_{abc}\xi, \phi_a^{(00)} \rangle| \lesssim \|b\|\|\langle y \rangle^{-3}\xi e^{\frac{2}{3}|y|^2}\|_{\infty}
$$

$$
|\langle L_{abc}\xi, \phi_a^{(ij)} \rangle| \lesssim (|a_\tau| + \|b\|)\|\langle y \rangle^{-3}\xi e^{\frac{2}{3}|y|^2}\|_{\infty}.
$$

In terms of the estimating functions $\beta$ and $M_1$, these estimates, after using the above estimate of $a_\tau$ and simplifying in $a$ and $c$, become

$$
\langle L_{abc}\xi, \phi_a^{(00)} \rangle \lesssim \beta^3 M_1
$$

$$
\langle L_{abc}\xi, \phi_a^{(ij)} \rangle \lesssim \beta^2 M_1 \left( \beta + |\Gamma_0| + \beta^2 A \right). \tag{2.85}
$$

Lastly, we estimate the inner products involving the nonlinearity. Due to (2.74), both $\langle \mathcal{N}, \phi_a^{(00)} \rangle$ and $\langle \mathcal{N}, \phi_a^{(ij)} \rangle$ are estimated by $O\left( \|\langle y \rangle^{-3}e^{\frac{2}{3}|y|^2}\xi\|_{\infty}^2 + \|\langle y \rangle^{-3}e^{\frac{2}{3}|y|^2}\xi\|_{\infty}^p \right)$. Writing this in terms of $\beta$ and $M_1$ and simplifying gives the estimate

$$
|\langle \mathcal{N}, \phi_a^{(00)} \rangle|, |\langle \mathcal{N}, \phi_a^{(ij)} \rangle| \lesssim \beta^4 M_1^2 + \beta^{2p} M_1^p. \tag{2.86}
$$

Estimates (2.80)-(2.85) and (2.86) imply that $\Gamma_0$ and $\Gamma_{ij}$ are of the order

$$
O \left( \beta(|\Gamma_0| + \sum_{i,j} |\Gamma_{ij}|) + \beta^3 + \beta^2 M_1 \left( \beta + |\Gamma_0| + \beta^2 A \right) + \beta^4 M_1^2 + \beta^{2p} M_1^p \right).
$$
By the facts that $\beta(\tau) \leq \beta_0 \ll 1$ and $A, M_1 \leq \beta^{-1/2}$, we obtain the estimates

$$|\Gamma_0| + \sum_{i,j} |\Gamma_{ij}| \lesssim \beta^3 + \beta^3 M_1 (1 + A) + \beta^4 M_1^2 + \beta^{2p} M_1^p$$

(2.87)

for the times $0 \leq \tau \leq T$.

Equations (2.73) and (2.87) yield the following corollary.

**Corollary 13.** Let $k_0 := \min\{1, 2p - 1\}$ and $k_3 := \min\{5/2, 2p\}$. Then for $m = 0$ and $3$

$$\| \langle y \rangle^{-m} e^{\frac{\beta}{2} |w|^2} \mathcal{F} \|_\infty \lesssim \beta^{k_m}(\tau)[1 + M_1 (1 + A) + M_1^2 + M_1^p].$$

(2.88)

### 2.9 Proof of Estimates (2.46)-(2.48)

Recall that $a = c - \frac{1}{2}$. Assume $B(\tau) \leq \beta^{-\frac{p}{(p-1)^2}}(\tau)$ for $\tau \in [0, T]$ which implies that $\tilde{\beta} \lesssim b \lesssim \beta$. We rewrite equation (2.77) as $\partial_{\tau} b = -\frac{4p}{(p-1)^2} b^2 + b \left( \frac{1}{2} - a - \frac{Tr b}{p-1} \right) + R_b$. By the definition of $A$, the second term on the right hand side is bounded by $\|b\| \beta^3 A \lesssim \beta^3 A$. Thus, using the bound for $R_b$ given in Proposition 12, we obtain (2.46).

To prove (2.47) we use the inequality $\beta I \lesssim b$ to obtain the estimate

$$\left\| -\partial_{\tau} b^{-1} + \frac{4p}{(p-1)^2} I \right\| \lesssim \beta + \beta M_1 (1 + A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p.$$ 

(2.89)

Since $\tilde{\beta}$ is a solution to $-\partial_{\tau} \tilde{\beta}^{-1} + 4p(p-1)^{-2} I = 0$, Equation (2.89) implies that

$$\left\| \partial_{\tau} (b^{-1} - \tilde{\beta}^{-1}) \right\| \lesssim \beta + \beta M_1 (1 + A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p.$$

Integrating this equation over $[0, \tau]$, multiplying the result by $\beta^{-1-\kappa}$ and using that $\tilde{\beta}(0) = b(0)$, gives the estimate

$$\beta^{-1-\kappa} \| \tilde{\beta} - b \| \lesssim \beta^{1-\kappa} \int_0^\tau \left( \beta + \beta M_1 (1 + A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p \right) ds,$$

where, recall, $\kappa := \min\{\frac{1}{2}, \frac{p-1}{2}\} < 1$. Hence, by the definition of $\beta$ and $B$ and the facts that $M_1$ and $A$ are increasing functions, (2.47) follows.
Define the quantity $\Gamma := \frac{1}{2} - a - \frac{2}{p-1} Tr b$. Differentiating $\Gamma$ with respect to $\tau$ and substituting for $\partial_\tau b$ and $a_\tau = c_\tau$. From equations (2.77) and (2.78), we obtain

$$\partial_\tau \Gamma = c(\Gamma + R_c) - \frac{2}{p-1} Tr \left( -\frac{4p}{(p-1)^2} b^2 - \frac{2b}{p-1} Tr b + (c-2a)b + R_b \right).$$

Replacing $b(c-2a)$ by $b\Gamma + \frac{2b}{p-1} Tr b$ and rearranging the resulting equation gives that

$$\partial_\tau \Gamma + \left[ a + \frac{1}{2} + \frac{2}{p-1} Tr b \right] \Gamma = \frac{8p}{(p-1)^3} Tr b^2 - (a + \frac{1}{2}) R_c - \frac{2}{p-1} R_b.$$

Let $\mu = \exp\left( \int_0^\tau \left( a + \frac{1}{2} + \frac{2}{p-1} Tr b \right) ds \right)$. We now integrate the above equation over $[0, \tau] \subseteq [0, T]$. Then the above equation implies that

$$\mu(\tau)\Gamma(\tau) - \mu(0)\Gamma(0) = \int_0^\tau \partial_\tau (\mu \Gamma) = \frac{8p}{(p-1)^3} \int_0^\tau \mu Tr b^2 ds - \int_0^\tau (a + \frac{1}{2}) \mu R_c ds - \int_0^\tau \frac{2}{p-1} \mu R_b ds.$$

Use the inequality $\|b\| \lesssim \beta$ and the estimates of $R_b$ and $R_c$ in Proposition 12 to obtain

$$|\Gamma| \lesssim \mu^{-1} \Gamma(0) + \mu^{-1} \int_0^\tau \mu \beta^2 ds + \mu^{-1} \int_0^\tau \mu \left( \beta^3 + \beta^3 M_1(1 + A) + \beta^4 M_1^2 + \beta^{2p} M_1^p \right) ds.$$

For our purpose, it is sufficient to use the less sharp inequality

$$|\Gamma| \lesssim \mu^{-1} \Gamma(0) + (1 + \beta(0) M_1(1 + A) + \beta^2(0) M_1^2 + \beta^{2p-2}(0) M_1^p) \mu^{-1} \int_0^\tau \mu \beta^2 ds.$$

The assumption that $A(\tau), B(\tau) \leq \beta^{-\frac{2}{p-1}}(\tau)$, implies that $a + \frac{1}{2} + \frac{2}{p-1} Tr b = 1 + O(\beta^2 A) \geq \frac{1}{2}$ and therefore $\beta^{-2} \mu^{-1} \lesssim \beta^{-2}(0)$ and $\int_0^\tau \mu(s) \beta^2(s) ds \lesssim \mu(\tau) \beta^2(\tau)$. The last two inequalities and the relation $\max_{s \leq \tau} \beta^{-2}(s)|\Gamma(s)| = A(\tau)$ lead to (2.48).

2.10 Rescaling of Fluctuations on a Fixed Time Interval

The coefficient in front of $|y|^2$ in the operator $L_{abc}$ (2.66), is time dependent, complicating the estimation of the semigroup generated by this operator. In this section we introduce the new time and space variables in such a way that the coefficient at $|y|^2$ in the new operator is constant (cf [12, 13, 20, 75]).
Let $T$ be given and let $t(\tau)$ be the inverse of the function $\tau(t) := \int_0^t \lambda^2(s) \, ds$. We approximate the scaling parameter $\lambda(t)$ over the time interval $[0, t(T)]$ by a new parameter $\lambda_1(t)$. We choose $\lambda_1(t)$ to satisfy for $t \leq t(T)$

$$\partial_t (\lambda_1^{-3} \partial_t \lambda_1) = 0 \text{ with } \lambda_1(t(T)) = \lambda(t(T)) \text{ and } \partial_t \lambda_1(t(T)) = \partial_t \lambda(t(T)).$$

We define $\alpha := \lambda_1^{-3} \partial_t \lambda_1 = a(T)$. This is an analog of the parameter $a$ and it is constant.

The last two conditions imply that $\lambda_1$ is tangent to $\lambda$ at $t = t(T)$. Define the new time and space variables as

$$z = \frac{\lambda_1}{\lambda} y \text{ and } \sigma = \sigma(t(\tau)) \text{ with } \sigma(t) := \int_0^t \lambda_1^2(s) \, ds$$

where $\tau \leq T$, $\sigma \leq S := \sigma(T)$ and $\lambda, \lambda_1$ are functions of $t(\tau)$. Now we introduce the new function $\eta(z, \sigma)$ by the equality

$$\lambda_1^{2p+1} e^{\frac{\alpha}{2} |z|^2} \eta(z, \sigma) = \lambda_1^{2p+1} e^{\frac{\alpha}{2} |y|^2} \xi(y, \tau). \quad (2.90)$$

Denote by $t(\sigma)$ the inverse of the function $\sigma(t)$. In the equation for $\eta(z, \sigma)$ derived below and in what follows the symbols $\lambda$, $a$ and $b$ stand for $\lambda(t(\sigma))$, $a(\tau(t(\sigma)))$ and $b(\tau(t(\sigma)))$, respectively. Substituting this change of variables into (2.65) gives the governing equation for $\eta$:

$$\partial_\sigma \eta = -L_\alpha \eta + W(a, b, \alpha) \eta + F(a, b, \alpha) + N(\eta, a, b, \alpha), \quad (2.91)$$

where

$$L_\alpha := L_0 + V, \quad L_0 := -\Delta_z + \frac{\alpha^2}{4} |z|^2 - \frac{(n+4)\alpha}{2}, \quad V := \frac{2p\alpha}{p-1} - \frac{2p\alpha}{p-1 + z^2 \beta}, \quad (2.92)$$

$$W(a, b, \alpha) := \lambda_1^{2p+1} \frac{p(a + \frac{1}{2})}{\lambda_1^2 p - 1 + \frac{2p}{\lambda_1^2} z^2 b \beta} - \frac{\lambda_1}{\lambda} \frac{2p\alpha}{p-1 + z^2 \beta}$$

$$F(a, b, \alpha) := \left( \lambda_1 \frac{2p-1}{\alpha} \right)^{p-1} e^{-\frac{\alpha}{2} |z|^2} e^{\frac{\alpha}{2} |y|^2} \mathcal{F}(a, b, c)$$

and

$$N(\eta, a, b, \alpha) := \left( \lambda_1 \frac{2p-1}{\alpha} \right)^{p-1} e^{-\frac{\alpha}{2} |z|^2} e^{\frac{\alpha}{2} |y|^2} \eta, \frac{\alpha}{2} |t|^2, b, c \right).$$
where, recall, \( c \) and \( a \) are related as \( c = a + \frac{1}{2} \) and \( \beta \) is defined in (2.44).

In the next statement we prove that the new parameter \( \lambda_1(t) \) is a good approximation of the old one, \( \lambda(t) \). We have

**Proposition 14.** If \( A(\tau) \leq \beta^{-\frac{a}{2}}(\tau) \) and \( \beta(0) \ll 1 \), then

\[
|\frac{\lambda}{\lambda_1}(t(\tau)) - 1| \lesssim \beta(\tau) \leq \beta(0). \tag{2.93}
\]

**Proof.** Differentiating \( \frac{\lambda}{\lambda_1} - 1 \) with respect to \( \tau \) (recall that \( \frac{dt}{d\tau} = \frac{1}{\lambda} \)) gives the expression

\[
\frac{d}{d\tau} \left( \frac{\lambda}{\lambda_1} - 1 \right) = \frac{\lambda}{\lambda_1} a - \frac{\lambda_1}{\lambda}^2 \alpha
\]

or, after some manipulations

\[
\frac{d}{d\tau} \left[ \frac{\lambda}{\lambda_1} - 1 \right] = 2 a \left( \frac{\lambda}{\lambda_1} - 1 \right) + \Gamma \tag{2.94}
\]

with

\[
\Gamma := a - \alpha - a \frac{\lambda_1}{\lambda} \left( \frac{\lambda}{\lambda_1} - 1 \right)^2 + (a - \alpha) \left( \frac{\lambda_1}{\lambda} - 1 \right).
\]

Observe that \( \frac{\lambda}{\lambda_1}(t(\tau)) - 1 = 0 \) when \( \tau = T \). Thus Equation (2.94) can be rewritten as

\[
\frac{\lambda}{\lambda_1}(t(\tau)) - 1 = - \int_T^\tau e^{-\int_\tau^s 2a(\rho)d\rho} \Gamma(\sigma)d\sigma. \tag{2.95}
\]

By the definition of \( A(\tau) \) and the definition \( \alpha = a(T) \) we have that, if \( A(\tau) \leq \beta^{-\frac{a}{2}}(\tau) \), then

\[
|a(\tau) - \alpha|, \quad |a(\tau) - \frac{1}{2}| \leq 2\beta(\tau) \tag{2.96}
\]

on the time interval \( \tau \in [0, T] \). Thus

\[
|\Gamma| \lesssim \beta + (1 + \frac{\lambda_1}{\lambda}) \left( \frac{\lambda}{\lambda_1} - 1 \right)^2 + \beta |\frac{\lambda}{\lambda_1} - 1|.
\]

which together with (2.95) and (2.96) implies (2.93). \qed
2.11 Estimate of the Propagators

Let \( \bar{P}^\alpha \) be the orthogonal projection operator (in the sense of the \( L^2 \) scalar product) onto the space spanned by the eigenvectors of \( L_0 \) corresponding to the smallest three eigenvalues and \( P^\alpha := 1 - \bar{P}^\alpha \). Thus \( P^\alpha \) is the orthogonal projection onto the orthogonal complement of the space spanned by the eigenvectors of \( L_0 \) corresponding to the smallest three eigenvalues. Denote by \( V^\alpha(\tau, \sigma) \) the propagator generated by the operator \(-P^\alpha L_\alpha P^\alpha\) on \( \text{Ran} P^\alpha \), where, recall, the operator \( L_\alpha \) is defined in (2.92). The main result of this section is the following theorem.

**Theorem 15.** For any function \( g \in \text{Ran} P^\alpha \) and for \( c_0 := \alpha - \epsilon \) with some \( \epsilon > 0 \) small we have

\[
\| \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} V^\alpha(\tau, \sigma) g \|_\infty \lesssim e^{-c_0(\tau - \sigma)} \| \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} g \|_\infty.
\]

The proof of this theorem is given after Lemma 20. We observe that in the \( L^2 \)-norm \( P^\alpha L_\alpha P^\alpha \geq (-\Delta + \frac{\alpha^2}{4}|z|^2 - \frac{n+4}{2} \alpha) P^\alpha \geq \frac{1}{2} \alpha P^\alpha \). However, this does not help in proving the weighted \( L^\infty \) bound above. Recall the definition of the operator \( L_0 := -\Delta + \frac{\alpha^2}{4}|z|^2 - \frac{(n+4)\alpha}{2} \) in (2.92) and define \( U_0(x, y) \) as the integral kernel of the operator \( e^{-\frac{\alpha|z|^2}{4}} e^{-rL_0} e^{\frac{\alpha|z|^2}{4}} \). We begin with

**Lemma 16.** For \( k = 0, 1, 2, 3, 4 \), any function \( g \) and \( r > 0 \) we have that

\[
\| \langle z \rangle^{-k} e^{\frac{\alpha|z|^2}{4}} e^{-L_0 r} g \|_\infty \lesssim e^{2\alpha r} \| \langle z \rangle^{-k} e^{\frac{\alpha|z|^2}{4}} g \|_\infty \quad (2.98)
\]

or equivalently

\[
e^{\frac{\alpha|z|^2}{2}} \int \langle x \rangle^{-k} U_0(x, y) e^{-\frac{\alpha}{2}|y|^2} \langle y \rangle^k dy \lesssim e^{2\alpha r}. \quad (2.99)
\]

**Proof.** We only prove the case \( k = 2 \). The cases \( k = 0, 4 \) are similar. The cases \( k = 1, 3 \) follow from \( k = 0, 2, 4 \) by an interpolation result. For the case \( k = 2 \), using that the integral kernel of \( e^{-rL_0} \) is positive and therefore \( \| e^{-rL_0} g \|_\infty \leq \| f^{-1} g \|_\infty \| e^{-rL_0} f \|_\infty \) for
any \( f > 0 \) and using that \( e^{-rL_0} e^{-\frac{\alpha}{4}|z|^2} = e^{2\alpha r} e^{-\frac{\alpha}{4}|z|^2} \) and \( e^{-rL_0}(\alpha|z|^2 - 1)e^{-\frac{\alpha}{4}|z|^2} = (\alpha|z|^2 - 1)e^{-\frac{\alpha}{4}|z|^2} \), we find that
\[
\| \langle z \rangle^{-2} e^{\frac{\alpha|z|^2}{4}} e^{-rL_0} g \|_\infty \leq \| \langle z \rangle^{-2} e^{\frac{\alpha|z|^2}{4}} e^{-rL_0} (|z|^2 + 1) \|_\infty \| \langle z \rangle^{-2} e^{\frac{\alpha|z|^2}{4}} g \|_\infty \\
= \| \langle z \rangle^{-2} [e^{2\alpha r} (\frac{1}{\alpha} + 1) + (|z|^2 - \frac{1}{\alpha})] \|_\infty \| \langle z \rangle^{-2} e^{\frac{\alpha|z|^2}{4}} g \|_\infty \\
\leq 2(\frac{1}{\alpha} + 1)e^{2\alpha r} \| \langle z \rangle^{-2} e^{\frac{\alpha|z|^2}{4}} g \|_\infty.
\]
This implies (2.98). To prove (2.99) we use that \( U_0(x,y) \) is, by definition, the integral kernel of the operator \( e^{-\frac{\alpha}{4}|z|^2} e^{-rL_0} e^{\frac{\alpha}{4}|z|^2} \), and take \( g(x) = \langle x \rangle k e^{-\frac{\alpha}{4}|x|^2} \) in (2.98) to obtain (2.99).

Next we prove a more refined bound on the free evolution \( e^{-rL_0} \).

**Lemma 17.** For any function \( g \) and positive constant \( r \) we have
\[
\| \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} e^{-rL_0} P^\alpha g \|_\infty \lesssim e^{-\alpha r} \| \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} g \|_\infty.
\]

**Proof.** First, we decompose the projection \( P^\alpha \) in a convenient way. We write the operator \( L_0 \) as
\[
L_0 = \sum_{k=1}^{n} L_0^{(k)} - 2\alpha, \quad \text{where} \quad L_0^{(k)} := -\partial^2 \ z_k + \frac{1}{4} \alpha^2 z_k^2 - \frac{\alpha}{2}.
\]

The spectra of the operators \( L_0^{(k)} \) are:
\[
\sigma \left( L_0^{(k)} \right) = \{ m\alpha | m = 0, 1, 2, \ldots \}. \tag{2.101}
\]

Let \( P_0^{(k)}, P_1^{(k)} \) and \( P_2^{(k)} \) be the orthogonal projections onto the eigenspaces of the operator \( L_0^{(k)} \) corresponding to the first, the second and third eigenvalues of \( L_0^{(k)} \), respectively, and let
\[
P_3^{(k)} := 1 - P_0^{(k)} - P_1^{(k)} - P_2^{(k)},
\]
\[
P_0^{(k)} := 1, \quad P_1^{(k)} := 1 - P_0^{(k)},
\]
\[
P_2^{(k)} := 1 - P_0^{(k)} - P_1^{(k)}.
\]
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Then for any \( k \), we have

\[
P_0^{(k)} + P_1^{(k)} + P_2^{(k)} + P_3^{(k)} = 1,
\]

\[
P_0^{(k)} = 1, \quad P_0^{(k)} + P_1^{(k)} = 1,
\]

\[
P_0^{(k)} + P_1^{(k)} + P_2^{(k)} = 1.
\]

(2.102)

Let \( \bar{i} = (i_1, i_2, \cdots, i_n) \), \( i_j = 0, 0', 1, 1', 2, 2' \), \( [\bar{i}] = \sum_{j=1}^{n} i_j \), where the primed numbers are counted as the usual ones, and \( P_{\bar{i}} = \Pi_{i_1}^{(1)} P_{i_2}^{(2)} \cdots P_{i_n}^{(n)} \). For every \( k \in \{1, \cdots, n\} \), we introduce the set

\[
I_k \equiv I_k^{(n)} = \{ \bar{i} = (i_1, \cdots, i_n) \mid \text{either } [\bar{i}] = 3 \text{ and } i_k \neq 0', 1', 2', \text{ or } [\bar{i}] < 3 \text{ and } i_j \neq 0', 1', 2' \forall 1 \leq j \leq n \}.
\]

Then we have the following lemma:

**Lemma 18.** For any \( 1 \leq k \leq n \), there exists a subset \( J_k \) of \( I_k \) such that

\[
\forall k \in \{1, 2, \cdots, n\}, \quad P_{\alpha} = \sum_{[\bar{i}] = 3, \bar{i} \in J_k} P_{\bar{i}}.
\]

(2.103)

For proof of this lemma please see Appendix B.3.

