ABRIKOsov LATTICE SOLUTIONS OF THE GINZBURG-LANDAU EQUATIONS OF SUPERCONDUCTIVITY

by

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Abstract

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In this thesis we study the Ginzburg-Landau equations of superconductivity, which are among the basic nonlinear partial differential equations of Theoretical and Mathematical Physics. These equations also have geometric interest as equations for the section and connection of certain principal bundles and are related to Seiberg-Witten equations used extensively in Differential Geometry. In 1957, Abrikosov suggested that for sufficiently high magnetic fields there exist solutions for which all physical quantities have the periodicity of a lattice, with the magnetic field penetrating the superconductor at the vertices of the lattice (Abrikosov lattice solutions). The corresponding phenomenon was confirmed experimentally and is among the most interesting aspects of superconductivity and is discussed in every book on the subject. In 2003, Abrikosov was awarded the Nobel Prize in Physics for this discovery.

Building on the previous results in the subject we prove the existence of such lattices in the case where each lattice cell contains a single quantum of magnetic flux, and in the general case reduce the problem to an n-dimensional problem, where n is the number of quanta of flux. We prove that for Type II superconductors, these solutions are stable, and in the case n = 1, we show that as the external magnetic field approaches the critical value at which superconductivity first appears, the lattice which minimizes the average free energy per lattice cell is the triangular lattice.
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Chapter 1

Introduction

This thesis deals with the Ginzburg-Landau equations for superconductivity on $\mathbb{R}^2$

$$\Delta_A \Psi = \kappa^2(|\Psi|^2 - 1)\Psi,$$

$$\text{curl}^{*} \text{curl} A = \text{Im}(\overline{\Psi} \nabla A \Psi).$$

Here $\Psi : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. $\nabla_A := \nabla - iA$ and $\Delta_A = -\nabla^{*}_A \nabla_A$ are the covariant gradient and covariant Laplacian. curl and curl$^*$ are given by curl $A = \partial_{x_1} A_2 - \partial_{x_2} A_1$ and curl$^*$ $f = (\partial_{x_2} f, -\partial_{x_1} f)$. $\kappa$ is a positive constant.

The Ginzburg-Landau equations first arose in the Ginzburg-Landau model of superconductivity, introduced by Ginzburg and Landau in 1950 [19] (reviews can be found in any text on superconductivity, e.g. [37, 42, 43]). It gives a macroscopic description of a superconducting material in terms of a complex-valued order parameter $\Psi$ where $n_s = |\Psi|^2$ gives the local density of (Cooper pairs of) superconducting electrons, and a vector field $A$ where $B = \text{curl} A$ is the magnetic field. The vector quantity $J = \text{Im}(\overline{\Psi} \nabla A \Psi)$ is the superconducting current. The parameter $\kappa$ depends on the material properties of the superconductor. For the Ginzburg-Landau equations on $\mathbb{R}^2$ the underlying geometry is a superconductor that fills all space but is homogeneous in one direction, in which case the original equations on $\mathbb{R}^3$ reduce to equations on $\mathbb{R}^2$. 
The equations also arise in particle physics as the Abelian-Higgs model, which is the simplest and perhaps most important ingredient of the standard model. Here $\Psi$ and $A$ are the Higgs and $U(1)$ gauge (electromagnetic) fields.

The Ginzburg-Landau equations are the Euler-Lagrange equations of the critical points of the Ginzburg-Landau energy functional,

$$E(\Psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla A \Psi|^2 + |\text{curl } A - H|^2 + \frac{\kappa^2}{2} (1 - |\Psi|^2)^2.$$ 

For superconductors, the functional $E$ represents the difference in the Helmholtz free energy of the superconducting and normal states, near the transition temperature, in the presence of an external magnetic field $H$. Alternatively, taking $H = 0$, one can view $E$ as the difference in the Gibbs free energy under the constraint that the average magnetic flux, given by

$$\Phi = \int_{\mathbb{R}^2} \text{curl } A,$$

is fixed. For particle physics, the functional represents the energy of a static configuration in the $U(1)$ Yang-Mills-Higgs classical gauge theory.

The Ginzburg-Landau equations admit several symmetries, i.e., transformations which map solutions to solutions. The most important of these is gauge symmetry: for any sufficiently regular function $\eta : \mathbb{R}^2 \to \mathbb{R}$, $\Psi \mapsto e^{i\eta} \Psi$, $A \mapsto A + \nabla \eta$. Pairs $(\Psi, A)$ that are gauge equivalent represent the same physical state. There is also the translation symmetry: for any $t \in \mathbb{R}^2$, $\Psi \mapsto \Psi(x + t)$, $A \mapsto A(x + t)$, and the rotation and reflection symmetry: for any $R \in O(2)$, $\Psi \mapsto \Psi(Rx)$, $A \mapsto R^{-1}A(Rx)$. This infinite dimensional symmetry group is one of the most interesting aspects that arise in the analysis of the Ginzburg-Landau equations.

An important property the Ginzburg-Landau equations is the quantization of magnetic flux. Finite energy states $(\Psi, A)$ are classified by their topological degree (the winding number of $\Psi$ at $\infty$),

$$\text{deg}(\Psi) := \text{degree} \left( \frac{\Psi}{|\Psi|} \bigg|_{|x|=\infty} : S^1 \to S^1 \right),$$
for $R \gg 1$. This is well defined because it clear from the definition of $\mathcal{E}$ that for a state to have finite energy, $|\Psi| \to 1$ as $|x| \to \infty$. For each such state we have the quantization of magnetic flux,

$$\Phi = 2\pi \text{deg}(\Psi) \in 2\pi \mathbb{Z},$$

which by Stokes theorem and the requirement that $|\Psi| \to 1$ and $|\nabla A \Psi| \to 0$ as $|x| \to \infty$. For Abrikosov lattice states (see below) the energy is infinite, but the flux quantization still holds for each lattice cell because of gauge-periodic boundary conditions.

The simplest solutions to the Ginzburg-Landau equations are the trivial ones corresponding to physically homogeneous states:

1. The perfect superconductor solution: $\Psi \equiv 1$ and $A \equiv 0$ (so the magnetic field $B = 0$),

2. The normal metal solution, where $\Psi = 0$ and the magnetic field $B$ is constant (as well, of course, as any gauge transformation of one of these solutions). The perfect superconductor is a solution only when the magnetic flux $\Phi = 0$. On the other hand, there is a normal solution for any value of $\Phi$, where $\Phi$ is then the strength of the applied external magnetic field.

Solving the Ginzburg-Landau equations near a flat interface between the normal and superconducting states shows that (in the units used here) the magnetic field varies on a length scale of 1, the penetration depth, while the order parameter varies on a length scale of $1/m_\kappa$, the coherence length, where $m_\kappa = \min(\kappa \sqrt{2}, 2)$ [14]. The two length scales coincide when $\kappa^2 = 1/2$. Considering a flat interface between the normal and superconducting states, one can show easily that at this point the surface tension changes sign from positive for $\kappa^2 < 1/2$ to negative for $\kappa^2 > 1/2$. This critical value $\kappa^2 = 1/2$ separates superconductors into two classes with different properties:

1. $\kappa^2 < 1/2$: Type-I superconductors exhibit first-order (discontinuous) phase transitions from the normal state to the superconducting state.
2. $\kappa^2 > 1/2$: Type-II superconductors exhibit second-order (continuous) phase transitions and the formation of vortex lattices.

One of the most interesting mathematical and physical phenomenon connected with Ginzburg-Landau equations is the presence of vortices in solutions. Roughly speaking, a vortex is a spatially localized structure in the solution, around which the order parameter has a nontrivial winding. In a superconductor, a vortex represents a localized defect where the normal state intrudes, and magnetic flux penetrates [20]. In the last decade or so, vortex solutions have become the object of intense mathematical study in several directions. One direction is to consider the singular limit (extreme Type II) $\kappa \to \infty$ (on a bounded domain), in which vortices become point defects whose locations are determined by some reduced finite-dimensional problem (see, for example, the books of Bethuel-Brezis-Hélein [11] for a model problem without magnetic field and Serfaty-Sandier [38]).

In the self-dual case $\kappa^2 = 1/2$, vortices effectively become non-interacting, and there is a rich multivortex solution family. Bogomolnyi [12] found the topological energy lower bound,

$$\mathcal{E}(\Psi, A)|_{\kappa^2 = 1/2} \geq \pi |\text{deg}(\Psi)|,$$

and showed that this bound is saturated (and hence the Ginzburg-Landau equations are solved) when certain first-order equations are satisfied. The mathematical implications of this self-duality, and consequent reduction to a first-order equations, were worked out by Taubes [41] who showed that for a given degree $n$, the family of solutions modulo gauge transformations is $2|n|$-dimensional, and the $2|n|$ parameters describe the locations of the zeros of the scalar field, i.e., the vortex centres. A review of this theory can be found in the book of Jaffe and Taubes [24].

A model for a vortex is given, for each degree $n \in \mathbb{Z}$, by a “radially symmetric” (or more precisely equivariant) solution of the Ginzburg-Landau equations of the form

$$\Psi^{(n)}(x) = f_n(r)e^{in\theta}, \quad A^{(n)}(x) = a_n(r)\nabla(n\theta),$$
where \((r, \theta)\) are the polar coordinates of \(x \in \mathbb{R}^2\). Note that \(\deg(\Psi^{(n)}) = n\). The pair \((\Psi^{(n)}, A^{(n)})\) is called the \(n\)-vortex (magnetic or Abrikosov in superconductivity, and Nielsen-Olesen or Nambu string in particle physics). For superconductors, this is a mixed state with the normal phase residing at the point where the vortex vanishes [25]. The existence of such solutions of the Ginzburg-Landau equations was already noticed by Abrikosov [1]. The \(n\)-vortex solution exhibits the length scales discussed above. Indeed, the asymptotics for the field components of the \(n\)-vortex are [24, 34]

\[
J^{(n)}(x) = n\beta_n K_1(r) [1 + o(e^{-m_\kappa r})] J \hat{x}
\]

\[
B^{(n)}(r) = n\beta_n K_1(r) [1 - \frac{1}{2r} + O(1/r^2)]
\]

\[
|1 - f_n(r)| \leq ce^{-m_\kappa r}
\]

\[
|f_n'(r)| \leq ce^{-m_\kappa r},
\]

as \(|x| \to \infty\), where \(J^{(n)} = \text{Im}(\overline{\Psi^{(n)}} \nabla_{A^{(n)}} \Psi^{(n)})\) is the \(n\)-vortex supercurrent, \(B^{(n)} = \text{curl} A^{(n)}\) is the \(n\)-vortex magnetic field, \(\beta_n > 0\) is a constant, and \(K_1\) is the modified Bessel function of order 1 of the second kind. The length scale of \(\Psi^{(n)}\) is \(1/m_\kappa\). Since \(K_1(r)\) behaves like \(e^{-r}/\sqrt{r}\) for large \(r\), we see that the length scale for \(J^{(n)}\) and \(B^{(n)}\) is 1.

The \(n\)-vortex is a critical point of the Ginzburg-Landau energy \(\mathcal{E}\), and the second variation of the energy,

\[
L^{(n)} := \text{Hess} \mathcal{E}(\Psi^{(n)}, A^{(n)}),
\]

is the linearized operator for the Ginzburg-Landau equations around the \(n\)-vortex, acting on the space \(X = L^2(\mathbb{R}^2, \mathbb{C}) \oplus L^2(\mathbb{R}^2; \mathbb{R}^2)\) [29]. The symmetry group of \(\mathcal{E}\), which is infinite dimensional due to gauge transformations, gives rise to an infinite-dimensional subspace \(L^{(n)} \subset X\), which we denote here by \(Z_{\text{sym}}\). We say the \(n\)-vortex is (linearly) stable if for some \(c > 0\),

\[
L^{(n)}|_{Z_{\text{sym}}} \geq c,
\]

and unstable if \(L^{(n)}\) has a negative eigenvalue. By this definition, a stable state is a local energy minimizer which is a strict minimizer in directions orthogonal to the infinitesimal
symmetry transformations. An unstable state is an energy saddle point. The basic result on vortex stability is the following [21]:

1. For Type-I superconductors, all \( n \)-vortices are stable.

2. For Type-II superconductors, the 1-vortices are stable, while \( n \)-vortices with \( |n| \geq 2 \) are unstable.

This stability behaviour was long conjectured [24], based on numerical computations [35], leading to a “vortex interaction” picture in which intervortex interactions are always attractive in the Type-I case, but become repulsive for like-signed vortices in the Type-II case [33]. This result agrees with the fact, mentioned above, that the surface tension is positive for \( \kappa^2 < 1/2 \) and negative for \( \kappa^2 > 1/2 \), so the vortices try to minimize their “surface” for \( \kappa^2 < 1/2 \) and maximize it for \( \kappa^2 > 1/2 \).

In 1957, Abrikosov predicted the existence of states of Type-II superconductors exhibiting vortices arrayed in a lattice pattern, now called *Abrikosov lattices*, within the Ginzburg-Landau theory [1]. (Due to a calculation error, Abrikosov concluded that the lattice which gives the minimum energy is the square lattice. The error was corrected by Kleiner et al. [27] who showed that it is, in fact, the triangular (also known as the hexagonal) lattice which minimizes the energy.) These lattice were later observed experimentally and have since played an important role in experimental work in superconductivity as well as having been the study of theoretical works (of the more mathematical studies, we mention the articles of Eilenberger [18] and Lasher [28]). In 2003, Abrikosov received the Nobel Prize for this discovery.

The rigorous investigation of Abrikosov solutions began soon after their discovery. Odeh [32] proved the existence of nontrivial minimizers and obtained a result concerning the bifurcation of solutions at the critical field strength. Barany et al. [10] investigated this bifurcation for certain lattices using equivariant bifurcation theory, and Takáč [40] adapted these results to study the zeros of the bifurcating solutions. Further results were
first proved in [15]. Except for a variational result of [32] (see also [15]), work done by both physicists and mathematicians has followed the general strategy of [1].

Among related results is a relation of the Ginzburg-Landau minimization problem, for a fixed, finite domain and for increasing Ginzburg-Landau parameter $\kappa^2$ and external magnetic field, to the Abrikosov lattice variational problem [4, 6]. Boundaries between superconducting, normal, and mixed phases have also been found [16, 15].

In Chapter 2 of this thesis we combine and extend the previous techniques to give a complete and self-contained proof of the existence of Abrikosov lattice solutions. To formulate our results we mention that lattices $\mathcal{L} \subset \mathbb{R}^2$ are characterized by the area $|\mathcal{L}|$ of the fundamental lattice cell $\Omega^\mathcal{L}$ (for details see Chapter 2). We will prove the following result, whose precise formulation will be given below (Theorem 8).

**Theorem 1.** Let $\mathcal{L}$ be a lattice with $|\Omega^\mathcal{L}| - \frac{2\pi}{\kappa^2} \ll 1$.

(I) If $|\Omega^\mathcal{L}| > \frac{2\pi}{\kappa^2}$, then there exists an $\mathcal{L}$-lattice solution, $(\Psi^\mathcal{L}, A^\mathcal{L})$. If $|\Omega^\mathcal{L}| \leq \frac{2\pi}{\kappa^2}$, then there is no $\mathcal{L}$-lattice solution in a neighbourhood of the branch of normal solutions.

(II) The solution $(\Psi^\mathcal{L}, A^\mathcal{L})$ is close to the branch of normal solutions and is unique, up to symmetry, in a neighbourhood of this branch.

(III) The solution $(\Psi^\mathcal{L}, A^\mathcal{L})$ is real analytic in $|\Omega^\mathcal{L}|$ in a neighbourhood of $\frac{2\pi}{\kappa^2}$.

(IV) The lattice shape for which the average energy per lattice cell is minimized approaches the triangular lattice as $|\Omega^\mathcal{L}| \to \frac{2\pi}{\kappa^2}$.

**Remark 2.**

(a) [32, 16] showed that for all $|\Omega^\mathcal{L}| > \frac{2\pi}{\kappa^2}$ there exists a global minimizer of $\mathcal{E}_{\Omega^\mathcal{L}}$.

(b) [32, 10] proved results related to the first part of (I).

(c) [28] proved partial results on (IV), which is generalization of an earlier result of [27] showing that the triangular lattice gives the minimum energy for the linearized problem.
All the rigorous results above deal with Abrikosov lattices with one quantum of magnetic flux per lattice cell. Partial results for higher magnetic fluxes were proven in [13, 7].

In Chapter 3 of the thesis we present our results concerning the stability of Abrikosov lattice solutions within the framework of the Gorkov-Eliashberg-Schmidt time-dependent Ginzburg-Landau equations. In particular we consider the linearized stability in terms of the Hessian of the Ginzburg-Landau equations. More precisely we consider the quadratic form induced by $E_{GL}''$ on two classes of perturbations and say that the solution is linearly stable if this quadratic form is positive.

We recall that the underlying geometry of the superconductor of the equations on $\mathbb{R}^2$ is a superconductor that fills all space but is homogeneous in one direction, and so, vortex solutions correspond to vortex lines in the superconductor. In general, therefore, the perturbations we consider are three-dimensional. For Type-II superconductors, however, vortices repel and therefore any solution with vortex lines that are not straight will be unstable, as the forces acting on the vortex lines will balance out only when the lines are straight. Limiting ourselves then to solutions of straight vortex lines, the homogeneity once again reduces the problem to a problem on $\mathbb{R}^2$.

We first consider perturbations that exhibit the same double periodicity as the lattice solutions themselves. For Type-II superconductors we prove the following result showing that all Abrikosov lattice solutions found in Theorem 1 that are sufficiently close to the normal solution are linearly stable under such perturbations.

**Theorem 3.** If $\kappa^2 > \frac{1}{2}$, then for $\mathcal{L}$ such that $|\Omega^\mathcal{L}|$ is sufficiently close to $\frac{2\pi}{\kappa^2}$, then

$$\langle v, E_{GL}''(\Psi^\mathcal{L}, A^\mathcal{L})v \rangle > 0$$

for all $\mathcal{L}$-lattice states $v \perp Z^\mathcal{L}$, where $Z^\mathcal{L}$ is the infinite subspace of zero mode arising from the gauge symmetry.

We also consider perturbations that have finite total energy. In this case we prove that there exists a constant $S^\mathcal{L}_\kappa$, for which we give an explicit expression, that determines
the instability of the Abrikosov lattice solutions, i.e., the Abrikosov solution is linearly stable if and only if $S^\mathcal{L}_\kappa > 0$ (we refer to Theorem 24 for a precise formulation of this result). We are confident that we will be able to determine the sign of this constant for Type-I and Type-II superconductors in the near future. Experimentally it is known that triangular Abrikosov lattices exist for Type-II superconductors. Hence we expect that $S^\mathcal{L}_\kappa > 0$ in this case.
Chapter 2

Existence of Abrikosov Lattice Solutions

The main goal of this chapter is to prove Theorems 1.

2.1 Abrikosov Lattice States

We begin by giving a mathematical definition of an Abrikosov lattice state and by discussing their basic properties. We recall that intuitively an Abrikosov lattice represents a superconductor whose physical properties are doubly periodic.

2.1.1 Lattices in $\mathbb{R}^2$

We first define lattices and introducing the terminology we will be using. A (Bravais) lattice $\mathcal{L}$ is a subset of $\mathbb{R}^2$ with the following properties.

