EXTENSIONS OF INPUT-OUTPUT STABILITY THEORY AND THE CONTROL OF AEROSPACE SYSTEMS

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
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Abstract

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This thesis is concerned with input-output stability theory. Within this framework, it is of interest how inputs map to outputs through an operator that represents a system to be controlled or the controller itself. The Small Gain, Passivity, and Conic Sector Stability Theorems can be used to assess the stability of a negative feedback interconnection involving two systems that each have specific input-output properties.

Our first contribution concerns characterization of the input-output properties of linear time-varying (LTV) systems. We present various theorems that ensure that a LTV system has finite gain, is passive, or is conic. We also consider the stability of various negative feedback interconnections.

Motivated by the robust nature of passivity-based control, we consider how to overcome passivity violations. This investigation leads to the hybrid conic systems framework whereby systems are described in terms of multiple conic bounds over different operating ranges. A special case relevant to systems that experience a passivity violation is the hybrid passive/finite gain framework. Sufficient conditions are derived that ensure the negative feedback interconnection of two hybrid conic systems is stable.

The input-output properties of gain-scheduled systems are also investigated. We show that a gain-scheduled system composed of conic subsystems has conic bounds as well. Using the conic bounds of the subsystems along with the scheduling signal properties, the overall conic bounds of the gain-scheduled system can be calculated. We also show that when hybrid very strictly passive/finite gain (VSP/finite gain) subsystems are gain-scheduled, the overall map is also hybrid VSP/finite gain.

Being concerned with the control of aerospace systems, we use the theory developed in this thesis to control two interesting plants. We consider passivity-based control of a spacecraft endowed with magnetic torque rods and reaction wheels. In particular, we synthesize a LTV input strictly passive controller. Using hybrid theory we control single- and two-link flexible manipulators. We present two controller synthesis schemes, each of which employs numerical optimization techniques and attempts to have the hybrid VSP/finite gain controllers mimic a $\mathcal{H}_2$ controller. One of our synthesis methods uses the Generalized Kalman-Yakubovich-Popov Lemma, thus realizing a convex optimization problem.
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Acronyms

LMI  linear matrix inequality
LTI  linear time-invariant
LTV  linear time-varying
ISP  input strictly passive
OSP  output strictly passive
VSP  very strictly passive
SSP  state strictly passive
PR   positive real
SPR  strictly positive real
BR   bounded real
KYP  Kalman-Yakubovich-Popov
LQR  linear quadratic regulator
GKYP Generalized Kalman-Yakubovich-Popov
FF   finite frequency
PID  proportional-integral-derivative
LPV  linear parameter-varying
SISO single-input single-output
MIMO multi-input multi-output
SQP  sequential quadratic programming
SDP  semidefinite programme
Chapter 1

Introduction

This thesis is about control theory. In particular, various novel results developed herein will be used to control aerospace systems. The theory we extended is of the input-output type, and is applicable to linear as well as nonlinear systems that may be time-invariant or time-varying. Although we will look at the control of spacecraft and flexible robotic manipulators, the theory developed in this thesis is equally applicable to other systems, as it is very general.

1.1 Some History

While partaking in fundamental research, it is important to know the history of the field in which one is working. Knowing the previous problems encountered, how they were overcome, and why open problems are still “open” and unsolved can help lead to novel and relevant research results. As such, let us review some history related to control theory.

1.1.1 Control and its Challenges

When discussing control theory or control systems, we are really talking about controlling some sort of system or plant. For example, an aircraft, a spacecraft, a robotic manipulator, an electric circuit, a water treatment processes, a distillation process, or even a refrigerator are all systems that must be controlled in order to achieve their desired objective(s). The aforementioned systems have a few things in common:

1. the systems have inputs that can alter the “state” of the system, and outputs that can be used to determine (or at least estimate) the current state of the system as well as how the system is performing;
2. the systems are complicated in that the inputs are related to the outputs by some nonlinear process;
3. the systems are uncertain in that we will never know exact parameter values (such as mass, stiffness, and damping, or resistance, inductance, and capacitance); and
4. there are disturbances that may alter the state of the system in an undesirable fashion, or may corrupt measurements.

The most difficult aspect of control, then, is to control the plant in the face of the unknown.
1.1.2 Lyapunov Stability and Input-Output Stability: A Historical Account

Given a plant to be controlled, it is imperative that the plant connected in feedback with the controller be stable, or what is the point of having the controller at all? So, what is stability, and how is it assessed?

Stability, in a broad sense, implies boundedness. That is, system states or outputs do not go to infinity as time evolves, but rather are “well behaved” (perhaps even converge) over time. Historically, two techniques have dominated stability analysis: Lyapunov techniques, and input-output techniques.

**Lyapunov Stability**

Aleksandr Lyapunov’s doctoral thesis entitled, “The General Problem of the Stability of Motion” was published in 1892.\(^1\)\(^2\) The focus of Lyapunov’s thesis was the stability of finite dimensional, nonlinear, mechanical systems. The main contribution of Lyapunov’s thesis was providing a means to assess stability of a nonlinear system (given a set of initial conditions) without explicitly solving the nonlinear differential equation describing the system.

After Lyapunov’s doctoral defense, his research interests shifted, and his thesis work found a new home in the hands of control systems designers and control theorists. It turned out that Lyapunov theory was rather practical (not just theoretically elegant) and applicable to many control problems. For instance, Lyapunov stability theory was first used by, fittingly, aerospace engineers at the Kazan Aviation Institute in the 1930’s. For example, using Lyapunov theory, N. G. Chetayev determined conditions that, when met, would ensure stability of a spin stabilized rocket.\(^3\) Later, Lure’s problem was tackled using Lyapunov theory. Sufficient conditions were found via Lyapunov techniques that would ensure a linear system connected in negative feedback with a nonlinearity was stable.

Lyapunov theory is, even in the year 2011, extremely popular and quite powerful. There are hundreds of books devoted to nonlinear stability analysis and nonlinear control using Lyapunov methods; for example, see Refs. \(^4\)–\(^7\), each of which are popular books published in recent years (e.g., in the 1990’s and 2000’s). Although Lyapunov stability analysis is quite popular, and is indeed useful (in fact, we will use Lyapunov’s direct method when it is appropriate), there are other methods, perhaps better methods, that can be used to determine if a closed-loop system is stable.

**Input-Output Stability**

All tools and methods have their shortcomings, and Lyapunov stability theory is no exception. For example, when assessing stability via Lyapunov’s direct method, one assumes there are no external inputs (i.e., disturbances or noise), but rather the system starts from a set of initial conditions away from the desired equilibrium. For some problems, this is quite natural; however, as discussed in Sec. 1.1.1, external inputs such as disturbances and noise are constantly being applied to the system being controlled. Therefore, it is perhaps fitting to investigate the relationship between a
system’s inputs and outputs (in the context of stability), rather than the stability of an equilibrium point given a set of initial conditions.

The input-output stability framework specifically considers the input-output relationship of systems. In particular, given a set of inputs that are bounded in some sense (to be specified in Sec. 3.1.2 of Chapter 3), the system is considered stable if bounded inputs lead to bounded outputs. In the context of control, of interest is the stability of a closed-loop system given that the plant and controller each have a specific input-output character.

Input-output stability theory was pioneered by two independent authors contemporaneously: Irwin Sandberg\(^8\text{–}^{12}\) and George Zames.\(^{13,14}\) At the time, Sandberg was a member of the technical staff in the Communication Sciences Research Division at Bell Laboratories located in Murray Hill, NJ, while Zames was part of the Department of Electrical Engineering at the Massachusetts Institute of Technology (MIT) in Cambridge, MA. Although Sandberg’s work appeared earlier (Zames even cites Refs. \(^8\text{–}^{10}\)) Zames’ work became more popular. Perhaps because Zames published his work in the *IEEE Transactions on Automatic Control*, while Sandberg published almost exclusively in the *Bell System Technical Journal*, or perhaps because Zames was (at the time) at MIT (Zames later went to McGill in Montreal, QC), while Sandberg was at Bell Labs, Zames’ work received more attention. As such, while Sandberg is referenced on occasion, Zames seems to be given most of the credit as the inventor of the input-output stability framework.

In Ref. 13, Zames outlined three ways of assessing closed-loop stability in terms of the input-output relationships of the plant and controller: a small gain result, a positivity result, and a conic sector result. These three results are now known as the Small Gain Theorem, the Passivity Theorem, and the Conic Sector Stability Theorem, and will be fully reviewed in Chapter 3. Interestingly, although the Conic Sector Stability Theorem is much more general (the other two theorems are actually special cases of the Conic Sector Stability Theorem), the Small Gain and Passivity Theorems have received much more attention in the literature over the years.

At the time (i.e., the late 1960’s and early 1970’s), Lyapunov theory was really the only nonlinear stability theory available. Assessing stability of a nonlinear system in terms of the relationship between inputs and outputs, as Zames proposed, was quite unique. The motivation to do so was generality, simplicity, and clarity:

“It is relatively easy to estimate conic bounds for simple interconnections, where it might be more difficult, say, to find Lyapunov functions.” - Zames, Ref. 13;

“The arguments and proofs in input-output theory are conceptually clearer than their Lyapunov stability counterparts.” - Vidyasagar, Ref. 5.

Various individuals, such as Jan Willems,\(^15\) Charles Desoer,\(^16\) Mathukumalli Vidyasagar,\(^5,16\) explored, extended, and applied the input-output framework in various ways. It was not until 1979 that another individual, Michael Safonov, generalized the input-output “conicity” property in his doctoral work at MIT.\(^{17,18}\) Safonov, with his doctoral supervisor Michael Athans, published
various papers that investigated the stability of systems in an input-output context; for example, see Refs. 19, 20.

Although Safonov presented a very nice extension of Zames’ original work (we will review Safonov’s “variable” cones in Sec. 3.5.3 in Chapter 3), the two most popular input-output stability techniques used throughout the 1980’s, 1990’s, and even today are the Small Gain Theorem and the Passivity Theorem. The Small Gain Theorem is actually at the heart of one of the most popular robust control methods, the $\mathcal{H}_\infty$ framework,\textsuperscript{21} while the Passivity Theorem has been employed in almost every branch of engineering: control of electric circuits,\textsuperscript{22} mechanical systems,\textsuperscript{23} flexible robotic manipulators,\textsuperscript{24} process control,\textsuperscript{25} etc. There are many excellent books devoted to passivity-based control.\textsuperscript{16,26,27}

Lyapunov versus Input-Output Stability Theory in the Context of Control and its Challenges

In Sec. 1.1.1, some of the realities that make control design challenging were listed. How do Lyapunov and input-output methods help us overcome such challenges? Both Lyapunov and input-output techniques are applicable to nonlinear as well as linear systems. Additionally, both techniques are useful for robustness analysis. However, when it comes to assessing stability (and for that matter, robust stability) in the presence of disturbances and noise, the input-output framework is simply better suited to the problem because inputs and their effect on outputs are explicitly considered in the problem formulation. Additionally, not only does the input-output framework explicitly consider inputs (i.e., disturbances and noise), but initial conditions can also be accommodated. The input-output framework, simply put, is much more natural, and fits the problem (i.e., design a controller so that bounded inputs yield bounded outputs) better.

1.2 Thesis Motivation, Objectives, and Outline

The title of this thesis is Extensions of Input-Output Stability Theory and the Control of Aerospace Systems. In this thesis we present various novel input-output results, and use these novel results to control aerospace systems. In particular, we consider spacecraft attitude control and flexible robotic manipulator control.

1.2.1 Motivation and Objectives

Input-Output Properties of Linear Time-Varying Systems

The attitude control of a spacecraft that employs magnetic as well as mechanical actuation (e.g., reaction wheels) is considered in this thesis. It can be shown that a generic spacecraft possesses a passive input-output map. As such, the Passivity Theorem can be used to guarantee robust closed-loop stability. A spacecraft equipped with magnetic torque rods uses the interaction of the Earth’s magnetic field and current carrying coils to create body torques. Because the Earth’s
magnetic field is not uniform, and the spacecraft is orbiting the Earth, the magnetic field as seen from the orbital frame of the spacecraft varies in an almost periodic fashion.

Given that a passive input-output map exists and the magnetic field is time-varying, linear time-varying passivity-based control schemes are of interest. One of the objectives of this thesis is to find sufficient conditions that ensure a linear time-varying system is input strictly passive so that it can stabilize a passive plant, such as the spacecraft that originally motivated the problem. Such linear time-varying input strictly passive controllers (which must be synthesized in some way) may be used to control other linear time-varying passive plants, not just the aerospace plant we consider.

In this thesis, we will investigate the input-output properties of linear time-varying systems. In particular, we are motivated to look at the passive and input strictly passive properties of linear time-varying systems, but we will also consider finite gain and conic properties.

**Overcoming Passivity Violations**

The Passivity Theorem can be used to realize robust closed-loop stability provided plant dynamics have a particular input-output map. Usually, the output of the system being controlled is some sort of rate. In the context of flexible manipulator control, a passive input-output map exists between joint torques (the inputs) and joint rates (the outputs). Usually, a manipulator is equipped with joint encoders, which provide a position measurement. Rate information is usually acquired by finite differencing the position data or using a derivative filter. Finite differencing will induce a slight time delay, while a derivative filter is dynamical. A time delay or additional dynamics will destroy the nominally passive map used as the basis for robust stabilization via the Passivity Theorem; passivity will be violated.

Given that the passive input-output map of a plant is often destroyed in practice, the Passivity Theorem can no longer be used as the basis for robust stabilization. It is perhaps possible to stabilize the closed-loop system if the control is designed to satisfy the requirements of the Small Gain theorem, but because of the inherent gain-limited nature of the Small Gain Theorem the performance would be prohibitively poor. As such, in the context of passivity violations, our objective is to somehow reintroduce some of the robustness and performance properties associated with high-gain, passivity-based control. How to synthesize a controller that will stabilize a plant that has experienced a passivity violation is also of interest.

**Gain-Scheduling**

When controlling a nonlinear system, such as a flexible robotic manipulator, designing a nonlinear controller that has some guaranteed robustness properties is quite challenging. For this reason (and others), linear time-invariant controllers are often used to control nonlinear plants.

In practice (especially in industry), better closed-loop performance can be achieved when many linear controllers are used and subsequently “gain-scheduled”. Gain-scheduling is a practical nonlinear design procedure wherein a set of linear “subcontrollers” optimally designed about
different operating points are simply interpolated when the nonlinear system is in between operating points.

Given that better performance can be realized when controllers are scheduled, we wish to show that when each of the subcontrollers have specific input-output properties, then the overall gain-scheduled controller also has some specific input-output properties. In particular, we will consider the scheduling of conic subcontrollers, as well as subcontrollers that are specifically designed to accommodate passivity violations.

1.2.2 Summary of Objectives

The objectives of this thesis are summarized below.

1. Regarding the input-output properties of linear time-varying systems, our objectives are to
   1.1. derive conditions that ensure a linear time-varying system has finite gain, is passive (i.e., purely passive or very, input, or output strictly passive; technical definitions will be given in Chapter 3), or has specific conic bounds,
   1.2. provide a means to synthesize a linear time-varying controller, and
   1.3. use the aforementioned theory and synthesis method to control the attitude of a spacecraft.

2. With respect to passivity violations, our objectives are to
   2.1. develop a control framework that allows for robust control of nominally passive systems which have had their passive nature partially destroyed, and
   2.2. synthesize controllers which are optimal in some sense, but simultaneously are guaranteed to stabilize systems which have experienced a passivity violation.

3. In the context of gain-scheduled control, our objectives are to
   3.1. characterize the conic bounds that a gain-scheduled system has when each subcontroller is conic and the scheduling signals have specific properties,
   3.2. similarly, show that when a set of subcontrollers (each of which are guaranteed to stabilized a plant that has experienced a passivity violation) are gain-scheduled in a particular, way the overall gain-scheduled controller is also guaranteed to stabilize the violated plant, and
   3.3. use a gain-scheduled controller to control a system (such as a flexible robotic manipulator) that has had its passive input-output map violated.

1.2.3 Outline

This thesis can be divided into two main parts: a theoretical part and an applications part. As such, in Part I we present our novel extensions of the input-output stability framework, and in Part II we consider applications of our theoretical developments.

Control theory is mathematical. As such, in Chapter 2 we review pertinent notions related to the input-output stability framework including function spaces (i.e., $L_2$ and $L_\infty$ spaces) and
norms. Additionally, in this thesis, numerical optimization will be used as a tool to synthesize controllers. The general form of nonlinear and convex optimization problems is reviewed in Chapter 2.

The foundation upon which we build is the input-output stability framework originally presented by George Zames, and generalized by others such as David Hill, Peter Moylan, and Michael Safonov. As such, Chapter 3, where “classic” input-output systems theory is reviewed, is quite an important chapter. In Chapter 3, we review operators, state-space representations, and $L_2$ stability. We also review system properties such as boundedness (i.e., a finite gain system), passivity, and conicity. Characterizing finite gain, passive, and conic properties in terms of a linear time-invariant systems transfer matrix or state-space realization is reviewed. The Bounded Real Lemma, Positive Real Lemma, Strictly Positive Real Lemma (which, is often referred to as the Kalman-Yakubovich-Popov Lemma), and Conic Sector Lemma are each presented. We review the dissipative systems framework, as well as stability in terms of the dissipative systems framework. The Small Gain, Passivity, and Conic Sector Stability Theorems are reviewed, and the Small Gain and Passivity Theorems are interpreted in terms of the Conic Sector Stability Theorem. We also briefly review Safonov’s variable cones. This chapter is an attempt to summarize all the pertinent results related to input-output stability theory in a cohesive, coherent manner.

Part I of this thesis begins with Chapter 4 where the input-output properties of linear time-varying systems are investigated. This chapter’s novel contribution is providing sufficient conditions that ensure a linear time-varying system has finite gain, is passive (either purely passive or very, input, or output strictly passive), or is conic with specific conic bounds. As mentioned in Sec. 1.2.1, our original motivation was to find conditions that ensure a linear time-varying system possesses specific passive properties, and as such can be used as a controller. Doing so inspired the other results, that being the sufficient conditions ensuring a linear time-varying system has finite gain or is conic. This chapter addresses Objective 1.1 listed in Sec. 1.2.2.

Chapter 5 presents the most significant contribution of this thesis, the hybrid input-output systems framework. Recall our second research motivation and objective discussed in Sec. 1.2.1: overcoming passivity violations. The solution method developed to overcome passivity violations is a special case of the hybrid input-output systems framework, called the hybrid passivity and finite gain systems (hybrid passive/finite gain systems) framework. A hybrid input-output system is a system with input-output characteristics that can be described in terms of multiple conic bounds, not just one set of conic bounds (as in the traditional conic sector framework). In a linear time-invariant sense, the input-output map of a system is divided up as a function of frequency into a set of regions where each region is classified as bounded, passive, or conic. The word “hybrid” is used to highlight that the system’s input-output map is not purely bounded, or passive, or conic (with one set of conic bounds that holds over all frequencies), but rather can be described in terms of many bounded, passive, or conic maps in a “hybrid”, “mixed”, or “blended” fashion. In this chapter, hybrid input-output systems theory is developed, but we also present a way to characterize the hybrid nature of a linear time-invariant system over a frequency band in terms of its transfer matrix or by using the Generalized Kalman-Yakubovich-Popov Lemma.
developed by Tetsuya Iwasaki and Shinji Hara. Additionally, we provide sufficient conditions that guarantee that the negative feedback interconnection of two hybrid input-output systems is $L_2$ stable. This chapter addresses Objective 2.1 in Sec. 1.2.2.

Often when controlling a nonlinear system (which could be passive, conic, hybrid passive/finite gain, etc.), a set of linear controllers (optimally designed about a linearization point) are used within a scheduling architecture so that better closed-loop performance can be achieved. This procedure is usually called gain-scheduling, and is often considered a “heuristic” nonlinear control method. In Chapter 6, we investigate the input-output properties of scheduled systems. In particular, we investigate the conic nature of a set of conic systems being scheduled. We show that if the systems being scheduled are conic, then the overall scheduled system is also conic, a novel result. Additionally, we investigate scheduling of hybrid very strictly passive/finite gain systems. We show that a set of hybrid very strictly passive/finite gain system gain-scheduled appropriately has hybrid very strictly passive/finite gain properties as well. This chapter addresses Objectives 3.1 and 3.2 stated in Sec. 1.2.2, and concludes Part I of this thesis.

Part II of this thesis begins with Chapter 7 where attitude control of a spacecraft is considered. In particular, the spacecraft uses both magnetic and mechanical actuation to apply body torques. As mentioned in Sec. 1.2.1, because the magnetic field changes in an (almost) periodic fashion relative to the position of the spacecraft on orbit, the linearized system naturally is endowed with some linear time-varying (specifically, linear periodic) properties. Motivated by this fact, and armed with the results of Chapter 4, we consider the design of a linear time-varying input strictly passive controller to stabilize the spacecraft (which, we show possesses a passive input-output map). We present a controller synthesis method as well as some simulation results. This chapter can be considered an applications chapter, where we are applying the theory developed in Chapter 4 to a practical engineering problem. The novel contributions of this chapter are employing the results of Chapter 4, and developing a controller synthesis method. This chapter addresses Objectives 1.2 and 1.3 listed in Sec. 1.2.2.

In Chapter 8, the tip position and rate of a single-link flexible manipulator is considered. We review $\mu$-tip theory, which is used to define a passive input-output map. We discuss how the nominally passive input-output map assumed to exist is in fact destroyed when rate information is acquired by a derivative filter, thus rendering the system hybrid with passive input-output properties at low frequency, and finite gain properties at high frequency where passivity has been violated. We present a controller synthesis technique where the controller is designed to approximate a traditional $H_2$ controller while simultaneously satisfying the stability requirements dictated by the Hybrid Passivity/Finite Gain Stability Theorem. We synthesize a hybrid very strictly passive/finite gain controller, and use it to control a single-link flexible manipulator test bed. The experimental results are successful, highlighting the success of the hybrid systems framework as well as the controller optimization method proposed. This chapter can be considered an applications chapter, where the hybrid passive/finite gain systems theory of Chapter 5 in applied in a linear time-invariant, single-input single-output context. The novel contributions of this chapter are the development of a controller synthesis technique, and experimental validation.
of the overall hybrid systems framework in a linear time-invariant, single-input single-output context. This chapter addresses Objective 2.2 in Sec. 1.2.2.

Our last chapter, Chapter 9, can also be considered an applications chapter. The hybrid passive/finite gain systems theory of Chapter 5 as well as the hybrid passive/finite gain gain-scheduling results of Chapter 6 are used to control the joint position and rate of a two-link flexible manipulator. This nonlinear system possesses a passive input-output map, but passivity is destroyed when rate information is derived from a derivative filter. The resultant system is rendered hybrid. We discuss how the hybrid passive/finite gain nature can be approximated by considering the linearized system. We also consider controller synthesis by posing a convex objective function and constraining the controller to be hybrid very strictly passive/finite gain by using the Generalized Kalman-Yakubovich-Popov Lemma. (The optimization constraints are expressed in terms of linear matrix inequalities.) The controller optimization objective function seeks to mimic the properties of a traditional $H_2$ controller. We synthesize two hybrid very strictly passive/finite gain controllers and use each controller within the scheduling architecture presented in Chapter 6 to control the two-link flexible manipulator system. The experimental results successfully demonstrate closed-loop control, with and without gain-scheduling. The novel contributions of this chapter are the controller optimization procedure using the Generalized Kalman-Yakubovich-Popov Lemma and the experimental validation in a nonlinear, multi-input multi-output context. This chapter addresses Objectives 2.2 and 3.3 stated in Sec. 1.2.2.
Chapter 2

Mathematical Preliminaries

In this chapter, we review some of the mathematical tools used within this thesis. The material presented here is by no means comprehensive; this chapter is merely a reference for the developments to come.

2.1 Function Spaces, Norms, and Inner Products

The material in this section is taken from Refs. 5, 6, 16, 21, 26, 28.

Vector valued functions \( u : \mathbb{R}^+ \to \mathbb{R}^m \) of dimension \( m \) are written

\[
\mathbf{u}(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}.
\]

The value of \( u \) at time \( t \in \mathbb{R}^+ \) is \( u(t) \in \mathbb{R}^m \). It is assumed that all functions \( u : \mathbb{R}^+ \to \mathbb{R}^m \) are piecewise continuous.

2.1.1 The \( L_2 \) Space and Extended \( L_2 \) Space

The space \( L_2 \) is the space of all square integral functions defined by

\[
L_2 := \left\{ u : \mathbb{R}^+ \to \mathbb{R}^m \left| \int_0^\infty u^T(t)u(t)dt < \infty \right. \right\}
\]

where \( u \) is an arbitrary function and \( u^T \) its transpose. The \( L_2 \) space is a Hilbert space\(^{28} \) where the inner product defines the norm:

\[
\langle u, v \rangle := \int_0^\infty u^T(t)v(t)dt, \quad \|u\|_2 := \sqrt{\langle u, u \rangle}
\]
where \( u \in L_2, v \in L_2 \), \( \langle \cdot, \cdot \rangle \) is the inner product, and \( \| \cdot \|_2 \) the \( L_2 \) norm. Using Parseval’s Theorem
\[
\langle u, v \rangle = \frac{1}{2\pi} \text{Re} \left\{ \int_{-\infty}^{\infty} u^H(j\omega)v(j\omega)d\omega \right\}, \quad \|u\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} u^H(j\omega)u(j\omega)d\omega}
\]

where \( u(j\omega) \) is the Fourier transform of the function \( u \) and \( u^H(j\omega) = u^T(-j\omega) \) is the complex-conjugate transpose of \( u(j\omega) \). Functions that are in \( L_2 \) can be thought of as “finite energy” functions. For example, \( u(t) = e^{-at}\sin(t) \) is in \( L_2 \) when \( 0 < a < \infty \), but is not in \( L_2 \) when \( -\infty < a \leq 0 \).

The truncation of a function \( u \), denoted \( u_T : \mathbb{R}^+ \rightarrow \mathbb{R}^m \), is
\[
u_T(t) := \begin{cases} u(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases} \quad T \in \mathbb{R}^+.
\]
The extended \( L_2 \) space is defined as
\[
L_{2e} := \{ u : \mathbb{R}^+ \rightarrow \mathbb{R}^m \mid u_T(t) \in L_2, \forall T \in \mathbb{R} \}
\]
where \( L_2 \subset L_{2e} \). The \( L_2 \) norm of a truncated function leads to the truncated \( L_2 \) norm:
\[
\|u_T\|_2 = \sqrt{\int_0^T u_T^T(t)u_T(t)dt} = \sqrt{\int_0^T u^T(t)u(t)dt} =: \|u\|_{2T}.
\]
The truncated inner product of two signals \( u \in L_{2e} \) and \( v \in L_{2e} \) is
\[
\langle u, v \rangle_T := \int_0^T u_T(t)v(t)dt = \int_0^T u_T^T(t)v_T(t)dt = \langle u_T, v_T \rangle.
\]
For pedagogical purposes, consider \( u(t) = e^{-at}\sin(t) \) once more; although the function \( u(t) = e^{-at}\sin(t) \) is not in \( L_2 \) when \( -\infty < a \leq 0 \), it is in \( L_{2e} \).

### 2.1.2 The \( L_\infty \) Space

The \( L_\infty \) space is defined as
\[
L_\infty := \{ u : \mathbb{R}^+ \rightarrow \mathbb{R}^m \mid \|u\|_\infty < \infty \}
\]
where
\[
\|u\|_\infty := \sup_{t \in \mathbb{R}^+} \left[ \max_{i=1}^{m} \left| u_i(t) \right| \right]
\]
is the \( L_\infty \) norm of the function \( u \). Functions that are in \( L_\infty \) are those that have a finite supremum. For example, \( u(t) = e^{-at}\sin(t) \) is in \( L_\infty \) when \( 0 \leq a < \infty \), but is not in \( L_\infty \) when \( -\infty < a < 0 \).
2.1.3 Useful Inequalities and Equalities Related to Norms

A useful inequality related to the inner product of $u \in L_2$ and $v \in L_2$ is the Cauchy-Schwarz inequality:

$$| \langle u, v \rangle_T | \leq \| u \|_{2T} \| v \|_{2T}, \quad \forall u, v \in L_2.$$  \hspace{1cm} (2.1)

The triangle inequality

$$\| u + v \|_{2T} \leq \| u \|_{2T} + \| v \|_{2T}, \quad \forall u, v \in L_2$$

and reverse triangle inequality

$$\| u \|_{2T} - \| v \|_{2T} \leq \| u - v \|_{2T}, \quad \forall u, v \in L_2$$

are also useful.

The following equality will be employed in various proofs:

$$\frac{1}{2} \left( \frac{1}{\sqrt{\alpha}} \| u \|_{2T} - \sqrt{\alpha} \| v \|_{2T} \right)^2 + \| u \|_{2T} \| v \|_{2T} = \frac{1}{2\alpha} \| u \|^2_{2T} + \frac{\alpha}{2} \| v \|^2_{2T} \quad (2.2)$$

where $\alpha > 0$ is an arbitrary scalar quantity.

2.2 Optimization Problems

In this thesis, controllers will be synthesized using various numerical optimization techniques. As such, we will briefly review the general form of nonlinear optimization problems, as well as convex optimization problems constrained by linear matrix inequalities (LMIs).

2.2.1 General Nonlinear Optimization Problems

Consider an objective function $J : \mathbb{R}^n \to \mathbb{R}$ to be minimized with respect to (w.r.t.) a set of design variables $x = [x_1 \cdots x_n]^T \in \mathbb{R}^n$, subject to (s.t.) a set of inequality constraints $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1 \cdots m$, and equality constraints $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1 \cdots p$. The objective function and inequality constraints may be nonlinear functions of $x$. This optimization problem is often written

$$\min_{w.r.t. \quad x} J(x)$$

s.t. $g_i(x) \geq 0, \quad i = 1 \cdots m,$

$h_j(x) = 0, \quad j = 1 \cdots p.$

A particular value of $x$ that satisfies the constraints is said to be a feasible value. The set $F \subseteq \mathbb{R}$ is the set of all feasible $x$ values, and the optimization problem is feasible if there exists at least one feasible value. The value $x^* \in F$ is said to be optimal value of $x$ if $\forall x \in F$, $J(x) \geq J(x^*)$. In addition to the inequality and equality constraints mentioned, there may be implicit constraints.
on the design variables $\mathbf{x}$. For example, $x_k > 0$ for $k = 1 \cdots n$ would be an implicit constraint on $\mathbf{x}$.

Often nonlinear optimization problems cannot be solved analytically, and are therefore solved numerically. There are various ways to solve nonlinear optimization problems. In this thesis, any nonlinear optimization problem posed will be solved using a gradient-based optimization scheme. In particular, a Sequential Quadratic Programming algorithm will be used.\textsuperscript{30}

2.2.2 Convex Optimization and Linear Matrix Inequalities

Convex optimization problems, especially those constrained by LMIs, are quite popular for a variety of reasons. For instance, they can be solve numerically with great efficiency, convergence to a global optima can be guaranteed, and very useful theory can be employed to transform a nonconvex or nonlinear problem into a convex problem.\textsuperscript{29,31} We will briefly review convex sets, functions, optimization problems, and constraints in the form of LMIs.

Convex Sets and Convex Functions

The set $\mathbb{X}^n \subseteq \mathbb{R}^n$ is convex if $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}^n$, $\forall \alpha \in [0, 1]$ the vector $\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2 \in \mathbb{X}^n$. For example, the set of symmetric matrices $\mathbb{S}^{n \times n} := \{ \mathbf{P} \in \mathbb{R}^{n \times n} | \mathbf{P} = \mathbf{P}^T \}$ is a convex set.

A symmetric matrix $\mathbf{P} \in \mathbb{S}^{n \times n}$ is positive definite if $\mathbf{x}^T \mathbf{P} \mathbf{x} > 0$, $\forall \mathbf{x} \in \mathbb{X}^n \\backslash \{ \mathbf{0} \}$, and positive semidefinite if $\mathbf{x}^T \mathbf{P} \mathbf{x} \geq 0$, $\forall \mathbf{x} \in \mathbb{X}^n$.

The function $\mathcal{J} : \mathbb{X}^n \rightarrow \mathbb{R}$ is convex if $\mathbb{X}^n \subseteq \mathbb{R}^n$ is convex (i.e., the domain of the function is convex) and $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}^n$, $\forall \alpha \in [0, 1]$

$$\mathcal{J}(\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2) \leq \alpha \mathcal{J}(\mathbf{x}_1) + (1 - \alpha)\mathcal{J}(\mathbf{x}_2).$$

For example, the functions\textsuperscript{29}

- $\mathcal{J} : \mathbb{R} \rightarrow \mathbb{R}^+$ where $\mathcal{J}(x) = x^2$,
- $\mathcal{J} : \mathbb{R}^+ \backslash \{0\} \rightarrow \mathbb{R}$ where $\mathcal{J}(x) = -\ln(x) = \ln(\frac{1}{x})$,
- $\mathcal{J} : \mathbb{R}^n \rightarrow \mathbb{R}$ where $\mathcal{J}(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{P} \mathbf{x}$, $\mathbf{P} \in \mathbb{R}^{n \times n}$, $\mathbf{P} = \mathbf{P}^T > 0$,
- $\mathcal{J} : \mathbb{R}^n \rightarrow \mathbb{R}$ where

$$\mathcal{J}(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{P} \mathbf{x} + \mathbf{q}^T \mathbf{x} + r = \frac{1}{2} \begin{bmatrix} \mathbf{x}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{q} \\ \mathbf{q}^T & 2r \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix},$$

$\mathbf{P} \in \mathbb{R}^{n \times n}$, $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{q} \in \mathbb{R}^n$, $r \in \mathbb{R}$,
- $\mathcal{J} : \mathbb{S}^{n \times n} \rightarrow \mathbb{R}$ where $\mathcal{J}(\mathbf{X}) = \ln \left( \det (\mathbf{X}^{-1}) \right)$, and
- any norm

are all convex. Additionally, given a set of positive scalars $\alpha_i > 0$, $i = 1 \cdots m$ and many convex functions $\mathcal{J}_i : \mathbb{R}^n \rightarrow \mathbb{R}$, their weighted sum $\mathcal{J}(\mathbf{x}) = \sum_{i=1}^{m} \alpha_i \mathcal{J}_i(\mathbf{x})$ is also convex.\textsuperscript{29}
A General Convex Optimization Problem

We will now consider a convex optimization problem. Consider a convex objective function $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}$ to be minimized with respect to a set of design variables $x \in \mathbb{R}^n$, subject to a set of convex inequality constraints $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1 \cdots m$, and a set of affine equality constraints $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1 \cdots p$. Often the affine equality constraints are written

$$h_j(x) = a_j^T x - b_j = 0, \quad j = 1 \cdots p,$$

where $a_j \in \mathbb{R}^n$. These affine equality constraints can also be written in matrix form:

$$\begin{bmatrix} a_1^T \\ \vdots \\ a_p^T \end{bmatrix} x - \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix} = Ax - b = 0,$$

where $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$. It follows that the convex optimization problem under discussion can be written

$$\min \quad \mathcal{J}(x)$$

w.r.t. $x$,

s.t. $g_i(x) \geq 0, \quad i = 1 \cdots m,$

$$Ax = b.$$

Linear Matrix Inequalities

Often constraints (especially constraints found in control problems) can be written in the following form:

$$\begin{align*}
F : \mathbb{R}^n &\to \mathbb{R}^{n \times n}, \\
F(x) &= F_0 + \sum_{i=1}^{m} x_i F_i \geq \epsilon I,
\end{align*}$$

(2.3)

(2.4)

where $F_0, F_i \in \mathbb{R}^{n \times n}$, $i = 1 \cdots m$ are symmetric matrices and $\epsilon \geq 0$. The above inequality is a LMI, and simply means that for any $x$, the matrix function $F$ is positive definite when $\epsilon > 0$, and positive semidefinite when $\epsilon = 0$. The inequality is linear in the variables $x_i$, hence the name Linear Matrix Inequality.

