A Time-Varying Feedback Approach to Reach Control on a Simplex

by

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University of Toronto

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Abstract

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This thesis studies the Reach Control Problem (RCP) for affine systems defined on simplices. The thesis focuses on cases when it is known that the problem is not solvable by continuous state feedback. Previous work has proposed (discontinuous) piecewise affine feedback to resolve the gap between solvability by open-loop controls and solvability by feedbacks. The first results on solvability by time-varying feedback are presented. Time-varying feedback has the advantage to be more robust to measurement errors circumventing problems of discontinuous controllers. The results are theoretically appealing in light of the strong analogies with the theory of stabilization for linear control systems. The method is shown to solve RCP for all cases in the literature where continuous state feedback fails, provided it is solvable by open loop control. Textbook examples are provided. The motivation for studying RCP and its relevance to complex control specifications is illustrated using a material transfer system.
Dedication

To Andrea for your love and support.
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I would like to thank Professor Mireille Broucke for giving me the opportunity to work on such an exciting and emerging field in control. Your insight, both mathematical and otherwise, is humbling. I would also like to thank the faculty, staff, and students of the System Control Group at the University of Toronto for such a wonderful learning experience (in the sense of Lyapunov). Lastly, I would like to thank my family for your encouragement.
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Chapter 1

Introduction

Modern systems control theory has primarily been focused on techniques such as regulation and tracking. These methods are limited in their ability to specify the desired system behaviour. They can be used to satisfy simple transient constraints such as percent overshoot or simple steady-state constraints such as stabilization of a closed-loop equilibrium. However, these common techniques fail to provide a rich set of specifications that are required by systems in fields with a high degree of complexity such as robotics. These specifications can be temporal in nature; for example, do task A three times and then do task B ad infinitum. Richer specifications can also arise from the desired behaviour of an interconnect of many system components or simply from a restriction of the system transient response. Therefore, there is a need from control systems designers for a technique that can accept a set of complex specifications and provide a control law that is guaranteed to satisfy these specifications.

Hybrid systems are dynamical systems that combine continuous and discrete behaviours. A class of hybrid systems called piecewise affine (PWA) systems are being developed in the control community to address the need for the control for complex specifications. PWA systems have been thoroughly discussed in the literature ranging as far afield as computational biology [16] and electrical systems [26]. The wide range of
applicability of PWA systems makes them a good candidate for the control for complex specifications.

The current methodology of using PWA feedbacks can be summarized as follows. The state space is partitioned into a polytope, the operating region, using the control specifications that can be expressed as a set of linear inequalities on the state variables. This polytope is further subdivided into simplices via triangulation. Each simplex corresponds to a state of a discrete automaton associated with the hybrid system. The transitions between the states of the discrete automaton encode the temporal component of the complex specification of the control system. Some of these transitions will be timed events or events driven by some form of communication or sensory input. However, we focus on events that occur when the state trajectory crosses the boundaries between contiguous simplices. The particular facet of a simplex that the trajectory is desired to cross is called the target set. Steering the state trajectory to this target set in finite time without first leaving the simplex is called the reach control problem.

The reach control problem (RCP) was first posed by Habets and van Schuppen in [17] in 2001. They formalize the first definitions of the RCP and the control synthesis for constructing affine feedbacks. Importantly, they also lay the foundation for what will later be introduced as the invariance conditions and the flow condition in [27]. Similar results, albeit less geometrical in nature, were obtained in [19]. The theoretical results were further developed in [10] where a clear division is presented when the RCP is solvable by affine feedbacks and when it is not. The results in [10] also present the equivalence of affine state feedbacks and continuous state feedbacks in the sense of the RCP. Finally, in [9], the reach control indices are also introduced which provide a deep understanding of the geometric structure of an affine control system defined on a simplex particularly with respect to the formation of equilibria.

A control synthesis procedure is presented in [9] for use when the RCP is solvable by open loop controls. Under certain assumptions on the triangulation, a further subdivision
Chapter 1. Introduction

of the simplex can result in a solution to the RCP using affine feedbacks defined for each
subsimplex. However, the resulting PWA feedbacks are discontinuous. That is, the
control input specified at a certain vertex is different for neighbouring subsimplices. This
can cause problems in real systems where sensor measurement errors are ever present.
Essentially, in a region surrounding the boundary of two neighbouring subsimplices, the
measurement error may lead the supervisory system to decide that the state is a member
of an incorrect subsimplex and therefore will select an incorrect affine feedback. It is
therefore possible for the invariance conditions to be violated and the RCP not solvable
using the resulting control. It is here where the results of this thesis begin.

1.1 Intuition of the Control Design

In this thesis we focus on the limitations of using piecewise affine feedback for an individual simplex. The results obtained in this thesis resolve the discontinuity introduced by piecewise affine feedback through the use of time-varying compensation and multi-affine feedbacks. The latter feedbacks move the set of equilibria within a simplex in an intelligent fashion. In this section we will briefly outline the intuition of the control design proposed herein. This intuition will be developed by introducing a simple two dimensional example. First, we will describe how equilibria can be moved within the simplex. Second, we will show how closed-loop trajectories generally flow in one direction. Lastly, the key realization is that we can slide the set of equilibria past trajectories, thereby allowing the trajectories to exit the simplex via the target set.

Consider the two dimensional simplex, \( S \), shown in Figure 1.1. \( S \) is a convex hull of vertices \( v_0, v_1, \) and \( v_2 \). Define \( F_0 \) to be the target denoted by the thick green line. We want all closed-loop trajectories to exit the simplex via this target set. The closed-loop vector field is denoted \( y_0 \) at \( v_0 \) and denoted \( b_1 \) and \( b_2 \) at \( v_1 \) and \( v_2 \) respectively. The invariance conditions are a set of bilinear inequalities that ensure that closed-loop
trajectories may not leave $S$ via $\mathcal{F}_1$ or $\mathcal{F}_2$. To satisfy the invariance conditions, we assume we have the freedom to pick $y_0$ as shown. Also, suppose that we are restricted in our choice of the closed-loop vector field at $v_1$ and $v_2$ while still satisfying the invariance conditions which results in $b_1$ and $b_2$.

![Figure 1.1: An example system on a simplex.](image)

When using affine feedbacks, the vector field at a given point in the simplex is determined by a convex combination of the vector fields at the vertices of the simplex. Therefore, the vector field along $\mathcal{F}_0$ is a convex combination of $b_1$ and $b_2$. Observe that $b_1$ and $b_2$ are linearly dependent. Therefore, at some point along this facet there will be an equilibrium point as shown in Figure 1.2(a). This equilibrium point was generated because of our lack of freedom in choosing the direction of the vector field on $\mathcal{F}_0$. Now, suppose we increase the magnitude of $b_1$ and decrease the magnitude of $b_2$. As we see in
Figure 1.2(b), the equilibrium point will move towards $v_2$. Now it can be seen that by changing the control inputs at $v_1$ and $v_2$ we can slide the equilibrium to any point on $\mathcal{F}_0$ while still satisfying the invariance conditions.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1a.png}
\caption{Equilibrium close to $v_1$.}
\end{subfigure} \hspace{0.05\textwidth}
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1b.png}
\caption{Equilibrium close to $v_2$.}
\end{subfigure}
\caption{Possible locations for equilibria.}
\end{figure}

We have now established that we can move closed-loop equilibria by modifying our choice of affine feedback. The next step in developing the intuition is to understand one property of the closed-loop vector field known as the flow-like condition. The vector field generated using the control placing the equilibrium near $v_1$ is shown in Figure 1.3(a). Note that all trajectories are guided to exit $\mathcal{P}^-$ via $\mathcal{H}^*$ and that the vector field always has a positive component in the $x_1$ direction. Similarly, the vector field generated using the control placing the equilibrium near $v_2$ is shown in Figure 1.3(b). Note that all trajectories exit $\mathcal{P}^+$ via $\mathcal{F}_0$ and, once again, that the vector field always has a positive component in the $x_1$ direction. The observation that the vector field along $\mathcal{H}^*$ always points with a positive component in the $x_1$ direction demonstrates the flow-like condition. This disallows trajectories from exiting $\mathcal{P}^+$ via $\mathcal{H}^*$. Essentially, trajectories in $\mathcal{P}^-$ will eventually enter $\mathcal{P}^+$ and then exit via $\mathcal{F}_0$. The process of continuously sliding an equilibrium from $v_1$ to $v_2$ is known as time-varying compensation and solves the reach control problem in some cases where a simple affine feedback cannot.
Figure 1.3: Plot of vector fields.

(a) Vector field with equilibrium close to $v_1$.

(b) Vector field with equilibrium close to $v_2$. 
1.2 Organization and Contributions

The thesis is a composition of mathematical background and results and the exposition of the mathematics through an extensive number of examples. The organization is as follows. In Chapter 2 the mathematical background is presented including notation, simplices and triangulation, \( \mathbb{M} \)-Matrices, convex analysis, nonsmooth analysis, and fixed point theory. The basic principles of reach control are presented in Chapter 3. This includes the formulation of the reach control problem, the definition of basic geometric properties, the development of invariance and flow conditions, and some basic results on the solvability of the RCP. Chapter 4 continues the theme of Chapter 3 and presents some more extreme cases where the RCP is solvable. Chapter 5 introduces the development of the reach control indices and their relationship to the existence of equilibria. Chapter 6 is the first examples chapter and presents examples where affine feedbacks are constructed and where the reach control indices are used to perform a subdivision and design discontinuous affine feedbacks. The main results of this thesis are developed in Chapter 7 where time-varying multi-affine feedbacks are introduced. Essentially time-varying affine feedbacks are used to interpolate between affine feedbacks in a manner that exploits the flow of the system. Chapter 8 presents a variety of examples using time-varying affine feedbacks to solve systems of higher dimension than previously solved. This higher dimension allows for interesting structure with respect to the reach control indices. Example 10 presents a novel procedure to determine the reach control indices and also provides a detailed procedure of applying a feedback transformation. Chapter 9 presents an application of time-varying multi-affine feedback to a material transfer system. Finally, Chapter 10 summarizes the thesis.

The contributions of this thesis are summarized.

- Example 7 of Chapter 6 highlights some of the crucial observations that drove our research. The example shows how equilibria arise in a four dimensional system when
using affine feedback. Lower dimensional examples cannot produce the appropriate behaviors to motivate the research of the thesis. The example provides the required intuition driving how the theory to follow is developed.

- Chapter 7 introduces the development of the flow-like condition that occurs when there are equilibria present in the simplex. Chapter 7 also presents a novel synthesis procedure for constructing time-varying affine feedbacks. This procedure is based upon the reach control indices.

- Chapter 8 presents examples using time-varying affine feedback to solve the RCP. A novel method for determining the reach control indices is presented in Example 10, where it is not possible to do so by inspection. A detailed feedback transformation is also presented in Example 10 through the use of the canonical simplex.

- Chapter 9 presents an application of time-varying affine feedback to a material transfer system.
Chapter 2

Background

2.1 Notation

Let \( \mathcal{K} \subset \mathbb{R}^n \) be a set. The complement of \( \mathcal{K} \) is \( \mathcal{K}^c := \mathbb{R}^n \setminus \mathcal{K} \), the closure is \( \overline{\mathcal{K}} \), and the interior is \( \mathcal{K}^o \). For a vector \( x \in \mathbb{R}^n \), the notation \( x \succ 0 \) \((x \succeq 0)\) means \( x_i > 0 \) \((x_i \geq 0)\) for \( 1 \leq i \leq n \). The notation \( x \prec 0 \) \((x \preceq 0)\) means \(-x \succ 0 \) \((-x \succeq 0)\). For a matrix \( A \in \mathbb{R}^{n \times n} \), the notation \( A \succ 0 \) \((A \succeq 0)\) means \( a_{ij} > 0 \) \((a_{ij} \geq 0)\) for \( 1 \leq i, j \leq n \). Notation \( \mathbf{0} \) denotes the subset of \( \mathbb{R}^n \) containing only the zero vector. The notation \( \mathcal{B} \) denotes the open unit ball, and \( \overline{\mathcal{B}} \) denotes its closure. The notation \( \text{co} \{ v_1, v_2, \ldots \} \) denotes the convex hull of a set of points \( v_i \in \mathbb{R}^n \). The notation \( \text{aff} \{ v_1, v_2, \ldots \} \) denotes the affine hull of a set of points \( v_i \in \mathbb{R}^n \). The symbol \( \mathcal{U} \) denotes a control type: we consider open-loop controls, continuous state feedback, affine feedback, and multiaffine feedback. Finally, \( T_S(x) \) denotes the Bouligand tangent cone to set \( \mathcal{S} \) at a point \( x \).
2.2 Simplices

The standard $n$-simplex is the subset of $\mathbb{R}^{n+1}$ given by

$$\Delta^n := \{ (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \mid \sum_i \alpha_i = 1, \alpha_i \geq 0 \}.$$ 

More generally, an $n$-dimensional simplex is an $n$-dimensional polytope with $n+1$ vertices. Let

$$V := \{ v_0, v_1, \ldots, v_n \}$$

be a set of $n + 1$ points in $\mathbb{R}^n$. We say $\{v_0, \ldots, v_n\}$ are affinely independent if they do not lie in an $(n - 1)$-dimensional plane in $\mathbb{R}^n$. Equivalently, $\{v_0, \ldots, v_n\}$ are affinely independent if $\{v_1 - v_0, \ldots, v_n - v_0\}$ are linearly independent. This provides another way to define a simplex: an $n$-dimensional simplex is the convex hull of $n + 1$ affinely independent points in $\mathbb{R}^n$. The affine hull of $V$, denoted $\text{aff} \{V\}$, is the smallest affine space containing $V$.

Suppose that $V$ is affinely independent and define the simplex

$$S := \text{co} \{ v_0, v_1, \ldots, v_n \}.$$ 

A face of $S$ is any sub-simplex which makes up its boundary. An $(n - 1)$-dimensional face of $S$ is called a facet. We denote the facets of $S$ by $F_0, \ldots, F_n$. Our numbering convention is such that

$$F_i = \text{co} \{ v \in V \setminus \{v_i\} \}, \quad i \in \{0, 1, \ldots, n\}.$$ 

That is, each facet is indexed by the vertex it does not contain. Let $h_i$ denote the unit normal vector to $F_i$, pointing out of $S$. An implicit description of $S$ is obtained using
these normal vectors. Namely, there exist \( \alpha_0, \ldots, \alpha_n \in \mathbb{R} \) such that

\[
S = \{ x \in \mathbb{R}^n \mid h_i \cdot x \leq \alpha_i, \quad \forall i \in \{0, \ldots, n\} \}.
\]

Finally we collect some useful properties about simplices.

**Lemma 2.1.** Let \( S \) be a simplex. Then the following hold:

(i) If \( x \in \text{co} \{v_1, \ldots, v_k\} \), then \( x \in F_j \), for \( k + 1 \leq j \leq n \).

(ii) \( h_j \cdot (v_i - v_0) = 0 \) for all \( 1 \leq i, j \leq n \) and \( j \neq i \).

(iii) \( h_j \cdot (v_i - v_k) = 0 \) for all \( 0 \leq i, k \leq n \) and \( j \neq i, k \).

(iv) \( h_i \cdot (v_i - v_0) < 0 \), for all \( 1 \leq i \leq n \).

(v) \( h_j \cdot (v_i - x) > 0 \) for all \( x \in S \setminus F_j \) and \( 1 \leq i, j \leq n \) and \( i \neq j \).

(vi) \( h_0 \cdot (v_i - v_0) > 0 \) for all \( 1 \leq i \leq n \).

(vii) The vectors \( \{v_1 - v_0, \ldots, v_n - v_0\} \) are a basis for \( \mathbb{R}^n \).

(viii) The vectors \( \{h_1, \ldots, h_n\} \) are a basis for \( \mathbb{R}^n \).

(ix) There exist \( \gamma_1 > 0, \ldots, \gamma_n > 0 \) such that \( h_0 = -\gamma_1 h_1 - \cdots - \gamma_n h_n \).

### 2.3 Placing Triangulation

In this section we describe a method to triangulate a polytopic space with respect to another space. Let \( P \) be a polytope and let \( O \) be an affine space of dimension less than \( n \). We assume \( P \cap O \) is a polytope with vertices \( V_O := \{o_1, \ldots, o_r\} \). We want to triangulate \( P \) with respect to \( O \).

First, we define an ordered point set \( V := \{v_1, \ldots, v_p\} \) such that \( P = \text{co} \ (V) \) and the first \( r \) points of \( V \) are \( V_O \). Note that not every element of \( V_O \) need be a vertex of \( P \). Now
we propose a triangulation of $\mathcal{P}$ which will have the feature that $\mathcal{O}$ can only lie in lower dimensional faces of simplices of the triangulation. We use a standard procedure called the placing triangulation (see [21, 20]). To describe this triangulation method we need a few definitions.

Suppose $V$ is a finite set of points such that $\mathcal{P} = \text{co} (V)$ is an $n$-dimensional polytope. A subdivision of $V$ is a finite collection $\mathcal{S} = \{ \mathcal{P}_1, \ldots, \mathcal{P}_q \}$ of $n$-dimensional polytopes such that: (1) The vertices of each $\mathcal{P}_i$ are drawn from $V$ (though not every point in $V$ need be used); (2) $\mathcal{P} = \bigcup_i \mathcal{P}_i$; (3) If $i \neq j$, then $\mathcal{P}_i \cap \mathcal{P}_j$ is a common (possibly empty) face of the boundaries of $\mathcal{P}_i$ and $\mathcal{P}_j$.

**Definition 2.1.** Let $x \in \mathbb{R}^n$, $\mathcal{P}$ an $n$-dimensional polytope, and $\mathcal{F}$ a facet of $\mathcal{P}$. The hyperplane $\mathcal{H} = \text{aff} (\mathcal{F})$ defines an open half-space containing $\text{int}(\mathcal{P})$. If $x$ is contained in the opposite open half-space, then $\mathcal{F}$ is said to be visible from $x$. (If $\mathcal{P}$ is a $k$-dimensional polytope in $\mathbb{R}^n$ with $k < n$ and $x \in \text{aff} (\mathcal{P})$, then the ambient space is viewed to be $\text{aff} (\mathcal{P})$.)

Now we can describe what it means to place a vertex. Let $\mathcal{S} = \{ \mathcal{P}_1, \ldots, \mathcal{P}_q \}$ be a subdivision of $V$ and $v \in \mathbb{R}^n$ such that $v \notin V$.

**Definition 2.2.** The subdivision $\mathcal{T}$ of $V \cup \{ v \}$ that results from placing $v$ is obtained as follows:

1. If $v \notin \text{aff} (V)$, then for each $\mathcal{P}_i \in \mathcal{S}$, include $\text{co} (\mathcal{P}_i \cup \{ v \})$ in $\mathcal{T}$.

2. If $v \in \text{aff} (V)$, then for each $\mathcal{P}_i \in \mathcal{S}$, $\mathcal{P}_i \in \mathcal{T}$ and if $\mathcal{F}$ is a facet of $\mathcal{P}_i$ that is contained in a facet of $\text{co} (V)$ visible from $v$, then $\text{co} (\mathcal{F} \cup \{ v \}) \in \mathcal{T}$.

**Theorem 2.1.** [21] Suppose $V$ is a finite set of points such that $V_{\mathcal{O}} \subset V$ and $\mathcal{P} = \text{co} (V)$ is an $n$-dimensional polytope. If the points of $V$ are ordered such that $\{o_1, \ldots, o_r\}$, the vertices of $\mathcal{P} \cap \mathcal{O}$, are listed first and if $\mathcal{T}$ is the subdivision obtained by placing the points of $V$ in order, then $\mathcal{T}$ is a triangulation of $V$ such that for every $n$-dimensional simplex $\mathcal{S} \in \mathcal{T}$, $\text{int}(\mathcal{S}) \cap \mathcal{O} = \emptyset$ and if $\mathcal{S} \cap \mathcal{O} \neq \emptyset$, then $\mathcal{S} \cap \mathcal{O}$ is a face of $\mathcal{S}$.
2.4 $\mathcal{M}$-Matrices

We say a matrix $M$ is a $\mathcal{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$ [4]. A special case of a $\mathcal{Z}$-matrix is a nonsingular $\mathcal{M}$. A nonsingular $\mathcal{M}$-matrix is a $\mathcal{Z}$-matrix with eigenvalues whose real parts are positive. The following theorem characterizes nonsingular $\mathcal{M}$-matrices. For an exhaustive treatment see [4], Chapter 6.

**Theorem 2.2.** Let $M \in \mathbb{R}^{k \times k}$ be a $\mathcal{Z}$-matrix. Then the following are equivalent:

(i) $M$ is a nonsingular $\mathcal{M}$-matrix.

(ii) Every real eigenvalue of $M$ is positive.

(iii) There exists a vector $\xi \succeq 0$ in $\mathbb{R}^k$ such that $M\xi \succ 0$.

(iv) The inequalities $y \succeq 0$ and $My \preceq 0$ have only the trivial solution $y = 0$, and $M$ is nonsingular.

(v) $M$ is monotone; that is, $My \succeq 0$ implies $y \succeq 0$ for all $y \in \mathbb{R}^k$.

(vi) $M$ is nonsingular and $M^{-1}$ is a non-negative matrix.

**Lemma 2.2.** If $M$ is a nonsingular $\mathcal{M}$-matrix, then every principal submatrix of $M$ is also a nonsingular $\mathcal{M}$-matrix.

**Theorem 2.3** (Theorem 6.4.6 [4]). Let $M \in \mathbb{R}^{k \times k}$ be a $\mathcal{Z}$-matrix. Then the following are equivalent:

(i) $M$ is an $\mathcal{M}$-matrix.

(ii) All principal minors of $M$ are nonnegative.

(iii) Every real eigenvalue of each principal submatrix of $M$ is nonnegative.
Definition 2.3 (Definition 2.1.2 [4]). An \( n \times n \) matrix \( A \) is cogradient to a matrix \( E \) if for some permutation matrix \( P \), \( PAP^T = E \). A is reducible if it is cogradient to

\[
E = \begin{bmatrix}
B & 0 \\
C & D
\end{bmatrix},
\]

where \( B \) and \( D \) are square matrices, or if \( n = 1 \) and \( A = 0 \). Otherwise, \( A \) is irreducible.

Theorem 2.4 (Theorem 6.4.16 [4]). Let \( A \) be a singular, irreducible \( M \)-matrix of order \( n \). Then

(i) \( A \) has rank \( n - 1 \).

(ii) There exists a vector \( x \succ 0 \) such that \( Ax = 0 \).

2.5 Convex Analysis

The following background is extracted from [25]. Let \( C \) be a non-empty convex set in \( \mathbb{R}^n \). We say that \( C \) recedes in direction \( v \in \mathbb{R}^n \) if

\[
(\exists x \in C) \{ x + \lambda v \mid \lambda \geq 0 \} \subset C.
\]

Equivalently, \( v \) is said to be a direction of recession of \( C \).

Let \( C_1 \) and \( C_2 \) be non-empty sets in \( \mathbb{R}^n \). A hyperplane \( H \) is said to separate \( C_1 \) and \( C_2 \) if \( C_1 \) is contained in one of the closed half-spaces associated with \( H \) and \( C_2 \) lies in the opposite closed half-space. It is said to separate \( C_1 \) and \( C_2 \) properly if \( C_1 \) and \( C_2 \) are not both actually contained in \( H \) itself. It is said to separate \( C_1 \) and \( C_2 \) strongly if there exists some \( \epsilon > 0 \) such that \( C_1 + \epsilon B \) is contained in one of the open half-spaces associated with \( H \), and \( C_2 + \epsilon B \) is contained in the opposite half-space.
Theorem 2.5. Let $C_1$ and $C_2$ be non-empty convex sets in $\mathbb{R}^n$. In order that there exist a hyperplane $H$ separating $C_1$ and $C_2$ strongly, it is necessary and sufficient that

$$\inf \{\|x_1 - x_2\| \mid x_1 \in C_1, \ x_2 \in C_2\} > 0.$$  \hspace{1cm} (2.1)

The relative interior of a convex set $C$ in $\mathbb{R}^n$, denoted by $ri\ C$, is defined as the interior of $C$ which results when $C$ is regarded as a subset of its affine hull $aff\ C$. A convex set $C$ is said to be relatively open if $ri\ C = C$.

Theorem 2.6. Let $C_1$ be a non-empty relatively open convex set in $\mathbb{R}^n$, and let $C_2$ be a non-empty affine set in $\mathbb{R}^n$ not meeting $C_1$. Then there exists a hyperplane $H$ containing $C_2$ such that one of the open half-spaces associated with $H$ contains $C_1$.

Theorem 2.7. Let $C_1$ and $C_2$ be non-empty disjoint closed convex sets in $\mathbb{R}^n$ having no common directions of recession. Then there exists a hyperplane separating $C_1$ and $C_2$ strongly.

Theorem 2.8. Let $C_1$ and $C_2$ be non-empty convex sets in $\mathbb{R}^n$ whose closures are disjoint. If either set is bounded, there exists a hyperplane separating $C_1$ and $C_2$ strongly.

2.6 Nonsmooth Analysis

Let $S \subset \mathbb{R}^n$ be a closed set. Define the distance function

$$d_S(x) := \inf \{\|x - y\| \mid y \in S\}.$$  

The Bouligand tangent cone (or simply tangent cone) to $S$ at $x$, denoted $T_S(x)$, is defined as

$$T_S(x) := \left\{ v \in \mathbb{R}^n \mid \lim_{t \downarrow 0} \inf \frac{d_S(x + tv)}{t} = 0 \right\}.$$  

The following is a useful characterization of the tangent cone (which is sometimes taken as the definition).

**Lemma 2.3.** Let $S$ be a closed set in $\mathbb{R}^n$. Let $\{x^k\}$ be a sequence such that $x^k \in S$ and $\lim_{k \to \infty} x^k = x$. Also, let $\{t^k\}$ be a sequence such that $t^k > 0$ and $\lim_{k \to \infty} t^k = 0$. Then

$$\lim_{k \to \infty} \frac{x^k - x}{t^k} \in T_S(x).$$

A set-valued map $\mathcal{Y} : \mathbb{R}^n \to 2^{\mathbb{R}^q}$ is said to be *upper semicontinuous* at $x \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x - x'\| < \delta \implies \mathcal{Y}(x') \subset \mathcal{Y}(x) + \epsilon \mathcal{B}.$$  

**Lemma 2.4.** Consider the set-valued map

$$\mathcal{Y}(x) := \{Ax + Bu + a \mid u \in \mathbb{R}^m\}.$$  

$\mathcal{Y}(x)$ is upper semicontinuous.

**Proof.** Let $L := \|A\|$ and fix $\epsilon > 0$. Select $\delta = \frac{\epsilon}{L}$. Let $x, x' \in \mathbb{R}^n$ be such that $\|x - x'\| < \delta$. Let $y' \in \mathcal{Y}(x')$ be arbitrary. There exists $u' \in \mathbb{R}^m$ such that $y' = Ax' + Bu' + a$. Now consider $y := Ax + Bu' + a \in \mathcal{Y}(x)$. We have

$$\|y - y'\| = \|Ax + Bu' + a - Ax' - Bu' - a\|$$

$$\leq \|A\| \|x - x'\|$$

$$\leq L \cdot \frac{\epsilon}{L} = \epsilon.$$  

Since $y' \in \mathcal{Y}(x')$ is arbitrary, we obtain

$$\mathcal{Y}(x') \subset \mathcal{Y}(x) + \epsilon \mathcal{B},$$
as desired.

Theorem 2.9. Consider the differential equation

\[ \dot{x} = f(x), \]

(2.2)

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is Lipschitz continuous. Let \( P \) be a compact, convex set. If for all \( x_0 \in P \), solutions of (2.2) satisfy \( x(t) \in P, \forall t \geq 0 \), then \( P \) contains an equilibrium of (2.2).

2.7 Fixed Point Theory

2.7.1 Sperner’s Lemma

Let \( T \) be a triangulation of \( n \)-dimensional simplex \( S \). A proper labeling of the vertices of \( T \) is as follows:

(P1) Vertices of the original simplex \( S \) have \( n + 1 \) distinct labels.

(P2) Vertices of \( T \) on a face of \( S \) are labeled using only the labels of the vertices forming the face.

Given a properly labeled triangulation of \( S \), we say a simplex in \( T \) is distinguished if its vertices have all \( n + 1 \) labels.

Lemma 2.5 (Sperner). Every properly labeled triangulation of \( S \) has an odd number of distinguished simplices.

Example 2.1. By way of example, consider the simplex \( S \) in Figure 2.1 and suppose the possible labels are \( a \) (blue), \( b \) (red), or \( c \) (green). The vertices each have a distinct label, so condition (P1) is met. Also, for the shown triangulation of \( S \), (P2) is satisfied.
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Figure 2.1: Example of Sperner’s Lemma

For example, along the left edge, vertices are labelled only by $a$ or $b$. Consequently there exists at least one distinguished subsimplex, shaded in the figure, with vertices with all three labels.
Chapter 3

Reach Control: Basic Principles

This thesis studies the reach control problem on simplices. The problem is for an affine system defined on a simplex to reach a prespecified facet of the simplex in finite time. In this chapter we explore basic principles which shape the features of the problem. These principles are derived from the geometry of the simplex and from convexity properties of affine systems. The proofs in this chapter are suppressed since these results have appeared in previous theses or papers.

3.1 Reach Control Problem

We study an $n$-dimensional simplex defined by

$$S := \text{co } \{v_0, \ldots, v_n\}$$

where $v_i \in \mathbb{R}^n$ are its vertices. Define the vertex set

$$V := \{v_0, \ldots, v_n\}.$$
We denote the \((n - 1)\)-dimensional facets by \(F_0, \ldots, F_n\), where the index of each facet is determined by the vertex it does not contain. Consider the affine control system defined on \(S\):

\[
\dot{x} = Ax + a + Bu, \quad x \in S,
\]

where \(A \in \mathbb{R}^{n \times n}, a \in \mathbb{R}^n, B \in \mathbb{R}^{n \times m}\), and \(\text{rank}(B) = m\). Let \(\phi_u(t, x_0)\) be the trajectory of (3.1) under a control \(u(t)\) starting from \(x_0 \in S\) and evaluated at time \(t\). We are interested in studying reachability of the target \(F_0\) from \(S\). By a control assignment \(U\) we mean either an assignment of an open-loop control \(u(t)\) for each initial condition \(x_0 \in S\), or a feedback control \(u(x)\) which is defined on all of \(S\). Let \(u_x\) denote the value of control used at a point \(x \in S\) associated with control assignment \(U\).

**Problem 3.1 (Reach Control Problem (RCP)).** Consider system (3.1) defined on \(S\). Find a control assignment \(U\) such that:

(i) For every \(x_0 \in S\) there exist \(T \geq 0\) and \(\gamma > 0\) such that \(\phi_u(t, x_0) \in S\) for all \(t \in [0, T]\), \(\phi_u(T, x_0) \in F_0\), and \(\phi_u(t, x_0) \notin S\) for all \(t \in (T, T + \gamma)\).