- $\mathcal{L}$ is discrete, i.e., it has no finite limit points.
- $\mathcal{L}$ is a subgroup of $\mathbb{R}^2$ as an additive group.
- $\mathcal{L}$ is not contained in any proper vector subspace of $\mathbb{R}^2$. 

These properties imply that $\mathcal{L}$ is the set of points of the form

$$\mathcal{L} = \{ m_1 t^{(1)} + m_2 t^{(2)} : m_1, m_2 \in \mathbb{Z} \},$$

for some linearly independent vectors $t^{(1)}, t^{(2)} \in \mathbb{R}^2$, called a basis of $\mathcal{L}$.

A cell of the lattice $\mathcal{L}$ is any parallelogram whose sides are elements of the lattice, i.e., a set $\Omega$ of the form

$$\Omega = \{ x + pt + p't' : 0 \leq p, p' \leq 1 \},$$

where $x \in \mathbb{R}^2$ and $t, t' \in \mathcal{L}$ are linearly independent. We denote the area of $\Omega$ by $|\Omega|$ and remark that $|\Omega| = |t \wedge t'|$, where the wedge product is defined by $x \wedge y = x_1 y_2 - x_2 y_1$.

Any lattice has a non-zero minimal cell area and we denote this by $|\mathcal{L}|$, which we consider as a measure of the size of the lattice.

To define the shape of the lattice, we identify $\mathbb{R}^2$ with $\mathbb{C}$ via the map $x \mapsto x_1 + ix_2$ and view $\mathcal{L} \subset \mathbb{C}$. It is well-known (see [5]) that any lattice $\mathcal{L} \subseteq \mathbb{C}$ can be given a basis $t^{(1)}, t^{(2)}$ such that the ratio $\tau = \frac{t^{(2)}}{t^{(1)}}$ satisfies the inequalities:

- $|\tau| \geq 1$.
- $\text{Im } \tau > 0$.
- $-\frac{1}{2} < \text{Re } \tau \leq \frac{1}{2}$, and $\text{Re } \tau \geq 0$ if $|\tau| = 1$.

(In effect, this means $\tau$ is in the fundamental domain of the modular group acting on the upper halfspace.) Although the basis is not unique, the value of $\tau$ is, and we take $\tau$ as a measure of the shape of the lattice $\mathcal{L}$.

Now, given a function $f$ on $\mathbb{R}^2$ that is $\mathcal{L}$-periodic, i.e., $f(x + t) = f(x)$ for all $t \in \mathcal{L}$, we define the average per lattice cell of $f$, $\langle f \rangle_{\mathcal{L}}$, to be

$$\langle f \rangle_{\mathcal{L}} := \frac{1}{|\Omega|} \int_{\Omega} f(x) \, d^2x,$$

where $\Omega$ is any cell of $\mathcal{L}$. It can easily be checked the right hand side of this definition is independent of the choice of $\Omega$. 

2.1.2 Abrikosov Lattice State

An Abrikosov lattice state is a pair \((\Psi, A) \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{C}) \times H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)\) for which there exists a lattice \(\mathcal{L}\) and a family of functions \(g_t \in H^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}), t \in \mathcal{L}\), such that

\[
\begin{align*}
\Psi(x + t) &= e^{ig_t(x)}\Psi(x), \\
A(x + t) &= A(x) + \nabla g_t(x).
\end{align*}
\] (2.1)

(We will refer to such pairs as (Abrikosov) \(\mathcal{L}\)-lattice states when it necessary to make \(\mathcal{L}\) explicit.) A lattice state is therefore a state whose translations are mutually gauge equivalent. As gauge equivalent states represent the same physical state, this definition captures the idea of a superconductor whose physical properties are doubly periodic. In particular we note that the superconducting charge density \(|\Psi|^2\), the superconducting current \(\text{Im}(\bar{\Psi}\nabla A\Psi)\), and the magnetic field curl \(A\) are all \(\mathcal{L}\)-periodic as can be easily verified.

2.1.3 Symmetries

We define a symmetry of Abrikosov lattices to be a group action on the set of Abrikosov lattices that preserves the property of being a solution of the Ginzburg-Landau equations. In particular we do not impose the requirement that a symmetry preserve the underlying lattice. We list the most important symmetries.

1. Gauge symmetry: given \(\eta \in H^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R})\), we define the map \(T_{\eta}\) to be

\[
(T_{\eta}\Psi(x), T_{\eta}A(x)) = (e^{i\eta(x)}\Psi(x), A(x) + \nabla \eta(x)).
\]

We note that \(T_{\eta}\) maps Abrikosov \(\mathcal{L}\)-lattices to Abrikosov \(\mathcal{L}\)-lattices.

2. Translation symmetry: given \(y \in \mathbb{R}^2\), we define the map \(T_{y}\) to be

\[
(T_{y}\Psi(x), T_{y}A(x)) = (\Psi(x + y), A(x + y)).
\]

Again, \(T_{y}\) maps Abrikosov \(\mathcal{L}\)-lattices to Abrikosov \(\mathcal{L}\)-lattices.
3. Rotation and reflection symmetry: given \( R \in O(\mathbb{R}^2) \), the group of orthogonal matrices on \( \mathbb{R}^2 \), we define the map \( T_U \) to be

\[
(T_R \Psi(x), T_R A(x)) = (\Psi(Rx), R^{-1} A(Rx)).
\]

In this case \( T_U \) maps Abrikosov \( \mathcal{L} \)-lattices to Abrikosov \( R^{-1} \mathcal{L} \)-lattices.

### 2.1.4 Energy of Abrikosov Lattices

Because the total energy of Abrikosov lattices is infinite we will instead consider the average energy per cell, \( \mathcal{E}_\mathcal{L} \), defined to be

\[
\mathcal{E}_\mathcal{L}(\Psi, A) := \left\langle \frac{1}{2} |\nabla \Psi|^2 + \frac{\kappa^2}{4} (1 - |\Psi|^2)^2 + |\text{curl} A|^2 \right\rangle_{\mathcal{L}}.
\]

We note that all the symmetries preserve the average energy per cell.

We begin by proving that the energy functional is well-defined for Abrikosov lattices.

**Proposition 4.** For any Abrikosov \( \mathcal{L} \)-lattice \((\Psi, A)\), \( \mathcal{E}_\mathcal{L}(\Psi, A) < \infty \).

**Proof.** Note that the closure of any cell \( \Omega \) is a compact subset of \( \mathbb{R}^2 \) with Lipschitz boundary. It then follows from the Sobolev embedding theorem that since \( \Psi \) and \( A \) are in \( H^1(\Omega) \), they are in \( L^p(\Omega) \) for any \( p \in [1, \infty) \) (see [2]). The finiteness of \( \mathcal{E}_\mathcal{L} \) then follows from the Cauchy-Schwarz inequality.

### 2.1.5 Flux Quantization

An important property of Abrikosov lattices is that their average magnetic flux is quantized:

**Proposition 5.** Let \((\Psi, A)\) be an Abrikosov \( \mathcal{L} \)-lattice such that \( \psi \neq 0 \). Then there exists \( n \in \mathbb{Z} \) such that

\[
\langle \text{curl} A \rangle_{\mathcal{L}} = \frac{2\pi n}{|\mathcal{L}|}.
\]
In the case where $\Psi$ is non-zero on the boundary of a cell $\Omega$ with $|\Omega| = |\mathcal{L}|$, we can define $\varphi$ on $\partial \Omega$ by the relation $\Psi = |\Psi|e^{i\varphi}$. Then we have $\text{Im}(\bar{\Psi}\nabla \Psi) = |\Psi|^2(\nabla \varphi - A)$, which implies that $\nabla \varphi - A$ is $\mathcal{L}$-periodic. Using Stoke’s theorem we then calculate

$$\langle \text{curl} A \rangle_{\mathcal{L}} = \frac{1}{|\Omega|} \oint_{\partial \Omega} A \cdot ds = \frac{1}{|\Omega|} \oint_{\partial \Omega} \nabla \varphi \cdot ds = \frac{2\pi n}{|\mathcal{L}|},$$

where the final step follows from the fact that $\Psi$ is single-valued. This argument demonstrates the relation between the average magnetic flux and the index of $\Psi$, and therefore between the flux and the number of zeros of $\Psi$ per cell.

For the general case, however, we need a more indirect proof. We begin with the following lemma.

**Lemma 6.** Let $(\Psi, A)$ be an Abrikosov $\mathcal{L}$-lattice such that $\Psi \neq 0$. For $t, t' \in \mathcal{L}$, define $K_{t, t'}$ by the formula

$$K_{t, t'} = g_t(x + t') - g_t(x) - g_{t'}(x + t) + g_{t'}(x),$$

where $x$ is such that $\Psi(x) \neq 0$. Then $K_{t, t'}$ is independent of the choice of $x$ and there exists $n \in \mathbb{Z}$ such that

$$K_{t, t'} = \frac{2\pi n}{|\mathcal{L}|} t \wedge t'.$$

**Proof.** Fix $t$ and $t' \in \mathcal{L}$. Using (2.1), we have the relations

$$\begin{cases}
\Psi(x + t + t') = e^{ig_t(x+t')}e^{ig_{t'}(x)}\Psi(x), \\
\Psi(x + t + t') = e^{ig_{t'}(x+t')}e^{ig_t(x)}\Psi(x).
\end{cases}$$

Therefore for any $x$ such that $\Psi(x) \neq 0$, we must have $g_t(x+t') - g_t(x) - g_{t'}(x+t) + g_{t'}(x) = 2\pi n_{t, t'}(x)$, for some $n_{t, t'}(x) \in \mathbb{Z}$. From (2.1) we also have the relations that

$$\begin{cases}
A(x + t + t') = A(x) + \nabla g_t(x + t') + \nabla g_{t'}(x), \\
A(x + t + t') = A(x) + \nabla g_{t'}(x + t) + \nabla g_t(x).
\end{cases}$$

Therefore we have $\nabla (g_t(x+t') - g_t(x) - g_{t'}(x+t) + g_{t'}(x)) = 0$, which means that $n_{t, t'}(x)$ is independent of $x$ and therefore $K_{t, t'} = 2\pi n_{t, t'}(x)$ is well-defined.
We now claim that $K_{t,t'}$ is an anti-symmetric bilinear form on $L$ taking values in $2\pi\mathbb{Z}$. This can easily be checked using (2.1) where necessary. Now since $t \wedge t'$ is the signed area of the cell with sides $t$ and $t'$, $\frac{2\pi}{|L|} t \wedge t' \in 2\pi\mathbb{Z}$, and one can easily check that this defines anti-symmetric bilinear form on $L$. But the group of such forms is isomorphic to $\mathbb{Z}$, so there exists $n \in \mathbb{Z}$ such that 

$$K_{t,t'} = \frac{2\pi n}{|L|} t \wedge t',$$

and the lemma is proven.

We now complete the proof of quantization of flux.

\textit{Proof of Proposition 5.} Since $\Psi \not\equiv 0$ we can find $x$ such that $\Psi(x) \neq 0$. Let $\Omega$ be the cell with the bottom-left corner at $x$ and sides parallel to vectors $t$ and $t'$ from $L$, labelled so that $|\Omega| = t \wedge t'$. Using (2.1), a simple calculation gives that 

$$\int_{\Omega} \text{curl} A \, dx = \oint_{\partial \Omega} A \cdot ds = g_t(x + t') - g_t(x) - g_{x+t'}(t) + g_t(x) = K_{t,t'}.$$

Applying the previous lemma we therefore have 

$$\langle \text{curl} A \rangle_L = \frac{1}{|\Omega|} \int_{\Omega} \text{curl} A \, dx = \frac{1}{|t \wedge t'|} \frac{2\pi n}{|L|} t \wedge t' = \frac{2\pi n}{|L|},$$

which proves the proposition.

One can check that the gauge and translation symmetries preserve the average flux per cell but for $R \in O(\mathbb{R}^2)$, we have 

$$\langle \text{curl} T_R A \rangle_{R^{-1}L} = (\det R) \langle \text{curl} A \rangle_L.$$

We note that under the constant $\langle A \rangle_L = b$, the flux quantization leads to a relation between the parameter $b$ and the size of the lattice $|L|$.

### 2.1.6 Reduction to a single cell

An important property of lattice states is that they are defined by their restriction to a single cell and can be reconstructed from this restriction using the lattice translations.
2.2 Fixing the Gauge and Rescaling

In this section we fix the gauge for solutions, $(\Psi, A)$, of (1.1) and then rescale them to eliminate the dependence of the size of the lattice on $b$. Our space will then depend only on the number of quanta of flux and the shape of the lattice.

The symmetries allows one to fix solutions to be of a desired form. We first use the rotation and reflection symmetry to assume that $b > 0$ and that the lattice $\mathcal{L}$ has a basis of the form

$$
t^{(1)} = r \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t^{(2)} = r \begin{pmatrix} \text{Re} \tau \\ \text{Im} \tau \end{pmatrix},
$$

where $r > 0$. From now we let $\Omega$ be the cell with side $t^{(1)}$ and $t^{(2)}$. We note that

$$|\Omega| = r^2 \text{Im} \tau.
$$

We will also use the following proposition to fix the gauge symmetry. This was introduced by [32] and proved in [40]. We provide an alternate proof in Section 2.10.

**Proposition 7.** Let $(\Psi, A)$ be an $\mathcal{L}$-lattice state, and let $b$ be the average magnetic flux per cell. Then there is a $\mathcal{L}$-lattice state $(\phi, A^b_0 + a)$ that is gauge-equivalent to a translation of $(\Psi, A)$, such that $A^b_0(x) = \frac{b}{2} Jx$, where $Jx = x^\perp := (-x_2, x_1)$, and $\phi$ and $a$ satisfy the following conditions:

(i) $a$ is doubly periodic with respect to $\mathcal{L}$: $a(x + t) = a(x)$ for all $t \in \mathcal{L}$;

(ii) $a$ has mean zero: $\langle a \rangle_{\mathcal{L}} = 0$;

(iii) $a$ is divergence-free: $\text{div} a = 0$;

(iv) $\phi(x + t) = e^{\frac{ib}{2} t^\perp x} \phi(x)$, for $t = t^{(1)}, t^{(2)}$.

Suppose now that we have a $\mathcal{L}$-lattice state $(\Psi, A)$. Let $b$ be the average magnetic flux per cell of the state and $n$ the quanta of flux per cell. From the quantization of the
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flux, we know that

\[ b = \frac{2\pi n}{|\Omega|} = \frac{2\pi n}{r^2 \text{Im} \tau}, \]

We set \( \sigma := \left( \frac{n}{b} \right)^{\frac{1}{2}} \). The last two relations give \( \sigma = \left( \frac{\text{Im} \tau}{2\pi} \right)^{\frac{1}{2}} r \). We now define the rescaling \((\psi, \alpha)\) to be

\[ (\psi(x), \alpha(x)) := (\sigma \Psi(\sigma x), \sigma A(\sigma x)). \]

Let \( L^r \) be the lattice spanned by \( t^{(1)} \) and \( t^{(2)} \) as above with \( r = r^\tau \), where

\[ r^\tau := \left( \frac{2\pi}{\text{Im} \tau} \right)^{\frac{1}{2}}. \tag{2.2} \]

We let \( \Omega^\tau \) be the cell with sides \( t^{(1)} \) and \( t^{(2)} \). We note that \( |\Omega^\tau| = 2\pi n \). We summarize the effects of the rescaling above:

(i) \((\psi, \alpha)\) is a \( L^\tau \)-lattice state.

(ii) \( \frac{1}{|\Omega^\tau|} \mathcal{E}_{L^\tau}(\psi, A) = \mathcal{E}_\lambda(\psi, \alpha) \), where \( \lambda = \frac{\kappa^2 n}{b} \) and

\[ \mathcal{E}_\lambda(\psi, \alpha) = \frac{\kappa^4}{2\pi \lambda^2} \int_{\Omega^\tau} \left( |\nabla \psi|^2 + |\text{curl} \alpha|^2 + \frac{\kappa^2}{2} (|\psi|^2 - \frac{\lambda}{\kappa^2})^2 \right) dx. \tag{2.3} \]

(iii) \( \Psi \) and \( A \) solve the Ginzburg-Landau equations if and only if \( \psi \) and \( a \) solve the rescaled Ginzburg-Landau equations

\[ (-\Delta_a - \lambda)\psi = -\kappa^2 |\psi|^2 \psi, \tag{2.4a} \]

\[ \text{curl}^* \text{curl} \alpha = \text{Im} \{ \bar{\psi} \nabla_a \psi \} \tag{2.4b} \]

for \( \lambda = \frac{\kappa^2 n}{b} \). The latter equations are valid on \( \Omega^\tau \) with the boundary conditions given in the next statement.

(iv) If \((\Psi, A)\) is of the form described in Proposition 7, then

\[ \alpha = A_0^n + a, \text{ where } A_0^n(x) := \frac{n}{2} J x, \tag{2.5} \]

where \( \psi \) and \( a \) satisfy
(a) $a$ is double periodic with respect to $L^\tau$,

(b) $\langle a \rangle_{L^\tau} = 0$,

(c) $\text{div} \ a = 0$,

(d) $\psi(x + t) = e^{i\frac{\pi}{2}\hat{x}x} \psi(x)$ for $t = t^{(1)}_\tau, t^{(2)}_\tau$.

We now introduce the spaces $\mathcal{H}_n(\tau)$ and $\vec{\mathcal{H}}(\tau)$ as follows. We define the Hilbert space $\mathcal{L}_n(\tau)$ to be the closure under the $L^2$-norm of the space of all smooth $\psi$ on $\Omega^\tau$ satisfying the quasiperiodic boundary condition (d) in part (iv) above. $\mathcal{H}_n(\tau)$ is then the space of all $\psi \in \mathcal{L}_n(\tau)$ whose (weak) partial derivatives up to order 2 are square-integrable.

Similarly, we define the Hilbert space $\vec{\mathcal{L}}(\tau)$ to be the closure of the space of all smooth $a$ on $\Omega^\tau$ that satisfy periodic boundary conditions, have mean zero, and are divergence free, and $\vec{\mathcal{H}}(\tau)$ is then the subspace of $\vec{\mathcal{L}}(\tau)$ consisting of those elements whose partial derivatives up to order 2 are square-integrable.

Our problem then is, for each $n = 1, 2, \ldots$, find $(\psi, a) \in \mathcal{H}_n(\tau) \times \vec{\mathcal{H}}(\tau)$ such that $(\psi, A^n_\lambda + a)$ solves the rescaled Ginzburg-Landau equations (2.4), and among these find the one that minimizes the average energy $E_\lambda$.

We will prove the following theorem for the case $n = 1$

**Theorem 8.**

(I) For every $b$ sufficiently close to but less than the critical value $b = \kappa^2$ and every lattice shape $\tau$, there exists an $\mathcal{L}^\tau$-lattice solution, $(\Psi_\lambda^\tau, A_\lambda^\tau)$ of the rescaled Ginzburg-Landau equations with one quantum of flux per cell.

(II) This solution is unique, up to the symmetries, in a neighbourhood of the normal solution.

(III) The family of these solutions is real analytic in $b$ in a neighbourhood of $b_c$. 

(IV) If \( \kappa^2 > 1/2 \), then the global minimizer \( L_b \) of the average energy per cell, \( E(L) \equiv \frac{1}{|\Omega^\tau|}E_{\Omega^\tau}(\Psi^L_b, A^L_b) \), approaches the \( L_{\text{triangular}} \) as \( b \to b_c \) in the sense that the shape \( \tau_b \) approaches \( \tau_{\text{triangular}} = e^{i\pi/3} \) in \( \mathbb{C} \).

