INVESTIGATION OF ACTIVE VIBRATION SUPPRESSION OF A FLEXIBLE
SATELLITE USING MAGNETIC ATTITUDE CONTROL

by

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Abstract

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The problem of attitude control of a flexible satellite using magnetic attitude control is investigated. The work is motivated by JC2Sat - a joint CSA and JAXA mission whose main purpose is a proof of concept of two satellites performing differential drag formation flying. The impact of additional flexible drag panels (of various sizes) on the attitude control is assessed. JC2Sat's attitude control system consists of three perpendicular magnetorquers and one reaction/bias-momentum wheel. Four Linear Quadratic Regulator controllers are compared, ranging in complexity from being time-invariant and assuming a rigid satellite, to being periodic and actively suppressing panel vibrations. These include the first controllers which use magnetic attitude control to actively suppress vibrations, and where the periodic vibration suppression controller is able to guarantee asymptotic stability of the linearized system. It was found that for larger panels, the controllers which actively suppressed the vibrations outperformed those that did not.

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Nomenclature

A, B, C State space matrices

b Local magnetic field vector ($\text{A}\cdot\text{m}^2$)

C_{ba} Matrix rotating a vector being expressed in frame a to frame b

c First moment of mass ($\text{kg}\cdot\text{m}$)

d Damping coefficient($\text{kg}\cdot\text{m}^2/\text{s}$)

e Eccentricity

F Forces acting on the satellite (N)

f Forces and torques acting on the satellite

$\vec{\mathcal{F}}_a$ Vectrix for frame a

$\mathcal{F}_a$ Reference frame a

h_w Bias momentum vector ($\text{N}\cdot\text{m}\cdot\text{s}$)

i Inclination (deg)

J Moment of inertia matrix ($\text{kg}\cdot\text{m}^2$)

k Spring coefficient ($\text{N}\cdot\text{m}/\text{rad}$)

M Mass matrix

m	Magnetic dipole moment ($\text{A}\cdot\text{m}^2$)
P	Matrix which solves the Riccati Equation
PR	Periodic rigid controller
PVS	Periodic vibration suppression controller
Q, R	LQR weighting matrices
Q_v, Q_w	Observer weighting matrices
$\underline{\mathbf{r}}_{\rightarrow}$	Position vector (m)
r	Distance of the satellite from the centre of the earth (km)
T	Kinetic energy or orbital period (J, s)
TIR	Time-invariant rigid controller
TIVS	Time-invariant vibration suppression controller
t_0	Initial time (s)
x, y, u	States, measurements and control inputs respectively
α	Right ascension of the ascending node (deg)
δ	Declination (deg)
ϵ, ϵ_4	Quaternions
λ	Vector of Lagrange multipliers
θ	Euler angles (rad)
τ	Torques acting on the satellite ($\text{N}\cdot\text{m}$)
ω	Angular velocity (rad/s)

ω	Argument of perigee (deg)
ω_0	Angular velocity of one rotation per orbit (rad/s)
ψ	Angle from periapsis direction (deg)
ζ	Damping ratio

Chapter 1

Introduction

The attitude control system is an integral part to many satellite missions. There are numerous ways in which the attitude of a satellite can be controlled, where each system has its advantages and disadvantages. Systems utilizing thrusters or reaction wheels have played a large part in the history of attitude control, however there are many other techniques which are becoming more prominent as technologies develop and missions adapt. One such system that has gained more relevance in the last decade, as smaller satellites become more prevalent, is the magnetic attitude control system[1].

Magnetic attitude control systems typically consist of three perpendicular current driven magnetic coils called magnetorquers. The combined magnetic moment that is produced creates a torque acting on the satellite by interacting with the Earth's magnetic field[2]. The magnetorquers can provide a very lightweight attitude control system, and because they have no moving parts, as other attitude control systems such as reaction wheels do, they can be extremely reliable and have great longevity[1]. There are however some drawbacks. The torque exerted is equal to the cross product of the satellite's magnetic moment and the local magnetic field vector. The torque available is hence constrained to lie in the plane orthogonal to the local magnetic field vector, meaning the satellite is always instantaneously underactuated about one axis. It is in fact only

the variability of the Earth's magnetic field throughout an orbit (which depends on the orbit inclination) that can guarantee on average controllability[2]. This gives rise to an inherently time-varying and non-linear control problem. Magnetic attitude control has been an active area of research for 40 years[3], however, the combination of magnetic attitude control and flexible satellites has not. There has been very little work on the control of flexible satellite using magnetic attitude control and it will be the focus of the work presented here.

1.1 Magnetic Attitude Control

The greatest challenge to magnetic attitude control is that the torque available is constrained to lie in the plane orthogonal to the local magnetic field vector, $\mathbf{b}_b$. Given a magnetic dipole moment, $\mathbf{m}$, the resulting torque acting on the satellite is

$$\boldsymbol{\tau}_m = \mathbf{m}^\times \mathbf{b}_b, \text{ where } \mathbf{m} = \begin{bmatrix} m_x & m_y & m_z \end{bmatrix}^\top, \text{ and } \mathbf{m}^\times = \begin{bmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{bmatrix}. \quad (1.1)$$

The vectors $\boldsymbol{\tau}_m$, $\mathbf{m}$ and $\mathbf{b}_b$ are all defined in the satellite body frame. Since only the portion of the magnetic dipole moment that is perpendicular to the magnetic field vector produces torque, to ensure only the perpendicular portion is produced by the magnetorquers, the magnetic dipole moment for a given desired torque, $\mathbf{u}$, is[2]

$$\mathbf{m} = \|\mathbf{b}_b\|^{-2} \mathbf{b}_b^\times \mathbf{u}. \quad (1.2)$$

A number of different control design approaches have been examined for this non-linear and time-varying system. A recent survey outlines the many varied methods[4]. The different methods are often grouped into linear and non-linear controllers. The linear controllers consider the nominal operation of the satellite about its linearization point. The non-linear controllers examine more global formulations. A second distinction discussed here is time-varying versus time-invariant controllers. The time-varying

controllers have the disadvantage of requiring the on-board storage of gain values corresponding to each point throughout a given orbit. The storage required can be an issue to the smaller satellites for which this research is most applicable. However, as is explained in Chapter 3, because of the periodic nature of the system, the storage can be greatly reduced using Fourier Series approximations for the time-varying gains.

One method among the linear control designs is to take advantage of the (quasi) periodic nature of the Earth's magnetic field. This involves using the linear state-space model of the system and solving the Periodic Riccati Equation (PRE) to get an optimal time-periodic set of control gains. Using periodic control theory, it has been shown that the linearized closed-loop system is asymptotically stable[5, 6]. A similar approach is also used for disturbance torque attenuation[7]. In one work[8], an infinite horizon, a finite horizon and a time-invariant controller are proposed and compared. It is found for circular orbits, the finite horizon controller performs much better than the infinite horizon one, which performs slightly better than the constant gain controller. Another time-invariant controller is proposed in Ref. [9] which uses a constant matrix of gain values that approximates the solution to the PRE. It however, does not guarantee asymptotic stability, unlike the time-varying solution.

Among the nonlinear controllers, a number of works[10, 11, 12] again use the periodicity assumption of the Earth's magnetic field and use Krasovskii-Lasalle type arguments to prove local asymptotic stability. The resulting controllers have the challenges of being time-varying. A different approach in Refs. [13] and [14], which does not make any periodicity assumptions, provides a time-invariant control law. It defines a criterion (which involves a function of the local magnetic field vector averaged over a given orbit) such that, if the criterion is met, a proportional-derivative (PD) type law can guarantee (almost) global attitude regulation. This is done for both an inertially pointing[13] and Earth pointing[14] satellite. A sufficient condition for stability in Refs. [13] and [14] is that the control gains are constrained by some upper bound. However, there is no

way of determining the upper bound other than by simulation. In the course of trying to identify the upper bound it is also shown that if a minimum level of an independent 3-axis control system, such as reaction wheels, is used, the gain limitation can be overcome[15]. The methods in the works mentioned here only treat magnetic control of rigid spacecraft. To the author's best knowledge, the only published work on using magnetic control of flexible satellites is an extension of the gain limited work in Refs. [13] and [14], such that when using a proportional-derivative type law there exists a similar bound such that asymptotic stability is guaranteed even in the presence of perturbations from flexible appendages[16]. It does not, however, attempt to actively suppress the flexible vibrations.

1.2 JC2Sat

The motivation for this work comes from the Japan Canada Joint Collaboration Satellite Formation Flying experiment (JC2Sat). It is a joint mission between the Canadian Space Agency (CSA) and the Japan Aerospace Exploration Agency (JAXA) whose mission, among other objectives, is a proof of concept of Differential Drag Formation Flying (DDFF). DDFF is accomplished by two or more satellites, in a low Earth orbit where there remains a very small amount of atmosphere, alternating between low and high drag configurations in such a way as to control their relative position. The details of DDFF are not discussed in detail here, but are described in Ref. [17]. Briefly, the differential drag of two satellites can be increased or decreased by changing the satellite's surface area in the along track direction. For JC2Sat, this is done by using drag panels whose surface area in the along track direction can be increased or decreased by changing the satellite's attitude. The time required for the two satellites to maneuver from an initial to final relative position is proportional to the differential drag per unit mass of the satellites which leads to the ideal satellite having a small mass and large panels. JC2Sat

is faced with very long times to perform relative position maneuvers (approximately 40 days to move from a 20 km to a 1 km separation). A design was proposed which included additional drag panels attached using highly flexible hinges in order to reduce these times. The flexible hinges were proposed because, among other reasons, their mass is much smaller than rigid, locking hinges. If the satellite maintained a constant attitude, the flexible vibrations would tend to dissipate. However, because the satellite needs to be able to change its attitude twice an orbit when performing relative maneuvers, unwanted vibrations can be induced, particularly if the flexibility of the satellite is not accounted for in the control design.

The attitude control system for JC2Sat uses a time-invariant controller which assumes the satellite is a rigid body. The first goal of the research is to assess the effect of the added flexibility while using a time-invariant controller designed assuming a rigid body. The second is to test if the performance of the attitude control system can be improved using controllers which are time-varying and/or actively suppress the panel vibrations. Each controller uses a Linear Quadratic Regulator (LQR) approach solving either the Periodic or Algebraic Riccati equations. Since only the attitude and angular velocity of the satellite body can be measured, to actively suppress the vibrations the panel states must be estimated using an observer. A steady-state Kalman filter is used to both estimate the panels' states as well as filter signal noise introduced into the system. The details of the control system are included in Chapter 3. The attitude control system for JC2Sat consists of three magnetorquers aligned along the axes of the body as well as a reaction wheel which provides torque and momentum bias about the pitch axis.

1.3 Flexible Satellites

The control of flexible satellites is an enormous body of research. The modeling and control of such satellites often involve either a Rayleigh-Ritz or Finite Element Analysis

approach. Based on the JC2Sat design, particularly the highly flexible nature of the panel hinges, neither approach is adopted. The system is treated as a system of rigid bodies, and the flexibility is represented by the flexible hinges between the bodies.

Two panel sizes are investigated. It will be shown by simulation that with the relatively small additional drag panels that all control designs performed well but there is little advantage in using the more complex control designs. Larger panels are also simulated to investigate the possibility of using much larger panels and also to examine if at some point the more complex control designs become advantageous.

1.4 Outline

The thesis is broken into six chapters. The Introduction includes the motivation for the research and a review of the relevant work that this research is built upon. Chapter 2 presents the model assumed for JC2Sat as well as the derivation of the equations of motion for the multibody system. Chapter 3 details how the equations of motion are used in the development of the various control laws. Chapter 4 outlines the simulation details including specific satellite and orbital parameters as well as environmental effects. Chapter 5 presents the simulation results under each proposed controller and offers discussions. The final chapter, Chapter 6, gives a summary of the research as well addresses possible extensions to be explored.

The work presented here investigates the attitude control of flexible satellites using magnetic attitude control, specifically for satellites performing differential drag formation flying. The affect of additional drag panels, of various sizes, is assessed using a control system which uses a combination of a single reaction/momentum-bias wheel and magnetic attitude control. Four controllers are compared, all of which use an LQR framework. The controllers vary in complexity from a controller which is time-invariant and assumes a rigid satellite, to one which is periodic and actively suppresses vibrations. To the author's

best knowledge, this is the first work which uses a controller that actively attempts to suppress vibrations using a control system which includes magnetic attitude control. The periodic controller which actively suppresses the vibrations is able to guarantee asymptotic stability of the linearized system. It is also shown that for large drag panels when using a controller which actively suppresses vibrations, the performance can be improved, both in terms of attitude and angular velocity accuracy as well as control effort, compared to a controller which assumes a rigid satellite.

Chapter 2

Satellite Dynamics

This chapter presents the physical model that is adopted for the satellite as well as the derivation of the equations of motion used for the simulation and control design. A Lagrangian approach is used in conjunction with the Natural Orthogonal Complement or Null Space method[18]. The Natural Orthogonal Complement method consists of defining a matrix of constraints for the system and then finding a matrix which lies in the null space of the constraint matrix, or in other words an orthogonal complement to the constraint matrix. This serves two purposes: the first is that it eliminates the need to solve for any non-working constraint forces/torques, and the second is that it provides a very elegant method of taking the uncoupled equations of motion for multiple unconstrained bodies, which are much simpler, and constraining them to give the coupled equations of motion of the entire system. A good introduction can be found in Refs. [18] and [19]. The Natural Orthogonal Complement method has also been used for modeling other non-holonomic mechanical systems[20], flexible multibody systems[21, 22] and for a Mars “tumbleweed” rover[23].

The chapter begins with a short introduction to vectrix notation followed by the description of the satellite model and a derivation of the dynamics of a single rigid body including translational motion. The constraints of the system are then derived and

following that is an explanation of the Natural Orthogonal Complement method and how it is applied to this particular multibody system, thus giving the resulting equations of motion of the satellite.

To help clarify the following sections, a few remarks on notation are made at the outset. Unbolded letters are used to indicate scalars, bolded lowercase letters are used to express vectors while bolded capital letters are reserved for matrices. $\mathbf{0}$ and $\mathbf{O}$ represent a column and matrix of zeros respectively, while $\mathbf{1}$ is an identity matrix and $\mathbf{1}_n$ is a column of zeros except with a 1 in the n 'th row. The size of the columns and matrices can be determined by their context.