Since

\[
\sum_{\bar{i} \in I_k, [\bar{i}] = j, i_k \neq 0', 1', 2'} P_{\bar{i}}
\]

is the eigenprojection corresponding to the \( j \)-th eigenvalue of \( L_0 \), \( j = 0, 1, 2 \), we have, by the definition of \( P^{\alpha} \) and Lemma [18] that

\[
\forall k \in \{1, 2, \cdots, n\}, \quad P^{\alpha} = \sum_{[\bar{i}] = 3, \bar{i} \in J_k} P_{\bar{i}}.
\]

(2.103)

Equations (2.100) and (2.103) give

\[
e^{-rL_0} P^{\alpha} = \sum_{[\bar{i}] = 3, \bar{i} \in J_k} e^{-rL_0} P_{\bar{i}} = e^{2ar} \sum_{[\bar{i}] = 3, \bar{i} \in J_k} \prod_{j=1}^{n} \left( e^{-rL_0^{(j)}} P_{ij}^{(j)} \right).
\]

(2.104)

Let \( z_0 = 1 \). By the inequality \( \langle z \rangle^3 \lesssim \sum_{k=0}^{n} |z_k|^3 \), we have

\[
\| \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} e^{-rL_0} P^{\alpha} \langle z \rangle^3 e^{-\frac{\alpha|z|^2}{4}} \|_{L^\infty \rightarrow L^\infty} \lesssim \| \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} e^{-rL_0} P^{\alpha} \sum_{k=0}^{n} |z_k|^3 e^{-\frac{\alpha|z|^2}{4}} \|_{L^\infty \rightarrow L^\infty} \lesssim \sum_{k=0}^{n} A_k,
\]

(2.105)
where \( A_k = \left\| \langle z \rangle^{-3} e^{\alpha|z|^2} e^{- r L_0 P \alpha |z_k|^3 e^{-\alpha|z|^2}} \right\|_{L^\infty \rightarrow L^\infty} \) for \( 0 \leq k \leq n \). Now by (2.104) and (2.107) imply the statement of the lemma.

Proposition 19. For any function \( g \) and positive constants \( \sigma \) and \( r \) we have

\[
\| \langle z \rangle^{-3} e^{\alpha|z|^2} U_\alpha(\sigma + r, \sigma) P^{\alpha} g \|_{\infty} \lesssim \left[ e^{2ar} r (1 + r) \beta^{1/2}(\sigma) + e^{-\sigma r} \right]\| \langle z \rangle^{-3} e^{\alpha|z|^2} g \|_{\infty}.
\]  

(2.108)

Proof. Let \( B_\lambda, \lambda \in \mathbb{R}/\mathbb{Z}^n \), be a collection of semi-open, disjoint boxes centered at \( \lambda \), of side \( R \), whose union is \( \mathbb{R}^n \). We take \( R \leq \frac{1 + r}{\sigma} \). Let \( g_\lambda(x) = g(x) \chi_\lambda(x) \), where \( \chi_\lambda(x) \) is the characteristic function of \( B_\lambda \). Then \( g(x) = \sum_\lambda g_\lambda(x) \). Let

\[
E(x, y) = \int_\sigma^\sigma + r e^{-J_\sigma^{\sigma + r} V(\sigma + s, \omega(s) + \omega_0(s)) ds} d\mu(\omega),
\]

(2.109)
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where \( d\mu(\omega) \) is an \( n \)-dimensional harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths \( \omega : [\sigma, \sigma + r] \to \mathbb{R} \) with the boundary condition \( \omega(\sigma) = \omega(\sigma + r) = 0 \) and

\[
\omega_0(s) = e^{\alpha(\tau-s)} \frac{e^{2\alpha\sigma} - e^{2\alpha s}}{e^{2\alpha\sigma} - e^{2\alpha \tau}} x + e^{\alpha(\sigma-s)} \frac{e^{2\alpha\tau} - e^{2\alpha s}}{e^{2\alpha\tau} - e^{2\alpha \sigma}} y. \tag{2.110}
\]

It is shown in Appendix that

\[
|\partial_y E(x, y)| \lesssim r\beta^\frac{1}{2}. \tag{2.111}
\]

Recall that \( U_\alpha(\tau, \sigma) \) is the evolution generated by \( -L_\alpha \). Let \( U(x, y) \) and \( U_0(x, y) \) be the integral kernels of the operators \( e^{-\frac{\alpha|x|^2}{4}} U_\alpha(\sigma + r, \sigma) e^{\frac{\alpha|y|^2}{4}} \) and \( e^{-\frac{\alpha|x|^2}{4}} e^{-rL_0} e^{\frac{\alpha|y|^2}{4}} \), respectively. By Feynmann-Kac formula \([A, 1]\), proved in Appendix, we have that \( U(x, y) = U_0(x, y) E(x, y) \). Then

\[
U_\alpha(\sigma + r, \sigma) P^\alpha g(x) = e^{\frac{\alpha|x|^2}{4}} \int U_0(x, y) E(x, y) e^{-\frac{\alpha|y|^2}{4}} P^\alpha g(y) dy \tag{2.112}
\]

\[
e^{\frac{\alpha|x|^2}{4}} \sum_\lambda \int U_0(x, y) E(x, y) e^{-\frac{\alpha|y|^2}{4}} P^\alpha g_\lambda(y) dy =: A(x) + B(x), \tag{2.113}
\]

where

\[
A(x) := \sum_\lambda e^{\frac{\alpha|x|^2}{4}} \int U_0(x, y) E(x, x_\lambda) e^{-\frac{\alpha|y|^2}{4}} P^\alpha g_\lambda(y) dy
\]

and

\[
B(x) := \sum_\lambda e^{\frac{\alpha|x|^2}{4}} \int U_0(x, y) [E(x, y) - E(x, x_\lambda)] e^{-\frac{\alpha|y|^2}{4}} P^\alpha g_\lambda(y) dy.
\]

First we estimate the function \( A \). We rewrite \( A(x) = e^{\frac{\alpha|x|^2}{4}} \int U_0(x, y) e^{-\frac{\alpha|y|^2}{4}} P^\alpha g_x(y) dy = (e^{-rL_0} P^\alpha g_x)(x) \) with \( g_x(y) = \sum_\lambda E(x, x_\lambda) g_\lambda(y) \). Now by Lemma \( 17 \) we have

\[
\left\| \langle x \rangle^{-3} e^{\alpha|x|^2} A \right\|_{\infty} = \left\| \langle x \rangle^{-3} e^{\frac{\alpha|x|^2}{4}} e^{-rL_0} P^\alpha g_x \right\|_{\infty} \lesssim e^{-\alpha r} \sup_y |\langle y \rangle^{-3} e^{-\frac{\alpha|y|^2}{4}} g_x(y)|.
\]

Since \( |E(x, x_\lambda)| \leq 1 \) and \( g_\lambda \)'s have disjoint supports, we obtain \( |g_x(y)| \leq \sum_\lambda |g_\lambda| = |g| \). The last two inequalities give

\[
\left\| \langle x \rangle^{-3} e^{\frac{\alpha|x|^2}{4}} A \right\|_{\infty} \lesssim e^{-\alpha r} \left\| \langle x \rangle^{-3} e^{\frac{\alpha|x|^2}{4}} g \right\|_{\infty}. \tag{2.114}
\]
Next we estimate the function $B$. Using $U_0(x, y) > 0$, (2.111), Mean Value Theorem and the fact that the diameters of $B_{\lambda}$ are not greater than $1 + r$, we obtain

$$|B(x)| \lesssim r(1 + r)\beta^2 e^{\frac{\alpha|z|^2}{4}} \int U_0(x, y)e^{-\frac{\alpha|y|^2}{4}} \sum_\lambda |P^{\alpha} g_\lambda(y)|dy$$

(2.115)

Thus by (2.115), Lemma 16 and the relation $|g| = \sum_\lambda |g_\lambda|$

$$\left\|\langle x \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} B\right\|_\infty \lesssim r(1 + r)\beta^2 e^{2\alpha r} \left\|\langle x \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} g\right\|_\infty.$$  

(2.116)

Combining (2.113), (2.114) and (2.116), we obtain the estimate (2.108). This proves Proposition 19.

We will also need the following lemma

**Lemma 20.**

$$\|\langle z \rangle^{-k} e^{\frac{\alpha|z|^2}{4}} U_\alpha(\tau, \sigma) g\|_\infty \leq e^{2\alpha(\tau-\sigma)} \|\langle z \rangle^{-k} e^{\frac{\alpha|z|^2}{4}} g\|_\infty$$  

(2.117)

with $k = 0$ or $3$.

**Proof.** By Equations (2.109) and (B.9) we have that $|U_\alpha(\tau, \sigma)|(x, y) \leq e^{-L_0(\tau-\sigma)}(x, y)$. Thus we have

$$\|\langle z \rangle^{-k} e^{\frac{\alpha|z|^2}{4}} U_\alpha(\tau, \sigma) g\|_\infty \leq \|\langle z \rangle^{-k} e^{\frac{\alpha|z|^2}{4}} e^{-L_0(\tau-\sigma)} |g|\|_\infty.$$  

(2.118)

Now we use Lemma 16 to estimate the right hand side to complete the proof.

**Proof of Theorem 15** Recall that $\bar{P}_{\alpha}$ is the projection on the span of the three first eigenfunctions of the operator $L_0$ and $P^{\alpha} := 1 - \bar{P}^{\alpha}$. We write

$$L_{\alpha} = P^{\alpha} L_0 P^{\alpha} + \bar{P}^{\alpha} L_0 \bar{P}^{\alpha} + E_1,$$

(2.119)

where the operator $E_1$ is defined as $E_1 := \bar{P}^{\alpha} L_0 P^{\alpha} + P^{\alpha} L_0 \bar{P}^{\alpha}$. Using that $\bar{P}^{\alpha} P^{\alpha} = 0$, we transform $E_1$ as

$$E_1 = -\bar{P}^{\alpha} \frac{2p_0\alpha\bar{z}\bar{\beta}z}{(p-1)(p-1+z/\bar{z})} P^{\alpha} - P^{\alpha} \frac{2p_0\alpha\bar{z}\beta}{(p-1)(p-1+z/\bar{z})} \bar{P}^{\alpha}.$$
This relation implies that

$$\|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} E_1 \eta(\sigma)\|_\infty \lesssim \beta^{1/2}(\tau(\sigma)) \|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(\sigma)\|_\infty. \quad (2.120)$$

We use the Duhamel principle to rewrite the propagator $V_\alpha(\sigma_1, \sigma_2)$ on $\text{Ran} \ P^\alpha$ as

$$V_\alpha(\sigma_1, \sigma_2)P^\alpha = U_\alpha(\sigma_1, \sigma_2) P^\alpha - \int_{\sigma_2}^{\sigma_1} U_\alpha(\sigma_1, s) E_1 V_\alpha(s, \sigma_2) P^\alpha ds. \quad (2.121)$$

Let $r = \sigma_1 - \sigma_2$, $g \in \text{Ran} P^\alpha$ and $\eta(\sigma_1) := V_\alpha(\sigma_1, \sigma_2) g$. We claim that if $e^{\alpha r} \leq \beta(\tau(\sigma_2))^{-1/32}$ then we have

$$\|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(\sigma_1)\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(\sigma_2)\|_\infty. \quad (2.122)$$

To prove the claim we compute each term on the right hand side of (2.121).

(A) Notice that $P^\alpha \eta(s) = \eta(s)$. We use Proposition 19 to obtain for $e^{\alpha r} \leq \beta(\tau(\sigma_2))^{-1/32}$ that

$$\|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} U_\alpha(\sigma_1, \sigma_2) g\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} g\|_\infty. \quad (2.123)$$

(B) By Lemma 20 and (2.120) we obtain

$$\|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \int_{\sigma_2}^{\sigma_1} U_\alpha(\sigma_1, s) E_1 \eta(s) ds\|_\infty \lesssim \int_{\sigma_2}^{\sigma_1} e^{2\alpha(\sigma_1-s)} \beta(\tau(s))^{1/2} \langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(s) ds. \quad (2.124)$$

Using the condition $e^{\alpha r} \leq \beta(\sigma_2)^{-1/32}$ and the relation $\beta(\tau(s)) \leq \beta(\tau(\sigma_2))$ for $s \geq \sigma_2$ again, we find

$$\|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \int_{\sigma_2}^{\sigma_1} U_\alpha(\sigma_1, s) E_1 \eta(s) ds\|_\infty \lesssim \int_{\sigma_2}^{\sigma_1} e^{-\alpha(\sigma_1-s)} \beta(\tau(s))^{1/2} \langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(s) ds. \quad (2.124)$$

Equations (2.121), (2.123) and (2.124) imply for $e^{\alpha r} \leq \beta^{-1/32}(\tau(\sigma_2))$ that (remember that $\eta(\sigma_2) = g$)

$$\|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(\tau)\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(\sigma_2)\|_\infty + \int_{\sigma_2}^{\tau} e^{-\alpha(\tau-s)} \beta(\tau(s))^{1/2} \langle z \rangle^{-3} e^{-\frac{\alpha|z|^2}{4}} \eta(s) ds. \quad (2.125)$$
Next, we define a function $K(r)$ as
\[ K(r) := \max_{0 \leq s \leq r} \langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} \eta(\sigma_2 + s). \]  
(2.126)

Then (2.125) implies that
\[ K(\sigma_1 - \sigma_2) \lesssim \|\langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} \eta(\sigma_2)\|_{\infty} + \int_{\sigma_2}^{\sigma_1} \beta(\tau(s)) \frac{1}{2} ds K(\sigma_1 - \sigma_2). \]

We observe that
\[ \int_{\sigma_2}^{\sigma_1} \beta(\tau(s)) \frac{1}{2} ds \leq 1/2 \]
if $e^{\alpha r} \leq \beta(\tau(\sigma_2))^{-1/32}$ and if $\beta(0)$ and, therefore, $\beta(\tau(s)) = \frac{1}{\beta(0) + \frac{1}{p-1} \tau(s)}$ are small. Thus we have
\[ K(\sigma_1 - \sigma_2) \lesssim \|\langle z \rangle^{-3} e^{\frac{\alpha|z|^2}{4}} \eta(\sigma_2)\|_{\infty}, \]
which together with Equation (2.126) implies (2.122). Writing
\[ V_\alpha(\tau, \sigma) = V_\alpha(\sigma_1, \sigma_2) V_\alpha(\sigma_2, \sigma_3) \cdots V_\alpha(\sigma_{m-1}, \sigma_m) \]
with $\sigma_1 = \tau$, $\sigma_m = \sigma$ and $|\sigma_i - \sigma_{i+1}| = r$ such that $e^{\alpha r} \leq \beta(\tau(\sigma_k)) \forall k$ and iterating (2.122) completes the proof of the theorem.

\[ \square \]

**2.12 Estimate of $M_1(\tau)$ (Equation (2.49))**

In this subsection we derive an estimate for $M_1(T)$ given in Equation (2.49). Given any time $\tau'$, choose $T = \tau'$ and pass from the unknown $\xi(y, \tau)$, $\tau \leq T$, to the new unknown $\eta(z, \sigma)$, $\sigma \leq S$, given in (2.90). Now we estimate the latter function. To this end we use Equation (2.91). Observe that the function $\eta$ is not orthogonal to the first three eigenvectors of the operator $L_0$ defined in (2.92). Thus we apply the projection $P^\alpha$ to Equation (2.91) to get
\[ \frac{d}{d\sigma} P^\alpha \eta = -P^\alpha L_0 P^\alpha \eta + P^\alpha \sum_{k=1}^{4} D_k, \]  
(2.127)
where we used the fact that $P^\alpha$ are $\tau$-independent and the functions $D_k \equiv D_k(\sigma)$, $k = 1, 2, 3, 4$, are defined as

\[ D_1 := -P^\alpha V \eta + P^\alpha V P^\alpha \eta, \quad D_2 := W(a, b, \alpha) \eta, \]

\[ D_3 := F(a, b, \alpha), \quad D_4 := N(\eta, a, b, \alpha), \]

recall the definitions of the functions $V$, $W$, $F$ and $N$ after (2.92).

**Lemma 21.** If $A(\tau), B_i(\tau) \leq \beta^{-\frac{\gamma}{2}}(\tau)$ for $\tau \leq T$ and $\|b_0\| \ll 1$, then we have

\[
\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_1(\sigma)\|_\infty \lesssim \beta^{3/2}(\tau(\sigma)) M_1(T),
\]  

\[
\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_2(\sigma)\|_\infty \lesssim \beta^{2+\frac{\gamma}{2}}(\tau(\sigma)) M_1(T),
\]

\[
\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_3(\sigma)\|_\infty \lesssim \beta^{\min\{5/2,2p\}}(\tau(\sigma))[1 + M_1(T)(1 + A(T)) + M_2(T) + M_3(T)],
\]

\[
\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_4\|_\infty \lesssim \beta^2(\tau(\sigma)) M_1(T) [\beta^{1/2}(\tau(\sigma)) M_1(T) + M_2(T)] + \beta^{2+\frac{\gamma}{2}}(\tau(\sigma)) M_4^{-1}(T) + M_2(T).
\]

**Proof.** In what follows we use the following estimates, implied by (2.93),

\[
\frac{\lambda_1}{\lambda}(t(\tau)) - 1 = O(\beta(\tau)), \quad \text{thus} \quad \frac{\lambda_1}{\lambda}(t(\tau)), \quad \frac{\lambda}{\lambda_1}(t(\tau)) \leq 2, \quad \langle z \rangle^{-3} \lesssim \langle y \rangle^{-3}
\]

where, recall that $z := \frac{\lambda_1}{\lambda} y$. We start with proving the following two estimates which will be used frequently below

\[
\|e^{\frac{\alpha z^2}{4}} \eta(\sigma)\|_\infty \lesssim \beta^{1/2}(\tau(\sigma)) M_1(\tau(\sigma)) + M_2(\tau(\sigma)) \lesssim \beta^{1/2}(\tau(\sigma)) M_1(T) + M_2(T),
\]

\[
\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma)\|_\infty \lesssim \beta^2(\tau(\sigma)) M_1(\tau(\sigma)) \lesssim \beta^2(\tau(\sigma)) M_1(T).
\]

Denote by $\chi_{\geq D}$ and $\chi_{\leq D}$ the characteristic functions of the sets $\{|x| \geq D\}$ and $\{|x| \leq D\}$:

\[
\chi_{\geq D}(x) := \begin{cases} 1, & \text{if } |x| \geq D \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \chi_{\leq D} := 1 - \chi_{\geq D}.
\]
Chapter 2. On blowup in nonlinear heat equations

We take $D := \frac{C}{\sqrt{\beta}}$ where $C$ is a large constant. Writing $1 = 1 - \chi_{\geq D} + \chi_{\geq D}$ and using the inequality $1 - \chi_{\geq D} \lesssim \beta^{-3/2}(\tau)\langle y \rangle^{-3}$, the relation between $\xi$ and $\eta$, see (2.90), and Estimate (2.132) we find

\[
\|e^{\frac{a|z|^2}{4}}\eta(\sigma)\|_{\infty} \lesssim \|e^{\frac{a(\tau(\sigma))|\eta|^2}{4}}\xi(\tau(\sigma))\|_{\infty} \lesssim \beta^{-3/2}(\tau(\sigma))\|\langle y \rangle^{-3}e^{\frac{a(\tau(\sigma))|\eta|^2}{4}}\xi(\tau(\sigma))\|_{\infty} \\
+\|e^{\frac{2|\eta|^2}{4}}\chi_{\geq D}\xi(\tau)\|_{\infty} \leq \beta^{1/2}(\tau(\sigma))M_1(\tau(\sigma)) + M_2(\tau(\sigma))
\]

(2.136)

which is (2.133). Similarly recall that $z = \frac{\lambda}{\chi}y$ which together with (2.90) and (2.132) yields

\[
\|\langle z \rangle^{-3}e^{\frac{a|z|^2}{4}}\eta(\sigma)\|_{\infty} \lesssim \|\langle y \rangle^{-3}e^{\frac{a(\tau(\sigma))|\eta|^2}{4}}\xi(\tau(\sigma))\|_{\infty} \lesssim \beta^2(\tau(\sigma))M_1(\tau(\sigma)) \leq \beta^2(\tau(\sigma))M_1(T).
\]

Thus we have (2.134).

Now we proceed directly to proving the lemma. First we rewrite $D_1$ as

\[
D_1(\sigma) = -P^\alpha \frac{2pa\bar{\beta}(\tau(\sigma))z}{(p-1)(p-1 + z\bar{\beta}(\tau(\sigma))z)}(1 - P^\alpha)\eta(\sigma).
\]

Now, using that $\langle z \rangle^{-1}\frac{zbz}{p-1+zbz} \lesssim \|b\|^{1/2}$ and that $\|b\| \lesssim \beta$, we obtain

\[
\|\langle z \rangle^{-3}e^{\frac{a|z|^2}{4}}D_1(\sigma)\|_{\infty} \lesssim \beta^{1/2}(\tau)\|\langle z \rangle^{-3}e^{\frac{a|z|^2}{4}}(1 - P^\alpha)\eta(\sigma)\|_{\infty}
\]

Next, due to the explicit form of $P^\alpha := 1 - P^\alpha$, i.e. $P^\alpha = |\phi_{0,\alpha}\rangle\langle \phi_{0,\alpha}| + \sum_{i=1}^{n} |\phi_{1,\alpha}^{(i)}\rangle\langle \phi_{1,\alpha}^{(i)}| + \sum_{1 \leq i \neq j \leq n} |\phi_{2,\alpha}^{(i)}\rangle\langle \phi_{2,\alpha}^{(i)}| + \sum_{1 \leq i \neq j \leq n} |\phi_{2,\alpha}^{(i)}\rangle\langle \phi_{2,\alpha}^{(i)}|$, where $\phi_{m,\alpha}$ are the normalized eigenfunctions of the operator $L_0 := -\Delta + \frac{a^2}{4}|z|^2 - \frac{n+4}{2}\alpha$, and decay properties of these eigenfunctions, we have for any function $g$

\[
\|\langle z \rangle^{-2}e^{\frac{a|z|^2}{4}}P^\alpha g\|_{\infty} \lesssim \|\langle z \rangle^{-3}e^{\frac{a|z|^2}{4}}g\|_{\infty}.
\]

(2.137)

Collecting the estimates above and using (2.134), we arrive at

\[
\|\langle z \rangle^{-3}e^{\frac{a|z|^2}{4}}D_1(\sigma)\|_{\infty} \lesssim \beta^{1/2}(\tau(\sigma))\|\langle z \rangle^{-3}e^{\frac{a|z|^2}{4}}\eta(\sigma)\|_{\infty} \lesssim \beta^{5/2}(\tau(\sigma))M_1(T).
\]
To prove (2.129) we recall the definition of $D_2$ and rewrite it as

$$D_2 = \left( \frac{\lambda^2}{\lambda_1^2} - 1 \right) \frac{p(a + \frac{1}{2})}{p - 1 + yby} + \frac{p(a - \alpha)}{p - 1 + yby} + \frac{p\left( \frac{\lambda^2}{\lambda_1^2} - 1 \right)yby}{(p - 1 + z\beta z)(p - 1 + yby)} + \frac{p(\alpha - \frac{1}{2})}{p - 1 + yby} - \frac{2p(\alpha - \frac{1}{2})}{p - 1 + z\beta z} + \frac{pz(\beta - b)z}{(p - 1 + z\beta z)(p - 1 + z\beta z)} \right) \eta.$$

Then Equations (2.93), (2.96) and the definition of $B$ in (2.43) imply

$$\| (z)^{-3} e^{\frac{\alpha t}{4} |z|^2} D_2(\sigma) \|_{\infty} \leq \beta \tau(\sigma) \| (z)^{-3} e^{\frac{\alpha t}{4} |z|^2} \eta(\sigma) \|_{\infty}.$$

Using (2.134) we obtain (2.129) (recall $\kappa := \min\{\frac{1}{2}, \frac{p-1}{2}\}$).