### 2.3 Asymptotics of solutions to (2.4)

In this section, assuming that equations (2.4), have a family, \((\psi_\epsilon, a_\epsilon, \lambda_\epsilon)\), of solutions depending on a small parameter \( \epsilon > 0 \), we establish some asymptotic properties of such a family. These properties will be needed below. Most of the results of this section were first stated in [1] (see also [13]). The main result of this section is the following

**Proposition 9.** If the equations (2.4) have a family, \((\psi_\epsilon, a_\epsilon, \lambda_\epsilon)\), \( \epsilon \to 0 \), of solutions of the form

\[
\psi_\epsilon = \epsilon \psi_0 + O(\epsilon^3), \quad a_\epsilon = \epsilon^2 a_1 + O(\epsilon^4), \quad \lambda_\epsilon = n + \epsilon^2 \lambda_1 + O(\epsilon^4),
\]

(2.6)

then \( \psi_0 \) and \( a_1 \) satisfy the equations

\[-\Delta_{A_0^\psi} \psi_0 = n \psi_0, \quad \text{and} \quad \text{curl} a_1 = H - \frac{1}{2} |\psi_0|^2, \quad \text{with} \quad H := \frac{1}{2} \langle |\psi_0|^2 \rangle,
\]

(2.7)

where \( \langle f \rangle \) stands for the average of a function \( f \) over the lattice cell \( \Omega^\tau \). Furthermore, we have

\[
E_{\lambda_\epsilon}(\psi_\epsilon, a_\epsilon) = \frac{\kappa^2}{2} + \frac{n^2 \kappa^4}{\lambda^2} - \frac{\kappa^4 \lambda^2 \epsilon^4}{2 \lambda^2 \left( (\kappa^2 - \frac{1}{2}) \beta(\psi_0) + \frac{1}{2} \right)} + O(\epsilon^6),
\]

(2.8)

where \( \alpha_\epsilon := A_0^\psi + a_\epsilon \) and \( \beta(\psi_0) \) is the Abrikosov function, which is defined by

\[
\beta(\psi_0) := \frac{\int_{\Omega^\tau} |\psi_0|^4}{\left( \int_{\Omega^\tau} |\psi_0|^2 \right)^2}.
\]

(2.9)

**Proof.** Plugging (2.6) into (2.4) and taking \( \epsilon \to 0 \) gives the first equation in (2.7) and

\[
\text{curl}^* \text{curl} a_1 = \text{Im}(\bar{\psi}_0 \nabla_{A_0^\psi} \psi_0).
\]

(2.10)

We show now that

\[
\text{Im}(\bar{\psi}_0 \nabla_{A_0^\psi} \psi_0) = -\frac{1}{2} \text{curl}^* |\psi_0|^2.
\]

(2.11)
(Recall, that for a scalar function, $f(x) \in \mathbb{R}$, \(\text{curl}^* f = (\partial_2 f, -\partial_1 f)\) is a vector.) It is easy to see (see (2.47), Section 2.9) that $\psi_0$ satisfies the first order equation

\[
((\nabla_{\mathcal{A}_0}^\alpha)_1 + i(\nabla_{\mathcal{A}_0}^\alpha)_2) \psi_0 = 0. \tag{2.12}
\]

Multiplying this relation by $\bar{\psi}_0$, we obtain $\bar{\psi}_0(\nabla_{\mathcal{A}_0}^\alpha)_1 \psi_0 + i\bar{\psi}_0(\nabla_{\mathcal{A}_0}^\alpha)_2 \psi_0 = 0$. Taking imaginary and real parts of this equation gives

\[
\text{Im } \bar{\psi}_0(\nabla_{\mathcal{A}_0}^\alpha)_1 \psi_0 = -\text{Re } \bar{\psi}_0(\nabla_{\mathcal{A}_0}^\alpha)_2 \psi_0 = -\partial_{x_2} \psi_0
\]

and

\[
\text{Im } \bar{\psi}_0(\nabla_{\mathcal{A}_0}^\alpha)_2 \psi_0 = \text{Re } \bar{\psi}_0(\nabla_{\mathcal{A}_0}^\alpha)_1 \psi_0 = \partial_{x_1} \psi_0,
\]

which, in turn, gives (2.11).

The equations (2.10) and (2.11) give the second equation in (2.7), with $H$ a constant of integration. $H$ has to be chosen so that $\int_\Omega \text{curl} a_1 = 0$, which gives the third equation in (2.7).

**Lemma 10.**

\[
(-\lambda_1 + H) \langle |\psi_0|^2 \rangle + \left( \kappa^2 - \frac{1}{2} \right) \langle |\psi_0|^4 \rangle = 0 \tag{2.13}
\]

and

\[
\mathcal{E}_\lambda(\psi_\epsilon, \alpha_\epsilon) = \frac{\kappa^2}{2} + \frac{n^2 \kappa^4}{\lambda^2} - \frac{\kappa^4 \lambda_1}{2 \lambda^2} \epsilon^4 \langle |\psi_0|^2 \rangle + O(\epsilon^6). \tag{2.14}
\]

**Proof.** Now we prove (2.13). We multiply the equation (2.4a) scalarly (in $L^2(\Omega'^r)$) by $\psi_0$, use that the operator $-\Delta_\mathcal{A}$ is self-adjoint and $(\Delta_\mathcal{A} - n)\psi_0 = 0$, substitute the expansions (2.6) and take $\epsilon = 0$, to obtain

\[
-\lambda_1 \int_{\Omega^r} |\psi_0|^2 + 2i \int_{\Omega^r} \bar{\psi}_0 a_1 \cdot \nabla_{\mathcal{A}_0} \psi_0 + \kappa^2 \int_{\Omega^r} |\psi_0|^4 = 0. \tag{2.15}
\]

This expression implies that the imaginary part of the second term on the l.h.s. of (2.15) is zero. (We arrive at the same conclusion by integrating by parts and using that $\text{div } a_1 = 0$.) Therefore

\[
2i \int_{\Omega^r} \bar{\psi}_0 a_1 \cdot \nabla_{\mathcal{A}_0} \psi_0 = -2 \int_{\Omega^r} a_1 \cdot \text{Im}(\bar{\psi}_0 \nabla_{\mathcal{A}_0} \psi_0) = -2 \int_{\Omega^r} a_1 \cdot \text{curl}^* \text{curl} a_1.
\]
Integrating the last term by parts, we obtain
\[
2i \int_{\Omega^r} \bar{\psi}_0 a_1 \cdot \nabla A_0^a \psi_0 = -\frac{1}{2} \int_{\Omega^r} |\psi_0|^4 + H \int_{\Omega^r} |\psi_0|^2.
\]
(2.16)
This equation together with (2.15) gives (2.13).

Now, we prove the statement (2.14) about the Ginzburg-Landau energy. Multiplying (2.4a) scalarly by \( \psi \) and integrating by parts gives
\[
\int_{\Omega^r} |\nabla \alpha \psi|^2 = \kappa^2 \int_{\Omega^r} \left( \lambda |\psi|^2 - \kappa^2 |\psi|^4 \right).
\]
Substituting this into the expression for the energy and using that \( |\Omega^r| = 2\pi \), we find
\[
E_{\lambda}(\psi, \alpha) = \frac{\kappa^4}{\lambda^2} \lambda^2 \frac{\lambda^2}{2\kappa^2} - \frac{\kappa^2}{2} |\psi|^4 + |\text{curl} \alpha|^2,
\]
(2.17)
where, recall, \( \langle f \rangle := \frac{1}{|\Omega^r|} \int_{\Omega^r} f \). Using the expansions (2.7) and the facts that \( \text{curl} A^a_0 = n \) and \( \langle \text{curl} a_1 \rangle = 0 \) gives
\[
E_{\lambda}(\psi_\epsilon, \alpha_\epsilon) = \frac{\kappa^2}{2} + \frac{n^2 \kappa^4}{\lambda^2} + \frac{\kappa^4}{\lambda^2} \epsilon^4 \left( -\frac{\kappa^2}{2} \langle |\psi_0|^4 \rangle + \langle |\text{curl} a_1|^2 \rangle \right) + O(\epsilon^6).
\]
(2.18)
Next, using the second equation in (2.7) in the form
\[
\text{curl} a_1 = -\frac{1}{2} |\psi_0|^2 + \frac{1}{2} \langle |\psi_0|^2 \rangle
\]
(2.19)
and substituting it into (2.18), we obtain
\[
E_{\lambda}(\psi_\epsilon, \alpha_\epsilon) = \frac{\kappa^2}{2} + \frac{n^2 \kappa^4}{\lambda^2} + \frac{\kappa^4}{\lambda^2} \epsilon^4 \left( -\frac{\kappa^2}{2} \langle |\psi_0|^4 \rangle - \frac{1}{4} \langle |\psi_0|^2 \rangle^2 \right) + O(\epsilon^6).
\]
(2.20)
Finally, using (2.13) and the definition \( H := \frac{1}{2} \langle |\psi_0|^2 \rangle \) gives (2.14).

Eqn (2.13), together with the definitions (2.9) and \( H := \frac{1}{2} \langle |\psi_0^a|^2 \rangle \) (see (2.7)), implies
\[
\lambda_1 \langle |\psi_0|^2 \rangle = \left( (\kappa^2 - \frac{1}{2}) \beta + \frac{1}{2} \right) \langle |\psi_0|^2 \rangle^2.
\]
We solve this equation for \( \langle |\psi_0|^2 \rangle \) to obtain
\[
\langle |\psi_0|^2 \rangle = \frac{\lambda_1}{(\kappa^2 - \frac{1}{2}) \beta + \frac{1}{2}}.
\]
(2.21)
This equation together with (2.14) yields (2.8).
2.4 Reformulation of the problem

In this section we reduce two equations (2.4) for $\psi$ and $\alpha$ to a single equation for $\psi$. Substituting $\alpha = A_0^n + a$, we rewrite (2.4) as

\begin{align}
(L^n - \lambda)\psi + 2ia \cdot \nabla A_0^n \psi + |a|^2 \psi + \kappa^2 |\psi|^2 \psi &= 0, \\
(M + |\psi|^2)a - \text{Im}(\bar{\psi} \nabla A_0^n \psi) &= 0,
\end{align}

where

\begin{align}
L^n := -\Delta A_0^n \quad \text{and} \quad M := \text{curl}^* \text{curl}.
\end{align}

The operators $L^n$ and $M$ are elementary and well studied. Their properties that will be used below are summarized in the following theorems, whose proofs may be found in Section 2.9.

**Theorem 11.** $L^n$ is a self-adjoint operator on $\mathcal{H}_n(\tau)$ with spectrum $\sigma(L^n) = \{(2k+1)n : k = 0, 1, 2, \ldots \}$, each eigenvalue being of (complex) multiplicity $n$. The lowest eigenvalue is given explicitly as

$$
\text{null}(L^n - n) = \left\{ e^{\frac{\imath \pi}{2} x_2(x_1 + ix_2)} \sum_{k=-\infty}^{\infty} c_k e^{ki\sqrt{2\pi \text{Im}\tau(x_1 + ix_2)}} \mid c_{k+n} = e^{i\pi \tau} e^{2ki\pi \tau} c_k \right\}$$

The infinite series converges for all $x \in \mathbb{R}^2$ since $\text{Im} \tau > 0$ and therefore the $c_k$ decay exponentially as $k \to \pm \infty$.

**Theorem 12.** $M$ is a strictly positive operator on $\mathcal{H}(\tau)$ with discrete spectrum.

We first solve the second equation (2.22b) for $a$ in terms of $\psi$, which we rewrite as

\begin{align}
(M + |\psi|^2)a - \text{Im}(\bar{\psi} \nabla A_0^n \psi) &= 0, \\
\end{align}

using the fact that $M$ is a strictly positive operator on $\mathcal{H}(\tau)$. A naive answer is

\begin{align}
a(\psi) = (M + |\psi|^2)^{-1} \text{Im}(\bar{\psi} \nabla A_0^n \psi).
\end{align}
A direct derivation of (2.25) is, however, surprisingly subtle, since one has to supplement (2.24) by the two equations \( \text{div } J_a = 0, \langle J_a \rangle = 0 \), where \( J_a := \text{Im}(\bar{\psi} \nabla A_0 + a) \). We show that these equations hold for any solution \((\psi, A_0 + a)\) of the first Ginzburg-Landau equation. Differentiating the equation \( E_\lambda(e^{is\chi} \psi, A_0 + a) + s \nabla \chi = E_\lambda(\psi, A_0 + a) \) with respect to \( s \) at \( s = 0 \), we obtain \[ \partial_\psi E_\lambda(\psi, A_0 + a) i \chi \psi + \partial_a E_\lambda(\psi, A_0 + a) \nabla \chi = 0. \] Since \( \partial_\psi E_\lambda(\psi, A_0 + a) = 0 \), this gives \[ 0 = \int_{\Omega} (Ma - J_a) \cdot \nabla \chi = \int_{\Omega} \text{div } J_a \chi. \] (2.26)

Since the last equation holds for any \( \chi \in H_1(\Omega^\tau, \mathbb{R}) \), we conclude that \( \text{div } J_a = 0 \).

Choosing \( \chi = h \cdot x, \forall h \in \mathbb{R}^2 \), in the first equation in (2.26), we find \( \langle J_a \rangle = 0 \).

Now, (2.24) can be rewritten as a fixed point problem \( a = M^{-1} J_a \), which has a unique solution in \( \mathcal{H}(\tau) \). The latter can be rewritten as (2.25).

We collect the elementary properties of the map \( a \) in the following proposition, where we identify \( \mathcal{H}_n(\tau) \) with a real Banach space using \( \psi \leftrightarrow \bar{\psi} := (\text{Re } \psi, \text{Im } \psi) \).

**Proposition 13.** The unique solution, \( a(\psi) \), of (2.22b) maps \( \mathcal{H}_n(\tau) \) to \( \mathcal{H}(\tau) \) and has the following properties:

(a) \( a(\cdot) \) is analytic as a map between real Banach spaces.

(b) \( a(0) = 0 \).

(c) For any \( \alpha \in \mathbb{R} \), \( a(e^{i\alpha} \psi) = a(\psi) \).

**Proof.** The only statement that does not follow immediately from the definition of \( a \) is (a). It is clear that \( \text{Im}(\bar{\psi} \nabla A_0 \psi) \) is real-analytic as it is a polynomial in \( \psi \) and \( \nabla \psi \), and their complex conjugates. We also note that \( (M - z)^{-1} \) is complex-analytic in \( z \) on the resolvent set of \( M \), and therefore, \( (M + |\psi|^2)^{-1} \) is analytic. (a) now follows. \( \Box \)

Now we substitute the expression (2.25) for \( a \) into (2.22a) to get a single equation \( F(\lambda, \psi) = 0 \), where the map \( F : \mathbb{R} \times \mathcal{H}_n(\tau) \rightarrow \mathcal{L}_n(\tau) \) is defined as

\[ F(\lambda, \psi) = (L^n - \lambda) \psi + 2ia(\psi) \cdot \nabla A_0 \psi + |a(\psi)|^2 \psi + \kappa^2 |\psi|^2 \psi. \] (2.27)
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The following proposition lists some properties of $F$.

**Proposition 14.**

(a) $F$ is analytic as a map between real Banach spaces,

(b) for all $\lambda$, $F(\lambda, 0) = 0$,

(c) for all $\lambda$, $D_\psi F(\lambda, 0) = L^n - \lambda$,

(d) for all $\alpha \in \mathbb{R}$, $F(\lambda, e^{i\alpha} \psi) = e^{i\alpha} F(\lambda, \psi)$.

(e) for all $\psi$, $\langle \psi, F(\lambda, \psi) \rangle \in \mathbb{R}$.

**Proof.** The first property follows from the definition of $F$ and the corresponding analyticity of $a(\psi)$. (b) through (d) are straightforward calculations. For (e), we calculate that

$$
\langle \psi, F(\lambda, \psi) \rangle = \langle \psi, (L^n - \lambda)\psi \rangle + 2i \int_{\Omega^r} \overline{\psi} a(\psi) \cdot \nabla \psi + 2 \int_{\Omega^r} (a(\psi) \cdot A_0^n)|\psi|^2 
+ \int_{\Omega^r} |a(\psi)|^2|\psi|^2 + \kappa^2 \int_{\Omega^r} |\psi|^4.
$$

The final three terms are clearly real and so is the first because $L^n - \lambda$ is self-adjoint. For the second term we calculate the complex conjugate and see that

$$
2i \int_{\Omega^r} \overline{\psi} a(\psi) \cdot \nabla \psi = -2i \int_{\Omega^r} \psi a(\psi) \cdot \nabla \overline{\psi} = 2i \int_{\Omega^r} (\nabla \psi \cdot a(\psi)) \overline{\psi},
$$

where we have integrated by parts and used the fact that the boundary terms vanish due to the periodicity of the integrand and that $\text{div} a(\psi) = 0$. Thus this term is also real and (e) is established.

\[ \square \]

### 2.5 Reduction to a finite-dimensional problem

In this section we reduce the problem of solving the equation $F(\lambda, \psi) = 0$ to a finite dimensional problem. We address the latter in the next section. We use the standard
method of Lyapunov-Schmidt reduction. Let $X := \mathcal{H}_n(\tau)$ and $Y := \mathcal{L}_n(\tau)$ and let $K = \text{null}(L^n - n)$. We let $P$ be the Riesz projection onto $K$, that is,

$$P := -\frac{1}{2\pi i} \oint_{\gamma} (L^n - z)^{-1} \, dz,$$

(2.28)

where $\gamma \subseteq \mathbb{C}$ is a contour around 0 that contains no other points of the spectrum of $L^n$. This is possible since 0 is an isolated eigenvalue of $L^n$. $P$ is a bounded, orthogonal projection, and if we let $Z := \text{null} P$, then $Y = K \oplus Z$. We also let $Q := I - P$, and so $Q$ is a projection onto $Z$.

The equation $F(\lambda, \psi) = 0$ is therefore equivalent to the pair of equations

$$PF(\lambda, P\psi + Q\psi) = 0,$$

(2.29)

$$QF(\lambda, P\psi + Q\psi) = 0.$$  

(2.30)

We will now solve (2.30) for $w = Q\psi$ in terms of $\lambda$ and $v = P\psi$. To do this, we introduce the map $G : \mathbb{R} \times K \times Z \to Z$ to be $G(\lambda, v, w) := QF(\lambda, v + w)$. Applying the Implicit Function Theorem to $G$, we obtain a real-analytic function $w : \mathbb{R} \times K \to Z$, defined on a neighbourhood of $(n, 0)$, such that $w = w(\lambda, v)$ is a unique solution to $G(\lambda, v, w) = 0$, for $(\lambda, v)$ in that neighbourhood. We substitute this function into (2.29) and see that the latter equation in a neighbourhood of $(n, 0)$ is equivalent to the equation

$$\gamma(\lambda, v) := PF(\lambda, v + w(\lambda, v)) = 0$$

(2.31)

(the bifurcation equation). Note that $\gamma : \mathbb{R} \times K \to \mathbb{C}$. We have shown that in a neighbourhood of $(n, 0)$ in $\mathbb{R} \times X$, $(\lambda, \psi)$ solves $F(\lambda, \psi) = 0$ if and only if $(\lambda, v)$, with $v = P\psi$, solves (2.31). Moreover, the solution $\psi$ of $F(\lambda, \psi) = 0$ can be reconstructed from the solution $v$ of (2.31) according to the formula

$$\psi = v + w(\lambda, v),$$

(2.32)

We note that $w$ and $\gamma$ inherit the symmetry of the original equation:
Lemma 15. For every $\alpha \in \mathbb{R}$, $w(\lambda, e^{i\alpha}v) = e^{i\alpha}w(\lambda, v)$ and $\gamma(\lambda, e^{i\alpha}v) = e^{i\alpha}\gamma(\lambda, v)$.

