Viscous relaxation times of the core and mantle of Mars from observations of tidal decay of the orbit of Phobos

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Abstract

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The orbit of Phobos exhibits an along-track acceleration, which suggests energy dissipation in the Mars-Phobos system. We hypothesize that the inferred dissipation occurs within Mars. We explore the response of a layered, incompressible Maxwell viscoelastic Mars to tidal forcing by Phobos using normal mode relaxation theory. Our results elucidate the general behavior of a tidally forced viscoelastic body, and have implications for the viscoelastic structure of Mars. We find the real and imaginary part of the degree-two tidal Love number for Mars to be 0.168 and -9.32x10^{-4} respectively. Models which satisfy these and other constraints have either: a fluid core with radius 2040 km and density 5410 kg/m^3; or an elastic inner core with radius 1200 km and density 6700 kg/m^3, along with a fluid outer core with thickness 850 km and density 4850 kg/m^3. These findings support previous hypotheses that Mars has at least a fluid outer core.
To Minocher and Shahnaz Pithawała
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# Contents

List of Figures vi  
List of Tables x  
1 Introduction 1  
   1.1 Background: Observations of secular acceleration of Phobos and the tidal Love number 1  
2 Method 6  
3 Results 12  
   3.1 Response of Simple Models 12  
   3.2 Response of Mars-like Models 14  
      3.2.1 Three Layer Models 15  
      3.2.2 Four Layer Models 15  
4 Discussion 17  
   4.1 Geological Implications 17  
   4.2 Remarks for Future Work 18  
5 Figures 19  
6 Tables 39  
Bibliography 44
5.1 Plot of -Im$[k_2]$ against Maxwell relaxation time (seconds) for a homogeneous viscoelastic model with $R = 3396$ km, $\mu = 10^{11}$ Pa, $\rho = 3933$ kg/m$^3$; forced at a period of 5.55 hours. ................................................................. 20

5.2 Schematic of 3- and 4-layer models where C is the core layer; M is the mantle layer; DL is the dissipative layer; and L is the lithosphere. ................................................................. 21

5.3 Plot of -Im$[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with one viscoelastic layer between two elastic layers. Viscoelastic layer (thickness 0.1R) at various locations ($s_1 = 0.10R - 0.70R$). ................................................................. 22

5.4 Plot of -Im$[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with one viscoelastic layer between two elastic layers. Viscoelastic layer (fixed midpoint 0.3R) of varying thickness (0.10R - 0.40R). ................................................................. 23

5.5 Plot of -Im$[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with one viscoelastic layer between two elastic layers. Viscoelastic layer (fixed midpoint 0.6R) of varying thickness (0.10R - 0.70R). ................................................................. 24

5.6 Plot of -Im$[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with a viscoelastic core (radius $s_1$) and two outer elastic layers. ................................................................. 25

5.7 4-layer model with an elastic core, two viscoelastic layers (LayerVE and mantle) and elastic lithosphere. $\rho_1 = 6700$ kg/m$^3$, $\rho_2 = 4500$ kg/m$^3$, $\rho_3 = 2900$ kg/m$^3$, $\rho_4 = 2900$ kg/m$^3$. $s_1=0.4R$, $s_2=0.65R$, $s_3=0.9117R$. ................................................................. 26

5.8 Contours of Re$[k_2]$ for 3-layer models with varying core and mantle relaxation times. Core radius and density are respectively: for (a) 0.461R and 6700 kg/m$^3$; for (b) 0.442R and 7000 kg/m$^3$; and for (c) 0.425R and 7300 kg/m$^3$. The red contour corresponds to Re$[k_2]$ calculated by Konopliv et al. [2011]. See Table 6.2 for other model parameters. ................................................................. 27
LIST OF FIGURES

5.9 Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 3Layer with varying core relaxation time and radius. Mantle and lithosphere elastic; other model parameters in Table 6.2. Plausible solution lies on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours (green circle). ................................. 28

5.10 Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 4E12U2 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.15R, inner core density = 6700 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours within the dark grey region. There are no plausible solutions for this model. ................................................................. 29

5.11 Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 4E22U2 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.15R, inner core density = 7000 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours within the dark grey region. There are no plausible solutions for this model. ................................................................. 30

5.12 Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 4E32U2 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.15R, inner core density = 7300 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours within the dark grey region. There are no plausible solutions for this model. ................................................................. 31

5.13 Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 4E12U3 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.25R, inner core density = 6700 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours (green circle) within the dark grey region. ...................... 32
5.14 Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model 4E22U3 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.25R, inner core density = 7000 kg/m$^3$. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[$k_2$] = 9.32x10$^{-4}$ and Re[$k_2$] = 0.168 contours (green circle) within the dark grey region. 33

5.15 Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model 4E32U3 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.25R, inner core density = 7300 kg/m$^3$. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[$k_2$] = 9.32x10$^{-4}$ and Re[$k_2$] = 0.168 contours (green circle) within the dark grey region. 34

5.16 Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model 4E13C with varying asthenosphere (dissipative layer) relaxation time and thickness. Elastic core density = 6700 kg/m$^3$ and fractional radius 0.461R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[$k_2$] = 9.32x10$^{-4}$ and Re[$k_2$] = 0.168 contours. There are no plausible solutions for this model. 35

5.17 Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model 4I13C with varying asthenosphere (dissipative layer) relaxation time and thickness. Effectively inviscid (relaxation time 100 s) core density = 6700 kg/m$^3$ and fractional radius 0.461R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[$k_2$] = 9.32x10$^{-4}$ and Re[$k_2$] = 0.168 contours. There are no plausible solutions for this model. 36

5.18 Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model 4I23C with varying asthenosphere (dissipative layer) relaxation time and thickness. Effectively inviscid (relaxation time 100 s) core density = 7000 kg/m$^3$ and fractional radius 0.442R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[$k_2$] = 9.32x10$^{-4}$ and Re[$k_2$] = 0.168 contours. There are no plausible solutions for this model. 37
5.19 Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model 4I33C with varying asthenosphere (dissipative layer) relaxation time and thickness. Effectively inviscid (relaxation time 100 s) core density = 7300 kg/m$^3$ and fractional radius 0.425R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[$k_2$] = 9.32x10$^{-4}$ and Re[$k_2$] = 0.168 contours. There are no plausible solutions for this model.
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Phobos Secular Acceleration</td>
<td>40</td>
</tr>
<tr>
<td>6.2</td>
<td>Model Constants</td>
<td>41</td>
</tr>
<tr>
<td>6.3</td>
<td>Results for Plausible 3- and 4-layer Models</td>
<td>42</td>
</tr>
</tbody>
</table>
1

Introduction

The orbit of Phobos has long been known to exhibit a secular acceleration in along-track motion \cite{Jacobson2010, Jones1989, Lainey2007, Sharpless1945, Sinclair1972}. This effect is generally attributed to dissipation of energy within Mars, associated with the small tidal bulge raised on Mars by Phobos \cite{Burns1972, Jeffreys1957, Redmond1964, Smith1976}. We investigate the possibility that all of this energy is dissipated within Mars.

The response of a planet to tidal forcing can be described by a superposition of its normal modes of oscillation. Each normal mode is a pattern of motion in which all parts of the planet oscillate sinusoidally in-phase at the same frequency. Additionally, each normal mode of oscillation is independent of the others (hence the commonly used term eigenmode) and depends on the material properties and structure of the body. The goal of this study is two-fold. Firstly, we investigate the behavior of a spherical, layered, incompressible, Maxwell body to tidal forcing using normal mode relaxation theory. We use a Maxwell viscoelastic rheology because it is the simplest model that results in an initial elastic response and a long-term viscous response – giving rise to a tidal phase lag. Secondly, we use what we’ve learned to seek simple models of Mars that satisfy observational constraints of bulk density, moment of inertia, and energy dissipation. These models may not represent a true Mars, but can serve as first-order constraints on the viscous relaxation times, densities, and thicknesses of interior layers. We discuss geological implications for Mars based on our findings, and conclude with a critique of our assumptions and directions for future studies.

