THREE DIMENSIONAL RADIO FREQUENCY CURRENT DENSITY IMAGING

by

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Biological tissues are generally conductive and knowing the current distribution in these tissues is of great importance in many biomedical applications. Radio frequency current density imaging (RF-CDI) is a technology that measures current density distributions at the Larmor frequency utilizing magnetic resonance imaging (MRI). RF-CDI computes the applied current density, $\mathbf{J}$, from the non-invasively measured magnetic field, $\mathbf{H}$, produced by $\mathbf{J}$. The previously implemented RF-CDI techniques could only image a single slice at a time. The previous method for RF current density reconstruction only computed one component of $\mathbf{J}$. Moreover, this reconstruction required an assumption about $\mathbf{H}$, which may be easily violated. These technical constraints have limited the potential biomedical applications of RF-CDI.

In this thesis, we address the limitations of RF-CDI mentioned above. First, we implement a multi-slice RF-CDI sequence with a clinical MRI system and characterize its sensitivity to MRI random noise. Second, we present a novel method to fully reconstruct all three components of $\mathbf{J}$ without relying on any assumption of $\mathbf{H}$. The central idea of our approach is to rotate the sample by 180 degrees in the horizontal plane to collect adequate MR data from two opposite sample orientations to compute one component of $\mathbf{J}$. Furthermore, this approach can be extended to reconstruct the other two components of $\mathbf{J}$ by adding one additional sample orientation in the horizontal plane. This method has been verified by simulations and electrolytic phantom experiments. We have therefore demonstrated for the first time the feasibility of imaging the magnitude and phase of all components of the RF current density vector.

The work presented in this thesis is expected to significantly enhance RF-CDI to image biological subjects. The current density vector $\mathbf{J}$ or any component of $\mathbf{J}$ can be measured over multiple slices without the compromise of motions of organs and tissues caused by gravitational force, thanks to the method of horizontal rotations. In addition, the reconstruction of the complex conductivity of biological tissues becomes possible due to the current advance in RF-CDI presented here.
Acknowledgments

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Finally, I wish to thank the members of my family. Words alone cannot express the gratitude I owe to my parents for their moral support and encouragement. I dedicate this thesis to my husband, Jie Cui. Without his love and support, this thesis would not have been possible.

Dinghui Wang
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<th>Description</th>
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<tbody>
<tr>
<td>AC</td>
<td>alternating current</td>
</tr>
<tr>
<td>CDI</td>
<td>current density imaging</td>
</tr>
<tr>
<td>CDII</td>
<td>current density impedance imaging</td>
</tr>
<tr>
<td>DC</td>
<td>direct current</td>
</tr>
<tr>
<td>EM</td>
<td>electro-magnetic</td>
</tr>
<tr>
<td>EIT</td>
<td>electrical impedance tomography</td>
</tr>
<tr>
<td>FDTD</td>
<td>finite difference time domain</td>
</tr>
<tr>
<td>FOV</td>
<td>field of view</td>
</tr>
<tr>
<td>LF</td>
<td>low frequency</td>
</tr>
<tr>
<td>LCP</td>
<td>left circularly polarized</td>
</tr>
<tr>
<td>MR</td>
<td>magnetic resonance</td>
</tr>
<tr>
<td>MRI</td>
<td>magnetic resonance imaging</td>
</tr>
<tr>
<td>PSD</td>
<td>pulse sequence diagram</td>
</tr>
<tr>
<td>RCP</td>
<td>right circularly polarized</td>
</tr>
<tr>
<td>RF</td>
<td>radio frequency</td>
</tr>
<tr>
<td>SAR</td>
<td>specific absorption rate</td>
</tr>
<tr>
<td>SNR</td>
<td>signal to noise ratio</td>
</tr>
<tr>
<td>VF</td>
<td>variable frequency</td>
</tr>
</tbody>
</table>
List of Symbols

\[ \vec{a}_x, \vec{a}_y, \vec{a}_z \] unit directional vectors of the Cartesian system

\[ \vec{a}_x, \vec{a}_y, \vec{a}_z \] unit directional vectors of the rotating frame

\[ M_x^y, M_y^y, M_z^y \] acquired \( x \), \( y \) and \( z \) components of \( \mathbf{M} \) when it is originally prepared along \( y \) axis

\[ \vec{B}_0 \] static magnetic field of a MR imager [Tesla] or [T]

\[ B_0 \] magnitude of \( \vec{B}_0 \) [T]

\[ B_1 \] rotating frame RF magnetic pulse or the rotary echo RF pulse [T]

\[ \vec{B} \] total rotating frame magnetic flux density [T]

\[ \vec{B}_x, \vec{B}_y \] rotating frame transverse magnetic flux density components [T]

\( \mathbf{C} \) matrix of initial magnetizations

\( \mathbf{C} \) complex MR image

\[ C_{y+}^x, C_{y-}^x \] complex MR image acquired for measuring \( M_{y+}^x, M_{y-}^x \)

\[ C_y^x \] difference of \( C_{y+}^x \) and \( C_{y-}^x \)

\[ C_{z+}^x, C_{z-}^x \] complex MR image acquired for measuring \( M_{z+}^x, M_{z-}^x \)

\[ C_z^x \] difference of \( C_{z+}^x \) and \( C_{z-}^x \)

\[ C_{x+}^y, C_{x-}^y \] complex MR image acquired for measuring \( M_{x+}^y, M_{x-}^y \)
$C_{z+}^y, C_{z-}^y$ complex MR image acquired for measuring $M_{z+}^y, M_{z-}^y$

\textbf{D} matrix of final magnetizations

$\bar{D}$ electric flux density in time domain [C/m$^2$]

\textbf{E} electric intensity vector in phasor domain [V/m]

$F_x, F_y, F_z$ $x, y$ and $z$ derivative filter noise factors

$f^{yz}$ factor associated with the derivative templates and $\Delta y, \Delta z$

\textbf{H} magnetic intensity vector in phasor form [A/m]

$\tilde{H}$ magnetic intensity vector in time domain [A/m]

$h_x, h_z, h_z$ magnitude of $H_x, H_y, H_z$ [A/m]

$H_x, H_y, H_z$ $x, y$ and $z$ components of $H$ [A/m]

$H_{xy}$ transverse magnetic intensity vector in phasor form [A/m]

$\tilde{H}_l$ LCP component of $H_{xy}$ in time domain [A/m]

$h_l$ magnitude of time harmonic $\tilde{H}_l$ [A/m]

$H_L$ scalar phasor associated with LCP component of $H_{xy}$ [A/m]

$H_L^0, H_L^\pi, H_L^{\frac{\pi}{2}}, H_L^{-\frac{\pi}{2}}$ $H_L$ in sample positions labeled as $0, \pi, \frac{\pi}{2}$ and $-\frac{\pi}{2}$

$H_R$ scalar phasor associated with RCP component of $H_{xy}$ [A/m]

$\tilde{H}_x, \tilde{H}_y$ rotating frame transverse magnetic intensity components [A/m]
\[ \tilde{H}_x^0, \tilde{H}_y^0, \tilde{H}_z^0, \tilde{H}_x^\pi, \tilde{H}_y^\pi, \tilde{H}_z^\pi \] measured in sample positions labeled as \( 0, \pi, \frac{\pi}{2} \) and \( -\frac{\pi}{2} \)

\[ \tilde{H}_x^0, \tilde{H}_y^0, \tilde{H}_z^0, \tilde{H}_x^\pi, \tilde{H}_y^\pi, \tilde{H}_z^\pi \] measured in sample positions labeled as \( 0, \pi, \frac{\pi}{2} \) and \( -\frac{\pi}{2} \)

\[ I \] imaginary part of the complex image \( C \)

\[ I_{y+}^x, I_{y-}^x \] imaginary parts of \( C_{y+}^x, C_{y-}^x \)

\[ I_{z+}^x, I_{z-}^x \] imaginary parts of \( C_{z+}^x, C_{z-}^x \)

\[ I_z \] current in the z direction [A]

\[ \mathbf{J} \] total current density vector in phasor form [A/m²]

\[ \bar{\mathbf{j}} \] total current density vector in time domain [A/m²]

\[ \bar{\mathbf{J}}_{\text{con}} \] conduction current density vector in time domain [A/m²]

\[ j_x, j_y, j_z \] magnitude of \( J_x, J_y, J_z \) [A/m²]

\[ J_I, J_R \] real and imaginary part of \( J_z \)

\[ J_{x}, J_{y}, J_{z} \] \( x, y \) and \( z \) components of \( \mathbf{J} \) [A/m²]

\[ J_{x}^{\text{rec}}, J_{y}^{\text{rec}}, J_{z}^{\text{rec}} \] reconstructed \( J_x, J_y, J_z \) with a relative constant angle

\[ J_{x}^{\text{true}}, J_{y}^{\text{true}}, J_{z}^{\text{true}} \] bench mark \( J_x, J_y, J_z \) with a relative constant angle

\[ \mathbf{M} \] macroscopic magnetization vector [A/m]

\[ \mathbf{M}_0 \] equilibrium magnetization vector [A/m]

\[ M_0 \] magnitude of \( \mathbf{M}_0 \) [A/m]
\[ M_x^x, M_y^y, M_z^z \] magnitude of the acquired \( \tilde{x} \), \( \tilde{y} \) and \( \tilde{z} \) components of \( M \) when it is originally prepared along the \( \tilde{x} \) axis [A/m]

\[ M_x^y, M_y^y, M_z^z \] magnitude of the acquired \( \tilde{x} \), \( \tilde{y} \) and \( \tilde{z} \) components of \( M \) when it is originally prepared along the \( \tilde{y} \) axis [A/m]

\[ M_x^z, M_y^z, M_z^z \] magnitude of the acquired \( \tilde{x} \), \( \tilde{y} \) and \( \tilde{z} \) components of \( M \) when it is originally prepared along the \( \tilde{z} \) axis [A/m]

\[ M_y^y, M_z^z \] \( \tilde{y} \) and \( \tilde{z} \) components of \( M \) for the measurement of \( \Gamma_x \) [A/m]

\[ M_y^z, M_z^z \] magnitude of the \( \tilde{y} \) and \( \tilde{z} \) components of \( M \) for the measurement of \( \Gamma_x \) [A/m]

\[ M_y^x, M_y^x \] measurements of \( M_y^x \) when \( M_y^x \) is stored in \(+z\), \(-z\) axis in the multi-slice RF-CDI sequence [A/m]

\[ M_z^x, M_z^x \] measurements of \( M_z^x \) when \( M_z^x \) is stored in \(+z\), \(-z\) axis in the multi-slice RF-CDI sequence [A/m]

\[ M_z^y, M_z^y \] \( \tilde{x} \) and \( \tilde{z} \) components of \( M \) for the measurement of \( \Gamma_y \) [A/m]

\[ M_z^x, M_z^z \] magnitude of the \( \tilde{x} \) and \( \tilde{z} \) components of \( M \) for the measurement of \( \Gamma_y \) [A/m]

\[ M_z^x, M_z^x \] measurements of \( M_z^x \) when \( M_z^x \) is stored in \(+z\), \(-z\) axis in the multi-slice RF-CDI sequence [A/m]

\[ M_z^y, M_z^y \] measurements of \( M_z^y \) when \( M_z^y \) is stored in \(+z\), \(-z\) axis in the multi-slice RF-CDI sequence [A/m]

\[ R \] real part of the complex image \( C \)
\( R_x^*, R_y^* \) real parts of \( C_{y+x}, C_{y-x} \)

\( R_z^*, R_{z+}^* \) real parts of \( C_{z+x}, C_{z-x} \)

\( \mathbf{R} \) rotation matrix

\( R \) real part of the complex image \( C \)

\( \mathbf{S} \) deformation matrix

\( t \) time [s]

\( T_1 \) longitudinal relaxation time [s]

\( T_2 \) transverse relaxation time [s]

\( T_2^* \) effective transverse relaxation time [s]

\( T_z \) total current on time [s]

\( T_E \) echo time [s]

\( T_R \) repetition time [s]

\( \tilde{x}, \tilde{y}, \tilde{z} \) rotating frame axes

\( \gamma \) gyromagnetic ratio [rad/Ts]

\( \gamma \) complex conductivity \( (\sigma + j\omega \epsilon) \) [S/m]

\( \Gamma_x, \Gamma_y \) net rotation angles proportional to \( \tilde{B}_x (\tilde{H}_x), \tilde{B}_y (\tilde{H}_y) \)

\( \delta I \) change of \( I \)

\( \delta I_x^*, \delta I_y^* \) change of \( I_x^*, I_y^* \)
\[ \delta I_{z^+}, \delta I_{z^-} \quad \text{change of } I_{z^+}, I_{z^-} \]

\[ \delta j_z \quad \text{change of } j_z \]

\[ \delta J_I \quad \text{change of } J_I \]

\[ \delta J_R \quad \text{change of } J_R \]

\[ \overline{\delta J_I} \quad \text{mean of } \delta J_I \]

\[ \overline{\delta J_R} \quad \text{mean of } \delta J_R \]

\[ \delta R \quad \text{change of } R \]

\[ \delta R_{y^+}^{z}, \delta R_{y^-}^{z} \quad \text{change of } R_{y^+}^{z}, R_{y^-}^{z} \]

\[ \delta R_{z^+}^{x}, \delta R_{z^-}^{x} \quad \text{change of } R_{z^+}^{x}, R_{z^-}^{x} \]

\[ \delta \Gamma_x \quad \text{change of } \Gamma_x \text{ [rad]} \]

\[ \overline{\delta \Gamma_x} \quad \text{mean of } \delta \Gamma_x \text{ [rad]} \]

\[ \delta \phi_z \quad \text{change of } \phi_z \text{ [rad]} \]

\[ \overline{\delta \phi_z} \quad \text{mean of } \delta \phi_z \text{ [rad]} \]

\[ \Delta B_0 \quad \text{inhomogeneity of the static magnetic field } \tilde{B}_0 \text{ [T]} \]

\[ \Delta x, \Delta y, \Delta z \quad \text{voxel dimensions [m]} \]

\[ \varepsilon \quad \text{electric permittivity [Farad/m]} \]

\[ \varepsilon_r \quad \text{relative electric permittivity} \]

\[ \sigma \quad \text{electric conductivity [S/m]} \]
\( \sigma \) standard deviation of Gaussian random noise

\( \sigma_{\Gamma_x}, \sigma_{\Gamma_y} \) standard deviation of \( \Gamma_x, \Gamma_y \) [rad]

\( \sigma_{\tilde{H}_x}, \sigma_{\tilde{H}_y} \) standard deviation of \( \tilde{H}_x, \tilde{H}_y \) [A/m]

\( \sigma_f \) standard deviation of \( J_z \) in either real or imaginary part [A/m²]

\( \sigma_{j_z} \) standard deviation of \( j_z \) [A/m²]

\( \sigma_\phi \) standard deviation of \( \phi_z \) [rad]

\( \varphi \) relative angle between the lab frame and the rotating frame [rad]

\( \theta_j \) angle from \( \tilde{H}_f \) to the \( \tilde{x} \) axis

\( \theta_x, \theta_y, \theta_z \) phase of \( J_x, J_y, J_z \)

\( \phi_x, \phi_y, \phi_z \) phase of \( H_x, H_y, H_z \)

\( \phi_R \) systematic phase error of MR images

\( \mu \) magnetic permeability [H/m]

\( \mu_0 \) magnetic permeability in free space [H/m]

\( \omega \) angular frequency or specifically the Larmor angular frequency [rad/s]
Chapter 1

Introduction

Current density imaging (CDI) is a technology that non-invasively measures current density distributions (Joy et al. 1989; Scott et al. 1992a). It utilizes magnetic resonance imaging (MRI) to map the magnetic fields produced by the applied electrical currents. From these magnetic quantities, the current density can be derived based on Maxwell’s electromagnetic equations. Radio frequency current density imaging (RF-CDI) (Scott et al. 1992a; Scott et al. 1995a; Scott et al. 1995b) measures current densities at the Larmor frequency of the MR imager. As the branch of CDI that works at the highest frequency range, RF-CDI is an imaging modality well suited for clinical and biomedical applications. More current can be tolerated without stimulating nerves and muscles at higher frequencies than at lower frequencies.

Previous applications of RF-CDI were limited to one slice. Furthermore, the previous RF-CDI method reconstructed only one component of the current density vector. In addition, this reconstruction was based on a restrictive assumption (referred to as the single orientation approximation) about the magnetic field produced by the applied current (Scott 1993). This thesis extends the ability of RF-CDI to measure the complete three-dimensional RF current density vector in a three-dimensional volume.

The objectives of this research were:
1. To implement the first multi-slice RF-CDI sequence and to characterize the random noise performance of the sequence.

2. To develop a new RF-CDI method to image one component of the current density vector field, without any assumptions of the magnetic field, using a 180-degree rotation of the sample in the horizontal plane.

3 To measure all three components of the current density vector field with an additional horizontal sample rotation.

This chapter first gives a brief review of the CDI technology. Then the motivation for developing full three-dimensional RF-CDI is described. The chapter concludes with the outline for the remainder of the dissertation.

1.1 Review of Current Density Imaging

1.1.1 Low Frequency CDI

Current density imaging (CDI) is classified according to the frequency range of the applied current, as low frequency CDI (LF-CDI), radio frequency CDI (RF-CDI) and variable frequency CDI (VF-CDI). The first CDI technique was implemented with pulsed direct currents (DC) whose frequencies lie in the range of 1-100 Hz. Therefore, it was named low frequency CDI (LF-CDI). For practical purposes, only the conductivity of the biological tissue enters into LF-CDI since, in this frequency range, the current due to the permittivity is negligible compared to the current due to the conductivity.
The applied DC current pulses yield pulsed magnetic fields. Only the longitudinal magnetic component, \textit{i.e.}, the component parallel to the static magnetic field $B_0$ of the MR imager, produces measurable effects on the nuclear magnetization. This magnetic component, though very small compared to the static magnetic field $B_0$, causes a measurable change of the magnitude of the static magnetic field. The local protons experience a slight shift of the resonant frequency proportional to the magnetic component field strength. The accumulative effect of this frequency shift is encoded in the phase of the complex MR images.

According to Ampere’s law, spatial derivatives of two magnetic components are required to compute one component of the current density. Since only the longitudinal magnetic component can be detected with MRI, the sample is imaged in two orthogonal positions (orientations) to reconstruct one component of the current density and in three orthogonal positions to reconstruct the total three-dimensional current density vector at each point in space.

LF-CDI was first developed in late 1980’s with a small bore 2T research MR system (Joy et al. 1989; Pesikan et al. 1990; Scott et al. 1991). Later, it was realized with a clinical 1.5 T MR system (DeMonte 2000). More recently, a 3D fast sequence with the ability for cardiac gating was implemented for live animal applications (DeMonte et al. 2004; DeMonte et al. 2006; DeMonte et al. 2008). LF-CDI has been exploited in many studies in the area of biomedical engineering. As early as its invention, its capability of imaging biological tissues was shown on a volunteer’s forearm (Joy et al. 1989). It has been
utilized to investigate the efficacy and safety of devices that involve current injection into the human body for diagnostic, therapeutic and other purposes. These researches include imaging the current density in tumor tissues of mice for different electrodes configurations (Miklavcic et al. 1998), measurement of current density in rabbit brain during transcranial electrotimulation (Joy et al. 1999), evaluation of skin burns under surface electrodes using a porcine skin-gel model, study of current pathways in porcine torso and heart for defibrillation electrodes (Patriciu et al. 2005), and analyzing the linearity of the measured current density corresponding to the applied current in live pigs for human electro-muscular incapacitation devices (DeMonte et al. 2008).

LF-CDI has also been employed as a tool to provide conductivity encoded contrast. It has been used to monitor chemical reactions where ions are released or dissolved (Beravs et al. 1999; Mikac et al. 2007). In one study, higher current density was observed in tumor tissue compared to normal tissue (Sersa et al. 1997), presumably because tumor tissue was more conductive. In another study, the ability of LF-CDI to diagnose osteoporosis was tested; the conductivity showed significant changes in the bones of patients with osteoporosis due to the lower calcium level (Beravs et al. 1997). Last but not least, the magnetic field and current density mapping techniques of LF-CDI have made possible the development of several important conductivity reconstruction methods (Hasanov et al. 2004; Hasanov et al. 2008; Khang et al. 2002; Kwon et al. 2002; Muftuler et al. 2004; Oh et al. 2003; Zhang 1992).
1.1.2 Radio Frequency CDI

Radio frequency CDI (RF-CDI) images sinusoidal current density at the Larmor frequency (the resonant frequency) of the MR imager (Scott et al. 1992a; Scott et al. 1995a; Scott, Joy et al. 1995b). It derives its name from the fact that the Larmor frequencies of most MR imagers are in the radio frequency (RF) range. For example, in a clinical 1.5 Tesla MR imager, the frequency of the applied current is approximately 64 MHz. At these frequencies, the displacement current can be comparable to the conduction current in biological tissues. Therefore, the current density measured by RF-CDI involves both conduction and displacement current density.

The Larmor frequency current produces a magnetic field at the nuclear magnetic resonant frequency and thus acts on the magnetization as does an RF B₁ pulse in MRI. In this case, unlike in that of LF-CDI, only the transverse magnetic components (the components perpendicular to B₀) have significant net effects on the magnetization while the influence of the longitudinal magnetic component is negligible. Furthermore, the magnetization is only sensitive to the part of the transverse RF magnetic field which is left circularly polarized (LCP) with respect to the direction of B₀. Previously implemented RF-CDI techniques evaluated the longitudinal component of the Larmor frequency current with only one sample orientation (position). However, the reconstruction method of these techniques relied on the single orientation approximation (described in detail in Section 2.2). In order to satisfy the assumption, in previous implementations of RF-CDI currents were forced to flow predominantly in a direction parallel to B₀ by careful configuration of the electrodes and positioning of the sample.
The theory and technology of RF-CDI was first developed and realized with a 2 T small bore magnetic resonance imager in early 1990’s (Scott et al. 1992a; Scott et al. 1995a; Scott et al. 1995b). Subsequently, RF-CDI was implemented with clinical MRI instrumentation (Carter 1995; Gerkis 1996; Yan 1997). Studies involving in-vitro and in-vivo applications of RF-CDI have been carried out since its invention. Since the RF current density and magnetic field mapping are directly relevant to dielectric heating at the frequency ranges of MRI, one of these applications is current density measurement in RF ablations (Scott et al. 2003; Scott et al. 2005). Another important potential application of RF-CDI is to measure the electrical activity inside a body. There have been early attempts to use RF-CDI in in-vivo animal experiments to reveal associated functional areas of the brain through the RF impedance changes of the related neuronal tissue caused by a stimulus (Beravs et al. 1999; Yoon et al. 1999; Yoon et al. 1998). All the above implementations of RF-CDI were constrained to a single slice and required the single orientation approximation as mentioned above.

1.1.3 Variable Frequency CDI

Variable frequency CDI (VF-CDI) was implemented to fill in the frequency gap between LF-CDI and RF-CDI (Weinrot 1998). The principle of VF-CDI is an extension of that of RF-CDI. A left circularly polarized (LCP) RF $B_1$ pulse at the Larmor frequency causes resonant phenomenon of the magnetization. For convenience, the resonant phenomenon is often described in the rotating frame (please refer to Appendix A) at the Larmor frequency. In this rotating frame, $\overline{B}_0$ disappears and the $B_1$ pulse appears to be fixed.
rather than rotating around $\vec{B}_0$. The effect of the $B_1$ field on the magnetization seen in this rotating frame is a rotation around the direction of $B_1$ with an angle proportional to the magnitude and the duration of $B_1$. In VF-CDI, an alternating current (AC) is applied at the same time as a rectangular RF pulse. The frequency, $f$, of the current pulse is determined by the magnitude, $B_1$, of the rectangular RF pulse as $f = \gamma B_1$ (the gyromagnetic ratio is $\gamma$). The magnetic field component along the $\vec{B}_0$ direction due to the AC current will have maximum net effect on the magnetization at this frequency.