2.1 Vectrix Introduction

Before deriving the dynamic equations of motion, a simple introduction to vectrix notation is provided here; for a more thorough explanation see Ref. [24]. Briefly, a vectrix, $\underline{\mathcal{F}}_a$, for frame $\mathcal{F}_a$, is defined as

$$\underline{\mathcal{F}}_a \triangleq \begin{bmatrix} \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}_2 & \hat{\mathbf{a}}_3 \end{bmatrix}^T$$

where $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$ and $\hat{\mathbf{a}}_3$ are the basis vectors defining $\mathcal{F}_a$. A vector, $\underline{\mathbf{v}}$, expressed in $\underline{\mathcal{F}}_a$ is

$$\underline{\mathbf{v}} = \mathbf{v}_a^T \underline{\mathcal{F}}_a = \underline{\mathcal{F}}_a^T \mathbf{v}_a, \text{ where } \mathbf{v}_a = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$$

are the components of $\underline{\mathbf{v}}$ expressed in $\underline{\mathcal{F}}_a$. The cross product between two vectors is

$$\underline{\mathbf{v}} \times \underline{\mathbf{w}} = \underline{\mathcal{F}}_a^T \mathbf{v}_a^\times \mathbf{w}_a, \text{ where } \mathbf{v}_a^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

To avoid confusion, the components of position vectors are written as $\mathbf{r}_a^{ba}$, where the superscripts refer to the vector pointing from the origin of $\mathcal{F}_a$ to that of $\mathcal{F}_b$ while the subscript refers to the components of the vector being expressed in $\mathcal{F}_a$. Similarly, rotation

matrix, $\mathbf{C}_{ba}$, transforms the components of a vector expressed in $\mathcal{F}_a$ to the components being expressed in $\mathcal{F}_b$. To clarify, if $\mathbf{r}_b^{ba}$ are the components of $\underline{\mathbf{r}}^{ba}$ expressed in $\mathcal{F}_b$ and $\mathbf{r}_a^{ba}$ are the components of $\underline{\mathbf{r}}^{ba}$ expressed in $\mathcal{F}_a$ then

$$\mathbf{r}_b^{ba} = \mathbf{C}_{ba} \mathbf{r}_a^{ba}.$$

For angular velocities sub and superscripts are similarly used where $\boldsymbol{\omega}_b^{ba}$ refers to the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_a$, defined in $\mathcal{F}_b$. Euler angle rates $\dot{\boldsymbol{\theta}}^{ba}$ are related to the angular velocity by $\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba} \dot{\boldsymbol{\theta}}^{ba}$ where $\mathbf{S}_b^{ba}$ corresponds to a 3-2-1 Euler sequence[24].

2.2 Satellite Model

The satellite model is shown in Figure 2.1. The satellite model consists of multiple rigid bodies constrained together. The proposed JC2Sat design considered here has the panels attached using tape spring hinges. Tape spring hinges provide a lightweight and simple deployment mechanism, but once deployed can be highly flexible because there is no locking mechanism. For small deflections, tape springs have been shown to behave like torsion springs[25]. As such, the hinges are treated as having torsion springs about the x and y axes of the two panel frames $\mathcal{F}_1$ and $\mathcal{F}_2$, which give the flapping and torsion bending represented by θ_x and θ_y respectively. The satellite contains a momentum/reaction wheel which is used to provide bias momentum and torque about the pitch axis.

2.3 Single Rigid Body Derivation

The Natural Orthogonal Complement method first derives the equations of motion for multiple unconstrained bodies. The equations of motion for a single rigid body will be done first to make the extension to multiple bodies clear. The vector from the inertial

Figure 2.1: Satellite showing panel bending about two axes.

frame, $\mathcal{F}_i$, to a mass element in a rigid body is given as

$$\underline{\mathbf{r}}^{\rho i} = \underline{\mathbf{r}}^{bi} + \underline{\boldsymbol{\rho}}^b$$

where $\underline{\boldsymbol{\rho}}^b$ is the vector from the origin of $\mathcal{F}_b$ to the mass element. Defining vector time derivatives as seen in $\mathcal{F}_i$ and $\mathcal{F}_b$ by $(\dot{})$ and $(^\circ)$ respectively, taking the inertial derivative yields

$$\dot{\underline{\mathbf{r}}}^{\rho i} = \dot{\underline{\mathbf{r}}}^{bi} + \underline{\boldsymbol{\omega}}^{bi} \times \underline{\boldsymbol{\rho}}^b$$

where we have made use of the fact that because it is a rigid body, the velocity of the mass element as seen from $\mathcal{F}_b$, $\underline{\boldsymbol{\rho}}^{\circ}$, is equal to zero. Defining a vector time derivative for a vector expressed in a particular frame as $(\dot{})$, expressing the vectors in $\mathcal{F}_i$ gives

$$\dot{\underline{\mathbf{r}}}_i^{\rho i} = \dot{\underline{\mathbf{r}}}_i^{bi} + \mathbf{C}_{bi}^\top \boldsymbol{\omega}_b^{bi \times} \boldsymbol{\rho}_b^b = \dot{\underline{\mathbf{r}}}_i^{bi} - \mathbf{C}_{bi}^\top \boldsymbol{\rho}_b^{b \times} \boldsymbol{\omega}_b^{bi}$$

where we have used the identity $\mathbf{v}^\times \mathbf{w} = -\mathbf{w}^\times \mathbf{v}$. The kinetic energy of the body, T_b , is defined as

$$T_b = \frac{1}{2} \int \dot{\underline{\mathbf{r}}}^{\rho i} \cdot \dot{\underline{\mathbf{r}}}^{\rho i} dm = \frac{1}{2} \int \dot{\underline{\mathbf{r}}}_i^{\rho i \top} \dot{\underline{\mathbf{r}}}_i^{\rho i} dm$$

where the integral is taken over the entire body. Putting the integral in matrix form gives

$$\begin{aligned}
 T_b &= \frac{1}{2} \int \begin{bmatrix} \dot{\mathbf{r}}_i^{bi\top} & \boldsymbol{\omega}_b^{bi\top} \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\mathbf{C}_{bi}^\top \boldsymbol{\rho}_b^{b\times} \\ \boldsymbol{\rho}_b^{b\times} \mathbf{C}_{bi} & -\boldsymbol{\rho}_b^{b\times} \boldsymbol{\rho}_b^{b\times} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i^{bi} \\ \boldsymbol{\omega}_b^{bi} \end{bmatrix} dm \\
 &= \frac{1}{2} \underbrace{\begin{bmatrix} \dot{\mathbf{r}}_i^{bi\top} & \boldsymbol{\omega}_b^{bi\top} \end{bmatrix}}_{\mathbf{v}_b^\top} \underbrace{\begin{bmatrix} \mathbf{1} m_b & -\mathbf{C}_{bi}^\top \mathbf{c}_b^\times \\ \mathbf{c}_b^\times \mathbf{C}_{bi} & \mathbf{J}_b \end{bmatrix}}_{\mathbf{M}_b} \begin{bmatrix} \dot{\mathbf{r}}_i^{bi} \\ \boldsymbol{\omega}_b^{bi} \end{bmatrix}
 \end{aligned}$$

where $\mathbf{m}_b$ is the body's mass, $\mathbf{c} = \int_b \boldsymbol{\rho}_b^{b\times} dm$ is the body's first moment of mass about the origin of $\mathcal{F}_b$ and $\mathbf{J}_b = \int_b -\boldsymbol{\rho}_b^{b\times} \boldsymbol{\rho}_b^{b\times} dm$ is the body's moment of inertia matrix about the origin of $\mathcal{F}_b$. In order to use Lagrange's equation, the kinetic energy term is expressed as a function of the holonomic generalized coordinates. To do so the kinetic energy is redefined using $\boldsymbol{\omega}_b^{bi} = \mathbf{S}_b^{bi} \dot{\boldsymbol{\theta}}^{bi}$ as

$$T_b = \frac{1}{2} \begin{bmatrix} \dot{\mathbf{r}}_i^{bi\top} & \dot{\boldsymbol{\theta}}^{bi\top} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{bi\top} \end{bmatrix} \underbrace{\mathbf{M}_b}_{\mathbf{\Pi}_b} \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{bi} \end{bmatrix}}_{\mathbf{\dot{q}}_b} \underbrace{\begin{bmatrix} \dot{\mathbf{r}}_i^{bi} \\ \dot{\boldsymbol{\theta}}^{bi} \end{bmatrix}}_{\mathbf{\dot{q}}_b}. \quad (2.1)$$

Assuming zero potential energy, the Lagrangian, $L = T_b$ and Lagrange's Equation can be written as

$$\frac{d}{dt} \left(\frac{\partial T_b}{\partial \dot{\mathbf{q}}_b} \right)^\top - \frac{\partial T_b}{\partial \mathbf{q}_b}^\top = \mathbf{\Pi}_b^\top \mathbf{f}_b$$

where $\mathbf{f}_b$ contains any generalized forces and torques acting on the body and where $\frac{\partial T_b}{\partial \mathbf{q}_b}$ denotes the partial derivative of T_b with respect to $\mathbf{q}_b$ and other partial derivatives are similarly defined. The details of $\mathbf{f}_b$ are outlined in Section 2.6. Expanding the left side of the previous equation gives

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial T_b}{\partial \dot{\mathbf{q}}_b} \right)^\top - \frac{\partial T_b}{\partial \mathbf{q}_b}^\top &= \frac{d}{dt} \left(\underbrace{\dot{\mathbf{q}}_b^\top \mathbf{\Pi}_b^\top}_{\mathbf{v}_b^\top} \mathbf{M}_b \mathbf{\Pi}_b \right)^\top - \frac{\partial T_b}{\partial \mathbf{q}_b}^\top \\
 &= \mathbf{\Pi}_b^\top \mathbf{M}_b \dot{\mathbf{v}}_b + \mathbf{\Pi}_b^\top \dot{\mathbf{M}}_b \mathbf{v}_b + \dot{\mathbf{\Pi}}_b^\top \mathbf{M}_b \mathbf{v}_b - \frac{\partial T_b}{\partial \mathbf{q}_b}^\top. \quad (2.2)
 \end{aligned}$$

Looking at the individual terms more closely we have

$$\dot{\mathbf{M}}_b = \begin{bmatrix} \mathbf{0} & -\dot{\mathbf{C}}_{bi}^T \mathbf{c}_b^\times \\ \mathbf{c}_b^\times \dot{\mathbf{C}}_{bi} & \mathbf{0} \end{bmatrix}, \quad \dot{\mathbf{\Pi}}_b = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{S}}_b^{bi} \end{bmatrix}.$$

The last term of Eq. (2.2) can be expanded as

$$\frac{\partial T_b}{\partial \mathbf{q}_b} = \begin{bmatrix} \frac{\partial T_b}{\partial \mathbf{r}_i^{bi}} & \frac{\partial T_b}{\partial \boldsymbol{\theta}^{bi}} \end{bmatrix} = \begin{bmatrix} \frac{\partial T_b}{\partial \mathbf{r}_i^{bi}} & \mathbf{0}^T \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial T_b}{\partial \mathbf{r}_i^{bi}} & \frac{\partial T_b}{\partial \boldsymbol{\omega}_b^{bi}} \end{bmatrix}}_{\frac{\partial T_b}{\partial \mathbf{v}_b}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{bi}}{\partial \boldsymbol{\theta}^{bi}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}^T & \frac{\partial T_b}{\partial \boldsymbol{\theta}^{bi}} \end{bmatrix}^*$$

where $\frac{\partial T_b}{\partial \mathbf{r}_i^{bi}} = \mathbf{0}^T$ and $\frac{\partial T_b}{\partial \mathbf{v}_b} = \mathbf{v}^T \mathbf{M}_b$. The $*$ denotes the differentiation with respect to $\boldsymbol{\theta}$ of all terms other than those containing $\boldsymbol{\omega}$. Using the identity $\frac{\partial(\mathbf{C}\mathbf{u})}{\partial \boldsymbol{\theta}} = (\mathbf{C}\mathbf{u})^\times \mathbf{S}$ [26] gives

$$\frac{\partial T_b}{\partial \boldsymbol{\theta}^{bi}}^* = \frac{\partial}{\partial \boldsymbol{\theta}^{bi}} \left(\frac{1}{2} m_b \dot{\mathbf{r}}_i^{biT} \dot{\mathbf{r}}_i^{bi} + \boldsymbol{\omega}_b^{biT} \mathbf{c}_b^\times \mathbf{C}_{bi} \dot{\mathbf{r}}_i^{bi} + \frac{1}{2} \boldsymbol{\omega}_b^{biT} \mathbf{J} \boldsymbol{\omega}_b^{bi} \right) = \boldsymbol{\omega}_b^{biT} \mathbf{c}_b^\times (\mathbf{C}_{bi} \dot{\mathbf{r}}_i^{bi})^\times \mathbf{S}_b^{bi}.$$

We can now write

$$\frac{\partial T_b}{\partial \mathbf{q}_b}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{bi}}{\partial \boldsymbol{\theta}^{bi}}^T \end{bmatrix} \mathbf{M}_b \mathbf{v}_b + \mathbf{\Pi}_b^T \underbrace{\begin{bmatrix} \mathbf{0} \\ (\mathbf{C}_{bi} \dot{\mathbf{r}}_i^{bi})^\times \mathbf{c}_b^\times \boldsymbol{\omega}_b^{bi} \end{bmatrix}}_{\boldsymbol{\alpha}_b}.$$

Substituting the above terms into Eq. (2.2) yields

$$\mathbf{\Pi}_b^T \mathbf{M}_b \dot{\mathbf{v}}_b + \mathbf{\Pi}_b^T \dot{\mathbf{M}}_b \mathbf{v}_b + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{S}}_b^{biT} \end{bmatrix} \mathbf{M}_b \mathbf{v}_b - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{bi}}{\partial \boldsymbol{\theta}^{bi}}^T \end{bmatrix} \mathbf{M}_b \mathbf{v}_b - \mathbf{\Pi}_b^T \boldsymbol{\alpha}_b = \mathbf{\Pi}_b^T \mathbf{f}_b.$$

Using the identity $\dot{\mathbf{S}}^T - \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\theta}}^T = \mathbf{S}^T \boldsymbol{\omega}^\times$ [27], and pulling out a $\mathbf{\Pi}_b^T$ from the above equation gives

$$\mathbf{\Pi}_b^T \mathbf{M}_b \dot{\mathbf{v}}_b + \mathbf{\Pi}_b^T \dot{\mathbf{M}}_b \mathbf{v}_b + \mathbf{\Pi}_b^T \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{bi \times} \end{bmatrix}}_{\boldsymbol{\Omega}_b} \mathbf{M}_b \mathbf{v}_b - \mathbf{\Pi}_b^T \boldsymbol{\alpha}_b = \mathbf{\Pi}_b^T \mathbf{f}_b. \quad (2.3)$$

Eq. (2.3) will be used later to express the equations of motion for multiple unconstrained rigid bodies. If the previous equation is premultiplied by $(\mathbf{\Pi}_b^T)^{-1}$ and the external forces are assumed to be zero, the resulting equation is

$$\mathbf{M}_b \dot{\mathbf{v}}_b + \dot{\mathbf{M}}_b \mathbf{v}_b + \boldsymbol{\Omega}_b \mathbf{M}_b \mathbf{v}_b - \boldsymbol{\alpha}_b = \mathbf{0}$$

which gives the equations of motion for a single rigid body with the translation motion defined in the inertial frame and with the body frame origin not necessarily at the centre of mass.