1.1 Background: Observations of secular acceleration of Phobos and the tidal Love number

Phobos is the innermost moon of Mars, and its rapid orbital evolution has been studied since its discovery in 1877 by Hall \cite{Hall1878}. Phobos’ orbit has a 9375 km semi-major axis, a mass of 1.05x10^{16}
1.1 Background: Observations of secular acceleration of Phobos
and the tidal Love number

kg [Jacobson, 2010] and a mean radius of 11.1 km [Seidelmann et al., 2007]. Its orbital period of
7.65 hours places the moon within the synchronous orbital distance [Bills et al., 2005]. However,
the sidereal spin period of Mars is 24 hr 37 min 22.663 sec [Bertelsen et al., 2004] and so the period
of relative motion is 11.10 hours. The semi-diurnal tidal forcing period of Phobos is then 5.55
hours. Mars has a second smaller moon, Deimos; it was also discovered in 1877 by Hall [1878]. It
has a 23 458 km semi-major axis, a mass of 1.51x10^{15} kg [Jacobson, 2010], and a mean radius of
6.2 km [Seidelmann et al., 2007]. We ignore the tide raised by Deimos because it does not influence
the motion of Phobos.

Phobos raises a tidal bulge on Mars that lags behind the line joining the centre of masses of
the two bodies. The phase-shifted bulge imparts a gravitational torque on Phobos. As a result,
the moon is accelerating both along-track and towards Mars. The corresponding fractional rate of
change in angular velocity is,
\[ \frac{1}{n} \frac{dn}{dt}, \]
where \( n \) is the orbital mean motion in longitude, or mean angular rate of 1128.8°/day [e.g. Sinclair,
1978]. For the Mars-Phobos binary orbit with respective masses \( M \) and \( m \) at an orbital semi-major
axis \( a \), the kinetic plus potential energy is given by,
\[ E = -\frac{GMm}{2a}. \] (1.2)

Using \( G = 6.67x10^{-11} \text{ N/m}^2/\text{kg}^2 \), \( M = 6.42x10^{23} \text{ kg} \) [Jacobson, 2008], \( m = 1.05x10^{16} \text{ kg} \), and \( a =
9 375 000 \text{ m} \), we find the total energy in the orbit to be \( E = -2.40x10^{22} \text{ J} \). The fractional rate of
energy loss is then
\[ \frac{1}{E} \frac{dE}{dt} = -\frac{1}{a} \frac{da}{dt}. \] (1.3)

Kepler’s third law gives \( a^3n^2 = G(M + m) \), which implies that
\[ \frac{3}{a} \frac{da}{dt} + \frac{2}{n} \frac{dn}{dt} = 0. \] (1.4)

The orbital energy loss is related to the observed along-track acceleration as,
\[ \frac{dE}{dt} = \frac{2}{3} \frac{E}{n} \frac{dn}{dt}. \] (1.5)

It is also related to the motion towards Mars via (1.4). Presented in Table 6.1 are the secular
orbital accelerations of Phobos as determined by various studies. Using the values for \( \frac{dn}{dt} \) from
Jacobson [2010], \( n \) from Sinclair [1978], \( E \) from above, and (1.5) we find an orbital energy loss
of 3.12 MW. This dissipation rate is diagnostic of internal structure but is not important to the
thermal state of Mars. Indeed, the energy dissipation per unit volume is only \( 1.94 \times 10^{-14} \text{ W/m}^3 \).
A viscoelastic body subjected to a tidal force experiences a stress, $\sigma(t)$ and a strain $\epsilon(t)$ that are out of phase; resulting in the dissipation of tidal energy. The imposed potential is related to the displacement field, whose gradient yields a tidal strain. The strain induces a new gravitational potential that can be described by the tidal Love number, $k$, named after A.E.H. Love [Munk and Macdonald, 1960]. If the imposed potential is small, the tidal Love number (at the surface) is the linear scaling factor defined by,

$$k = \frac{U_{\text{induced}}}{U_{\text{imposed}}},$$

(1.6)

where $U_{\text{induced}}$ is the additional potential caused by the redistribution of Martian mass under Phobos’ tidal forcing; and $U_{\text{imposed}}$ is the perturbing tidal potential. The total perturbed potential is then,

$$U_{\text{tot}} = U_{\text{induced}} + U_{\text{imposed}};$$

(1.7)

and from (1.6)

$$U_{\text{tot}} = U_{\text{imposed}}(1 + k).$$

(1.8)

In the Fourier domain (as will be shown in § 3.1), the tidal Love number for a viscoelastic body is complex, and can be written as:

$$k = \text{Re}[k] + i\text{Im}[k]$$

(1.9)

with the real and imaginary parts corresponding to the elastic and viscous deformation respectively. By measuring a body’s tidal Love number, and having knowledge of the deforming force field, we implicitly gather information about the viscoelastic properties of the body.

The imposed tidal potential on a point on the surface of a primary with mass $M$ and radius $R$ due to a secondary mass $m$ orbiting at a distance $a$ in a circular path on the equatorial plane can be written in a spherical coordinate system with spherical harmonics of degree $n$ as,

$$U_{\text{imposed}} = -\frac{Gm}{a} \sum_{n=2}^{\infty} \left( \frac{R}{a} \right)^n P_n(\cos[\theta])$$

(1.10)

where $\theta$ is the separation angle between the secondary mass and the point on the surface of the primary. Following Redmond and Fish [1964]; Taylor and Margot [2010] we can write the induced potential (at the center of the secondary) as,

$$U_{\text{induced}} = -\frac{Gm}{a} \sum_{n=2}^{\infty} \left( \frac{R}{a} \right)^{2n+1} k_n P_n(\cos[\theta])$$

(1.11)

where $k_n$ is the complex tidal Love number of the primary at degree $n$. The gradient of the induced potential imparts a tangential force on the orbiting secondary and results in a torque that is related
1.1 Background: Observations of secular acceleration of Phobos and the tidal Love number

to the induced potential as,

\[ \Gamma_n = -m \frac{\delta}{\delta \theta_{\theta=\gamma}} U_{\text{induced}}, \quad (1.12) \]

where the derivative with respect to \( \theta \) is evaluated at \( \theta = \gamma \) which represents the phase lag angle between \( U_{\text{induced}} \) and \( U_{\text{imposed}} \). Expanding to \( n = 6 \) we get,

\[ \Gamma = -\frac{Gm^2}{a} \left( \frac{3R^5 \text{Re}[k_2]\sin[2\gamma]}{2a^5} + \frac{3R^7 \text{Re}[k_3](\sin[\gamma] + 5 \sin[3\gamma])}{8a^7} \right. \\
+ \left. \frac{5R^9 \text{Re}[k_4](2 \sin[2\gamma] + 7 \sin[4\gamma])}{16a^9} + \frac{15R^{11} \text{Re}[k_5](2 \sin[\gamma] + 7 \sin[3\gamma] + 21 \sin[5\gamma])}{128a^{11}} \right) \\
+ \frac{21R^{13} \text{Re}[k_6](5 \sin[2\gamma] + 12 \sin[4\gamma] + 33 \sin[6\gamma])}{256a^{13}} + \ldots \quad (1.13) \]

From Efroimsky [2011] we establish that the phase lag is related to \( k_n \) via

\[ \gamma = \arctan \left( \frac{-\text{Im}[k_n]}{\text{Re}[k_n]} \right). \]

Additionally, we assert that the power series expansion of the \( \text{Re}[k_n] \sin[n\gamma] \) term in (1.13), is very nearly equal to \(-n \text{Im}[k_n]\). Similarly, the term \( \text{Re}[k_n] \sin[\gamma] \) can be written as \(-\text{Im}[k_n]\).