(ii) There exists \(\varepsilon > 0\) such that for all \(x \in S\), and for all \(u_x\) corresponding to the control assignment \(U\), \(\|Ax + a + Bu_x\| > \varepsilon\).

(iii) Feedback \(u(x)\) satisfies the invariance conditions (3.5) on \(F_0\).

Condition (i) of RCP ensures that trajectories of (3.1) starting from initial conditions in \(S\) reach the target \(F_0\) in finite time, while not first leaving \(S\). Conditions (ii) and (iii) holds automatically if condition (i) is met using affine or continuous state feedback. However, a distinction arises when studying solvability of RCP by open-loop and discontinuous controls, when conditions (ii) and (iii) cannot be deduced from (i). Condition (ii) is a robustness condition that ensures that trajectories leave \(S\) with sufficient speed. Both conditions (i) and (ii) preclude the possibility of closed-loop equilibria arising in \(S\), especially on \(F_0\). Instead condition (iii) guarantees that trajectories do not “spray out” of
cone(S) when they exit S. Finally, we remark that conditions (ii) and (iii) do not appear explicitly in the main contribution of the thesis, the multiaffine feedback design because they actually only arise in discussions about open-loop control, which is not the focus of the thesis. Condition (ii) is needed in the proof of Theorem BASICS:nec2 (although the proof is not presented because it is found in the literature). Condition (iii) is used in the proof of Lemma 7.1.

Definition 3.1. A point \( x_0 \in S \) can reach \( F_0 \) with constraint in \( S \) with control type \( U \), denoted by \( x_0 \xrightarrow{S} F_0 \), if there exists a control \( u \) of type \( U \) such that properties (i)-(ii) of Problem 3.1 hold. We write \( S \xrightarrow{S} F_0 \) by control type \( U \) if for every \( x_0 \in S \), \( x_0 \xrightarrow{S} F_0 \) with control of type \( U \).

Let \( B = \text{Im} \, B \), the image of \( B \). Define the set

\[
O := \{ x \in \mathbb{R}^n : Ax + a \in B \}. \tag{3.2}
\]

Notice that the vector field \( Ax + a + Bu \) can vanish at any \( x \in O \) for an appropriate choice of \( u \in \mathbb{R}^m \), so \( O \) is the set of all possible equilibrium points of (3.1). Thus, if \( x_0 \) is an equilibrium of (3.1) under feedback control, then \( x_0 \in O \). We also define the set of possible equilibrium points of (3.1) on \( S \) by

\[
\mathcal{G} := S \cap O. \tag{3.3}
\]

Associated with \( \mathcal{G} \) is its vertex index set

\[
I_{\mathcal{G}} := \{ i : v_i \in V \cap \mathcal{G} \}.
\]

The following lemma provides a characterization of \( O \).

Lemma 3.1.
(i) If $\text{Im} \ (A) \subseteq B$ and $a \notin B$, then $\mathcal{O} = \emptyset$;

(ii) If $\text{Im} \ (A) \subseteq B$ and $a \in B$, then $\mathcal{O} = \mathbb{R}^n$;

(iii) Otherwise, $\mathcal{O}$ is an affine space.

**Lemma 3.2.** If $(A, B)$ is controllable, then $\mathcal{O}$ is an affine subspace with $\dim(\mathcal{O}) = m$.

### 3.2 Necessary Conditions

In this section we investigate necessary conditions for solvability of RCP using open-loop controls. We say that a function $u : [0, \infty) \rightarrow \mathbb{R}^m$ is an *open-loop control* if it is bounded on any compact interval and it is measurable. By Caratheodory’s theorem, solutions of (3.1) using open-loop controls exist and are unique.

Let $h_i \in \mathbb{R}^n$ be the unit normal vector to each facet $F_i$ pointing outside of $S$. Define the index sets

$$ I := \{1, \ldots, n\} $$

and $I_i := I \setminus \{i\}$, where $I_0 = I$. Define the closed, convex cone $C_i$ at $v_i \in V$ by

$$ C_i := \{ y \in \mathbb{R}^n : h_j \cdot y \leq 0, \ j \in I_i \}.$$

Also define

$$ \text{cone}(S) := C_0 = \text{cone}\{v_1 - v_0, \ldots, v_n - v_0\}.$$

Note that $\text{cone}(S)$ is the tangent cone to $S$ at $v_0$.

**Definition 3.2.** We say the *invariance conditions* are solvable if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that

$$ Av_i + a + Bu_i \in C_i, \quad i \in \{0, \ldots, n\}.$$
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Equivalently,

$$h_j \cdot (Av_i + a + Bu_i) \leq 0, \quad i \in \{0, \ldots, n\}, \quad j \in I_i. \quad (3.4)$$

We will see in the sequel that the invariance conditions (3.4) are required for constructing affine feedbacks. For general state feedbacks, stronger conditions are needed.

**Definition 3.3.** We say a state feedback $u(x)$ satisfies the invariance conditions if for all $j \in I$ and $x \in F_j$,

$$h_j \cdot (Ax + Bu(x) + a) \leq 0. \quad (3.5)$$

**Example 3.1.** Figure 3.1 illustrates the definitions so far for the case $n = 3$ and $m = 2$. We have a simplex $\mathcal{S}$ with normal vectors $h_i$ to each facet $F_i$. Depicted by a shaded section is $\text{cone} (\mathcal{S})$, the tangent cone at $v_0$. The space $\mathcal{B}$ is copied to $v_0$, and in this view we see that $\mathcal{B} \cap \text{cone} (\mathcal{S}) = \emptyset$. That is, $\mathcal{B}$ does not “dip” into the tangent cone at $v_0$. The
set \( \mathcal{O} \) intersects \( \mathcal{S} \) along the face \( \overline{v_1v_2} \), and this forms \( \mathcal{G} \). It is interpreted as the set of possible equilibria of the system. We know that in \( \mathcal{G} \), the only velocity vectors available to the closed loop system are vectors in \( \mathcal{B} \). This is depicted by placing copies of \( \mathcal{B} \) at each of the vertices of \( \mathcal{G} \). Two velocity vectors \( b_1 \) and \( b_2 \) are shown, and these clearly satisfy the invariance conditions at \( v_1 \) and \( v_2 \), respectively. At vertices not in \( \mathcal{G} \), the drift term \( Ax + a \) becomes relevant, and the figure depicts closed-loop velocity vectors at \( v_0, v_3 \not\in \mathcal{G} \) which satisfy their respective invariance conditions. The invariance conditions can be interpreted in terms of the cones \( \mathcal{C}_i \). Consider vertex \( v_3 \) where \( \mathcal{C}_3 \) is depicted by a shaded region. This cone is shaped like an open book whose spine is parallel to the face \( \overline{v_0v_3} \) and whose cover and back cover lie in \( \mathcal{F}_2 \) and \( \mathcal{F}_1 \), respectively. The invariance condition at \( v_3 \) is satisfied if the closed-loop velocity vector \( Av_3 + Bu_3 + a \) lies in \( \mathcal{C}_3 \).

The invariance conditions are central to the solution of RCP on simplices, as seen by the next result.

**Theorem 3.1** ([9]). Suppose \( \mathcal{S} \xrightarrow{S} \mathcal{F}_0 \) by open-loop controls. Then the invariance conditions (3.4) are solvable.

The next result says that if a controller abides by the invariance conditions at a vertex of \( \mathcal{S} \), then an equilibrium of the controlled system may not be automatically induced at that vertex.

**Theorem 3.2** ([9]). If \( \mathcal{S} \xrightarrow{S} \mathcal{F}_0 \) by open-loop controls, then for all vertices \( v_i \in \mathcal{O} \),

\[
\mathcal{B} \cap \mathcal{C}_i \neq \emptyset .
\]

### 3.3 Affine Systems

In this section we explore properties of affine systems on compact, convex sets. First we examine the relationship between positively invariant sets, existence of equilibria,
trajectories leaving the set in finite time. Second we explore the way in which trajectories exit a compact, convex set when invariance conditions such as (3.4) hold.

Let $\mathcal{P} \subset \mathbb{R}^n$ be a compact, convex set and consider the affine system defined on $\mathcal{P}$:

$$\dot{x} = Ax + a, \quad x \in \mathcal{P},$$  \hspace{1cm} (3.6)

where $A \in \mathbb{R}^{n \times n}$ and $a \in \mathbb{R}^n$. Let $\phi(t, x_0)$ denote the unique trajectory of (3.6) starting from $x_0 \in \mathcal{P}$.

**Lemma 3.3.** Consider the affine system (3.6) defined on a compact, convex set $\mathcal{P}$. Suppose there exists $x_0 \in \mathcal{P}$ such that $\phi(t, x_0) \in \mathcal{P}$ for all $t \geq 0$. Then the set

$$\Phi := \text{co} \{ \phi(t, x_0) \mid t \geq 0 \} \subset \mathcal{P}$$

is a positively invariant set. Moreover, $\Phi$ contains an equilibrium of (3.6), i.e. there exists $\overline{x} \in \Phi$ such that $A\overline{x} + a = 0$.

**Lemma 3.4.** Consider the affine system (3.6) defined on a compact, convex set $\mathcal{P}$. We have $Ax + a \neq 0$ for all $x \in \mathcal{P}$ if and only if there exists $\xi \in \mathbb{R}^n$ such that

$$\xi \cdot (Ax + a) < 0, \quad x \in \mathcal{P}.$$  \hspace{1cm} (3.7)

We call the equation (3.7) a flow condition for system (3.6) on $\mathcal{P}$. A consequence of the existence of a flow condition on a compact, convex set is that all trajectories of (3.6) originating in the set eventually leave it.

**Lemma 3.5.** Consider the affine system (3.6) defined on a compact, convex set $\mathcal{P}$. Suppose that for all $x \in \mathcal{P}$, $Ax + a \neq 0$. Then for each $x_0 \in \mathcal{P}$, the trajectory starting at $x_0$ eventually leaves $\mathcal{P}$, i.e. there exists $t_1 > 0$ such that $\phi(t_1, x_0) \notin \mathcal{P}$.

The existence of a flow condition is also related to the set $\mathcal{O}$ for an affine control
system. To see this, consider again the control system (3.1). The next result shows that a flow condition naturally arises on any compact, convex set that does not intersect $\mathcal{O}$. Importantly, the control input plays no role and, moreover, the origin of the vector $\xi$ can be made explicit.

**Lemma 3.6.** Consider the affine control system (3.1) defined on a compact, convex set $\mathcal{P}$. If $\mathcal{P} \cap \mathcal{O} = \emptyset$, then there exists $\xi \in \text{Ker } B^T$ such that

$$\xi \cdot (Ax + a) < 0, \quad \forall x \in \mathcal{P}.$$ 

**Lemma 3.7.** Consider the affine control system (3.1) defined on a compact, convex set $\mathcal{P}$. If $\text{ri}(\mathcal{P}) \cap \mathcal{O} = \emptyset$, then there exists $\xi \in \text{Ker } B^T$ such that

$$\xi \cdot (Ax + a) < 0, \quad \forall x \in \text{ri } \mathcal{P}.$$ 

Finally, we examine the way that trajectories of an affine system exit a compact, convex set when invariance conditions such as (3.4) or (3.5) hold. The following result is quite standard and the proof is omitted.

**Lemma 3.8.** Consider the affine system (3.6) defined on a compact, convex set $\mathcal{P}$. Suppose additionally that $\mathcal{P}$ is a polytope with facets $\{\mathcal{F}_0, \ldots, \mathcal{F}_k\}$. Let $h_i$ be the outward normal vector of $\mathcal{F}_i$. Suppose that for some facet $\mathcal{F}_i$ the following conditions hold:

$$h_i \cdot (Ax + a) \leq 0, \quad \forall x \in \mathcal{F}_i. \quad (3.8)$$

Then all trajectories originating in $\mathcal{P}$ that leave $\mathcal{P}$ do so via a facet $\mathcal{F}_j$, $j \neq i$.

The next result shows that for affine systems, if trajectories exit from the proper target facet, than the invariance conditions hold. The argument is a minor adaptation of Theorem 3.1, where it is shown that the invariance conditions must be solvable (but need not actually hold).
Lemma 3.9. Consider the affine system (3.6) defined on a compact, convex set \( \mathcal{P} \). Suppose additionally that \( \mathcal{P} \) is a polytope with facets \( \{ \mathcal{F}_0, \ldots, \mathcal{F}_k \} \). Let \( h_i \) be the outward normal vector of \( \mathcal{F}_i \). Suppose that \( \mathcal{P} \rightarrow \mathcal{F}_0 \). Then

\[
h_i \cdot (Ax + a) \leq 0, \quad \forall x \in \mathcal{F}_i, i = 1, \ldots, k. \tag{3.9}
\]

3.4 Affine Feedback

In this section we use the properties of affine systems to propose a solution of RCP by affine feedback. It relies on a synthesis procedure described in the next result.

Lemma 3.10. Consider two sets of points \( V = \{ v_0, \ldots, v_n \mid v_i \in \mathbb{R}^n \} \) and \( \{ u_0, \ldots, u_n \mid u_i \in \mathbb{R}^m \} \). Suppose \( V \) is affinely independent. Then there exist unique matrices \( K \in \mathbb{R}^{m \times n} \) and \( g \in \mathbb{R}^m \) such that

\[
u_i = Kv_i + g, \quad i \in \{0, \ldots, n\}.
\]

Proof. We must show there exist matrices \( K \) and \( g \) such that

\[
\begin{bmatrix}
v_0^T & 1 \\
\vdots & \vdots \\
v_n^T & 1
\end{bmatrix}
\begin{bmatrix}
K^T \\
g^T
\end{bmatrix}
= \begin{bmatrix}
v_0^T \\
\vdots \\
v_n^T
\end{bmatrix}.
\tag{3.10}
\]

If the \((n+1) \times (n+1)\) left-hand matrix is full rank, then multiplying by its inverse yields the unique solutions \( K \) and \( g \). However,

\[
\text{rank } \begin{bmatrix}
v_0^T & 1 \\
\vdots \\
v_n^T & 1
\end{bmatrix} = 1 + \text{rank } \begin{bmatrix}
v_1^T - v_0^T \\
v_2^T - v_0^T \\
\vdots \\
v_n^T - v_0^T
\end{bmatrix} = 1 + n.
\]
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The last equality follows since the points \{v_0, \ldots, v_n\} are affinely independent if and only if \{v_1 - v_0, \ldots, v_n - v_0\} are linearly independent.

Using the results in Section 3.3 and the synthesis procedure of Lemma 3.10, we obtain necessary and sufficient conditions for solvability of RCP by affine feedback.

Theorem 3.3. Given the system (3.1) and an affine feedback \(u(x) = Kx + g\), where \(K \in \mathbb{R}^{m \times n}, g \in \mathbb{R}^m\), and \(u_0 = u(v_0), \ldots, u_n = u(v_n)\), the closed-loop system satisfies \(S \rightarrow F_0\) if and only if

(a) The invariance conditions (3.4) hold.

(b) There is no equilibrium in \(S\).

Proof. 

(\(\Rightarrow\)) If \(S \rightarrow F_0\) by affine feedback, then clearly the closed-loop system does not have equilibria in \(S\), for otherwise trajectories starting at an equilibrium would not leave \(S\). The invariance conditions (3.4) hold by Lemma 3.9.

(\(\Leftarrow\)) By assumption, for vertex set \(V\) there exist inputs \(\{u_0, \ldots, u_n\}\) satisfying the invariance conditions (3.4). Invoking Lemma 3.10, there exists an affine feedback \(u = Kx + g\) such that the invariance conditions are satisfied at the vertices. The resulting closed-loop system is

\[
\dot{x} = (A + BK)x + (Bg + a) = \tilde{A}x + \tilde{a}.
\]

By assumption, \(\tilde{A}x + \tilde{a} \neq 0\) for all \(x \in S\), so by Lemma 3.5, all trajectories leave \(S\) in finite time. From (3.4),

\[
h_j \cdot (\tilde{A}v_i + \tilde{a}) \leq 0, \quad i \in \{0, \ldots, n\}, \quad j \in I_i.
\]  

(3.11)
By convexity

\[ h_i \cdot (\hat{A}x + \hat{a}) \leq 0, \quad \forall x \in F_i, \quad i \in I. \] (3.12)

By Lemma 3.8 trajectories cannot leave \( S \) via \( F_1, \ldots, F_n \). This proves condition (i) of RCP. For condition (ii), since \( \|\hat{A}x + \hat{a}\| \neq 0 \) for all \( x \in S \), \( S \) is compact, and \( x \mapsto \|\hat{A}x + \hat{a}\| \) is continuous, there exists \( \varepsilon > 0 \) such that \( \|\hat{A}x + \hat{a}\| > \varepsilon \) for all \( x \in S \). \( \square \)

Theorem 3.3 gives conditions for the solvability of RCP which are primarily of theoretical interest. In practice, these conditions do not realize a synthesis procedure, for one cannot guarantee that a proposed affine feedback satisfying the invariance conditions does not place closed-loop equilibria in \( S \). On the other hand, existence of a flow condition guarantees there are no equilibria in \( S \) by Lemma 3.4, so one can replace the requirement of no closed-loop equilibria with existence of a flow condition. This observation leads to an alternative set of necessary and sufficient conditions for solvability of RCP.

**Theorem 3.4.** \( S \xrightarrow{\text{S}} F_0 \) by affine feedback if and only if there exist \( u_0, \ldots, u_n \in \mathbb{R}^m \) and \( \xi \in \mathbb{R}^n \) such that

(a) The invariance conditions (3.4) hold.

(b) The flow condition holds: \( \xi \cdot (A v_i + B u_i + a) < 0, \quad i \in \{0, \ldots, n\} \).

Theorem 3.4 can be viewed as a computational solution to the problem. If the invariance and flow conditions can be solved simultaneously for the unknowns \( \xi \in \mathbb{R}^n \) and \( u_i \in \mathbb{R}^m \), then an affine feedback can be constructed by the procedure of Lemma 3.10. Unfortunately, this approach relies on solving bilinear inequalities, which are known to be \( NP \)-hard [29].
Chapter 4

Triangulations and Affine Feedback

In this chapter we investigate RCP under the assumption of a special triangulation of the state space. This triangulation is selected in order to be able to obtain more geometric necessary and sufficient conditions for solvability of RCP. Some necessary conditions have already been presented in Section 3.2. This chapter instead describes geometric sufficient conditions for solvability by affine feedback which are existing in the literature [10].

4.1 Affine Feedback

Consider again a simplex \( S = \text{co} \{v_0, v_1, \ldots, v_n\} \) and an affine system defined on \( S \)

\[
\dot{x} = Ax + Bu + a, \quad x \in S,
\]

where \( A \in \mathbb{R}^{n \times n}, a \in \mathbb{R}^{n}, B \in \mathbb{R}^{n \times m}, \) and \( \text{rank}(B) = m. \)

We begin by exploring geometric sufficient conditions for existence of affine feedbacks solving RCP in terms of the sets \( \mathcal{O} \) and \( \mathcal{G} \), defined in (3.2) and (3.3), respectively. We have seen in Theorem 3.3 that the invariance conditions by themselves are generally not enough to establish that RCP is solvable by affine feedback. However, there is one extreme case when the invariance conditions are also sufficient to solve the problem.
These depend on combining Theorem 3.3 with the fact that $O$ is the only place in the state space where equilibria can appear.

**Theorem 4.1.** Suppose $G = \emptyset$. If the invariance conditions are solvable, then $S \xrightarrow{S} F_0$ by affine feedback.

In general it is difficult to extend results such as Theorem 4.1. Instead, we consider a special triangulation of the state space, first discussed in Section 2.3, which significantly simplifies the problem of finding geometric necessary and sufficient conditions for solvability of RCP. It is based on the following assumption.

**Assumption 4.1.** Simplex $S$ and system (3.1) satisfy the following condition: if $G \neq \emptyset$, then $G$ is a $\kappa$-dimensional face of $S$, where $0 \leq \kappa \leq n$.

**Remark 4.1.1.** We have discussed that there are three possibilities for $O$. If $O = \emptyset$, then one applies Theorem 4.1. If $O$ is the entire state space then we will see later that there are easily derived necessary and sufficient conditions for solvability. The only interesting case is when $O$ is a $\kappa$-dimensional affine subspace with $\kappa < n$. This case arises, for example, when $(A, B)$ is controllable, and then the placing triangulation can be applied.

We can now find several new sufficient conditions for existence of an affine feedback to solve RCP. The main idea is to exploit the fact that if the state space is triangulated with respect to $O$, then using Lemma 3.6, a flow condition fulfilling Theorem 3.4 is (almost) achievable on $S$.

**Theorem 4.2.** Suppose Assumption 4.1 holds and $G \neq \emptyset$. Suppose the following conditions hold.

1. The invariance conditions (3.4) are solvable.

2. $B \cap \text{cone}(S) \neq 0$.

Then $S \xrightarrow{S} F_0$ by affine feedback.
One can also obtain sufficient conditions for existence of affine feedback even when $B \cap \text{cone}(S) = 0$. Of course, this will only be possible if $v_0 \notin \mathcal{G}$ by Theorem 3.2. This relies on the idea that there are enough degrees of freedom in $B$ with respect to $\mathcal{G}$. We make the following assumptions.

**Assumption 4.2.**

(A1) $\mathcal{G} = \text{co} \{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < m$.

(A2) $B \cap \text{cone}(S) = 0$.

(A3) There exists a linearly independent set $\{b_i \in B \cap C_i \mid i \in I_\mathcal{G}\}$.

The important new assumption is (A3) which says that $B$ and $\mathcal{G}$ are arranged with respect to each other so that there are enough degrees of freedom in $B$ both to span a $\kappa+1$-dimensional subspace of $B$ and at the same time satisfy all the invariance conditions for the vertices of $\mathcal{G}$. For this to work, it is of course necessary that $\kappa < m$. It is helpful to obtain some intuition as to why linear independence is a central property which determines whether or not an affine feedback exists in the case when $B \cap \text{cone}(S) = 0$.

Consider Figure 4.1 in which $B \cap \text{cone}(S) = 0$ and $\mathcal{G} = C_1 \cup C_2$. Suppose we find a linearly independent set $\{b_1, b_2 \mid b_i \in B \cap C_i\}$ as shown in the figure (a copy of $B$ is attached at each vertex). Then along $[v_1, v_2]$ we can always choose a feedback $u(x)$ such that the closed-loop vector field

$$y(x) := Ax + B u(x) + a = c_1(x)b_1 + c_2(x)b_2, \quad c_i(x) \geq 0 \quad (4.2)$$

continuously interpolates between $y(v_1) = b_1$ and $y(v_2) = b_2$. This is also evident from examining the figure. Indeed with this controller, the invariance conditions are guaranteed to hold not only at $v_1$ and $v_2$ (by the definition of $C_i$) but also on the open interval...
(v_1, v_2). Namely, because \( c_i(x) \geq 0 \),

\[
h_j \cdot y(x) \leq 0, \quad j = 3, \ldots, n.
\]

Now consider the opposite situation as depicted in Figure 4.2. Again we have \( \mathcal{B} \cap \text{cone}(\mathcal{S}) = 0 \) and \( \mathcal{G} = v_1v_2 \), but in this case \( m = 1 \). Thus, for every choice of \( b_i \in \mathcal{B} \cap \mathcal{C}_i \), \( i = 1, 2 \), we obtain that \( \{b_1, b_2\} \) are linearly dependent. Pick any \( b_1 \in \mathcal{B} \cap \mathcal{C}_1 \) with \( b_1 \neq 0 \), as shown. Then we know that for any \( b_2 \in \mathcal{B} \cap \mathcal{C}_2 \),

\[
b_2 = c_1b_1, \quad c_1 \in \mathbb{R}.
\]

Now check the invariance conditions at \( v_2 \). In particular, we have the invariance condition

\[
h_1 \cdot b_2 = c_1(h_1 \cdot b_1) \leq 0.
\]

If we have assumed that \( \mathcal{B} \cap \text{cone}(\mathcal{S}) = 0 \), then it must be that \( h_1 \cdot b_1 > 0 \), for otherwise
we would have $0 \neq b_1 \in \mathcal{B} \cap \text{cone}(S)$. Then from (4.3) we obtain that $c_1 \leq 0$. This is illustrated in Figure 4.2, where $b_2$ points in the opposite direction of $b_1$. Consider a continuous vector field $y(x)$ on $\mathcal{S}$ and suppose we assign $y(v_1) = b_1$. Then we know that $y(x)$ is irrevocably constrained to be $y(v_2) = c_1 b_1$ with $c_1 \leq 0$. Now suppose that $y(x)$ continuously interpolates between $y(v_1)$ and $y(v_2)$ along $[v_1, v_2]$ using only $\{b_1, b_2\}$. Then along $[v_1, v_2]$, $y(x)$ has the form:

$$y(x) = c(x)b_1,$$

where $c(x)$ is a continuous function of $x \in [v_1, v_2]$ with $c(v_1) > 0$ and $c(v_2) \leq 0$. By the Intermediate Value Theorem, there exists $\pi \in [v_1, v_2]$ such that $c(\pi) = 0$. Thus, there is an equilibrium along $[v_1, v_2]$ for the closed loop system.

Therefore, it is clear that $y(x)$ cannot simply interpolate between $\{b_1, b_2\}$ along $[v_1, v_2]$ and other directions in $\mathcal{B}$ must be invoked. This argument can now be carried on inductively to higher dimensions and in each dimension one finds that more degrees of freedom are needed in $\mathcal{B}$ to carrying out the continuous assignment of the vector field. Finally, the procedure either terminates with exhausting all the vertices of $\mathcal{G}$, without first exhausting the degrees of freedom in $\mathcal{B}$, or instead one first exhausts all the usable degrees of freedom in $\mathcal{B}$. This question of which is exhausted first determines a sharp boundary between existence of affine feedbacks and existence of equilibria.

**Theorem 4.3.** Suppose Assumption 4.1 holds and $\mathcal{G} = \text{co} \{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < m$. Suppose the following conditions hold.

1. The invariance conditions (3.4) are solvable.

2. There exists a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_\mathcal{G}\}$.

Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.
Figure 4.2: Not enough degrees of freedom in $\mathcal{B}$. 

Chapter 5

Reach Control Indices

The reach control indices are defined under Assumption 4.1 in the situation when it is known that RCP is not solvable by continuous state feedback but it is still solvable by open-loop control. Based on the results of [10], as presented in Chapter 4, the following situation must be studied.

(A1) \( \mathcal{G} = \text{co} \{v_1, \ldots, v_{\kappa+1}\} \), with \( 0 \leq \kappa < n \).

(A2) \( \mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0} \).

(A3) The maximum number of linearly independent vectors in any set \( \{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap C_i\} \) (with only one vector for each \( \mathcal{B} \cap C_i, i \in I_G \)) is \( \hat{m} \) with \( 1 \leq \hat{m} \leq \kappa \).

This chapter gives a derivation of the reach control indices, which were first introduced in [9]. Here we provide more detailed discussions than can be provided in [9]. The intuition gained from this more detailed discussion allows us to then understand how a control strategy using time-varying compensation should work. The time-varying compensation strategy will exploit the “wiggle room” suggested by the indices. We will further clarify this so-called wiggle room by an extensive study of numerical examples. Finally, we note that certain proofs are suppressed if they are already in the literature.
The chapter begins with some preliminaries on $\mathcal{M}$-matrices, whose role is to provide a concise algebraic way to capture the consequences of the invariance conditions on the system dynamics.

### 5.1 Preliminaries

We introduce a family of matrices that concisely characterize the invariance conditions on $\mathcal{G}$. Let $1 \leq p \leq q \leq \kappa + 1$ and define

$$
M_{p,q} := \begin{bmatrix}
(h_p \cdot b_p) & (h_p \cdot b_{p+1}) & \cdots & (h_p \cdot b_q) \\
\vdots & \vdots & & \vdots \\
(h_q \cdot b_p) & (h_q \cdot b_{p+1}) & \cdots & (h_q \cdot b_q)
\end{bmatrix} \in \mathbb{R}^{(q-p+1) \times (q-p+1)}. \quad (5.1)
$$

Define the matrices

$$
H_{p,q} := [h_p \cdots h_q] \in \mathbb{R}^{n \times (q-p+1)}, \quad Y_{p,q} := [b_p \cdots b_q] \in \mathbb{R}^{n \times (q-p+1)}.
$$

Then

$$
M_{p,q} = H_{p,q}^T Y_{p,q}.
$$

Recall from Section 2.4 that $M$ is a $\mathcal{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_\mathcal{G}$, each $M_{p,q}$ is a $\mathcal{Z}$-matrix. Also under the condition that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$, we will see in Lemma 5.7 that certain matrices of the form $M_{p,q}$ are also nonsingular $\mathcal{M}$-matrices.
5.2 Reach Control Indices

We know that for each $v_i \in \mathcal{G}$ and for all $u \in \mathbb{R}^m$, $Av_i + Bu_i + a \in \mathcal{B}$. Select any $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i = 1, \ldots, \kappa + 1$ and write the list

$$\{b_1, \ldots, b_{\kappa+1}\}.$$

We are interested in a list that contains the maximal number of linearly independent vectors with the stipulation that only one vector may be selected from each cone $\mathcal{B} \cap \mathcal{C}_i$, $i = 1, \ldots, \kappa + 1$. Naturally, a maximal number exists, so let it be $\hat{m} \leq m$. Suppose $\hat{m}$ such vectors have been identified. We reorder the indices $\{1, \ldots, \kappa + 1\}$ (leaving the indices $0, \kappa + 2, \ldots, n$ the same) so that $\{b_1, \ldots, b_{\hat{m}}\}$ are linearly independent. We call $\{b_1, \ldots, b_{\hat{m}}\}$ a maximal set with respect to $\mathcal{G}$. Now we have a list

$$\{b_1, \ldots, b_{\hat{m}}, b_{\hat{m}+1}, \ldots, b_{\kappa+1}\},$$

and we define

$$\hat{\mathcal{B}} := \text{span}\{b_1, \ldots, b_{\hat{m}}\}. \tag{5.2}$$

By assumption (A3), $\hat{m} \leq \kappa$, so we introduce integer $p \geq 1$ satisfying

$$\hat{m} + p = \kappa + 1.$$

Since $\hat{\mathcal{B}} \subset \mathcal{B}$, $\hat{\mathcal{B}} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. We will derive reach control indices based on $\hat{\mathcal{B}}$; at the end we show the result does not depend on the choice of vectors that realize a maximal set with respect to $\mathcal{G}$. Since the ensuing derivations regard $\hat{\mathcal{B}}$, we make the following assumption.

**Assumption 5.1.** Simplex $\mathcal{S}$ and system (3.1) satisfy the following conditions.

(R1) $\mathcal{G} = \text{co} \{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$. 