2.4 Constraints

Before taking advantage of the Natural Orthogonal Complement Method, the constraints of the system must first be developed. The first goal of the section is to develop a matrix, referred to in the literature as the natural orthogonal complement or projection matrix, that relates the *independent* generalized coordinate rates to the *dependent* generalized coordinate rates. The second goal is to develop a matrix containing the constraints of the system.

The six generalized coordinate rates for the spacecraft body (free to move in space) are

$$\dot{\mathbf{q}}_b^T = [\dot{\mathbf{r}}_i^{biT} \quad \dot{\boldsymbol{\theta}}^{biT}].$$

The 24 generalized coordinate rates for the four uncoupled bodies (including the central body, panel 1, panel 2 and the momentum wheel) are then expressed as

$$\dot{\hat{\mathbf{q}}}^T = [\dot{\mathbf{r}}_i^{biT} \quad \dot{\boldsymbol{\theta}}^{biT} \quad \dot{\mathbf{r}}_i^{1iT} \quad \dot{\boldsymbol{\theta}}^{1iT} \quad \dot{\mathbf{r}}_i^{2iT} \quad \dot{\boldsymbol{\theta}}^{2iT} \quad \dot{\mathbf{r}}_i^{wiT} \quad \dot{\boldsymbol{\theta}}^{wiT}]$$

where $\dot{\mathbf{r}}_i^{bi}$, $\dot{\mathbf{r}}_i^{1i}$, $\dot{\mathbf{r}}_i^{2i}$ and $\dot{\mathbf{r}}_i^{wi}$ are the components of vectors expressing translational velocity in the inertial frame. $\dot{\boldsymbol{\theta}}^{bi}$, $\dot{\boldsymbol{\theta}}^{1i}$, $\dot{\boldsymbol{\theta}}^{2i}$ and $\dot{\boldsymbol{\theta}}^{wi}$ are Euler angle rates. Since the bodies are constrained such that the panels can only rotate about their x and y axes and the momentum wheel can rotate about only the pitch axis a number of the coordinates are no longer independent. These include the yaw rotation of the panels, the roll and yaw rotation of the wheel and the translational motion of all three. $\dot{\hat{\mathbf{q}}}$ will be referred to as the vector of *dependent* coordinate rates. The remaining 11 degrees of freedom are now included in the vector of *independent* coordinate rates, expressed as

$$\dot{\boldsymbol{\eta}}^T = [\dot{\mathbf{r}}_i^{biT} \quad \boldsymbol{\omega}_b^{biT} \quad \dot{\boldsymbol{\theta}}^{1bT} \quad \dot{\boldsymbol{\theta}}^{2bT} \quad \dot{\theta}_y^{wb}], \quad \text{where } \dot{\boldsymbol{\theta}}^{1bT} = [\dot{\theta}_x^{1b} \quad \dot{\theta}_y^{1b}] \quad \text{and} \quad \dot{\boldsymbol{\theta}}^{2bT} = [\dot{\theta}_x^{2b} \quad \dot{\theta}_y^{2b}].$$

Here the x and y subscripts refer to the axes about which rotation occurs. For the panels this refers to their x and y axes. $\dot{\theta}_y^{wb}$ is the rate of rotation of the momentum wheel with respect to the body about the y or pitch axis.

To develop the constraint equations, the translational and rotational motions of the panels and momentum wheel must be written in two ways. The first is to express the *dependent* coordinates as a function of the *independent* coordinates. The second is to express the constraints such that, as a function of the *dependent* coordinates, they are equal to zero. They will be expressed in the following matrix form respectively as

$$\dot{\mathbf{q}} = \mathbf{\Gamma} \dot{\boldsymbol{\eta}}, \quad \boldsymbol{\Xi} \dot{\mathbf{q}} = \mathbf{0}. \quad (2.4)$$

This will be done first for panel 1. The procedure is identical for panel 2. The translational constraint for the momentum wheel is the same as for panel 1 and 2, however, the rotational constraint is slightly different and will be outlined after panel 1. The location of panel 1 is given as

$$\underline{\mathbf{r}}_{\rightarrow}^{1i} = \underline{\mathbf{r}}_{\rightarrow}^{bi} + \underline{\mathbf{r}}_{\rightarrow}^{1b}.$$

Taking the derivative and expressing each term in the appropriate frame gives

$$\begin{aligned} \dot{\underline{\mathbf{r}}}_{\rightarrow}^{1i} &= \dot{\underline{\mathbf{r}}}_{\rightarrow}^{bi} + \underline{\boldsymbol{\omega}}_{\rightarrow}^{bi} \times \underline{\mathbf{r}}_{\rightarrow}^{1b} \\ \dot{\mathbf{r}}_i^{1i} &= \dot{\mathbf{r}}_i^{bi} - \mathbf{C}_{bi}^T \mathbf{r}_b^{1b \times} \boldsymbol{\omega}_b^{bi}. \end{aligned} \quad (2.5)$$

Writing it as a function of generalized coordinate rates equal to zero gives

$$\dot{\mathbf{r}}_i^{bi} - \mathbf{C}_{bi}^T \mathbf{r}_b^{1b \times} \mathbf{S}_b^{bi} \dot{\boldsymbol{\theta}}^{bi} - \dot{\mathbf{r}}_1^{1i} = \mathbf{0}. \quad (2.6)$$

To develop the rotational constraint for the two bodies we write the angular velocity of panel 1 as

$$\begin{aligned} \underline{\boldsymbol{\omega}}_{\rightarrow}^{1i} &= \underline{\boldsymbol{\omega}}_{\rightarrow}^{bi} + \underline{\boldsymbol{\omega}}_{\rightarrow}^{1b} \\ \boldsymbol{\omega}_1^{1i} &= \mathbf{C}_{1b} \boldsymbol{\omega}_b^{bi} + \boldsymbol{\omega}_1^{1b}. \end{aligned} \quad (2.7)$$

Because of the yaw constraint we have

$$\boldsymbol{\omega}_1^{1b} = \mathbf{1}_1 \dot{\theta}_x^{1b} + \mathbf{C}_x(\theta_x^{1b}) \mathbf{1}_2 \dot{\theta}_y^{1b} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \cos(\theta_x^{1b}) \\ 0 & -\sin(\theta_x^{1b}) \end{bmatrix}}_{\tilde{\mathbf{S}}_1^{1b}} \dot{\boldsymbol{\theta}}^{1b}$$

where $\mathbf{C}_x(\theta_x^{1b})$ is a principle rotation about the x axis by an angle of θ_x^{1b} . A 3-2-1 Euler sequence is again used here. Note that Eq. (2.7) when including the yaw constraint can also be written in two ways:

$$\dot{\boldsymbol{\theta}}^{1i} = \mathbf{S}_1^{1i-1} \mathbf{C}_{1b} \boldsymbol{\omega}_b^{bi} + \mathbf{S}_1^{1i-1} \tilde{\mathbf{S}}_1^{1b} \dot{\boldsymbol{\theta}}^{1b} \quad (2.8)$$

and

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_1^{1b-1} \mathbf{S}_1^{1i} \dot{\boldsymbol{\theta}}^{1i} - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_1^{1b-1} \mathbf{C}_{1b} \mathbf{S}_b^{bi} \dot{\boldsymbol{\theta}}^{bi} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \dot{\boldsymbol{\theta}}^{1b} = 0. \quad (2.9)$$

Eqs. (2.5) and (2.8) are used to construct $\mathbf{\Gamma}$ while Eqs. (2.6) and (2.9) are used to construct $\mathbf{\Xi}$. Noting that the angular velocity of the reaction wheel, $\boldsymbol{\omega}_w^{wb} = \mathbf{1}_2 \dot{\theta}_y^{wb}$, corresponding to the roll and yaw constraint, and following a similar procedure as above, the rotational constraint for the reaction wheel can be expressed in two ways

$$\dot{\boldsymbol{\theta}}^{wi} = \mathbf{S}_w^{wi-1} \mathbf{C}_{wb} \boldsymbol{\omega}_b^{bi} + \mathbf{S}_w^{wi-1} \mathbf{1}_2 \dot{\theta}_y^{wb} \quad (2.10)$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{S}_w^{wb-1} \mathbf{S}_w^{wi} \dot{\boldsymbol{\theta}}^{wi} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{S}_w^{wb-1} \mathbf{C}_{wb} \mathbf{S}_b^{bi} \dot{\boldsymbol{\theta}}^{bi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\boldsymbol{\theta}}^{wb} = \mathbf{0}. \quad (2.11)$$

Eqs. (2.5), (2.8) and (2.10), as well as the equivalent equations for panel 2, are placed in

matrix form to give

$$\dot{\mathbf{q}} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{bi-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{1b \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1^{1i-1} \mathbf{C}_{1b} & \mathbf{S}_1^{1i-1} \tilde{\mathbf{S}}_1^{1b} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{2b \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2^{2i-1} \mathbf{C}_{2b} & \mathbf{0} & \mathbf{S}_2^{2i-1} \tilde{\mathbf{S}}_2^{2b} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{wb \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_w^{wi-1} \mathbf{C}_{wb} & \mathbf{0} & \mathbf{0} & \mathbf{S}_w^{wi-1} \mathbf{1}_2 \end{bmatrix}}_{\mathbf{\Gamma}} \dot{\boldsymbol{\eta}}.$$

The same is done for Eqs. (2.6), (2.9) and (2.11) and the equivalent equations for panel 2 to give

$$\underbrace{\begin{bmatrix} \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{1b \times} \mathbf{S}_b^{bi} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_1^{1b-1} \mathbf{C}_{1b} \mathbf{S}_b^{bi} & \mathbf{0} & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_1^{1b-1} \mathbf{S}_1^{1i} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{2b \times} \mathbf{S}_b^{bi} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_2^{2b-1} \mathbf{C}_{2b} \mathbf{S}_b^{bi} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_2^{2b-1} \mathbf{S}_2^{2i} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{wb \times} \mathbf{S}_b^{bi} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{S}_w^{wb-1} \mathbf{C}_{wb} \mathbf{S}_b^{bi} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{S}_w^{wb-1} \mathbf{S}_w^{wi} \end{bmatrix}}_{\mathbf{\Xi}} \dot{\mathbf{q}} = \mathbf{0}$$

There are two reasons for writing the above equations in this form. The first is that the $\mathbf{\Xi}$ matrix can be added to Lagrange's Equations along with a vector of Lagrange multipliers to include the system constraints. The second and more important reason is that $\mathbf{\Gamma}$ lies in the null space of $\mathbf{\Xi}$ such that

$$\mathbf{\Xi} \dot{\mathbf{q}} = \mathbf{\Xi} \mathbf{\Gamma} \dot{\boldsymbol{\eta}} = \mathbf{0}. \quad (2.12)$$

To show this, consider that Eq. (2.12) must hold for all $\dot{\boldsymbol{\eta}}$. It follows that $\mathbf{\Xi} \mathbf{\Gamma} = \mathbf{\Gamma}^\top \mathbf{\Xi}^\top = \mathbf{0}$. This can easily be verified by evaluating the multiplication. This result will be used in the following section.

2.5 Natural Orthogonal Complement

The advantage of the Natural Orthogonal Complement method is that it can take the decoupled equations of motion for multiple unconstrained bodies, constrain them together, eliminate the need to determine constraint forces, and give the equations of motion for the multibody system. Recalling the kinetic energy of a single body can be written as shown in Eq. (2.1), the kinetic energy of the four unconstrained bodies can be written as

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \hat{\mathbf{\Pi}}^T \underbrace{\begin{bmatrix} \mathbf{M}_b & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M}_w \end{bmatrix}}_{\hat{\mathbf{M}}} \underbrace{\begin{bmatrix} \mathbf{\Pi}_b & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{\Pi}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{\Pi}_w \end{bmatrix}}_{\hat{\mathbf{\Pi}}} \dot{\mathbf{q}}.$$

In order to constrain the bodies together, the constraint equations and Lagrange multipliers, $\boldsymbol{\lambda}$, must be appended to Lagrange's Equation as [18]

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}} \right)^T - \frac{\partial T}{\partial \mathbf{q}}^T = \boldsymbol{\Xi}^T \boldsymbol{\lambda} + \hat{\mathbf{\Pi}}^T \hat{\mathbf{f}}^{ext} \quad (2.13)$$

where $\hat{\mathbf{f}}^{ext}$ contains all external forces and torques acting on the system. These include all forces and torques other than those required to satisfy the kinematical constraints. The details of $\hat{\mathbf{f}}^{ext}$ are outlined in Section 2.6. Note that zero potential energy has been assumed here. Although there is potential energy in the hinges when they are deflected, it is ignored here and instead included as a restoring force. This is also outlined in Section 2.6. Evaluating the derivatives in Eq. (2.13) is straight forward because the bodies are uncoupled in the kinetic energy term. By using the equation for the single rigid body, Eq. (2.3), we can easily write the following as the equations of motion for the system

$$\hat{\mathbf{\Pi}}^T \hat{\mathbf{M}} \dot{\mathbf{v}} + \hat{\mathbf{\Pi}}^T \dot{\mathbf{M}} \mathbf{v} + \hat{\mathbf{\Pi}}^T \hat{\boldsymbol{\Omega}} \mathbf{M} \mathbf{v} - \hat{\mathbf{\Pi}}^T \hat{\boldsymbol{\alpha}} = \boldsymbol{\Xi}^T \boldsymbol{\lambda} + \hat{\mathbf{\Pi}}^T \hat{\mathbf{f}}^{ext} \quad (2.14)$$

where

$$\hat{\mathbf{v}} = \begin{bmatrix} \mathbf{v}_b \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_w \end{bmatrix}, \quad \hat{\boldsymbol{\Omega}} = \begin{bmatrix} \boldsymbol{\Omega}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Omega}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Omega}_w \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{\alpha}} = \begin{bmatrix} \boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_w \end{bmatrix}.$$

To eliminate the need to solve for the Lagrange multipliers, we take advantage of the previously mentioned null space of $\boldsymbol{\Xi}$ by premultiplying Eq. (2.14) by $\boldsymbol{\Gamma}^\top$ to give

$$\boldsymbol{\Gamma}^\top \hat{\boldsymbol{\Pi}}^\top \hat{\mathbf{M}} \dot{\hat{\mathbf{v}}} + \boldsymbol{\Gamma}^\top \hat{\boldsymbol{\Pi}}^\top \hat{\mathbf{M}} \dot{\hat{\mathbf{v}}} + \boldsymbol{\Gamma}^\top \hat{\boldsymbol{\Pi}}^\top \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \hat{\mathbf{v}} - \boldsymbol{\Gamma}^\top \hat{\boldsymbol{\Pi}}^\top \hat{\boldsymbol{\alpha}} = \underbrace{\boldsymbol{\Gamma}^\top \boldsymbol{\Xi}^\top}_{\mathbf{0}} \boldsymbol{\lambda} + \boldsymbol{\Gamma}^\top \hat{\boldsymbol{\Pi}}^\top \hat{\mathbf{f}}^{ext}. \quad (2.15)$$