The orbital angular momentum of the system is given by,

\[ L = I n, \quad (1.14) \]

where \( I \) is the moment of inertia \( ma^2 \). The torque is related to the change in angular momentum \((L)\) of the system as

\[ \Gamma = \frac{dL}{dt} = \frac{dI}{dt} n + I \frac{dn}{dt}. \quad (1.15) \]

Thus we have,

\[ \frac{dI}{dt} = m \frac{d}{dt} (a^2) = 2ma \frac{da}{dt}. \quad (1.16) \]

Using equation 1.4 we can write

\[ \frac{dI}{dt} = -\frac{4}{3} ma^2 \frac{dn}{dt}. \quad (1.17) \]

Finally we have the torque in terms of \( dn/dt \),

\[ \frac{dn}{dt} = -\frac{3}{ma^2} \Gamma_n. \quad (1.18) \]

With \( M \gg m \), we proceed to write equation (1.18) in terms of orbital elements \((m, M, a, R,\)
1.1 Background: Observations of secular acceleration of Phobos and the tidal Love number

and \( n \) and \( \text{Im}[k_n] \).

\[
\frac{1}{n} \frac{dn}{dt} = -\frac{3mnR^5}{a^5M} \left( 3\text{Im}[k_2] + 6 \left( \frac{R}{a} \right)^2 \text{Im}[k_3] + 10 \left( \frac{R}{a} \right)^4 \text{Im}[k_4] \right. \\
+ \left. 15 \left( \frac{R}{a} \right)^6 \text{Im}[k_5] + 21 \left( \frac{R}{a} \right)^8 \text{Im}[k_6] + \ldots \right) ,
\]

(1.19)

where we have expanded to degree \( n = 6 \) and simplified all trigonometric terms. It is apparent that inclusion of higher order terms in the calculation of \( \frac{dn}{dt} \) increases the rate of tidal evolution. If the terms \( k_2, k_3, \ldots \) are almost equal, the effect of higher order terms is significant for the Mars-Phobos system, with \( R/a \sim 0.3 \). However, for a homogeneous elastic body, the expression for \( k_n \) (1.20) shows that \( k_2 \) dominates over higher order terms.

\[
k_n = \frac{3}{2(n-1)} \left( \frac{\gamma}{\gamma + \mu w_n} \right) ;
\]

(1.20)

\[
w_n = \frac{2n^2 + 4n + 3}{n} ,
\]

with effective gravitational rigidity \( \gamma = 4\pi\rho^2R^2/3 \). Based on this result we assume that \( k_2 \) is the major contributor in (1.19) and truncate our calculations at \( n = 2 \). The real part of the degree-two tidal Love number has been measured with good precision by Konopliv et al. [2011], who find \( \text{Re}[k_2] = 0.168 \pm 0.009 \). Using the orbital acceleration of Phobos, \( dn/dt \), observed by Jacobson [2010] and truncating equation (1.19) at degree-two we find \( \text{Im}[k_2] \),

\[
\text{Im}[k_2] = -9.32 \times 10^{-4} .
\]

We now have constraints on both real and imaginary parts of the degree-two tidal Love number and can seek models that satisfy these values. In the following section we outline the method used to compute \( k_2 \) for a given model.
Method

We investigate the behavior of a layered, Maxwell viscoelastic body subject semi-diurnal tidal forcing by Phobos at a period of 5.55 hours. In our incompressible, Maxwell N-layer model, we have 4N parameters. Each layer contributes a bounding radius, density, rigidity, and viscosity. However, we only have 5 direct constraints with which to solve the system, they are: the mass, mean radius, mean moment of inertia, and the real and imaginary parts of the degree-two tidal Love number. Thus, models with 2 or more layers are underconstrained. We attempt to find a 4N-5 dimensional manifold in which acceptable models exist. A model is deemed plausible if it satisfies mean density and MOI; and both real and imaginary parts of $k_2$. Additionally, we constrain the mantle viscosity using the viscosity of olivine at temperatures hypothesized for the mantle of Mars [Breuer and Spohn, 2006]. The resulting value varies between $10^{19}$ and $10^{21}$ Pa s for a wet to dry rheology [Karato and Wu, 1993].

The forced response of a Maxwell viscoelastic body can be determined from the forcing function and knowledge of the spatiotemporal patterns of its free oscillations. The response of the body depends on its material properties (density, rigidity, and viscosity), described by $k$. If the forced response is observed, and the forcing function is known – we can implicitly determine the density, rigidity, and viscosity of the forced body. For the Mars-Phobos system, we have observed the secular acceleration of Phobos, which we attribute to the tidal bulge raised on Mars by Phobos. We infer the tidal forcing of Mars by Phobos with the above equations and thus estimate the interior structure of Mars from its tidal Love number $k_2$. We use normal mode relaxation theory, as described by Alterman et al. [1959], Wu and Peltier [1982], and Sabadini and Vermeersen [2004] to calculate $k_2$ for a given model undergoing semi-diurnal tidal forcing with a period of 5.55 hours. We model Mars as an incompressible, laterally homogeneous, Maxwell viscoelastic sphere. Unlike Peltier [1974]; Sabadini and Vermeersen [2004]; Wu and Peltier [1982], who used Laplace transformed Lamé parameters, we use the approach of Bills et al. [2005] and a complex effective rigidity to solve the equivalent elastic problem in the Fourier domain.
Our internal structure model does not have any lateral variations in material properties. As a result, the induced elastic deformation is symmetric about the line connecting the centers of mass of Phobos and Mars. We expand the pertinent spatial functions (the spheroidal radial and tangential displacement components, the divergence of the displacement, and the perturbation of the gravitational potential) in spherical harmonics. The radial dependance is then a polynomial in radius, \( r \), and the angular dependance includes Legendre polynomials of degree \( n \). Thus we have the radial displacement \( u \), the tangential displacement \( v \), the divergence of the displacement \( \nabla \cdot \mathbf{u} \) (herein \( \Delta \)), and the perturbation in the gravitational potential \( \phi_1 \) as functions of the scalars \( U_n \), \( V_n \), \( \chi_n \), and \( \phi_n \), which depend only on the harmonic degree \( n \) and on the radial distance \( r \).

\[
\begin{align*}
    u &= \sum_{n=0}^{\infty} U_n(r) P_n(\cos \theta) \\
v &= \sum_{n=0}^{\infty} V_n(r) \partial_\theta P_n(\cos \theta) \\
\Delta &= \sum_{n=0}^{\infty} \chi_n(r) P_n(\cos \theta) \\
\phi_1 &= -\sum_{n=0}^{\infty} \phi_n(r) P_n(\cos \theta)
\end{align*}
\]