Proof. We first check that $w(\lambda, e^{i\alpha}v) = e^{i\alpha}w(\lambda, v)$. We note that by definition of $w$, $G(\lambda, e^{i\alpha}v, w(\lambda, e^{i\alpha}v)) = 0$, but by the symmetry of $F$, we also have $G(\lambda, e^{i\alpha}v, e^{i\alpha}w(\lambda, v)) = e^{i\alpha}G(\lambda, v, w(\lambda, v)) = 0$. The uniqueness of $w$ then implies that $w(\lambda, e^{i\alpha}v) = e^{i\alpha}w(\lambda, v)$.

We can now verify that

$$\gamma(\lambda, e^{i\alpha}v) = PF(\lambda, e^{i\alpha}v + w(\lambda, e^{i\alpha}v)) = e^{i\alpha}PF(\lambda, v + w(\lambda, v))) = e^{i\alpha}\gamma(\lambda, v).$$

We will also need the following property of $w$ below.

Lemma 16.

$$w(\lambda, v) = O(|v|^3).$$

Proof. Since $w$ is real-analytic and $w(\lambda, e^{i\alpha}v) = e^{i\alpha}w(\lambda, v)$, we have that $w(\lambda, 0) = 0$ and $D_vw(\lambda, 0) = 0$. Now we differentiate the relation $QF(\lambda, v + w(\lambda, v)) = 0$ with respect to $v$ to obtain

$$QD_vF(\lambda, v + w(\lambda, v))(I + D_vw(\lambda, v)) = 0.$$ 

At $v = 0$, we then have $Q(L^n - \lambda)(I + D_vw(\lambda, 0)) = 0$, and since $Q$ commutes with $L^n$, we then have $(L^n - \lambda)D_vw(\lambda, 0)) = 0$. But for $\lambda$ in a neighbourhood of 1, $(L^n - \lambda)$ is invertible on the range of $D_vw(\lambda, 0)$ and therefore $D_vw(\lambda, 0) = 0$. The lemma now follows.

Solving the bifurcation equation (2.31) is a subtle problem unless $n = 1$. The latter case is tackled in section 2.6.

We conclude this section with mentioning an approach to finding solutions to the bifurcation equation (2.31) for any $n$. For a fixed $n$, we define the first reduced energy $E_\lambda(\psi) := E_\lambda(\psi, A)$, where $A = A_0^n + a$, with $A_0^n(x) := \frac{n}{2}x^1$ and $a(\psi) = (M + |\psi|^2)^{-1} \text{Im}(\bar{\psi}\nabla A_0^n \psi)$ (see (2.5) and (2.25)). Critical points of this energy solve the equation $F(\lambda, \psi) = 0$.

Next, we introduce the finite dimensional effective Ginzburg-Landau energy

$$e_\lambda(v) := E_\lambda(v + w(\lambda, v)).$$
It is straightforward to show that
(i) \( e_\lambda(v) \) has a critical point \( v_0 \) iff \( E_\lambda(u) \) has a critical point \( u_0 = v_0 + w(\lambda, v_0) \);
(ii) Critical points, \( v_0 \), of \( e_\lambda(v) \) solve the equation (2.29);
(iii) \( e_\lambda(v) \) is gauge invariant, \( e_\lambda(e^{i\alpha}v) = e_\lambda(v) \). One can further find the leading behaviour of \( e_\lambda(v) \) in \( v \).

### 2.6 Bifurcation theorem for \( n = 1 \)

In this section we look at the case \( n = 1 \), and look for solutions near the trivial solution. For convenience we drop the (super)index \( n = 1 \) from the notation. We will see that as \( b = \kappa^2 \lambda \) decreases past the critical value \( b = \kappa^2 \), a branch of non-trivial solutions bifurcates from the trivial solution. More precisely, we have the following result.

**Theorem 17.** For every \( \tau \) there exists a branch, \( (\lambda_s, \psi_s, \alpha_s) \), \( s \in \mathbb{C} \) with \( |s|^2 < \epsilon \) for some \( \epsilon > 0 \), of nontrivial solutions of the rescaled Ginzburg-Landau equations (2.4), unique (apart from the trivial solution \( (1,0,A_0) \)) in a sufficiently small neighbourhood of \( (1,0,A_0) \) in \( \mathbb{R} \times \mathcal{H}(\tau) \times \tilde{\mathcal{H}}(\tau) \), and s.t.

\[
\begin{align*}
\lambda_s &= 1 + g_\lambda(|s|^2), \\
\psi_s &= s\psi_0 + s g_\psi(|s|^2), \\
\alpha_s &= A_0 + g_A(|s|^2),
\end{align*}
\]

where \((L-1)\psi_0 = 0\), \( g_\psi \) is orthogonal to \( \text{null}(L-1) \), \( g_\lambda : [0,\epsilon) \to \mathbb{R} \), \( g_\psi : [0,\epsilon) \to \mathcal{H}(\tau) \), and \( g_A : [0,\epsilon) \to \tilde{\mathcal{H}}(\tau) \) are real-analytic functions such that \( g_\lambda(0) = 0 \), \( g_\psi(0) = 0 \), \( g_A(0) = 0 \) and \( g_\lambda'(0) > 0 \). Moreover,

\[ g_\lambda'(0) = \left( \kappa^2 - \frac{1}{2} \right) \frac{\int_{\Omega^c} |\psi_0|^4}{\int_{\Omega} |\psi_0|^2} + \frac{1}{4\pi} \int_{\Omega^c} |\psi_0|^2. \]  

(2.33)

**Proof.** The proof of this theorem is a slight modification of a standard result from the bifurcation theory. It can be found in Section 2.8, Theorem 21, whose hypotheses are...
satisfied by $F$ as shown above (see also [32, 10]). The latter theorem gives us a neighbour- 
hood of $(1,0)$ in $\mathbb{R} \times \mathcal{H}(\tau)$ such that the only non-trivial solutions are given by

\[
\begin{align*}
\begin{cases}
\lambda_s = 1 + g_\lambda(|s|^2), \\
\psi_s = s\psi_0 + sg_\psi(|s|^2).
\end{cases}
\end{align*}
\]

Recall that $a(\psi)$ is defined in (2.25). We now define $\tilde{g}_A(s) = a(\psi_s)$, which is real-analytic 
and satisfies $\tilde{g}_A(-t) = a(-\psi_t) = \tilde{g}_A(t)$, and therefore is really a function of $t^2, g_A(t^2)$. 
Hence $A_s = A_0 + g_A(|s|^2)$.

Finally, (2.33) follows from (2.13) with $n = 1$ and the relation $|\Omega^\tau| = 2\pi$. \hfill \Box

Theorem 17 implies (I) - (III) of Theorem 8. \hfill \Box

Finally, we mention

Lemma 18. Recall that $\text{Im } \tau > 0$. Let $(\lambda_s, \psi_s, \alpha_s)$ be the solution branch constructed 
above and let $m_\tau = \sqrt{\text{Im } \tau}^{-1}
\begin{pmatrix}
1 & \text{Re } \tau \\
0 & \text{Im } \tau
\end{pmatrix}$.

Then $(\lambda_s, \tilde{\psi}_s, \tilde{\alpha}_s)$, where the functions $(\tilde{\psi}_s, \tilde{\alpha}_s)$ are defined on a 
$\tau$-independent square lattice and are given by

\[
\begin{align*}
\tilde{\psi}_s(x) &= \psi_s(m_\tau x), \\
\tilde{\alpha}_s(x) &= M^*_\tau \alpha_s(m_\tau x),
\end{align*}
\]

depend $\mathbb{R}$-analytically on $\tau$.

We sketch the proof of this lemma. The transformation above maps functions on a 
lattice of the shape $\tau$ into functions on a $\tau$-independent square lattice, but leads to a 
slightly more complicated expression for the Ginzburg-Landau equations. Namely, let $U_\tau \psi(x) := \psi(m_\tau x)$ and $V_\tau a(x) := m_\tau^* a(m_\tau x)$. Applying $U_\tau$ and $V_\tau$ to the equations 
(2.22), we conclude that $(\tilde{\psi}_s, \tilde{\alpha}_s)$ satisfy the equations

\[
(L^\tau_n - \lambda)\psi + 2i(m_\tau^*)^{-1}a \cdot (m_\tau^*)^{-1} \nabla A_0^\tau \psi + |(m_\tau^*)^{-1}a|^2 \psi + \kappa^2 |\psi|^2 \psi = 0,
\]

(2.35a)
\[(M_\tau + |\psi|^2)a + \tilde{F}_\tau^a(\psi) = 0, \quad (2.35b)\]

where

\[L^n_\tau := -U_\tau \Delta_{A^n_0} U_\tau^{-1} \text{ and } M_\tau := V_\tau \text{ curl}^* \text{ curl} V_\tau^{-1}. \quad (2.36)\]

Here we used that \(V_\tau A^n_0 = A^n_0\) and \(U_\tau \nabla \psi = (m^\tau)^{-1}U_\tau \psi\). (The latter relation is a straightforward computation and the former one follows from the facts that for any matrix \(m\), \((mx)^\perp = (\det m)(m^\perp)^{-1}x^\perp\), and that in our case, \(\det m_\tau = 1\).) Note that the gauge in the periodicity condition will still depend on \(\text{Im} \tau\). These complications, however, are inessential and the same techniques as above can be applied in this case.

The important point here is to observe that the function \(\psi_0\), constructed in Section B, the function \(w(\lambda, s\psi_0)\), where \(w(\lambda, v)\) is the solution of (2.30), and the bifurcation equation (2.31) depend on \(\tau\) real-analytically. We leave the details of the proof to the interested reader.

### 2.7 Proof of Theorem 8

In this section, we continue with the case \(n = 1\) and prove Theorem 8, which, as was mentioned above, is a precise restatement of Theorem 1 of Introduction. Theorem 17 implies, after rescaling to the original variables, the statements (I)-(III) of Theorem 8. It remains to prove the statement (IV).

We fix a lattice shape \(\tau\) and denote the functions \(\psi_0\), \(\psi_s\) and \(\alpha_s\) given in Theorem 17 by \(\psi^{\tau}_0\), \(\psi^{\tau}_s\) and \(\alpha^{\tau}_s\), respectively. Recall that \(b = \frac{\kappa^2}{\lambda}\). Since the function \(g_\lambda(|s|^2)\) given in Theorem 17 obeys \(g_\lambda(0) = 0\) and \(g'_\lambda(0) \neq 0\), the function \(b_s = \kappa^2(1 + g_\lambda(|s|^2))^{-1} =: \kappa^2 + g_b(|s|^2)\) can be inverted to obtain \(|s| = s(b)\). Absorbing \(\hat{s} = \frac{s}{|s|}\) into \(\psi^{\tau}_0\), we can define the family \((\psi^{\tau}_s(b), \alpha^{\tau}_s(b), b^{\tau}_s(b))\) of \(L^\tau\)-periodic solutions of the Ginzburg-Landau equations parameterized by average magnetic flux \(b\). Clearly, \(\psi^{\tau}_s(b), \alpha^{\tau}_s(b), b^{\tau}_s(b)\) are analytic in \(b\). We note the relation between the new perturbation parameter \(\mu := \kappa^2 - b\) and the bifurcation
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Parameter $|s|^2$:

$$\mu = \frac{g_\lambda(|s|^2)}{\lambda} = g_\lambda'(0)\kappa^2 + O(|s|^4). \quad (2.37)$$

Using the real-analyticity of the function $g_b$, $g^\tau$, and $g^\tau_A$, we can express $\lambda(\mu) := \kappa^2/b$, $\psi^\tau(\mu) := \psi^\tau_s(b)$ and $\alpha^\tau(\mu) := \alpha^\tau_s(b)$ as

$$\lambda(\mu) = 1 + \frac{1}{\kappa^2}\mu + O(\mu^2) \quad (2.38)$$
$$\psi^\tau(\mu) = \mu^{1/2}\psi^\tau_0 + \mu^{3/2}\psi^\tau_1 + O(\mu^{5/2}) \quad (2.39)$$
$$\alpha^\tau(\mu) = A_0 + \mu a^\tau_1 + O(\mu^2). \quad (2.40)$$

We identify the expansions $(2.38) - (2.40)$ with the expansion $(2.6)$ of Proposition 9 with $\epsilon = \mu^{1/2}$ (so that $\lambda_1 = \frac{1}{\kappa^2}$).

Now, we define the energy of the state $(\psi^\tau(\mu), \alpha^\tau(\mu), \lambda(\mu))$:

$$E_b(\tau) := E_\lambda(\mu)(\psi^\tau(\mu), \alpha^\tau(\mu)). \quad (2.41)$$

The equation $(2.8)$ of this proposition together with $(2.41)$ gives for $n = 1$

$$E_b(\tau) = \frac{\kappa^2}{2} + \frac{\kappa^4}{\lambda^2} - \frac{\kappa^4\lambda^2\mu^2}{2\lambda^2((\kappa^2 - \frac{1}{2})\beta(\tau) + \frac{1}{2})} + O(\mu^3), \quad (2.42)$$

where $\beta(\tau) \equiv \beta(\psi^\tau_0)$ is of the Abrikosov function,

$$\beta(\tau) := \frac{\int_{\Gamma^\tau} |\psi^\tau_0|^4}{\left(\int_{\Gamma^\tau} |\psi^\tau_0|^2\right)^2}. \quad (2.43)$$

Here, recall, $\psi^\tau_0$ is a non-zero element in the nullspace of the operator $L^n - 1$ acting on $\mathcal{H}_n(\tau)$. Since the nullspace is a one-dimensional complex subspace, $\beta$ depends only on $\tau$. The next result establishes a relation between the minimizers of the energy and Abrikosov function.

**Theorem 19.** In the case $\kappa > \frac{1}{\sqrt{2}}$, the minimizers, $\tau_b$, of $\tau \mapsto E_b(\tau)$ are related to the minimizer, $\tau_*$, of $\beta(\tau)$, as $\tau_b - \tau_* = O(\mu^{1/2})$. In particular, $\tau_b \to \tau_*$ as $b \to \kappa^2$.

**Proof.** To prove the theorem we note that $E_b(\tau)$ is of the form $E_b(\tau) = e_0 + e_1\mu + e_2(\tau)\mu^2 + O(\mu^3)$. The first two terms are constant in $\tau$, so we consider $\bar{E}_b(\tau) = e_2(\tau) + $
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$O(\mu)$. $\tau_b$ is also the minimizer of $\tau \mapsto \tilde{E}_b(\tau)$ and $\tau_*$, of $e_2(\tau)$. We have the expansions

$$\tilde{E}_b(\tau_*) - \tilde{E}_b(\tau_b) = \frac{1}{2} \tilde{E}_b''(\tau_b)(\tau_* - \tau_b)^2 + O((\tau_* - \tau_b)^3)$$

and

$$\tilde{E}_b(\tau_*) - \tilde{E}_b(\tau_b) = -\frac{1}{2} e_2''(\tau_b)(\tau_* - \tau_b)^2 + O((\tau_* - \tau_b)^3) + O(\mu),$$

which imply the desired result. That concludes the proof of the theorem.

The following result was discovered numerically in the physics literature and proven in [3] using earlier result of [31]:

**Theorem 20.** The function $\beta(\tau)$ has exactly two critical points, $\tau = e^{i\pi/3}$ and $\tau = e^{i\pi/2}$. The first is minimum, whereas the second is a maximum.

Theorems 19, 20 imply the remaining, (IV), statement of Theorem 8.

### 2.8 Bifurcation with Symmetry

In this section we present a variant of a standard result in Bifurcation Theory.

**Theorem 21.** Let $X$ and $Y$ be complex Hilbert spaces, with $X$ a dense subset of $Y$, and consider a map $F : \mathbb{R} \times X \to Y$ that is analytic as a map between real Banach spaces. Suppose that for some $\lambda_0 \in \mathbb{R}$, the following conditions are satisfied:

1. $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$,

2. $D_{\psi}F(\lambda_0, 0)$ is self-adjoint and has an isolated eigenvalue at 0 of (geometric) multiplicity 1,

3. For non-zero $v \in \text{null} D_{\psi}F(\lambda_0, 0)$, $\langle v, D_{\lambda,\psi}F(\lambda_0, 0)v \rangle \neq 0$,

4. For all $\alpha \in \mathbb{R}$, $F(\lambda, e^{i\alpha}\psi) = e^{i\alpha}F(\lambda, \psi)$.

5. For all $\psi \in X$, $\langle \psi, F(\lambda, \psi) \rangle \in \mathbb{R}$. 
Then \((\lambda_0, 0)\) is a bifurcation point of the equation \(F(\lambda, \psi) = 0\). In fact, there is a family of non-trivial solutions, \((\lambda, \psi)\), unique in a neighbourhood of \((\lambda_0, 0)\) in \(\mathbb{R} \times X\), and this family has the form

\[
\begin{cases}
\lambda = \phi_\lambda(|s|^2), \\
\psi = sv + s\phi_\psi(|s|^2),
\end{cases}
\]

for \(s \in \mathbb{C}\) with \(|s| < \epsilon\), for some \(\epsilon > 0\). Here \(v \in \text{null } D\psi F(\lambda_0, 0)\), and \(\phi_\lambda : [0, \epsilon) \to \mathbb{R}\) and \(\phi_\psi : [0, \epsilon) \to X\) are unique real-analytic functions, such that \(\phi_\lambda(0) = \lambda_0\), \(\phi_\psi(0) = 0\).