To simplify the above equation, we define

$$\hat{\boldsymbol{\Pi}} \boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{1b \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{1b} & \tilde{\mathbf{S}}_1^{1b} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_b^{2b \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{2b} & \mathbf{0} & \tilde{\mathbf{S}}_2^{2b} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{bi}^\top \mathbf{r}_w^{wb \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{wb} & \mathbf{0} & \mathbf{0} & \mathbf{1}_2 \end{bmatrix} = \boldsymbol{\Lambda}.$$

Note that $\hat{\mathbf{v}}$, which contains 24 coordinates, can be expressed by the 11 independent coordinates contained in $\dot{\boldsymbol{\eta}}$. In order to have the equations of motion in terms of only the independent coordinates we first note that

$$\hat{\mathbf{v}} = \hat{\boldsymbol{\Pi}} \dot{\mathbf{q}} = \hat{\boldsymbol{\Pi}} \boldsymbol{\Gamma} \dot{\boldsymbol{\eta}} = \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}}, \quad \text{and} \quad \dot{\hat{\mathbf{v}}} = \boldsymbol{\Lambda} \ddot{\boldsymbol{\eta}} + \dot{\boldsymbol{\Lambda}} \dot{\boldsymbol{\eta}}.$$

Substituting this into Eq. (2.15) finally gives

$$\boldsymbol{\Lambda}^\top \hat{\mathbf{M}} \boldsymbol{\Lambda} \ddot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^\top \hat{\mathbf{M}} \dot{\boldsymbol{\Lambda}} \dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^\top \dot{\hat{\mathbf{M}}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^\top \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} - \boldsymbol{\Lambda}^\top \hat{\boldsymbol{\alpha}} = \boldsymbol{\Lambda}^\top \hat{\mathbf{f}}^{ext} \quad (2.16)$$

which are the equations of motion for the four constrained bodies. The details of $\hat{\mathbf{f}}^{ext}$ are described in the next section.

2.6 Force Vectors

Let

$$\mathbf{f} = \begin{bmatrix} \mathbf{F} \\ \boldsymbol{\tau} \end{bmatrix}$$

where $\mathbf{F}$ is the total force acting on the body in inertial coordinates and $\boldsymbol{\tau}$ is the total torque acting on the body in the body coordinates. Let the generalized coordinates be

$$\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \boldsymbol{\theta} \end{bmatrix}$$

where $\mathbf{r}$ denotes inertial position expressed in inertial coordinates, and $\boldsymbol{\theta}$ denotes an Euler sequence describing the inertial attitude of the body. We note again that

$$\boldsymbol{\omega} = \mathbf{S}\dot{\boldsymbol{\theta}}$$

where $\mathbf{S}$ is the kinematical matrix relating $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\theta}}$. Consider the virtual displacement

$$\delta\mathbf{q} = \begin{bmatrix} \delta\mathbf{r} \\ \delta\boldsymbol{\theta} \end{bmatrix}.$$

$\delta\mathbf{r}$ represents a virtual physical translation $\delta\mathbf{r}$ while $\delta\boldsymbol{\theta}$ represents a virtual (infinitesimal) rotation given by $\mathbf{S}\delta\boldsymbol{\theta}$. The virtual work performed by the external force and torque is given by

$$\begin{aligned} \delta\mathbf{W} &= \mathbf{F}^\top \delta\mathbf{r} + \boldsymbol{\tau}^\top \mathbf{S}\delta\boldsymbol{\theta} \\ &= \begin{bmatrix} \delta\mathbf{r}^\top & \delta\boldsymbol{\theta}^\top \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{S}^\top \boldsymbol{\tau} \end{bmatrix} \end{aligned} \quad (2.17)$$

$$= \delta\mathbf{q}^\top \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^\top \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \boldsymbol{\tau} \end{bmatrix} \quad (2.18)$$

$$= \delta\mathbf{q}^\top \boldsymbol{\Pi}^\top \mathbf{f}. \quad (2.19)$$

Therefore, the generalized force in Lagrange's equation is given by

$$\boldsymbol{\Pi}^\top \mathbf{f}$$

for an unconstrained body. Defining

$$\hat{\mathbf{f}} = \begin{bmatrix} \mathbf{f}_b \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_w \end{bmatrix}$$

the equations of motion for the unconstrained system become

$$\hat{\Pi}^T \hat{\mathbf{M}} \dot{\hat{\mathbf{v}}} + \hat{\Pi}^T \dot{\hat{\mathbf{M}}} \hat{\mathbf{v}} + \hat{\Pi}^T \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \hat{\mathbf{v}} - \hat{\Pi}^T \hat{\boldsymbol{\alpha}} = \hat{\Pi}^T \hat{\mathbf{f}}.$$

Now let us decompose the force vectors as follows:

$$\mathbf{f}_b = \begin{bmatrix} \mathbf{F}_b \\ \boldsymbol{\tau}_b \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{F}_b &= \mathbf{F}_b^{ext} + \mathbf{F}_b^1 + \mathbf{F}_b^2 + \mathbf{F}_b^w \\ \boldsymbol{\tau}_b &= \boldsymbol{\tau}_b^{ext} + \boldsymbol{\tau}_b^1 + \boldsymbol{\tau}_b^2 + \boldsymbol{\tau}_b^w \end{aligned}$$

where $\mathbf{F}_b^{ext}$ are external forces acting on the central body, $\mathbf{F}_b^1$ are constraint forces applied to the body by panel 1, $\mathbf{F}_b^2$ are constraint forces applied to the body by panel 2 and $\mathbf{F}_b^w$ are constraint forces applied to the body by the wheel. All quantities are similarly defined. Note that $\boldsymbol{\tau}_b^1$, $\boldsymbol{\tau}_b^2$ and $\boldsymbol{\tau}_b^w$ contain only the torques required to maintain the kinematic constraints. Similarly, for panel 1,

$$\mathbf{f}_1 = \begin{bmatrix} \mathbf{F}_1 \\ \boldsymbol{\tau}_1 \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{F}_1 &= \mathbf{F}_1^{ext} + \mathbf{F}_1^b = \mathbf{F}_1^{ext} - \mathbf{F}_b^1 \\ \boldsymbol{\tau}_1 &= \boldsymbol{\tau}_1^{ext} + \boldsymbol{\tau}_1^b = \boldsymbol{\tau}_1^{ext} - \mathbf{C}_{1b} \boldsymbol{\tau}_b^1. \end{aligned}$$

$\mathbf{F}_1^{ext}$ and $\boldsymbol{\tau}_1^{ext}$ are for external forces and torques acting on panel 1 respectively. $\mathbf{F}_1^b$ is the force applied to panel 1 by the body and $\boldsymbol{\tau}_1^b$ is the torque applied to the panel by the body. By Newton's 3rd law,

$$\mathbf{F}_1^b = -\mathbf{F}_b^1, \quad \boldsymbol{\tau}_1^b = -\mathbf{C}_{1b}\boldsymbol{\tau}_b^1.$$

Likewise,

$$\mathbf{f}_2 = \begin{bmatrix} \mathbf{F}_2 \\ \boldsymbol{\tau}_2 \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{F}_2 &= \mathbf{F}_2^{ext} + \mathbf{F}_2^b = \mathbf{F}_2^{ext} - \mathbf{F}_b^2 \\ \boldsymbol{\tau}_2 &= \boldsymbol{\tau}_2^{ext} + \boldsymbol{\tau}_2^b = \boldsymbol{\tau}_2^{ext} - \mathbf{C}_{2b}\boldsymbol{\tau}_b^2 \end{aligned}$$

and

$$\mathbf{f}_w = \begin{bmatrix} \mathbf{F}_w \\ \boldsymbol{\tau}_w \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{F}_w &= \mathbf{F}_w^{ext} + \mathbf{F}_w^b = \mathbf{F}_w^{ext} - \mathbf{F}_b^w \\ \boldsymbol{\tau}_w &= \boldsymbol{\tau}_w^{ext} + \boldsymbol{\tau}_w^b = \boldsymbol{\tau}_w^{ext} - \mathbf{C}_{wb}\boldsymbol{\tau}_b^w. \end{aligned}$$

Therefore, we have

$$\hat{\mathbf{f}} = \hat{\mathbf{f}}^{ext} + \hat{\mathbf{f}}^{const}$$

where

$$\hat{\mathbf{f}}^{ext} = \begin{bmatrix} \mathbf{F}_b^{ext} \\ \tau_b^{ext} \\ \mathbf{F}_1^{ext} \\ \tau_1^{ext} \\ \mathbf{F}_2^{ext} \\ \tau_2^{ext} \\ \mathbf{F}_w^{ext} \\ \tau_w^{ext} \end{bmatrix}, \quad \hat{\mathbf{f}}^{const} = \begin{bmatrix} \mathbf{F}_b^1 + \mathbf{F}_b^2 + \mathbf{F}_b^w \\ \tau_b^1 + \tau_b^2 + \tau_b^2 \\ \mathbf{F}_1^b \\ \tau_1^b \\ \mathbf{F}_2^b \\ \tau_2^b \\ \mathbf{F}_w^b \\ \tau_w^b \end{bmatrix}.$$

$\hat{\mathbf{f}}^{ext}$ are external forces and torques acting on the system and $\hat{\mathbf{f}}^{const}$ are the inertial forces and torques required to satisfy the kinematical constraints. Note that we include all external control torques, dissipative torques and spring torques in $\hat{\mathbf{f}}^{ext}$. The generalized force acting on the system is then

$$\hat{\Pi}^T \hat{\mathbf{f}} = \hat{\Pi}^T \hat{\mathbf{f}}^{ext} + \hat{\Pi}^T \hat{\mathbf{f}}^{const}.$$

The generalized constraint force satisfies

$$\Xi^T \boldsymbol{\lambda} = \hat{\Pi}^T \hat{\mathbf{f}}^{const}$$

where $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers[28]. Therefore, the equations of motion for the constrained system become

$$\hat{\Pi}^T \hat{\mathbf{M}} \dot{\hat{\mathbf{v}}} + \hat{\Pi}^T \dot{\hat{\mathbf{M}}} \hat{\mathbf{v}} + \hat{\Pi}^T \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \hat{\mathbf{v}} - \hat{\Pi}^T \hat{\boldsymbol{\alpha}} = \Xi^T \boldsymbol{\lambda} + \hat{\Pi}^T \hat{\mathbf{f}}^{ext}.$$

Pre-multiplying by $\boldsymbol{\Gamma}^T$, this becomes

$$\boldsymbol{\Lambda}^T \hat{\mathbf{M}} \boldsymbol{\Lambda} \ddot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^T \dot{\hat{\mathbf{M}}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^T \dot{\hat{\mathbf{M}}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^T \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} - \boldsymbol{\Lambda}^T \hat{\boldsymbol{\alpha}} = \boldsymbol{\Lambda}^T \hat{\mathbf{f}}^{ext}$$

which is Eq. (2.16) from before. Let us now further examine $\hat{\mathbf{f}}^{ext}$. $\mathbf{F}_b^{ext}$, $\mathbf{F}_1^{ext}$, $\mathbf{F}_2^{ext}$ and $\mathbf{F}_w^{ext}$ are all external forces acting on the body, panel 1, panel 2 and wheel respectively. We can split τ_1^{ext} and τ_2^{ext} into two components: the torque from external sources such as

gravity-gradient, aerodynamic, etc. and the dissipative and spring torques at the hinge. Thus

$$\begin{aligned}\boldsymbol{\tau}_1^{ext} &= \hat{\boldsymbol{\tau}}_1^{ext} + \boldsymbol{\tau}_1^{rest} + \boldsymbol{\tau}_1^{damp} \\ \boldsymbol{\tau}_2^{ext} &= \hat{\boldsymbol{\tau}}_2^{ext} + \boldsymbol{\tau}_2^{rest} + \boldsymbol{\tau}_2^{damp}.\end{aligned}$$

$\boldsymbol{\tau}_w^{ext}$ can be similarly decomposed into

$$\boldsymbol{\tau}_w^{ext} = \hat{\boldsymbol{\tau}}_w^{ext} + \boldsymbol{\tau}_w^{cont}$$

where $\hat{\boldsymbol{\tau}}_w^{ext}$ are torques due to external sources and $\boldsymbol{\tau}_w^{cont}$ is a control torque applied to the wheel. Finally, $\boldsymbol{\tau}_b^{ext}$ can also be decomposed as

$$\boldsymbol{\tau}_b^{ext} = \hat{\boldsymbol{\tau}}_b^{ext} + \boldsymbol{\tau}_b^{cont} - \mathbf{C}_{b1}\boldsymbol{\tau}_1^{rest} - \mathbf{C}_{b1}\boldsymbol{\tau}_1^{damp} - \mathbf{C}_{b2}\boldsymbol{\tau}_2^{rest} - \mathbf{C}_{b2}\boldsymbol{\tau}_2^{damp} - \mathbf{C}_{bw}\boldsymbol{\tau}_w^{cont}$$

where $\hat{\boldsymbol{\tau}}_b^{ext}$ are torques from external sources such as disturbances. $\boldsymbol{\tau}_b^{cont}$ are control torques from magnetorquers and all other terms are due to Newton's 3rd law. Now let us examine the term $\boldsymbol{\Lambda}^T \hat{\mathbf{f}}^{ext}$.

$$\begin{aligned}\boldsymbol{\Lambda}^T \hat{\mathbf{f}}^{ext} &= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{r}_b^{1b \times} \mathbf{C}_{bi} & \mathbf{C}_{1b}^T & \mathbf{r}_b^{2b \times} \mathbf{C}_{bi} & \mathbf{C}_{2b}^T & \mathbf{r}_b^{wb \times} \mathbf{C}_{bi} & \mathbf{C}_{wb}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{S}}_1^{1bT} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{S}}_2^{2bT} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{F}_b^{ext} \\ \boldsymbol{\tau}_b^{ext} \\ \mathbf{F}_1^{ext} \\ \boldsymbol{\tau}_1^{ext} \\ \mathbf{F}_2^{ext} \\ \boldsymbol{\tau}_2^{ext} \\ \mathbf{F}_w^{ext} \\ \boldsymbol{\tau}_w^{ext} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}_b^{ext} + \mathbf{F}_1^{ext} + \mathbf{F}_2^{ext} + \mathbf{F}_w^{ext} \\ \boldsymbol{\tau}_b^{ext} + \mathbf{r}_b^{1b \times} \mathbf{C}_{bi} \mathbf{F}_1^{ext} + \mathbf{C}_{b1} \boldsymbol{\tau}_1^{ext} + \mathbf{r}_b^{2b \times} \mathbf{C}_{bi} \mathbf{F}_2^{ext} + \mathbf{C}_{b2} \boldsymbol{\tau}_2^{ext} + \mathbf{r}_b^{wb \times} \mathbf{C}_{bi} \mathbf{F}_w^{ext} + \mathbf{C}_{bw} \boldsymbol{\tau}_w^{ext} \\ \tilde{\mathbf{S}}_1^{1bT} \boldsymbol{\tau}_1^{ext} \\ \tilde{\mathbf{S}}_2^{2bT} \boldsymbol{\tau}_2^{ext} \\ \mathbf{1}_2^T \boldsymbol{\tau}_w^{ext} \end{bmatrix}.\end{aligned}$$