We can express the response of the equivalent elastic model to a tidal force at a radial distance \( r \) from the center of the sphere as a vector \( \mathbf{y} \), which has six components \cite{Love1911, Sabadini2004, Wu1982}:

\[
\begin{align*}
y_1 &= U_n \\
y_2 &= V_n \\
y_3 &= \Pi_n + 2 \mu \partial_r U_n \\
y_4 &= \mu (\partial_r V_n - \frac{V_n}{r} + \frac{U_n}{r}) \\
y_5 &= -\phi_n \\
y_6 &= -\partial_r \phi_n - \frac{(n+1)}{r} \phi_n + 4\pi G \rho_0 U_n
\end{align*}
\]

where \( U_n \) is the radial deformation, \( V_n \) is the tangential deformation, \( y_3 \) is the radial stress, \( y_4 \) is the tangential stress, \( y_5 \) is the perturbed gravitational potential, \( y_6 \) is the potential stress, \( \lambda \) and \( \mu \) are the Lamé parameters, \( \Pi_n = \lambda \chi_n \), and derivatives with respect to \( r \) are denoted by \( \partial_r \). We solve for the incompressible case (\( \nabla \cdot \mathbf{u} = 0 \)) with homogeneous layers (\( \partial_r \rho_0 = 0 \)). With the solution vector \( \text{(2.5)} \), the momentum and Laplace equations for the incompressible case can be written in
matrix form,

\[ \frac{d}{dr} y = A \cdot y \]  \hspace{1cm} (2.6)\

where

\[
A(r) = \begin{pmatrix}
-\frac{2}{r} & \frac{n(n+1)}{r} & 0 \\
\frac{4/(3\mu r) - \rho_0 g}{r} & \frac{-n(n+1)}{r} \left( \frac{6\mu}{r} - \rho_0 g \right) & 0 \\
\frac{-1/\left( 6\mu \left( 1 + \frac{n}{r} \right) \right) - \rho_0 g}{r} & \frac{2(2n^2 + 2n - 1)\mu}{r^2} & -\frac{1}{r} \\
\frac{-4\pi G \rho_0}{r} & \frac{4\pi G \rho_0 (n+1)}{r} & 0 \\
0 & 0 & 0 \\
\frac{1}{r} & \frac{\rho_0 (n+1)}{r} & \rho_0 \\
\frac{-n(n+1)}{r} & \frac{\rho_0}{r} & 0 \\
0 & \frac{-n}{r} & \frac{1}{r}
\end{pmatrix}
\]

with gravity \( g = 4\pi G \rho_0 r/3 \). For each of the \( N \) layers of the Mars model, where material properties within each layer are constant, we can write the solutions of (2.6) as six eigenvectors comprising the fundamental solution matrix \( Y^i(r) \) for the \( i^{th} \) layer:

\[
Y^i(r) = \begin{pmatrix}
\frac{n r^{n+1}}{(n+3) r^{n+1}} & \frac{n r^{n-1}}{r^{n-1}} \\
\frac{r^n (n \rho_0 g r + 2\mu (n^2 - n - 3))}{2\mu (n+2) r^n} & \frac{r^n-2 (\rho_0 g r + 2\mu (n-1))}{2\mu (n-1) r^{n-2}} \\
\frac{4\pi G \rho_0 n r^{n+1}}{r} & \frac{4\pi G \rho_0 r^{n-1}}{r} \\
\frac{n(n+1)r^{-n}}{r^{-n}} & \frac{(2-n)r^{-n}}{r^{-n}} \\
\frac{-2\mu (n+1)(3n-1)+g(n+1)\rho_0}{2\mu (n+2) r^{n+1}} & \frac{2\mu (n^2-1)}{2\mu (n+2) r^{n+1}} \\
\frac{(2n+1)r^{n-1}}{r^{n-1}} & \frac{4\pi G \rho_0 (n+1)}{r^{n-1}} \\
\frac{(1+n)r^{-n-2}}{r^{-n-2}} & \frac{0}{r^{-n-2}} \\
\frac{-2\mu (1+n)(2+n)+g(n+1)\rho_0}{2\mu (n+2) r^{n+2}} & \frac{\rho_0 (r^{-n-1})}{r^{-n-1}} \\
\frac{4\pi G \rho_0 (1+n)}{4\pi G \rho_0 (1+n)} & 0
\end{pmatrix} \hspace{1cm} (2.7)
We will also make use of the inverse of the above matrix in (2.16).

\[
\mathbf{Y}^i_{\text{inv}}(r) = \begin{pmatrix}
\frac{(1+n)r^{1-n}(-2\mu(2+n)+gr\rho)}{2\mu(3+8n+4n^2)} & \frac{n(1+n)(2+n)n^{1-n}}{(1+2n)(3+2n)} \\
\frac{2\mu(-1+4n^2)}{4G\pi r^{1-n} \rho} & \frac{(n-1)(1+n)^2 r^{1-n}}{4n^2-1}
\end{pmatrix}
\]

The columns of \(\mathbf{Y}^i(r)\) are eigenvector solutions of the system (2.6). That is, taking a radial derivative of \(\mathbf{y}\) is equivalent to multiplying \(\mathbf{y}\) by the corresponding eigenvalue of the coefficient matrix \(\mathbf{A}\). The solutions matrix is related to \(\mathbf{y}\) via a 3-component integration constant, \(\mathbf{C}\) as:

\[
\mathbf{y}^i(r) = \mathbf{Y}^i(r) \cdot \mathbf{C}^i
\]

The only Lamé parameter to enter the fundamental matrix is \(\mu\) (elastic rigidity) because we assumed an incompressible body.

Wu and Peltier [1982] solved the problem using Laplace transformed variables, and their expression for the Laplace transformed effective rigidity, \(\tilde{\mu}(s)\) became,

\[
\tilde{\mu}(s) = \frac{\mu s}{s + \mu/\eta}
\]

where \(\eta\) is the viscosity and \(s\) is the Laplace variable. It is apparent that the limits \(s \to 0\) and \(\eta \to 0\) in (2.10) are equivalent; the shear modulus \(\tilde{\mu}(s)\) vanishes for both. In these cases, the Maxwell
body behaves like an inviscid fluid [Wu and Peltier, 1982].

Peltier [1974] used a Laplace transform to convert the partial differential equations of a Maxwell viscoelastic sphere to an equivalent set of ordinary differential equations for the elastic sphere. Since our forcing function is strictly periodic, we will follow Bills et al. [2005] and use a Fourier transform. Thus, a body subjected to tidal forcing with frequency $\omega$ will undergo stress $\sigma(t)$ and strain $\epsilon(t)$:

$$
\sigma(t) = Se^{i\omega t} \\
\epsilon(t) = E e^{i\omega t}
$$

The Correspondence Principle states that the time dependent stress-strain relation of a viscoelastic material can be transformed to an equivalent static (elastic) relation between the Fourier transformed stress and strain [e.g. Read, 1950]. We can find the equivalent elastic solution with a complex effective rigidity, $\mu(\tau \omega)$, that relates the stress and strain functions in (2.11) in the frequency domain,

$$
S = \mu(\tau \omega)E
$$

$$
\mu(\tau \omega) = \mu \left( \frac{i \tau \omega}{1 + i \tau \omega} \right)
$$

where $\tau$ is the Maxwell relaxation time (defined as the time required for viscous strain to equal initial elastic strain), given by the ratio of viscosity to elastic rigidity. Equation (2.12) has a Hookean form, and thus the complex rigidity (2.13) converts the viscoelastic problem into an equivalent elastic problem [Bills et al., 2005]. The real part of (2.13) corresponds to the in-phase (elastic) behavior and the imaginary part corresponds to the out-of-phase (viscous) behavior. Letting $\tau \to 0$ results in $\mu(\tau \omega) \to 0$, equivalent to the limits $s \to 0$ and $\eta \to 0$ in (2.10). Therefore, by allowing the relaxation time of a layer to approach 0, we can approximate an inviscid fluid. Conversely, by allowing the relaxation time of a model layer to approach infinity, we can approximate an elastic material. Of course, these limits depend on the forcing frequency, $\omega$ since we always deal with the product $\tau \omega$. We discuss the limits for the case of tidal forcing by Phobos in the next section.

