COMPACT DYNAMICAL FOLIATIONS

by

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Abstract

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According to the work of Dennis Sullivan, there exists a smooth flow on the 5-sphere all of whose orbits are periodic although there is no uniform bound on their periods. The question addressed in this thesis is whether such an example can occur in the partially hyperbolic context. That is, does there exist a partially hyperbolic diffeomorphism of a compact manifold such that all the leaves of its center foliation are compact although there is no uniform bound for their volumes. We will show that the answer to the previous question under the very mild hypothesis of dynamical coherence is no.

The thesis is organized as follows. In the first chapter we give the necessary background and results in partially hyperbolic dynamics needed for the rest of the work, studying in particular the geometry of the center foliation. Chapter two is devoted to a general discussion of compact foliations. We give proof or sketches of all the relevant results used. Chapter three is the core of the thesis, where we establish the non existence of Sullivan's type of examples in the partially hyperbolic domain, and generalize to diffeomorphisms whose center foliation has arbitrary dimension. The last chapter is devoted to applications of the results of chapter three, where in particular it is proved that if the center foliation of a dynamically coherent partially hyperbolic diffeomorphism is compact and without holonomy, then it is plaque expansive.

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I could thank here all the people that helped me during this project, but since this is in some sense the culmination of my curricular studies, I'll use the opportunity to thank (some of) the persons that helped me along the way. The order, of course, is not important.

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Proof of what all this and other people have taught me, I hope, can be found here.

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Notations

We collect here the notations used throughout this thesis. For necessary background see [Ste51],[God91].

Manifolds and Vector Bundles *Manifold* will mean $\mathcal{C}^\infty$ manifold. Riemannian metrics will be assumed of differentiability class at least $\mathcal{C}^2$. A *closed manifold* means a compact manifold without boundary. A submanifold means an injectively immersed manifold unless otherwise stated.

Vector bundles considered will be continuous sub-bundles of the tangent bundle of some manifold M . In particular, for a vector bundle E over M we will not explicitly write the projection.

A k dimensional vector bundle E over M is said to be orientable if its structure group can be reduced to $GL^+(k, \mathbb{R})$, the group of k by k matrices with positive determinant. If E is not orientable then M has a canonical two-fold covering

$$\pi : \widetilde{M} \rightarrow M$$

such that π^*E is orientable.

If E is a vector bundle over a closed Riemannian manifold M and $T : E \rightarrow E$ is a vector bundle isomorphism we will denote by $\|T\|$ the uniform norm, i.e.

$$\|T\| = \max\{\|T_x\| : x \in M\}$$

Foliations By a $\mathcal{C}^{1,0}$ *foliation atlas of codimension q* on a manifold M we mean an atlas

$\mathcal{A} = \{U_i, h_i : U_i \rightarrow \mathbb{R}^p \times \mathbb{R}^q\}_i$ such that the homeomorphisms giving the change of coordinates $h_{ij} : h_j(U_i \cap U_j) \rightarrow h_i(U_i \cap U_j)$ are of the form

$$h_{ij}(u, v) = (a_{ij}(u, v), b_{ij}(v))$$

where $a_{ij} : h_j(U_i \cap U_j) \rightarrow \mathbb{R}^p$ is of class at least $\mathcal{C}^1$, and $b_{ij} : h_j(U_i \cap U_j) \rightarrow \mathbb{R}^q$ is continuous. A $\mathcal{C}^{1,0}$ *foliation* on a manifold M is a maximal $\mathcal{C}^{1,0}$ foliation atlas (with respect to the inclusion)¹. Given a $\mathcal{C}^{1,0}$ foliation atlas it determines a unique $\mathcal{C}^{1,0}$ foliation (the maximal foliation atlas in which it is contained). A $\mathcal{C}^{1,0}$ foliation $\mathcal{F}$ is said to be a $\mathcal{C}^1$ foliation if there exists a foliation atlas for $\mathcal{F}$ whose changes of coordinates are $\mathcal{C}^1$. In cases where it is not necessary to emphasize the regularity properties of $\mathcal{C}^{1,0}$ foliations, we will refer to them simply as *foliations*.

The sets U_i are said to be *foliated cubes* or simply *cubes* if $h_i(U_i)$ is a cube in $\mathbb{R}^{p+q}$.

A foliation atlas $\mathcal{F} = \{U_i, h_i\}_i$ is *nice* if

1. Every U_i is a cube.
2. If $U_i \cap U_j \neq \emptyset$ then there exists a cube in the foliation determined by $\mathcal{F}$ that contains $cl(U_i \cup U_j)$.

If $\mathcal{F}$ is a foliation atlas, there is always a nice refinement. We therefore will always assume that foliation atlases are nice.

If (U_i, h_i) is a coordinate chart of a foliation $\mathcal{F}$ with U_i a cube, $x \in M$ and

$\pi_q : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ denotes the projection, the set $P_x = \{y \in U_i : \pi_q h_i(y) = \pi_q h_i(x)\}$

is the plaque of U_i containing x . Two points $x, y \in M$ are said related if there exists a finite chain of plaques $P_1, \dots, P_k$ such that

1. $x \in P_1, y \in P_k$.

¹This is also called *lamination with smooth leaves* in the literature.

2. for every $i = 1, \dots, k - 1$ we have $P_i \cap P_{i+1} \neq \emptyset$.

Being related is an equivalence relation. The classes of this relation are called the leaves of the foliation: they are injectively immersed $\mathcal{C}^1$ submanifolds of M . If L is a leaf, we will usually say that it belongs to the foliation and we will denote this by writing $L \in \mathcal{F}$.

A foliation is said to be compact if every leaf is a compact submanifold of the ambient manifold.

Holonomy Let P, P' be plaques of a codimension q foliation $\mathcal{F}$ and assume that $P \cap P' \neq \emptyset$. Take $x \in P, x' \in P'$ and consider two small discs D, D' of dimension q centered at x, x' respectively and such that they are transverse to the leaves of $\mathcal{F}$. Then there exist relative open subsets $U \subset D, U' \subset D'$ containing the points x, x' and a homeomorphism $h : U \rightarrow U'$ such that $h(x) = x'$ and for every $y \in U$ the points y and $h(y)$ are in the same leaf. The map h is said to be the *holonomy transport* from U to U' . Likewise, if x, x' are in the same leaf there exist well defined holonomy transports from some transversal U containing x to a transversal U' containing x' . The set of maps obtained under this procedure forms a pseudogroup called the holonomy pseudogroup.

If $x \in M$, the sub-pseudogroup of the holonomy pseudogroup that fixes x is referred to as the holonomy pseudogroup at x . Every element of this pseudogroup is obtained by taking a loop inside the leaf containing x , covering this loop by plaques and considering the holonomy transport that this chain of plaques determines. If two loops define the same class in the fundamental group of the leaf in which x is contained, then the two corresponding holonomy transports defined by these loops have the same germ. Denoting by L the leaf containing x and picking a small transversal T containing x , one has a representation

$$hol : \pi_1(L, x) \rightarrow G_x(T)$$

where $G_x(T)$ denotes the group of germs of homeomorphisms of T that fix x . Changing the transversal T changes the representation hol to an equivalent one. Likewise if the base point x is changed inside L .

The image of hol is called the holonomy group of L at x . One says that L has *finite holonomy* if the holonomy group of L at x is finite.

Orientation Let $\mathcal{F}$ be a foliation on M . The set of tangent spaces of the leaves forms a vector bundle $T\mathcal{F}$ over M . If $T\mathcal{F}$ is orientable we say that the foliation $\mathcal{F}$ is orientable. If $\mathcal{F}$ is not orientable, and considering the two-fold covering $\pi : \widetilde{M} \rightarrow M$ that orients $T\mathcal{F}$, one obtains an orientable foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ such that π sends leaves of $\widetilde{\mathcal{F}}$ onto leaves of $\mathcal{F}$.

A leaf $\widetilde{L} \in \widetilde{\mathcal{F}}$ has finite holonomy if and only if $L = \pi(\widetilde{L}) \in \mathcal{F}$ has finite holonomy.

Foliated bundles By a foliated bundle we mean a continuous fiber bundle $p : E \rightarrow B$ with typical fiber F , where E, B, F are manifolds and such that its structure group is discrete. In this case there exists a foliation $\mathcal{F}_E$ of E transverse to the fibers of p , where every leaf L is a covering of B .

The holonomy of $\mathcal{F}_E$ has the following simple characterization: if $x \in B$ and $F_x = \pi^{-1}(x)$ then there exists a representation

$$\psi : \pi_1(B, x) \rightarrow Homeo(F_x)$$

such that every map in the image of ψ sends each leaf of $\mathcal{F}_E$ to itself.

Given a leaf L and a point $y \in L$ such that $p(y) = x$ one has a representation

$$\psi \circ p_* : \pi_1(L, y) \rightarrow Homeo(F_x).$$

Taking the germ at y of the maps in the image of $\psi \circ p_*$ one obtains the holonomy group of L at y . Note that in this case the holonomy transports are defined globally in the fiber F_x .

As will be explained in Chapter 2, the representation ψ determines the bundle E up to isomorphism, and given a representation of the form

$$\psi : \pi_1(B, x) \rightarrow \text{Homeo}(F)$$

there exists a foliated bundle $p : E \rightarrow B$ with typical fiber F such that ψ coincides with the total holonomy representation. Given a foliated bundle with typical fiber F and total holonomy representation ψ , we will use the notation $E = F \times_\psi B$.

Chapter 1

Partially Hyperbolic Diffeomorphisms

In this chapter we give the necessary definitions and preliminaries of dynamical systems that will be used throughout the whole work. The first part of the material in this chapter is standard, but we recommend the reader to glance through this chapter to get used to the definitions and notation. In the last part, we discuss completeness of the invariant foliations and local product structure.

The study of chaotic dynamical systems started with H. Poincaré in the last part of the XIX century, who was interested in understanding the trajectories of celestial bodies. He realized that small changes in the initial conditions of the orbit could lead to very different orbit types, and developed many qualitative methods to understand the behavior of such systems. Later in the XX century the area grew very rapidly, playing a role of central importance in both applied and theoretical mathematics.

It was S. Smale who realized that a good model for understanding chaotic dynamical systems were the so called hyperbolic systems, which comprised a reasonable compromise between simplicity and generality.

Here we are interested in a generalization of hyperbolic systems, where besides the

hyperbolic directions we will allow a neutral one. The presence of this direction permits a very rich type of structure in these systems, which naturally makes their study harder than hyperbolic systems. Nonetheless, it is believed that these systems are “usual” inside the chaotic ones¹, which together with the beauty of their complexity makes their study one of the most active research areas in dynamical systems today. We now give the definition.

Let M be a closed Riemannian manifold. We say that a diffeomorphism $f : M \rightarrow M$ is *partially hyperbolic in the strong sense*² if there exists a continuous splitting of the tangent bundle into a Whitney sum of the form

$$TM = E^u \oplus E^c \oplus E^s$$

where neither of the bundles E^s nor E^u are trivial, and such that

1. All bundles E^u, E^s, E^c are df -invariant.
2. For all $x \in M$ and for all unit vectors $v^\sigma \in E_x^\sigma$ ($\sigma = s, u, c$)

$$\|d_x f(v^s)\| < \|d_x f(v^c)\| < \|d_x f(v^u)\|.$$

3. $\lambda = \max\{\|df|E^s\|, \|df^{-1}|E^u\|\} < 1$.

The bundles E^s, E^u, E^c are the *stable*, *unstable* and *center* bundle respectively. We also define the bundles $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^s \oplus E^c$.

As a harmless abuse of language, we will refer to these diffeomorphisms as partially hyperbolic. The case when $E^c = 0$ corresponds to f being *completely hyperbolic* or *Anosov*.

¹See for example [PS97].

²In contrast, one says that f is weakly partially hyperbolic if there exists a df -invariant continuous splitting $TM = E \oplus F$ such that the derivative of f uniformly contracts E or uniformly expands F , a constant $0 < C < 1$ and a positive integer n with the property that for every $x \in M$ we have $\|d_x f^n|E_x\| \cdot \|(d_x f)^{-n}|F_{f^{-n}x}\| < C$.

The degree of differentiability of the map f is an important issue when studying its properties (in particular its metric properties). However here we only need to assume that f is of class $\mathcal{C}^1$, and we will keep this assumption for the rest of the thesis.

1.1 Examples

We start by giving some examples of partially hyperbolic diffeomorphisms. For properties of these (and other) examples we refer the reader to [HHU07].

1.1.1 Ergodic Automorphisms of the Torus

The simplest way to construct a partially hyperbolic map is as follows. Consider a matrix $A \in SL(n, \mathbb{Z})$, and denote by f_A the map induced by A on the Torus $\mathbb{T}^n$. Then if we denote by E^s, E^c and E^u the direct sum of the eigenspaces corresponding to the eigenvalues of norm less than, equal to and bigger than one respectively, we easily see that f_A is partially hyperbolic with respect to these bundles.

As an interesting fact, we note the following. It is well known (and easy to show) that the automorphism f_A is ergodic with respect to the Lebesgue measure if and only if none of the eigenvalues of A is a root of unity. Let us assume that is the case. We claim that f_A is partially hyperbolic in the strong sense. The proof is as follows.

Since the product of the eigenvalues of A is equal to 1, one sees that it suffices to show that there exists one eigenvalue outside the unit circle. We will assume then that all eigenvalues of A are on the unit circle, and we will show that this implies that every eigenvalue is a root of the unity, obtaining a contradiction to the fact that f_A is ergodic.

Recalling that the eigenvalues of A^k are the k^{th} -powers of the eigenvalues of A , one

obtains that for every positive integer k

$$\text{tr}(A^k) = \sum_{i=1}^n \lambda_i^k \in \mathbb{Z}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A .

Now, by compactness, there exist positive integers $(k_l)_l^\infty$ such that

$$(\lambda_1^{k_l}, \dots, \lambda_n^{k_l}) \xrightarrow{l \rightarrow \infty} (1, \dots, 1)$$

and hence $\sum_{i=1}^n \lambda_i^{k_l} \xrightarrow{l} n$.

This implies that for big enough l the sum $\sum_{i=1}^n \lambda_i^{k_l}$ is equal to n , and since each eigenvalue has norm one, we conclude that each one of them is necessarily equal to one.

1.1.2 Time 1-maps of Hyperbolic flows

A classic example of partially hyperbolic map is obtained by taking a surface S of negative sectional curvature, and considering the time one map of the geodesic flow $\{\phi_t\}$ acting in the unit tangent bundle of S .

To check that we really get a partially hyperbolic map, we should recall the well known fact that $\{\phi_t\}$ is an *Anosov flow*, meaning that there exist two bundles E^s and E^u invariant under the flow, a number $0 < \lambda < 1$ and a constant $C > 0$ so that for $t \geq 0$

1. $\|d_x \phi_t(v)\| \leq C \lambda^t \|v\|$ if $v \in E_x^s$
2. $\|d_x \phi_{-t}(v)\| \leq C \lambda^t \|v\|$ if $v \in E_x^u$

These bundles correspond to the horocycle directions and are perpendicular to each other with respect to the Riemannian metric on S . If we denote by E^c the line bundle parallel to the flow direction, we easily get that the time one map of the flow (in fact, any map ϕ_t for $t \neq 0$) is partially hyperbolic. Of course we get the same conclusion if $\{\phi_t\}$ is an Anosov flow.