We are interested in LMIs for a few reasons. For example, many LMIs can be combined to create one “big” LMI. In control problems, combining many LMIs into one large LMI is equivalent to combining various constraints into one “big” constraint. Additionally, LMIs are convex. To see that LMIs are convex, recall that the function $F : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is convex if the domain of $F$ is
convex and $\forall x', x'' \in \mathbb{R}^n$, $\alpha \in [0, 1]$ the inequality

$$F(\alpha x' + (1 - \alpha)x'') \leq \alpha F(x') + (1 - \alpha)F(x'')$$

holds. As such, consider the following:

$$F(\alpha x' + (1 - \alpha)x'') = F_0 + \sum_{i=1}^{m} (\alpha x'_i + (1 - \alpha)x''_i) F_i$$

$$= F_0 - \alpha F_0 + \alpha F_0 + \alpha \sum_{i=1}^{m} x'_i F_i + (1 - \alpha) \sum_{i=1}^{m} x''_i F_i$$

$$= \alpha F_0 + \alpha \sum_{i=1}^{m} x'_i F_i + (1 - \alpha) F_0 + (1 - \alpha) \sum_{i=1}^{m} x''_i F_i$$

$$= \alpha F(x') + (1 - \alpha)F(x'').$$

Thus, as shown above, the function $F$ describing a LMI is a convex function.

**Convex Optimization Problems Constrained by Linear Matrix Inequalities**

Consider the following optimization problem:

$$\begin{align*}
\min & \quad J(x) \\
\text{w.r.t.} & \quad x, \\
\text{s.t.} & \quad F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq \epsilon 1
\end{align*}$$

where $F_0, F_i \in \mathbb{R}^{n \times n}$, $i = 1 \cdots m$ are symmetric matrices and $\epsilon \geq 0$ as before. The objective function $J: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, and the constraints are in the form of a LMI. Optimization problems of this form can be solved efficiently by interior point methods.\textsuperscript{29-31}

**Forming LMIs: A Simple Example**

Let us illustrate how to form an LMI through an example.\textsuperscript{31} Given constant matrices $A \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ (where $Q = Q^T > 0$) find a symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that the inequality

$$PA + A^TP + Q < 0$$

holds. The elements of $P$ are our design variables, and although the inequality is linear in the matrix $P$, it does not look like the LMI in Eq. (2.4). For simplicity, let us take $n = 2$ so that each matrix has $2 \times 2$ in dimension, and $x = [p_1 \ p_2 \ p_3]^T$. Next, we will write $P$ in terms of a basis $E_i \in \mathbb{R}^{2 \times 2}$, $i = 1 \cdots 3$ as follows:

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. $$
Note that the matrices $E_i$ are linearly independent (and symmetric), thus forming a basis for the symmetric matrix $P$. We can now write $PA + A^T P + Q < 0$ as

$$p_1 \left( E_1 A + A^T E_1 \right) + p_2 \left( E_2 A + A^T E_2 \right) + p_3 \left( E_4 A + A^T E_4 \right) + Q < 0.$$  

By defining $F_0 = -Q$ and $F_i = -E_i A - A^T E_i = F_i^T$ we can now write

$$F_0 + \sum_{i=1}^{3} p_i F_i > 0$$

which is equivalent to Eq. (2.4) for some small value of $\epsilon > 0$. Additionally, note that the domain of the LMI is the set of symmetric matrices. The set of all symmetric matrices is a convex set.

The Schur Complement

Sometimes a matrix inequality that is not linear can be converted into a (larger) LMI using the Schur complement.\textsuperscript{27,32}

**Lemma 2.2.1.** The matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = P^T > 0$$

if and only if

$$P_{22} > 0, \quad P_{11} - P_{12} P_{22}^{-1} P_{12}^T > 0.$$  

Similarly, the matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = Q^T \geq 0$$

if and only if

$$Q_{22} \geq 0, \quad Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T \geq 0, \quad Q_{12} \left( I - Q_{22} Q_{22}^{-1} \right) = 0$$

where $Q_{22}^{-1}$ is the Moore-Penrose pseudoinverse of $Q_{22}$.\textsuperscript{31}

### 2.3 Summary

In this chapter, we have reviewed various mathematical definitions and results. In particular, we have reviewed norms, inner products, and various norm inequalities. We also reviewed how constrained optimization problems are posed. In particular, we reviewed convex optimization problems where constraints are in the form of LMIs.
Chapter 3

Classic Input-Output Stability Theory

The purpose of this chapter is to review classic input-output theory, and in particular the classic Small Gain, Passivity, and Conic Sector Stability Theorems. In this thesis each of the previously mentioned theorems are used or extended in some way; as such, we will state and discuss each in a moderate amount of detail. Before each input-output stability theorem is discussed, the notion of a system, and in particular a system’s input-output representation, will be reviewed. We will discuss operators (which are used to represent the input-output mapping of a system), as well as specific kinds of systems (e.g., linear time-invariant systems) and some of their properties (e.g., finite gain, positive realness, conicity). Stability of system interconnections will also be considered in the context of the dissipative systems framework.

3.1 System Operators and Input-Output Stability

3.1.1 Operators, Causality, and Adjoints

Consider an input space $L_2e(U) \subseteq L_2e$ and an output space $L_2e(Y) \subseteq L_2e$ where $u \in L_2e(U)$ and $y \in L_2e(Y)$.\textsuperscript{26} Associated with these input and output spaces is an operator $\mathcal{G} : L_2e(U) \rightarrow L_2e(Y)$ such that $y(t) = (\mathcal{G}u)(t)$. An operator $\mathcal{G} : L_2e(U) \rightarrow L_2e(Y)$ is causal if

$$
(\mathcal{G}u)_T(t) = (\mathcal{G}u_T)(t), \quad u \in L_2e(U), \quad \forall T \in \mathbb{R}^+.
$$

Causality means that past and present outputs do not depend on future inputs. Physical systems, such as those considered in this thesis, are represented by causal operators that map an input $u \in L_2e(U)$ to an output $y \in L_2e(Y)$. We always assume a system is causal.

The adjoint of the operator $\mathcal{G} : L_2e(U) \rightarrow L_2e(Y)$ can be defined using the inner product:\textsuperscript{21}

$$
\langle (\mathcal{G}u), y \rangle_T = \int_0^T (\mathcal{G}u)^T(t)y(t)dt = \int_0^Tu^T(t)(\mathcal{G}^\sim y)(t)dt = \langle u, (\mathcal{G}^\sim y) \rangle_T,
$$

17
∀u ∈ L_2(U), ∀y ∈ L_2(Y) where G^\sim : L_2e(Y) → L_2e(U) is the operator adjoint. An operator is self-adjoint if (G^\sim)^\sim = G.

3.1.2 L_2 Stability

A system represented by the operator G : L_2e(U) → L_2e(Y) mapping inputs u ∈ L_2e(U) to outputs y ∈ L_2e(Y) via y(t) = (Gu)(t) is said to be input-output L_2 stable, or simply L_2 stable, if

u ∈ L_2(U) ⇒ y ∈ L_2(Y).

A simple or intuitive way to interpret L_2 stability is in terms of input and output energy; “finite energy inputs lead to finite energy outputs”. In essence, this thesis is about finding conditions that ensure L_2 stability of various feedback interconnections. In particular, consider the negative feedback interconnection of systems G_1 : L_2e(E_1) → L_2e(Y_1) and G_2 : L_2e(E_2) → L_2e(Y_2) presented in Fig. 3.1, where u_1 ∈ L_2e(U_1) and u_2 ∈ L_2e(U_2) are external disturbances (including physical disturbances and signal noise). Often G_1 is the plant to be controlled and G_2 is the controller to be designed. The closed-loop system can be represented by y(t) = (Gu_2)(t) where

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.1.png}
\caption{The negative feedback interconnection of two systems G_1 and G_2.}
\end{figure}

inputs u = [u_1^T \ u_2^T]^T ∈ L_2e(U) are mapped to outputs y = [y_1^T \ y_2^T]^T ∈ L_2e(Y) through the operator G : L_2e(U) → L_2e(Y). Our main or most important objective, then, is to design G_2 so that the closed-loop system G is input-output stable. Of course, other requirements such as the need for “robust” input-output stability, or “optimal” regulation/control of the outputs y may have to be considered while designing G_2.

3.2 State-Space Realizations

Consider a mapping y(t) = (Gu)(t) where inputs u ∈ L_2e(U) are mapped to outputs y ∈ L_2e(Y) through an operator G : L_2e(U) → L_2e(Y). Often speaking in terms of a system’s input-output map is too general; as such, we will consider various finite-dimensional state-space realizations of the system G : L_2e(U) → L_2e(Y).
3.2.1 Nonlinear Systems

If $\mathcal{G}$ is a nonlinear system, it may be expressed in terms of the following finite-dimensional state-space realization:

$$
\dot{x}(t) = f(x(t), u(t)), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (3.1a)
$$

$$
y(t) = g(x(t), u(t)), \quad y \in \mathbb{R}^m, \quad (3.1b)
$$

where $x : \mathbb{R}^+ \to \mathbb{R}^n$ is the system state, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. We assume that the state-space realization is

1. completely reachable from the origin, that is for any $t_1$ and $x(t_1)$ there exists a $t_0 \leq t_1$ and an input $u \in L_2(U)$ such that (s.t.) the states can move from $x(t_0) = 0$ to $x(t_1)$ in the finite time interval $t \in [t_0, t_1]$, and

2. zero-state observable, that is $u(t) = 0$ and $y(t) = 0$ together imply that $x(t) = 0$.

3.2.2 Linear Time-Varying Systems

If $\mathcal{G}$ is a linear time-varying (LTV) system, it may be expressed in terms of the following finite-dimensional state-space realization:

$$
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (3.2a)
$$

$$
y(t) = C(t)x(t) + D(t)u(t), \quad y \in \mathbb{R}^m, \quad (3.2b)
$$

where $A : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$, $B : \mathbb{R}^+ \to \mathbb{R}^{n \times m}$, $C : \mathbb{R}^+ \to \mathbb{R}^{m \times n}$, and $D : \mathbb{R}^+ \to \mathbb{R}^{m \times m}$ are appropriately dimensioned real matrices that are continuous and bounded over the time interval of interest. It is assumed that the system is both controllable and observable.\(^{33,34}\) The solution to Eq. (3.2) is

$$
y(t) = C(t)\Phi(t, t_0)x_0 + C(t)\int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t) = (\mathcal{G}u)(t) \quad (3.3)
$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ is the state-transition matrix and $x_0 = x(t_0)$ is the initial system state. The state-transition matrix satisfies

$$
\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = 1, \quad \Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0), \quad \Phi^{-1}(t, \tau) = \Phi(\tau, t).
$$

3.2.3 Linear Time-Invariant Systems

If $\mathcal{G}$ is a linear time-invariant (LTI) system, it may be expressed in terms of the following finite-dimensional state-space realization:

$$
\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (3.4a)
$$

$$
y(t) = Cx(t) + Du(t), \quad y \in \mathbb{R}^m, \quad (3.4b)
$$
where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, \) and \( D \in \mathbb{R}^{m \times m} \) are constant matrices. It is assumed that the state-space realization is controllable and observable. The solution to Eq. (3.4) is

\[
y(t) = C e^{A(t-t_0)} x_0 + C \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau + D u(t) = (Gu)(t)
\]

where \( \Phi(t,\tau) = e^{A(t-\tau)} \) is the state-transition matrix. If we assume \( x_0 = 0 \) we can represent this LTI system in terms of a transfer matrix \( G(s) \in \mathbb{C}^{m \times m} \):

\[
y(s) = G(s)u(s), \quad G(s) = C(sI - A)^{-1} B + D.
\]

A transfer matrix is simply a matrix composed of real rational transfer functions of the complex variable \( s \). It is assumed that the transfer functions that compose \( G(s) \) are biproper (relative degree zero) or strictly proper (relative degree −1). The transfer matrix is strictly proper if \( D = 0 \).

### 3.3 Finite Gain Systems

**Definition 3.3.1.** A square system \( y(t) = (Gu)(t), \ G : L_{2e}(U) \to L_{2e}(Y), \ u \in L_{2e}(U), \ y \in L_{2e}(Y) \) possesses finite gain (finite \( L_2 \) gain) if there exist real constants \( 0 < \gamma < \infty \) and \( \beta' \) such that

\[
\| y \|_{2T} \leq \gamma \| u \|_{2T} + \beta', \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+.
\]

If \( \beta' = 0 \) the system is said to have finite \( L_2 \) gain with zero bias. In general, \( \beta' \) is positive.

Recall the definition of \( L_2 \) stability presented in Sec. 3.1 on page 17; the system \( G : L_{2e}(U) \to L_{2e}(Y) \) is \( L_2 \) stable if \( u \in L_2(U) \) leads to \( y \in L_2(Y) \). Hence, if the system has finite \( L_2 \) gain then it is \( L_2 \) stable. We can see this by starting with Eq. (3.6) and letting \( T \to \infty \), which leads to

\[
\| y \|_2 \leq \gamma \| u \|_2 + \beta', \quad \forall u \in L_2(U).
\]

That is, \( u \in L_2(U) \) leads to \( y \in L_2(Y) \); "bounded inputs lead to bounded outputs".

Determining if a system has finite gain, that is to say is \( L_2 \) stable, can be somewhat difficult in most situations, if not impossible altogether. For LTI systems, however, assessing \( L_2 \) stability via computation of gain is feasible, as discussed next.

### 3.3.1 Linear Time-Invariant Finite Gain Systems

Let us discuss how to compute the “largest” gain of a LTI system in terms of a transfer matrix \( G(s) \) and a minimal state-space realization of the form presented in Eq. (3.4). Note that a LTI system represented by \( G(s) \) must have all poles in the open left half plane for it to have finite gain.
Frequency Domain Calculation of Gain

Given a stable LTI system with transfer matrix \( G(s) \), the gain of a system, \( \gamma \), can be computed in terms of the supremum of the maximum singular value of \( G(j\omega) \). To see this, consider the squared \( L_2 \) norm of \( y \) manipulated in the following manner:

\[
\|y\|_2^2 = \int_0^\infty y^T(t)y(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} y^H(j\omega)y(j\omega)d\omega
\]

where we have transferred into the frequency domain using Parseval’s Theorem. Using the relation \( y(s) = G(s)u(s) \) gives

\[
\|y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} u^H(j\omega)G^H(j\omega)G(j\omega)u(j\omega)d\omega \leq \sup_{\omega \in \mathbb{R}} \sigma^2 \{G(j\omega)\} \frac{1}{2\pi} \int_{-\infty}^{\infty} u^H(j\omega)u(j\omega)d\omega = \gamma^2 \|u\|_2^2
\]

where \( \sigma \{G(j\omega)\} = \left[ \hat{\lambda} \left\{G^H(j\omega)G(j\omega)\right\}\right]^{\frac{1}{2}} \) is the maximum singular value and \( \hat{\lambda} \{\cdot\} \) is the maximum eigenvalue. We then have \( \|y\|_2 \leq \gamma \|u\|_2 \). If \( 0 < \gamma < \infty \) then the system has finite gain. Note that there is no \( \beta' \) present because, in using \( G(s) \) for computation, we have implicitly assumed zero initial conditions.

If we write \( \|y\|_2 \leq \gamma \|u\|_2 \) as

\[
\frac{\|y\|_2}{\|u\|_2} \leq \gamma
\]

we can proceed to define the induced \( L_2 \) gain, or the \( H_\infty \) norm of the transfer matrix \( G(s) \):

\[
\sup_{u \in L_2, u \neq 0} \frac{\|y\|_2}{\|u\|_2} = \|G\|_\infty
\]

where \( \|G\|_\infty \) is the \( H_\infty \) norm of \( G(s) \). The \( H_\infty \) norm is the smallest gain \( \gamma \).

The Bounded Real Lemma

Having presented a means to determine the gain of a system in terms of the frequency response of a transfer matrix, we now turn to a method that employs the state-space realization of a system.

Lemma 3.3.1 (Bounded Real Lemma\textsuperscript{27,35}). A stable LTI system described by Eqs. (3.4) and (3.5) that is controllable and observable has gain less than \( 0 < \gamma < \infty \) if and only if (iff) there exist real matrices \( P = P^T > 0 \), \( L \), and \( W \) of appropriate dimension such that

\[
PA + A^TP = -\frac{1}{\gamma}C^TC - LT^TL,
\]

\[
PB = -\frac{1}{\gamma}C^TD - LT^TW
\]

\[
\frac{1}{\gamma}D^TD = \gamma I - W^TW.
\]

If a LTI system with a minimal state-space realization satisfies Lemma 3.3.1 for some \( \gamma \), then it satisfies Eq. (3.6) for that value of \( \gamma \). To find the largest value of \( \gamma \) (i.e., the supremum), a
bisection search may be done, for example.

Traditionally, the bounded real (BR) Lemma takes $\gamma = 1/2^2$ although we are not following tradition by letting $\gamma$ take on any finite value, our version of the Bounded Real Lemma is more general. A similar Bounded Real Lemma is presented in Ref. 36.

### 3.4 Passive Systems

**Definition 3.4.1.** A square system $y(t) = (Gu)(t)$, $G : L_{2e}(U) \to L_{2e}(Y)$, $u \in L_{2e}(U)$, $y \in L_{2e}(Y)$ is **very strictly passive** (VSP) if there exist real constants $\delta > 0$, $\epsilon > 0$ and $\beta$ such that

$$
\int_0^T y^T(t)u(t)dt \geq \delta \|u\|^2_{2T} + \epsilon \|y\|^2_{2T} + \beta, \forall u \in L_{2e}(U), \forall T \in \mathbb{R}^+.
$$

(3.7)

A VSP system is often referred to as an input-output strictly passive system, or an input strictly passive system with finite gain. The system $y(t) = (Gu)(t)$, $G : L_{2e}(U) \to L_{2e}(Y)$, $u \in L_{2e}(U)$, $y \in L_{2e}(Y)$ is

- **input strictly passive** (ISP) when Eq. (3.7) holds with $\delta > 0$ and $\epsilon = 0$,
- **output strictly passive** (OSP) when Eq. (3.7) holds with $\delta = 0$ and $\epsilon > 0$, and
- **passive** when Eq. (3.7) holds with $\delta = \epsilon = 0$.

**Definition 3.4.2.** A square system $y(t) = (Gu)(t)$, $G : L_{2e}(U) \to L_{2e}(Y)$, $u \in L_{2e}(U)$, $y \in L_{2e}(Y)$ is **state strictly passive** (SSP) if there exist $\psi(\cdot) > 0$ and $\beta$ such that

$$
\int_0^T y^T(t)u(t)dt \geq \int_0^T \psi(x(t))dt + \beta, \forall u \in L_{2e}, \forall T \geq 0.
$$

(3.8)

The variable $\beta$ is less than or equal to zero (i.e., $\beta \leq 0$), and is usually related to the initial energy in the system. Also, $-\beta' = \beta$.

Some authors, for example Ref. 4, call a SSP system a strictly passive system, which we feel is confusing when compared with, for example, Ref. 16. In Ref. 16 an input strictly passive system is called a strictly passive system. Hence, we utilize a more verbose nomenclature to distinguish between each kind of passivity.

Many mechanical systems have an input-output map that is passive; generally the map is passive for collocated inputs and outputs such as a force and velocity, or a torque and angular velocity. As such, even complicated nonlinear systems such as spacecraft and flexible robotic manipulators can be shown to be passive, or even strictly passive (either VSP, OSP, or ISP). In a LTI context, passivity can also be shown in a rather straightforward manner, as discussed next.

### 3.4.1 Passive Linear Time-Invariant Systems

Given a LTI system with transfer matrix $G(s)$ or a minimal state-space realization $(A, B, C, D)$ (as presented in Eq. (3.4) on page 20), we would like to find conditions which, when met, ensure a system is passive, VSP, OSP, or ISP.
Positive Real and Strictly Positive Real Transfer Matrices

Let us find conditions that ensure a LTI system is passive in terms of the frequency response of $G(s)$. By assuming the system's initial conditions are zero, it follows that $\beta = 0$. To start, consider the following time-domain integral, and subsequent conversion into the frequency domain via Parseval’s Theorem:

$$\int_{0}^{T} y^T(t)u(t)dt = \int_{0}^{\infty} y^T_r(t)u_r(t)dt$$

$$= \frac{1}{2\pi} \text{Re} \int_{-\infty}^{\infty} y^H_r(j\omega)u_r(j\omega)d\omega$$

$$= \frac{1}{2\pi} \text{Re} \int_{-\infty}^{\infty} \left[ \frac{1}{2} y^H_r(j\omega)u_r(j\omega) + \frac{1}{2} u^H_r(j\omega)y_r(j\omega) \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} u^H_r(j\omega) \left[ G^H(j\omega) + G(j\omega) \right] u_r(j\omega)d\omega$$

Referring to Eq. (3.7), for the LTI system $G$ with transfer matrix $G(s)$ to be passive we require

$$G^H(j\omega) + G(j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}.$$  

Similarly,

$$G^H(j\omega) + G(j\omega) \geq 2\delta \mathbf{1}, \quad \forall \omega \in \mathbb{R}$$

where $\delta > 0$ for the system to be ISP. As such, we are lead to the definition of positive real and strictly positive real transfer matrices.

**Definition 3.4.3** (Positive Real Transfer Matrix$^{22}$). A rational transfer matrix $G(s) \in \mathbb{C}^{m \times m}$ is **positive real** (PR) if

1. all elements of $G(s)$ are analytic in $\text{Re}\{s\} > 0$, and
2. $G^H(s) + G(s) \geq 0$ in $\text{Re}\{s\} > 0$, or equivalently

   2.1. poles on the imaginary axis are simple and have nonnegative-definite residues, and
   2.2. $G^H(j\omega) + G(j\omega) \geq 0$, $\forall \omega \in \mathbb{R}$ with $j\omega$ not a pole of any element of $G(j\omega)$.

As previously mentioned, an LTI system with a PR transfer matrix is passive. Additionally, a PR system is minimum phase, that is, all zeros will be in the closed left half plane. As such, a PR system is at least marginally stable (e.g., when all poles are on the imaginary axis) and minimum phase. When transfer functions such as $g(s)$ are under consideration, it is simple to visually determine if $g(s)$ is PR; PR transfer functions have a phase response that satisfies $-\frac{\pi}{2} \leq \arg g(j\omega) \leq \frac{\pi}{2}$.

**Definition 3.4.4** (Strictly Positive Real Transfer Matrix$^{37}$). A stable rational transfer matrix $G(s) \in \mathbb{C}^{m \times m}$ is **strictly positive real** (SPR) if $G(s - \delta)$ is PR for some $\delta > 0$; that is, if
1. All elements of $G(s)$ are analytic in $\text{Re } \{s\} \geq 0$,
2. $G^H(j\omega) + G(j\omega) > 0$, $\omega \in \mathbb{R}$, and
3. 3.1. $Z = G^T(\infty) + G(\infty) \geq 0$, or
    3.2. $\lim_{\omega \to \infty} \omega^2[G^H(j\omega) + G(j\omega)] > 0$ if $Z$ is singular.

Some comments; the above definition is actually for “strong” SPR systems. A system is said to be “weak” SPR if conditions 1 and 2 hold, but not 3. When we say “SPR”, we mean strong SPR.

If $G(s)$ is biproper with a positive definite $D$ matrix then $Z > 0$, that is condition 3.1 will be satisfied. As such, the system will also be ISP. If $G(s)$ is biproper with a positive semidefinite $D$, or $G(s)$ strictly proper, then $Z \geq 0$ and condition 3.2 must hold. In this case, $G(s) + \delta I$ is ISP where $\delta > 0$. Also, because a SPR transfer matrix must be stable, it follows that $G(s)$ corresponds to a VSP system when $Z > 0$, and $G(s) + \delta I$ corresponds to a VSP system when $Z \geq 0$ and 3.2 holds. This is because if $G(s)$ is ISP and has finite gain (i.e., is stable), then $G(s)$ is VSP.

When a LTI system takes the form of a transfer function $g(s)$, the transfer function is SPR if it is Hurwitz, has a phase response that satisfies $-\pi/2 < \arg g(j\omega) < \pi/2$, and either has a strictly positive feedthrough matrix (i.e., the transfer function is biproper) or $\lim_{\omega \to \infty} \omega^2 \text{Re } \{g(j\omega)\} > 0$ if there is no feedthrough matrix (i.e., the transfer function is strictly proper).

**State-Space Characterization of Positive Real and Strictly Positive Real Systems**

We will now move on to a state-space characterization of PR and SPR systems.

**Lemma 3.4.1** (Positive Real Lemma$^{27}$). A LTI system described by Eqs. (3.4) and (3.5) that is controllable and observable is positive real iff there exist real matrices $P = P^T > 0$, $L$, and $W$ of appropriate dimension such that

$$ PA + A^T P = -L^T L, $$
$$ PB = C^T - L^T W, $$
$$ D + D^T = W^T W. $$

If a system with a minimal state-space realization satisfies Lemma 3.4.1 then it is passive.

**Lemma 3.4.2** (Strictly Positive Real Lemma$^{6,27,38}$). A LTI system described by Eqs. (3.4) and (3.5) that is controllable and observable is strictly positive real iff there exist real matrices $P = P^T > 0$, $L$, and $W$ of appropriate dimension, and $\nu > 0$ such that

$$ PA + A^T P = -L^T L - 2\nu P, $$
$$ PB = C^T - L^T W, $$
$$ D + D^T = W^T W. $$

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The above lemma, when satisfied, ensures a system with a minimal state-space realization is SPR. As mentioned previously, if the system is SPR it is stable (i.e., $\lambda_i \{A\} < 0$, $i = 1 \cdots n$). If the system is strictly proper (i.e., $D = 0$), the SPR Lemma reduces to

$$PA + A^T P = -L^T L - 2\nu P,$$
$$PB = C^T.$$

Historically, Lemma 3.4.2 has been called the Kalman-Yakubovich-Popov Lemma (the KYP Lemma). However, in more modern literature the Kalman-Yakubovich-Popov Lemma is a lemma that is more general, providing conditions that ensure a system is PR, SPR, or BR.

At first glance, Lemmas 3.3.1, 3.4.1, and 3.4.2 look quite similar, and they are. However, each lemma tells us something different. If a system has gain less than $\gamma$, then the system will satisfy Lemma 3.3.1; that is, the system has finite gain. On the other hand, a passive system (which may not have finite gain) would satisfy Lemma 3.4.1, which is to say the system is PR. If a system satisfies Lemma 3.4.2, not only is the system SPR, but owing to the fact that the system must be stable, the system would satisfy Lemma 3.3.1 for some $\gamma$ as well.

### 3.5 Conic Systems

The notion of a dynamic system (possibly nonlinear) having an input-output map confined to a conic region was first laid out by George Zames in Refs. 13 and 14. This idea is a generalization or abstraction of how memoryless nonlinearities behave, where “memoryless” means the system has no dynamics.

**Definition 3.5.1.** Consider a memoryless nonlinearity $y(t) = \phi(u(t), t)$ where $\phi : \mathbb{R}^m \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $\phi(0, t) = 0$, $\forall t \in \mathbb{R}^+$, and two constants $a$ and $b$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\}$ where $b > a$. The memoryless nonlinearity is said to be inside the conic sector $[a, b]$, written $\phi \in \text{cone}[a, b]$, if

$$\left[u(t) - \frac{1}{b}\phi(u(t), t)\right]^T \left[\phi(u(t), t) - au(t)\right] \geq 0, \quad \forall u \in \mathbb{R}^m$$

which, once expanded, is equivalent to

$$-\frac{1}{b} \phi^T(u(t), t)\phi(u(t), t) + (1 + \frac{a}{b})\phi(u(t), t)u(t) - au^T(t)u(t) \geq 0, \quad \forall u \in \mathbb{R}^m.$$  

To illustrate, consider the following memoryless nonlinearity:

$$y(t) = \phi(u(t)) = \tanh(5u(t)), \quad \forall u \in \mathbb{R}.$$  

The input-output map of this memoryless nonlinearity is shown in Fig. 3.2(a). Notice, as highlighted in Fig. 3.2(b), that the input-output map of $\phi$ is confined to the first and third quadrants. In particular, $\phi$ is lower bounded by a line of zero slope, and upper bounded the line...
(a) A memoryless nonlinearity; \( y(t) = \tanh(5u(t)) \).

(b) Conic bounds confining the input-output map of the memoryless nonlinearity.

Figure 3.2: A memoryless nonlinearity and conic bounds.

\( y(t) = 5u(t) \). We say that \( \phi \) is inside the conic sector, or simply inside “the cone” \([0, 5]\).

Now, let us generalize the notion of a conic sector to include systems that have dynamics.

**Definition 3.5.2.** A square system \( y(t) = (G u)(t) \), \( G : L^2_e(U) \rightarrow L^2_e(Y) \), \( u \in L^2_e(U) \), \( y \in L^2_e(Y) \) is said to be inside the conic sector \([a, b]\), written \( G \in \text{cone}[a, b] \), if there exist real constants \( a \in \mathbb{R} \), \( b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\} \) where \( b > a \), and \( \beta \) such that

\[
-\frac{1}{b} \|y\|_{2T}^2 + \left(1 + \frac{a}{b}\right) \langle y, u \rangle_T - a \|u\|_{2T}^2 \geq \beta, \quad \forall u \in L^2_e(U), \quad \forall T \in \mathbb{R}^+.
\]  

(3.9)

If inequality Eq. (3.9) holds for some \([a + \Delta, b - \Delta]\) where \( \Delta \in \mathbb{R}^+ \setminus \{0\} \), we say that the system \( G \) is strictly inside the conic sector \([a, b]\), and write \( G \in \text{cone}(a, b) \).

When \( 0 < b < \infty \) inequality Eq. (3.9) can be rewritten in terms of a cone centre and cone radius, \( c \) and \( r \), where \( c = (b + a)/2 \) and \( r = (b - a)/2 \) (or where \( a = c - r \) and \( b = c + r \)); the system is said to be inside the conic sector \([c - r, c + r]\) \((G \in \text{cone}[c - r, c + r])\) if there exist real constants \( c \in \mathbb{R}, \quad r \geq 0 \), and \( \beta' \) such that

\[
\|y - cu\|_{2T} \leq r \|u\|_{2T} + \beta', \quad \forall u \in L^2_e(U), \quad \forall T \in \mathbb{R}^+.
\]  

(3.10)

It is straightforward to show that inequalities Eqs. (3.9) and (3.10) are equivalent when \( 0 < b < \infty \).

Not only can cones be used to represent the input-output behavior of a system, but they can be used to represent model uncertainty. For example, the cone center may represent a nominal system mapping, while the cone radius represents model uncertainty.\(^{20,41,42}\)

Zames’ work, although seminal, is a bit vague with respect to the permissible values that \( a \) and \( b \) (and, by extension, \( c \) and \( r \)) may take on. Admittedly, at one point Zames discusses the possibility of \( b \rightarrow \infty \), but neglects to discuss how \( c \) and \( r \) change as \( b \rightarrow \infty \). The bounds presented here are those given by Safonov and his coauthors.\(^{18,40}\) Presently, it seems standard
to follow Safonov’s convention, although some authors constrain $b$ to be less than infinity (i.e., $0 < b < \infty$) which leads to conservatism and loss of generality; for example, see Refs. 43 and 44.

By generalizing the idea that a system $\mathcal{G}$ has an input-output map confined to a conic sector we lose the ability to draw the conic sector exactly, as is possible when dealing with memoryless nonlinearities. However, in order to gain insight into how a system behaves, we can still draw a conic sector representing the input-output behavior of $\mathcal{G}$, but such a conic sector is simply an abstraction. For example, consider the cone with bounds $[a, b]$ where $0 < b < \infty$, $-\infty < a < 0$ presented in Fig. 3.3(a); $a$ corresponds to the slope of the lower line defining the cone, while $b$ corresponds to the slope of the upper line defining the cone. We can equivalently express this cone in terms of a cone centre and radius, as shown in Fig. 3.3(b). These abstract pictures that

![Conic sector](image)

represent the input-output behavior of a system $\mathcal{G}$ are quite useful when it comes to assessing closed-loop stability of two or more systems connected in a negative feedback loop, which is why we, and others, frequently use them.

Before closed-loop stability can be assessed, the cone that defines a system must be determined. In particular, we would like to know if a system $\mathcal{G}$ together with $[a, b]$ satisfies Eq. (3.9) (or, Eq. (3.10)) for some particular $a$ and $b$ (or $c$ and $r$) values. Doing so will be discussed in a LTI context next.

### 3.5.1 Linear Time-Invariant Conic Systems

As just mentioned, we would like to know if a system is confined to a particular conic sector. We shall consider LTI systems that can be represented in terms of a transfer matrix $G(s)$ or minimal state-space realization $(A, B, C, D)$ (as presented in Eq. (3.4) on page 20).

#### Conic Systems in the Frequency Domain

Given a system with transfer matrix $G(s)$ and some conic bounds $[a, b]$, we would like to know if $G(s)$ together with $[a, b]$ satisfies Eq. (3.9). Starting with Eq. (3.9), moving to the frequency
domain by employing Parseval's Theorem, and using the relation $y(s) = G(s)u(s)$ we have the following:

$$-rac{1}{b} \|y\|^2_{2T} + (1 + \frac{a}{b}) \langle y, u \rangle_T - a \|u\|^2_{2T} \geq 0,$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} u_T^H(j\omega) \left[ -\frac{1}{b} G^H(j\omega) G(j\omega) \right] u_T(j\omega) d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} u_T^H(j\omega) \left( \frac{1}{2} (1 + \frac{a}{b}) \left[ G^H(j\omega) + G(j\omega) \right] \right) u_T(j\omega) d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} u_T^H(j\omega) \left[ -a1 \right] u_T(j\omega) d\omega \geq 0.$$

It follows that $G(s)$ is in the conic sector $[a, b]$ if

$$-\frac{1}{b} G^H(j\omega) G(j\omega) + \frac{1}{2} (1 + \frac{a}{b}) \left[ G^H(j\omega) + G(j\omega) \right] - a1 \geq 0, \quad \forall \omega \in \mathbb{R} \quad (3.11)$$

holds, which is equivalent to

$$\text{He} \left\{ [I - \frac{1}{b} G(j\omega)]^H [G(j\omega) - a1] \right\} \geq 0, \quad \forall \omega \in \mathbb{R}$$

where $\text{He} \{H(s)\} = \frac{1}{2} \left[ H(s) + \overline{H(s)} \right]$ is the Hermitian part of the transfer matrix $H(s)$.

### State-Space Characterization of Conic Systems

Next, we present a lemma that characterizes a LTI conic system in terms of its state-space realization.

**Lemma 3.5.1** (Conic Sector Lemma). A LTI system described by Eqs. (3.4) and (3.5) that is controllable and observable lies in the conic sector $[a, b]$ iff there exist real matrices $P = P^T > 0$, $L$, and $W$ of appropriate dimension such that

$$PA + A^TP = -\frac{1}{b} C^T C - L^T L,$$

$$PB = \frac{1}{2} (1 + \frac{a}{b}) C^T - \frac{1}{b} C^T D - L^T W,$$

$$\frac{1}{b} (1 + \frac{a}{b}) [D + DT] = W^TW + \frac{b}{a} D^TD + a1.$$

The above lemma is not an exact replica of the result presented in Ref. 43; ours is slightly different owing to the fact that we permit $b$ to go to infinity.

Lemma 3.5.1 can be used to determine if a LTI system is conic with bounds $a$ and $b$. Given a LTI system with minimal state-space realization $(A, B, C, D)$ and bounds $a$ and $b$, if $P = P^T > 0$, $L$, and $W$ can be found which satisfy Lemma 3.5.1 then $G \in \text{cone}[a, b]$.

### 3.5.2 Conic Representation of Finite Gain and Passive Systems

Thus far we have discussed three kinds of input-output mappings a system $G$ may have. The system $G$ may have a finite gain input-output mapping, a passive input-output mapping (either
VSP, ISP, OSP, or purely passive), or a conic input-output mapping. Each kind of input-output property is relevant and important; however, the conic representation is particularly powerful in that finite gain systems and passive systems can each be expressed in terms of specific cones. That is to say, finite gain and passive systems are each a special kind of conic system. We will show the relation between finite gain, passive, and conic systems next.