Simulating fluid layers by letting $\tau$ approach zero allows us to propagate the fields in (2.5) across all our model layers without separate boundary conditions for fluid layers (e.g. in the core). At internal boundaries, the solution vector on the side of layer $i$ is equivalent to that on the side of layer $i + 1$ with the continuity of the solution vector:

$$
y^i(r_s) = y^{i+1}(r_s),
$$

where $r_s$ is the radius at the boundary between layers $i$ and $i + 1$. 
Additionally, the solution \( y \) in the \( i^{th} \) layer \((i \geq 2)\) at its upper boundary \( r_i \) is given by the solution at its lower boundary \( r_{i-1} \) via,

\[
y^i(r_i) = Y^i(r_i) \cdot Y^i_{\text{inv}}(r_{i-1}) \cdot y^i(r_{i-1}).
\]  

(2.15)

For the innermost layer \((i = 1)\), the fourth, fifth, and sixth solution vectors of \( Y^i(r) \) are irregular as \( r \to 0 \). Therefore, in the first layer, we use only the three regular solutions given by \( Y_{\text{reg}} \) (a 6x3 matrix of the regular solution vectors of \( Y^i(r) \) when \( i = 1 \)). We use the matrices \( Y_{\text{reg}}, Y^i(r), \) and \( Y^i_{\text{inv}}(r) \); and the relations (2.9), (2.14), and (2.15) to propagate the solutions through \( N \) layers up to the surface. The 3-component vector integration constant, \( C \) in (2.9), is then determined by a set of boundary conditions at the surface [Sabadini and Vermeersen, 2004]. At the free surface, we find the linear combination of the three propagated solution vectors which satisfy appropriate boundary conditions (given by 2.18) on stress and potential stress \((y_3, y_4, y_6)\). The displacement and perturbed potential components of \( y \) \((y_1, y_2, y_5)\) are unconstrained at the surface. We thus solve for \( y \) at the surface \((r=R)\) using,

\[
y(R) = \left( Y_{\text{reg}} \prod_{i=2}^{N} Y^i(r_i) Y^i_{\text{inv}}(r_{i-1}) \right) \cdot (P Y^N(r_N) Y^N_{\text{inv}}(r_{N-1}) \cdot y^N(r_{N-1}))^{-1} \cdot b \]  

(2.16)

where \( P \) is a 3x6 matrix which selects the stress and potential stress components of the propagated set of vectors.

\[
P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(2.17)

\[
b = \begin{pmatrix} 0 \\ 0 \\ -2n+1 \end{pmatrix}
\]  

(2.18)

Then at the surface we have,

\[
\begin{pmatrix} y_1 \\ y_2 \\ y_5 \end{pmatrix} = \begin{pmatrix} U_n \\ V_n \\ -\phi_n \end{pmatrix}
\]  

(2.19)

and

\[
\begin{pmatrix} y_3 \\ y_4 \\ y_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2n+1 \end{pmatrix}
\]  

(2.20)

Recall that \( y_5 \) is the total perturbed gravitational potential (\( U_{\text{tot}} \) in (1.7)). In (1.7) we have assumed that the response is proportional to the imposed potential. Moreover, we have assumed that the imposed potential is small enough that the response is linear in proportion. Using the method described above for \( n = 2 \) we can take (1.8) and set the imposed potential to unity; this allows us to express \( y_5 \) in terms of \( k_2 \).
3

Results

3.1 Response of Simple Models

We begin our investigation by analyzing the response of simple hypothetical models to tidal forcing of Mars by Phobos, starting with the homogeneous elastic model. The analytical expression for $k_2$ for a homogeneous elastic body is:

$$k_2 = \frac{3}{2} \left( \frac{2\gamma}{2\gamma + 19\mu} \right)$$  \hspace{1cm} (3.1)

with effective gravitational rigidity $\gamma = 4\pi \rho^2 R^2 / 3$. Since the body is elastic, there is no phase lag between stress and strain rate, so dissipation is zero. For a Maxwell viscoelastic sphere we substitute our Fourier transformed rigidity (2.13) in place of $\mu$. The expression for $k_2$ for a homogeneous viscoelastic model can then be written in terms of the real and imaginary parts as in (1.9). The real part is

$$\text{Re}[k_2] = \frac{3}{2} \left( \frac{2\gamma}{2\gamma + 19\mu} \right) \left( 1 + \frac{38\gamma\mu}{4\gamma^2 + (2\gamma + 19\mu)^2 \tau^2 \omega^2} \right)$$  \hspace{1cm} (3.2)

and the imaginary part is

$$\text{Im}[k_2] = -\frac{57\gamma\mu\tau\omega}{4\gamma^2 + (2\gamma + 19\mu)^2 \tau^2 \omega^2}.$$  \hspace{1cm} (3.3)

The imaginary part reaches a minimum for

$$\tau\omega = \left( \frac{2\gamma}{2\gamma + 19\mu} \right),$$  \hspace{1cm} (3.4)

thus the peak magnitude of $\text{Im}[k_2]$ is

$$\frac{57\mu}{2(2\gamma + 19\mu)}.$$  \hspace{1cm} (3.5)

Figure 5.1 shows the response of a homogeneous viscoelastic model to tidal forcing by Phobos. The model has a radius, $R = 3396$ km; an elastic rigidity, $\mu = 10^{11}$ Pa; a density, $\rho = 3933$ kg/m$^3$; and is forced at a period of 5.55 hours. We note that the peak magnitude of $\text{Im}[k_2]$ ($\sim 0.71$) occurs for
Maxwell relaxation time $\tau \sim 10^{2.2}$ seconds. In the homogeneous case, there is only one normal mode of deformation in response to the forcing. In layered models, each additional non-trivial boundary (across which the material properties are different) contributes several additional modes [Wu and Peltier, 1982]. Each mode contributes to the real (elastic) and imaginary (viscous) response. The sum of all the modes gives the response of the body as a whole.

Next we add a perturbing layer to our homogeneous models. We begin with a three layer model, which has an elastic core and lithosphere, and a viscoelastic mantle (denoted LayerVE). Each layer is defined by bounding radii, which we write as fractions of mean planetary radius, $R$ (Figure 5.2). The thickness of each layer is given by the difference of bounding radii. The layer densities are (from the interior out) 6700, 4500, and 2900 kg/m$^3$, respectively. Elastic rigidity, $\mu$, for all layers is $10^{11}$ Pa. We compute $k^2$ at the surface and explore the result of changing LayerVE’s location, thickness and Maxwell relaxation time.

First we fix the thickness of LayerVE to be 0.1R (339.6 km, defined as the difference between $s_2$ and $s_1$) and vary its location by increasing $s_1$. Figure 5.3 is a logarithmic plot of $-\text{Im}[k^2]$ vs. Maxwell relaxation time ($\tau$) for LayerVE. In general, for a given $\tau$, the magnitude of $\text{Im}[k^2]$ increases with increasing $s_1$. We see two local maxima, which are the result of the summation of the eigenmodes. As $s_1$ decreases, the maxima associated with lower values of $\tau$ shrinks in magnitude. The peak magnitude occurs at lower $\tau$ as $s_1$ increases. A series expansion of the expression for $\text{Im}[k^2]$ in $s_1$ shows that $-\text{Im}[k^2]$ scales as $s_1^2$ for low values of $\tau$, and as $s_1^2-s_1^3$ for large $\tau$. In Figure 5.3 we see that as $\tau$ increases or decreases past its optimum value, $-\text{Im}[k^2]$ approaches zero. LayerVE is effectively elastic for large $\tau$ and inviscid for small $\tau$. We observe similar behavior for a thicker LayerVE.