We remark that there exist two categories in some sense of transitive Anosov flows, which in turn give two different types of partially hyperbolic diffeomorphisms. On the one hand there exist the mixing ones (like the example corresponding to the geodesic flow in a surface of negative sectional curvature), and on the other the ones where the distribution $E^s \oplus E^u$ is integrable (see [Pla72]). As an example of this last situation one can consider the flow obtained by suspending a completely hyperbolic map by a constant function.

One of the main difference between these two types of examples is the fact that in the first case the partially hyperbolic map is *accessible*, meaning that given two points in the manifold there exists a piecewise $\mathcal{C}^1$ curve which connects the two points, and whose derivative is always tangent to either the stable or the unstable bundle.

1.1.3 Direct and skew products

Take two partially hyperbolic diffeomorphisms $f : M \rightarrow M$ and $g : N \rightarrow N$, and assume that for every $x \in M, y \in N$

1. $\|d_x f|_{E^s}\| < m(d_y g)$, where m denotes the conorm³.
2. $\|d_y g\| < m(d_x f|_{E^u})$.

Then one can see that $F = f \times g : M \times N \rightarrow M \times N$ is partially hyperbolic where the bundles are given by

- $E_F^s = E_f^s$
- $E_F^u = E_f^u$
- $E_F^c = E_f^c \oplus TN$

(here we identify $T(M \times N) = TM \oplus TN$).

³The conorm of a matrix A is defined as $m(A) = \inf\{\|Av\| : \|v\| = 1\}$.

As a generalization of the previous example consider the case where N is a compact Lie group, and suppose that a differentiable function $\theta : M \rightarrow N$ is given.

Then the *skew product* of f by θ is the map $F : M \times N \rightarrow M \times N$ given by

$$F(x, y) = (f(x), \theta(x) \cdot y)$$

Note that in these examples all center leaves are compact.

As a concrete example of this last situation we consider the following map studied by Bonatti and Wilkinson: take the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and consider the Thom map $f_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$. If $g : \mathbb{T}^2 \rightarrow \mathbb{R}$ commutes with $-I$ we define F_g as the skew product of f_A by g .

Consider also the involution $J : \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{T} \rightarrow \mathbb{T}^3$ defined by

$$J(x, t) = (-x, t + 1/2)$$

Then $N = \mathbb{T}^3/J$ is a compact 3-manifold and the quotient map $p : \mathbb{T}^3 \rightarrow J$ is a covering map. One sees that $\pi : N \rightarrow \mathbb{S}^2$ is a Seifert bundle with 4 exceptional leaves, where π is defined by the following diagram

$$\begin{array}{ccccc} \mathbb{T}^2 \times \mathbb{T} & \longrightarrow & \mathbb{T}^2 & \xrightarrow{-I} & \mathbb{S}^2 \\ p \downarrow & & \pi \nearrow & & \\ N & & & & \end{array}$$

Now F_g commutes with the involution, and thus defines a diffeomorphism $G : N \rightarrow N$. Since the map p is a finite covering map it is easy to see that G is a partially hyperbolic diffeomorphism, where the invariant bundles are obtained as the push forward of the ones of F_g by p . In particular E_G^c is *not orientable*.

Note that all center leaves of G are also compact. ■

The study of partially hyperbolic diffeomorphisms was started by Hirsch, Pugh and Shub in 1970 (see [HPS70]) under the general setting of *normal hyperbolicity*⁴:

Definition. A diffeomorphism $f : M \rightarrow M$ is said to be normally hyperbolic to the invariant⁵ foliation $\mathcal{F}$ if f is partially hyperbolic and $\mathcal{F}$ integrates the center bundle of f .

We will also say that a foliation $\mathcal{F}$ is normally hyperbolic if there exists a map f normally hyperbolic to $\mathcal{F}$. This is motivated since we will be interested in determining the geometric properties imposed on a foliation by the existence of a map normally hyperbolic to it. The classic reference for normal hyperbolicity, and one which will be used all along this work is [HPS77].

While completely hyperbolic systems is a well developed subject in dynamical systems, much less is known about partially hyperbolic diffeomorphisms. The difficulty is, of course, the presence of a third direction where already the behavior of the derivative is unknown; one can have points where the derivative expands, and points where the derivative contracts as long as these expansions and contractions are dominated by the corresponding expansions and contractions in the stable and unstable bundle respectively.

Thus, to obtain a general theorem one is usually led to impose some type of restriction on the action of the tangent map in the center bundle.

1.2 Invariant Foliations

The stable manifold theorem (see [HPS77]) implies that the bundles E^u and E^s are integrable to continuous foliations $\mathcal{W}^u, \mathcal{W}^s$ whose leaves are of the same degree of differentia-

⁴The term “partially hyperbolic” however comes from the important work of Brin and Pesin ([BP74]).

⁵i.e. for every leaf $L \in \mathcal{F}$, $f(L) \in \mathcal{F}$.

bility that f is. These leaves are homeomorphic to Euclidean spaces of the corresponding dimension. Nonetheless, the transversal regularity of those foliations is only Hölder (see [PSW97]).

The integrability of the center bundle E^c , on the other hand, cannot be asserted in general as the example in [Sma67] shows (see also [HHU07]). Establishing necessary and sufficient conditions that guarantee this property remains one of most important problems in the area. Some partial results in this matter can be found in [HHU08],[Bri03] and [BW05].

In any case, for most known examples the center bundle is integrable, and since we want to discuss the geometric properties of the center foliation we are going to assume that E^c is integrable from now on.

Standing Hypothesis for the rest of the work: f is a partially hyperbolic diffeomorphism with invariant center foliation $\mathcal{W}^c$.

1.2.1 Local Product Structure

Given a point $x \in M$ and a positive number $\gamma > 0$ we will denote by $W_\gamma^s(x)$ the disc of size γ inside the leaf $W^s(x)$. Similarly for $W_\gamma^u(x)$.

Since the foliation $\mathcal{W}^c$ is normally hyperbolic one can use the results of [HPS77] to show that there exists $\gamma > 0$ such that for each center leaf L both

$$W_\gamma^s(L) = \bigcup_{x \in L} W_\gamma^s(x)$$

$$W_\gamma^u(L) = \bigcup_{x \in L} W_\gamma^u(x)$$

are immersed submanifolds tangent to $E^s \oplus E^c$, $E^u \oplus E^c$ respectively.

We also define

$$W^s(L) = \bigcup_{n \geq 0} f^{-n} W_\gamma^s(f^n L) = \bigcup_{x \in L} W^s(x)$$

$$W^u(L) = \bigcup_{n \geq 0} f^n W_\gamma^u(f^{-n} L) = \bigcup_{x \in L} W^u(x)$$

Note that $W_\gamma^s(L) \subset W^s(L)$, $W_\gamma^u(L) \subset W^u(L)$ are open (with the induced topology).

It follows by definition that for a given center leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are subfoliated by the corresponding strong foliation (either $\mathcal{W}^s$ or $\mathcal{W}^u$). The condition of being foliated by the center foliation however, is much more subtle. We give the following definition:

Definition. The submanifolds $W^s(L)$ and $W^u(L)$ are said to be *complete* if they are subfoliated by the center foliation. The center foliation is complete if for every center leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are complete

For example, if the center foliation is $\mathcal{C}^1$ then it is complete (Theorem 7.6 in [HPS77]). For a discussion of this matter in the context of 3-manifolds see [BW05].

In the case of compact leaves the fact that $W^s(L)$ is complete essentially means that “leaves don’t escape to infinity”. To formalize this idea we establish the following simple but important lemma:

Lemma 1.1. *Suppose that L, L' are compact center leaves with $L' \subset W^s(L)$, and assume that $W^s(L')$ is complete. Then $L \subset W^s(L')$, and in particular given any $x \in L$ we have $W^s(x) \cap L' \neq \emptyset$.*

The proof of this is obvious.

If one of the leaves is periodic under the diffeomorphisms we can obtain a more general result, which will be important later on.

Proposition 1.2. *Suppose that L, L' are compact center leaves with $L' \subset W^s(L)$, and assume that*

1. $W^s(L)$ is complete.

2. L is periodic under f .

Then for any point $x \in L$ we have $W^s(x) \cap L' \neq \emptyset$. In particular $W^s(L')$ is also complete.

Proof. Before starting with the proof we recall some results about the unwrapping bundle.

For each center leaf $\tilde{L}$ let $U_{\tilde{L}} \subset TM$ be its tubular neighborhood. The tubular neighborhood of $\tilde{L}$ is an immersed submanifold of M but not embedded in general: we construct $U_{\tilde{L}}$ in the tangent bundle to avoid self intersections. The image of $U_{\tilde{L}}$ under the exponential map will be referred as the *geometric tubular neighborhood of $\tilde{L}$* , to distinguish it from the tubular neighborhood in the tangent bundle. If $\epsilon > 0$, we will denote by $U_{\tilde{L}}(\epsilon)$ the ϵ -disc subbundle of the normal bundle of $\tilde{L}$.

Let $\mathcal{A} = \coprod_{\tilde{L} \in \mathcal{W}^c} L$, and denote by $i : \mathcal{A} \rightarrow M$ the inclusion. Note that f acts naturally on the zero section of the bundle

$$\zeta = \coprod_{\tilde{L} \in \mathcal{W}^c} U_{\tilde{L}}.$$

We denote this induced map by $\tilde{f}$, and we identify the zero section of ζ with $\mathcal{A}$. By theorem 6.1 of [HPS77] there exist $\epsilon > 0$ such that $\tilde{f}$ is defined in the ϵ -disc bundle

$$\zeta(\epsilon) = \coprod_{\tilde{L} \in \mathcal{W}^c} U_{\tilde{L}}(\epsilon)$$

and

$$\begin{array}{ccc} \zeta(\epsilon) & \xrightarrow{\tilde{f}} & \zeta \\ \text{expoi} \downarrow & & \downarrow \text{expoi} \\ M & \xrightarrow{f} & M \end{array}$$

The set $\mathcal{A}$ is $\tilde{f}$ -invariant, and $\tilde{f}$ is normally hyperbolic to $\mathcal{A}$. We remark that $\zeta(\epsilon)$ is not invariant in general.

Back to the proof of the Proposition, we will assume with no loss of generality that L is fixed under f . By hypothesis,

$$L' \subset \bigcup_{z \in L} W^s(z)$$

and in particular there exists a nonempty subset $S \subset L$ such that

$$\forall x' \in L' \exists z \in S \text{ s.t. } x' \in W^s(z).$$

Figure 1.1: After enough iterations both center leaves are very close, and hence the local stable manifold of each point in one leaf intersects the other

Then by compactness of L' we conclude that

$$r = \sup\{dist_{W^s(z)}(z, w) : z \in S, w \in L'\} < \infty$$

where $dist_{W^s(z)}(z, \cdot)$ denotes the distance measured inside the stable manifold of z .

Since L is compact, there exist a $\delta < \epsilon$ such that

$$\exp : U_L(\delta) \rightarrow M$$

is injective. Then there exists a positive integer N such that $f^N(L')$ is contained in the δ -geometric tubular neighborhood of L . We lift $f^N(L')$ to $U_L(\delta)$.

The map $\pi : U_L(\delta) \rightarrow L$ is a retraction (note that no branching can occur inside $U_L(\delta)$), and restricted to $f^N(L')$ it is an immersion since it is transverse to $\mathcal{W}^s$. Hence $\pi(f^N(L')) = f^N(L)$. But this means that for any point $x \in f^N(L)$, $W_\epsilon^s(x) \cap f^N(L') \neq \emptyset$, and thus the same is true inside M . Apply f^{-N} to conclude the statement of the proposition. ■

To discuss this topic further we need another notion:

Definition. A partially hyperbolic diffeomorphism is *dynamically coherent* if the bundles E^c, E^{cu} and E^{cs} are integrable to $\mathcal{C}^{1,0}$ foliations $\mathcal{W}^c, \mathcal{W}^{cu}, \mathcal{W}^{cs}$ and such that

$$\mathcal{W}^c = \{W^{cu} \cap W^{cs} : W^{cu} \in \mathcal{W}^{cu}, W^{cs} \in \mathcal{W}^{cs}\}.$$

As explained in [BW08], it follows that

1. $\mathcal{W}^s$ sub-foliates $\mathcal{W}^{cs}$
2. $\mathcal{W}^u$ sub-foliates $\mathcal{W}^{cu}$

Since we are assuming that the center bundle E^c integrates to an invariant foliation, we easily conclude from the previous remark that the foliations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ are also invariant.

In the case of a dynamically coherent partially hyperbolic diffeomorphism, given a point $x \in L$ it follows that $W^s(L) \subset W^{cs}(x)$ is an open submanifold, and likewise $W^u(L) \subset W^{cu}(x)$ is open. Completeness of $W^s(L)$ is the same as metric completeness inside $W^{cs}(x)$.

Proposition 1.3. *Suppose that $W^s(L)$ is complete. Then for any $x \in L$ we have $W^s(L) = W^{cs}(x)$. Similarly if $W^u(L)$ is complete and $x \in L$ then $W^u(L) = W^{cu}(x)$.*

Proof. It suffices to show that $W^s(L)$ is closed inside the center stable manifold where it is contained.

Take a sequence $(z_n)_n$ in $W^s(L) \subset W^{cs}(x)$ such that

$$z_n \xrightarrow[n \rightarrow \infty]{} z$$

(convergence inside $W^{cs}(x)$). Now consider a bi-foliated cube⁶ centered in z and let P the plaque of the cube containing z .

For sufficiently big n all the terms of the sequence belong to the cube and they are very close to x , and hence there exist stable manifolds of points in L that intersect P .

Since it was assumed that $W^s(L)$ is complete, we conclude that $W^c(z) \subset W^s(L)$. Hence $W^s(L)$ is closed. ■

In the case of 3-manifolds there is a stronger version of the previous Proposition (not assuming that the center foliation is compact) in [BW05].

Remark. If the center foliation of f is complete the results of [BW08] imply that f is dynamically coherent. The foliations that integrate E^{cs} and E^{cu} are $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ respectively.

Definition. We say that a normally hyperbolic foliation $\mathcal{W}^c$ has *local product structure* if there exists some $\eta > 0$ such that if $d(x, y) < \eta$ and P_x, P_y denote plaques of $\mathcal{W}^c$ centered at x , of radius η then $W_{2\eta}^s(L_x)$ meets $W_{2\eta}^u(L_y)$ along a plaque of $\mathcal{W}^c$ of radius at least $\eta/2$ (see section 7 in [HPS77]).

⁶That is, a foliated cube for both foliations $\mathcal{W}^c$ and $\mathcal{W}^s$.

If the system has local product structure one can specify locally each plaque as follows.

For a point $x \in M$ we define the sets $H_x = W_{2\eta}^u(P_x)$, $V_x = W_{2\eta}^s(P_x)$. Given any two points $x, y \in M$ such that $d(x, y) < \eta$ both intersections $H_x \cap V_y$ and $V_x \cap H_y$ consist of center plaques. Taking η small, there will be only one center plaque in the intersection $H_x \cap V_y$ meeting $W_{2\eta}^u(x)$, and likewise there exists only one center plaque of $V_x \cap H_y$ intersecting $W_{2\eta}^s(x)$.

