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Damped-driven Hamiltonian PDE
§1. Damped-driven Hamiltonian PDE

We are interested in the following class of equations:

\[ \langle \text{Hamiltonian PDE} \rangle = \nu \text{-small damping} + \kappa_\nu \langle \text{force} \rangle, \]

(\*)

where \( \nu \ll 1 \) and the scaling constant \( \kappa_\nu \) is such that solutions stay of order one as \( \nu \to 0 \) and \( t \gg 1 \). The constant \( \kappa_\nu \) is unknown, to find it is a part of the problem.

Equations (\*) describe turbulence in various physical media. E.g., water turbulence, if (\*) are the Navier-Stokes equations and optical turbulence if (\*) is a CGL equation (i.e., if the Hamiltonian PDE is a nonlinear Schrödinger equation).

All equations will be considered in the finite-volume case. The force may be deterministic or random. I am the most interested in the random case. The objects and the constructions I consider make sense in the deterministic case as well, but then I can prove much less.
§2. Navier-Stokes equations

Very important examples of equations

\[ \langle \text{Hamiltonian PDE} \rangle = \nu \text{-small damping} + \kappa_\nu \langle \text{force} \rangle, \quad (\ast) \]

are given by the Navier-Stokes equations (NSE) in dimension 2 and 3 and Burgers equation (often regarded as a 1d NSE). Now the Hamiltonian PDE is the Euler equations in the NSE case, and the Hopf equation in the Burgers case. In the theory of turbulence they deal with eq. (\ast), where \( \nu \) is \( 10^{-30} - 10^{-10} \) – very small indeed!

Concerning \( \kappa_\nu \):

- for 1d Burgers \( \kappa_\nu = 1 \),
- for 2d NSE \( \kappa_\nu = \sqrt{\nu} \),
- for 3d NSE nothing is clear. Kolmogorov conjectured \( \kappa_\nu = 1 \).

For 3d NSE we know about the problem nothing (including the right scaling constant \( \kappa_\nu \)). For 2d NSE we know something, and there was significant progress after 2000. Now I briefly review it:
Consider 2d NSE under periodic boundary conditions:

\[
\dot{u} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = \sqrt{\nu} \text{(random force)}, \quad \text{div} \ u = 0, \quad (NSE)
\]

where \( x \in \mathbb{T}^2 \) and \( \int u \, dx = \int \text{force} \, dx = 0 \). The random force is smooth in \( x \). As a function of time this is a nondegenerate white noise. Let \( \mathcal{H} \) be a suitable space of divergence-free vector fields \( u(x) \) such that \( \int u \, dx = 0 \). Then:

- (NSE) has a unique statistical equilibrium. This is a measure \( \mu_{\nu} \) in the space \( \mathcal{H} \) such that the distribution \( \mathcal{D}(u(t)) \) of any solution \( u(t) \) of (NSE) converges to \( \mu_{\nu} \) as \( t \to \infty \). \( \mu_{\nu} \) is called a stationary measure.
- The set of stationary measures \( \{ \mu_{\nu}, 0 < \nu \leq 1 \} \) is pre-compact in the space of measures in \( \mathcal{H} \), and every limiting measure \( \mu_0 = \lim_{\nu_j \to 0} \mu_{\nu_j} \) is an invariant measure of the 2d Euler equation (i.e. of \( (NSE)_{\nu=0} \))
- This limiting measures \( \mu_0 \) are “genially infinite-dimensional”: \( \mu_0 \)-measure of any finite-dimensional subset of \( \mathcal{H} \) equals zero.
The limiting invariant measure(s) $\mu_0$ describes the 2d turbulence. But it is unknown if $\mu_0$ is unique, i.e. if it depends on the sequence $\nu_j \to 0$, and how to calculate it.