We now fix the midpoint of LayerVE and vary its thickness. Figures 5.4 and 5.5 are for LayerVE centered at radius 0.3R and 0.6R respectively. As layer thickness increases, results look more like the homogeneous viscoelastic model, i.e. the curves have one peak instead of two. This is to be expected since as the thickness of LayerVE increases, the model approaches a homogeneous viscoelastic sphere. The behavior of $\text{Im}[k^2]$ for varying LayerVE thickness depends on the relaxation time. For $\tau = 1$ second, a series expansion in thickness shows that $-\text{Im}[k^2]$ scales as $s_1^2$ with a negative phase shift in $s_1$. For $\tau = 10^8$ seconds, $-\text{Im}[k^2]$ scales as $s_1$.

We also investigate the response of a 3-layer model with a viscoelastic core, and elastic mantle and lithosphere. We maintain the same densities and elastic rigidities as above. The outermost layer has a thickness of 300km ($s_2/R = 0.9117$). In Figure 5.6 we see that the magnitude of $\text{Im}[k^2]$ increases with $s_1$. Peak magnitude of $\text{Im}[k^2]$ occurs for lower relaxation times as core radius increases. A series expansion of $\text{Im}[k^2]$ in $s_1$ shows that $-\text{Im}[k^2]$ scales as $s_1^2$. Additionally, $-\text{Im}[k^2]$ approaches zero at two limits in $\tau$, these are the inviscid and elastic limits – where $\text{Im}[k^2]$ approaches zero.
3.2 Response of Mars-like Models

We extend our 3-layer model results to models with an additional layer. If the new layer is juxtaposed with an identical layer, the response is the same as the equivalent 3-layer model. Although the added layer contributes extra eigenmodes to the forced response, the net result is similar in behavior to the 3-layer models. Figure 5.7 displays LayerVE relaxation time curves for models with varying mantle rheology. It is evident that for mantle relaxation times greater than $10^8$ s, the mantle is effectively elastic. For viscoelastic mantles with low relaxation times, the curves never approach zero with changes in LayerVE’s $\tau$. This is because the mantle is contributing to the magnitude of $\text{Im}[k_2]$ even when LayerVE contributes very little. The bulk response is dominated by a combination of LayerVE and the viscoelastic mantle.

3.2 Response of Mars-like Models

We now approximate Mars with spherical, incompressible, laterally homogeneous, 3- and 4-layer models with the values for Martian density, rigidity, mean moment of inertia (MOI), etc. listed in Table 6.2. The results of the models offer insight into the viscoelastic structure of Mars such that it can yield the observed acceleration of Phobos. We compute the complex degree-two tidal Love number ($k_2$) for each model and compare it to observed values.

Three layer models consist of a core, mantle, and elastic lithosphere. Four layer models consist of a core, two interior layers, and an elastic lithosphere (see Figure 5.2). In all cases, the elastic lithosphere has a density of 2900 kg/m$^3$ and a thickness of 300 km. From Phillips et al. [2008], we choose a global 300 km lithospheric thickness. This is one of the largest elastic thickness estimates for a given locale on Mars. We believe that if we can find a successful model with a globally thick elastic lithosphere, we can find successful models for any lower thickness. This is because as the elastic volume of the body increases, the dissipation of orbital energy due to the tide-induced bulge decreases. All layers have the same elastic rigidity, $10^{11}$ Pa. A Martian mantle with a hypothesized viscosity of at least $10^{19}$ Pa s [Breuer and Spohn, 2006; Karato and Wu, 1993] corresponds to a Maxwell relaxation time of at least $10^8$ s, which we found to be effectively elastic. For some investigations we fix the core density to nominal values of either 6700, 7000, or 7300 kg/m$^3$ (Table 6.2), whereas in others we compute the core density. We can compute at most two layer densities given the bulk density, MOI, and bounding radii. For 4-layer models, models are named thusly: the first character describes the number of layers in the model; the second character describes the core state (I = inviscid, E = elastic); the third character describes the core density (1 = 6700 kg/m$^3$, 2 = 7000 kg/m$^3$, 3 = 7300 kg/m$^3$); the fourth character describes the position of the dissipative layer (2 = above core, 3 = under lithosphere); the fifth character describes the core radius (C = constrained, U = unconstrained); and a sixth character follows if the core radius is unconstrained and describes the thickness of the dissipative layer (2 = 0.15R, 3 = 0.25R). Therefore, a 4-layer
model with an elastic inner core density of 6700 kg/m$^3$, a constrained core radius, and a dissipative layer above the core has the name $4E12C$.

### 3.2 Response of Mars-like Models

#### 3.2.1 Three Layer Models

Figure 5.1 shows that a homogeneous viscoelastic Mars would require a relaxation time of 70 hours to yield the observed acceleration of Phobos (and the corresponding -Im[$k_2$]). For a mantle with an elastic rigidity of $10^{11}$ Pa, this relaxation time gives a very low value for the bulk viscosity of Mars ($\sim 10^{16}$ Pa s). We consequently propose that there exists a layer within Mars that is responsible for the bulk of the energy dissipation.

We first investigate three 3-layer models, each with a different core density (6700, 7000, and 7300 kg/m$^3$) and respective radius (0.461R, 0.442R, and 0.425R). We chose these densities as a sample set from the values cited by Khan and Connolly [2008] in their Table 5. The corresponding radii are the upper limits on the core size before a density inversion occurs in the model. In these models, the core and mantle are viscoelastic. The red contours in Figure 5.8 show the observed value for the real part of $k_2$ [Konopliv et al., 2011]. We see that for these particular models, the mantle relaxation time required to yield the observed Re[$k_2$] = 0.168 is on the order of 1000 seconds. For an elastic rigidity of $10^{11}$ Pa, this relaxation time corresponds to a viscosity of $10^{14}$ Pa s; and is far below the range of hypothesized mantle viscosities (see § 2). We investigate varying the size and relaxation time of the core in model 3Layer, which has an elastic mantle and lithosphere (Figure 5.9). The core density is calculated for each bounding radius, $s_1/R$. The red contours show Re[$k_2$] values, the black contours show -Im[$k_2$] values. The observed real and imaginary parts of $k_2$ occur for core radius of $\sim 2070$ km and relaxation time on the order of 10 seconds. The resulting core and mantle densities are $\sim 5410$ kg/m$^3$ and $\sim 3780$ kg/m$^3$ respectively. The short relaxation time of the core renders it effectively inviscid at the semi-diurnal forcing period of 5.55 hours.

#### 3.2.2 Four Layer Models

From our three layer investigation we found that a core with a very short relaxation time can accommodate the observed tidal energy dissipation. The following models have an additional layer, which we call the dissipative layer (DL) since it ought to have a short Maxwell relaxation time. We place this layer in two locations: either above the core, or beneath the lithosphere. In the former case, we seek DL densities similar to the core density (because of similar chemical composition) in order to simulate an outer core. Similarly, for DL beneath the lithosphere, we seek densities similar to the mantle in order to simulate an asthenosphere.

Models $4E12U2$, $4E22U2$, and $4E32U2$ are 4-layer models with an elastic inner core of density 6700, 7000, and 7300 kg/m$^3$ respectively. We allow the inner core radius ($s_1/R$) to vary while the outer core thickness is fixed to 0.15R. We vary the Maxwell relaxation time of DL and plot contours
of the real and imaginary parts of \( k_2 \) (Figures 5.10 – 5.12). Recall that densities are calculated for two layers (DL and the mantle), while the core and lithospheric density are fixed. The light grey shaded regions indicate where density inversions occur, whereas the dark grey shaded region indicates where DL has a density that is similar to the elastic core. We choose this range (i.e. \( 1000 \text{ kg/m}^3 + \rho_{\text{mantle}} \leq \rho_{\text{outercore}} \leq \rho_{\text{innercore}} - 500 \text{ kg/m}^3 \)) based on first order analogue scaling of terrestrial values from Stacey [1969]. The chosen range allows for a representative yet geologically sound solution space. Successful models lie within the dark grey region at the intersection of the observed Re\([k_2]\) and -Im\([k_2]\) contours. We see that for DL (outer core) 0.15R thick, there are no solutions which yield the observed Re\([k_2]\). Models 4E12U3, 4E22U3, and 4E32U3 are the same as the previous three except DL (outer core) has a thickness of 0.25R. Figure 5.13 shows that model 4E12U3 is successful for an inner core radius of \( \sim 1250 \) km, and an outer core thickness of \( \sim 850 \) km. The outer core has a density \( \sim 4850 \text{ kg/m}^3 \), and a relaxation time on the order of 3 seconds (Table 6.3). The outer core is then effectively inviscid, and the total core radius (outer plus inner) is nearly equal to the radius of the core in the plausible solution of model 3Layer. From Figures 5.14 and 5.15 we see that for models 4E22U3 and 4E32U3, the intersection of the observed \( k_2 \) contours lies just outside the range of our plausible outer core densities. From our simple 3-layer investigation we surmise that a thicker outer core or a less dense inner core can also satisfy the conditions on density and observed \( k_2 \) values.