Let us now look at this term by term.

$$\mathbf{F}_{s/c}^{ext} = \mathbf{F}_b^{ext} + \mathbf{F}_1^{ext} + \mathbf{F}_2^{ext} + \mathbf{F}_w^{ext}$$

are just all external forces acting on the spacecraft.

$$\begin{aligned} & \boldsymbol{\tau}_b^{ext} + \mathbf{r}_b^{1b \times} \mathbf{C}_{bi} \mathbf{F}_1^{ext} + \mathbf{C}_{b1} \boldsymbol{\tau}_1^{ext} + \mathbf{r}_b^{2b \times} \mathbf{C}_{bi} \mathbf{F}_2^{ext} \\ & + \mathbf{C}_{b2} \boldsymbol{\tau}_2^{ext} + \mathbf{r}_w^{wb \times} \mathbf{C}_{bi} \mathbf{F}_w^{ext} + \mathbf{C}_{bw} \boldsymbol{\tau}_w^{ext} \\ & = \hat{\boldsymbol{\tau}}_b^{ext} + \boldsymbol{\tau}_b^{cont} - \mathbf{C}_{b1} \boldsymbol{\tau}_1^{rest} - \mathbf{C}_{b1} \boldsymbol{\tau}_1^{damp} - \mathbf{C}_{b2} \boldsymbol{\tau}_2^{rest} - \mathbf{C}_{b2} \boldsymbol{\tau}_2^{damp} - \mathbf{C}_{bw} \boldsymbol{\tau}_w^{cont} \\ & + \mathbf{r}_b^{1b \times} \mathbf{C}_{bi} \mathbf{F}_1^{ext} + \mathbf{C}_{b1} \hat{\boldsymbol{\tau}}_1^{ext} + \mathbf{C}_{b1} \boldsymbol{\tau}_1^{rest} + \mathbf{C}_{b1} \boldsymbol{\tau}_1^{damp} \\ & + \mathbf{r}_b^{2b \times} \mathbf{C}_{bi} \mathbf{F}_2^{ext} + \mathbf{C}_{b2} \hat{\boldsymbol{\tau}}_2^{ext} + \mathbf{C}_{b2} \boldsymbol{\tau}_2^{rest} + \mathbf{C}_{b2} \boldsymbol{\tau}_2^{damp} \\ & + \mathbf{r}_b^{wb \times} \mathbf{C}_{bi} \mathbf{F}_w^{ext} + \mathbf{C}_{bw} \hat{\boldsymbol{\tau}}_w^{ext} + \mathbf{C}_{bw} \boldsymbol{\tau}_w^{cont} \end{aligned}$$

The right side of the equation then becomes

$$= \hat{\boldsymbol{\tau}}_{s/c}^{ext} + \boldsymbol{\tau}_b^{cont}$$

where

$$\hat{\boldsymbol{\tau}}_{s/c}^{ext} = \boldsymbol{\tau}_b^{ext} + \mathbf{C}_{b1} \hat{\boldsymbol{\tau}}_1^{ext} + \mathbf{r}_b^{1b \times} \mathbf{C}_{bi} \mathbf{F}_1^{ext} + \mathbf{C}_{b2} \hat{\boldsymbol{\tau}}_2^{ext} + \mathbf{r}_b^{2b \times} \mathbf{C}_{bi} \mathbf{F}_2^{ext} + \mathbf{C}_{bw} \hat{\boldsymbol{\tau}}_w^{ext} + \mathbf{r}_b^{wb \times} \mathbf{C}_{bi} \mathbf{F}_w^{ext}$$

is the total torque from external sources on the spacecraft about the origin of $\mathcal{F}_b$. Continuing to look at individual terms we have

$$\begin{aligned} \tilde{\mathbf{S}}_1^{1b \top} \boldsymbol{\tau}_1^{ext} &= \tilde{\mathbf{S}}_1^{1b \top} \hat{\boldsymbol{\tau}}_1^{ext} + \tilde{\mathbf{S}}_1^{1b \top} [\boldsymbol{\tau}_1^{rest} + \boldsymbol{\tau}_1^{damp}] \\ \tilde{\mathbf{S}}_2^{2b \top} \boldsymbol{\tau}_2^{ext} &= \tilde{\mathbf{S}}_2^{2b \top} \hat{\boldsymbol{\tau}}_2^{ext} + \tilde{\mathbf{S}}_2^{2b \top} [\boldsymbol{\tau}_2^{rest} + \boldsymbol{\tau}_2^{damp}] \\ \mathbf{1}_2^\top \boldsymbol{\tau}_w^{ext} &= \mathbf{1}_2^\top \hat{\boldsymbol{\tau}}_w^{ext} + \mathbf{1}_2^\top \boldsymbol{\tau}_w^{cont} \end{aligned}$$

Now, we have

$$\begin{aligned} \boldsymbol{\tau}_1^{rest} &= \begin{bmatrix} -k_{1x}\theta_x^{1b} \\ -k_{1y}\theta_y^{1b} \\ 0 \end{bmatrix}, \quad \boldsymbol{\tau}_1^{damp} = \begin{bmatrix} -d_{1x}\dot{\theta}_x^{1b} \\ -d_{1y}\dot{\theta}_y^{1b} \\ 0 \end{bmatrix}, \quad \boldsymbol{\tau}_2^{rest} = \begin{bmatrix} -k_{2x}\theta_x^{2b} \\ -k_{2y}\theta_y^{2b} \\ 0 \end{bmatrix}, \quad \boldsymbol{\tau}_2^{damp} = \begin{bmatrix} -d_{2x}\dot{\theta}_x^{2b} \\ -d_{2y}\dot{\theta}_y^{2b} \\ 0 \end{bmatrix}, \\ \boldsymbol{\tau}_w^{cont} &= \begin{bmatrix} 0 \\ \tau_w^{cont} \\ 0 \end{bmatrix} \end{aligned}$$

where k_{1x} , k_{1y} , k_{2x} and k_{2y} are the spring coefficients about the x and y axes of panels 1 and 2 respectively and d_{1x} , d_{1y} , d_{2x} and d_{2y} are the damping coefficients of the springs about the x and y axes of panels 1 and 2 respectively. The damping coefficients are $d_{1x} = 2\zeta\sqrt{k_{1x}/J_{1x}}$ where ζ is the damping ratio and J_{1x} is the moment of inertia of panel 1 about its x axis. The other damping coefficients are similarly defined. Now,

$$\tilde{\mathbf{S}}_1^{1b\top} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x^{1b}) & -\sin(\theta_x^{1b}) \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ \cos(\theta_x^{1b})b \\ 0 \end{bmatrix}$$

and

$$\mathbf{1}_2^\top \begin{bmatrix} 0 \\ \tau_w^{cont} \\ 0 \end{bmatrix} = \tau_w^{cont}.$$

Summarizing, we have

$$\boldsymbol{\Lambda}^\top \hat{\mathbf{f}}^{ext} = \mathbf{f}_{sys} + \mathbf{f}_{cont} + \mathbf{f}_{dist}$$

where

$$\mathbf{f}_{sys} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -k_{1x}\theta_x^{1b} - d_{1x}\dot{\theta}_x^{1b} \\ -\cos(\theta_x^{1b})[k_{1y}\theta_y^{1b} + d_{1y}\dot{\theta}_y^{1b}] \\ -k_{2x}\theta_x^{2b} - d_{2x}\dot{\theta}_x^{2b} \\ -\cos(\theta_x^{2b})[k_{2y}\theta_y^{2b} + d_{2y}\dot{\theta}_y^{2b}] \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{f}_{cont} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\tau}_b^{cont} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\tau}_w^{cont} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{dist} = \begin{bmatrix} \mathbf{F}_{s/c}^{ext} \\ \hat{\boldsymbol{\tau}}_{s/c}^{ext} \\ \tilde{\mathbf{S}}_1^{1b\top} \hat{\boldsymbol{\tau}}_1^{ext} \\ \tilde{\mathbf{S}}_2^{2b\top} \hat{\boldsymbol{\tau}}_2^{ext} \\ \mathbf{1}_2^\top \hat{\boldsymbol{\tau}}_w^{ext} \end{bmatrix}. \quad (2.20)$$

$\mathbf{f}_{sys}$ contains damping and restoring forces, $\mathbf{f}_{cont}$ contains control torque terms and $\mathbf{f}_{dist}$ contains all external disturbances.

The equations of motion for the entire system, including forces can now be expressed as

$$\boldsymbol{\Lambda}^\top \hat{\mathbf{M}} \boldsymbol{\Lambda} \ddot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^\top \hat{\mathbf{M}} \dot{\boldsymbol{\Lambda}} \dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^\top \dot{\hat{\mathbf{M}}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}^\top \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \boldsymbol{\Lambda} \dot{\boldsymbol{\eta}} - \boldsymbol{\Lambda}^\top \hat{\boldsymbol{\alpha}} = \mathbf{f}_{sys} + \mathbf{f}_{cont} + \mathbf{f}_{dist}. \quad (2.21)$$

These equations, which govern the multibody system were derived using Lagrangian methods and the Natural Orthogonal Complement method. With the equations of motion in hand, it makes it possible to simulate the satellite dynamics using numerical integration. It also gives a starting point to develop the linearized system and the control design which is presented in the following chapter.

Chapter 3

Control Design

This chapter outlines the various control laws that are compared. This includes a brief introduction to the controllers and their ability to stabilize the system as well as other advantages and disadvantages. Four linear controllers are compared, all of which use a state space framework. The first and simplest is a time-invariant controller which assumes the satellite to be a rigid body. A similar control law is to be used for the JC2Sat. The second is also time-invariant, but takes into account the flexibility of the satellite by attempting to actively suppress the panel vibrations by using an observer to estimate the unmeasured panels states. An advantage of time-invariant controllers is that only one matrix of gain values is required to be stored on board the satellite. Two more complicated, time-varying controllers, take advantage of the roughly periodic nature of the Earth's magnetic field to determine a time periodic matrix of gain values. One drawback is the large amount of on board data storage required to save the gain values over time, which may make the controller infeasible[29]. However, the periodic nature of the gain values makes it possible to replace the matrices of gain values using Fourier Series approximations[30] which greatly reduces the data storage requirements. These two controllers are similar to the first two, with one treating the satellite as rigid and the other trying to actively control the vibrations. The four controllers will be referred

to as time-invariant rigid (TIR), time-invariant vibration suppression (TIVS), periodic rigid (PR) and periodic vibration suppression (PVS) respectively.

3.1 Linearization

In order to design each of the controllers, the system must first be represented in state space form and linearized. This section presents the state space model used for each of the control law designs. Before proceeding, a quick note is needed on the choice of attitude parameters. Although the derivation of the system dynamics uses Euler angles, to avoid any singularities, the simulation and control design uses Euler parameters[24] (or quaternions) $\boldsymbol{\epsilon} = [\epsilon_1 \ \epsilon_2 \ \epsilon_3]^\top$ and ϵ_4 which satisfy the following relations

$$\boldsymbol{\epsilon}^\top \boldsymbol{\epsilon} + \epsilon_4^2 = 1, \ \dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\boldsymbol{\epsilon}^\times + \epsilon_4 \mathbf{1})\boldsymbol{\omega}, \text{ and } \dot{\epsilon}_4 = -\frac{1}{2}\boldsymbol{\epsilon}^\top \boldsymbol{\omega}.$$

For the linearized model, the small angle and small rate approximations, small $\boldsymbol{\epsilon}$, $\epsilon_4 \approx 1$ and small $\boldsymbol{\omega}$ are used, leading to the linearized kinematics $\dot{\boldsymbol{\epsilon}} \approx \frac{1}{2}\boldsymbol{\omega}$ and $\dot{\epsilon}_4 \approx 0$. For a rigid body with the origin of the body frame at the centre of mass, the linearization begins with Euler's equation,

$$\mathbf{J}_r \dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times (\mathbf{J}_r \boldsymbol{\omega} + \mathbf{h}_w) = \boldsymbol{\tau}$$

where $\mathbf{J}_r$ is the moment of inertia matrix of the satellite body and panels combined if the entire satellite was treated as rigid. $\mathbf{h}_w = [0 \ h_w \ 0]^\top$ is the bias momentum vector and h_w is the angular momentum of the bias momentum wheel about the pitch axis. Here, all vectors are expressed in the body fixed frame. Noting that h_w will change when the wheel applies a torque, for the purposes of the control design it is assumed that h_w is constant since the change in h_w is small compared to its nominal value. As shown in Figure 3.1 for a nadir pointing satellite in a circular orbit, the linearization point has a constant angular velocity about the pitch axis, $-\omega_0$, equal to one rotation per orbit. The linearization hence assumes $\boldsymbol{\epsilon}(t) = \boldsymbol{\epsilon}_0 + \delta\boldsymbol{\epsilon}(t)$ and $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega}(t)$ where $\boldsymbol{\epsilon}_0 = \mathbf{0}$

Figure 3.1: Satellite orbital frame.

and $\boldsymbol{\omega}_0 = [0 \ -\omega_0 \ 0]^\top$. A figure showing the orbital frame about which the system is linearized is shown in Figure 3.1. The state space representation is

$$\begin{aligned} \dot{\mathbf{x}} &= \underbrace{\begin{bmatrix} \mathbf{O} & 0.5 \cdot \mathbf{1} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \delta \epsilon \\ \delta \omega \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{O} \\ \mathbf{J}_r^{-1} \begin{bmatrix} \mathbf{1}_2 & \mathbf{1} \end{bmatrix} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} u_w \\ \mathbf{u}_m \end{bmatrix}}_{\mathbf{u}}, \\ \mathbf{y} &= \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} \end{aligned} \quad (3.1)$$

where

$$\mathbf{A}_{22} = \mathbf{J}_r^{-1} \begin{bmatrix} 0 & 0 & h_w + \omega_0(J_{rz} - J_{ry}) \\ 0 & 0 & 0 \\ -h_w + \omega_0(J_{ry} - J_{rx}) & 0 & 0 \end{bmatrix}. \quad (3.2)$$

J_{rx} , J_{ry} and J_{rz} are the moment of inertias about the x, y and z axes respectively, assuming a principal axes frame for $\mathbf{J}_r$. Note from $\mathbf{C}$ there is full state-feedback. u_w and $\mathbf{u}_m$ are the desired control torques provided by the reaction wheel and magnetorquers respectively. The actual wheel torque, $\boldsymbol{\tau}_w = [0 \ u_w \ 0]^\top$, is the same as the desired torque. Due to the restriction of the magnetic control, the actual magnetic torque, $\boldsymbol{\tau}_m$, is calculated using Eqs. (1.1) and (1.2). Eqs. (3.1) and (3.2) are used in the design of the TIR controller.