**Proof.** The analysis of Section 2.5 reduces the problem to the one of solving the bifurcation equation (2.31). Since the projection \(P\), defined there, is rank one and self-adjoint, we have

\[P\psi = \frac{1}{\|v\|^2} \langle v, \psi \rangle v, \text{ with } v \in \text{null } D\psi F(\lambda_0, 0).\] (2.44)

We can therefore view the function \(\gamma\) in the bifurcation equation (2.31) as a map \(\gamma : \mathbb{R} \times \mathbb{C} \to \mathbb{C}\), where

\[\gamma(\lambda, s) = \langle v, F(\lambda, sv_0 + w(\lambda, sv)) \rangle.\]

We now look for non-trivial solutions of this equation, by using the Implicit Function Theorem to solve for \(\lambda\) in terms of \(s\). Note that if \(\gamma(\lambda, t) = 0\), then \(\gamma(\lambda, e^{i\alpha}t) = 0\) for all \(\alpha\), and conversely, if \(\gamma(\lambda, s) = 0\), then \(\gamma(\lambda, |s|) = 0\). So we need only to find solutions of \(\gamma(\lambda, t) = 0\) for \(t \in \mathbb{R}\). We now show that \(\gamma(\lambda, t) \in \mathbb{R}\). Since the projection \(Q\) is self-adjoint, and since \(Qw(\lambda, v) = w(\lambda, v)\) we have

\[\langle w(\lambda, tv), F(\lambda, tv + w(\lambda, tv)) \rangle = \langle w(\lambda, tv), QF(\lambda, tv + w(\lambda, tv)) \rangle = 0.\]

Therefore, for \(t \neq 0\),

\[\langle v, F(\lambda, tv + \Phi(\lambda, tv)) \rangle = t^{-1} \langle tv + w(\lambda, tv), F(\lambda, tv + w(\lambda, tv)) \rangle,
\]

and this is real by condition (5) of the theorem. Thus we can restrict \(\gamma\) to a function \(\gamma_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\).
By a standard application of the Implicit Function Theorem to $t^{-1} \gamma_0(\lambda, t) = 0$, in which (1)-(3) are used (see for example [8]), there is $\epsilon > 0$ and a real-analytic function $\tilde{\phi}_\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $\tilde{\phi}_\lambda(0) = \lambda_0$ and if $\gamma_0(\lambda, t) = 0$ with $|t| < \epsilon$, then either $t = 0$ or $\lambda = \phi_\lambda(t)$. Recalling that $\gamma(\lambda, e^{i\alpha}t) = e^{i\alpha} \gamma(\lambda, t)$, we have shown that if $\gamma(\lambda, s) = 0$ and $|s| < \epsilon$, then either $s = 0$ or $\lambda = \phi_\lambda(|s|)$.

We also note that by the symmetry, $\tilde{\phi}_\lambda(-t) = \tilde{\phi}_\lambda(|t|) = \tilde{\phi}_\lambda(t)$, so $\tilde{\phi}_\lambda$ is an even real-analytic function, and therefore must in fact be a function solely of $|t|^2$. We therefore set $\phi_\lambda(t) = \tilde{\phi}_\lambda(\sqrt{t})$, and so $\phi_\lambda$ is real-analytic.

We now define $\phi_\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ to be

$$
\phi_\psi(t) = \begin{cases} 
t^{-1}w(\phi_\lambda(t), tv) & t \neq 0, \\
0 & t = 0,
\end{cases}
$$

(2.45)

$\phi_\psi$ is also real-analytic and satisfies $s\phi_\psi(|s|^2) = w(\phi_\lambda(|s|^2), sv)$ for any $s \in \mathbb{C}$ with $|s|^2 < \epsilon$.

Now we know that there is a neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times \text{null} D_\psi F(\lambda_0, 0)$ such that in that neighbourhood $F(\lambda, \psi) = 0$ if and only if $\gamma(\lambda, s) = 0$ where $P\psi = sv$. By taking a smaller neighbourhood if necessary, we have proven that $F(\lambda, \psi) = 0$ in that neighbourhood if and only if either $s = 0$ or $\lambda = \phi_\lambda(|s|^2)$. If $s = 0$, we have $\psi = sv + s\phi_\psi(|s|^2) = 0$ which gives the trivial solution. In the other case, $\psi = sv + s\phi_\psi(|s|^2)$ and that completes the proof of the theorem.

2.9 The Operators $L^n$ and $M$

In this section we prove Theorems 12 and 11. The proofs below are standard.

Proof of Theorem 12. The fact that $M$ is positive follows immediately from its definition. We note that its being strictly positive is the result of restricting its domain to elements having mean zero.
Proof of Theorem 11. First, we note that $L^n$ is clearly a positive self-adjoint operator. To see that it has discrete spectrum, we first note that the inclusion $H^2 \hookrightarrow L^2$ is compact for bounded domains in $\mathbb{R}^2$ with Lipschitz boundary (which certainly includes lattice cells). Then for any $z$ in the resolvent set of $L^n$, $(L^n - z)^{-1} : L^2 \to H^2$ is bounded and therefore $(L^n - z)^{-1} : L^2 \to L^2$ is compact.

In fact we find the spectrum of $L^n$ explicitly. We introduce the harmonic oscillator annihilation and creation operators, $\alpha^n$ and $(\alpha^n)^*$, with

$$\alpha^n := (\nabla A_0^n)_1 + i(\nabla A_0^n)_2 = \partial_{x_1} + i\partial_{x_2} + \frac{n}{2} x_1 + \frac{in}{2} x_2.$$  

(2.46)

One can verify that these operators satisfy the following relations:

1. $[\alpha^n, (\alpha^n)^*] = 2 \text{curl} A_0^n = 2n;$

2. $L^n - n = (\alpha^n)^* \alpha^n.$

As for the harmonic oscillator (see for example [22]), this gives the explicit information about $\sigma(L)$ as stated in the theorem. Furthermore, the second and the third properties imply

$$\text{null}(L^n - n) = \text{null} \alpha^n. \quad (2.47)$$

We can now prove the following.

**Proposition 22.** null $L^n - n$ is given by

$$\text{null}(L^n - n) = \left\{ e^{\frac{n}{2}|x|^2} \sum_{k=-\infty}^{\infty} c_k e^{k \sqrt{2\pi \text{Im} \tau(x_1 + ix_2)}} \mid c_{k+n} = e^{in\pi \tau} e^{2k\pi \tau} c_k \right\} \quad (2.48)$$

and therefore, in particular, dim$_{\mathbb{C}}$ null $L^n = n$.

**Proof.** We find $\alpha^n$. A simple calculation gives the following operator equation

$$e^{\frac{n}{4}|x|^2} \alpha^n e^{-\frac{n}{4}|x|^2} = \partial_{x_1} + i\partial_{x_2}.$$
This immediately proves that $\psi \in \text{null } \alpha^n$ if and only if $\xi = e^{\frac{2}{3} |x|^2} \psi$ satisfies $\partial_{z_1} \xi + i \partial_{z_2} \xi = 0$. We now identify $x \in \mathbb{R}^2$ with $z = x^1 + ix^2 \in \mathbb{C}$ and see that this means that $\xi$ is analytic. We therefore define the entire function $\Theta$ to be

$$\Theta(z) = e^{-\frac{n(\tau \pi)^2}{4 \pi^2} z^2} \xi \left( \frac{\tau^\pi z}{\pi} \right).$$

The quasiperiodicity of $\psi$ transfers to $\Theta$ as follows.

$$\Theta(z + \pi) = \Theta(z),$$

$$\Theta(z + \pi \tau) = e^{-2inz} e^{-in\pi\tau} \Theta(z).$$

To complete the proof, we now need to show that the space of the analytic functions which satisfy these relations form a vector space of dimension $n$. It is easy to verify that the first relation ensures that $\Theta$ have a absolutely convergent Fourier expansion of the form

$$\Theta(z) = \sum_{k=-\infty}^{\infty} c_k e^{2kiz}.$$

We recall that $\text{Im } \tau > 0$ and therefore this series does converge. The second relation, on the other hand, leads to relation for the coefficients of the expansion. Namely, we have

$$c_{k+n} = e^{in\pi \tau} e^{2k\pi \tau} c_k$$

And that means such functions are determined solely by the values of $c_0, \ldots, c_{n-1}$ and therefore form an $|n|$-dimensional vector space.

This completes the proof of Theorem 11.

### 2.10 Fixing the Gauge

We provide here an alternate proof of Proposition 7, largely based on ideas in [18]. We begin by defining the function $B : \mathbb{R} \to \mathbb{R}$ to be

$$B(\zeta) = \frac{1}{r} \int_{0}^{r} \text{curl} A(\xi, \zeta) \, d\xi.$$
It is clear that \( b = \frac{1}{r \tau^2} \int_0^{r \tau^2} B(\zeta) \, d\zeta \). A calculation shows that \( B(\zeta + r \tau^2) = B(\zeta) \).

We now define \( P = (P_1, P_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) to be

\[
P_1(x) = bx_2 - \int_0^{x_2} B(\zeta) \, d\zeta,
\]
\[
P_2(x) = \int_{\frac{x_1}{\tau^2} x_2}^{x_1} \text{curl} \, A(\xi, x_2) \, d\xi + \frac{\tau \wedge x}{\tau^2} B(x_2).
\]

A calculation shows that \( P \) is doubly-periodic with respect to \( \mathcal{L} \).

We now define \( \eta' : \mathbb{R}^2 \to \mathbb{R} \) to be

\[
\eta'(x) = \frac{b}{2} x_1 x_2 - \int_0^{x_1} A_1(\xi, 0) \, d\xi - \int_0^{x_2} A_2(x_1, \zeta) - P_2(x_1, \zeta) \, d\zeta.
\]

\( \eta' \) satisfies

\[
\nabla \eta = -A + A_0 + P.
\]

Now let \( \eta'' \) be a doubly-periodic solution of the equation \( \Delta \eta'' = -\text{div} \, P \). Also let \( C = (C_1, C_2) \) be given by

\[
C = -\frac{1}{|\Omega|} \int_\Omega P + \nabla \eta \, dx,
\]
where \( \Omega \) is any fundamental cell, and set \( \eta''' = C_1 x_1 + C_2 x_2 \).

We claim that \( \eta = \eta' + \eta'' + \eta''' \) is such that \( A + \nabla \eta \) satisfies (i) - (iii) of the proposition. We first note that \( A + \nabla \eta = A - A + A_0 + P + \nabla \eta'' + C \). By the above, \( A' = P + \nabla \eta'' + C \) is periodic. We also calculate that \( \text{div} \, A' = \text{div} \, P + \Delta \eta'' = 0 \). Finally \( \int A' = \int P + \nabla \eta - C = 0 \).

All that remains is to prove (iv). This will follow from a gauge transformation and translation of the state. We note that

\[
A_0(x + t) + A'(x + t) = A_0(x) + A'(x) + \frac{b}{2} \begin{pmatrix} -t_2 \\ t_1 \end{pmatrix}.
\]

This means that \( A_0(x + t) + A'(x + t) = A_0(x) + A'(x) + \nabla g_t(x) \), where \( g_t(x) = \frac{b}{2} t \wedge x + C_t \) for some constant \( C_t \). To establish (iv), we need to have it so that \( C_t = 0 \) for \( t = r, r\tau \).
First let \( l \) be such that \( r \land l = -\frac{C_r}{b} \) and \( r\tau \land l = -\frac{C_{\tau r}}{b} \). This \( l \) exists as it is the solution to the matrix equation

\[
\begin{pmatrix}
0 & r \\
-\tau_2 r & -\tau_1 r
\end{pmatrix}
\begin{pmatrix}
 l_1 \\
 l_2
\end{pmatrix}
= \begin{pmatrix}
 -\frac{C_r}{b} \\
 -\frac{C_{\tau r}}{b}
\end{pmatrix},
\]

and the determinant of the matrix is just \( r^2 \tau_2 \), which is non-zero because \((r,0)\) and \(r\tau\) form a basis of the lattice. Let \( \zeta(x) = \frac{b}{2} l \land x \). A straightforward calculation then shows that \( e^{i\zeta(x)} \psi(x + l) \) satisfies (iv) and that \( A(x + l) + \nabla \zeta(x) \) still satisfies (i) through (iii). This proves the proposition.
Chapter 3

Stability of Abrikosov Lattice Solutions

In this chapter we study the stability of the Abrikosov lattice solutions of the Ginzburg-Landau equations found in Chapter 2 under different classes of perturbations.

3.1 Gorkov-Eliashberg-Schmidt equations

We begin by discussing the framework of the stability problem, i.e., by discussing the time-dependent Ginzburg-Landau equations of superconductivity (the Gorkov-Eliashberg-Schmidt equations):

\[
\begin{align*}
\gamma \partial_{t\Phi} \Psi & = \Delta_A \Psi + \kappa^2 (1 - |\Psi|^2) \Psi, \\
\sigma (A + \nabla \Phi) & = - \text{curl}^* \text{curl} A + \text{Im}(\overline{\Psi} \nabla_A \Psi),
\end{align*}
\]  

(3.1)

where $\Psi : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}$ is the order parameter, $A : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$, $\Phi : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ are the vector and scalar potentials, respectively, $\gamma$ is a complex number, and $\sigma$ is a real number. $\partial_{t\Phi}$ is defined to be $\partial_t + i \Phi$.

The second equation is based on Ampère’s law (a Maxwell equation with $-\partial_t E$ ne-
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glected) and Faraday’s law,

\[ \text{curl } B = J \quad \text{and} \quad \text{curl } E + \partial_t B = 0, \]

where \( B, E \) and \( J \) are the magnetic field, the electric field and the total current. To this one adds the constitutive relations \( J = J_s + J_n \), where \( J_s = \text{Im}(\overline{\Psi} \nabla \Psi) \), \( J_n \) are the superconducting and normal current, respectively, and \( J_n = \sigma E \), where \( \sigma \) is the conductivity coefficient (in general, the conductivity tensor), and the expression of the magnetic and electric fields in terms of the vector and magnetic potentials: \( B = \text{curl } A \) and \( E = -\nabla \Phi - \partial_t A \).

Multiplying the second equation in (3.1) by \( \sigma^{-1} \) and taking div of the result we obtain

\[ \Delta \Phi = -\partial_t \text{div } A + \sigma^{-1} \text{div}[\text{Im}(\overline{\Psi} \nabla \Psi)]. \tag{3.2} \]

This gives an equation for \( \Phi \), which can be easily solved. The solution is determined up to a harmonic function on \( \mathbb{R}^2 \). We fix the solution (up to a constant) by demanding that \( \Phi \) is bounded. In what follows we always assume that \( \Phi \) is a bounded solution of the equation (3.2) and, in particular, is a function of \( \Psi \) and \( A \), and we do not list it among unknowns.

The Gorkov-Eliashberg-Schmidt equations (3.1) admit several symmetries, that is, transformations which map solutions to solutions.

**Gauge symmetry:** for any sufficiently regular function \( \eta : \mathbb{R}^2 \to \mathbb{R} \),

\[ \Psi \mapsto e^{i\eta} \Psi, \quad A \mapsto A + \nabla \eta, \quad \Phi \mapsto \Phi - \partial_t \eta; \]

**Translation symmetry:** for any \( h \in \mathbb{R}^2 \),

\[ \Psi(x, t) \mapsto \Psi(x + h, t), \quad A(x, t) \mapsto A(x + h, t), \quad \Phi(x, t) \mapsto \Phi(x + h, t); \]

**Rotation and reflection symmetry:** for any \( R \in O(2) \)

\[ \Psi(x, t) \mapsto \Psi(Rx, t), \quad A(x, t) \mapsto R^{-1} A(Rx, t), \quad \Phi(x, t) \mapsto \Phi(Rx, t). \]
In order to keep the notation as simple as possible we consider only the case $\gamma = \sigma = 1$. In this case (3.1) is the $L^2$–gradient flow, in the sense that
\[
\begin{pmatrix}
\partial_t \Psi + i\Phi \Psi \\
\partial_t A + \nabla \Phi
\end{pmatrix} = -E'_\Omega(\Psi, A),
\]
(3.3)
for the Ginzburg-Landau energy functional
\[
E_\Omega(\Psi, A) := \frac{1}{2} \int_\Omega \left\{ |\nabla_A \Psi|^2 + (\text{curl } A)^2 + \frac{\kappa^2}{2} (1 - |\Psi|^2)^2 \right\},
\]
(3.4)
with either $\Omega = \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^2$, and with appropriate boundary conditions. (Note that the right hand side of (3.1) is the $L^2$–gradient of $E_\Omega(\Psi, A)$.)

### 3.2 Linearized stability of static solutions

We wish to study the stability of certain static solutions $(\Psi^*, A^*, \Phi^*)$ of (3.1). The latter asserts that solutions of (3.1) with initial conditions close, in a certain norm, to $(\Psi^*, A^*, \Phi^*)$ (with possibly modified parameters) remain close as $t \to \infty$. Proving such a result is an open problem. We address the simpler problem of linearized stability.

*Linearized stability* can be stated as follows. Let $L := E''_{GL}(\Psi^*, A^*)$, the Hessian of the energy. We consider the operator $L$ on a suitable space of $w = (\xi, \alpha)$ with the inner product
\[
\langle w, w' \rangle = \int_\Omega \text{Re } \xi \xi' + \alpha \cdot \alpha'.
\]
(3.5)
$L$ is a real-linear operator satisfying (see [21]) $\langle w', Lw \rangle = \langle Lw', w \rangle$ in the inner product (3.5). We say that the solution $u^* = (\Psi^*, A^*)$ is *linearly stable* if and only if
\[
\text{null } L = Z,
\]
(3.6)
\[
\langle w, Lw \rangle > 0, \forall w \perp Z.
\]
(3.7)
Here $Z$ is the span of the zero modes of $L$ (see below for identification of these modes).
To explain the meaning of this definition, we linearize (3.1) around the solution \((\Psi_*, A_*, \Phi_*)\) to obtain the real-linear equation

\[
\begin{pmatrix}
(\partial_t + i\phi)\xi \\
\partial_t\alpha + \nabla\phi
\end{pmatrix}
= -L
\begin{pmatrix}
\xi \\
\alpha
\end{pmatrix},
\]

(3.8)

Linearized stability should imply that any solution of (3.8), with an initial condition \(w_0 \perp Z\) satisfies \(\|w\| \to 0\), as \(t \to \infty\). Indeed, differentiating the expression \(\|w\|^2\) with respect to time and using (3.8) for \(\partial_tw\), we obtain

\[
\frac{1}{2} \partial_t\|w\|^2 = -\langle w, Lw \rangle.
\]

Equations (3.6) and (3.7) imply that the right hand side is positive, which indicates that the desired result might be true. In fact, this statement would be true if we had the stronger bound \(\langle w, Lw \rangle \geq c\|w\|^2\), \(c > 0\).

### 3.3 Review of existence results

We now turn to the more specific case of the Abrikosov lattice solutions found in the first chapter, \(u_\epsilon = (\psi_\epsilon, a_\epsilon)\) and consider the rescaled Ginzburg-Landau equations (2.4).

We recall the properties of these Abrikosov lattice solutions:

1. For any lattice shape \(\tau\), there exists \(\epsilon_0 > 0\) and a family \((\lambda_\epsilon, \psi_\epsilon, a_\epsilon) \in \mathbb{R} \times C^\infty(\mathbb{R}^2; \mathbb{C}) \times C^\infty(\mathbb{R}^2; \mathbb{R}^2)\), defined for \(\epsilon \in [0, \epsilon_0]\) such that for all \(\epsilon \in [0, \epsilon_0]\), \(u_\epsilon = (\psi_\epsilon, a_\epsilon)\) is a solution of

\[
\begin{pmatrix}
\Delta_\epsilon \psi + \lambda_\epsilon \psi - \kappa^2|\psi|^2\psi \\
- \text{curl}^* \text{curl} a + \text{Im}(\bar{\psi}\nabla_\epsilon \psi)
\end{pmatrix}
= 0.
\]

2. Set \(t^{(1)}\) and \(t^{(2)}\) to be

\[
t^{(1)} = r \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t^{(2)} = r \begin{pmatrix} \text{Re} \tau \\ \text{Im} \tau \end{pmatrix}, \quad \text{where } r = \left(\frac{2\pi}{\text{Im} \tau}\right)^{\frac{1}{2}}, \quad (3.9)
\]
and set $\Omega$ to be

$$\Omega = \{ p_1 t^{(1)} + p_2 t^{(2)} : 0 \leq p_1, p_2 < 1 \}.$$ 

Then $\psi_\epsilon(t + t^{(j)}) = e^{i(\psi_\epsilon(x) + \alpha_\epsilon(x))}$ for $j = 1, 2$, and $a_\epsilon(x) = a_0(x) + \alpha_\epsilon(x)$ with $a_0(x) = \frac{1}{2} Jx$ and $\alpha_\epsilon$ is such that $\alpha_\epsilon(t + t^{(j)}) = \alpha(x)$ for $j = 1, 2$, div $\alpha_\epsilon = 0$ and $\langle \alpha_\epsilon \rangle_\Omega = 0$.