Now we prove (2.130). By (2.132) and the relation between $D_3$, $F$ and $\mathcal{F}$ we have

$$\| (z)^{-3} e^{\frac{\alpha t}{4} |z|^2} D_3(\sigma) \|_{\infty} \leq \| (y)^{-3} e^{\frac{\alpha t}{4} |y|^2} F(a, b, c)(\tau(\sigma)) \|_{\infty}$$

which together with (2.88) implies (2.130).

Lastly we prove (2.131). By the relation between $D_4$, $N$ and $\mathcal{N}$ and the estimate in (2.74) we have

$$\| (z)^{-3} e^{\frac{\alpha t}{4} |z|^2} D_4(\sigma) \|_{\infty} \leq \| (y)^{-3} e^{\frac{\alpha t}{4} |y|^2} N(\xi(\tau(\sigma)), b(\tau(\sigma)), c(\tau(\sigma))) \|_{\infty}$$

$$\leq \| (y)^{-3} e^{\frac{\alpha t}{4} |y|^2} \xi(\tau(\sigma)) \|_{\infty} \| e^{\frac{\alpha t}{4} |y|^2} \xi(\tau(\sigma)) \|_{\infty} + \| e^{\frac{\alpha t}{4} |y|^2} \xi(\tau(\sigma)) \|_{\infty}^{p-1}.$$

Using (2.136) and the definition of $M_1$ we complete the proof.

Below we will need the following lemma. Recall that $S := \sigma(t(T))$.

**Lemma 22.** If $A(\tau) \leq \beta^{-\frac{5}{4}}(\tau)$, then for any $c_1, c_2 > 0$ there exists a constant $c(c_1, c_2)$ such that

$$\int_{0}^{S} e^{-c_1(S-\sigma)} \beta^{c_2}(\tau(t(\sigma))) d\sigma \leq c(c_1, c_2) \beta^{c_2}(T). \quad (2.138)$$

**Proof.** We use the shorthand $\tau(\sigma) \equiv \tau(t(\sigma))$, where, recall $t(\sigma)$ is the inverse of $\sigma(t) = \int_{0}^{t} \lambda_1^2(k) dk$ and $\tau(t) = \int_{0}^{t} \lambda_2^2(k) dk$. By Proposition 14 we have that $\frac{1}{2} \leq \frac{\lambda}{\lambda_1} \leq 2$ provided that $A(\tau) \leq \beta^{-\frac{5}{4}}(\tau)$. Hence

$$\frac{1}{4} \sigma \leq \tau(\sigma) \leq 4\sigma \quad (2.139)$$
which implies \( \frac{1}{\beta(0) + \sigma} \lesssim \frac{1}{\beta(0)} \). By a direct computation we have

\[
\int_0^S e^{-c_1(S - \sigma)} \beta(\tau(\sigma)) d\sigma \leq c(c_1, c_2) \frac{1}{(1 + 4 \sigma S)^{c_2}}.
\]

(2.140)

Using (2.139) again we obtain \( 4S \geq \tau(S) = T \geq \frac{1}{4} S \) which together with (2.140) implies (2.138).

By Proposition 15, Equation (2.134) and the slow decay of \( \beta(\tau) \) we obtain

\[
K_1 \lesssim e^{-c_0 S} \|\langle z \rangle^{-3} e^{\frac{a|z|^2}{4}} P^\alpha \eta(S)\|_\infty \lesssim e^{-c_0 S} \|\langle z \rangle^{-3} e^{\frac{a|z|^2}{4}} \eta(0)\|_\infty \lesssim \beta^2(T) M_1(0).
\]

(2.142)

By Proposition 15, Equations (2.128)- (2.131) and \( \int_0^S e^{-c_0(S - \sigma)} \beta^2(\tau(\sigma)) d\sigma \lesssim \beta^2(T) \) (see Lemma 22) we have

\[
K_2 \lesssim \beta^2(T) \{ \beta^2(0)[1 + M_1(T) A(T) + M_1^2(T)] + [M_2(T) M_1(T) + M_1(T) M_0^{p-1}(T)] \}.
\]

(2.143)

Equation (2.90) and the definitions of \( S \) and \( T \) imply that \( \lambda_1(t(S)) = \lambda(t(T)), z = y, \eta(S) = \xi(T), \) and \( P^\alpha \xi = \xi, \) consequently

\[
\|\langle z \rangle^{-3} e^{\frac{a|z|^2}{4}} P^\alpha \eta(S)\|_\infty = \|\langle y \rangle^{-3} e^{\frac{a|y|^2}{4}} \xi(T)\|_\infty.
\]

(2.144)
Collecting the estimates (2.141)-(2.144) and using the definition of $M_1$ in (2.45) we have

$$M_1(T) := \sup_{\tau \leq T} \beta^{-2}(\tau)\|y\|^{-3}e^{\frac{a|y|^2}{4}} \xi(\tau)\|_{\infty}$$

$$\lesssim M_1(0) + \beta^2(0)[1 + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] + M_2(T)M_1(T) + M_1(T)M_2^{p-1}(T)$$

which together with the fact that $T$ is arbitrary implies Equation (2.49).

2.13 Estimate of $M_2$ (Equation (2.50))

The following lemma is proven similarly to the corresponding parts of Lemma 21 and therefore it is presented without a proof.

**Lemma 23.** If $A(\tau), B(\tau) \leq \beta^{-\frac{p}{2}}(\tau)$ and $\|b_0\| \ll 1$ and $D_k(\sigma), k = 2, 3, 4,$ are the same as in Lemma 21, then

$$\|e^{\frac{\alpha}{4}|z|^2} D_2(\sigma)\|_{\infty} \lesssim \beta^2(\tau(\sigma))[\beta^{1/2}(\tau(\sigma))M_1(T) + M_2(T)];$$

$$\|e^{\frac{\alpha}{4}|z|^2} D_3(\sigma)\|_{\infty} \lesssim \beta^{\min\{1,2p-1\}}(\tau(\sigma))[1 + M_1(T)(1 + A(T)) + M_1^2(T) + M_1^p(T)];$$

$$\|e^{\frac{\alpha}{4}|z|^2} D_4(\sigma)\|_{\infty} \lesssim \beta(\tau(\sigma))M_1^2(T) + M_2^2(T) + \beta^{p/2}(\tau(\sigma))M_1^p(T) + M_2^p(T).$$

To estimate $M_2$ it is convenient to treat the $z$-dependent part of the potential in (2.92) as a perturbation. Let the operator $L_0$ be the same as in (2.91). Rewrite (2.91) to have

$$\eta(S) = e^{-(L_0 + \frac{2p\alpha}{p-1})S} \eta(0) + \int_0^S e^{-(L_0 + \frac{2p\alpha}{p-1})(S-\sigma)}(V_2 \eta(\sigma) + \sum_{k=2}^4 D_k(\sigma))d\sigma,$$

where, recall $S := \sigma(t(T)), V_2$ is the operator given by

$$V_2 := \frac{2p\alpha}{p-1 + z\beta z},$$
and the terms $D_n$, $n = 2, 3, 4$, are the same as in (2.127). Lemma 16 implies that
\[
\|e^{\frac{\alpha|y|^2}{4}}e^{-(L_0 + \frac{2\alpha}{p-1})s}g\|_\infty = e^{-\frac{2\alpha}{p-1}s}\|e^{\frac{\alpha|y|^2}{4}}e^{-L_0s}g\|_\infty \lesssim e^{-\frac{2\alpha}{p-1}s}\|e^{\frac{\alpha|y|^2}{4}}g\|_\infty
\]
for any function $g$ and time $s \geq 0$. Hence we have
\[
\|e^{\frac{\alpha|z|^2}{4}}\eta(S)\|_\infty \lesssim K_0 + K_1 + K_2
\]
where the functions $K_i$ are given by
\[
K_0 := e^{-\frac{2\alpha}{p-1}S}\|e^{\frac{\alpha|z|^2}{4}}\eta(0)\|_\infty;
\]
\[
K_1 := \int_0^S e^{-\frac{2\alpha}{p-1}(S-\sigma)}\|e^{\frac{\alpha|z|^2}{4}}V_2\eta(\sigma)\|_\infty d\sigma;
\]
\[
K_2 := \sum_{n=2}^4 \int_0^S e^{-\frac{2\alpha}{p-1}(S-\sigma)}\|e^{\frac{\alpha|z|^2}{4}}D_n\|_\infty d\sigma.
\]
We estimate the $K_n$’s, $n = 0, 1, 2$.

(K0) We start with $K_0$. By (2.133) and the decay of $e^{-\frac{2\alpha}{p-1}S}$ we have
\[
K_0 \lesssim M_2(0) + \beta^{1/2}(0)M_1(0).
\]

(K1) By the definition of $V_2$ we have
\[
\|e^{\frac{\alpha|z|^2}{4}}V_2\eta(\sigma)\|_\infty \lesssim \|\frac{1}{\beta(\tau(\sigma))}\langle z \rangle^{-2}e^{\frac{\alpha|z|^2}{4}}\eta(\sigma)\|_\infty.
\]
Moreover by the relation between $\xi$ and $\eta$ in Equation (2.90) and Proposition 14 we have
\[
\max_{0 \leq \sigma \leq S} \|e^{\frac{\alpha|y|^2}{4}}V_2\eta(\sigma)\|_\infty \lesssim \max_{0 \leq \tau \leq T} \|\frac{1}{\beta(y)}\langle y \rangle^{-2}e^{\frac{\alpha|y|^2}{4}}\xi(\tau)\|_\infty
\]
\[
\leq \max_{0 \leq \tau \leq T} \frac{1}{\beta} (\|\langle y \rangle^{-3}e^{\frac{\alpha|y|^2}{4}}\xi(\tau)\|_\infty)^{\frac{2}{3}} (\|e^{\frac{\alpha|\eta|^2}{4}}\xi(\tau)\|_\infty)^{\frac{1}{3}}
\]
\[
\leq \beta^{\frac{2}{3}}(0)M_1^\frac{2}{3}(T)M_2^\frac{1}{3}(T).
\]
Therefore we obtain
\[
K_1 \lesssim \max_{0 \leq \sigma \leq S} \|e^{\frac{\alpha|y|^2}{4}}V_2\eta(\sigma)\|_\infty \int_0^S e^{-\frac{2\alpha}{p-1}(S-\sigma)}d\sigma
\]
\[
\lesssim \beta^{\frac{2}{3}}(0)M_1^\frac{2}{3}(T)M_2^\frac{1}{3}(T).
\]
(K2) By the definitions of $D_k$, $k = 2, 3, 4$, and Equations (2.145)-(2.147) we have

$$
\sum_{k=2}^{4} \|e^{\frac{\alpha|z|^2}{4}} D_k(\sigma)\|_\infty \lesssim \beta^\frac{5}{2}(\tau(\sigma))[1 + M_2(T) + M_1(T)A(T) + M_1^p(T) + M_2^p(T)] + M_2^2(T) + M_2^p(T)
$$

and consequently

$$
K_2 \lesssim \beta^\frac{5}{2}(0)[1 + M_2(T) + M_1(T)A(T) + M_1^p(T) + M_2^p(T)] + M_2^2(T) + M_2^p(T). \tag{2.152}
$$

Collecting the estimates (2.149)-(2.152) we have

$$
\|e^{\frac{\alpha|z|^2}{4}} \eta(S)\|_\infty \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + \beta^\frac{5}{2}(0)M_1^2(T)M_2^\frac{3}{2}(T)
$$

$$
+ \beta^\frac{5}{2}(0)[1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] + M_2^2(T) + M_2^p(T). \tag{2.153}
$$

The relation between $\xi$ and $\eta$ in Equation (2.90) implies

$$
\|e^{\frac{\alpha|z|^2}{4}} \xi(T)\|_\infty = \|e^{\frac{\alpha|z|^2}{4}} \eta(S)\|_\infty
$$

which together with (2.153) gives

$$
M_2(T) \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + \beta^\frac{5}{2}(0)M_1^2(T)M_2^\frac{3}{2}(T) + M_2^2(T) + M_2^p(T)
$$

$$
+ \beta^\frac{5}{2}(0)[1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)].
$$

Since $T$ is an arbitrary time, the proof of the estimate (2.50) for $M_2$ is complete.
Chapter 3

Stability of spherical collapse under mean curvature flow

3.1 Introduction

We study the behavior of mean curvature flow (MCF) of hypersurfaces in $\mathbb{R}^{n+1}$ with initial conditions close to spheres. Given an initial simple, closed hypersurface $S_0$ in $\mathbb{R}^{n+1}$ the MCF determines a family $\{S_t | t \geq 0\}$ of closed hypersurfaces in $\mathbb{R}^{n+1}$, given by immersions $X(\cdot, t) : \Omega \to \mathbb{R}^{n+1}$, satisfying the following evolution equation:

$$\frac{\partial X}{\partial t} = -H(X)\nu(X),$$

where $\Omega \subset \mathbb{R}^{n+1}$ is a fixed $n$–dimensional hypersurface, $\nu(X)$ and $H(X)$ are the outward unit normal vector and mean curvature at $X \in S_t$, respectively. We show that if $S_0$ is close to an Euclidean $n$-sphere in the norm $H^s$, $s > \frac{n}{2} + 1$, then the solution $S_t$ collapses to a round point in a finite time. Due to the translation and dilation symmetry of (3.1), it suffices to consider $S_0$ close to the standard $n$-sphere (the Euclidean sphere with radius 1 and center at the origin).

The mean curvature flow is the steepest descent flow for the area functional. It arises
in applications, such as models of annealing metals \cite{73} and other problems involving phase separation and moving interfaces (\cite{27, 57, 11}). It has been recently successfully applied by Huisken and Sinestrari to topological classification of surfaces and submanifolds (\cite{54, 55}, see also \cite{61}). It is closely related to the Ricci and inverse mean curvature flow.

Mean curvature flow was first studied by Brakke \cite{9}. Evans and Spruck \cite{28} constructed a unique weak solution of the nonlinear PDE for certain functions whose zero level set evolves in time according to its mean curvature. Similar results were obtained by Chen, Giga and Goto \cite{15} and by Ambrosio and Soner \cite{3}. The short-time existence in Hölder spaces was proven in \cite{9, 52, 28, 56}. Ecker and Huisken \cite{24} established interior estimates and the short time existence for initial surface which is locally Lipschitz continuous follows from this immediately. Higher codimension mean curvature flows were studied by Mu-Tao Wang \cite{85} (see also \cite{64}). For more results on the existence, uniqueness and regularity of the solution one can see \cite{9, 29, 30, 58, 17}.

The question of the long time existence is, as usual, more subtle. Ecker and Huisken \cite{25} showed longtime existence for mean curvature flow in the case of linearly growing graphs. Later they \cite{24} proved that if the initial surface is a locally Lipschitz continuous entire graph over $\mathbb{R}^n$ then the solution will exist for all times.

However, the most interesting aspect of the mean curvature flow is formation of singularities, with two canonical examples being solutions with initial conditions being Euclidean spheres or cylinders. In the latter cases, the solution is again a family of spheres or cylinders collapsing to their center or axis, with the radii evolving as $\sqrt{2n(t_* - t)}$ or $\sqrt{2(n-1)(t_* - t)}$. The behaviour of solutions in the last two cases are fairly different. In the former case, the first result here was due to Gage and Hamilton \cite{40}, who showed that initial convex plane curves shrink to a ‘round’ point, i.e. approach asymptotically circles of radii $\sqrt{2(t_* - t)}$. Later, Grayson \cite{48} showed that any embedded plane curves always
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Shrink smoothly until they are convex, and then to points by the evolution theorem of convex curves. For higher dimensions the latter result does not hold. In a seminal work, [52], Huisken showed that under mean curvature flow a convex hypersurface in \( \mathbb{R}^n \), \( n \geq 3 \), shrinks smoothly to a point, getting spherical in the limit. For more results see [18, 19, 62, 63, 88].

Our main result is as follows.

**Theorem 24.** Let \( \Omega \) be the standard \( n \)-dimensional sphere and let a surface \( S_0 \), defined by an immersion \( x_0 \in H^s(\Omega) \), for some \( s > \frac{n}{2} + 1 \), be close to \( \Omega \), in the sense that \( \|x_0 - 1\|_{H^s} \ll 1 \). Then there exist \( t_* < \infty \) and \( z_* \in \mathbb{R}^{n+1} \), s.t. (3.1) has the unique solution, \( S_t, t < t_* \), and this solution contracts to the point \( z_* \), as \( t_* \to \infty \). Moreover, \( S_t \) is defined by an immersion \( x(\cdot, t) \in H^s(\Omega) \), with the same \( s \), of the form

\[
x(\omega, t) = z(t) + R(\omega, t)\omega,
\]

for some \( z(t) \in \mathbb{R}^{n+1} \) and \( R(\cdot, t) \in H^s(\Omega) \), satisfying \( z(t) = z_* + O((t_* - t)^{\frac{1}{2} + \frac{1}{2s} (n + \frac{1}{2} - \frac{1}{n})}) \) and

\[
R(\omega, t) = \lambda(t)(\sqrt{n \frac{n}{a(t)}} + \xi(\omega, t)),
\]

with \( \lambda(t), a(t) \) and \( \xi(\cdot, t) \) which satisfy

\[
\lambda(t) = \sqrt{2a(t)(t_* - t)} + O((t_* - t)^{\frac{1}{2} + \frac{1}{2s} (1 - \frac{1}{n})}),
\]

\[
a(t) = -\lambda(t)\dot{\lambda}(t) = a_* + O((t_* - t)^{\frac{1}{2} + \frac{1}{2s} (1 - \frac{1}{n})}) \text{ and } \|\xi(\cdot, t)\|_{H^s} \lesssim (t_* - t)^{\frac{1}{n}}. \text{ Moreover, } |z_*| \ll 1.
\]

**Remark 2.** If the initial condition \( x_0 \) is invariant under the transformation \( x_i \to -x_i \) for any \( i = 1, \ldots, n + 1 \), then \( z(t) = 0 \) and the proof below simplifies considerably.
above is that, unlike spheres, the cylinders are not stable under the mean curvature flow. For instance, it follows from results of \cite{42} that for an open set of initial conditions arbitrarily close to a cylinder which have an arbitrary shallow ‘waist’, the solution to the mean curvature flow forms a ‘neck’ which pinches in a finite time.

The form of expression (3.2) above is a reflection of a large class of symmetries of the mean curvature flow:

- (3.1) is invariant under rigid motions of the surface, i.e. $X \mapsto RX + a$, where $R \in O(n+1)$, $a \in \mathbb{R}^{n+1}$ and $X = X(u,t)$ is a parametrization of $S_t$, is a symmetry of (3.1).

- (3.1) is invariant under the scaling $X \mapsto \lambda X$ and $t \mapsto \lambda^{-2}t$ for any $\lambda > 0$.

Our approach utilizes these symmetries in an essential way. It uses the rescaling of the equation (3.1) by a parameter $\lambda(t)$ whose behaviour is determined by the equation itself and a series of differential inequalities for a Lyapunov-type functions.

This chapter is organized as follows. We rescale the equation (3.1) in Section 3.2 by introducing collapse variables, designed so that the new equation has global solutions and reformulate the main theorem in terms of the rescaled surfaces. Then in Section 3.3 we derive the equation for the surface as a normal graph over a sphere. In Section 3.4 we introduce a notion of the ‘center’ of a surface, close to a unit sphere in $\mathbb{R}^{n+1}$ and show that such a center exists. We will show in Section 3.9 that the centers $z(t)$ of the solutions to (3.1) converge to the collapse point, $z_*$, of Theorem 24. In Section 3.5 we reparametrize the solutions of the new equation by isolating the leading term and a perturbation. In Section 3.6 we use the Lyapunov-Schmidt type decomposition to derive equations for the parameters and perturbation. In Section 3.7 we discuss the spectrum of the linearized equation. In Section 3.8 we introduce certain Lyapunov functionals and derive differential inequalities for them. These inequalities are used in Section 3.9 to
derive a priori bounds on Sobolev norms of the perturbation. In Section 3.9 we prove the main theorem.

**Notation.** The relation \( f \lesssim g \) for positive functions \( f \) and \( g \) signifies that there is a numerical constant \( C \), s.t. \( f \leq Cg \).