The effect of this AC field can be viewed in the rotating frame as an analogy to the effect of the $B_1$ pulse in the laboratory frame. Similarly to the use of a single rotating frame in MRI, it is preferable to discuss the movement of the magnetization in VF-CDI using a second frame which rotates with respect to the first frame. This second rotating frame rotates around the direction of the $B_1$ pulse at the frequency of the applied AC current. It is called the double rotating frame (Scott 1993). Much of the theory and techniques of RF-CDI can be modified for VF-CDI. But unlike RF-CDI, in VF-CDI, the frequency of the AC current is variable by adjusting the magnitude of the $B_1$ pulse. Constrained by the highest magnitude of the $B_1$ pulse that can be produced by a MR imager, the highest frequency that can be imaged by VF-CDI is below 10 kHz. On the other hand, the higher the AC frequency the higher the magnitude of the $B_1$ pulse and the higher the power it deposited on the object. Therefore, in VF-CDI there is a trade-off between the specific absorption rate (SAR) and the frequency of the current density to be measured.

A different approach was later attempted for imaging AC currents in the kHz range, and termed alternating current CDI (AC-CDI) (Mikac et al. 2001). This technique can be
regarded as an extension of LF-CDI. It utilizes the hardware platform of LF-CDI and applies multiple short DC current pulses between multiple 180° RF B\textsubscript{1} pulses. The frequency of the measured current density is limited by the shortest duration of the 180° RF B\textsubscript{1} pulse that can be produced by the MR imager. The 180° RF B\textsubscript{1} pulses also contribute to extra RF power deposited on the object being imaged. One advantage of using the 180° RF B\textsubscript{1} pulses is that they reduce the signal loss due to the effect of the diffusion.

1.2 Motivation

The noninvasiveness and the high spatial resolution inherited from MRI makes CDI an ideal imaging modality to measure electrical and magnetic fields inside a biological body. LF-CDI is a well developed CDI technique. Quantitative measurements of current density vectors and current flow have been successfully obtained in in-vitro and in-vivo animal experiments. Furthermore, current density impedance imaging at low frequencies (LFCDI) has also been developed based on LF-CDI (Hasanov et al. 2004; Hasanov et al. 2008). However, it is difficult for LF-CDI to achieve good signal to noise ratio (SNR) in a living body without interfering with the electrical activities of nerves, skeleton muscles and the heart. Therefore, in biomedical and clinical applications, it may be preferable to apply radio frequency currents since higher currents can be tolerated without stimulating nerves and muscles (Geddes 1983). The specific absorption rate (SAR) should be calculated in order to monitor the distribution of absorbed RF power. There have been several studies involving estimation of SAR for RF-CDI experiments (Beravs et al. 1999; Beravs et al. 2000). The results suggest that substantial sensitivity of RF-CDI can be
achieved while satisfying the regulatory limits of SAR and temperature changes.

Moreover, working at a different frequency range from LF-CDI, RF-CDI can provide different information on the electric properties of tissues. At low frequencies, the current flow in tissues mainly due to the extracellular fluid. This is because cell membranes offer a significant barrier to current flow and thus a cell is far less conductive compared to its surrounding electrolyte (Foster et al. 1989). On the other hand, applied radio frequency currents develop much smaller potential differences across cell membranes and hence the intracellular fluid plays an important role in the current flow. Therefore, RF-CDI may facilitate better understanding of the fundamental physical or physiological processes.

Finally, the RF-CDI techniques may be implemented using the available hardware of the MRI system. Although in the past and present implementations of RF-CDI the Larmor frequency currents are injected through electrodes by additional hardware, Scott (Scott 1993; Scott et al. 1995a) also suggested that current may also be induced by coils.

There are severe limitations of the RF-CDI methods implemented prior to this work. First, only a single slice can be measured in one scan. For multiple slices or for imaging a three-dimensional volume, the long imaging time often makes previous RF-CDI implementations impractical. Second, the previous reconstruction method of RF-CDI yields only one component of the current density vector. Thirdly, the validity of the method depends on a very restrictive condition, the single orientation approximation. This condition may be satisfied by careful arrangement of the electrodes and return wires,
and choice of special geometries in phantom experiments. However, it need not hold in biomedical applications. Finally, the incomplete measurement of the magnetic field severely limits the usefulness of previous RF-CDI implementations in the important next development of the reconstruction of the complex conductivity.

It was suggested in (Scott et al. 1995a) that in principle, if one could make measurements of the sample in three orthogonal orientations, it may be possible to remove the single orientation constraint. However, this approach encountered some difficulties which have not been addressed until now. Moreover, in the approach of orthogonal rotations, all three positions are required even to recover one component of the current density. Motions of organs and tissues tend to occur during the flip of the body because of gravity.

Therefore, in order to enhance CDI’s potential for biological and clinical uses, it is essential to address the technical constraints of the earlier implementations of RF-CDI mentioned above. There are two major tasks. First, a multi-slice RF-CDI imaging technique is required to speed up the imaging procedure. Second, a new reconstruction method of RF-CDI is expected to recover complete 3D current density vectors without any constraints of the electromagnetic fields.

1.3 Thesis Outline

The thesis is divided into 6 chapters. Chapter 2 reviews previous RF-CDI experimental methods to determine the magnetic component and the previous method of calculating current density from these measurements. It provides a new derivation of the
electromagnetic considerations for RF-CDI introduced by Scott et al. (Scott 1993; Scott et al. 1995a). This serves as the theoretical basis for the new reconstruction method that is presented in Chapters 4 and 5. Two major measurement methods of RF-CDI, namely, the polar decomposition method and the rotating frame method (also as known as the rotary echo method) are also briefly discussed.

Chapter 3 describes the implementation of the first multi-slice RF-CDI sequence. This chapter is an extension of the implementation of multi-slice RF-CDI presented in the MASc thesis by Wang (Wang 2004). New results from phantom experiments are demonstrated. The random noise performance of the sequence is analyzed by theoretical derivation, simulation as well as phantom experiments.

In Chapters 4 and 5 we present the novel method to measure the magnitude and phase of components of the current density vector without any constraints on the electromagnetic fields produced by the current. Chapter 4 demonstrates our new approach to computing one component of the current density vector. First, the effect of the violation of the single orientation approximation is discussed with illustrations of computer simulations. After that, the new method of the reconstruction is formulated based on a single 180-degree sample rotation in the horizontal plane. This method is then tested by simulation models and phantom experiments.

Chapter 5 extends the reconstruction method described by Chapter 4 to recover all three components of the current density vector. With one additional sample position in the
horizontal plane, the full magnetic vector field can be recovered. This then makes it possible to calculate all three components of the current density vector field. The mathematical expression is first derived, followed by its verification by numerical simulation and preliminary experimental testing. Finally, the technical constraints of the new approach are discussed.

Chapter 6 provides the main conclusions of this work. Considerations for biomedical applications and directions for future work are also addressed in this chapter.
Chapter 2

Background of RF-CDI

This chapter gives a new derivation of the fundamental theory of RF-CDI, as it was developed by Scott et al (Scott 1993; Scott et al. 1995a). This forms the basis for Chapter 4 and Chapter 5. RF-CDI technology is composed of two major parts, the imaging methods used to measure the magnetic components produced by the externally applied current and the reconstruction method used to calculate the current density from the measured magnetic components. Both aspects are reviewed and discussed in this chapter. For convenience, the electromagnetic field is discussed intensively in the phasor domain throughout this thesis. The connection between the previous expressions and the new ones presented here are shown in Appendix B.

2.1 Electromagnetic Consideration

The principle of RF-CDI technology is based on Maxwell’s equations in an isotropic, linear and reciprocal media. When current at the Larmor frequency is applied to the sample, it produces a sinusoidally time-varying electromagnetic field. To distinguish the vector quantities in the time domain and the phasor domain, arrowed letters such as $\vec{V}$ are used for the instantaneous vector quantities and bold letters such as $\mathbf{V}$ are used for the vector phasor notations. Let $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ be the unit directional vectors of the Cartesian coordinate system. For a general time harmonic vector

$$\vec{V} = v_x \cos(\omega t + \theta_x)\vec{a}_x + v_y \cos(\omega t + \theta_y)\vec{a}_y + v_z \cos(\omega t + \theta_z)\vec{a}_z,$$  \hspace{1cm} (2.1)

the corresponding phasor expression is
\[ \mathbf{V} = V_x \mathbf{a}_x + V_y \mathbf{a}_y + V_z \mathbf{a}_z \]
\[ = v_x e^{j\theta} \mathbf{a}_x + v_y e^{j\theta} \mathbf{a}_y + v_z e^{j\theta} \mathbf{a}_z. \]

(2.2)

In other words, \( \vec{V} = \text{Re}(\mathbf{V} e^{j\omega t}) \), where \( \text{Re}(\bullet) \) denotes the real part of a complex quantity.

For an isotropic and linear media, Ampere’s circuital law indicates

\[ \nabla \times \mathbf{H} = (\sigma + j\omega \varepsilon)\mathbf{E} = \mathbf{J}, \]

(2.3)

Where:

- \( \mathbf{H}, \mathbf{E} \) and \( \mathbf{J} \) are, respectively and in phasor form, the magnetic intensity, the electrical intensity and the current density,
- \( \mathbf{J} \) is the total current density, including both the conduction current density \( \sigma \mathbf{E} \) and the displacement current density \( j\omega \varepsilon \mathbf{E} \),
- \( \sigma \) and \( \varepsilon \) are real parameters, representing the conductivity and the permittivity respectively,
- \( \omega \) is the frequency of the applied current, which is also the Larmor frequency of the MR imager.

In the laboratory frame, the static magnetic field \( \mathbf{B}_0 \) is in the direction of \( z \) axis.

Generally, the transverse component of the magnetic field produced by the current can be decomposed into a right circularly polarized (RCP) field and a left circularly polarized (LCP) field

\[ \mathbf{H}_{xy} = H_x \mathbf{a}_x + H_y \mathbf{a}_y \]
\[ = \frac{1}{2} (H_x + jH_y) (\mathbf{a}_x - j\mathbf{a}_y) + \frac{1}{2} (H_x - jH_y) (\mathbf{a}_x + j\mathbf{a}_y). \]

(2.4)
The first term in (2.4) is the RCP component with respect to the $z$ axis, and the second term is the LCP component. This can be demonstrated by writing the vector expression in the time domain for the second term.

$$
\hat{H}_l = \text{Re}\left(\frac{1}{2} (H_x - jH_y)(\tilde{a}_x + j\tilde{a}_y)e^{j\omega t}\right)
$$

$$
= \text{Re}(h_l e^{jh_l} (\tilde{a}_x + j\tilde{a}_y)e^{j\omega t}), \quad (2.5)
$$

$$
= h_l (\cos(\omega t + \theta_l)\tilde{a}_x - \sin(\omega t + \theta_l)\tilde{a}_y)
$$

where $h_l$ is a non negative real number and $\theta_l$ is real. We can see that the vector $\hat{H}_l$ rotates clockwise (in the left hand direction) with respect to the $z$ axis in the transverse plane with an angular frequency $\omega$ when $t$ increases. It has a magnitude of $h_l$ and makes an angle $-\theta_l$ with respect to the $x$ axis at $t = 0$ as shown in Figure 2.1. Define $H_L$ and $H_R^1$ as

$$
H_L = \frac{1}{2} (H_x - jH_y) \quad (2.6)
$$

and

$$
H_R = \frac{1}{2} (H_x + jH_y), \quad (2.7)
$$

then

$$
H_x = H_L + H_R \quad (2.8)
$$

and

$$
H_y = j(H_L - H_R) \quad (2.9)
$$

According to (2.3), (2.8) and (2.9), we can calculate $J_z$ from $H_L$ and $H_R$


\footnote{1 \(H_L\) and \(H_R\) are equivalent to \(H^+\) and \((H^-)^*\) in some MRI notations (Houl 2000).}
\[ J_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_z}{\partial y} = j(\frac{\partial H_L}{\partial x} - \frac{\partial H_R}{\partial x}) - (\frac{\partial H_L}{\partial y} + \frac{\partial H_R}{\partial y}) \]  \hspace{1cm} (2.10)

This is the basic equation that we will use in the remainder of the thesis. Alternatively, notice that since \( \mu \approx \mu_0 \) for biological tissues, \( \nabla \cdot \mathbf{H} = 0 \), i.e.

\[ \frac{\partial H_L}{\partial x} + \frac{\partial H_L}{\partial y} + \frac{\partial H_L}{\partial z} = 0 \]  \hspace{1cm} (2.11)

Combining (2.8) ~ (2.11), we obtain

\[ J_z = 2j \frac{\partial H_L}{\partial x} - 2 \frac{\partial H_R}{\partial y} + j \frac{\partial H_z}{\partial z} \]  \hspace{1cm} (2.12)

or

\[ = -2j \frac{\partial H_R}{\partial x} - 2 \frac{\partial H_L}{\partial y} + j \frac{\partial H_z}{\partial z} \]  \hspace{1cm} (2.12a)

Thus, to compute \( J_z \), we have three options, to measure \( H_L, H_R \) utilizing (2.10) or either of \( H_L, H_R \) and \( H_z \) utilizing (2.12) or (2.12a). With one sample position, none of these is possible without apriori knowledge of \( H_L, H_R \) or \( H_z \).

2.2 The Single Orientation Reconstruction

Only the LCP component of the magnetic field yielded by the RF current in the transverse plane contributes to the resonance of magnetization (Freeman et al. 1971) and hence can be measured by imaging. More details about the measurements of the LCP component are provided in section 2.3. In brief, the LCP component of the magnetic field corresponding to the applied current is measured in the rotating frame defined by the transmitter coil. In other words, a 0° phase RF pulse created by the transmitter coil defines the \( \hat{x} \) axis of the rotating frame which rotates around \( \vec{B}_0 \) clockwise at the Larmor
frequency $\omega$ as shown in Figure 2.1. In this rotating frame, $\tilde{H}_l$ defined in (2.5) remains static. Let $\varphi$ be the relative angle between the rotating frame $(\tilde{x}, \tilde{y})$ and the laboratory frame $(x, y)$ at $t=0$ as shown in Figure 2.1. ($\tilde{H}_x, \tilde{H}_y$) are the projections of $\tilde{H}_l$ to the two rotating frame axes $\tilde{x}$ and $\tilde{y}$. They are real scalars extracted by RF-CDI imaging methods. According to (2.5), $H_L$ defined in (2.6) is related to the rotating frame components $\tilde{H}_x$ and $\tilde{H}_y$ by

$$H_L = h e^{j\varphi} = |\tilde{H}_l| \left( \cos \theta_1 + j \sin \theta_1 \right) = (\tilde{H}_x - j\tilde{H}_y) e^{-j\varphi}. \quad (2.13)$$

Generally, $\varphi$ in (2.13) is a spatial dependent parameter. However, ideally, for a coil designed to produce uniform $B_1$ such as a birdcage coil, this angle is a global constant when the coil is unloaded. We expect that the interaction between the sample and the coil is not significant enough to deviate $\varphi$ much from the value of the unloaded coil when the
static field $B_0$ is not very high (e.g. 1.5T) and a relatively small sample is imaged. For a constant $\varphi$, inserting (2.13) to (2.12), we obtain

$$J \varepsilon e^{j\varphi} = \left\{ 2\left( \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} \right) + 2j \left( \frac{\partial \tilde{H}_z}{\partial x} + \frac{\partial \tilde{H}_y}{\partial y} \right) \right\} + j \frac{\partial H_z}{\partial z} e^{j\varphi}. \quad (2.14)$$

When

$$\left| \frac{\partial H_z}{\partial z} \right| << |J_z|, \quad (2.15)$$

the last term can be ignored (Scott et al. 1995a). Condition (2.15) is the single orientation approximation. With this condition the component of the current density vector along the $z$ direction is estimated by

$$J \varepsilon e^{j\varphi} = 2\left( \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} \right) + 2j \left( \frac{\partial \tilde{H}_z}{\partial x} + \frac{\partial \tilde{H}_y}{\partial y} \right). \quad (2.16)$$

Thus we see that, given the complete magnetic field, $H$, the complete current density vector field, $J$, can be computed. Secondly, given the LCP transverse components of an $H$, for which (2.15) holds, the longitudinal component of $J$ can be evaluated.

### 2.3 Measurements of Rotating Frame Magnetic Components

We now discuss how physical measurements of $H$ can be obtained using magnetic resonance imaging\(^2\) (MRI). The applied Larmor frequency current produces a magnetic field, of which the LCP component acts like a RF pulse in MRI. Ideally, the effects of the LCP component on the magnetization can be seen, in a rotating frame at the Larmor frequency, as a rotation around the LCP component by an angle proportional to both its

\(^2\) Some relevant concepts and terminologies of MRI are listed in Appendix A.
amplitude and duration. However, the inhomogeneity of the static magnetic field and the influence of $T_1$ and $T_2$ relaxation complicate the problem. There have been two major RF-CDI imaging methods developed to measure the LCP magnetic component, the polar decomposition imaging method (Scott et al. 1992a) and the rotary echo imaging method (also as known as the rotating frame imaging) (Scott et al. 1995b). From the MRI perspective, it is more convenient to discuss the magnetic flux density, $\mathbf{B}$, instead of the magnetic intensity $\mathbf{H}$. In tissues, $\mathbf{B} = \mu_0 \mathbf{H}$, where the constant $\mu_0$ is the permeability in free space. We denote the LCP component of the magnetic flux density as $(\tilde{B}_x, \tilde{B}_y)$.

2.3.1 The Polar Decomposition Imaging Method

The goal of the polar decomposition method is to measure two RF magnetic components and one DC magnetic component. In addition to $(\tilde{B}_x, \tilde{B}_y)$, the method is capable to determine the inhomogeneity of the static magnetic field $\Delta B_0$. When the current is applied, the total effective magnetic field is $\tilde{B} = (\tilde{B}_x, \tilde{B}_y, \Delta B_0)$ and consequently the magnetization will rotate around the axis defined by this field accompanied by the distortion due to the relaxations. The essence of the polar decomposition imaging method is to compute the rotation matrix given a sufficient number of pairs of the known initial and the measured final magnetization using polar decomposition (Onishchik et al. 2002). The computation is based on the singular value decomposition of the measured matrix of final magnetization vectors (Scott et al. 1992a). Then, the direction and the magnitude of $\tilde{B}$ can be determined from the rotation axis and the rotation angle, respectively.
In practice, the magnetization $M_0$ is prepared along the $\hat{x}$, the $\hat{y}$ and the $\hat{z}$ axes respectively for three initial starting points. After each preparation, the same amount of RF current is applied to the object, so that the various initial magnetizations rotate around the same axis by the same angle. Then the components of the final magnetization after the rotation are measured by MRI. The initial magnetizations can be expressed as column vectors and combined in a matrix $C$ as

$$C = M_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  \hfill (2.17)

Similarly, the final magnetizations after the current being applied can be denoted in a matrix $D$ as

$$D = \begin{bmatrix} M_x^z & M_y^z & M_z^z \\ M_x^y & M_y^y & M_z^y \\ M_x^x & M_y^x & M_z^x \end{bmatrix},$$  \hfill (2.18)

where the superscript denotes the axis along which the magnetization has been initially prepared and the subscript denotes the measured magnetization component. If

$$\gamma |\vec{B}| >> T_2^{-1},$$  \hfill (2.19)

the transformation from the initial magnetization to the final magnetization can be approximated in terms of a rotation plus a deformation (Scott et al. 1992a). In other words, $D$ can be decomposed as

$$D = RS = SR,$$  \hfill (2.20)
where \( \mathbf{R} \) is the rotation matrix and the deformation part is described by \( \mathbf{S} \). \( \mathbf{C} \) is absorbed in \( \mathbf{S} \). The polar decomposition theorem is utilized to determine the rotation matrix \( \mathbf{R} \).

In principle, the axis of the rotation gives the direction of \( \tilde{\mathbf{B}} \) and the rotation angle is proportional to the magnitude of \( \tilde{\mathbf{B}} \). A phase unwrapping process is required because the rotation angle extracted from \( \tilde{\mathbf{B}} \) is the principal angle that lies only in the range of \( -\pi \) to \( \pi \). Furthermore, the rotation axis can not be uniquely determined from \( \mathbf{R} \) either, because the reversed axis and the supplementary rotation angle will result in the same final position. This gives rise to another unwrapping problem, which is interleaved with the first phase unwrapping procedure. This unique unwrapping problem remained the major obstacle for the implementation of the polar decomposition imaging method until recently a new approach has been proposed to solve the problem (Ma et al. 2008).

2.3.2 The Rotary Echo Imaging Method

The rotary echo imaging method used in this thesis was developed by Scott (Scott et al. 1995b) to measure \( (\tilde{B}_x, \tilde{B}_y) \). The essential feature of this method is a large RF pulse called the rotary echo \( \mathbf{B}_1 \) pulse that is applied simultaneously with the current to constrain the rotation axis of the magnetization. For instance, to measure \( \tilde{B}_x \), we apply \( \mathbf{B}_1 \) in the direction of the \( \tilde{x} \) axis. The effective magnetic field in the rotating frame is \( (\tilde{B}_x + B_1, \tilde{B}_y, \Delta B_0) \). When \( B_1 \gg \tilde{B}_y, \Delta B_0 \), the effects of the magnetic components \( \tilde{B}_y \) and \( \Delta B_0 \), which are perpendicular to \( \mathbf{B}_1 \), can be neglected relative to \( \mathbf{B}_1 \). To a good approximation, the magnetization rotates around the \( \tilde{x} \) axis in the \( \tilde{y}\tilde{z} \) plane with the flip
angle proportional to $B_1 + \tilde{B}_x$ (Figure 2.2 (a)). In order to cancel out $B_1$ effect in the final twist angle $\Gamma_x$, the direction of $B_1$ is changed to $-\tilde{x}$ in the middle of the RF current pulse duration. Therefore, $\Gamma_x$ is proportional to the strength of $\tilde{B}_x$ (Figure 2.2 (b)).

Figure 2.2 Schematic illustration for the single-slice rotary echo RF-CDI method to show how $\tilde{B}_x$ is measured.

(a) The effect of $B_1 + \tilde{B}_x$ is to rotate the magnetization $\textbf{M}$ in the $\tilde{y}\tilde{z}$ plane
(b) $B_1$ reverses its direction in the middle the current duration. The total effect of $B_1 - \tilde{B}_x$ is to rotate $\textbf{M}$ back towards the $\tilde{z}$ axis. The final angle between $\textbf{M}$ and $\tilde{z}$, $\Gamma_x$, is proportional to $\tilde{B}_x$.
(c) A 90° pulse rotates the $\tilde{y}\tilde{z}$ plane to the transverse plane in order to measure $\Gamma_x$. 
The $\bar{y} \bar{z}$ plane can then be rotated to the transverse plane to measure the angle $\Gamma_y$ which is encoded in the collected MRI phase images (Figure 2.2 (c)). This approach, namely, the plane rotation approach, is employed in the single-slice rotary echo RF-CDI sequence (Scott et al. 1995b). Alternatively, the $\bar{y}$ and $\bar{z}$ components of the final magnetization in Figure 2.2 (b) can be measured separately. This is the basic measurement scheme for the multi-slice RF-CDI sequence that will be discussed in detail in Chapter 3.

2.4 Discussion

It might be an advantage of RF-CDI that with only one sample position, one component of current density can be reconstructed, if the single orientation assumption (2.15) is satisfied. In pervious phantom implementations, the geometry of the phantoms and the positions of the electrodes and return wires were deliberately designed so that the dominant direction the current flow is parallel to the $z$ axis to guarantee the condition (2.15). However, it is not easy to apply the same strategy in biomedical applications. Moreover, it is difficult to check the validity of assumption (2.15) given only the measured LCP magnetic field components. Consequently, the interpretation of the experimental results will be complicated.

The two RF-CDI imaging methods both have advantages and limitations. The polar decomposition method is an attractive mathematical tool to recover the LCP magnetic components and the inhomogeneity of the static magnetic field as well. The polar decomposition imaging method, however, has not been widely utilized for RF-CDI due to the difficulty arising from the unwrapping postprocessing described in section 2.3.1. In
previous implementations, the rotation angle was constrained in the range of $-2\pi$ to 0, and thus the product of the amplitude and the duration of the RF current could not exceed an upper limit. Therefore, this method is very sensitive to noise. Additionally, the amplitude of the RF current has a lower limit to satisfy condition (2.19). In practice, short and high current pulses were applied in previous implementations.