See


Let us consider other Damped-Driven Hamiltonian PDE. Maybe for them we could tell more?
§3. CGL equations

Consider the damped-driven NLS equations (they are responsible for the optical turbulence):

\[ \dot{u} + i \Delta u - i |u|^2 u = \nu \Delta u + \sqrt{\nu} \text{ (random force)}, \quad (CGL_1) \]

\( d := \dim x = 1, 2, 3 \). We are interested in solutions for \( \nu \ll 1 \) on long time-intervals. So we pass to the slow time

\[ \tau = \nu t \]

and re-write the equation in the slow time:

\[ \frac{\partial u}{\partial \tau} + \nu^{-1} i (\Delta u - |u|^2 u) = \Delta u + \text{ (random force)}. \quad (CGL_2) \]

For \( d \leq 3 \) \((CGL_1)\) has a unique stationary measure \( \mu_\nu \), which also is a unique stationary measure for \((CGL_2)\). As for the NSE, for sequences \( \nu_j \to 0 \) we have limits
\[ \mu_{\nu_j} \rightarrow \mu_0 \text{ as } \nu_j \rightarrow 0, \text{ where } \mu_0 \text{ is an invariant measure for NLS equation} \]

\[ \dot{u} + i\Delta u - i|u|^2u = 0. \]  \hspace{1cm} (NLS)

See


For \( d = 1 \) the NLS equation is integrable and we know about the inviscid limit much more, see

[SK, A. Piatnitski 08] JMPA (2008),


Now consider small oscillations in a damped-driven NLS in the presence of an external electric field. Then the Hamiltonian PDE is the linear Schrödinger equation with potential. We assume that the damping is nonlinear:
§2. Damped-driven linear Schrödinger equation.

I write the equation in the slow time $\tau = \nu t$:

\begin{equation}
\frac{\partial u}{\partial \tau} + \nu^{-1} i ( -\Delta u + V(x)u ) = \Delta u - \gamma_R |u|^{2p} u - i \gamma_I |u|^{2q} u + \text{(random force)},
\end{equation}

$x \in \mathbb{T}^d$, $\gamma_R, \gamma_I \geq 0$, $\gamma_R + \gamma_I = 1$; $V(x) \in C^\infty(\mathbb{T}^d; \mathbb{R})$.

- If $d = 1$, then $p, q \geq 0$ are any,
- if $d \geq 2$, then some restrictions are needed. For example:
- $d = 2$, $\gamma_R > 0$, $p \geq q \geq 0$ any.
**RANDOM FORCE.** Let $\{e_r(x), r \geq 1\}$ be the trigonometric basis of $L_2(\mathbb{T}^d; \mathbb{C})$. Then

$$
\text{(random force)} = \frac{d}{d\tau} \sum_{j=1}^{\infty} b_j \bar{\beta}_j(\tau) e_j(x).
$$

Here each process $\bar{\beta}_j = \beta_j(\tau) + i\beta_{-j}(\tau)$ is a complex Brownian motion. The constants $b_j$ all are non-zero and fast decay to zero when $j$ grows. So the force is smooth in $x$. 
\[
\frac{\partial u}{\partial \tau} + \frac{i}{\nu} (-\Delta u + V(x)u) = \Delta u - \gamma_R |u|^{2p} u - i\gamma_I |u|^{2q} u + \text{(random force)},
\]

Denote \( \mathcal{H}^r = \mathcal{H}^r(\mathbb{T}^d; \mathbb{C}) \).

**Theorem 1.** If \( u_0 \in \mathcal{H}^r, r > d/2 \), then \( \exists! \) solution \( u(\tau), \tau \geq 0 \), of (1).

So (1) defines a RDS in \( \mathcal{H}^r \). Let \( u(0) = u_0 \) be random, \( u_0 = u_0^\omega \). Then \( D(u_0) \) is a measure in \( \mathcal{H}^r \). Calculate solution \( u^\omega(\tau) \) and the measure \( D(u(\tau)) \) – distribution of this solution at time \( \tau \).

I recall that a measure \( \mu \) in \( \mathcal{H}^r \) is called a stationary measure for eq. (1) if \( D(u_0) = \mu \) implies that \( D(u(t)) \equiv \mu \).