### 3.2 Rescaled equation

Instead of the surface \( S_t \), it is convenient to consider the new, rescaled surface \( \tilde{S}_\tau = \lambda^{-1}(t)(S_t - z(t)) \), where \( \lambda(t) \) and \( z(t) \) are some differentiable functions to be determined later, and \( \tau = \int_0^t \lambda^{-2}(s)ds \). The new surface is described by \( y \), which is, say, an immersion of some fixed \( n \)-dimensional hypersurface \( \Omega \subset \mathbb{R}^{n+1} \), i.e. \( y(\cdot, \tau) : \Omega \rightarrow \mathbb{R}^{n+1} \), or a local parametrization of \( \tilde{S}_\tau \), i.e. \( y(\cdot, \tau) : U \rightarrow S_t \). Thus the new collapse variables are given by

\[
y(\omega, \tau) = \lambda^{-1}(t)(X(\omega, t) - z(t)) \quad \text{and} \quad \tau = \int_0^t \lambda^{-2}(s)ds.
\]

(3.3)

Let \( \dot{\lambda} = \frac{\partial \lambda}{\partial t} \) and \( \frac{\partial z}{\partial \tau} \) be the \( \tau \)-derivative of \( z(t(\tau)) \), where \( t(\tau) \) is the inverse function of \( \tau(t) = \int_0^t \lambda^{-2}(s)ds \). Using that \( \frac{\partial X}{\partial t} = \frac{\partial y}{\partial \tau} + \dot{\lambda} y + \lambda \frac{\partial y}{\partial \tau} = \lambda^{-2} \frac{\partial z}{\partial \tau} + \dot{\lambda} y + \lambda^{-1} \frac{\partial y}{\partial \tau} \) and \( H(\lambda y) = \lambda^{-1} H(y) \), we obtain from (3.1) the equation for \( y, \lambda \) and \( z \):

\[
\frac{\partial y}{\partial \tau} = -H(y)\nu(y) + ay - \lambda^{-1} \frac{\partial z}{\partial \tau} \quad \text{and} \quad a = -\lambda \dot{\lambda}.
\]

(3.4)

In what follows we take \( \Omega \) to be the unit sphere centered at the origin. In this case, the equation (3.4) has static solutions (\( a = \) a positive constant, \( z = 0 \), \( y(\omega) = \sqrt{\frac{\pi}{a}} \omega \)).

Standard results on the local well-posedness for the mean curvature flow (see e.g. [67], Theorem 8.3, and also [26, 38]) imply that for an initial condition \( y_0 \in C^\alpha, \alpha > 1 \), and given functions \( a(\tau), z(\tau) \in C^1 \cap L^\infty(\mathbb{R}) \), there is \( T > 0 \), s.t. (3.4) has a unique solution, \( y \in C^\alpha \), on the time interval \([0, T]\) and either \( T = \infty \) or \( T < \infty \) and \( \|y\|_{C^\alpha} \rightarrow \infty \) and \( \tau \rightarrow T \). Here \( C^\alpha \) is the space of \([\alpha](= \text{the integer part of } \alpha) \) times differentiable functions.
on $\Omega$ (or $U$), whose highest derivatives are Hölder continuous with the index $\alpha - [\alpha]$. This result extends also to $H^s(\Omega)$ with $s > \frac{n}{2} + 1$.

Our goal is to prove the following result.

**Theorem 25.** Let $\rho_0 \in H^s(\Omega)$ satisfy $\|\rho_0 - 1\|_H^s \ll 1$ for some $s > \frac{n}{2} + 1$, and let $\lambda_0 > 0$ and $|z_0| \ll 1$. Then (3.4) with initial data $(y_0 = \rho_0(\omega), \lambda_0, z_0)$ has a unique solution $(y, \lambda, z) \forall \tau$, with $y(\omega, \tau)$ of the form $y(\omega, \tau) = \rho(\omega, \tau)\omega$, with $\rho(\cdot, \tau) \in H^s(\Omega)$,

$$\rho(\omega, \tau) = \sqrt{\frac{n}{a(\tau)}} + \xi(\omega, \tau),$$

and $\lambda(\tau)$ and $z(\tau)$ satisfying $\lambda(\tau) = \lambda_0 e^{-\int_0^\tau a(s)ds}$ and $|z(\tau) - z_0| \lesssim e^{-(n+1) \frac{1}{2} \tau}$, for some $a(\tau) = a_\ast + O(e^{-(1 - \frac{1}{2n})\tau})$, with $|a_\ast - n| \leq \frac{1}{2}$ and $|z_\ast| \ll 1$.

This Theorem together with (3.3) implies Theorem 24 (see Section 3.9).

### 3.3 Differential equation for $\rho$

In what follows $g_{ij}$ is the metric induced on $\Omega$ by the inner product in $\mathbb{R}^{n+1}$ and $\Delta$ is the Laplace-Beltrami operator in this metric. Furthermore we define $(Hess \rho)_{ij} = \frac{\partial^2 \rho}{\partial u_i \partial u_j} - \Gamma^k_{ij} \frac{\partial \rho}{\partial u_k}$, $k = \frac{1}{2} g^{km} (\frac{\partial g_{km}}{\partial u_i} + \frac{\partial g_{kn}}{\partial u_j} - \frac{\partial g_{jn}}{\partial u_k})$, and $\nabla^k \rho = g^{kn} \frac{\partial \rho}{\partial u_n}$ in a local parametrization $x = x(u)$ (so that $g_{ij} := \frac{\partial x^k}{\partial u_i} \frac{\partial x^k}{\partial u_j}$), where the summation over the repeated indices is assumed. (Note that $(Hess)_{ij} = \nabla_i \nabla_j$, where $\nabla_i \rho = \frac{\partial \rho}{\partial u_i}$ and $(\nabla \omega)_j = \frac{\partial \omega_j}{\partial u_i} - \Gamma^k_{ij} \omega_k$.)

**Proposition 26.** Let $\tilde{S}_\tau = \lambda^{-1}(t)(S_t - z(t))$ be defined by an immersion $y(\omega, \tau) = \rho(\omega, \tau)\omega$ of $\Omega$ for some functions $\rho(\cdot, \tau) : \Omega \rightarrow \mathbb{R}^+$, differentiable in their arguments and let $z(\tau) \in C^1(\mathbb{R}^+, \mathbb{R}^{n+1})$. Then $\tilde{S}_\tau$ satisfies (3.4) if and only if $\rho$ and $z$ satisfy the equation

$$\frac{\partial \rho}{\partial \tau} = G(\rho) + a \rho - \lambda^{-1} z_r \cdot \omega + \lambda^{-1} \tilde{z}_r \cdot \nabla \rho \rho,$$

where $\tilde{z}_{rk} = \frac{\partial z^i}{\partial u^r} z^j_{\tau}$, $z_r := \frac{\partial z}{\partial \tau}$ and

$$G(\rho) = \frac{1}{\rho^2} \Delta \rho - \frac{n}{\rho} - \frac{\nabla \rho \cdot Hess(\rho) \nabla \rho}{\rho^2(\rho^2 + |\nabla \rho|^2)} + \frac{|\nabla \rho|^2}{\rho(\rho^2 + |\nabla \rho|^2)}.$$  

(3.6)
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Proof. We rewrite \( (3.4) \) as

\[
\frac{\partial y}{\partial \tau} \cdot \nu(y) = -H(y) + ay \cdot \nu(y) - \lambda^{-1} \frac{\partial z}{\partial \tau} \cdot \nu(y). \quad (3.7)
\]

Recall that in our representation \( y = \rho(\omega, \tau)\omega, \; \omega \in \Omega \). We extend \( \rho \) to \( \mathbb{R}^{n+1} \setminus \{0\} \) by \( \tilde{\rho}(x, \tau) = \rho(\alpha(x), \tau) \), where \( \alpha : \mathbb{R}^{n+1} \to \Omega, \; \alpha(x) := \frac{x}{|x|} \). Then \( y = \tilde{\rho}(x, \tau) \hat{x} \) and we can write \( \tilde{S}_\tau = \{ x \in \mathbb{R}^{n+1} : \varphi(x, \tau) = \{ 0 \} \} \), where \( \varphi(x, \tau) = |x| - \tilde{\rho}(x, \tau) \). Now, \( \nu(y) = \frac{\nabla_x \varphi}{|\nabla_x \varphi|} \)

and \( \tilde{H} := \text{div}_x(\frac{\nabla_x \varphi}{|\nabla_x \varphi|}) \), where \( \nabla_x \) is the standard gradient in \( x \). Therefore

\[
\frac{\partial \tilde{\rho}}{\partial \tau} = \tilde{G}(\tilde{\rho}) + a\tilde{\rho} - \lambda^{-1} z_\tau \cdot (\hat{x} - \nabla_x \tilde{\rho}) \text{ on } S_t, \quad (3.8)
\]

where \( \tilde{G}(\tilde{\rho}) = -\tilde{H} \sqrt{1 + |\nabla_x \tilde{\rho}|^2} \).

Let \( \text{Hess}_x \) be the operator-valued matrix with the entries \( (\text{Hess}_x)_{ij} := \partial_{x_i} \partial_{x_j} \). We compute \( \tilde{H} \):

\[
\tilde{H} = \text{div}(\frac{\hat{x} - \nabla_x \tilde{\rho}}{\sqrt{1 + |\nabla_x \tilde{\rho}|^2}}) = \frac{n}{|x|} - \Delta_x \tilde{\rho} + \frac{1}{|x|} \hat{x} \cdot \nabla_x \tilde{\rho} + \frac{1}{|x|^2} \nabla_x \tilde{\rho} \cdot \text{Hess}_x(\tilde{\rho}) \nabla_x \tilde{\rho}. \quad (3.9)
\]

Since \( \tilde{\rho}(\lambda x) = \tilde{\rho}(x) \), we have that \( x \cdot \nabla_x \tilde{\rho} = 0 \). Differentiating this equation with respect to \( x_i \) we find that \( x \cdot \nabla_x \partial_{x_i} \tilde{\rho} = -\partial_{x_i} \tilde{\rho} \), and therefore \( x \cdot \nabla_x |\nabla_x \tilde{\rho}|^2 = 2|\nabla_x \tilde{\rho}|^2 \). Plugging this into \( (3.9) \) gives

\[
\tilde{H} = \frac{n}{|x|} - \Delta_x \tilde{\rho} + \frac{1}{|x|} |\nabla_x \tilde{\rho}|^2 + \frac{1}{|x|^2} \nabla_x \tilde{\rho} \cdot \text{Hess}_x(\tilde{\rho}) \nabla_x \tilde{\rho}. \quad (3.10)
\]

Let \( r = |x| \). We note first that due to the well-known representation (see [14])

\[
\Delta_x = r^{-n} \partial_r r^n \partial_r + \frac{1}{r^2} \Delta \text{ on } \mathbb{R}^{n+1}, \quad (3.11)
\]

we have that \( \Delta_x \tilde{\rho} \big|_{S_t} = \frac{1}{\rho} \Delta \tilde{\rho} \). Furthermore, since \( |x| \frac{\partial \rho}{\partial x} \) is homogeneous of degree 0,

\[
\partial_{x_i} \tilde{\rho} = \frac{\partial \rho}{\partial x^k} |_{\Omega} \frac{\partial \rho}{\partial x^i} = \frac{1}{|x|^n} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} \Omega_{jk} \nabla^k \rho = \frac{1}{|x|^n} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} \frac{\partial \rho}{\partial x^k} \nabla^k \rho = \frac{1}{|x|^n} \frac{\partial x^i}{\partial x^k} \nabla^k \rho \text{ and therefore } \nabla_x \tilde{\rho} \cdot z_\tau = \frac{1}{|x|} |x| \frac{\partial x^i}{\partial x^k} \nabla^k \rho. \text{ Next, we need the following lemma which is proved in the appendix.}
\]

Lemma 27.

\[
|\nabla_x \tilde{\rho}|^2 = \frac{1}{|x|^2} |\nabla \rho|^2, \quad (3.12)
\]

\[
\nabla_x \tilde{\rho} \cdot \text{Hess}_x(\tilde{\rho}) \nabla_x \tilde{\rho} = \frac{1}{|x|^4} (\nabla \tilde{\rho} \cdot \text{Hess}(\rho) \nabla \rho). \quad (3.13)
\]
This lemma, together with equations \( H = \tilde{H}_{\tilde{S}_r} \), (3.10) and (3.11), gives \( H(\rho) := -\frac{\rho}{\sqrt{\rho^2 + |\nabla \rho|^2}} G(\rho) \) and therefore \( \tilde{G}(\tilde{\rho})|_{\tilde{S}_r} = G(\rho) \). This, together with (3.8), gives (3.5) - (3.6). Hence if \( \tilde{S}_r \), defined by the immersion \( y(\omega, \tau) = \rho(\omega, \tau) \) of \( \Omega \), satisfies (3.4) then \( \rho(\tau) \) satisfies (3.5) - (3.6). Reversing the steps we see that if \( \rho \) satisfies (3.5) - (3.6), then the immersion \( y(\omega, \tau) = \rho(\omega, \tau) \) satisfies (3.4).

3.4 Collapse center

In this section we introduce a notion of the 'center' of a surface, close to a unit sphere, \( \Omega \), in \( \mathbb{R}^{n+1} \), and show that such a center exists. We will show in Section 3.9 that the centers \( z(t) \) of the solutions \( S_t \) to (3.1) converge to the collapse point, \( z_* \), of Theorem 24.

For a closed surface \( S \), given by an immersion \( x : \Omega \to \mathbb{R}^{n+1} \), we define the center, \( z \), by the relations \( \int_{\Omega} ((x - z) \cdot \omega) \omega^j = 0 \), \( j = 1, \ldots, n+1 \). The reason for this definition will become clear in Section 3.7.

Proposition 28. Assume a surface \( S \) is given by an immersion \( x : \Omega \to \mathbb{R}^{n+1} \), with \( y := \lambda^{-1}(x - \bar{z}) \in H^1(\Omega, \mathbb{R}^{n+1}) \) close, in the \( H^1(\Omega, \mathbb{R}^{n+1}) \)-norm, to the identity \( 1 \), for some \( \lambda \in \mathbb{R}^+ \) and \( \bar{z} \in \mathbb{R}^{n+1} \). Then there exists \( z \in \mathbb{R}^{n+1} \) such that \( \int_{\Omega} ((x - z) \cdot \omega) \omega^j = 0 \), \( j = 1, \ldots, n+1 \).

Proof. By replacing \( x \) by \( x^{\text{new}} \), if necessary, we may assume that \( \bar{z} = 0 \) and \( \lambda = 1 \). Let \( x \in H^1(\Omega, \mathbb{R}^{n+1}) \). The relations \( \int_{\Omega} ((x - z) \cdot \omega) \omega^j = 0 \forall j \) are equivalent to the equation \( F(x, z) = 0 \), where \( F(x, z) = (F_1(x, z), \ldots, F_{n+1}(x, z)) \), with \( F_j(x, z) = \int_{\Omega} ((x - z) \cdot \omega) \omega^j, \ j = 1, \ldots, n+1 \).

Clearly \( F \) is a \( C^1 \) map from \( H^1(\Omega, \mathbb{R}^{n+1}) \times \mathbb{R}^{n+1} \) to \( \mathbb{R}^{n+1} \). We notice that \( F(1, 0) = 0 \). We solve the equation \( F(x, z) = 0 \) near \((1, 0)\), using the implicit function theorem. To this end we calculate the derivatives \( \partial_z F_j = -\int_{\Omega} \omega^i \omega^j = -\frac{1}{n+1} \delta_{ij} |\Omega| \) for \( j = 1, \ldots, n+1 \).
The above relations allow us to apply implicit function theorem to show that for any \( x \) close to \( 1 \), there exists \( z \), close to 0, such that \( F(x, z) = 0 \).

Assume we have a family, \( x(\cdot, t) : \Omega \to \mathbb{R}^{n+1}, \ t \in [0, T] \), of immersions and functions \( \bar{z}(t) \in \mathbb{R}^{n+1} \) and \( \lambda(t) \in \mathbb{R}^+ \), s.t. \( \lambda^{-1}(t)(x(\omega, t) - \bar{z}(t)) \), in the \( H^1(\Omega, \mathbb{R}^{n+1}) \)-norm, to the identity \( 1 \) (i.e. a unit sphere). Then Proposition 28 implies that there exists \( z(t) \in \mathbb{R}^{n+1} \), s.t.

\[
\int_{\Omega} ((x(\omega, t) - z(t)) \cdot \omega) \omega^j = 0, \ j = 1, \ldots, n + 1. \tag{3.14}
\]

Furthermore, if \( g(\omega, \tau) := \lambda^{-1}(t)(x(\omega, t) - z(t)) = \rho(\omega, \tau) \omega \), where \( \tau = \tau(t) \) is given in \( 3.3 \), then we conclude that

\[
\int_{\Omega} \rho(\omega, \tau) \omega^j = 0, \ j = 1, \ldots, n + 1. \tag{3.15}
\]

To apply the above result to the immersion \( x(\cdot, t) : \Omega \to \mathbb{R}^{n+1} \), solving \( 3.1 \), we pick \( \bar{z}(t) \) to be a piecewise constant function constructed iteratively, starting with \( \bar{z}(t) = 0 \) for \( 0 \leq t \leq \delta \) for \( \delta \) sufficiently small (this works due to our assumption on the initial conditions), and \( \bar{z}(t) = z(\delta) \) for \( \delta \leq t \leq \delta + \delta' \) and so forth (see Section 3.9). This gives \( z(t) \in \mathbb{R}^{n+1} \), s.t. \( 3.14 \) holds. This is \( z(t) \) we use in \( 3.3 \).

### 3.5 Reparametrization of solutions

The next proposition will be used to reparametrize the initial condition for \( 3.5 \).

Proposition 29. If \( \| \rho - \sqrt{\frac{n}{a^n}} \|_{L^1} \leq \delta := \frac{(n-\frac{1}{2})^{n/2} \sqrt{\pi} |\Omega|}{6a^n} \) for some \( n - \frac{1}{2} < a < n + \frac{1}{2} \),

then there exists \( a = a(\rho) \) s.t.

\[
\rho - \sqrt{\frac{n}{a^n}} \perp 1 \text{ in } L^2(\Omega). \tag{3.16}
\]

Moreover, \( |a(\rho) - a| \lessapprox \| \rho - \sqrt{\frac{n}{a^n}} \|_{L^1} \text{ and } \| \rho - \sqrt{\frac{n}{a^n}} \|_{H^s} \lessapprox \| \rho - \sqrt{\frac{n}{a^n}} \|_{H^s}. \)
Proof. The orthogonality conditions on the fluctuation can be written as $F(\rho, a) = 0$, where $F : L^2(\Omega) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined as $F(\rho, a) = \int_{\Omega} (\rho - \sqrt{a})$.

Note first that the mapping $F$ is $C^1$ and $F(\sqrt{a_n}, a) = 0 \forall a$. We compute the linear map $\partial_a F(\rho, a)$:

$$ (\partial_a F)(\rho, a) = \int_{\Omega} \frac{\sqrt{n}}{2} a^{-3/2} = \frac{\sqrt{n}}{2a^{3/2}} |\Omega|. $$

(3.17)

Hence $\partial_a F(\rho, a)$ is invertible. Thus, by implicit function theorem, the equation $F(\rho, a) = 0$ has a unique solution for $a$ in a neighbourhood of the point $(\sqrt{a_\#}, a_\#)$.

To obtain estimates on the neighbourhood above we follow through the proof of the implicit function theorem. We expand the function $F(\rho, a)$ in $a$ around $a_\#$:

$$ F(\rho, a) = F(\rho, a_\#) + \partial_a F(\rho, a_\#)(a - a_\#) + R(\rho, a), $$

(3.18)

where $R(\rho, a)$ satisfies

$$ |R(\rho, a)| \leq \frac{1}{2} \sup |\partial_a^2 F| r^2 \leq \frac{3\sqrt{n} |\Omega|}{8(n - 1/2)^{5/2}} r^2 $$

(3.19)

and

$$ |R(\rho, a') - R(\rho, a)| = |F(\rho, a') - F(\rho, a) - \partial_a F(\rho, a_\#)(a' - a)| $$

$$ = |\int_0^1 \partial_a F(\rho, sa' + (1-s)a) ds - \partial_a F(\rho, a_\#)(a' - a)| $$

$$ = |\int_0^1 \partial_a F(\rho, sa' + (1-s)a) ds - \partial_a F(\rho, a_\#)||a' - a| $$

$$ \leq \sup |\partial_a^2 F(\rho, a)| r|a' - a| $$

$$ \leq \frac{3\sqrt{n} |\Omega|}{8(n - 1/2)^{5/2}} r|a' - a|, $$

(3.20)

provided $|a' - a_\#|, |a - a_\#| \leq r$. Using (3.18), we rewrite the equation $F(\rho, a) = 0$ as a fixed point problem $a - a_\# = \Phi_\rho(a - a_\#)$, where $\Phi_\rho(a - a_\#) = -\partial_a F(\rho, a_\#)^{-1}[F(\rho, a_\#) + R(\rho, a)]$. Choose $r = (\frac{2}{\delta} \frac{(n-1/2)^{5/2}}{3\sqrt{n} |\Omega|})^{1/2} = \frac{2(n-1/2)^{5/2}}{3\sqrt{n} |\Omega|} r^{1/2}$. Then if $|a - a_\#| \leq r$, we have by (3.19)

$$ |\Phi_\rho(a - a_\#)| \leq \frac{2a_\#^{3/2}}{\sqrt{n} |\Omega|} (\delta + \frac{3\sqrt{n} |\Omega|}{8(n - 1/2)^{5/2}} r^2) = \frac{4a_\#^{3/2}}{\sqrt{n} |\Omega|} \delta = r. $$

Moreover, by (3.20)

$$ |\Phi_\rho(a' - a_\#) - \Phi_\rho(a - a_\#)| = |\partial_a F(\rho, a_\#)^{-1}||R(\rho, a') - R(\rho, a)| $$$$ \leq \frac{2a_\#^{3/2}}{\sqrt{n} |\Omega|} \frac{3\sqrt{n} |\Omega|}{8(n - 1/2)^{5/2}} r |a' - a| = \frac{1}{2} |a' - a|. $$
we conclude that this fixed point problem has a unique solution satisfying the estimates
\[ |a - a_\#| \leq r, \text{ provided } \|\rho - \sqrt{\frac{n}{a_\#}}\|_{L^1} \leq \delta. \]
\]

Unfortunately, we cannot apply Proposition 29 directly to solutions, \(\rho(\omega, \tau)\), of the
equation (3.5), since the parameter-function \(a(\tau)\) which would come out of Proposition
29 would have to be equal to the \(a(\tau)\) entering (3.5). To overcome this problem, we
'deconstruct' \(\rho(\omega, \tau)\), using that it originates as \(\rho(\omega, \tau) := \lambda(t) - R(\omega, t)\) (see Equation (3.3) and Theorem 25) and \(\tau = \tau(t)\) given by (3.3), and
we construct an orthogonal decomposition for \(R(\omega, t)\).