(R2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$. 

(R3) $\hat{\mathcal{B}} = \text{span}\{b_1, \ldots, b_{\hat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$, $\hat{m} \leq \kappa$. 

(R4) $\mathcal{B} \cap \mathcal{C}_i \neq 0$, $i = 1, \ldots, \kappa + 1$. 

By the maximality of $\{b_1, \ldots, b_{\hat{m}}\}$, there is no $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in \{\hat{m} + 1, \ldots, \kappa + 1\}$, such that $\{b_1, \ldots, b_{\hat{m}}, b_i\}$ is a linearly independent set. In particular, this means that for each $i = \hat{m} + 1, \ldots, \kappa + 1$ and for each $b_i \in \mathcal{B} \cap \mathcal{C}_i$

$$b_i \in \text{span}\{b_1, \ldots, b_{\hat{m}}\}.$$ 

That is, 

$$\mathcal{B} \cap \mathcal{C}_i \subset \hat{\mathcal{B}} \cap \mathcal{C}_i, \quad i = \hat{m} + 1, \ldots, \kappa + 1.$$ 

Since also $\hat{\mathcal{B}} \cap \mathcal{C}_i \subset \mathcal{B} \cap \mathcal{C}_i$, $i = 1, \ldots, \kappa + 1$, we conclude that 

$$\hat{\mathcal{B}} \cap \mathcal{C}_i = \mathcal{B} \cap \mathcal{C}_i, \quad i = \hat{m} + 1, \ldots, \kappa + 1. \quad (5.3)$$ 

By Assumption (R4), we obtain 

$$\hat{\mathcal{B}} \cap \mathcal{C}_i \neq 0, \quad i = 1, \ldots, \kappa + 1.$$ 

Now consider cone $\mathcal{B} \cap \mathcal{C}_{\hat{m}+1}$. Among the fixed vectors $\{b_1, \ldots, b_{\hat{m}}\}$, find the smallest (non-empty) subset, say (w.l.o.g. by reordering indices $1, \ldots, \hat{m}$), $\{b_1, \ldots, b_{r_{\hat{m}}-1}\}$ such that 

$$\mathcal{B} \cap \mathcal{C}_{\hat{m}+1} \subset \text{span}\{b_1, \ldots, b_{r_{\hat{m}}-1}\}. \quad (5.4)$$ 

This amounts to choosing the smallest subspace that contains the cone $\mathcal{B} \cap \mathcal{C}_{\hat{m}+1}$ and, additionally, is generated by basis vectors among $\{b_1, \ldots, b_{\hat{m}}\}$ only. We define 

$$\mathcal{V}_{\hat{m}+1} := \text{span}\{b_1, \ldots, b_{r_{\hat{m}}-1}\}$$
to be the container subspace of $B \cap C_{\hat{m}+1}$.

**Lemma 5.1.** For each $i = \hat{m} + 1, \ldots, \kappa + 1$, there exists a unique minimal container subspace $V_i$ for $B \cap C_i$.

The proof relies on a standard fact about subspace intersection.

**Lemma 5.2.** Let $\{v_1, \ldots, v_r\}$ be a linearly independent set. Given $1 \leq q \leq p \leq r$, define

$$V_1 = \text{span}\{v_1, \ldots, v_p\}$$
$$V_2 = \text{span}\{v_q, \ldots, v_r\}.$$

Then

$$V_1 \cap V_2 = \text{span}\{v_q, \ldots, v_p\}.$$

**Proof.** First, it is clear that $\text{span}\{v_q, \ldots, v_p\} \subset V_1 \cap V_2$. Now we show the converse. Let $0 \neq \beta \in V_1 \cap V_2$. Then there exist coordinates $c_1, \ldots, c_p$ (not all zero) and $d_q, \ldots, d_p$ (not all zero) such that

$$\beta = c_1 v_1 + \cdots + c_p v_p$$
$$\beta = d_q v_q + \cdots + d_r v_r.$$

Then

$$0 = c_1 v_1 + \cdots + c_{q-1} v_{q-1} + (c_q - d_q)v_q + \cdots + (c_p - d_p)v_p - d_{p+1} v_{p+1} - \cdots - d_r v_r.$$

Since $\{v_1, \ldots, v_r\}$ are linearly independent, we obtain $c_i = 0$, $i = 1, \ldots, q - 1$ and $d_i = 0$, $i = p + 1, \ldots, r$. Thus, $\beta \in \text{span}\{v_q, \ldots, v_p\}$ as desired. \qed

**Proof of Lemma 5.1.** We prove the result only for $i = \hat{m} + 1$. Let $\mathcal{V} \neq \mathcal{V}'$ be two container subspaces for $B \cap C_{\hat{m}+1}$; that is, $B \cap C_{\hat{m}+1} \subset \mathcal{V}$ and $B \cap C_{\hat{m}+1} \subset \mathcal{V}'$. Then $B \cap C_{\hat{m}+1} \subset \mathcal{V} \cap \mathcal{V}'$,
and $V \cap V' \neq 0$, by Assumption (R4). Also we know that neither subspace is a subset of the other, otherwise one of them is not a container subspace of smallest dimension. Now both $V$ and $V'$ are generated by basis elements in the list \{${b_1, \ldots, b_\hat{m}}$\}. Without loss of generality, let $V = \text{span}\{b_1, \ldots, b_p\}$ and $V' = \text{span}\{b_q, \ldots, b_r\}$. From the observations above that $V \nsubseteq V'$, $V' \nsubseteq V$, and $V \cap V' \neq 0$, we have $1 < q < p < r$. By Lemma 5.1, we get $V \cap V' = \text{span}\{b_q, \ldots, b_p\}$, with $p - q + 1 < \min\{p, r - q + 1\}$. This contradicts the claim that $V$ and $V'$ are container subspaces for $B \cap C_{\hat{m}+1}$ of smallest dimension generated by the basis \{${b_1, \ldots, b_\hat{m}}$\}. \hfill $\Box$

The next result says there is always a vector in $B \cap C_{\hat{m}+1}$ that depends on all the basis vectors of its container subspace.

**Lemma 5.3.** Suppose Assumption 5.1 and (5.4) hold. There exists $b_{\hat{m}+1} \in B \cap C_{\hat{m}+1}$ such that

\[
b_{\hat{m}+1} = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}, \quad c_i \neq 0, \quad i = 1, \ldots, r_1 - 1.
\]

**Proof.** Suppose w.l.o.g. that for all $b \in B \cap C_{\hat{m}+1}$,

\[
b = c_1 b_1 + \cdots + c_{r_1-2} b_{r_1-2}.
\]

That is, no $b \in B \cap C_{\hat{m}+1}$ depends on $b_{r_1-1}$. Then $B \cap C_{\hat{m}+1} \subset \text{span}\{b_1, \ldots, b_{r_1-2}\}$, which contradicts the definition of $r_1$. Therefore, for each $i \in \{1, \ldots, r_1 - 1\}$ there exists $\beta_i \in B \cap C_{\hat{m}+1}$ such that

\[
\beta_i = \mu_{i,1} b_1 + \cdots + \mu_{i,r_1-1} b_{r_1-1}
\]

with $\mu_{i,1} \neq 0$. Now we show inductively that $b$ satisfying (5.5) exists. Let $j = 1$. Set

\[
b = \beta_1 \in B \cap C_{\hat{m}+1}.
\]
Then $b$ satisfies $c_1 := \mu_1^1 \neq 0$. Next suppose there exists $b \in \mathcal{B} \cap \mathcal{C}_{\hat{m}+1}$ such that

$$b = c_1 b_1 + \cdots + c_j b_j + \cdots + c_{r_1-1} b_{r_1-1},$$

and $c_i \neq 0$, for $i = 1, \ldots, j$. Moreover, by reordering indices $j+1, \ldots, r_1-1$, it can be assumed that $c_{j+1} = 0$ (for if this is not possible, then $b$ satisfies (5.5) and the induction is finished). Let

$$\overline{c} := \min_{i \in \{1, \ldots, j\}} |c_i| > 0.$$ 

Consider $\beta_{j+1} = \mu_{i_j}^{j+1} b_1 + \cdots + \mu_{r_1-1}^{j+1} b_{r_1-1}$, with $\mu_{j+1}^{j+1} \neq 0$. We choose $\alpha \in (0, 1)$ such that

$$\alpha |\mu_i^{j+1}| < \overline{c}, \quad i = 1, \ldots, j.$$ 

(5.6)

Now let

$$b' := b + \alpha \beta_{j+1} \in \mathcal{B} \cap \mathcal{C}_{\hat{m}+1}.$$ 

Then $b' = c_1' b_1 + \cdots + c_{r_1-1}' b_{r_1-1}$, where, using (5.6),

$$c_i' = c_i + \alpha \mu_i^{j+1} \neq 0, \quad i = 1, \ldots, j$$

$$c_{j+1}' = \alpha \mu_{j+1}^{j+1} \neq 0.$$ 

Therefore $b'$ replaces $b$ and the induction step is complete. \qed

The following result shows that $\hat{\mathcal{B}}$ contains all information about any set of cones that share degrees of freedom in $\mathcal{B}$ in the sense of Lemma 5.3.

**Lemma 5.4.** Suppose Assumption 5.1 and (5.4) hold. Then

$$\mathcal{B} \cap \mathcal{C}_i = \hat{\mathcal{B}} \cap \mathcal{C}_i, \quad i = 1, \ldots, r_1-1, \hat{m}+1.$$ 

**Proof.** First, it is clear that $\hat{\mathcal{B}} \cap \mathcal{C}_i \subset \mathcal{B} \cap \mathcal{C}_i$ for $i = 1, \ldots, r_1-1, \hat{m}+1$. Now we show the
converse. Consider w.l.o.g. \( i = 1 \). Suppose there exists \( b_1 \in B \cap C_1 \) and \( b_1 \notin \hat{B} \cap C_1 \). Let \( b_{m+1} \) be as in (5.5). Then \( \{b_1, b_2, \ldots, b_m, b_{m+1}\} \) is a linearly independent set containing one more vector than \( \hat{B} \), a contradiction.

It is useful at this stage to swap the indices \( \hat{m} + 1 \) and \( r_1 \), so (5.4) and (5.5) become respectively

\[
\hat{B} \cap C_{r_1} \subset \text{span}\{b_1, \ldots, b_{r_1-1}\} \tag{5.7}
\]

\[
(\exists b_{r_1} \in B \cap C_{r_1}) \ b_{r_1} = c_1b_1 + \cdots + c_{r_1-1}b_{r_1-1}, \quad c_i \neq 0. \tag{5.8}
\]

We can now write

\[
\hat{B} = \text{span}\{b_1, \ldots, b_{r_1-1}, \tilde{b}_{r_1}, b_{r_1+2}, \ldots, b_{m+1}\}.
\]

The overbar on \( \tilde{b}_{r_1} \) means it depends on all the previous \( r_1 - 1 \) vectors in the list. Also

\[
\{b_1, \ldots, b_{r_1-1}, b_{r_1+2}, \ldots, b_{m+1}\}
\]

are linearly independent.

Lemma 5.3 finds a “maximal vector” in the cone \( B \cap C_{r_1} \) that depends on all basis vectors of the container subspace. This dependency is restricted by the constraint that \( B \cap \text{cone}(S) = 0 \).

**Lemma 5.5.** Suppose Assumption 5.1 and (5.7)-(5.8) hold. Then the coefficients in (5.8) satisfy \( c_i < 0 \), \( i = 1, \ldots, r_1 - 1 \).

**Proof.** Suppose w.l.o.g. (by reordering indices \( \{1, \ldots, r_1 - 1\} \)), there exists \( 1 \leq p \leq r_1 - 1 \) such that \( c_i > 0 \) for \( i = 1, \ldots, p \) and \( c_i < 0 \) for \( i = p + 1, \ldots, r_1 - 1 \). Consider the vector

\[
\beta := b_{r_1} - c_{p+1}b_{p+1} - \cdots - c_{r_1-1}b_{r_1-1} = c_1b_1 + \cdots + c_pb_p.
\]
Notice that $\beta \neq 0$ since $\{b_1, \ldots, b_p\}$ are linearly independent. Since $b_i \in B \cap C_i$, $i \in \{1, \ldots, r_1\}$, we have

$$h_j \cdot \beta = h_j \cdot \left(b_{r_1} - c_{p+1}b_{p+1} - \cdots - c_{r_1-1}b_{r_1-1}\right) \leq 0, \quad i = 1, \ldots, p, r_1 + 1, \ldots, n.$$ 

Also

$$h_j \cdot \beta = h_j \cdot (c_1b_1 + \cdots + c_pb_p) \leq 0, \quad i = p + 1, \ldots, n.$$

In sum, $h_j \cdot \beta \leq 0$, $i \in I$; that is, $\beta \in B \cap \text{cone}(S)$. By Assumption (R2), $\beta = 0$, a contradiction.

The dependency of cones on a limited number of vectors in $B$ places restrictions on the orientation of those vectors with respect to $S$.

**Lemma 5.6.** Suppose Assumption 5.1 and (5.7)-(5.8) hold. Then

$$h_j \cdot b_i = 0, \quad i = 1, \ldots, r_1, \quad j \in I \setminus \{1, \ldots, r_1\}. \quad (5.9)$$

**Proof.** Let $b_{r_1}$ be as in (5.8). By Lemma 5.5, $c_i < 0$. Since $b_{r_1} \in B \cap C_{r_1}$,

$$h_j \cdot b_{r_1} = h_j \cdot (c_1b_1 + \cdots + c_{r_1-1}b_{r_1-1}) \leq 0, \quad j \in I \setminus \{1, \ldots, r_1\}.$$

Every term in the sum is non-negative, since $b_i \in B \cap C_i$ and $c_i < 0$, and so we obtain

$$h_j \cdot b_i = 0, \quad i = 1, \ldots, r_1, \quad j = r_1 + 1, \ldots, n.$$ 

\[\Box\\

**Lemma 5.7.** Suppose Assumption 5.1 and (5.7)-(5.8) hold. Then $M_{1,r_1-1}$ is a nonsingular $\mathcal{M}$-matrix.

**Proof.** First we show $M_{1,r_1-1} = H_{1,r_1-1}^T Y_{1,r_1-1}$ is nonsingular. Suppose there exists $c \in$
$\mathbb{R}^{n-1}$ such that $H_{1,r_1-1}^TY_{1,r_1-1}c = 0$. Let $y := Y_{1,r_1-1}c$. By assumption

$$h_j \cdot y = 0, \quad j = 1, \ldots, r_1 - 1.$$ 

Also by Lemma 5.6

$$h_j \cdot y = 0, \quad j = r_1 + 1, \ldots, n.$$ 

Thus, either $y \in B \cap \text{cone}(S)$ or $-y \in B \cap \text{cone}(S)$. By Assumption (R2), $y = 0$. However, $y = c_1b_1 + \cdots + c_{r_1-1}b_{r_1-1}$ and $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent, so $c = 0$. We conclude that $M_{1,r_1-1}$ is nonsingular.

Second, we know $M_{1,r_1-1}$ is a $\mathcal{Z}$-matrix. It has a positive diagonal; that is, $(M_{1,r_1-1})_{ii} > 0$ for $i = 1, \ldots, r_1 - 1$. For if not, we would have $h_j \cdot b_i \leq 0$ for all $j = 1, \ldots, n$, which implies $0 \neq b_i \in B \cap \text{cone}(S)$, a contradiction. Also, because $h_j \cdot b_i \leq 0$, $j \neq i$, the off-diagonal entries are non-positive, i.e. $(M_{1,r_1-1})_{ji} \leq 0$ for $j \neq i$.

Finally, we show $M_{1,r_1-1}$ satisfies (iv) of Theorem 2.2. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ with $c \neq 0$ and $c \succeq 0$ such that $M_{1,r_1-1}c \preceq 0$. Define the vector $\bar{y} = Y_{1,r_1-1}c \in B$. Note that $\bar{y} \neq 0$ because $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent. Then $M_{1,r_1-1}c = H_{1,r_1-1}^TY_{1,r_1-1}c = H_{1,r_1-1}^T\bar{y} \leq 0$ implies $h_j \cdot \bar{y} \leq 0$ for $j = 1, \ldots, r_1 - 1$. Also, since $c_i \geq 0$ and $b_i \in B \cap C_i$,

$$h_j \cdot \bar{y} = \sum_{i=1}^{r_1-1} c_i(h_j \cdot b_i) \leq 0, \quad j = r_1, \ldots, n.$$ 

This implies $0 \neq \bar{y} \in B \cap \text{cone}(S)$, a contradiction. Therefore, $M_{1,r_1-1}$ has the property that the only solution of the inequalities $c \succeq 0$ and $M_{1,r_1-1}c \preceq 0$ is $c = 0$.

In sum, $M_{1,r_1-1}$ is a nonsingular $\mathcal{Z}$-matrix satisfying (iv) of Theorem 2.2. By Theorem 2.2, $M_{1,r_1-1}$ is a nonsingular $\mathcal{M}$-matrix.

Remark 5.0.1. An important feature of the formula (5.8) with $c_i < 0$ is that, using it, any $b_i$, $i \in \{1, \ldots, r_1\}$, can be expressed as a negative linear combination of the other
vectors \( \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_r\} \). This means Lemma 5.7 can be deduced for other combinations of \( r_1 - 1 \) indices, not just \( \{1, \ldots, r_1 - 1\} \). In particular, if we begin with the singular matrix \( M_{r_1, r_1} \in \mathbb{R}^{r_1 \times r_1} \), and if the \( i \)th row and column are removed, the resulting submatrix is a nonsingular \( \mathcal{M} \)-matrix.

Next consider the cone \( \mathcal{B} \cap \mathcal{C}_{\hat{m}+2} \). There exists a smallest subspace generated by the (fixed) basis \( \{b_1, \ldots, b_{r_1 - 1}, b_{r_1 + 1}, \ldots, b_{\hat{m}+1}\} \) which contains \( \mathcal{B} \cap \mathcal{C}_{\hat{m}+2} \). By independently reordering each index set \( \{1, \ldots, r_1 - 1\} \) and \( \{r_1 + 1, \ldots, \hat{m} + 1\} \) we have

\[
\mathcal{B} \cap \mathcal{C}_{\hat{m}+2} \subset \text{span}\{b_{\rho_2}, \ldots, b_{r_1 - 1}, b_{r_1 + 1}, \ldots, b_{\rho_2+r_2-2}\}
\]

where \( \rho_2 \geq 1 \) and \( \rho_2 + r_2 - 2 \leq \hat{m} + 1 \). Now we swap the indices \( \rho_2 + r_2 - 1 \) and \( \hat{m} + 2 \), so that

\[
\mathcal{B} \cap \mathcal{C}_{\rho_2+r_2-1} \subset \text{span}\{b_{\rho_2}, \ldots, b_{r_1 - 1}, b_{r_1 + 1}, b_{\rho_2+r_2-2}\}.
\] (5.10)

The next result shows that because of the restrictions placed on \( \mathcal{B} \) by Lemma 5.6, the container subspaces of dependent cones split into independent subspaces, thus yielding a decomposition of \( \mathcal{B} \) with respect to \( \mathcal{G} \).

**Lemma 5.8.** Suppose Assumption 5.1, (5.7), and (5.10) hold. Then \( \rho_2 = r_1 + 1 \).

**Proof.** Suppose by way of contradiction that \( 1 \leq \rho_2 \leq r_1 - 1 \). Applying Lemma 5.6 we obtain

\[
h_j \cdot b_i = 0, \quad i = 1, \ldots, r_1, \quad j = r_1 + 1, \ldots, n \] (5.11)

\[
h_j \cdot b_i = 0, \quad i = \rho_2, \ldots, \rho_2 + r_2 - 1, \quad j = 1, \ldots, \rho_2 - 1, \rho_2 + r_2, \ldots, n. \] (5.12)

Consider

\[
M_{\rho_2,r_1-1} = H_{\rho_2,r_1-1}^T Y_{\rho_2,r_1-1},
\]

where \( Y_{\rho_2,r_1-1} = [b_{\rho_2} \cdots b_{r_1-1}] \). By Lemma 5.7, \( M_{1,r_1-1} \) is a nonsingular \( \mathcal{M} \)-matrix. By
applying Lemma 2.2 in succession, we have that $M_{p_2, r_1-1}$ is also a nonsingular $\mathcal{M}$-matrix. By Theorem 2.2(iii) there exists $c' = (c'_{p_2}, \ldots, c'_{r_1-1})$ such that $c' \preceq 0$ and $M_{p_2, r_1-1}c' < 0$. Define $\beta := Y_{p_2, r_1-1}c' \neq 0$. The statement $H_{p_2, r_1-1}^T\beta = M_{p_2, r_1-1}c' < 0$ gives

$$h_j \cdot \beta < 0, \quad j = \rho_2, \ldots, r_1 - 1.$$ 

Also, by (5.11)-(5.12)

$$h_j \cdot \beta = h_j \cdot (c'_{p_2}b_{p_2} + \cdots + c'_{r_1-1}b_{r_1-1}) = 0, \quad j = 1, \ldots, \rho_2 - 1, r_1 + 1, \ldots, n.$$ 

This implies $\beta \in \mathcal{B} \cap \cone(\mathcal{S})$. By Assumption (R2) we obtain $\beta = 0$, a contradiction. □

By Lemma 5.8, (5.10) becomes

$$\mathcal{B} \cap \mathcal{C}_{r_1+r_2} \subset \span\{b_{r_1+1}, b_{r_1+r_2-1}\}. \quad (5.13)$$

By iterating on Lemmas 5.6, 5.8, and our index swap, we can show that the decomposition obtained so far extends to all cones $\mathcal{B} \cap \mathcal{C}_i$ associated with $\mathcal{G}$. First, we note that the procedure generates a specially ordered list of velocity vectors associated with vertices in $\mathcal{G}$. This list has the form

$$\{b_1, \ldots, b_{r_1-1}, \overline{b}_{r_1}, b_{r_1+1}, \ldots, b_{r_1+r_2-1}, \overline{b}_{r_1+r_2}, \ldots, b_{r_1+\cdots+r_p-1+1}, \ldots, b_{r_1}, \overline{b}_r, b_{r+1}, \ldots, b_\kappa+1\}, \quad (5.14)$$

where $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and

$$r := r_1 + \cdots + r_p.$$ 

The vectors that do not have an overbar are provided by Assumption (R3). The vectors that have overbars are provided by Lemma 5.3. To streamline the notation, define for
Using these indices, the list (5.14) can be written as:

\[
\{b_1, \ldots, b_{m_1+r_1-2}, \bar{b}_{m_1+r_1-1}, b_{m_1+r_1}, \ldots, b_{m_p}, \ldots, \bar{b}_{m_p+r_p-1}, b_{r+1}, \ldots, b_{\kappa+1}\}. \tag{5.15}
\]

Now we summarize the properties of this list:

- If we remove the overbarred vectors in (5.15) we obtain a shortened list containing \( \hat{m} \) vectors:

\[
\{b_1, \ldots, b_{m_1+r_1-2}, \ldots, b_{m_p}, \ldots, b_{m_p+r_p-2}, b_{r+1}, \ldots, b_{\kappa+1}\}. \tag{5.16}
\]

The span of these vectors is precisely \( \hat{B} \), and the only difference with (5.2) is that the indices are different because of our index swaps. In particular, (5.16) is a maximal set with respect to \( \mathcal{G} \).

- As mentioned, the vectors \( \bar{b}_{m_1+r_1-1}, \ldots, \bar{b}_{m_p+r_p-1} \) are generated by Lemma 5.3. This means that each \( \bar{b}_{m_i+r_i-1}, i = 1, \ldots, p \) depends on all of the previous \( r_i - 1 \) vectors in the list. In turn this means that if any one vector is removed from a sublist of the form

\[
\{b_{m_i}, \ldots, b_{m_i+r_i-1}\}, \quad i \in \{1, \ldots, p\}
\]

then the remaining vectors are linearly independent.

- A number of relationships between the integers \( \kappa, \hat{m}, p, \) and \( r \) are implied by the construction. First, by definition \( p = \kappa + 1 - \hat{m} \). Also, by construction \( \hat{m} \leq \kappa \). Next we observe that each of the “excess” \( p \) vertices of \( \mathcal{G} \), namely \( v_{m_1+r_1-1}, \ldots, v_{m_p+r_p-1} \), has an associated non-zero velocity vector by (R4). By Lemma 5.8, each of these \( p \) vertices uses up at least one exclusive vector in the basis for \( \hat{B} \). So we need at least
$p$ independent vectors in $\hat{B}$. That is,

$$\hat{m} \geq p = \kappa + 1 - \hat{m}.$$ 

Thus, in order for (R4) to hold it is necessary that

$$\hat{m} \geq \frac{\kappa + 1}{2}. \quad (5.17)$$

This condition can be interpreted to say that RCP is only solvable if there are sufficient inputs.

- Since each of the $p$ lists $\{b_{m_i}, \ldots, b_{m_i+r_i-1}\}, i \in \{1, \ldots, p\}$, comprises $r_i - 1$ independent vectors in $\hat{B}$, and there are a total of $\hat{m}$ such vectors, we deduce that

$$(r_1 - 1) + \cdots + (r_p - 1) - p \leq \hat{m} \iff r - p \leq \hat{m}.$$ 

- The construction does not make any statements about the “extra” linearly independent vectors $\{b_{r+1}, \ldots, b_{\kappa+1}\}$ in $\hat{B}$. Moreover, the cones $\hat{B} \cap C_i, i = r + 1, \ldots, \kappa + 1$ do not enjoy the same properties as $\hat{B} \cap C_i, i = 1, \ldots, r$. In particular, Lemma 5.4 cannot in general be deduced for these cones. Fortunately, this fact does not affect the application of the reach control indices for control design.

We can now summarize the results of the container subspace construction. Invoking Lemma 5.4,

$$\hat{B} \cap C_{r_1} = B \cap C_{r_1} \subset B_1 := \text{span}\{b_1, \ldots, b_{r_1}\}, \quad (5.18)$$

$$\hat{B} \cap C_{r_1+r_2} = B \cap C_{r_1+r_2} \subset B_2 := \text{span}\{b_{r_1+1}, \ldots, b_{r_1+r_2}\}, \quad (5.19)$$

$$\vdots$$

$$\hat{B} \cap C_{r_1+\cdots+r_p} \subset B_p := \text{span}\{b_{r_1+\cdots+r_p-1+1}, \ldots, b_r\}. \quad (5.20)$$
One can now extract a geometric consequence of our decomposition that has relevance to the usable directions in $B$ at points in $G$ which are not vertices. For example, at any point in $x \in \text{co}\{v_1, \ldots, v_r\}$, the only usable directions in $B$ to satisfy the invariance conditions (3.5) are those already in $B_1$. We state the result only for $k = 1$.

**Lemma 5.9.** Suppose Assumption 5.1 and (5.18)-(5.20) hold. There does not exist $\beta \in B$ such that $\{b_1, \ldots, b_{r_1-1}, \beta\}$ are linearly independent and

$$h_j \cdot \beta \leq 0, \quad j = r_1 + 1, \ldots, n.$$ (5.21)

**Proof.** By Lemma 5.7, $M_{1,r_1-1}$ is a nonsingular $M$-matrix. Applying Theorem 2.2(iii) there exists $c' = (c'_1, \ldots, c'_{r_1-1})$ such that $c' \preceq 0$ and $M_{1,r_1-1}c' \prec 0$. Define $b'_{r_1} := Y_{1,r_1-1}c'$.

The vector $HT_{r_1}b'_{r_1} \in \mathbb{R}^n$ has the following sign pattern:

$$(-, \ldots, -, *, 0, \ldots, 0)$$ (5.22)

where the $*$ appears in the $(r_1)$th component and the zero components are due to Lemma 5.6. In particular $b'_{r_1} \in B \cap C_{r_1}$ and the first $r_1 - 1$ invariance conditions are strictly negative. Now suppose we find a non-zero vector $\beta \in B$ such that (5.21) holds and $\{b_1, \ldots, b_{r_1-1}, \beta\}$ are linearly independent. Then for $\alpha > 0$ we can form

$$b''_{r_1} := b'_{r_1} + \alpha \beta.$$ 

Using (5.22) and (5.21), $\alpha$ can be selected sufficiently small so that $h_j \cdot b''_{r_1} \leq 0$ for all $j = 1, \ldots, r_1 - 1, r_1 + 1, \ldots, n$. That is, $b''_{r_1} \in B \cap C_{r_1}$. Moreover, with $\beta \neq 0$,

$$\{b_1, \ldots, b_{r_1-1}, b''_{r_1}\}$$

is a linearly independent set. This contradicts (5.18), where $b_{r_1}$ depends on $\{b_1, \ldots, b_{r_1-1}\}$. 


Theorem 5.1. Suppose Assumption 5.1 holds. Then there exist integers \( r_1, \ldots, r_p \geq 0 \) and a decomposition of \( B \) into \( p \) subsets such that

\[
\begin{align*}
B \cap C_i & \subset \text{span}\{b_1, \ldots, b_{r_1}\}, & i = 1, \ldots, r_1, \quad (5.23) \\
B \cap C_i & \subset \text{span}\{b_{r_1+1}, \ldots, b_{r_1+r_2}\}, & i = r_1 + 1, \ldots, r_1 + r_2, \quad (5.24) \\
& \vdots & \vdots \\
B \cap C_i & \subset \text{span}\{b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_r\}, & i = r_1 + \cdots + r_{p-1} + 1, \ldots, r. \quad (5.25)
\end{align*}
\]

Proof. We consider only \( k = 1 \). We must show \( B \cap C_i \subset \text{span}\{b_1, \ldots, b_{r_1}\} \), for \( i = 1, \ldots, r_1 \). (By (5.18) we already have \( B \cap C_{r_1} \subset \text{span}\{b_1, \ldots, b_{r_1-1}\} \)). Consider any \( i \in \{1, \ldots, r_1\} \) and any \( \beta_i \in B \cap C_i \) such that

\[
\beta_i = c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta
\]

where \( c_i \in \mathbb{R} \) and \( \beta \in B \). W.l.o.g. assume that \( \beta \) is independent of \( \{b_1, \ldots, b_{r_1-1}\} \), otherwise the \( c_i \)'s can be redefined. From the invariance conditions associated with \( v_i \) and by Lemma 5.6, we have

\[
h_j \cdot \beta_i = h_j \cdot (c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta) = h_j \cdot \beta \leq 0, \quad j = r_1 + 1, \ldots, n.
\]

By Lemma 5.9, \( \beta = 0 \). Hence, for any \( i \in \{1, \ldots, r_1\} \) and any \( \beta_i \in B \cap C_i \), \( \beta_i \in \text{span}\{b_1, \ldots, b_{r_1}\} \), as desired. \( \square \)

As before, the lists in (5.23)-(5.25) have the property that any vector in a list on the right is dependent on all the other vectors in its list. Also, if any vector is removed from a list, the remaining vectors are linearly independent. In particular, the \( k \)th list has \( r_k - 1 \) linearly independent vectors of \( B \). We can say that \( B \) has been decomposed into
Chapter 5. Reach Control Indices

$p$ independent cycles of dependency.