**Finite Gain Systems**

Let us start by showing that a system\[ y(t) = (Gu)(t), \quad G : L_{2e}(U) \rightarrow L_{2e}(Y), \quad u \in L_{2e}(U), \quad y \in L_{2e}(y) \] that is \( G \in \text{cone}[-\gamma, \gamma] \) for some \( 0 < \gamma < \infty \) is a finite gain system satisfying inequality Eq. (3.6) on page 20. To do so, we will let \( a = -\gamma \) and \( b = \gamma \) where \( 0 < \gamma < \infty \), then manipulate inequality Eq. (3.9) on page 26 in the following way:

\[
-\frac{1}{\gamma} \|y\|_{2T}^2 + \gamma \|u\|_{2T}^2 \geq \beta.
\]

Rearranging we have

\[
\|y\|_{2T}^2 \leq \gamma^2 \|u\|_{2T}^2 + \beta^2
\]

which is equivalent to inequality Eq. (3.6). This demonstrates that a system \( G \) with conic bounds \([−\gamma, γ] \), \( 0 < \gamma < \infty \) is in fact a finite gain system.

The finite gain system \( G \in \text{cone}[-\gamma, \gamma] \) is drawn in terms of its cone in Fig. 3.4(a). Notice that, in terms of Eq. (3.10) \( c = 0 \), \( r = \gamma \); the cone has a centre equal to zero and radius \( \gamma \).

Given that a finite gain system is a special kind of conic system, it is expected Lemmas 3.3.1 and 3.5.1 are related in some way. Recall that a system that has gain less than \( \gamma \) will satisfy Lemma 3.3.1, and a system that is conic with bounds \([a, b] \) will satisfy Lemma 3.5.1. If \( a = -\gamma \) and \( b = \gamma \) are the conic bounds, then Lemma 3.5.1 collapses down to the conditions given by Lemma 3.3.1.

**Passive Systems**

Next we will show that a system\[ y(t) = (Gu)(t), \quad G : L_{2e}(U) \rightarrow L_{2e}(Y), \quad u \in L_{2e}(U), \quad y \in L_{2e}(y) \] that is \( G \in \text{cone}[a, b] \) for some \( a \in \mathbb{R}^+ \) and \( b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\} \) values where \( b > a \) satisfies inequality Eq. (3.7) on page 22 for some \( \delta \geq 0 \) and \( \epsilon \geq 0 \) parameters. In particular, VSP, ISP, OSP, and purely passive mappings can be captured depending on the values of \( a \) and \( b \).

We will focus on the VSP case; the ISP, OSP, and passive cases follow in a straightforward manner. First, we will rearrange inequality Eq. (3.9) as follows:

\[
(1 + \frac{a}{b}) \langle y, u \rangle_T \geq a \|u\|_{2T}^2 + \frac{1}{b} \|y\|_{2T}^2 + \beta.
\]

Next, letting \( 0 < a \) and \( 0 < b < \infty \) inequality Eq. (3.12) can be written

\[
\langle y, u \rangle_T \geq \left( \frac{ab}{a+b} \right) \|u\|_{2T}^2 + \left( \frac{1}{a+b} \right) \|y\|_{2T}^2 + \frac{\beta}{1+b}.
\]
This expression is equivalent to Eq. (3.7) where \( \delta = \frac{ab}{a+b} > 0 \) and \( \epsilon = \frac{1}{a+b} > 0 \), and hence represents a VSP system. Therefore, \( \mathcal{G} \in \text{cone}[a,b] \) where \( 0 < a < b < \infty \) is a VSP system.

The range of values that \( a \) and \( b \) may take for \( \mathcal{G} \in \text{cone}[a,b] \) to capture an ISP, OSP, and passive input-output map are tabulated in Table 3.1. Additionally, Table 3.1 includes the \( \delta \) and \( \epsilon \) values in terms of \( a \) and \( b \).

<table>
<thead>
<tr>
<th>VSP</th>
<th>ISP</th>
<th>OSP</th>
<th>Passive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; a )</td>
<td>( 0 &lt; b &lt; \infty )</td>
<td>( a = 0 )</td>
<td>( a = 0 )</td>
</tr>
<tr>
<td>( \delta = \frac{ab}{a+b} )</td>
<td>( \delta = a )</td>
<td>( \delta = 0 )</td>
<td>( \delta = 0 )</td>
</tr>
<tr>
<td>( \epsilon = \frac{1}{a+b} )</td>
<td>( \epsilon = 0 )</td>
<td>( \epsilon = \frac{1}{b} )</td>
<td>( \epsilon = 0 )</td>
</tr>
</tbody>
</table>

Table 3.1: The values \( a \) and \( b \) may take for a conic mapping to represent a VSP, ISP, OSP, or passive input-output mapping.

Figs. 3.4(b), 3.4(c), and 3.4(d) depict typical VSP, ISP, and passive cones. Passive systems will always have \( a \) and \( b \) parameters that force the representative cone to lie in the first and third quadrants of the \( L_{2e}(U) \times L_{2e}(Y) \) space.

\[ |y|_2 \in L_{2e}(Y) \]
\[ |u|_2 \in L_{2e}(U) \]

(a) Finite gain cone where \( a = -\gamma = -b \).

\[ |y|_2 \in L_{2e}(Y) \]
\[ |u|_2 \in L_{2e}(U) \]

(b) VSP cone where \( 0 < a < b < \infty \).

\[ |y|_2 \in L_{2e}(Y) \]
\[ |u|_2 \in L_{2e}(U) \]

(c) ISP cone where \( 0 < a \) and \( b = \infty \).

\[ |y|_2 \in L_{2e}(Y) \]
\[ |u|_2 \in L_{2e}(U) \]

(d) Passive cone where \( a = 0 \) and \( b = \infty \).

Figure 3.4: Conic representation of finite gain and passive systems.
Recall that there was a direct relation between Lemmas 3.3.1 and 3.5.1; similarly, there is a relation between Lemmas 3.4.1 and 3.5.1. Given a system, when \( a = 0 \) and \( b = \infty \) the system is passive, which in a LTI context means the system is PR. When \( a = 0 \) and \( b = \infty \), Lemma 3.5.1 collapses down to the conditions stated in Lemma 3.4.1, which is to say the system is PR as we would expect.

### 3.5.3 Variable Conic Sectors

Consider a square system \( y(t) = (\mathcal{G}u)(t) \) with conic bounds defined by either \( a \) and \( b \) (as in Eq. (3.9)), or a cone centre \( c \) and radius \( r \) (as in Eq. (3.10)). In Ref. 18 (specifically, see pg. 36 and 47) Safonov generalizes Zames’ cones by letting \( a \) and \( b \), or the cone centre and radius, be represented by *operators*. Doing so allows one to describe the input-output map \( y(t) = (\mathcal{G}u)(t) \) in terms of a *variable* conic sector. Such variable cones are also discussed in Refs. 17–20. The word “variable” is our own, and we use it to distinguish Safonov’s cones from Zames’ cones.

**Definition 3.5.3.** Consider a square system \( y(t) = (\mathcal{G}u)(t), \mathcal{G} : L_{2e}(U) \to L_{2e}(Y), u \in L_{2e}(U), y \in L_{2e}(Y) \). Also consider a set of operators \( \mathcal{F}_{11} : L_{2e}(Y) \to L_{2e}, \mathcal{F}_{12} : L_{2e}(U) \to L_{2e}, \mathcal{F}_{21} : L_{2e}(Y) \to L_{2e}, \mathcal{F}_{22} : L_{2e}(U) \to L_{2e} \) where \( (\mathcal{F}_{kl})0 = 0 \) for \( k, l = 1, 2 \). For convenience we write

\[
\mathcal{F} = \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{bmatrix}
\]

where \( \mathcal{F} : L_{2e}(Y) \times L_{2e}(U) \to L_{2e} \). If there exist \( \beta \) such that

\[
(\mathcal{F}_{11}y + \mathcal{F}_{12}u, \mathcal{F}_{21}y + \mathcal{F}_{22}u)_T \geq \beta, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+
\]

then the system is said to be inside the variable conic sector \( \mathcal{F} \).

As a special case, consider when

\[
\mathcal{F} = \begin{bmatrix} -\frac{1}{b} & 1 \\ 1 & -a \end{bmatrix}
\]

where \( a \in \mathbb{R}^+, b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\} \) and \( b > a \). In this case \( \mathcal{F} \) is a constant matrix and Eq. (3.13) reduces to Eq. (3.9) on page 26.

### 3.6 Dissipative Systems

In Secs. 3.3, 3.4, and 3.5, finite gain, passive, and conic systems were each defined and discussed. Each kind of system, or specifically each kind of input-output behavior is important and useful, although one is perhaps better suited than another for a particular problem.

The classic dissipative systems framework developed by Hill and Moylan provides a convenient way to express various input-output properties, and assess stability of various feedback interconnections. It is convenient in that finite gain, passive, and conic systems can each be described
as special cases of a more general input-output description. As such, we will make use of the dissipative systems framework in this thesis. What follows is a review of the dissipative systems framework; the material in this section is taken from Refs. 6, 45–47.

Consider a mapping \( y(t) = (G u)(t) \) where inputs \( u \in L_{2e}(U) \) are mapped to outputs \( y \in L_{2e}(Y) \) through an operator \( G : L_{2e}(U) \to L_{2e}(Y) \). To be general, we will assume the system is nonlinear with a finite-dimensional state-space realization equivalent to that presented in Eq. (3.1), that is the system has a finite-dimensional state-space realization of the form

\[
\dot{x}(t) = f(x(t), u(t)), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,
\]

\[
y(t) = g(x(t), u(t)), \quad y \in \mathbb{R}^m.
\]

We assume the state-space realization is completely reachable from the origin, and zero-state observable. The system is also permitted to be LTV or LT1, and as such would have a minimal state-space realization equivalent to Eqs. (3.2) and (3.4).

Associated with the system is a supply rate \( w : L_{2e}(U) \times L_{2e}(Y) \to L_{2e} \) and a continuously differentiable function, the storage function, \( \phi : \mathbb{R}^n \to \mathbb{R}^+ \). The system described by Eq. (3.1) is said to be dissipative with respect to the supply rate \( w \) if\(^6\)

\[
\int_0^T w(u(t), y(t)) dt = \langle y, Q y \rangle_T + 2 \langle y, S u \rangle_T + \langle u, R u \rangle_T \geq \phi(x(T)) - \phi(x(0)) \geq -\phi(x(0))
\]

holds \( \forall T \in \mathbb{R}^+ \) where \( w(u(t), y(t)) \) is evaluated along the trajectories of Eq. (3.1) given \( u \in L_{2e}(U) \). The system is said to be \((Q, S, R)\)-dissipative. The matrices \( Q \in \mathbb{R}^{m \times m}, S \in \mathbb{R}^{m \times m}, \) and \( R \in \mathbb{R}^{m \times m} \) are symmetric (and thus square); in Refs. 45–47 they are permitted to be rectangular, but we are interested primarily in square systems.

The supply rate can be thought of as the power input to the system, while the storage function represents the energy stored by the system at a particular time; for example, \( \phi(x(T)) \) represents the energy stored at time \( T \). The integral of the supply rate represents the energy supplied to the system over \( t \in [0, T] \). If the system has initial conditions that are quiescent then \( \phi(x(0)) = 0 \).

As previously mentioned, finite gain, passive, and conic systems can be described in terms of the dissipative inequality of Eq. (3.15). In particular, the inequalities in Eqs. (3.6), (3.7), and (3.9) (and even Eq. (3.10)) defining finite gain, passive, and conic systems can be recovered for particular \( Q, S, \) and \( R \) matrices. This will be shown next.

### 3.6.1 Finite Gain Systems in Terms of the Dissipation Inequality

Assume there exist a constant \( 0 < \gamma < \infty \). A square system \( y(t) = (G u)(t) \), \( G : L_{2e}(U) \to L_{2e}(Y) \), \( u \in L_{2e}(U), \ y \in L_{2e}(Y) \) that is \((-\gamma^{-1}1, 0, \gamma1)\)-dissipative possesses finite gain and is thus \( L_2 \) stable. To see this, we substitute \( Q = -\gamma^{-1}1, S = 0, \) and \( R = \gamma1 \) into the dissipation inequality
in Eq. (3.15) and rearranging as follows:

\[
\begin{align*}
\langle y, Qy \rangle_T + 2 \langle y, Su \rangle_T + \langle u, Ru \rangle_T & \geq -\phi(x(0)), \\
-\gamma^{-1} \|y\|_{2T}^2 + \gamma \|u\|_{2T}^2 & \geq -\phi(x(0)), \\
\|y\|_{2T}^2 & \leq \gamma^2 \|u\|_{2T}^2 + \gamma \phi(x(0)), \\
\|y\|_{2T} & \leq \sqrt{\gamma^2 \|u\|_{2T}^2 + \gamma \phi(x(0))} \leq \gamma \|u\|_{2T} + \sqrt{\gamma \phi(x(0))}.
\end{align*}
\]

Comparing the above with inequality Eq. (3.6) we see that a \((-\gamma^{-1}1, 0, \gamma1)\)-dissipative system is a system that has gain that is finite, and is thus \(L_2\) stable.

### 3.6.2 Passive Systems in Terms of the Dissipation Inequality

Assume there exist constants \(\delta > 0\) and \(\epsilon > 0\). A square system \(y(t) = (Gu)(t), G : L_2c(U) \rightarrow L_2c(Y), u \in L_2c(U), y \in L_2c(Y)\) that is \((-\epsilon1, \frac{1}{2}1, -\delta1)\)-dissipative is VSP. To show this we substitute \(Q = -\epsilon1, S = \frac{1}{2}1, R = -\delta1\) into the dissipation inequality in Eq. (3.15) and rearranging in the following way:

\[
\begin{align*}
\langle y, Qy \rangle_T + 2 \langle y, Su \rangle_T + \langle u, Ru \rangle_T & \geq -\phi(x(0)), \\
-\epsilon \|y\|_{2T}^2 + \langle y, u \rangle_T - \delta \|u\|_{2T}^2 & \geq -\phi(x(0)), \\
\langle y, u \rangle_T & \geq \delta \|u\|_{2T}^2 + \epsilon \|y\|_{2T}^2 - \phi(x(0)).
\end{align*}
\]

Comparing the above with inequality Eq. (3.7) we see that a \((-\epsilon1, \frac{1}{2}1, -\delta1)\)-dissipative system is a VSP system. ISP, OSP, and passive systems are also \((Q, S, R)\)-dissipative where \(Q, S, \) and \(R\) take on specific values, as shown in Table 3.2. Substitution of any \((Q, S, R)\) triplet from Table 3.2 into Eq. (3.15) will yield the corresponding passivity inequality expressed in Eq. (3.7).

<table>
<thead>
<tr>
<th></th>
<th>Q</th>
<th>S</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>VSP</td>
<td>(-\epsilon1)</td>
<td>(\frac{1}{2}1)</td>
<td>(-\delta1)</td>
</tr>
<tr>
<td>ISP</td>
<td>0</td>
<td>(\frac{1}{2}1)</td>
<td>(-\delta1)</td>
</tr>
<tr>
<td>OSP</td>
<td>(-\epsilon1)</td>
<td>(\frac{1}{2}1)</td>
<td>0</td>
</tr>
<tr>
<td>Passive</td>
<td>0</td>
<td>(\frac{1}{2}1)</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.2: \(Q, S,\) and \(R\) values for VSP, ISP, OSP, and passive systems.

### 3.6.3 Conic Systems in Terms of the Dissipation Inequality

Assume there exist constants \(a\) and \(b\) where \(a \in \mathbb{R}, b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\}\) and \(b > a\). A square system \(y(t) = (Gu)(t), G : L_2c(U) \rightarrow L_2c(Y), u \in L_2c(U), y \in L_2c(Y)\) that is \((-\frac{1}{b}1, \frac{1}{2}(1 + \frac{a}{b})1, -a1)\)-dissipative is inside the conic sector \([a, b]\), that is \(G \in \text{cone}[a, b]\). This can be shown in a similar fashion as before: we substitute \(Q = -\frac{1}{b}1, S = \frac{1}{2}(1 + \frac{a}{b})1,\) and \(R = -a1\) into the dissipation
inequality in Eq. (3.15) and rearrange as shown below:

\[
\begin{align*}
\langle y, Qy \rangle_T + 2 \langle y, Su \rangle_T + \langle u, Ru \rangle_T & \geq -\phi(x(0)), \\
- \frac{1}{h} \|y\|^2_{2T} + (1 + \frac{\alpha^2}{h}) \langle y, u \rangle_T - a \|u\|^2_{2T} & \geq -\phi(x(0)).
\end{align*}
\]

Comparing the above with Eq. (3.9) it should be clear that a \((-\frac{1}{h}I, \frac{1}{h}(1 + \frac{\alpha^2}{h})I, -aI)\)-dissipative system is a conic system with bounds \([a, b]\).

As previously discussed in Sec. 3.5, if \(0 < b < \infty\) the inequality of Eq. (3.9) can be written in terms of an inequality involving a cone centre and cone radius as shown in Eq. (3.10). It follows then that a square system \(G\) that is \((-I, cI, -(c^2 - r^2)I)\)-dissipative is interior conic with bounds \([c - r, c + r]\) where \(c \in \mathbb{R}, r \geq 0\). This can be shown by substituting \(Q = -I, S = cI,\) and \(R = -(c^2 - r^2)I\) into the dissipation inequality in Eq. (3.15) and rearranging in the following manner:

\[
\begin{align*}
\langle y, Qy \rangle_T + 2 \langle y, Su \rangle_T + \langle u, Ru \rangle_T & \geq -\phi(x(0)), \\
- \|y\|^2_{2T} + 2c \langle y, u \rangle_T - (c^2 - r^2) \|u\|^2_{2T} & \geq -\phi(x(0)), \\
\|y\|^2_{2T} - 2c \langle y, u \rangle_T + c^2 \|u\|^2_{2T} & \leq r^2 \|u\|^2_{2T} + \phi(x(0)), \\
\|y - cu\|_{2T} & \leq \sqrt{r^2 \|u\|^2_{2T} + \phi(x(0))} \leq r \|u\|_{2T} + \frac{\phi(x(0))}{\beta'}.
\end{align*}
\]

Comparing the above with Eq. (3.10) it should be clear that a \((-I, cI, -(c^2 - r^2)I)\)-dissipative system is a conic system with bounds \([c - r, c + r]\).

### 3.7 Stability of Interconnected Dissipative Systems

**Theorem 3.7.1.** Consider the negative feedback interconnection of \(G_1 : L_{2e} \rightarrow L_{2e}\) and \(G_2 : L_{2e} \rightarrow L_{2e}\) presented in Fig. 3.1 on page 18. The admissible inputs \(e_1(t) = u_1(t) - y_2(t)\) and \(e_2(t) = u_2(t) + y_1(t)\) are in \(L_{2e}\). Each system can be described using the state-space equations of Eq. (3.1). Additionally, each system is \((Q_i, S_i, R_i)\)-dissipative with respect to a supply rate \(w_i\) and has a corresponding storage function \(\phi_i\) according to Eq. (3.15) (where \(i = 1\) corresponds to \(G_1\) and \(i = 2\) corresponds to \(G_2\)). The negative feedback interconnection is input-output stable \((L_2\) stable) if

\[
\begin{bmatrix}
-(Q_1 + R_2) & S_1 - S_2^T \\ S_1^T - S_2 & -(Q_2 + R_1)
\end{bmatrix} > 0,
\]

that is, \(u_1, u_2 \in L_2 \Rightarrow y_1, y_2 \in L_2\).

Proof of the above theorem is outlined in Ref. 47. A Lyapunov proof that yields an equivalent result can be found in Ref. 46 (which is represented in Ref. 6), and a similar proof specific to the interconnection of passive systems is shown in Ref. 5.
3.7.1 The Small Gain Theorem

**Corollary 3.7.1** (Small Gain Theorem\(^5,6,16,26\)). Consider the negative feedback interconnection of \(G_1 : L_{2e} \to L_{2e}\) and \(G_2 : L_{2e} \to L_{2e}\) presented in Fig. 3.1 on page 18 where \(G_1\) is \((-\gamma_1^{-1}, 0, \gamma_1)\)-dissipative and \(G_2\) is \((-\gamma_2^{-1}, 0, \gamma_2)\)-dissipative (where \(0 < \gamma_1 < \infty\) and \(0 < \gamma_2 < \infty\)). If \(\gamma_1 \gamma_2 < 1\) the negative feedback interconnection is \(L_2\) stable.

*Proof.* By Theorem 3.7.1, provided

\[
\begin{bmatrix}
-(Q_1 + R_2) & S_1 - S_2^T \\
S_1^T - S_2 & -(Q_2 + R_1)
\end{bmatrix} = \begin{bmatrix}
-(-\gamma_1^{-1} + \gamma_2)I & 0 \\
0 & -(-\gamma_2^{-1} + \gamma_1)I
\end{bmatrix}
\]

is positive definite, the closed-loop system is \(L_2\) stable. Clearly the above is positive definite provided \(-1 + \gamma_1 \gamma_2 < 0\), that is \(\gamma_1 \gamma_2 < 1\). This result is known as the Small Gain Theorem. \(\square\)

3.7.2 The Passivity Theorem

**Corollary 3.7.2** (The Passivity Theorem\(^5,6,16,26\)). Consider the negative feedback interconnection of \(G_1 : L_{2e} \to L_{2e}\) and \(G_2 : L_{2e} \to L_{2e}\) presented in Fig. 3.1 on page 18 where \(G_1\) is \((-\epsilon_1 I, \frac{1}{2} I, -\delta_1)\)-dissipative and \(G_2\) is \((-\epsilon_2 I, \frac{1}{2} I, -\delta_2)\)-dissipative where \(\delta_i, \epsilon_i \in \mathbb{R}^+, i = 1, 2\). If \(\delta_1 + \epsilon_2 > 0\) and \(\delta_2 + \epsilon_1 > 0\), the negative feedback interconnection is \(L_2\) stable.

*Proof.* By Theorem 3.7.1, provided

\[
\begin{bmatrix}
-(Q_1 + R_2) & S_1 - S_2^T \\
S_1^T - S_2 & -(Q_2 + R_1)
\end{bmatrix} = \begin{bmatrix}
(\epsilon_1 + \delta_2)I & 0 \\
0 & (\epsilon_2 + \delta_1)I
\end{bmatrix}
\]

is positive definite, the closed-loop system is \(L_2\) stable. The above is positive definite provided \(\delta_1 + \epsilon_2 > 0\) and \(\delta_2 + \epsilon_1 > 0\). This result is known as the Passivity Theorem. \(\square\)

Notice that \(\delta_1 + \epsilon_2 > 0\) and \(\delta_2 + \epsilon_1 > 0\) may hold for a variety of interconnections; for example, when \(G_1\) and \(G_2\) are

- both VSP (i.e., \(\delta_1 > 0, \epsilon_1 > 0, \delta_2 > 0, \epsilon_2 > 0\)),
- both ISP (i.e., \(\delta_1 > 0, \epsilon_1 = 0, \delta_2 > 0, \epsilon_2 = 0\)),
- both OSP (i.e., \(\delta_1 = 0, \epsilon_1 > 0, \delta_2 = 0, \epsilon_2 > 0\)), and
- passive and VSP (i.e., \(\delta_1 = 0, \epsilon_1 = 0, \delta_2 > 0, \epsilon_2 > 0\)).

In fact, \(\delta_1 + \epsilon_2 > 0\) and \(\delta_2 + \epsilon_1 > 0\) may hold for some particular \(\delta_i\)’s and \(\epsilon_i\)’s that are not even positive. Such a situation is actually a special case of the next corollary (Corollary 3.7.3).

There is another theorem, called the “Weak Passivity Theorem”, which, as hinted by its name, is weaker than the general Passivity Theorem presented in Corollary 3.7.2. The Weak Passivity Theorem is useful in its own right, and we will make use of it in Chapter 7. As such, we state it next.
Theorem 3.7.2 (The Weak Passivity Theorem). Consider the negative feedback interconnection of \( G_1: L_{2e} \to L_{2e} \) and \( G_2 : L_{2e} \to L_{2e} \) presented in Fig. 3.1 on page 18. Specifically, let \( \mathbf{u}_2(t) = 0 \) so that

\[
\mathbf{e}_1(t) = \mathbf{u}_1(t) - (G_2 \mathbf{e}_2)(t), \quad \mathbf{e}_2(t) = (G_1 \mathbf{e}_1)(t).
\]

It is assumed that \( \mathbf{u}_1 \in L_{2e} \). If \( G_1 \) is passive and \( G_2 \) is ISP with ISP parameter \( \delta_2 > 0 \), the input-output map from \( \mathbf{u}_1 \) to \( \mathbf{y}_1 \) is \( L_2 \) stable, that is \( \mathbf{u}_1 \in L_2 \Rightarrow \mathbf{y}_1 \in L_2 \).

The proof is simple and short, so we will present it.

**Proof.** Consider the following inner product involving \( \mathbf{y}_1 \) and \( \mathbf{u}_1 \) and subsequent expansion using \( \mathbf{u}_1(t) = \mathbf{e}_1(t) + (G_2 \mathbf{e}_2)(t) \):

\[
\langle \mathbf{y}_1, \mathbf{u}_1 \rangle_T = \langle \mathbf{y}_1, \mathbf{e}_1 \rangle_T + \langle \mathbf{y}_1, G_2 \mathbf{e}_2 \rangle_T \geq \beta_1 + \langle \mathbf{e}_2, G_2 \mathbf{e}_2 \rangle_T \geq \delta_2 \| \mathbf{e}_2 \|_{2T}^2 + \beta
\]

where \( \beta = \beta_1 + \beta_2 \). Using the Cauchy-Schwarz inequality (see Eq. (2.1) on page 12) we then have

\[
\| \mathbf{y}_1 \|_{2T} \| \mathbf{u}_1 \|_{2T} \geq \langle \mathbf{y}_1, \mathbf{u}_1 \rangle_T \geq \delta_2 \| \mathbf{e}_2 \|_{2T}^2 + \beta = \delta_2 \| \mathbf{y}_1 \|_{2T}^2 + \beta.
\]

We now use the norm identity shown in Eq. (2.2) on page 12; by adding \( \frac{1}{2} \left( \frac{1}{\sqrt{\alpha}} \| \mathbf{u}_1 \|_{2T} - \sqrt{\alpha} \| \mathbf{y}_1 \|_{2T} \right)^2 \) to the right hand side of the above expression we have

\[
\frac{1}{2\alpha} \| \mathbf{u}_1 \|_{2T}^2 + \frac{\alpha}{2} \| \mathbf{y}_1 \|_{2T}^2 \geq \delta_2 \| \mathbf{y}_1 \|_{2T}^2 + \beta,
\]

\[
\frac{1}{2\alpha} \| \mathbf{u}_1 \|_{2T}^2 \geq \left( \delta_2 - \frac{\alpha}{2} \right) \| \mathbf{y}_1 \|_{2T}^2 + \beta,
\]

\[
\| \mathbf{u}_1 \|_{2T}^2 \geq 2\alpha \left( \delta_2 - \frac{\alpha}{2} \right) \| \mathbf{y}_1 \|_{2T}^2 + 2\alpha\beta.
\]

Letting \( \alpha = \delta_2 \), defining \( -\beta' := \beta \) (recall, \( \beta \leq 0 \)), then taking the square root of both sides, after simplification we are left with

\[
\| \mathbf{y}_1 \|_{2T} \leq \gamma \| \mathbf{u}_1 \|_{2T} + \beta'
\]

where \( \gamma = 1/\delta_2 \) and \( \beta' = \sqrt{2\beta\delta_2^{-1}} \). Letting \( T \to \infty \) yields \( L_2 \) stability between \( \mathbf{u}_1 \in L_2 \) and \( \mathbf{y}_1 \in L_2 \).

The Weak Passivity Theorem is “weaker” than the general passivity theorem because nothing is said about \( \mathbf{y}_2 \in L_2 \). In particular, there is no guarantee that \( \mathbf{y}_2 \in L_2 \). However, the general Passivity Theorem does guaranteed \( L_2 \) stability from \( \mathbf{u}_1, \mathbf{u}_2 \in L_2 \) to \( \mathbf{y}_1, \mathbf{y}_2 \in L_2 \) (and hence \( \mathbf{e}_1, \mathbf{e}_2 \in L_2 \) as well), which is why it is sometimes called the “Strong” Passivity Theorem.

3.7.3 The Conic Sector Theorem

**Corollary 3.7.3 (The Conic Sector Theorem).** Consider the negative feedback interconnection of \( G_1 : L_{2e} \to L_{2e} \) and \( G_2 : L_{2e} \to L_{2e} \) presented in Fig. 3.1 on page 18 where \( G_1 \) is \((-\frac{1}{b_1}1, \frac{1}{2}(1 + \frac{a_1}{b_1})1, -a_11\)-dissipative and \( G_2 \) is \((-\frac{1}{b_2}1, \frac{1}{2}(1 + \frac{a_2}{b_2})1, -a_21\)-dissipative where \( a_i \in \mathbb{R}, b_i \in \mathbb{R}^+ \cup \mathbb{R}^- \).
\{\infty\} \setminus \{0\} \text{ and } b_i > a_i, \ i = 1, 2. \text{ If the matrix inequality of Eq. (3.16) holds for some } Q_i, S_i, \text{ and } R_i, \text{ the negative feedback interconnection is } L_2 \text{ stable.}

**Proof.** By Theorem 3.7.1, provided

\[
\begin{bmatrix}
-(Q_1 + R_2) & S_1 - S_2^T \\
S_1^T - S_2 & -(Q_2 + R_1)
\end{bmatrix} = \begin{bmatrix}
\left(\frac{1}{b_1} + a_2\right)I & \frac{1}{2} \left(\frac{a_1}{b_1} - \frac{a_2}{b_2}\right)I \\
\frac{1}{2} \left(\frac{a_1}{b_1} - \frac{a_2}{b_2}\right)I & \left(\frac{1}{b_2} + a_1\right)I
\end{bmatrix}
\]

is positive definite, the closed-loop system is \( L_2 \) stable. The above matrix will be positive definite if

\[
\frac{1}{b_1} + a_2 > 0, \quad (3.17a)
\]

\[
\left(\frac{1}{b_1} + a_2\right)\left(\frac{1}{b_2} + a_1\right) - \frac{1}{4} \left(\frac{a_1}{b_1} - \frac{a_2}{b_2}\right)^2 > 0. \quad (3.17b)
\]

This result is known as the Conic Sector Stability Theorem, or to be concise, simply the Conic Sector Theorem.

**Special Cases**

We will consider some special cases next. Consider the case where \( a_1 = -\gamma_1 = -b_1 \) and \( a_2 = -\gamma_2 = -b_2 \) where \( 0 < \gamma_i < \infty, \ i = 1, 2. \) From Corollary 3.7.3, the closed-loop will be stable provided

\[
\frac{1}{\gamma_1} - \gamma_2 > 0, \quad \left(\frac{1}{\gamma_1} - \gamma_2\right)\left(\frac{1}{\gamma_2} - \gamma_1\right) > 0.
\]

These requirements are equivalent to \( \gamma_1 \gamma_2 < 1, \) the same condition stipulated by the Small Gain Theorem.

Next, consider when \( a_1 = 0 \) and \( b_1 = \infty; \) what values must \( a_2 \) and \( b_2 \) take for the closed-loop to be stable? From Eq. (3.17a) \( a_2 > 0 \) if \( b_1 = \infty. \) Given that \( a_1 = 0, b_1 = \infty, \) and \( a_2 > 0, \) for \( L_2 \) stability

\[
\frac{a_2}{b_2} > \frac{1}{4} \left(\frac{a_2}{b_2}\right)^2
\]

must hold in order for Eq. (3.17b) to hold. Clearly then \( 0 < b_2 < \infty. \) Recall that a passive system is a conic system with \( a_1 = 0 \) and \( b_1 = \infty, \) while a VSP system is a conic system with \( a_2 > 0 \) and \( 0 < b_2 < \infty; \) therefore, the conditions for stability of this special case are the same conditions stipulated by the Passivity Theorem; the negative feedback interconnection of a passive system (i.e., \( a_1 = 0 \) and \( b_1 = \infty \)) and a VSP system (i.e., \( a_2 > 0 \) and \( 0 < b_2 < \infty \)) is \( L_2 \) stable.

Readers should not be misled into thinking stability can only be guaranteed via the Small Gain Theorem or the Passivity Theorem. There are some other unique interconnections that can be shown to be stable via the Conic Sector Theorem, but not the Small Gain nor Passivity Theorems. For example, consider the case where \( G_1 \) is interior conic with \( a_1 < 0 \) and \( 0 < b_1 < \infty \)
and \( \mathcal{G}_2 \) is interior conic with
\[
a_2 = \frac{\Delta - 1}{b_1}, \quad b_2 = \frac{\Delta - 1}{a_1}, \quad \forall \Delta \in (0, 1)
\]
If Eqs. (3.17a) and (3.17b) hold for these \( a_i \) and \( b_i \) values (where \( i = 1, 2 \)) the closed-loop is \( L_2 \) stable. To show this, consider the following expression related to Eq. (3.17a):
\[
\frac{1}{b_1} + a_2 = \frac{1}{b_1} + \frac{\Delta - 1}{b_1} = \frac{\Delta}{b_1}.
\]
Therefore, Eq. (3.17a) will be satisfied \( \forall \Delta \in (0, 1), \forall b_1 \in (0, \infty) \). Next, consider the following expression related to Eq. (3.17b):
\[
\begin{align*}
\left( \frac{1}{b_1} + a_2 \right) \left( \frac{1}{b_2} + a_1 \right) - \frac{1}{4} \left( \frac{a_1 b_2 - a_2 b_1}{b_1 b_2} \right)^2 &= \left( \frac{\Delta}{b_1} \right) \left( \frac{a_1}{\Delta - 1} + a_1 \right) - \frac{1}{4} \left[ \frac{(\Delta - 1) - (\Delta - 1)}{b_1 b_2} \right]^2 \\
&= \left( \frac{\Delta^2}{\Delta - 1} \right) \left( \frac{a_1}{b_1} \right).
\end{align*}
\]
Because \( \frac{\Delta^2}{\Delta - 1} < 0, \forall \Delta \in (0, 1) \) and \( \frac{a_1}{b_1} < 0 \) (because \( a_1 < 0 \) and \( 0 < b_1 < \infty \)), it follows that Eq. (3.17b) holds when \( a_2 = \frac{\Delta - 1}{b_1} \) and \( b_2 = \frac{\Delta - 1}{a_1} \) when \( \forall \Delta \in (0, 1) \). Thus the negative feedback interconnection of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) where \( a_1 < 0, 0 < b_1 < \infty, a_2 = \frac{\Delta - 1}{b_1}, \) and \( b_2 = \frac{\Delta - 1}{a_1} \) is \( L_2 \) stable.
Specifically, note that as \( \Delta \to 0, \mathcal{G}_2 \in \text{cone}(-\frac{1}{b_1}, -\frac{1}{a_1}), \) that is to say \( \mathcal{G}_2 \) is strictly inside the cone described by \( a_2 = -\frac{1}{b_1}, b_2 = -\frac{1}{a_1} \). This special case is also discussed in Refs. 43 and 44. A controller synthesis method based on these requirements is discussed in Refs. 48 and 49.

### 3.7.4 A Geometric Interpretation of the Conic Sector Theorem

Although the Conic Sector Theorem is somewhat abstract, it has a very simple geometric interpretation in terms of a “topological separation” of subspaces (such as the space \( L_{2e} \times L_{2e} \)). We will illustrate this interpretation next.