For any time \(t_0\) and constant \(\delta > 0\) we define \(I_{t_0, \delta} := [t_0, t_0 + \delta]\) and
\[
\mathcal{A}_{t_0, \delta} := C^1(I_{t_0, \delta}, [n - \frac{1}{2}, n + \frac{1}{2}]).
\]
Let \(\lambda(t)\) be positive, differentiable function and denote \(R_\lambda(\omega, t) := \lambda(t)^{-1}R(\omega, t)\). For
any function \(a \in \mathcal{A}_{t_0, \delta}\) and \(\lambda_0 > 0\), we define the positive function
\[
\lambda(a, \lambda_0)(t) := (\lambda_0^2 - 2\int_{t_0}^t a(s)\,ds)^{1/2},
\]
i.e. \(\lambda(a, \lambda_0)(t)\) satisfies \(\lambda(t)\partial_t \lambda(t) = a(t)\) and \(\lambda(t_0) = \lambda_0\). Suppose \(R\) is such that
\[
\sup_{t \in I_{t_0, \delta}} \|R_\lambda(a, \lambda_0)(t) - \sqrt{\frac{n}{a(t)}}\| \ll 1, \quad (3.21)
\]
for some \(a \in \mathcal{A}_{t_0, \delta}\) and \(\lambda_0 > 0\). We define the set
\[
\mathcal{U}_{t_0, \delta, \lambda_0} := \{ R \in C^1(I_{t_0, \delta}, L^2(\Omega)) \mid (3.21) \text{ holds for some } a(t) \}.
\]
In what follows, all inner products are the \(L^2\) inner products.

**Proposition 30.** Suppose \(\lambda_0^{-2}\delta \ll 1\). Then there exists a unique \(C^1\) map \(g : \mathcal{U}_{t_0, \delta, \lambda_0} \to \mathcal{A}_{t_0, \delta}\), such that for \(t \in I_{t_0, \delta}\), any \(R \in \mathcal{U}_{t_0, \delta, \lambda_0}\) can be uniquely represented in the form
\[
R_\lambda(\omega, t) = \sqrt{\frac{n}{a(t)}} + \xi(\omega, t), \quad (3.22)
\]
with \(g(R)(t) = a(t), \xi(\cdot, t) \perp 1 \text{ in } L^2(\Omega), \text{ and } \lambda(t) = \lambda(a, \lambda_0)\).
Proof. In this proof we write \( \lambda(a) \) instead of \( \lambda(a, \lambda_0) \) and \( R_{\lambda(a)}(t) \) for the function \( \omega \rightarrow R_\lambda(\omega,t) \). Define the \( C^1 \) map \( G : C^1(I_{t_0,\delta}, [n - \frac{1}{2}, n + \frac{1}{2}]) \times C^1(I_{t_0,\delta}, L^2(\Omega)) \rightarrow C^1(I_{t_0,\delta}, \mathbb{R}) \) as

\[
G(a, R)(t) := \left( R_{\lambda(a)}(t) - \sqrt{\frac{n}{a(t)}}, 1 \right).
\]

The orthogonality condition on the fluctuation can be written as \( G(a, R) = 0 \). We solve this equation using the implicit function theorem. Note first that \( G(a, \sqrt{\frac{\pi}{a}}) = 0, \forall a \).

Next, we compute

\[
\partial_a G = \left( \sqrt{\frac{n}{2} a^{-3/2}}, 1 \right) + \langle \partial_a R_{\lambda(a)}, 1 \rangle.
\]

Note that \( \partial_a R_{\lambda(a)} \alpha = \lambda(t)^{-2} R_{\lambda(a)} \int_{t_0}^t \alpha(s) ds \). Using this expression and the inequality \( \lambda(t) \geq \frac{a_0}{\sqrt{2}} \), provided that \( \delta \leq (4\|a\|_\infty)^{-1} \lambda_0^2 \), we estimate

\[
\|\partial_a R_{\lambda(a)} \alpha\|_\infty \lesssim \delta \lambda_0^2 \|R_\lambda\|_\infty \|\alpha\|_\infty.
\]

So \( \partial_a R_{\lambda(a)} \) is small, if \( \delta \ll (\lambda_0^2 \|R_\lambda\|_\infty)^{-1} \). This shows that \( \partial_a G(a, R) \) is invertible, provided \( R_{\lambda(a)} \) is close to \( \sqrt{\frac{\pi}{a}} \). Hence the implicit function theorem implies that for any \( \bar{a} \in \mathcal{A}_{t_0,\delta} \) there exists a neighborhood \( U_{\bar{a}} \) of \( \sqrt{\frac{\pi}{\bar{a}}} \) in \( C^1(I_{t_0,\delta}, L^2(\Omega)) \) and a unique \( C^1 \) map \( g : V_{\bar{a}} := \{ R \in C^1(I_{t_0,\delta}, L^2(\Omega)) \mid R_{\lambda(\bar{a})} \in U_{\bar{a}} \} \rightarrow \mathcal{A}_{t_0,\delta} \), such that \( G(g(R), R) = 0 \) for all \( R \in V_{\bar{a}} \).

Proceeding as in the proof of Proposition 29, we obtain a quantitative description of the neighbourhood, which implies the statement of Proposition 30.

Let an immersion \( x(\cdot, t) : \Omega \rightarrow \mathbb{R}^{n+1} \) satisfy (3.1) and let \( z(t) \in \mathbb{R}^{n+1} \) be as in Section 3.4, i.e. such that (3.14) holds. We apply Proposition 30 to \( R(\omega, t) = (x(\omega, t) - z(t)) \cdot \omega \) to obtain \( a(\tau) \) and \( \xi(\omega, \tau) \) s.t. \( \rho(\omega, \tau) \equiv R_{\lambda(a, \lambda_0)}(\omega, t) \) satisfies

\[
\rho(\omega, \tau) = \sqrt{\frac{n}{a(\tau)}} + \xi(\omega, \tau),
\]

with \( \xi \perp 1 \). (Here in some functions we changed the time \( t \) to \( \tau = \tau(t) \), given in (3.3).)

This together with (3.15) implies that \( \int_{\Omega} (\rho - \sqrt{\frac{\pi}{a}}) \omega^j = 0, \ j = 0, \ldots, n+1 \), where we use the notation \( \omega^0 = 1 \), or

\[
\rho - \sqrt{\frac{n}{a}} \perp \omega^j, \ j = 0, \ldots, n+1, \text{ in } L^2(\Omega).
\]
3.6 Lyapunov-Schmidt decomposition

Let $\rho$ solve \eqref{eq:3.5} and assume it can be written as $\rho(\omega, \tau) = \rho_a(\tau) + \xi(\omega, \tau)$, with $\rho_a = \sqrt{\frac{n}{a}}$ and $\xi \perp \omega^j$, $j = 0, \ldots, n + 1$. Plugging this into equation \eqref{eq:3.5}, we obtain the equation

$$\frac{\partial \xi}{\partial \tau} = -L_a \xi + \mathcal{N}(\xi) + \lambda^{-1} \bar{z}_\tau \cdot \frac{\nabla \xi}{\rho_a + \xi} + \mathcal{F}, \quad \text{(3.25)}$$

where $L_a = -\partial J(\rho_a)$, $\mathcal{N}(\xi) = J(\rho_a + \xi) - J(\rho_a) - \partial J(\rho_a) \xi$, $\mathcal{F} = -\partial_\tau \rho_a - \lambda^{-1} z_\tau \cdot \omega$ and, recall, $z_\tau = \frac{\partial a}{\partial \tau}$ and $\bar{z}_\tau k = \partial x_i \partial u^k z_{i, \tau}$. Let $a_\tau = \frac{\partial a}{\partial \tau}$. We compute

$$L_a = \frac{a}{n} (-\Delta - 2n),$$

$$\mathcal{N}(\xi) = -\frac{(\rho_a + \rho) \xi}{\rho^2 \rho_a^2} \Delta \xi - \frac{n \xi^2}{\rho^2} + \frac{|\nabla \xi|^2}{\rho(\rho^2 + |\nabla \xi|^2)} - \frac{\nabla \xi \cdot \text{Hess}(\xi) \nabla \xi}{\rho^2 (\rho^2 + |\nabla \xi|^2)},$$

$$\mathcal{F} = \frac{\sqrt{n}}{2} a^{-3/2} a_\tau - \lambda^{-1} z_\tau \cdot \omega.$$

Now, we project \eqref{eq:3.25} onto $\text{span}\{\omega^j, j = 0, \ldots, n + 1\}$. By

$$\xi \perp \omega^j, \quad j = 0, \ldots, n + 1, \quad \text{(3.27)}$$

we have

$$\frac{\sqrt{n}}{2} a^{-3/2} a_\tau |\Omega| \leq \langle \mathcal{N}(\xi) + \lambda^{-1} \bar{z}_\tau \cdot \frac{\nabla \xi}{\rho_a + \xi}, 1 \rangle,$$

$$-c \lambda^{-1} z_{\tau}^j = \langle \mathcal{N}(\xi) + \lambda^{-1} \bar{z}_\tau \cdot \frac{\nabla \xi}{\rho_a + \xi}, \omega^j \rangle, \quad j = 1, \ldots, n + 1,$$

where $c := \int_{\Omega} (\omega^j)^2 = \frac{1}{n+1} |\Omega|$. Indeed, this equation follows from

- $\langle \partial_\tau \xi, \omega^j \rangle = -\langle \xi, \partial_\tau \omega^j \rangle = 0$, $j = 0, \ldots, n + 1$;
- $\langle L_a \xi, \omega^j \rangle = \langle \xi, L_a \omega^j \rangle = 0$, $j = 0, \ldots, n + 1$;
- $\langle \mathcal{F}, 1 \rangle = \frac{\sqrt{n}}{2} a^{-3/2} a_\tau, 1 \rangle = \frac{\sqrt{n}}{2} a^{-3/2} a_\tau |\Omega|$;
- $\langle \mathcal{F}, \omega^j \rangle = -\lambda^{-1} \langle z_\tau \cdot \omega, \omega^j \rangle = -c \lambda^{-1} z_{\tau}^j$, $j = 1, \ldots, n + 1$.

Equations \eqref{eq:3.28} and \eqref{eq:3.29} give

$$|a^{-3/2} a_\tau| \lesssim |\langle \mathcal{N}(\xi) + \lambda^{-1} \bar{z}_\tau \cdot \frac{\nabla \xi}{\rho_a + \xi}, 1 \rangle| \lesssim \|\mathcal{N}(\xi)\|_{L^1} + \lambda^{-1} |z_\tau||\nabla \xi||_{L^1}. \quad \text{(3.30)}$$
and
\[ \lambda^{-1}|z_\tau| \lesssim |\left\langle N(\xi) + \lambda^{-1}z_\tau, \frac{\nabla \xi}{\rho_a + \xi}, \omega \right\rangle| \leq \|N(\xi)\|_{L^1} + \lambda^{-1}|z_\tau|\|\nabla \xi\|_{L^1}. \quad (3.31) \]

Next, we estimate \(N(\xi)\). Using (3.26), where, recall, \(\rho = \rho_a + \xi\), and assuming that \(|\xi| \leq \frac{1}{2}\rho_a\), we have that
\[ \|N(\xi)\|_{L^1} \lesssim (\|\nabla \xi\|_{L^4}^2 + \|\xi\|_{H^1})\|\xi\|_{H^2}. \quad (3.32) \]

This together with (3.30) and (3.31) gives
\[ |a^{-3/2}a_\tau| \lesssim (\|\nabla \xi\|_{L^4}^2 + \|\xi\|_{H^1})\|\xi\|_{H^2} \quad (3.33) \]

and
\[ |z_\tau| \lesssim \lambda(\|\nabla \xi\|_{L^4}^2 + \|\xi\|_{H^1})\|\xi\|_{H^2}, \quad (3.34) \]

provided that \(\|\xi\|_{H^1} \ll 1\).

### 3.7 Linearized operator

The linearization of the map \(-J(\rho)\) at \(\rho_a = \sqrt{\frac{n}{a}}\) is the operator \(L_a := -\partial J(\sqrt{\frac{n}{a}}) = \frac{a}{n}(-\Delta - 2n)\). The spectrum of \(-\Delta\) is well known (see [83]): \(\{l(l+n-1)\mid l = 0, 1, \ldots\}\).

Let \(H_l\) be the space of all the eigenfunctions corresponding to the eigenvalue \(l(l+n-1)\) of \(-\Delta\). Then \(\dim H_l = \binom{n+l}{n} - \binom{n+l-2}{n}\). Moreover, \(H_0 = \text{span}\{1\}\) and \(H_1 = \text{span}\{\frac{x^1}{|x|}, \ldots, \frac{x^{n+1}}{|x|}\}\). Hence the spectrum of \(L_a\) is \(\{\frac{a}{n}(l(l+n-1)-2n)\mid l = 0, 1, \ldots\}\) and
\[ L_a \omega^j = -2a\delta_{j0}, \quad j = 0, \ldots, n+1. \quad (3.35) \]

The conclusions above imply that
\[ \langle \xi, L_a \xi \rangle \geq \frac{2a}{n}\|\xi\|^2 \quad \text{if} \ \xi \perp \omega^j, \quad j = 0, \ldots, n+1. \quad (3.36) \]
Now, (3.36) is the reason why we need the conditions (3.24).

Observe that the eigenfunctions $\omega^j$ are related to the zero modes of the operator $L_\alpha := \partial J(\rho_\alpha)$, where $\alpha = (R, z)$, and $\rho_\alpha$ is the map from $\Omega$ to $\mathbb{R}$ satisfying $|\rho_\alpha(x)\hat{x} - z| = R$.

Note that, if $S_{R,z}$ denotes the sphere in $\mathbb{R}^{n+1}$ of radius $R$, centered at $z \in \mathbb{R}^{n+1}$ and $\text{graph}(\rho) := \{\rho(\omega) : \omega \in \Omega\}$ for $\rho : \Omega \to \mathbb{R}^+$, then $S_\alpha = \text{graph}(\rho_\alpha)$. Hence $J(\rho_\alpha) = 0$ for any $\alpha$. Indeed, differentiating $J(\rho_\alpha) = 0$ we find $\partial J(\rho_\alpha) \partial_\alpha \rho_\alpha + \partial_\alpha J(\rho_\alpha) = 0$, which implies $L_\alpha \partial_R \rho_\alpha = -2a \partial_R \rho_\alpha$, $L_\alpha \partial_z \rho_\alpha = 0$, i.e. $\partial_\alpha \rho_\alpha$ are eigenfunctions of the operator $L_\alpha$.

On the other hand, the equation for $\rho_\alpha$ implies $\rho_\alpha(x)^2 + |z|^2 - 2\rho_\alpha(x) \cdot \hat{x} = R^2$ and therefore

$$\rho_\alpha(\hat{x}) = z \cdot \hat{x} + \sqrt{R^2 - |z|^2 + (z \cdot \hat{x})^2},$$

where, recall, $\hat{x} = \frac{x}{|x|}$. Differentiating the former relation with respect to $R$ and $z^j$, we obtain

$$\partial_R \rho_\alpha(x) = \frac{R}{\rho_\alpha(x) - z \cdot \hat{x}} \quad \text{and} \quad \partial_z \rho_\alpha(x) = \frac{\rho_\alpha(x) \hat{x}^j - z^j}{\rho_\alpha(x) - z \cdot \hat{x}}.$$

Hence we have that

$$\partial_R \rho_\alpha(x) = 1 + O(|z|), \quad \partial_z \rho_\alpha(x) = \hat{x}^j + O(|z|).$$

Since $L_\alpha = L_a + O(|z|)$, these equations and $L_\alpha \partial_R \rho_\alpha = -2a \partial_R \rho_\alpha$, $L_\alpha \partial_z \rho_\alpha = 0$ imply (3.35). This relates the zero modes (3.37) to the eigenfunctions in (3.35). Finally, we note that $\partial_\alpha \rho_\alpha$ are tangent vectors of the manifold of spheres, $\{S_{R,z} | R \in \mathbb{R}^+, \ z \in \mathbb{R}^{n+1}\}$.

### 3.8 Lyapunov functional

Let function $\xi$ obey (3.25) and (3.27). Using that (3.27) implies $\langle L_a \xi, \xi \rangle \geq \frac{2a}{n} \|\xi\|^2$, we derive in this section some differential inequalities for certain Sobolev norms of such a $\xi$. These inequalities allow us to prove a priori estimates for these Sobolev norms. For $k \geq 1$, we define the functional $\Lambda_k(\xi) = \frac{1}{2} \langle \xi, L_a^k \xi \rangle$. 
**Proposition 31.** There exist constants $c > 0$ and $C > 0$ such that

$$ca^k \|\xi\|^2_{H^k} \leq \Lambda_k(\xi) \leq Ca^k \|\xi\|^2_{H^k}.$$ 

**Proof.** By a standard computation, we see that there exists a $C > 0$ such that $\langle \xi, L^k_a \xi \rangle \leq Ca^k \|\xi\|^2_{H^k}$. We prove the lower bound below. Recall that $\langle \xi, L_a \xi \rangle \geq \frac{2a}{n} \|\xi\|^2$. From the definition of $L_a$ we also have $\langle \xi, L_a \xi \rangle = C_1 a \|\nabla \xi\|^2 - C_2 a \|\xi\|^2$ for some $C_1 > 0$ and $C_2 > 0$. These two inequalities imply that

$$\langle \xi, L_a \xi \rangle = \mu \langle \xi, L_a \xi \rangle + (1 - \mu) \langle \xi, L_a \xi \rangle$$

$$\geq \mu C_1 a \|\nabla \xi\|^2 - \mu C_2 a \|\xi\|^2 + (1 - \mu) C a \|\xi\|^2$$

$$= \mu C_1 a (\|\nabla \xi\|^2 + \|\xi\|^2)$$

provided that $\mu = \frac{C}{C + C_1 + C_2}$, where $C = \frac{2a}{n}$.

For the general case, observe that $L_a$ is a self-adjoint operator and $L^k_a$ has the same eigenfunctions as $L_a$ with eigenvalues $\{\frac{a^k}{n^l}(l(l + n - 1) - n)^k : l = 0, 1, \cdots\}$. Hence, by (3.36), $\langle \xi, L^k_a \xi \rangle \geq (\frac{2a}{n})^k \|\xi\|^2$. On the other hand, we have as before $\langle \xi, L^k_a \xi \rangle \geq (\frac{a}{n})^k \|\xi\|^2_{H^k} - C \|\xi\|^2$. Then proceeding as above we find $\langle \xi, L^k_a \xi \rangle \geq a^k \|\xi\|^2_{H^k}$, which is the lower bound in the proposition. \hfill \Box

**Proposition 32.** Let $k > \frac{n}{2} + 1$. Then there exists a constant $C > 0$ such that

$$\partial_\tau \Lambda_k(\xi) \leq -\frac{a}{n} \Lambda_k(\xi) - \left[\frac{1}{2} - C(\Lambda_k(\xi)^{1/2} + \Lambda_k(\xi)^k)\right] \|L^k_a \xi\|^2.$$ \hspace{1cm} (3.39)

**Proof.** We have

$$\frac{1}{2} \partial_\tau \langle \xi, L^k_a \xi \rangle = \langle \partial_\tau \xi, L^k_a \xi \rangle + \frac{1}{2} \langle \xi, (\partial_\tau L^k_a) \xi \rangle.$$ \hspace{1cm} (3.40)

First, from (3.25)

$$\langle \partial_\tau \xi, L^k_a \xi \rangle = -\langle L_a \xi, L^k_a \xi \rangle + \langle N(\xi), L^k_a \xi \rangle + \left(\lambda^{-1} z_\tau \cdot \frac{\nabla \xi}{\rho_a + \xi}, L^k_a \xi \right) + \langle F, L^k_a \xi \rangle.$$ \hspace{1cm} (3.41)

We consider each term on the right hand side. We have by (3.36)

$$\langle L_a \xi, L^k_a \xi \rangle = \frac{1}{2} \|L^{k+1}_a \xi\|^2 + \frac{1}{2} \langle L^k_a \xi, L_a L^k_a \xi \rangle$$

$$\geq \frac{1}{2} \|L^{k+1}_a \xi\|^2 + \frac{a}{n} \langle L^k_a \xi, L^k_a \xi \rangle.$$ \hspace{1cm} (3.42)
To estimate the next term we need the following inequality proven in Appendix B:
\[ \| L_{\frac{k-1}{2}}^k N(\xi) \| \lesssim (\Lambda_{\frac{k}{2}}^1(\xi) + \Lambda_{\frac{k-1}{2}}^k(\xi)) \| L_{\frac{k+1}{2}}^k \xi \|. \] (3.43)

This estimate implies that
\[ |\langle N(\xi), L_{\frac{k-1}{2}}^k \xi \rangle| = |\langle L_{\frac{k-1}{2}}^k N(\xi), L_{\frac{k+1}{2}}^k \xi \rangle| \leq \| L_{\frac{k-1}{2}}^k N(\xi) \| \| L_{\frac{k+1}{2}}^k \xi \| \] (3.44)

From (3.44) and Proposition 31 we obtain that
\[ \langle \lambda^{-1} \tilde{z}_r \cdot \nabla \xi, L_{\frac{k-1}{2}}^k \xi \rangle = \langle L_{\frac{k-1}{2}}^k (\lambda^{-1} \tilde{z}_r \cdot \nabla \xi), L_{\frac{k+1}{2}}^{k+1} \xi \rangle \leq C(\Lambda_{\frac{k}{2}}^1(\xi) + \Lambda_{\frac{k-1}{2}}^k(\xi)) \| L_{\frac{k+1}{2}}^{k+1} \xi \|^2. \] (3.45)

We have, by (3.35), (3.24) (i.e. \( L_a \omega^j = -2a \delta_{j0}, \langle \omega^j, \xi \rangle = 0 \)) and the self-adjointness of \( L_a \), that \( \langle \omega^j, L_a^k \xi \rangle = 0, \ j = 0, \ldots, n + 1 \), and therefore
\[ \langle \mathcal{F}, L_a^k \xi \rangle = 0. \] (3.46)

Finally, we have using (3.33)
\[ \langle \xi, (\partial_{\tau} L_a^k) \xi \rangle = \frac{ka_r}{a} \langle \xi, L_a^k \xi \rangle \leq C(\| \xi \|_{H^k}^2 + \| \xi \|_{H^k}^{2k}) \| L_{\frac{k+1}{2}}^{k+1} \xi \|^2. \] (3.47)

Relations (3.40)-(3.47) yield (3.39).

\[ 3.9 \ \text{Proof of Theorem 25} \]

We begin with reparametrizing the initial condition. Applying Proposition 28 to the immersion \( x_0(\omega) \) and the number \( \lambda_0 = 1 \), we find \( z_0 \in \mathbb{R}^{n+1} \), s.t. \( \int_{\Omega} \rho_0(\omega) \omega_j = 0, \ j = 1, \ldots, n + 1 \), where \( \rho_0(\omega) = (x_0(\omega) - z_0) \cdot \omega \). Then we use Proposition 29 for \( \rho_0(\omega) \) to obtain \( a_0 \) and \( \xi_0(\omega) \), s.t. \( \rho_0 = \rho_a + \xi_0 \), with \( \xi_0 \perp 1 \). Here, recall, \( \rho_a = \sqrt{\frac{n}{a}} \). The last two statements imply that \( \xi_0 \perp \omega^j, \ j = 0, \ldots, n + 1 \), where, recall, \( \omega^0 = 1 \). If
Moreover, by (3.33), $\Lambda_k(\xi_0) \frac{1}{2} + \Lambda_k(\xi_0)^k \leq \frac{1}{10C}$, $\Lambda_k(\xi_0) \ll 1$ and $|a_0 - n| \leq \frac{1}{10}$ (see Proposition [29]), where the constant $C$ is the same as in Proposition [32].