The rotary echo imaging approach of RF-CDI measurement employs a rotary echo $B_1$ field, whose magnitude must be much larger than that produced by the applied current. This requirement effectively limits the magnitude of the current that can be applied. Furthermore, there are nonlinear errors associated with the finite magnitude of the rotary echo pulse (Scott et al. 1995b). Nevertheless, the rotary echo imaging method exhibits higher dynamic range, and hence is less sensitive to noise. Its practical implementation requires fewer data acquisitions than the polar decomposition method. Previous implementations of both RF-CDI imaging methods were limited to image one slice at one scan. Chapter 3 presents the first realization of a multi-slice RF-CDI technique based on the rotary echo imaging method.
Chapter 3

Multi-slice RF-CDI Measurement

Previously implemented RF-CDI techniques (Scott et al. 1992a; Scott et al. 1995b) are only able to map the LCP components of the magnetic field yielded by the RF current in a single slice after each RF current pulse. This chapter presents the first implementation of the RF-CDI imaging technique that is capable of collecting data of the LCP magnetic components in multiple slices after each RF current pulse to speed up the measurement. In this chapter, we use the reconstruction method based on the single orientation approximation (2.15) to reconstruct one component of the RF current density. We will show later in Chapters 4 and 5 that the multi-slice RF-CDI imaging method can also be used for reconstruction methods not based on (2.15).

3.1 Characteristics of the Pulse Sequence

The multi-slice RF-CDI sequence is based on the Z projection approach proposed by Scott, G.C. (Scott 1993). This sequence belongs to the rotary echo imaging method discussed in the previous chapter in section 2.2.2. To emphasize, Figure 3.1 (a)-(c) demonstrates the concept again with an example in which the rotary echo field $B_1$ is along the $x$ direction in the rotating frame to estimate the LCP component $\tilde{B}_x$ produced by the applied current. By inverting the direction of $B_1$ in the middle of the RF current pulse, the final flip angle $\Gamma_x$ is proportional to the strength of $\tilde{B}_x$.

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1 Contents of this chapter have been published in (Wang et al. 2009). © 2009 IEEE. Reprinted, with permission, from IEEE Transactions on Medical Imaging, "Multislice radio-frequency current density imaging" by D. Wang, T.P. Demonte, W. Ma, M. L. G. Joy, and A. I. Nachman.
To recover $\Gamma_x$, the plane rotation approach (Scott et al. 1995b) rotates the $\hat{y}\hat{z}$ plane to the transverse plane, spoils the out of slice magnetization, and then rephases and measures the phase of the in slice magnetization. The time window that can be used for the measurement is thus constrained by the $T_2$ relaxation and hence data from only one slice are collected after one current pulse. Figure 3.1 illustrates how the Z projection sequence recovers $\Gamma_x$ in multiple slices by separately measuring the $\hat{y}$ and the $\hat{z}$ components of the final $M$. These are represented by $M^x_y$ and $M^x_z$. The superscript $x$ here denotes the
magnetic field component to be evaluated from $\mathbf{M}$ and the subscript indicates the component of the final $\mathbf{M}$ after rotation about the $\tilde{x}$ axis. As shown in Figure 3.1 (d), $\mathbf{M}_x^t$ is stored in the longitudinal direction for future measurement by a nonselective RF pulse. The remaining magnetization in the transverse plane is then spoiled (Figure 3.1 (e)). Finally (Figure 3.1 (f)), a series of selective $90^\circ$ pulses rotate the $\mathbf{M}_y^t$ component into the transverse plane, slice by slice, and data are collected. $\mathbf{M}_z^t$ is measured in a similar fashion after another current pulse. By these means, more slices can be imaged.

![Figure 3.2](image.png)

Figure 3.2 The multi-slice RF-CDI pulse sequence diagram. $B_1$ is the rotary echo pulse. The P pulse represents a set of hard pulses. A gradient recalled echo multi-slice acquisition is employed to collect the data. The spoilers, $S_1$ and $S_2$, dephase the magnetization component remaining in the transverse plane, and the spoilers $S_3$~$S_6$ dephase any remaining transverse magnetization from the prior readout. After the readout of the last slice, spoilers $S_7$ and $S_8$ dephase any remaining transverse magnetization.
because the time window for measurement is limited by the longer $T_1$ relaxation rather than the $T_2$ relaxation.

The pulse sequence diagram is shown in Figure 3.2. The rotary echo pulse $B_1$ and the $P$ pulse cycle according to the magnetization components to be measured. To cancel the error of longitudinal relaxation, the magnetic components are stored along alternating $\pm z$ axes. The phase cycling steps are shown in Table 3.1.

Table 3.1 Phase cycling of the multi-slice RF-CDI sequence

<table>
<thead>
<tr>
<th>Cycling Number</th>
<th>Component Estimated</th>
<th>Magnetization Measured</th>
<th>B$_1$ Phase</th>
<th>P Pulse</th>
<th>P Phase</th>
<th>Complex Image</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\tilde{H}_x$</td>
<td>$M_y$</td>
<td>0 ($\tilde{x}$)</td>
<td>90</td>
<td>180 ($-\tilde{x}$)</td>
<td>$C_{y+}$</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{H}_x$</td>
<td>$M_y$</td>
<td>0 ($\tilde{x}$)</td>
<td>90</td>
<td>0 ($\tilde{x}$)</td>
<td>$C_{y-}$</td>
</tr>
<tr>
<td>3</td>
<td>$\tilde{H}_y$</td>
<td>$M_x$</td>
<td>-90 ($\tilde{y}$)</td>
<td>90</td>
<td>90 ($-\tilde{y}$)</td>
<td>$C_{x+}$</td>
</tr>
<tr>
<td>4</td>
<td>$\tilde{H}_y$</td>
<td>$M_x$</td>
<td>-90 ($\tilde{y}$)</td>
<td>90</td>
<td>-90 ($\tilde{y}$)</td>
<td>$C_{x-}$</td>
</tr>
<tr>
<td>5</td>
<td>$\tilde{H}_z$</td>
<td>$M_z$</td>
<td>0 ($\tilde{x}$)</td>
<td>~</td>
<td>~</td>
<td>$C_{z+}$</td>
</tr>
<tr>
<td>6</td>
<td>$\tilde{H}_z$</td>
<td>$M_z$</td>
<td>0 ($\tilde{x}$)</td>
<td>180</td>
<td>0 ($\tilde{x}$)</td>
<td>$C_{z-}$</td>
</tr>
<tr>
<td>7</td>
<td>$\tilde{H}_z$</td>
<td>$M_z$</td>
<td>-90 ($\tilde{y}$)</td>
<td>~</td>
<td>~</td>
<td>$C_{z+}$</td>
</tr>
<tr>
<td>8</td>
<td>$\tilde{H}_z$</td>
<td>$M_z$</td>
<td>-90 ($\tilde{y}$)</td>
<td>180</td>
<td>0 ($\tilde{x}$)</td>
<td>$C_{z-}$</td>
</tr>
</tbody>
</table>

3.2 Postprocessing

There are altogether eight phase cycles in the sequence (Table 3.1). Four complex images, $C_{y+}$, $C_{y-}$, $C_{x+}$ and $C_{x-}$, are acquired to estimate the rotating frame component $\tilde{H}_x$, and the other four are for $\tilde{H}_y$. The superscript in $C_{y+}$ indicates that the rotating
field component being measured is $\tilde{H}_x$, while the subscript denotes that the magnetization component $M_x$ is stored in the $+z$ axis. Other complex variables are defined likewise.

From these images, $\tilde{H}_x$ is derived from the measured twist angle $\Gamma_x$ by

$$
\tilde{H}_x = \frac{-\Gamma_x}{\gamma T,\mu_0},
$$

(3.1)

where $\gamma$ is the gyromagnetic ratio and $T_i$ is the duration of the current. $\Gamma_x$ is computed through arc tangent (\tan^{-1}) or arc cotangent (\cot^{-1}) operations

$$
\Gamma_x = \begin{cases}
\tan^{-1}\left[\text{Re}(C_x^y / C_x^z)\right] & \text{if } |M_x^z| \geq |M_x^z| \\
\cot^{-1}\left[\text{Re}(C_x^y / C_x^z)\right] & \text{if } |M_x^z| < |M_x^z|
\end{cases}
$$

(3.2)

where

$$
\begin{align*}
C_x^y &= C_{x+} - C_{x-} \\
C_x^z &= C_{x+} - C_{x-}
\end{align*}
$$

(3.3)

Next, the reconstructed rotation angle $\Gamma_x$ is unwrapped by a quality-guided two-dimensional phase unwrapping (Ghiglia et al.1998). Now, let us consider the first equation in (3.3) only, because the second one can be treated in the same way. Disregarding noise, within the discussion of this chapter, it is sufficient to write $C_{y+}^x$ and $C_{y-}^x$ as

$$
C_{y+}^x = M_{y+}^x e^{i\phi_R}
$$

(3.4)

$$
C_{y-}^x = M_{y-}^x e^{i\phi_R}
$$

(3.5)

where $\phi_R$ is the systematic phase error. Let us define time $t = 0$ as the moment immediately after the rotary echo $B_1$ and the RF current. $M_{y+}^x(0)$ and $M_{y-}^x(0)$ are the
values of the magnetization component along the \( \hat{y} \) axis and the \( z \) axis at \( t = 0 \), respectively. Then,

\[
M_{y}^{x} = M_{y}^{x}(t) = \left( M^{0} (1 - e^{-t/T_{1}}) + M_{y}^{x}(0) e^{-t/T_{1}} \right) e^{-t/T_{2}^*}
\]

(3.6)

and

\[
M_{y}^{x} = M_{y}^{x}(t) = \left( M^{0} (1 - e^{-t/T_{1}}) - M_{y}^{x}(0) e^{-t/T_{1}} \right) e^{-t/T_{2}^*}.
\]

(3.7)

In these equations, \( T_{1} \) and \( T_{2}^* \) are the longitudinal relaxation constant and the effective transverse relaxation constant, respectively. \( t \) is the time between the event that \( M_{y}^{x} \) is stored in the \( \pm z \) axis and the event that it is rotated to the transverse plane. Also, \( M^{0} \) is the equilibrium value of magnetization and \( T_{E} \) is the echo time for MRI data acquisition. \( M_{z}^{x} \) and \( M_{z}^{x} \) can be expressed similarly as (3.6) and (3.7) and hence \( C_{z}^{x} \) is calculated. \( \tilde{H}_{y} \) is estimated in the same way.

After the rotating frame components \( \tilde{H}_{x} \) and \( \tilde{H}_{y} \) are calculated, current density can be computed according the RF-CDI reconstruction method. In this chapter, the multi-slice RF current density in the \( z \) direction is estimated using the single orientation reconstruction (2.16).

### 3.3 Sensitivity Analysis

The thermal noise of MRI defines a fundamental limit of the sensitivity of CDI measurements. Therefore, investigating the effect of the thermal noise effect on CDI is crucial in interpretation of the measurements. A general equation for current density
noise was derived for LF-CDI and single-slice RF-CDI based on the fact that the current
density information is encoded in the MRI phase images used in these techniques (Scott
1993; Scott et al. 1995b). However, this formula cannot apply to multi-slice RF-CDI
directly because the multi-slice sequence does not extract magnetic field components
from phase images only. Therefore, the current density noise for the multi-slice RF-CDI
is derived herein from a different point of view.

Write the MR complex images in real and imaginary form as

\[ C = R + iI. \]  \hspace{1cm} (3.8)

For different images acquired in different phase cycles, \( C, R \) and \( I \) will have subscripts
and superscripts the same as in (3.3). Now add independent Gaussian random noise with
zero mean and variance \( \sigma^2 \) to the real and the imaginary parts and denote the random
noise as \( \delta R \) and \( \delta I \). The measured twist angle \( \Gamma_+ + \delta \Gamma_+ \) can then be expanded in a
Taylor series around the true value \( \Gamma_+ \) in terms of \( R_{xy}^+, I_{xy}^+, R_{yz}^+ \) and so on. With large MR
signal to noise ratio (SNR), truncating Taylor series up to the linear term is a good
approximation for \( \delta \Gamma_+ \) (Haacke et al. 1999). Using this approach and combining (3.3)-(3.7),
from the first equation of (3.2), one can obtain

\[
\delta \Gamma_+ = \frac{A}{|C_z|^2} \left( \text{Re}(C_z^+)(\delta R_{y+}^x - \delta R_{y-}^x) + \text{Im}(C_z^+)(\delta I_{y+}^x - \delta I_{y-}^x) \right)
+ \frac{A}{|C_z^+|^2} \left( (2B \text{Re}(C_z^+)-\text{Re}(C_y^+))(\delta R_{yz}^+ - \delta R_{yz}^-) \right), \hspace{1cm} (3.9)
+ \frac{A}{|C_z^-|^2} \left( (2B \text{Im}(C_z^-)-\text{Im}(C_y^-))(\delta I_{yz}^+ - \delta I_{yz}^-) \right)
\]
where \( B = \frac{M^x_y(0)}{M^z_y(0)} \) and \( A = \frac{1}{1 + B^2} \).

Therefore, the mean of \( \partial \Gamma_x \) is

\[
\overline{\partial \Gamma_x} = 0, \quad (3.10)
\]

and the variance of \( \partial \Gamma_x \) is given by

\[
\overline{\partial \Gamma_x^2} = \frac{e^{2\mu T_i + T_e/T_z^*}}{2\left(M^x_y(0)^2 + M^z_y(0)^2\right)} \sigma^2. \quad (3.11)
\]

Notice that \( M^x_y(0)^2 + M^z_y(0)^2 = |M|^2 \), in which \( |M| \) is the magnetization at time \( t = 0 \).

If we define the reference image as the MR image with no current applied and the magnetization turned to the transverse plane at time 0, then \( |M|e^{-T_e/T_z^*} \) is the signal of the reference image magnitude. Define \( SNR = |M|e^{-T_e/T_z^*}/\sigma \) as the SNR of the reference image. Then, the standard deviation of \( \Gamma_x \) is

\[
\sigma_{\Gamma_x} = \frac{e^{\mu T_i}}{\sqrt{2}SNR}. \quad (3.12)
\]

\( \sigma_{\Gamma_y} \) is the same as \( \sigma_{\Gamma_x} \). Due to (3.1), \( \sigma_{\Gamma_x} \) and \( \sigma_{\Gamma_y} \) this can be converted to the standard deviation of \( \tilde{H}_x \) and \( \tilde{H}_y \) as

\[
\sigma_{\tilde{H}_x} = \frac{e^{\mu T_i}}{\sqrt{2\gamma T_c \mu_0 SNR}}. \quad (3.13)
\]

Using the reconstruction method (2.16), for multi-slice RF-CDI, the standard deviation of the current density \( J_z \) in either the real or the imaginary part is
where $F_x$ and $F_y$ are the derivative noise template weighting factors, while $\Delta x$ and $\Delta y$ are the pixel dimensions. For the cases that use $3 \times 3$ Sobel templates in the postprocessing, $F_x = F_y = \sqrt{3}/4$ (Scott et al. 1992b). The value of $t$ in equation (3.14) is different for each slice. It increases as the slice acquisition number increases. Therefore, equation (3.14) predicts that the current noise grows exponentially depending on $T_1$ as the slice acquisition number increases. Please note the similarity between (3.14) and equation (21) in (Scott et al. 1995c). In fact, if we define $SNR = |M| e^{-T_e/\gamma} / \sigma$, then (3.14) is consistent with equation (21) in (Scott et al. 1995b).

Sometimes, it is more convenient to evaluate the noise in the magnitude and phase of $J_z$. Similarly to the statistical relationships between real, imaginary and magnitude, phase images in MRI, when the SNR of the current density is high, the standard deviation of noise evaluated in current density magnitude is approximately the standard deviation of current density noise evaluated in real or imaginary parts of current density, and the standard deviation of current phase noise is close to the reciprocal of current density SNR in radians (Scott et al. 1995b). The detailed explanation follows.

Let $J_R$, $J_I$ be the real part and the imaginary part of $J_z$, and $j_z$, $\phi_z$ be the magnitude and phase of $J_z$ in noiseless case, then,

$$j_z = \left( J_R^2 + J_I^2 \right)^{1/2}$$

(3.15)
and

\[
\tan \phi_z = \frac{J_L}{J_R}. \tag{3.16}
\]

Define \( \delta J_R \) and \( \delta J_I \) as the random noise in \( J_R \) and \( J_I \) and \( \delta j_z \) the noise in \( j_z \). When \( \delta j_z \ll j_z \), from (3.15) we obtain

\[
j_z + \delta j_z = \left( (J_R + \delta J_R)^2 + (J_I + \delta J_I)^2 \right)^{1/2} \approx j_z + \frac{J_R \delta J_R + J_I \delta J_I}{j_z}. \tag{3.17}
\]

Because of (3.10), \( \overline{\delta J_R} = \overline{\delta J_I} = 0 \). Therefore, \( \overline{\delta j_z} = 0 \). If \( \delta J_R \) and \( \delta J_I \) are uncorrelated, according to (3.17),

\[
\sigma_{j_z} = \left( \text{var}(\delta j_z) \right)^{1/2} = \sigma_j. \tag{3.18}
\]

Using the similar approach, differentiate (3.16), we have

\[
\delta \phi_z \sec^2 \phi_z = \frac{\delta J_I}{J_R} - \delta J_R \frac{J_I}{J_R^2}, \tag{3.19}
\]

where \( \delta \phi_z \) is defined as noise in \( \phi_z \). Hence, \( \overline{\delta \phi_z} = 0 \) and the standard deviation of \( \delta \phi_z \) is

\[
\sigma_\phi = \left( \overline{\delta \phi_z^2} \right)^{1/2} = \left( \left( \frac{J_I}{J_R} \right)^2 \left( \overline{\delta J_R^2} \frac{J_I}{J_R^2} + \overline{\delta J_I^2} \right) \cos^4 \phi_z \right)^{1/2} = \frac{\sigma_J}{j_z}. \tag{3.20}
\]

### 3.4 Experiment and Simulation Testing

#### 3.4.1 Experimental Methods

The experiments were performed with a clinical 1.5 Tesla GE® SIGNA LX MR system. A head coil was used for both transmitter and receiver. The block diagram in Figure 3.3 illustrates the hardware and pulse sequence connections in multi-slice RF-CDI
experiments. By running the multi-slice RF-CDI sequence, the MRI system provides signals such as the trigger signals and signals related to the RF pulses, which are used as input to the RF current control box. Two types of RF current control boxes have been employed according to different versions of MR imager (Carter 1995; Scott et al. 2003). The RF current control box produces the blanking signal and RF signals with the desired frequency and duration, which are in turn used as input to the amplifier to obtain the RF current. The role of the matching network is to tune the impedance of the load to the characteristic impedance of the cable to maximize the power delivered to the phantom. In order to prevent any damage to the RF amplifier caused by the reflected power (Yoon, 2003), an isolator (ferrite circulator with -80 dB isolation, EMR Corp., Phoenix, USA) was placed between the amplifier and the matching network.

Figure 3.3 Block diagram of experimental setup for multi-slice RF-CDI. The current control box and the matching network are custom built. The RF amplifier is AMT 3206.
Two cylindrical phantoms as shown in Figure 3.4 were used to test the multi-slice RF-CDI sequence. One was a single-chamber phantom with diameter 38 mm and height 110 mm. The phantom was filled with doped saline, which was made up of 0.154 M (0.9 g/100 ml) NaCl, and 4 mM (0.1 g/100 ml) or 8 mM (0.2 g/100 ml) CuSO₄·5H₂O of distilled water. Conductivity was \( \sigma \approx 1.48 \text{ S/m} \) and relative permittivity \( \varepsilon_r \approx 78.36 \). \( T_1 \) was approximately 340 ms and 160 ms, respectively. The other phantom was multi-chambered. It was composed of three plastic tubes with a diameter of 86 mm and height of 150 mm. The center chamber was filled with solution with 0.9 g/100 ml NaCl and 0.1 g/100 ml CuSO₄·5H₂O giving \( \sigma \approx 1.48 \text{ S/m} \) and \( \varepsilon_r \approx 78.36 \). The outer two chambers were filled with solution of same concentration of CuSO₄·5H₂O but without NaCl, which resulted in \( \sigma \approx 0.036 \text{ S/m} \) and \( \varepsilon_r \approx 78.36 \). \( T_1 \) was about 340 ms in all three chambers. Each phantom was placed in the centre area of the coil with its longitudinal axis parallel to static magnetic field \( \vec{B}_0 \) to satisfy the single orientation condition (2.15).

Multi-slice RF-CDI experiments were performed to reconstruct current density images for both axial and sagittal slices. The sequence was also tested on the single-chamber phantom without RF current source connected to evaluate the effect of random noise from the MR system by computing the spatial mean and standard deviation of current density. Since \( 3 \times 3 \) Sobel templates were used to calculate the spatial derivatives, every third pixel in one plane was included in the estimation to avoid statistical dependence (Yoon 2003).
Figure 3.4 Experimental Phantoms. Upper: illustration of the cross section of the single-chamber phantom (left) and the multi-chamber phantom (right). Lower: illustration of the numerical model for the single –chamber phantom (left) and the multi-chamber (right).
3.4.2 Simulation Methods

Numerical computation was employed to facilitate predicting and analyzing the experimental results and to verify the current noise formula (3.14). The electromagnetic fields produced by RF current for both phantoms were simulated by 3-D FDTD (Taflove et al. 2000). The spatial resolutions for the simulation were $\Delta x = \Delta y = 0.9375\text{mm}$ and $\Delta z = 2\text{mm}$ for the multi-chamber phantom and $\Delta x = \Delta y = \Delta z = 2\text{mm}$ for the single-chamber phantom. Mur’s second order absorbing boundary conditions were implemented on the boundaries to prevent reflections (Taflove et al. 2000). The electrodes and wires were represented by setting all tangential components of the electric field to zero. The acrylic containers were given the values $\sigma = 0\text{S/m}$ and $\varepsilon_r = 2.66$. In each simulation, a sinusoidal current at frequency 64 MHz was placed in one cell along the return wires to approximate the applied current at the Larmor frequency for a 1.5 T MR imager in time domain. The simulation models are illustrated in Figure 3.4. The steady-state fields in time domain from FDTD were then converted to phasor form by discrete Fourier transform. The rotating frame magnetic components were calculated from the transverse magnetic field and current density $J_z$ was computed by (2.16). In order to compare the simulated and experimental results of the multi-chamber phantom, the simulation results were normalized such that the means of the magnitude and the phase of $J_z$ in the centre chamber of the centre slice are the same between simulation and experiment.
In order to estimate the effects of MR random noise, eight MR images were simulated for each slice of the single-chamber phantom based on the calculated rotating frame components. The magnitude of the reference MR image was normalized to unity and zero mean independent normally distributed noise with the standard deviation $1/\text{SNR}$ was added to both real and imaginary parts of each complex image. After current density was reconstructed according to (2.16), it was compared to the noiseless current density. With the differences of the two sets of images, current noise mean and standard deviation were evaluated (Wang et al. 2006).

### 3.4.3 Results

In the experiment with the multi-chamber phantom, a rubber tube filled with doped saline was attached on the outmost wall of the phantom spirally as a position mark for different slices. Figure 3.5 illustrates results for slice 1, slice 3 and slice 5 out of a total of five axial slices. The white arrows in Figure 3.5 (a) indicate the positions of the transections of the externally placed tube. Noiseless current density images of five axial slices at the centre were simulated for the multi-chamber phantom. The reconstructed current density $J_z$ according to (2.16) is very close to $J_z$ calculated directly from FDTD simulation and its distributions in the five slices are very similar. Figure 3.6 shows magnitude and phase images of the reconstructed current density of the center slice.