Consider the negative feedback interconnection of two conic systems \( \mathcal{G}_1 : L_{2e}(E_1) \to L_{2e}(Y_1) \) and \( \mathcal{G}_2 : L_{2e}(E_2) \to L_{2e}(Y_2) \) as shown in Fig. 3.1 on page 18. In particular, \( \mathcal{G}_1 \in \text{cone}[a_1, b_1] \) and \( \mathcal{G}_2 \in \text{cone}[a_2, b_2] \) as depicted in Figs. 3.5(a) and 3.5(b). To simplify our discussion we will set \( u_1 \) and \( u_2 \) to zero and consider the positive feedback interconnection of \( \mathcal{G}_1 : L_{2e}(Y_2) \to L_{2e}(Y_1) \) and \( -\mathcal{G}_2 : L_{2e}(Y_1) \to L_{2e}(Y_2) \). (The geometric interpretation we are about to present can be argued with nonzero \( u_1 \) and \( u_2 \), but doing so would require a lengthy discussion, and is beyond the scope of this simple overview. Interested readers may see Refs. 13 or 18.) From Ref. 13, if \( \mathcal{G}_2 \in \text{cone}[a_2, b_2] \) then \( -\mathcal{G}_2 \in \text{cone}[-b_2, -a_2] \). We are abusing notation slightly by letting the upper part of the cone describing \( -\mathcal{G}_2 \) take on positive or negative values (depending on \( a_2 \)). Our modified feedback interconnection is shown in Fig. 3.6, while the cones describing
\( e_1 \parallel e_2 \) is a positive cone,

\( T \in L^2(e)(E_1) \) and

\( T \in L^2(e)(Y_1) \) are positive cones,

\( a_1, b_1 < 0, 0 < b_1 < \infty \).

\( a_2, b_2 < 0, 0 < b_2 < \infty \).

Figure 3.5: Two conic systems, \( G_1 \) and \( G_2 \), to be interconnected.

\( G_1 : L^{2e}(Y_2) \to L^{2e}(Y_1) \) and \( -G_2 : L^{2e}(Y_1) \to L^{2e}(Y_2) \) are shown in Figs. 3.7(a) and 3.7(b). Notice that cone\([a_1, b_1]\) \( \subset L^{2e}(Y_2) \times L^{2e}(Y_1) \), while cone\([-b_2, -a_2]\) \( \subset L^{2e}(Y_1) \times L^{2e}(Y_2) \). By reconfiguring the axis of cone\([-b_2, -a_2]\) to match the axis of cone\([a_1, b_1]\) (i.e., letting the horizontal axis be \( L^{2e}(Y_2) \) and the vertical axis be \( L^{2e}(Y_1) \)) we create what is called the complement of the cone representing \( G_2 \), written \( \bar{G}_2 \), as shown in Fig. 3.7(c). We can also write \( \bar{G}_2 \notin \text{cone}\left[-\frac{1}{b_2}, -\frac{1}{a_2}\right] \), which is to say the cone representing \( \bar{G}_2 \) is outside cone\([-\frac{1}{b_2}, -\frac{1}{a_2}] \). The cone representing \( G_1 \) and the complementary cone representing \( \bar{G}_2 \) (i.e., the cone \( \bar{G}_2 \)) can now be drawn together, as shown in Fig. 3.7(d). The fact that \( G_1 \) and \( \bar{G}_2 \) have conic representations that do not intersect is not a coincidence; the cones form a topological separation within the space \( L^{2e}(Y_2) \times L^{2e}(Y_1) \).

Figure 3.6: Modified feedback interconnection.

Armed with this new insight, let us return to the Small Gain Theorem and the Passivity Theorem and attempt to interpret each theorem in terms of the topological separation of cones. We will start with the Small Gain Theorem. Consider two systems, \( G_1 \in \text{cone}\left[-\gamma_1, \gamma_1\right] \) and
$y_1 \in L^2(Y_1)$

$y_2 \in L^2(Y_2)$

(a) $\mathcal{G}_1 \in \text{cone}[a_1, b_1]$.

(b) $-\mathcal{G}_2 \in \text{cone}[-b_2, -a_2]$.

$y_1 \in L^2(Y_1)$

$y_2 \in L^2(Y_2)$

(c) The complement of $\mathcal{G}_2 \in \text{cone}[a_2, b_2]$, $\bar{\mathcal{G}}_2 \notin \text{cone}[-\frac{1}{b_2}, -\frac{1}{a_2}]$.

(d) Topological separation of each cone.

Figure 3.7: Illustration of the Conic Sector Theorem.

$\mathcal{G}_2 \in \text{cone}[-\gamma_2, \gamma_2]$, as shown in Figs. 3.8(a) and 3.8(b). Forming the complement of the cone representing $\mathcal{G}_2$, then drawing both $\mathcal{G}_1 \in \text{cone}[-\gamma_1, \gamma_1]$ and $\bar{\mathcal{G}}_2 \notin \text{cone}[-\frac{1}{\gamma_2}, \frac{1}{\gamma_2}]$ together we see that the cones do not intersect provided $\gamma_2 < \frac{1}{\gamma_1}$. Therefore, in terms of cones, the Small Gain Theorem enforces a topological separation of the cone representing $\mathcal{G}_1$ and complement of the cone representing $\mathcal{G}_2$.\(^{18}\)

Next, consider the Passivity Theorem and the interconnection of a passive system $\mathcal{G}_1 \in \text{cone}[0, \infty]$ and a VSP system $\mathcal{G}_2 \in \text{cone}[a_2, b_2]$ (where $0 < a_2 < b_2 < \infty$). The conic representation of each system are shown in Figs. 3.9(a) and 3.9(b). The cone representing $\mathcal{G}_1$ and the complement of the cone representing $\mathcal{G}_2$ are drawn together in Fig. 3.9(c). Clearly there is a topological separation of the cone representing $\mathcal{G}_1$ and complement of the cone representing $\mathcal{G}_2$ provided $0 < a_2 < b_2 < \infty$. 

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$\|y_1\|_{2T} \in L_{2\nu}(Y_1)$

(a) $G_1 \in \text{cone}[\gamma_1, -\gamma_1]$ where $0 < \gamma_1 < \infty$.

$\|e_1\|_{2T} \in L_{2\nu}(E_1)$

$\gamma_1$

$-\gamma_1$

$\|y_2\|_{2T} \in L_{2\nu}(Y_2)$

(b) $G_2 \in \text{cone}[\gamma_2, -\gamma_2]$ where $0 < \gamma_2 < \infty$.

$\|e_2\|_{2T} \in L_{2\nu}(E_2)$

$\gamma_2$

$-\gamma_2$

$\|y_3\|_{2T} \in L_{2\nu}(Y_3)$

(c) Cones do not intersect provided $\gamma_1 \gamma_2 < 1$.

Figure 3.8: Illustration of the Small Gain Theorem using cones.
(a) A passive system, $\mathcal{G}_1 \in \text{cone}[0, \infty]$.  

(b) A VSP system, $\mathcal{G}_1 \in \text{cone}[a_2, b_2]$, $0 < a_2 < b_2 < \infty$.  

(c) Topological separation.

Figure 3.9: Illustration of the Passivity Theorem using cones.

### 3.8 Summary

This chapter presented an overview of classic input-output stability theory. In particular, the notion of $L_2$ stability, and systems that possess finite gain, passive, and conic input-output maps are defined and discussed. We review the dissipative systems framework, and stability of two dissipative systems interconnected within a negative feedback interconnection. This thesis builds upon the ideas and results presented in this chapter, which is why we have discussed each of these topics in detail.
Part I

Extensions of Input-Output Stability Theory
Chapter 4

State-Space Realizations of Linear Time-Varying Systems

This chapter focuses on LTV systems described by Eqs. (3.2) and (3.3). We first investigate sufficient conditions based on the state-space realization of a system that (when met) ensure the system has finite gain, is passive, or has conic input-output properties. Doing so is motivated by the PR Lemma and SPR Lemma; in essence, we seek to state equivalent theorems for LTV systems.

Additional attention will be given to passive LTV systems, and the stability of various feedback interconnections involving passive systems and sector bounded nonlinearities. There are various results related to passive LTV systems in the literature\textsuperscript{50–54} but most are focused on the synthesis of passive circuit networks, rather than the stability of feedback interconnections.

Although the stability of mechanical systems that have time-varying mass, damping, and stiffness has previously been investigated, the results rely on Lyapunov methods;\textsuperscript{55–58} a passivity-based approach to stability of LTV mechanical systems has not been fully explored. Additionally, the synthesis of time-varying controllers for time-varying systems (in particular, mechanical systems) has yet to be fully investigated in a passivity-based context. As such, we will investigate the control of a simple mass-spring-damper system and a LTV controller synthesis method.

4.1 Linear Time-Varying Systems

As discussed in Sec. 3.2.2 on page 19, the LTV systems we are concerned with have the following finite-dimensional state-space realization:

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,
\]

\[
y(t) = C(t)x(t) + D(t)u(t), \quad y \in \mathbb{R}^m,
\]

where \(A : \mathbb{R}^+ \to \mathbb{R}^{n \times n}, B : \mathbb{R}^+ \to \mathbb{R}^{n \times m}, C : \mathbb{R}^+ \to \mathbb{R}^{m \times n},\) and \(D : \mathbb{R}^+ \to \mathbb{R}^{m \times m}\) are appropriately dimensioned real matrices that are continuous and bounded over the time interval of interest. An alternate output is \(z(t) = L(t)x(t) + W(t)u(t)\) where \(z : \mathbb{R}^+ \to \mathbb{R}^m, L : \mathbb{R}^+ \to \mathbb{R}^{m \times n},\)
and \( \mathbf{W} : \mathbb{R}^+ \to \mathbb{R}^{m \times m} \). It is assumed that \((\mathbf{A}(\cdot), \mathbf{B}(\cdot))\) is controllable, and both \((\mathbf{C}(\cdot), \mathbf{A}(\cdot))\) and \((\mathbf{L}(\cdot), \mathbf{A}(\cdot))\) are observable.\textsuperscript{33,34} The theorems and lemmas presented in this chapter do not require the system to be controllable and observable, but we impose this additional requirement so that the state-space realization is minimal. A minimal state-space realization is desired because, eventually, synthesis of controllers will be done, and we desire controllers that have a minimal state-space realization so that they are efficient from a computational point of view.

In a mechanical context, Eq. (3.2) may represent a system with time-varying mass, damping, or stiffness, where the inputs are forces and the outputs are velocities. In an electrical circuit context, Eq. (3.2) may represent the interconnection of time-varying passive circuit components, such as resistors, capacitors, inductors, gyrators, and transformers, where the inputs are port currents and the outputs are port voltages.

### 4.2 State-Space Realization of Linear Time-Varying Finite Gain Systems

Recall from Sec. 3.3 on page 20 (Definition 3.3.1) that a square system \( \mathbf{y}(t) = (\mathcal{G}\mathbf{u})(t), \mathcal{G} : L_{2e}(U) \to L_{2e}(Y), \mathbf{u} \in L_{2e}(U), \mathbf{y} \in L_{2e}(Y) \) possesses finite gain if there exist real constants \( 0 < \gamma < \infty \) and \( \beta' \) such that\textsuperscript{5,26}

\[
\|\mathbf{y}\|_{2T} \leq \gamma \|\mathbf{u}\|_{2T} + \beta', \quad \forall \mathbf{u} \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+.
\]

Alternatively, if for real constants \( 0 < \gamma < \infty \) and \( \beta \) the inequality

\[
\gamma \|\mathbf{u}\|_{2T}^2 \geq \frac{1}{\gamma} \|\mathbf{y}\|_{2T}^2 + \beta, \quad \forall \mathbf{u} \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+
\]  

(4.2)

holds, then Eq. (3.6) holds as well (see Sec. 3.6.1, page 32).

The following theorem provides a state-space characterization of the finite gain nature of a LTV system.

**Theorem 4.2.1.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable possesses finite gain if there exist continuous, bounded \( \mathbf{P}(t) = \mathbf{P}^T(t) > 0, \mathbf{L}(\cdot), \mathbf{W}(\cdot) \), and \( 0 < \gamma < \infty \) such that

\[
\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{P}(t) = -\frac{1}{\gamma} \mathbf{C}^T(t)\mathbf{C}(t) - \mathbf{L}^T(t)\mathbf{L}(t),
\]

(4.3a)

\[
\mathbf{P}(t)\mathbf{B}(t) = -\frac{1}{\gamma} \mathbf{C}^T(t)\mathbf{D}(t) - \mathbf{L}^T(t)\mathbf{W}(t),
\]

(4.3b)

\[
\frac{1}{\gamma} \mathbf{D}^T(t)\mathbf{D}(t) = \mathbf{1} - \mathbf{W}^T(t)\mathbf{W}(t).
\]

(4.3c)

**Proof.** To be concise we will neglect writing the temporal argument of the input and output signals, and time-varying matrices. Consider the following Lyapunov-like function and its temporal
derivative:

\[ V = \frac{1}{2} x^T P x, \]
\[ \dot{V} = \frac{1}{2} x^T P \dot{x} + \frac{1}{2} x^T \dot{P} x + \frac{1}{2} \dot{x}^T \dot{P} x = \frac{1}{2} x^T \left( \dot{P} + PA + A^T P \right) x + x^T P Bu. \]

Substituting Eqs. (4.3a) and (4.3b) yields

\[ \dot{V} = -\frac{1}{2 \gamma} x^T C^T C x - \frac{1}{2} x^T C^T D u - \frac{1}{2} \left( x^T L^T L x + x^T + 2 x^T L^T W u \right). \]

Noting that \( y^T y = x^T C^T C x + 2 x^T C^T D u + u^T D^T D u \) we can now write

\[ \dot{V} = -\frac{1}{2 \gamma} y^T y + \frac{1}{2 \gamma} u^T D^T D u - \frac{1}{2} \left( x^T L^T L x + x^T + 2 x^T L^T W u \right). \]

Using Eq. (4.3c) then gives

\[ \dot{V} = -\frac{1}{2 \gamma} y^T y + \frac{1}{2 \gamma} u^T u - \frac{1}{2} z^T z. \]

By integrating \( \dot{V} \) between 0 and \( T \) and manipulating slightly we have

\[
\int_0^T \dot{V} dt = V(T) - V(0) \geq -V(0), \\
\frac{\gamma}{2} \| u \|_{2T}^2 - \frac{1}{2 \gamma} \| y \|_{2T}^2 - \frac{1}{2} \| z \|_{2T}^2 \geq -V(0), \\
\gamma \| u \|_{2T}^2 \geq \gamma^{-1} \| y \|_{2T}^2 + \| z \|_{2T}^2 + \frac{\beta}{2V(0)} \geq \gamma^{-1} \| y \|_{2T}^2 + \beta,
\]

which is equivalent to Eq. (4.2).

Notice that if the matrices \( A(\cdot), B(\cdot), C(\cdot), \) and \( D(\cdot) \) are time invariant and \( \dot{P}(t) = 0, \) Lemma 3.3.1 on page 21 is recovered from Theorem 4.2.1. What Theorem 4.2.1 tells us is that if Eqs. 4.3 are satisfied then the system has gain less than \( \gamma. \) Unfortunately, much like Theorem 3.3.1, Theorem 4.2.1 does not tell us what the largest value of \( \gamma \) is (i.e., the supremum). Finding the largest gain would have to be done by, for example, a bisection search.

### 4.2.1 Boundary Condition and Solution of \( P(\cdot) \)

Eq. (4.3a) is a first order, ordinary matrix differential equation in \( P(\cdot) \). To be solved, a boundary condition must be known. To determine the boundary condition, consider the following; starting with Eq. (4.3a) we will pre- and post-multiply by \( \Phi^T(t,0) \) and \( \Phi(t,0): \)

\[
\Phi^T(t,0) \dot{P}(t) \Phi(t,0) + \Phi^T(t,0) P(t) A(t) \Phi(t,0) + \Phi^T(t,0) A^T(t) P(t) \Phi(t,0) = -\Phi^T(t,0) Q(t) \Phi(t,0)
\]
where \( Q(t) = \gamma^{-1}C^T(t)C(t) + L^T(t)L(t) \geq 0 \). Recalling that \( \Phi(t,0) = A(t)\Phi(t,0) \) (see Sec. 3.2.2 starting on page 19) we have

\[
\Phi^T(t,0)\Phi(t,0) + \Phi^T(t,0)P(t)\Phi(t,0) + \Phi^T(t,0)P(t)\Phi(t,0) = -\Phi^T(t,0)Q(t)\Phi(t,0)
\]

which is just

\[
\frac{d}{dt} \left[ \Phi^T(t,0)P(t)\Phi(t,0) \right] = -\Phi^T(t,0)Q(t)\Phi(t,0).
\]

Integrating both sides between \( t \) and \( T \) gives

\[
\int_t^T \frac{d}{d\tau} \left[ \Phi^T(\tau,0)P(\tau)\Phi(\tau,0) \right] d\tau = -\int_t^T \Phi^T(\tau,0)Q(\tau)\Phi(\tau,0) d\tau,
\]

\[
\Phi^T(T,0)P(T)\Phi(T,0) - \Phi^T(t,0)P(t)\Phi(t,0) = -\int_t^T \Phi^T(\tau,0)Q(\tau)\Phi(\tau,0) d\tau.
\]

Pre- and post-multiplying both sides by \( \Phi^{-T}(t,0) \) and \( \Phi^{-1}(t,0) \) and rearranging yields

\[
P(t) = \Phi^{-T}(t,0)\Phi^T(T,0)P(T)\Phi(T,0)\Phi^{-1}(t,0) + \Phi^{-T}(t,0) \int_t^T \Phi^T(\tau,0)Q(\tau)\Phi(\tau,0) d\tau \Phi^{-1}(t,0)
\]

\[
= \Phi^T(T,t)P(T)\Phi(T,t) + \Phi^T(0,t) \int_t^T \Phi^T(\tau,0)Q(\tau)\Phi(\tau,0) d\tau \Phi(0,t).
\]

Each of the terms in the matrix equation above are symmetric. The first term on the right hand side of Eq. (4.4) is positive definite provided \( P(T) \) is positive definite, and the second term is positive semidefinite. For \( P(\cdot) \) to be positive definite \( \forall t \in [0,T] \), \( P(T) \) must be positive definite. To find \( P(\cdot) \) over \( t = 0 \) to \( t = T \), Eq. (4.3a) would be integrated backwards in time (numerically) given \( P(T) = P^T(T) > 0 \).

### 4.3 State-Space Realizations of Passive Linear Time-Varying Systems

Recall from Sec. 3.4 starting on page 22 the definition of a VSP system (Definition 3.4.1): a square system \( y(t) = (G(u))(t), G : L_{2e}(U) \rightarrow L_{2e}(Y), u \in L_{2e}(U), y \in L_{2e}(Y) \) is VSP if there exist real constants \( \delta > 0, \epsilon > 0 \) and \( \beta \) such that

\[
\int_0^T y^T(t)u(t)dt \geq \delta \|u\|_{2T}^2 + \epsilon \|y\|_{2T}^2 + \beta, \quad \forall u \in L_{2e}(U), \forall T \in \mathbb{R}^+.
\]

When \( \delta = \epsilon = 0 \) the system is passive; when \( \delta > 0 \) and \( \epsilon = 0 \) the system is ISP; and when \( \delta = 0 \) and \( \epsilon > 0 \) the system is OSP. Similarly, a square system \( y(t) = (G(u))(t), G : L_{2e}(U) \rightarrow L_{2e}(Y), u \in L_{2e}(U), y \in L_{2e}(Y) \) is SSP if there exist \( \psi(\cdot) > 0 \) and \( \beta \) such that

\[
\int_0^T y^T(t)u(t)dt \geq \int_0^T \psi(x(t))dt + \beta, \quad \forall u \in L_{2e}, \forall T \in \mathbb{R}^+.
\]
What follows are sufficient conditions that ensure a LTV system with minimal state-space realization given by Eqs. (3.2) and (3.3) is either VSP, ISP, OSP, SSP, or just passive. The results relevant to LTV systems that are VSP, ISP, OSP, or SSP are new; conditions equivalent to ours for passive LTV systems are given in Refs. 52–54.

4.3.1 Systems With a Feedthrough Matrix

We will first consider the case where \( D(t) \neq 0 \). We will begin with VSP systems.

**Theorem 4.3.1.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable with \( D(t) > 0 \) is VSP if there exist continuous, bounded \( P(t) = P^T(t) > 0, L(\cdot), W(\cdot), \) and \( \delta > 0, \epsilon > 0 \) such that

\[
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -2\epsilon C^T(t)C(t) - L^T(t)L(t),
\]

\[
P(t)B(t) = C^T(t) - 2\epsilon C^T(t)D(t) - L^T(t)W(t),
\]

\[
\frac{1}{2} \left[ D(t) + D^T(t) \right] = \frac{1}{2} W^T(t)W(t) + \epsilon D^T(t)D(t) + \delta I.
\]

**Proof.** As before, we will neglect writing the temporal argument signals and time-varying matrices. Consider the following Lyapunov-like function and its temporal derivative:

\[
V = \frac{1}{2} x^T P x,
\]

\[
\dot{V} = \frac{1}{2} x^T P \dot{x} + \frac{1}{2} \dot{x}^T P x + \frac{1}{2} x^T P \dot{x}
\]

\[
= \frac{1}{2} x^T (\dot{P} + PA + A^T P) x + x^T P Bu.
\]

Using Eqs. (4.5a) and (4.5b), \( \dot{V} \) becomes

\[
\dot{V} = -\epsilon x^T C^T C x - \frac{1}{2} x^T L^T L x + x^T C^T u - 2\epsilon x^T C^T D u - x^T L^T W u.
\]

The feedthrough matrix \( D \) is square, and can be broken up into symmetric and skew-symmetric parts: \( D = \frac{1}{2} (D + D^T) + \frac{1}{2} (D - D^T) \) where \( u^T (D - D^T) u = 0 \). Given that \( y = Cx + Du \) and \( y^T u = x^T C^T u + \frac{1}{2} u^T (D^T + D) u \) we can write \( \dot{V} \) as

\[
\dot{V} = y^T u - \epsilon x^T C^T C x - 2\epsilon x^T C^T D u - \frac{1}{2} u^T (D + D^T) u - \frac{1}{2} x^T L^T L x - x^T L^T W u.
\]

Using Eq. (4.5c) leads to

\[
\dot{V} = y^T u - \delta u^T u - \epsilon y^T y - \frac{1}{2} z^T z.
\]
Integrating $\dot{V}$ from 0 to $T$ gives

\[
\int_0^T \dot{V} dt = V(T) - V(0) \geq \beta,
\]

\[
\int_0^T \dot{V} dt \geq \delta \|u\|_{2T}^2 + \epsilon \|y\|_{2T}^2 + \frac{1}{2} \|z\|_{2T}^2 + \beta
\]

which completes the proof. \qed

Notice, if we break up $D(\cdot)$ and write it as $D(t) = \bar{D}(t) + \delta I > 0$ where $\bar{D}(t) > 0$ we can equivalently write Eq. (4.5c) as $\frac{1}{2} [\bar{D}(t) + \bar{D}^T(t)] = \frac{1}{2} W^T(t)W(t) + cD^T(t)D(t)$.

Eq. (4.5a) is a first order, ordinary matrix differential equation in $P(\cdot)$. As discussed in Sec. 4.2.1 the terminal condition $P(T) > 0$ must be provided, then Eq. (4.5a) would be integrated backward in time to find the time history of $P(\cdot)$ between $t = 0$ and $t = T$.

Next, we will consider ISP systems.

**Theorem 4.3.2.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable with $D(t) > 0$ is ISP if there exist continuous, bounded $P(t) = P^T(t) > 0$, $L(\cdot)$, $W(\cdot)$, and $\delta > 0$ such that

\[
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -L(t)L(t), \tag{4.6a}
\]

\[
P(t)B(t) = C^T(t) - L^T(t)W(t), \tag{4.6b}
\]

\[
\frac{1}{2} [D(t) + D^T(t)] = \frac{1}{2} W^T(t)W(t) + \delta I. \tag{4.6c}
\]

**Proof.** The proof is essentially equivalent to the proof of Theorem 4.3.1; starting with the Lyapunov-like function $V = \frac{1}{2}x^TPx$, taking its temporal derivative, then substituting Eqs. (4.6a), (4.6b), (4.6c) and appropriately integrating between 0 and $T$ yields the desired result. \qed

Note, Eq. (4.6c) can be written as $\dot{D}(t) + \dot{D}^T(t) = W^T(t)W(t)$. OSP systems are considered next.

**Theorem 4.3.3.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable is OSP if there exist continuous, bounded $P(t) = P^T(t) > 0$, $L(\cdot)$, $W(\cdot)$, and $\epsilon > 0$ such that

\[
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -2\epsilon C^T(t)C(t) - L^T(t)L(t), \tag{4.7a}
\]

\[
P(t)B(t) = C^T(t) - 2\epsilon C^T(t)D(t) - L^T(t)W(t), \tag{4.7b}
\]

\[
\frac{1}{2} [D(t) + D^T(t)] = \frac{1}{2} W^T(t)W(t) + \epsilon D^T(t)D(t). \tag{4.7c}
\]
Proof. The proof follows from Theorem 4.3.1, and is omitted in the interest of space. \(\square\)

An OSP system is one that is passive and possesses finite gain. Determining if a system has finite gain can be assessed via other means; for example, see Theorem 4.2.1 in Sec. 4.2.

We now consider SSP systems.

**Theorem 4.3.4.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable is SSP if there exist continuous, bounded \(P(t) = P^T(t) > 0, L(\cdot), W(\cdot),\) and \(\nu > 0\) such that

\[
\begin{align*}
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) &= -L^T(t)L(t) - 2\nu P(t), \quad (4.8a) \\
P(t)B(t) &= C^T(t) - L^T(t)W(t), \quad (4.8b) \\
D(t) + D^T(t) &= W^T(t)W(t). \quad (4.8c)
\end{align*}
\]

Proof. Starting with \(V = \frac{1}{2}x^TPx\), taking the temporal derivative of \(V\), then substituting Eqs. (4.8a), (4.8b), and (4.8c) gives

\[\dot{V} = y^Tu - \nu x^TPx - \frac{1}{2}x^TL^TLx - x^TL^TWu - \frac{1}{2}u^TW^TWu.\]

Integrating between 0 and \(T\) leads to

\[
\int_0^T y^Tu dt \geq \int_0^T \nu x^TPx dt + \frac{1}{2}\|z\|^2_{2T} + \beta \geq \int_0^T \psi(x) dt + \beta
\]

which completes the proof. \(\square\)

Comparing to Lemma 3.4.2 on page 24, Theorem 4.3.4 is similar to the SPR Lemma (which is traditionally called the KYP Lemma) for LTI systems.\(^4\)\(^27\)

For completeness, passive systems are considered next.

**Theorem 4.3.5.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable is passive if there exist continuous, bounded \(P(t) = P^T(t) > 0, L(\cdot), W(\cdot),\) such that

\[
\begin{align*}
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) &= -L^T(t)L(t), \quad (4.9a) \\
P(t)B(t) &= C^T(t) - L^T(t)W(t), \quad (4.9b) \\
D(t) + D^T(t) &= W^T(t)W(t). \quad (4.9c)
\end{align*}
\]

Proof. The proof follows from Theorem 4.3.1, and is omitted in the interest of space. \(\square\)
Essentially the same result (i.e., Theorem 4.3.5 above) can be found in Refs. 52–54, although the proofs presented in Refs. 52–54 are different than the proof presented here.

Theorems 4.3.1, 4.3.2, 4.3.3, 4.3.4, and 4.3.5 provide a means to assess the passive characteristics of a system. Given a system with \((A(\cdot), B(\cdot), C(\cdot), D(\cdot))\) matrices, if the system satisfies Theorem 4.3.1 then the system is VSP, if the system satisfies Theorem 4.3.2 then it is ISP, and so on. Not only is assessing the passive nature of a system important (e.g., given a LTV plant to be controlled, determining if the plant is passive may be done via Theorem 4.3.5), but being able to design a LTV system that is VSP, ISP, OSP, or SSP is also relevant, especially in the context of controller design and stability via the Passivity Theorem. In fact, we will use a LTV ISP system as a controller in Chapter 7.

### 4.3.2 Systems Without a Feedthrough Matrix

We will now consider systems with \(D(t) = 0\). Doing so is important because many physical systems (in particular mechanical, aerospace, and electrical systems) do not have feedthrough matrices.

A system that does not have a feedthrough matrix cannot be VSP or ISP; \(D > 0\) for a system to be VSP or ISP. Hence, we will only consider OSP, SSP, and passive systems. We will start with OSP systems.

**Corollary 4.3.1.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable with \(D(t) = 0\) is OSP if there exist continuous, bounded \(P(t) = P^T(t) > 0\), and \(Q(t) = Q^T(t) > 0\) such that

\[
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -Q(t), \quad (4.10a)
\]

\[
P(t)B(t) = C^T(t). \quad (4.10b)
\]

**Proof.** Using \(V = \frac{1}{2}x^TPx\), its temporal derivative, Eqs. (4.10a) and (4.10b), and the relation \(y = Cx\) it can be shown that

\[
\dot{V} = -\frac{1}{2}x^T\left(\dot{P} + PA + A^TP\right)x + x^TPBu = -\frac{1}{2}x^TQx + y^Tu.
\]

By integrating between 0 and \(T\) we have

\[
\int_0^T y^T u dt \geq \int_0^T \frac{1}{2}x^TQx dt + \beta. \quad (4.11)
\]

Under the assumption that \(C\) and \(Q\) are bounded we can write

\[
C^T C \leq \overline{\alpha}I < \infty, \quad 0 < \underline{q}I \leq Q \leq \overline{q}I < \infty.
\]
It follows that

\[ y^T y = x^T C^T C x \leq \alpha x^T x, \quad x^T Q x \geq q x^T x \implies x^T Q x \geq \frac{q}{\epsilon} y^T y. \]

Using the above inequality we arrive at

\[
\int_0^T y^T u dt \geq \frac{q}{2T} \|y\|^2_{2T} + \beta
\]

which completes the proof.

Note, Eq. (4.11) is a SSP inequality where \( \psi(x(t)) = \frac{1}{2} x^T(t) Q(x(t)) x(t). \) Thus, a system with \( D(t) = 0 \) that is SSP is OSP.

Corollary 4.3.1 is similar to the SPR Lemma (a.k.a., the KYP Lemma) for LTI systems that do not have a feedthrough matrix.

Passivity will now be considered.

**Corollary 4.3.2.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable with \( D(t) = 0 \) is passive if there exist continuous, bounded \( P(t) = P^T(t) > 0 \), and \( Q(t) = Q^T(t) \geq 0 \) such that

\[
\begin{align*}
\dot{P}(t) + P(t) A(t) + A^T(t) P(t) &= -Q(t), \quad (4.12a) \\
P(t) B(t) &= C^T(t). \quad (4.12b)
\end{align*}
\]

**Proof.** Following the same steps outlined in Corollary 4.3.1, we arrive at Eq. (4.11) and find

\[
\int_0^T y^T u dt \geq \beta
\]

owing to the fact \( Q \) is positive semidefinite.

\[ \Box \]

### 4.4 Linear Time-Varying Conic Systems

From Sec. 3.5 on page 25 (specifically, recalling Definition 3.5.2 on page 25) the square system \( y(t) = (G u)(t), \ G : L_{2e}(U) \to L_{2e}(Y), \ u \in L_{2e}(U), \ y \in L_{2e}(Y) \) is inside the conic sector \([a,b]\) if there exist real constants \( a \in \mathbb{R}, b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\} \) where \( b > a \), and \( \beta \) such that

\[-\frac{1}{b} \|y\|^2_{2T} + (1 + \frac{a}{b}) \langle y, u \rangle_T - a \|u\|^2_{2T} \geq \beta, \ \forall u \in L_{2e}(U), \ \forall T \in \mathbb{R}^+.
\]

**Theorem 4.4.1.** A LTV system described by Eqs. (3.2) and (3.3) that is controllable and observable lies inside the cone \([a,b]\) (i.e., \( G \in \text{cone}[a,b] \)) if there exist continuous, bounded
\( P(t) = P^T(t) > 0, \ L(\cdot), \) and \( W(\cdot) \) such that

\[
\begin{align*}
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) &= -\frac{1}{\delta} C^T(t)C(t) - L^T(t)L(t), \quad (4.13a) \\
P(t)B(t) &= \frac{1}{2} \left(1 + \frac{a}{\delta}\right) C^T(t)C(t) - \frac{1}{\delta} C^T(t)D(t) - L^T(t)L(t)W(t), \quad (4.13b) \\
\frac{1}{2} \left(1 + \frac{a}{\delta}\right) \left[ D(t) + D^T(t) \right] &= W^T(t)W(t) + \frac{1}{\delta} D^T(t)D(t) + aI. \quad (4.13c)
\end{align*}
\]

**Proof.** Consider the following Lyapunov-like function and its derivative:

\[
\begin{align*}
V &= x^T P x, \\
\dot{V} &= x^T \left( PA + A^T P + \dot{P} \right) x + 2x^TPBu.
\end{align*}
\]

Using Eqs. (4.13a) and (4.13b) we can write

\[
\dot{V} = -\frac{1}{\delta} x^T C^T C x - x^T L^T L x + \left(1 + \frac{a}{\delta}\right) x^T C^T u - \frac{2}{\delta} x^T C^T D u - 2x^T L^T W u.
\]

Note that \( u^T y = u^T C x + u^T D u = u^T C x + \frac{1}{2} u^T \left( D + D^T \right) u, \) which leads to

\[
\dot{V} = \left(1 + \frac{a}{\delta}\right) y^T u - \frac{1}{2} \left(1 + \frac{a}{\delta}\right) u^T \left( D + D^T \right) u - \frac{1}{\delta} x^T C^T C x - \frac{2}{\delta} x^T C^T D u - x^T L^T L x - 2x^T L^T W u.
\]

Using Eq. (4.13c) gives

\[
\begin{align*}
\dot{V} &= \left(1 + \frac{a}{\delta}\right) y^T u - \frac{1}{\delta} x^T C^T C x - \frac{2}{\delta} x^T C^T D u - \frac{1}{\delta} u^T D^T D u - au^T u - x^T L^T L x - 2x^T L^T W u - u^T W^T W u \\
&= \left(1 + \frac{a}{\delta}\right) y^T u - \frac{1}{\delta} y^T y - au^T u - z^T z.
\end{align*}
\]

Integrating \( \dot{V} \) between 0 and \( T \) while assuming zero initial conditions gives

\[
\begin{align*}
\int_0^T \dot{V} dt &= V(T) - V(0) \geq \beta, \\
\left(1 + \frac{a}{\delta}\right) \int_0^T y^T u dt &\geq \frac{1}{\delta} \int_0^T y^T y dt + a \int_0^T u^T u dt + \int_0^T z^T z dt + \beta \\
&\geq \frac{1}{\delta} \int_0^T y^T y dt + a \int_0^T u^T u dt + \beta
\end{align*}
\]

which is equivalent to Eq. (3.9) on page 26. \( \square \)

Ref. 43 presents a similar theorem, but applicable to LTI systems. Additionally, notice that Theorem 4.4.1 collapses to Lemma 3.5.1 on page 28 when the system in question is LTI.
4.5 Closed-Loop Stability of Passive Linear Time-Varying Systems

We will now discuss the stability of passive LTV systems. In an LTI context, it can be shown that a passive (i.e., PR) system can be stabilized by a system that is SPR. To do so, Lyapunov’s direct method is used along with the Positive Real Lemma (Lemma 3.4.1, page 24), the SPR Lemma (Lemma 3.4.2, page 24), and the Krasovskii-LaSalle theorem. We will consider the LTV equivalent: the negative feedback interconnection of a passive LTV system and an OSP LTV system. Specifically, we will consider the case where each system does not have a feedthrough matrix. Because we are dealing with nonautonomous systems, the Krasovskii-LaSalle theorem cannot be used. Instead we will use Barbalat’s Lemma. When Barbalat’s Lemma is combined with Lyapunov’s direct method, a “Lyapunov-like” lemma can be stated. Consider a Lyapunov function $V$ and the following conditions:

1. $V$ is lower bounded;
2. $\dot{V}$ is negative semidefinite; and
3. $\ddot{V}$ is bounded.

If these conditions are met, then $\dot{V} \to 0$ as $t \to \infty$. Often convergence of a systems states can be shown using this Lyapunov-like lemma.