One sees then that the plaque through y is specified by these two points in $W_{2\eta}^u(x)$ and $W_{2\eta}^s(x)$.

Definition. This types of coordinates will be referred to as *transverse local coordinates at x* .

Figure 1.2: Local Product Structure

In particular, if D is a small transverse disc to $\mathcal{W}^c$ centered at x the transverse local coordinates allow us to define a continuous system of coordinates on D . Namely, there exists an open embedding $\Psi_x^D : D \rightarrow W_\eta^u(x) \times W_\eta^s(x)$ such that if P_y denotes the plaque through y then

$$\Psi_x^D(y) = (u \text{ coordinate of } P_y, s \text{ coordinate of } P_y)$$

We will show that if the center bundle is integrable, then local product structure is equivalent to dynamical coherence. We first note the following simple Proposition:

Proposition 1.4. *Let f be a partially hyperbolic diffeomorphism and assume that f is dynamically coherent. Then f has local product structure.*

Proof. By definition of dynamic coherence if $x \in W^{cs}(y) \cap W^{cu}(z)$, then the whole center leaf of x is contained in the intersection. The number η of the local product structure is found since the angle between the bundles E^c, E^u, E^s is uniformly bounded away from zero. ■

It is an interesting fact that the uniformity in the local product structure of a foliation that integrates E^c implies dynamical coherence, which seems to be a stronger assertion⁷.

Proposition 1.5. *Let f be a partially hyperbolic diffeomorphism with center foliation $\mathcal{W}^c$, and assume that it has local product structure. Then f is dynamically coherent.*

Proof (After C. Pugh). We will show that both bundles E^{cs}, E^{cu} integrate to laminations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ and

$$\mathcal{W}^c = \{W^{cu} \cap W^{cs} : W^{cu} \in \mathcal{W}^{cu}, W^{cs} \in \mathcal{W}^{cs}\}.$$

Let $\gamma < \eta/2$ where η is the constant of the local product structure, and fix a plaquation $\mathcal{P} = \{P_x : x \in M\}$ of the foliation⁸ $\mathcal{W}^c$, where each plaque P_x has radius γ .

Fix x and consider a point $y \in W_\gamma^s(P_x)$ such that $d(x, y) < \gamma/2$. By hypothesis, the submanifolds $W_\gamma^s(P_x)$ and $W_\gamma^u(P_y)$ meet in a center plaque of diameter at least γ , and

⁷Beware. Things are not always as they seem.

⁸i.e. for every $x \in M$, P_x is a plaque of $\mathcal{W}^c$ centered at x .

this plaque is contained in $W_\gamma^s(P_x)$. Note that the size of this plaque is bounded from below.

This implies that there exists some number $0 < \delta < \gamma$ which only depends on η such that for any point $x \in M$ and any point $y \in W_\gamma^s(L_x)$, y has a neighborhood inside $W_\gamma^s(L_x)$ of radius bigger than equal δ and consisting of center plaques.

Fix a center leaf L and consider now a point $y \in W^s(L)$. Some iterate of $f^N(y)$ will be close to $f^N(L)$, and since the number δ does not depend on the leaf, one concludes that $f^N(y)$ has a neighborhood of size δ consisting of center plaques inside $f^N(W^s(L)) = W^s(f^N(L))$. Applying f^{-N} and by invariance of the center foliation we finally conclude that y has a small neighborhood foliated by center leaves inside $W^s(L)$.

In particular, if L, L' are two center leaves then the intersection $W^s(L) \cap W^s(L')$ is an open set in each submanifold.

For a set V we define the set

$$K(V) = \bigcup_{x \in V} W^s(L_x).$$

If L is a fixed leaf, we consider the sets

$$F_0(L) = K(L), F_1(L) = K(F_0(L)), \dots, F_n = K(F_{n-1}(L)).$$

Since the intersection of the stable manifolds of center leaves is open, we conclude that for every n the set $F_n(L)$ is an immersed submanifold, tangent to E^{cs} . Consider the family $\mathcal{F}^{cs} = \{F(L) = \bigcup_{n \geq 0} F_n(L) : L \in \mathcal{W}^c\}$: as before the set $F(L)$ is a submanifold tangent to E^{cs} , and since by construction the family $\mathcal{F}^{cs}$ is a partition of M that integrates the bundle E^{cs} , it is a foliation. Note that each leaf of $\mathcal{F}^{cs}$ is foliated by center leaves.

Likewise we obtain a foliation $\mathcal{F}^{cu}$ that integrates E^{cu} , and is subfoliated by the center foliation. One concludes that the leaves of $\mathcal{F}^{cs}$ meet the leaves of $\mathcal{F}^{cu}$ along center leaves. ■

We say that a k - dimensional continuous bundle E is plaquewise uniquely integrable if

it integrates to a foliation $\mathcal{F}$, and given any k -dimensional immersed disk D everywhere tangent to E it follows that D is contained in a single leaf of $\mathcal{F}$. It is obvious that plaquewise unique integrability implies local product structure. Hence we recover the following theorem of [BW08].

Corollary 1.6. *Suppose that $f : M \rightarrow M$ is a partially hyperbolic diffeomorphism such that E^c is plaquewise uniquely integrable. Then f is dynamically coherent.*

Appendix: The Definition of Dynamical Coherence

In this appendix we discuss the notion of dynamical coherence. The original definition of dynamical coherence appeared in [PS00], and was the equal to our definition 1.2.1 except for the fact that it was required that the foliations $\mathcal{W}^c, \mathcal{W}^s$ subfoliate $\mathcal{W}^{cs}$, and the foliations $\mathcal{W}^c, \mathcal{W}^u$ subfoliate $\mathcal{W}^{cu}$. Later in [BW08] it was shown that $\mathcal{W}^s$ automatically subfoliates $\mathcal{W}^{cs}$, and $\mathcal{W}^u$ subfoliates $\mathcal{W}^{cu}$.

As was established in 1.5 dynamical coherence is, in the case where the center foliation integrates to an invariant foliation, equivalent to local product structure. Thus the aforementioned problem of dynamical coherence can be formulated in two parts.

- a) Existence of an invariant center foliation.
- b) Existence of local product structure.

In [Sma67] S. Smale gives an example of a partially hyperbolic diffeomorphism where the center bundle is not involutive (and hence, not integrable). More strikingly, in a recent example F. Rodriguez-Hertz, A. Rodriguez-Hertz and R. Ures shown that the answer to the first question is negative even when the bundle is involutive. See [HHUa]. Thus, **a)** should be reformulated in

- a') Find natural conditions for the existence of an invariant center foliation.

Chapter 2

Compact Foliations

A significant question for a general compact foliation is whether the Riemannian volume of the leaves is uniformly bounded or not. It is known that for this type of foliation the structure near every leaf is very simple, and thus one can hope to give a classification of this type of foliation.

On the other hand, in the absence of the condition of uniform volume bound for the leaves there exist very pathological examples which prevent any possibility of classification at all ([Vog77]).

The history of the problem of deciding whether for a compact foliation the volume of the leaves is uniformly bounded goes back to G. Reeb who gave in his thesis an example of a flow on a non-compact manifold whose orbits were all periodic, but the time of return (i.e., the volume of each leaf) was not locally bounded. Note that since the manifold is not compact it is trivial to arrange an example where the volume is not globally bounded from above.

This example led A. Haefliger to ask if in a compact manifold the time of return was necessarily bounded, or, equivalently, if there could be an example of a compact manifold with a compact foliation having locally unbounded volume.

Later D.B.A. Epstein with a very intricate argument showed that in a compact three manifold this phenomenon could not happen (see [Eps72]). However in 1976 D. Sullivan gave an example of a compact flow in S^5 where the time of return was not bounded ([Sul76]), and a similar type of example was given by D.B.A. Epstein and E. Vogt in a manifold of dimension 4 ([EV78]).

It is important to point out that the regularity of the foliation is not an issue for these examples: all mentioned examples can be made analytic.

In this chapter we recall some of the properties of compact foliations, establishing in particular some equivalent conditions for a compact foliation to have uniformly bounded volume in the leaves.

2.1 Equivalent conditions to uniform volume bounds

The aim of this section is to prove the following theorem of D.B.A. Epstein [Eps76] (see also [Mil75])

Theorem 2.1. *Let $\mathcal{F}$ be a compact foliation on a (not necessarily compact) manifold N , and consider $\pi : M \rightarrow M/\mathcal{F}$.*

Then the following properties are equivalent.

1. π is closed.
2. $M/\mathcal{F}$ is Hausdorff.
3. Every leaf $L \subset \mathcal{F}$ has arbitrarily small foliated neighborhoods.

4. For every $K \subset M$ compact, its saturation is compact.

If N is compact, the previous conditions are also equivalent to:

5. All leaves of $\mathcal{F}$ have finite holonomy.

6. The volume of the leaves is uniformly bounded from above.

If furthermore the foliation $\mathcal{F}$ is a dynamically coherent normally hyperbolic foliation then the previous conditions are also equivalent to

7. For every periodic leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are complete.

Definition. If a compact foliation satisfies the conditions of the previous theorem we say that it is *uniformly compact*.

For clarity, we split the proof the proof into several parts, except for the last part which will be proved in the next chapter. Throughout this section N, π and $\mathcal{F}$ will have the meaning given in the previous theorem. We point out that no assumption on the transverse regularity is made, but only that the leaves are of class at least $\mathcal{C}^1$.

The simplest equivalence is among the first four conditions. We discuss them now.

Proof. We will prove the chain of implications $1 \Rightarrow 2 \Rightarrow 3$, and $1 \Leftrightarrow 4$.

$1 \Rightarrow 2$ First suppose that the map π is closed. We fix a leaf L and consider compact neighborhood U of L . Since the projection is closed, the set $V = U - \text{sat}(\partial U)$ is open. We claim that it is also saturated.

For, if L' intersects V then intersects $\text{int}(U)$, and not ∂U . Hence (by connectedness) $L' \subset V$. This implies that L has arbitrarily small saturated neighborhoods, and this implies that the quotient is Hausdorff.

$2 \Rightarrow 3$ Immediate.

$3 \Rightarrow 1$ Now suppose that $N/\mathcal{F}$ is Hausdorff, and consider $C \subset N$ closed.

If x is in $N \setminus C$, one can find a compact neighborhood U of x that does not intersect C . Then since the quotient is Hausdorff we get that $\text{sat}(\partial U)$ is closed. Applying a similar argument like in the part we can construct a foliated neighborhood of x in the complement of C .

We conclude that $\pi(C^c)$ is open, and hence $\pi(C)$ is closed.

$1 \Rightarrow 4$ Assume that π is closed. Then we have a continuous function $\pi : N \rightarrow N_{\mathcal{F}} = N/\mathcal{F}$ so that

- (a) N is locally compact and Hausdorff.
- (b) $N_{\mathcal{F}}$ is Hausdorff (by 3).
- (c) for every point L the set $\pi^{-1}(L)$ is compact.
- (d) π is closed.

These conditions imply that the map π is proper.

$4 \Rightarrow 1$ If π is proper we take $A \subset N$ closed and denote by B its saturation. We claim that B is closed.

Let $x \in \overline{B}$. Then, since the projection is proper, one can find a compact saturated neighborhood U of x . Note that $U \cap A$ is compact, and hence $\text{sat}(U \cap A)$ is also compact.

If $x \notin \text{sat}(U \cap A)$, then $U \setminus \text{sat}(U \cap A)$ is a saturated neighborhood of x which does not intersect A hence $x \notin \overline{B}$, a contradiction.

■

2.1.1 Local Stability

To go further we first recall the classic stability theorem of Reeb:

Theorem 2.2. *Let L be a compact leaf in $\mathcal{F}$ with finite holonomy.*

Then there exists a disc V , an open foliated neighborhood U of L , and a representation $\psi : \pi_1(L) \rightarrow \text{Homeo}(V)$ such that U is homeomorphic to a foliated bundle of the form $V \times_\psi L$.

For a proof see for example [ES56]. We point out that V can be taken arbitrarily small.

The previous theorem implies in particular that for a uniformly compact foliation there is a good “*local model*” around each leaf. In fact if the foliation is C^1 one can identify the holonomy group of L as a subgroup of the orthogonal transformations $\mathcal{O}(k)$ acting on the disc V .

To see this we first assume that the foliation is C^∞ . Then note that since the holonomy group acts as a finite group of diffeomorphisms in V , there exists an invariant Riemannian metric under this action. Then one uses the theorem of R. Palais [Pal70] which says that any C^1 action of a finite group is equivalent to a C^∞ one, and notes that under equivalent actions the local model remains the same.

Note the following immediate consequence of Reeb’s Stability Theorem:

Corollary 2.3. *If for every leaf L the holonomy group is finite, then $N/\mathcal{F}$ is Hausdorff.*

Hence we have proved that $5 \Rightarrow 1$ in Theorem 2.1.

Another useful Corollary of Reeb’s Stability is the following.

Corollary 2.4. *Let $\mathcal{F}$ be a compact foliation. Then the set of leaves with trivial holonomy is open and dense.*

Proof. It is well known that for any foliation the set of points with trivial holonomy comprise a generic set in the manifold, hence dense. See for example [EMT77].

By Reeb's Stability Theorem, if L is a compact leaf with trivial holonomy then there exist a foliated neighborhood of the form $V \times_\psi L$ where V is a transverse disc (of complementary dimension) to the foliation. It's easy to see that if L' is any compact leaf in this neighborhood, the holonomy group is a subgroup to the corresponding holonomy group to L . Hence L' also has trivial holonomy, and thus the set of points with trivial holonomy is an open set in N . ■

2.1.2 The volume function

To establish the equivalence of 5 and 6 in 2.1 we will discuss the volume function induced by the Riemannian metric. Let us point out that the results obtained do not depend on the metric chosen.

We will use the following lemma, which has importance on its own:

Lemma 2.5. *Fix a compact leaf L of a foliation $\mathcal{F}$. Then given $\epsilon > 0$ and $n \in \mathbb{N}$ there exist V a neighborhood of L and a collection of foliated cubes $\{U_i\}_{i=1}^k$ such that*

1. $L \subset V \subset \bigcup_{i=1}^k U_i$.

2. *If L' is another leaf of $\mathcal{F}$ so that $L' \cap V \neq \emptyset$, then either*

- (a) $\text{vol}(L') > n \cdot \text{vol}(L)$, or

- (b) *there exists a number $1 \leq m \leq n$ so that*

$$|\text{vol}(L') - m \cdot \text{vol}(L)| < \epsilon$$

and for every i , $L' \cap U_i$ consists of exactly m plaques.

A proof can be found in [Eps76].

Proposition 2.6. *Consider a foliation $\mathcal{F}$ and fix a compact leaf $L \in \mathcal{F}$. Then the following two assertions are equivalent.*

1. *The holonomy group of L is finite.*
2. *There exist a neighborhood V of L where the Riemannian volume of each leaf intersecting V is uniformly bounded.*

Proof. First we assume that the holonomy group of L is finite, and we denote by m the number of elements of this group.