Now we use the local existence result for the mean curvature flow. For $\delta > 0$ sufficiently small, the solution, $x(\omega, t)$, in the interval $[0, \delta]$, stays sufficiently close to the standard sphere $\Omega$. Hence we can apply Proposition [28] with $\bar{z}(t) = 0$, to this solution in order to find $z(t)$, s.t.

$$\int_{\Omega} ((x(\omega, t) - z(t)) \cdot \omega) \omega_j = 0, \ j = 1, \ldots, n + 1, \text{ and } z(0) = z_0.$$ 

By Proposition [26] $y(\omega, \tau) := \lambda(t)^{-1}(x(\omega, t) - z(t)) = \rho(\omega, \tau)\omega$, with $\rho(\omega, \tau) = (x(\omega, t) - z(t)) \cdot \omega$ and $\lambda(t)$ satisfying (3.5). Finally we apply Proposition [30] to $R(\omega, t) := (x(\omega, t) - z(t)) \cdot \omega = \lambda(t)\rho(\omega, \tau)$ to obtain $a(\tau)$ and $\xi(\omega, \tau)$ s.t. $\rho(\omega, \tau) = \rho_a(\tau) + \xi(\omega, \tau)$, with $\xi \perp \omega_j$, $j = 0, \ldots, n + 1$. We repeat this procedure on the interval $[\delta, \delta + \delta']$ with $\bar{z}(t) := z(\delta)$ and so forth. This gives $T_1 > 0$, $z(t(\tau))$, $\rho(\omega, \tau)$, $a(\tau)$ and $\xi(\omega, \tau)$, $\tau \leq T_1$, s.t. $x(\omega, t) = z(t) + \lambda(t)\rho(\omega, \tau(t))$ and $\rho(\omega, \tau) = \rho_a(\tau) + \xi(\omega, \tau)$, with $\rho$ and $\lambda$ satisfying (3.5) and $\xi \perp \omega_j$, $j = 0, \ldots, n + 1$.

Now, let

$$T = \sup\{\tau > 0 : \Lambda_k(\xi(\tau))^\frac{1}{2} + \Lambda_k(\xi(\tau))^k \leq \frac{1}{5C}, \ |a(\tau) - n| \leq \frac{1}{2}\}.$$ 

By continuity, $T > 0$. Assume $T < \infty$. Then $\forall \tau \leq T$ we have by Proposition [32] that $\partial_t \Lambda_k(\xi) \leq -\frac{a}{n} \Lambda_k(\xi)$. We integrate this equation to obtain $\Lambda_k(\xi) \leq \Lambda_k(\xi_0) e^{-\int_0^T \frac{a(t)}{n} ds} \leq \Lambda_k(\xi_0) e^{-(1 - \frac{1}{n})T} \leq \Lambda_k(\xi_0)$. This implies

$$\Lambda_k(\xi(T))^\frac{1}{2} + \Lambda_k(\xi(T))^k \leq \Lambda_k(\xi_0)^\frac{1}{2} + \Lambda_k(\xi_0)^k \leq \frac{1}{10C}.$$ 

Moreover, by (3.33),

$$|a(\tau)^{-\frac{1}{2}} - a(0)^{-\frac{1}{2}}| \leq \frac{1}{2} \int_0^T |a(s)^{-\frac{3}{2}} a_r(s)| ds \lesssim \int_0^T \Lambda_k(\xi) ds \leq \Lambda_k(\xi_0) \int_0^T e^{-(1 - \frac{1}{n})s} ds \ll 1,$$

(3.48)
and by (3.34)

\[ |z(\tau) - z(0)| \leq \int_0^\tau |z_\tau(s)| ds \lesssim \int_0^\tau \lambda(s) \Lambda_k(\xi) ds \leq \Lambda_k(\xi_0) \int_0^\tau e^{-\lambda s} ds = 1. \] (3.49)

Hence \(|a(T) - n| \leq \frac{1}{4}\), and therefore proceeding as above we see that there exists \(\delta > 0\) such that \(\Lambda_k(\xi(t)) \leq \frac{1}{\delta^2}\) and \(|a(t) - n| \leq \frac{1}{2}\), for \(t \leq T + \delta\), a contradiction!

So \(T = \infty\) and \(\Lambda_k(\xi) \leq \Lambda_k(\xi_0) e^{-\int_0^\tau \frac{\alpha(s)}{n} ds}\). By Proposition 31, we know that \(\|\xi\|^2_{H^k} \lesssim \Lambda_k(\xi_0) e^{-\int_0^\tau \frac{\alpha(s)}{n} ds}\).

Observe that \(\lambda_0^2 - \lambda(t)^2 = 2 \int_0^t a(\tau(s)) ds\). Let \(t_*\) be the zero of the function \(\lambda_0^2 - 2 \int_0^t a(\tau(s)) ds\). Since \(|a(t) - n| \leq \frac{1}{2}\), we have \(t_* < \infty\) and \(\lambda(t) \to 0\) as \(t \to t_*\). Similarly to (3.48), we know that

\[ |a(\tau_2)^{-1/2} - a(\tau_1)^{-1/2}| \lesssim \int_{\tau_1}^{\tau_2} e^{-\frac{1}{2\pi} s} ds \to 0, \]

as \(\tau_1, \tau_2 \to \infty\). Hence there exists \(a_* > 0\), such that \(|a(\tau) - a_*| \lesssim e^{-\frac{1}{2\pi} r}\). Similar arguments show that there exists \(z_* \in \mathbb{R}^{n+1}\) such that \(|z(\tau) - z_*| \lesssim e^{-\frac{1}{2\pi} r}\). Then

\[ \lambda^2 = \lambda_0^2 - 2 \int_0^t a(\tau(s)) ds = 2 \int_0^t a(\tau(s)) ds = 2a_*(t_* - t) + o(t_* - t)\].

The latter relation implies that \(\tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a_*} \int_0^t \frac{ds}{(\tau_* - s)(1 + o(1))}\) and therefore \(e^{-\frac{1}{2\pi} r} = O((t_* - t) \frac{1}{2a_*})\).

So \(\lambda(t) = \sqrt{2a_* (t_* - t)} + O((t_* - t) \frac{1}{2a_*})\), \(\rho(\omega, \tau(t)) = \frac{n}{a(\tau(t))} + \xi(\omega, \tau(t))\), and \(\|\xi(\omega, \tau(t))\|_{H^k} \lesssim (t_* - t) \frac{1}{2a_*}\). The latter inequality with \(k = s\), together with estimates on \(a, z\) and \(\lambda\) obtained above and the relation \(R(\omega, t) = \lambda(t) \rho(\omega, t)\), proves Theorem 25.
Appendix A

Feynmann-Kac Formula

In this appendix we present, for the reader’s convenience, a proof of the Feynmann-Kac
formula \( U(x, y) = U_0(x, y)E(x, y) \) and the estimate (2.111) used in section 2.11(cf. [10]).
For stochastic calculus proofs of similar formulae see [21, 47, 51, 59, 79].

Let \( L_0 := -\Delta_y + \frac{\alpha^2}{4}|y|^2 - \frac{\alpha}{2} \) and \( L := L_0 + V \) where \( V \) is a multiplication operator
by a function \( V(y, \tau) \), which is bounded and Lipschitz continuous in \( \tau \). Let \( U(\tau, \sigma) \) and
\( U_0(\tau, \sigma) \) be the propagators generated by the operators \(-L\) and \(-L_0\), respectively. The
integral kernels of these operators will be denoted by \( U(\tau, \sigma)(x, y) \) and \( U_0(\tau, \sigma)(x, y) \).

**Theorem 33.** The integral kernel of \( U(\tau, \sigma) \) can be represented as

\[
U(\tau, \sigma)(x, y) = U_0(\tau, \sigma)(x, y) \int e^{\int_\sigma^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega)
\]  
(A.1)

where \( d\mu(\omega) \) is a probability measure (more precisely, a conditional harmonic oscillator,
or Ornstein-Uhlenbeck, probability measure) on the continuous paths \( \omega : [\sigma, \tau] \to \mathbb{R}^n \) with
\( \omega(\sigma) = \omega(\tau) = 0 \), and \( \omega_0(\cdot) \) is the path defined as

\[
\omega_0(s) = e^{\alpha(\tau-s)} \frac{e^{2\alpha \sigma} - e^{2\alpha s}}{e^{2\alpha \sigma} - e^{2\alpha \tau}} x + e^{\alpha(\sigma-s)} \frac{e^{2\alpha \tau} - e^{2\alpha s}}{e^{2\alpha \tau} - e^{2\alpha \sigma}} y.
\]  
(A.2)

**Remark 3.** \( d\mu(\omega) \) is the Gaussian measure with mean zero and covariance \((-\partial_y^2 + \alpha^2)^{-1},\)
normalized to 1. The path $\omega_0(s)$ solves the boundary value problem

$$(-\partial_s^2 + \alpha^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x.$$  

(A.3)

Below we will also deal with the normalized Gaussian measure $d\mu_{xy}(\omega)$ with mean $\omega_0(s)$ and covariance $(-\partial_s^2 + \alpha^2)^{-1}$. This is a conditional Ornstein-Uhlenbeck probability measure on continuous paths $\omega : [\sigma, \tau] \to \mathbb{R}^n$ with $\omega(\sigma) = y$ and $\omega(\tau) = x$ (see e.g. [47, 51, 79]).

Now, assume in addition that the function $V(y, \tau)$ satisfies the estimates

$$V \leq 0 \text{ and } |\partial_y V(y, \tau)| \lesssim \beta^{1/2}(\tau)$$  

(A.4)

where $\beta(\tau)$ is a positive function. Then Theorem 33 implies Equation (2.111) by the following corollary.

**Corollary 34.** Under (A.4),

$$|\partial_y \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega)| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau)$$

Proof. By Fubini’s theorem

$$\partial_y \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega) = \int \partial_y \left[ e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} \right] d\mu(\omega)$$

Equation (A.4) implies

$$|\partial_y \int_\sigma^\tau V(\omega_0(s) + \omega(s), s)ds| \leq |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau), \text{ and } e^{\int_\sigma^\tau V(\omega_0(s) + \omega(s), s)ds} \leq 1.$$  

Thus

$$|\partial_y \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega)| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau) \int d\mu(\omega) = |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau)$$

to complete the proof.  

Proof of Theorem 33. We begin with the following extension of the Ornstein-Uhlenbeck process-based Feynman-Kac formula to time-dependent potentials:

\[ U(\tau, \sigma)(x, y) = U_0(\tau, \sigma)(x, y) \int e^{\int_0^\tau V(\omega(s), s)ds} d\mu_{xy}(\omega). \]  

(A.5)

where \( d\mu_{xy}(\omega) \) is the conditional Ornstein-Uhlenbeck probability measure described in Remark 3 above. This formula can be proven in the same way as the one for time independent potentials (see [17], Equation (3.2.8)), i.e. by using the Kato-Trotter formula and evaluation of Gaussian measures on cylindrical sets. Since its proof contains a slight technical wrinkle, for the reader’s convenience we present it below.

Now changing the variable of integration in (A.5) as \( \omega = \omega_0 + \tilde{\omega} \), where \( \tilde{\omega}(s) \) is a continuous path with boundary conditions \( \tilde{\omega}(\sigma) = \tilde{\omega}(\tau) = 0 \), using the translational change of variables formula \( \int f(\omega) d\mu_{xy}(\omega) = \int f(\omega_0 + \tilde{\omega}) d\mu(\tilde{\omega}) \), which can be proven by taking \( f(\omega) = e^{i\langle \omega, \zeta \rangle} \) and using (A.3) (see [47], Equation (9.1.27)) and omitting the tilde over \( \omega \) we arrive at (A.1).  

There are at least three standard ways to prove (A.5): by using the Kato-Trotter formula, by expanding both sides of the equation in \( V \) and comparing the resulting series term by term and by using Ito’s calculus (see [51, 79, 78, 47]). The first two proofs are elementary but involve tedious estimates while the third proof is based on a fair amount of stochastic calculus. For the reader’s convenience, we present the first elementary proof of (A.5).

Before starting proving (A.5) we establish an auxiliary result. We define the operator \( \mathcal{K} \) as

\[
\mathcal{K}(\sigma, \delta) := \int_0^\delta U_0(\sigma + \delta, \sigma + s)V(\sigma + s, \cdot)U_0(\sigma + s, \sigma)ds - U_0(\sigma + \delta, \sigma) \int_0^\delta V(\sigma + s, \cdot)ds \quad \text{(A.6)}
\]

Lemma 35. For any \( \sigma \in [0, \tau] \) and \( \xi \in C_0^\infty \) we have, as \( \delta \to 0^+ \),

\[
\sup_{0 \leq \sigma \leq \tau} \| \frac{1}{\delta} \mathcal{K}(\sigma, \delta)U(\sigma, 0)\xi \|_2 \to 0. \quad \text{(A.7)}
\]
Proof. If the potential term, $V$, is independent of $\tau$, then the proof is standard (see, e.g. [78]). We use the property that the function $V$ is Lipschitz continuous in time $\tau$ to prove (A.7). The operator $K$ can be further decomposed as

$$K(\sigma, \delta) = K_1(\sigma, \delta) + K_2(\sigma, \delta)$$

with

$$K_1(\sigma, \delta) := \int_0^\delta U_0(\sigma + \delta, \sigma + s) V(\sigma, \cdot) U_0(\sigma + s, \sigma) ds - \delta U_0(\sigma + \delta, \sigma) V(\sigma, \cdot)$$

and

$$K_2(\sigma, \delta) := \int_0^\delta U_0(\sigma + \delta, \sigma + s) [V(\sigma + s, \cdot) - V(\sigma, \cdot)] U_0(\sigma + s, \sigma) ds - U_0(\sigma + \delta, \sigma) \int_0^\delta [V(\sigma + s, \cdot) - V(\sigma, \cdot)] ds.$$

Since $U_0(\tau, \sigma)$ are uniformly $L^2$-bounded and $V$ is bounded, we have $U(\tau, \sigma)$ is uniformly $L^2$-bounded. This together with the fact that the function $V(\tau, y)$ is Lipschitz continuous in $\tau$ implies that

$$\|K_2(\sigma, \delta)\|_{L^2 \to L^2} \lesssim 2 \int_0^\delta s ds = \delta^2.$$

We rewrite $K_1(\sigma, \delta)$ as

$$K_1(\sigma, \delta) = \int_0^\delta U_0(\sigma + \delta, \sigma + s) [V(\sigma, \cdot) U_0(\sigma + s, \sigma) - 1] - [U_0(\sigma + s, \sigma) - 1] V(\sigma, \cdot) ds.$$

Let $\xi(\sigma) = U(\sigma, 0) \xi$. We claim that for a fixed $\sigma \in [0, \tau]$,

$$\|K_1(\sigma, \delta) \xi(\sigma)\|_2 = o(\delta). \tag{A.8}$$

Indeed, the fact $\xi_0 \in C_0^\infty$ implies that $L_0 \xi(\sigma), \ L_0 V(\sigma) \xi(\sigma) \in L^2$. Consequently (see [77])

$$\lim_{s \to 0^+} \frac{(U_0(\sigma + s, \sigma) - 1) g}{s} \to L_0 g,$$

for $g = \xi(\sigma)$ or $V(\sigma, y) \xi(\sigma)$ which implies our claim. Since the set of functions $\{\xi(\sigma) | \sigma \in [0, \tau]\} \subset L_0 L^2$ is compact and $\|\frac{1}{\delta} K_1(\sigma, \delta)\|_{L^2 \to L^2}$ is uniformly bounded, we have (A.8) as $\delta \to 0$ uniformly in $\sigma \in [0, \tau]$.

Collecting the estimates on the operators $K_i, \ i = 1, 2$, we arrive at (A.7). \qed
Lemma 36. Equation (A.5) holds.

Proof. In order to simplify our notation, in the proof that follows we assume, without losing generality, that $\sigma = 0$. We divide the proof into two parts. First we prove that for any fixed $\xi \in C_0^\infty$ the following Kato-Trotter type formula holds

$$U(\tau, 0)\xi = \lim_{m \to \infty} \prod_{0 \leq k \leq m-1} U_0\left(\frac{k + 1}{m}\tau, \frac{k}{m}\tau\right)e^{\int_{\frac{k}{m}}^{\frac{k+1}{m}} V(y,s)ds} \xi$$

in the $L^2$ space. We start with the formula

$$U(\tau, 0) - \prod_{0 \leq k \leq m-1} U_0\left(\frac{k + 1}{m}\tau, \frac{k}{m}\tau\right)e^{\int_{\frac{k}{m}}^{\frac{k+1}{m}} V(y,s)ds}$$

$$= \prod_{0 \leq k \leq m-1} U\left(\frac{k + 1}{m}\tau, \frac{k}{m}\tau\right) - \prod_{0 \leq k \leq m-1} U_0\left(\frac{k + 1}{m}\tau, \frac{k}{m}\tau\right)e^{\int_{\frac{k}{m}}^{\frac{k+1}{m}} V(y,s)ds}$$

$$= \sum_{0 \leq j \leq m} \prod_{j \leq k \leq m-1} U_0\left(\frac{k + 1}{m}\tau, \frac{k}{m}\tau\right)e^{\int_{\frac{k}{m}}^{\frac{k+1}{m}} V(y,s)ds} A_j U\left(\frac{j}{m}\tau, 0\right)$$

with the operator

$$A_j := U_0\left(\frac{j + 1}{m}\tau, \frac{j}{m}\tau\right)e^{\int_{\frac{j}{m}}^{\frac{j+1}{m}} V(y,s)ds} - U\left(\frac{j + 1}{m}\tau, \frac{j}{m}\tau\right).$$

We observe that $\|U_0(\tau, \sigma)\|_{L^2 \to L^2} \leq 1$, and moreover by the boundness of $V$, the operator $U(\tau, \sigma)$ is uniformly bounded in $\tau$ and $\sigma$ in any compact set. Consequently

$$\|U(\tau, 0) - \prod_{0 \leq k \leq m-1} U_0\left(\frac{k + 1}{m}\tau, \frac{k}{m}\tau\right)e^{\int_{\frac{k}{m}}^{\frac{k+1}{m}} V(y,s)ds} \|_2$$

$$\leq m \max \|A_j + \mathcal{K}\left(\frac{k}{m}, \frac{1}{m}\right)\|_{L^2 \to L^2} + m \max \|\mathcal{K}\left(\frac{j}{m}, \frac{1}{m}\right)\|_{L^2 \to L^2}$$

A.10 where, recall the definition of $\mathcal{K}$ from (A.6). Now we claim that

$$\|A_j + \mathcal{K}\left(\frac{k}{m}, \frac{1}{m}\right)\|_{L^2 \to L^2} \lesssim \frac{1}{m^2}.$$  

Indeed, by the Duhamel principle we have

$$U\left(\frac{j + 1}{m}\tau, \frac{j}{m}\tau\right) = U_0\left(\frac{j + 1}{m}\tau, \frac{j}{m}\tau\right) + \int_{\frac{j}{m}}^{\frac{j+1}{m}} U_0\left(\frac{j + 1}{m}\tau, s\right)V(y,s)U\left(\frac{j}{m}\tau, s\right)ds.$$
Appendix A. Feynmann-Kac Formula

Iterating this equation on $U(s, \frac{k}{m} \tau)$ and using the fact that $U(s, t)$ is uniformly bounded if $s, t$ is on a compact set, we obtain

$$
\left\| U\left( \frac{j+1}{n} \tau, \frac{j}{n} \tau \right) - U_0\left( \frac{j+1}{n} \tau, \frac{j}{n} \tau \right) 
- \int_0^{\frac{j}{n} \tau} U_0\left( \frac{j+1}{n} \tau, s \right) V(y, s) U_0\left( s, \frac{j}{n} \tau \right) ds \right\|_{L^2 \to L^2} \lesssim \frac{1}{n^2}.
$$

On the other hand we expand $e^{\int_{\tau}^{\tau+\frac{(j+1)}{n}} V(y, s) ds}$ and use the fact that $V$ is bounded to get

$$
\left\| U_0\left( \frac{j+1}{n} \tau, \frac{j}{n} \tau \right) e^{\int_{\tau}^{\tau+\frac{(j+1)}{n}} V(y, s) ds} - U_0\left( \frac{j+1}{n} \tau, \frac{j}{n} \tau \right) 
- U_0\left( \frac{j+1}{n} \tau, \frac{j}{n} \tau \right) \int_{\tau}^{\tau+\frac{(j+1)}{n}} V(y, s) ds \right\|_{L^2 \to L^2} \lesssim \frac{1}{n^2}.
$$

By the definition of $K$ and $A_j$ we complete the proof of (A.11). Equations (A.7), (A.10) and (A.11) imply (A.9). This completes the first step.