The spatial mean and standard deviation of each compartment were estimated for the five slices from experiment. The results are shown in Figure 3.7. As a comparison, MRI SNR was estimated for each chamber of every slice from acquired MR images. The corresponding random noises were added to the simulated data by means described in
Figure 3.5  Experimental results for the multi-chamber phantom. The images are displayed from left to right for slice 1, slice 3 and slice 5. The imaging parameters are: $\text{FOV} = 0.24 \text{ m}$, $\text{frequency samples} = 256$, $\text{phase encoding steps} = 256$, $\text{slice thickness} = 5.0 \text{ mm}$, $\text{slice gap} = 5.0 \text{ mm}$, $T_e = 10 \text{ ms}$, $T_p \approx 500 \text{ ms}$, $T_c = 8 \text{ ms}$, $B_1 = 23 \mu \text{T}$. (a) MR magnitude image for one phase cycle (cycle 4). (b) Unwrapped twist angle for $\vec{H}_y$ in degrees. (c) Reconstructed current density magnitude in logarithm plot (the cross indicates the direction of the current). (d) Reconstructed current density phase in degrees.
Figure 3.6. RF-CDI simulation results for the multi-chamber phantom at the centre slice. Left: magnitude of current density $J_z$ in logarithm scale. Right: phase of current density $J_z$ in degrees.

Figure 3.7 Comparison of the experimental results and the simulated results for the multi-chamber phantom. Dots indicate the average values and the bars correspond to the standard deviations for each chamber (the inner, the middle and the outer ones). (a) Statistical estimation of magnitude of $J_z$ for experimental data (left) and simulated data (right). (b) Statistical estimation of $J_z$ phase for experiment data (left) and simulated data (right).
section 3.4.2. The spatial mean and standard deviation of current magnitude and phase are also shown in Figure 3.7. We can see that the results from the experiment and the simulation exhibit similar patterns.

It is difficult for single-slice RF-CDI to image the RF current density, $J_z$, at points on any plane that is not perpendicular to $\vec{B}_0$, because the magnetic components on adjacent slices are also needed for computing spatial derivatives in (2.16). Another advantage of multi-slice RF-CDI is that it can also image current density in slices parallel to $\vec{B}_0$, such as sagittal or coronal slices. To illustrate this, five sagittal slices were imaged with the single-chamber phantom in order to obtain the derivative information for computing the RF current density for the three inner slices. Figure 3.8 illustrates the results for the three sagittal slices. The corresponding simulation shows uniform distribution of the magnitude and phase of $J_z$ in the slices. In Figure 3.8, the signal loss close to the electrodes is caused by the high conductivity of the copper electrodes.

Standard deviations of current density magnitude and phase noise were evaluated by simulation at current density levels 25, 50 and 100 A/m$^2$ for 7 slices for the single-chamber phantom. $T_1 = 160$ ms and MRI SNR is 100 for all three cases. The results are shown in Figure 3.9 (a). We can see that the current noise evaluated from the simulations matches the predicted $\sigma_J$ (solid line) and $\sigma_\phi$ (dash-dot line) from (3.14), (3.18) and (3.20) very well. Current noise increases exponentially as the slice number increases.
The single-chamber cylindrical phantom was placed inside the imager with no cables, matching network or RF current source connected to exclude the inherent nonlinear errors in the rotating frame method and the noise from the current source. Therefore, the reconstructed current density images were expected to reflect mainly the influence of MRI random noise. Figure 3.9 (b) shows the results of estimated current density noise from seven slices. $T_1$ was about 160 ms for these experiments. Standard deviation was evaluated in the real part of the reconstructed current density. Current density noise increases with respect to the slice acquisition order, and the increasing trend is close to the predicted exponential curves. Because of different imaging parameters, MRI SNR was around 112 for one experiment and 28 for the other two. The values of $\Delta x$ and $\Delta y$ in the first one are double the sizes of the other two. Therefore, as predicted, the current noise increases by a factor of 8 in the latter case.
Figure 3.9 Noise evaluation from simulation and experiments. The horizontal axis indicates the slice number in acquisition order, while the current density magnitude noise and phase noise are plotted in log scales. (a) Noise estimations for current density magnitude and phase at different current density level from simulation. (b) Experimental current noise evaluation with zero current applied to the phantom.
3.5 Discussion

As predicted by (3.14), current density noise increases as $t$ (or slice acquisition number) increases if other parameters remain the same for each slice. However, this effect is not shown in Figure 3.7 for two reasons. First, since the two copper electrodes are attached to the inner two chambers of the phantom, the MR signal in the centre cylinder is lower than in the other two chambers. Furthermore, the MR signal in the center cylinder in slices close to either electrode is much lower than that in the slices further away from the electrodes (Figure 3.5 (a)). Hence the MR SNR is lower in these regions. Second, $T_1 (340 \text{ ms})$ is much longer than the time required to acquire one line in k-space for all slices after the P pulse.

Generally, the degree of spatial nonuniformity does not reflect only the random noise effects in RF-CDI experiments even if the true current density distribution is approximately uniform. Other artifacts can contribute to the spatial nonuniformities. The most likely contributing artifact is the nonlinear errors associated with finite rotary echo $B_1$ magnitude. This artifact is well discussed and demonstrated in (Scott et al 1995b). To evaluate the random current density noise alone, a more practical alternative is to calculate the standard deviations of current density for the same pixel from a set of RF-CDI experiments with the same amount of current applied (Yoon 2003).

The sensitivity to MR random noise of each slice is dependent on the longitudinal relaxation time constant $T_1$ of the material to be imaged. Therefore, in biomedical applications, the relaxation rates of tissues to be imaged will constrain the number of
slices that can be imaged. Typically, the $T_1$ relaxation constant lies into the range of 200 to 1000 ms at 1.5 T (Stark et al 1988). Assume that the criterion for multi-slice imaging is that the current density noise for the last acquired slice is no more than twice that of the first acquired slice and the duration between data acquisition between two slices is 20 ms. As a rough estimation, for most biological tissues that are commonly imaged by MRI, multi-slice RF-CDI will be able to image around 7 to 35 slices.

Given current duration $T_1$, MRI SNR and other imaging parameters, the lower limit of RF current density noise for multi-slice RF-CDI will approach that of the single-slice RF-CDI sequence using four phase cycles. Since eight phase cycles are used in the multi-slice sequence, the imaging time for the multi-slice RF-CDI is about double that of the single-slice RF-CDI to image one slice. If more slices are required, the imaging time for the single-slice RF-CDI sequence will be multiplied by the number of the slices while the time needed for the multi-slice RF-CDI sequence will remain the same. For example, in order to image five slices, the ratio of the total imaging time of the single-slice RF-CDI sequence to that of the multi-slice RF-CDI sequence will be 5/2. Therefore, the single-slice sequence is faster when only one slice is required and the multi-slice sequence is faster when three or more slices are required.
Chapter 4

RF-CDI Reconstruction with a Single 180-degree Rotation¹

Up to this point, RF current density has been reconstructed using equation (2.16), which is based on the single orientation approximation (2.15). This chapter presents a new reconstruction method that can fully recover the RF current density component along the static magnetic field without the constraint imposed by (2.15). In the next chapter, this method will be extended to measure the full RF current density vector field.

4.1 Validity of the Single Orientation Approximation

Since the single orientation approximation (2.15) is critical for the validity of the previous RF-CDI reconstruction method, we start by examining the robustness of this condition and what causes the violation of this condition. First, let us examine the validity of condition (2.15) by an FDTD model. As shown in Figure 4.1, the model is composed of a porcine heart suspended inside a 10×10×10 cm³ saline filled cubic phantom. The geometry of the heart was segmented from MR images of a live pig. It was assumed that the chambers of the heart and the aorta were filled with blood. The conductivity and the relative permittivity of each part are listed in Table 4.1. The dielectric properties at 64 MHz were obtained from known published values².

¹ Parts of the material presented in this chapter and Chapter 5 are to be published in (Wang et al. 2010).
© 2010 IEEE. Reprinted, with permission, from IEEE Transactions on Medical Imaging, "Radio frequency current density imaging based on a 180-degree rotation with feasibility study of full current density vector reconstruction" by D. Wang, W. Ma, T.P. Demonte, A. I. Nachman and M. L. G. Joy.
² http://niremf.ifac.cnr.it/tissprop/htmlclie/htmlclie.htm provided by Italian National Research Council, Institute for Applied Physics. The calculation is based on the data and methods in (Cole & Cole 1941) and (Gabriel et al. 1996; Gabriel et al. 1996b; Gabriel et al. 1996a).
Table 4.1 Dielectric properties in the model

<table>
<thead>
<tr>
<th></th>
<th>Saline</th>
<th>Plastic Container</th>
<th>Heart</th>
<th>Blood</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conductivity (S/m)</td>
<td>1.48</td>
<td>0</td>
<td>0.678</td>
<td>1.207</td>
</tr>
<tr>
<td>Relative Permittivity</td>
<td>80.4</td>
<td>2.58</td>
<td>106.51</td>
<td>86.45</td>
</tr>
</tbody>
</table>

Figure 4.1 FDTD simulation of a porcine heart inside a saline phantom. (a) The FDTD model. (b) The model in a 2D projection view. The horizontal lines indicate the positions of the slices 2.3 cm, 3.2 cm and 4.1 cm below the top of the phantom. (c) Conductivity and permittivity of a slice about 3.2 cm below the top of the phantom.
The resolution of the simulation is $1.875 \times 1.875 \times 1.875$ mm$^3$. The current at 64 MHz is injected electrodes on the surface of the phantom. The steady state response was then converted to the frequency domain by the fast Fourier transform. A model of the cubic phantom filled with saline only was also simulated for comparison.

Some of the results are presented in Figure 4.2. Figure 4.2 (a), (c) and (d) compare the map of $|\hat{\partial H_z / \hat{\partial} z|/|J_z|}$ for the model without the heart and the model with the heart in slices 3.2 cm, 2.3 cm and 4.1 cm below the top of phantom. When the phantom is filled with saline only, the single orientation approximation (2.15) is violated in the regions close to the boundaries of the phantom, which are bright in Figure 4.2 (a), (c) and (d). Condition (2.15) is approximately valid in the dark regions that include the centre part of the slices. Note, however, once the heart is included in the model, the single orientation approximation becomes invalid in some regions inside and around the heart due to the inhomogeneous conductivity and permittivity. Figure 4.2 (b) gives a quantitative estimate of the local dominancy of $J_z$ by plotting the map of $|J_z| / \sqrt{|J_x|^2 + |J_y|^2}$.

Suppose we want to image another current density component and thus place the phantom in the position as shown in Figure 4.3 (a). The single orientation approximation is violated in a band across the centre (Figure 4.3(b)). This pattern is completely different from the one of the previous position (Figure 4.2 (a)). Therefore, there are few regions where the reconstructed components are both correct. Comparing Figure 4.3(b) with (c), we can see that the single orientation approximation (2.15) tends to be valid in the regions where $J_z$ is much larger than the transverse current density component.
However, this tendency is not as obvious in Figure 4.2. In fact, the degree of the local dominancy of $J_z$ does not always correlate to the correctness of the calculation. This is more evident in the model of Figure 4.4 in section 4.3.1, in which large error occurs in regions where $J_z$ is about ten times larger than the transverse current density component.

Figure 4.2 Simulation results of the model shown in Figure 4.1. (a) The map of $|\partial H_z / \partial z| / J_z$ without the heart (left) and with the heart (right) for the slice 3.2 cm below the top of the phantom. (b) The map of $|J_z| / (|J_x|^2 + |J_y|^2)^{1/2}$ with without the heart (left) and with the heart (right). (c) The map of $|\partial H_z / \partial z| / J_z$ without the heart (left) and with the heart (right) for the slice 2.3 cm below the top of the phantom. (d) The map of $|\partial H_z / \partial z| / J_z$ without the heart (left) and with the heart (right) for the slice 4.1 cm below the top of the phantom.
We can learn from these examples that (2.15) fails because of the electrical properties of the object and how current is applied to the object. However, local variations in the electrical properties and local current density will not be sufficient to predict whether or not condition (2.15) is met.

Now let us analyze the single orientation approximation (2.15) theoretically. To satisfy this condition, ideally,

\[
\frac{\partial H_z}{\partial z} = 0. \tag{4.1}
\]

Since \( \nabla \cdot \mathbf{H} = 0 \), (4.1) is equivalent to

\[
\frac{\partial H_z}{\partial z} = 0.
\]
\[
\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0.
\] (4.2)

In other words, if we define two vector fields \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) as

\[
\mathbf{H}_1 = H_z \hat{\mathbf{a}}_z,
\] (4.3)

and

\[
\mathbf{H}_2 = H_x \hat{\mathbf{a}}_x + H_y \hat{\mathbf{a}}_y,
\] (4.4)

\( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) satisfy \( \nabla \cdot \mathbf{H}_1 = 0 \) and \( \nabla \cdot \mathbf{H}_2 = 0 \). This means that the magnetic field \( \mathbf{H} \) which satisfies condition (2.15) can be approximated as a sum of two independent magnetic fields \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \). \( \mathbf{H}_1 \) only has \( z \) component and \( \mathbf{H}_2 \) only has \( x \) and \( y \) components.

This property of the magnetic field \( \mathbf{H} \), however, provides little practical insight. It is difficult to predict whether and where the condition (2.15) is violated from the measured magnetic field because neither \( H_z \) nor \( H_x \) and \( H_y \) can be determined using the previous RF-CDI methods. A sufficient condition for (2.15) is that the current flows in the \( z \) direction (the direction of \( \vec{B}_0 \)) not only in the region of interest but also in the neighboring space. This can be seen from the Biot-Savart law,

\[
\vec{H} = \frac{1}{4\pi} \int \frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^2} dV',
\] (4.5)

which indicates that the magnetic vector at each point of the field is a spatial convolution of the current density and a standard kernel. When the current density at every point is along the \( z \) direction the integral in (4.5) guarantees that the corresponding magnetic field is perpendicular to the \( z \) axis. In this case, \( \mathbf{H}_1 \approx 0 \). For this reason, in previous phantom
experiments (Scott et al. 1992a; Scott et al. 1995a; Scott et al. 1995b; Wang et al. 2009),
current was forced to flow predominantly in the \( z \) direction.

This experimental strategy to satisfy (2.15) implies that the application of the previous RF-CDI reconstruction method to biological objects will be limited. Since the current is required to flow globally in one direction, there will be little flexibility to choose the current pathways. Furthermore, it will be more difficult than in phantom experiments to control the current flow by carefully arranging the electrodes and return wires because of the inhomogeneous electrical properties of biological objects.

4.2 Theoretical Basis for the New Reconstruction Method

Recall from the electromagnetic consideration for RF-CDI in Chapter 2 that the current density component \( J_z \) can be computed by the magnetic components transverse to the \( z \) direction, that is,

\[
J_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j(\frac{\partial H_L}{\partial x} - \frac{\partial H_R}{\partial x}) - (\frac{\partial H_L}{\partial y} + \frac{\partial H_R}{\partial y}),
\]

(4.6)

in which

\[
H_L = \frac{1}{2}(H_x - jH_y)
\]

(4.7)

and

\[
H_R = \frac{1}{2}(H_x + jH_y).
\]

(4.8)
$H_L$ and $H_R$ are the left circularly polarized (LCP) and the right circularly polarized (RCP) components with regard to the $+z$ direction. On the other hand, $H_x$ and $H_y$ can be expressed by $H_L$ and $H_R$ as

$$H_x = H_L + H_R, \quad (4.9)$$

and

$$H_y = j(H_L - H_R). \quad (4.10)$$

According to (4.6), both the LCP and RCP components of the magnetic field yielded by the current are required to compute $J_z$.

In a slightly different form, we can write

$$J_z = 2j \frac{\partial H_L}{\partial x} - 2 \frac{\partial H_L}{\partial y} + j \frac{\partial H_R}{\partial z}. \quad (4.11)$$

If the last term on the right side is negligible, $J_z$ can be calculated using only by $H_L$. This is the theoretical basis for the single orientation reconstruction.

Now consider rotating the sample 180 degrees as illustrated in Figure 4.4. In the original position, the sample frame is the same as the laboratory frame. Let $\tilde{H}_x^0$ and $\tilde{H}_y^0$ be the imaged LCP magnetic components at this position, then,

$$H_L = (\tilde{H}_x^0 - j\tilde{H}_y^0)e^{-j\varphi}, \quad (4.12)$$

where $\varphi$ be the relative angle between the rotating frame $\{\tilde{x}, \tilde{y}\}$ and the laboratory frame $\{x, y\}$ at $t = 0$ (please refer to Figure 2.1).
Assuming the rotation is around the laboratory frame $x$ axis, then after the 180 degree rotation, both the $z$ and $y$ axes flip signs with regard the laboratory frame. The original RCP component in the transverse plane according to the laboratory frame becomes the LCP component after rotation. Therefore, with the same RF-CDI data acquisition, one should be able to measure the RCP field. Replace $H_R$ with $H_L^\pi$ in (4.8), we have

$$H_R = H_L^\pi = \left( \tilde{H}_x^\pi - j\tilde{H}_y^\pi \right) e^{-j\phi},$$  \hspace{1cm} (4.13)

where the superscript $\pi$ in $H_L^\pi$, $\tilde{H}_x^\pi$ and $\tilde{H}_y^\pi$ indicates that they are the new components after the 180-degree rotation. Assume that $\phi$ is spatially constant and does not change after rotation. Inserting (4.12) and (4.13) into (4.6), we then obtain

Figure 4.4 Illustration of sample rotation in the $yz$ plane. The arrow on the animal represents a right circularly polarized component at the original position. It becomes left circularly polarized after the rotation.
\[ J_z e^{j\varphi} = \left( \frac{\partial \tilde{H}_y^0}{\partial x} + \frac{\partial \tilde{H}_x^0}{\partial y} - \frac{\partial \tilde{H}_y^z}{\partial x} - \frac{\partial \tilde{H}_x^z}{\partial y} \right) + j \left( \frac{\partial \tilde{H}_x^0}{\partial x} + \frac{\partial \tilde{H}_y^0}{\partial y} - \frac{\partial \tilde{H}_x^z}{\partial x} + \frac{\partial \tilde{H}_y^z}{\partial y} \right) \, (4.14) \]

Therefore, using the two positions in Figure 4.4, \( J_z \) can be reconstructed with a relative angle \( \varphi \) with respect to the laboratory frame. In the remainder of the chapter, the reconstructed \( J_z e^{j\varphi} \) is denoted as \( J_z^{\text{rec}} \) and the benchmark \( J_z e^{j\varphi} \) from the forward modeling is denoted as \( J_z^{\text{true}} \). The magnitude and phase of \( J_z e^{j\varphi} \) are referred as \( j_z \) and \( \varphi_z \) respectively as in Chapter 3.

### 4.3 Verification by Simulation

Three simulations and experiments using a phantom were performed. Their purpose was to demonstrate that with a single rotation of the object, the proposed methods will avoid the error that occurs in the previous methods (Scott et al. 1992a; Scott et al. 1995a; Scott et al. 1995b) at points where condition (2.15) does not hold. In the simulated models (Figure 4.5 (a), Figure 4.3 (a) and Figure 4.8 (b)), the magnetic field components were computed by 3-D FDTD and converted to the phasor form at 64 MHz. Then current density components were calculated by Ampere’s law. The 3×3 Sobel templates and the 2×1 template \([-1 1]\) were used in the calculation of the spatial derivatives for Ampere’s law in Section 4.3.1 and Section 4.3.2, respectively. The relative angle \( \varphi \) between the rotating frame defined by the transmit coil and the laboratory frame was assumed spatially constant and invariant with respect to rotations. In the reconstruction process, the rotating frame components \( (\tilde{H}_x^0, \tilde{H}_y^0) \) were calculated by (4.7) and (4.12). Then the
whole set of electromagnetic field vectors was rotated 180 degrees with respect to the $x$ axis of the laboratory frame and $(\vec{H}_x, \vec{H}_y)$ were calculated similarly. Finally, the current density was computed by the two reconstruction methods (2.16) and (4.14) respectively. In the reconstruction, the $3 \times 3$ Sobel templates were used for the calculation of spatial derivatives for data from simulation as well as data from experimental measurements described in Section 4.4.

4.3.1 Testing of the 180-degree Rotation Method

We chose to validate the single rotation method using the phantom model and orientation with respect to $\vec{B}_0$ shown in Figure 4.5 (a). This choice resulted in magnetic fields that violated condition (2.15) in the bright regions shown in Figure 4.6 (c). In these regions, current was expected to flow mainly in the $z$ direction (See Figure 4.6 (e) (f)). However, the $H_z$ field produced by the top and bottom return wires (Figure 4.5 (a)) changed rapidly along the $z$ direction and thus was the major cause of the violation of (2.15) in this case.

The spatial resolution of the simulation was $\Delta x = \Delta y = \Delta z = 2$ mm. The dimensions of the container were $92 \times 92 \times 42$ mm$^3$. The thickness of the walls of the container was 6 mm. The top and bottom electrodes made contact with the conductive material. The plastic container was assigned relative permittivity $\varepsilon_r = 2.58$ and conductivity $\sigma = 0$ S/m. The conductive material inside the container was composed of two parts. The outer part had constant relative permittivity $\varepsilon_r = 80$ and conductivity $\sigma = 1.5$ S/m, while
conductivity of the inner part varied linearly in one direction (the $x$ direction in Figure 4.5) from 0.5 S/m to 0.9 S/m and the relative permittivity $\varepsilon_r = 40$ was a constant. The distribution of the electrical properties for the centre transverse plane (1.6 cm below the top electrode) is shown in Figure 4.5 (b).

For the centre slice shown in Figure 4.5 (b), $J_{z}^{\text{rec}}$ was computed by (2.16) and (4.14) respectively and compared to $J_{z}^{\text{true}}$ from the forward modeling (Figure 4.6 (a) and (b)). Relative errors were calculated by $\left| J_{z}^{\text{rec}} - J_{z}^{\text{true}} \right|/\left| J_{z}^{\text{true}} \right|$ (Figure 4.6 (d)). The map of $\left| \partial H_z / \partial z \right|/\left| J_z \right|$ was also computed (Figure 4.6 (c)). As shown in Figure 4.6 (c), $\left| \partial H_z / \partial z \right|$ is significant compared to $\left| J_z \right|$ in the regions near the top of the slice. Therefore, in these regions $J_{z}^{\text{rec}}$ computed from (2.16) deviates from the $J_{z}^{\text{true}}$ (Figure 4.6 (a) and (b)), while $J_{z}^{\text{rec}}$ calculated by (4.14) matches $J_{z}^{\text{true}}$.

![Figure 4.5 Model of the simulation. (a) A model of a rectangular container filled with spatial varying conductive material. (b) Distribution of the conductivity (left) and the relative permittivity (right) on the centre transverse plane (the $xy$ plane) of the model.](image)
Figure 4.6 Results of the simulation. (a) Magnitude of $J^\text{true}_z$ from the forward modeling (left), the $J^\text{rec}_z$ calculated by (2.16) (middle) and the $J^\text{rec}_z$ computed by (4.14) (right). (b) Phase of $J^\text{true}_z$ (left), the $J^\text{rec}_z$ calculated by (2.16) (middle) and the $J^\text{rec}_z$ calculated by (4.14) (right). (c) Map of $|\partial H_z/\partial z|/|J_z|$. (d) Relative error of $J^\text{rec}_z$ calculated by (2.16) (left) and $J^\text{rec}_z$ calculated by (4.14) (right). (e) Magnitude of $J^\text{true}_x$ and $J^\text{true}_y$. (f) The map of $|J^\text{true}_x|/\sqrt{|J^\text{true}_x|^2 + |J^\text{true}_y|^2}$. 

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Figure 4.7 Current density reconstruction using sample rotation for model shown in Figure 4.3 (a). (a) Magnitude of $J_{z,\text{true}}$ in the slice 3.2 cm below the top of the phantom. (b) phase of $J_{z,\text{true}}$. (c) The magnitude of $J_{z,\text{rec}}$ computed by (2.16) (left) and by (4.14) (right). (d) Phase of $J_{z,\text{rec}}$ computed by (2.16) (left) and by (4.14) (right). (e) Relative difference of $J_{z,\text{true}}$ and $J_{z,\text{rec}}$ computed by (2.16) (left) and by (4.14) (right).