**Bogolyubov-Krylov Principle.** For equations with dissipation a stationary measure “always” exists.

Applying the methods, developed in the last 10 years to study the randomly forced 2d NSE, one can show that a stationary measure for eq. (1) is unique:
Theorem 2 (SK-Shirikyan, Harier, Odasso, Shirikyan). Under the imposed assumptions eq. (1) has a unique stationary measure $\mu^\nu$. For any solution $u(\tau)$ of (1) we have

$$\text{dist}(\mathcal{D}(u(\tau)), \mu^\nu) \to 0 \quad \text{as} \quad \tau \to \infty.$$ 

PROBLEMS. a) What happens to solutions $u^\nu(\tau, x)$ as $\nu \to 0$?
b) What happens to a stationary measure $\mu^\nu$ as $\nu \to 0$?

It is easy to see that the set of stationary measures $\{\mu^\nu, 0 < \nu \leq 1\}$ is pre-compact in the space of measure. So $\mu^{\nu_j} \rightharpoonup \mu^0$ as $\nu_j \to 0$. As before, one can check that $\mu^0$ is an invariant measure for the linear Schrödinger equation

$$(\text{Lin.Schrödinger}) \quad \dot{u} + i(-\Delta u + V(x)u) = 0.$$ 

But which one? - This equation has plenty of invariant measures! And does the limit $\mu^0$ depends on the sequence $\nu_j \to 0$?

I recall that in the case of NSE we do not have answers to these questions.
§3. \( \nu \)-coordinates.

Consider \( A = -\Delta + V(x) \). Assume that \( V \geq 1 \). Let \( \xi_1, \xi_2, \ldots \) be its \( L_2 \)-normalised real eigenfunctions and \( 1 \leq \lambda_1 \leq \lambda_2 \leq \ldots \) – corresponding eigenvalues. Assume that the spectrum \( \{\lambda_1, \lambda_2, \ldots\} \) is nonresonant in the sense that

\[
\sum \lambda_j \cdot s_j \neq 0 \quad \forall s \in \mathbb{Z}^\infty, \ 0 < |s| < \infty.
\]

This is a mild restriction on the potential \( V \): non-resonant potentials are typical both in the sense of Borel and in the sense of measure.

**FOURIER TRANSFORM.** For any \( u(x) \in \mathcal{H} = L_2(\mathbb{T}^d, \mathbb{C}) \) decompose it in the \( \xi \)-basis:

\[
u_j \xi_1 + v_2 \xi_2 + \ldots, \quad v_j \in \mathbb{C}.
\]

Denote \( \mathbf{v} = (v_1, v_2, \ldots) \). These are the (complex) \( \nu \)-coordinates. Consider the map

\[\Psi : u(\cdot) \mapsto \mathbf{v}.
\]

This an unitary transformation \( \mathcal{H} \to l_2 \) which for any \( r \) defines an isomorphism of \( \mathcal{H}^r \) with a suitable weighted \( l_2 \)-space.
In the $v$-coordinates (Lin.Schrödinger) reeds

$$\frac{\partial}{\partial \tau} v_j + i \nu^{-1} \lambda_j v_j = 0 \quad \forall j.$$ 

Consider the action-angle variables for these equations

$$I_j = \frac{1}{2} |v_j|^2, \quad \varphi_j = \text{Arg} v_j \in S^1; \quad I = (I_1, \ldots) \in \mathbb{R}^\infty_+, \quad \varphi = (\varphi_1, \ldots) \in \mathbb{T}^\infty.$$

In these coordinates the equations become

$$\dot{I}_j = 0, \quad \dot{\varphi}_j + \nu^{-1} \lambda_j = 0 \quad \forall j.$$

So: a) $I_j$'s are integrals of motion for (Lin.Schrödinger),

b) invariant measures for (Lin.Schrödinger) are all measures of the form $p(I) dI d\varphi$.