**Theorem 4.5.1.** Consider a passive system $G_1$ satisfying Corollary 4.3.2 and an OSP system $G_2$ satisfying Corollary 4.3.1 interconnected as in Fig. 3.1 on page 18 with $u_1(t) = u_2(t) = 0$. If both $Q_i$ ($i = 1, 2$) are continuous and bounded, the closed-loop system is globally asymptotically stable.

**Proof.** Consider the following Lyapunov function and its temporal derivative:

$$V = \frac{1}{2} x_1^T P_1 x_1 + \frac{1}{2} x_2^T P_2 x_2,$$

$$\dot{V} = \frac{1}{2} x_1^T \left( \dot{P}_1 + P_1 A_1 + A_1^T P_1 \right) x_1 + x_1^T P_1 B_1 e_1$$

$$+ \frac{1}{2} x_2^T \left( \dot{P}_2 + P_2 A_2 + A_2^T P_2 \right) x_2 + x_2^T P_2 B_2 e_2.$$

Using Corollary 4.3.1 and 4.3.2 on pages 51 and 52, along with the fact that $e_1 = -y_2$ and $e_2 = y_1$ gives

$$\dot{V} = -\frac{1}{2} x_1^T Q_1 x_1 + x_1^T C_1^T e_1 - \frac{1}{2} x_2^T Q_2 x_2 + x_2^T C_2^T e_2$$

$$= -\frac{1}{2} x_1^T Q_1 x_1 - x_1^T C_1 T e_1 - \frac{1}{2} x_2^T Q_2 x_2 + y_2 y_1$$

$$= -\frac{1}{2} x_1^T Q_1 x_1 - \frac{1}{2} x_2^T Q_2 x_2.$$

Clearly, $\dot{V}$ is negative semidefinite owing to the positive semidefinite nature of $Q_1$ and the boundedness of both $Q_1$ and $Q_2$. The negative semidefinite nature of $\dot{V}$ implies $V(t) \leq V(0)$, and combined with the fact $V$ is bounded from below by zero in turn implies $x_1$ and $x_2$ are bounded.
Next consider the temporal derivative of $\dot{V}$:

$$\ddot{V} = -x_1^T Q_1 \dot{x}_1 - \frac{1}{2} x_1^T Q_1 x_1 - x_2^T Q_2 \dot{x}_2 - \frac{1}{2} x_2^T Q_2 x_2$$

$$= x_1^T Q_1 B_1 C_2 x_2 - x_2^T Q_2 B_2 C_1 x_1 - \sum_{i=1}^{2} \left[ \frac{1}{2} x_i^T \left( Q_i A_i + A_i^T Q_i \right) x_i + \frac{1}{2} x_i^T Q_i x_i \right].$$

Each term in $\ddot{V}$ is bounded, thus $\ddot{V}$ is bounded. Then, via Barbalat’s Lemma, $\dot{V} \to 0$ as $t \to \infty$, which implies $x_2 \to 0$, which then implies $\dot{x}_2 \to 0$, which finally implies $x_1 \to 0$. Hence, all states converge to the origin and the system is globally asymptotically stable.

Theorem 4.5.1 is a new result. Notice that the symmetry of the traditional strong passivity theorem is maintained. It does not matter which system, $G_1$ or $G_2$, is passive as long as the other is OSP. One can show a passive system and a SSP system connected in negative feedback, both of which have nonzero feedthrough matrices, is globally asymptotically stable using a similar proof. Also, it can easily be shown that the negative feedback interconnection of two OSP systems with no feedthrough matrices, or two SSP systems with feedthrough matrices is globally asymptotically stable.

Next we will consider the presence of a sector bounded, memoryless nonlinearity within a feedback loop. Consider the negative feedback interconnection of a dynamic linear time-varying system and a sector bounded, memoryless nonlinearity as shown in Fig. 4.1. The sector bounded, memoryless nonlinearities we are considering are those that satisfy

$$\phi(0, t) = 0, \forall t \in \mathbb{R}^+, \quad (4.14a)$$

$$y^T(t) \phi(y(t), t) \geq 0, \forall y \in \mathbb{R}^m, \forall t \in \mathbb{R}^+. \quad (4.14b)$$

We will now show that the negative feedback interconnection of a SSP LTV system and a sector bounded, memoryless nonlinearity is globally asymptotically stable.

Theorem 4.5.2. Assuming no external inputs, the negative feedback interconnection of a SSP LTV system and a sector bounded, memoryless nonlinearity satisfying the properties of Eq. (4.14)
is globally asymptotically stable.

Proof. We will make use of Eq. (4.8) from Theorem 4.3.4 on page 50. The control is simply $u(t) = -\phi(y, t)$. Consider the following Lyapunov function and its temporal derivative:

$$
V = \frac{1}{2} x^T P x,
$$
$$
\dot{V} = \frac{1}{2} x^T \left( \dot{P} + PA + A^T P \right) x + x^T PBu
$$
$$
= -\frac{1}{2} x^T \left( L^T L + 2\nu P \right) x + x^T \left( C^T - L^T W \right) u
$$
$$
= -\nu x^T P x + y^T u - \frac{1}{2} (Lx + Wu)^T (Lx + Wu)
$$
$$
\leq -\nu x^T P x - y^T \phi(y, t)
$$
$$
\leq -\nu \frac{1}{2} x^T x \quad (\text{Note: } 0 < q^1 \leq Q \leq p^1 < \infty)
$$
$$
< 0 \quad (x \neq 0).
$$

Thus, the system is globally asymptotically stable.

Next we will show that the negative feedback interconnection of an OSP LTV system with $D(t) = 0$ and a sector bounded, memoryless nonlinearity is globally asymptotically stable.

Theorem 4.5.3. Assuming no external inputs, the negative feedback interconnection of an OSP LTV system with $D(t) = 0$ and a sector bounded, memoryless nonlinearity satisfying the properties of Eq. (4.14) is globally asymptotically stable.

Proof. We are assuming no feedthrough matrix is present, thus we will make use of Eq. (4.10) from Corollary 4.3.1. Consider the following Lyapunov function and its temporal derivative:

$$
V = \frac{1}{2} x^T P x,
$$
$$
\dot{V} = \frac{1}{2} x^T \left( \dot{P} + PA + A^T P \right) x + x^T PBu
$$
$$
= -\frac{1}{2} x^T Q x - x^T C^T \phi(y, t)
$$
$$
= -\frac{1}{2} x^T Q x - y^T \phi(y, t)
$$
$$
\leq -\frac{1}{2} q^2 x^T x \quad (\text{Note: } 0 < q^1 \leq Q \leq p^1 < \infty)
$$
$$
< 0 \quad (x \neq 0).
$$

Thus, the system is globally asymptotically stable.
4.6 Control of a Nonlinear, Time-Varying System

4.6.1 The System Differential Equation and its Output Strictly Passive Nature

Consider a mass-spring-damper system described by the following second-order, nonlinear, time-varying differential equation:

\[ m\ddot{q}(t) + c(t)\dot{q}(t) + [k_1 + k_2q^2(t)]q(t) = u(t), \]

where \( m > 0, k_1 > 0, \) and \( k_2 > 0 \) are constant and \( c(\cdot) \) is time-varying. Specifically,

\[ c(t) = \zeta[\cos(\omega t) + 1] + \epsilon, \]

where \( \zeta \geq \epsilon > 0 \) and \( \omega \in (0, 2\pi) \). We assume that both \( \zeta \) and \( \epsilon \) are quite small. Although we have deliberately specified the system to be a mass-spring-damper system, this differential equation could describe a resistor-capacitor-inductor circuit as well. This nonlinear, time-varying system has an OSP input-output mapping between \( u(\cdot) \) and \( \dot{q}(\cdot) \) which can be shown in the following way. Consider the system Hamiltonian and its temporal derivative:

\[
\begin{align*}
H(t) &= \frac{1}{2}m\dot{q}^2(t) + \frac{1}{2}k_1q^2(t) + \frac{1}{4}k_2q^4(t), \\
\dot{H}(t) &= m\ddot{q}(t)\dot{q}(t) + k_1q(t)\ddot{q}(t) + k_2q^3(t)\dot{q}(t) \\
&= u(t)\dot{q}(t) - c(t)q^2(t).
\end{align*}
\]

(We really should be writing \( H(q(t), \dot{q}(t), t) \), not \( H(t) \), but we do so to be concise.) Integrating \( \dot{H}(\cdot) \) between 0 and \( T \) yields

\[
\begin{align*}
\int_0^T \dot{H}(t)\,dt &= H(T) - H(0) \geq -H(0), \\
\int_0^T u(t)\dot{q}(t)\,dt &\geq \int_0^T c(t)\dot{q}^2(t)\,dt - \underbrace{H(0)}_{\beta} \geq \epsilon \int_0^T \dot{q}^2(t)\,dt + \beta.
\end{align*}
\]

Thus, the system is OSP.

When both \( \zeta \) and \( \epsilon \) are quite small the unforced/uncontrolled system response (i.e., the velocity response), although damped, is oscillatory. To illustrate this, consider the unforced/uncontrolled system response of the system shown in Fig. 4.2(a). The initial conditions are \( q(0) = 1 \) (m) and \( \dot{q}(0) = 1 \) (m/s), and the numerical values for the mass, stiffness, and damping parameters are given in Table 4.6.1. In Fig. 4.2(b) is the damping coefficient versus time; notice that the amount of damping is almost zero at some points in time. In Fig. 4.2(a) we see that the system clearly oscillates significantly, an undesirable characteristic. As such, we will design a controller to regulate \( \dot{q}(\cdot) \) to zero.
Figure 4.2: Unforced and uncontrolled system response and system damping vs. time.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1 (kg)</td>
</tr>
<tr>
<td>$k_1$</td>
<td>7.5 (N/m)</td>
</tr>
<tr>
<td>$k_2$</td>
<td>10 (N/m)</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>$5 \times 10^{-3}$ (N/s)</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$5 \times 10^{-5}$ (N/s)</td>
</tr>
</tbody>
</table>

Table 4.1: System parameters used in simulation.

4.6.2 Controller Design Inspired by Optimal Control

Recall that the strong version of the passivity theorem states that the negative feedback interconnection of two systems is $L_2$ stable if $\delta_1 + \epsilon_2 > 0$ and $\delta_2 + \epsilon_1 > 0$. Given that the system we want to control is OSP (and thus has a positive definite $\epsilon_1$ associated with it), by designing an OSP controller we can guarantee closed-loop stability. In particular, we will design an OSP LTV controller using the Linear Quadratic Regulator (LQR) formulation and Corollary 4.3.1 stated on page 51.

To design an OSP LTV controller, we need a linear time-varying plant model. As such, we will linearize our nonlinear, time-varying system and place it in a first-order form:

\[
\begin{align*}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{A}(t) \\ \mathbf{B}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \\
y(t) &= \begin{bmatrix} \mathbf{C}(t) \end{bmatrix} x(t).
\end{align*}
\]

Notice the time-varying nature of $\mathbf{A}(\cdot)$ owing to the time-varying damping in the system. In this example $\mathbf{B}(\cdot)$ and $\mathbf{C}(\cdot)$ are actually not time-varying. However, the controller synthesis technique presented next is applicable to systems that do possess time-varying $\mathbf{B}(\cdot)$ and $\mathbf{C}(\cdot)$ matrices. As
such, we write $B(\cdot)$ and $C(\cdot)$ as if they were time-varying to emphasize they can be time-varying.

Consider the following controller:

\[
\begin{align*}
\dot{x}_c(t) &= A_c(t)x_c(t) + B_c(t)y(t) \\
-u(t) &= C_c(t)x_c(t)
\end{align*}
\]

where $x_c \in \mathbb{R}^{n_c}$, $u, y \in \mathbb{R}^{m}$ and the time-varying matrices $A_c(\cdot)$, $B_c(\cdot)$, and $C_c(\cdot)$ are appropriately dimensioned real matrices that are continuous and bounded over the time interval of interest. It is well known that given the performance index $J$

\[
J = x^T(T)Sx(T) + \int_0^T \left[ x^T(t)M(t)x(t) + u^T(t)N(t)u(t) \right] dt,
\]

where $S = S^T \geq 0$, $M(\cdot) = M^T(\cdot) \geq 0$, and $N(\cdot) = N^T(\cdot) > 0$, one can derive an optimal state feedback $C_c(t) = N^{-1}B^T(t)X(t)$. The matrix $X(\cdot)$ can be found by solving the matrix Riccati equation

\[
-\dot{X}(t) = M(t) + A^T(t)X(t) + X(t)A(t) - X(t)B(t)N^{-1}(t)B^T(t)X(t)
\]

backward in time from $t = T$ to $t = 0$ where $X(T) = S$. This is the well known LQR solution for time-varying systems.

Given that we have designed $C_c(\cdot)$ via a LQR formulation, we must now design $A_c(\cdot)$ and $B_c(\cdot)$ such that the controller is OSP. To do so, we will let $A_c(t) = A(t) - B(t)C_c(t)$. It now remains to find $B_c(\cdot)$. By employing Corollary 4.3.1 we have $B_c(t) = P^{-1}(t)C_c^T(t)$ where $P(\cdot)$ is found by solving

\[
\dot{P}(t) + P(t)A_c(t) + A_c^T(t)P(t) = -Q(t)
\]

backwards in time from $t = T$ to $t = 0$ given the boundary condition $P(T) > 0$, as discussed in Sec. 4.2.1.

We will now synthesize (numerically) an OSP LTV controller using the procedure outlined above. The system parameters given in Table 4.6.1 will be used during synthesis (and simulation). The following weights will be used for synthesis: $M(t) = Q(t) = \mu(t)\text{diag}\{10, 1\}$, $\mu(t) = 10[\cos(\omega t) + 1] + \epsilon$, $N = 1$ (note, $N$ is not time-varying, although it could be).

We will numerically solve for the matrix $P(\cdot)$. To do so, we pick $P(T) = \frac{1}{10} I$ and then iteratively solve

\[
P(t_{k-1}) = P(t_k) + (t_k - t_{k-1}) \left[ P(t_k)A_c(t_k) + A_c^T(t_k)P(t_k) + Q(t_k) \right]
\]

backward from $t = T$ to $t = 0$. In a similar fashion, $X(\cdot)$ can be solved backward in time given $X(T) = S = \frac{1}{10} I$.

The maximum and minimum eigenvalues of $X(\cdot)$ are shown in Fig. 4.3(a), as are those of $P(\cdot)$ in Fig. 4.3(b). All eigenvalues are positive as expected (although they are very small at $T = 16$ (s), they are positive). Notice that both both $X(\cdot)$ and $P(\cdot)$ reach a steady state quite quickly. The position and velocity of the closed-loop system versus time as released from initial conditions of
(a) Maximum and minimum eigenvalues of $X(t)$ vs. time. (b) Maximum and minimum eigenvalues of $P(t)$ vs. time.

Figure 4.3: Maximum and minimum eigenvalues of $X(t)$ and $P(t)$ vs. time.

1 (m) and 1 (m/s) are shown in Fig. 4.4. This controlled response is a vast improvement over the uncontrolled response shown in Fig. 4.2(a). What this shows is that the nonlinear, time-varying system being considered can be easily controlled using an OSP LTV controller.

Figure 4.4: Position and velocity vs. time when controlled by OSP LTV controller.

### 4.6.3 Memoryless Nonlinear Control

Rather than using a dynamic compensator, we will now employ Theorem 4.5.3 and stabilize the plant with a sector bounded, memoryless nonlinear control of the form

$$u(t) = -\phi(\dot{q}(t), t) = -\phi(y(t), t) = -\tanh(y(t))(1 + 4e^{-\alpha t}).$$

This memoryless nonlinearity is confined to the first and third quadrant of the $(y(t), \phi(y(t), t))$ plane for all positive time, and $\phi(0, t) = 0$, $\forall t \in \mathbb{R}^+$ as well. Fig. 4.5(a) shows the profile of the
nonlinearity at times $t = 0, 2.5, 5, 7.5, 10$ (s). The position and velocity of the system versus time as released from initial conditions of 1 (m) and 1 (m/s) are shown in Fig. 4.5(b).

Figure 4.5: Memoryless nonlinearity and controlled response.

### 4.7 Summary

In this chapter the input-output properties of LTV systems were investigated, along with the stability of various passive LTV systems connected in feedback. We presented various theorems and corollaries that provide sufficient conditions that ensure a LTV system has finite gain, is passive, or has conic bounds. Turning our attention to passive systems, we investigated the stability of various systems connected in feedback. In particular, we showed that a passive LTV system and a OSP LTV system (each of which do not have a feedthrough matrix) connected in a negative feedback interconnection is globally asymptotically stable. The control of a nonlinear, time-varying mass-spring-damper system was also considered. We presented a simple controller synthesis method.

In Chapter 7 we will employ some of the theory developed in this chapter to control the attitude of a spacecraft that is actuated by magnetic torque rods as well as reaction wheels.
Chapter 5

Hybrid Input-Output Systems

5.1 A Motivating Example

Robust Control via the Passivity Theorem in an Ideal World

Consider the simple mass-spring system shown in Fig. 5.1. We will ignore the sensor and actuator dynamics associated with measurement of \( y \in \mathbb{R} \), the velocity of the first cart, and application of the force \( u \in \mathbb{R} \). The physical parameters of the system are provided in Table 5.1. This system is governed by the following second-order, linear matrix differential equation:

\[
\ddot{q}(t) + K\dot{q}(t) = \bar{b}u(t)
\]

where \( \bar{b} = [1 \ 0]^T \), \( \dot{q} = [q_1 \ q_2]^T \), \( M = M^T > 0 \) is the mass matrix, \( K = K^T > 0 \) is the stiffness matrix, and \( u \in \mathbb{R} \) is the system input (a force). It can be shown that the map \( u \rightarrow y \) where

![Figure 5.1: A mass-spring system consisting of two masses and two springs.](image)

Table 5.1: Mass and spring values for simple mass-spring system.

<table>
<thead>
<tr>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( k_1 )</th>
<th>( k_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (kg)</td>
<td>0.5 (kg)</td>
<td>10 (N/m)</td>
<td>15 (N/m)</td>
</tr>
</tbody>
</table>
\[ y(t) = \mathbf{b}^T \dot{\mathbf{q}}(t) = \dot{q}_1 \] is a passive one. It follows that the system transfer function

\[ \frac{y(s)}{u(s)} = g(s) = C(s1 - A)^{-1}B \]

is PR, as indicated by the phase response of \( g(s) \) being bounded by \( \pm 90^\circ \), which is shown in Fig. 5.2(a). In fact, even when the mass distribution or stiffness parameters are significantly perturbed, the map \( u \rightarrow y \) remains passive. To illustrate this point, consider the mass-spring system but with \( m_1 = 2 \) kg and \( m_2 = 1 \) kg, a significant perturbation to the original model. Let \( g'(s) \) be the system transfer function with these new mass values. The frequency response of \( g'(s) \) shown in Fig. 5.2(b). The phase response of \( g'(s) \) is still bounded by \( \pm 90^\circ \) clearly indicating the system is still PR.

![Diagram](image1)

(a) Ideal system with a passive input-output map.

![Diagram](image2)

(b) Perturbed system; input-output map is still passive.

![Diagram](image3)

(c) Sensor frequency response; \( f(s) = a/(s + a) \).

![Diagram](image4)

(d) “Violated” system; input-output map is no longer passive.

Figure 5.2: The nature of passivity violations.

The fact that the system map \( u \rightarrow y \) remains passive given significant perturbations to the mass distribution (or stiffness) is the basis for robust stability via the Passivity Theorem. Any VSP controller is guaranteed to stabilize the mass-spring system for any mass or stiffness values.
For example, a controller of the form

$$-u(s) = \left[ \frac{K_p}{s} + g_{VSP}(s) \right] y(s)$$

where $0 < K_p < \infty$ is the proportional control gain and $g_{VSP}(s)$ is a VSP transfer function would realize robust stabilization with respect to uncertain mass and stiffness.

**Violation of Passivity**

Previously we explicitly ignored sensor and actuator dynamics. In reality, some sort of velocity sensor would be used to measure the velocity of the first cart. Consider the following sensor model:

$$y_1(s) = f(s)y(s).$$

The sensor measures $y(s)$, the true velocity of the first cart, and provides the control system with $y_1(s)$. If $f(s) = 1$ the sensor would not induce any sort of gain or phase distortion of the measurement $y(s)$. Unfortunately, such “ideal” sensors do not exist; real sensors are dynamical, having gain and phase that change as a function of frequency.

Consider the following sensor model:

$$f(s) = \frac{a}{s + a}.$$  

We will, for simplicity, continue to ignore actuator dynamics. The frequency response of $f(s)$ is shown in Fig. 5.2(c). The sensor is “well behaved” at low frequency; the gain is unity and there is (essentially) no phase lag. However, at high frequency the gain rolls off while the phase is lagging by 90°.

Now consider the frequency response of $y_1 = g_1(s)u_1(s)$ shown in Fig. 5.2(d) where $g_1 = f(s)g'(s)$ and $u_1(s) = u(s)$. The system is no longer phase bounded by ±90° over all frequencies; passivity has been violated. Robust closed-loop stability can no longer be guaranteed by the Passivity Theorem.

Although the presence of $f(s)$ has destroyed the passive nature of the system, the overall effect of $f(s)$ is not as drastic as perhaps thought. Consider the frequency response of $g_1(s)$ in Fig. 5.2(d) once again. Let us split the frequency response up into two regions: a low frequency region and a high frequency region. The divide between the low and high frequency regions is the “critical frequency” $\omega_c$, which is approximately 12 rad/s. At low frequency the phase response of $g_1(s)$ is still bounded by ±90°, although the gain is infinite; the system is PR over a finite frequency range. Additionally, gain at high frequency is bounded, while the phase approaches −180°; the system is BR (bounded real, that is, has gain that is finite) over a finite frequency range. In effect, the plant $g_1(s)$ can be segmented into two parts: a low frequency passive (i.e., PR) part, and a high frequency finite gain (i.e., BR) part.
Hybrid Input-Output Systems

The reminder of this chapter is devoted to characterizing and ensuring closed-loop stability of continuous systems that have an input-output map that can be described by multiple properties, i.e., finite gain, passive, and conic properties. We will call such systems hybrid input-output systems. The word “hybrid” is used to highlight the fact that the systems in question (along with the stability criteria to be developed) are not described in terms of one kind of input-output characteristic; they are described in terms of finite gain, passive, and conic properties in a “hybrid”, “mixed”, or “blended” fashion.

The previous example involving control of a mass-spring system was used to highlight the fact that at low frequency the system is “still” passive, while at high frequency the sensor dynamics violate passivity, but at the same time the system has gain that is finite. The system has a passive input-output map, and a finite gain input-output map when passivity has been destroyed; the system is said to be a hybrid passive/finite gain system. The mass-spring system is linear, and as such thinking in terms of the systems frequency response is quite natural. We will show that a general system (i.e., nonlinear or time-varying) having an input-output map that is described in a hybrid fashion is also permissible. However, because speaking in terms of a system’s frequency response is so easy and convenient, we will often refer to bandwidth or frequency ranges even when the systems we are dealing with are nonlinear.

Other authors have investigated characterizing and stabilizing system that have input-output maps that have multiple properties. In particular, Refs. 61 and 62 were the starting point of our investigation. In Refs. 61 and 62 “mixed” systems are developed, and a stability criterion is provided. Mixed systems are those that have an input-output map that i) has gain strictly less than one over an operating range, ii) has a mixture of passivity (either VSP, ISP, OSP, or purely passive characteristics) and gain strictly less than one over another operating range, and iii) has a purely passive nature over a third operating range. For closed-loop stability of a plant $G_1$ being controlled by $G_2$, the mixed systems framework requires that both the plant and the controller i) have gain strictly less than one over an operating range, ii) are both passive and have gain strictly less than one over a second operating range, and iii) are both passive over a third operating range. In a LTI context, the “operating ranges” can be thought of as a frequency bands, but again, as in our hybrid framework the systems could be nonlinear or time-varying. A similar result only applicable to linear systems is that of Ref. 63 where the stability of a linear systems is guaranteed by both passivity and small gain arguments in different frequency bands.

Eventually we will be concerned with designing controllers to satisfy hybrid properties. Iwasaki and Hara (along with their coauthors) have investigated in great detail LTI systems that are PR, SPR, and BR within a frequency band. They have developed what they refer to as a Generalized KYP (GKYP) Lemma. We will make use of the GKYP Lemma for identification of hybrid properties, as well as controller synthesis in Chapter 9.
5.2 Hybrid Dissipative Systems

In Sec. 3.6 (starting on page 31) the “classic” dissipative systems framework developed by Hill and Moylan was summarized. What is quite nice about the classic dissipative systems framework is that traditional finite gain, passive, and conic systems can each be conveniently represented in terms of the matrices $Q$, $S$, and $R$. Also, the Small Gain, Passivity, and Conic Sector Theorem can each be interpreted in terms of the interconnection of two dissipative systems.

For our purposes, the classic dissipative systems framework is somewhat restrictive; we desire a more general dissipative systems framework. As such, we will redefine the dissipative inequality originally presented in Eq. (3.15). Our only modification will be to replace the matrices $Q$, $S$, and $R$ with the bounded linear operators $Q$, $S$, and $R$. This will allow us to characterize hybrid finite gain, passive, and conic systems.

Consider a mapping $y(t) = (G u)(t)$ where inputs $u \in L_{2e}(U)$ are mapped to outputs $y \in L_{2e}(Y)$ through an operator $G : L_{2e}(U) \rightarrow L_{2e}(Y)$. To be general, we will assume the system is nonlinear with a finite-dimensional state-space realization equivalent to that presented in Eq. (3.1), that is, the system $y(t) = (G u)(t)$ has a finite-dimensional state-space realization of the form

$$
\dot{x}(t) = f(x(t), u(t)), \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m,
$$

$$
y(t) = g(x(t), u(t)), \; y \in \mathbb{R}^m.
$$

We assume the state-space realization is completely reachable from the origin, and zero-state observable.45,46 The system is also permitted to be LTV or LTI, and as such would have a minimal state-space realization equivalent to Eqs. (3.2) and (3.4).

As in the classic dissipative systems framework, the system has associated with it a supply rate $w : L_{2e}(U) \times L_{2e}(Y) \rightarrow L_{2e}$ and a continuously differentiable function, the storage function, $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^+$. The system $y(t) = (G u)(t)$ with state-space realization described by Eq. (3.1) is said to be dissipative with respect to the supply rate $w$ if

$$
\int_0^T w(u(t), y(t)) dt = (y, Q y)_{\mathbb{R}} + 2 (y, S u)_{\mathbb{R}} + (u, R u)_{\mathbb{R}} \geq -\phi(x(0)) \tag{5.2}
$$

holds $\forall T \in \mathbb{R}^+$ where $w(u(t), y(t))$ is evaluated along the trajectories of Eq. (3.1) given $u \in L_{2e}(U)$ and $Q : L_2 \rightarrow L_2$, $S : L_2 \rightarrow L_2$, and $R : L_2 \rightarrow L_2$ are bounded linear operators. The system is said to be $(Q, S, R)$-dissipative, or a hybrid dissipative system. By transferring the truncation to the signals $u$ and $y$ and using Parseval’s theorem Eq. (5.2) can be written as

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} y^H(\omega)Q(\omega)y(\omega)d\omega + \frac{1}{2\pi} \text{Re} \left\{ \int_{-\infty}^{\infty} y^H(\omega)S(\omega)u(\omega)d\omega \right\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} u^H(\omega)R(\omega)u(\omega)d\omega \geq -\phi(x(0)) \tag{5.3}
$$

$\forall u \in L_{2e}(U), \forall T \in \mathbb{R}^+$ where $Q(\cdot)$, $S(\cdot)$, and $R(\cdot)$ are the Fourier transforms of the impulse
responses of the operators $Q$, $S$, and $R$. Specifically note that we have Fourier transformed the
impulse responses of the operators $Q$, $S$, and $R$, the input signal $u$, and output signal $y$; we
have not attempted to Fourier transform the (possibly nonlinear) operator $G$ describing the map
$y(t) = (Gu)(t)$.

The difference between this new dissipative systems definition and the classic dissipative
systems definition is that $Q$, $S$, and $R$ are bounded linear operators, not constant matrices. If
the operators $Q$, $S$, and $R$ are specified to be constant matrices then, as expected, the classic
dissipative systems definition is recovered.

5.3 The Causal, Bounded, Linear Time-Invariant Operators $A_i$

Before defining hybrid input-output systems (e.g., hybrid conic systems or hybrid passive/finite
gain systems), we must consider a set of operators that will be used to define the operators $Q$, $S$, and $R$. In turn, the operators $Q$, $S$, and $R$ will define the hybrid input-output nature of a
particular system represented by the operator $G$.

Consider the causal, bounded, LTI operators $A_i : L_2 \rightarrow L_2$, $i = 1 \cdots N$. The operators are
MIMO, and will be deliberately written as $A_i = A_i^1$ where $A_i : L_2 \rightarrow L_2$, $i = 1 \cdots N$ are SISO
causal, bounded, LTI operators. The Laplace transforms of the impulse responses of the operators
$A_i$ are the transfer matrices $A_i(s) \in H_\infty$. Naturally then $A_i(s) = A_i(s)1$ where $A_i(s) \in H_\infty$. Together the operators $A_i$ (and $A_i$) have various properties, as listed below.

**Property 5.3.1.** The operators $A_i$ satisfy

\[
\sum_{i=1}^{M} A_i^\dagger A_i = 1 \quad (5.4)
\]

\[
\Leftrightarrow \sum_{i=1}^{M} A_i^\dagger A_i = 1
\]

\[
\Leftrightarrow \sum_{i=1}^{M} A_i^H(s)A_i(s) = 1
\]

\[
\Leftrightarrow \sum_{i=1}^{M} A_i(-s)A_i(s) = 1
\]

\[
\Leftrightarrow \sum_{i=1}^{M} \alpha_i(\omega) = 1 \quad (5.5)
\]

where $1$ is the identity operator and

\[
\alpha_i(\omega) = A_i(-j\omega)A_i(j\omega) = |A_i(j\omega)|^2.
\]

Equivalence of Eqs. (5.4) and (5.5), as well as each of the intermediate summations can be
shown by considering the truncated inner product of an arbitrary signal \( \mathbf{v} \in L_{2c} \) with itself:

\[
\langle \mathbf{v}, \mathbf{v} \rangle_T = \langle \mathbf{v}_T, \mathbf{1}_T \mathbf{v}_T \rangle = \sum_{i=1}^{M} \langle \mathbf{v}_T, \mathcal{A}_i^* \mathcal{A}_i \mathbf{v}_T \rangle = \sum_{i=1}^{M} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{v}_T^H(j\omega) \mathcal{A}_i^H(j\omega) \mathcal{A}_i(j\omega) \mathbf{v}_T(j\omega) d\omega
\]

\[
= \sum_{i=1}^{M} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{v}_T^H(j\omega) (\alpha_i(j\omega) \mathbf{1}) \mathbf{v}_T(j\omega) d\omega.
\]

In simplifying we have used Parseval’s Theorem.

**Property 5.3.2.** The functions \( \alpha_i \) may only be zero or one, depending on the frequency \( \omega \). To be specific, \( \alpha_i : \mathbb{R} \to \{0, 1\} \) where

\[
\alpha_1(\omega) = \begin{cases} 
1 & \forall \omega \in \Omega_1 := \{ \omega \in \mathbb{R} : |\omega| < \omega_1 \} \\
0 & \forall \omega \notin \Omega_1
\end{cases},
\]

\[
\alpha_i(\omega) = \begin{cases} 
1 & \forall \omega \in \Omega_i := \{ \omega \in \mathbb{R} : \omega_{i-1} \leq |\omega| < \omega_i \} \\
0 & \forall \omega \notin \Omega_i
\end{cases}, \quad i = 2 \cdots M - 1,
\]

\[
\alpha_M(\omega) = \begin{cases} 
1 & \forall \omega \in \Omega_M := \{ \omega \in \mathbb{R} \cup \{\infty\} : \omega_{M-1} \leq |\omega| \} \\
0 & \forall \omega \notin \Omega_M
\end{cases}
\]

where \( \omega_M = \infty \), and \( 0 < \omega_1 < \omega_2 < \cdots < \omega_{M-1} < \omega_M \). The \( \omega_i \)'s are referred to as the “critical frequencies”. If \( M = 2 \), there is only one critical frequency, \( \omega_c \).

The functions \( \alpha_i \), and by extension the operators \( \mathcal{A}_i \), take on a very important role in the hybrid input-output systems framework. Each \( \alpha_i \) can be thought of as an ideal filter; the function \( \alpha_1 \) can be thought of as an ideal low pass filter, each \( \alpha_i \) for \( i = 2 \cdots M - 1 \) can be thought of as ideal band pass filters, and \( \alpha_M \) can be thought of as an ideal high pass filter. The functions \( \alpha_i \) allow us to “divide” a system’s input-output map up into a set of “intermediate” input-output maps that have specific properties.

**Property 5.3.3.** For an arbitrary signal \( \mathbf{v} \in L_{2c} \) the inner product of \( (\mathcal{A}_i^* \mathcal{A}_i \mathbf{v}_T)(\cdot) \) and \( (\mathcal{A}_j^* \mathcal{A}_j \mathbf{v}_T)(\cdot) \) where \( i \neq j \) is zero, that is

\[
\langle \mathcal{A}_i^* \mathcal{A}_i \mathbf{v}_T, \mathcal{A}_j^* \mathcal{A}_j \mathbf{v}_T \rangle = 0, \quad \forall \mathbf{v} \in L_{2c}, \quad \forall T \in \mathbb{R}^+.
\]

**Property 5.3.4.** For an arbitrary signal \( \mathbf{v} \in L_{2c} \) the inner product of \( (\mathcal{A}_i^* \mathcal{A}_i \mathbf{v}_T)(\cdot) \) and \( (\mathcal{A}_j^* \mathcal{A}_j \mathbf{v}_T)(\cdot) \) is

\[
\langle \mathcal{A}_i^* \mathcal{A}_i \mathbf{v}_T, \mathcal{A}_j^* \mathcal{A}_j \mathbf{v}_T \rangle = \langle \mathcal{A}_i \mathbf{v}_T, \mathcal{A}_j \mathbf{v}_T \rangle = \| \mathcal{A}_i \mathbf{v}_T \|^2_2, \quad \forall \mathbf{v} \in L_{2c}, \quad \forall T \in \mathbb{R}^+.
\]

Both Property 5.3.3 and Property 5.3.4 are a result of Properties 5.3.1 and 5.3.2. To see why the operators \( \mathcal{A}_i \) adhere to Property 5.3.3 and Property 5.3.4, consider the following truncated
inner product involving an arbitrary signal \( v \in L^2 \), and subsequent manipulation:

\[
\langle v, v \rangle_T = \langle 1v_T, 1v_T \rangle \\
= \int_0^\infty (1v_T)^T(t)(1v_T)(t)dt \\
= \int_0^\infty \left[ \sum_{i=1}^M (A_i^\sim A_i v_T)^T(t) \right] dt \text{ (using Eq. (5.4))} \\
= \int_0^\infty \left[ \sum_{i=1}^M (A_i^\sim A_i v_T)^T(t)(A_i^\sim A_i v_T)(t) \right] dt \text{ (simplified using Property 5.3.3)} \\
= \int_0^\infty v_T^T(t)(A_i^\sim A_i v_T)(t)dt \text{ (simplified using Property 5.3.4)} \\
= \int_0^\infty v_T^T(t)(1v_T)(t)dt \\
= \langle v_T, v_T \rangle
\]

As we shall see in the next section, the operators \( A_i \) (which satisfy Properties 5.3.1, 5.3.2, 5.3.3, and 5.3.4) will be used to define the operators \( Q, S, R \).

### 5.4 Hybrid Conic Systems

We will first consider hybrid conic systems. Other types of hybrid systems, such as hybrid passive/finite gain systems, will be shown to be a special kind of hybrid conic system. Additionally, we will discuss how hybrid conic systems can be used to describe or approximate Safonov’s variable cones,\(^{18}\) which we discussed in Sec. 3.5.3, page 31.