We consider a neighborhood V as in 2.5 corresponding to $\epsilon = 1$. Then we see that if a leaf L' intersects V we have

$$\text{vol}(L') < m \cdot \text{vol}(L) + 1$$

Conversely, suppose that there exists a neighborhood U where the volume of the leaves is bounded. With no loss of generality we can assume that U is small enough so that there exists a transverse retraction $r : U \rightarrow L$.

Take V and $\{U_i\}_{i=1}^k$ as in 2.5, and by intersecting if necessary we assume that V and the cubes U_i lie inside U .

Consider n sufficiently big so that $\text{vol}(L') < n \cdot \text{vol}(L)$. Then for every leaf L' that intersects V one has that $L' \cap U_i$ consist at most of n plaques.

We then consider a small transverse disc T inside V so that each element of the holonomy group of L can be represented by an embedding $g : T \rightarrow r^{-1}(x)$ (this is possible since the holonomy group of L is finite).

It follows that for every leaf L' that intersects V , the number of elements in $L' \cap r^{-1}(x)$ is finite. Hence by taking T small enough one sees that the holonomy group $G(L)$ induces a representation

$$\psi : G(L) \rightarrow \text{Sym}(s)$$

where $Sym(s)$ denotes the symmetric group in s letters and $s \leq n$. Furthermore every element of $G(L)$ preserves T .

Since the leaf L is compact, its fundamental group is finitely generated and thus there exist finitely many representations of the previous form. Considering now the set

$$A = \bigcap \{\ker(\psi) : \psi \text{ representation of } G(L) \text{ in } Sym(s)\}$$

and observing that A acts trivially on T , we conclude that

$$G(L)/A \simeq G(L)$$

is finite. ■

We conclude that in the case of a compact manifold N , we have that 5 and 6 are equivalent.

2.1.3 Montgomery's Theorem

Finally we need to prove that the fact of every leaf has arbitrarily small foliated neighborhoods (equivalently, $N/\mathcal{F}$ is Hausdorff) implies that the holonomy of every leaf is finite. We are going to use the following result of D. Montgomery extended by D.B.A. Epstein and K. Millett ([Mon37],[Eps76])

Theorem 2.7. *Let G be a group that acts effectively on a connected manifold N by homeomorphisms and such that every point has a finite orbit. Then G is finite.*

This theorem also will be used in the next chapter.

Proof. Instead of giving the complete proof we will limit ourselves to deriving the general assertion for groups from the classical theorem of D. Montgomery which covers the case when G is cyclic.

Note first that the aforementioned theorem implies that every element g of G has finite order. Newman's theorem (see [Dre69]) implies that for $g \in G \setminus \{1\}$ the set $Fix(g)$ is nowhere dense (and is, of course, closed).

Take a countable subgroup $H < G$: we will show that H is finite. Since an infinite group necessarily has a countable infinite subgroup we conclude that G is finite. Suppose that H is infinite, and take $x \in N$. Since the orbit of x is finite and H is infinite we can find an element $g \in H \setminus \{1\}$ such that $gx = x$

This implies that $N = \bigcup \{Fix(g) : g \in H\}$; we have thus written the manifold N as a countable union of closed nowhere dense sets, which is a contradiction by Baire's Theorem. ■

Proposition 2.8. *Let $\mathcal{F}$ be a compact foliation on a compact manifold N , and assume that every leaf L has arbitrarily small foliated neighborhoods. Then the holonomy group of every leaf is finite.*

Proof. Let V be a small saturated neighborhood of a leaf L , and consider D a transverse disc to the foliation such that each element contained in the holonomy group is represented by an homeomorphism defined in D . This is possible since the leaf L has arbitrarily small foliated neighborhoods. We conclude that the group $G(L)$ acts as a group of homeomorphisms in the open connected manifold D , and each orbit under $G(L)$ is finite.

By the theorem 2.7 the group is finite. ■

This concludes the proof of Theorem 2.1 by showing that $5 \Rightarrow 1$.

2.2 Foliated bundles

In this section we apply Montgomery's theorem to conclude that if $\mathcal{F}_E$ is the foliation transverse to the fibers of a foliated bundle E over B , and $\mathcal{F}_E$ is compact then it is

uniformly compact. See page 4 for the definition of foliated bundle and related notions.

We will denote by $p : E \rightarrow B$ the fibration, and for a given point $x \in B$ the fiber $p^{-1}(x)$ will be denoted by F_x . Recall that for the case of foliated bundles there exists a notion of *total holonomy*, meaning that if we fix a point $x \in B$ there exists a representation

$$\psi : \pi_1(B, x) \rightarrow \text{Homeo}(F_x)$$

Given any point $y \in L \cap p^{-1}(x)$ we can factor the holonomy representation of L at y by first composing with $p_* : \pi_1(L, y) \rightarrow \pi_1(B, x)$ and then taking the germ at y . In particular, the holonomy transports whose germs generate $G(L)$ are defined in the whole fiber F_x .

In fact, this map total holonomy determines the bundle up to equivalence. The following theorem is true:

Theorem 2.9. *Suppose that $p : E \rightarrow B$, $p' : E' \rightarrow B$ are two foliated bundles with typical fiber F and denote by ψ, ψ' their corresponding total holonomy representations. Then E is equivalent to E' (as bundles) if and only if the representations ψ and ψ' are equivalent (as representations).*

For a proof see [CC00].

One is led to inquire if given manifolds B, F and a representation

$$\psi : \pi_1(B) \rightarrow \text{Homeo}(F)$$

there exists a foliated bundle E over B with typical fiber F and having ψ as its holonomy representation. This indeed is true as we now explain.

Let $\widehat{B}$ denote the universal cover of B , and consider the action of $\pi_1(B, x)$ in $\widehat{B} \times F$ given by

$$g \cdot (\tilde{x}, p) = (g \cdot \tilde{x}, \psi(g)(p))$$

(in the first coordinate the action is the natural one of $\pi_1(B, x)$ in $\widehat{B}$).

Define $E = \widehat{B} \times F / \sim$ to be the orbit space of the previous action, and $\tilde{\gamma} : \widehat{B} \times F \rightarrow E$ the quotient map. Then $\tilde{\gamma}$ induces a map $\gamma : E \rightarrow B$. One shows that E is a foliated bundle, and the corresponding foliation is obtained by projecting the “horizontal” foliation $\{\tilde{y} \times F\}_{\tilde{y} \in \tilde{B}}$. The total holonomy of the bundle is equivalent (as a representation) to ψ .

The previous construction is called the *suspension of the representation* ψ . For details see [CC00].

We conclude with the promised proof of the fact that for foliated bundles, the natural foliation is compact if and only if it is uniformly compact.

Proposition 2.10. *Suppose that the foliation $\mathcal{F}$ is compact. Then it is uniformly compact.*

Proof. Fix a leaf $L \in \mathcal{F}$, and take a point $y \in L$. We will show that $G(L)$ is finite.

As we explained before each element $g \in G(L)$ can be represented by an homeomorphism $\hat{g} : F_x = p^{-1}(x) \rightarrow F_x$ that fixes y , and for each leaf $L' \in \mathcal{F}$ sends the set $L' \cap F_x$ to itself. These sets are finite since $\mathcal{F}$ is compact.

Applying the Theorem of Montgomery 2.7 we conclude that $G(L)$ is finite.

■

Appendix: Sullivan's Example

In this Appendix we give a brief exposition of Sullivan's example of an analytic flow on a compact 5 manifold such that all its trajectories are periodic, but there is no upper bound in the period. The version that we present here is due to W. Thurston. For details see [Sul76] and [Eps]. I thank Amie Wilkinson for reading a preliminary version of this part and pointing out some mistakes.

Let $K = \mathbb{T}^2 \times \mathbb{S}^1$ the unit tangent bundle of the flat 2 torus. A point $p = (x, \theta) \in K$ will be thought as a unit vector with direction θ anchored at the point $x \in \mathbb{R}^2$. Now given two positive real numbers (r, v) define the following flow on K .

If $p \in K$ take the circle of radius r tangent at p (this is defined up to integer translations) and consider its unit tangent vector field. This defines an embedded circle in K passing through p . Consider the constant flow in the plane circle going counter-clockwise having speed v , and lift this to a flow on K .

Note that we have obtained a family of periodic flows on K . We denote by $X_{(r,v)}$ the tangent vector to this flow.

Now consider the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

and let Γ the subgroup of elements in G whose entries are integers. Then Γ is a

co-compact lattice and hence $N = G/\Gamma$ is a compact manifold.

The manifold N is a principal S^1 -bundle over the 2-torus. Note that the 1-form $\omega = dz - ydx$ defined on H is left invariant, and thus induces a connection form ω on N . The curvature of this form will be represented by the 2-form $\Omega = dx \wedge dy$ on the torus.

As it is well known, the geodesics on N are spirals that project to circles on $\mathbb{T}^2$, and if the projection has radius r , the distance from points in the same fiber of the corresponding geodesic is $d = \pi r^2$. Note that in $\mathbb{T}^2$ a given point goes around its corresponding circle in time $\tau = 2\pi r/v$.

Define M as the pullback of $N \rightarrow \mathbb{T}^2$ given by

$$\begin{array}{ccc} & & N \\ & & \downarrow \\ K = \mathbb{T}^2 \times S^1 & \longrightarrow & \mathbb{T}^2 \end{array}$$

We now want to lift the flow by circles on K to a flow by circles on M . To do this we proceed as follows. The tangent space of M splits as the direct sum of the horizontal space with respect to sum of the trivial connection and ω , and the tangent spaces of the circles of the fiberings $K \rightarrow \mathbb{T}^2, N \rightarrow \mathbb{T}^2$. We specify a vector field $Z_{(r,v)}$ on M by giving its coordinates with respect to the aforementioned decomposition. Using coordinates, if $((x, \theta), n) \in M$ the coordinates of $Z_{(r,v)}$ are

$$(v(x, \theta), v/r, -rv/2)$$

Note that the third coordinate is a “push” in the vertical direction of $N \rightarrow \mathbb{T}^2$, and since this component is precisely $-d/\tau$, we conclude that the orbits of $Z_{(r,v)}$ are closed. Also, $Z_{(r,v)}$ covers $X_{(r,v)}$.

Consider $M \times S^1$ and denote its coordinates by $((x, \theta), n, u)$. Now define a vector field S on $M \times S^1$ using the previous vector field $Z(r, v)$ and taking $v = \sin 2u, v/r = 2\sin^2 u$. Hence S has coordinates

$$(\sin(2u)x, \sin(2u)\theta, \sin^2(u), -2\cos^2(u), 0)$$

Then all trajectories of S are also closed, and since $r = |\cot gu|$ we conclude that the length of the circles goes to infinity as $u \rightarrow 0, \pi$. Also note that when $u = 0, \pi$ then the vector field is constant hence all trajectories have the same length.

Chapter 3

Main Results

This chapter is devoted to the proof of the following theorems:

Theorem A. *Let $\mathcal{W}^c$ be a compact dynamically coherent normally hyperbolic foliation integrating the center bundle of the partially hyperbolic map f , and assume that for every f -periodic center leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are complete. Then $\mathcal{W}^c$ is uniformly compact.*

Conversely, if $\mathcal{W}^c$ is uniformly compact then it is complete.

Definition. A dynamically coherent partially hyperbolic diffeomorphism is said to be *centrally transitive* if there exist a center leaf whose forward orbit is dense.

Theorem B. *Let $f : M \rightarrow M$ be a centrally transitive partially hyperbolic diffeomorphism whose center foliation $\mathcal{W}^c$ is compact. Then $\mathcal{W}^c$ is uniformly compact.*

Theorem C. *Let $f : M \rightarrow M$ be a dynamically coherent partially hyperbolic diffeomorphism whose center foliation is compact, and assume that either*

1. $\mathcal{W}^c$ is one dimensional, or

2. f is a center isometry.

Then $\mathcal{W}^c$ is uniformly compact.

We will first discuss the components common to all proofs and then we will go over the specific parts to each theorem. Some results about the existence of periodic center leaves are also obtained.

Throughout this chapter we will assume that every partially hyperbolic diffeomorphism is dynamically coherent without explicitly mentioning it. During the proofs however we will use the more natural hypothesis of local product structure, although this is equivalent to dynamical coherence by Proposition 1.5.

Standing hypothesis for the chapter: f is a dynamically coherent partially hyperbolic diffeomorphism.

We start by noticing the following simple but very useful proposition.

Proposition 3.1. *Fix a leaf $L \in \mathcal{W}^c$ and let $G(L), G^s(L), G^u(L)$ be the holonomy groups of L inside the manifolds $M, W^s(L), W^u(L)$ respectively. Then $G(L)$ is isomorphic to $G^s(L) \times G^u(L)$.*

Proof. Take a disc $D = D(x) \cap E^c$ such that $T_x D = E_x^s \oplus E_x^u$.

Recall that the holonomy groups are defined up to a conjugation (see [God91]), and changing the transversal where the holonomy group is defined is reflected by changing the holonomy group by a conjugate one.

This means that we are free to choose any transversal that we like (inside the corresponding manifold) to compute the holonomy group.

For computing $G^s(L)$ and $G^u(L)$ we will take small discs $W_\epsilon^s(x) \subset W^s(x)$ and $W_\epsilon^u(x) \subset W^u(x)$ such that the projections $\widetilde{\mathcal{W}}^s(x), \widetilde{\mathcal{W}}^u(x)$ of those discs lie inside D . Using the transversal coordinates at x given by the local product structure the assertion is clear (see figure 3.1).

Figure 3.1: Transverse coordinates in D .

■

It follows that if $G^s(L)$ and $G^u(L)$ are finite then $G(L)$ is finite. In particular, if we can show that the foliation $\mathcal{W}^c$ is uniformly compact when restricted to each manifold $W^s(L), W^u(L)$ then it is uniformly compact in the whole manifold.

Corollary 3.2. *Assume that E^c has codimension one inside both E^{cs} and E^{cu} . Then the foliation $\mathcal{W}^c$ is uniformly compact. Moreover, if the center foliation is orientable then all center leaves are without holonomy.*

For an ad-hoc proof of the previous Corollary when $\dim M = 3$ see [BW05].

Proof. We use the fact that if $\mathcal{F}$ is codimension one foliation in a (not necessarily compact) manifold V , then all holonomy groups are finite. See [Hae62]. Furthermore, since the holonomy maps are represented by local homeomorphisms of $\mathbb{R}$, one can show that every holonomy group has order at most two, and in fact has order two precisely when it

contains an element which changes the orientation of $\mathbb{R}$. Hence if the foliation is oriented, all holonomy groups are trivial.

The statement follows by applying the previous remark and Proposition 3.1 to the manifolds $W^s(L), W^u(L)$. ■

It will be important to study the holonomy maps for the f -periodic leaves. Those give the simplest relation between the topology of the foliation and the dynamics under f . So first we need to show that we can find lots of periodic leaves.