**Definition 5.1.** The integers \(\{r_1, \ldots, r_p\}\) are called the *reach control indices* of system (3.1) with respect to simplex \(S\).

### 5.3 Equilibria

The importance of the reach control indices stems from their ability to isolate closed-loop equilibria when using continuous state feedback. Define for \(k = 1, \ldots, p\)

\[
m_k := r_1 + \cdots + r_{k-1} + 1,
\]

\[
I_{G_k} := \{m_k, \ldots, m_k + r_k - 1\},
\]

\[
G_k := \text{co} \{v_{m_k}, \ldots, v_{m_k + r_k - 1}\}.
\]

The following lemma gives the direct consequence of Lemma 5.9 in the case of continuous state feedback.

**Lemma 5.10 ([10]).** Suppose Assumption 5.1 holds. Let \(u(x)\) be a continuous state feedback satisfying the invariance conditions (3.5). Then for each \(k = 1, \ldots, p\) and \(x \in G_k\),

\[
h_j \cdot y(x) = 0, \quad j \in I \setminus \{m_k, \ldots, m_k + r_k - 1\}.
\]

**Proof.** We prove the result only for \(k = 1\). Let \(y(x) = Ax + Bu(x) + a\) be the closed-loop vector field on \(S\) satisfying the invariance conditions (3.5). For \(x \in G_1\), we have

\[
y(x) = c_1(x)b_1 + \cdots + c_{r_1}(x)b_{r_1} + \beta(x),
\]

where \(\beta(x) \in B\). We may assume w.l.o.g. (as in the proof of Theorem 5.1) that \(\beta(x)\) is
independent of \(\{b_1, \ldots, b_{r_1}\}\). From (3.5) we know

\[
h_j \cdot y(x) \leq 0, \quad j = r_1 + 1, \ldots, n.
\]

Using (5.30) and Lemma 5.6, these conditions become

\[
h_j \cdot \beta(x) \leq 0, \quad j = r_1 + 1, \ldots, n.
\]

By Lemma 5.9, this implies \(\beta(x) = 0\). Therefore, for each \(x \in G_1\),

\[
h_j \cdot y(x) = 0, \quad j = r_1 + 1, \ldots, n.
\]

\[\square\]

Lemma 5.10 describes the fundamental geometric property that forces the existence of an equilibrium.

**Theorem 5.2 ([10]).** Suppose Assumption 5.1 holds. Let \(u(x)\) be a continuous state feedback satisfying the invariance conditions (3.5). Then each \(G_k, k = 1, \ldots, p\), contains an equilibrium of the closed-loop system.

**Proof.** We consider only \(k = 1\). Let \(I_{G_1} := \{1, \ldots, r_1\}\) and define the closed-loop vector field

\[
y(x) := Ax + Bu(x) + a, \quad x \in S.
\]

Now we show how to obtain a proper labeling of \(G_1\), as defined in Section 2.7.1. We begin by defining the sets

\[
Q_i := \{x \in G_1 \mid h_i \cdot y(x) > 0\}, \quad i \in I_{G_1}.
\]

Observe that \(v_i \in Q_i\) and \(v_i \notin Q_j, i, j \in I_{G_1}, i \neq j\), for otherwise, we would have
\( y(v_i) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) \) which either contradicts that \( \mathcal{B} \cap \text{cone}(\mathcal{S}) = 0 \) or implies \( y(v_i) \) is an equilibrium. Therefore, either the proof concludes with an equilibrium on a vertex of \( G_1 \), or we can infer that inclusion in a set \( Q_i \) provides a distinct label for the vertices \( v_i \in G_1 \). This satisfies (L1) of a proper labeling of \( G_1 \). Next, let \( \mathcal{T} \) be any triangulation of \( G_1 \) and consider a vertex \( v \) of \( \mathcal{T} \) which is not a vertex of \( G_1 \) and lies in \( \partial G_1 \). W.l.o.g. let \( v \in \text{co} \{v_1, \ldots, v_l\} \) for some \( 2 \leq l \leq r_1 - 1 \). Then it must be that \( v \in Q_k \) for some \( 1 \leq k \leq l \), by the same reasoning that otherwise \( y(v) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) \). Clearly this labeling of \( v \) satisfies the second condition (L2) for a proper labeling. Finally, for vertices \( v \) of \( \mathcal{T} \) in the interior of \( G_1 \), any label \( Q_i \) such that \( h_i \cdot y(v) > 0 \) can be used (at least one such exists because if all \( h_i \cdot y(v) \leq 0 \), \( i \in I_{G_1} \), it implies \( h_i \cdot y(v) \leq 0 \) for all \( i = 1, \ldots, n \) or \( y(v) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) \)).

Now for each \( \alpha > 0, \alpha \in \mathbb{Z} \), define a triangulation \( \mathcal{T}^\alpha \) of \( G_1 \) such that each simplex of \( \mathcal{T}^\alpha \) has diameter \( \frac{1}{\alpha} \). Apply Sperner’s lemma for each \( \mathcal{T}^\alpha \) to obtain a distinguished simplex \( \text{co} \{v_1^\alpha, \ldots, v_l^\alpha\} \) and its baricenter \( x^\alpha \). The set \( \{x^\alpha\} \) defines a bounded sequence in \( G_1 \) which has a convergent subsequence, again denoted \( \{x^\alpha\} \). We have \( \lim_{\alpha \to \infty} x^\alpha = \bar{x} \in G_1 \), since \( G_1 \) is closed. Also, by construction \( v_i^\alpha \to \bar{x}, i \in I_{G_1} \). By Sperner’s lemma we know that \( h_i \cdot y(v_i^\alpha) > 0, i \in I_{G_1} \), so by continuity of \( y(x) \) this implies \( h_i \cdot y(\bar{x}) \geq 0, i \in I_{G_1} \). Combined with (5.29), we obtain that \( -y(\bar{x}) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) = 0 \), which implies \( \bar{x} \in G_1 \) is an equilibrium of the closed-loop system \( \dot{x} = y(x) \). □
Chapter 6

Examples I

In this chapter, preliminary examples are presented which illustrate RCP and its solution according to the theory developed in the previous chapters. Examples in which RCP is solvable using affine feedback are first presented. We then focus on examples where continuous state feedback cannot solve the problem (due to existence of closed-loop equilibria within the simplex). In these cases, the reach control indices can be defined, and we illustrate in a particular fourth-order example the information they give about closed-loop equilibria.

6.1 Affine feedback

Simple examples in which $\mathcal{G} = \emptyset$ and the invariance conditions are solvable are first presented. By, Theorem 4.1 we have $\mathcal{S} \rightarrow \mathcal{F}_0$ by affine feedback.
6.1.1 Example 1

Consider the simplex $S$ determined by $v_0 = (0, 0)$, $v_1 = (0.5, 1)$ and $v_2 = (1, 1)$. Let the affine dynamics on $S$ be defined as

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

The set of points $O$ where $Ax + a \in B$ is given by

$$O = \{ x \mid x_2 - x_1 + 1 = 0 \}.$$ 

Hence $S \cap O = \emptyset$. The normal vectors of the simplex $S$ are $h_0 = (0, 1)$, $h_1 = (1, -1)$ and $h_2 = (-1, 0.5)$. This results in the following invariance conditions for $S$.

At $v_0$

$$h_1 \cdot y_0 \leq 0 \implies (1, -1) \cdot (1, 1 + u_0) \leq 0$$

$$h_2 \cdot y_0 \leq 0 \implies (-1, 0.5) \cdot (1, 1 + u_0) \leq 0$$

At $v_1$

$$h_2 \cdot y_1 \leq 0 \implies (-1, 0.5) \cdot (1.5, 1 + u_1) \leq 0$$

At $v_2$

$$h_1 \cdot y_2 \leq 0 \implies (1, -1) \cdot (1, 1 + u_2) \leq 0$$

In order to satisfy the invariance conditions the control inputs at the vertices of $S$ can be chosen as $u_0 = 0.5$, $u_1 = -2$ and $u_2 = 5$. The affine feedback law can be solved using
the following equation.

\[ u_i = Fv_i + g \quad i = 0, \ldots, 2 \]

\[
\begin{bmatrix}
-v_0^T - 1 \\
-v_1^T - 1 \\
-v_2^T - 1
\end{bmatrix}
\begin{bmatrix}
F^T \\
g^T
\end{bmatrix}
= \begin{bmatrix}
u_0 \\
u_1 \\
u_2
\end{bmatrix}.
\]

Solution of the above yields an affine feedback law:

\[ u = \begin{bmatrix} 14 & -9.5 \end{bmatrix} x + 0.5 \quad x \in S \]

Simulation of the closed-loop system is shown in Figure 6.1. The vector field satisfies the invariance conditions and points out of \( S \) only though the exit facet \( F_0 \). All trajectories originating within \( S \) will leave via \( F_0 \). Therefore we have \( S \xrightarrow{S} F_0 \) by affine feedback.

![Figure 6.1: Example 1: Closed loop vector field on \( S \)](image-url)
6.1.2 Example 2

Consider another simplex $S$ determined by $v_0 = (0, 1)$, $v_1 = (1, 0)$ and $v_2 = (-1, 0)$, and also consider the affine dynamics:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

We have

$$\mathcal{O} = \{ x \mid x_2 = -1 \}.$$

Thus $S \cap \mathcal{O} = \emptyset$. The normal vectors of the simplex $S$ are $h_0 = (0, -1)$, $h_1 = (-1, 1)$ and $h_2 = (1, 1)$. The corresponding invariance conditions for $S$ are:

At $v_0$

$$h_1 \cdot y_0 \leq 0 \implies (-1, 1) \cdot (2, 1 + u_0) \leq 0$$

$$h_2 \cdot y_0 \leq 0 \implies (1, 1) \cdot (2, 1 + u_0) \leq 0$$

At $v_1$

$$h_2 \cdot y_1 \leq 0 \implies (1, 1) \cdot (1, 1 + u_1) \leq 0$$

At $v_2$

$$h_1 \cdot y_2 \leq 0 \implies (-1, 1) \cdot (1, 1 + u_2) \leq 0$$

In order to satisfy the invariance conditions we choose the control inputs at the vertices as $u_0 = -4$, $u_1 = -4$ and $u_2 = 0$. The choice of control inputs at the vertices yields an affine feedback law

$$u = \begin{bmatrix} -2 & -2 \end{bmatrix} x - 2 \quad x \in S$$
Simulation of the closed-loop system is shown in Figure 6.2. The vector field satisfies the invariance conditions and points out of $S$ only though the exit facet $F_0$. Note the equilibrium point located on the set $O$ which is outside of $S$. All trajectories originating within $S$ will leave via $F_0$, and RCP is therefore solved using affine feedback.

![Figure 6.2: Example 2: Closed loop vector field on $S$.](image)

### 6.1.3 Example 3

Let the simplex $S$ be determined by the vertices $v_0 = (0, 1)$, $v_1 = (1, 0)$ and $v_2 = (-1, 0)$. The affine system on $S$ is as follows

$$
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

We have

$$
O = \{ x \mid x_2 = 0 \}.
$$

Hence $S \cap O = \mathcal{O} = \text{co} \{ v_1, v_2 \} = F_0$. The normal vectors are $h_1 = (-1, 1)$ and $h_2 = (1, 1)$. The invariance conditions are:
At $v_0$

$$ h_1 \cdot y_0 \leq 0 \implies (-1, 1) \cdot (1, u_0) \leq 0 $$

$$ h_2 \cdot y_0 \leq 0 \implies (1, 1) \cdot (1, u_0) \leq 0 $$

At $v_1$

$$ h_2 \cdot y_1 \leq 0 \implies (1, 1) \cdot (0, u_1) \leq 0 $$

At $v_2$

$$ h_1 \cdot y_2 \leq 0 \implies (-1, 1) \cdot (0, u_2) \leq 0 $$

In order to satisfy the invariance conditions we choose the control inputs at the vertices of $S$ as $u_0 = -4$, $u_1 = -1$ and $u_2 = -1$. The control input choices yield an affine feedback law

$$ u = \begin{bmatrix} 0 & -3 \end{bmatrix} x - 1 \quad x \in S $$

Simulation of the closed-loop system is given in Figure 6.3. The closed loop vector field satisfies the invariance conditions and points out of $S$ only though the exit facet $F_0$. Although $G = F_0$, the condition $B \cap \text{cone}(S) \neq 0$ ensures that the closed loop vectors at the vertices of $G$ both point outward of $F_0$ in the direction of $B$ guaranteeing no equilibrium. All trajectories originating within $S$ will leave via $F_0$ and RCP is solved using the affine feedback.
6.2 Reach Control Indices

We consider the situation when RCP is not solvable by continuous state feedback, and the reach control indices can be defined. Thus, Assumption 5.1 holds. For these examples, the solution method of [9] can be used, but the resulting controller will be discontinuous. We are primarily interested in the characteristics of the reach control indices.

6.2.1 Example 4

Consider a simplex $S$ in $\mathbb{R}^2$ determined by $v_0 = (-1, 1)$, $v_1 = (1, 0)$ and $v_2 = (0, 0)$. The affine dynamics are

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have

$$\mathcal{O} = \{ x \mid x_2 = 0 \}.$$

Hence $S \cap \mathcal{O} = \mathcal{G} = \text{co} \{ v_1, v_2 \} = \mathcal{F}_0$, with $\kappa = 1$ and $m = 1$. Also it can be easily verified that $\mathcal{B} \cap \text{cone}(S) = \mathbf{0}$. By the results of [10], the problem is not solvable by
continuous state feedback due to the existence of equilibria on $\mathcal{S}$. This can be illustrated clearly as follows. Suppose we choose the control inputs at the vertices of $\mathcal{S}$ to satisfy the invariance conditions as $u_0 = -\frac{3}{4}, u_1 = -1$ and $u_2 = 1$. The resulting affine feedback law is

$$u = \begin{bmatrix} -2 & -3.75 \end{bmatrix} x + 1 \quad x \in \mathcal{S}$$

A simulation of the closed-loop system is shown in Figure 6.4. The vector field satisfies the invariance conditions however there exists an equilibrium point on the set $\mathcal{G} = \text{co} \{v_1, v_2\}$.

![Figure 6.4: Example 4: Closed loop vector field on $\mathcal{S}$](image)

Now we investigate the reach control indices. By inspection, $b_1 := (0, -1) \in \mathcal{B} \cap \mathcal{C}_1$ and $\mathcal{B} = \text{span}\{b_1\}$. Thus, $\mathcal{B}$ exhibits a single dependent cycle with respect to $\mathcal{G}$ with $b_2 := -b_1 \in \mathcal{B} \cap \mathcal{C}_2$ and $p = \kappa + 1 - m = 1$. The single reach control index is $r_1 = 2$. 
6.2.2 Example 5

Now we demonstrate a case in $\mathbb{R}^3$ with simplex $S$ determined by $v_0 = (0,1,0)$, $v_1 = (1,0,0)$, $v_2 = (-1,0,0)$ and $v_3 = (0,0,1)$. The affine dynamics on $S$ in $\mathbb{R}^3$ are defined as

$$
\dot{x} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

With

$$
\mathcal{O} = \{x \mid x_2 = 0, x_3 = 0\}.
$$

The set $\mathcal{G} = \text{co} \{v_1, v_2\}$ and we note $\kappa = 1$, $m = 1$ and $\mathcal{B} \cap \text{cone}(S) = \emptyset$ thereby rendering RCP unsolvable using a continuous state feedback.

We now we investigate the reach control indices. Since $b_1 := (-2,-1,0) \in \mathcal{B} \cap C_1$, $\mathcal{B} = \text{span}\{b_1\}$ and $b_2 := -b_1 \in \mathcal{B} \cap C_2$. Thus Assumption 5.1 holds. There is a single reach control index $r_1 = 2$.

6.2.3 Example 6

Consider a simplex in $\mathbb{R}^3$ determined by $v_0 = (1,-1,1)$, $v_1 = (1,-1,0)$, $v_2 = (-1,0,0)$, and $v_3 = (0,-1,-1)$. The affine dynamics are

$$
\dot{x} = \begin{bmatrix} 4 & 8 & -4 \\ -1 & -2 & 1 \\ 6 & 12 & -6 \end{bmatrix} x + \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} u
$$

We have

$$
\mathcal{O} = \{x \mid x_1 + 2x_2 - x_3 = -1\}.
$$
so \( \mathcal{G} = \text{co} \{v_1, v_2, v_3\} \). It can be verified that the invariance conditions are solvable at each vertex. However, \( \mathcal{B} \cap \mathcal{C}_3 = \emptyset \), so RCP is not solvable by Theorem 3.2. This can also be verified by formula (5.17). We have

\[
m = \hat{m} < \frac{\kappa + 1}{2} = \frac{3}{2}.
\]  

(6.1)

6.2.4 Example 7

Consider the simplex \( \mathcal{S} \) in \( \mathbb{R}^4 \) defined by the following vertices \( v_0 = (0, 0, 0, 0), v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 0) \) and \( v_4 = (0, 0, 0, 1) \). Consider also the affine dynamics on \( \mathcal{S} \)

\[
\dot{x} = \begin{bmatrix} -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

We have

\[
\mathcal{O} = \{x \mid x_1 + x_2 + x_3 + x_4 - 1 = 0\}.
\]

Hence \( \mathcal{G} = \mathcal{S} \cap \mathcal{O} = \text{co} \{v_1, v_2, v_3, v_4\} = \mathcal{F}_0 \), with \( \kappa = 3 \) and \( m = 2 \). It can be easily shown that \( \mathcal{B} \cap \text{cone}(\mathcal{S}) = \emptyset \). With \( \kappa > m \) and \( \mathcal{B} \cap \text{cone}(\mathcal{S}) = \emptyset \), RCP is not solvable using a continuous state feedback.

By inspection it is revealed \( b_1 := (-2, 1, 0, 0) \in \mathcal{B} \cap \mathcal{C}_1, b_3 := (0, 0, 2, -1) \in \mathcal{B} \cap \mathcal{C}_3 \) and \( \mathcal{B} = \text{span}\{b_1, b_3\} \). \( \mathcal{B} \) splits into two dependent cycles with respect to \( \mathcal{G} \), with \( b_2 := -b_1 \in \mathcal{B} \cap \mathcal{C}_2 \) and \( b_4 := -b_3 \in \mathcal{B} \cap \mathcal{C}_4 \). Thus, \( p = 2 \), and the reach control indices are \( r_1 = 2, \ r_2 = 2 \).

We now linger further on this example, since all the interesting information concerning reach control indices first appears in dimension \( n = 4 \) with two inputs. Note that any
$x \in S$ can be expressed as $x = \lambda_0 v_0 + \cdots + \lambda_n v_n$ where $\sum \lambda_i = 1$. Via the reach control indices, we have now established that if we want to satisfy the invariance conditions for $S$ we must have that $y(v_1) = c_1 b_1$, $y(v_2) = c_2 b_2$, $y(v_3) = c_3 b_3$, and $y(v_4) = c_4 b_4$ where $c_i > 0$. If we construct an affine feedback $u(x) = Kx + g$ with these velocity vectors, the vector field at any point $x \in S$ can then be written

$$y(x) = y(\lambda_0 v_0 + \cdots + \lambda_n v_n)$$

$$= \lambda_0 y(v_0) + \lambda_1 y(v_1) + \lambda_2 y(v_2) + \lambda_3 y(v_3) + \lambda_4 y(v_4)$$

$$= \lambda_0 y(v_0) + \lambda_1 c_1 b_1 + \lambda_2 c_2 b_2 + \lambda_3 c_3 b_3 + \lambda_4 c_4 b_4$$

$$= \lambda_0 y(v_0) + (\lambda_1 c_1 - \lambda_2 c_2) b_1 + (\lambda_3 c_3 - \lambda_4 c_4) b_3.$$  

Since $y(v_0)$, $b_1$, and $b_3$ are linearly independent, if $x$ is an equilibrium then

$$\lambda_0 = 0, \quad \lambda_1 c_1 - \lambda_2 c_2 = 0, \quad \lambda_3 c_3 - \lambda_4 c_4 = 0, \quad \sum_{i=0}^{n} \lambda_i = 1.$$ 

If we restrict our attention to $G_1$ by letting $\lambda_1 + \lambda_2 = 1$ and by solving the above system of equations we obtain the location of the equilibrium in $G_1$ as

$$\bar{x}_1 = \frac{c_2}{c_1 + c_2} v_1 + \frac{c_1}{c_1 + c_2} v_2.$$ 

Similarly for $G_2$ we obtain

$$\bar{x}_2 = \frac{c_4}{c_3 + c_4} v_3 + \frac{c_3}{c_3 + c_4} v_4.$$ 

By convexity, any point on $\text{co} \{\bar{x}_1, \bar{x}_2\}$ is also an equilibrium. We now obtain the set of equilibria in $S$ to be $\bar{X} = \{x \in \mathbb{R}^n | x = (1-\gamma)\bar{x}_1 + \gamma \bar{x}_2, \gamma \in [0, 1]\}$ which appear whenever the invariance conditions are satisfied.
Chapter 7

Time-Varying Compensation

In this chapter we develop our method for time-varying compensation. It is based on observations about the reach control indices, especially driven by Example 7 in the previous chapter. We require two additional results before we can begin. One is an additional necessary condition for solvability by open-loop control that we’ll exploit. The second is a result on the precise location of equilibria when using affine feedbacks.

7.1 Necessary Conditions Revisited

In this section we see that an additional necessary condition for solvability by open-loop control can be discovered and exploited for our later design work. Therefore the conditions in Assumption 5.1 can be appended by one new condition without any loss of generality. We consider in this chapter the following assumptions.

**Assumption 7.1.** Simplex $\mathcal{S}$ and system (3.1) satisfy the following conditions.

(D1) $\mathcal{G} = \text{co} \{v_1, \ldots, v_{\kappa+1}\}$, where $m \leq \kappa < n$.

(D2) $\mathcal{B} \cap \text{cone}({\mathcal{S}}) = \mathbf{0}$.

(D3) $\mathcal{B} \cap \mathcal{C}, \neq \mathbf{0}, \quad i \in I_{\mathcal{G}}$. 

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(D4) \( \exists \{r_1, \ldots, r_p\} \) such that (5.23)-(5.25) hold.

(D5) \( \mathcal{B}_k \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}, \quad k = 1, \ldots, p. \)

Conditions (D1)-(D2) define the problem setup. The necessity of (D3) is proved in Theorem 3.2, (D4) is proved in Theorem 5.1, and the necessity of (D5) is proved next.

**Lemma 7.1.** If \( \mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0 \) by open-loop controls, then \( \mathcal{B}_k \not\subset \mathcal{H}_0 \) for each \( k = 1, \ldots, p. \)

**Proof.** We consider only \( k = 1 \). Define

\[
\hat{\mathcal{F}}_0 := \text{co} \{v_1, \ldots, v_{r_1}\}.
\]

Let \( x \in \hat{\mathcal{F}}_0 \). By condition (iii) of RCP, for any open-loop control values \( \{u_x\} \) solving RCP,

\[
h_j \cdot (Ax + Bu_x + a) \leq 0, \quad x \in \mathcal{F}_j.
\]

By the same proof as in Lemma 5.10, we also have that

\[
h_j \cdot (Ax + Bu_x + a) = 0, \quad j = r_1 + 1, \ldots, n.
\]

Suppose by way of contradiction that \( \mathcal{B}_1 \subset \mathcal{H}_0 \). Then

\[
h_0 \cdot (Ax + Bu_x + a) = 0.
\]

Now we observe that for any \( z \in \hat{\mathcal{F}}_0 \)

\[
T_{\hat{\mathcal{F}}_0}(z) = \{y \in \mathbb{R}^n \mid h_j \cdot y = 0, h_1 \cdot y \leq 0, j = 0, r_1 + 1, \ldots, n, z, y \in \mathcal{F}_i\}.
\]

We conclude that \( Ax + Bu_x + a \in T_{\hat{\mathcal{F}}_0}(x) \), for all \( x \in \hat{\mathcal{F}}_0 \). By Theorem 3.8 of [12], this implies \( \hat{\mathcal{F}}_0 \) is a strongly invariant set, a contradiction to the statement that \( \mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0 \) by open-loop controls.

\( \square \)
7.2 Equilibria Revisited

In this section we show that Lemma 5.3 indicates where equilibria appear if we try to solve RCP using affine feedback when Assumption 7.1 holds with $m \leq \kappa$. To that end consider an affine controller $u(x) = Kx + g$ and closed-loop vector field

$$y(x) := Ax + Bu(x) + a$$

such that the invariance conditions hold at all vertices of $S$ and, moreover, for vertices in $G$, we have selected control values so that

$$y(v_i) = b_i, \quad i = 1, \ldots, \kappa + 1,$$

where the $b_i$'s are provided by (5.23)-(5.25). The next result shows that $S$ contains a $(\kappa - m)$-dimensional subset of equilibria of the closed-loop system.

**Lemma 7.2.** Suppose Assumption 7.1 holds with $m \leq \kappa$. For each $k = 1, \ldots, p$ define

$$\tau_k := \frac{-c_{m_k}v_{m_k} - \cdots - c_{m_k+r_k-2}v_{m_k+r_k-2} + v_{m_k+r_k-1}}{1 - c_{m_k} - \cdots - c_{m_k+r_k-2}} \in G_k, \quad (7.1)$$

where the constants $c_i$ are provided by (5.5). Also define

$$\mathcal{E} := \text{aff} \{\tau_1, \ldots, \tau_p\} \cap S.$$

Let $u(x)$ and $y(x)$ be as above. Then

$$y(x) = 0, \quad x \in \mathcal{E},$$

$$y(x) \neq 0, \quad x \not\in \mathcal{E}.$$

**Proof.** Let $d_k := 1 - c_{m_k} - \cdots - c_{m_k+r_k-2} > 0$. By the convexity of $y(x)$ we have for each
where the last equality follows from Lemma 5.3. Again by convexity it follows that
\( y(\overline{x}) = 0 \) for all \( x \in \text{aff} \{\overline{x}_1, \ldots, \overline{x}_p\} \).

Now suppose there is \( \overline{x} \in S \setminus E \) such that \( y(\overline{x}) = 0 \). Then \( \overline{x} \in G \) so
\[
\overline{x} = \alpha_1 v_1 + \cdots + \alpha_{\kappa+1} v_{\kappa+1}, \quad \alpha_i \geq 0, \quad \sum \alpha_i = 1. 
\]
By convexity of \( y(x) \),
\[
0 = y(\overline{x}) = \sum_i \alpha_i y(v_i) = \sum \alpha_i b_i. \tag{7.2}
\]
We can order this expression according the reach control indices. First define
\[
\beta_k := \alpha_{m_k} b_{m_k} + \cdots + \alpha_{m_k+r_k-1} b_{m_k+r_k-1}, \quad k = 1, \ldots, p.
\]
Clearly \( \beta_k \in B_k, k = 1, \ldots, p \). Equation (7.2) becomes
\[
0 = \beta_1 + \cdots + \beta_p + \alpha_{r+1} b_{r+1} + \cdots + \alpha_{\kappa+1} b_{\kappa+1}.
\]
By Theorem 5.1 the vectors
\[
\{\beta_1, \ldots, \beta_p, b_{r+1}, \ldots, b_{\kappa+1}\}
\]
are linearly independent. This implies \( \beta_k = 0, k = 1, \ldots, p, \) and \( \alpha_{r+1} = \cdots = \alpha_{\kappa+1} = 0. \)
Consider $\beta_1$. We have

$$\alpha_{m1}b_{m1} + \cdots + \alpha_{m1+r_1-1}b_{m1+r_1-1} = 0.$$ 

Using (5.5) and collecting terms, we obtain

$$(\alpha_{m1} + \alpha_{m1+r_1-1}c_{m1})b_{m1} + \cdots + (\alpha_{m1+r_1-2} + \alpha_{m1+r_1-1}c_{m1+r_2})b_{m1+r_2} = 0.$$ 

Because $\{b_{m1}, \ldots, b_{m1+r_1-2}\}$ are linearly independent, we get

$$\alpha_{m1} = -\alpha_{m1+r_1-1}c_{m1}, \ldots, \alpha_{m1+r_1-2} = -\alpha_{m1+r_1-1}c_{m1+r_2}.$$ 

Using the above expressions and (7.1), we have

$$z_1 := \alpha_{m1}v_{m1} + \cdots + \alpha_{m1+r_1-1}v_{m1+r_1-1} = (\alpha_{m1+r_1-1}d_1)x_1.$$ 

The above procedure can be repeated to generate $z_k = (\alpha_{m_k+r_k-1}d_k)x_k$, $k = 2, \ldots, p$. Finally, we obtain

$$\bar{x} = (\alpha_{m1+r_1-1}d_1)x_1 + \cdots + (\alpha_{m_p+r_p-1}d_p)x_p$$

with $\sum_k \alpha_{m_k+r_k-1}d_k = 1$. That is, $\bar{x} \in \mathcal{E}$, a contradiction. 

\[\square\]

### 7.3 A Flow-like Condition

The development of the time-varying compensator is divided into two parts. First we establish that a flow-like condition holds on $\mathcal{S}$ which has desirable properties relative to the sub-simplices $\mathcal{G}_k$, $k = 1, \ldots, p$. Second, we propose a time-varying compensator whose role in essence is to “move” the set of equilibria generated by affine feedback in
a direction opposite to the direction indicated by the flow-like condition. In this section
the flow-like condition is developed. We give an overview of how it is derived.

Consider each $G_k$ and its associated set of velocity vectors in $B_k$. By assumption (D5), $B_k$ is not parallel to $H_0$. Consequently one can find a vector in $\text{Ker } (B^T)$, call it $\beta_k$, with the property that there exists a hyperplane with normal vector $\beta_k$ that strongly separates at least two vertices in $G_k$. This is the content of Lemma 7.8. Simultaneously this $\beta_k$ trivially contributes toward a flow-like condition on $G$ since $\beta_k \cdot (Ax + Bu(x) + a) = 0$ for any $x \in G$ and any control $u(x)$. By taking a linear combination of the $\beta_k$’s, one gets both the vertex separation property and the flow-like condition using a single vector $\xi^1$. This is done in Lemma 7.9. The fact that the vertex separation property of $\beta_i$ is not corrupted by the presence of $\beta_j$, $j \neq i$, in the expression for $\xi^1$ is a result of the special form of the $\beta_k$’s. This form is derived in Lemma 7.6. It relies on the properties of the simplex summarized in Lemma 2.1 and on Lemma 5.7 that each principal submatrix of $M_{m_k,m_k+r_k-1}$ is a nonsingular $\mathcal{M}$-matrix. Lemmas 7.4 and 7.5 present the essential algebraic tool that $M_{m_k,m_k+r_k-1}$ is a singular, irreducible $\mathcal{M}$-matrix.