3. The map $\epsilon \to (\lambda_\epsilon, \psi_\epsilon, \alpha_\epsilon)$ is a real analytic map from $[0, \epsilon_0)$ to $\mathbb{R} \times L^2(\Omega; \mathbb{C}) \times L^2(\Omega; \mathbb{R}^2)$ and

$$\begin{aligned}
\lambda_\epsilon &= 1 + \epsilon^2 \lambda_1 + o(\epsilon^4), \\
\psi_\epsilon &= \epsilon \psi_0 + \epsilon^3 \psi_1 + o(\epsilon^5), \\
\alpha_\epsilon &= \epsilon^2 a_1 + o(\epsilon^4).
\end{aligned}$$

Moreover we have the relations

$$\langle \text{Im}(\bar{\psi}_0 \nabla_0 \psi_0) \cdot (\text{curl}^* \text{curl})^{-1} \text{Im}(\bar{\psi}_0 \nabla_0 \psi_0) \rangle = \frac{1}{4} \langle |\psi_0|^4 \rangle - \frac{1}{4} \langle |\psi_0|^2 \rangle^2.$$ 

$$\lambda_1 \langle |\psi_0|^2 \rangle = \left( \kappa^2 - \frac{1}{2} \right) \langle |\psi_0|^4 \rangle + \frac{1}{2} \langle |\psi_0|^2 \rangle^2.$$ 

### 3.4 Explicit form of $L$ and its properties

The Hessian $L_\epsilon = \mathcal{E}_{GL}''(\psi_\epsilon, a_\epsilon)$, or more accurately, the linearization of the rescaled Ginzburg-Landau equations, is explicitly given by

$$L_\epsilon v = \begin{pmatrix}
-\Delta_a \xi - \lambda_\epsilon \xi & + 2\kappa^2 |\psi_\epsilon|^2 \xi & + \kappa^2 \psi_\epsilon^2 \xi & + 2i \alpha \cdot \nabla_\alpha \psi_\epsilon & + i \psi_\epsilon \text{div} \alpha \\
\text{curl}^* \text{curl} \alpha & + |\psi_\epsilon|^2 \alpha & - \text{Im}(\bar{\psi}_\epsilon \nabla_\alpha \xi + \bar{\xi} \nabla_\alpha \psi_\epsilon) & \end{pmatrix},$$

where $v = (\xi, \alpha)$.

It is clear that $L$ is a real linear operator and one can easily show that it is symmetric, i.e.,

$$\langle v, L_\epsilon v' \rangle = \langle L_\epsilon v, v' \rangle$$

with respect to the inner product (3.5).
\( L \) also has zero modes arising from the gauge and translation symmetries of the Ginzburg-Landau equations. Explicitly the translation modes \( T_i, i = 1, 2 \), are given by

\[
T_i = \begin{pmatrix}
(\nabla a \psi)_{\ell_1 i}
\cr
-(\text{curl} a_{\ell}) e_i^\perp
\end{pmatrix},
\]

and the gauge modes \( G_\gamma, \gamma \) sufficiently regular, are given by

\[
G_\gamma = \begin{pmatrix}
i\gamma \psi \\
\nabla \gamma
\end{pmatrix}.
\]

\(3.11\)

3.5 Results

We will first study the simpler problem of stability of \( u_\epsilon \) under perturbations that exhibit the same type of double periodicity as the solutions. We therefore define our space of perturbations \( \mathcal{H}_{\text{per}} \) to be the set of \( v = (\xi, \alpha) \in L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \) satisfying \( \xi(x + t^{(j)}) = e^{i\frac{\ell}{2}(j) \wedge x} \xi(x) \) for \( j = 1, 2 \) and \( \alpha(x + t^{(j)}) = \alpha(x) \) for \( j = 1, 2 \).

We denote by \( L_{\epsilon}^{\text{per}} \) the operator \( L_\epsilon \) acting on \( \mathcal{H}_{\text{per}} \). We will prove the following result for this operator.

**Theorem 23.** Let \( Z_\epsilon \) be the (infinite) subspace spanned by the gauge modes \( G_\gamma \) for \( \gamma \in H^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}) \) such that \( \gamma(x + t^{(i)}) = \gamma(x) \). If \( \kappa^2 > \frac{1}{2} \), then for \( \epsilon \) sufficiently small

\[
\inf_{v \perp Z_\epsilon, \|v\| = 1} \langle v, L_{\epsilon}^{\text{per}} v \rangle > 0.
\]

We will also consider a more natural class of perturbations by removing the requirement that the perturbations have the same double periodicity as the solutions. We define the space of perturbations \( \mathcal{H} \) to simply be the set of \( v = (\xi, \alpha) \in L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \).

We now consider \( L_{\epsilon} \) on \( \mathcal{H} \). To state our result we define the functions \( \phi_{k\sigma_1}, k \in [0, 2\pi)^2, \sigma_1 = \pm 1 \), to be

\[
\phi_{k\sigma_1}(x) = e^{-(k_1 + ik_2)x_2} e^{-\frac{1}{2}x_1^2 + \frac{1}{2}x_1 x_2} \sum_{m=-\infty}^{\infty} c_m e^{im(x_1 + ix_2)},
\]
where the \( c_m \in \mathbb{C} \) satisfy
\[
c_{m+2} = e^{-\tau_2(k_1+ik_2)} e^{i\pi\tau} e^{im\pi \tau} c_m,
\]
and \( c_0 = 1, \ c_1 = 0 \) for \( \sigma_1 = 1 \) and \( c_0 = 0, \ c_1 = 1 \) for \( \sigma_1 = -1 \). We note that the series converges because \( \text{Im} \ \tau > 0 \) and thus the \( c_m \) decay exponentially as \( m \to \pm \infty \).

We now define the stability constants \( S_{\kappa, \tau} \) to be
\[
S_{\kappa, \tau} = \inf_{k, \sigma} (2\kappa^2 - 1) [-\beta(\tau) \langle |\phi_{01}|^2 \rangle \langle |\phi_{k\sigma_1}|^2 \rangle + 2 \langle |\phi_{01}|^2 |\phi_{k\sigma_1}|^2 \rangle + \sigma_2 \text{Re} \langle \bar{\phi}_{01}^2 \phi_{k\sigma_1}^2 \rangle]
\]
\[
- 2k \cdot \text{Im} \langle \bar{\phi}_{k\sigma_1} \nabla \phi_{k\sigma_1} \rangle, \quad (3.12)
\]
where \( \sigma = (\sigma_1, \sigma_2) \).

Our main result is the following.

**Theorem 24.** Fix \( \kappa \) and \( \tau \). Let \( Z_\epsilon \) be the (infinite) subspace spanned by the gauge modes \( G_\gamma \) for \( \gamma \in H^2(\mathbb{R}^2; \mathbb{R}) \). Then for sufficiently small \( \epsilon \),
\[
\inf_{v \perp Z_\epsilon, \|v\| = 1} \langle v, L_\epsilon v \rangle > 0 \ (\text{resp.} \ < 0) \quad \text{if and only if} \quad S_{\kappa, \tau} > 0 \ (\text{resp.} \ < 0) .
\]

We are confident that there are relations between the expression in the definition of \( S \) for different \( \phi \) (due to symmetries discussed and used in the proof below) and that we can compute these numbers. We plan to do this in the near future.

The rest of this chapter is devoted to the proof of these two theorems.

### 3.6 Gauge Fixing

To deal with the infinite dimensional kernel of \( L_\epsilon \), we follow [21] and first restrict the class of perturbations to the space of those \( v = (\xi, \alpha) \) that are orthogonal to the gauge zero-modes (3.11). After integration by parts this leads to the condition that
\[
\text{Im} (\bar{\psi}_\epsilon \xi) = \text{div} \alpha. \quad (3.13)
\]
We now consider a modified quadratic form $\tilde{L}_\epsilon$ defined by
\[
\langle v, \tilde{L}_\epsilon v \rangle = \langle v, L_\epsilon v \rangle + \int (\text{Im}(\bar{\psi}_\epsilon \xi) - \text{div} \alpha)^2.
\]
It is clearly true that $\tilde{L}_\epsilon$ agrees with $L_\epsilon$ on the subspace defined by the gauge condition (3.13), but it has the advantage of shifting the essential spectrum away of zero, as can be seen below. It is straightforward to show that $\tilde{L}_\epsilon$ is explicitly given by
\[
\tilde{L}_\epsilon v = \begin{pmatrix}
-\Delta_a \xi - \lambda_\epsilon \xi + (2\kappa^2 + \frac{1}{2})|\psi_\epsilon|^2 \xi + (\kappa^2 - \frac{1}{2})\psi_\epsilon^2 \bar{\xi} + 2i\alpha \cdot \nabla_a \psi_\epsilon \\
-\Delta \alpha + |\psi_\epsilon|^2 \alpha - 2 \text{Im}(\bar{\xi} \nabla_a \psi_\epsilon)
\end{pmatrix}.
\] (3.14)
The goal is now to show that $\langle v, \tilde{L}_\epsilon v \rangle > 0$ for all $v$.

### 3.7 Complexification

It will be convenient to complexify the spaces $H^\text{per}$ and $H$. We do this explicitly only for the latter space as the process is the same. We first identify $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with the function $\alpha^C = \alpha_1 - i\alpha_2 : \mathbb{R}^2 \to \mathbb{C}$. In order to express all quantities in terms of the complexified field we introduce the operator $\partial_a^C$, defined to be
\[
\partial_a^C = \partial_{x_1} - i\partial_{x_2} - ia^C.
\]
We also introduce the conjugate of an operator: given an operator $A$, $\bar{A}$ denotes the operator $C A^C$ where $C$ denotes complex conjugation.

Straightforward calculations show that
\[
2i\alpha \cdot \nabla_a \psi_\epsilon = -i(\partial_{a_0}^C \psi_\epsilon)\alpha^C + i(\partial_{a_0} \psi_\epsilon)\bar{\alpha}^C.
\]
Also
\[
- \text{Im}(\bar{\xi} \nabla_a \psi_\epsilon)^C = \frac{i}{2}(\partial_{a_0}^C \psi_\epsilon)\xi + \frac{i}{2}(\partial_{a_0} \psi_\epsilon)\bar{\xi}.
\]
We also note that $-\partial_{a_0}^C$ and $-\partial_{a_0}^*$ are respectively the creation and annihilation operators associated to $\Delta_{a_0}$. In particular we have
\[
\partial_{a_0}^* \psi_0 = 0.
\]
There is also the important fact that

\[ \text{Im}(\bar{\psi}_0 \nabla a_0 \psi_0) = -\frac{i}{2} \bar{\psi}_0 (\partial_{a_0} \psi_0). \]

We now drop the superscript from the notation and define the complex Hilbert space \( \mathcal{K} \) consisting of \( v = (\xi, \phi, \alpha, \omega) \) where \( \alpha \) and \( \omega \) are divergence-free and have mean-zero. \( \mathcal{H} \) is embedded in \( \mathcal{K} \) via the injection

\[
\begin{pmatrix}
\xi \\
\alpha
\end{pmatrix} \rightarrow \begin{pmatrix}
\xi \\
\bar{\xi} \\
\alpha \\
\bar{\alpha}
\end{pmatrix},
\]

and through this embedding \( \tilde{L}_\epsilon \) induces an operator on \( \mathcal{K} \), which we denote by \( K_\epsilon \) and which is given by

\[
K_\epsilon = \begin{pmatrix}
-\Delta + |\psi_\epsilon|^2 & (\kappa^2 - \frac{1}{2})\psi_\epsilon^2 & -i(\partial^*_a \psi_\epsilon) & i(\partial_a \psi_\epsilon) \\
(\kappa^2 - \frac{1}{2})\bar{\psi}_\epsilon^2 & -\Delta - \lambda_\epsilon + (2\kappa^2 + \frac{1}{2})|\psi_\epsilon|^2 & -i(\partial_{a_0} \psi_\epsilon) & i(\partial_{a_0}^* \psi_\epsilon) \\
i(\partial_{a_0}^* \psi_\epsilon) & i(\partial_{a_0} \psi_\epsilon) & -\Delta + |\psi_\epsilon|^2 & 0 \\
-i(\partial_{a_0} \psi_\epsilon) & -i(\partial_{a_0}^* \psi_\epsilon) & 0 & -\Delta + |\psi_\epsilon|^2
\end{pmatrix}.
\]

We will need the following simple relation between \( L_\epsilon \) and its complexification

\[
\langle v, L_\epsilon v \rangle = \langle v^C, K_\epsilon v^C \rangle,
\]

where \( v^C \) is the the vector corresponding to \( v \) in the embedding of \( \mathcal{H} \) given above.

### 3.8 Proof of Theorem 23

We now focus on the case of periodic perturbations and prove Theorem 23. From the above it is clearly sufficient to show that for all \( v \in \mathcal{K}^{\text{per}} \),

\[
\langle v, K_\epsilon^{\text{per}} v \rangle > 0.
\]
We first prove the following lemma about the unperturbed operator $K_{0}^{per}$, which is explicitly given by

\[
K_{0}^{per} = \begin{pmatrix}
-\Delta_{0} - 1 & 0 & 0 & 0 \\
0 & -\Delta_{0} - 1 & 0 & 0 \\
0 & 0 & -\Delta & 0 \\
0 & 0 & 0 & -\Delta
\end{pmatrix}.
\]

**Lemma 25.** $K_{0}^{per}$ is a positive self-adjoint operator with discrete spectrum. It has a zero eigenvalue of multiplicity 2 and the kernel is spanned by the elements $v_{1} = (\psi_{0}, 0, 0, 0)$ and $v_{2} = (0, \bar{\psi}_{0}, 0, 0)$.

**Proof.** This operator is simply the operators $L^{n}$ with $n = 1$ and $M$, which we studied previously in section 2.9, with the difference that $M$ no longer acts on divergence-free mean-zero fields. \qed

We now expand $K_{\epsilon}^{per}$ in powers of $\epsilon$. We have $K_{\epsilon}^{per} = K_{0}^{per} + \epsilon W_{0}^{per} + \epsilon^{2} W_{1}^{per} + o(\epsilon^{3})$. By standard perturbation theory (see e.g. [26, 36, 23]), the spectrum of $K_{\epsilon}^{per}$ consists of eigenvalues of the same total multiplicities in an $\epsilon$-neighbourhood of the eigenvalues of $K_{0}^{per}$. Thus it suffices to determine the two lowest eigenvalues of $K_{\epsilon}^{per}$.

To do this we use the Feshbach-Schur map argument (see e.g. [22, 9]) with the projection $P$ given by orthogonal projection onto null $K_{0}^{per}$. This argument implies that $\lambda \in \sigma(K_{\epsilon}^{per})$ if and only if $\lambda \in \sigma(F_{P}(\lambda))$, where

\[
F_{P}(\lambda) := PK_{\epsilon}^{per} P - PK_{\epsilon}^{per} \bar{P}(PK_{\epsilon}^{per} \bar{P} - \lambda)^{-1} \bar{P}K_{\epsilon}^{per} P,
\]

and $\bar{P} = 1 - P$. It is straightforward to show that

\[
\|W_{\epsilon}^{per}\| \lesssim 1.
\]

We know that $\sigma(\bar{P}K_{0}^{per} \bar{P}) \subset [\nu_{0}, \infty)$ for some $\nu_{0} > 0$ and therefore, by standard perturbation theory we have that

\[
\sigma(PK_{\epsilon}^{per} P) \subset [c, \infty),
\]
with \( c = \nu_0 + O(\epsilon) \). We now write \( W_\epsilon^{\text{per}} = W_0^{\text{per}} + \epsilon W_1^{\text{per}} + o(\epsilon^2) \). We note that we have the relation \( K_0^{\text{per}} P = PK_0^{\text{per}} = 0 \) and the fact that by (3.16), \( \| PK_\epsilon^{\text{per}} \bar{P} \| = O(\epsilon) \) and by (3.17), \( \| (\bar{PK}_\epsilon^{\text{per}} \bar{P} - \lambda)^{-1} \| \lesssim 1 \), provided \( \lambda < c \), so assuming that \( \lambda = O(\epsilon) \) we have

\[
\mathcal{F}(\lambda) = \epsilon P W_0^{\text{per}} P + \epsilon^2 (P W_1^{\text{per}} P - P W_0^{\text{per}} P (PK_0^{\text{per}} P)^{-1} P W_0^{\text{per}} P) + O(\epsilon^3).
\]

We are thus led to considering the lower order operators in \( \mathcal{F}_P(\lambda) \).

For the order \( \epsilon \) operator, \( W_0^{\text{per}} \) is explicitly given by

\[
W_0^{\text{per}} = \begin{pmatrix}
0 & 0 & i(\partial_{a_0} \psi_0) \\
0 & 0 & -i(\overline{\partial_{a_0} \psi_0}) \\
i(\partial_{a_0} \psi_0) & 0 & 0 \\
-i(\overline{\partial_{a_0} \psi_0}) & 0 & 0
\end{pmatrix}.
\] (3.18)

Here we have used the fact that \( \partial^* \psi_0 = 0 \).