For the PR controller the $\mathbf{B}(t)$ matrix is now time varying to take into account the

local magnetic field vector. The $\mathbf{A}$ and $\mathbf{C}$ matrices are the same as in the previous case, however the new $\mathbf{B}(t)$ and $\mathbf{u}$ for the PR controller are

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{J}_r^{-1} \begin{bmatrix} \mathbf{1}_2 & -\hat{\mathbf{b}}_b^\times \end{bmatrix} \end{bmatrix}, \text{ and } \mathbf{u} = \begin{bmatrix} u_w \\ \hat{\mathbf{m}} \end{bmatrix}$$

where $\hat{\mathbf{b}}_b = \frac{\mathbf{b}_b}{\|\mathbf{b}_b\|}$ and $\mathbf{m} = \|\mathbf{b}_b\| \hat{\mathbf{m}}$. To ensure the magnetic dipole moment lies in the plane orthogonal to $\mathbf{b}_b$, the perpendicular component of it, $\mathbf{m}_\perp = \hat{\mathbf{b}}_b^\times \hat{\mathbf{b}}_b^\times \mathbf{m}$, is used.

For the TIVS and PVS controllers the equations of motion for the entire system are used in the control design. They are repeated here for convenience,

$$\Lambda^\top \hat{\mathbf{M}} \Lambda \ddot{\boldsymbol{\eta}} + \underbrace{\Lambda^\top \hat{\mathbf{M}} \dot{\Lambda} \dot{\boldsymbol{\eta}} + \Lambda^\top \hat{\mathbf{M}} \Lambda \dot{\boldsymbol{\eta}} + \Lambda^\top \hat{\boldsymbol{\Omega}} \hat{\mathbf{M}} \Lambda \dot{\boldsymbol{\eta}} + \Lambda^\top \hat{\boldsymbol{\alpha}}}_{\text{negligible}} = \mathbf{f}_{sys} + \mathbf{f}_{cont} + \mathbf{f}_{dist}. \quad (3.3)$$

When small angles and angular rates are assumed, the terms within the underbrace are negligible when compared to the system mass matrix, $(\Lambda^\top \hat{\mathbf{M}} \Lambda)$. The system mass matrix contains the terms which relate to the translational motion of the body as well as the rotational motion of the reaction wheel. Since neither are used in the control design (the reaction wheel is assumed to be constant for the control design), a system mass matrix, which is trimmed such that it only corresponds to the attitude and angular velocity of the the body and the panel deflections and their rates, is defined as $\tilde{\mathbf{M}}$. It is evaluated at the linearization point of zero attitude and angular velocity error and zero panel deflections and panel rates. For large panel deflections, the satellite body will have a rigid body heave component to its motion and the translational and rotational motion will be slightly coupled. For the control design, assuming small deflections, this coupling is assumed to be zero. The states which are used for feedback (those which correspond to $\tilde{\mathbf{M}}$) are now included in the state vector such that $\mathbf{x}^\top = [\delta \boldsymbol{\epsilon}^\top \ \tilde{\boldsymbol{\theta}}^{1b\top} \ \tilde{\boldsymbol{\theta}}^{2b\top} \ \delta \boldsymbol{\omega}^\top \ \dot{\tilde{\boldsymbol{\theta}}}^{1b\top} \ \dot{\tilde{\boldsymbol{\theta}}}^{2b\top}]$. This enables the addition of both the hinge restoring and damping forces to the state space model.

The linearized model for the TIVS controller is now

$$\begin{aligned}\dot{\mathbf{x}} &= \underbrace{\begin{bmatrix} \mathbf{O} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} \mathbf{O} \\ \tilde{\mathbf{M}}^{-1} \begin{bmatrix} \mathbf{1}_2 & \mathbf{1} \\ \mathbf{O} \end{bmatrix} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}, \\ \mathbf{y} &= \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{1} & \mathbf{O} \end{bmatrix}}_{\mathbf{C}} \mathbf{x} = \begin{bmatrix} \delta\epsilon \\ \delta\omega \end{bmatrix}\end{aligned}$$

where $\mathbf{A}_{12} = \text{diag}\{0.5 \ 0.5 \ 0.5 \ 1 \ 1 \ 1 \ 1\}$. $\mathbf{A}_{21} = \tilde{\mathbf{M}}^{-1} \cdot \text{diag}\{0 \ 0 \ 0 \ -k_{1x} \ -k_{1y} \ -k_{2x} \ -k_{2y}\}$ includes the restoring forces due to the hinges where again the k terms are the spring coefficients. The restoring force terms are from $\mathbf{f}_{sys}$ defined in Eq. (2.20) but evaluated at the linearization point where $\cos(\theta_x^{1b}) = \cos(\theta_x^{2b}) = 0$. $\mathbf{A}_{22}$ includes the hinge damping forces as well as the terms due to the momentum bias and the one rotation per orbit due to the desired nadir pointing attitude so that

$$\mathbf{A}_{22} = \tilde{\mathbf{M}}^{-1} \begin{bmatrix} 0 & 0 & h_w + \omega_0(J_{bz} - J_{by}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -h_w + \omega_0(J_{by} - J_{bx}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_{1x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_{1y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d_{2x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_{2y} \end{bmatrix}$$

where the d terms are again the damping coefficients and are also from Eq. (2.20) where $\mathbf{f}_{sys}$ is evaluated at the linearization point. J_{bx} , J_{by} and J_{bz} are the moments of inertia of the central body about its x , y and z axes respectively. Note from $\mathbf{C}$ that only the attitude and angular velocity are measured (the panel deflections are not). For the PVS controller, the local magnetic field vector is included into the $\mathbf{B}(t)$ matrix in the same

fashion as the PR controller, where

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{M}}^{-1} \begin{bmatrix} \mathbf{1}_2 & -\hat{\mathbf{b}}_b^\times \\ \mathbf{0} \end{bmatrix} \end{bmatrix}.$$

3.2 Linear Quadratic Regulator

All four controllers use the Linear Quadratic Regulator (LQR) framework. To do this, full state-feedback is initially assumed and unmeasurable states are later obtained using an estimator. The performance objective to be minimized is defined as

$$\mathcal{J} = \frac{1}{2} \int_0^\infty (\mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u}) dt$$

where $\mathbf{Q}$ is positive-semidefinite, $\mathbf{R}$ is positive-definite, and the linearized system in state-space form is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A} \mathbf{x} + \mathbf{B}(t) \mathbf{u}, \\ \mathbf{y} &= \mathbf{C} \mathbf{x}. \end{aligned}$$

For the linear systems discussed in this work, $\mathbf{B}$ can be time-varying or time-invariant, whereas, $\mathbf{A}$ and $\mathbf{C}$ are always time-invariant. It can be shown[31] that the optimal control that minimizes the performance index is given by

$$\mathbf{u}(t) = -\mathbf{R}^{-1} \mathbf{B}^\top(t) \mathbf{P}(t) \mathbf{x} \quad (3.4)$$

where $\mathbf{P}(t)$, which is symmetric and positive semi-definite, is the solution to the Riccati Equation[32] given here

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t) \mathbf{A} + \mathbf{A}^\top \mathbf{P}(t) - \mathbf{P}(t) \mathbf{B}(t) \mathbf{R}^{-1} \mathbf{B}^\top(t) \mathbf{P}(t) + \mathbf{Q}. \quad (3.5)$$

The time-invariant and periodic controllers lead to two specific solutions to Eq. (3.5). The time-invariant case, with constant $\mathbf{B}$ and $\mathbf{P}$, leads to the Algebraic Riccati Equation (ARE)

$$\mathbf{P} \mathbf{A} + \mathbf{A}^\top \mathbf{P} - \mathbf{P} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

Here, if $(\mathbf{A}, \mathbf{B})$ is stabilizable and $(\mathbf{Q}^{\frac{1}{2}}, \mathbf{A})$ is detectable then the ARE has a unique positive semi-definite solution $\mathbf{P}$ and moreover, the feedback given by Eq. (3.4) stabilizes $(\mathbf{A}, \mathbf{B})$ [31]. This, however, does not guarantee the stabilization of the true non-linear system, particularly when it is time-varying as it is when using magnetic attitude control. The Algebraic Riccati Equation is solved using the Matlab command `lqr`. For the periodic cases, it is assumed that $\mathbf{B}(t) = \mathbf{B}(t + T)$ is T -periodic, where T is the orbital period. Under these conditions, the optimal control is determined using the solution to the Periodic Riccati Equation (PRE), which is the same as Eq. (3.5) but with $\mathbf{P}(t) = \mathbf{P}(t + T)$. The solution to the PRE is found by numerically integrating the PRE backward in time, given a set of terminal conditions, until $\mathbf{P}(t)$ converges to a T -periodic solution. The feedback given by Eq. (3.4) stabilizes the periodic closed-loop system. This applies to the time-varying system unlike the previous control law, however, it remains unable to guarantee global stability of the non-linear system.

3.3 Observer Design

As mentioned in section 3.1, only the body states, and not the panel states can be measured. In order to actively suppress the panel vibrations, the panel states must be estimated. An observer is used to accomplish this as well as being used to filter noise. Hence, a full state observer is used for even the rigid controllers which measure the full state vector. The observer takes the following form[31]

$$\dot{\hat{\mathbf{x}}}_e = \mathbf{A}\hat{\mathbf{x}}_e + \mathbf{B}\boldsymbol{\tau} + \mathbf{L}(\mathbf{C}\hat{\mathbf{x}}_e - \mathbf{C}\mathbf{x}_e) \quad (3.6)$$

where $\mathbf{x}_e$ is the error vector or the difference between the actual states, $\mathbf{x}$, and the desired ones and $\hat{\mathbf{x}}_e$ is the estimate of the error vector. The distinction between $\mathbf{x}_e$ and $\mathbf{x}$ is made because during the half of the orbit where the desired attitude is pitched forward to increase the drag coefficient $\mathbf{x}_e$ and $\mathbf{x}$ are different. They are however the same while the desired attitude is nadir pointing. Note that there is not always a distinction between

the applied and desired torque, but because of the restriction on the available torque when using magnetic attitude control, the applied torque is not in general equal to the desired one, and in this case $\boldsymbol{\tau}$ is the applied torque. In order to select $\mathbf{L}$, a Steady State Kalman Filter is used. If the measurement noise and process noise are assumed to be zero mean Gaussian noise, then it can be shown[31] that the optimal $\mathbf{L}$ is given by

$$\mathbf{L} = -\mathbf{P}_e \mathbf{C}^T \mathbf{Q}_v^{-1}$$

where $\mathbf{P}_e$ is the solution to a similar Algebraic Riccati Equation to the one mentioned earlier,

$$\mathbf{P}_e \mathbf{A}^T + \mathbf{A} \mathbf{P}_e - \mathbf{P}_e \mathbf{C}^T \mathbf{Q}_v^{-1} \mathbf{C} \mathbf{P}_e + \mathbf{Q}_w = \mathbf{O}$$

where $\mathbf{Q}_w$ and $\mathbf{Q}_v$ are the covariances of the process noise and measurement noise respectively. The assumption of zero mean Gaussian noise on the measurement signal is reasonable, however the same cannot be said for assuming the process noise is also zero mean Gaussian. Hence, the values of $\mathbf{Q}_v$ and $\mathbf{Q}_w$ (particularly $\mathbf{Q}_w$) are tuned using simulation results. The above assumes the system $(\mathbf{C}, \mathbf{A})$ is observable. Because both drag panels are identical, the spring coefficients are assumed to be slightly different in order to ensure observability.

3.4 Fourier Series Approximation

As mentioned in Section 3.2 the periodic controllers use a solution to the Periodic Riccati equation, which is time varying and periodic. The time varying nature of the control gains requires the additional storage of the $\mathbf{P}(t)$ matrices over the orbit. The periodic nature of $\mathbf{P}(t)$ makes it possible to be easily approximated using Fourier Series Approximations[30]. Each entry of the $\mathbf{P}(t)$ matrix is converted from a set of values into an equation that is a function of the position in the orbit. Since the $\mathbf{P}(t)$ values are given at discrete time

values Δt , $\mathbf{P}(t)$ can be approximated over an orbit by the following

$$\mathbf{P}_{FS}(t_n) = \frac{1}{2}\mathbf{A}_0 + \sum_{p=1}^K [\mathbf{A}_p \cos(\omega_p t) + \mathbf{B}_p \sin(\omega_p t)] \quad (3.7)$$

where the time, time step and trigonometric arguments are respectively given by

$$t_n = n\Delta t, \quad \Delta t = \frac{T}{N}, \quad \text{and} \quad \omega_p t = \frac{2\pi p n}{T}$$

where T is the orbital period, N is the number of time steps per orbit and K is the number of coefficients used to approximate the function. The larger the number of coefficients the better the approximation is. The coefficients are determined as a function of $\mathbf{P}(t)$ and are given as

$$\begin{aligned} \mathbf{A}_p &= \frac{2}{N} \sum_{n=1}^N \mathbf{P}(t_n) \cos\left(\frac{2\pi p n}{N}\right), \quad p = 1, \dots, K \\ \mathbf{B}_p &= \frac{2}{N} \sum_{n=1}^N \mathbf{P}(t_n) \sin\left(\frac{2\pi p n}{N}\right), \quad p = 1, \dots, K \\ \mathbf{A}_o &= \frac{1}{N} \sum_{n=1}^N \mathbf{P}(t_n). \end{aligned}$$

The amount of on board storage required can be further reduced because $\mathbf{P}$ is symmetric and hence only the lower triangle of $\mathbf{P}$ is needed. Also, by observing the $\mathbf{B}^T \mathbf{P}$ term in Eq. (3.4) and noticing that the upper half of the $\mathbf{B}$ matrices are zero, only the lower half of the lower triangle of $\mathbf{P}$ is needed. The Fourier Series approximations are used in the simulations to determine the control torque.