In the second step we compute the integral kernel, $G_m(x, y)$, of the operator

$$
G_m := \prod_{0 \leq k \leq m-1} U_0\left( \frac{k+1}{m} \tau, \frac{k}{m} \tau \right) e^{\int_{\frac{k}{m} \tau}^{\frac{(k+1)}{m} \tau} V(,s) ds}
$$

in (A.9). By the definition, $G_m(x, y)$ can be written as

$$
G_m(x, y) = \int \cdots \int \prod_{0 \leq k \leq m-1} U_{\frac{m}{m}}(x_{k+1}, x_k) e^{\int_{\frac{k}{m} \tau}^{\frac{(k+1)}{m} \tau} V(x_k, s) ds} dx_1 \cdots dx_{m-1} \quad (A.12)
$$

with $x_m := x$, $x_0 := y$ and $U_{\tau}(x, y) \equiv U_0(0, \tau)(x, y)$ is the integral kernel of the operator $U_0(\tau, 0) = e^{-L_0 \tau}$. We rewrite (A.12) as

$$
G_m(x, y) = U_{\tau}(x, y) \int e^{\sum_{k=0}^{m-1} \int_{\frac{k}{m} \tau}^{\frac{(k+1)}{m} \tau} V(x_k, s) ds} d\mu_m(x_1, \ldots, x_m), \quad (A.13)
$$

where

$$
d\mu_m(x_1, \ldots, x_m) := \prod_{0 \leq k \leq m-1} U_{\frac{m}{m}}(x_{k+1}, x_k) \frac{U_{\tau}(x, y)}{U_{\tau}(x, y)} dx_1 \cdots dx_{k-1}.
$$

Since $G_m(x, y)|_{V=0} = U_{\tau}(x, y)$ we have that $\int d\mu_m(x_1, \ldots, x_m) = 1$. Let $\Delta := \Delta_1 \times \ldots \times \Delta_m$, where $\Delta_j$ is an interval in $\mathbb{R}$. Define a cylindrical set

$$
P^m_{\Delta} := \{ \omega : [0, \tau] \to \mathbb{R}^n \mid \omega(0) = y, \omega(\tau) = x, \omega(k\tau/m) \in \Delta_k, 1 \leq k \leq m-1 \}.\]
By the definition of the measure $d\mu_{xy}(\omega)$, we have $\mu_{xy}(P^m_\Delta) = \int_{\Delta} d\mu_m(x_1, \ldots, x_m)$. Thus, we can rewrite (A.13) as

$$G_m(x, y) = U_\tau(x, y) \int e^\sum_{k=0}^{m-1} \int_{\frac{k}{m}}^{\frac{k+1}{m}} V(\omega_{\frac{k}{m}}, s) ds \ d\mu_{xy}(\omega), \quad (A.14)$$

By the dominated convergence theorem the integral on the right hand side of (A.14) converges in the sense of distributions as $n \to \infty$ to the integral on the right hand side of (A.5). Since the left hand side of (A.14) converges to the left hand side of (A.5), also in the sense of distributions (which follows from the fact that $G_m$ converges in the operator norm on $L^2$ to $U(\tau, \sigma)$), (A.5) follows.

Note that on the level of finite dimensional approximations the change of variables formula can be derived as follows. It is tedious, but not hard, to prove that

$$\prod_{0 \leq k \leq m-1} U_m(x_{k+1}, x_k) = e^{-\alpha \frac{(x-e^{-\alpha \tau} y)^2}{2(1-e^{-2\alpha \tau})}} \prod_{0 \leq k \leq m-1} U_m(y_{k+1}, y_k)$$

with $y_k := x_k - \omega_0(\frac{k}{m} \tau)$. By the definition of $\omega_0(s)$ and the relations $x_0 = y$ and $x_m = x$ we have

$$G_m(x, y) = U_\tau(x, y) G_m^{(1)}(x, y) \quad (A.15)$$

where

$$G_m^{(1)}(x, y) := \frac{1}{4\pi \sqrt{\alpha(1-e^{-2\alpha \tau})}} \int \cdots \int \prod_{0 \leq k \leq n-1} U_n(y_{k+1}, y_k) e^{\int_{\frac{k}{m}}^{\frac{k+1}{m}} V(y_k+\omega_0(\frac{k}{m}), s) ds} \ dy_k \cdots dy_1. \quad (A.16)$$

Since $\lim_{m \to \infty} G_m^{(1)}(x, y)$ exists by (A.7), we have $\lim_{m \to \infty} G_m^{(1)}(x, y)$ (in the weak limit) exists also. As shown in [47], $\lim_{m \to \infty} G_m^{(1)} = \int e^{\int_0^\tau V(\omega_0(s)+\omega(s), s) ds} d\mu(\omega)$ with $d\mu$ being the (conditional) Ornstein-Uhlenbeck measure on the set of path from 0 to 0. This completes the derivation of the change of variables formula.

**Remark 4.** In fact, Equations (A.9), (A.15) and (A.16) suffice to prove the estimate in Corollary 34.
Appendix B

Detailed computations and proofs in Chapter 2

B.1 Equation (2.25): Computation of $A_1$

Here through some examples we show how to compute the matrix $A_1$. We have

$$
\langle \partial_a V_{\mu}, \varphi_{a_2}^{ij} \rangle = \lambda^{-n+p-1} \left( \frac{a+\frac{2}{p-1}}{p-1} \right)^{\frac{1}{p-1}} \int e^{-\frac{2}{p} |y|^2} \phi_a^{(ij)} dy + O(\|b\|)
$$

$$
= \frac{\lambda^{-n+p-1} (a+\frac{2}{p-1})^{\frac{1}{p-1}}}{2} \int e^{-\frac{2}{p} |y|^2} \phi_a^{(ij)} dy + O(\|b\|),
$$

Similarly we can compute all the other entries.

$$
\langle \partial_{b_i} V_{\mu}, \varphi_{a_2}^{ij} \rangle = \lambda^{-n+p-1} \left( \frac{a+\frac{2}{p-1}}{p-1} \right)^{\frac{1}{p-1}} \int e^{-\frac{2}{p} |y|^2} \phi_a^{(ij)} dy + O(\|b\|)
$$

$$
= -\lambda^{-n+p-1} \left( \frac{a+\frac{2}{p-1}}{p-1} \right)^{\frac{1}{p-1}} \int e^{-\frac{2}{p} |y|^2} \phi_a^{(ij)} dy + O(\|b\|)
$$

$$
= \begin{cases}
-\lambda^{-n+p-1} \left( \frac{a+\frac{2}{p-1}}{p-1} \right)^{\frac{1}{p-1}} \frac{3}{2} \left( \frac{2\pi}{a} \right)^n + O(\|b\|), & \text{if } i = j, \\
-\lambda^{-n+p-1} \left( \frac{a+\frac{2}{p-1}}{p-1} \right)^{\frac{1}{p-1}} \frac{1}{a} \left( \frac{2\pi}{a} \right)^n + O(\|b\|), & \text{if } i \neq j.
\end{cases}
$$
B.2 Derivation of Equation \((2.65)-(2.71)\)

Instead of using \((2.62)\) to derive the equation for \(\xi\), it is more convenient to work directly with \((2.16)\). Let \(v = V_{ab} + e^{\frac{a}{4}}|y|^2 \xi\), then we have

\[
\begin{align*}
\partial_r v &= \frac{1}{p-1} (\frac{a+\frac{1}{2}}{p-1+aryl})^{\frac{1}{p-1}} - \frac{a+\frac{1}{2}}{p-1+aryl} + e^{\frac{a}{4}}|y|^2 \frac{a}{4} |y|^2 \xi + e^{\frac{a}{4}}|y|^2 \xi, \\
\partial_y v &= -\frac{1}{p-1} (\frac{a+\frac{1}{2}}{p-1+aryl})^{\frac{1}{p-1}} \sum_i \frac{biy_j}{p-1+aryl} + e^{\frac{a}{4}}|y|^2 \frac{a}{2} y_j \xi + e^{\frac{a}{4}}|y|^2 \partial_y \xi, \\
\partial_{yy} v &= \frac{1}{(p-1)^2} (\frac{a+\frac{1}{2}}{p-1+aryl})^{\frac{1}{p-1}} (\frac{2 \sum_i b_i y_j}{p-1+aryl})^{2} - \frac{1}{p-1} (\frac{a+\frac{1}{2}}{p-1+aryl})^{\frac{1}{p-1}} \sum_i 2b_i (p-1+aryl) - \frac{2a}{p-1} W_{abc} - \frac{2a}{p-1} \xi \\
&\quad + e^{\frac{a}{4}}|y|^2 (\frac{a}{2} + \frac{a^2}{4} y_j^2) \xi + ayie^{\frac{a}{4}}|y|^2 \partial_y \xi + e^{\frac{a}{4}}|y|^2 \partial_{yj} \xi. \\
\end{align*}
\]

Plugging \((B.1)\) into \((2.16)\) we obtain

\[
\begin{align*}
\partial_r \xi + \frac{a+\frac{1}{2}}{4} |y|^2 \xi &= \frac{a+\frac{1}{2}}{p-1+aryl} (\frac{2 \sum_i b_i y_j}{p-1+aryl})^{2} W_{abc} - \sum_i 2b_i (p-1+aryl) - \frac{2a}{p-1} W_{abc} - \frac{2a}{p-1} \xi \\
&\quad + \Delta \xi + \frac{2by}{(p-1+aryl)} W_{abc} - \frac{a^2}{2} |y|^2 \xi - a \sum_i y_i \partial_y \xi - \frac{2a}{p-1} W_{abc} - \frac{2a}{p-1} \xi \\
&\quad + e^{\frac{(p-1)a}{4}} |y|^2 W_{abc} + \xi |p-1| (W_{abc} + \xi). \\
\end{align*}
\]

It follows that

\[
\begin{align*}
\partial_r \xi &= \left(\Delta - \frac{1}{4} (a^2 + a) |y|^2 + \frac{na}{2} - \frac{2a}{p-1} + \frac{pc}{p-1+aryl}\right) \xi \\
&\quad + \left[|W_{abc} + \xi |p-1| (W_{abc} + \xi) - W_{abc}^p - p W_{abc}^{p-1} \xi \right] e^{\frac{(p-1)a}{4}} |y|^2 \\
&\quad + \left(\frac{c}{p-1+aryl} - \frac{2a}{(p-1)(p-1+aryl)} + \frac{4p}{(p-1)^2(p-1+aryl)^2} - \frac{cr/c}{p-1}\right) W_{abc}. \\
\end{align*}
\]

Rearranging the terms on the r.h.s. we obtain the equations \((2.65)-(2.71)\) for \(\xi\).

B.3 Proof of Lemma 18

\textit{Proof.} We prove this result by induction in the dimension \(n\). For \(n = 1\), the result is straightforward since \(1 = P_0^{(1)} + P_1^{(1)} + P_2^{(1)} + P_3^{(1)}\).

Assume the statement of the lemma is true for all dimensions \(m \leq n - 1\) and we will prove it for dimension \(n\). By symmetry we only need to prove it for the case \(k = n\). We
have by the assumption,

\[ 1 = \sum_{\vec{i} \in J'_1} P_{\vec{i}} \]  \hspace{1cm} (B.2)

where \( J'_1 \subset I^{(n-1)}_1 \). We claim the following relations

\[ P_0^{(n)} = \sum_{\vec{i} \in J'_1} P_{\vec{i}} P_0^{(n)} , \]  \hspace{1cm} (B.3)

\[
P_1^{(n)} = P_0^{(1)} \cdots P_0^{(n-1)} P_1^{(n)} + \sum_{j=1}^{n-1} P_0^{(1)} \cdots P_0^{(j-1)} P_{0'}^{(j)} P_{1'}^{(j+1)} \cdots P_0^{(n-1)} P_1^{(n)}
\]

\[
= P_0^{(1)} \cdots P_0^{(n-1)} P_1^{(n)} + \sum_{k=1}^{n-1} \sum_{t=1,2} P_0^{(1)} \cdots P_0^{(k-1)} P_{0'}^{(k)} P_{1'}^{(k+1)} \cdots P_0^{(n-1)} P_1^{(n)} + \sum_{k<l} P_0^{(1)} \cdots P_0^{(k-1)} P_{0'}^{(k)} P_{1'}^{(k+1)} \cdots P_0^{(l-1)} P_{0'}^{(l+1)} P_{1'}^{(l+1)} \cdots P_0^{(n-1)} P_1^{(n)} ,
\]  \hspace{1cm} (B.4)

\[ P_2^{(n)} = P_0^{(1)} \cdots P_0^{(n-1)} P_2^{(n)} + \sum_{j=1}^{n-1} P_0^{(1)} \cdots P_0^{(j-1)} P_{0'}^{(j)} P_{0'}^{(j+1)} \cdots P_0^{(n-1)} P_2^{(n)} , \]  \hspace{1cm} (B.5)

\[ P_3^{(n)} = P_{0'}^{(1)} \cdots P_{0'}^{(n-1)} P_3^{(n)} . \]  \hspace{1cm} (B.6)

In fact, \((B.3)\) and \((B.6)\) follow directly from \((B.2)\). Moreover, using the second relation in \((2.102)\) we obtain

\[ 1 = P_0^{(1)} \cdots P_0^{(n-1)} = P_0^{(1)} \cdots P_0^{(n-2)} (P_0^{(n-1)} + P_{1'}^{(n-1)}) \]

\[ = P_0^{(1)} \cdots P_0^{(n-2)} P_0^{(n-1)} + P_0^{(1)} \cdots P_0^{(n-2)} P_{1'}^{(n-1)} \]

\[ = P_0^{(1)} \cdots P_0^{(n-3)} (P_0^{(n-2)} + P_{1'}^{(n-2)}) P_0^{(n-1)} + P_0^{(1)} \cdots P_0^{(n-2)} P_{1'}^{(n-1)} \]

\[ = \cdots \]

\[ = P_0^{(1)} \cdots P_0^{(n-1)} + \sum_{i=1}^{n-1} P_0^{(1)} \cdots P_0^{(j-1)} P_{0'}^{(j)} P_{0'}^{(j+1)} \cdots P_0^{(n-1)} , \]

and

\[ P_0^{(1)} \cdots P_0^{(j-1)} P_{1'}^{(j)} = P_0^{(1)} \cdots P_0^{(j-2)} (P_0^{(j-1)} + P_{1'}^{(j-1)}) P_{1'}^{(j)} \]

\[ = P_0^{(1)} \cdots P_0^{(j-2)} P_{1'}^{(j-1)} P_{1'}^{(j)} + P_0^{(1)} \cdots P_0^{(j-2)} P_{1'}^{(j-1)} P_{1'}^{(j)} \]

\[ = P_0^{(1)} \cdots P_0^{(j-3)} (P_0^{(j-2)} + P_{1'}^{(j-2)}) P_{0'}^{(j-1)} P_{1'}^{(j)} + P_0^{(1)} \cdots P_0^{(j-2)} P_{1'}^{(j-1)} P_{1'}^{(j)} \]

\[ = \cdots \]

\[ = P_0^{(1)} \cdots P_0^{(j-1)} P_{1'}^{(j)} + \sum_{k<j} P_0^{(1)} \cdots P_0^{(k-1)} P_{0'}^{(k)} P_{0'}^{(k+1)} \cdots P_0^{(j-1)} P_{1'}^{(j)} . \]
(B.5) follows readily from (B.7). Finally, using (B.7) and (B.8) we arrive at (B.4). Thus by (B.3)-(B.6) and the relation 1 = \( P_0^{(n)} + P_1^{(n)} + P_2^{(n)} + P_3^{(n)} \) we find

\[
1 = \sum_{i' \in J_1'} P_{i'} P_0^{(n)} + P_0^{(1)} \cdots P_0^{(n-1)} P_1^{(n)} \\
+ \sum_{k<l} P_{0'}^{(1)} \cdots P_{0'}^{(k-1)} P_{0'}^{(k+1)} \cdots P_{0'}^{(l-1)} P_{0'}^{(l)} P_{0'}^{(l+1)} \cdots P_{0'}^{(n-1)} P_1^{(n)} \\
+ \sum_{k=1}^{n-1} \sum_{l=1,2} P_{0'}^{(1)} \cdots P_{0'}^{(k-1)} P_{l}^{(k)} P_{0'}^{(k+1)} \cdots P_{0'}^{(n-1)} P_1^{(n)} \\
+ \sum_{j=1}^{n-1} P_{0'}^{(1)} \cdots P_{0'}^{(j-1)} P_{1}^{(j)} P_{0'}^{(j+1)} \cdots P_{0'}^{(n-1)} P_2^{(n)} + P_1^{(1)} \cdots P_0^{(n-1)} P_3^{(n)}.
\]

Therefore we obtain 1 = \( \sum_{i \in J_n} P_i \), where

\[
J_n = \{ i = (i_1, \ldots, i_{n-1}, 0)| (i_1, \ldots, i_{n-1}) \in J_1' \} \cup \{(0, \ldots, 0, k) : k = 1, 2 \} \\
\cup (\bigcup_{1 \leq k < t \leq n-1} \{ (i_1, \ldots, i_{n-1}, 1) : i_k = i_l = 1', i_j = 0' \text{ if } j < k, i_j = 0 \text{ if } k < j < n \}) \\
\cup (\bigcup_{k=1}^{n-1} \{ (i_1, \ldots, i_{n-1}, 1) : i_j = l \delta_{jk} \forall 1 \leq j \leq n-1, l = 1, 2') \} \\
\cup (\bigcup_{k=1}^{n-1} \{ (i_1, \ldots, i_{n-1}, 2) : i_k = 1', i_j = 0' \text{ if } j < k, i_j = 0 \text{ if } k < j < n \}).
\]

Obviously this \( J_n \) is a subset of \( I_n \). This proves Lemma 18.

\[\square\]

**B.4 Proof of (2.106) in the case \( i_j = 3 \)**

Let \( L_0 = -\Delta + \frac{\alpha^2}{4} x^2 - \frac{\sigma}{2} \) and \( U_0(x, y) \) be the integral kernel of \( U_\sigma := e^{\frac{\alpha^2}{4} x^2} e^{-r L_0} e^{-\frac{\sigma}{2} x^2} \).

By a standard formula (see [79, 47]) we have

\[
U_0(x, y) = 4\pi (1 - e^{-2\sigma r})^{-\frac{1}{2}} \sqrt{\alpha} e^{2\sigma r} e^{-\frac{1}{2\alpha} (x - e^{-\sigma r} y)^2}.
\]

Define a new function \( f := e^{-\frac{\alpha^2}{4}} P_3 g \). The definitions above imply

\[
U_\sigma(\sigma + r, \sigma) P_3 g = \int e^{\frac{\alpha^2}{4}} U_0(x, y) f(y) dy.
\] \hspace{1cm} (B.9)

Integrate by parts on the right hand side of (B.9) to obtain

\[
U_\sigma(\sigma + r, \sigma) P_3 g = e^{\frac{\alpha^2}{4}} \int \partial_y^3 U_0(x, y) f^{(-3)}(y) dy.
\] \hspace{1cm} (B.10)
where $f^{(-m-1)}(x) := \int_{-\infty}^{x} f^{(-m)}(y)dy$ and $f^{(-0)} := f$. By the facts that $f = e^{-\alpha x^2/4}P_3g$ and $P_3g \perp y^m e^{-\alpha x^2/4}$, $m = 0, 1, 2$, we have that $f \perp 1, y, y^2$. Therefore by integration by parts we have

$$f^{(-m)}(y) = \int_{-\infty}^{y} f^{(-m+1)}(x)dx = -\int_{y}^{\infty} f^{(-m+1)}(x)dx, \ m = 1, 2, 3.$$ 

Moreover by the definition of $f^{(-m)}$ and the equation above we have

$$|f^{(-m)}(y)| \lesssim \langle y \rangle^{3-m} e^{-\alpha y^2} \|\langle y \rangle^{-3} e^{\alpha y^2} P_3g\|_\infty.$$ 

Using the explicit formula for $U_0(x, y)$ given above we find

$$|\partial_y^k U_0(x, y)| \lesssim \frac{e^{-\alpha kr}}{(1 - e^{-2\alpha r})^k (|x| + |y| + 1)^k U_0(x, y).}$$

Collecting the estimates above and using Equation (B.10), we have the following result

$$\langle x \rangle^{-3} e^{\alpha x^2/4} |U_0(\sigma + r, \sigma) P_3g(x)|$$

$$\lesssim \frac{1}{(1 - e^{-2\alpha r})^3} \langle x \rangle^{-3} e^{\alpha x^2/4} \int (|x| + |y| + 1)^3 e^{-3\alpha r} U_0(x, y) |f^{(-3)}(y)| dy$$

$$\lesssim \frac{e^{-3\alpha r}}{(1 - e^{-2\alpha r})^3} e^{\alpha x^2/4} \int \langle x \rangle^{-3} U_0(x, y) e^{-\alpha y^2} \langle y \rangle^3 dy \|\langle y \rangle^{-3} e^{\alpha y^2} P_3g\|_\infty.$$
Appendix C

Proof of Lemma 27

**Proof.** Recall the notation \( \alpha(x) = \frac{x}{|x|} \). Let \( \beta : U \to \mathbb{R}^{n+1} \) be a local parametrization of \( \Omega \), and we denote \( \rho \) in the local coordinates, i.e. \( \rho \circ \beta \), again as \( \rho : U \to \mathbb{R} \). We write \( \tilde{\rho} := \rho \circ \alpha = \rho \circ \beta \circ \beta^{-1} \circ \alpha \), which we rewrite as \( \tilde{\rho} = \rho \circ \sigma \), where \( \sigma := \beta^{-1} \circ \alpha : \mathbb{R}^{n+1} \to U \).