**Lemma 26.** \( PW_0^{\text{per}} P = 0 \).

**Proof.** A simple calculation shows that

\[
W_0^{\text{per}} v_1 = \begin{pmatrix}
0 \\
0 \\
0 \\
-i\psi_0(\overline{\partial_{a_0} \psi_0})
\end{pmatrix},
\]

\[
W_0^{\text{per}} v_2 = \begin{pmatrix}
0 \\
0 \\
i\overline{\psi_0}(\partial_{a_0} \psi_0) \\
0
\end{pmatrix}.
\]

It is clear then that \( \langle v_j, W_0^{\text{per}} v_k \rangle = 0 \) for \( j, k = 1, 2 \), and the lemma follows. \( \Box \)

We now turn to the \( \epsilon^2 \) order operator, which we will represent as a \( 2 \times 2 \) matrix using the basis \( \beta = \{ v_1, v_2 \} \). We first note that \( W_1^{\text{per}} \) is explicitly given by

\[
W_1^{\text{per}} = \begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix},
\] (3.19)

where

\[
A_1 = \begin{pmatrix}
-\lambda_1 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 - i\overline{a_1} \partial^*_{a_0} + i\overline{a}_1 \partial_{a_0} \\
(\kappa^2 - \frac{1}{2})\overline{\psi_0}^2 \\
(\kappa^2 - \frac{1}{2})\psi_0^2 \\
-\lambda_1 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 + i\overline{a}_1 \overline{\partial^*_{a_0}} - i\overline{a_1} \overline{\partial_{a_0}}
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
and

\[ A_2 = \begin{pmatrix} |\psi_0|^2 & 0 \\ 0 & |\psi_0|^2 \end{pmatrix}. \]

**Lemma 27.** In terms of the basis \( \beta \), \( PW_{1}^{\text{per}} P - PW_{0}^{\text{per}} \overline{P(K_{0}^{\text{per}} P)}^{-1}\overline{PW}_{0}^{\text{per}} P \) is represented by the matrix

\[
\langle |\psi_0|^2 \rangle \beta(\tau) \begin{pmatrix} \kappa^2 - \frac{1}{2} + \frac{1}{\beta(\tau)} & \kappa^2 - \frac{1}{2} \\ \kappa^2 - \frac{1}{2} & \kappa^2 - \frac{1}{2} + \frac{1}{\beta(\tau)} \end{pmatrix}.
\]

**Proof.** We begin with \( PW_{1}^{\text{per}} P \), which is represented by the matrix \( M_{1} \) given by

\[
M_{1}^{jk} = \frac{\langle v_{j}, W_{1}^{\text{per}} v_{k} \rangle}{\|v_{j}\|^2}.
\]

Using the fact that \( \partial \overline{a_{0}} \psi_0 = 0 \), we calculate that

\[
W_{1}^{\text{per}} v_{1} = \begin{pmatrix} -\lambda_{1} \psi_0 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 \psi_0 + i\overline{a_{1}} \overline{\partial a_{0}} \psi_0 \\ (\kappa^2 - \frac{1}{2})|\psi_0|^2 \overline{\psi_0} \\ 0 \\ 0 \end{pmatrix},
\]

\[
W_{1}^{\text{per}} v_{2} = \begin{pmatrix} -\lambda_{1} \overline{\psi}_0 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 \overline{\psi}_0 - i a_{1} \overline{\partial a_{0}} \psi_0 \\ (\kappa^2 - \frac{1}{2})|\psi_0|^2 \overline{\psi}_0 \\ 0 \\ 0 \end{pmatrix}.
\]

We now note that \( i\overline{a_{1}} \partial a_{0} \psi_0 = 2i a_{1} \cdot \nabla a_{0} \psi_0 \), and therefore

\[
\langle i\overline{a_{1}} \overline{\psi}_0 \partial a_{0} \psi_0 \rangle = \langle 2i a_{1} \cdot \overline{\psi}_0 \nabla a_{0} \psi_0 \rangle = -2 \langle a_{1} \cdot \text{Im}(\overline{\psi}_0 \nabla a_{0} \psi_0) \rangle = -\frac{1}{2} \langle |\psi_0|^4 \rangle + \frac{1}{2} \langle |\psi_0|^2 \rangle^2.
\]
We now calculate that
\[
\left\langle \psi_0, -\lambda_1 \psi_0 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 \psi_0 + i\bar{a}_1 \partial_{\alpha_0} \psi_0 \right\rangle = -\lambda_1 \left\langle |\psi_0|^2 \right\rangle + (2\kappa^2 + \frac{1}{2}) \left\langle |\psi_0|^4 \right\rangle - \frac{1}{2} \left\langle |\psi_0|^2 \right\rangle^2
\]
\[
= -\kappa^2 \left\langle |\psi_0|^4 \right\rangle - \frac{1}{2} \left\langle |\psi_0|^2 \right\rangle^2 + (2\kappa^2 + \frac{1}{2}) \left\langle |\psi_0|^4 \right\rangle - \frac{1}{2} \left\langle |\psi_0|^2 \right\rangle^2
\]
\[
= (\kappa^2 + \frac{1}{2}) \left\langle |\psi_0|^2 \right\rangle.
\]
By similar calculations we can see that $M^1$ is
\[
\left\langle |\psi_0|^2 \right\rangle \beta(\tau) \begin{pmatrix} \kappa^2 + \frac{1}{2} & \kappa^2 - \frac{1}{2} \\ \kappa^2 - \frac{1}{2} & \kappa^2 + \frac{1}{2} \end{pmatrix}.
\]

We now turn to $PW^*_0 P (PK^*_0 P)^{-1} PW^*_0 P$ and we let $M^2$ be its matrix representation. We note that $PW^*_0 P = W^*_0 P$, and therefore using the above we calculate that
\[
(\bar{PK}^*_0 \bar{P})^{-1} \bar{PW}^*_0 v_1 = \begin{pmatrix} 0 \\ 0 \\ i\Delta^{-1} \bar{\psi}_0(\bar{\alpha}_{\alpha_0} \psi_0) \end{pmatrix}, \quad (\bar{PK}^*_0 \bar{P})^{-1} \bar{PW}^*_0 v_2 = \begin{pmatrix} 0 \\ 0 \\ -i\Delta^{-1} \bar{\psi}_0(\bar{\alpha}_{\alpha_0} \psi_0) \end{pmatrix}.
\]
We then have
\[
W^*_0 \bar{P} (\bar{PK}^*_0 \bar{P})^{-1} \bar{PW}^*_0 v_1 = \begin{pmatrix} -\bar{\psi}_0(\bar{\alpha}_{\alpha_0} \psi_0) \Delta^{-1} \bar{\psi}_0(\bar{\alpha}_{\alpha_0} \psi_0) \\ 0 \\ 0 \end{pmatrix},
\]
\[
W^*_0 \bar{P} (\bar{PK}^*_0 \bar{P})^{-1} \bar{PW}^*_0 v_2 = \begin{pmatrix} -\bar{\psi}_0(\bar{\alpha}_{\alpha_0} \psi_0) \Delta^{-1} \bar{\psi}_0(\bar{\alpha}_{\alpha_0} \psi_0) \\ 0 \\ 0 \end{pmatrix}.
\]
Now we know that

$$a_1 = ((\text{curl}^* \text{curl})^{-1} \text{Im}(\bar{\psi}_0 \nabla_0 \psi_0))^C = \frac{i}{2} \Delta^{-1} \bar{\psi}_0 (\partial_{aa} \psi_0).$$

Therefore

$$\langle \psi_0, - (\partial_{aa} \psi_0) \Delta^{-1} \bar{\psi}_0 (\partial_{aa} \psi_0) \rangle = \langle -2i \bar{\psi}_0 (\partial_{aa} \psi_0) \bar{a}_1 \rangle = \langle |\psi_0|^4 \rangle - \langle |\psi_0|^2 \rangle^2.$$

After a similar calculation, it follows that $M^2$ is

$$\langle |\psi_0|^2 \rangle (\beta(\tau) - 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The lemma now follows as $PW_{1}^{\text{per}} P - PW_{0}^{\text{per}} \bar{P} (\bar{P} K_{0}^{\text{per}} P)^{-1} \bar{P} W_{0}^{\text{per}} P$ is represented by the matrix $M^1 - M^2$. 

It can be easily checked that eigenvalues of the $\epsilon^2$ order operator are

$$\mu_1 = \langle |\psi_0|^2 \rangle (\beta(\tau)(2\kappa^2 - 1) + 1), \quad \mu_2 = \langle |\psi_0|^2 \rangle.$$

Since $\kappa^2 > \frac{1}{2}$ it is clear that both eigenvalues are strictly positive. The Feschbach-Schur operator now implies that the two lowest eigenvalues of $K_{\epsilon}^{\text{per}}$ are positive. We have seen above that the other eigenvalue are greater or equal to some $c > 0$, and it therefore follows that $K_{\epsilon}^{\text{per}}$ is a positive operator, and the proof of Theorem 23 is complete.

### 3.9 Bloch Decomposition

We now turn to the operator $K_{\epsilon}$. The analysis here will be very similar to the Bloch theory (or Floquet theory) of Schrödinger operators with periodic potentials (see [36, 17], as well as [39, 30] for the application of this theory to other equations). The basic idea of the analysis is to decompose the space $K$ as the direct integral of spaces on a compact domain in such a way that the operator $K_{\epsilon}$ likewise is decomposed as the direct integral of operators on these spaces.
The basic ingredient is to construct a unitary representation on $K$ that commutes with $K_\epsilon$. To do this we begin by defining the translation operator $S_t$, $t \in \mathbb{R}^2$, by $S_t f(x) = f(x + t)$. We also define the magnetic translation operator $T_t$ to be $T_t = e^{-\frac{i}{2} t \wedge \alpha} S_t$. It is straightforward to show that $S_t$ and $T_t$ are unitary operators and that they satisfy the relations

$$S_{t+s} = S_t S_s, \quad T_{t+s} = e^{-\frac{i}{2} t \wedge s} T_t T_s.$$ 

We also note that $T_t$ commutes with both $\nabla_{\mathbf{a}_0}$ and $\Delta_{\mathbf{a}_0}$.

We now wish to define a representation for the discrete group $\mathbb{Z}^2$. We recall the lattice vectors $t^{(1)}$ and $t^{(2)}$ given by (3.9) above. Now given $\alpha \in \mathbb{Z}^2$, we first define $t^\alpha := 2\alpha_1 t^{(1)} + \alpha_2 t^{(2)}$, and then define the operator $\rho_\alpha$ on $K$ to be

$$\rho_\alpha = T_{t^\alpha} \oplus T_{-t^\alpha} \oplus S_{t^\alpha} \oplus S_{-t^\alpha}.$$ 

We have the following proposition.

**Proposition 28.** $\rho_\alpha$ defines a unitary representation on $K$ that satisfies $\rho_\alpha K_\epsilon = K_\epsilon \rho_\alpha$.

**Proof.** We note that for $\alpha, \beta \in \mathbb{Z}^2$,

$$t^\alpha \wedge t^\beta = (\alpha \wedge \beta) 2t^{(1)} \wedge t^{(2)} = 2(\alpha \wedge \beta) r^2 \sin \theta = 4\pi (\alpha \wedge \beta),$$

and it follows that

$$T_{t^\alpha + t^\beta} = T_{t^\alpha} T_{t^\beta}.$$ 

Thus $T_{t^\alpha}$ is a group representation. It now follows that $\rho_\alpha$ as defined is a group representation of $\mathbb{Z}^2$ on $K$. A straightforward calculation shows that it commutes with $K_\epsilon$ for all $\epsilon$. \qed

We now decompose the space $K$. First note that the dual group of $\mathbb{Z}^2$ is $\mathbb{T}^2$, the torus $[0, 2\pi)^2$ under addition mod$2\pi$. We recall that the dual group to $\mathbb{Z}^2$ is the group consisting of all characters of $\mathbb{Z}^2$, i.e., all homomorphisms from $\mathbb{Z}^2 \to U(1)$. Explicitly, for $k \in \mathbb{T}^2$, we have the character $\chi_k$ given by

$$\chi_k(\alpha) = e^{ik \cdot \alpha}.$$
In order to decompose the space, we first define $2\Omega$ to be the lattice cell spanned by $2t^{(1)}$ and $t^{(2)}$ (which is not $2\Omega$ in the normal sense). Now for each $k \in \mathbb{T}^2$, we define the Hilbert space $\mathcal{H}_k := L^2(2\Omega)$, and we then define the Hilbert space $\mathcal{H}$ to be the direct integral

$$\mathcal{H} = \int_{\mathbb{T}^2} \mathcal{H}_k \frac{dk}{4\pi^2}.$$  

Here $dk$ is the standard Lebesgue measure on $\mathbb{T}^2$. We now have the following Bloch decomposition result.

**Proposition 29.**

(a) Define $U : \mathcal{K} \to \mathcal{H}$ on smooth functions with compact domain by the formula

$$(Uv)_k(x) = \sum_{\alpha \in \mathbb{Z}^2} \chi_k^{-1}(\alpha) \rho_\alpha v(x).$$

Then $U$ extends uniquely to a unitary operator.

(b) For $k \in \mathbb{T}^2$, let $K_{ek}$ be the operator $K_e$ acting on $\mathcal{H}_k$ with domain consisting of those $v \in \mathcal{H}_k \cap H^2$ such that $v$ satisfies the boundary conditions $\rho_\alpha v(x) = \chi_k(\alpha)v(x)$ for the basis elements $\alpha = (1,0), (0,1)$. Then

$$UK_eU^{-1} = \int_{\mathbb{T}^2} K_{ek} \frac{dk}{4\pi^2}.$$  \hspace{1cm} (3.20)

and

$$\sigma(K_e) = \bigcup_{k \in \mathbb{T}^2} \sigma(K_{ek}).$$  \hspace{1cm} (3.21)

**Proof.** For (a), we begin by showing that $U$ is an isometry on smooth functions with
compact domain. Using Fubini’s theorem we calculate

\[ \| Uv \|_{\mathcal{H}}^2 = \int_{\mathbb{T}^2} \| (Uv)_k \|_{\mathcal{H}_k}^2 \frac{dk}{4\pi^2} \]

\[ = \int_{\mathbb{T}^2} \int_{2\Omega} \left| \sum_{\alpha \in \mathbb{Z}^2} \chi_k^{-1}(\alpha) \rho_\alpha v(x) dx \right|^2 \frac{dk}{4\pi^2} \]

\[ = \int_{2\Omega} \left( \sum_{\alpha, \beta \in \mathbb{Z}^2} \rho_\alpha v(x) \overline{\rho_\beta v(x)} \right) \int_{\mathbb{T}^2} \chi_k^{-1}(\alpha) \chi_k(\beta) \frac{dk}{4\pi^2} \right) dx \]

\[ = \int_{2\Omega} \sum_{\alpha \in \mathbb{Z}^2} |\rho_\alpha v(x)|^2 dx \]

\[ = \int_{\mathbb{R}^2} |v(x)|^2 dx \]

\[ = \| v \|_K^2. \]

Therefore \( U \) extends to an isometry on all of \( \mathcal{K} \). To show that \( U \) is in fact a unitary operator we define \( U^* : \mathcal{H} \to \mathcal{K} \) by the formula

\[ U^* g(x + 2\alpha_1 t^{(1)} + \alpha_2 t^{(2)}) = \int_{\mathbb{T}^2} \chi_k(\alpha) g_k(x) \frac{dk}{4\pi^2}, \]

where \( \alpha = (\alpha_1, \alpha_2) \) and \( x \in 2\Omega \). Straightforward calculations show that \( U^* \) is the adjoint of \( U \) and that it too is an isometry, proving that \( U \) is a unitary operator.

For (b), we need to first show that \( (Uv)_k \) is in the domain of \( K_{ek} \). For \( (Uv)_k \) we have

\[ \rho_\alpha (Uv)_k(x) = \sum_{\beta \in \mathbb{Z}^2} \chi_k^{-1}(\beta) \rho_\alpha \rho_\beta v(x) \]

\[ = \sum_{\beta \in \mathbb{Z}^2} \chi_k^{-1}(\beta) \rho_{\alpha + \beta} v(x) \]

\[ = \chi_k(\alpha) \sum_{\beta \in \mathbb{Z}^2} \chi_k^{-1}(\alpha + \beta) \rho_{\alpha + \beta} v(x) \]

\[ = \chi_k(\alpha) (Uv)_k(x). \]
Therefore we have that
\[
(K_{\epsilon k}(U v)_k)(x) = \sum_{\alpha \in \mathbb{Z}^2} \chi_k^{-1}(\alpha) K_{\epsilon \lambda} \rho_\alpha v(x)
\]
\[
= \sum_{\alpha \in \mathbb{Z}^2} \chi_k^{-1}(\alpha) \rho_\alpha K_v(x)
\]
\[
= (U K_v)_k(x),
\]
which establishes (3.20). The relation (3.21) follows from the general theory (see [36] for details).

\[
\square
\]

3.10 Proof of Theorem 24

We now use the analysis of the previous section to study the spectrum of \(K_{\epsilon k}\) and prove Theorem 24.

We now analyze the operators \(K_{\epsilon k}\). Since each \(K_{\epsilon k}\) has a different domain, it will be convenient to consider unitarily equivalent operators that share a common domain. We therefore define the unitary operator \(V_k : \mathcal{H}_0 \to \mathcal{H}_k\) to be
\[
V_k v(x) = e^{ik \cdot x} v(x).
\]

It is easy to check that \(v \in \mathcal{D}(K_{\epsilon 0})\) if and only if then \(V_k v \in \mathcal{D}(K_{\epsilon k})\). We now set \(N_{\epsilon k} = V_k^{-1} K_{\epsilon k} V_k\). This operator is given by
\[
\begin{pmatrix}
-\Delta_{\epsilon -k} - \lambda_{\epsilon} + (2\kappa^2 + \frac{1}{2})|\psi_{\epsilon}|^2 & (\kappa^2 - \frac{1}{2})\psi_{\epsilon} & -i(\partial_{a,\epsilon}^\ast \psi_{\epsilon}) & i(\partial_{a,\epsilon} \psi_{\epsilon}) \\
(\kappa^2 - \frac{1}{2})\overline{\psi}_{\epsilon} & -\Delta_{\epsilon -k} - \lambda_{\epsilon} + (2\kappa^2 + \frac{1}{2})|\psi_{\epsilon}|^2 & -i(\partial_{a,\epsilon} \psi_{\epsilon}) & i(\partial_{a,\epsilon}^\ast \psi_{\epsilon}) \\
i(\overline{\partial}_{a,\epsilon} \psi_{\epsilon}) & i(\partial_{a,\epsilon}^\ast \psi_{\epsilon}) & -\Delta_{-k} + |\psi_{\epsilon}|^2 & 0 \\
-i(\partial_{a,\epsilon}^\ast \psi_{\epsilon}) & -i(\partial_{a,\epsilon} \psi_{\epsilon}) & 0 & -\Delta_{-k} + |\psi_{\epsilon}|^2
\end{pmatrix}.
\]

In particular we note that \(N_{\epsilon 0} = K_{\epsilon 0}\) and therefore \(\mathcal{D}(N_{\epsilon k}) = \mathcal{D}(K_{\epsilon 0})\).

We again begin by considering the unperturbed operator \(N_{0k}\) (i.e., \(\epsilon = 0\), which is
explicitly given by
\[
N_{0k} = \begin{pmatrix}
-\Delta_{a_0-k} - 1 & 0 & 0 & 0 \\
0 & -\Delta_{a_0-k} - 1 & 0 & 0 \\
0 & 0 & -\Delta_{-k} & 0 \\
0 & 0 & 0 & -\Delta_{-k}
\end{pmatrix}
\]

We have the following lemma.

**Lemma 30.** The spectrum of \( N_{0k} \) is given by
\[
\sigma(\hat{N}_k) = \{0, 2, 4, \ldots\} \cup \left\{ \frac{r^2}{4} (\alpha_1^2|\tau|^2 - 4\alpha_1\alpha_2\tau_1 + 4\alpha_2^2) + r (k_1\alpha_1\tau_2 + k_2(-\alpha_1\tau_1 + 2\alpha_2)) + |k|^2 : \alpha \in \mathbb{Z}^2 \right\}.
\]
The first set consists of eigenvalues of multiplicity 4. The zero eigenfunctions of the first set are of the form \( (\phi_1, \bar{\phi}_2, 0, 0) \), where \( \phi_i \) is of the form
\[
\phi_i(x) = e^{-(k_1+ik_2)x_2}e^{-\frac{1}{2}x_2^2+\frac{1}{2}x_1x_2} \sum_{m=-\infty}^{\infty} c_m e^{\frac{im\pi}{r}(x_1+ix_2)}
\]
for some \( c_m \in \mathbb{C} \) satisfying the relations
\[
c_{m+2} = e^{-r\tau_2(k_1+ik_2)}e^{i\pi\tau}e^{im\pi\tau}c_m.
\]

**Proof.** The proof of this lemma is very similar to the proofs in section 2.9 and we proceed similarly. We begin by considering the operator \(-\Delta_{-k}\) on \( L^2(\Omega; \mathbb{C}) \) with periodic boundary conditions. Using standard methods one can show that this is a positive self-adjoint operator with discrete spectrum. To obtain more detailed information on the spectrum we consider the orthonormal basis given by
\[
e_\alpha(x) = \frac{1}{\sqrt{4\pi}} e^{\frac{|x|^2}{4\pi}} e^{-i\alpha(\tau \wedge x) + 2\alpha_2 x_2}
\]
for \( \alpha \in \mathbb{Z}^2 \). Using this basis one can show that \(-\Delta_{-k}\) is unitarily equivalent to the multiplication operator \( M \) on \( \ell^2(\mathbb{Z}^2; \mathbb{C}) \) given by
\[
(M\xi)_\alpha = \frac{r^2}{4} (\alpha_1^2|\tau|^2 - 4\alpha_1\alpha_2\tau_1 + 4\alpha_2^2) + r (k_1\alpha_1\tau_2 + k_2(-\alpha_1\tau_1 + 2\alpha_2)) + |k|^2.
\]
The spectrum of this operator is then just the range of the function.