Figure 4.8 Structure of the multi-chamber phantom. (a) Illustration of the cross-section of the phantom. (b) An FDTD simulation model resembling the phantom.
Our new reconstruction method was also tested on the model with porcine heart in saline
shown in Figure 4.3 (a). The results for the slice 3.2 cm below the top of the phantom are
illustrated in Figure 4.7. Again, $J_z^{rec}$ computed by (2.16) agrees with $J_z^{true}$ where (2.15)
is satisfied, but deviates from $J_z^{true}$ where the condition (2.15) is violated. On the other
hand, $J_z^{rec}$ computed by (4.14) matches $J_z^{true}$ as expected.

### 4.3.2 Simulation for Phantom Experiment

A multi-chamber phantom shown in Figure 4.8 was used for the RF-CDI experimental
testing. The center chamber was filled with solution with 0.9g/100ml NaCl and
0.1g/100ml CuSO4•5H2O giving $\sigma \approx 1.48$ S/m and $\varepsilon_r \approx 78.36$. The outer two chambers
were filled with solution of 0.1g/100ml CuSO4•5H2O, which resulted in $\sigma \approx 0.036$ S/m
and $\varepsilon_r \approx 78.36$. Copper plate electrodes were attached to both ends of the inner two
chambers. The current was expected to flow mainly in the axial direction of the phantom
due to the configuration of electrodes. The phantom had a diameter of 8.6 cm and height
of 15 cm. An FDTD model resembling the phantom was simulated (Figure 4.7 (b)). The
spatial resolution in the simulation was $\Delta x = \Delta y = 0.9375$ mm and $\Delta z = 2$ mm. $\Delta z$ was
then interpolated to 0.9375 mm. When oblique slices of the phantom were imaged in the
experiments, the correspondent field components in simulation were obtained by vector
rotation and linear interpolation. The results of the simulation are shown in Figures 4.9-4.11.
4.4 Experimental Testing

The experiments were performed using a clinical 1.5 Tesla GE® EXCITE MR scanner. A body coil was used for both transmitting and receiving. The phantom was placed in the centre of the coil. The Larmor frequency current was originated from the RF output of the imager. This signal was connected as the input to the RF control box (Scott et al. 2003) to produce RF current with desired phase and blanking signal which was in turn inputted to the RF amplifier (NMRplus model 5T300, Communication Power Corporation). The output of the amplifier was connected to the phantom through coaxial cable and an impedance matching network. Two sets of experiments were performed. In each set, data were acquired by the multi-slice RF-CDI sequence for two sample positions that were related by a 180-degree rotation as indicated in Figure 4.4. The current density component along the direction of \( \vec{B}_0 \), that is, \( J_z \), in the laboratory frame, was estimated by (2.16) and by (4.14) respectively and compared to the results from the simulation.

In the first set of experiments, the longitudinal axis of the phantom was aligned with \( \vec{B}_0 \) for the original position (Figure 4.8 (a)) and then rotated to -\( \vec{B}_0 \). Three axial slices were imaged close to the middle of the phantom. Other imaging parameters were: field of view FOV = 0.24 m, frequency samples=256, phase encoding steps =256, slice thickness =10.0 mm, slice gap = 0 mm. Figure 4.9 demonstrates the results for the centre slice. The results from the simulation were normalized such that the total current \( I_z \) of the centre slice was equal to that from the experiment for the reconstruction by (2.16). This gave rise to a complex scaling factor which was used for all sets of phantom.
Figure 4.9 Simulation and experimental results of the first alignment of the phantom. (a) Phantom placed with its longitudinal axis parallel to $\vec{B}_0$. (b) Map of $|\partial H_z / \partial z| / j_z$. (c) Simulated current density magnitude $j_z$: true $j_z$ (left), $j_z$ reconstructed by (2.16) (centre) and by (4.1r) (right) shown in logarithm scale. (d) Simulated current density phase $\phi_z$: true $\phi_z$ (left), $\phi_z$ reconstructed by (2.16) (centre) and by (4.14) (right). (e) Experimental $j_z$: $j_z$ reconstructed by (2.16) (left) and by (4.14) (right) shown in logarithm scale. (f) Experimental $\phi_z$: $\phi_z$ reconstructed by (2.16) (centre) and by (4.14) (right).
Figure 4.10 Simulation and experimental results of the second set of experiments. (a) Phantom placed with its longitudinal axis parallel to $\vec{B}_0$. (b) Magnitude (left) and phase (right) of $j(\partial H_z / \partial z)e^{j\phi}$. (c) True $j_z$ (left) and $\phi_z$ (right). (d) $j_z$ (left) and $\phi_z$ (right) evaluated by (2.16) for simulated data. (e) $j_z$ (left) and $\phi_z$ (right) evaluated by (4.14) for simulated data. (f) $j_z$ (left) and $\phi_z$ (right) evaluated by (2.16) for experimental data. (g) $j_z$ (left) and $\phi_z$ (right) evaluated by (4.14) for experimental data.
Figure 4.11 Simulation and experimental results of the third alignment of the phantom. (a) Phantom placed with its longitudinal axis 45° parallel to $\vec{B}_0$. (b) Map of $|\partial H_z / \partial z|/j_z$. (c) Simulated $j_z$: true $j_z$ (left), $j_z$ reconstructed by (2.16) (centre) and by (4.14) (right) shown in logarithm scale. (d) Simulated $\phi_z$: true $\phi_z$ (left), $\phi_z$ reconstructed by (2.16) (centre) and by (4.14) (right). (e) Experimental $j_z$: $j_z$ reconstructed by (2.16) (left) and by (4.14) (right) shown in logarithm scale. (f) Experimental $\phi_z$: $\phi_z$ reconstructed by (2.16) (centre) and by (4.14) (right).
alignment. As shown in Figure 4.9 (b), \( \left| \frac{\partial H_z}{\partial z} \right|/|J_z| \) is low and hence the electromagnetic fields satisfy the single orientation approximation (2.15). Therefore, the results by both reconstruction methods are consistent with the \( J_z^{true} \) distribution predicted directly by the simulation.

In the second pair of experiments, the longitudinal axis of the phantom was perpendicular to \( \vec{B}_0 \). Figure 4.10 (a) shows one of the two anti-parallel positions. Three sagittal slices were imaged to compute the derivatives along the \( y \) axis of the laboratory frame of the centre slice. Other imaging parameters are the same as the first experimental set. The results are shown in Figure 4.10. As the \( y \) axis of the sample frame is along the \( z \) axis of the laboratory frame in these two positions, \( J_z^{true} \) in the laboratory frame is close to 0 (the true \( J_z^{true} \) is lower than 1.5 A/m\(^2\) in the centre slice) as shown in Figure 4.10 (c). Therefore, the \( J_z^{rec} \) evaluated by (2.16) is essentially the error term \( j \frac{\partial H_z}{\partial z} e^{j\omega} \) in (2.14). This can be clearly seen by comparing Figure 4.10 (d) (f) to (b). On the other hand, \( J_z^{rec} \) computed by (4.14) in the simulation is as small as \( J_z^{true} \). In fact, \( J_z^{true} \) is so small that the numerical error is magnified and as a result \( J_z \) in Figure 4.10 (c) and (e) differ significantly from each other in some area. \( J_z^{rec} \) evaluated by (4.14) in the experiment is close to 0 and consequently phase of \( J_z^{rec} \) is very noisy (Scott et al. 1995b; Wang et al. 2009).
In the third pair of experiments, the longitudinal axis of the phantom was placed in 45 degrees and -45 degrees to \( \vec{B}_0 \) for the two positions (Figure 4.11 (a)). Three axial slices were images. Other imaging parameters are: field of view \( \text{FOV} = 0.24 \) m, frequency samples=256, phase encoding steps =256, slice thickness =5.0 mm, slice gap = 0 mm. Only one slice is shown in Figure 4.11. The map in Figure 4.11 (b) indicates that there are regions in the middle and the outer chamber where \( |\frac{\partial H_z}{\partial z}|/|J_z| \) is greater than 0.5. Therefore, large relative error tends to occur in these regions when \( J_z \) is estimated by (2.16). Both the experimental and the simulation results indicate that \( |J_z| \) computed by (2.16) has a diagonal error pattern in the middle chamber and outer chamber, while \( |J_z| \) calculated by the new method (4.14) in these regions are more uniform and similar to the expected one.

4.5 Discussion of Noise

MRI random noise effects on the reconstruction with a single 180-degree rotation can be evaluated according to (3.14) and (4.12). Since the slice acquisition order is generally not the same for the two sample positions, it is convenient to absorb the \( e^{i/T_i} \) in the MRI SNR term. In other words, \( \text{SNR} = |\mathbf{M}|e^{-T_i/T_i} / \sigma \). Then, the noise in either the real or the imaginary part of \( J_z^{\text{rec}} \) is

\[
\sigma_j = \frac{1}{\sqrt{2\gamma \mu_0 T_c}} \left( \frac{1}{\text{SNR}^2} \right)^2 \left( \frac{1}{\text{SNR}^2} \right)^2 \left( \frac{F_x}{\Delta x} \right)^2 + \left( \frac{F_y}{\Delta y} \right)^2 \right)^{\frac{1}{2}}, \quad (4.15)
\]
where $SNR^0$ and $SNR^\pi$ are the SNRs for the original position and the second position. We can see that if $SNR^\pi = SNR^0$, then the random noise in $J_z$ decreases to $1/\sqrt{2}$ of the single orientation reconstruction.
Chapter 5

Three Dimensional Current Density Vector Reconstruction

In chapter 4, it has been demonstrated that with a 180-degree rotation of the sample, the current density component along the $\tilde{B}_0$ direction can be fully recovered. A natural question to ask is how to extend this method for imaging all three components of the current density vector field. This chapter introduces a new method for imaging the full RF current density vector field using an additional sample position. A theoretical derivation of the method is presented. The feasibility of this new approach is tested on simulated and experimental data.

5.1 Theory

As discussed in Chapter 4, using the imaging positions shown in Figure 4.4, the $H_x$ and $H_y$ components in the sample frame can be calculated with a relative angle $\phi$. With another pair of positions perpendicular to the first pair, the $H_x$ and $H_z$ components in the sample frame can be evaluated. In fact, the derivation below will show that one additional sample position suffices for obtaining the full vector field $\mathbf{H}$. Figure 5.1 demonstrates the possible positions for measuring the $\mathbf{H}$ field. Since the rotation from position 1 to any other positions in Figure 5.1 is around the $x$ axis of the laboratory frame, the rotation angle alone can be used to uniquely denote each position as labeled in the figure. For positions 1 and 2, we have
\[ H_l^0 = (\tilde{H}_l^0 - j\tilde{H}_l^0) e^{-j\varphi} = \frac{1}{2} (H_x - jH_y), \quad (5.1) \]

and
\[ H_l^{\pi} = (\tilde{H}_l^{\pi} - j\tilde{H}_l^{\pi}) e^{-j\varphi} = \frac{1}{2} (H_x + jH_y), \quad (5.2) \]

in which superscripts 0, \( \pi \), \( \pi / 2 \) and \( -\pi / 2 \) correspond to different positions. Therefore,
\[ H_x = H_l^0 + H_l^{\pi} \quad (5.3) \]

and
\[ H_y = j(H_l^0 - H_l^{\pi}). \quad (5.4) \]

Similarly, for positions 3 and 4, we can obtain
\[ H_l^{\pi/2} = \left(\tilde{H}_l^{\pi/2} - j\tilde{H}_l^{\pi/2}\right) e^{-j\varphi} = \frac{1}{2} (H_x - jH_z). \quad (5.5) \]

and
\[ H_l^{-\pi/2} = \left(\tilde{H}_l^{-\pi/2} - j\tilde{H}_l^{-\pi/2}\right) e^{-j\varphi} = \frac{1}{2} (H_x + jH_z). \quad (5.6) \]

Then,
\[ H_x = H_l^{\pi/2} + H_l^{-\pi/2} \quad (5.7) \]

and
\[ H_z = j(H_l^{\pi/2} - H_l^{-\pi/2}). \quad (5.8) \]

Note that, the information obtained from measurements in four positions is redundant as the component \( H_x \) is calculated twice. Any three of the positions in Figure 5.1 provide enough information to determine the field \( \mathbf{H} \). For instance, without using the
information from the fourth position in Figure 5.1, we can obtain, in view of (5.3), (5.7)-(5.8),

\[ H_c = j(2H_L^+ - H_L^0 - H_L^-). \]  

(5.9)

Figure 5.1 Demonstration of four possible sample positions to reconstruct current density vectors. The gray planes are possible imaging slices (Axial for positions 1 and 2, and sagittal for positions 3 and 4).
Therefore, $H_z e^{j\omega}$, as well as the corresponding current density vector can be calculated from three sample positions. In the remainder of the chapter, we denote $(J_x^{\text{rec}}, J_y^{\text{rec}}, J_z^{\text{rec}})$ the reconstructed $(J_x e^{j\omega}, J_y e^{j\omega}, J_z e^{j\omega})$, and the bench mark $(J_x e^{j\omega}, J_y e^{j\omega}, J_z e^{j\omega})$ obtained from the simulation of the forward problem by $(J_x^{\text{true}}, J_y^{\text{true}}, J_z^{\text{true}})$, respectively.

### 5.2 Testing on Simulated Data

We have verified the current density vector reconstruction scheme on the model in Figure 4.5. The first three sample positions shown in Figure 5.1 were used to compute the current density vector. The components $H_x e^{j\omega}$ and $H_y e^{j\omega}$ were reconstructed from the rotating frame components of the first two positions (labeled by angles 0 and $\pi$ in Figure 5.1) using equations (4.9)-(4.10) and (4.12)-(4.13). Then $H_z e^{j\omega}$ was calculated by (5.5) and (5.9). The reconstructed components of the magnetic field are essentially identical to those from the forward modeling as shown in Figure 5.2. As $J_z^{\text{rec}}$ is already shown in Chapter 4. Figure 5.3 shows the reconstructed magnitude and phase of the other two components of the current density vector $\mathbf{J}$ in the centre slice, for the noiseless case. They are essentially identical to the bench mark values directly from the simulation.
Figure 5.2 Reconstruction results of the magnetic vector field for the center slice in the simulation. (a) Simulation model. (b) Magnitude of $H_x e^{j\phi}$ from the forward modeling (left) and reconstruction (right). (c) Phase of $H_x e^{j\phi}$ from the forward modeling (left) and reconstruction (right). (d) Magnitude of $H_y e^{j\phi}$ from the forward modeling (left) and reconstruction (right). (e) Phase of $H_y e^{j\phi}$ from the forward modeling (left) and reconstruction (right). (f) Magnitude of $H_z e^{j\phi}$ from the forward modeling (left) and reconstruction (right). (g) Phase of $H_z e^{j\phi}$ from the forward modeling (left) and reconstruction (right).
5.3 Testing on Experimental Data

The multi-chamber phantom was used for experimental testing of the RF current density vector reconstruction method. The first and second pairs of phantom alignments described in section 4.4 constitute the four positions in Figure 5.1. Therefore, there is enough data to reconstruct the current density vector in the centre slice. The field components are represented here in the sample frame shown in Figure 5.4 (a). In order to compute the rotating frame magnetic field components in each position, the rotating angles $\Gamma_x$ and $\Gamma_y$ were 2-D phase unwrapped within each slice and then unwrapped in the third direction. The unwrapped angles were globally calibrated by multiples of $\pi$ to
obtain the absolute values of \((H_x e^{i\phi}, H_y e^{i\phi}, H_z e^{i\phi})\). This calibration, however, is not necessary for the computation of the current density vector field, since only the derivatives of the magnetic components are involved. In practice a current of low magnitude (which yields a magnetic field that does not require unwrapping) can be used to obtain the correct multiples of \(\pi\) when the absolute value of \((H_x e^{i\phi}, H_y e^{i\phi}, H_z e^{i\phi})\) is required. The reconstructed \((H_x e^{i\phi}, H_y e^{i\phi}, H_z e^{i\phi})\) for the centre slice are plotted in Figure 5.4 and compared to the benchmark values from the forward modeling. Due to the configuration of electrodes and return wires, the current flows predominantly in the \(z\) direction of the phantom frame. Compared to \(J_z\), the other two components, \(J_x\) and \(J_y\) are negligible. The current density components directly from the forward modeling in the centre slice perpendicular to the axis of the phantom are shown in Figure 5.5.

The current density vector was calculated using the data from positions 1-3 (experiment 1) and 2-4 (experiment 2) respectively. The reconstruction by all the four positions was also tested and denoted as experiment 3. From positions 1 and 2, \((H_x e^{i\phi}, H_y e^{i\phi})\) can be recovered and from positions 3 and 4, \((H_x e^{i\phi}, H_z e^{i\phi})\) can be recovered. Let us denote the first reconstructed \(H_x e^{i\phi}\) by \(H_{x1}\) and the second one by \(H_{x2}\). In experiment 3, we used \(H_{x2}\) and \(H_y e^{i\phi}\) to compute \(J_z^{\text{rec}}\), and \(H_{x1}\) and \(H_z e^{i\phi}\) to compute \(J_y^{\text{rec}}\). Figure 5.6 shows the reconstructed results for these three sets of experimental data. The mean of the current density vector components was estimated in each chamber for reconstructions by different positions and compared to the results of the simulation in Figure 5.7.
Figure 5.4 Magnetic intensity vector reconstruction results of the center slice in the experiment. (a) An FDTD model of the multi-chamber phantom. (b) Magnitude of \( H_x e^{i\varphi} \) directly from simulation (left) and reconstructed in experiment (right). (c) Phase of \( H_x e^{i\varphi} \) directly from simulation (left) and from reconstruction (right). (d) Magnitude of \( H_y e^{i\varphi} \) directly from simulation (left) and from reconstruction (right). (e) Phase of \( H_y e^{i\varphi} \) directly from simulation (left) reconstructed in experiment (right). (f) Magnitude of \( H_z e^{i\varphi} \) directly from simulation (left), reconstructed by positions 1-3 (middle) and by positions 3 and 4 (right).
Figure 5.5 RF current density components in the centre slice from simulation. (a) Magnitude of $J_{z \text{true}}$ in logarithm scale and phase of $J_{z \text{true}}$ in degree. (b) $|J_{x \text{true}}|$ (left) and $|J_{y \text{true}}|$ (right) (note the scale is $[0 \ 1]$ A/m²). The images are shown in the $xy$ plane in the sample frame.

Figure 5.6 RF current density vector reconstructed from the experimental data. Magnitude of $J_{z \text{rec}}$ in logarithm scale, phase of $J_{z \text{rec}}$ in degree, magnitude of $J_{x \text{rec}}$ and $J_{y \text{rec}}$ in logarithm scale for the reconstructions using positions 1-3 (a), positions 2-4 (b) and all four positions are shown from left to right.
Figure 5.7 Comparison of the current density vector reconstructed by different sample positions and the simulated current density vector. Experiment 1 is the reconstruction using positions 1-3 in Figure 5.1. Experiment 2 is the reconstruction using positions 2-4. Experiment 3 is the reconstruction using positions 1-4. (a) Magnitude of averaged $J_z^{true}$ and $J_z^{rec}$ in the centre, middle and outer chambers. (b) Phase of averaged $J_z^{true}$ and $J_z^{rec}$. (c) Magnitude of averaged $J_x^{true}$ and $J_x^{rec}$. (d) Magnitude of averaged $J_y^{true}$ and $J_y^{rec}$. Note that the scale in (a) is 0-900 A/m$^2$ and the scale in (c) and (d) is 0-100 A/m$^2$. 

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5.4 Discussion

5.4.1 Random noise effects and artifacts

The random noise behavior of the current density components that only require two sample positions is described by (4.15). For the purpose of evaluating the MRI random noise effect on the reconstructed current density components involving three sample positions, it is more convenient to express the current density components in terms of the measured rotating frame magnetic field. For example, for the reconstruction using positions 1-3, we have, according to equations (5.5)-(5.9), and (4.9)-(4.10), (4.12)-(4.13),

\[
J_{x}^{\text{rec}} = J_{x}e^{ip} = \left\{ \frac{\partial \hat{H}_{y}^{0}}{\partial z} - \frac{\partial \hat{H}_{y}^{\pi}}{\partial y} + \frac{\partial \hat{H}_{y}^{0}}{\partial y} + 2 \frac{\partial \hat{H}_{y}^{\pi}}{\partial y} \right\}
\]

and

\[
J_{y}^{\text{rec}} = J_{y}e^{ip} = \left\{ -\frac{\partial \hat{H}_{x}^{0}}{\partial z} - \frac{\partial \hat{H}_{x}^{\pi}}{\partial z} - \frac{\partial \hat{H}_{x}^{0}}{\partial x} + 2 \frac{\partial \hat{H}_{x}^{\pi}}{\partial x} \right\}
\]

If SNRs are similar for adjacent pixels involved in derivative calculations for each position, current density noise for \( J_{y}^{\text{rec}} \) can be estimated by
\[
\sigma_{J_y} = \frac{1}{\sqrt{2} \gamma \mu_0 T_c} \left( \left( \frac{1}{SNR^0} \right)^2 + \left( \frac{1}{SNR^\pi} \right)^2 \right) \left( \frac{F_y}{\Delta z} \right)^2 + \left( \frac{F_z}{\Delta x} \right)^2 + 4 \left( \frac{1}{SNR^2} \right)^2 \left( \frac{F_x}{\Delta x} \right)^2 \right)^{1/2},
\]

(5.12)

where \(SNR^0\), \(SNR^\pi\) and \(SNR^{\pi/2}\) are the MRI SNRs for sample positions 1-3 defined similarly as in (4.11). If \(F_x = F_z\) and \(\Delta x = \Delta z\), then \(\sigma_{J_y}\) will be \(\sqrt{2}\) of the noise in the current component reconstructed by two positions and the same as the noise in the current component reconstructed by one position using the single orientation assumption.

If the numerical templates used to calculate the derivatives along the \(z\) direction and the \(y\) direction do not overlap, then the current density noise of \(J_{y,\text{rec}}\), \(\sigma_{J_y}\), can be estimated similarly to (5.12). Otherwise, it can be expressed in a more general form as

\[
\sigma_{J_y} = \frac{1}{\sqrt{2} \gamma \mu_0 T_c} \left( \left( \frac{1}{SNR^0} \right)^2 + \left( \frac{1}{SNR^\pi} \right)^2 \right) \left( f^{\gamma \zeta} \right) + 4 \left( \frac{1}{SNR^2} \right)^2 \left( \frac{F_y}{\Delta y} \right)^2 \right)^{1/2},
\]

(5.13)

where \(f^{\gamma \zeta}\) is a factor associated with the templates used to compute derivative and \(\Delta y, \Delta z\). In the reconstruction demonstrated in this chapter, the 3x3 Sobel templates are employed in the \(yz\) plane to calculate the derivatives. Therefore,

\[
f^{\gamma \zeta} = \frac{1}{4} \left( \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right) + \frac{1}{32} \left( \frac{1}{\Delta y} + \frac{1}{\Delta z} \right)^2 + \frac{1}{32} \left( \frac{1}{\Delta y} - \frac{1}{\Delta z} \right)^2.
\]

(5.14)

This will cause another increase of current noise for a factor of 1.15 compared to (5.12) if \(\Delta y = \Delta z\). In practice if noise minimization is important, overlaps of the templates can be
avoided, for instance, by taking the derivative along $y$ in the $xy$ plane and, the derivative along $z$ in the $yz$ plane.

As a rough estimation, the MRI SNR of each chamber of the multi-chamber phantom used in the experiments is estimated by the average SNR of the three slices. The results are shown in Table 5.1. The predicted standard deviations of current density noise for experiment 1, 2 and 3 are summarized in Table 5.2.