So far we have discussed the strong separation of vertices in each $G_k$ and a flow-like condition on $G$ in terms of vector $\xi^1$. There remains the question of a flow-like condition for $S \setminus G$. This condition arrives automatically from convex analysis, as given in Lemma 7.10. Finally the flow-like conditions based on $\xi^1$ and $\xi^2$ are put together into one flow-like condition using a vector $\xi^*$ in Theorem 7.1. This $\xi^*$ is adjusted to retain the vertex separation property, the main property upon which the time-varying compensator is based.

Lemma 7.3. Suppose Assumption 7.1 holds. For each $k = 1, \ldots, p$

$$B_k \perp \text{span}\{h_{m_1}, \ldots, h_{m_k+r_k-1}, h_{m_k+1}, \ldots, h_n\}$$  \hspace{1cm} (7.3)

$$B_k \subset \text{span}\{v_{m_k} - v_0, \ldots, v_{m_k+r_k-1} - v_0\}.$$  \hspace{1cm} (7.4)
Proof. Equation (7.3) is a restatement of Lemma 5.6. Next applying Lemma 2.1(iii), we can decompose the state space as
\[
\mathbb{R}^n = \text{span}\{v_{m_k} - v_0, \ldots, v_{m_k + r_k - 1} - v_0\} \oplus \text{span}\{h_{m_1}, \ldots, h_{m_k + r_k - 1}, h_{m_{k+1}}, \ldots, h_n\}.
\]
Combined with (7.3), we obtain (7.4).

Lemma 7.4. Suppose Assumption 7.1 holds. For each \(k = 1, \ldots, p\), \(M_{m_k, m_k + r_k - 1} \in \mathbb{R}^{r_k \times r_k}\) is irreducible.

Proof. Suppose not. Then by the definition of reducibility there exists a permutation matrix \(P\) such that
\[
PM_{m_k, m_k + r_k - 1}P^T = \begin{bmatrix} M_1 & 0 \\ \ast & M_2 \end{bmatrix}
\]
where \(M_1 \in \mathbb{R}^{\rho \times \rho}\) and \(M_2 \in \mathbb{R}^{(r_k - \rho) \times (r_k - \rho)}\) for some \(1 \leq \rho \leq r_k - 1\). Without loss of generality suppose we have reordered the indices \(\{m_k, \ldots, m_k + r_k - 1\}\) in accordance with the permutation matrix \(P\). Then
\[
M_{m_k, m_k + r_k - 1} = \begin{bmatrix} M_{m_k, m_k + \rho - 1} & H_{m_k, m_k + \rho - 1}^T Y_{m_k + \rho, m_k + r_k - 1} \\ \ast & M_{m_k + \rho, m_k + r_k - 1} \end{bmatrix}
\]
and \(H_{m_k, m_k + \rho - 1}^T Y_{m_k + \rho, m_k + r_k - 1} = 0\). The latter gives
\[
h_j \cdot b_i = 0, \quad i = m_k + \rho, \ldots, m_k + r_k - 1, \quad j = m_k, \ldots, m_k + \rho - 1.
\]
Combining (7.5) with Lemma 5.6 we get
\[
h_j \cdot b_i = 0, \quad i = m_k + \rho, \ldots, m_k + r_k - 1, \quad j \in I \setminus \{m_k + \rho, \ldots, m_k + r_k - 1\}.
\]
Consider \(M_{m_k + \rho, m_k + r_k - 1}\). By Lemma 5.7 and Remark 5.0.1 it is a nonsingular \(M\)-
matrix. By Theorem 2.2(iii) there exists $c \in \mathbb{R}^{r_k - \rho}$, $c \neq 0$, with $c \preceq 0$ such that $M_{m_k + \rho, m_k + r_k - 1}c < 0$. Let $y := Y_{m_k + \rho, m_k + r_k - 1}c$. Note that $y \neq 0$ since $\{b_{m_k + \rho}, \ldots, b_{m_k + r_k - 1}\}$ are linearly independent for any $\rho \geq 1$ by Theorem 5.1. Then we have $M_{m_k + \rho, m_k + r_k - 1}c = H_{m_k + \rho, m_k + r_k - 1}^T Y_{m_k + \rho, m_k + r_k - 1}c = H_{m_k + \rho, m_k + r_k - 1}^T y < 0$. That is, 

$$h_j \cdot y < 0, \quad j = m_k + \rho, \ldots, m_k + r_k - 1.$$ 

Also from (7.6) 

$$h_j \cdot y = 0, \quad j \in I \setminus \{m_k + \rho, \ldots, m_k + r_k - 1\}.$$ 

We conclude $0 \neq y \in B \cap \text{cone}(\mathcal{S})$, a contradiction of Assumption 7.1. 

**Lemma 7.5.** $M_{m_k, m_k + r_k - 1}$ is a singular $\mathcal{M}$-matrix.

**Proof.** By Lemma 2.1(viii), $\text{rank}(H_{m_k, m_k + r_k - 1}) = r_k$. Also we know $\text{rank}(Y_{m_k, m_k + r_k - 1}) = r_k - 1$. Therefore, $\text{rank}(M_{m_k, m_k + r_k - 1}) = r_k - 1$, so it is clearly singular. Now we prove that $M_{m_k, m_k + r_k - 1}$ is an $\mathcal{M}$-matrix. By Lemma 5.7 and Remark 5.0.1, each principal submatrix of $M_{m_k, m_k + r_k - 1}$ (formed by removing the $i$th row and column from $M_{m_k, m_k + r_k - 1}$) is a nonsingular $\mathcal{M}$-matrix. By Theorem 2.2(ii), every real eigenvalue of a nonsingular $\mathcal{M}$-matrix is positive. Therefore, every real eigenvalue of each principal submatrix of $M_{m_k, m_k + r_k - 1}$ is positive. We conclude by Theorem 2.3(iii) that $M_{m_k, m_k + r_k - 1}$ is an $\mathcal{M}$-matrix. 

Referring to Assumption 7.1, define the matrix 

$$\hat{B} = [\ b_1 \cdots \ b_{k+1} \]. \quad (7.7)$$ 

Here we assume the columns of $\hat{B}$ are ordered according to Theorem 5.1. Recall the definitions of $I_{\hat{G}_k}$ and $G_k$ from (5.27) and (5.28).
Lemma 7.6. Suppose Assumption 7.1 holds. For each $k \in \{1, \ldots, p\}$ there exists $\beta_k \in \text{Ker}(\hat{B}^T)$ such that

(i) $\beta_k = d_{m_k} h_{m_k} + \cdots + d_{m_k + r_k - 1} h_{m_k + r_k - 1}$, with $d_i < 0$.

(ii) $\beta_k \cdot (v_i - v_0) = 0$, $i \in I \setminus I_{\tilde{g}_k}$.

(iii) $\beta_k \cdot (v_i - v_j) = 0$, $i, j \in I_{\tilde{g}} \setminus I_{\tilde{g}_k}$.

Proof. By Lemmas 7.4 and 7.5, $M_{m_k, m_k + r_k - 1}$ is a singular, irreducible $\mathcal{M}$-matrix; therefore, so is $M^T_{m_k, m_k + r_k - 1}$. By Theorem 2.4, there exists $d_n < 0$ such that $M^T_{m_k, m_k + r_k - 1} d_n = 0$. Define

$$\beta_k := H_{m_k, m_k + r_k - 1} d_n.$$

This gives the form (i). Next we show $\beta_k \in \text{Ker}(\hat{B}^T)$. First, we have

$$M^T_{m_k, m_k + r_k - 1} d_n = Y^T_{m_k, m_k + r_k - 1} H_{m_k, m_k + r_k - 1} d_n = Y^T_{m_k, m_k + r_k - 1} \beta_k = 0.$$

That is,

$$\beta_k \cdot b_i = 0, \quad i = m_k, \ldots, m_k + r_k - 1.$$

Also from (7.3)

$$\beta_k \cdot b_i = 0, \quad i = 1, \ldots, m_{k-1} + r_{k-1} - 1, m_{k+1}, \ldots, \kappa + 1.$$

We conclude $\beta \in \text{Ker}(\hat{B}^T)$. The statement (ii) follows immediately from Lemma 2.1(ii). The statement (iii) follows immediately from Lemma 2.1(iii). \hfill \square

Lemma 7.7. Suppose Assumption 7.1 holds. Consider $\beta_1, \ldots, \beta_p$ from Lemma 7.6. Then

$$\text{Ker}(\hat{B}^T) = \text{span}\{\beta_1, \ldots, \beta_p, h_{\kappa+2}, \ldots, h_n\}.$$
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Proof. By construction \( \beta_k \in \text{span}\{h_{m_k}, \ldots, h_{m_k+r_k-1}\} \) and so by Lemma 2.1(viii), \( \{\beta_1, \ldots, \beta_p, h_{n+1}, \ldots, h_n\} \) are linearly independent. From (7.3), \( \hat{B} \perp \text{span}\{h_{n+2}, \ldots, h_n\} \). Thus, \( h_{n+2}, \ldots, h_n \in \text{Ker} (\hat{B}^T) \). Also from Lemma 7.6, \( \beta_1, \ldots, \beta_p \in \text{Ker} (\hat{B}^T) \). Now \( \text{rank}(\hat{B}) = \hat{m} \), so \( \dim(\text{Ker} (\hat{B}^T)) = n - \hat{m} \). But \( p + n - (\kappa + 2) + 1 = n - \hat{m} \). Thus, \( \{\beta_1, \ldots, \beta_p, h_{n+2}, \ldots, h_n\} \) is a basis of \( \text{Ker} (B^T) \). \( \square \)

Lemma 7.8. Suppose Assumption 7.1 holds. Consider \( \beta_1, \ldots, \beta_p \) from Lemma 7.6. For each \( k \in \{1, \ldots, p\} \), there exist \( i_k, j_k \in I_{G_k} \) such that

\[
\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0.
\]

(7.8)

Proof. Fix \( k \in \{1, \ldots, p\} \) and suppose by way of contradiction that for every \( \beta \in \text{Ker} (\hat{B}^T) \) and \( i, j \in I_{G_k} \), \( \beta \cdot (v_i - v_j) = 0 \). This implies \( (v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k}) \in \hat{B} \). Suppose w.l.o.g. that

\[
v_{m_k+1} - v_{m_k} = b' + b''
\]

where \( b' \in B_k \) and \( 0 \neq b'' \in \text{span}\{b_{m1}, \ldots, b_{mk-1+r_k-1}, b_{mk+1}, \ldots, b_{n}\} \). Then by Lemma 2.1(iii) and by Lemma 5.6

\[
0 = h_j \cdot (v_{m_k+1} - v_{m_k}) = h_j \cdot b' + h_j \cdot b'' = h_j \cdot b'', \quad j = m_1, \ldots, m_{k-1} + r_k - 1, m_{k+1}, \ldots, n.
\]

By construction \( \{b_{m1}, \ldots, b_{mk+r_k-2}, b''\} \) are linearly independent. This contradicts Lemma 5.9. Thus, \( b'' = 0 \). This argument can be repeated for each \( v_{m_k+i} - v_{m_k}, \ i = 1, \ldots, r_k - 1 \) to get \( (v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k}) \in B_k \). By Lemma 2.1(vii), \( \{(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k})\} \) are a basis for \( B_k \), so \( B_k \subset H_0 \). This contradicts Assumption 7.1.

We deduce that for each \( k \in \{1, \ldots, p\} \), there exist \( i_k, j_k \in I_{G_k} \) and \( \beta \in \text{Ker} (\hat{B}^T) \)
such that $\beta \cdot (v_{i_k} - v_{j_k}) \neq 0$. By Lemma 7.7, $\beta \in \text{Ker} (\hat{B}^T)$ can be expressed as

$$\beta = \alpha_1 \beta_1 + \cdots + \alpha_p \beta_p + \alpha_{\kappa+2} h_{\kappa+2} + \cdots + \alpha_n h_n.$$  

By Lemma 7.6(iii) and Lemma 2.1(iii)

$$0 \neq \beta \cdot (v_{i_k} - v_{j_k}) = (\alpha_1 \beta_1 + \cdots + \alpha_p \beta_p + \alpha_{\kappa+2} h_{\kappa+2} + \cdots + \alpha_n h_n) \cdot (v_{i_k} - v_{j_k})$$

$$= \alpha_k \beta_k \cdot (v_{i_k} - v_{j_k})$$

implying that $\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0$. \hfill $\square$

In light of Lemma 7.8, we assume without loss of generality (by reordering the indices within each group $I_{\mathcal{G}_k}$) that for $k = 1, \ldots, p$

$$v_{m_k} \in \arg\max_{i \in I_{\mathcal{G}_k}} \beta_k \cdot v_i, \quad v_{m_k+r_k-1} \in \arg\min_{i \in I_{\mathcal{G}_k}} \beta_k \cdot v_i. \quad (7.9)$$

We also define the sets

$$\mathcal{E}^0 := \text{co} \{v_{m_1}, v_{m_2}, \ldots, v_{m_p}\}$$

$$\mathcal{E}^\infty := \text{co} \{v_{m_1+r_1-1}, v_{m_2+r_2-1}, \ldots, v_{m_p+r_p-1}\}.$$

**Lemma 7.9.** Suppose Assumption 7.1 holds. Consider $\beta_1, \ldots, \beta_p \in \text{Ker} (\hat{B}^T)$ from Lemma 7.6. Define

$$\xi^1 := \xi_1^1 \beta_1 + \cdots + \xi_p^1 \beta_p \quad (7.10)$$

and

$$\mathcal{H} := \{x \in \mathbb{R}^n : \xi^1 \cdot (x - v_0) = 1\}. \quad (7.11)$$

There exist real $\xi_1^1, \ldots, \xi_p^1 > 0$ such that $\mathcal{H}$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$.

**Proof.** Let $x \in \mathcal{E}^0$ and $y \in \mathcal{E}^\infty$. Then there exist $\alpha_i \geq 0$, $\sum_{i=1}^p \alpha_i = 1$, and $\gamma_i \geq 0$, \ldots
\[ \sum_{i=1}^{p} \gamma_i = 1 \text{ such that } x = \gamma_1 v_{m_1} + \cdots + \gamma_p v_{m_p} \text{ and } y = \alpha_1 v_{m_1+r_1-1} + \cdots + \alpha_p v_{m_p+r_p-1}. \]

For \( k = 1, \ldots, p \) define

\[ \Pi_k := \beta_k \cdot (v_{m_k} - v_0), \quad \pi_k := \beta_k \cdot (v_{m_k+r_k-1} - v_0). \]

By Lemma 7.6 we may write \( \beta_k = d_{m_k} h_{m_k} + \cdots + d_{m_k+r_k-1} h_{m_k+r_k-1} \) with \( d_i < 0 \). By Lemma 2.1(iii)-(iv) we have that

\[ \beta_k \cdot (v_i - v_0) = d_i h_i \cdot (v_i - v_0) > 0, \quad i \in I_{\tilde{\gamma}_k}. \]

By Lemma 7.8 we know that \( \Pi_k \neq \pi_k \); thus,

\[ 0 < \pi_k < \Pi_k, \quad k = 1, \ldots, p. \]

Select \( \xi_1^k \in \left( \frac{1}{\Pi_k}, \frac{1}{\pi_k} \right) \neq \emptyset \) for \( k = 1, \ldots, p \). Then

\[ \xi^1 \cdot (x - v_0) = (\xi_1^1 \beta_1 + \cdots + \xi_p^1 \beta_p) \cdot (\lambda_1 (v_{m_1} - v_0) + \cdots + \lambda_p (v_{m_p} - v_0)) \]
\[ = \gamma_1 \xi_1 \beta_1 \cdot (v_{m_1} - v_0) + \cdots + \gamma_p \xi_p \beta_p \cdot (v_{m_p} - v_0) \]
\[ = \gamma_1 \xi_1 \Pi_1 + \cdots + \gamma_p \xi_p \Pi_p \]
\[ \geq \min_k \{\xi_k \Pi_k\} > 1. \quad (7.12) \]

Similarly,

\[ \xi^1 \cdot (y - v_0) = (\xi_1^1 \beta_1 + \cdots + \xi_p^1 \beta_p) \cdot (\alpha_1 (v_{m_1+r_1-1} - v_0) + \cdots + \alpha_p (v_{m_p+r_p-1} - v_0)) \]
\[ = \alpha_1 \xi_1 \beta_1 \cdot (v_{m_1+r_1-1} - v_0) + \cdots + \alpha_p \xi_p \beta_p \cdot (v_{m_p+r_p-1} - v_0) \]
\[ = \alpha_1 \xi_1 \pi_1 + \cdots + \alpha_p \xi_p \pi_p \]
\[ \leq \max_k \{\xi_k \pi_k\} < 1. \quad (7.13) \]
Thus, $H$ strongly separates $E^0$ and $E^\infty$.

**Lemma 7.10.** Suppose Assumption 7.1 holds. Let $P := \text{co} \{v_0, v_{\kappa+2}, \ldots, v_n\}$. There exists $\xi^2 \in \text{Ker} \left( \hat{B}^T \right)$ such that

$$
\xi^2 \cdot (Ax + a) > 0, \quad x \in P.
$$

(7.14)

**Proof.** Observe that $P$ is compact and convex and that $P \cap O = \emptyset$. The image of $P$ under the affine map $x \mapsto Ax + a$, denoted $C_1 = AP + a$ is also compact and convex. We observe that $C_1 \cap \hat{B} = \emptyset$. For suppose not. Then there is a point $x \in P$ such that $Ax + a \in \hat{B}$. Then $x \in O$, by definition, which contradicts $P \cap O = \emptyset$. Note that both $C_1$ and $\hat{B}$ are convex sets, and that $C_1$ is bounded. By Theorem 2.8 there exists a hyperplane $H_2$ separating $\hat{B}$ and $C_1$ strongly. This implies $\hat{B}$ is parallel to $H_2$ since $\hat{B}$ is a subspace. Let $\xi^2$ be the normal vector to $H_2$ pointing to the side containing $P$. Then, $\xi^2 \in \text{Ker} \hat{B}^T$ and $\xi^2 \cdot (Ax + a) > 0$ for all $x \in P$. \qed

**Theorem 7.1.** Suppose Assumption 7.1 holds. Also suppose

\[ Av_i + a \in C_i, \quad i = 0, \kappa + 2, \ldots, n \]  
(7.15)

\[ Av_i + a \in \hat{B}, \quad i = 1, \ldots, \kappa + 1. \]  
(7.16)

Then there exists $0 \neq \xi^* \in \text{Ker} \left( \hat{B}^T \right)$ such that

$$
\xi^* \cdot (Ax + a) \geq 0, \quad x \in S,
$$

(7.17)

and such that $H^*$ strongly separates $E^0$ and $E^\infty$ where

$$
H^* := \{ x \in \mathbb{R}^n \mid \xi^* \cdot (x - v_0) = 1 \}.
$$

(7.18)
Proof. Consider $\xi^1$ given by Lemma 7.9 and $\xi^2$ given by Lemma 7.10. Define

$$\xi^* := (1 - \lambda)\xi^1 + \lambda\xi^2, \quad \lambda \in (0, 1). \quad (7.19)$$

Since $\beta_k = H_{m_k,m_k+r_{k-1}}d$ with $d \prec 0$ and by (7.15), we have for each $k = 1, \ldots, p$,

$$\beta_k \cdot (Av_i + a) \geq 0, \quad i = 0, \kappa + 2, \ldots, n.$$  

Therefore,

$$\xi^1 \cdot (Av_i + a) \geq 0, \quad i = 0, \kappa + 2, \ldots, n.$$  

Thus, for $i = 0, \kappa + 2, \ldots, n$

$$\xi^* \cdot (Av_i + a) = (1 - \lambda)\xi^1 \cdot (Av_i + a) + \lambda\xi^2 \cdot (Av_i + a) > 0.$$  

Since $\xi^1, \xi^2 \in \text{Ker}(\hat{B}^T)$, then $\xi^* \in \text{Ker}(\hat{B}^T)$. Thus, by (7.16)

$$\xi^* \cdot (Av_i + a) = 0, \quad i \in I_G.$$  

By convexity, $\xi^* \cdot (Av_i + a) \geq 0$ for all $x \in \mathcal{S}$ and for all $\lambda \in (0, 1)$.

Let $x \in \mathcal{E}^0$ and $y \in \mathcal{E}^\infty$, and let $\gamma_i$ and $\alpha_i$ be as in the proof of Lemma 7.9. Using (7.12) and (7.13)

$$\xi^* \cdot (x - v_0) = (1 - \lambda)\xi^1 \cdot (x - v_0) + \lambda\xi^2 \cdot (x - v_0)$$

and

$$\xi^* \cdot (y - v_0) = (1 - \lambda)\xi^1 \cdot (y - v_0) + \lambda\xi^2 \cdot (x - v_0).$$  

Since functions $f_1(x) = \xi^1 \cdot (x - v_0)$ and $f_2(x) = \xi^2 \cdot (x - v_0)$ are continuous, they achieve
a minimum and maximum on each compact set $E^0$ and $E^\infty$. This means we can select
\[ \lambda \in (0, 1) \] close enough to 0 such that
\[ \xi^* \cdot (y - v_0) < 1 < \xi^* \cdot (x - v_0), \quad x \in E^0, y \in E^\infty. \]  
(7.20)

With this choice of $\lambda$, $H^*$ strongly separates $E^0$ and $E^\infty$. \hfill \square

\section{7.4 Time-Varying Compensator}

The time-varying compensator will be constructed so as to exploit the flow-like condition
(7.17) and the separation property (7.18) of Theorem 7.1. First, we define two affine
feedbacks $u^0(x)$ and $u^\infty(x)$ that place equilibria at $E^0$ and $E^\infty$, respectively. Then we
define a time-varying compensator $u(x, \alpha)$ with additional state $\alpha \in \mathbb{R}$. This compensator
simply interpolates between $u^0(x)$ and $u^\infty(x)$ as $\alpha$ varies from 0 to 1. By construction
when $\alpha = 0$, all closed-loop equilibria are in $E^0$. When $\alpha = 1$, they are in $E^\infty$. Thus,
as $\alpha$ varies from 0 to 1, the set of closed-loop equilibria crosses from one side of $H^*$ to
the other in a direction with decreasing $\xi^*$ component. We can say that trajectories flow
downstream according to (7.17) while equilibria flow upstream. Informally, no trajectory
can be “stuck” at an equilibrium, and this ultimately allows all trajectories to exit $S$, as
shown in Theorem 7.2.

Suppose the invariance conditions for $S$ are solvable; thus, there exist inputs $u^0_0, \ldots, u^0_n \in \mathbb{R}^m$ such that (3.4) hold. Let $y^0_i := Av_i + Bu^0_i + a$, for $i = 0, \ldots, n$. We choose
$u^0_{m_1}, u^0_{m_2}, \ldots, u^0_{m_p} \in \mathbb{R}^m$ such that
\[ y^0_i = 0, \quad i \in \{m_1, m_2, \ldots, m_p\} \]  
(7.21)
\[ y^0_i = b_i, \quad i \in \{1, \ldots, \kappa + 1\} \setminus \{m_1, m_2, \ldots, m_p\}. \]  
(7.22)
Finally, construct the associated affine feedback

\[ u^0(x) = K^0 x + g^0, \]

and let \( \phi^0(t,x_0) \) denote trajectories of the closed-loop system. Note that the closed-loop system has equilibria at \( v_{m_1}, \ldots, v_{m_p} \).

Next we define a symmetrical controller \( u^\infty(x) \) which is identical to \( u^0(x) \) except that it places equilibria at \( v_{m_k + r_k - 1}, k = 1, \ldots, p \). Let \( y^\infty_i := A v_i + B u^\infty_i + a, \) for \( i = 0, \ldots, n \).

First set \( u^\infty_i = u^0_i, i = 0, \kappa + 2, \ldots, n \). Then we choose \( u^\infty_1, \ldots, u^\infty_{\kappa+1} \in \mathbb{R}^m \) such that

\[
\begin{align*}
y^\infty_i &= b_i, \quad i \in \{1, \ldots, \kappa + 1\} \setminus \{m_1 + r_1 - 1, m_2 + r_2 - 1, \ldots, m_p + r_p - 1\} \quad (7.23) \\
y^\infty_i &= 0, \quad i \in \{m_1 + r_1 - 1, m_2 + r_2 - 1, \ldots, m_p + r_p - 1\}. \quad (7.24)
\end{align*}
\]

Finally, construct the associated affine feedback

\[ u^\infty(x) = K^\infty x + g^\infty, \]

and let \( \phi^\infty(t,x_0) \) denote trajectories of the closed-loop system. Note that the closed-loop system has equilibria at \( v_{m_k + r_k - 1}, k = 1, \ldots, p \).

**Lemma 7.11.** There exist \( \xi^0, \xi^\infty \in \mathbb{R}^n \) such that

\[
\begin{align*}
\xi^0 \cdot (Ax + Bu^0(x) + a) &> 0, \quad x \in S \setminus \mathcal{E}^0, \quad (7.25) \\
\xi^\infty \cdot (Ax + Bu^\infty(x) + a) &> 0, \quad x \in S \setminus \mathcal{E}^\infty. \quad (7.26)
\end{align*}
\]

**Proof.** We consider only (7.25), since the proof for (7.26) is completely analogous. First, we claim

\[ Ax + Bu^0(x) + a \neq 0, \quad \forall x \in S \setminus \mathcal{E}^0. \]
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Suppose not. Then there exists $\bar{x} \in S \setminus E^0$ such that

$$A\bar{x} + Bu^0(\bar{x}) + a = 0.$$ 

Since $\bar{x} \in G$, there exist $\lambda_1, \ldots, \lambda_{\kappa+1} \geq 0$ with $\sum \lambda_i = 1$ and not all $\lambda_i, i \in I_G \setminus \{m_1, \ldots, m_p\}$ equal to zero such that

$$\bar{x} = \sum_{i=1}^{\kappa+1} \lambda_i v_i.$$ 

By convexity of $y^0(x) := Ax + Bu^0(x) + a$ and the fact that $y^0(v_i) = 0$ for $i = m_1, \ldots, m_p$, we have

$$0 = y^0(\bar{x}) = \sum_{i=1}^{\kappa+1} \lambda_i (Av_i + Bu^0(v_i) + a) = \sum_{i \in I_G \setminus \{m_1, \ldots, m_p\}} \lambda_i b_i,$$

with not all $\lambda_i$’s equal to zero. However, Theorem 5.1 implies that $\{b_i \mid i \in I_G \setminus \{m_1, \ldots, m_p\}\}$ are linearly independent. Thus, we reach a contradiction.

Now let $P := 0$ and $S' := \text{co} \{v_i \mid i \in \{0, \ldots, n\} \setminus \{m_1, \ldots, m_p\}\}$. Since $y^0(x)$ is affine, $P' := \{y \in \mathbb{R}^n \mid y = y^0(x), x \in S'\}$ is compact and convex. By the argument above $P \cap P' = \emptyset$, so by the Separating Hyperplane Theorem 2.8, there exists $\xi^0 \in \mathbb{R}^n$ such that

$$\xi^0 \cdot (Ax + Bu^0(x) + a) > 0, \quad x \in S'.$$

Let $x \in S \setminus E^0$. That is, there exist $\lambda_0, \ldots, \lambda_n \geq 0$ with $\sum \lambda_i = 1$ and not all $\lambda_i, i \in I_G \setminus \{m_1, \ldots, m_p\}$ equal to zero such that

$$x = \sum_{i=0}^{n} \lambda_i v_i.$$ 

By convexity of $y^0(x)$ we have

$$\xi^0 \cdot y^0(x) = \sum_{i=0}^{n} \lambda_i \xi^0 \cdot y^0(v_i).$$
Since not all \( \lambda_i, \ i \in I_G \ \{ m_1, \ldots, m_p \} \) are equal to zero, \( \xi^0 \cdot y^0(v_i) > 0 \) for \( v_i \in S' \), and \( \xi^0 \cdot y^0(v_i) = 0 \) for \( v_i \in Z^0 \), we get \( \xi^0 \cdot y^0(x) > 0 \), as desired.

Now we extend the state \( x \) by an additional state \( \alpha \in \mathbb{R} \) with dynamics

\[
\dot{\alpha} = -c \alpha + c, \quad \alpha(0) = 0, \tag{7.27}
\]

where \( c > 0 \) is a to-be-determined constant. Construct the extended state vector \( x_e := (x, \alpha) \) and define a multi-affine feedback

\[ u(x, \alpha) := (1 - \alpha)u^0(x) + \alpha u^\infty(x). \]

Evidently the role of \( u(x, \alpha) \) is to interpolate from \( u^0(x) \) to \( u^\infty(x) \) as \( \alpha \) varies from 0 to 1. Define the closed-loop system

\[ y(x, \alpha) := Ax + Bu(x, \alpha) + a. \]

### 7.4.1 Main Results

The next result guarantees that using \( u(x, \alpha) \), closed-loop trajectories cannot exit from restricted facets \( F_1, \ldots, F_n \).

**Lemma 7.12.** For each \( \alpha \in [0, 1] \), \( y(x, \alpha) \) satisfies the invariance conditions (3.5).

**Proof.** Let \( y^0(x) := Ax + Bu^0(x) + a \) and \( y^\infty(x) := Ax + Bu^\infty(x) + a \). We have that

\[
y(x, \alpha) = (1 - \alpha)(Ax + a + Bu^0(x)) + \alpha(Ax + a + Bu^\infty(x))
= (1 - \alpha)y^0(x) + \alpha y^\infty(x).
\]

By construction, using \( u^0(x) \) or \( u^\infty(x) \) the invariance conditions (3.4) are satisfied. That is, \( h_j \cdot y^0(v_i) \leq 0 \) and \( h_j \cdot y^\infty(v_i) \leq 0 \) for \( i \in \{0, \ldots, n\} \) and \( j \in I_i \). Therefore, for each
\[ h_j \cdot y(v_i, \alpha) = (1 - \alpha)h_j \cdot y^0(v_i) + \alpha h_j \cdot y^\infty(v_i) \leq 0, \quad i \in \{0, \ldots, n\}, \quad j \in I_i. \]

Finally, by convexity of \( y(x, \alpha) \), the invariance conditions (3.5) hold for all \( x \in S \).

The following is the main result of the thesis.

**Theorem 7.2.** Suppose Assumption 7.1 holds and suppose the invariance conditions (3.4) for \( S \) are solvable. There exists \( c > 0 \) sufficiently small such that \( S \xrightarrow{\hat{s}} F_0 \) using feedback \( u(x, \alpha) \).