We wish to find out:

a’) how behave quantities $I_j(u(\tau))$, where $u(\tau)$ is a solution for the CGL equation (1);

b’) which invariant measures $p(I) dI d\varphi$ may be obtained as the inviscid limits of stationary measures for eq. (1).

To do this let us write eq. (1) in the $v$-variables:
\[ \frac{\partial}{\partial \tau} v + i\nu^{-1} \text{diag} \{\lambda_j\} v = \Psi\left( \Delta u - \gamma_R |u|^{2p} u - \gamma_I |u|^{2q} u \right) + \Psi((\text{random force})) , \]

where \( v = (v_1, v_2, \ldots) \) and \( u := \Psi^{-1}(v) \). Pass to the action-angles:

\[
\frac{\partial}{\partial \tau} I_j + 0 = F_j(I, \phi) + (\text{random force})_j, \quad j \geq 1,
\]

\[
\frac{\partial}{\partial \tau} \phi_j + \nu^{-1} \lambda_j = \ldots, \quad j \geq 1.
\]

Now we are in the setting of the averaging theory. Accordingly we expect to be able to find

\[
\lim_{\nu \to 0} I^\nu(\tau), \quad 0 \leq \tau \leq T, \quad \text{as a solution of the averaged } I\text{-equations:}
\]

\[
\frac{\partial}{\partial \tau} I_j(\tau) = \langle F_j \rangle(I) + \langle (\text{random force})_j \rangle, \quad j \geq 1.
\]

Here \( \langle F_j \rangle(I) = \int_{\mathbb{T}^\infty} F_j(I, \phi) \, d\phi \) and \( \langle (\text{random force})_j \rangle \) is defined by similar stochastic rules. In infinite dimensions this idea is not good and does not work well since \( I(\tau) \in \mathbb{R}_+^\infty \) – this is a very bad phase-space, and the averaged \( I\)-equations are singular!

To study the limiting dynamics of \( I(\tau) \) we consider other - non-singular - limiting equations.
§4. Effective equations.

Denote $\hat{A} = \text{diag}\{\lambda_j, j \geq 1\}$ and consider new equation:

\[
\frac{\partial}{\partial \tau} v = -\hat{A}v + \mathcal{L}v + R(v) + \text{(random force)}.
\]

Here

- $\mathcal{L} = \text{diag}\{l_k\}$ – some bounded linear operator, constructed in terms of the Fourier transform of the potential $V(x)$, and such that $-\hat{A} + \mathcal{L} \leq -\frac{1}{2} \hat{A}$.

- The nonlinearity $R(v)$ is constructed from the dissipative term of eq. (1) $-\gamma_R |u|^{2p} u$; this is a locally Lipschitz analytic mapping.

- $(\text{random force})_k = \frac{\partial}{\partial \tau} Y_k \beta_k(\tau)$, where $Y_k = (\sum_l b_l^2 |\Psi_{kl}|^2)^{1/2}$.

Eq. (2) is called the Effective Equation. This is a semilinear heat equation with a non-local nonlinearity, written in terms of Fourier coefficients of a solution.
Consider rotations $\Phi_\theta$, $\theta \in \mathbb{T}^\infty$, of vectors $v$:

$$\Phi_\theta(v) = v', \quad v'_j = e^{i\theta_j} v_j.$$ 

The Effective Equation (2) is rotation-invariant: if $v(\tau)$ is its solution, then for any $\theta \in \mathbb{T}^\infty$ the rotated curve $\Phi_\theta(v(\tau))$ also is a solution.

The Effective Equation is independent from the Hamiltonian part $-i\gamma I_\varepsilon u^2 u$ of the perturbation in eq. (1). If we add to eq. (1) any other Hamiltonian term $i \nabla_u \tilde{h}(u)$, this will not change the Effective Equation.

