NON-PRICE INFORMATION AND PRICE SUSTAINABILITY IN THE KOOPMANS-BECKMANN PROBLEM

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In an important paper in Location Theory, Koopmans-Beckmann [7] developed a model of the optimal assignment of N plants (or firms) to N sites. The model seeks a locational assignment of plants to sites which maximizes the aggregate profitability of the assignment taking into account the freight costs associated with intermediate goods transfers between pairs of plants. These intermediate goods flows create an interdependency in the locational pattern which raises the question of whether a (decentralized) market economy can generate prices for land (or more specifically for each site) which would sustain a socially optimal (i.e., highest total profit) assignment.

Koopmans and Beckmann are quite negative on this question. They conjecture [7, p. 69] that "no price system on plants, on locations, and on commodities in all locations, that is regarded as given by plant owners, say, will sustain any assignment. There will always be an incentive for someone to seek a location other than the one he holds." Several authors have gone on subsequently to assert that this conjecture shows the necessity of centralized planning in a mixed market economy.¹ After all, they would argue, if decentralized decision-making is incapable of sustaining an optimal assignment then public planners should enforce that assignment and thereby make firms collectively better off.

In view of the social importance of this conclusion, it is surprising that few people have questioned the reasoning used by Koopmans and Beckmann. The purpose of this paper is to show that Koopmans and Beckmann have made an implausible assumption regarding the existence of non-price information. They argue that they seek an optimal assignment which is sustainable on the basis of (land) price information alone. While we agree with this desire, the game-theoretic nature of their model makes the exclusion of non-price information implausible. Further, when non-price information is introduced into the $N = 2$ case, an optimum-sustaining set of land of prices can in fact be found. It is thus concluded that the Koopmans-Beckmann model may not be an appropriate means of proving the price sustainability conjecture.

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¹See for example Koopmans [6, p. 154] and Beckmann [1, p. 95].

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The remainder of this paper is divided into two sections. In the first, the Koopmans-Beckman model and its conclusions are reviewed and some earlier criticisms noted. In the second section, the game-theoretic nature of their model is discussed and the plausibility and implications of introducing non-price information is assessed.

1. THE KOOPMANS-BECKMANN MODEL AND ITS PROOF

Koopmans and Beckmann [7] formulated a problem in which N factories were to be assigned to an equal number of sites in order to maximize overall profitability. This assignment problem is complicated by two factors: (i) some sites are more suited for some factories than others, and (ii) relative locations are important because some plants sell intermediate goods to each other. They define their assignment problem in the following well-known manner [7, p. 67].²

$$(1.a) \quad \text{Maximize: } \sum_{k,i} a_{ki} x_{ki} - \sum_{k,s} \sum_{i,j} c_{ij} x_{ks}^{ij}$$

subject to:

$$(1.b) \quad x_{ki} b_{ks} + \sum_j x_{ks}^{ji} = x_{si} b_{ks} + \sum_j x_{ks}^{ij} \quad (\text{for all } k, s, i)$$

$$(1.c) \quad \sum_i x_{ki} = 1 \quad (\text{for all } k)$$

$$(1.d) \quad \sum_k x_{ki} = 1 \quad (\text{for all } i)$$

$$(1.e) \quad x_{ki} \geq 0, x_{ks}^{ij} \geq 0, x_{ks}^{ii} = 0 \quad (\text{for all } k, s, i, j)$$

$$(1.f) \quad x_{ki} = 0 \text{ or } 1 \quad (\text{for all } k, i)$$

where a_{ki} is the semi-net revenue of factory k at site i .

x_{ki} is the intensity of operating plant k at site i ; i.e., this variable takes on a value of 1 if an assignment is made and 0 if it is not.

c_{ij} is the transportation cost in shipping a unit weight between site i and site j .

x_{ks}^{ij} is the flow from plant k at site i to plant s at site j .

b_{ks} is the weight flow from plant k required to operate plant s at a unit intensity level.

For each possible assignment there is a "semi-net revenue" a_{ki} which is the annual gross revenues earned at site i from plant k less the annual cost of primary inputs (utilities, structures, equipment, and labor for example). It should be emphasized that there is a distinct and important difference between semi-net revenue a_{ki} and the profit of plant k at location i , π_{ki} .³ In addition each factory k at

²Their model is also often written as an equivalent quadratic assignment problem. In their paper, Koopmann and Beckmann present both the linear and quadratic versions but for simplicity we use only the former.