**Proof.** Since the invariance conditions are solvable, we can apply an affine feedback transformation \( u_1(x) = K_1 x + g_1 + w \) with \( w \in \mathbb{R}^m \) a new exogenous input, such that

\[
Av_i + Bu_1(v_i) + a \in C_i, \quad i = 0, \kappa + 2, \ldots, n
\]
\[
Av_i + Bu_1(v_i) + a = b_i, \quad i = 1, \ldots, \kappa + 1,
\]

and \( b_i \in B \cap C_i, \ i = 1, \ldots, \kappa + 1 \), are selected according to Theorem 5.1. Let \( \hat{B} \) be as in (7.7) and let \( \hat{B} = \text{Im} (\hat{B}) \). Clearly, \( Av_i + Bu_1(v_i) + a \in \hat{B} \) for \( i = 1, \ldots, r \). Also, \( Av_i + Bu_1(v_i) + a \in \hat{B} \) for \( i = r + 1, \ldots, \kappa + 1 \). For suppose not; w.l.o.g. suppose \( b_{r+1} \in B \setminus \hat{B} \). Then \( \{b_1, \ldots, b_{m_1+r_1-2}, \ldots, b_{m_p}, \ldots, b_{m_p+r_p-2}, b_{r+1}\} \) is a set of \( \hat{m} + 1 \) linearly independent vectors, contradicting the definition of \( \hat{m} \).

Consider the new system

\[ \dot{x} = \hat{A}x + Bw + \hat{a} = (A + BK_1)x + Bw + a + Bg_1, \]

and let \( \hat{O} := \{x \in \mathbb{R}^n \mid \hat{A}x + \hat{a} \in B\} \) and \( \hat{G} = \hat{O} \cap S \). We claim \( \hat{G} = G \). (**Proof of claim:** Let \( v_i \in G \). Then \( Av_i + a \in B \), so \( Av_i + B(K_1 v_i + g_1) + a = \hat{A}v_i + \hat{a} \in B \). Thus, \( v_i \in \hat{G} \). Conversely, let \( v_i \in \hat{G} \). Then \( \hat{A}v_i + \hat{a} \in B \). Thus, \( Av_i + a = \hat{A}v_i + \hat{a} - B(K_1 v_i + g_1) \in B \).
Therefore, \( v_i \in \mathcal{G} \). The result now follows by taking the convex hull of \( \{v_1, \ldots, v_{\kappa+1}\} \).

The above considerations show that the reach control indices are the same for the transformed system, since they only involve \( \mathcal{B} \) and the vertices in \( \mathcal{G} \). Therefore, we can assume without loss of generality that the above transformation has been applied to the original system, and the new system is relabeled as (3.1). It satisfies Assumption 7.1, and additionally (7.15) and (7.16). Therefore, Theorem 7.1 applies.

Now consider \( \mathcal{H}^* \) given by (7.18). Define the associated open half-spaces \( \mathcal{H}^- := \{ x \in \mathbb{R}^n \mid \xi^* \cdot x < 1 \} \) and \( \mathcal{H}^+ := \{ x \in \mathbb{R}^n \mid \xi^* \cdot x > 1 \} \). Define the compact, convex sets \( \mathcal{P}^- := \overline{\mathcal{S} \cap \mathcal{H}^-} \) and \( \mathcal{P}^+ := \overline{\mathcal{S} \cap \mathcal{H}^+} \). Notice that \( \mathcal{P}^- \subset \mathcal{S} \setminus \mathcal{E}^0 \) and \( \mathcal{P}^+ \subset \mathcal{S} \setminus \mathcal{E}^\infty \) by Theorem 7.1 and the convention (7.20).

Next we define \( u^0(x) \) and \( u^\infty(x) \). In particular, for \( i \in \{0, \kappa+2, \ldots, n\} \), set \( u^0(v_i) = u^\infty(v_i) = 0 \), since the invariance conditions are already satisfied at these vertices. For \( i \in I_G \), assign \( u^0(v_i) \) and \( u^\infty(v_i) \) according to (7.21)-(7.22) and (7.23)-(7.24), respectively.

Observe that \( Bu^0(x), Bu^\infty(x) \in \hat{\mathcal{B}} \) for all \( x \in \mathcal{S} \).

First we discuss the behavior of trajectories using \( u^0(x) \) and \( u^\infty(x) \). By Lemma 7.11, a flow condition holds on \( \mathcal{P}^- \) using \( u^0(x) \). Since \( \mathcal{P}^- \) is compact, by a standard argument all trajectories \( \phi^0(t, x_0) \) exit \( \mathcal{P}^- \) in finite time. Since the invariance conditions hold using \( u^0(x) \), trajectories only exit \( \mathcal{P}^- \) via \( \mathcal{F}_0 \) or \( \mathcal{H}^* \). Similarly, by Lemma 7.11, a flow condition holds on \( \mathcal{P}^+ \) using \( u^\infty(x) \). Since \( \mathcal{P}^+ \) is compact, all trajectories \( \phi^\infty(t, x_0) \) exit \( \mathcal{P}^+ \) in finite time. Since the invariance conditions hold using \( u^\infty(x) \), trajectories only exit \( \mathcal{P}^+ \) via \( \mathcal{F}_0 \) or \( \mathcal{H}^* \). Using (7.17) and that \( Bu^\infty(x) \in \hat{\mathcal{B}} \), we have

\[
\xi^* \cdot (\dot{A}x + Bu^\infty(x) + \bar{a}) = \xi^* \cdot (\dot{A}x + \bar{a}) \geq 0, \quad x \in \mathcal{S}.
\]

Therefore trajectories \( \phi^\infty(t, x_0) \) cannot exit through \( \mathcal{H}^* \). Hence all trajectories \( \phi^\infty(t, x_0) \) starting in \( \mathcal{P}^+ \) exit \( \mathcal{S} \) in finite time.

Now we consider the controller \( u(x, \alpha) \) with associated trajectories \( \phi(t, x_0) \). For \( c > 0 \)
sufficiently small, \( u(x, \alpha) \) is sufficiently close to \( u^0(x) \) for a sufficiently long time interval \([0, \tau_1]\) so that (7.25) holds on \( P^- \) and all trajectories \( \phi(t, x_0) \) initialized in \( P^- \) either exit \( S \) or enter \( P^+ \) in a finite time \( \tau < \tau_1 \). Trajectories initialized in \( P^+ \) either exit \( S \) or they remain in \( P^+ \) (since they cannot cross over to \( P^- \) by (7.17)).

There exists \( \tau_2 > \tau_1 \) when all trajectories remaining in \( S \) are in \( P^+ \) and \( u(x, \alpha) \) is sufficiently close to \( u^\infty(x) \) such that the flow condition (7.26) takes effect on \( P^+ \). Thus, all trajectories must exit \( P^+ \), and they do so through \( F_0 \) and not \( H^* \), again, because of (7.17). Thus, \( S \xrightarrow{\delta} F_0 \) using feedback \( u(x, \alpha) \).

\( \square \)
Chapter 8

Examples II

8.1 Time-Varying Compensation

8.1.1 Example 8

We continue Example 5 and solve this problem using a time-varying affine feedback. Consider a simplex $S \in \mathbb{R}^3$ determined by the convex hull of $v_0 = (0, 1, 0)$, $v_1 = (-1, 0, 0)$, $v_2 = (1, 0, 0)$, and $v_3 = (0, 0, 1)$. The affine dynamics on $S$ are

$$
\dot{x} = Ax + a + Bu = \begin{bmatrix}
2 & 0 & 0 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} u
$$

Reach Control Indices

We have

$$\mathcal{O} = \{x|x_2 = 0, x_3 = 0\}.$$ 

Hence, $\mathcal{G} = \mathcal{O} \cap S = \text{co} \{v_1, v_2\}$. The normal vectors for $S$ are $h_0 = (0, -1, 0)$, $h_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $h_2 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $h_3 = (0, 0, -1)$. The invariance conditions for $S$ are
Chapter 8. Examples II

At $v_0$

\[ h_1 \cdot y_0 \leq 0 \implies u \geq -1 \]
\[ h_2 \cdot y_0 \leq 0 \implies u \leq \frac{1}{3} \]
\[ h_3 \cdot y_0 \leq 0 \implies u \in \mathbb{R} \]

At $v_1$

\[ h_1 \cdot y_2 \leq 0 \implies u \geq 1 \]
\[ h_3 \cdot y_2 \leq 0 \implies u \in \mathbb{R} \]

At $v_2$

\[ h_2 \cdot y_1 \leq 0 \implies u \leq -1 \]
\[ h_3 \cdot y_1 \leq 0 \implies u \in \mathbb{R} \]

At $v_3$

\[ h_1 \cdot y_3 \leq 0 \implies u \geq -1 \]
\[ h_2 \cdot y_3 \leq 0 \implies u \leq \frac{1}{3} \]

The invariance conditions are solvable. We must apply a feedback transformation of the form presented in [9] to prepare the example for time-varying feedback. Let

\[ u = K_1 x + g_1 + G_1 w \]
\[ = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} x + w \]
resulting in new affine dynamics

\[
\dot{x} = \dot{A}x + \dot{a} + \dot{B}w = (A + BK_1)x + BG_1w
\]

\[
= \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} x + \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} w.
\]

By inspection it is revealed that \(b_1 := (2, 1, 0) \in \mathcal{B} \cap \mathcal{C}_1\) and \(\hat{B} = \text{span}\{b_1\}\). We also define \(b_2 := -b_1 = (-2, -1, 0) \in \mathcal{B} \cap \mathcal{C}_2\) and we have one dependent cycle, \(\mathcal{G} = \mathcal{G}_1 = \text{co}\{v_1, v_2\}\).

**Flow-Like Condition**

We will develop the flow-like condition in the same order as the theory developed in Section 7.3. Firstly, we have the following \(M\)-matrix

\[
M_{1,2}^{T} = Y_{1,2}^{T}H_{1,2} = \frac{1}{2} \begin{bmatrix}
3 & -3 \\
-1 & 1
\end{bmatrix}
\]

whose kernel is spanned by \(c_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}^{T}\). We now have

\[
\beta_1 = \begin{bmatrix}
-h_1 \\
-h_2
\end{bmatrix} c_1 = \begin{bmatrix}
1 & -2 & -2
\end{bmatrix}^{T}.
\]

We now have a basis for \(\text{Ker}(B^{T}) = \text{span}\{\beta_1, -h_3\}\). We can order the vertices with respect to \(\beta_1\). In fact, as far as the control design is concerned, we can stop developing the flow-like condition here since only \(\beta_1\) is required to design following feedbacks. However,
we will continue the development. We obtain

\[ v_2 = v_{m_1} \in \arg \max_{i \in I_{G_1}} \beta_1 \cdot v_i, \quad v_1 = v_{m_1 + r_1 - 1} \in \arg \min_{i \in I_{G_1}} \beta_1 \cdot v_i. \]

We use these to define the sets

\[ E^0 = \{v_2\}, \quad E^\infty = \{v_1\}. \]

We now define

\[ \Pi_1 = \beta_1 \cdot (v_{m_1} - v_0) = 3, \quad \pi_1 = \beta_1 \cdot (v_{m_1 + r_1 - 1} - v_0) = 1. \]

We select \( c_1 \in (\frac{1}{\Pi_1}, \frac{1}{\pi_1}) = (\frac{1}{3}, 1) \). We can now determine that

\[ \xi^1 = \beta_1 c_1 = \frac{1}{2} \beta_1 = \begin{bmatrix} \frac{1}{2} & -1 & -1 \end{bmatrix}^T. \]

We check that indeed \( \xi^1 \) separates \( E^0 \) and \( E^\infty \)

\[ \xi^1 \cdot v_2 = \frac{1}{2}, \quad \xi^1 \cdot v_1 = -\frac{1}{2}. \]

We now need to find \( \xi^2 \in \text{Ker} \ (B^T) \) such that it provides a flow condition on \( P = \text{co} \ \{v_0, v_3\} \). We have the following velocity vectors

\[ y_0 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T, \quad y_6 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T. \]

We see that

\[ \xi^2 = \frac{3}{2} \beta_1 - h_3 = \begin{bmatrix} \frac{3}{2} & -3 & -2 \end{bmatrix}^T. \]
provides the flow condition on $\mathcal{P}$. Lastly we require $\xi^*$. We need $\lambda \in (0,1)$ close enough to 0 such that $\xi^*$ still separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$. We find that

$$\xi^* = (1 - \lambda)\xi^1 + \lambda \xi^2 = \frac{3}{4} \xi^1 + \frac{1}{4} \xi^2 = \begin{bmatrix} \frac{3}{4} & -\frac{5}{4} & -\frac{3}{2} \end{bmatrix}^T.$$  

provides the flow-like condition we have been seeking.

**Designing the Control**

First we design $w^0(x) = K^0 x + g^0$ such that

$$y^0(v_0) = \hat{A} v_0 + \hat{a} \quad \Rightarrow w_0^0 = 0,$$

$$y^0(v_{m_1+r_1-1}) = y^0(v_1) = b_1 \quad \Rightarrow w_1^0 = 1,$$

$$y^0(v_{m_1}) = y^0(v_2) = 0 \quad \Rightarrow w_2^0 = 0,$$

$$y^0(v_3) = \hat{A} v_3 + \hat{a} \quad \Rightarrow w_3^0 = 0.$$  

By solving

$$\begin{bmatrix} K^0 & g^0 \end{bmatrix} = \begin{bmatrix} w_0^0 & \ldots & w_3^0 \end{bmatrix} \begin{bmatrix} v_0 & \ldots & v_3 \\ 1 & \ldots & 1 \end{bmatrix}^{-1}$$

we find that

$$w^0(x) = K^0 x + g^0 = \begin{bmatrix} -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2} \end{bmatrix} x + \frac{1}{2}.$$  

Now we design $w^\infty(x) = K^\infty x + g^\infty$ such that

$$y^\infty(v_0) = \hat{A} v_0 + \hat{a} \quad \Rightarrow w_0^\infty = 0,$$

$$y^\infty(v_{m_1+r_1-1}) = y^\infty(v_1) = 0 \quad \Rightarrow w_1^\infty = 0,$$

$$y^\infty(v_{m_1}) = y^\infty(v_2) = b_2 \quad \Rightarrow w_2^\infty = -1,$$

$$y^\infty(v_3) = \hat{A} v_3 + \hat{a} \quad \Rightarrow w_3^\infty = 0.$$
By solving
\[
\begin{bmatrix}
K^\infty & g^\infty \\
\end{bmatrix} = \begin{bmatrix}
w_0^\infty & \ldots & w_3^\infty \\
\end{bmatrix} \begin{bmatrix}
v_0 & \ldots & v_3 \\
1 & \ldots & 1 \\
\end{bmatrix}^{-1} 
\]
we find that
\[
w^\infty(x) = K^\infty x + g^\infty = \left[ -\frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right] x - \frac{1}{2}.
\]

We now extend the state with \(\alpha\) and we obtain
\[
w(x, \alpha) = (1 - \alpha)w^0(x) + \alpha w^\infty
\]
where
\[
\alpha(t) = 1 - e^{-ct}.
\]

Since we have used a feedback transformation, we must determine our actual input \(u\) to be
\[
u(x, \alpha) = K_1 x + g_1 + G_1 w(x, \alpha)
\]
\[
= K_1 x + g_1 + G_1 [(1 - \alpha)w^0(x) + \alpha w^\infty]
\]
\[
= (1 - \alpha) [(K_1 + G_1 K_0) x + (g_1 + G_1 g_0)] + \alpha [(K_1 + G_1 K^\infty) x + (g_1 + G_1 g^\infty)]
\]
\[
= (1 - \alpha)u^0(x) + \alpha u^\infty(x)
\]
\[
= \frac{1}{2}(1 - \alpha) \left( \begin{bmatrix} -3 & -1 & -1 \end{bmatrix} x + 1 \right) + \frac{1}{2} \alpha \left( \begin{bmatrix} -3 & 1 & 1 \end{bmatrix} x - 1 \right).
\]
Chapter 8. Examples II

Location of the Equilibria

The vector field in $S$ is

$$y(x, \alpha) = Ax + a + Bu(x, \alpha)$$
$$= Ax + a + B \left[ (1 - \alpha)u^0(x) + \alpha u^\infty(x) \right]$$
$$= (1 - \alpha) \left[ Ax + a + Bu^0(x) \right] + \alpha \left[ Ax + a + Bu^\infty(x) \right]$$
$$= (1 - \alpha) y^0(x) + \alpha y^\infty(x)$$
$$= (1 - \alpha) y^0(\lambda_0 v_0 + \ldots + \lambda_3 v_3) + \alpha y^\infty(\lambda_0 v_0 + \ldots + \lambda_3 v_3)$$
$$= (1 - \alpha) \left[ \lambda_0 y^0(v_0) + \ldots + \lambda_3 y^0(v_3) \right] + \alpha \left[ \lambda_0 y^\infty(v_0) + \ldots + \lambda_3 y^\infty(v_3) \right]$$
$$= (1 - \alpha) \left[ \lambda_0 y(v_0) + \lambda_1 b_1 + \lambda_3 y(v_3) \right] + \alpha \left[ \lambda_0 y(v_0) + \lambda_2 b_2 + \lambda_3 y(v_3) \right]$$
$$= \lambda_0 y(v_0) + (1 - \alpha) \lambda_1 b_1 + \alpha \lambda_2 b_2 + \lambda_3 y(v_3)$$
$$= \lambda_0 y(v_0) + [(1 - \alpha) \lambda_1 - \alpha \lambda_2] b_1 + \lambda_3 y(v_3).$$

We know that $y(v_0)$, $b_1$, and $y(v_3)$ are linearly independent. Therefore, $y(x, \alpha) = 0$ when

$$\lambda_0 = 0$$
$$\lambda_1 = \alpha$$
$$\lambda_2 = 1 - \alpha$$
$$\lambda_3 = 0$$
$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

This is equivalent to writing $\lambda_1 = \alpha$ and $\lambda_2 = 1 - \alpha$ and $\lambda_0 = \lambda_3 = 0$. Therefore the location of the equilibrium, $\bar{x} \in S$ is $\bar{x} = \alpha v_1 + (1 - \alpha)v_2$.

Simulation

We simulate the control $u(x, \alpha)$. Figure 8.1 provides a plot of the evolution of the level values $\xi^* \cdot \phi(t, v_i)$ versus time. Each blue trajectory corresponds to $\phi(t, v_i)$ for $i = 0, \ldots, n$. 
and are labeled on the trajectory at the time that the trajectory exits $S$. The magenta level value corresponds to a vertex in $E^0$ and the cyan level value corresponds to a vertex in $E^\infty$. The red plot is the location of $\bar{x}_1(t)$. Notice that all blue trajectories move up level sets of $\xi^*$ and the red trajectory moves down level sets of $\xi^*$ as we would expect. Also, since $h_0 \cdot b_2 > 0$ the trajectory $\phi(t, v_2)$ exits immediately.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8.1}
\caption{Simulation of control $u(x, \alpha)$ for Example 8. Level values $\xi^* \cdot \phi(t, v_i)$ versus time.}
\end{figure}

8.1.2 Example 9

We present an example with two dependent cycles with respect to $G$. We also discuss how the structure of the invariance conditions can be inspected to determine the decomposition of $G$ into these cycles. Consider the simplex $S \in \mathbb{R}^4$ determined by the convex hull of $v_0 = (-1, -1, 0, -1)$, $v_1 = (0, 0, 1, 0)$, $v_2 = (0, -1, 0, 0)$, $v_3 = (0, -1, 1, 0)$, and
$v_4 = (0, -1, 0, 1)$. The affine dynamics on $\mathcal{S}$ are

$$
\dot{x} = Ax + a + Bu = \begin{bmatrix}
-4 & 4 & -6 & -3 \\
-5 & -2 & 2 & 3 \\
-4 & 1 & -4 & 3 \\
-8 & 3 & -4 & -3 \\
\end{bmatrix} x + \begin{bmatrix}
7 \\
-3 \\
3 \\
5 \\
\end{bmatrix} + \begin{bmatrix}
-1 & 2 \\
1 & 0 \\
1 & 3 \\
-1 & 1 \\
\end{bmatrix} u.
$$

**Reach Control Indices**

We have

$$\mathcal{O} = \{ x | x_1 = 0 \}.$$

Hence, $\mathcal{G} = \mathcal{O} \cap \mathcal{S} = \text{co} \{ v_1, v_2, v_3, v_4 \}$. The normal vectors for $\mathcal{S}$ are $h_0 = (1, 0, 0, 0)$, $h_1 = (0, -1, 0, 0)$, $h_2 = (-2, 0, 1, 1)$, $h_3 = (0, 1, -1, 0)$, and $h_4 = (1, 0, 0, -1)$. The invariance conditions for $\mathcal{S}$ are

At $v_0$

$$
\begin{align*}
    & h_1 \cdot y_0 \leq 0 \implies u_1 \geq -1 \\
    & h_2 \cdot y_0 \leq 0 \implies u_1 \leq 2 \\
    & h_3 \cdot y_0 \leq 0 \implies u_2 \geq \frac{2}{3} \\
    & h_4 \cdot y_0 \leq 0 \implies u_2 \leq 3 
\end{align*}
$$

At $v_1$

$$
\begin{align*}
    & h_2 \cdot y_1 \leq 0 \implies u_1 \leq 1 \\
    & h_3 \cdot y_1 \leq 0 \implies u_2 \geq 0 \\
    & h_4 \cdot y_1 \leq 0 \implies u_2 \leq 0 
\end{align*}
$$
Chapter 8. Examples II

At $v_2$

$$h_1 \cdot y_2 \leq 0 \implies u_1 \geq 1$$
$$h_3 \cdot y_2 \leq 0 \implies u_2 \geq -1$$
$$h_4 \cdot y_2 \leq 0 \implies u_2 \leq -1$$

At $v_3$

$$h_1 \cdot y_3 \leq 0 \implies u_1 \geq -1$$
$$h_2 \cdot y_3 \leq 0 \implies u_1 \leq -1$$
$$h_4 \cdot y_3 \leq 0 \implies u_2 \leq 1$$

At $v_4$

$$h_1 \cdot y_4 \leq 0 \implies u_1 \geq -2$$
$$h_2 \cdot y_4 \leq 0 \implies u_1 \leq -2$$
$$h_3 \cdot y_4 \leq 0 \implies u_2 \geq -1$$

The invariance conditions are solvable and have an interesting structure. Solvability of the invariance conditions requires that $u_2 = 0$ at $v_1$ and $u_2 = -1$ at $v_2$. This means that we lose the same degree of freedom at each of $v_1$ and $v_2$. We will see later that the convex hull of $v_1$ and $v_2$ forms $G_1$. It is also interesting to note that $u_1$ is bounded below at $v_1$ and bounded above at $v_2$. We have similar results for $v_3$ and $v_4$ where we lose the freedom of choice of $u_1$. We will see later that the convex hull of $v_3$ and $v_4$ forms $G_2$. Despite the structure of the invariance conditions, it is not immediately apparent what the reach control indices are for this example. We apply a feedback transformation of
the form presented in [9] to aid the discussion. Let

\[ u = K_1 x + g_1 + G_1 w \]

\[ = \begin{bmatrix} 4 & 2 & -2 & -3 \\ -1 & -1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} w \]

resulting in new affine dynamics

\[ \dot{x} = \hat{A} x + \hat{a} + \hat{B} w = (A + BK_1) x + (a + Bg_1) + BG_1 w \]

\[ = \begin{bmatrix} -10 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ -13 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 & -6 \\ 1 & -2 \end{bmatrix} w. \]

By inspection it is revealed that \( b_1 := (1, -1, -1, 1) \in \mathcal{B} \cap \mathcal{C}_1, b_3 := (-4, 0, -6, -2) \in \mathcal{B} \cap \mathcal{C}_3, \) and \( \hat{\mathcal{B}} = \text{span} \{b_1, b_3\}. \) We also define \( b_2 := -2b_1 = (-2, 2, 2, -2) \in \mathcal{B} \cap \mathcal{C}_2 \) and \( b_4 := -\frac{1}{2}b_3 = (2, 0, 3, 1) \in \mathcal{B} \cap \mathcal{C}_4, \) thus \( \mathcal{B} \) is split into two dependent cycles with respect to \( \mathcal{G} \)

\[ \mathcal{G}_1 = \text{co} \{v_1, v_2\} \]

\[ \mathcal{G}_2 = \text{co} \{v_3, v_4\}. \]

Note that we have deliberately selected \( b_2 \) such that \( b_2 \neq -b_1 \) and \( b_4 \) such that \( b_4 \neq -b_3. \) This allows us to demonstrate the position of equilibria in a more general case since the position of the equilibria in each cycle are not directly proportional to the evolution of the dynamic variable \( \alpha. \)
Flow-Like Condition

We will develop the flow-like condition in the same order as the theory developed in Section 7.3. Firstly, we have the following $M$-matrices

\[
M_{1,2}^T = Y_{1,2}^T H_{1,2} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \quad M_{3,4}^T = Y_{3,4}^T H_{3,4} = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}
\]

whose kernels are spanned by $c_1 = \begin{bmatrix} 1 \\ 1 \frac{1}{2} \end{bmatrix}^T$ and $c_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}^T$ respectively. We now have

\[
\beta_1 = \begin{bmatrix} -h_1 \\ -h_2 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ 1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \end{bmatrix}^T,
\]

\[
\beta_2 = \begin{bmatrix} -h_3 \\ -h_4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -3 \\ -1 \frac{1}{3} \frac{1}{3} \end{bmatrix}^T.
\]

We now have a basis for $\text{Ker} \ (B^T) = \text{span}\{\beta_1, \beta_2\}$. We can order the vertices with respect to $\beta_1$ and $\beta_2$. In fact, as far as the control design is concerned, we can stop developing the flow-like condition here since only $\beta_1$ and $\beta_2$ are required to design following feedbacks. However, we will continue the development. We obtain

\[
v_1 = v_{m_1} \in \arg\max_{i \in I_{G_1}} \beta_1 \cdot v_i, \quad v_2 = v_{m_1 + r_1 - 1} \in \arg\min_{i \in I_{G_1}} \beta_1 \cdot v_i,
\]

\[
v_4 = v_{m_2} \in \arg\max_{i \in I_{G_2}} \beta_2 \cdot v_i, \quad v_3 = v_{m_2 + r_2 - 1} \in \arg\min_{i \in I_{G_2}} \beta_2 \cdot v_i.
\]

We use these to define the sets

\[
\mathcal{E}^0 = \text{co} \ \{v_1, v_4\}, \quad \mathcal{E}^\infty = \text{co} \ \{v_2, v_3\}.
\]
We now define
\[
\Pi_1 = \beta_1 \cdot (v_{m_1} - v_0) = 1, \quad \pi_1 = \beta_1 \cdot (v_{m_1+r_1-1} - v_0) = \frac{1}{2}, \\
\Pi_2 = \beta_2 \cdot (v_{m_2} - v_0) = 3, \quad \pi_2 = \beta_2 \cdot (v_{m_2+r_2-1} - v_0) = 1.
\]
We select \( c_1 \in \left( \frac{1}{\Pi_1}, \frac{1}{\pi_1} \right) = \left( 1, 2 \right) \) and \( c_2 \in \left( \frac{1}{\Pi_2}, \frac{1}{\pi_2} \right) = \left( \frac{1}{3}, 1 \right). \) We can now determine that
\[
\xi^1 = \beta_1 c_1 + \beta_2 c_2 = \frac{5}{4} \beta_1 + \frac{1}{2} \beta_2 = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{1}{8} & \frac{7}{8} \end{bmatrix}^T.
\]
We check that indeed \( \xi^1 \) separates \( \mathcal{E}^0 \) and \( \mathcal{E}^\infty \)
\[
\xi^1 \cdot v_1 = -\frac{1}{8}, \quad \xi^1 \cdot v_4 = \frac{1}{8}, \\
\xi^1 \cdot v_2 = -\frac{6}{8}, \quad \xi^1 \cdot v_3 = -\frac{7}{8}.
\]
We now need to find \( \xi^2 \in \text{Ker} \left( B^T \right) \) such that it provides a flow condition on \( \mathcal{P} = \text{co} \left\{ v_0, v_6 \right\}. \) We have the following velocity vector
\[
y_0 = \frac{1}{3} \begin{bmatrix} 10 & 1 & 3 & 13 \end{bmatrix}^T.
\]
We see that
\[
\xi^2 = \xi^1 = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{1}{8} & \frac{7}{8} \end{bmatrix}^T.
\]
provides the flow condition on \( \mathcal{P}. \) Lastly we require \( \xi^*. \) We need \( \lambda \in (0, 1) \) close enough to 0 such that \( \xi^* \) still separates \( \mathcal{E}^0 \) and \( \mathcal{E}^\infty. \) We find that
\[
\xi^* = (1 - \lambda)\xi^1 + \lambda \xi^2 = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{1}{8} & \frac{7}{8} \end{bmatrix}^T.
\]
provides the flow-like condition we have been seeking.
Designing the Control

First we design $w^0(x) = K^0 x + g^0$ such that

\[ y^0(v_0) = \hat{A}v_0 + \hat{a} \quad \Rightarrow w^0_0 = (0, 0), \]
\[ y^0(v_{m_1}) = y^0(v_1) = 0 \quad \Rightarrow w^0_1 = (0, 0), \]
\[ y^0(v_{m_1+r_1-1}) = y^0(v_2) = b_2 \quad \Rightarrow w^0_2 = (-2, 0), \]
\[ y^0(v_{m_2+r_2-1}) = y^0(v_3) = b_3 \quad \Rightarrow w^0_3 = (0, 1), \]
\[ y^0(v_{m_2}) = y^0(v_4) = 0 \quad \Rightarrow w^0_4 = (0, 0). \]

By solving

\[
\begin{bmatrix}
K^0 & g^0
\end{bmatrix} = \begin{bmatrix} w^0_0 & \ldots & w^0_4 \end{bmatrix} \begin{bmatrix} v_0 & \ldots & v_4 \\ 1 & \ldots & 1 \end{bmatrix}^{-1}
\]

we find that

\[ w^0(x) = K^0 x + g^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

Now we design $w^\infty(x) = K^\infty x + g^\infty$ such that

\[ y^\infty(v_0) = \hat{A}v_0 + \hat{a} \quad \Rightarrow w^\infty_0 = (0, 0), \]
\[ y^\infty(v_{m_1}) = y^\infty(v_1) = b_1 \quad \Rightarrow w^\infty_1 = (1, 0), \]
\[ y^\infty(v_{m_1+r_1-1}) = y^\infty(v_2) = 0 \quad \Rightarrow w^\infty_2 = (0, 0), \]
\[ y^\infty(v_{m_2+r_2-1}) = y^\infty(v_3) = 0 \quad \Rightarrow w^\infty_3 = (0, 0), \]
\[ y^\infty(v_{m_2}) = y^\infty(v_4) = b_4 \quad \Rightarrow w^\infty_4 = (0, -\frac{1}{2}). \]

By solving

\[
\begin{bmatrix}
K^\infty & g^\infty
\end{bmatrix} = \begin{bmatrix} w^\infty_0 & \ldots & w^\infty_4 \end{bmatrix} \begin{bmatrix} v_0 & \ldots & v_4 \\ 1 & \ldots & 1 \end{bmatrix}^{-1}
\]
we find that
\[
w^\infty(x) = K^\infty x + g^\infty = \begin{bmatrix} -4 & 0 & 2 & 2 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix}.
\]

We now extend the state with \(\alpha\) and we obtain
\[
w(x, \alpha) = (1 - \alpha)w^0(x) + \alpha w^\infty
\]
where
\[
\alpha(t) = 1 - e^{-ct}.
\]

Since we have used a feedback transformation, we must determine our actual input \(u\) to be
\[
u(x, \alpha) = K_1 x + g_1 + G_1 w(x, \alpha)
\]
\[
= K_1 x + g_1 + G_1 [(1 - \alpha)w^0(x) + \alpha w^\infty]
\]
\[
= (1 - \alpha) [(K_1 + G_1 K_0) x + (g_1 + G_1 g_0)]
\]
\[
+ \alpha [(K_1 + G_1 K_\infty) x + (g_1 + G_1 g_\infty)]
\]
\[
= (1 - \alpha)u^0(x) + \alpha u^\infty(x)
\]
\[
= (1 - \alpha) \left( \begin{bmatrix} 4 & 1 & -2 & -3 \\ -1 & 3 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)
\]
\[
+ \alpha \left( \begin{bmatrix} 8 & 2 & -4 & -5 \\ -3 & -1 & 2 & 2 \end{bmatrix} x + \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right)
\]
Location of the Equilibria

The vector field in $S$ is

$$y(x, \alpha) = Ax + a + Bu(x, \alpha)$$

$$= Ax + a + B \left[(1 - \alpha)u^0(x) + \alpha u^\infty(x)\right]$$

$$= (1 - \alpha) \left[Ax + a + Bu^0(x)\right] + \alpha \left[Ax + a + Bu^\infty(x)\right]$$

$$= (1 - \alpha) y^0(x) + \alpha y^\infty(x)$$

$$= (1 - \alpha) y^0(\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4)$$

$$+ \alpha y^\infty(\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4)$$

$$= (1 - \alpha) \left[y^0(\lambda_0 v_0) + \lambda_1 y^0(v_1) + \lambda_2 y^0(v_2) + \lambda_3 y^0(v_3) + \lambda_4 y^0(v_4)\right]$$

$$+ \alpha \left[y^\infty(\lambda_0 v_0) + \lambda_1 y^\infty(v_1) + \lambda_2 y^\infty(v_2) + \lambda_3 y^\infty(v_3) + \lambda_4 y^\infty(v_4)\right]$$

$$= (1 - \alpha) \left[y^0(\lambda_0 v_0) + \lambda_1 b_1 + \lambda_3 b_3\right] + \alpha \left[y^\infty(\lambda_0 v_0) + \lambda_2 b_2 + \lambda_4 b_4\right]$$

We know that $b_1$, $b_3$, and $y^0(v_0)$ are linearly independent. Therefore, $y(x, \alpha) = 0$ when

$$\lambda_0 = 0$$

$$(1 - \alpha)\lambda_1 - 2\alpha\lambda_2 = 0$$

$$(1 - \alpha)\lambda_3 - \frac{1}{2}\alpha\lambda_4 = 0$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1.$$
are labeled on the trajectory at the time that the trajectory exits $\mathcal{S}$. The magenta level values correspond to vertices in $\mathcal{E}^0$ and the cyan level values correspond to vertices in $\mathcal{E}^\infty$. The red plots show the locations of $\bar{x}_1(t)$ and $\bar{x}_2(t)$. Notice that all blue trajectories move up level sets of $\xi^*$ and the red trajectory moves down level sets of $\xi^*$ as we would expect.