**Definition 5.4.1.** Consider a square system \( y(t) = (G)u(t), G : L2e(U) \rightarrow L2e(Y), u \in L2e(U), y \in L2e(Y) \) with state-space form equivalent to Eq. (3.1). The system is dissipative with respect to a supply rate and has a corresponding storage function as in Eq. (5.2). If the bounded LTI operators \( Q, S, R \) satisfy

\[
Q = \sum_{i=1}^M (-\frac{1}{b_i})A_i^\sim A_i, \quad S = \frac{1}{2} \sum_{i=1}^M (1 + \frac{a_i}{b_i})A_i^\sim A_i, \quad R = \sum_{i=1}^M (-a_i)A_i^\sim A_i \tag{5.7}
\]

for some \( a_i \in \mathbb{R}, b_i \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\}, b_i > a_i \), where the operators \( A_i \) satisfy Properties 5.3.1, 5.3.2, 5.3.3, and 5.3.4, then the system is said to be a hybrid conic system satisfying the inequality

\[
\sum_{i=1}^M \left( -\frac{1}{b_i} \|A_i y\|_{2T}^2 + (1 + \frac{a_i}{b_i}) \langle A_i y, A_i u \rangle_T - a_i \|A_i u\|_{2T}^2 \right) \geq -\phi(x(0)), \quad \forall u \in L2e(U), \quad \forall T \in \mathbb{R}^+.
\tag{5.8}
\]

Notice that the operators \( Q, S, R \) are self-adjoint. Also notice that the units of \( Q \) and \( R \) are consistent; the operator \( Q \) has units of one over gain, while the operator \( R \) has units of gain. The operator \( S \) is unitless.
Previously in Sec. 5.3 we mentioned that the operators $\mathcal{A}_i$ can be thought of as or interpreted as filters. The operators $\mathcal{A}_i$ can be interpreted as a filter because, as shown in Eq. (5.8), the operators $\mathcal{A}_i$ are filtering both the inputs and outputs, $u$ and $y$. In particular, the operators $\mathcal{A}_i$ are filtering the mapping $u \rightarrow y$ into different regions, each described by a different set of conic bounds.

Recall that the hybrid dissipative inequality in Eq. (5.2) can also be written as in Eq. (5.3) where $Q(\cdot)$, $S(\cdot)$, and $R(\cdot)$ are the Fourier transforms of the impulse responses of the operators $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{R}$. In terms of hybrid conic systems, the Fourier transforms of the impulse responses of the operators presented in Eq. (5.7) are

$$Q(\omega) = \left( \sum_{i=1}^{M} (-\frac{1}{b_i}) \alpha_i(\omega) \right) \mathbf{1}, \quad S(\omega) = \frac{1}{2} \left( \sum_{i=1}^{M} (1 + \frac{\alpha_i(\omega)}{b_i}) \alpha_i(\omega) \right) \mathbf{1}, \quad R(\omega) = \left( \sum_{i=1}^{M} (-a_i) \alpha_i(\omega) \right) \mathbf{1}.$$  

(5.9)

In turn, Eq. (5.3) can then be written

$$\sum_{i=1}^{M} \left( \frac{1}{2} \int_{-\infty}^{\infty} \left(-\frac{1}{b_i}\right) \alpha_i(\omega) y_r^H(j\omega)y_r(j\omega)d\omega + \frac{1}{2\pi} 1 \right) \left( \int_{-\infty}^{\infty} (1 + \frac{\alpha_i(\omega)}{b_i}) \alpha_i(\omega) y_r^H(j\omega)u_r(j\omega)d\omega \right)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} (-a_i) \alpha_i(\omega) u_r^H(j\omega)u_r(j\omega)d\omega \right) \geq -\phi(x(0))$$  

(5.10)

$\forall u \in L_2(U)$ and $\forall T \in \mathbb{R}^+$.  

5.4.1 Linear Time-Invariant Hybrid Conic Systems

Consider a system that is LTI with transfer matrix $G(s) = C(sI - A)^{-1}B + D$ where $(A, B, C, D)$ is a minimal state-space realization. We would like to know if the system satisfies various conic bounds over the finite frequency intervals $\Omega_i$, $i = 1 \cdots M$, and is thus a hybrid conic system. Essentially, what we need is a condition that when met confirms that the system is conic with bounds $a_i$ and $b_i$ over the finite frequency range $\Omega_i$.

Although calculation of the hybrid conic bounds (i.e., the bounds $[a_i, b_i]$ over each $\Omega_i$) for LTI hybrid conic system is possible (as we will show next), when it comes to nonlinear systems or LTV systems, to our knowledge calculation of the hybrid conic bounds over each $\Omega_i$ is not possible.

Hybrid Conic Systems in the Frequency Domain

Consider the LTI system $G(s)$ and a set of conic bounds $[a_i, b_i]$, $i = 1 \cdots M$. Much like the traditional conic case discussed in Sec. 3.5.1 (starting on page 27), we would like to know if $G(s)$ together with the bounds $[a_i, b_i]$ satisfy Eq. (5.10). Such LTI hybrid conic systems are those that have a frequency response that is confined to the conic region $[a_i, b_i]$ over the finite frequency interval $\Omega_i$.

To derive hybrid conic conditions on the transfer matrix $G(s)$, consider Eq. (5.10) where we
will substitute \( y(j\omega) = G(j\omega)u(j\omega) \) in for \( y(j\omega) \) and manipulate in the following way:

\[
\sum_{i=1}^{M} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} u_{r}^{H}(j\omega) \left[ -\alpha_{i}(\omega) \frac{1}{\beta_{i}} \mathbf{G}^{H}(j\omega) \mathbf{G}(j\omega) \right] u_{r}(j\omega) d\omega \right. \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} u_{r}^{H}(j\omega) \left( \alpha_{i}(\omega) \frac{1}{2} \left( 1 + \frac{\alpha}{\beta_{i}} \right) \left[ \mathbf{G}^{H}(j\omega) + \mathbf{G}(j\omega) \right] \right) u_{r}(j\omega) d\omega \\
\left. + \frac{1}{2\pi} \int_{-\infty}^{\infty} u_{r}^{H}(j\omega) \left[ -\alpha_{i}(\omega) a_{i} \right] u_{r}(j\omega) d\omega \right) \geq 0,
\]

\[
\sum_{i=1}^{M} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_{i}(\omega) u_{r}^{H}(j\omega) \left[ -\frac{1}{\beta_{i}} \mathbf{G}^{H}(j\omega) \mathbf{G}(j\omega) + \frac{1}{2} \left( 1 + \frac{\alpha}{\beta_{i}} \right) \left[ \mathbf{G}^{H}(j\omega) + \mathbf{G}(j\omega) \right] - a_{i} \mathbf{1} \right] u_{r}(j\omega) d\omega \right) \geq 0.
\]

It follows that the system is a hybrid conic system with hybrid cones defined by \([a_{i}, b_{i}], i = 1 \cdots M \) if

\[
- \frac{1}{\beta_{i}} \mathbf{G}^{H}(j\omega) \mathbf{G}(j\omega) + \frac{1}{2} \left( 1 + \frac{\alpha}{\beta_{i}} \right) \left[ \mathbf{G}^{H}(j\omega) + \mathbf{G}(j\omega) \right] - a_{i} \mathbf{1} \geq 0, \quad \forall \omega \in \Omega_{i}.
\] (5.11)

If Eq. (5.11) holds \( \forall \omega \in \Omega_{i} \) the system is conic with bounds \( a_{i} \) and \( b_{i} \). However, Eq. (5.11) may hold for many \( a_{i} \) and \( b_{i} \) pairs; determining the limits of the conic region, that is the smallest/largest \( a_{i} \) and \( b_{i} \), may have to be done in an iterative manner. Notice how similar Eqs. (3.11) (page 28) and (5.11) are; Eq. (3.11) is used to define conicity over all frequencies (that is, \( \forall \omega \in \mathbb{R} \)), while Eq. (5.11) defines conicity in a hybrid manner, over a finite frequency range (that is, \( \forall \omega \in \Omega_{i} \)).

State-Space Characterization of Hybrid Conic Systems

It is possible to determine if a LTI system with transfer matrix \( \mathbf{G}(s) \) satisfies hybrid conic bounds given Eq. (5.11). To do so, Eq. (5.11) would be evaluated at a finite number of points over the relevant portion of the frequency domain (i.e., the domains \( \Omega_{i} \)) given some bounds \( a_{i} \) and \( b_{i} \). Although this is definitely an option, gridding over the frequency domain and evaluating the definiteness of Eq. (5.11) is not ideal. It would be convenient if, for example, it were possible to evaluate the hybrid conic nature of a system based only on its minimal state-space realization. It turns out that the Generalized KYP Lemma (the GKYP Lemma) is just the tool we need to do so.

The GKYP Lemma allows one to characterize the conic nature (and also the PR, SPR, and BR nature) of a system over a finite frequency interval \( \Omega_{i} \). In our work, we are concerned with a low frequency region captured by \( \Omega_{1} \), a set of intermediate regions captured by \( \Omega_{i} \) where \( i = 2 \cdots M - 1 \), and a high frequency region captured by \( \Omega_{M} \). The GKYP Lemma has various versions that are each applicable to such finite frequency regions.

Before stating various versions of the GKYP Lemma, we will define the matrix

\[
\mathbf{\Pi}_{c} := \begin{bmatrix}
\frac{1}{\beta} \mathbf{1} & -\frac{1}{\beta} \left( 1 + \frac{\alpha}{\beta} \right) \mathbf{1} \\
-\frac{1}{\beta} \left( 1 + \frac{\alpha}{\beta} \right) \mathbf{1} & a \mathbf{1}
\end{bmatrix}
\]
and the modified finite frequency domains

\[ \Omega'_i := \Omega_i \setminus \{ \omega \in \mathbb{R} : \det(j\omega I - A) = 0 \}, \quad i = 1 \cdots M. \]

We will start with the frequency domain \( \Omega_1 = \{ \omega \in \mathbb{R} : |\omega| < \omega_1 \} \). The following lemma is the low frequency version of the GKYP Lemma\(^{65} \) specialized to conic systems.

**Lemma 5.4.1.** The system \( G(s) = C(sI - A)^{-1}B + D \) is conic \( \forall \omega \in \Omega'_1 \) if \( \exists P, Q \in \mathbb{R}^{n \times n} \) where \( P = PT \) and \( Q = QT \geq 0 \) such that

\[
\begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P \\ P & (\omega_1 - \bar{\omega}_1)^2Q \end{bmatrix} \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & 1 \end{bmatrix} \Pi_c \begin{bmatrix} C & D \\ 0 & 1 \end{bmatrix} \leq 0 \quad (5.12)
\]

where \( \bar{\omega}_1 \) is a trivially small number that effectively transforms \( |\omega| \leq (\omega_1 - \bar{\omega}_1) \) into the strict inequality \( |\omega| < \omega_1 \).

To be clear, both \( P \) and \( Q \) are symmetric, and although \( Q \) must be positive semidefinite, \( P \) could be negative semidefinite, positive semidefinite, or indefinite. Additionally, if \( \omega_1 \to \infty \) then \( P = PT > 0, Q = 0 \), and an LMI that is equivalent to the traditional Conic Sector Lemma (Lemma 3.5.1, page 28) is recovered.\(^{66} \)

Next we will consider the intermediate frequency domains \( \Omega_i = \{ \omega \in \mathbb{R} : \omega_{i-1} \leq |\omega| < \omega_i \} \), \( i = 2 \cdots M - 1 \). The following lemma is the GKYP Lemma specialized to conic systems over an intermediate frequency range.\(^{66\cdots68} \)

**Lemma 5.4.2.** The system \( G(s) = C(sI - A)^{-1}B + D \) is conic \( \forall \omega \in \Omega'_i \) if \( \exists P, Q \in \mathbb{C}^{n \times n} \) where \( P = PT \) and \( Q = QT \geq 0 \) such that

\[
\begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P + j\bar{\omega}_i Q \\ P - j\bar{\omega}_i Q & -\omega_{i-1}(\omega_i - \bar{\omega}_i)Q \end{bmatrix} \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & 1 \end{bmatrix} \Pi_c \begin{bmatrix} C & D \\ 0 & 1 \end{bmatrix} \leq 0 \quad (5.13)
\]

where \( \bar{\omega}_i = (\omega_{i-1} + (\omega_i - \bar{\omega}_i))/2 \) and \( \bar{\omega}_i \) is a trivially small number that effectively transforms \( \omega_{i-1} \leq |\omega| \leq (\omega_i - \bar{\omega}_i) \) into \( \omega_{i-1} \leq |\omega| < \omega_i \).

Although \( (A, B, C, D) \) are all real matrices, \( P \) and \( Q \) are generally complex matrices.

Last we will consider the high frequency range captured by \( \Omega_M = \{ \omega \in \mathbb{R} \cup \{\infty\} : \omega_M \leq |\omega| \} \). The following lemma is the GKYP Lemma specialized to conic systems over a high frequency range.\(^{67} \)

**Lemma 5.4.3.** The system \( G(s) = C(sI - A)^{-1}B + D \) is conic \( \forall \omega \in \Omega'_M \) if \( \exists P, Q \in \mathbb{C}^{n \times n} \) where \( P = PT \) and \( Q = QT \geq 0 \) such that

\[
\begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} Q & P \\ P & -\omega_M^2Q \end{bmatrix} \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & 1 \end{bmatrix} \Pi_c \begin{bmatrix} C & D \\ 0 & 1 \end{bmatrix} \leq 0. \quad (5.14)
\]

In Lemmas 5.4.1, 5.4.2, and 5.4.3 the system \( G(s) \) is permitted to have poles on the imaginary axis, which is why the frequency range of interest is \( \Omega'_i \) and not simply \( \Omega_i \). If \( G(s) \) does not have
poles on the imaginary axis within the finite frequency range $\Omega_i$ then the non strict LMIs in Eqs. (5.12), (5.13), and (5.14) are replaced with strict LMIs, $Q \geq 0$ is replaced with $Q > 0$, and $\Omega'_i$ is replaced with $\Omega_i$.\textsuperscript{65-68}

Using the GKYP Lemma to determine if a system is conic over a finite frequency domain (i.e., any domain $\Omega_i$) has not been presented in the literature before; the results in the literature are specialized to finite frequency PR, SPR, and BR systems. We are the first to characterize hybrid conic systems (that is finite frequency conic systems) using the GYKP Lemma.

5.4.2 Variable Conic Sectors and Their Relation to Hybrid Conic Systems

In Sec. 3.5.3, starting on page 31, systems that can be described in terms of a variable conic sector were reviewed. Although the idea of a system having an input-output map that can be described in terms of an operator $\mathcal{F}$ embedded within an inner product is very general, calculation of the operator $\mathcal{F}$ is somewhat difficult (or, some would argue, impossible). Hybrid conic systems, although still very general, can be thought of as a special kind of variable conic system. This will be illustrated next.

Consider a system $\mathbf{y}(t) = (\mathcal{G}\mathbf{u})(t)$ described by the variable conic sector (see Eq. (3.13), page 31)

$$\langle \mathcal{F}_{11} \mathbf{y}_T + \mathcal{F}_{12} \mathbf{u}_T, \mathcal{F}_{21} \mathbf{y}_T + \mathcal{F}_{22} \mathbf{u}_T \rangle \geq \beta, \quad \forall \mathbf{u} \in L_2(U), \quad \forall T \in \mathbb{R}^+$$

where

$$\mathcal{F}_{11} = Q = \sum_{i=1}^{M} (-\frac{1}{b_i}) \mathcal{A}_i^\circ \mathcal{A}_i, \quad \mathcal{F}_{12} = \mathcal{F}_{21} = 1, \quad \mathcal{F}_{22} = R = \sum_{j=1}^{M} (-a_j) \mathcal{A}_j^\circ \mathcal{A}_j. \quad (5.15)$$

Substitution of these particular $\mathcal{F}_{kl}$, $k, l = 1, 2$ operators into Eq. (3.13) (i.e., the variable conic inequality) gives

$$\left\langle \left( \sum_{i=1}^{M} (-\frac{1}{b_i}) \mathcal{A}_i^\circ \mathcal{A}_i \mathbf{y}_T \right) + \mathbf{u}_T, \mathbf{y}_T + \left( \sum_{j=1}^{M} (-a_j) \mathcal{A}_j^\circ \mathcal{A}_j \mathbf{u}_T \right) \right\rangle \geq \beta$$
which can be significantly simplified using Properties 5.3.1, 5.3.3, and 5.3.4 in the following way:

\[
\left\langle \sum_{i=1}^{M} \left( -\frac{1}{b_i} \right) \mathbf{A}_i^{\sim} \mathbf{A}_i \mathbf{y}_T, \mathbf{y}_T \right\rangle + \left\langle \sum_{i=1}^{M} \left( -\frac{1}{b_i} \right) \mathbf{A}_i^{\sim} \mathbf{A}_i \mathbf{y}_T, \sum_{j=1}^{M} (a_j) \mathbf{A}_j^{\sim} \mathbf{A}_j \mathbf{u}_T \right\rangle \\
+ \left( \mathbf{u}_T, \mathbf{y}_T \right) + \left( \mathbf{u}_T, \sum_{j=1}^{M} (a_j) \mathbf{A}_j^{\sim} \mathbf{A}_j \mathbf{u}_T \right) \geq \beta,
\]

\[
\sum_{i=1}^{M} \left( -\frac{1}{b_i} \right) \| \mathbf{A}_i \mathbf{y}_T \|_2^2 + \sum_{i=1}^{M} \frac{a_i}{b_i} \left( \mathbf{A}_i^{\sim} \mathbf{A}_i \mathbf{y}_T, \mathbf{A}_i^{\sim} \mathbf{A}_i \mathbf{u}_T \right) \\
+ \left( \mathbf{y}_T, \sum_{i=1}^{M} \mathbf{A}_i^{\sim} \mathbf{A}_i \mathbf{u}_T \right) + \sum_{i=1}^{M} (a_i) \| \mathbf{A}_i \mathbf{u}_T \|_2^2 \geq \beta,
\]

\[
\sum_{i=1}^{M} \left( -\frac{1}{b_i} \| \mathbf{A}_i \mathbf{y}_T \|_2^2 + (1 + \frac{a_i}{b_i}) \left( \mathbf{A}_i \mathbf{y}_T, \mathbf{A}_i \mathbf{u}_T \right) - a_i \| \mathbf{A}_i \mathbf{u}_T \|_2^2 \right) \geq \beta,
\]

\forall \mathbf{u} \in L_{2e}(U) \text{ and } \forall T \in \mathbb{R}^+. \text{ Comparing the last line above to the definition of a hybrid conic system in Eq. (5.8), page 69, we see that a system described by a variable conic sector } \mathbf{F} \text{ where the operators } \mathbf{F}_{ki}, k, l = 1, 2 \text{ take the form presented in Eq. (5.15) describes a hybrid conic system. It follows then that a hybrid conic system (or a hybrid dissipative system for that matter) can be thought of as a special kind of variable conic system.

5.4.3 Approximating Variable Cones with Hybrid Cones

Consider a system } \mathbf{y}(t) = (\mathbf{G} \mathbf{u})(t) \text{ that lies in the variable conic sector } \mathbf{F}, \text{ as in Eq. (3.13). In general, for a system } \mathbf{y}(t) = (\mathbf{G} \mathbf{u})(t) \text{ (which may be nonlinear and time-varying) it is generally not possible to find the operators } \mathbf{F}_{ki}, k, l = 1, 2 \text{ that together with } \mathbf{y}(t) = (\mathbf{G} \mathbf{u})(t) \text{ satisfy Eq. (3.13). However, using hybrid cones it is possible to arbitrarily approximate the variable conic sector } \mathbf{F}. \text{ We will illustrate how through an example, as discussed next.}

Consider the variable finite gain cone abstractly depicted in Fig. 5.3(a). The variable cone represents a finite gain mapping } \mathbf{u} \rightarrow \mathbf{y}, \text{ but the gain is not constant. Recall that gain usually represents a “worst case” amplification. In reality a physical system will not amplify every signal by the maximum gain. The variable cone in Fig. 5.3(a) is a better representation of what a real physical system does; for some inputs the amplification of the signal } \mathbf{u} \in L_{2e} \text{ is small, but for others the amplification is large.}

Without a means to find or calculate the variable conic sector } \mathbf{F}, \text{ we will approximate it using a hybrid conic representation. In particular we will use } M \text{ finite gain cones to approximate the variable finite gain cone. The hybrid finite gain cones have bounds } a_i = -b_i = -\gamma_i \text{ where } 0 < \gamma_i < \infty. \text{ Using Eq. (5.8) (page 69) the system can be described by}

\[
\sum_{i=1}^{M} \left( -\gamma_i^{-1} \| \mathbf{A}_i \mathbf{y}_T \|_2^2 + \gamma_i \| \mathbf{A}_i \mathbf{u}_T \|_2^2 \right) \geq -\phi(\mathbf{x}(0)), \text{ } \forall \mathbf{u} \in L_{2e}(U), \text{ } \forall T \in \mathbb{R}^+.
\]
Each term $-\gamma_i^{-1} \|A_i y\|_2^2 + \gamma_i \|A_i u\|_2^2$ is an approximation of the variable cone over a range, where the range is captured by the filtering effect of $A_i$. The gain over these ranges can be calculated as

$$\sup_{u \in L_2, u \neq 0} \frac{\|A_i y\|_2}{\|A_i u\|_2} = \gamma_i$$

where we have assumed zero initial conditions (thus $\phi(x(0)) = \beta = 0$). The gain $\gamma_i$ can be thought of as a “filtered gain”, because it is calculated by comparing the ratio of the filtered signals $(A_i u_T)(\cdot)$ and $(A_i y_T)(\cdot)$.

Note that by approximating the variable cone we are being conservative. However, our approximation approaches the actual variable cone as $M$ is increased toward infinity. In this way, a hybrid conic system can be thought of a kind of “Ritz” or “Taylor series” approximation of a variable cone; the more cones (with constant bounds) used to approximate a variable cone, the better the approximation.

![Diagram](attachment:variable_cone_diagram.png)

(a) A finite gain variable conic sector.  
(b) First approximation of the finite gain conic sector using $A_1$.  
(c) Second approximation of the finite gain conic sector using $A_2$.  
(d) Third approximation of the finite gain conic sector using $A_3$.

Figure 5.3: Variable finite gain cone approximated using a three hybrid finite gain cones.

To solidify some of the ideas discussed, let us look at a very specific approximation of the variable finite gain cone in Fig. 5.3(a). In particular, we will use three finite gain cones to
approximate the variable finite gain cone. It follows that

\[-\gamma_1^{-1} \|A_1 y_t\|^2 + \gamma_1 \|A_1 u_t\|^2 - \gamma_2^{-1} \|A_2 y_t\|^2 + \gamma_2 \|A_2 u_t\|^2 - \gamma_3^{-1} \|A_3 y_t\|^2 + \gamma_3 \|A_3 u_t\|^2 \geq -\phi(x(0))\]

is our hybrid finite gain representation of the variable finite gain cone. The three finite cones used to approximate the viable finite gain cone are (abstractly) depicted in Figs. 5.3(b), 5.3(c), and 5.3(d). For signals filtered by $A_1$, the cone in Fig. 5.3(b) is representative of the input-output map. For signals filtered by $A_2$, the cone in Fig. 5.3(c) is representative of the input-output map. And for signals filtered by $A_3$, the cone in Fig. 5.3(d) is representative of the input-output map.

### 5.5 Hybrid Passive and Finite Gain Systems

This chapter began with an example; a nominally passive system experienced a violation of passivity. The mapping, although no longer passive, could then be described in terms of passive characteristics and finite gain characteristics in disjoint regions.

In the previous section, we explored the notion of a system being described by $M$ cones in terms of a hybrid conic input-output mapping. In what follows, we will develop hybrid passive/finite gain systems; systems that can be described in terms of a passive cone (i.e., either VSP, OSP, ISP, or just passive), and a finite gain cone. In this case $M = 2$. This type of system description will enable a description of systems which have passive properties, and finite gain properties when passivity has been violated.

To be clear it is best if we specialize our notation slightly. When discussing hybrid passive/finite gain systems that have two input-output regimes, Eqs. (5.4) and (5.5) (page 67) will be rewritten as

\[A^\sim A + B^\sim B = 1 \iff A^\sim A + B^\sim B = 1 \iff \alpha(\omega) + \beta(\omega) = 1\]  

(5.16)

where (for simplicity) we have let $A := A_1$ and $B := A_2$, and $\alpha(\omega) := \alpha_1(\omega)$ and $\beta(\omega) := \alpha_2(\omega)$. Also, $A = A_1$ and $B = B_1$. The Laplace transforms of the impulse responses of the causal, bounded, LTI operators $A$ and $B$ ($A$ and $B$) are the transfer matrices $A(s) \in H_\infty$ and $B(s) \in H_\infty$ (are the transfer functions $A(s) \in H_\infty$ and $B(s) \in H_\infty$). Properties 5.3.1, 5.3.2, 5.3.3, and 5.3.4 hold as well. In particular,

\[\alpha(\omega) = \begin{cases} 1, & \forall \omega \in \Omega_l \\ 0, & \forall \omega \in \Omega_h \end{cases}, \quad \beta(\omega) = \begin{cases} 0, & \forall \omega \in \Omega_l \\ 1, & \forall \omega \in \Omega_h \end{cases}\]  

(5.17)

where

\[\Omega_l = \{\omega \in \mathbb{R} : |\omega| < \omega_c\}, \quad \Omega_h = \{\omega \in \mathbb{R} \cup \{\infty\} : |\omega| \geq \omega_c\}\]

and $\omega_c$ is the critical frequency (i.e., $\omega_1 = \omega_c$ when $M = 2$). (The subscript “l” means “low”, and “h” means “high”.) Next, we define hybrid passive/finite gain systems.

**Definition 5.5.1.** Consider a square system $y(t) = (G u)(t)$, $G : L_{2e}(U) \to L_{2e}(Y)$, $u \in L_{2e}(U)$,
y \in L_2e(Y)$ with state-space form equivalent to Eq. (3.1). The system is dissipative with respect to a supply rate and has a corresponding storage function as in Eq. (5.2). If the bounded LTI operators $Q$, $S$, and $R$ satisfy

$$Q = -\left[ \varepsilon A^\top A + \gamma^{-1}B^\top B \right], \quad S = \frac{1}{2}A^\top A, \quad R = [-\delta A^\top A + \gamma B^\top B]$$  \hspace{1cm} (5.18)

for some nonnegative constants $\delta$, $\epsilon$, and $\gamma$ where the operators $A$ and $B$ satisfy Properties 5.3.1, 5.3.2, 5.3.3, and 5.3.4, then the system is said to be a hybrid passive/finite gain system satisfying the inequality

$$\langle Ay, Au \rangle_T - \delta \|Au\|^2_{2T} - \varepsilon \|Ay\|^2_{2T} + \gamma \|Bu\|^2_{2T} - \gamma^{-1} \|By\|^2_{2T} \geq -\phi(x(0)), \quad \forall u \in L_2e(U), \quad \forall T \in \mathbb{R}^+.$$  \hspace{1cm} (5.19)

A hybrid passive/finite gain system is one that has an input-output map that can be described as a passive map and as a finite gain map when passivity no longer holds. $\Omega_l$ is the low frequency regime where a passive input-output holds, and $\Omega_h$ is the high frequency regime where passivity has been violated (but the map has finite gain characteristics), and $\omega_c$ is the critical frequency that divides the two regimes.

Using Eq. (5.3) a hybrid passive/finite gain system can also be expressed by

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{y}_T^\top(-j\omega)Q(\omega)\mathbf{y}_T(j\omega)d\omega + \frac{1}{2\pi} \text{Re} \int_{-\infty}^{\infty} \mathbf{y}_T^\top(-j\omega)S(\omega)\mathbf{u}_T(j\omega)d\omega$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{u}_T^\top(-j\omega)R(\omega)\mathbf{u}_T(j\omega)d\omega \geq -\phi(x(0))$$ \hspace{1cm} (5.20)

where the Fourier transforms of the impulse responses of the operators $Q$, $S$, and $R$ presented in Eq. (5.18) are

$$Q(\omega) = -\left[ \varepsilon \alpha(\omega) + \gamma^{-1}(1 - \alpha(\omega)) \right] \mathbf{1}, \quad S(\omega) = \frac{1}{2} \alpha(\omega) \mathbf{1}, \quad R(\omega) = [-\delta \alpha(\omega) + \gamma(1 - \alpha(\omega))] \mathbf{1}. \hspace{1cm} (5.21)$$

In Eq. (5.21) we have used the relation $\beta(\omega) = 1 - \alpha(\omega)$ from Eq. (5.16).

The frequency variable $\alpha : \mathbb{R} \to \{0, 1\}$ is a theoretical abstraction used to distinguish between passive system characteristics and nonpassive but still finite gain system characteristics. When the system in question possesses a passive input-output map, $\alpha(\omega) = 1$. When the system fails to possess a passive input-output map, that is the system has experienced a passivity violation, but the map has finite gain, $\alpha(\omega) = 0$. The divide occurs at a critical frequency, $\omega_c$, that is used to define $\alpha$.

The constants $\delta$ and $\epsilon$ depend on the passive nature of the system when passivity holds, and $\gamma$ depends on the finite gain nature of the system when passivity is violated. Notice in Eq. (5.21) that units of $Q(\cdot)$ and $R(\cdot)$ are consistent: $\epsilon$ and $\gamma^{-1}$ have units of one over gain, while $\delta$ and $\gamma$ have units of gain.
When $\alpha(\omega) = 1$ Eq. (5.20) becomes
\[
\frac{1}{2\pi} \Re \int_{-\omega_c}^{\omega_c} y^T_T(-j\omega)u_T(j\omega)d\omega \geq \frac{\delta}{2\pi} \int_{-\omega_c}^{\omega_c} u^T_T(-j\omega)u_T(j\omega)d\omega + \frac{\gamma}{2\pi} \int_{-\omega_c}^{\omega_c} y^T_T(-j\omega)y_T(j\omega)d\omega \tag{5.22}
\]
and the input-output map is said to be

- passive if $\delta = \epsilon = 0$,
- VSP, or input strictly passive with finite gain if $\delta > 0$ and $\epsilon > 0$,
- input strictly passive (ISP) if $\delta > 0$ and $\epsilon = 0$, and
- output strictly passive (OSP) if $\delta = 0$ and $\epsilon > 0$.

If a passive input-output map no longer exists, that is passivity has been violated, $\alpha(\omega) = 0$ and Eq. (5.20) becomes
\[
\frac{1}{\gamma 2\pi} \int_{-\infty}^{-\omega_c} y^T_T(-j\omega)y_T(j\omega)d\omega + \frac{1}{\gamma 2\pi} \int_{\omega_c}^{\infty} y^T_T(-j\omega)y_T(j\omega)d\omega \leq \frac{\gamma}{2\pi} \int_{-\infty}^{-\omega_c} u^T_T(-j\omega)u_T(j\omega)d\omega + \frac{\gamma}{2\pi} \int_{\omega_c}^{\infty} u^T_T(-j\omega)u_T(j\omega)d\omega
\]
which simplifies to
\[
\frac{1}{\gamma \pi} \int_{\omega_c}^{\infty} y^T_T(-j\omega)y_T(j\omega)d\omega \leq \frac{\gamma}{\pi} \int_{\omega_c}^{\infty} u^T_T(-j\omega)u_T(j\omega)d\omega. \tag{5.23}
\]
The input-output map is said to be a finite gain input-output map.

It is worth noting that the input-output signals of the system in question, $u$ and $y$, are Fourier transformed, but not the plant operator $G : L_{2e} \rightarrow L_{2e}$. The hybrid passivity/finite gain framework holds for both linear and nonlinear systems, although it is perhaps more intuitive to envision the hybrid character of a plant in terms of a frequency response which is a linear concept. Additionally, notice that if a passivity violation does not occur, that is the system is passive over all frequencies, $\omega_c \rightarrow \infty$ and from Eq. (5.22) the traditional passivity inequality is recovered (see Eq. (3.7), page 22). Similarly, if a system is not passive but does have finite gain, $\omega_c = 0$ and from Eq. (5.23) the finite gain inequality is recovered (see Eq. (3.6), page 20).

### 5.5.1 Approximating $\alpha$ and $\beta$.

The parameter $\alpha$ divides system mapping characteristics into two regions; when a system possesses a passive input-output map (regardless of gain characteristics) and when a system does not possess a passive input-output map but possesses finite gain. In reality, $\alpha$ is simply a mathematical abstraction but it is one that helps us distill and understand various relations between the system(s) in question. It will be advantageous to try and understand exactly what $\alpha$ represents, and if an approximation of $\alpha$ provides us with additional insight.

Previously in Eq. (5.17) we defined $\alpha$ to be one or zero as a function of frequency. This is essentially how an ideal low-pass filter is defined, as presented in Fig. 5.4. In reality an ideal
low-pass filter can only be approximated, yet the hybrid system inequality (which is a dissipative inequality) presented in Eq. (5.19) is defined using an exact low-pass filter (and exact high-pass filter). However, as we will show, with the right approximation we can get arbitrarily close to the exact inequality representing a hybrid system.

Figure 5.4: Frequency response of an ideal low-pass filter.

Consider the construction of a $N^{th}$ order low-pass Butterworth filter, which is an approximation of an ideal low-pass filter:

$$\alpha_N(\omega) = A_N(-j\omega)A_N(j\omega) = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}$$

where

$$\lim_{N \to \infty} \alpha_N(\omega) = \alpha(\omega)$$

For example, the magnitude responses of $N = 10$ and $N = 30$ low-pass Butterworth filters are shown in Figs. 5.5(a) and 5.5(b).

Knowing that $\alpha_N(\omega) = A_N(-j\omega)A_N(j\omega)$, $A_N(-s)A_N(s)$ is

$$A_N(-s)A_N(s) = \frac{1}{1 + \left(\frac{-s^2}{\omega_c^2}\right)^N}.$$  

It follows that the causal transfer function representing a $N^{th}$ order low-pass Butterworth filter can be represented by

$$A_N(s) = \frac{1}{\prod_{k=1}^{N} \frac{(s-s_k)}{\omega_c}}$$

where $s_k = \omega_c e^{j(2k+N-1)\pi} / \omega_c$, $|s_k/\omega_c| = 1$, Re $\{s_k\} < 0$ and $k = 1, \cdots, N$. Note that $A_N(s) \in \mathcal{H}_\infty$, and similarly $A_N(-s)$ is analytic in Re $\{s\} < 0$ and therefore $A_N(-s) \in \mathcal{H}_\infty$. The operator $A_N$ represents the action of a $N^{th}$ order low-pass Butterworth filter in the time domain.
In a similar fashion, we can approximate \( \beta \) as \( \beta_N(\omega) = B_N(-\omega c j) B_N(\omega c j) \). In order to construct \( B_N(s) \), let \( B_N\left(\frac{s}{\omega_c}\right) = A_N\left(\frac{\omega_c}{s}\right) \). We can now write

\[
B_N\left(\frac{\omega}{\omega_c}\right) B_N\left(\frac{\omega}{\omega_c}\right) = A_N\left(\frac{-\omega_c}{s}\right) A_N\left(\frac{\omega_c}{s}\right).
\]

Letting \( s = j\omega \) we have

\[
B_N\left(-\frac{j\omega}{\omega_c}\right) B_N\left(\frac{j\omega}{\omega_c}\right) = A_N\left(-\frac{\omega_c}{j\omega}\right) A_N\left(\frac{\omega_c}{j\omega}\right)
= A_N\left(\frac{j\omega_c}{\omega}\right) A_N\left(-\frac{j\omega_c}{\omega}\right)
= \frac{1}{1 + (\frac{\omega_c}{\omega})^{2N}}
= \frac{\omega^{2N}}{\omega^{2N} + \omega_c^{2N}}.
\]

Therefore,

\[
B_N(-s) B_N(s) = \frac{1}{1 + \left(\frac{\omega_c}{\omega}\right)^{2N}}.
\]

Having defined both \( A_N(-s) A_N(s) \) and \( B_N(-s) B_N(s) \), we are able to show that the approximate form of the equality presented in Eq. (5.16) also holds:

\[
A_N(-s) A_N(s) + B_N(-s) B_N(s) = 1 \iff A_N(-j\omega) A_N(j\omega) + B_N(-j\omega) B_N(j\omega) = 1.
\]

It follows that \( A_N^\sim A_N + B_N^\sim B_N = 1, \ A_N^\sim A_N + B_N^\sim B_N = 1 \) where \( A_N = A_N 1 \) and \( B_N = B_N 1 \). The operator \( B_N \) is the time domain equivalent of a \( N \)th order high-pass Butterworth filter.
Based on the construction of $A_N$ and $B_N$ we can approximate Eq. (5.19) as

$$
\langle A_N y_T, A_N u_T \rangle - \delta \langle A_N u_T, A_N u_T \rangle - \epsilon \langle A_N y_T, A_N y_T \rangle
+ \gamma \langle B_N u_T, B_N u_T \rangle - \gamma^{-1} \langle B_N y_T, B_N y_T \rangle \geq -\phi(x(0))
$$

Any inner products containing the operator $A_N$ are associated with the passive nature of the system, and inner products containing the operator $B_N$ are associated with the finite gain nature of the system when passivity is violated. The LTI operators $A_N$ and $B_N$ filter the signals within the inner products. The inner products containing $A_N$ have their signals $u_T$ and $y_T$ filtered by a LTI low-pass filter, and the inner products containing $B_N$ have their signals filtered by a LTI high-pass filter. It is clear that as $N \to \infty$ we can exactly represent the hybrid system inequality presented in Eq. (5.19).