³Heffley [5, p. 312].

site i ships a non-negative weight, x_{ks}^{ij} , of its output to factory s at location j . The unit weight cost of shipment from site i to site j is c_{ij} . Thus, the aggregate profitability (1.a) of an assignment of factories to sites is the aggregated semi-net revenue less the aggregated transport costs on intermediate good shipments.

Total profitability is maximized subject to certain constraints of which (1.b) is that if factory k locates at site i the required weight of its product that must be shipped to factory s is in fact shipped wherever s locates. The other conditions assert that each factory must be allocated a site (1.c), that exactly one plant be allocated to each site (1.d), that x_{ki} be an integer of value zero or unity (1.f), and that there be no intrasite flows (1.e).

It should be noted that, without (1.f), we have a simple linear program for which it is possible to generate a dual program.⁴ If this integer constraint is deleted and the remaining constraints are treated as suitably-defined inequalities, the well-known dual is as follows.⁵

$$(2.a) \quad \text{Minimize: } \sum_i v_i + \sum_j r_j$$

subject to:

$$(2.b) \quad u_{ks}^j - u_{ks}^i \leq c_{ij} \quad (\text{for all } k, s, i, j)$$

$$(2.c) \quad r_i \geq a_{ki} - v_k + \sum_s b_{ks} u_{ks}^i - \sum_s b_{sk} u_{sk}^i \quad (\text{for all } k, i)$$

$$(2.d) \quad r_i \geq 0 \quad (\text{for all } i)$$

$$(2.e) \quad v_k \geq 0, \quad (\text{for all } k)$$

$$(2.f) \quad u_{ks}^i \geq 0 \quad (\text{for all } k, s, i)$$

where r_i is the shadow price on site i ,

v_k is the marginal profitability of plant k , and

u_{ks}^i is the opportunity cost of supplying plant k 's output to plant s at site i .

Does the analysis of Koopmans and Beckmann suggest anything about the equivalence of centralized and decentralized land allocation? This is a confused point because of their indirect line of argument. Koopmans-Beckmann began by restricting themselves to the case where the semi-net revenue for any plant k at every site is the same.

$$(3) \quad a_{ki} = a_k \quad (\text{for all } i, k)$$

They argue that the only optimal solution to the linear programming problem (1), omitting the integer constraint (1.f) but including (3), is always a fractional assignment [(1/ N) of each factory to each site]. The dual solution (2) thus *cannot*

⁴This search for optimal integer assignments by relying on the linear programming dual may seem somewhat circuitous. Why not instead look for the dual to the integer programming problem itself? The answer is simply that an analytical form for the integer programming dual does not yet exist. As Bell and Shapiro [2] indicate, iterative estimates of this dual are possible for specific numerical examples but these are of limited use in looking for general solutions.

⁵Heffley [4, pp. 1158–1161].

hold for any integer solution because an integral assignment is never optimal under (3). They assert [7, p. 69] that this indicates that no locational rents can be formed which could sustain an optimal assignment in a decentralized economy where (3) holds. No proof is presented for this latter assertion as has been noted by Hartwick [3, p. 160].

In addition, Koopmans and Beckmann suggest in the quotation cited earlier that the above conclusion holds even when assumption (3) is deleted. Heffley [4, 5] presents some counter-examples wherein the semi-net revenues at different locations are not the same and in which a set of optimum-sustaining locational rents can be found which also satisfy (2).

Thus, in the work of both Hartwick and Heffley, some reservations have been expressed about the validity and generality of the Koopmans-Beckmann price-sustainability conjecture. However, no question has been raised about the issue of price versus non-price information and it is to this issue which we now turn.

2. INFORMATION AND EFFICIENT LOCATION

Let us initially consider the two-plant, two-site ($N = 2$) version of the Koopmans-Beckmann problem. Suppose that each firm has been allocated to a site (firm 1 to site 1 and firm 2 to site 2) but that firm 1 is considering moving to site 2. Because there are only two sites, firm 1 can move to site 2 only by forcing firm 2 to relocate to site 1. Thus, it would seem reasonable to expect that firm 1 would anticipate this effect on firm 2 if indeed there are only two sites available. Certainly, one would not expect firm 1 to ignore the impact of its own relocation on firm 2's choice of site. This however is precisely what Koopmans and Beckmann assume [7, p. 71].

One can lay the failure of a price system in the 2-plant case, and perhaps also in the N -plant case considered before, to the fact that a move of one plant is evaluated without considering the effect on everybody's transportation costs of the subsequent moves of other plants that are needed to make room for the first move. . . We have made some explorations along these lines . . . no rent system was found that even halfway meets the requirement of informational decentralization.