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<th>Table 5.1 Estimated MRI SNR</th>
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<td><strong>Position 1</strong></td>
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<table>
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<th>Table 5.2 Predicted current density noise</th>
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<td><strong>Experiment 1</strong></td>
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<td><strong>$\sigma_{j_x}$ (A/m$^2$)</strong></td>
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Statistical assessments show that $J_z^{true}$ is approximately uniform within each chamber, while the reconstructed current density components display much more fluctuation within each chamber. The random noise in $J_z^{rec}$ was estimated to be in the range of 6.8 A/m$^2$ to 9.9 A/m$^2$ for different experiments as shown in Table 5.2. The evaluated spatial standard deviation of $|J_z^{rec}|$ is slightly larger than the predicted current random noise in the outer two chambers, and two to three times that of the predicted random noise in the center chamber.

The components $J_x^{true}$ and $J_y^{true}$ are essentially zero compared to $J_z^{true}$. Although the magnitude of either averaged $J_x^{rec}$ or averaged $J_y^{rec}$ is less than 2.5% of the magnitude of the averaged $J_z^{rec}$, they are larger than the results of the simulation, especially in the centre chamber and for experiment 1 (Figure 5.7). Another observation from Figure 5.7 is that the phase difference in the centre chamber and the outer two chambers in the experiments is consistently slightly smaller than that of the simulation. We discuss below several possible factors, other than random noise effects, which may contribute to the observed differences between the experiments and simulation and among experiments using different sample positions.

Firstly, the numerical model does not perfectly match the experimental phantom. For instance, the configuration of the return wires and electrical properties might differ slightly from the real phantom. Secondly, it is possible that there are slight shifts of the slices in different orientations. The resulting registration inaccuracies can cause
observable error. Thirdly, the nonlinear effects due to the finite amplitude of the rotary echo pulse in the imaging sequence can be another source of errors (Scott et al. 1995b). The nonlinear errors are more evident for the measurement in position 1 and 2, not only because the maximum $|\vec{B}_x|$ and $|\vec{B}_y|$ are higher, but also because the spatial gradients of $|\vec{B}_x|$ and $|\vec{B}_y|$ are larger in these two positions. Since the gradients of $|\vec{B}_x|$ and $|\vec{B}_y|$ are the largest in the centre chamber, the error in $J^{\text{rec}}_z$ is most significant there. Moreover, the nonlinear errors are passed to $J^{\text{rec}}_x$ and $J^{\text{rec}}_y$ in experiment 1, because the computation involves the measured $|\vec{B}_x|$ and $|\vec{B}_y|$ from position 1 and 2 according to equations (5.10) and (5.11). The nonlinear errors, however, may be avoided by the developing polar decomposition approach for RF-CDI measurement (Ma et al. 2008). Finally, the coupling between sample and coil is another likely cause. This will be discussed in Section 5.4.2.

5.4.2 Imaging conditions

Although theoretically current density vector can be reconstructed by sample rotations, it is worth emphasizing the experimental conditions required for the rotational approach. The first one is regarding the angle $\varphi$ between the rotating frame defined by the transmitter coil and the laboratory frame. A map of $\varphi$ for each sample position will be sufficient for computing current density vectors with arbitrary transmitter coil. However, due to lack of knowledge of distribution of $\varphi$ in generic coils, the requirement for $\varphi$ at current stage is that its spatial variation in the region of interest and the difference among the imaging positions are negligible. To satisfy this condition, a coil designed to transmit
a uniform RF $B_1$ field in a volume greater than the region of interest should be used. These experiments were performed using a 1.5 Tesla magnet; a magnet greater than 3 Tesla would present new challenges due to $B_1$ non-uniformity and increased coupling effects.

Another important issue is sample and coil coupling. There are two aspects of sample and coil coupling. First, RF $B_1$ field produced by the transmit coil is influenced by the interaction of sample and coil. In RF-CDI experiments, we are mainly concerned with the uniformity of the direction of $B_1$ and the consistency of the $B_1$ direction among different sample positions as discussed above. Second, the current applied to the sample might change from one position to another due to the change of relative positions between the sample and coil. The coupling issues of sample and coil precluded the approach to reconstruct current density vector through three orthogonal orientations from realizing in a small bore imager (Scott et al. 1995a). The coupling is no longer significant because in our experiment setup, since the bore of the imager is much larger than the sample, which thus occupies a small portion of the volume inside the body coil. Nevertheless, the coupling effects need to be further investigated. Since the relative position between the sample and the coil after the 180-degree rotation is similar to the original relative position, the difference of coupling from one position to another is expected to be smaller than the one experienced by the earlier suggested orthogonal rotations. When a third position perpendicular to the first two positions is included to recover the whole current density vector field, it is helpful to use four imaging positions instead of three to untangle the different coupling patterns among orthogonal positions. As mentioned in Section 5.3, we
denote $H_{s_1}$ the value of $H_x e^{j\phi}$ evaluated by position 1 and 2 and $H_{s_2}$ the value of $H_x e^{j\phi}$ evaluated by position 3 and 4. A comparison of $H_{s_1}$ and $H_{s_2}$ can provide a good measure of the consistency among different positions. Furthermore, to compensate for the different coupling, the ratio of $H_{s_1}$ and $H_{s_2}$ may be utilized to scale $H_y e^{j\phi}$ or $H_z e^{j\phi}$ to yield one consistent $(H_x e^{j\phi}, H_y e^{j\phi}, H_z e^{j\phi})$ set.
Chapter 6

Discussion, Future Work and Conclusions

A new method of imaging the full RF current density vector field in three dimensions using an MR imager has been introduced and tested. The method uses only rotations of the sample in the horizontal plane. This chapter addresses future challenges in biomedical applications of RF-CDI. In addition, potential development of impedance imaging based on RF-CDI is discussed. The chapter closes with the main conclusions of the dissertation.

6.1 Discussion

6.1.1 Patient Safety

If RF-CDI is to be used for human imaging, the heating effects from RF power dissipated in the subject needs to be monitored for concerns of safety. In the experiments presented in this thesis, the RF power delivered to the phantoms from the applied current is around 0.5 to 14 W during the duration of $T_r$. The specific absorption rate (SAR) of the saline ($\sigma = 1.48 \text{ S/m}$) due to the current is estimated around 0.2-3.6 W/kg, which is comparable to the generally recommended limits for peak local SAR (2-4 W/kg). Usually, tissues have lower conductivity than the saline used in the experiments. This will result in higher SAR if the same amount of current is applied. Nevertheless, as estimated in (Scott 1993), good SNR of RF-CDI images can be obtained for human tissues within the peak local SAR limit 2 W/kg due to the current only. However, the RF pulses in the imaging
sequences contribute to SAR as well, especially the rotary echo $B_1$ pulse, which may be the dominant source of RF power deposited. Therefore, careful evaluation should be conducted when these sequences are used for biomedical applications. Alternatively, it will be beneficial to develop multi-slice RF-CDI pulse sequences which do not require the rotary echo pulse.

6.1.2 Practical Consideration for Sample Rotations

Measurement of one current density component should be possible in a conventional MR imager, since in both positions, the patient can be aligned with her/his longitudinal axis parallel to $\vec{B}_0$. However, the application of the full current density vector reconstruction that utilizes an additional position perpendicular to the first pair will be limited by the height of the patient. Open concept MRI is a possible solution to this problem.

Alternatively, oblique, rather than orthogonal positions might be used instead. As shown in Figure 6.1, assuming that the angle between the longitudinal axis of the object and $\vec{B}_0$ is $\theta \left(0 < \theta < \frac{\pi}{2}\right)$, then similar to (5.1)-(5.2) and (5.5)-(5.6), we can obtain

$$H_L^0 = (\vec{H}_x^0 - j\vec{H}_y^0)e^{-j\phi} = \frac{1}{2}(H_x - jH_y), \tag{6.1}$$

and

$$H_L^\pi = (\vec{H}_x^\pi - j\vec{H}_y^\pi)e^{-j\phi} = \frac{1}{2}(H_x + jH_y), \tag{6.2}$$

$$H_L^\theta = (\vec{H}_x^\theta - j\vec{H}_y^\theta)e^{-j\phi} = \frac{1}{2}(H_x - j\left(H_y \cos \theta + H_z \sin \theta\right)) \tag{6.3}$$

and
\[ H_L^{-\theta} = (\tilde{H}_x^{-\theta} - j\tilde{H}_y^{-\theta})e^{-j\varphi} = \frac{1}{2}(H_x + j(H_y \cos \theta + H_z \sin \theta)) \] . \quad (6.4)

Therefore, \( H_z e^{j\varphi} \) can be evaluated by

\[ H_z \sin \theta = j(2H_L^0 - (1 + \cos \theta)H_L^0 - (1 - \cos \theta)H_L^\pi) . \quad (6.5) \]

One apparent issue with this approach is the reduction of the SNR of the measured \( H_z e^{j\varphi} \).

As a rough estimation, assuming the MRI SNR is the same for each sample position, then the SNR of the \( H_z \) calculated by (6.5) will be proportional to \( \sin \theta \). In other words, the SNR of \( H_z \) will reduce to a factor of 0.71 for \( \theta = 45^\circ \) and a factor of 0.5 for \( \theta = 30^\circ \), compared to that of \( \theta = 90^\circ \).
Figure 6.1 Demonstration of reconstructing full current density vectors with oblique positions.
6.2 Future Research Directions

6.2.1 Rationale for Diagnostic Radio Frequency Impedance Imaging

The electrical conductivity, $\sigma$, and the relative permittivity, $\varepsilon_r$, are important properties of tissues that have been of interest for many researches for over a century. This interest stems from the fact that they are tissue specific and also vary with frequency and tissue state due to several biophysical and biochemical mechanisms (Foster & Schwan 1989). The Larmor frequencies of most MR systems lie in the range of 5-300 MHz. In this frequency range, the imaginary part of the complex conductivity is comparable to the real part of the complex conductivity in many tissues so that both are measureable. Figure 6.2 plots the estimated typical dielectric properties in the frequency range of 5-300 MHz for several human tissues\(^1\). The frequencies at the lower end of this range overlaps with the $\beta$ dispersion of soft tissues, which is characterized by slow increase of the conductivity $\sigma$ and rapid decrease of the relative permittivity $\varepsilon_r$. The $\beta$ dispersion occurs between 10 kHz and 100 MHz (typically between 100 kHz and 10 MHz) (Foster & Schwan 1989). The major cause of the $\beta$ dispersion is the capacitive charging of the cellular membranes. A minor cause might be the dipolar orientation of tissue protein. Therefore, knowledge about the dielectric properties in the frequencies of the $\beta$ dispersion could provide an insight into the fundametnals of tissue physiology. The evaluation of the complex conductivity at 100-300 MHz is also of special interest in the area of MRI for the calculation of $B_1$ field inhomogeneity and power deposition, especially for high field

\(^1\) The data are obtained from Italian National Research Council Institute for Applied Physics http://niremf.ifac.cnr.it/tissprop/. Their calculation is based on (Cole & Cole 1941; Gabriel et al. 1996; Gabriel et al. 1996b; Gabriel et al. 1996a).
imaging. In addition, in-vitro measurements (Foster & Schepps 1981; Joines et al. 1994; Surowiec et al. 1988) suggest that both the conductivity and the relative permittivity tend to be consistently greater in malignant tissue than in normal tissue for the same type throughout frequencies 5-300 MHz. The observations indicate that impedance imaging within this range may be utilized to facilitate the diagnoses for tumors.

6.2.2 Potential Techniques for RF Impedance Imaging

Several tomography techniques have been developed to map the dielectric distribution at various frequencies but there is no mature impedance imaging modality that covers the 5-300 MHz frequency range. Electric impedance tomography (EIT) works best using electric fields around 50 kHz (Cheney et al. 1999; Henderson & Webster 1978; Price 1979). Magnetic resonance imaging – electric impedance tomography (MRI-EIT) (Khang et al. 2002; Oh et al. 2003) and current density impedance imaging (CDII) (Hasanov et al. 2004; Hasanov et al. 2008) measure conductivity at frequencies up to 1 kHz. The working frequency for microwave tomography (Bond et al. 2003; Francois et al. 1998; Meaney et al. 2000) is above 300 MHz, typically in GHz range.

RF-CDI provides some contrast mechanism of the conductivity in the sense that the ratio of tangential current density must equal the ratio of complex conductivity on either side of a boundary (Scott 1993). It is, however, generally difficult to quantitatively characterize the complex conductivity of tissues in a RF-CDI image, because one can not guarantee that current only flows parallel to the boundaries.
Figure 6.2 Estimated dielectric properties of body tissues. The conductivity $\sigma$ and the relative permittivity $\varepsilon_r$ are plotted in (a) and (b). The complex conductivity is defined as

$$\gamma = \sigma + j\omega\varepsilon_0\varepsilon_r.$$  

The real part of $\gamma$ is $\sigma$ as in (a) and the imaginary part is $\omega\varepsilon_0\varepsilon_r$ as in (c). The angle of $\gamma$ in degree is plotted in (d).
Iterative reconstruction methods have succeeded in reconstruction of the static conductivity (Khang, et al. 2002; Oh et al. 2003). These methods can be formulated in a way such that the measured quantities of the electromagnetic components are compared to predicted values of those same quantities obtained by forward modeling of electromagnetic fields. The resulting observed discrepancies are then used to make improvement in the system model parameters, i.e. the conductivity at each point. Recently, there have been preliminary reports on an iterative method for computation of complex conductivity (Katscher et al. 2006). However, its feasibility and underlying assumptions need to be further demonstrated.

Several explicit reconstruction methods have also been proposed and attempted for imaging the complex conductivity based on measurements of magnetic fields (Bulumulla et al. 2009; Katscher et al. 2009; Nachman et al. 2007; Sekino et al. 2008; Wiesinger et al. 2006; Zhang et al. 2010). They are all derived from the following Maxwell’s equations

\[ \nabla \times \mathbf{E} = -j \omega \mu_0 \mathbf{H}, \tag{6.6} \]

\[ \nabla \times \mathbf{H} = (\sigma + j \omega \epsilon) \mathbf{E}, \tag{6.7} \]

and

\[ \nabla \cdot \mathbf{H} = 0. \tag{6.8} \]

Taking the curl of (6.7) and substituting it to (6.6), we obtain

\[ \nabla \times (\nabla \times \mathbf{H}) = \frac{\nabla \gamma}{\gamma} \times (\nabla \times \mathbf{H}) - j \omega \mu_0 \gamma \mathbf{H}, \tag{6.9} \]

where \( \gamma = \sigma + j \omega \epsilon \), representing the complex conductivity. If \( \gamma \) can be considered as a constant, then using \( \nabla \cdot \mathbf{H} = 0 \) and \( \nabla \times (\nabla \times \mathbf{H}) = \nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} \), (6.9) becomes
Helmholtz equation

\[ \nabla^2 \mathbf{H} = -j \omega \mu_0 \gamma \mathbf{H}, \tag{6.10} \]

which can be separated as three scalar equations

\[
\begin{aligned}
\nabla^2 H_x &= -j \omega \mu_0 \gamma H_x \\
\nabla^2 H_y &= -j \omega \mu_0 \gamma H_y, \\
\nabla^2 H_z &= -j \omega \mu_0 \gamma H_z,
\end{aligned}
\tag{6.11}
\]

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \). Equation (6.11) indicates that the complex conductivity can be calculated for homogeneous media from any component of the magnetic field if this component is not zero. Equation (6.11) can also be applied to any linear combinations of \( H_x, H_y \) and \( H_z \), such as \( H_L = \frac{1}{2}(H_x - jH_y) \), the LCP component of \( \mathbf{H}_{xy} \). It is the mathematical principle employed in (Sekino et al. 2008).

Katscher et al. (Katscher et al. 2009) derived a reconstruction formula by the similar means. By substituting (6.7) and \( \gamma = \sigma + j \omega \varepsilon \) to the integral form of (6.6)

\[
\oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -j \omega \mu_0 \int_{\mathcal{A}} \mathbf{H} \cdot d\mathbf{A}, \tag{6.12}
\]

one can obtain

\[
\oint_{\partial A} \frac{1}{\gamma} \nabla \times \mathbf{H} \cdot d\mathbf{l} = -j \omega \mu_0 \int_{\mathcal{A}} \mathbf{H} \cdot d\mathbf{A}. \tag{6.13}
\]

If \( \gamma \) is constant on the surface \( \mathcal{A} \), then \( \gamma \) can be calculated by

\[
\gamma = -\frac{\oint_{\partial A} \nabla \times \mathbf{H} \cdot d\mathbf{l}}{-j \omega \mu_0 \int_{\mathcal{A}} \mathbf{H} \cdot d\mathbf{A}}. \tag{6.14}
\]
In fact, (6.14) is essentially the same as (6.10), because applying Stokes theorem to (6.13), we have

\[
\gamma = -j \omega \mu_0 \int_A \mathbf{H} \cdot d\mathbf{A} = -j \omega \mu_0 \int_A \mathbf{H} \cdot d\mathbf{A} = -j \omega \mu_0 \int_A \mathbf{H} \cdot d\mathbf{A}.
\]  

(6.15)

Obviously, (6.15) can be obtained by integrating both sides of (6.10) on the surface \(A\).

On the other hand, Nachman et al. proposed a reconstruction method obtained in a different way (Nachman et al. 2007). Taking the scalar product of (6.9) with \((\nabla \times \mathbf{H})\) and using \(\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) = 0\), we obtain

\[
(\nabla \times (\nabla \times \mathbf{H})) \cdot (\nabla \times \mathbf{H}) = -j \omega \mu_0 \gamma \mathbf{H} \cdot (\nabla \times \mathbf{H}).
\]

(6.16)

Since \(\nabla \cdot \mathbf{H} = 0\), the complex conductivity \(\gamma\) can be calculated by

\[
\gamma = \sigma + j \omega \varepsilon = -j \frac{(\nabla^2 \mathbf{H}) \cdot (\nabla \times \mathbf{H})}{\omega \mu_0 \mathbf{H} \cdot (\nabla \times \mathbf{H})}
\]

(6.17)

in regions where \(\mathbf{H} \cdot (\nabla \times \mathbf{H}) \neq 0\). Compared to (6.10), (6.17) has the advantage that it can be used in inhomogeneous media. In the implementations of both (6.10) and (6.17), one can expect poor signal to noise ratio where \(\mathbf{H}\) or \(\nabla \times \mathbf{H}\) (the total current density \(\mathbf{J}\)) is close to constant because in those cases \(\nabla^2 \mathbf{H}\) would be close to 0. Since both methods involve second derivatives, the implementation of the numerical calculation of these derivatives needs to be carefully addressed in practice to prevent the corruption of the reconstruction due to noise.
More recently, Zhang et al. (Zhang et al. 2010) proposed a reconstruction method for complex conductivity using two magnetic fields $\mathbf{H}_1$ and $\mathbf{H}_2$. We can learn the basic idea of this method by expanding (6.9) to three equations of complex scalars. If both $\gamma$ and $\nabla \gamma$ are regarded as unknowns, there are total four complex scalar unknowns. Therefore, if two $\mathbf{H}$ fields are available, there will be enough information to solve the equations.

This method as well as equations (6.10) and (6.14) was originally proposed for the reconstruction of the complex conductivity based on MRI $B_1$ mapping techniques (Katscher et al. 2009; Wiesinger, et al. 2006, Zhang et al. 2010). Computing complex conductivity via $B_1$ mapping will be more suitable for high field MRI, because when $\vec{B}_0$ is high ($\geq 3T$), $B_1$ field produced by the transmission coil of the MR imager tends to be inhomogeneous. The feasibility of the implementation of these methods, however, depends upon future development of the MRI $B_1$ mapping techniques (Zhang et al. 2010). Although there are well established techniques for measuring transmission $B_1$ magnitude (Akoka et al. 1993; Cunningham et al. 2006), almost all existing $B_1$ phase mapping methods are only able to measure the transmit or receive $B_1$ phase relative to the field of a reference coil (Van de Moortele et al. 2005). In other words, it is difficult to accurately measure any component of $\mathbf{H}$ or the LCP component of $\mathbf{H}_{xy}$ with respect to $\vec{B}_0$ (in the $z$ direction of the laboratory frame). Therefore, at current stage, the implementation of the complex conductivity reconstruction relies on three assumptions (Katscher et al. 2009):

1. The RCP component of the transmit $B_1$ field with respect to $\vec{B}_0$ is zero.

2. $H_z$ is 0.
3. The transmitting and receiving fields differ only by a 90 degree phase shift. These conditions, while true in an unloaded birdcage coil, need to be further investigated in a loaded coil especially for high field imaging.

On the other hand the reconstruction methods can be applied directly to the absolute $H$ field measured by the method used for RF-CDI that is described in Chapter 4 and Chapter 5. This removes the assumptions 1 and 2 mentioned above. Since RF-CDI magnetic field measurement technique requires that the direction of the transmit $B_1$ fields is approximately uniform, it is better suited with MR system of $B_0<3T$ and thus measures $\gamma$ at lower frequencies. The relative angle between the rotating frame and the laboratory frame, $\varphi$, will not influence the calculation as far as it can be regarded as a constant. Two or more currents may also be employed to satisfy the condition $H \cdot (\nabla \times H) \neq 0$ for the implementation of (6.17).

6.3 Conclusions

In the first part of this dissertation, we have implemented the first multi-slice radio frequency current density imaging (RF-CDI) technique. Electrolytic phantom studies indicate that the multi-slice RF-CDI sequence is capable of imaging current densities in multiple slices for various slice planes. This capability provides the essential measurements of the magnetic components required for the reconstruction of the current density vector. The characteristics of the random noise behavior have been analyzed and verified by simulations and experiments. The results imply that the number of slices that
can be imaged is constrained by $T_1$ values. This number has been roughly estimated to be between 7 and 35 for biological tissues.

In the second part of the dissertation, we have presented the first RF-CDI method that measures the full radio frequency (RF) current density vector. Previously implemented RF-CDI techniques measured the current density component along the direction of $\vec{B}_0$ with one sample orientation. However, the reconstruction relied on a restrictive assumption of the magnetic field produced by the applied current, the single orientation approximation. The violation of this approximation has been illustrated in both simulation models and in experiments, ranging from subtle differences to significant errors. Our new reconstruction method for RF-CDI has been developed based on sample reorientations. Results from electrolyte phantom experiments and simulation models have shown that this method can accurately determine any component or all three components of the current density vector without any restrictive assumptions of the magnetic field produced by the applied current. This method presents an important practical advantage in biomedical applications: motions of organs and tissues caused by the gravitational force can be avoided since our approach relies only on horizontal rotations.

We have thus advanced RF-CDI technology to measure the three dimensional current density vector fields in a three dimensional volume. However, technical limitations exist and they need to be taken into account in optimal design of the experiments. This makes it possible to further develop RF-CDI for biomedical and clinical imaging. In addition, by
providing the full magnetic vector field, the advance in RF-CDI presented in this thesis makes possible the reconstruction of the complex conductivity of biological tissues.
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Appendix A

Fundamental Concepts and Terminologies in MRI

This appendix does not intend to teach the readers the basics of MRI. Instead, it lists some concepts and terminologies in MRI that is most relevant to the description of the RF-CDI sequences. The knowledge of MRI techniques can be found in MRI text books, such as (Haacke et al. 1999; Liang & Lauterbur 1999)

A.1 Resonance and Relaxation

A fundamental property of nuclei is that those with odd atomic weights and/or odd atomic numbers, such as the nucleus of the hydrogen atom (proton), possess an angular momentum, often called spin. In MRI, an ensemble of nuclei of the same type is referred to as a nuclear spin system. When placed in an external magnetic field, a spin produces nuclear magnetism. To describe the collective behavior of a spin system, a macroscopic magnetization vector $\mathbf{M}$ is introduced, which is the vector sum of all the microscopic magnetic moments in a volume.

In the absence of external magnetic field, $\mathbf{M} = 0$. In the presence of an external magnetic field $\vec{B}_0$, a spin will precess left-handedly about the direction of $\vec{B}_0$ (the $z$ axis by convention) at Larmor frequency, $\omega_0 = \gamma B_0$, where $\gamma$ is the gyromagnetic ratio of the nucleus (usually proton in MRI). The macroscopic effect is the generation of a bulk magnetization vector pointing along the direction of $\vec{B}_0$. If another magnetic field, $\vec{B}_1(t)$, rotates at the Larmor frequency in the transverse plane, $\mathbf{M}$ will rotate towards the
transverse plane, while keeping precessing about the $z$ axis. This phenomenon is called resonance.