Proposition 3.3. *Let f be a dynamically coherent partially hyperbolic diffeomorphism with (not necessarily compact) center foliation $\mathcal{W}^c$. If we define the sets*

$$\mathcal{P}^s = \bigcup \{W^{cs}(x) : \text{there exists a periodic center leaf } L \subset W^{cs}(x)\}$$

$$\mathcal{P}^u = \bigcup \{W^{cu}(x) : \text{there exists a periodic center leaf } L \subset W^{cu}(x)\}$$

then both $\mathcal{P}^s$ and $\mathcal{P}^u$ are dense in M .

Proof. We will prove the statement for $\mathcal{P}^s$. The other case is analogous.

Fix any point x . Then there exist two positive iterates of x , say $f^n(x)$ and $f^{n+m}(x)$ such that

$$d(f^n(x), f^{n+m}(x)) < \frac{\eta}{10}$$

where η is the constant of the local product structure.

Take a disc $D = W_\epsilon^u(f^n(x))$ with ϵ so small that in the ball $B_\epsilon(f^n(x))$ we have coordinates given by the local product structure. Note that these coordinates are built by “stacks” of sets of the form $W^s(P)$ where P is a center plaque passing through a point of $W_\epsilon^u(f^n(x))$.

We then define a map from D to itself by first iterating m times and then projecting back to $W^u(f^n(x))$ using the sets $W^s(P)$. Under this procedure the disc D “engulfs” itself, and since this map is clearly continuous, it has a fixed point $p \in D$.

But this means that $W^s(L_p)$ is periodic under f . Note that ϵ can be chosen so that $d(p, f^n(x)) < \eta/2$.

Figure 3.2: The existence of a fixed point for the cs holonomy map

We apply the same procedure to the map f^{-1} , but now considering the disc $D' = W_\epsilon^s(f^{n+m}(x))$ to find a point q with the properties

- $W^u(L_q)$ is periodic
- $d(q, f^n(x)) < \eta/2$

Now since $d(p, q) < \eta$ we know that there exists a center plaque in $W^s(L_p) \cap W^u(L_q)$. This plaque is of course periodic under f , and thus the whole center leaf where it is

contained is periodic. Let $\tilde{L}$ be this leaf. Observe that $\tilde{L} \subset W^{cs}(p)$.

Since $p \in W^u(f^n(x))$ it follows that the (also periodic) leaf $f^{-n}(\tilde{L})$ is contained in $W^{cs}(f^{-n}(p))$. Note that $f^{-n}(p)$ can be taken arbitrarily close to the point x (by taking n sufficiently big).

This proves the proposition. ■

3.1 Completeness and uniform volume bounds

In this part we establish Theorem A. We first discuss the relation among different groups of center leaves when these center leaves are inside the same stable or unstable leaf. Most of the times we only discuss only one of these possibilities, the other being completely analogous. From now on we will assume that the foliation $\mathcal{W}^c$ is compact.

Standing hypothesis: $\mathcal{W}^c$ is a compact foliation.

Lemma 3.4. *Suppose that L, L' are center leaves satisfying*

1. $L \subset W^u(L')$.
2. $W^u(L')$ is complete.
3. $G^s(L')$ is finite.

Then $G^s(L)$ is also finite.

Proof. First suppose that L is in the δ -neighborhood of L' with δ so small that the local product structure is defined inside this neighborhood.

Then by Theorem 2.2 one can find arbitrarily small foliated neighborhoods U' of L' inside $W^s(L')$. Using these, we now construct arbitrarily small neighborhoods U of L inside $W^s(L)$, and this by Theorem 2.1 implies that $G^s(L)$ is finite as well.

The idea is to construct the foliated neighborhood by pieces. Take a point $x \in L$: since we are assuming that $L \subset W^s(L')$ there exists $y \in L'$ such that $x \in W_\delta^s(y)$. If we consider a small plaque centered at y , by using the local product structure and the fact that L' has arbitrarily small foliated neighborhoods inside $W^s(L')$ we can conclude that there exists a plaque $P_x \subset L$ centered at x and a foliation cube C_x of $\mathcal{W}^c|_{W^{cs}(x)}$ with the properties

1. C_x is homeomorphic to $P_x \times D^s$ where D^s is the unit disc of dimension s .
2. The “transverse size” of the C_x is small, where the transverse size is defined as¹

$$\sup\{dist_{W^s(u)}(u, v) : u, v \in C_x, u \in W^s(v)\}.$$

Figure 3.3: Construction of the foliated neighborhood of L in $W^s(L)$.

Now consider another point $z \in P_x$. By proceeding as before, we can find P_z, C_z with similar properties as the ones corresponding to x . Note then that by taking a smaller

¹Recall that $dist_{W^s(u)}$ denotes the distance measured inside the stable manifold of u .

transverse size for the cube of x if necessary, we can construct a neighborhood C of $P_x \cup P_z$ such that

- C is homeomorphic to $(P_x \cup P_z) \times D^s$.
- The transverse size of C is small.

It follows then by compactness of L that by using the previous procedure we can construct a foliated neighborhood U of L whose transverse size is as small as we want.

If L is not completely contained in the δ -neighborhood of L' , by taking pre-images and using the fact that

$$r = \sup\{dist_{W^s(y)}(x, y) : x \in L, y \in L', x \in W^s(y)\} < \infty$$

we conclude that for some $N > 0$ the leaves $f^{-N}(L)$ and $f^{-N}(L')$ are δ close, hence $G^s(f^{-N}(L))$ is finite. Since iterating does not change the cardinality of the holonomy groups, we obtain that $G^s(L)$ is also finite. ■

We are now seeking conditions that guarantee that the holonomy group $G^s(L)$ of a center leaf L is finite. The next proposition is crucial.

Proposition 3.5. *Suppose that L is a periodic center leaf, and $W^s(L)$ is complete. Then for every leaf $L' \subset W^s(L)$ we have*

1. L' is a finite covering of L .
2. The group $G^s(L')$ is finite.

We note the following general lemma:

Lemma 3.6. *Suppose that L is a center leaf and denote by W the strong stable manifold of a point $x \in L$. Then if $\sup\{vol(f^n(L)) : n \geq 0\} < \infty$, the number of intersections of W with L is bounded.*

Proof. Fix a finite foliation atlas $\mathcal{A} = \{U_i\}_1^k$ by bi-distinguished cubes with respect to the foliations $\mathcal{W}^c$ and $\mathcal{W}^s$.

If $y \in L \cap W \setminus \{x\}$ then for some positive n we have that $f^n(y)$ will be in the same cube of $\mathcal{A}$ that $f^n(x)$ is. Now $f^n(y)$ belongs to the stable manifold of $f^n(x)$, and since $f^n(W)$ is transverse to $f^n(L)$, we conclude that $f^n(x)$ and $f^n(y)$ are not contained in the same plaque of $f^n(L)$.

Altogether, this implies that

$$\text{vol}(f^n(L)) \geq 2m$$

where $m = \inf\{\text{vol}(P) : P \text{ plaque of the atlas } \mathcal{A}\} > 0$.

The same argument shows that if there are r points in $L \cap W$ then for some large enough n one has $\text{vol}(f^n(L)) \geq rm$. But $\text{vol}(f^n(L))$ is bounded, so r cannot be arbitrarily big.

■

Proof of 3.5. First we observe that if $x \in L$ then

$$W^s(x) \cap L = \{x\}$$

To see this we observe that by 3.6 there exist finitely many points contained in the intersection of L and $W^s(x)$.

Figure 3.4: After enough iterates all points of the intersection are in the same foliated cube.

Denote by $x_0 = x, \dots, x_k$ the points in this intersection. Then if we fix a foliated cube centered in x and subordinate to both foliations $\mathcal{W}^s$ and $\mathcal{W}^c$, we see that if $x_i \neq x_j$ then x_i and x_j are in different plaques.

But under iterations these points are approaching $f^n(x)$, so all have to coincide.

Now take any other leaf $L' \subset W^s(L)$, and fix a point $x \in L$. By 1.2 we know that $W^s(x) \cap L' \neq \emptyset$. Let y be any point in this intersection.

Using the fact that $\mathcal{W}^s$ sub-foliates $W^s(L)$ and the previous remark, we can use the stable leaves to define a projection from a neighborhood of the point y inside L' to a neighborhood of the point x inside L . It is a homeomorphism from one neighborhood to the other, which implies that L' is a covering of L .

Note that all coverings are obtained by projecting along the stable leaves.

We want to show now that $G^s(L)$ is finite. To define the holonomy group we fix once and for all a transversal $W = W^s(x)$ with $x \in L$. The elements of $G^s(L)$ are represented by germs of local homeomorphisms $g : W \rightarrow W$.

Fix one of these maps, and note that is obtained by taking a closed loop $\alpha \in L$ and lifting this loop to nearby leaves. Given this closed loop α we can extend the map g to the whole transversal W as follows:

Given $y \in W$ consider the lift $\tilde{\alpha}_y$ of the curve α to the leaf $W^c(y)$, such that $\tilde{\alpha}_y(o) = y$. Then define $g(y) = \tilde{\alpha}_y(1)$.

This procedure clearly defines a continuous extension of g to the whole W , with the property that for every point $y \in W$, the points y and $g(y)$ are in the same center leaf.

Our next claim is that for every other center leaf L' the number of points in $W \cap L'$ is finite.

Suppose not, then by compactness of L' we can find a sequence of points in $W \cap L'$ converging to a point in L' . In particular the plaques of W have to accumulate near this limit point. But then using the holonomy along W and Lemma 1.2 we conclude that $W \cap L$ is also infinite, a contradiction.

Consider the group $G^s(L)$: this group acts in the connected manifold W by homeomorphisms (since every element can be extended globally) and the orbit of every point in W under this action is finite by the previous observation.

Then by 2.7 we conclude that $G^s(L)$ is finite.

Finally let L' be any other leaf. Since $G^s(L)$ is finite we have by 2.2 that there exists a foliated neighborhood U of L where the holonomy of every leaf inside U is finite.

To see this last point one can argue as follows: take a small transversal $D \subset W$ such that every element of $G^s(L)$ sends D to itself. This can be achieved since U is foliated.

If $L' \subset U$, then one sees that $G^s(L')$ is a subset² of $G^s(L)$, and hence it is finite.

In general, if L' is not inside U one takes n positive integer big enough so that $f^n(L') \subset U$ and concludes that $G^s(f^n(L'))$ is finite: this as we have seen before implies that $G^s(L')$ is finite.

■

²It also follows easily that it is a subgroup since every closed loop in L' is the lift a closed loop in L .

Of course, there is a similar statement for $L' \subset W^u(L)$.

The ideas of the last part of the previous proposition are also useful to prove the following:

Proposition 3.7. *Suppose that L is a periodic leaf with finite holonomy. Then*

1. *Both $W^s(L)$ and $W^u(L)$ are complete.*
2. *If $L' \subset W^s(L)$ then $G^s(L')$ is finite, and if $L' \subset W^u(L)$ then $G^u(L')$ is finite.*

Proof. We work with $W^s(L)$. The case of $W^u(L)$ is similar.

To prove the first part recall that by 1.3 it suffices to show that $W^s(L)$ is foliated by the center foliation.

Take a point $x \in W^s(L)$ then there exists a point $y \in L$ such that $x \in W^s(y)$.

Since $G^s(L)$ is finite there exists a foliated neighborhood of L inside $W^s(L)$. Under positive iterations x is going to enter this neighborhood, and hence all its center leaf is contained in $W^s(L)$. Using that $W^s(L)$ is f -invariant we conclude that $W^c(x) \subset W^s(L)$.

To prove the second part we use the same argument as in the last part of 3.5 to conclude that near L every center leaf has finite holonomy. But every leaf inside $W^s(L)$ has an iterate which is very close to L , and hence the claim follows since under iterations the cardinality of the holonomy groups is preserved.

■

The next corollary is immediate but interesting enough to point out.

Corollary 3.8. *Suppose that $W^s(L)$ contains a periodic center leaf L' with finite holonomy. Then $W^s(L) = W^s(L')$ and in particular $W^s(L)$ is complete.*

Proof of Theorem A. First we assume that for every periodic center leaf L both $W^s(L)$ and $W^u(L)$ are complete and we want to show that every center leaf has finite holonomy. By Proposition 3.1, it suffices to prove that for every center leaf L both groups $G^s(L)$ and $G^u(L)$ are finite.

We fix then a center leaf L and consider $G^s(L)$. By lemma 3.4 we only need to find one center leaf in $L' \subset W^u(L)$ such that $G^s(L')$ is finite. But this follows from Propositions 3.3 and 3.5.

To establish the converse, denote by $\pi : M \rightarrow N = M/\mathcal{W}^c$ the projection onto the quotient of M by the leaves of the center foliation. The space N is a compact Hausdorff space, and it is thus metrizable where a compatible metric is given as follows: if K, K' are two center leaves then the distance between them as points in N is

$$dist^*(\pi(K), \pi(K')) = \inf\{d(x, y) : x \in K, y \in K'\}$$

(see [Bou98]).

Fix a leaf $L \in \mathcal{W}^c$, and consider any other leaf L' such that $L' \cap W^s(L) \neq \emptyset$.

Let ϵ be such that every center leaf has an (abstract) ϵ -tubular neighborhood. If we can show that some iterate of L' is inside the ϵ -tubular neighborhood of the iterate of L , then we can guarantee that all points $y \in L'$ are in the stable manifold of some point of L . Note that in principle, we cannot assert the converse (i.e. every stable manifold of a point in L intersects L'). To see this, we proceed as follows.

By theorem 2.2 we can conclude that every point $p \in N$ has a neighborhood $V(\delta_x, x)$ so that $\pi^{-1}(p)$ is homeomorphic to a foliated bundle with transverse section $V(\delta_x)$, and since the space N is compact we can find finitely many points $p_1, \dots, p_k$ and $0 < \delta < \epsilon/4$ satisfying

$$N = \cup_{i=1}^k V(\delta, p_i).$$

Finally let α the Lebesgue number of the previous covering. By hypothesis there

exist $x \in L, y \in L'$ in the same strong stable manifold. Thus, for some positive iterate N the points $f^N(x)$ and $f^N(y)$ are at distance less than α , and hence contained in the same $V(\delta, p_i)$ for some $i = 1, \dots, k$. This neighborhood is foliated, so both center leaves $L_N = W^c(f^N(x)), L'_N = W^c(f^N(y))$ are completely contained in it. Note that

$$\text{dist}(\pi L_N, \pi L'_N) \leq \epsilon/2 < \epsilon$$

so L'_N is contained in the ϵ -tubular neighborhood of L_N , as we wanted to prove. ■

In [BW05] C. Bonatti and A. Wilkinson asked whether for a dynamically coherent partially hyperbolic diffeomorphism the center stable and center unstable manifolds of periodic center leaves were complete. When the center foliation is compact Theorem A completely characterizes the answer. Furthermore, one obtains the following consequence.

Corollary 3.9. *Assume that f is a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation and such that for every periodic center leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are complete. Then the same is true for every leaf (i.e. the center foliation is complete).*

3.2 The centrally transitive case

Throughout this section we assume that f is a centrally transitive partially hyperbolic diffeomorphism.

We note that in this case we can improve proposition 3.3.

Proposition 3.10. *Define the sets*

$$\mathcal{R}^s = \bigcup \{W^s(L) : L \text{ is a periodic center leaf with trivial holonomy}\}$$

$$\mathcal{R}^u = \bigcup \{W^u(L) : L \text{ is a periodic center leaf with trivial holonomy}\}$$

Then both $\mathcal{R}^s$ and $\mathcal{R}^u$ are dense in M .