### 5.5.2 Linear Time-Invariant Hybrid Passive/Finite Gain Systems

Given a LTI system with transfer matrix $G(s) = C(sI - A)^{-1}B + D$ where $(A, B, C, D)$ is a minimal state-space realization, we would like to know if the system is a hybrid passive/finite gain system. The hybrid passive/finite gain nature of the system depends on the $\delta$, $\epsilon$, and $\gamma$ values (as well as $\omega_c$), which can be determined from the systems frequency response, or via the GKYP Lemma. As was the case for hybrid conic systems, we are not aware of a way to calculate the $\delta$, $\epsilon$, and $\gamma$ values for nonlinear hybrid systems nor LTV hybrid systems.

#### Hybrid Passive/Finite Gain Systems in the Frequency Domain

We will consider calculation of $\delta$, $\epsilon$, and $\gamma$ in the frequency domain. Hybrid ISP system properties will be considered first. To see how the ISP parameter $\delta$ can be computed, consider the following:

$$
\frac{1}{2\pi} \Re \int_{-\omega_c}^{\omega_c} y_T^T(-j\omega)u_T(j\omega) d\omega = \frac{1}{4\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega) \left[ G^T(-j\omega) + G(j\omega) \right] u_T(j\omega) d\omega
\geq \frac{1}{2} \inf_{-\omega_c < \omega < \omega_c} \lambda \left\{ G^T(-j\omega) + G(j\omega) \right\} \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega)u_T(j\omega) d\omega.
$$

(5.24)

Therefore, $\delta = \frac{1}{2} \inf_{-\omega_c < \omega < \omega_c} \lambda \left\{ G^T(-j\omega) + G(j\omega) \right\}$ where $\lambda \{ \cdot \}$ is the minimum eigenvalue.

Next hybrid VSP system characteristics will be considered, and in particular the calculation of the OSP parameter $\epsilon$. Recall that a VSP system is also an ISP system that possess finite gain. The so-called passive system gain can be calculated as follows:

$$
\frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} y_T^T(-j\omega)y_T(j\omega) d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega)G^T(-j\omega)G(j\omega)u_T(j\omega) d\omega
\leq \sup_{-\omega_c < \omega < \omega_c} \sigma^2 \left\{ G(j\omega) \right\} \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega)u_T(j\omega) d\omega.
$$

(5.25)
where $\bar{\sigma}\{\cdot\}$ is the maximum singular value. Therefore, $\kappa = \sup_{-\omega_c < \omega < \omega_c} \bar{\sigma}\{G(j\omega)\}$ and is referred to as the passive system gain. To show that a hybrid ISP, finite gain system is hybrid VSP, start with the hybrid ISP inequality presented in Eq. (5.24), let $\bar{\delta} = \delta/2$, and substitute Eq. (5.25) as follows:

$$\frac{1}{2\pi} \Re \int_{-\omega_c}^{\omega_c} y_T^T(-j\omega)u_T(j\omega) d\omega \geq \frac{\bar{\delta}}{2\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega)u_T(j\omega) d\omega + \frac{\bar{\delta}}{2\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega)u_T(j\omega) d\omega \geq \frac{\bar{\delta}}{2\pi} \int_{-\omega_c}^{\omega_c} u_T^T(-j\omega)u_T(j\omega) d\omega + \frac{\bar{\delta}}{2\pi} \int_{-\omega_c}^{\omega_c} y_T^T(-j\omega)y_T(j\omega) d\omega.$$  

(5.26)

Hence, a hybrid ISP, finite gain system is clearly hybrid VSP where the OSP parameter is $\bar{\delta} \kappa^2$. Therefore, stating a hybrid system has $\delta > 0$ and $0 < \kappa < \infty$ implies $\delta > 0$ and $\epsilon > 0$.

Finally, the finite gain parameter associated with an input-output map that is no longer passive can be calculated as follows:

$$\frac{1}{\pi} \int_{\omega_c}^{\infty} y_T^T(-j\omega)y_T(j\omega) d\omega \leq \sup_{\omega \geq \omega_c} \bar{\sigma}^2\{G(j\omega)\} \frac{1}{\pi} \int_{\omega_c}^{\infty} u_T^T(-j\omega)u_T(j\omega) d\omega.$$  

Thus, $\gamma = \sup_{\omega \geq \omega_c} \bar{\sigma}\{G(j\omega)\}$.

**State-Space Characterization of Hybrid Passive/Finite Gain Systems**

The GKYP Lemma can be used to determine if a system has hybrid passive/finite gain properties or hybrid VSP/finite gain properties, as discussed next. In preparation for our discussion, we will define

$$\Pi_p := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and

$$\Omega'_l := \Omega_l \setminus \{\omega \in \mathbb{R} : \det(j\omega I - A) = 0\}.$$  

We will begin our discussion with LTI systems that are passive within a low frequency bandwidth, i.e., those that satisfy Eq. (5.22) on page 78 with $\delta = 0$ and $\epsilon = 0$. A LTI system that is passive $\forall \omega \in \Omega_l$ is PR $\forall \omega \in \Omega_l$, also called finite frequency PR (FF PR). A transfer matrix $G(s) \in \mathbb{C}^{n \times n}$ is PR $\forall \omega \in \Omega_l$ if

$$\begin{bmatrix} G(s) \\ 1 \end{bmatrix}^H \Pi_p \begin{bmatrix} G(s) \\ 1 \end{bmatrix} \leq 0, \quad \forall \omega \in \Omega'_l.$$  

(5.27)

This condition can also be written in terms of a LMI using the GKYP lemma.

**Lemma 5.5.1.** The system $G(s) = C(s I - A)^{-1}B + D$ is PR $\forall \omega \in \Omega'_l$ if $\exists P, Q \in \mathbb{R}^{n \times n}$ where
\( P = P^T \) and \( Q = Q^T \geq 0 \) such that

\[
\begin{bmatrix}
A & B \\
1 & 0
\end{bmatrix}^T
\begin{bmatrix}
-Q & P \\
0 & (\omega_c - \bar{\omega}_c)^2 Q
\end{bmatrix}
\begin{bmatrix}
A & B \\
1 & 0
\end{bmatrix}
+ \begin{bmatrix}
C & D \\
0 & 1
\end{bmatrix}^T \Pi_p \begin{bmatrix}
C & D \\
0 & 1
\end{bmatrix} \leq 0. \tag{5.28}
\]

where \( \bar{\omega}_c \) is a trivially small number that effectively transforms \(|\omega| \leq (\omega_c - \bar{\omega}_c)\) into the strict inequality \(|\omega| < \omega_c\).

If Lemma 5.5.1 holds, then \( \delta \) and \( \epsilon \) associated with Eqs. (5.18) and (5.19) are both zero. It is worth noting that the FF PR definitions given in Eqs. (5.27) and (5.28) permit poles on the imaginary axis, including the origin.\(^{65} \) Additionally, if \( \omega_c \to \infty \) then \( P = P^T > 0, \ Q = 0, \) and the traditional PR Lemma (Lemma 3.4.1, page 24) is recovered.

Next we will consider systems that are ISP and have finite gain within a low frequency bandwidth, that is, systems that are VSP in a low frequency bandwidth. Such a system must satisfy Eq. (5.22), that is there must exist \( \delta > 0 \) and \( \epsilon > 0 \). In particular, if there exists \( \delta > 0 \) and \( 0 < \kappa < \infty \) where \( \kappa \) satisfies Eq. (5.25), then Eq. (5.22) will be satisfied. In terms of LTI systems, a LTI system that is ISP with finite gain \( \forall \omega \in \Omega_l \) (i.e., VSP \( \forall \omega \in \Omega_l \)) is SPR \( \forall \omega \in \Omega_l \), or called FF SPR. A transfer matrix \( G(s) \in \mathbb{C}^{n \times n} \) is SPR \( \forall \omega \in \Omega_l \) if all the poles of \( G(s) \) are in the open left half plane and\(^{67} \)

\[
\begin{bmatrix}
G(s) \\
1
\end{bmatrix}^H \Pi_p \begin{bmatrix}
G(s) \\
1
\end{bmatrix} < 0, \ \forall \omega \in \Omega_l.
\]

This strict inequality can be written as a LMI using the GKYP lemma.\(^{64,68,71} \)

**Lemma 5.5.2.** The system \( G(s) = C(sI - A)^{-1}B + D \) is SPR \( \forall \omega \in \Omega_l \) if \( \exists P, Q \in \mathbb{R}^{n \times n} \) where \( P = P^T \) and \( Q = Q^T > 0 \) such that

\[
\begin{bmatrix}
A & B \\
1 & 0
\end{bmatrix}^T
\begin{bmatrix}
-Q & P \\
0 & (\omega_c - \bar{\omega}_c)^2 Q
\end{bmatrix}
\begin{bmatrix}
A & B \\
1 & 0
\end{bmatrix}
+ \begin{bmatrix}
C & D \\
0 & 1
\end{bmatrix}^T \Pi_p \begin{bmatrix}
C & D \\
0 & 1
\end{bmatrix} < 0. \tag{5.29}
\]

If Lemma 5.5.2 holds, then \( \delta \) and \( \epsilon \) associated with Eqs. (5.18) and (5.19) are both strictly greater than zero.

Let us now move on to discussing the properties of hybrid systems at high frequency. A system that has finite gain above \( \omega_c \) is BR above \( \omega_c \). Such a system satisfies Eq. (5.23) with \( 0 < \gamma < \infty \). A transfer matrix \( G(s) \in \mathbb{C}^{n \times n} \) is BR \( \forall \omega \in \Omega_h \) with gain \( \gamma \) if all the poles of \( G(s) \) are in the open left half plane and\(^{66} \)

\[
\begin{bmatrix}
G(s) \\
1
\end{bmatrix}^H \Pi_b \begin{bmatrix}
G(s) \\
1
\end{bmatrix} \leq 0, \ \forall \omega \in \Omega_h \text{ where } \Pi_b := \begin{bmatrix}
1 & 0 \\
0 & -\gamma^2 1
\end{bmatrix}.
\]

This inequality can be written as a LMI using the GKYP lemma.
Lemma 5.5.3. The system $G(s) = C(sI - A)^{-1}B + D$ is BR ∀ω ∈ Ωb with gain γ if ∃P, Q ∈ ℝⁿ×ⁿ where $P = P^T$ and $Q = Q^T ≥ 0$ such that

$$
\begin{bmatrix}
A & B \\
1 & 0
\end{bmatrix}^T
\begin{bmatrix}
P & Q \\
-\omega_c^2Q & 1
\end{bmatrix}
\begin{bmatrix}
A & B \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
C & D \\
0 & 1
\end{bmatrix}
\Pi_b
\begin{bmatrix}
C & D \\
0 & 1
\end{bmatrix} ≤ 0.
$$

(5.30)

If Lemma 5.5.3 holds, γ associated with Eqs. (5.18) and (5.19) is finite and strictly positive.

Of interest to us are systems that are hybrid, possessing passive or VSP properties below $\omega_c$, and finite gain properties above $\omega_c$ at high frequency. In particular, a LTI system $G(s)$ that is passive below $\omega_c$ and finite gain above $\omega_c$ is termed a hybrid PR/BR system, or a finite frequency PR/BR (FF PR/BR) system. FF PR/BR systems generally describe plants that have experienced a passivity violation. A LTI system $G(s)$ that is ISP and has finite gain below $\omega_c$ (that is, is VSP below $\omega_c$), and has a finite gain mapping above $\omega_c$ is termed hybrid SPR/BR or FF SPR/BR. FF SPR/BR systems will be used as controllers to control FF PR/BR systems where stability will be guaranteed via the Hybrid Passivity/Finite Gain Stability Theorem, which is a special case of the Hybrid Conic Sector Stability Theorem discussed in Sec. 5.6. Note that although a FF SPR/BR system has finite gain over all frequencies, the particular gain at low frequency is different then the particular gain at high frequency.

### 5.6 Closed-Loop Stability of Hybrid Input-Output Systems

Having defined hybrid conic and passive/finite gain systems, we now turn our attention to the stability of a hybrid plant connected in (negative) feedback with a hybrid controller. Each system $G_j : L_{2e} → L_{2e}$, $j = 1, 2$, is a hybrid system, either a hybrid conic or a hybrid passive/finite gain system. If the both the plant and control are hybrid conic systems then, from Eqs. (5.7) (page 69) and (5.9) (page 70), they have associated with them the operators

$$
Q_j = \sum_{i=1}^{M}(−\frac{1}{b_{ji}})A_i^{-}A_i, \quad S_j = \frac{1}{2}\sum_{i=1}^{M}(1 + \frac{a_i}{b_{ji}})A_i^{-}A_i, \quad R_j = \sum_{i=1}^{M}(−a_{ji})A_i^{-}A_i
$$

or, equivalently

$$
Q_j(ω) = \left(\sum_{i=1}^{M}(−\frac{1}{b_{ji}})\alpha_i(ω)\right)1, \quad S_j(ω) = \frac{1}{2}\left(\sum_{i=1}^{M}(1 + \frac{a_i}{b_{ji}})\alpha_i(ω)\right)1, \quad R_j(ω) = \left(\sum_{i=1}^{M}(−a_{ji})\alpha_i(ω)\right)1
$$

where $a_{ji} ∈ ℝ$, $b_{ji} ∈ ℝ^+ \cup \{∞\} \setminus \{0\}$, $b_{ji} > a_{ji}$. If the two systems are specifically hybrid passive/finite gain systems then, from Eqs. (5.18) (page 77) and (5.21) (page 77), they have associated with them the operators

$$
Q_j = −[ε_jA^{-}A + \gamma_j^{-1}B^{-}B], \quad S_j = \frac{1}{2}A^{-}A, \quad R_j = [−δ_jA^{-}A + \gamma_jB^{-}B]
$$
or, equivalently

\[
\begin{align*}
Q_j(\omega) &= -\left[ \epsilon_j \alpha(\omega) + \gamma_j^{-1}(1 - \alpha(\omega)) \right] 1, \\
S_j(\omega) &= \frac{1}{2} \alpha(\omega)1, \\
R_j(\omega) &= [-\delta_j \alpha(\omega) + \gamma_j(1 - \alpha(\omega))] 1
\end{align*}
\]

where \(\delta_j, \epsilon_j,\) and \(\gamma_j\) are nonnegative constants.

Before presenting our stability proof, let us define the following inner product:

\[
\langle \mathbf{v}_T, \mathbf{w}_T \rangle = \int_0^\infty \mathbf{v}_T^T(t)\mathbf{w}_T(t)dt = \frac{1}{2\pi} \text{Re} \left\{ \int_{-\infty}^\infty \mathbf{v}_T^T(j\omega)\mathbf{w}_T(j\omega)d\omega \right\} =: \langle \mathbf{v}_T, \mathbf{w}_T \rangle_{\omega}
\]

where \(\mathbf{v} \in L_{2e}\) and \(\mathbf{w} \in L_{2e}\) are arbitrary functions.

**Theorem 5.6.1.** Consider the negative feedback interconnection of \(\mathcal{G}_1 : L_{2e} \rightarrow L_{2e}\) and \(\mathcal{G}_2 : L_{2e} \rightarrow L_{2e}\) presented in Fig. 3.1 on page 18. The admissible inputs \(e_1(t) = u_1(t) - y_2(t)\) and \(e_2(t) = u_2(t) + y_1(t)\) are in \(L_{2e}\). Each system can be described the state-space equations of Eq. (3.1). Additionally, each system is \((Q_j, S_j, R_j)\)-dissipative with respect to a supply rate \(w_j\) and has a corresponding storage function \(\phi_j\), according to Eq. (3.15) (where \(j = 1\) corresponds to \(\mathcal{G}_1\) and \(j = 2\) corresponds to \(\mathcal{G}_2\)). The negative feedback interconnection is \(L_2\) stable if Eq. (5.7) (or Eq. (5.18)) describes the operators \(Q_j, S_j,\) and \(R_j\) and

\[
\left\langle \begin{bmatrix} y_{1,T} \\ y_{2,T} \end{bmatrix}, \begin{bmatrix} -Q_1 - R_2 & S_1 - S_2^\gamma \\ S_1^\gamma - S_2 & -(Q_2 + R_1) \end{bmatrix}, \begin{bmatrix} y_{1,T} \\ y_{2,T} \end{bmatrix} \right\rangle > 0,
\]

that is, \(u_1, u_2 \in L_2 \Rightarrow y_1, y_2 \in L_2\).

**Proof.** Consider

\[
\langle y_{1,T}, Q_1 y_{1,T} \rangle + 2 \langle y_{1,T}, S_1 e_{1,T} \rangle + \langle e_{1,T}, R_1 e_{1,T} \rangle
\]
\[
+ \langle y_{2,T}, Q_2 y_{2,T} \rangle + 2 \langle y_{2,T}, S_2 e_{2,T} \rangle + \langle e_{2,T}, R_2 e_{2,T} \rangle \geq -\phi(x_1(0)) - \phi(x_2(0)).
\]

Using the relationships \(e_1(t) = u_1(t) - y_2(t)\) and \(e_2(t) = u_2(t) + y_1(t)\) we can write

\[
\begin{align*}
- \left\langle \begin{bmatrix} y_{1,T} \\ y_{2,T} \end{bmatrix}, \begin{bmatrix} -Q_1 - R_2 & S_1 - S_2^\gamma \\ S_1^\gamma - S_2 & -(Q_2 + R_1) \end{bmatrix}, \begin{bmatrix} y_{1,T} \\ y_{2,T} \end{bmatrix} \right\rangle \\
+ 2 \left\langle \begin{bmatrix} y_{1,T} \\ y_{2,T} \end{bmatrix}, \begin{bmatrix} S_1 & \frac{1}{2}(R_2^\gamma + R_2) \\ -\frac{1}{2}(R_2^\gamma + R_1) & S_2 \end{bmatrix}, \begin{bmatrix} u_{1,T} \\ u_{2,T} \end{bmatrix} \right\rangle \\
+ \left\langle \begin{bmatrix} u_{1,T} \\ u_{2,T} \end{bmatrix}, \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \begin{bmatrix} u_{1,T} \\ u_{2,T} \end{bmatrix} \right\rangle \geq \beta
\end{align*}
\]

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which, when recalling that the $\mathcal{R}_i$ operators are self-adjoint, can be simplified to

$$\langle u_T, \mathcal{R}u_T \rangle \geq \langle y_T, Qy_T \rangle - 2 \langle y_T, Su_T \rangle + \beta$$

where

$$S = \begin{bmatrix} S_1 & R_2 \\ -R_1 & S_2 \end{bmatrix}.$$ 

By using Parseval’s Theorem we can write

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} u_T(j\omega)R(\omega)u_T(j\omega)d\omega \geq \frac{1}{2\pi} \int_{-\infty}^{\infty} y_T(j\omega)Q(\omega)y_T(j\omega)d\omega - \frac{1}{2\pi} \text{Re} \left\{ \int_{-\infty}^{\infty} y_T(j\omega)S(\omega)u_T(j\omega)d\omega \right\} + \beta$$

which equivalently can be written

$$\langle u_T, Ru_T \rangle \geq \langle y_T, Qy_T \rangle - 2 \langle y_T, Su_T \rangle + \beta.$$  \hspace{1cm} (5.31)

Before continuing, let us define and discuss a few matrices that will be used in our proof. To start, notice that the matrix $R(\cdot)$ is diagonal and is composed of $R_1(\cdot)$ and $R_2(\cdot)$:

$$R(\omega) = \begin{bmatrix} R_1(\omega) & 0 \\ 0 & R_2(\omega) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{M} (-a_{11})\alpha_i(\omega) & 0 \\ 0 & \sum_{i=1}^{M} (-a_{22})\alpha_i(\omega) \end{bmatrix}.$$ 

By taking the absolute value of each element within $R(\cdot)$ we can define

$$\bar{R}(\omega) = \begin{bmatrix} \sum_{i=1}^{M} |a_{11}|\alpha_i(\omega) & 0 \\ 0 & \sum_{i=1}^{M} |a_{22}|\alpha_i(\omega) \end{bmatrix}$$

which is positive semidefinite. ($\bar{R}(\cdot)$ is positive semidefinite because each $a_{ji}$ term could be zero.)

Now, turning our attention to $S(\cdot)$, note that

$$S(\omega) = \begin{bmatrix} S_1(\omega) & R_2(\omega) \\ -R_1(\omega) & S_2(\omega) \end{bmatrix} = \begin{bmatrix} 1/2 \sum_{i=1}^{M} (1 + \frac{a_{11}}{b_{ii}})\alpha_i(\omega) & 0 \\ 0 & \sum_{i=1}^{M} (-a_{22})\alpha_i(\omega) \end{bmatrix}.$$ 

which always has full rank. Even when $a_{11} = -b_{11} = -\gamma_1$ and $a_{22} = -b_{22} = -\gamma_2$, where $0 < \gamma_1 < \infty$ and $0 < \gamma_2 < \infty$ for $i = 1 \cdots M$ the matrix $S(\cdot)$ will not be rank deficient owing to the fact the columns of $S(\cdot)$ will be linearly independent.

Next, assuming $Q(\cdot) > 0$ (i.e., $\langle y_T, Qy_T \rangle > 0$ and hence $\langle y_T, Qy_T \rangle_{\omega} > 0$) the inverse of $Q(\cdot)$, $Q^{-1}(\cdot)$, exits, as do $Q^{1/2}(\omega)$ and $Q^{-1/2}(\cdot)$. Because $Q(\cdot) > 0$, we have that $Q^{-1}(\cdot) > 0$, $Q^{1/2}(\cdot) > 0$, and $Q^{-1/2}(\cdot) > 0$. Using $Q^{-1/2}(\cdot)$ we can define

$$\bar{S}(\omega) = Q^{-1/2}(\omega)S(\omega).$$
Given that $S(\cdot)$ has full column rank and our definition of $\bar{S}(\cdot)$ we know that
\[
\bar{S}^H(\omega)\bar{S}(\omega) = S^H(\omega)Q^{-1}(\omega)S(\omega) > 0
\]
because $Q^{-1}(\omega) > 0$, which we have assumed.

We will now continue simplifying Eq. (5.31); using $Q(\omega)^{\frac{1}{2}}Q(\omega)^{-\frac{1}{2}} = 1$, $Q(\omega) = Q(\omega)^{\frac{1}{2}}Q(\omega)^{\frac{1}{2}}$ and $\bar{R}(\cdot)$, Eq. (5.31) can be written
\[
\langle u_T, \bar{R}u_T \rangle_\omega \geq \langle u_T, S^{\bar{H}}Su_T \rangle_\omega 
\]
We will add $\langle u_T, S^{\bar{H}}Su_T \rangle_\omega$ to both sides of the above equation and simplify:
\[
\langle u_T, \left( \bar{R} + S^{\bar{H}} \right)u_T \rangle_\omega 
\geq \langle u_T, Q^\frac{1}{2}Q^\frac{1}{2}y_T \rangle_\omega - 2 \langle y_T, Q^\frac{1}{2}Q^{-\frac{1}{2}}Su_T \rangle_\omega + \langle u_T, S^{\bar{H}}Su_T \rangle_\omega + \beta
\]
\[
= \langle Q^\frac{1}{2}y_T, Q^\frac{1}{2}y_T \rangle_\omega - 2 \langle \bar{R}y_T, Q^{-\frac{1}{2}}Su_T \rangle_\omega + \langle u_T, S^{\bar{H}}Su_T \rangle_\omega + \beta
\]
\[
= \langle Q^\frac{1}{2}y_T - \bar{S}u_T, Q^\frac{1}{2}y_T - \bar{S}u_T \rangle_\omega + \beta
\]
The structure of $\bar{R}(\cdot) + S^{\bar{H}}(\cdot)S(\cdot)$ is one that is positive definite provided $Q(\cdot)$ is positive definite. Therefore, we can write
\[
\bar{\zeta} \langle u_T, u_T \rangle_\omega \geq \langle u_T, \left( \bar{R} + S^{\bar{H}} \right)u_T \rangle_\omega
\]
where
\[
\bar{\zeta} = \sup_{\omega \in \mathbb{R}} \bar{\lambda} \left\{ \bar{R}(\omega) + S^{\bar{H}}(\omega)S(\omega) \right\}
\]
and $\bar{\lambda} \{ \cdot \}$ is the largest eigenvalue. We now can write
\[
\bar{\zeta} \langle u_T, u_T \rangle_\omega \geq \langle Q^\frac{1}{2}y_T - \bar{S}u_T, Q^\frac{1}{2}y_T - \bar{S}u_T \rangle_\omega + \beta
\]
which, by transforming back into the time domain, is equivalent to
\[
\bar{\zeta} \langle u_T, u_T \rangle_\omega - \beta \geq \left\langle Q^\frac{1}{2}y_T - \bar{S}u_T, Q^\frac{1}{2}y_T - \bar{S}u_T \right\rangle
\]
\[
= \left\| Q^\frac{1}{2}y_T - \bar{S}u_T \right\|^2_2.
\]
Noting that $-\beta \geq 0$, by taking the square root of both sides of the above equation and subsequently simplifying we have
\[
\sqrt{\bar{\zeta}} \left\| u_T \right\|_2 + \sqrt{-\beta} \geq \left\| Q^\frac{1}{2}y_T - \bar{S}u_T \right\|_2.
\]
(5.32)
By using the reverse triangle inequality (see Sec. 2.1.3, page 12) the right hand side of Eq. (5.32)
can be written as
\[
\sqrt{\zeta} \| u_T \|_2 + \sqrt{-\beta} \geq \| Q^2 y_T \|_2 - \| S u_T \|_2,
\]
\[
\sqrt{\zeta} \| u_T \|_2 + \| S u_T \|_2 + \sqrt{-\beta} \geq \| Q^2 y_T \|_2.
\] (5.33)

Noting that
\[
\| Q^{-\frac{1}{2}} \|_\infty \| Q^2 y_T \|_2 \geq \| Q^{-\frac{1}{2}} Q^2 y_T \|_2 = \| y_T \|_2,
\]
\[
\| Q^2 y_T \|_2 \geq \| Q^{-\frac{1}{2}} \|_\infty^{-1} \| y_T \|_2
\]
and
\[
\| \bar{S} \|_\infty \| u_T \|_2 \geq \| \bar{S} u_T \|_2
\]

it follows that Eq. (5.33) can be written
\[
\left( \sqrt{\zeta} + \| \bar{S} \|_\infty \right) \| u_T \|_2 + \sqrt{-\beta} \geq \| Q^{-\frac{1}{2}} \|_\infty^{-1} \| y_T \|_2,
\]
\[
\| Q^{-\frac{1}{2}} \|_\infty \left( \sqrt{\zeta} + \| \bar{S} \|_\infty \right) \| u_T \|_2 + \| Q^{-\frac{1}{2}} \|_\infty \sqrt{-\beta} \geq \| y_T \|_2,
\]
\[
\gamma \| u_T \|_2 + \beta' \geq \| y_T \|_2.
\]

By letting \( T \to \infty \) proves that \( y \in L_2 \) if \( u \in L_2 \), that is \( u_1, u_2 \in L_2 \Rightarrow y_1, y_2 \in L_2 \).

Notice that in our proof we have allowed the systems \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) to have nonzero initial conditions. Should the initial conditions be quiescent, \( \beta = 0 \).

5.6.1 The Structure and Positive Definite Nature of \( Q \)

Theorem 5.6.1 states that the negative feedback interconnection of two hybrid conic systems or two hybrid passive/finite gain systems is \( L_2 \) stable provided

\[
Q = \begin{bmatrix}
-(Q_1 + R_2) & S_1 - S_2^2 \\
S_1^2 - S_2 & -(Q_2 + R_1)
\end{bmatrix}
\]
is positive definite (i.e., \( \langle y_T, Q y_T \rangle > 0 \)), or equivalently provided

\[
Q(\omega) = \begin{bmatrix}
-(Q_1(\omega) + R_2(\omega)) & S_1(\omega) - S_2^2(\omega) \\
S_1^2(\omega) - S_2(\omega) & -(Q_2(\omega) + R_1(\omega))
\end{bmatrix}
\] (5.34)
is positive definite (i.e., $\langle y_T, Qy_T \rangle_\omega > 0$). Let us evaluate in detail the structure and positive definite nature of $Q(\cdot)$ (equivalently, $Q$).

Positive Definite Nature of $Q$ when the Systems are Hybrid Conic Systems

When both $\mathcal{G}_1$ and $\mathcal{G}_2$ are hybrid conic systems, the matrices $Q_j(\cdot)$, $S_j(\cdot)$, and $R_j(\cdot)$ (where $j = 1, 2$) have the following form (see Eq. (5.9), page 70):

$$Q_j(\omega) = \left( \sum_{i=1}^{M} (-\frac{1}{b_{ji}}) \alpha_i(\omega) \right) 1, \quad S_j(\omega) = \frac{1}{2} \left( \sum_{i=1}^{M} (1 + \frac{a_{ji}}{b_{ji}}) \alpha_i(\omega) \right) 1, \quad R_j(\omega) = \left( \sum_{i=1}^{M} (-a_{ji}) \alpha_i(\omega) \right) 1$$

(5.35)

where $a_{ji} \in \mathbb{R}$, $b_{ji} \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\}$, $b_{ji} > a_{ji}$, $j = 1, 2$, $i = 1 \cdots M$. Recall from Property 5.3.1 (specifically, the equality in Eq. (5.5) on page 67) and Property 5.3.2 that $\sum_{i=1}^{M} \alpha_i(\omega) = 1$ and $\alpha_i : \mathbb{R} \to \{0, 1\}$. We can think of each $\alpha_i$ as being “on” or “off”, thus turning “on” or “off” specific input-output properties of the systems $\mathcal{G}_1$ and $\mathcal{G}_2$. What $Q_j(\cdot)$, $S_j(\cdot)$, and $R_j(\cdot)$ define are a series of cones representing the system’s input-output map of $G_j$ over a set of regions. Over the first region where $\alpha_1(\omega) = 1$, $\forall \omega \in \Omega_1$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined by a set of conic bounds $[a_{11}, b_{11}]$ and $[a_{21}, b_{21}]$. Over the last region where $\alpha_M(\omega) = 1$, $\forall \omega \in \Omega_M$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined by a set of conic bounds $[a_{1M}, b_{1M}]$ and $[a_{2M}, b_{2M}]$. In general, for $i = 1 \cdots M$, over the $i$th region, $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined by a set of conic bounds $[a_{1i}, b_{1i}]$ and $[a_{2i}, b_{2i}]$.

Note, also, that the roles of $\mathcal{G}_1$ and $\mathcal{G}_2$ may switch; prototypically, $\mathcal{G}_1$ is the plant while $\mathcal{G}_2$ is the controller. However, $\mathcal{G}_2$ could be the plant while $\mathcal{G}_1$ is the controller.

**Corollary 5.6.1.** The matrix $Q(\cdot)$ presented in Eq. (5.34) is positive definite if

$$\frac{1}{b_{1i}} + a_{2i} > 0, \quad \left( \frac{1}{b_{1i}} + a_{2i} \right) \left( \frac{1}{b_{2i}} + a_{1i} \right) - \frac{1}{4} \left( \frac{a_{1i}}{b_{1i}} - \frac{a_{2i}}{b_{2i}} \right)^2 > 0, \quad \forall i = 1 \cdots M.$$

**Proof.** Substituting the $Q_j(\cdot)$, $S_j(\cdot)$, and $R_j(\cdot)$ expressions of Eq. (5.35) into Eq. (5.34) gives

$$Q(\omega) = \left[ \frac{1}{2} \left[ \sum_{i=1}^{M} \left( \frac{a_{1i}}{b_{1i}} - \frac{a_{2i}}{b_{2i}} \right) \alpha_i(\omega) \right] 1 \frac{1}{2} \left[ \sum_{i=1}^{M} \left( \frac{a_{1i}}{b_{1i}} + \frac{a_{2i}}{b_{2i}} \right) \alpha_i(\omega) \right] 1 \right]$$

$$= \sum_{i=1}^{M} \alpha_i(\omega) \left[ \frac{1}{2} \left( \frac{a_{1i}}{b_{1i}} + \frac{a_{2i}}{b_{2i}} \right) 1 \frac{1}{2} \left( \frac{a_{1i}}{b_{1i}} - \frac{a_{2i}}{b_{2i}} \right) 1 \right] .$$

It follows then that the closed-loop system will be $L_2$ stable provided that $\mathcal{G}_1$ and $\mathcal{G}_2$ collectively satisfy

$$\frac{1}{b_{1i}} + a_{2i} > 0, \quad \left( \frac{1}{b_{1i}} + a_{2i} \right) \left( \frac{1}{b_{2i}} + a_{1i} \right) - \frac{1}{4} \left( \frac{a_{1i}}{b_{1i}} - \frac{a_{2i}}{b_{2i}} \right)^2 > 0, \quad \forall i = 1 \cdots M$$

(5.36)

over each input-output region defined by each $\alpha_i$ so that $Q(\cdot) > 0, \forall \omega \in \mathbb{R}$. 

Theorem 5.6.1 on page 85 combined with Corollary 5.6.1 will be referred to as the Hybrid
Conic Sector Theorem.

Recall, starting on page 36, Sec. 3.7.3 where the traditional Conic Sector Theorem is stated. If we compare the stability requirements of the traditional Conic Sector Theorem shown in Eqs. (3.17a) and (3.17a) on page 37 and the stability requirements presented in Eq. (5.36), they are clearly very similar. In fact, if our hybrid systems are only defined by one conic region so that \( \omega_1 \to \infty, \alpha_1(\omega) = 1, \forall \omega \in \mathbb{R} \) (and hence \( \alpha_k(\omega) = 0, \forall \omega \in \mathbb{R} \) for \( k = 2, \ldots, M \)) then the stability requirements dictated by the traditional Conic Sector Theorem are recovered.

**Positive Definite Nature of \( Q \) when the Systems are Hybrid Passive/Finite Gain Systems**

Recall Sec. 5.5 (which starts on page 76) where hybrid passive/finite gain systems were developed. A hybrid passive/finite gain system can be thought of as a special kind of hybrid conic system where the system input-output map can be defined in terms of a passive input-output map (i.e., a passive cone, that is either VSP, ISP, OSP, or purely passive properties), and a finite gain input-output map (i.e., a finite gain cone) when passivity has been violated. When the system in question has a passive input-output map \( \alpha(\omega) = 1, \forall \omega \in \Omega_1 \), and when the system in question no longer has a passive input-output map but the map possesses finite gain \( \alpha(\omega) = 0, \forall \omega \in \Omega_h \).