A **rotating frame** is defined as a coordinate system whose transverse plane is rotating left-handedly at an angular frequency $\omega$. The rotating frame referred to in this thesis is always the Larmor-rotating frame with $\omega = \omega_0$. In the rotating frame, instead of precessing in the laboratory frame, the magnetization $\mathbf{M}$ will remain in a plane perpendicular to the transverse plane. $\tilde{B}_j(t)$ is usually denoted as $B_j$ field in the rotating frame.

The $B_j$ field is always called the **RF pulse**, because it is short-lived and oscillates in the radio frequency (RF) range. If an RF pulse rotates $\mathbf{M}$ about $B_j$ field in the rotating frame by an angle $\alpha$, it is commonly called an $\alpha$ pulse. For example, a 90 pulse is an RF pulse that rotates $\mathbf{M}$ by 90 degrees, and a 180 pulse rotates $\mathbf{M}$ by 180 degrees.

After $\mathbf{M}$ has been perturbed from its equilibrium state by an RF pulse, it will return to this state, provided the RF pulse is removed and sufficient time is given. This process is characterized by a precession of $\mathbf{M}$ about the $\tilde{B}_o$ field, called **free precession**, a recovery of the longitudinal magnetization $M_z$, called the **longitudinal relaxation**, and the destruction of the transverse magnetization $M_{xy}$, called the **transverse relaxation**. The longitudinal relaxation arises from the interaction between the spins and the atomic neighborhood and is characterized by a time constant $T_1$. The transverse relaxation is
caused by the dephasing of the different spins and is characterized by a time constant \( T_2 \).

The values of \( T_1 \) and \( T_2 \) depend on the tissue composition, structure, and surroundings. For a given spin system, \( T_1 \) is always longer than \( T_2 \).

### A.2 Spatial Localization

Two major types of signal localization are used in MRI, namely, **selective excitation** and **spatial encoding**. Modern MRI systems provide three orthogonal gradients, which play a central role of the localization methods. Two-dimensional (2-D) imaging can be accomplished by **slice-selective excitation** and spatial information encoding.

A **slice-selection gradient** field and a **selective RF pulse** are essential to selectively excite spins in a slice. A selective RF pulse is frequency-selective and could only excite the spins whose Larmor frequencies are within a narrow bandwidth. To make the RF pulse spatially selective, a slice-selection gradient which varies linearly along the slice-selective direction is added to the homogeneous \( \vec{B}_0 \) field during the excitation period. Thus, only the spins in the desired slice are excited by the RF pulse.

After spins in a slice have been activated by a selective pulse, spatial information within the slice can be encoded into the signal to be detected during the free precession period. There are two ways of encoding spatial information: **frequency encoding** and **phase encoding**. Frequency encoding is realized by adding a linear gradient field, called **frequency encoding gradient**, along the **frequency encoding direction**, during the data
acquisition period. This makes the frequency of the magnetic resonance (MR) signal vary linearly along the frequency encoding direction.

Phase encoding is done by frequency encoding the signal for a short time interval before the data acquisition period. During the phase encoding interval, a linear gradient field called **phase encoding gradient** is applied along the **phase encoding direction**, which is usually orthogonal to the frequency encoding direction. A phase-encoded signal has the form of a non-encoded signal with a position-dependent initial phase angle, which can be adjusted with a variable phase-encoding gradient strength or phase-encoding interval from one excitation to another.

2-D spatial information is encoded in the detected complex MR signal. A MR image can be obtained by the 2-D Fourier transform, so it is also a complex image, including the **magnitude image** and the **phase image**.

**A.3 Pulse Sequence Diagram**

**Pulse sequence** diagram (PSD) is a graphic tool to describe the timing of RF pulses and gradient waveforms in an imaging scheme. Figure A.1 shows a diagram of a gradient-echo sequence. The timing is described for the RF pulses, the slice-selective gradient, the frequency encoding gradient, and the phase encoding gradient from the top to bottom, respectively.
$T_E$ is the symbol for **echo time**, defined as the time between the centre of the RF pulse and the centre of data acquisition period. The time between repetitions of the basic sequence in an imaging sequence is called **repetition time**, denoted by $T_R$. 

Figure A.1 Gradient-echo pulse sequence diagram.
Appendix B

Notes on RF-CDI Theory

The purpose of this appendix is to provide more detailed derivation of the RF-CDI reconstruction theory. The relation between the notations used in this thesis and previous expressions in (Scott et al. 1995a) is also discussed.

B.1 Field Representation

The principle of RF-CDI technology is based on Maxwell’s equations. For an isotropic, linear and reciprocal media, the Ampere’s circuital law indicates

\[ \nabla \times \vec{H} = \vec{j}_{\text{con}} + \frac{\partial \vec{D}}{\partial t} = \vec{j} \tag{B.1} \]

where \( \vec{H} \) is the magnetic intensity, \( \vec{D} \) is the electric flux density, \( \vec{j}_{\text{con}} \) is the conduction current density and \( \vec{j} \) represents the total current density, including both conduction and displacement current density. The magnetic intensity is also determined by its divergence

\[ \nabla \cdot \vec{H} = 0. \tag{B.2} \]

When current at Larmor frequency is applied to the sample, it will produce a sinusoidally time-varying electromagnetic field. Assume that the Cartesian coordinate system is placed such that the static magnetic field \( \vec{B}_0 \) is in the direction of \( z \) axis, then the total current density distribution \( \vec{j} \) and magnetic intensity \( \vec{H} \) are functions of position and can be expressed as
\[ \vec{J} = j_x \cos(\omega t + \phi_x) \vec{a}_x + j_y \cos(\omega t + \phi_y) \vec{a}_y + j_z \cos(\omega t + \phi_z) \vec{a}_z \] (B.3)

and

\[ \vec{H} = h_x \cos(\omega t + \theta_x) \vec{a}_x + h_y \cos(\omega t + \theta_y) \vec{a}_y + h_z \cos(\omega t + \theta_z) \vec{a}_z \] (B.4)

where \( \omega \) is the Larmor frequency. \( j, h, \phi, \theta \) are real functions. In phasor form, these can also be represented as

\[ \vec{J} = J_x \vec{a}_x + J_y \vec{a}_y + J_z \vec{a}_z \] (B.5)

\[ \vec{H} = H_x \vec{a}_x + H_y \vec{a}_y + H_z \vec{a}_z \] (B.6)

with

\[ J_i = j_i e^{j\omega t}, \]

\[ H_i = h_i e^{j\omega t} (i = x, y, z) \] (B.7)

In corresponding phasor expression, equations (B.1) and (B.2) become

\[ \nabla \times \vec{H} = \vec{J} \] (B.8)

and

\[ \nabla \cdot \vec{H} = 0. \] (B.9)

\( \vec{J} \) in (B.8) is the total current density, i.e.,

\[ \vec{J} = (\sigma + j\omega\varepsilon) \vec{E}, \] (B.10)

in which real parameters \( \sigma \) and \( \varepsilon \) represent conductivity and permittivity respectively.

**B.2 Transverse Magnetic Field**

Now, consider the magnetic intensity in the transverse plane \( \vec{H}_{xy} = H_x \vec{a}_x + H_y \vec{a}_y \).
Generally, it is an elliptically polarized field, which can always be treated as the summation of a right circularly polarized field and a left circularly polarized field:

\[
\mathbf{H}_{xy} = H_x \mathbf{\hat{a}}_x + H_y \mathbf{\hat{a}}_y \\
= \frac{1}{2} (H_x + jH_y) (\mathbf{\hat{a}}_x + j\mathbf{\hat{a}}_y) + \frac{1}{2} (H_x - jH_y) (\mathbf{\hat{a}}_x + j\mathbf{\hat{a}}_y)
\]

The first term in (B.11) is the right circularly polarized part and the second term is the left circularly polarized field. This can be demonstrated by the instantaneous vector expression for the second term.

\[
\tilde{H}_l = \text{Re} \left( \frac{1}{2} (H_x + jH_y) (\mathbf{\hat{a}}_x + j\mathbf{\hat{a}}_y) e^{j\omega t} \right) \\
= \text{Re} (h_l e^{j\beta_l (\mathbf{\hat{a}}_x + j\mathbf{\hat{a}}_y)} e^{j\omega t}) \\
= h_l (\cos(\omega t + \theta_l) \mathbf{\hat{a}}_x - \sin(\omega t + \theta_l) \mathbf{\hat{a}}_y)
\]

We can see clearly that the vector \(\tilde{H}_l\) rotates left-handed in the transverse plane with an angular frequency \(\omega\). It has a magnitude of \(h_l\) and is oriented \(-\theta_l\) with respect to \(x\) axis at \(t = 0\) as shown in Figure B.1.
Suppose a coordinate system with the direction vectors in the transverse plane denoted by \( \mathbf{a}_x \) and \( \mathbf{a}_y \) is rotating left-handedly around the \( z \) axis at angular velocity \( \omega \). At \( t = 0 \), \( \mathbf{a}_x \) is in an angle \( \varphi \) with respect to \( \mathbf{a}_x \) as shown in Figure B.1. In this coordinate system (or rotating frame), \( \mathbf{H}_i \) could be written as

\[
\mathbf{H}_i = \mathbf{H}_p \mathbf{a}_x + \mathbf{H}_y \mathbf{a}_y
\]  

(B.13)

in which \( \mathbf{H}_p \) and \( \mathbf{H}_y \) are both real scalars. Instead of rotating, the left circularly polarized field \( \mathbf{H}_i \) would remain static in the rotating frame in an angle \( -\theta_i - \varphi \) with respect to \( \mathbf{a}_x \).

From (B.12)–(B.13)
\[
\frac{1}{2} (H_x - jH_y) = h e^{j\phi}
\]
\[
= h_x \cos \theta_j + j h_y \sin \theta_j
\]
\[
= \left( \tilde{H}_x - j \tilde{H}_y \right) e^{-j\phi}
\]

\(\tilde{H}_x\) and \(\tilde{H}_y\) can be computed from the detected magnetic flux density components \(\tilde{B}_x\) and \(\tilde{B}_y\) respectively if they are defined in the left rotating frame determined by the transmitter coil. The goal of RF-CDI is to construct the RF using the measured information of \(\tilde{H}_x\) and \(\tilde{H}_y\).

**Connection to Greig Scott’s expression**

In *Electromagnetic Considerations for RF Current Density Imaging* (Scott et al. 1995a), the formulas for rotating frame components are equation (13) and (14)

\[
(13) \quad \tilde{H}_x = \frac{1}{2} \left[ h_x \cos(\theta_x - \Psi) + h_y \cos(\theta_y - \Psi) \right]
\]

\[
(14) \quad \tilde{H}_y = \frac{1}{2} \left[ h_x \cos(\theta_x - \Psi) - h_y \cos(\theta_y - \Psi) \right].
\]

Expanding above (B.14), we have

\[
\tilde{H}_x = \frac{1}{2} \left[ h_x \cos(\theta_x + \varphi) + h_y \cos(\theta_y + \varphi) \right]
\]

and

\[
\tilde{H}_y = \frac{1}{2} \left[ h_x \cos(\theta_x + \varphi) - h_y \cos(\theta_y + \varphi) \right]
\]

Notice that, \(\Psi\) in the above equations is equivalent to \(-\varphi\) in (B.14).
B.3 Single Orientation Approximation

Partially expand (B.8) and expand (B.9), we obtain

\[ J_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \]  
(B.15)

\[ \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \]  
(B.16)

If \( \phi \) is a global constant, According to (B.14), (B.15) can be rewritten as

\[ J_z = 2j \left( \frac{\partial H_x}{\partial x} - j \frac{\partial H_y}{\partial x} \right) e^{-j\phi} - 2 \left( \frac{\partial H_x}{\partial y} - j \frac{\partial H_y}{\partial y} \right) e^{-j\phi} - j \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) \]

\[ = 2 \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) e^{-j\phi} + 2j \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) e^{-j\phi} - j \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) \]  
(B.17)

Combine (B.17) with (B.16), we have

\[ J_z e^{j\phi} = 2 \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) + 2j \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) + j \frac{\partial H_z}{\partial z} e^{j\phi} \]  
(B.18)

or

\[ j_z \cos(\omega t + \phi_z + \phi) = \frac{2 \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \cos \omega t - 2 \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) \sin \omega t}{\partial e} \]

\[ - \frac{\partial}{\partial z} (h_z \sin(\omega t + \theta_z + \phi)) \]  
(B.19)

which implies that when \( \left| \frac{\partial H_z}{\partial z} \right| \ll j_z \), the current density in the z direction can be evaluated by

\[ J_z = j_z e^{j\phi_z + \phi} = 2 \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) + 2j \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) \]  
(B.20)
Connection to Greig Scott’s expression

Compare (B.19) to equations (20) and (21) in (Scott et al. 1995a).

If we set \( \omega t = -\phi \) and \( \varphi = -\Psi \) in above (B.19), then

\[
 j_z \cos(\phi_z - \Psi - \phi) = 2\left(\frac{\partial \tilde{H}_z}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\right) \cos \phi + 2\left(\frac{\partial \tilde{H}_x}{\partial x} + \frac{\partial \tilde{H}_y}{\partial y}\right) \sin \phi - \frac{\partial}{\partial z} \left(h_z \sin(\theta_z - \Psi - \phi)\right).
\]

Rearrange the above equation, we have

\[
2\left(\frac{\partial \tilde{H}_z}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\right) \cos \phi + 2\left(\frac{\partial \tilde{H}_x}{\partial x} + \frac{\partial \tilde{H}_y}{\partial y}\right) \sin \phi = j_z \cos(\phi_z - \Psi - \phi) + \frac{\partial}{\partial z} \left(h_z \sin(\theta_z - \Psi - \phi)\right)
\]

which is exactly (20) + (21) in (Scott et al. 1995a).

B.4 Reconstruction with one 180-degree rotation

Now consider rotating the sample around lab frame \( x \) axis 180 degrees as illustrated in Figure 2. The original right circularly polarized component in the transverse plane according to the imager coordinates becomes the left circularly polarized component after rotation. Therefore, with the same RF-CDI data acquisition, we should be able to measure the right circularly polarized field. We have

\[
\frac{1}{2}(H_x + jH_y) = (\tilde{H}_x^\pi - j\tilde{H}_y^\pi)e^{-j\psi}, \quad (B.21)
\]
where the superscript $\pi$ in $\tilde{H}^\pi_x$ indicates that it is obtained after rotation. Similarly, superscript 0 is used to indicate the data collected at the original position. Assume that $\phi$ does not change after rotation, then due to (B.14) and (B.21),

$$H_x = \left( \tilde{H}^0_x + \tilde{H}^\pi_x \right) - j \left( \tilde{H}^0_y + \tilde{H}^\pi_y \right) e^{-j\varphi}$$  \hspace{1cm} (B.22)$$

and

$$H_y = \left( \tilde{H}^0_y - \tilde{H}^\pi_y \right) - j \left( \tilde{H}^0_x - \tilde{H}^\pi_x \right) e^{-j\varphi}. \hspace{1cm} (B.23)$$

Figure B.2  Illustration of sample rotation in $yz$ plane

The arrow on the cylinder represents a right circularly polarized component at the original position. It becomes left circularly polarized after the rotation.
More details on rotation

The magnetic intensity in the transverse plane before rotation is

\[ \mathbf{H}_{xy}(x, y, z) = H_x(x, y, z)\hat{a}_x + H_y(x, y, z)\hat{a}_y. \]

Assuming the sample has been rotated around the \( x \) axis for 180 degrees and denote the transverse magnetic intensity after the rotation \( \mathbf{H}^R_{xy}(x, y, z) \), then

\[ \mathbf{H}^R_{xy}(x, y, z) = H_x(x, y, z)\hat{a}_x + H_y^R(x, y, z)\hat{a}_y = H_x(x, -y, -z)\hat{a}_x - H_y(x, -y, -z)\hat{a}_y. \]

Therefore, we have (B.21). If the rotation is around the \( y \) axis,

\[ \mathbf{H}^R_{xy}(x, y, z) = H_x(x, y, z)\hat{a}_x + H_y^R(x, y, z)\hat{a}_y = -H_x(-x, y, -z)\hat{a}_x + H_y(-x, y, -z)\hat{a}_y. \]

(B.21) becomes

\[ -\frac{1}{2}(H_x + jH_y) = (\tilde{H}_x - j\tilde{H}_y)e^{-j\varphi} \quad \text{(B.21a)} \]

and

\[ H_x = \left( (\tilde{H}_x - \tilde{H}_y) - j(\tilde{H}_y - \tilde{H}_x) \right)e^{-j\varphi} \quad \text{(B.22a)} \]

\[ H_y = \left( (\tilde{H}_y + \tilde{H}_y) + j(\tilde{H}_y + \tilde{H}_y) \right)e^{-j\varphi} \quad \text{(B.23a)} \]

Insert (B.22) and (B.23) to (B.15), we obtain

\[ J_{z}e^{j\varphi} = \left( \frac{\partial \tilde{H}_y^0}{\partial x} - \frac{\partial \tilde{H}_y^0}{\partial y} - \frac{\partial \tilde{H}_y^0}{\partial x} - \frac{\partial \tilde{H}_y^0}{\partial y} \right) \]

\[ + j \left( \frac{\partial \tilde{H}_x^0}{\partial x} + \frac{\partial \tilde{H}_y^0}{\partial y} - \frac{\partial \tilde{H}_x^0}{\partial x} + \frac{\partial \tilde{H}_y^0}{\partial y} \right), \]

(B.24)

which means \( J_z \) can be reconstructed with an relative angle \( \varphi \) with respect to the lab frame.
B.5 How to Extract $H_z$ with a Third Orientation

Figure B.3 demonstrates the possible orientations to measure the complete $H$ field. Let’s use superscripts $0, \pi, \pi/2$ and $-\pi/2$ to distinguish the measurable rotating frame components. Assume the green surface is the imaging plane (Axial for orientation 1 and 2, and sagittal for orientation 3 and 4). We’ve already known that $H_x$ and $H_y$ can be calculated by (B.22) and (B.23). Assume $\varphi$ is the same constant for all orientations. Similar to (B.22) and (B.23), with orientation 3 and 4, we’ll have

$$H_x = \left( \widetilde{H}_x \frac{\pi}{2} + \widetilde{H}_x \frac{-\pi}{2} \right) - j \left( \widetilde{H}_y \frac{\pi}{2} + \widetilde{H}_y \frac{-\pi}{2} \right) e^{-j\varphi}$$  \hspace{1cm} (B.25)

and

$$-H_z = \left( \widetilde{H}_y \frac{\pi}{2} - \widetilde{H}_y \frac{-\pi}{2} \right) + j \left( \widetilde{H}_x \frac{\pi}{2} - \widetilde{H}_x \frac{-\pi}{2} \right) e^{-j\varphi}.$$ \hspace{1cm} (B.26)
Since $H_x$, $\tilde{H}_x^\pi$ and $\tilde{H}_y^\pi$ are known after three orientations, $\tilde{H}_x^{-\pi/2}$ and $\tilde{H}_y^{-\pi/2}$ can be solved by (B.25) because $H_x$ is complex scalar and all the rotating frame components are real. Therefore we can obtain the components for orientation 4 without doing the imaging.

$$\begin{cases} 
\tilde{H}_x^{-\pi/2} = \text{real}(H_x e^{i\varphi}) - \tilde{H}_x^\pi \\
\tilde{H}_y^{-\pi/2} = -\text{imag}(H_x e^{i\varphi}) - \tilde{H}_y^\pi 
\end{cases} \tag{B.27}$$

Substituting (B.27) to (B.28), we can calculate $H_z$. 

Figure B.3 Possible orientations for complete magnetic field reconstruction. Rotations are around the $x$ axis of the lab frame.
Figure B.4 Possible orientations for complete magnetic field reconstruction.
Rotations are around the $y$ axis of the lab frame.

If the rotation is around the $y$ axis of the lab frame as shown in Figure B.4, according to (B.23a) and (B.24a), we will obtain the following equations.

\[
\begin{align*}
H_y &= \left( \frac{\pi}{2} \tilde{H}_y + \frac{\pi}{2} \tilde{H}_x \right) + j \left( \frac{\pi}{2} \tilde{H}_x + \frac{\pi}{2} \tilde{H}_y \right) e^{-j\phi}, \\
H_x &= \left( \frac{\pi}{2} \tilde{H}_x - \frac{\pi}{2} \tilde{H}_y \right) - j \left( \frac{\pi}{2} \tilde{H}_y - \frac{\pi}{2} \tilde{H}_x \right) e^{-j\phi}.
\end{align*}
\]

\[
\begin{align*}
\tilde{H}_y^{\pi/2} &= \text{real}(H_y e^{j\phi}) - \tilde{H}_y^{\pi/2} \\
\tilde{H}_x^{\pi/2} &= \text{imag}(H_y e^{j\phi}) - \tilde{H}_x^{\pi/2}
\end{align*}
\]
B.6 Reconstruction based on orthogonal rotations

(Scott et al. 1995a) proposed a method to reconstruct the complete $\mathbf{H}$ field by three orthogonal sample orientations. Rewrite (B.14)

$$\frac{1}{2}(H_x - jH_y) = (\tilde{H}_x^{xy} - j\tilde{H}_y^{xy}) e^{-j\phi} \quad \text{(B.28)}$$

The subscripts on the left side of the equation denote the components in the sample coordinate system while the subscripts on the right side indicate the components in the rotating frame defined by transmitter and receive coils. The superscripts on the right side denote the plane in the sample coordinate system which is positioned perpendicular to $\tilde{B}_0$. For the other two orientations, we could obtain

$$\frac{1}{2}(H_y - jH_z) = (\tilde{H}_y^{yz} - j\tilde{H}_y^{yz}) e^{-j\phi} \quad \text{(B.29)}$$

$$\frac{1}{2}(H_z - jH_x) = (\tilde{H}_z^{zx} - j\tilde{H}_z^{zx}) e^{-j\phi} \quad \text{(B.30)}$$

Thus there is sufficient information to extract current density distribution. We can write the above equations as

$$\frac{1}{2} \begin{bmatrix} 1 & -j & 0 \\ 0 & 1 & -j \\ -j & 0 & 1 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \tilde{H}_x^{xy} - j\tilde{H}_y^{xy} \\ \tilde{H}_y^{yz} - j\tilde{H}_y^{yz} \\ \tilde{H}_z^{zx} - j\tilde{H}_z^{zx} \end{bmatrix} e^{-j\phi}$$

The solution is

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} e^{j\phi} = \begin{bmatrix} 1-j & 1+j & -1+j \\ -1+j & 1-j & 1+j \\ 1+j & -1+j & 1-j \end{bmatrix} \begin{bmatrix} \tilde{H}_x^{xy} - j\tilde{H}_y^{xy} \\ \tilde{H}_y^{yz} - j\tilde{H}_y^{yz} \\ \tilde{H}_z^{zx} - j\tilde{H}_z^{zx} \end{bmatrix} \quad \text{(B.31)}$$
Appendix C

RF-CDI Experiment Instructions

The goal of this document is to provide a brief guide to setup RF-CDI experiments with the new RF current control box (Zhang 2006).

C.1 Overview of Experiment Setup

Figure C.1 Schematic experiment layout
Figure C.1 illustrates the schematic experiment layout for RF-CDI experiments. The RF-CDI sequence is downloaded to the clinical MRI system. By running the sequence, the MRI system produces signals such as the trigger signals and RF pulses, which are used as input to the RF current control box. The outputs of the RF current control box are the blanking signal and RF signal with the desired frequency, phase, amplitude and duration, which are in turn input to the amplifier. The output of the amplifier (AMT 3206 RF amplifier(1994)) is then connected to the phantom through a matching network. The role of the matching network is to tune the impedance of the load to 50 ohms, which is the characteristic impedance of the cable. Therefore, the power delivered to the phantom can be maximized.

C.2 Setup for RF current control box

Figure C.2 shows the front view and back view of the RF current control box. On the front of the box, there are four BNC connectors: RF Out, Blanking, Trigger and RF In. Trigger and RF In are input connections, which receive the trigger signal and the signal of RF pulse from the MR imager. RF out and Blanking are output connections, which are to be connected to the RF amplifier. The details about how to connect the control box will be described in the section of hardware connection.