3.5 Summary

This chapter presents the details of the various control designs used to stabilize the system. It outlines how the unmeasurable states are estimated and also how signal noise is filtered. The results of linearizing the equations of motion of the system are presented in state space form and it is shown how the on board data storage requirements can be

reduced. This chapter and the previous one provide the background needed to present the simulation details and results presented in Chapters 4 and 5 respectively.

Chapter 4

Simulation Scenarios and Controller Design

This chapter presents the details of the numerous simulations that were performed. This includes the magnetic field and orbital models, the desired attitude as well the disturbance torques and signal noise added to the system. It also includes the satellite properties and control weighting values. Lastly, the solution to the Periodic Riccati Equation is examined. As mentioned in Section 1.2, the additional drag panels are added to the satellite in order to decrease the amount of time required to perform relative position maneuvers. It is also mentioned in Section 1.3 that two sizes of panels are simulated. The first set of panels is similar to those in the proposed JC2Sat design. This is to evaluate the design and assess if the additional flexibility in the system will adversely affect the attitude performance. It will be shown that the more complex controllers are not able to significantly improve the performance for the smaller panels because their higher natural frequencies are outside the controller bandwidth and the vibrations damp away relatively quickly. The second, a larger set, of panels is examined for two reasons. The first is to determine if the times for relative position maneuvers could be further reduced without significantly decreasing the attitude performance. The second is to examine if a satellite

with larger panels, because of their lower natural frequencies and larger mass relative to the mass of the satellite body, could benefit from the more complex controllers actively damping the vibrations.

As mentioned in Chapter 3, four controllers are examined, each under two sets of initial conditions. The first set of initial conditions are the ideal ones with zero initial attitude, angular velocity or panel angle errors such that $\epsilon = \mathbf{0}$, $\omega_b^{bi} = \mathbf{0}$, and $\tilde{\theta}_1^{1b} = \tilde{\theta}_2^{2b} = \mathbf{0}$. This is to ensure the controllers do not introduce any unwanted motion under nominal operation. The ‘worst case’ initial conditions have initial angular velocity errors as well as initial panel deflections such that $\epsilon = \mathbf{0}$, $\omega_b^{bi} = [0.02 \ 0.02 \ 0.02]^\top$ (rad/s), $\tilde{\theta}_1^{1b} = [20^\circ \ 20^\circ]^\top$ and $\tilde{\theta}_2^{2b} = [20^\circ \ -15^\circ]^\top$. These conditions are to test if the controllers are able to suppress panel vibrations while at the same time recovering from initial angular velocity errors. It is also to test the performance of the linear controllers away from their linearization point. Before getting to the results, a few details of the simulation environment are included here.

4.1 Satellite Orbit

It should be mentioned that for simulation purposes, the attitude dynamics and orbital dynamics are uncoupled. The orbital dynamics are simulated in order to determine the local magnetic field vector as a function of time, which is outlined in the following section. For each simulation the satellite orbit is assumed to be Keplerian with zero eccentricity, at an altitude of 700km, and an inclination of 87° . The right ascension of the ascending node and the initial time are zero. Before discussing the details of the satellite orbit, let us first define a few reference frames. Let $\mathcal{F}_e$ be the earth-centered inertial frame (ECI) as shown in Figure 4.1. Let $\mathcal{F}_p$ be the perifocal frame, which has $\hat{\mathbf{p}}_1$ pointed toward the ascending node, $\hat{\mathbf{p}}_3$ is normal to the orbital plane pointing in the same direction as the angular momentum vector and the origin is the same as $\mathcal{F}_e$. The position vector of the

Figure 4.1: Satellite position vector in the heliocentric inertial frame.

satellite from the origin of $\mathcal{F}_e$ expressed in $\mathcal{F}_e$ is[2]

$$\mathbf{r}_e^{se}(t) = \mathbf{C}_{pe}^T \begin{bmatrix} r \cos \psi \\ r \sin \psi \\ 0 \end{bmatrix}$$

where r is the magnitude of the distance from the center of the Earth and ψ is the angle of the satellite from the periapsis direction. The matrix rotating vectors from the perifocal frame to the earth-centered inertial frame is

$$\mathbf{C}_{pe} = \begin{bmatrix} c_\alpha c_\omega - s_\omega c_i s_\alpha & s_\alpha c_\omega + c_i s_\omega c_\alpha & s_i s_\omega \\ -c_\alpha s_\omega - c_\omega c_i s_\alpha & -s_\alpha s_\omega + c_\omega c_i c_\alpha & s_i c_\omega \\ s_\alpha s_i & -s_i c_\alpha & c_i \end{bmatrix}$$

where α is the right ascension of the ascending node, i is the inclination and ω is the argument of perigee. Since the eccentricity, e , is zero then r is constant and we can write

$$\psi = \sqrt{\frac{\mu}{r^3}}(t - t_0)$$

where μ is the standard gravitational parameter for the earth. We now have an expression for the position of the satellite as a function of time expressed in $\mathcal{F}_e$. Knowing $\mathbf{r}_e^{se}(t)$ will allow the local magnetic field vector be determined as a function of time, which is explained in the next section.

4.2 Magnetic Field Model

In order to simulate the magnetic attitude control, a model of the Earth's magnetic field is assumed. The same model is also used in solving the solution to the Periodic Riccati Equation. The model used is the Tilted Dipole model. This section outlines the details of the model as well as how the local magnetic field vector is determined as a function of the position of a satellite. For further details, see Ref. [2]. The majority of the Earth's magnetic field can be represented as the gradient of a scalar potential function, V , where the local magnetic field vector, $\mathbf{b}$, is given as

$$\mathbf{b} = -\nabla V$$

and V can be represented as a series of spherical harmonics,

$$V(r, \theta, \phi) = a \sum_{n=1}^k \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^n [g_n^m \cos(m\phi) + h_n^m \sin(m\phi)] P_n^m(\theta)$$

where g_n^m and h_n^m are Gaussian coefficients, r , θ and ϕ are the geocentric distance, coelevation and East longitude from Greenwich respectively, P_n^m are the associated Legendre functions and a is the equatorial radius of the Earth which, as adopted by the International Geomagnetic Reference Field (IGRF), is 6371.2 km. In order to simplify the model, V can be expanded to the first degree to obtain a tilted dipole model. It can be shown that the dipole field in local tangential coordinates is given by

$$\begin{aligned} b_r &= 2 \left(\frac{a}{r}\right)^3 [g_1^0 \cos \theta + (g_1^1 \cos \phi + h_1^1 \sin \phi) \sin \theta] \\ b_\theta &= \left(\frac{a}{r}\right)^3 [g_1^0 \sin \theta - (g_1^1 \cos \phi + h_1^1 \sin \phi) \cos \theta] \\ b_\phi &= \left(\frac{a}{r}\right)^3 [g_1^1 \cos \phi - h_1^1 \sin \phi] \end{aligned}$$

The components expressed in $\mathcal{F}_e$ can be expressed as $\mathbf{b}_e = [b_x \ b_y \ b_z]^\top$, where

$$\begin{aligned} b_x &= (b_r \cos \delta + b_\theta \sin \delta) \cos \alpha - B_\phi \sin \alpha \\ b_y &= (b_r \cos \delta + b_\theta \sin \delta) \sin \alpha + B_\phi \cos \alpha \\ b_z &= (b_r \sin \delta - b_\theta \cos \delta) \end{aligned}$$

and where $\delta = 90^\circ - \theta$ is the declination. ϕ is determined by

$$\phi = \alpha - \alpha_G$$

where α_G is the the right ascension of the Greenwich meridian which takes into account the rotation of the earth and hence it's magnetic field with respect to the earth-centered inertial (ECI) frame, $\mathcal{F}_e$. If we assume that at time, $t=0$, the Greenwich meridian points in the direction of the vernal equinox then

$$\alpha_G = \frac{d\alpha_G}{dt}t$$

where $d\alpha_G/dt$ is the average rotation of the Earth (360.9856469 deg/day). From Figure 4.1 it can be seen that

$$\tan \alpha = \left(\frac{r_2}{r_1} \right), \quad \tan \delta = \left(\frac{r_3}{\sqrt{r_1^2 + r_2^2}} \right).$$

Since $\mathbf{r}_e^{se}(t)$ is know, then $\mathbf{b}_e(t)$ can easily be determined by taking the inverses of the above equations so long as care is taken to ensure the signs of r_1 and r_2 are considered to ensure the correct quadrant is found.

In order to express $\mathbf{b}_e(t)$ in $\mathcal{F}_b$, $\mathcal{F}_b$ must be related to $\mathcal{F}_e$. This is done through a succession of rotation matrices. The body frame, as shown in Figure 2.1, under nominal operating conditions has the roll or x axis pointing in the along track direction, the pitch or y axis pointing normal to the orbital plane and the yaw or z axis nadir pointing. The inertial frame used in the derivation of the dynamics is chosen to have the same orientation as $\mathcal{F}_e$. The local magnetic field vector can now be expressed in the body frame as

$$\begin{aligned} \mathbf{b}_b &= \mathbf{C}_{be} \mathbf{b}_e \\ &= \mathbf{C}_{bp} \mathbf{C}_{pe} \mathbf{b}_e \end{aligned}$$

which gives the local magnetic field vector in the body frame. Note that since $\mathcal{F}_e$ and $\mathcal{F}_i$ have the same orientation then we have $\mathbf{C}_{be} = \mathbf{C}_{bi}$.

4.3 Satellite Properties

As mentioned in section 2.2, the satellite model consists of multiple rigid bodies constrained together using hinges that are treated as torsion springs. The details of the satellite properties are shown in Table 4.1. The rigid satellite moment of inertias refer to the satellite body and panels treated as if they were a single rigid body.

Property	Value
Body height, length and width (mm)	320, 361, 170
Panel length and width - large and small (mm)	1280, 1444 - 320, 361
Body and panel mass - large and small (kg)	15.5, 2.8 - 0.119
Body moment of inertia ($\text{kg}\cdot\text{m}^2$)	$\text{diag}\{0.52, 0.55, 0.47\}$
Large panel moment of inertia ($\text{kg}\cdot\text{m}^2$)	$\text{diag}\{0.38, 0.49, 0.87\}$
Small panel moment of inertia ($\text{kg}\cdot\text{m}^2$)	$\text{diag}\{0.0010, 0.0013, 0.0023\}$
Rigid satellite moment of inertia with large panels ($\text{kg}\cdot\text{m}^2$)	$\text{diag}\{5.5, 2.4, 6.3\}$
Rigid satellite moment of inertia with small panels ($\text{kg}\cdot\text{m}^2$)	$\text{diag}\{0.59, 0.59, 0.53\}$
Momentum bias ($\text{N}\cdot\text{m}\cdot\text{s}$)	-0.10
Large panel 1 flapping and torsion spring coefficients ($\text{N}\cdot\text{m}/\text{rad}$)	0.012, 0.012
Large panel 2 flapping and torsion spring coefficients ($\text{N}\cdot\text{m}/\text{rad}$)	0.006, 0.006
Small panel 1 flapping and torsion spring coefficients ($\text{N}\cdot\text{m}/\text{rad}$)	0.0024, 0.0024
Small panel 2 flapping and torsion spring coefficients ($\text{N}\cdot\text{m}/\text{rad}$)	0.0012, 0.0012
Damping ratio	0.005

Table 4.1: Satellite properties.

4.4 Simulation Details

The simulations were performed using a fourth order Runge-Kutta integrator in Matlab using a 0.1 second time step. In order to test the robustness of the system, signal noise is added as well as errors to the satellite properties and control torques. Gaussian noise with a standard deviation of 0.5° is added to the attitude measurements. Arbitrary 5%

adjustments are made to the satellite moment of inertia, the panel spring constants, and the applied control torque. The change in desired attitude about the pitch axis twice an orbit required during relative position maneuvers is 60° . For the first half of the orbit the desired pitch is zero with the satellite nadir pointing and during the second half of the orbit the desired attitude is pitched -60° about the pitch or y axis. A gravity-gradient torque is included as well as a residual magnetic dipole moment equal to $[0.1 \ -0.1 \ 0.3]^\top (\text{A}\cdot\text{m}^2)$. The residual magnetic dipole moment is based on a high-fidelity JC2Sat model, and is by far the greatest disturbance torque. Both the solar pressure and atmospheric drag torques are significantly less and are hence not included in the simulation. What is not omitted from the simulation is the need to measure the local magnetic field vector. For JC2Sat, the measurement is done using a three-axis magnetometer (TAM). Because the magnetic dipole from the magnetorquers interferes with the measurement, TAM's are either located far enough away from the magnetorquers (on a boom for instance) or the magnetorquers are momentarily turned off for the measurement to be taken. The latter is true for JC2Sat. The magnetorquer control system uses discrete pulse modulation. The pulse width, $\Delta t_{MTQ} = 0.15\text{sec}$. After removing the command voltage, a pause of 0.25sec is taken before taking the TAM readings. The time to take the TAM readings, $\Delta t_{TAM} = 0.1 \text{ sec}$. The timing is shown in Figure 4.2.

Figure 4.2: Magnetorquer and TAM timing.

The weightings for the observer designs are set to $\mathbf{Q}_w = \mathbf{Q}_v = 50 \cdot 10^{-3} \cdot \mathbf{1}$ for all the controllers where the size of $\mathbf{Q}_w$ and $\mathbf{Q}_v$ depends on the specific controller. For both the no panel and small panel cases, the rigid controller has the weightings $\mathbf{Q} = \text{diag}\{5 \ 5 \ 5 \ 1 \ 1 \ 1\}$ and $\mathbf{R} = \text{diag}\{10^5 \ 10^{9.3} \ 10^{9.3} \ 10^{9.3}\}$. For the vibration suppression

controllers with the small panels $\mathbf{Q} = \text{diag}\{5 \ 5 \ 5 \ 50 \ 50 \ 50 \ 50 \ 1 \ 1 \ 1 \ 50 \ 50 \ 50 \ 50\}$ and $\mathbf{R} = \text{diag}\{10^5 \ 10^{8.7} \ 10^{8.7} \ 10^{8.7}\}$. For the large panels, the control weightings are set to $\mathbf{Q} = \text{diag}\{5 \ 5 \ 5 \ 1 \ 1 \ 1\}$ and $\mathbf{R} = \text{diag}\{10^5 \ 10^{9.3} \ 10^{9.3} \ 10^{9.3}\}$ for both the time-invariant and periodic controllers. For the vibration suppression controllers $\mathbf{Q} = \text{diag}\{5 \ 5 \ 5 \ 50 \ 50 \ 50 \ 50 \ 1 \ 1 \ 1 \ 50 \ 50 \ 50 \ 50\}$ and $\mathbf{R} = \text{diag}\{10^5 \ 10^{8.6} \ 10^{8.6} \ 10^{8.6}\}$. The magnetorquers for the satellite with no panels or small panels saturate at $1.5 \text{ A}\cdot\text{m}^2$ while for the large panels saturate at $5 \text{ A}\cdot\text{m}^2$. The larger saturation limit is to account for the satellite's additional moment of inertia due to the larger panels.