We now seek plausible models with DL beneath the lithosphere, acting as a putative asthenosphere. We first look at model 4E13C with an elastic core with density 6700 kg/m\(^3\) and respective fractional radii 0.461R. The resulting asthenosphere density is similar to the mantle. Figure 5.16 shows that the contours of the real and imaginary parts of the observed \( k_2 \) do not intersect. At the relaxation time yielding the observed Re\([k_2]\), the corresponding -Im\([k_2]\) value is too high – thus resulting in too much energy dissipation. Figures 5.17-5.19 also show that for models with an effectively inviscid core (\( \tau = 100 \text{ s} \)) the relaxation time required to yield the observed Re\([k_2]\) corresponds to a value for -Im\([k_2]\) that is much too high. Changing the thickness (and so density) of the asthenosphere does not alter the result.
4

Discussion

4.1 Geological Implications

Studies by Yoder et al. [2003]; Zharkov and Gudkova [1993, 2005] found that the large degree-two tidal Love number is indicative of at least a fluid outer core, if not a completely liquid core. From our 3-layer models we conclude that for a mantle of geologically reasonable viscosity ($10^{19} - 10^{21}$ Pa s) and an elastic rigidity of $10^{11}$ Pa, the Maxwell relaxation time is long enough that the behavior of the mantle is effectively elastic. Since the mantle and lithosphere are elastic, the core must dissipate most of the energy. The resulting core relaxation time is on the order of a few seconds. Thus, we confirm the above authors’ initial hypothesis that for a 3-layer Mars, the core must be effectively fluid on the time-scale of the semi-diurnal tide (5.55 hours). We also found that an effectively fluid layer above an elastic inner core in a 4-layer model can accommodate the energy dissipation needed to cause the observed secular acceleration of Phobos. Our model was for an outer core 850 km thick, but we postulate that a thicker layer could also match the plausible model criteria. Additionally, we believe that an elastic inner core with a lower density would also yield a plausible model. To a first order we confined our outer core models to a particular range in densities (see §3.2.2). We have not employed the results of any geochemical models of Mars to validate the range, however the resulting density ratios broadly match terrestrial values. The failure of models with a dissipative, asthenosphere-type layer to match constraints has an important implication. We would expect that even for an effectively elastic asthenosphere, a model with an effectively inviscid core would yield the observed $k_2$ values since the model would be a simple extension of 3Layer with an inviscid core, elastic mantle, and elastic lithosphere. However we found that introducing an asthenosphere-type layer (with a density between that of the mantle and lithosphere), regardless of its relaxation time, does not allow for the observed real and imaginary parts of $k_2$. 
4.2 Remarks for Future Work

Our approach of using an equivalent elastic model to solve for the viscoelastic problem restricts all perturbation to be controlled by elastic forces. We found that our models require an effectively fluid layer to yield the observed orbital decay of Phobos. However, for a fluid the elastic rigidity is zero and so there are no elastic forces which control the fluid layer’s perturbation. Consequently, our assumption for constant elastic rigidity ($10^{11}$ Pa) fails in the fluid layer. Therefore, we propose that fluid layers be treated as purely viscous.

The assumption that energy dissipation occurs solely within Mars was made because of poor constraints on the rheological behavior of Phobos. Based on its high porosity, it is possible that Phobos is a rubble-pile [e.g. Britt et al., 2002]). However, some studies suggest that the moon was initially competent, and may still be tidally rigid [e.g. Asphaug and Melosh, 1993; Yoder, 1982]. Without a consensus on at least a rheological template for Phobos, any further study of its contribution to tidal dissipation would be highly speculative. If Phobos is to be modeled as a rubble-pile, new mathematical machinery would need to be implemented since the method used in this study would fail.

Additionally, future work should investigate the intricacies of the modal responses; specifically which boundaries trigger which modes. We believe that understanding the fundamental behavior of the excitation of individual modes can lead to better interpretation of the results. One such method [e.g. Tromp and Mitrovica, 1999], is to analyze the radial dependent shear kernel to identify boundaries to which a mode belongs.
5

Figures
Figure 5.1: Plot of $-\text{Im}[k_2]$ against Maxwell relaxation time (seconds) for a homogeneous viscoelastic model with $R = 3396$ km, $\mu = 10^{11}$ Pa, $\rho = 3933$ kg/m$^3$; forced at a period of 5.55 hours.
Figure 5.2: Schematic of 3- and 4-layer models where $C$ is the core layer; $M$ is the mantle layer; $DL$ is the dissipative layer; and $L$ is the lithosphere.
Figure 5.3: Plot of $-\text{Im}[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with one viscoelastic layer between two elastic layers. Viscoelastic layer (thickness 0.1R) at various locations ($s_1 = 0.10R - 0.70R$).
Figure 5.4: Plot of $-\text{Im}(k^2)$ against Maxwell relaxation time (seconds) for a 3-layer model with one viscoelastic layer between two elastic layers. Viscoelastic layer (fixed midpoint 0.3R) of varying thickness (0.10R - 0.40R).
Figure 5.5: Plot of $-\text{Im}[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with one viscoelastic layer between two elastic layers. Viscoelastic layer (fixed midpoint $0.6R$) of varying thickness ($0.10R \ldots 0.70R$).
Figure 5.6: Plot of $-\text{Im}[k_2]$ against Maxwell relaxation time (seconds) for a 3-layer model with a viscoelastic core (radius $s_1$) and two outer elastic layers.
Figure 5.7: 4-layer model with an elastic core, two viscoelastic layers (LayerVE and mantle) and elastic lithosphere. $\rho_1 = 6700 \text{ kg/m}^3$, $\rho_2 = 4500 \text{ kg/m}^3$, $\rho_3 = 2900 \text{ kg/m}^3$, $\rho_4 = 2900 \text{ kg/m}^3$. $s_1=0.4R$, $s_2=0.65R$, $s_3=0.9117R$. 