![Simulation of control $u(x, \alpha)$ for Example 9. Level values $\xi^* \cdot \phi(t, v_i)$ versus time.](image)

Figure 8.2: Simulation of control $u(x, \alpha)$ for Example 9. Level values $\xi^* \cdot \phi(t, v_i)$ versus time.

### 8.1.3 Example 10

In this example we apply a feedback transformation in more detail than in previous examples. We accomplish this via a coordinate transformation to the canonical simplex and analyze our system in that domain. We also introduce another procedure to decompose $\mathcal{G}$ into is dependent cycles which exploits the structure of the canonical simplex.
Chapter 8. Examples II

Consider a simplex \( S \in \mathbb{R}^6 \) determined by the convex hull of \( v_0 = (1, -1, 1, 1, 0), v_1 = (1, -1, 0, 0, 1, 1), v_2 = (0, -1, 0, 0, 1, 1), v_3 = (0, 1, 1, 1, 0, 0), v_4 = (1, -1, 0, 1, 1, 1), v_5 = (1, 1, 0, 1, 0, -1), \) and \( v_6 = (-1, 1, 1, 1, 1, 0) \). The affine dynamics on \( S \) are

\[
\dot{x} = Ax + a + Bu = \begin{bmatrix}
-2 & -3 & -2 & 2 & -12 & 6 \\
4 & 6 & 5 & -14 & 10 & 1 \\
0 & -2 & -2 & 7 & -4 & 3 \\
1 & 0 & -1 & 3 & -1 & 3 \\
-2 & -3 & -2 & 7 & -4 & -1 \\
-4 & -3 & -1 & 9 & 0 & -4 \\
\end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} u.
\]

The normal vectors for \( S \) are \( h_0 = (0, \frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}), h_1 = (-1, -1, -\frac{1}{2}, 1, -1, -\frac{1}{2}), h_2 = (1, 1, \frac{1}{2}, 0, 1, \frac{1}{2}), h_3 = (0, 0, -\frac{1}{2}, 0, 1, -\frac{1}{2}), h_4 = (0, 0, \frac{1}{2}, -1, 0, -\frac{1}{2}), h_5 = (0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}), \) and \( h_6 = (0, -\frac{1}{2}, 0, 0, -1, 0) \).

Reach Control Indices

We have

\[
\mathcal{O} = \left\{ x \in \mathbb{R}^6 : \begin{bmatrix} 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix} x = \begin{bmatrix} 1 \\
1 \end{bmatrix} \right\}.
\]

Hence, \( \mathcal{G} = \mathcal{O} \cap S = \text{co} \{v_1, v_2, v_3, v_4, v_5\} \). We must now decompose \( \mathcal{G} \) into its cycles \( \mathcal{G}_1, \ldots, \mathcal{G}_p \). To accomplish this we will solve a series of feasibility problems. Define the matrix of normal vectors \( H_i = [h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n] \) which represent the invariance conditions at vertex \( v_i \). We seek \( u_{ij} \) such that \( H_i^T Bu_{ij} \leq 0 \) and \(-H_j^T Bu_{ij} \leq 0 \). If we can find such a \( u_{ij} \) then \( Bu_{ij} \) satisfies invariance conditions at \( v_i \) and \(-Bu_{ij} \) satisfies invariance conditions at \( v_j \). Note that if \( u_{ij} \) is feasible then the symmetric control \( u_{ji} = -u_{ij} \). We apply this procedure for all \( i \in I_\mathcal{G} \) and all \( j \in I_\mathcal{G} \setminus \{i\} \). We find \( u_{12} = (1, -1, -1), u_{13} = (1, 0, -2), u_{23} = (-2, 3, 1), u_{45} = (0, -1, 0) \) and all other \( u_{ij} \) are either infeasible or symmetric.
We have feasible \( u_{12}, u_{13}, \) and \( u_{23} \). Define

\[
\begin{align*}
b_1 &= Bu_{12} = (-2, 0, 1, 1, 0, -1), \\
b_2 &= Bu_{23} = (4, 4, 6, 6, -2, -6), \\
b_3 &= Bu_{31} = (2, -4, -9, -9, 2, 9).
\end{align*}
\]

The set \( \{b_1, b_2, b_3\} \) is linearly dependent, but has the property that if any vector is removed from the set, the remaining vectors are linearly independent. We have found our first cycle \( G_1 = \text{co} \{v_1, v_2, v_3\} \). We also have a feasible \( u_{45} \). Define \( b_4 = Bu_{45} \) and \( b_5 = Bu_{54} \). The set \( \{b_4, b_5\} \) is obviously linearly dependent and any singleton is linearly independent. We have found our second cycle \( G_2 = \text{co} \{v_4, v_5\} \). Since \( \mathcal{G} = \text{co} \{G_1, G_2\} \) we have successfully decomposed \( \mathcal{G} \). We now have that \( p = 2, r_1 = 3, \) and \( r_2 = 2 \).

**Coordinate Transformation**

We introduce a coordinate transformation from the general simplex, coordinates \( x \), to the canonical simplex, coordinates \( z \). This is useful because it simplifies the design of the feedback transformation to follow. Define

\[
Q = \begin{bmatrix}
(v_1 - v_0) & \ldots & (v_6 - v_0)
\end{bmatrix}
\]

and

\[ q = v_0. \]

Since we have assumed that the \( v_i \) are affinely independent, \( Q \) is invertible. Let \( z = (\lambda_1, \ldots, \lambda_6) \). For any \( x \in \mathcal{S} \) we can write

\[
x = \lambda_0 v_0 + \ldots + \lambda_6 v_6 = Qz + q
\]
where $\sum_{i=0}^{6} \lambda_i = 1$. We now have the mapping between general and canonical coordinate systems. Note that $v_0$ maps to $(0, 0, 0, 0, 0, 0)$ and $v_i$ maps to the elementary basis vector $e_i$ for $i = 1, \ldots, 6$. We will choose the normal vectors for the canonical simplex to be $\tilde{h}_0 = (1, 1, 1, 1, 1, 1)$ and $\tilde{h}_i = -e_i$. There is an important relationship between the normal vectors for the general simplex and the normal vectors for the canonical simplex. Namely,

$$
\begin{bmatrix}
  h_0 & \ldots & h_n
\end{bmatrix} = (Q^T)^{-1} \begin{bmatrix}
  \tilde{h}_0 & \ldots & \tilde{h}_n
\end{bmatrix}.
$$

Note that if the above relation does not hold for a set of normal vectors for the general simplex and a set of normal vectors for the canonical simplex, the invariance conditions are not equivalent. We now transform the dynamics in $x$ coordinates to dynamics in $z$ coordinates.

$$
\dot{x} = Ax + a + Bu
$$

$$
Q\dot{z} = A(Qz + q) + a + Bu
$$

$$
\dot{z} = Q^{-1}AQz + Q^{-1}(a + Aq) + Q^{-1}Bu
$$

$$
\dot{z} = \tilde{A}z + \tilde{a} + \tilde{B}u
$$

**Feedback Transformation**

Now that we have obtained the representation for the system on the general simplex in the coordinates of the canonical simplex we can proceed with the feedback transformation. We split his design into four parts: 1. design of $g_1$, 2. design of the first $p$ columns of $K_1$, 3. design of the last $n - p$ columns of $K_1$, and 4. design of $G_1$. Let

$$
u = K_1 x + g_1 + G_1 w
$$

$$
= K_1(Qz + q) + g_1 + G_1 w
$$

$$
= K_1^*z + g_1^c + G_1 w.
$$
Then

\[
\dot{z} = \tilde{A}z + \tilde{a} + \tilde{B}u \\
= (\tilde{A} + \tilde{B}K_1^c)z + \tilde{a} + \tilde{B}g_1^c + \tilde{B}G_1w \\
= \hat{A}z + \hat{a} + \hat{B}w.
\]

We need to find $K_1^c$, $g_1^c$, and $G_1$ such that

\[
\hat{A}e_i + \hat{a} = 0, \quad i \in I_{\tilde{G}} \\
\hat{A}e_i + \hat{a} \in C_i \setminus B, \quad i \notin I_{\tilde{G}} \setminus \{0\} \\
\hat{A}0 + \hat{a} \in C_0 \setminus B, \\
\hat{B} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}.
\]

To design $g_1^c$, we exploit the fact that $v_0$ maps to $(0,0,0,0,0,0)$ on the canonical simplex. Therefore, at $(0,0,0,0,0,0)$ we need $\hat{a} = \hat{a} + \hat{B}g_1^c \in \text{cone}\{S\} \setminus B$. We select $g_1^c = \frac{1}{3}(-5, 0, 5)$.

For vertices on $G$ we need $\hat{A}e_i + \hat{a} = 0$. This is equivalent to requiring $\hat{B}K_1^c e_i = -\hat{A}e_i - \hat{a} - \hat{B}g_1^c$. Therefore, for each vertex we need to assign $K_1^c e_i$ which is simply the corresponding column of $K_1^c$. For vertices off $G$, not including $v_0$, we need $\hat{A}e_i + \hat{a} \in C_i \setminus B$ which is equivalent to finding $K_1^c e_i$ which satisfies $r \leq 0$ and $\text{sum}(r) < 0$ where $r = H_i^T \left((\hat{A} + \hat{B}K_1^c)e_i + \hat{a} + \hat{B}g_1^c\right)$. Many solutions satisfy these constraints, but we will choose the solution that minimizes the Euclidean norm of $K_1^c e_i$ subject to $r \leq 0$ and $r \leq -1$. We now have

\[
K_1^c = \frac{1}{12} \begin{bmatrix}
-12 & -12 & 8 & 12 & -20 & 16 \\
24 & 24 & -3 & -66 & 18 & -15 \\
12 & 12 & -5 & -6 & 26 & -13
\end{bmatrix}.
\]
Lastly, we must find \( G_1 \). We need \( G_1 \) to transform \( B \) into \[
\begin{bmatrix}
b_1 & b_2 & b_4
\end{bmatrix}
\] which are given from the discussion regarding the decomposition of \( \mathcal{G} \). Therefore we select

\[
G_1 = \begin{bmatrix}
u_{12} & u_{23} & u_{45}
\end{bmatrix} = \begin{bmatrix}
1 & -2 & 0 \\
-1 & 3 & -1 \\
-1 & 1 & 0
\end{bmatrix}.
\]

We have now found \( K_1^c, g_1^c \), and \( G_1 \). We must now convert these results back to the general simplex to find that

\[
\dot{x} = Ax + a + Bu = \hat{A}x + \hat{a} + \hat{B}w
\]

\[
= \begin{bmatrix}
0 & -3 & -10 & 0 & -26 & 10 \\
0 & 6 & 4 & 0 & 20 & -4 \\
0 & -6 & -8 & 0 & -28 & 8 \\
0 & -3 & -7 & 0 & -20 & 7 \\
0 & -3 & -1 & 0 & -7 & 1 \\
0 & 3 & 7 & 0 & 20 & -7
\end{bmatrix} x + \frac{1}{6} \begin{bmatrix}
26 \\
-20 \\
28 \\
20 \\
7 \\
-20
\end{bmatrix} + \begin{bmatrix}
-2 & 4 & 0 \\
0 & 4 & 4 \\
1 & 6 & -1 \\
1 & 6 & 0 \\
0 & -2 & -2 \\
-1 & -6 & -3
\end{bmatrix} w.
\]

**Flow-Like Condition**

We will develop the flow-like condition in the same order as the theory developed in Section 7.3. Firstly, we have the following \( \mathcal{M} \)-matrices

\[
M_{1,3}^T = Y_{1,3}^T H_{1,3} = \begin{bmatrix}
3 & -2 & 0 \\
0 & 6 & -2 \\
-9 & 0 & 2
\end{bmatrix}, \quad M_{4,5}^T = Y_{4,5}^T H_{4,5} = \begin{bmatrix}
1 & -2 \\
-1 & 2
\end{bmatrix}
\]
whose kernels are spanned by $c_1 = \begin{bmatrix} 1 & \frac{3}{2} & \frac{9}{2} \end{bmatrix}^T$ and $c_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ respectively. We now have

$$\beta_1 = \begin{bmatrix} -h_1 & -h_2 & -h_3 \end{bmatrix} c_1 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 2 & -1 & -5 & 2 \end{bmatrix}^T,$$

$$\beta_2 = \begin{bmatrix} -h_4 & -h_5 \end{bmatrix} c_2 = \begin{bmatrix} 0 & 0 & -\frac{3}{4} & 1 & 0 & \frac{1}{4} \end{bmatrix}^T.$$ 

We now have a basis for $\text{Ker} (B^T) = \text{span}\{\beta_1, \beta_2, -h_6\}$. We can order the vertices with respect to $\beta_1$ and $\beta_2$. In fact, as far as the control design is concerned, we can stop developing the flow-like condition here since only $\beta_1$ and $\beta_2$ are required to design following feedbacks. However, we will continue the development. We obtain

$$v_3 = v_{m_1} \in \arg\max_{i \in I_{G_1}} \beta_1 \cdot v_i,$$

$$v_4 = v_{m_2} \in \arg\max_{i \in I_{G_2}} \beta_2 \cdot v_i,$$

$$v_1 = v_{m_1+r_1-1} \in \arg\min_{i \in I_{G_1}} \beta_1 \cdot v_i,$$

$$v_5 = v_{m_2+r_2-1} \in \arg\min_{i \in I_{G_2}} \beta_2 \cdot v_i.$$

We use these to define the sets

$$E^0 = \text{co} \{v_3, v_4\}, \quad E^\infty = \text{co} \{v_1, v_5\}.$$ 

We now define

$$\Pi_1 = \beta_1 \cdot (v_{m_1} - v_0) = \frac{9}{2}, \quad \pi_1 = \beta_1 \cdot (v_{m_1+r_1-1} - v_0) = 1,$$

$$\Pi_2 = \beta_2 \cdot (v_{m_2} - v_0) = 1, \quad \pi_2 = \beta_2 \cdot (v_{m_2+r_2-1} - v_0) = \frac{1}{2}.$$ 

We select $c_1 \in \left(\frac{1}{\Pi_1}, \frac{1}{\pi_1}\right) = \left(\frac{2}{9}, 1\right)$ and $c_2 \in \left(\frac{1}{\Pi_2}, \frac{1}{\pi_2}\right) = (1, 2)$. We can now determine that

$$\xi^1 = \beta_1 c_1 + \beta_2 c_2 = \frac{1}{2} \beta_1 + \frac{3}{2} \beta_2 = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} & 1 & -\frac{5}{2} & \frac{11}{8} \end{bmatrix}^T.$$
We check that indeed $\xi^1$ separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$

$$\xi^1 \cdot v_3 = \frac{5}{8}, \quad \xi^1 \cdot v_4 = -\frac{1}{8},$$

$$\xi^1 \cdot v_1 = -\frac{9}{8}, \quad \xi^1 \cdot v_5 = -\frac{13}{8}.$$

We now need to find $\xi^2 \in \text{Ker } (B^T)$ such that it provides a flow condition on $\mathcal{P} = \text{co } \{v_0, v_6\}$. We have the following velocity vectors

$$y_0 = \frac{1}{3} \begin{bmatrix} -20 & 8 & -16 & -14 & -1 & 14 \end{bmatrix}^T,$$

$$y_6 = \frac{1}{3} \begin{bmatrix} -26 & 20 & -28 & -20 & -7 & 20 \end{bmatrix}^T.$$

We see that

$$\xi^2 = \frac{1}{4} \beta_1 + \frac{3}{4} \beta_2 - h_6 = \frac{1}{16} \begin{bmatrix} -2 & 6 & -1 & 8 & -4 & 11 \end{bmatrix}^T.$$

provides the flow condition on $\mathcal{P}$. Lastly we require $\xi^*$. We need $\lambda \in (0, 1)$ close enough to 0 such that $\xi^*$ still separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$. We find that

$$\xi^* = (1 - \lambda)\xi^1 + \lambda \xi^2 = \frac{3}{4} \xi^1 + \frac{1}{4} \xi^2 = \frac{1}{8} \begin{bmatrix} -2 & 0 & -1 & 8 & -16 & 11 \end{bmatrix}^T.$$

provides the flow-like condition we have been seeking.
First we design $w^0(x) = K^0 x + g^0$ such that

\[
\begin{align*}
g^0(v_0) &= \hat{A}v_0 + \hat{a} \quad \Rightarrow w^0_0 = (0, 0, 0), \\
y^0(v_1) &= b_1 \quad \Rightarrow w^0_1 = (1, 0, 0), \\
y^0(v_2) &= b_2 \quad \Rightarrow w^0_2 = (0, 1, 0), \\
y^0(v_3) &= 0 \quad \Rightarrow w^0_3 = (0, 0, 0), \\
y^0(v_4) &= 0 \quad \Rightarrow w^0_4 = (0, 0, 0), \\
y^0(v_5) &= b_5 \quad \Rightarrow w^0_5 = (0, 1, 0), \\
y^0(v_6) &= \hat{A}v_6 + \hat{a} \quad \Rightarrow w^0_6 = (0, 0, 0).
\end{align*}
\]

By solving

\[
\begin{bmatrix} K^0 & g^0 \end{bmatrix} = \begin{bmatrix} w^0_0 & \ldots & w^0_6 \end{bmatrix} \begin{bmatrix} v_0 & \ldots & v_6 \\ 1 & \ldots & 1 \end{bmatrix}^{-1}
\]

we find that

\[
w^0(x) = K^0 x + g^0 = \frac{1}{2} \begin{bmatrix} 2 & 2 & 1 & -2 & 2 & 1 \\ -2 & -2 & -1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} x + \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.
\]
Now we design \( w^\infty(x) = K^\infty x + g^\infty \) such that

\[
\begin{align*}
y^\infty(v_0) &= \hat{A}v_0 + \hat{a} \quad \Rightarrow w_0^\infty = (0,0,0), \\
y^\infty(v_1) &= 0 \quad \Rightarrow w_1^\infty = (0,0,0), \\
y^\infty(v_2) &= b_2 \quad \Rightarrow w_2^\infty = (0,1,0), \\
y^\infty(v_3) &= b_3 \quad \Rightarrow w_3^\infty = (-3,-1,0), \\
y^\infty(v_4) &= b_4 \quad \Rightarrow w_4^\infty = (0,0,1), \\
y^\infty(v_5) &= 0 \quad \Rightarrow w_5^\infty = (0,0,0), \\
y^\infty(v_6) &= \hat{A}v_6 + \hat{a} \quad \Rightarrow w_6^\infty = (0,0,0).
\end{align*}
\]

By solving

\[
\begin{bmatrix} K^\infty & g^\infty \end{bmatrix} = \begin{bmatrix} w_0^\infty & \ldots & w_6^\infty \end{bmatrix} \begin{bmatrix} v_0 & \ldots & v_6 \\ 1 & \ldots & 1 \end{bmatrix}^{-1}
\]

we find that

\[
w^\infty(x) = K^\infty x + g^\infty = \frac{1}{2} \begin{bmatrix} 0 & 0 & -3 & 0 & 6 & -3 \\ -2 & -2 & -2 & 0 & 0 & -2 \\ 0 & 0 & -1 & 2 & 0 & 1 \end{bmatrix} x + \frac{1}{2} \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}.
\]

We now extend the state with \( \alpha \) and we obtain

\[
w(x, \alpha) = (1 - \alpha)w^0(x) + \alpha w^\infty
\]

where

\[
\alpha(t) = 1 - e^{-ct}.
\]

Since we have used a feedback transformation, we must determine our actual input \( u \)
to be

\[ u(x, \alpha) = K_1 x + g_1 + G_1 w(x, \alpha) \]

\[ = K_1 x + g_1 + G_1 [(1 - \alpha) w^0(x) + \alpha w^\infty] \]

\[ = (1 - \alpha) [(K_1 + G_1 K_0) x + (g_1 + G_1 g_0)] \]

\[ + \alpha [(K_1 + G_1 K_\infty) x + (g_1 + G_1 g_\infty)] \]

\[ = (1 - \alpha) u^0(x) + \alpha u^\infty(x) \]

\[ \frac{1}{12} (1 - \alpha) \left[ \begin{array}{cccccc} 24 & 24 & 26 & 0 & 16 & 34 \\ -24 & -24 & -33 & -54 & -30 & -45 \\ -12 & -12 & -17 & 6 & 2 & -25 \end{array} \right] x + \left[ \begin{array}{c} -62 \\ 117 \\ 29 \end{array} \right] \]

\[ + \frac{1}{12} \alpha \left[ \begin{array}{cccccc} 12 & 12 & 14 & 12 & 16 & 22 \\ -12 & -12 & -15 & -78 & -18 & -39 \\ 0 & 0 & 1 & -6 & -10 & -7 \end{array} \right] x + \left[ \begin{array}{c} -62 \\ 111 \\ 35 \end{array} \right] \]

**Simulation**

We simulate the control \( u(x, \alpha) \). Figure 8.3 provides a plot of the evolution of the level values of \( \xi^* \cdot \phi(t, v_i) \) versus time. Each blue trajectory corresponds to \( \phi(t, v_i) \) for \( i = 0, \ldots, n \) and are labeled on the trajectory at the time that the trajectory exits \( S \). The magenta level values correspond to vertices in \( E^0 \) and the cyan level values correspond to vertices in \( E^\infty \). The red plots show the locations of \( \bar{x}_1(t) \) and \( \bar{x}_2(t) \). Notice that all blue trajectories move up level sets of \( \xi^* \) and the red trajectory moves down level sets of \( \xi^* \) as we would expect.

**8.1.4 Example 11**

In this example we explore a seventh order system on a general simplex. At the end of the discussion for this example, we provide an alternative and more intuitive expression
Figure 8.3: Simulation of control $u(x, \alpha)$ for Example 10. Level values $\xi^* \cdot \phi(t, v_i)$ versus time.

for the location of the equilibria in $S$ as a function of $\alpha$. Consider a simplex $S \in \mathbb{R}^7$ determined by the convex hull of $v_0 = (0, 1, 0, 1, -1, -1, 0), v_1 = (1, 1, -1, -1, 0, -1, -1), v_2 = (0, 0, 0, 0, 0, 0, 0), v_3 = (1, 1, 0, -1, 0, 0, -1), v_4 = (1, 0, 1, -1, 1, 1, 0), v_5 = (0, 1, -1, 0, 0, -1, -1).
The normal vectors for $S$ are $h_0 = (0, 0, -1, 0, 0, 1, 0), h_1 = 0, -1, 1, 1, 0, 0, -2), h_2 = (-1, 2, 1, -1, 1, -1, 1), h_3 = (1, -2, 1, 1, -2, -1), h_4 = (-1, 2, -2, -1, -1, 2, 2), h_5 = (0, 1, -1, -1, -1, 1, 2), h_6 = (1, -1, 0, 1, 0, -1), and h_7 = (0, -1, 1, 0, 0, -1, 1). We have

$$\mathcal{O} = \left\{ x \in \mathbb{R}^7 : \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

and $\mathcal{G} = \text{co}\{v_1, v_2, v_3, v_4, v_5, v_6\}$. We perform a decomposition of $\mathcal{G}$ using the procedure outlined in Section 8.1.3 and obtain that $\mathcal{G}_1 = \text{co} \{v_1, v_3\}$ and $\mathcal{G}_2 = \text{co} \{v_1, v_5, v_6\}$.