What happens in this duopoly problem if firm 1 does take into account the forced move of firm 2? Let us examine the profit levels foreseen by each plant in each location in this case. These profit levels are given as follows:

$$(4.a) \quad \pi_{11} = a_{11} - b_{21}c_{21} - r_1$$

$$(4.b) \quad \pi_{12} = a_{12} - b_{21}c_{21} - r_2$$

$$(4.c) \quad \pi_{22} = a_{22} - b_{12}c_{12} - r_2$$

$$(4.d) \quad \pi_{21} = a_{21} - b_{12}c_{12} - r_1$$

where π_{ki} is the profit in assignment factory k to site i .

These conditions are derived assuming that transfer costs are symmetrical (i.e., the same in each direction) between the two sites. Note in these conditions that the firm performing the profit calculation assumes that the other firm occupies the remaining site.

The restrictions on r_1 and r_2 which lead to one plant per site are either

$$(5.a) \quad a_{21} - a_{22} \leq r_1 - r_2 \leq a_{11} - a_{12}$$

when plant i is assigned to site i , or

$$(5.b) \quad a_{11} - a_{12} \leq r_1 - r_2 \leq a_{21} - a_{22}$$

when plant i is assigned to site j . One of these two conditions can be met for any set of semi-net revenues (a_{ki} 's). Thus, we are led to the conclusion that when firm 1 anticipates firm 2's reactions, a set of locational rents (r_1, r_2) can be found to hold for any optimal assignment.⁶

Koopmans and Beckmann of course reach exactly the opposite conclusion about price sustainability. They are not alone in this as both Heffley and Hartwick also find the same result in the $N = 2$ case. The assumption that each firm ignores the locational response of the other is thus crucial to each paper's conclusion about this failure of the price system.

Why did Koopmans and Beckmann assume such myopic behavior on the part of the firm? The answer is apparent from the above quotation. They seek a set of site rents which would be optimum-sustaining and at the same time constitute the only information used by the firms. Conjectures about the relocation of a displaced firm would constitute additional information. Thus, they conclude that price (site rent) information alone cannot preserve an optimal assignment.

We certainly have no argument in principle with their desire to restrict informational requirements to prices. However, the structure of their model with its N sites and N firms is inconsistent with this desire at least for small values of N (i.e., duopoly and oligopoly situations). Every relocation must involve at least one dislocation and a firm must know at least that much even if it does not necessarily (as in $N > 2$ cases) know where every firm will end up.⁷ In this sense, the Koopmans-Beckmann problem in reality would most likely be a game theory problem. However, by focusing exclusively on price information, the authors are trying to assume away its essence, its gaming attributes.

What is the Koopmans-Beckmann model really telling us? Reconsider for the moment the two-firm model. In reality, there are usually a large (more than two) number of sites at which each firm could locate. Suppose that the profit level of each firm was affected (positively) only by proximity to the other firm. The profit-maximizing solution for each firm would then be to locate adjacent to the other where intermediate freight costs were minimized. This is what the Koopmans-Beckmann model is really telling us. Because there are only two sites and because assignments must be integral, exactly adjacent locations in the model are impossible in general. Koopmans and Beckmann conclude that this shows up

⁶When (3) is invoked, (5.a) and (5.b) yield $r_1 = r_2$ implying that the optimal solution is sustained by putting the same rent on each location. This is not surprising. If transportation costs are symmetrical, semi-net revenues are spatially invariant, and site rents are the same, each firm is indifferent to the assignment of sites. Neither firm would desire to relocate in this case.

⁷When there are three or more firms, the prospective mover has to make some assumptions about where the remaining firms will locate. Under specific assumptions about relocation behavior, it should be possible to derive whether or not a set of equilibrium-sustaining rents exist. This reinforces our conclusion that non-price information cannot be deleted from the structure of the Koopmans-Beckmann problem.

the failure of informational decentralization. Our own interpretation is that this shows up the inconsistency of assuming only price information in a duopoly ($N = 2$) or oligopoly ($N > 2$ but not large) situation. It is in this sense that we argue that the structure of the Koopmans-Beckmann model is inconsistent for the kind of conclusion that they want to draw.

A conjecture that transportation costs for intermediate goods shipments do create a locational interdependency leading to price system failure may yet prove to be correct. Our argument is that the Koopmans-Beckmann paper does not provide this proof. Indeed, we suspect that any "proof" based on a discrete number of sites (and firms) wherein nonprice information is ignored will tell more about this assumption than about the failure of a price system.

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