4.5 Solving the PRE

In Chapter 3 the details of the four controllers are outlined. Two of those controllers are time-varying using the solution of the Periodic Riccati Equation (PRE) repeated here

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A} + \mathbf{A}^\top\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}\mathbf{B}^\top(t)\mathbf{P}(t) + \mathbf{Q}$$

where $\mathbf{P}(t)$ is used to determine the time-varying control gains. $\mathbf{P}(t)$ is found by numerically integrating the PRE backwards in time given some terminal condition $\mathbf{P}(t_f)$. The terminal condition for both the PR and PVS controllers is $\mathbf{P}(t_f) = \mathbf{1}$. The local magnetic field vector, $\mathbf{b}_b$, contained in the $\mathbf{B}$ matrices is determined assuming the satellite follows the desired attitude. Figure 4.3 shows the maximum eigenvalue of $\mathbf{P}(t)$ as a function of orbit for the PVS controller designed for small panels. The solution quickly converges to an approximately periodic one. It is only approximate because, due to the rotation of the earth, the ground track and hence magnetic field vector is slightly different each orbit. This results in $\mathbf{P}(t)$ not being truly periodic, however, the deviation is small and $\mathbf{P}(t)$ is taken from a single orbit, one where the ground track is the same as the first orbit of the simulations. Also included in the plot is the maximum eigenvalue of $\mathbf{P}$ for the TIVS controller. The eigenvalue of $\mathbf{P}(t)$ oscillates about the constant eigenvalue of $\mathbf{P}$, which lies slightly below the average of the max eigenvalue of $\mathbf{P}(t)$. This is also the

Figure 4.3: Maximum eigenvalue of $\mathbf{P}(t)$ versus orbit.

case for the individual elements of $\mathbf{P}$ and $\mathbf{P}(t)$ - the individual elements of $\mathbf{P}(t)$ oscillate about the corresponding elements of $\mathbf{P}$ and the average of the elements $\mathbf{P}(t)$ being slightly higher. This leads to a higher average control output using the PVS controller than the TIVS one for a given error for the chosen LQR weightings. The same can be said for the periodic controllers designed assuming a rigid satellite, as well as for the larger panels.

As mentioned in Section 3.4, the solution to the Periodic Riccati Equation is replaced by a Fourier Series approximation given by Eq. (3.7). For a high accuracy approximation $K=10$ is used for each element of $\mathbf{P}(t)$. As an example, Figure 4.4 shows the solution to $\mathbf{P}(t)$ compared the the Fourier Series approximation for the 8th diagonal element of $\mathbf{P}(t)$ which corresponds to the x axis of the body frame, $\mathbf{P}(8, 8)$. As the figure shows, the approximation agrees well with the true solution. This is typical for all elements of $\mathbf{P}(t)$.

Figure 4.4: P(8,8) Fourier Series Comparison.

Chapter 5

Results

A number of results are presented in this chapter. The first set compares the performance of the satellite with small additional panels to the original rigid satellite both using the TIR controller. The second set of results compares all four controllers with small panels and the last set compares the controllers with large panels. As mentioned, each set of results has two sets of initial conditions, an ideal set matching nominal operating conditions and a worse case set with initial angular velocity and panel deflections. The four controllers again are time-invariant rigid (TIR), rigid periodic (PR), time-invariant vibration suppression (TIVS) and periodic vibration suppression (PVS). The first set of results are shown in Table 5.1. Under both sets of initial conditions, the rigid satellite performed better than the satellite with small panels, both in terms of attitude and

Performance Criteria	Ideal IC's		Worst Case IC's	
	No Panels	Small Panels	No Panels	Small Panels
θ rms error ($^{\circ}$)	1.35	1.41	1.50	1.57
ω rms error ($^{\circ}/s$)	$1.52 \cdot 10^{-2}$	$1.58 \cdot 10^{-2}$	$6.01 \cdot 10^{-2}$	$6.32 \cdot 10^{-2}$
Average $\ \mathbf{m}\ $ (A·m ²)	0.174	1.75	0.195	0.202
Average u_w (N·m)	$4.64 \cdot 10^{-5}$	$4.66 \cdot 10^{-5}$	$4.74 \cdot 10^{-5}$	$4.77 \cdot 10^{-5}$

Table 5.1: Performance comparison with and without small additional panels.

angular velocity accuracy as well as control expenditure. This is not surprising as the satellite with the additional panels has a slightly higher moment of inertia. More importantly, the performance is only slightly worse in all areas, implying that including additional panels in order to increase the differential drag would have been a viable choice, at the expense of a small decrease in performance and a small increase in control system expenditure. The performance of the two are compared in Figures 5.1, 5.2 and 5.3 which compare ϵ , ω and $\mathbf{m}$ respectively. The figures shown are for worst case initial conditions. The performance with the ideal initial conditions is very similar the those shown. It is essentially the same as the later orbits, after the errors have been settled, but throughout the entire four orbits. The errors in ϵ_2 and ω_y each half orbit correspond to the satellite changing its desired pitch angle to change its drag coefficient. As can be seen, there is very little difference due to the additional panels that can be observed from the figures. The second set of results examine if the performance could be improved using higher complexity controllers so that any adverse effects of the additional panels could be overcome. These are shown in Table 5.2. All the controllers performed well in the sense that they were able to stabilize the worst case initial conditions, and not induce any large errors under normal operation. The controllers also exhibited the same general performance for both sets of initial conditions. The periodic controllers, both rigid and vibration suppression, had smaller errors in attitude and angular velocity at the expense of higher control output, as compared to their time-invariant counterparts. Recalling Figure 4.3, which showed that the periodic controller would, on average, give a higher control effort for a given error, this is expected. The periodic control gains were computed using the same control weightings that were used for the time-invariant cases. Numerous other sets of control gains were attempted comparing the periodic controllers to the time-invariant ones, and no set found that either outperformed the other in terms of both attitude and angular velocity errors as well as control effort. Essentially it was found that neither controller performed better than the other, there was merely trade off

between accuracy and control effort. So although the periodic controllers can guarantee to stabilize the linearized system whereas their time-invariant counterparts cannot, there does not appear to be any advantage in terms of controller performance.

There also does not appear to be a significant difference between the rigid and vibration suppression controllers. Comparing the TIVS and TIR, there is little difference between the two. The same can be said for the PVS and PR controllers. This can be explained by examining the amount of panel vibrations experienced in the simulations as shown in Figure 5.4. Even with the rigid controller, the panel vibrations do not continue for a long period. This is due to their higher natural frequency which leads to the vibrations damping out more quickly. This does not allow the vibration suppression controllers to have any significant advantage. Because there is not a large difference in performance, plots of the performance for the other controllers would show almost identical results as those on the right side of Figures 5.1-5.3, and are hence not included.

The last set of results are for large additional panels. These are shown in Table 5.3. All the controllers again performed well. All are able to recover from poor initial conditions, and do not induce a great amount of unwanted vibrations under normal operation. Again, the periodic controllers have greater attitude and angular velocity accuracy than their time-invariant counterparts at the expense of additional control output. Again, various control gains were attempted with neither controller outperforming the other. The difference between the performance for the large and small panels is that, for the large panels, the vibration suppression controllers outperform the rigid controllers for both the periodic and time-invariant cases. The vibrations suppression controllers are able to have smaller attitude and angular velocity errors, while using less control output. This difference can be explained by looking at the panel vibrations for the PR and PVS controllers shown in Figure 5.5. Not only is the PVS controller able to suppress the panel vibrations more quickly, it also induces smaller deflections each half orbit when the pitch angle is changed. The same is observed when comparing the PR and TIVS controllers.

Again, only the performance of the PR and PVS controllers are shown as there is little difference between the controllers that can be seen from the plots. Figures 5.6, 5.7 and 5.8 again compare ϵ , ω and $\mathbf{m}$ respectively.

The results of a number of simulations have been presented here. Briefly, it was found that having additional panels added to the satellite slightly increased the attitude and angular velocity errors while also having slightly increased control output. These increases being small imply the additional panels could have been added without having a large adverse effect on the mission. For both panel sizes the periodic controllers were not able to have increased accuracy without additional control effort as compared to their rigid counterparts. The performance of the periodic and time-invariant controllers was found to be essentially the same. Hence, there does not appear to be a benefit in terms of controller performance using the more complex periodic controllers, particularly when considering the additional on-board storage requirements. For the small additional panels, there was also no increase in performance by using the vibration suppression controllers compared to the rigid ones because the panels vibrations quickly damped away regardless of the controller. This was not however the case with the large additional panels, where the vibration suppression controllers were able to outperform the rigid controllers both in terms of attitude and angular velocity errors as well as control output. At a certain panel size, actively suppressing the panels becomes advantageous, and the controller choice would be mission specific.

Figure 5.1: ϵ errors for the TIR controller with no panels (left) and small panels (right) with worst case initial conditions.

Figure 5.2: ω errors for the TIR controller with no panels (left) and small panels (right) with worst case initial conditions.

Figure 5.3: $\mathbf{m}$ for the TIR controller with no panels (left) and small panels (right) with worst case initial conditions.

Ideal IC's	TIR	PR	TIVS	PVS
θ rms error ($^\circ$)	1.41	1.04	1.50	1.36
ω rms error ($^\circ/\text{s}$)	$1.58 \cdot 10^{-2}$	$1.34 \cdot 10^{-2}$	$1.54 \cdot 10^{-2}$	$1.51 \cdot 10^{-2}$
Average $\ \mathbf{m}\ $ ($\text{A} \cdot \text{m}^2$)	0.175	0.301	0.169	0.182
Average u_w ($\text{N} \cdot \text{m}$)	$4.66 \cdot 10^{-5}$	$5.66 \cdot 10^{-5}$	$5.11 \cdot 10^{-5}$	$5.13 \cdot 10^{-5}$
Worst Case IC's	TIR	PR	TIVS	PVS
θ rms error ($^\circ$)	1.57	1.15	1.74	1.62
ω rms error ($^\circ/\text{s}$)	$6.32 \cdot 10^{-2}$	$2.40 \cdot 10^{-2}$	$4.50 \cdot 10^{-2}$	$3.97 \cdot 10^{-2}$
Average $\ \mathbf{m}\ $ ($\text{A} \cdot \text{m}^2$)	0.202	0.352	0.207	0.227
Average u_w ($\text{N} \cdot \text{m}$)	$4.77 \cdot 10^{-5}$	$5.74 \cdot 10^{-5}$	$6.59 \cdot 10^{-5}$	$6.60 \cdot 10^{-5}$

Table 5.2: Performance summary with small additional panels.

Figure 5.4: Small panel 2 deflections for the TIR controller (left) and TIVS controller (right) with worst case initial conditions.

Ideal IC's	TIR	PR	TIVS	PVS
θ rms error ($^{\circ}$)	2.45	1.90	1.71	1.58
ω rms error ($^{\circ}/s$)	$2.32 \cdot 10^{-2}$	$2.31 \cdot 10^{-2}$	$1.61 \cdot 10^{-2}$	$1.61 \cdot 10^{-2}$
Average $\ \mathbf{m}\ $ (A $\cdot$ m 2)	1.62	0.204	0.209	0.236
Average u_w (N $\cdot$ m)	$10.11 \cdot 10^{-5}$	$12.21 \cdot 10^{-5}$	$5.85 \cdot 10^{-5}$	$5.95 \cdot 10^{-5}$
Worst Case IC's	TIR	PR	TIVS	PVS
θ rms error ($^{\circ}$)	4.35	3.96	3.12	2.83
ω rms error ($^{\circ}/s$)	$15.18 \cdot 10^{-2}$	$14.39 \cdot 10^{-2}$	$10.23 \cdot 10^{-2}$	$9.30 \cdot 10^{-2}$
Average $\ \mathbf{m}\ $ (A $\cdot$ m 2)	0.414	0.447	0.474	0.507
Average u_w (N $\cdot$ m)	$11.74 \cdot 10^{-5}$	$12.16 \cdot 10^{-5}$	$7.12 \cdot 10^{-5}$	$7.24 \cdot 10^{-5}$

Table 5.3: Performance summary with large additional panels.

Figure 5.5: Large panel 2 deflections for the PR controller (left) and PVS controller (right) with worst case initial conditions.

Figure 5.6: ϵ errors with large panels for the PR controller (left) and PVS controller (right) with worst case initial conditions.

Figure 5.7: ω errors with large panels for the PR controller (left) and PVS controller (right) with worst case initial conditions.

Figure 5.8: $\mathbf{m}$ with large panels for the PR controller (left) and PVS controller (right) with worst case initial conditions.

Chapter 6

Conclusions

The control of a flexible satellite using magnetic attitude control, performing differential drag formation flying was investigated. The use of magnetic attitude control results in an inherently non-linear, time-varying control problem. The motivation for the work comes from the JC2Sat mission, where additional panels were being considered in order to increase the available differential drag. The goals of the research included assessing the impact of additional panels on the attitude control for various panel sizes and to examine if the performance could be increased using controllers with increased complexity. All controllers were of the Linear Quadratic Regulator form. The simplest controller was time-invariant and assumed a rigid satellite. The more complex controllers included a controller that also assumed a rigid satellite but used time-periodic control gains to account for the periodic nature of the Earth's magnetic field, a time-invariant controller which actively suppressed panel vibrations by estimating the unmeasured panel states using an observer and finally one that was periodic and used active vibration suppression.

It was found that the additional small panels degraded JC2Sats performance only slightly, and that the panels would have been a viable design choice using the time-invariant rigid controller. It was also found that with small additional panels, controllers which actively suppressed the panel vibrations did not increase performance over rigid

ones. The smaller panels tended to have their vibrations damp away quickly enough that the active vibration controllers did not have any advantage. There was also no significant difference between the performance of the periodic controllers compared to the time-invariant ones. Larger drag panels were also investigated and it was found that again, the time-invariant rigid controller was able to recover from poor initial conditions and maintain reasonable attitude stability and that the periodic controllers did not relate to any significant improvement in performance. It was found, however, that the performance with the large panels could be improved in terms of both attitude accuracy and control system output when actively suppressing panel vibrations. It is believed that under certain conditions, namely a satellite with high flexibility with low frequency and low damping, active vibration suppression could be advantageous.

To the authors best knowledge, this has been the first published work into active vibration suppression using magnetic attitude control. However, since it is based on the JC2Sat, it is based on a specific satellite design and attitude control system. Further work that could be done in the area of active vibration suppression using magnetic attitude control could include, but is of course not limited to, including flexible bodies into the modeling and control design, as opposed to a system of rigid bodies attached with flexible hinges. Active suppression using only magnetorquers with no reaction wheel or bias momentum could also be investigated.

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