A feedback transformation needs to be performed. Unlike the previous example we
will simply provide $K_1$, $g_1$, and $G_1$. Let

$$u = K_1x + g_1 + G_1w$$

$$= \frac{1}{24} \begin{bmatrix} 0 & 36 & -120 & -12 & -72 & 108 & 84 \\ 6 & 25 & -90 & -21 & -54 & 57 & 109 \\ -60 & -24 & 312 & 36 & 204 & -252 & -240 \\ -12 & 29 & -54 & -15 & -36 & 63 & 29 \\ -12 & 12 & 24 & 0 & 12 & 0 & -36 \end{bmatrix} x + \frac{1}{4} \begin{bmatrix} 4 \\ 4 \\ -14 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} 0 & 4 & 4 & 4 \\ -2 & 5 & 1 & 1 \\ 0 & -12 & -8 & -4 \\ 2 & 3 & 5 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} w.$$ 

Note that $G_1 \in \mathbb{R}^{5 \times 4}$ since $B$ has an extra degree of freedom that we don’t need. The
transformed system is now

\[
\dot{x} = \hat{A}x + \hat{a} + \hat{B}w
\]

\[
= \frac{1}{3} \begin{bmatrix}
0 & 3 & -3 & 0 & 0 & 3 & 3 \\
0 & -17 & 6 & 0 & 0 & -6 & -17 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -29 & 18 & 0 & 0 & -18 & -29 \\
0 & 23 & -12 & 0 & 0 & 12 & 23 \\
0 & 20 & -9 & 0 & 0 & 9 & 20 \\
0 & -6 & 6 & 0 & 0 & -6 & -6
\end{bmatrix} x + \frac{1}{3} \begin{bmatrix}
3 & 1 & -1 & -1 \\
2 & 3 & 1 & -2 \\
3 & 0 & -3 & 5 \\
4 & 1 & 0 & 3 \\
-1 & -2 & 0 & -2 \\
2 & -2 & -2 & 2 \\
3 & -1 & -2 & 5
\end{bmatrix} + w.
\]

Flow-Like Condition

We will develop the flow-like condition in the same order as the theory developed in Section 7.3. Firstly, we have the following \(M\)-matrices

\[
M_{1,3}^T = Y_{1,3}^T H_{1,3} = \begin{bmatrix}
9 & 0 & -2 \\
0 & 3 & -1 \\
-18 & 0 & 4
\end{bmatrix}, \quad M_{4,6}^T = Y_{4,6}^T H_{4,6} = \begin{bmatrix}
1 & -2 & 0 \\
0 & 4 & -1 \\
0 & -8 & 2
\end{bmatrix}
\]
whose kernels are spanned by $c_1 = \begin{bmatrix} 1 & \frac{3}{2} & 9 \frac{3}{2} \end{bmatrix}^T$ and $c_2 = \begin{bmatrix} 1 & \frac{1}{2} & 2 \end{bmatrix}^T$ respectively. We now have

$$\beta_1 = \begin{bmatrix} -h_1 & -h_2 & -h_3 \end{bmatrix}, \quad c_1 = \begin{bmatrix} -3 & 7 & -7 & -4 & -6 & \frac{31}{2} & 5 \end{bmatrix}^T,$$

$$\beta_2 = \begin{bmatrix} -h_4 & -h_5 & -h_6 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{5}{2} & -1 \end{bmatrix}^T.$$ 

We now have a basis for $\text{Ker}(B^T) = \text{span}\{\beta_1, \beta_2, -h_7\}$. We can order the vertices with respect to $\beta_1$ and $\beta_2$. In fact, as far as the control design is concerned, we can stop developing the flow-like condition here since only $\beta_1$ and $\beta_2$ are required to design following feedbacks. However, we will continue the development. We obtain

$$v_3 = v_{m_1} \in \arg \max_{i \in I_{G_1}} \beta_1 \cdot v_i, \quad v_1 = v_{m_1 + r_1 - 1} \in \arg \min_{i \in I_{G_1}} \beta_1 \cdot v_i,$$

$$v_6 = v_{m_2} \in \arg \max_{i \in I_{G_2}} \beta_2 \cdot v_i, \quad v_5 = v_{m_2 + r_2 - 1} \in \arg \min_{i \in I_{G_2}} \beta_2 \cdot v_i.$$

We use these to define the sets

$$\mathcal{E}^0 = \text{co} \{v_3, v_6\}, \quad \mathcal{E}^\infty = \text{co} \{v_1, v_5\}.$$

We now define

$$\Pi_1 = \beta_1 \cdot (v_{m_1} - v_0) = \frac{9}{2}, \quad \pi_1 = \beta_1 \cdot (v_{m_1 + r_1 - 1} - v_0) = 1,$$

$$\Pi_2 = \beta_2 \cdot (v_{m_2} - v_0) = 2, \quad \pi_2 = \beta_2 \cdot (v_{m_2 + r_2 - 1} - v_0) = \frac{1}{2}.$$

We select $c_1 \in \left(\frac{1}{\Pi_1}, \frac{1}{\pi_1}\right) = \left(\frac{2}{9}, 1\right)$ and $c_2 \in \left(\frac{1}{\Pi_2}, \frac{1}{\pi_2}\right) = \left(\frac{1}{2}, 2\right)$. We can now determine that

$$\xi^1 = \beta_1 c_1 + \beta_2 c_2 = \frac{1}{2} \beta_1 + \beta_2 = \frac{1}{4}\begin{bmatrix} -10 & 12 & -4 & -10 & -6 & 11 & 6 \end{bmatrix}^T.$$
We check that indeed $\xi^1$ separates $E^0$ and $E^\infty$

$$\xi^1 \cdot v_3 = \frac{3}{2}, \quad \xi^1 \cdot v_6 = -\frac{5}{4},$$

$$\xi^1 \cdot v_1 = -\frac{1}{4}, \quad \xi^1 \cdot v_5 = -\frac{1}{4}.$$ 

We now need to find $\xi^2 \in \text{Ker } (B^T)$ such that it provides a flow condition on $P = \text{co } \{v_0, v_7\}$. We have the following velocity vectors

$$y_0 = \frac{1}{3} \begin{bmatrix} 0 & -11 & 0 & -11 & 11 & 0 \end{bmatrix}^T,$$

$$y_7 = \frac{1}{3} \begin{bmatrix} 3 & -17 & 0 & -29 & 23 & 20 & -6 \end{bmatrix}^T.$$ 

We see that

$$\xi^2 = \beta_1 + \beta_2 - 3h_7 = \begin{bmatrix} -4 & \frac{19}{2} & -\frac{15}{2} & -\frac{9}{2} & -\frac{9}{2} & 11 & 7 \end{bmatrix}^T.$$ 

provides the flow condition on $P$. Lastly we require $\xi^*$. We need $\lambda \in (0, 1)$ close enough to 0 such that $\xi^*$ still separates $E^0$ and $E^\infty$. We find that

$$\xi^* = (1 - \lambda)\xi^1 + \lambda\xi^2 = \frac{1}{2} \xi^1 + \frac{1}{2} \xi^2 = \frac{1}{8} \begin{bmatrix} -26 & 50 & -34 & -28 & -24 & 55 & 34 \end{bmatrix}^T.$$ 

provides the flow-like condition we have been seeking.
Designing the Control

First we design \( w^0(x) = K^0 x + g^0 \) such that

\[
\begin{align*}
\text{y}^0(v_0) &= \hat{A}v_0 + \hat{a} \quad \Rightarrow w^0_0 = (0, 0, 0, 0), \\
y^0(v_1) &= b_1 = (-7, 0, 9, 14, -7, 2, 7) \quad \Rightarrow w^0_1 = (3, 2, 0, 0), \\
y^0(v_2) &= b_2 = (1, 3, 0, 1, -2, -2, -1) \quad \Rightarrow w^0_2 = (0, 1, 0, 0), \\
y^0(v_3) &= (0, 0, 0, 0, 0, 0) \quad \Rightarrow w^0_3 = (0, 0, 0, 0), \\
y^0(v_4) &= b_4 = (-1, 1, -3, 0, 0, -2, -2) \quad \Rightarrow w^0_4 = (0, 0, 1, 0), \\
y^0(v_5) &= b_5 = (-1, -2, 5, 3, -2, 2, 5) \quad \Rightarrow w^0_5 = (0, 0, 0, 1), \\
y^0(v_6) &= (0, 0, 0, 0, 0, 0) \quad \Rightarrow w^0_6 = (0, 0, 0, 0), \\
y^0(v_7) &= \hat{A}v_7 + \hat{a} \quad \Rightarrow w^0_7 = (0, 0, 0, 0).
\end{align*}
\]

By solving

\[
\begin{bmatrix}
K^0 & g^0
\end{bmatrix} = \begin{bmatrix}
w^0_0 & \ldots & w^0_7
\end{bmatrix} \begin{bmatrix}
v_0 & \ldots & v_7 \\
1 & \ldots & 1
\end{bmatrix}^{-1}
\]

we find that

\[
w^0(x) = K^0 x + g^0 = \frac{1}{2} \begin{bmatrix}
0 & 3 & -3 & -3 & 0 & 0 & 6 \\
1 & 0 & -3 & -1 & -1 & 1 & 3 \\
1 & -2 & 2 & 1 & 1 & -2 & -2 \\
0 & -1 & 1 & 1 & -1 & -2 & 0
\end{bmatrix} x + \frac{1}{2} \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}.
\]
Chapter 8. Examples II

Now we design \( w^\infty(x) = K^\infty x + g^\infty \) such that

\[
\begin{align*}
g^\infty(v_0) &= \hat{A}v_0 + \hat{a} \quad \Rightarrow w_0^\infty = (0, 0, 0, 0), \\
g^\infty(v_1) &= (0, 0, 0, 0, 0, 0) \quad \Rightarrow w_1^\infty = (0, 0, 0, 0), \\
g^\infty(v_2) &= b_2 = (1, 3, 0, 1, -2, -2, -1) \quad \Rightarrow w_2^\infty = (0, 1, 0, 0), \\
g^\infty(v_3) &= b_3 = -2b_1 = (14, 0, -18, -28, 14, -4, -14) \quad \Rightarrow w_3^\infty = (-6, -4, 0, 0), \\
g^\infty(v_4) &= b_4 = (-1, 1, -3, 0, 0, -2, -2) \quad \Rightarrow w_4^\infty = (0, 0, 1, 0), \\
g^\infty(v_5) &= (0, 0, 0, 0, 0, 0) \quad \Rightarrow w_5^\infty = (0, 0, 0, 0), \\
g^\infty(v_6) &= b_6 = -2b_5 = (2, 4, -10, -6, 4, -4, -10) \quad \Rightarrow w_6^\infty = (0, 0, 0, -2), \\
g^\infty(v_7) &= \hat{A}v_7 + \hat{a} \quad \Rightarrow w_7^\infty = (0, 0, 0, 0).
\end{align*}
\]

By solving

\[
\begin{bmatrix}
K^\infty & g^\infty
\end{bmatrix} = \begin{bmatrix}
w_0^\infty & \ldots & w_7^\infty
\end{bmatrix}^{-1} \begin{bmatrix}
v_0 & \ldots & v_7 \\
1 & \ldots & 1
\end{bmatrix}
\]

we find that

\[
w^\infty(x) = K^\infty x + g^\infty = \frac{1}{2} \begin{bmatrix}
6 & -12 & 6 & 6 & -12 & -6 \\
5 & -10 & 3 & 5 & 3 & -7 & -5 \\
1 & -2 & 2 & 1 & 1 & -2 & -2 \\
2 & -2 & 0 & 2 & 0 & 0 & -2
\end{bmatrix} x + \frac{1}{2} \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}.
\]

We now extend the state with \( \alpha \) and we obtain

\[
w(x, \alpha) = (1 - \alpha)w^0(x) + \alpha w^\infty
\]

where

\[
\alpha(t) = 1 - e^{-\epsilon t}.
\]
Since we have used a feedback transformation, we must determine our actual input \( u \) to be

\[
u(x, \alpha) = K_1 x + g_1 + G_1 w(x, \alpha)
\]

\[
= K_1 x + g_1 + G_1 \left[ (1 - \alpha)w^0(x) + \alpha w^\infty \right]
\]

\[
= (1 - \alpha) \left[ (K_1 + G_1 K_0)x + (g_1 + G_1 g_0) \right]
\]

\[
+ \alpha \left[ (K_1 + G_1 K_\infty)x + (g_1 + G_1 g_\infty) \right]
\]

\[
= (1 - \alpha)u^0(x) + \alpha u^\infty(x)
\]

\[
= \frac{1}{24} (1 - \alpha) \begin{pmatrix}
24 & 0 & -120 & 0 & -60 & 84 & 72 \\
24 & -2 & -108 & -12 & -63 & 63 & 106 \\
-120 & 36 & 360 & 36 & 204 & -228 & -276 \\
12 & 8 & -60 & -18 & -21 & 33 & 44 \\
0 & -12 & 48 & 12 & 24 & -24 & -60
\end{pmatrix} x 
\]

\[
+ \frac{1}{24} \alpha \begin{pmatrix}
96 & -132 & -60 & 84 & -24 & 0 & -24 \\
54 & -65 & -75 & 27 & -42 & 18 & 58 \\
-288 & 408 & 156 & -192 & 72 & 48 & 12 \\
102 & -181 & 39 & 99 & 42 & -102 & -100 \\
0 & -12 & 48 & 12 & 24 & -24 & -60
\end{pmatrix} x
\]
Location of the Equilibria

The vector field in $S$ is

$$y(x, \alpha) = (1 - \alpha)y^0(x) + \alpha y^\infty(x)$$

$$= (1 - \alpha)y^0 \left( \sum_{i=0}^{7} \lambda_i v_i \right) + \alpha y^\infty \left( \sum_{i=0}^{7} \lambda_i v_i \right)$$

$$= (1 - \alpha) \sum_{i=0}^{7} \lambda_i y^0(v_i) + \alpha \sum_{i=0}^{7} \lambda_i y^0(v_i)$$

$$= (1 - \alpha) \left[ \lambda_0 y^0(v_0) + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_4 b_4 + \lambda_5 b_5 + \lambda_7 y^0(v_7) \right]$$

$$+ \alpha \left[ \lambda_0 y^\infty(v_0) + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 + \lambda_6 b_6 + \lambda_7 y^\infty(v_7) \right]$$

$$= \lambda_0 y(v_0) + [(1 - \alpha) \lambda_1 - 2\alpha \lambda_3] b_1 + \lambda_2 b_2 + \lambda_4 b_4 + [(1 - \alpha) \lambda_5 - 2\alpha \lambda_6] b_5 + \lambda_7 y(v_7).$$

Since $y(v_0), b_1, b_2, b_4, b_5, y(v_7)$ are linearly independent, their coefficients must be zero if $x$ is an equilibrium. We immediately obtain that

$$\lambda_0 = 0$$

$$(1 - \alpha) \lambda_1 - 2\alpha \lambda_3 = 0$$

$$\lambda_2 = 0$$

$$\lambda_4 = 0$$

$$(1 - \alpha) \lambda_5 - 2\alpha \lambda_6 = 0$$

$$\lambda_7 = 0$$

$$\sum_{i=0}^{7} \lambda_i = 1.$$
Essentially we are isolating $\bar{x}_{G_1}$, the equilibrium on $G_1$. we have

$$
\lambda_1 = \frac{2\alpha}{1 + \alpha}, \\
\lambda_3 = \frac{1 - \alpha}{1 + \alpha}, \\
\bar{x}_{G_1} = \lambda_1 v_1 + \lambda_3 v_3.
$$

We do the same for $\bar{x}_{G_2}$ to obtain

$$
\lambda_5 = \frac{2\alpha}{1 + \alpha}, \\
\lambda_6 = \frac{1 - \alpha}{1 + \alpha}, \\
\bar{x}_{G_2} = \lambda_5 v_5 + \lambda_6 v_6.
$$

The set of all equilibria in $\mathcal{S}$ is $\{\bar{x}_{G_1}, \bar{x}_{G_2}\}$. Now we have a more intuitive expression for an equilibrium in $\mathcal{S}$.

$$
\bar{x}(t) = (1 - \gamma) \left[ \frac{2\alpha}{1 + \alpha} v_1 + \frac{1 - \alpha}{1 + \alpha} v_3 \right] + \gamma \left[ \frac{2\alpha}{1 + \alpha} v_5 + \frac{1 - \alpha}{1 + \alpha} v_6 \right], \quad \gamma \in [0, 1].
$$

**Simulation**

We simulate the control $u(x, \alpha)$. Figure 8.4 provides a plot of the evolution of the level values $\xi^* \cdot \phi(t, v_i)$ versus time. Each blue trajectory corresponds to $\phi(t, v_i)$ for $i = 0, \ldots, n$ and are labeled on the trajectory at the time that the trajectory exits $\mathcal{S}$. The magenta level values correspond to vertices in $E^0$ and the cyan level values correspond to vertices in $E^\infty$. The red plots show the locations of $\bar{x}_1(t)$ and $\bar{x}_2(t)$. Notice that all blue trajectories move up level sets of $\xi^*$ and the red trajectory moves down level sets of $\xi^*$ as we would expect.
Figure 8.4: Simulation of control $u(x, \alpha)$ for Example 11. Level values $\xi^* \cdot \phi(t, v_i)$ versus time.
Chapter 9

Application

9.1 Material Transfer System

We apply the multiaffine feedback technique developed in the thesis to a simplified model of a material transfer system. Instead of focusing on a single simplex, we turn our attention to a polytope. In developing this example, we will see that new techniques are required to generate continuous vector fields even in the most simple of problems.

Consider the conveyor system shown graphically in Figure 9.1. Its objective is to move goods in a production facility between two locations autonomously, in this case two conveyors, and it is constrained to move on a linear track. We choose a double integrator to model the dynamics of this system and the state space representation is

\[
\dot{x} = Ax + a + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]  

We impose the following specifications on System 9.1. First, we have a safety specification limiting the speed of the cart: \(|x_2| \leq 3\) m/s. Second, we want to limit the motion of the cart such that it does not contact the conveyor: \(|x_1| \leq 3\) m. These specifications partition the state space, \(\mathbb{R}^2\), into a square region. However, we don’t want to have a
Figure 9.1: Schematic of material transfer system.

non-zero speed when adjacent to the conveyors, otherwise the cart would cause damage to the system. Thus, we add the deceleration specification: $|x_1 + x_2| \leq 3$. The last specification that we need to add is the liveness specification. We want the cart to move from position A to position B and back again. In other words, we do not want the cart to have non-zero speed around $x = 0$ and therefore we partition out this region with: $\|x\|_1 \geq 1$. We summarize our constraints as follows:

1. Safety:
   
   (a) Position: $|x_1| \leq 3$
   
   (b) Speed: $|x_2| \leq 3$
   
   (c) Deceleration: $|x_1 + x_2| \leq 3$

2. Liveness: $|x_1| + |x_2| \geq 1$.

Before developing our solution to this problem, we remark that if the safety and liveness constraints were not viewed as hard constraints, then an approximate and ad hoc control design method would be to apply a switching controller that switches between
several desired reference speeds of a cruise controller. As the cart approaches either conveyor belt, the reference speeds would be reduced in steps to zero. The advantage of this design is that it is simple and intuitive. The disadvantage is that a safety guarantee cannot be provided. Moreover, the design may not be robust to large variations in system parameters, such as mass, since the switching regime is designed by trial and error for a nominal plant.

We now explain the design methodology based on piecewise affine and multiaffine feedback. The specifications comprise a set of linear inequalities which generate a polytope $\mathcal{P}$. We have the following set of vertices for $\mathcal{P}$: $v_1 = (0, 3)$, $v_2 = (0, 1)$, $v_3 = (1, 0)$, $v_4 = (3, 0)$, $v_5 = (3, -3)$, $v_6 = (0, -1)$, $v_7 = (0, -3)$, $v_8 = (-1, 0)$, $v_9 = (-3, 0)$ and $v_{10} = (-3, 3)$. $\mathcal{P}$ and its vertices are shown in Figure 9.2.

Figure 9.2: Polytope $\mathcal{P}$ shaded in grey and shown with its vertices $v_i$. 
\( \mathcal{P} \) is a polytope in the sense of mathematical topology in which it can be defined as a union of simplices. Therefore we can triangulate \( \mathcal{P} \) into a set of full-dimensional simplices. We will perform this triangulation using only the vertices of \( \mathcal{P} \) and no further subdivision. We naively triangulate \( \mathcal{P} \) into the ten simplices \( \mathcal{S}_1 = \text{co} \{ v_2, v_1, v_3 \} \), \( \mathcal{S}_2 = \text{co} \{ v_1, v_4, v_3 \} \), \( \mathcal{S}_3 = \text{co} \{ v_4, v_5, v_3 \} \), \( \mathcal{S}_4 = \text{co} \{ v_3, v_5, v_6 \} \), \( \mathcal{S}_5 = \text{co} \{ v_5, v_7, v_6 \} \), \( \mathcal{S}_6 = \text{co} \{ v_6, v_7, v_8 \} \), \( \mathcal{S}_7 = \text{co} \{ v_7, v_9, v_8 \} \), \( \mathcal{S}_8 = \text{co} \{ v_9, v_{10}, v_8 \} \), \( \mathcal{S}_9 = \text{co} \{ v_8, v_{10}, v_2 \} \), and \( \mathcal{S}_{10} = \text{co} \{ v_{10}, v_1, v_2 \} \). Note that \( \mathcal{P} = \bigcup_{i=1}^{10} \mathcal{S}_i \). The triangulation of \( \mathcal{P} \) is shown in Figure 9.3.

![Figure 9.3: Triangulation of \( \mathcal{P} \) into simplices \( \mathcal{S}_i \).](image)

On each simplex we will develop a control law which solves the RCP for that particular simplex. As such, each control is only valid in a certain portion of the state space. To handle the switching between the controllers we must have a supervisor to coordinate when each control is valid. We can model this supervisor as a discrete event system.
The states of the DES coincide with the membership of the continuous time state to a particular simplex and these are labelled $S_i$ for $i = 1, \ldots, 10$. We also add two more states to the DES to introduce a time delay for loading and offloading the material transfer system and these are labelled $D_1$ and $D_2$. The time delay would be implemented by stabilizing an equilibrium corresponding to a position sufficient near each conveyor belt and a velocity of zero. We do not include this design here since the approach is standard, but it is clear that such a controller could be included as part of the suite of controllers that the supervisor invokes. We must now define the transitions between the twelve states of the DES. We first define the exit facet, or target set, for each simplex. For $i = 1, \ldots, 9$ the exit facet for simplex $S_i$ is defined by $F^{S_i}_0 := S_i \cap S_{i+1}$ and for simplex $S_{10}$ we define the exit facet to be $F^{S_{10}}_0 := S_{10} \cap S_1$. We list the following definitions of the DES transitions

- $\sigma_1 :=$ when $x$ leaves $F^{S_1}_0$ and enters $S_2$
- $\sigma_2 :=$ when $x$ enters $F^{S_2}_0$
- $\sigma_3 :=$ when material is offloaded from the cart
- $\sigma_4 :=$ when $x$ leaves $F^{S_3}_0$ and enters $S_4$
- $\sigma_5 :=$ when $x$ leaves $F^{S_4}_0$ and enters $S_5$
- $\sigma_6 :=$ when $x$ leaves $F^{S_6}_0$ and enters $S_6$
- $\sigma_7 :=$ when $x$ leaves $F^{S_6}_0$ and enters $S_7$
- $\sigma_8 :=$ when $x$ enters $F^{S_7}_0$
- $\sigma_9 :=$ when material is loaded onto the cart
- $\sigma_{10} :=$ when $x$ leaves $F^{S_8}_0$ and enters $S_9$
- $\sigma_{11} :=$ when $x$ leaves $F^{S_{10}}_0$ and enters $S_{10}$
- $\sigma_{12} :=$ when $x$ leaves $F^{S_{10}}_0$ and enters $S_1$

The state and transition diagram of the DES is shown in Figure 9.4.
9.1.1 Affine State Feedbacks

By examining System 9.1 we have that

\[ \mathcal{O} = \{ x : x_2 = 0 \} \]

and

\[ \mathcal{G}_{S_i} = S_i \cap \mathcal{O}. \]

It can be checked that invariance conditions are solvable for every \( S_i \) in \( \mathcal{P} \). It is obvious that \( \mathcal{G}_{S_5} = \emptyset \) and \( \mathcal{G}_{S_{10}} = \emptyset \). Therefore, by Theorem 4.1, there exists an affine state feedback solving the RCP for \( S_5 \) and \( S_{10} \). From examining Figure 9.3 we can see that \( \mathcal{G}_{S_i} \neq \emptyset \) and \( \mathcal{B} \cap \text{cone}\{S_i\} \neq 0 \) for \( i \in \{1, 3, 4, 6, 8, 9\} \). Therefore, by Theorem 4.2,
there exists an affine state feedback solving the RCP for $S_1, S_3, S_4, S_6, S_8,$ and $S_9$. The control laws for simplices where the RCP is solvable simply using affine state feedback are summarized as follows

$$u_1(x) = \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} \end{bmatrix} x + \frac{7}{2}$$

$$u_3(x) = \begin{bmatrix} 0 & -\frac{2}{3} \end{bmatrix} x - 1$$

$$u_4(x) = \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \end{bmatrix} x - \frac{7}{5}$$

$$u_5(x) = \begin{bmatrix} -1 & -\frac{5}{2} \end{bmatrix} x - \frac{7}{2}$$

$$u_6(x) = \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} \end{bmatrix} x - \frac{7}{2}$$

$$u_8(x) = \begin{bmatrix} 0 & -\frac{2}{3} \end{bmatrix} x + 1$$

$$u_9(x) = \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \end{bmatrix} x + \frac{7}{5}$$

$$u_{10}(x) = \begin{bmatrix} -1 & -\frac{5}{2} \end{bmatrix} x + \frac{7}{2}$$

### 9.1.2 Time-Varying Affine State Feedbacks

It remains to design control laws for $S_2$ and $S_7$ where affine state feedbacks will fail to yield a solution to the RCP. We have that $G_{S_i} \neq \emptyset$ and $\mathcal{B} \cap \text{cone}(S_i) = 0$ for $i \in \{2, 7\}$. For $S_2$ and $S_7$ one can check that Assumption 7.1 is satisfied. Then by Theorem 7.2, the RCP is solvable by time-varying affine feedback. By inspecting the affine state feedback control laws listed above one can see that there is a symmetry between controls $u_1(x)$ and $u_6(x)$, $u_3(x)$ and $u_8(x)$, $u_4(x)$ and $u_9(x)$, and $u_5(x)$ and $u_{10}(x)$. The symmetry is that $g_i = -g_j$ and $K_i = K_j$ for these pairs. Therefore, since we know a solution exists for $S_2$ and this solution is symmetric for $S_7$, we will proceed with the design for the
time-varying affine control law for only $S_2$.

$S_2$ is generated by the convex hull of $v_0 = (0, 3)$, $v_1 = (3, 0)$, and $v_2 = (1, 0)$. The normal vectors for $S_2$ are $h_0 = (0, -\frac{1}{3})$, $h_1 = (-\frac{1}{2}, -\frac{1}{6})$, and $h_2 = (\frac{1}{2}, \frac{1}{2})$. We have that $\mathcal{O} = \{x : x_2 = 0\}$ and therefore $\mathcal{G}_{S_2} = co \{v_1, v_2\}$. Since $\kappa = 1$ there can only be one dependent cycle with respect to $\mathcal{G}_{S_2}$. System 9.1 for $S_2$ requires a feedback transformation and we select

$$u = K_1 x + g_1 + G_1 w = \begin{bmatrix} 0 & -2 \end{bmatrix} x + w.$$ 

and obtain the transformed system

$$\dot{x} = \tilde{A} x + \tilde{a} + \tilde{B} w = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w.$$ 

At $v_1$ we select $w_1 = -1$ which generates $b_1 = (0, -1)$ and at $v_2$ we select $w_2 = 1$ which generates $b_2 = (0, 1)$. Figure 9.5 presents $S_2$ and these geometric properties graphically.

Now we find the flow-like condition on $S_2$. We have that

$$\text{Ker } \hat{B}^T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$ 

We find $\beta_1 \in \text{Ker } \hat{B}^T$ where $\beta_1 = (1, 0)$. We find that $\xi^* = \xi_1 = \xi_2 = \beta_1 = (1, 0)$. We have that $y_0 = (3, -6)$. It can be checked that $\xi^*$ provides a flow condition on $\mathcal{P} = \{v_0\}$ and also separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$. We can now order the vertices in $S_2$. We now have that $\pi_1 = \beta_1 \cdot v_2 = 1$ and $\Pi_1 = \beta_1 \cdot v_1 = 3$.

We design $w_0(x)$ such that $y_1 = 0$ and $y_2 = b_2$. This results in

$$w_0(x) = K_0 x + g_0 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} x + \frac{3}{2}.$$ (9.2)
We design $w_\infty(x)$ such that $y_2 = 0$ and $y_1 = b_1$ and this results in

$$w_\infty(x) = K_\infty x + g_\infty = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{6} \end{bmatrix} x + \frac{1}{2}.$$  \hfill (9.3)

We now extend the state with $\alpha_2$ and we obtain

$$w(x, \alpha_2) = (1 - \alpha_2)w^0(x) + \alpha_2 w^\infty$$

where

$$\alpha_2(t) = 1 - e^{-ct}.$$ 

Since we have used a feedback transformation, we must determine our actual input $u$ to
be

\[ u_2(x, \alpha_2) = K_1 x + g_1 + G_1 w(x, \alpha_2) \]

\[ = (1 - \alpha_2) [(K_1 + G_1 K_0) x + (g_1 + G_1 g_0)] + \alpha_2 [(K_1 + G_1 K_\infty) x + (g_1 + G_1 g_\infty)] \]

\[ = (1 - \alpha_2) u^0(x) + \alpha_2 u^\infty(x) \]

\[ = (1 - \alpha_2) \left( \left[ -\frac{1}{2} \ -\frac{5}{2} \right] x + \frac{3}{2} \right) + \alpha_2 \left( \left[ -\frac{1}{2} \ -\frac{13}{6} \right] x + \frac{1}{2} \right) \]

Using the symmetry of \( u_2(x, \alpha_2) \) and \( u_7(x, \alpha_7) \) we find that

\[ u_7(x, \alpha_7) = (1 - \alpha_7) \left( \left[ -\frac{1}{2} \ -\frac{5}{2} \right] x + \frac{3}{2} \right) + \alpha_7 \left( \left[ -\frac{1}{2} \ -\frac{13}{6} \right] x - \frac{1}{2} \right). \]

### 9.1.3 Simulation

The control system for the materials transfer system was simulated using Matlab. Figures 9.6 to 9.8 present three different simulations for \( c = 100 \), \( c = 1 \), and \( c = 0.01 \) for the time-varying affine feedbacks. The plots contain an overlay of the boundaries of the simplices in black. The blue is a plot of the closed-loop vector field using Matlab’s `streamslice`. For \( S_2 \) and \( S_7 \) where time-varying affine feedback is used, the `streamslice` was generated with \( \alpha = 0 \). The red and green represent the trajectories from ten initial conditions corresponding to the vertices of \( P \). Each trajectory has a red and green portion where the red portion represents the trajectory traversing the initial fifteen simplices it encounters and the green portion represents the final fifteen simplices. The green portion illustrates the location of the closed-loop limit cycle generated in this affine system using (time-varying) affine feedbacks. The only noticeable different between the simulations in Figures 9.6 and 9.7 is the slightly large width of the limit cycle for the case when \( c = 1 \). However, there is a significant difference in the simulation in Figure 9.8 where the limit cycle no longer has a pleasing convex shape. It can be seen that the trajectory roughly follows the `streamslice` when \( \alpha = 0 \). This is due to the relatively slow speed of the
equilibria in $S_2$ and $S_7$. The closed-loop vector field changes very slowly as compared to the system state which causes the system state to approach the slow moving equilibria near $v_{m_i+r_i-1}$ for $i = 1, 2$. Since there is no guideline for designing $c$, other than selecting $c$ small enough, the heuristics outlined here may present some way of determining what an appropriate range of values for $c$ may be.

Figure 9.6: Simulation of material transfer system with $c = 100$ for time-varying affine feedbacks.
Figure 9.7: Simulation of material transfer system with $c = 1$ for time-varying affine feedbacks.
Figure 9.8: Simulation of material transfer system with $c = 0.01$ for time-varying affine feedbacks.
Chapter 10

Conclusions

Modern system dynamics are becoming more complex for a variety of reasons including, for example, an increasing degree of interconnectedness. Similarly, the demands on systems are increasing in complexity due to the specifications imposed upon them. Traditional control methodologies such as regulation and tracking may not be able to satisfy these complex specifications. For example, specifications can include safety and liveness and can be temporal in nature. Piecewise affine feedback is an approach which permits controls designers to apply a much richer set of specifications to an affine system and guarantees that these specifications will be achieved.

Previous work developed the geometric structure of the reach control problem. This material was reviewed in Chapters 2 to 5. Chapter 2 focussed on the mathematical background required to deeply understand the reach control problem. Chapter 3 introduced the basics of the reach control problem. Chapter 4 further developed the discussion of using affine feedbacks to solve the reach control problem. The reach control indices and their relationship to closed-loop equilibria were discussed in Chapter 5. Preliminary examples of designing piecewise-affine feedbacks and examples involving the reach control indices were presented in Chapter 6.

The main results of this thesis are developed in Chapter 7. These results provide
an alternative to the subdivision and piecewise-affine approach developed in [15]. The benefit of the time-varying affine feedback approach is that it resolves the issue of the discontinuities found in the subdivision approach. The benefits come with the small online cost of computing the dynamics of an additional state, $\alpha$, and an offline cost of computing the flow-like condition, $\xi^*$. The state, $\alpha$, is used to interpolate between two affine feedbacks, $u_0(x)$ and $u_\infty(x)$, which generates a multi-affine feedback. The flow-like condition allows one to order the vertices in each cycle of $G$ to allow for the synthesis of $u_0(x)$ and $u_\infty(x)$. Results show that there exists a $c$ sufficiently small (i.e. if the dynamics of $\alpha$ are slow enough) such that the reach control problem is solvable using time-varying affine feedback. This is used in cases where there are not enough degrees of freedom in $\hat{B}$ to solve the reach control problem using ordinary affine feedback.

In-depth examples are explored in Chapter 8. These examples primarily showcase the development of time-varying affine feedbacks. However, in Example 10, an algorithmic approach is used to extract the reach control indices as opposed to a by inspection technique. The benefit is that the reader can then use this method to extract the reach control indices from a general example. Example 10 also showcases the application of a feedback transformation that aids in the synthesis of affine feedbacks. Finally, an application of piecewise affine feedbacks and time-varying affine feedbacks is presented as an application to a material transfer system in Chapter 9. It is important to note that in even the most basic of systems all aspects of the reach control problem can be encountered which justifies the development of the time-varying compensators developed in this thesis.
Bibliography