-Im$k_2$ = 9.32 x 10^{-4}
**Figure 5.8:** Contours of Re[k_2] for 3-layer models with varying core and mantle relaxation times. Core radius and density are respectively: for (a) 0.461R and 6700 kg/m^3; for (b) 0.442R and 7000 kg/m^3; and for (c) 0.425R and 7300 kg/m^3. The red contour corresponds to Re[k_2] calculated by Konopliv et al. [2011]. See Table 6.2 for other model parameters.
**Figure 5.9**: Tidal Love number contours (-Im[$k_2$], in black; and Re[$k_2$], in red) for model *s*Layer with varying core relaxation time and radius. Mantle and lithosphere elastic; other model parameters in Table 6.2. Plausible solution lies on the crossing of the Im[$k_2$] = 9.32x10^{-4} and Re[$k_2$] = 0.168 contours (green circle).
Figure 5.10: Tidal Love number contours (−Im[k₂], in black; and Re[k₂], in red) for model 4E12U2 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.15R, inner core density = 6700 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours within the dark grey region. There are no plausible solutions for this model.
Figure 5.11: Tidal Love number contours (\(-\text{Im}[k_2]\), in black; and \(\text{Re}[k_2]\), in red) for model 4E22U2 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.15R, inner core density = 7000 kg/m\(^3\). Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the \(\text{Im}[k_2] = 9.32 \times 10^{-4}\) and \(\text{Re}[k_2] = 0.168\) contours within the dark grey region. There are no plausible solutions for this model.
Figure 5.12: Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 4E32U2 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.15R, inner core density = 7300 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10^{-4} and Re[k₂] = 0.168 contours within the dark grey region. There are no plausible solutions for this model.
**Figure 5.13:** Tidal Love number contours (-Im$[k_2]$, in black; and Re$[k_2]$, in red) for model $4E12U3$ with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.25R, inner core density = 6700 kg/m$^3$. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im$[k_2] = 9.32 \times 10^{-4}$ and Re$[k_2] = 0.168$ contours (green circle) within the dark grey region.
Figure 5.14: Tidal Love number contours (-Im[k₂], in black; and Re[k₂], in red) for model 4E22U3 with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.25R, inner core density = 7000 kg/m³. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im[k₂] = 9.32x10⁻⁴ and Re[k₂] = 0.168 contours (green circle) within the dark grey region.
Figure 5.15: Tidal Love number contours (-Im$[k_2]$, in black; and Re$[k_2]$, in red) for model $4E32U3$ with varying outer core (dissipative layer) relaxation time and inner core radius. Outer core thickness = 0.25R, inner core density = 7300 kg/m$^3$. Other model parameters in Table 6.2. Light grey regions mark where density inversions occur. Dark grey region marks where the dissipative layer has densities appropriate for an outer core. Plausible solution lie on the crossing of the Im$[k_2] = 9.32 \times 10^{-4}$ and Re$[k_2] = 0.168$ contours (green circle) within the dark grey region.
Figure 5.16: Tidal Love number contours (-Im[k^2], in black; and Re[k^2], in red) for model 4E13C with varying asthenosphere (dissipative layer) relaxation time and thickness. Elastic core density = 6700 kg/m^3 and fractional radius 0.461R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[k^2] = 9.32x10^{-4} and Re[k^2] = 0.168 contours. There are no plausible solutions for this model.
Figure 5.17: Tidal Love number contours (−Im[k₂], in black; and Re[k₂], in red) for model 4H3C with varying asthenosphere (dissipative layer) relaxation time and thickness. Effectively inviscid (relaxation time 100 s) core density = 6700 kg/m³ and fractional radius 0.461R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[k₂] = 9.32×10⁻⁴ and Re[k₂] = 0.168 contours. There are no plausible solutions for this model.
Figure 5.18: Tidal Love number contours (-Im[k2], in black; and Re[k2], in red) for model 4I23C with varying asthenosphere (dissipative layer) relaxation time and thickness. Effectively inviscid (relaxation time 100 s) core density = 7000 kg/m³ and fractional radius 0.442R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[k2] = 9.32x10⁻⁴ and Re[k2] = 0.168 contours. There are no plausible solutions for this model.
Figure 5.19: Tidal Love number contours (−Im[\(k_2\)], in black; and Re[\(k_2\)], in red) for model 4I33C with varying asthenosphere (dissipative layer) relaxation time and thickness. Effectively inviscid (relaxation time 100 s) core density = 7300 kg/m³ and fractional radius 0.425R. Other model parameters in Table 6.2. Plausible solutions lie on the crossing of the Im[\(k_2\)] = 9.32x10^{-4} and Re[\(k_2\)] = 0.168 contours. There are no plausible solutions for this model.
6

Tables
Table 6.1: Phobos Secular Acceleration

<table>
<thead>
<tr>
<th>( \frac{dn}{dt} \times 10^{-3} ) (deg yr(^{-2}))</th>
<th>Source</th>
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<tr>
<td>1.882 ± 0.171</td>
<td>Sharpless [1945]</td>
</tr>
<tr>
<td>1.326 ± 0.118</td>
<td>Sinclair [1978]</td>
</tr>
<tr>
<td>1.249 ± 0.018</td>
<td>Jacobson et al. [1989]</td>
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<tr>
<td>1.367 ± 0.006</td>
<td>Bills et al. [2005]</td>
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<tr>
<td>1.270 ± 0.015</td>
<td>Lainey et al. [2007]</td>
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<tr>
<td>1.270 ± 0.003</td>
<td>Jacobson [2010]</td>
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### Table 6.2: Model Constants

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>Mean Moment ((I/MR^2))</td>
<td>0.365†</td>
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<tr>
<td>Elastic Rigidity</td>
<td>(10^{11}) Pa ‡</td>
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<tr>
<td>Lithosphere Thickness</td>
<td>300 km *</td>
</tr>
<tr>
<td>Planetary Equatorial Radius (R)</td>
<td>3,396,000 m **</td>
</tr>
<tr>
<td>Mars-Phobos Semi-major Axis ((a))</td>
<td>9,375,000 m **</td>
</tr>
<tr>
<td>Bulk Density</td>
<td>3933 kg/m³ x</td>
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<tr>
<td>Nominal Core Density</td>
<td>6700, 7000, or 7300 kg/m³ xx</td>
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<tr>
<td>Total Energy in Mars-Phobos Binary Orbit ((E))</td>
<td>(2.44 \cdot 10^{22}) J</td>
</tr>
<tr>
<td>Semi-diurnal Tidal Period</td>
<td>(\sim 5.55) hours</td>
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<tr>
<td>Core Radius ((s1)) for Constrained Models</td>
<td>0.461R, 0.442R, or 0.425R</td>
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<tr>
<td>Nominal Lithosphere Density</td>
<td>2900 kg/m³</td>
</tr>
</tbody>
</table>

† *Yoder et al.* [2003]
‡ *Bills et al.* [2005]
* *Phillips et al.* [2008]
** *Seidelmann et al.* [2007]
× *Arvidson et al.* [1980]
xx Selected values from Table 5 of *Khan and Connolly* [2008]
<table>
<thead>
<tr>
<th>Model</th>
<th>Log($\tau_{\text{core}}$) s</th>
<th>s1/R</th>
<th>$\rho_{\text{core}}$ kg/m$^3$</th>
<th>$\rho_{\text{mantle}}$ kg/m$^3$</th>
<th>Figure</th>
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<tr>
<td>3Layer</td>
<td>0.97</td>
<td>0.609</td>
<td>5410</td>
<td>3780</td>
<td>5.9</td>
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<td>4 Layer Models – Unconstrained Core Radius</td>
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<tr>
<td>Model</td>
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<td>s1/R</td>
<td>$\rho_{\text{mantle}}$ kg/m$^3$</td>
<td>$\rho_{\text{diss-layer}}$ kg/m$^3$</td>
<td>Figure</td>
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<tr>
<td>4E12U3</td>
<td>0.43</td>
<td>0.365</td>
<td>3830</td>
<td>4850</td>
<td>5.13</td>
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Declaration

This thesis documents research and analyses conducted by Taronish M. Pithawala under the supervision of Dr. Rebecca R. Ghent at the University of Toronto, Toronto, Canada in accordance with the requirements for the degree of Master of Applied Science. Assistance with research and theoretical underpinnings was provided by Dr. Bruce G. Bills of the Jet Propulsion Laboratory, Pasadena, California, USA.

This research has not previously been presented in identical or similar form to any other Canadian or foreign examination board.

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Bibliography


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