Example. If $p = 2$, then

$$R(v)_k = -\gamma R v_k \sum_l |v_l|^2 L_{kl},$$

where $L_{kl}$ is some explicit tensor. So Effective Equation takes the form

$$\frac{\partial}{\partial \tau} v_k = (-\lambda_k + l_k) v_k - \gamma R v_k \sum_l |v_l|^2 L_{kl} + Y_k \frac{\partial}{\partial \tau} \bar{\beta}_k(\tau).$$
Equation (1):

\[
\frac{\partial u}{\partial \tau} + \frac{i \nu}{\nu} (-\Delta u + V(x)u) = \Delta u - \gamma_R |u|^{2p} u - i\gamma_I |u|^{2q} u + \text{(random force)}.
\]

Effective Equation (2):

\[
\frac{\partial}{\partial \tau} v = -\hat{A}v + \mathcal{L}v + R(v) + \text{(random force)}.
\]

**Theorem 3.** Let \( u^\nu(\tau), 0 \leq \tau \leq T \), be a solution of eq. (1), \( u^\nu(0) = u_0 \), and \( v^\nu(\tau) = \Psi(u^\nu(\tau)) \). Let \( v^0(\tau) \) be a solution of Effective Equation (2) such that \( v^0(0) = v_0 = \Psi(u_0) \). Then

\[
I(v^\nu(\tau)) \rightarrow I(v^0(\tau)), \quad 0 \leq \tau \leq T,
\]

as \( \nu \to 0 \), in probability.
Let $\mu^\nu$ be the unique stationary measure of eq. (1), $\Psi \circ \mu^\nu$ – this measure, written in the $\nu$-variables, and $I \circ \Psi \circ \mu^\nu$ – corresponding distribution of the actions.

**Theorem 4.** Effective Equation (2) has a unique stationary measure $m_0$ and

$$I \circ \Psi \circ \mu^\nu \rightarrow I \circ m_0 \quad \text{as} \quad \nu \rightarrow 0.$$  

How behave angles of the stationary solutions? Their distribution is described by the measure $\varphi \circ \Psi \circ \mu^\nu$

**Theorem 5.** $\varphi \circ \Psi \circ \mu^\nu \rightarrow d\varphi$, where $d\varphi$ is the Haar measure on $\mathbb{T}^\infty$.

(This is the *random phase approximation*.)
Since (2) is rotation-invariant, then its unique stationary measure $m_0$ is rotation-invariant, $\Phi_\theta \circ m_0 = m_0$. So distributions of angles $\varphi$ for the measure $m_0$ also is the Haar measure, $\varphi \circ m_0 = d\varphi$. This observation and Theorems 4, 5 imply

**Theorem 6.** \( \Psi \circ \mu^{\nu} \rightharpoonup m_0 \) as \( \nu \to 0 \). i.e., the unique stationary measure of (1), written in the \( v \)-variables, converges to the unique stationary measure of the Effective Equation (2).

That is, for ANY solution \( u^{\nu}(\tau) \) of (1) we have

\[
\lim_{\nu \to 0} \lim_{\tau \to \infty} \mathcal{D}(\Psi(u^{\nu}(\tau)) = m_0.
\]

**Example.** If $\gamma_R = 0$, then the Effective Equation is linear. Its unique stationary measure $m_0$ is the Gaussian measure which is the law of the Ornstein-Uhlenbeck process

\[
\int_{-\infty}^{0} e^{s(-\hat{A} + \mathcal{L})} \cdot (\text{diag}\ \{Y_j\}) \ d\bar{\beta}(s).
\]
§5. Equations without dissipation.

Consider 1d-equations without dissipation (but with the friction). Let $p = 1$:

$$\frac{\partial u}{\partial \tau} + \nu^{-1} i (-u_{xx} + V(x)u) = -\gamma_R |u|^2 u - i \gamma_I |u|^{2q}u + \text{(random force)}.$$ 

Now the effective equation is the system

$$\frac{\partial}{\partial \tau} v_k = -\gamma_R v_k \sum_l |v_l|^2 L_{kl} + Y_k \frac{\partial}{\partial \tau} \bar{\beta}_k(\tau), \quad k \geq 1.$$ 

Theorems 1-6 remain true.

For all these results see