Figure 3.5: In any center unstable disk there exist points whose center leaf has trivial holonomy (in the manifold).

Proof. The idea of the proof is the same as the proposition 3.3. We indicate the relevant changes.

Take x and consider a very small ϵ -disc $D = W_\epsilon^{cu}(x)$. In particular we require that ϵ is so small that D is contained in a foliation cube with respect to the foliation $\mathcal{W}^c$.

Since the system is centrally transitive there exists a center leaf $L = W^c(z_0)$ such that its forward orbit is dense. Then there exist some positive integer n and a point $z \in f^n(L)$ such that local stable manifold of z intersects D . Denote by y the point of intersection. See figure 3.5.

Since the set of points whose center leaves have trivial holonomy is open (2.4), we can find a point a foliated neighborhood N such that every center leaf inside N has trivial holonomy. Note that we can find arbitrarily large iterates of L that lie completely inside N , and since the distance between $f^n(z)$ and $f^n(y)$ tends to zero, we will find arbitrarily large iterates of y contained in N . Note that since N is foliated, the corresponding center leaves of each of those iterates will be also contained inside N .

Then we can repeat the argument in 3.3 and obtain the periodic center leaf P with a point inside N , and hence it has trivial holonomy. Note that by 3.8 $W^s(P)$ is complete,

and intersects the unstable manifold of some positive iterate of y . From this it follows that $W^s(P) \cap D \neq \emptyset$, and hence the claim. ■

Remark. In fact, for Theorem B we only need to show that given any point $x \in M$ we can find arbitrarily close points $y \in W^{cu}(x), z \in W^{cs}(x)$ such that L_y, L_z have finite holonomy. We gave the proof of the previous proposition due to its interest on its own.

Proof of Theorem B. Take a leaf L . Again by Proposition 3.7 it suffices to show that $G^s(L)$ is finite, and for this we will construct arbitrarily small foliated neighborhoods of L inside $W^s(L)$ as we did in Lemma 3.4. Note that since we are not assuming that $W^s(L)$ is complete we can not use Proposition 1.2.

The construction of the foliated neighborhood of L inside $W^s(L)$ is the same, except that now we use Propositions 3.10 and 3.7 to find arbitrarily close to each point $x \in L$ a point y such that $G^s(W^c(y))$ is finite (in fact it is trivial). Then again we construct the foliated neighborhood of L by gluing small neighborhoods of the plaques of L . ■

Definition. A center leaf is said to be *bi-transitive* if its forward and backwards orbit is dense.

The set of bi-transitive center leaves will be shown to be dense (see 4.12). We have the following.

Corollary 3.11. *Let f be centrally transitive partially hyperbolic diffeomorphism and L be a center leaf that is either periodic or bi-transitive. Then both $W^s(L)$ and $W^u(L)$ are complete.*

Therefore, for a centrally transitive map the set of points x such that $W^s(L_x)$ and $W^u(L_x)$ are complete is a dense set.

Proof. The case when L is periodic follows from Theorem B and 3.7.

We prove the second case by using Proposition 1.3, i.e. we show that if the forward and backwards orbit of L is dense then $W^s(L)$ and $W^u(L)$ are foliated by the center foliation.

Suppose then that $y \in W^s(z)$ with $z \in L$. Since the set of points whose center leaf has trivial holonomy is an open set by 2.4, we can find a foliated neighborhood U such that every leaf inside U has trivial holonomy.

As in the proof of Theorem A, to show that $W^c(y) \subset W^s(L)$ it suffices to show that for some positive iterate the points $f^N(y)$ and $f^N(z)$ are inside U . But this follows immediately since the forward orbit of L is dense and U is foliated.

The case of $W^u(L)$ is similar. ■

Note that in the previous proof we have showed in fact that the set of points whose center leaves are bi-transitive is contained in the set of points whose center leaves have trivial holonomy.

We have seen that the presence of recurrence allows us to find enough periodic center leaves with finite holonomy to spread out the finiteness of the holonomy to the whole manifold. Using this idea we can prove the following.

Theorem 3.12. *Let f be a dynamically coherent partially hyperbolic diffeomorphism whose central foliation is compact, and assume it preserves a probability measure ν . If ν is positive on open sets, the center foliation is uniformly compact.*

Proof. We will show that the set of periodic center leaves with trivial holonomy is dense in M . Then we proceed as in the proof of Theorem B.

Take U open in M : we want to show that there exist a point $x \in U$ such that L_x is periodic. Since the set of leaves with trivial holonomy is open and dense by 2.4, we can

assume that every center leaf intersecting U has trivial holonomy.

Using the same arguments as in 3.10, it suffices to find a point in U that is recurrent: that is, a point $y \in U$ such that for arbitrarily big positive iterates we have $f^n(y) \in U$. But this is guaranteed by Poincaré's Recurrence Theorem (see [Mañ83]), and the fact that ν is positive on open sets. ■

We are now ready to state the following Corollary, which gives a partial answer to the question of whether the center foliation of a dynamically coherent partially hyperbolic diffeomorphism is complete.

Corollary 3.13. *Assume that f is a dynamically coherent partially hyperbolic diffeomorphism whose center foliation is compact, and either*

1. *f is transitive.*
2. *f is volume preserving*
3. *The center foliation is one dimensional.*

Then $\mathcal{W}^c$ is complete.

This Corollary is direct consequence of Theorems A, B and C.

3.3 The one dimensional case

Here we prove Theorem C, and in particular we conclude that the foliation corresponding to Sullivan's example is not the center foliation of any dynamically coherent partially hyperbolic diffeomorphism. More precisely, we have the following.

Definition. A $\mathcal{C}^1$ -flow in a compact manifold is a *Sullivan Flow* if every orbit is periodic but there is no upper bound in the periods.

Theorem C implies that:

Corollary 3.14. *A one dimensional normally hyperbolic foliation does not support a Sullivan flow whose orbits coincide with the leaves of the foliation.*

Proof. Since the orbits of the flow forms a $\mathcal{C}^1$ foliation, this foliation is uniquely integrable. Hence it is dynamically coherent by Corollary 1.6, so we can apply Theorem C to conclude the claim. ■

We notice that in the proof of Theorem B we have established:

Lemma 3.15. *Suppose that the set of center stable manifolds corresponding to periodic center leaves with finite holonomy is dense, and likewise for the set of center unstable manifolds. Then the center foliation is uniformly compact.*

We obtain Theorem C as a Corollary from the following more general theorem:

Theorem 3.16. *Suppose that f is a partially hyperbolic diffeomorphism with compact center foliation, and if L is a periodic leaf of period m denote by $\phi_L = f_*^m : \pi_1(L) \rightarrow \pi_1(L)$.*

If for every periodic leaf L the map ϕ_L is periodic, then the foliation $\mathcal{W}^c$ is uniformly compact.

Before we start with the proof let us point out the following. Given a compact center leaf L , its fundamental group is finitely generated. Thus if $r > 0$ there exist finitely many homotopy classes $\alpha \in \pi_1(L)$ with representatives with length less than equal r .

Proof. By 3.3 and the previous lemma, we only need to show that in this case, all periodic leaves have finite holonomy.

We then take a periodic center leaf L , and by passing to an iterate we assume it is fixed under f . Consider also a strong stable manifold $W = W^s(x)$ where $x \in L$. The holonomy group is represented by germs of homeomorphisms at the point x .

We fix a loop α in L whose holonomy representation is given by a local homeomorphisms $g : D \subset W \rightarrow W$, where D is a small disc centered at x of radius ϵ . We claim that we can extend g to an homeomorphism from the whole W to itself.

To see this recall that g only depends on the homotopy class of α inside $\pi_1(L, x)$ ([God91]), and by hypothesis

$$r = \sup_{n \in \mathbb{Z}} \{length[f^n(\alpha)]\} = \max_{n \in \mathbb{Z}} \{length(\phi_L^n([\alpha]))\} < \infty$$

where $[\alpha]$ denotes the homotopy class in the fundamental group of the corresponding leaf, and $length[\alpha] = \inf\{length(\beta) : [\beta] = [\alpha], \beta \text{ loop inside } L \text{ at } x\}$

Now, every holonomy map can be obtained by the following procedure: if the map is going to represent the loop α we fix an atlas subordinate to the foliation $\mathcal{W}^c$ and then we cover the α by plaques of this atlas. Then the image of a point y close to x under the map corresponding to α is obtained by following the same chain of plaques of seeing where the final plaque intersects the transversal.

In particular we see that since the atlas can be taken finite, that for a fixed size r there exists a definite value ϵ where all holonomy corresponding to classes of loops whose lengths are less than equal to r are defined in discs of size at least ϵ .

Going back to our map g , now it is easy to see that we can extend its domain to a disc of size $\lambda\epsilon$, where $\lambda > 1$ is the contraction of the stable manifold. To extend g in a disc of $D(x, \lambda\epsilon)$ we iterate by f , apply the holonomy corresponding to $\phi_L([\alpha])$, and then pre-iterate back by f . Clearly this defines a continuous extension of g to $D(x, \lambda\epsilon)$. The same argument shows that we can extend g to the whole W .

By lemma 3.6, the orbit of every point of W by g is finite (note that lemma 3.6 does not require completeness of the manifold $W^s(L)$). But then by Theorem 2.7 we conclude that $G^s(L)$ is finite.

Likewise $G^u(L)$ is finite, and hence by 3.1 $G(L)$ is finite. ■

Corollary 3.17. *Let f be a partially hyperbolic diffeomorphism whose center foliation $\mathcal{W}^c$ is compact and one dimensional.*

Then $\mathcal{W}^c$ is uniformly compact. In particular for every periodic center leaf L the manifolds $W^s(L)$ and $W^u(L)$ are complete.

Proof. Fix a periodic leaf and take $x \in L$. Since $\pi_1(L, x) \simeq \mathbb{Z}$, the induced action of f in $\pi_1(L, x)$ is either Id or $-Id$. From this and the previous theorem follows that the center foliation is uniformly compact. The last remark is consequence of Theorem A. ■

To finish with the last part of Theorem C, we recall that a partially hyperbolic map f is said a *center isometry* if for every $x \in M$ we have

$$v \in E_x^c \Rightarrow \|df(v)\| = \|v\|$$

It is known in this case that f is dynamically coherent (see [Bri03]). From 3.16 immediately follows the second part of Theorem C:

Corollary 3.18. *If f is a center isometry whose center foliation $\mathcal{W}^c$ is compact, then $\mathcal{W}^c$ is uniformly compact and thus complete.*

Chapter 4

Applications

Now we will apply the results of the previous chapter to obtain information about the map f . In the first part we will discuss plaque expansiveness and stability of the center foliation. In the second part we will study the map induced by f on the space obtained by collapsing the center leaves to points.

4.1 Plaque expansive foliations

Our first application is to show that if $\mathcal{W}^c$ is compact then all foliations $\mathcal{W}^c, \mathcal{W}^{cs}, \mathcal{W}^{cu}$ are *plaque expansive*.

We recall some definitions.

Definition. Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism¹.

1. A sequence $\underline{x} = \{x_n\}_{-N}^N$ where $N \in \mathbb{N} \cup \{\infty\}$ is called a δ -pseudo-orbit for f if $d(fx_n, x_{n+1}) \leq \delta$ for every $n = -N, \dots, N-1$.
2. The pseudo-orbit $\underline{y} = \{y_n\}_{-N}^N$ ϵ -shadows the pseudo-orbit $\underline{x} = \{x_n\}_{-N}^N$ if $\text{dist}(x_n, y_n) < \epsilon$ for every $n = -N, \dots, N-1$.

¹Of course these notions can also be defined for a general homeomorphism acting on a metric space. We won't have the opportunity to use these more general notions though.

3. We say that the pseudo-orbit $\underline{x}$ *respects the foliation $\mathcal{F}$* or it is *subordinate to the foliation $\mathcal{F}$* if for every $n \in \{-N, \dots, N-1\}$, $f(x_n)$ and x_{n+1} are in the same leaf of $\mathcal{F}$.

The relevant theorem involving pseudo-orbits is the following.

Theorem 4.1 (Shadowing). *Let $f : M \rightarrow M$ be a dynamically coherent partially hyperbolic map. Then there exists a constant $C(f) > 1$ only depending on f such that if δ is sufficiently small then any δ -pseudo-orbit can be $C(f)\delta$ -shadowed by a $C(f)\delta$ -pseudo-orbit subordinate to the foliation $\mathcal{W}^c$. That is, $f(x_n)$ and x_{n+1} lie always in the same center plaque.*

In particular, if $E^c = 0$ (i.e. the system is hyperbolic) we recover the classical shadowing theorem. Namely pseudo-orbits can be shadowed by true orbits.

This theorem is a generalization of classical shadowing theorem due to Hirsch, Pugh and Shub (see theorem 7A-2 in [HPS77]) , where it is proved under the same hypothesis that given $\epsilon > 0$ there exists $\delta > 0$ such that any δ -pseudo-orbit can be ϵ -shadowed by a ϵ -pseudo-orbit subordinate to the foliation $\mathcal{W}^c$. Since we are presenting a slightly different version (in particular, we obtain the relation between δ and ϵ) we will give the proof.

It will be convenient to have a notation for the following construction. Let η be the constant of the local product structure, and suppose we are given $x, y \in M$ such that $d(x, y) < \delta$, where $\delta \leq \eta$. Then we can define the *bracket between x and y* as

$$[x, y] = W_\delta^s(x) \cap W_\delta^{cu}(y)$$

see figure 4.1.

Taking δ so small that the exponential is defined and is nearly an isometry and since the angle between E^c, E^u and E^c, E^s is bounded away from zero, one sees that for some constant $A > 0$ depending only on the angles we have

Figure 4.1: The definition of the bracket between x and y .

$$d(x, [x, y]) < A\delta \quad (4.1)$$

$$d(y, [x, y]) < A\delta \quad (4.2)$$

Remark. Since we are going to deal with exponential estimates, it would be no loss in generality in assuming that $A = 2$. For the proof of 4.1 we will use A since we want to get a precise estimate in the dependence between δ and ϵ . However, to simplify the presentation later on we will assume that $A = 2$ without further notice.

Proof of 4.1. Denote by λ the weakest contraction in the stable bundle.

Lemma 4.2. *We can assume that λ is as small as we want.*

Proof. To see this observe that if m is a positive integer and $g = f^m$, then $\lambda(g) \leq \lambda(f)^m$. Suppose that we have proved the Theorem for the iterate g , i.e. there exist a constant

$C(g)$ such that if δ is sufficiently small then we can shadow any δ -pseudo-orbit for g by a $C(g)\delta$ -pseudo-orbit that respects the foliation $\mathcal{W}^c$.