Now, writing \( u = u(x) \equiv \sigma(x) \), we define \( \tilde{g}^{ij} = \partial u_i \partial x_j \). We claim

\[
\tilde{g}^{ij}(x) g_{jk} = \frac{1}{|x|^2} \delta_{ik}. \tag{C.1}
\]

Indeed, since \( \beta(\sigma(x)) = \alpha(x) \), we have

\[
(\partial u^i / \partial x^m) (\partial u^m / \partial x^j) = \frac{\partial \alpha^i}{\partial x^j} = \frac{1}{|x|} (\delta_{ij} - x^i x^j / |x|^2). \tag{C.2}
\]

Note that \( \sigma \) is homogeneous of degree 0, so \( x \cdot \nabla x \sigma = 0 \). This together with (C.2) implies that

\[
\tilde{g}^{ij}(x) g_{jk} = \frac{\partial u^i / \partial x^m (\partial u^m / \partial \sigma) \circ \sigma (\partial u^n / \partial \sigma) \circ \sigma)}{\partial x^i} = \delta_{ik}. \tag{C.3}
\]

Since \( |x| \partial u^i / \partial x^m \) is homogeneous of degree 0, we have that \( |x| \partial u^i / \partial x^m = \frac{\partial u^i}{\partial x^m} |\Omega| \), and therefore

\[
\frac{\partial u^i}{\partial m} = \frac{1}{|x|} (\partial u^i / \partial x^m |\Omega|). \tag{C.4}
\]

Using \( \sigma \circ \beta = 1_U \), we compute that \( (\partial u^i / \partial x^j \circ \beta) \partial u^j / \partial x^k = \delta_{ik} \), which is equivalent to \( \partial u^i / \partial x^j (\partial x^j / \partial \sigma) = \delta_{ik} \). This gives us

\[
\frac{\partial u^i}{\partial x^m} \frac{\partial u^m}{\partial u^k} = \frac{1}{|x|} (\frac{\partial u^i}{\partial x^m} |\Omega|) \frac{\partial u^m}{\partial u^k} = \frac{1}{|x|} \delta_{ik}. \tag{C.4}
\]
and \((C.1)\) we compute 

\[ \partial \rho \]

Now, by \((C.3)\) and \((C.4)\), we have the equation \((C.1)\).

Using the relations \(\frac{\partial \rho}{\partial u} = g_{ij} \nabla^j \rho\) (this follows from the definition of \(\nabla \rho\)), \(\frac{\partial \tilde{\rho}}{\partial x^i} = \frac{\partial u^j}{\partial x^i} \frac{\partial \rho}{\partial u^j}\), and \((C.1)\), we compute

\[
\left| \nabla_x \tilde{\rho} \right|^2 = \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \tilde{\rho}}{\partial x^i} \\
= \frac{\partial u^k}{\partial x^i} \frac{\partial \rho}{\partial u^k} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^k} = \tilde{g}^{kl}(x) \frac{\partial \rho}{\partial u^k} \frac{\partial \tilde{\rho}}{\partial x^i} \\
= \tilde{g}_{kl}(x)g_{km}(u)\nabla^m \rho g_{ln}(u)\nabla^n \rho = \frac{1}{|x|^2} \nabla^l \rho g_{ln} \nabla^n \rho \\
= \frac{1}{|x|^2} |\nabla \rho|^2.
\]

This gives \((3.12)\).

Now we prove \((3.13)\). We have

\[
\nabla_x \tilde{\rho} \cdot \text{Hess}_x(\tilde{\rho}) \nabla_x \tilde{\rho} = \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial^2 \tilde{\rho}}{\partial x^j \partial x^i} = \frac{\partial u^m}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \\
= \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} + \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} + \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \\
=: A + B.
\]

Then

\[
A = \tilde{g}^{ml} \tilde{g}^{kn} g_{mp} \nabla^p \rho \frac{\partial^2 \rho}{\partial u^l \partial u^k} g_{nu} \nabla^q \rho = \frac{1}{|x|^4} \nabla^l \rho \frac{\partial^2 \rho}{\partial u^l \partial u^k} \nabla^k \rho
\]

and

\[
B = \frac{1}{2} \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^j} + \frac{1}{2} \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^j} \\
= \frac{1}{2} \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^j} \\
= \frac{1}{2} \tilde{g}^{ml} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^j}.
\]

Now, by \(\frac{\partial \rho}{\partial u^i} = g_{ij} \nabla^j \rho\), \(B = B_1 = B_2 = B_3\), where

\[
B_1 = \frac{1}{2} \tilde{g}^{ml} g_{mr} \tilde{g}^{kn} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^j} \frac{\partial \rho}{\partial u^m} (\nabla \rho)^r (\nabla \rho)^s = \frac{1}{2|x|^2} g_{ks} \frac{\partial \tilde{\rho}}{\partial u^i} \frac{\partial \rho}{\partial u^m} (\nabla \rho)^r (\nabla \rho)^s; \\
B_2 = \frac{1}{2} \tilde{g}^{ml} g_{ns} g_{nr} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \rho}{\partial u^m} \frac{\partial \tilde{\rho}}{\partial x^j} \frac{\partial \rho}{\partial u^m} (\nabla \rho)^r (\nabla \rho)^s = \frac{1}{2|x|^2} g_{ns} \frac{\partial \tilde{\rho}}{\partial u^i} \frac{\partial \rho}{\partial u^m} (\nabla \rho)^r (\nabla \rho)^s; \\
B_3 = \frac{1}{2} \tilde{g}^{ml} g_{kr} g_{ns} \frac{\partial \tilde{\rho}}{\partial x^i} \frac{\partial \rho}{\partial u^m} \frac{\partial \rho}{\partial u^m} (\nabla \rho)^r (\nabla \rho)^s.
\]

Hence

\[
B = -\frac{1}{|x|^4} \Gamma^p_{rs} \frac{\partial \rho}{\partial u^r} \nabla^r \rho \nabla^s \rho,
\]

\((C.8)\).
Appendix C. Proof of Lemma 27

where $\Gamma_{rs} = \frac{|x|^2}{2} (g_{ks} \frac{\partial \tilde{g}^{kp}}{\partial u^r} + g_{mr} \frac{\partial \tilde{g}^{po}}{\partial u^s} - |x|^2 \tilde{g}^{pl} g_{kr} g_{ns} \frac{\partial \tilde{g}^{kn}}{\partial u^r}).$ Using that

$$\frac{\partial}{\partial u^r} (g_{ks} \tilde{g}^{kp}) = \frac{\partial}{\partial u^r} \left( \frac{1}{|x|^2} \delta_{sp} \right) = 0$$

(points $x \in \mathbb{R}^{n+1}$ are parameterized by $\beta(u)$ and $|x|$), we compute $g_{ks} \frac{\partial \tilde{g}^{kp}}{\partial u^r} = \frac{\partial}{\partial u^r} (g_{ks} \tilde{g}^{kp}) - \tilde{g}^{kp} \frac{\partial g_{ks}}{\partial u^r} = -\tilde{g}^{kp} \frac{\partial g_{ks}}{\partial u^r}$. This gives

$$\Gamma_{rs} = \frac{|x|^2}{2} (\tilde{g}^{kp} \frac{\partial g_{ks}}{\partial u^r} + \tilde{g}^{kp} \frac{\partial g_{kr}}{\partial u^s} - |x|^2 \tilde{g}^{pl} g_{kr} \frac{\partial g_{ns}}{\partial u^r})$$

$$= \frac{|x|^2}{2} (\tilde{g}^{kp} \frac{\partial g_{ks}}{\partial u^r} + \tilde{g}^{kp} \frac{\partial g_{kr}}{\partial u^s} - \tilde{g}^{pk} \frac{\partial g_{rs}}{\partial u^k}).$$

Since $|x| = 1$, and therefore $\tilde{g}^{pk} = g^{kp}$ on $\Gamma$, we have that on $\Gamma$

$$\Gamma_{rs} = \frac{1}{2} g^{kp} \left( \frac{\partial g_{ks}}{\partial u^r} + \frac{\partial g_{kr}}{\partial u^s} - \frac{\partial g_{rs}}{\partial u^k} \right),$$

which coincides with our definition for $\Gamma_{rs}$ at the beginning of Section 3.3.

Equations (C.6), (C.7) and (C.8) give (3.13). This finishes the proof of the lemma.  \[\square\]
Appendix D

Proof of (3.43)

Lemma 37. Let $k > \frac{n}{2} + 1$ and assume that $|\xi| \leq \frac{1}{2}v_\alpha$. Then

$$\|L^{\frac{k-1}{2}}_\alpha N(\xi)\| \lesssim (\Lambda_k^{1/2}(\xi) + \Lambda_k^k(\xi))\|L^{\frac{k+1}{2}}_\alpha \xi\|.$$  \hspace{1cm} (D.1)

Proof. Assume first that $k$ is an integer. Then $\|L^{\frac{k-1}{2}}_\alpha \eta\| \simeq \|\eta\|_{H^{k-1}} \simeq \|\eta\|_{L^2} + \|\nabla^{k-1}\eta\|_{L^2}$. Now, by the expression for $N(a,b,\xi)$ in (3.26), which we recall here,

$$N(\xi) = -\left(\frac{\rho + \rho}{\rho^2 \rho^2_\alpha}\right)\xi \Delta \xi - \frac{n\xi^2}{\rho^2} + \frac{|\nabla \xi|^2}{\rho^2} - \frac{\nabla \xi \cdot Hess(\xi) \nabla \xi}{\rho^2} - \frac{\nabla \xi \cdot Hess(\xi) \nabla \xi}{\rho^2} + \frac{|\nabla \xi|^2}{\rho^2},$$  \hspace{1cm} (D.2)

the term $|\nabla^{k-1} N(\xi)|$ is bounded above by terms of the form $|\xi^t(\nabla \xi)^r(\nabla^{\alpha_1} \xi)\cdots(\nabla^{\alpha_s} \xi)|$, where

$$0 \leq t, r \leq k+1, 1 \leq s \leq k, t+r+s \geq 2, 2 \leq \alpha_1 \leq \cdots \leq \alpha_s \leq k-s+2, \alpha_1+\cdots+\alpha_s \leq k+s.$$  \hspace{1cm} (D.3)

Note that the last two conditions in (D.3) imply that $s \leq k$. Then by Hölder’s inequality we have

$$\|\nabla^{k-1} N(\xi)\| \leq \|\nabla \xi\|_{L^\infty} \|\nabla^{\alpha_1} \xi\|_{L^{p_1}} \cdots \|\nabla^{\alpha_s} \xi\|_{L^{p_s}},$$

where $\frac{1}{p_1} + \cdots + \frac{1}{p_s} = \frac{1}{2}$.

Since $k > \frac{n}{2} + 1$, we have, by the Sobolev embedding theorem, that $\|\xi\|_{L^\infty} + \|\nabla \xi\|_{L^\infty} \lesssim \|\xi\|_{H^k}$. Moreover, we choose $p_i$ so that $k-\alpha_i > \frac{n}{2} - \frac{n}{m}$ for all $i = 1, \cdots, s - 1$ and
Appendix D. Proof of (3.43)

\[ k + 1 - \alpha_s > \frac{n}{2} - \frac{n}{p_s} \] (this choice implies \( \sum_{j=1}^{s} \alpha_j < \frac{n}{2} + 1 + (k - \frac{n}{2})s \), which is compatible with (D.3)). Then, using the Sobolev embedding theorem again, we have \( \| \nabla^\alpha \xi \|_{L^{p_i}} \leq \| \xi \|_{H^k} \), for \( i = 1, \cdots, s - 1 \), and \( \| \nabla^s \xi \|_{L^{p_s}} \leq \| \xi \|_{H^{k+1}} \). Combining these estimates gives us

\[ \| L_a \frac{k-1}{2} A(\xi) \| \lesssim \| \xi \|_{H^{k-1}} \| \xi \|_{H^k}. \]

Now from \( 1 \leq r + s - 1 \leq 2k \) and Proposition 31 we obtain (D.1). Furthermore, one can easily check that \( k \) can be taken arbitrary close to \( \frac{n}{2} + 1 \) (this means that one is able to satisfy \( 1 \geq \alpha_i - \frac{n}{p_i} \) for \( i = 1, \cdots, s - 1 \), \( 2 \geq \alpha_s - \frac{n}{p_s} \) and \( \alpha_i \geq 2, \forall i \)).

If \( k \) is not integer, we proceed as follows. Let \( \beta = k - [k] \in (0, 1) \). We use the space \( \tilde{H}^\beta \) with the norm

\[ \| f \|_{\tilde{H}^\beta} = \| f \|_{L^2} + \int \frac{dh}{|h|^{n+\beta}} \| \Delta_h f \|_{L^2}, \]

where \( \Delta_h f(x) = f(x + h) - f(x) \). We have the embeddings

\[ \| f \|_{H^\beta} \lesssim \| f \|_{\tilde{H}^\beta} \lesssim \| f \|_{H^{\beta'}}, \beta < \beta'. \] (D.4)

Let us prove the first embedding:

\[ (-\Delta + 1)^{\beta/2} f(x) = C_\beta f(x) + \int (f(x - y) - f(x)) G_\beta(y) dy, \]

where \( C_\beta \) is an analytic continuation of \( C_\beta := \int G_\beta(x) dx \) with \( \text{Re}(\beta) < n \) and \( G_\beta(y) := \int e^{iy\cdot k} (|k|^2 + 1)^{\beta/2} dk \). Note that \( G_\beta(y) \sim |y|^{-n-\beta} \) as \( |y| \to 0 \) and is exponentially decaying at \( \infty \). So

\[ \| f \|_{H^\beta} = \| (-\Delta + 1)^{\beta/2} f \|_{L^2} \leq C_\beta \| f \|_{L^2} + \int \frac{dy}{|y|^{n+\beta}} \| \Delta_y f \|_{L^2} \lesssim \| f \|_{\tilde{H}^\beta}, \]

which proves the first embedding in (D.4).

For the second embedding, let \( \varphi = (-\Delta + 1)^{\beta'/2} f \). Then

\[ f = (-\Delta + 1)^{-\beta'/2} \varphi = \int \tilde{G}_{\beta'}(x - y) \varphi(y) dy, \]
where \( \tilde{G}_{\beta'}(y) := \int e^{iyk}(|k|^2 + 1)^{-\beta'/2} dk \). Note that \( \tilde{G}_{\beta'}(y) \sim |y|^{-n+\beta'} \) as \( |y| \to 0 \) and is exponentially decaying at \( \infty \). Let \( \beta < \beta'' < \beta' \). Then

\[
\int_{|h| \leq 1} \frac{dh}{|h|^{n+\beta}} \| \Delta_h f \|_{L^2} = \int_{|h| \leq 1} \frac{dh}{|h|^{n+\beta}} \int_{|x-y| \leq 2} (\tilde{G}_{\beta'}(x+h-y) - \tilde{G}_{\beta'}(x-y))\varphi(y)dy + \int_{|x-y| \geq 2} (\tilde{G}_{\beta'}(x+h-y) - \tilde{G}_{\beta'}(x-y))\varphi(y)dy \|_{L^2} \\
\lesssim \int_{|h| \leq 1} \frac{dh}{|h|^{n+\beta}} \| |h|^{\beta''} \int_{|x-y| \leq 2} |x-y|^{-n+\beta''} |\varphi(y)|dy \|_{L^2} \\
+ |h| \int_{|x-y| \geq 2} |x-y|^{-n+\beta'-1} |\varphi(y)|dy \|_{L^2} \\
\lesssim \| \varphi \|_{L^2} = \| f \|_{H^{\beta'}}
\]

and

\[
\int_{|h| \geq 1} \frac{dh}{|h|^{n+\beta}} \| \Delta_h f \|_{L^2} \leq 2 \| f \|_{L^2} \int_{|h| \geq 1} \frac{dh}{|h|^{n+\beta}} \lesssim \| f \|_{H^{\beta'}}.
\]

This proves the second embedding in (D.4).

Using (D.4), we obtain

\[
\| \prod_{j=1}^s \xi_j \|_{H^\beta} \lesssim \int \frac{dh}{|h|^{n+\beta}} \| \Delta_h \prod_{j=1}^s \xi_j \|_2 \\
\leq \sum_{i=1}^s \int \frac{dh}{|h|^{n+\beta}} \| \prod_{j=1}^{i-1} \xi_j \Delta_h \xi_i \prod_{j=i+1}^s \xi_j \|_2 \\
\leq \sum_{i=1}^s \left( \prod_{j \neq i} \| \xi_j \|_{p_j^{(i)}} \right) \int \frac{dh}{|h|^{n+\beta}} \| \Delta_h \xi_i \|_{p_i^{(i)}}
\]

where \( T_hf(x) = f(x+h), \sum_{j=1}^s \frac{1}{p_j^{(i)}} = \frac{1}{2} \). Using appropriate embeddings, we conclude finally that

\[
\| \prod_{j=1}^s \xi_j \|_{H^\beta} \lesssim \sum_{i=1}^s \left( \prod_{j \neq i} \| \xi_j \|_{H_j^{(i)}} \right) \]

(D.6)

where \( c_j^{(i)} > \frac{n}{2} - \frac{n}{p_j^{(i)}} \forall j \neq i \) and \( c_1^{(i)} - \beta > \frac{n}{2} - \frac{n}{p_1^{(i)}} \). Similarly as before we know that \( \sum_{j=1}^s c_j^{(i)} - \beta > \frac{n}{2}(s-1) \), which guarantees the existence of \( p_j^{(i)} \).

For \( k \) not an integer, we write

\[
\| \mathcal{N}(\xi) \|_{H^{k-1}} \sim \| (-\Delta + 1)^{\beta/2} \nabla^m \mathcal{N}(\xi) \|_{L^2},
\]

(D.7)

where \( m = [k] - 1 \) and \( \beta = k - [k] \in (0, 1) \). \( \nabla^m \mathcal{N}(\xi) \) is treated as before to obtain

\[
\nabla^m \mathcal{N}(\xi) \sim \xi^l (\nabla \xi)^r \nabla^{\alpha_1} \xi \cdots \nabla^{\alpha_s} \xi,
\]

(D.8)
where \( t \leq m + 2, \ r \leq m + 2, \ 2 \leq \alpha_j \leq m - s + 3, \ \sum_{j=1}^s \alpha_j \leq m + 1 + s, \ s \leq m + 1 \) and \( t + r + s \geq 2. \)

If \( \alpha_j < m + 2 \ \forall j, \) then, using \((D.6)\) with \( \xi_j = \nabla^{\alpha_j} \xi \ \forall j, \ c_j^{(i)} + \alpha_j = k \ \forall j \neq i \) and \( c_i^{(i)} + \alpha_i = k + 1, \) we find

\[
\| \xi^t (\nabla \xi)^r \prod_{j=1}^s \nabla^{\alpha_j} \xi \|_{H^b} \lesssim \| \xi \|_{H^{b-1}} \| \xi \|_{H^{b+1}}. \tag{D.9}
\]

We use this estimate, together with \((D.7)\) and \((D.8)\), to obtain

\[
\| \mathcal{N}(\xi) \|_{H^{b-1}} \lesssim \sum_{i=1}^{2|k|} \| \xi \|_{H^b}^i \| \xi \|_{H^{b+1}}. \tag{D.10}
\]

If \( \alpha_s = m + 2 \) and therefore \( s = 1, \) then we let \( f = \xi^t (\nabla \xi)^r \) and proceed as

\[
(-\Delta + 1)^{\beta/2} f \nabla^{m+2} \xi = f (-\Delta + 1)^{\beta/2} \nabla^{m+2} \xi + [( -\Delta + 1)^{\beta/2}, f ] \nabla^{m+2} \xi. \tag{D.11}
\]

The first term on the r.h.s. is easy to estimate:

\[
\| f (-\Delta + 1)^{\beta/2} \nabla^{m+2} \xi \| \leq \| f \|_{\infty} \| \xi \|_{H^{b+1}} \tag{D.12}
\]

To estimate the second term in the r.h.s. we note that

\[
[( -\Delta + 1)^{\beta/2}, f ] \eta = \int ( f(x) - f(y) ) G_{\beta}(x - y) \eta(y) dy = \int \eta(x-z)(f(x-z) - f(x)) G_{\beta}(z) dz.
\]

Using this representation we obtain for \( \beta' > \beta, \)

\[
\| [( -\Delta + 1)^{\beta/2}, f ] \eta \|_2 \leq \sup_z \| \eta(x-z)(f(x-z) - f(x)) G_{\beta}(z) \|_{L^2(dz)} \leq \sup_z \| \eta(x-z)(f(x-z) - f(x)) G_{\beta}(z) \|_{L^2(dx)} \leq \| \eta \|_q \sup_z \| \Delta f \|_{|z|^p} \|_p,
\]

where \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \) Similar to \((D.5)\), we have

\[
\sup_z \left\| \frac{1}{|z|^q} \Delta z f \right\|_{H^b} \lesssim \| f \|_{H^{b+\gamma'}}, \ \gamma' > \gamma.
\]
Using this estimate and Sobolev embedding theorem, we find

$$
\|((\Delta + 1)^{\beta/2}, f) \eta\|_2 \lesssim \|\eta\|_{H^a} \sup_z \frac{\Delta_z f}{|z|^{\beta'}} \|H^b \lesssim \|\eta\|_{H^a} \|f\|_{H^{b+\beta''}},
$$

where $\beta'' > \beta'$, $a > \frac{n}{2} - \frac{n}{q}$, $b > \frac{n}{2} - \frac{n}{p}$. Taking $f = \xi^t(\nabla \xi)^r$ and $\eta = \nabla^{m+2} \xi$, $a = \beta$, we find

$$
\|\left((\Delta + 1)^{\beta/2}, f\right) \nabla^{m+2} \xi\| \leq \|\xi^t(\nabla \xi)^r\|_{H^{r+\beta''}} \|\xi\|_{H^{b+1}}.
$$

Note that $\beta'' + r > n - \frac{n}{2} = \frac{n}{2}$. Let $\beta'' + r = j$. As before, we estimate

$$
\|\xi^t(\nabla \xi)^r\|_{H^j} \lesssim \sum_{j_1 + \ldots + j_{t+r} = j} \|\nabla^{j_1} \xi \ldots \nabla^{j_t} \xi \nabla^{j_{t+1}+1} \xi \ldots \nabla^{j_{t+r}+1} \xi\|_2 \lesssim \|\xi\|_{H^{j+1}}^{t+r} \forall j > \frac{n}{2}.
$$

Since $k > \frac{n}{2} + 1$, we can take $j = k - 1$ and so

$$
\|\left((\Delta + 1)^{\beta/2}, f\right) \nabla^{m+2} \xi\| \leq \|\xi\|_{H^k}^{t+r} \|\xi\|_{H^{k+1}},
$$

where, recall, $f = \xi^t(\nabla \xi)^r$. This inequality together with (D.7), (D.11) and (D.12) implies (D.10) also in this case. As was mentioned above, (D.10) implies (D.1).
Bibliography