We now turn to the operator $-\Delta_{a_0 - k}$ on $L^2(\Omega; \mathbb{C})$ with quasiperiodic boundary conditions. This is a positive self-adjoint operator with discrete spectrum. We use the harmonic oscillator annihilation and creation operators, $c_k$ and $c_k^*$, defined by

$$c_k = \partial_{x_1} + i\partial_{x_2} + \frac{1}{2}(x_1 + ix_2) + i(k_1 + ik_2).$$

These operators satisfy the relations

1. $[c_k, c_k^*] = 2$;
2. $-\Delta_{a_0 - k} - 1 = c_k^*c_k$.

This gives that $\sigma(-\Delta_{a_0 - k}) = \{1, 3, 5, \ldots\}$ and that each eigenvalue is of the same multiplicity. Furthermore, the properties imply that null$(-\Delta_{a_0 - k} - 1) = \text{null } c_k$ and so we study the latter.

Given $\phi : 2\Omega \to \mathbb{C}$ we define $\Theta_k : 2\Omega \to \mathbb{C}$ to be

$$\Theta_k(x) = e^{ik \cdot x}e^{\frac{1}{2}x_1^2 - \frac{i}{2}x_1x_2}\phi(x).$$

One can show that $\phi \in \text{null } c_k$ if and only if $\Theta_k$ is a solution of the Riemann equation $(\partial_{x_1} + i\partial_{x_2})\Theta_k = 0$. We now identify $x \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. Since $\phi$ satisfies “periodic boundary conditions”, $\Theta_k$ satisfies the following boundary conditions.

$$\Theta_k(z + 2r) = e^{2ik r}\Theta_k(z),$$

$$\Theta_k(z + r \tau) = e^{ir(k_1 \tau_1 + k_2 \tau_2)}e^{-i\pi \tau}e^{-\frac{2\pi i}{r}z}\Theta_k(z).$$

These conditions allow us to extend $\Theta_k$ to an entire function on all of $\mathbb{C}$. Moreover, the first relation ensures that $\Theta_k$ has an absolutely convergent Fourier expansion of the form

$$\Theta_k(z) = e^{ik_{1z}} \sum_{m=-\infty}^{\infty} c_me^{\frac{\pi m \tau}{r}z}.$$
The second relation, on the other hand, leads to relation for the coefficients of the expansion. Namely, we have

\[ c_{m+2} = e^{-r\tau_2(k_1+ik_2)}e^{i\pi\tau}e^{im\pi\tau}c_m. \] (3.25)

This means that \( \Theta_k \) is determined by the value of \( c_0, c_1 \), and therefore the set of all \( \Theta_k \) forms a 2-dimensional vector space. This implies that every eigenvalue of \( -\Delta_{\alpha_0-k} \) is of multiplicity 2.

The proposition now follows from the above considerations and the form of \( N_{0k} \).

In order to analyze the operators \( N_{ek} \) we will need to make explicit use of certain symmetries that correspond to the fact that we have complexified the operator and that the representation of \( \mathbb{Z}^2 \) is in terms of the double cell. We recall that \( C \) denotes complex conjugation. We define the operator \( R \) to be

\[ R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]

Now let \( \gamma = CR. \)

We also define the operator \( \tau \) to be \( T_{t(1)} \oplus T_{t(1)}^\dagger \oplus S_{t(1)} \oplus S_{t(1)} \) (similar to \( \rho_\alpha \) above). We note that in order to translate elements of \( D(K_0) \) we first extend them to the entire plane using the boundary conditions and after translating restrict the state once again to \( 2\Omega \). That \( \tau \) is well-defined follows from the fact that the operators \( T_t \) commute with each other.

We have the following proposition.

**Proposition 31.** For all \( \epsilon \geq 0 \),

(a) \( [N_{ek}, \gamma] = 0. \)

(b) \( [N_{ek}, \tau] = 0. \)
(c) $[\gamma, \tau] = 0, \gamma^2 = 1, \tau^2 = 1$.

**Proof.** The relations involving $\gamma$ are straightforward to prove, and (b) follows from the proof of the Bloch decomposition above. It remains to show that $\tau^2 = 1$. This follows from the fact that $T_2^{(1)} = T_{2(1)}$ and similarly $S_2^{(1)} = S_{2(1)}$, so $\tau^2 = \rho_\alpha$ with $\alpha = (1,0)$, but $\rho_\alpha$ acts as the identity on $\mathcal{D}(K_\epsilon)$. \qed

Now the operator $\tau$ is unitary and therefore has eigenvalues $\pm 1$. The operator $\gamma$ is only real-linear, but nevertheless it too has eigenvalues $\pm 1$ and the corresponding subspaces span the entire space $\mathcal{H}_k$. To see this we note that

$$
\begin{pmatrix}
\psi \\
\phi \\
\alpha \\
\omega
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
\psi + \bar{\phi} \\
\bar{\psi} + \phi \\
\alpha + \bar{\omega} \\
\bar{\alpha} + \omega
\end{pmatrix}
+ \frac{1}{2}
\begin{pmatrix}
\psi - \bar{\phi} \\
-\bar{\psi} + \phi \\
\alpha - \bar{\omega} \\
-\bar{\alpha} + \omega
\end{pmatrix}.
$$

Since $\gamma$ and $\tau$ commute, $\mathcal{H}_k$ is a direct sum of the four common eigenspaces of these operators, which we denote as $V_\sigma$, $\sigma = (\sigma_1, \sigma_2)$, $\sigma_i = \pm 1$.

The properties (a) - (c) of the above lemma imply that the common eigenspaces $V_\sigma$ of the operators $\gamma$ and $\tau$ are invariant under the operator $N_{\epsilon k}$. Hence we can restrict $N_{\epsilon k}$ to these invariant subspaces. We now prove the following lemma.

**Lemma 32.** There exists a basis $v_{k \sigma}$ of null $N_{0k}$ such that $v_\sigma \in V_\sigma$, and therefore the restriction of $N_{0k}$ to each $V_\sigma$ has a one-dimensional kernel.

**Proof.** We first note that given any $\phi \in \text{null}(-\Delta_{a_0-k} - 1)$, we have

$$
\gamma
\begin{pmatrix}
\phi \\
\pm \bar{\phi} \\
0 \\
0
\end{pmatrix}
= \pm
\begin{pmatrix}
\phi \\
\pm \bar{\phi} \\
0 \\
0
\end{pmatrix}.
$$
It therefore suffices to find \( \phi_{\pm} \) such that \( T_{t(1)} \phi_{\pm} = \pm \phi_{\pm} \). To do this we recall from the proof of Lemma 30 that any \( \phi \in \text{null}(\Delta_{a_0-k} - 1) \) is of the form

\[
\phi(x) = e^{-(k_1 + ik_2) x_2} e^{\frac{1}{2} x_2^2 + i x_1 x_2} \sum_{m=\pm \infty} c_m e^{i m \pi r} (x_1 + i x_2).
\]

Using the fact that \( t^{(1)} \wedge x = r x_2 \), it then follows that

\[
T_{t(1)} \phi(x) = e^{-\frac{1}{2} r x_2} e^{-(k_1 + ik_2) x_2} e^{\frac{1}{2} x_2^2 + i x_1 x_2} e^{\frac{1}{2} r x_2} \sum_{m=\pm \infty} e^{i m \pi} c_m e^{i m \pi r} (x_1 + i x_2).
\]

We recall that \( \phi \) is defined by \( c_0 \) and \( c_1 \). We now define \( \phi_+ \) by the conditions that \( c_0 = 1 \) and \( c_1 = 0 \). Then \( c_m = 0 \) for all odd \( m \) and \( e^{i m \pi} = 1 \) when \( c_m \neq 0 \), so it follows that

\[
T_{t(1)} \phi_+(x) = \phi_+(x).
\]

Similarly we define \( \phi_- \) by the conditions that \( c_0 = 0 \) and \( c_1 = 1 \). Then \( c_m = 0 \) for all even \( m \) and \( e^{i m \pi} = -1 \) when \( c_m \neq 0 \), so it follows that

\[
T_{t(1)} \phi_-(x) = -\phi_-(x).
\]

Then \( v_{(1,1)} = (\phi_+, \bar{\phi}_+, 0, 0) \), \( v_{(1,-1)} = (\phi_+, -\bar{\phi}_+, 0, 0) \), \( v_{(-1,1)} = (\phi_-, \bar{\phi}_-, 0, 0) \), and \( v_{(-1,-1)} = (\phi_-, -\bar{\phi}_-, 0, 0) \) form the desired basis.

To study the restriction, \( N_{ek\sigma} \), of \( N_{ek} \) to each \( V_{\sigma} \), we proceed as in the proof of Theorem 23. Using standard perturbation theory together with the Feshbach-Schur map argument, we see that \( \lambda \) is an eigenvalue of \( N_{ek\sigma} \) if and only if \( \lambda \) is an eigenvalue of the Feshbach-Schur map

\[
\mathcal{F}_\sigma(\lambda) := P_\sigma N_{ek} P_\sigma - P_\sigma N_{ek} \tilde{P}_\sigma (\tilde{P}_\sigma N_{ek} \tilde{P}_\sigma - \lambda)^{-1} \tilde{P}_\sigma N_{ek} P_\sigma,
\]

where \( P_\sigma \) is projection onto the vector \( v_\sigma \). We only need to analyze the behaviour of the four lowest eigenvalues of \( \mathcal{F}_\sigma(\lambda) \) as the other eigenvalues are all positive and bounded.
away from 0 uniformly in \( \epsilon \) (for small \( \epsilon \)). Thus we consider \( \lambda \) of the order \( \epsilon \), and writing \( N_{\epsilon k} \) in the form \( N_{\epsilon k} = N_{0k} + \epsilon W_{0k} + \epsilon^2 W_{1k} + o(\epsilon^3) \), we have the expansion

\[
\mathcal{F}_\sigma(\lambda) = \epsilon P_\sigma W_{0k}P_\sigma + \epsilon^2 (P_\sigma W_{1k}P_\sigma - P_\sigma W_{0k} \overline{P_\sigma (P_\sigma N_{0k} \overline{P_\sigma})^{-1} \overline{P_\sigma} W_{0k} P_\sigma}) + O(\epsilon^3).
\]

We recall that \( P_\sigma \) is a rank 1 projection and therefore we are interested in the sign of

\[
\langle v_\sigma, \mathcal{F}_\sigma(\lambda)v_\sigma \rangle.
\]

Explicitly \( W_{0k} \) is given by the same expression as \( W_0^{\text{per}} \) above (3.18) and it is straightforward to show that \( P_\sigma W_{0k} P_\sigma = 0 \).

We now let \( H_\sigma = P_\sigma (W_{1k} - W_{0k} \overline{P_\sigma (P_\sigma N_{0k} \overline{P_\sigma})^{-1} \overline{P_\sigma} W_{0k} P_\sigma}) \) and consider \( \langle v_\sigma, H_\sigma v_\sigma \rangle \).

First note that \( W_{1k} \) is given explicitly by

\[
W_{1k} = \left(\begin{array}{ccc}
-\lambda_1 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 + B_k & (\kappa^2 - \frac{1}{2})\psi_0^2 & 0 \\
(\kappa^2 - \frac{1}{2})\overline{\psi}_0^2 & -\lambda_1 + (2\kappa^2 + \frac{1}{2})|\psi_0|^2 + B_k & 0 \\
0 & 0 & |\psi_0|^2
\end{array}\right),
\]

where

\[
B_k = -ia_1 \partial_{a_0}^* + i\overline{a}_1 \partial_{a_0} + i(k_1 - ik_2)\partial_{a_1}^* - i(k_1 + ik_2)\partial_{a_1}.
\]

Recalling the definitions of \( c_k \) and \( c_k^* \) (3.22), we note that

\[
B_k = ia_1(\partial_{x_1} + i\partial_{x_2} + \frac{1}{2}x_1 + \frac{i}{2}x_2) + i\overline{a}_1(\partial_{x_1} - i\partial_{x_2} - \frac{1}{2}x_1 + \frac{i}{2}x_2)
\]

\[
- i(k_1 - ik_2)(\partial_{x_1} + i\partial_{x_2}) + i(k_1 - ik_2)i\overline{a}_1 - i(k_1 + ik_2)(\partial_{x_1} - i\partial_{x_2}) + i(k_1 + ik_2)i\overline{a}_1
\]

\[
= ia_1 c_k - i\overline{a}_1 c_k^* - 2i(k_1 \partial_{x_1} + k_2 \partial_{x_2}).
\]

We now calculate, using the fact that \( c_k \phi_+ = 0 \), that

\[
\langle v_1, W_{1k} v_1 \rangle = \langle -2\lambda_1 |\phi_+|^2 + (4\kappa^2 + 1)|\psi_0|^2 |\phi_+|^2 + (2\kappa^2 - 1) \text{Re}(\overline{\psi}_0^2 \phi_+^2)
\]

\[
- 2 \text{Re}(i\overline{a}_1 \phi_+ c_k^* \phi_+ + 2i\overline{\phi}_+(k_1 \partial_{x_1} \phi_+ + k_2 \partial_{x_2} \phi_+))\rangle.
\]
We can simplify this expression by using integration by parts to note that
\[
\langle -2 \text{Re}(i\bar{a}_1 \phi_+ c_1^* \phi_+) \rangle = 2 \langle |\phi_+| \text{curl} a_1 \rangle \\
= 2 \left( \langle |\phi_+| (-\frac{1}{2} |\psi_0|^2 + \frac{1}{2} \langle |\psi_0|^2 \rangle) \right) \\
= \langle |\psi_0|^2 \rangle \langle |\phi_+|^2 \rangle - \langle |\psi_0|^2 |\phi_+|^2 \rangle.
\]

Thus, using the fact that \( \lambda_1 = (\kappa^2 - \frac{1}{2})\beta(\tau) \langle |\psi_0|^2 \rangle + \frac{1}{2} \langle |\psi_0|^2 \rangle \), we have
\[
\langle v_1, W_{1k} v_1 \rangle = -(2\kappa^2 - 1)\beta(\tau) \langle |\psi_0|^2 \rangle \langle |\phi_+|^2 \rangle + 4\kappa^2 \langle |\psi_0|^2 |\phi_+|^2 \rangle + (2\kappa^2 - 1) \text{Re} \langle \bar{\psi}_0^2 \phi_+^2 \rangle \\
+ 2 \text{Re} \left( i\bar{\phi}_+(k_1 \partial_{x_1} \phi_+ + k_2 \partial_{x_2} \phi_+) \right).
\]

Now we also calculate that
\[
\langle v_1, W_{0k} \bar{P}_1 (\bar{P}_1 N_{0k} \bar{P}_1)^{-1} \bar{P}_1 W_{0k} v_1 \rangle = \text{Re} \left( 2\bar{\phi}_+ \langle \partial_{aa} \psi_0 \rangle (\bar{\Delta}_{-k})^{-1} (\phi_+ \langle \partial_{aa} \psi_0 \rangle) \right).
\]

To simplify this expression we first note that \( \bar{\Delta}_{-k} = q_k^* q_k \), where we set \( q_k = \partial_{x_1} + i\partial_{x_2} + ik_1 - k_2 \). Since \( q_k \) and \( q_k^* \) commute we have
\[
q_k (\bar{\Delta}_{-k})^{-1} q_k^* = 1. \quad (3.26)
\]

We now calculate that
\[
q_k^* (\phi_+ \bar{\psi}_0) = (\partial_{x_1} \phi_+ + i\partial_{x_2} \phi_+ + ik_1 \phi_+ - k_2 \phi_+) \bar{\psi}_0 + \phi_+ (\partial_{x_1} \bar{\psi}_0 + i\partial_{x_2} \bar{\psi}_0) \\
= (ck_k \phi_+ - \frac{1}{2} x_1 \phi_+ + \frac{i}{2} \phi_+) \bar{\psi}_0 + \phi_+ \left( \partial_{x_1} \bar{\psi}_0 - i\partial_{x_2} \bar{\psi}_0 \right) \\
= \phi_+ \left( \partial_{x_1} \bar{\psi}_0 + i\partial_{x_2} \bar{\psi}_0 - \frac{1}{2} x_1 \bar{\psi}_0 - \frac{i}{2} \bar{\psi}_0 \right), \\
= \phi_+ (\partial_{aa} \bar{\psi}_0).
\]

These last two relations yield
\[
\langle v_1, W_{0k} \bar{P}_1 (\bar{P}_1 N_{0k} \bar{P}_1)^{-1} \bar{P}_1 W_{0k} v_1 \rangle = 2 \text{Re} \langle |\psi_0|^2 |\phi_+|^2 \rangle.
\]

Finally, we have that
\[
\langle v_{k(1,1)}, H_{1} v_{k(1,1)} \rangle = (2\kappa^2 - 1) [-\beta(\tau) \langle |\psi_0|^2 \rangle \langle |\phi_+|^2 \rangle + 2 \langle |\psi_0|^2 |\phi_+|^2 \rangle + \text{Re} \langle \bar{\psi}_0^2 \phi_+^2 \rangle] \\
- 2k \cdot \text{Im} \langle \bar{\phi}_+ \nabla \phi_+ \rangle.
\]
Similar calculations establish that

\[
\langle v_{k\sigma}, H_{\sigma} v_{k\sigma} \rangle = (2\kappa^2 - 1)\left[ -\beta(\tau) \langle |\psi_0|^2 \rangle \langle |\phi_{\sigma_1}|^2 \rangle + 2 \langle |\psi_0|^2|\phi_{\sigma_1}|^2 \rangle + \sigma_2 \text{Re} \langle \bar{\psi}_0^2 \phi_{\sigma_1}^2 \rangle \right]
- 2k \cdot \text{Im} \langle \bar{\phi}_{\sigma_1} \nabla \phi_{\sigma_1} \rangle,
\]

which is the expression found in (3.12) for \( S_{\kappa,\tau} \). Therefore, if we have \( \langle v_{k\sigma}, H_{\sigma} v_{k\sigma} \rangle > 0 \) for all \( k \) and all \( \sigma \), bounded uniformly in \( k \) away from 0, then \( S_{\kappa,\tau} > 0 \) and it follows that \( K_\epsilon > 0 \) by the perturbation argument above. Similarly if \( \langle v_{k\sigma}, H_{\sigma} v_{k\sigma} \rangle < 0 \) for some \( k \) and \( \sigma \), then \( S_{\kappa,\tau} < 0 \) and also \( K_\epsilon < 0 \). Thus Theorem 24 is proven.
Bibliography