Let $\underline{x}$ be a δ -pseudo-orbit for f , and denote by L maximum of the Lipschitz constants of f and f^{-1} . Note that $L > 1$ since points in the same unstable manifold are separated by more than $1/\lambda > 1$ after one iteration. It follows that the pseudo-orbit $\{x_{km}\}_k$ obtained by taking only the multiples of m is a $D\delta$ -pseudo-orbit for g where

$$D = \frac{L^m - 1}{L - 1}.$$

If δ is small enough then we will be able to $C(g)D\delta$ -shadow this pseudo-orbit by say, $\{y_{km}\}_k$. Then we “complete” this pseudo-orbit to

$$\underline{y} = \{\dots, y_{-m}, fy_{-m}, \dots, f^{m-1}y_{-m}, y_0, fy_0, \dots, f^{m-1}y_0, y_m, \dots\}$$

and we get an $C(g)D\delta$ -pseudo-orbit for f which $(C(g)D)^2\delta$ -shadows $\underline{x}$. ■

So without loss of generality we can assume that if A is the constant given in the definition of the bracket, then $r = \lambda A < 1$. Denote by $R = \sum_{n \geq 0} r^n$. The number δ will be taken very small (how small will be specified later).

First assume that we have a finite δ -pseudo-orbit $\underline{x} = \{x_0, \dots, x_N\}$, and define the points $y_0, \dots, y_N$ by

- $y_0 = x_0$
- $y_n = [x_n, f(y_{n-1})] = W_\delta^s(x_n) \cap W_\delta^{cu}(f(y_{n-1}))$ for $n = 1, \dots, N$.

Note that $d(x_1, y_1) < A\delta$. We want to estimate $d(x_n, y_n)$; suppose then that we have proved that $d(x_{n-1}, y_{n-1}) < A\delta(1 + r + \dots + r^{n-1})$. Then we get

$$\begin{aligned} d(f(y_{n-1}), x_n) &\leq d(f(y_{n-1}), f(x_{n-1})) + d(f(x_{n-1}), x_n) \\ &\leq \lambda d(y_{n-1}, x_{n-1}) + \delta \quad (\text{since } y_n \in W^s(x_n)) \\ &\leq A\lambda\delta(1 + r + \dots + r^{n-1}) + \delta < \delta(1 + r + \dots + r^n). \end{aligned}$$

and hence by (4.1)

$$\begin{aligned} d(x_n, y_n) &= d(x_n, [x_n, f(y_{n-1})]) \\ &\leq Ad(x_n, f(y_{n-1})) \\ &< A\delta(1 + r + \dots r^n) < AR\delta. \end{aligned}$$

We have thus constructed a sequence $\underline{y} = \{y_0, \dots, y_N\}$ satisfying

1. $d(f(y_n), x_{n+1}), d(f(y_n), y_{n+1}) \leq R\delta$
2. $\text{dist}(x_n, y_n) \leq AR\delta$
3. $\underline{y}$ is subordinate to $\mathcal{W}^{cu}$.

It follows by dynamical coherence that if our original pseudo-orbit $\underline{x}$ is subordinate to $\mathcal{W}^{cs}$ then $\underline{y}$ is subordinate to $\mathcal{W}^c$.

Now we apply the same argument to the ARL -pseudo-orbit for f^{-1} , $\underline{y}^{-1} = \{y_N, y_{N-1}, \dots, y_0\}$ and get another pseudo-orbit $\underline{z}^{-1} = \{z_N, z_{N-1}, \dots, z_0\}$ with the properties

1. $d(f^{-1}y_{n+1}, z_n), d(f^{-1}z_{n+1}, z_n) \leq A(RL)^2\delta$
2. $d(z_n, y_n) \leq (ARL)^2\delta$
3. $\underline{z}^{-1}$ is subordinate to $\mathcal{W}^c$.

Finally we end up with a $(AR)L^3$ -pseudo-orbit $\underline{z} = \{z_0, \dots, z_N\}$ for f subordinate to $\mathcal{W}^c$. Notice that

$$d(x_n, z_n) \leq d(x_n, y_n) + d(z_n, y_n) \leq AR\delta + (ARL)^2\delta = C(f)\delta.$$

so we have proved the theorem in the case where the pseudo-orbit is finite.

Now suppose that our pseudo-orbit $\underline{x}$ is infinite (for example bi-infinite). The previous argument allows us to find for every N a $C\delta$ -pseudo-orbit $\underline{z}^N$ which $C\delta$ -shadows the segment $\{x_{-N}, \dots, x_N\}$.

Since M is compact we can find a subsequence $\{N_k\}_k$ such that $z_n^{N_k} \xrightarrow{n \rightarrow \infty} z_n$. The sequence $\underline{z} = \{z_n\}_n$ is a $C\delta$ -pseudo-orbit which $C\delta$ -shadows $\underline{x}$, for numbers δ so that $C\delta < \eta$. ■

Definition. We say that a foliation $\mathcal{F}$ is *plaque expansive* if there exists $\xi > 0$ such that if $\underline{x} = \{x_n\}_n, \underline{y} = \{y_n\}_n$ are two ξ -pseudo-orbit respecting $\mathcal{F}$ and satisfying $\text{dist}(x_n, y_n) < \xi$ for every $n \in \mathbb{Z}$, we have that x_n and y_n are always in the same plaque of $\mathcal{F}$.

If the center foliation of f is plaque expansive, one says simply that f is *plaque expansive*.

This abuse of language is justified by the following proposition:

Proposition 4.3. *Let f be a dynamically coherent partially hyperbolic diffeomorphism. Then $\mathcal{W}^c$ is plaque expansive if and only if $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are plaque expansive.*

Proof. Suppose that $\mathcal{W}^c$ is plaque expansive and take two ϵ -pseudo-orbits respecting the foliation $\mathcal{W}^{cs}$. We suppose that ϵ is smaller than the constant of local product structure.

We define $z_0 = [x_o, y_o] = W_{2\epsilon}^s(x_o) \cap W_{2\epsilon}^{cu}(y_o)$. Since z_0 is the stable manifold of x_o we obtain that $\text{dist}(f(z_0), f(x_o)) < 2\lambda\epsilon$.

Now we define $z_1 = [x_1, f(y_0)] = [x_1, f(z_0)]$. Note that

$$d(x_1, f(z_0)) \leq d(x_1, f(x_o)) + \text{dist}(f(x_o), (z_0)) < \epsilon + 2\lambda\epsilon$$

and hence

$$d(z_1, x_1) \leq 2\epsilon(1 + 2\lambda)$$

Observe also that z_1 is in the same center that $f(z_0)$ because $x_1 \in W^{cs}(f(x_o))$.

Assume that we have found points $z_0, \dots, z_n$ so that

Figure 4.2: Construction of the sequence subordinate to the center foliation

1. $d(z_n, x_n) \leq 2\epsilon(1 + 2\lambda + \cdots + (2\lambda)^{n-1})$
2. $z_n \in W^s(x_n) \cap W^{cu}(y_n)$
3. $z_n \in W^c(z_{n-1})$.

We then define $z_{n+1} = [x_{n+1}, f(y_n)] = [x_{n+1}, f(z_n)]$, and note as before that $z_{n+1} \in W^c(z_n)$. Also

$$\begin{aligned} d(x_{n+1}, f(z_n)) &\leq \epsilon + 2\epsilon(1 + 2\lambda + \cdots + (2\lambda)^{n-1})\lambda \\ &\leq \epsilon(1 + 2\lambda + \cdots + (2\lambda)^n). \end{aligned}$$

and hence

$$\begin{aligned} d(z_{n+1}, x_{n+1}) &\leq 2\epsilon(1 + 2\lambda + \cdots + (2\lambda)^n) \\ d(z_{n+1}, f(z_n)) &\leq 2\epsilon(1 + 2\lambda + \cdots + (2\lambda)^n). \end{aligned}$$

By invariance, the foliation $\mathcal{W}^{cs}$ is also the center stable foliation for any iterate of f , so one can proceed as in Theorem 4.1 and assume that $2\lambda < 1$. Denote $M = \sum_{n \geq 0} (2\lambda)^n$ and choose ϵ so small that

$$(2M + 1)\epsilon < \text{expansivity constant of } \mathcal{W}^c.$$

Then our previous procedure defines a $2M\epsilon$ pseudo-orbit $\{z_n\}_{n \geq 0}$ subordinate to $\mathcal{W}^c$, satisfying for every $n \geq 0$

$$\text{dist}(z_n, y_n) < \epsilon(2M + 1).$$

We define z_n for $n < 0$ as follows: let $p = W_\epsilon^u(y_0) \cap W^c(z_0)$, and define $z_{-1} = f^{-1}(p)$. Finally define $z_{-n} = f^{-1}(z_{-n+1})$.

Since $z_{-n} \in W^u(y_{-n})$ one sees that for every $n > 0$

$$\text{dist}(z_{-n}, y_{-n}) < \epsilon$$

After all we have constructed a pseudo-orbit $\{z_n\}_{n \geq 0}$ subordinate to $\mathcal{W}^c$, and so that

$$\text{for every } n \in \mathbb{Z} \quad \text{dist}(z_n, y_n) < \text{expansivity constant of } \mathcal{W}^c$$

Hence, since we were assuming that $\mathcal{W}^c$ is plaque expansive, $z_0 \in W^c(y_0)$.

But this implies that $x_0 \in W^{cs}(y_0)$, and we conclude that $\mathcal{W}^{cs}$ is plaque expansive. Similarly $\mathcal{W}^{cu}$ is plaque expansive.

The converse is immediate. ■

Plaque expansiveness is an important device to attack stability problems as we will see later. Note that if the center foliation is plaque expansive we have the following corollary to 4.1.

Corollary 4.4. *Under the same hypothesis of theorem 4.1, if the center foliation of f is plaque expansive then there exists $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ then any bi-infinite δ -pseudo-orbit $\underline{x}$ can be $C\delta$ -shadowed by a $C\delta$ -pseudo-orbit $\underline{y}$ which respects $\mathcal{W}^c$. If $\underline{z}$ is any other $C\delta$ -pseudo-orbit which $C\delta$ -shadows $\underline{x}$ and respects $\mathcal{W}^c$, then y_n and z_n are always in the same plaque of $\mathcal{W}^c$.*

Corollary 4.5. *Let f be a dynamically coherent partially hyperbolic diffeomorphism and suppose that the non-wandering set of f is equal to M . Then the set of points whose central leaf is periodic is dense in M .*

Proof. Take any point x and let U be an arbitrary neighborhood of x . Since x is non-wandering there exists a positive integer k such that $f^k(U) \cap U \neq \emptyset$.

Now define the pseudo-orbit obtained by taking the sequence $[xf(x) \dots f^n(x)]$, and copying this block one after the other (in both directions).

By 4.1 there exists a pseudo-orbit $\{y_n\}_n$ subordinate to $\mathcal{W}^c$ and close to pseudo-orbit defined before. Now consider the pseudo-orbit $\{z_n\}$ defined by shifting everything k places to the left, i.e. $z_n = y_{k+n}$.

Then $\{z_n\}$ also shadows the first pseudo-orbit, and hence, since the system is plaque expansive, z_n and y_n are always in the same plaque. In particular y_0 and $f^k(y_0)$ are in the same plaque.

This means that the leaf through y_0 is periodic. ■

The hypotheses of the previous Theorem are fulfilled for example if the partially hyperbolic map f transitive and plaque expansive.

We now study the stability properties of compact normally hyperbolic foliations in the case where all leaves have trivial holonomy.

Definition. We say that a foliation $\mathcal{F}$ is *without holonomy* if for every leaf $L \in \mathcal{F}$ we have that $G(L) = \{0\}$.

Theorem 4.6. *Let f be a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation, and assume that for every center L leaf the submanifolds $W^s(L)$ and $W^u(L)$ are complete.*

Then

1. *All foliations $\mathcal{W}^c$, $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are plaque expansive.*
2. *If $\Omega(f) = M$ then the periodic center leaves are dense.*

Proof. By 4.3 it suffices to show that $\mathcal{W}^c$ is plaque expansive. The argument given here is an elaboration of 8.2 in [HPS77].

Consider the space $N = M/\mathcal{F}$. As was explained during the proof of Theorem A the space N is a compact metric space, with distance given by the formula

$$\text{dist}^*(\pi(L), \pi(L')) = \inf\{d(x, y) : x \in L, y \in L'\}.$$

We recall the construction of the unwrapping bundle. For each center leaf L let $U_L \subset TM$ be its tubular neighborhood; the tubular neighborhood of L is an immersed submanifold of M but not embedded in general. We construct U_L in the tangent bundle to avoid self intersections. The image of U_L under the exponential map will be referred as the *geometric tubular neighborhood of L* , to distinguish it from the tubular neighborhood in the tangent bundle. If $\epsilon > 0$, we will denote by $U_L(\epsilon)$ the ϵ -disc subbundle of the normal bundle of L .

Let $\mathcal{A} = \coprod_{L \in \mathcal{W}^c} L$, and denote by $i : \mathcal{A} \rightarrow M$ the inclusion. Note that f acts naturally on the zero section of the bundle

$$\zeta = \coprod_{L \in \mathcal{W}^c} U_L.$$

We denote this induced map by $\tilde{f}$, and we identify the zero section of ζ with $\mathcal{A}$. By theorem 6.1 of [HPS77] there exist $\epsilon > 0$ such that $\tilde{f}$ is defined in the ϵ -disc bundle

$$\zeta(\epsilon) = \coprod_{L \in \mathcal{W}^c} U_L(\epsilon)$$

and

$$\begin{array}{ccc} \zeta(\epsilon) & \xrightarrow{\tilde{f}} & \zeta \\ \text{expoi} \downarrow & & \downarrow \text{expoi} \\ M & \xrightarrow{f} & M \end{array}$$

The set $\mathcal{A}$ is $\tilde{f}$ -invariant, and $\tilde{f}$ is normally hyperbolic to $\mathcal{A}$. Also, this zero section is a maximal invariant set for $\tilde{f}$.

By theorem 2.2 each leaf $L \in \mathcal{W}^c$ has a foliated neighborhood $E(L)$ homeomorphic to a foliated bundle. Let $\nu(L)$ the transverse diameter of $E(L)$.

The projection $\pi : M \rightarrow N$ is open and since the space N is compact, there exist center leaves $L_1, \dots, L_r$ such that

$$M = \cup_{i=1}^r E(L_i)$$

One can take $\max\{\nu(L_i)\} < \epsilon/2$. Finally let α the Lebesgue number of the covering $\{\pi(E(L_i)) : i = 1, \dots, r\}$. We now claim that f is α -leaf expansive (the reader can provide the definition), and this clearly implies that it is plaque expansive.

If not, there exist two center leaves L, L' such that for every $n \in \mathbb{Z}$

$$\text{dist}^*(\pi(f^n L), \pi(f^n L')) < \alpha.$$

By definition of the metric in the quotient one sees that for every $n \in \mathbb{Z}$ the leaves $f^n(L)$ and $f^n(L')$ are always in the same local foliated bundle, and hence by election of α and ϵ ,

$f^n(L')$ is always in the ϵ – geometric tubular neighborhood of $f^n(L)$.

Now we use the hypothesis of the center foliation to be without holonomy: fix $n \in \mathbb{Z}$ and consider

$$\exp : U_{f^n L} \rightarrow M.$$

Then $\exp^{-1}(f^n L')$ contains a diffeomorphic copy of $f^n L'$ (in particular, there is no branching for this copy). This follows from the fact that both leaves $f^n L, f^n L'$ are in the same foliated neighborhood $E(L_{j_n})$, and hence $f^n L'$ is inside the image of the set where the exponential is an embedding.