The LCD display, keypad and the rotary encoder are served as user interface to set the parameters to control the properties of the output RF signal such as the duration and amplitude of the RF current. These parameters will be discussed in detail in the section of parameters setup.
On the back of the box, there is a red button whose function is to reset. Please notice that this button should only be pushed when the RF-CDI sequence is stopped. After reset, a new set of parameters can be entered to the box. One does not need to reset the box between scans if the parameters are not to be changed.

Figure C.2 RF current control box
C.2.1 Hardware Connections

The hardware connections are shown in Fig.3. The RF input to the control box is connected to the RF output signal from the MR imager, which can be found from the panel on the wall besides the console of MR imager. The trigger input for the RF current control box is connected to the DABOUT7 output from the MR imager, which can be reached by a BNC connector from the back room of the GE MRI system. The trigger signal serves not only the start command for the control box but also the control word for

Figure C.3 Hardware connection schematic

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the relative phase between the RF input and the RF output. An illustration of the trigger signal is shown in Figure C.4. The blanking output and the RF output should be connected to the AMT 3206 RF amplifier. The switch for the blanking on the amplifier should be on H, which means high level is for blanking. The RF OUT from the amplifier is then connected to the impedance matching network and phantom through an isolator, whose function is to prevent power reflection to the amplifier.

C.2.2 Parameters Setup

There are 6 parameters for the RF current control box. They are listed in Table C.1. SWITC is a control variable to select the trigger input. When it is 1, the trigger should be the one from DABOUT7 with the waveform as shown in Figure C.4. An alternative is 2, which means the trigger signal to be checked is a simpler one without the function for selecting relative phase of RF current. With this second option, the trigger input of the box should be connected to the trigger output from the MR imager.
Table C.1 Parameters for RF Current Control Box

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Meaning</th>
<th>Unit</th>
<th>Typical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>SWITC</td>
<td>options of trigger input</td>
<td>N/A</td>
<td>1</td>
</tr>
<tr>
<td>Ams</td>
<td>the duration of the first half of RF current</td>
<td>ms</td>
<td>4</td>
</tr>
<tr>
<td>Bms</td>
<td>the duration of the second half of RF current</td>
<td>ms</td>
<td>4</td>
</tr>
<tr>
<td>Cgap</td>
<td>the gap between Ams and Bms</td>
<td>us</td>
<td>0~100</td>
</tr>
<tr>
<td>Delay</td>
<td>the delay between the end of trigger and RF signal</td>
<td>us</td>
<td>565</td>
</tr>
<tr>
<td>Eadj</td>
<td>control the amplitude of the output RF</td>
<td>N/A</td>
<td>011111111111</td>
</tr>
</tbody>
</table>

Ams and Bms are the durations of the first half and the second half of the RF current, respectively. They should be equal to the first half and the second of the rotary echo RF pulse from the RF input. Cgap is the gap between Ams and Bms, which should cover the transition of the reversing in the middle of the rotary echo pulse. During Cgap, there is no RF output.

Delay is the time delay between the trigger and the output of RF signal, and also the blanking signal. An oscilloscope should be used to tune the parameter Delay. Figure C.5 shows the result when the Delay is 565 us for the multi-slice RF-CDI sequence. The green signal is the input RF signal (RF pulse from the imager), while the yellow one is the output RF signal from the RF current control box. The pink square wave is the blanking output from the box.
The Eadj is to adjust the magnitude of the RF output of the box by tuning one input of the multiplier at the RF input stage inside the box. The range of the parameter is from $0 \sim 2.048$ in the resolution of $2^{-11}$. It is shown on the LCD display (Figure C.6) in binary form with all 0’s the minimum value 0 and all 1’s the maximum value 2.048.
The values of the parameters can be changed and entered by the keypad or the rotary encoder (shown in Figure C.1). The keypad has not been labeled yet. The value or function presented by each key is shown in Figure C.7. The cursor on the LCD display can be moved by the four directional arrows. The last two parameters should be shown by moving the cursor down to the second page. After all the parameters are set, the cursor should be moved to the * in the first row (Figure C.6 Top) and the Enter key should be pushed. The LCD display will then shown either sequence a or sequence b depending on whether the SWITC is 1 or 2. This means the RF control box is ready for work. The key Esc has not been assigned any function yet.

Figure C.6 The parameters on LCD display
Top: parameters 1 to 4. Bottom: parameters 3 to 6. Also note that SWITC and Eadj should not have any units although there are units shown on the screen.
C.3 Operation on GE’s 1.5T Exciter MR imaging system

The typical console input values for the interface control variables (CVs) to perform a multi-slice RF-CDI experiment are listed in the Table C.2. The detailed description for the names and meanings of the CVs can be found in chapter 7 of (1997). Table C.3 shows an example of the setup for specific CVs associated with the multi-slice RF-CDI. These CVs can be accessed by clicking the ‘Research Operation’ button and then the ‘Display CVs’ button.

Before running Prescan, make sure that the amplifier is turned down (the hardware is connected and ready for an experiment). Turn on the amplifier when the Prescan has completed and then press SCAN button to run the multi-slice RF-CDI sequence.
<table>
<thead>
<tr>
<th></th>
<th>Parameter Label</th>
<th>value</th>
<th></th>
<th>Parameter Label</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Patient Information</strong></td>
<td>Patient ID</td>
<td><em>ms_rfcdi</em>²</td>
<td>Patient Name</td>
<td>Name</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Weight (kg)</td>
<td>70</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Patient Position</strong></td>
<td>Patient Position</td>
<td>Supine</td>
<td>Patient Entry</td>
<td>Head First</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Coil</td>
<td><em>Head Coil</em></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Imaging Parameters</strong></td>
<td>Plane</td>
<td><em>Axial</em></td>
<td>Mode</td>
<td>2D</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Pulse Sequence</td>
<td>GRE</td>
<td>Imaging Options</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Psd Name</td>
<td>/research/dhwang \ /rfcdi11ms</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Scan Timing</strong></td>
<td>TE</td>
<td>10ms</td>
<td>TR</td>
<td>20ms³</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FOV</td>
<td>24cm</td>
<td>Start(S/I)</td>
<td>S10</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Slice Thickness</td>
<td>2 mm</td>
<td>End(S/I)</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Spacing</td>
<td>3mm</td>
<td>NEX</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td><strong>Acquisition Timing</strong></td>
<td>Freq</td>
<td>256</td>
<td>Freq Dir</td>
<td>A/P</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Phase</td>
<td>256</td>
<td>Auto Center Freq</td>
<td>Water</td>
<td></td>
</tr>
</tbody>
</table>

a Italic words indicate a typical input, not the unique input.

b This is not actual TR, but the time between adjacent selective 90s.
### Table C.3 Typical Setup for Specific CVs Associated with RF-CDI

<table>
<thead>
<tr>
<th>CV</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>tlead_rot_echo</td>
<td>the time before the start of rotary echo $B_i$</td>
<td>500 $\mu$s</td>
</tr>
<tr>
<td>a_rot_echo</td>
<td>amplitude of the rotary echo pulse</td>
<td>1$^a$</td>
</tr>
<tr>
<td>rot_echo_dur</td>
<td>the duration of the current</td>
<td>8 ms</td>
</tr>
<tr>
<td>t_wait$^b$</td>
<td>the extra waiting time for the recover of the magnetization</td>
<td>250 ms</td>
</tr>
</tbody>
</table>

---

$^a$ Fraction of maximum (23 $\mu$T or 980Hz) rotary echo strength. It might need to be reduced when rot_echo_dur increases.

$^b$ TR=(15+number of slices $\times$ 20 + t_wait )ms.
Appendix D

MRI Random Noise and Its Effects on RF-CDI

D.1 Random Noise in MRI Complex Images

There are many factors that influence the signal to noise ratio (SNR) of a MRI image. The random noise of an MRI experiment derives not only from the resistance of the receiving coil resistance, but also the thermally generated, randomly fluctuating noise currents in the sample. Generally, the random noise in each channel of a quadrature detector is assumed to be white Gaussian noise with zero mean. Since the real and imaginary images are computed from the detected k-space data by discrete Fourier transform which is a linear and orthogonal transform, the Gaussian Characteristics is preserved in the real and the imaginary images and the variance of the noise can be assume uniform over the whole field of view. Moreover, the noise is uncorrelated among voxels as well as in the real and imaginary part of each voxel (Gudbjartsson & Patz 1995).

D.1.1 Random Noise in MRI Magnitude Images

Write the complex MRI signal of a voxel as

\[ M = (s_R + x) + j(s_I + y), \]

(D.1)

where \( s_R \) and \( s_I \) are the real and imaginary parts of the true signal and \( x \) and \( y \) are normal distributed independent random variables with mean 0 and standard deviation \( \sigma \). The magnitude of the complex voxel signal is
\[ m = |M| = \sqrt{(s_R + x)^2 + (s_I + y)^2} \]  \hspace{1cm} (D.2)

Assume the magnitude of the noiseless signal is

\[ A = \sqrt{s_R^2 + s_I^2} , \]  \hspace{1cm} (D.3)

then the probability density function for \( m \) is a Rician distribution as

\[
P_m(m) = \begin{cases} 
\frac{m}{\sigma^2} e^{-\frac{(m^2 + A^2)}{2\sigma^2}} I_0\left(\frac{Am}{\sigma^2}\right), & m \geq 0 \\
0, & m < 0 
\end{cases} \]  \hspace{1cm} (D.4)

where

\[ I_0(\eta) = \frac{1}{\pi} \int_0^\pi e^{\eta \cos \theta} d\theta \]

is the modified Bessel function of the first kind and zero\(^{th}\) order. When \( A = 0 \), (D.4) reduces to Rayleigh distribution

\[
P_m(m) = \begin{cases} 
\frac{m}{\sigma^2} e^{-\frac{m^2}{2\sigma^2}}, & m \geq 0 \\
0, & m < 0 
\end{cases} \]  \hspace{1cm} (D.5)

The mean and the variance for this distribution are (Papoulis & Pillai 1991)

\[ \bar{m} = \sigma \sqrt{\frac{\pi}{2}} \]  \hspace{1cm} (D.6)

and

\[ \sigma_m = (2 - \frac{\pi}{2})\sigma^2 . \]

These relations are often utilized to evaluate the MRI noise. For instance, a common way to estimate \( \sigma \) is through measuring the average value of a region outside the object where there is no signal and then diving the factor \( \sqrt{\frac{\pi}{2}} \approx 1.25 \) (Haacke et al. 1999).
When the SNR of MRI image, $A/\sigma$, is high (greater than 3), (D.4) starts to approximate the Gaussian distribution. When $A \gg \sigma$, 

$$P_m(m) \approx \frac{1}{\sqrt{2\pi \sigma}} e^{-(m - \sqrt{A^2 + \sigma^2})^2 / 2\sigma^2}.$$  \hspace{1cm} (D.7)

Figure D.1 shows $P_m(m/\sigma)$ for different MRI SNR $A/\sigma$. It is worth noting that the mean of the Rician distribution, $\bar{m}/\sigma$ is greater than $A/\sigma$, due to the nonlinear transform of (D.2).

![Graph showing $P_m(m/\sigma)$ for different MRI SNR $A/\sigma$. The vertical lines indicate the means of the distribution.](image)

Figure D.1 $P_m(m/\sigma)$ for different MRI SNR $A/\sigma$ (adapted from Figure 1 in (Gudbjartsson & Patz 1995) © 1995 John Wiley&Sons. Reprinted with permission). The vertical lines indicate the means of the distribution.

When the signal is much greater than the noise, the mean of the magnitude of the voxel signal is approximately the magnitude of the true signal 

$$\bar{m} = \sqrt{s_R^2 + s_I^2} = A$$  \hspace{1cm} (D.8)
and the variance of the magnitude \( m \) equals the variance of \( x \) or \( y \) (Haacke, Brown, Thompson, & Venkatesan 1999).

\[
\sigma_m^2 \approx \sigma^2
\]  

(D.9)

**D.1.2 Noise in MRI Phase Images**

Denote \( \theta \) the phase of MRI noiseless images, i.e.,

\[
\tan \theta = \frac{s_I}{s_R}.
\]  

(D.10)

Define the phase noise as \( \delta \theta \).

\[
\tan(\theta + \delta \theta) = \frac{s_I + y}{s_R + x}.
\]  

(D.11)

The distribution of \( \delta \theta \) is

\[
P_{\delta \theta}(\delta \theta) = \frac{1}{2\pi} e^{-\delta \theta^2/2\sigma^2} \left( 1 + \frac{A}{\sigma} \sqrt{2\pi} \cos \delta \theta e^{\delta \theta^2/2\sigma^2} \int_{-\infty}^{A \cos \delta \theta} e^{-x^2/2} dx \right).
\]  

(D.12)

When \( A = 0 \), (D.12) reduces to uniform distribution

\[
P_{\delta \theta}(\delta \theta) = \frac{1}{2\pi}.
\]  

(D.13)

For this case, the standard deviations of the phase noise is

\[
\sigma_\theta = \sqrt{\frac{2\pi^2}{3}}.
\]  

(D.14)

For large MRI SNR, \( A \gg \sigma \), (D.12) can be approximated by a zero mean Gaussian distribution,

\[
P_{\delta \theta}(\delta \theta) \approx \frac{1}{\sqrt{2\pi(\sigma / A)^2}} e^{-\delta \theta^2/2(\sigma / A)^2}.
\]  

(D.15)

The standard deviation of this distribution is
\[ \sigma_y = \frac{1}{\text{SNR}} \] (D.16)

**D.2 Random Noise in RF-CDI**

There are two magnetic field mapping methods for RF-CDI, the polar decomposition method and the rotary echo method. The random noise effects are discussed here only based on the rotary echo measurement, for it is the only method utilized in this thesis.

**D.2.1 Noise Effects of RF-CDI Measurements**

One feature of the rotary echo method is that the rotating frame magnetic components are linearly dependent on rotation angle \( \Gamma_x \) and \( \Gamma_y \). Because the single-slice RF-CDI measures the rotation angle completely in the phase images, \( \tilde{H}_x \) and \( \tilde{H}_y \) bear the same distribution as described in (D.12). Although it is a complicated equation, for \( A/\sigma > 3 \), it starts to resemble zero mean Gaussian distribution (Gudbjartsson & Patz 1995). Therefore, for a single-slice RF-CDI measurement with reasonable SNRs (>3), Gaussian distribution is a good approximation for the noisy \( \tilde{H}_x \) and \( \tilde{H}_y \). The standard deviation is given by (Scott 1993)

\[ \sigma_{\tilde{H}} = \frac{1}{k \gamma T_c \mu_0 \text{SNR}} \] (D.17)

where \( k \) is a factor associated with the phase cycling details in the measurement. \( \tilde{H}_x \) or \( \tilde{H}_y \) are uncorrelated for different voxels. \( \tilde{H}_x \) and \( \tilde{H}_y \) are uncorrelated to each other in each voxel.
On the other hand, the multi-slice RF-CDI sequence does not calculate the rotation angle directly from the MRI phase image. Instead, as discussed in Chapter III, it is computed by

\[
\Gamma_x = \begin{cases} 
\tan^{-1}\left[ \text{Re}(C_y/C_z^*) \right] & \text{if } |M_z| \geq |M_y| \\
\cot^{-1}\left[ \text{Re}(C_z/C_y^*) \right] & \text{if } |M_z| < |M_y|
\end{cases}
\]  
(D. 18)

where \( C_y^* \) etc. represent the complex MRI images. Write the noisy \( C_y^* \) as

\[
C_y^* = m_y^*e^{j(\theta + \delta\theta_y)} ,
\]  
(D.19)

where \( m_y^* \) has the Rician distribution as in (D.4) and the distribution of \( \delta\theta_y \) is described by (D.12). Insert the expression of (D.19) to (D.18), the first equation in (D.18), one can obtain

\[
\Gamma_x = \tan^{-1}\left[ \frac{m_y^*e^{j\delta\theta_y}}{m_z^*e^{j\delta\theta_z}} \right] = \tan^{-1}\left( \frac{m_y^*\cos(\delta\theta_y - \delta\theta_z)}{m_z^*} \right). 
\]  
(D.20)

When SNR for \( C_y^* \) and \( C_z^* \) is big (usually \( |\Gamma_x| \) close to \( \pi/4 \), \( \cos(\delta\theta_y - \delta\theta_z) \approx 1 \), and the probability density functions for both \( m_y^* \) and \( m_z^* \) resemble Gaussian distribution. In this case, \( \Gamma_x \) can be predicted to have similar distribution as the phase of MRI complex images. Other than that, it is hard to completely characterize the noise in the rotation angle without intensive mathematical derivation or numerical curve fitting. However, as discussed in Chapter III, when MRI SNR is large for images without current, the mean and standard deviation of \( \bar{H}_x \) and \( \bar{H}_y \) can be approximated by

\[
\overline{\delta H} = 0.  
\]  
(D.21)

\[
\sigma_H = \frac{e^{iT_\gamma}}{\sqrt{2T_\gamma\mu_0\text{SNR}}}. 
\]  
(D.22)
D.2.2 Noise in Reconstructed RF Current Density Images

The single orientation reconstruction is used as an example to discuss the current density noise. Current density noise associated with other reconstruction methods can be evaluated similarly. The single orientation reconstruction formula is

\[
J_{z}^{rec} = 2 \left( \frac{\partial \hat{H}_{y}}{\partial x} - \frac{\partial \hat{H}_{x}}{\partial y} \right) + 2j \left( \frac{\partial \hat{H}_{x}}{\partial x} + \frac{\partial \hat{H}_{y}}{\partial y} \right).
\]  

(D.23)

The derivatives in (D.23) are calculated by numerical templates. In other words, the real part or the imaginary part of \( J_{z}^{rec} \) is a linear combination of \( \hat{H}_{x} \) and \( \hat{H}_{y} \) at various voxels. Since \( \hat{H}_{x} \) and \( \hat{H}_{y} \) are uncorrelated to each other, and \( \hat{H}_{x} \) or \( \hat{H}_{y} \) is uncorrelated for different voxels, variance of the real part or imaginary part of \( J_{z}^{rec} \) is the sum of the variance of weighted \( \hat{H}_{x} \) or \( \hat{H}_{y} \) for different voxels. For the single-slice RF-CDI measurement, the real part and the imaginary part of \( J_{z}^{rec} \) are approximately Gaussian distributed if MRI SNR is greater than 3.

![2D Sobel operators for computation of numerical derivatives.](image)

Figure D.2 2D Sobel operators for computation of numerical derivatives. Left: labels for spatial positions. Middle: Sobel template for derivatives in the \( x \) direction. Right: Sobel template for derivatives in the \( y \) direction.
It is not straightforward to discuss the current noise in terms of current magnitude and current phase if the real part and the imaginary part are correlated. To simplify the situation, one can always select the numerical templates such that there is no overlapping voxels for the derivative calculation in the $x$ and the $y$ directions to guarantee that the real and the imaginary part are uncorrelated. However, the non-overlapping of the templates is not a necessary condition for the un-correlation.

For instance, if 2D sobel templates as shown in Figure D.2 are used for the derivative calculation, then

\[
\text{Re}(J_{\psi}^{\text{rec}}) = \frac{1}{4\Delta x} \left(-\hat{H}_y^1 - 2\hat{H}_y^4 - \hat{H}_y^7 + \hat{H}_y^3 + 2\hat{H}_y^6 + \hat{H}_y^9\right) + \frac{1}{4\Delta y} \left(\hat{H}_x^1 + 2\hat{H}_x^2 + \hat{H}_x^3 - \hat{H}_x^7 - 2\hat{H}_x^8 - \hat{H}_x^9\right),
\]

(D.24)

and

\[
\text{Im}(J_{\psi}^{\text{rec}}) = \frac{1}{4\Delta x} \left(-\hat{H}_x^1 - 2\hat{H}_x^4 - \hat{H}_x^7 + \hat{H}_x^3 + 2\hat{H}_x^6 + \hat{H}_x^9\right) + \frac{1}{4\Delta y} \left(-\hat{H}_y^1 - 2\hat{H}_y^2 - \hat{H}_y^3 + 2\hat{H}_y^8 + \hat{H}_y^9\right),
\]

(D.25)

where the superscripts in the rotating frame components denote the positions of the voxels involved in the computation as shown in Figure D.2. Now, let us calculate the covariance of the real part and the imaginary part of $J_{\psi}^{\text{rec}}$. Since $\delta\hat{H}_x$ and $\delta\hat{H}_y$ have zero mean, the covariance is
\[ \text{Cov}(\text{Re}(J_{rec}^z), \text{Im}(J_{rec}^z)) = E\left(\frac{1}{4\Delta x}(-\delta \tilde{H}_x^1 - 2\delta \tilde{H}_x^4 - \delta \tilde{H}_y^7 + \delta \tilde{H}_y^3 + 2\delta \tilde{H}_y^6 + \delta \tilde{H}_y^9) \right) \]
\[ + \frac{1}{4\Delta y}(\delta \tilde{H}_x^1 + 2\delta \tilde{H}_x^4 + \delta \tilde{H}_y^3 - \delta \tilde{H}_y^7 - 2\delta \tilde{H}_y^8 - \delta \tilde{H}_y^9)) \times \]
\[ (\frac{1}{4\Delta x}(-\delta \tilde{H}_x^1 - 2\delta \tilde{H}_x^4 - \delta \tilde{H}_y^7 + \delta \tilde{H}_y^3 + 2\delta \tilde{H}_y^6 + \delta \tilde{H}_y^9)) \]
\[ + \frac{1}{4\Delta y}(-\delta \tilde{H}_y^1 - 2\delta \tilde{H}_y^2 - \delta \tilde{H}_y^3 + \delta \tilde{H}_y^7 + 2\delta \tilde{H}_y^8 + \delta \tilde{H}_y^9))\]  

(D.26)

Because \( \delta \tilde{H}_x \) or \( \delta \tilde{H}_y \) is uncorrelated for different voxels, and \( \delta \tilde{H}_x \) and \( \delta \tilde{H}_y \) are uncorrelated in the same voxel,

\[ \text{Cov}(\text{Re}(J_{rec}^z), \text{Im}(J_{rec}^z)) = \frac{1}{16\Delta x\Delta y}E\left[(\delta \tilde{H}_x^1)^2 - (\delta \tilde{H}_y^7)^2 - (\delta \tilde{H}_y^3)^2 + (\delta \tilde{H}_y^9)^2 \right] \]
\[ - (\delta \tilde{H}_x^1)^2 + (\delta \tilde{H}_y^3)^2 - (\delta \tilde{H}_x^4)^2 - (\delta \tilde{H}_y^9)^2 \]  
\[ = \frac{1}{16\Delta x\Delta y} \left[ \sigma_{\tilde{H}_x}^2 - \sigma_{\tilde{H}_y}^2 - \sigma_{\tilde{H}_y}^2 + \sigma_{\tilde{H}_y}^2 \right] \]
\[ - \sigma_{\tilde{H}_x}^2 + \sigma_{\tilde{H}_y}^2 + \sigma_{\tilde{H}_y}^2 - \sigma_{\tilde{H}_y}^2 \]
\[ = 0 \]  

(D.27)

Therefore, the real part and the imaginary part of \( J_{rec}^z \) are uncorrelated in this case. Then, when SNR of current density is high, the standard deviation of the current density magnitude is the same as \( \sigma_J \) evaluated in the real or the imaginary part, and the standard deviation of current phase noise is the reciprocal of current density SNR. Furthermore, for the reconstruction based on the single-slice RF-CDI measurement, the distribution of the current density magnitude should resemble the Rice density function and the distribution of the current density phase should be similar to (D.12).
Appendix E

List of Publications

E.1 Journal Papers


E.2 Conference Papers


Nachman, A. I., **Wang, D.,** Ma, W., & Joy, M. L. G., 2007 "A local formula for inhomogeneous complex conductivity as a function of the RF magnetic field", *Unsolved problems and Unmet Needs session, the 15th Science Meeting of International Society for Magnetic Resonance in Medicine.*