But then we see that the set $\{\tilde{f}^n L'\}_{n \in \mathbb{Z}}$ is invariant, contradicting the maximality of $\mathcal{A}$.

The second part follows from 4.5 . ■

Note that as the example in page 11 shows, the condition that the center foliation is without holonomy is necessary to assert leaf expansiveness.

In the case of 3-manifolds we have the following Corollary. I thank Raul Ures for a simplification in the proof.

Corollary 4.7. *Suppose that f is a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation on a 3-dimensional manifold. Then*

1. *There exist a finite covering $\tilde{M}$ of M such that if we denote by $\tilde{f}$ the lift of f to $\tilde{M}$, then the center foliation of $\tilde{f}$ is without holonomy.*
2. *The center foliation of f is plaque expansive.*

Proof. Note that since the covering is finite and M is compact, the second part follows from the first one and the previous theorem.

By [HHUb], and since we are assuming dynamical coherence, no center stable or center unstable manifold can be compact. We can then pass to a quadruple covering and assume that the center foliation is both orientable and transversely orientable. Then one sees that every center stable and center unstable leaf is homeomorphic to a cylinder, with induced foliation homeomorphic to the horizontal foliation by circles. Hence the holonomy inside each center stable or center unstable leaf is trivial, and by 3.1 all holonomy is trivial. ■

In higher dimensions the previous argument doesn't work, but there is still the possibility that passing to a finite covering one could "straight up" all leaves. We pose the following

Conjecture: Let f be a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation on a manifold M . Then there exist a finite covering $\widetilde{M}$ of M such that if we denote by $\widetilde{f}$ the lift of f to $\widetilde{M}$, then the center foliation of $\widetilde{f}$ is without holonomy.

Now we are ready to attack the problem of the stability of the center foliation. For doing that we need some definitions:

Definition.

1. Let f, g be partially hyperbolic diffeomorphisms on M . We say that they are *centrally conjugate* if there exists a homeomorphism $h : M \rightarrow M$ sending center leaves of f onto center leaves of g and so that for every $x \in M$

$$h(W_f^c(x)) = W_g^c(hx)$$

The map h is called a *center conjugacy*.

2. The system $(f, \mathcal{W}^c)$ is structurally stable if f has a $\mathcal{C}^1$ neighborhood U so that for every $g \in U$ we have that g is partially hyperbolic and centrally conjugate to f .

The following important theorem is shown in [HPS77] (Theorem 7.1)

Theorem 4.8. *Let f be partially hyperbolic and assume that it is plaque expansive. Then $(f, \mathcal{W}^c)$ is structurally stable, and for g sufficiently close to f the center conjugacy can be taken $\mathcal{C}^0$ close to the identity and $\mathcal{C}^1$ nearly isometric when restricted to each center leaf.*

The next corollary follows directly from the previous two theorems.

Corollary 4.9. *Suppose that f is a dynamically coherent partially hyperbolic diffeomorphism whose center foliation is compact and without holonomy, and either*

1. $\mathcal{W}^c$ is one dimensional, or
2. f is transitive.

Then $(f, \mathcal{W}^c)$ is structurally stable.

If g is a diffeomorphism close to f , then by the previous theorem it is a partially hyperbolic diffeomorphism centrally conjugate to f . Since the leaf conjugacy in particular defines a bijection between the holonomy groups of the leaves f and g , it follows that g preserves a uniformly compact foliation.

4.2 The dynamics in the quotient

In this part we continue investigating the behavior of the map $g : N \rightarrow N$ induced by f where $N = M/\mathcal{W}^c$. We assume that the center foliation of f is uniformly compact, and hence N is metrizable.

Using theorem 4.1 we deduce.

Corollary 4.10. *The map g has the pseudo-orbit shadowing property, namely given $\epsilon > 0$ there exists $\delta > 0$ such that if $\{p_n\}_{n \in \mathbb{Z}}$ is a δ -pseudo-orbit then there exists a point $p \in N$ such that for every $n \in \mathbb{Z}$ we have*

$$\text{dist}_N(g^n(p), p_n) < \epsilon$$

If the center foliation is without holonomy, then the point p is unique.

This theorem allows us to give a simple proof of the following fact.

Theorem 4.11. *Let f be a dynamically coherent partially hyperbolic diffeomorphism whose center foliation is compact and such that E^c has codimension 1 in both E^s and E^u . Assume further that E^c is orientable². Then f fibers over a map $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ which is conjugate to a hyperbolic diffeomorphism.*

Proof. By 3.2 all holonomy groups of the center leaves are trivial, and hence by Sebastiani's theorem [Seb68] the quotient N is a compact 2-manifold and the projection $\pi : M \rightarrow N$ is a locally trivial fibration .

But N supports an expansive map with the pseudo-orbit shadowing property, and hence by a theorem of K. Hiraide and J. Lewowicz N is the 2-torus and g is conjugate to a pseudo-Anosov map. But any pseudo-Anosov map in the two torus is conjugate to an Anosov map. For an account of these facts see for example [Hir92] or [Lew03].

■

In the case when f is centrally transitive, it follows that the map g is transitive. Recall that if $h : X \rightarrow X$ is an homeomorphism on a metric space X , a point $x \in X$ is said to be *bi-transitive* if its forward and backward orbits are dense. The set B_h of bi-transitive points is a $\mathcal{G}^\delta$ set in X . Hence if X is complete then B_h is dense in X .

Recalling also that the projection $\pi : M \rightarrow N$ is open, we conclude the following.

²Note that we can always achieve this by passing to a two-fold covering.

Proposition 4.12. *Assume that f is a centrally transitive partially hyperbolic diffeomorphism. Then the set of bi-transitive center leaves is a residual set in M (and hence, a dense set).*

We recall the following definition due to D. Ruelle. A Smale space consists of a compact metric space X together with a homeomorphism $h : X \rightarrow X$, and a continuous function $[\cdot, \cdot] : \Delta(\epsilon) \rightarrow X$ where $\Delta(\epsilon)$ is the ϵ neighborhood of the diagonal in $X \times X$, and satisfying the following properties:

1. for every $x, y, z \in X$ we have

$$(a) \quad [x, x] = x$$

$$(b) \quad [[x, y], z] = [x, z]$$

$$(c) \quad [x, [y, z]] = [x, z]$$

If $0 < \delta < \epsilon$ define

$$V^s(x, \delta) = \{y : y = [y, x] \text{ and } d(x, y) < \delta\}$$

$$V^u(x, \delta) = \{y : y = [x, y] \text{ and } d(x, y) < \delta\}$$

One can show that for some small δ the bracket defines a homeomorphism from $V^s(x, \delta) \times V^u(x, \delta)$ onto an open neighborhood of x .

2. h preserves $[\cdot, \cdot]$.
3. there exists $\lambda > 0$ such that for every $n > 0$ we have

$$d(h^n(x), h^n(y)) \leq \lambda^n d(x, y) \text{ if } x, y \in V^s(z, \delta)$$

$$d(h^{-n}(x), h^{-n}(y)) \leq \lambda^n d(x, y) \text{ if } x, y \in V^u(z, \delta)$$

We assume now that the center foliation of f is without holonomy. Going back to the case of $g : N \times N \rightarrow N$, we now show that the local product structure defines a bracket as follows:

take points $p = \pi(L), q = \pi(L')$ in N so that $dist_N(p, q) < \eta$, where η the constant of the local product structure. Then there exists points $x \in L$, and $y \in L'$ whose distance in the manifold is less than η , so in particular we can define

$$[p, q] = \pi(W^s(L) \cap W^u(L'))$$

This bracket clearly continuous and well defined (here we use that the foliation is without holonomy), and satisfies the conditions (a), (b), (c) given in the previous definition. We now show

Proposition 4.13. *The triple $\{N, g, [\cdot, \cdot]\}$ is a Smale space.*

Proof. Since f preserves the foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ it is clear that g preserves the bracket.

Note that

$$V^s(p, \delta) = \{\pi(L) : dist_N(L, \pi^{-1}(p)) < \delta, L \text{ center leaf inside } W^s(W^c(\pi^{-1}(p)))\}$$

and similarly for $V^u(p, \delta)$.

But then if we take λ the expansivity constant of the stable foliation, we see that for $q, q' \in V^s(p, \delta)$ and n positive integer we have for some constant $C > 0$

$$dist_N(g^n(q), g^n(q')) < C\lambda^n dist_N(p, q)$$

and similarly for points in $V^u(p, \delta)$. It is well known then that one can change the metric for an equivalent one to get $C = 1$.

■

In particular since every map corresponding to a Smale space is expansive we obtain another proof of the fact that g is expansive.

Smale spaces have a very rich structure. In particular we obtain the following:

Theorem 4.14. *Suppose that f satisfies the hypotheses of Theorem B and that the center foliation is without holonomy. Then*

1. *There exist a finite partition $N_1, \dots, N_k$ of N , where each N_i is compact and such that f cyclically permutes them. Moreover, $g^k : N_i \rightarrow N_i$ is topologically mixing.*
2. *There exist a transitive subshift of finite type Σ_A , and a continuous bounded to one projection $\theta : \Sigma_A \rightarrow N$. Moreover θ is one to one on the pre-image of the bi-transitive points (and hence, on a residual set).*

For a discussion of these results in the context of Smale spaces see [Rue04].

Let us assume now that the f is a centrally transitive dynamically coherent partially hyperbolic diffeomorphism whose center foliation is uniformly compact. If $n \in \mathbb{N}$ we define by $N_n^c(f)$ the number of periodic center leaves with period less than equal to n . Arguments as in 4.6 show that this number is finite. We can then define the *central zeta function* of f as the formal power series

$$\zeta_f^c(z) = \sum_{n=1}^{\infty} \frac{N_n^c(f)}{n} z^n.$$

Using the previous theorem we have.

Corollary 4.15. *Assume that the center foliation is without holonomy. Then the central zeta function of f is rational.*

The proof follows from the existence of the symbolic model for the map g . See [Shu86] for the details.

We finish this section noticing another Corollary to the previous theorem. Denote by $h_{top}(g)$ the topological entropy of the map $g : N \rightarrow N$.

Corollary 4.16. *Suppose that the center foliation is without holonomy or that M is a 3-manifold. Then*

$$h_{top}(g) = \limsup_{n \rightarrow \infty} \frac{\log N_n^c(f)}{n}.$$

Furthermore, if f is a center isometry or E^c is one dimensional then

$$h_{top}(f) = \limsup_{n \rightarrow \infty} \frac{\log N_n^c(f)}{n}.$$

Proof. Suppose first that $\mathcal{W}^c$ is without holonomy.

Clearly a center leaf is periodic if and only if it is a periodic point for the map g . By theorem 4.14 there exist subshift of finite type and a bounded to one map $\theta : \Sigma_A \rightarrow N$ that semiconjugates the shift map on Σ_A and g . The theorem then follows from the similar statement for the shift map and the fact that the preimage of the periodic points for g under θ comprises the set of periodic points for the shift.

In the case of 3-manifolds we use the previous argument together with Corollary 4.7.

To prove the second part note that clearly $h_{top}(f) \geq h_{top}(g)$.

By Bowen's theorem (see [Bow71]) we also have that

$$h_{top}(f) \leq h_{top}(g) + \sup\{h_{top}(f|L) : L \in \mathcal{W}^c\}$$

where $h_{top}(f|L)$ denotes the entropy of f restricted to L . It suffices then to show that for every $L \in \mathcal{W}^c$ we have $h_{top}(f|L) = 0$, since in this case

$$h_{top}(f) = h_{top}(g) = \limsup_{n \rightarrow \infty} \frac{\log N_n^c(f)}{n}.$$

Assume first that E^c is one dimensional, and hence all leaves L are circles. A classical result states that for a diffeomorphism of the circle its topological entropy is zero. The

proof of this result only uses that a diffeomorphism of the circle preserves the order of the points and the circle has finite length. Hence we can apply the same argument to the orbit of a circle $\{f^n(L)\}_{n \in \mathbb{Z}}$ to conclude $h_{top}(f|L) = 0$. See [Wal82].

If f is a center isometry, the result trivially follows. ■

In the 3-dimensional case, the previous Corollary also follows from the results in [HHTU10].

4.3 Type of the leaves

Whenever we have a foliation on a manifold, a natural question is what is the relation among the different leaves. Of course one cannot expect them to be homeomorphic in general, so a reasonable thing to ask is whether they share the same universal cover.

Definition. A foliation $\mathcal{F}$ on a manifold M is said to be of *uniform type* if all leaves of $\mathcal{F}$ have homeomorphic universal covers.

For example if ϕ is a flow on M without singularities, then the foliation induced by the orbits of ϕ is of uniform type. More generally, if G is a Lie group acting on M and the action is effective and locally free³, then the orbit-foliation is of uniform type.

Now we assume that $f : M \rightarrow M$ is a dynamically coherent partially hyperbolic diffeomorphism with compact foliation $\mathcal{W}^c$. We have the following.

Proposition 4.17.

1. Assume that for every periodic center leaf $K \in \mathcal{W}^c$ the submanifolds $W^s(K)$ and $W^u(K)$ are complete. Then for every center leaf L the foliation $\mathcal{W}^c$ restricted to $W^s(L)$ or $W^u(L)$ is of uniform type.

³An action of a Lie group is locally free if all stabilizers are discrete.

2. If f is centrally transitive then $\mathcal{W}^c$ is of uniform type in M .

Proof. By Corollary 3.9 we have that the center foliation is complete.

Now suppose that L, L' are center leaves and $L' \subset W^s(L)$. During the last part of the proof of Theorem A we showed that in this case there exists a center leaf $\tilde{L}$ and a foliated neighborhood U of $\tilde{L}$ such that U is homeomorphic to a foliated bundle with base $\tilde{L}$, and for some positive iterate the leaves $f^N(L)$ and $f^N(L')$ are contained inside U .

Then $f^N(L)$ and $f^N(L')$ are coverings of $\tilde{L}$, hence they have homeomorphic universal covers which implies that L and L' have homeomorphic universal covers. Likewise if $L' \subset W^u(L)$.

Assume now that f is transitive and fix a leaf $\tilde{L}$. Consider a foliated neighborhood U of $\tilde{L}$ as in the previous part.

Now if L is any leaf, we know that there exists a forward transitive point $z \in W^s(L)$ (see Proposition 3.10 and in particular Figure 3.5). We conclude then that the center leaf containing z and $\tilde{L}$ have homeomorphic universal covers. But by the previous part $W^c(z)$ and L have also homeomorphic universal covers. We conclude that all leaves in the manifold have homeomorphic universal covers. ■

Another condition that guarantees that the center foliation is of uniform type is the following. A partially hyperbolic diffeomorphism is said *accessible* if given $x, y \in M$ there exists a piecewise $\mathcal{C}^1$ curve $c : [0, 1] \rightarrow M$ whose tangent is always contained in E^s or E^u and such that $c(0) = x, c(1) = y$.

Corollary 4.18. *If f is an accessible partially hyperbolic diffeomorphism whose center foliation is uniformly compact, then all center leaves have homeomorphic universal covers.*

The proof is obvious.

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