When both \( G_1 \) and \( G_2 \) are hybrid passive/finite gain systems, the matrices \( Q_j(\cdot), S_j(\cdot), \) and \( R_j(\cdot) \) (where \( j = 1, 2 \)) have the following form (see Eq. (5.21), page 77):

\[
Q_j(\omega) = -\left[ \epsilon_j \alpha(\omega) + \gamma_j^{-1}(1 - \alpha(\omega)) \right] 1, \quad S_j(\omega) = \frac{1}{2} \alpha(\omega) 1, \quad R_j(\omega) = [-\delta_j \alpha(\omega) + \gamma_j(1 - \alpha(\omega))] 1
\]

(5.37)

where \( \delta_j, \epsilon_j, \) and \( \gamma_j \) are nonnegative constants. The parameters \( \delta_1, \epsilon_1, \) and \( \gamma_1 \) are associated with \( G_1 \), while \( \delta_2, \epsilon_2, \) and \( \gamma_2 \) are associated with \( G_2 \).

**Corollary 5.6.2.** The matrix \( Q(\cdot) \) presented in Eq. (5.34) is positive definite if \( \epsilon_1 + \delta_2 > 0, \epsilon_2 + \delta_1 > 0 \) and \( \gamma_1 \gamma_2 < 1 \).

**Proof.** Substituting the \( Q_j(\cdot), S_j(\cdot), \) and \( R_j(\cdot) \) equalities of Eq. (5.37) into Eq. (5.34) gives

\[
Q(\omega) = \begin{bmatrix}
[(\epsilon_1 + \delta_2)\alpha(\omega) + (\gamma_1^{-1} - \gamma_2)(1 - \alpha(\omega))] 1 & 0 \\
0 & [(\epsilon_2 + \delta_1)\alpha(\omega) + (\gamma_2^{-1} - \gamma_1)(1 - \alpha(\omega))] 1
\end{bmatrix}
\]

When \( \alpha(\omega) = 1, Q(\cdot) \) becomes

\[
Q(\omega)|_{\alpha(\omega)=1} = \begin{bmatrix}
(\epsilon_1 + \delta_2) 1 & 0 \\
0 & (\epsilon_2 + \delta_1) 1
\end{bmatrix}
\]

Clearly, when \( \epsilon_1 + \delta_2 > 0 \) and \( \epsilon_2 + \delta_1 > 0 \), \( Q(\omega)|_{\alpha(\omega)=1} \) is positive definite. When \( \alpha(\omega) = 1, Q(\cdot) \)
becomes

\[ Q(\omega)|_{\alpha(\omega)=0} = \begin{bmatrix} \gamma_1^{-1}(1 - \gamma_1 \gamma_2) & 0 \\ 0 & \gamma_2^{-1}(1 - \gamma_1 \gamma_2) \end{bmatrix}. \]

When \( \gamma_1 \gamma_2 < 1 \), \( Q(\omega)|_{\alpha(\omega)=0} \) is positive definite. \( \square \)

Theorem 5.6.1 on page 85 combined with Corollary 5.6.2 will be referred to as the Hybrid Passivity and Finite Gain Theorem, or the Hybrid Passivity/Finite Gain Stability Theorem.

It is very interesting that \( \epsilon_1 + \delta_2 > 0 \), \( \epsilon_2 + \delta_1 > 0 \), and \( \gamma_1 \gamma_2 < 1 \) are exactly the same as the inequalities stipulating stability via the traditional Passivity and Small Gain Theorems (see Secs. 3.7.1 and 3.7.2, page 35), yet in the traditional sense the parameters are defined globally (e.g., over all frequencies in a LTI sense), unlike in the hybrid sense.

Although the gain inequalities \( \gamma_1 \gamma_2 < 1 \) can only be satisfied one way (in the region where passivity is violated), the passivity inequalities can be satisfied one of many ways; when the plant is passive and the control is VSP (i.e., \( \epsilon_1 = \delta_1 = 0 \) and \( \epsilon_2 > 0, \delta_2 > 0 \)), when the plant is input strictly passive and the control is input strictly passive (i.e., \( \epsilon_1 = 0, \delta_1 > 0 \) and \( \epsilon_2 = 0, \delta_2 > 0 \)), when the plant is output strictly passive and the control is output strictly passive (i.e., \( \epsilon_1 > 0, \delta_1 = 0 \) and \( \epsilon_2 > 0, \delta_2 = 0 \)), and when \( \epsilon_1 + \delta_2 > 0 \) and \( \epsilon_2 + \delta_1 > 0 \).

Notice the symmetry that is similar to that of the traditional Passivity and Small Gain Theorems. By symmetry we refer to the fact that the plant and control may switch roles. Should the plant be passive with gain \( \gamma_1 \) when passivity is violated and the control be VSP with gain \( \gamma_2 \) when passivity is violated such that \( \gamma_1 \gamma_2 < 1 \), then the feedback interconnection is \( L_2 \) stable. The converse may also be true, if the control is passive with gain \( \gamma_2 \) when passivity is violated and the plant is VSP with gain \( \gamma_1 \) when passivity is violated such that \( \gamma_1 \gamma_2 < 1 \), the feedback interconnection is \( L_2 \) stable. Other symmetrical switches are permitted as well.
5.7 Summary

This chapter commenced with an example: in Sec. 5.1 a nominally passive plant had its passive nature destroyed by a simple sensor, thus rendering the plant hybrid in the sense that the frequency response could be described in terms of a PR (passive) region and a BR (finite gain) region. This notion of “dividing up” a system’s input-output map was then extended. In particular, after presenting the hybrid dissipative systems framework, we defined hybrid conic systems. Hybrid conic systems are those that have an input-output map that can be described in terms of a set of intermediate maps, where each of the intermediate maps has specific conic bounds. Interestingly, hybrid conic systems can be used to approximate the variable cones developed by Safonov. We then looked at a special type of hybrid conic system: a hybrid passive/finite gain system where the input-output map can be described in terms of a passive map, and a finite gain map when passivity has been violated. Our initial motivating example in Sec. 5.1 was the inspiration for investigating hybrid passive/finite gain systems, and hybrid conic systems where the result of generalizing the idea of a hybrid input-output map further. Stability of hybrid systems connected in a negative feedback loop was also discussed. The theory presented in this chapter, although similar to Refs. 61, 62 and 63, is new.
Chapter 6

Scheduling of Conic and Hybrid VSP/Finite Gain Systems

When controlling nonlinear systems, such as multi-link flexible manipulators or a spacecraft, linear controllers are often used. Linear controllers are used mainly because they are simple and there is a plethora of design and synthesis techniques available (e.g., PID, LQR, $\mathcal{H}_2$, $\mathcal{H}_\infty$, etc.\textsuperscript{35}). Additionally, in some cases robust closed-loop stability of a nonlinear plant connected in a negative feedback loop with a linear controller can be ensured. Such is the case when the nonlinear plant is passive and the controller is a linear VSP controller (i.e., a SPR controller), as robust stability of the closed-loop is guaranteed via the Passivity Theorem.

Although designing a linear controller is straightforward, given a nonlinear plant it is natural to assume that better closed-loop control can be achieved if the controller were nonlinear as well. Although there are nonlinear control design techniques (e.g., feedback linearization, sliding mode control, backstepping, and others; see Ref. 4), designing a nonlinear controller that is robust, and perhaps optimal, is not trivial.

A compromise between using a simple linear controller and trying to synthesize a nonlinear controller is to design and employ a gain-scheduled controller. A gain-scheduled controller is one that is composed of a family of linear controllers. Which controller is active depends the scheduling signal, which in turn depends on the plant output, the plant states, time, or any combination of the plant output, states, and time. Designing a simple gain-scheduled controller is rather trivial: linearize the nonlinear system about $N$ operating points, then design $N$ linear controllers. During operation, when the nonlinear plant is in between operating points, simply interpolate the adjacent controller parameters (i.e., the gains). The interpolation scheme is historically chosen to be linear; however, it is feasible that nonlinear scheduling signals could be used as well.

Some of the first gain-scheduled controllers were employed on aircraft in the 1950’s.\textsuperscript{72} Aircraft dynamics are quite nonlinear, and change as a function of altitude, velocity, angle of attack, and other variables. The dynamics of an aircraft are significantly different during take-off and landing (i.e., at low altitude and low velocity) than they are at cruise (i.e., at high altitude and high
velocity). To design a gain-scheduling controller for an aircraft, a set of controller gains would be chosen for take-off and landing, while another set of controller gains would be chosen for cruise. The controller gains are then simply linearly interpolated as a function of altitude and velocity.

Although control via gain-scheduling has been used in industry for literally decades, and still is, it was not until the late 1980’s and early 1990’s that control theorists took a serious look at the closed-loop stability of systems employing gain-scheduling. Some of the first papers include Refs. 73–75 where stability of gain-scheduled systems is assessed within a linear parameter-varying (LPV) framework; Refs. 76 and 77 consider the design of gain-scheduled controllers in a LMI context. Each of the aforementioned papers assess stability via Lyapunov arguments.

\[ G_1 \]

\[ H_1 \]

\[ u_1 \]

\[ y_1 \]

\[ s_1 \]

\[ s_{N+1} \]

\[ y_2 \]

\[ u_2 \]

\[ H_N \]

\[ \mathcal{G}_1 \]

\[ \mathcal{H}_1 \]

\[ \vdots \]

\[ \vdots \]

\[ \mathcal{H}_N \]

Figure 6.1: Gain-scheduled control of a nonlinear passive plant $\mathcal{G}_1$. The signals $s_i, i = 1 \cdots N$, are the scheduling signals, and each $\mathcal{H}_i, i = 1 \cdots N$ is a SPR controller optimally designed about a particular set-point of the linearized nonlinear system $\mathcal{G}_1$.

Although assessing the stability of some sort of LPV system via a Lyapunov method is most definitely valid, it would be ideal if an input-output stability proof relevant to gain-scheduling were available. Such a stability result would be much more general, being applicable to any kind of system with an input-output character such as passivity or conicity. There is one such input-output stability result particular to gain-scheduling: in Ref. 78 it is shown that a family of SPR systems (i.e., SPR controllers) gain-scheduled in a particular way is in fact ISP. As such, a nonlinear passive plant is guaranteed to be stabilized by this gain-scheduled SPR controller via the weak version of the Passivity Theorem (see Theorem 3.7.2 on page 36). The general control architecture is shown in Fig. 6.1 where $\mathcal{G}_1$ is the nonlinear passive plant being controlled, the signals $s_i, i = 1 \cdots N$, are the scheduling signals, and each $\mathcal{H}_i, i = 1 \cdots N$ are SPR controllers. Ref. 78 is extended in Ref. 79; a family of SPR systems gain-scheduled in a particular fashion is shown to be VSP; the strong version of the Passivity Theorem can be used to guarantee closed-loop stability of any nonlinear passive plant when the controller is the gain-scheduled SPR controller. In Refs. 78 and 79 flexible robotic manipulators (which have a passive input-output map) were controlled by the gain-scheduled SPR controller, where each SPR “subcontroller” or “basis controller” is designed about a particular manipulator set-point. In general, however, the results of Refs. 78 and 79 can be used to control any nonlinear passive plant, not just flexible
robotic manipulators.

The purpose of this chapter is to investigate the input-output properties of systems being scheduled. The systems being scheduled, which we will refer to as the “subsystems” or “basis systems”, are most likely a set of LTI controllers (that are optimal about a set-point), and the overall scheduled system will most likely be used as a controller for some sort of nonlinear plant. Specifically, we will look at the conic bounds of a gain-scheduled system when the subsystems being scheduled are conic. We will also look at the scheduling of hybrid VSP/finite gain systems, and the hybrid VSP/finite gain nature of the overall gain-scheduled system. Naturally, our motivation is to design a gain-scheduled controller that has some guaranteed properties, such as conicity or hybridness. We will begin by reviewing the results of Refs. 78 and 79, as the scheduling results pertaining to conic systems and hybrid passive/finite gain systems are a direct extension of Refs. 78 and 79.

6.1 Scheduling of Passive Systems

Recall from Sec. 3.4, Definition 3.4.1 on page 22: a square system \( y(t) = (\mathcal{H}u)(t), \) \( \mathcal{H} : L_{2e}(U) \rightarrow L_{2e}(Y) \), \( u \in L_{2e}(U), \) \( y \in L_{2e}(Y) \) is VSP if there exist real constants \( \delta > 0 \) and \( \epsilon > 0 \) such that

\[
\int_0^T y^T(t)u(t)dt \geq \delta \|u\|^2_T + \epsilon \|y\|^2_T, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+.
\]

We are assuming that the initial conditions of the system are quiescent, and hence \( \beta = 0 \).

Consider a system mapping \( u \rightarrow y \) made up of \( i = 1 \cdots N \) subsystems, each of the form \( y_i(t) = (\mathcal{H}_i u_i)(t), \) \( \mathcal{H}_i : L_{2e}(U_i) \rightarrow L_{2e}(Y_i) \), \( u_i \in L_{2e}(U_i), y_i \in L_{2e}(Y_i) \) as shown in Fig. 6.2. The subsystems are not necessarily linear, nor time-invariant; the subsystems \( \mathcal{H}_i \) could be nonlinear and time-varying, nonlinear and time-invariant, LTV, or LTI. Each subsystem is VSP (ISP). We will show that when the VSP (ISP) subsystems are scheduled appropriately, the overall map \( u \rightarrow y \) is VSP (ISP). When we say “scheduled appropriately”, what we mean is that i) the scheduling signals operate on both the input and output of the subsystems \( \mathcal{H}_i \), as shown in Fig. 6.2, and ii) the scheduling signals have certain properties, as discussed next.

The scheduling signals, \( s_i \), may be functions of anything provided i) they are square integrable,
$s_i \in L_2$, and ii) possess finite gain, $s_i \in L_\infty$ (i.e., $0 < \|s_i\|_\infty < \infty$). In the context of controlling some sort of nonlinear plant, the scheduling signals may be functions of the plant output, the plant states, time, or a combination of all three. Although each $s_i$ could be a function of many variables, for simplicity we will express it as a function of time only. Additionally, the scheduling signals do not have to be linear functions of any variable; they may be nonlinear functions of the plant output, state, or time. Nonlinear scheduling signals were explored in Ref. 79.

Referring to Fig. 6.2, the scheduling signals act on the inputs and outputs of the subsystems in the following way:

$$y(t) = \sum_{i=1}^{N} s_i(t)y_i(t), \quad u_i(t) = s_i(t)u(t). \quad (6.1)$$

Additionally, we require that together the scheduling signals satisfy

$$\sum_{i=1}^{N} s_i^2(t) \geq \nu > 0, \quad s_i \in L_2 \cap L_\infty \quad (6.2)$$

which ensures that at least one scheduling signal is active at all times.

We will now prove that when the subsystems being scheduled are VSP, the overall gain-scheduled system is also VSP.

**Theorem 6.1.1.** Referring to Fig. 6.2 where the scheduling signals satisfy Eqs. (6.1) and (6.2), if each of the subsystems $\mathcal{H}_i : L_2c \rightarrow L_2c$ is VSP, that is to say ISP with finite gain where $\delta_i > 0$ and $0 < \gamma_i < \infty$, then the overall system map $u \rightarrow y$ is VSP with ISP parameter $\delta = \nu \min_{i=1 \ldots N} \delta_i > 0$ and gain $0 < \gamma < \infty$ where $\gamma = \sum_{i=1}^{N} \gamma_i \sigma_i^2$ and $\sigma_i := \|s_i\|_\infty$.

**Proof.** Our proof, which is essentially taken from Ref. 79, consists of two parts. First we will show that the scheduled system is ISP, and then show that the scheduled system possesses finite gain. Combining the ISP and finite gain nature of the scheduled system yields the desired result: a VSP system.

To show that the scheduled system is ISP, consider the following integral involving $u$ and $y$, and subsequent manipulation using Eqs. (6.1) and (6.2):

$$\int_0^T u^T(t)y(t)dt = \int_0^T u^T(t)\sum_{i=1}^{N} s_i(t)y_i(t)dt$$

$$= \sum_{i=1}^{N} \int_0^T s_i(t)u^T(t)y_i(t)dt$$

$$= \sum_{i=1}^{N} \int_0^T u_i^T(t)y_i(t)dt$$

$$\geq \sum_{i=1}^{N} \delta_i \int_0^T u_i^T(t)u_i(t)dt$$

where we have used the fact each subsystem satisfies $\int_0^T u_i^T(t)y_i(t)dt \geq \delta_i \int_0^T u_i^T(t)u_i(t)dt$. Con-
continuing we have

\[
\int_0^T u^T(t)y(t)dt \geq \sum_{i=1}^N \delta_i \int_0^T u_i^T(t)u_i(t)dt \\
\geq \sum_{i=1}^N \delta_i \int_0^T s_i^2(t)u^T(t)u(t)dt \\
\geq v \left[ \sum_{i=1}^N \delta_i \right] \int_0^T u^T(t)u(t)dt \\
\geq \delta \int_0^T u^T(t)u(t)dt
\]

where \( \delta = v \min_{i=1 \ldots N} \delta_i \). Thus, when each subsystem is VSP the overall scheduled system is ISP.

Next, we will show that the scheduled system has finite gain. To start, consider the following:

\[
\|y\|_{2T} = \left\| \sum_{i=1}^N s_i y_i \right\|_{2T} \\
\leq \sum_{i=1}^N \|s_i y_i\|_{2T} \quad \text{(via the triangle inequality)} \\
\leq \sum_{i=1}^N \sigma_i \|y_i\|_{2T}
\]

(6.3)

where \( \sigma_i = \|s_i\|_{\infty} \). Each subsystem has finite gain, that is \( \|y_i\|_{2T} \leq \gamma_i \|u_i\|_{2T} \). Using this fact and the inequality in Eq. (6.3) we have that

\[
\|y\|_{2T} \leq \sum_{i=1}^N \sigma_i \|y_i\|_{2T} \leq \sum_{i=1}^N \gamma_i \sigma_i \|u_i\|_{2T} = \sum_{i=1}^N \gamma_i \sigma_i \|s_i u\|_{2T} \leq \sum_{i=1}^N \gamma_i \sigma_i^2 \|u\|_{2T}
\]

which shows that the overall scheduled system satisfies

\[
\|y\|_{2T} \leq \gamma \|u\|_{2T}
\]

where the gain is \( \gamma = \sum_{i=1}^N \gamma_i \sigma_i^2 < \infty \). Thus, the scheduled system has finite gain. The gain \( \gamma \) is finite because each \( \gamma_i \) and \( \sigma_i \) are finite.

Combining the ISP and finite gain properties of the scheduled system leads to the conclusion that the scheduled system is VSP.

\[\square\]

**Corollary 6.1.1.** Referring to Fig. 6.2 where the scheduling signals satisfy Eqs. (6.1) and (6.2), if each of the subsystems \( \mathcal{H}_i : L_{2e} \to L_{2e} \) is ISP with \( \delta_i > 0 \) then the overall system map \( u \to y \) is ISP with \( \delta = v \min_{i=1 \ldots N} \delta_i \).
Proof. The proof is identical to the first part of the proof presented in Theorem 6.1.1.

Proof of this theorem can be found in Ref. 78 as well.

Corollary 6.1.2. Referring to Fig. 6.2 where the scheduling signals satisfy Eqs. (6.1) and (6.2), if each of the subsystems $\mathcal{H}_i : L_{2e} \rightarrow L_{2e}$ is passive then the overall system map $u \rightarrow y$ is passive.

Proof. The proof is essentially the same as the proof presented in Theorem 6.1.1. First note that each subsystem satisfies $\int_0^T u_i^T(t)y_i(t)dt \geq 0$. Next, consider

$$\int_0^T u(t)y(t)dt = \int_0^T u(t) \sum_{i=1}^N s_i(t)y_i(t)dt = \sum_{i=1}^N \int_0^T s_i(t)u_i^T(t)y_i(t)dt = \sum_{i=1}^N \int_0^T u_i^T(t)y_i(t)dt \geq 0$$

which shows that the overall scheduled system is passive.

If the subsystems are a combination of VSP, ISP, OSP, and passive systems then at best the overall gain-scheduled system is passive. Should just one subsystem not have finite gain (i.e., be ISP, not VSP), then the overall scheduled system cannot have finite gain, and hence can at best be ISP. If just one subsystem is not ISP, then the overall scheduled system can not be ISP (but may be have finite gain). If one subsystem is not ISP and another subsystem does not have finite gain, the gain-scheduled system can at best be just passive.

### 6.2 Scheduling of Conic Systems

We will now consider the scheduling of conic subsystems. We will show that when a set of conic subsystems are gain-scheduled, the over all gain-scheduled system is also conic, a new result.

From Sec. 3.5 (page 25) we have Definition 3.5.2, the definition of a conic system: a square system $y(t) = (\mathcal{H}u)(t)$, $\mathcal{H} : L_{2e}(U) \rightarrow L_{2e}(Y)$, $u \in L_{2e}(U)$, $y \in L_{2e}(Y)$ is said to be inside the conic sector $[a,b]$, if there exist real constants $a \in \mathbb{R}$, $b \in \mathbb{R}^+ \cup \{\infty\} \setminus \{0\}$ where $b > a$ such that

$$-\frac{1}{b} \|y\|_{2T}^2 + \left(1 + \frac{a}{b}\right) \langle y, u \rangle_T - a \|u\|_{2T}^2 \geq 0, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+.$$ 

Above we have set $\beta = 0$ corresponding to quiescent initial conditions. When $0 < b < \infty$ the above inequality can be rewritten in terms of a cone centre and cone radius, $c$ and $r$, where $c = (b + a)/2$ and $r = (b - a)/2$ (or where $a = c - r$ and $b = c + r$); the system is said to be inside the conic sector $[c - r, c + r]$ if there exist real constants $c \in \mathbb{R}$, $r \geq 0$ such that

$$\|y - cu\|_{2T} \leq r \|u\|_{2T}, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+. \quad (6.4)$$

Again, the system initial conditions are taken to be zero.

Consider a system mapping $u \rightarrow y$ made up of $i = 1 \cdots N$ subsystems, each of the form $y_i(t) = (\mathcal{H}_i u_i)(t)$, $\mathcal{H}_i : L_{2e}(U_i) \rightarrow L_{2e}(Y_i)$, $u_i \in L_{2e}(U_i)$, $y_i \in L_{2e}(Y_i)$ as shown in Fig. 6.3. Each $\mathcal{H}_i$ is not necessarily linear, but in most practical applications will be LTI. Each subsystem
Figure 6.3: Scheduled system to be analyzed with scheduling signals $s_i$.

has an associated set of bounds $a_i$ and $b_i$, or equivalently, an associated cone centre, and cone radius, $c_i$ and $r_i \geq 0$. In particular, let us assume that $a_i < 0$ and $0 < b_i < \infty$ so that we can explicitly express each system $\mathcal{H}_i$ in terms of $c_i$ and $r_i$. When $0 < a_i < b_i < \infty$ or $b_i = \infty$ and $a_i \in \mathbb{R}$, the results of Sec. 6.1 are applicable. We will show that when the conic subsystems (i.e., $\mathcal{H}_i \in \text{cone}[c_i - r_i, c_i + r_i]$) are scheduled appropriately the overall map $u \rightarrow y$ is conic with cone centre $c$ and cone radius $r$.

The scheduling signals, $s_i$, may be functions of anything provided i) they are square integrable, $s_i \in L_{2e}$, and ii) possess finite gain, $s_i \in L_{\infty}$ (i.e., $0 < \|s_i\|_{\infty} < \infty$). As before, the scheduling signals do not have to be a linear function of any variable; they may be nonlinear functions of time, for example.

As shown in Fig. 6.3 the scheduling signals are applied to the input and output of each subsystem as follows:

$$\bar{y}_i(t) = s_i(t)y_i(t), \quad u_i(t) = s_i(t)u(t). \quad (6.5)$$

The output of the complete system is then

$$y(t) = \sum_{i=1}^{N} \bar{y}_i(t) \quad (6.6)$$

Because the gain associated with each $s_i$ must be finite we have that

$$\|\bar{y}_i\|_{2T} = \|s_iy_i\|_{2T} \leq \sigma_i \|y_i\|_{2T} \quad \text{and} \quad \|u_i\|_{2T} = \|s_iu\|_{2T} \leq \sigma_i \|u\|_{2T} \quad (6.7)$$

where $\sigma_i = \|s_i\|_{\infty} < \infty$. Also,

$$\|u_i\|_{2T} = \|s_iu\|_{2T} \geq \nu_i \|u\|_{2T} \quad (6.8)$$

where $\nu_i := \inf_{t \in \mathbb{R}^+} |s_i(t)| \geq 0$. Note that the minimum of each $s_i$ need not be exactly zero. We also assume that

$$\sum_{i=1}^{N} s_i^2(t) \geq v > 0, \quad s_i \in L_{2e} \cap L_{\infty} \quad (6.9)$$

which ensures that at least one scheduling signal is active at all times.
We will now show that a set of conic subsystems with \( c_i \in \mathbb{R} \) and \( r_i \geq 0 \) scheduled appropriately is also conic.

**Theorem 6.2.1.** Consider the scheduling architecture presented in Fig. 6.3 where \( s_i \) are the scheduling signals and \( H_i \) are the subsystems being scheduled. Each of the \( N \) subsystems are conic with cone centers \( c_i \) and cone radii \( r_i \). Provided \( s_i \in L_2 \cap L_{\infty} \) and satisfy Eqs. (6.5) to (6.9), the input-output map \( \mathbf{u} \to \mathbf{y} \) is conic with cone centre \( c = \sum_{i=1}^{N} c_i \) and cone radius \( r = \sum_{i=1}^{N} r_i \) where \( \bar{c}_i = c_i \sigma_i^2 \), i) \( \bar{r}_i = r_i \sigma_i^2 \) if \( r_i > |c_i| \), and ii) \( \bar{r}_i = \sqrt{r_i^2 \sigma_i^4 + c_i^2 \sigma_i^4 (\sigma_i^2 - \nu_i^2)} \) if \( |c_i| \geq r_i \) where \( \sigma_i = \|s_i\|_{\infty} < \infty \) and \( \nu_i = \inf_{t \in \mathbb{R}^+} |s_i(t)| \geq 0 \).

**Proof.** Our proof will be executed in two steps. We will show that

1. the subsystem mappings \( \mathbf{u} \to \bar{y}_i \) are conic with cone centres \( \bar{c}_i \) and cone radii \( \bar{r}_i \), then show that
2. the map \( \mathbf{u} \to \mathbf{y} \) is conic with cone centre \( c \) and cone radius \( r \).

We being with Step 1 and Case i), where \( r_i > |c_i| \). To start, consider the input-output relation of a subsystem:

\[
\|\mathbf{y}_i - c_i \mathbf{u}_i\|_{2T} \leq r_i \|\mathbf{u}_i\|_{2T},
\|\mathbf{y}_i\|_{2T}^2 - 2c_i \langle \mathbf{y}_i, \mathbf{u}_i \rangle_T + c_i^2 \|\mathbf{u}_i\|_{2T}^2 \leq r_i^2 \|\mathbf{u}_i\|_{2T}^2.
\] (6.10)

Assuming that \( r_i > |c_i| \), and therefore \( r_i^2 - c_i^2 > 0 \), and using Eqs. (6.5) and (6.7) we have

\[
\frac{1}{\sigma_i^2} \|\mathbf{y}_i\|_{2T}^2 - 2c_i \langle \mathbf{y}_i, \mathbf{u}_i \rangle_T \leq (r_i^2 - c_i^2) \sigma_i^2 \|\mathbf{u}_i\|_{2T}^2,
\|\mathbf{y}_i\|_{2T}^2 - 2c_i \sigma_i^4 \langle \bar{y}_i, \mathbf{u}_i \rangle_T + c_i^2 \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2 \leq r_i^2 \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2,
\|\mathbf{y}_i - c_i \mathbf{u}_i\|_{2T}^2 \leq r_i \|\mathbf{u}_i\|_{2T},
\|\mathbf{y}_i - c_i \mathbf{u}_i\|_{2T} \leq \bar{r}_i \|\mathbf{u}_i\|_{2T}
\]

where \( \bar{c}_i = c_i \sigma_i^2 \) and \( \bar{r}_i = r_i \sigma_i^2 \). Thus, comparing with Eq. (6.4), when \( r_i > |c_i| \) the map \( \mathbf{u} \to \bar{y}_i \) is conic with \( \bar{c}_i = c_i \sigma_i^2 \) and \( \bar{r}_i = r_i \sigma_i^2 \).

Consider now Step 1 and Case ii), where \( |c_i| \geq r_i \). Starting at Eq. (6.10) and employing Eqs. (6.7) and (6.8) we have

\[
\frac{1}{\sigma_i^2} \|\mathbf{y}_i\|_{2T}^2 - 2c_i \langle \bar{y}_i, \mathbf{u}_i \rangle_T + c_i^2 \nu_i^2 \|\mathbf{u}_i\|_{2T}^2 \leq r_i^2 \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2,
\|\mathbf{y}_i\|_{2T}^2 - 2c_i \sigma_i^4 \langle \bar{y}_i, \mathbf{u}_i \rangle_T + c_i^2 \nu_i^2 \sigma_i^2 \|\mathbf{u}_i\|_{2T}^2 \leq r_i^2 \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2.
\]

Adding \( c_i^2 \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2 \) to both sides of the inequality gives

\[
\|\mathbf{y}_i\|_{2T}^2 - 2c_i \sigma_i^4 \langle \bar{y}_i, \mathbf{u}_i \rangle_T + c_i^2 \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2 + c_i^2 \nu_i^2 \sigma_i^2 \|\mathbf{u}_i\|_{2T}^2 \leq (r_i^2 + c_i^2) \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2,
\|\bar{y}_i - c_i \mathbf{u}_i\|_{2T}^2 \leq (r_i^2 + c_i^2) \sigma_i^4 \|\mathbf{u}_i\|_{2T}^2.
\]
Bringing \( c_i^2 \nu_i^2 \sigma_i^2 \|u\|_{2T}^2 \) to the other side gives
\[
\|\bar{y}_i - c_i \sigma_i^2 u\|_{2T}^2 \leq [r_i^2 \sigma_i^4 + c_i^2 \sigma_i^2 (\sigma_i^2 - \nu_i^2)] \|u\|_{2T}^2
\]
and therefore
\[
\|\bar{y}_i - \bar{c}_i u\|_{2T} \leq \bar{r}_i \|u\|_{2T}
\]
where \( \bar{r}_i = \sqrt{r_i^2 \sigma_i^4 + c_i^2 \sigma_i^2 (\sigma_i^2 - \nu_i^2)} \). Therefore, comparing with Eq. (6.4), when \( |c_i| \geq r_i \) the map \( u \to \bar{y}_i \) is conic with \( \bar{c}_i = c_i \sigma_i^2 \) and \( \bar{r}_i = \sqrt{r_i^2 \sigma_i^4 + c_i^2 \sigma_i^2 (\sigma_i^2 - \nu_i^2)} \).

Now for Step 2; we will show that the map \( u \to y \) where \( y = \sum_{i=1}^N \bar{y}_i \) (Eq. (6.6)) is conic with cone centre \( c = \sum_{i=1}^N \bar{c}_i \) and cone radius \( r = \sum_{i=1}^N \bar{r}_i \). Following Zames (See Ref. 13, Sec. 4.2, property iii), which is proven in Appendix A of the same paper) we have
\[
\|y - cu\|_{2T} = \left\| \sum_{i=1}^N \bar{y}_i - \left[ \sum_{i=1}^N \bar{c}_i \right] u \right\|_{2T}
\]
\[
= \left\| \sum_{i=1}^N (\bar{y}_i - \bar{c}_i u) \right\|_{2T}
\]
\[
\leq \sum_{i=1}^N \| (\bar{y}_i - \bar{c}_i u) \|_{2T} \quad (\text{via the triangle inequality})
\]
\[
\leq \sum_{i=1}^N \bar{r}_i \|u\|_{2T}
\]
\[
= r \|u\|_{2T}.
\]
Thus, the map \( u \to y \) is conic. \( \square \)

Notice that the map \( u \to \bar{y}_i \) is conic with cone centre \( \bar{c}_i \) and cone radius \( \bar{r}_i \). The cone radius \( \bar{r}_i \) takes on different values depending on whether or not \( r_i > |c_i| \) or \( |c_i| \geq r_i \). Also notice that the inequality \( \bar{r}_i \geq 0 \) holds owing to the fact that \( r_i \geq 0, \sigma_i \geq 0, \) and \( \sigma_i^2 - \nu_i^2 \geq 0 \), while \( \bar{c}_i \) could be negative or positive depending on each \( c_i \).

The supremum or infimum of \( s_i \) (i.e., \( \sigma_i \) and \( \nu_i \)) determines how the cone centre and radius of the map \( u \to y \) “morphs”, that is “grows” or “shrinks”, as compared to the original cones describing the subsystems \( u_i \to y_i \). Consider the case where each \( u_i \to y_i \) subsystem has \( r_i > |c_i| \), and thus \( u \to y \) has \( c = \sum_{i=1}^N c_i \sigma_i^2 \) and \( r = \sum_{i=1}^N r_i \sigma_i^2 \). If \( \sigma_i = 1 \), then the cone center and radius of \( u \to y \) grows only as a result of summing each of the subsystem’s cone centers and radii, i.e., \( c = \sum_{i=1}^N c_i \) and \( r = \sum_{i=1}^N r_i \). If \( \sigma_i > 1 \) the cone center and radius of \( u \to y \) grows a larger amount than the \( \sigma_i = 1 \) scenario because each \( \sigma_i \) enlarges each subsystem cone. When \( \sigma_i < 1 \) the cone describing \( u \to y \) could shrink because each \( \sigma_i \) reduces the size of each subsystem cone.

To illustrate how cones do or do not morph, consider two time-dependent scheduling signals \( s_1 \) and \( s_2 \) that will schedule two subsystems \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). If the scheduling signals are linear functions of time, as shown in Fig. 6.4(a), then \( \sigma_1 = \sigma_2 = 1, \nu_1 = \nu_2 = 0, \) and thus \( c = c_1 + c_2 \) and \( r = r_1 + r_2 \); the resultant cone describing \( u \to y \) grows. If the scheduling signals are
nonlinear functions of time, as depicted in Fig. 6.4(b) where \( \sigma_1 = \sigma_2 = \sigma' > 1 \) and \( \nu_1 = \nu_2 = 0 \), then \( c = (c_1 + c_2)(\sigma')^2 \) and \( r = (r_1 + r_2)(\sigma')^2 \); the resultant cone is enlarged a greater amount than the case where the scheduling signals have a supremum of 1. Fig. 6.5 depicts two subsystem cones (drawn in lighter shades), and the resultant overall cone (in a darker shade), which is enlarged. Such is the case when \( \sigma_i \geq 1 \), where the resultant cone will be enlarged a greater amount when \( \sigma_i > 1 \) compared to when \( \sigma_1 = 1 \).

Figure 6.5: Resultant cone when scheduling two subsystems.

Theorem 6.2.1 is a new result. What we have shown is that a set of conic subsystems gain-scheduled in a specific way (i.e., the architecture of Fig. 6.3 where \( s_i \in L_{2c} \cap L_{\infty} \)) has conic properties. The conic bounds of the gain-scheduled system depend on the original conic bounds of each subsystem, as well as the scheduling signals. Each conic subsystem \( \mathcal{H}_i \) does not have to be a LTI system; the subsystems could be LTV (i.e., a linear system that satisfies Theorem
4.4.1 on page 52) or nonlinear, for example. Similarly, the scheduling signals may be functions of anything; for example, a function of time, the output of the plant, or both.

6.3 Scheduling of Hybrid Passive/Finite Gain Systems

In the previous two sections we considered the scheduling of classic passive and conic subsystems. We now turn our attention to the scheduling of hybrid VSP/finite gain systems. It will be shown that a set of hybrid VSP/finite gain subsystems of the form $y_i = (H_i u_i)(t)$ (where $\delta_i > 0$, $0 < \kappa_i < \infty$, and $0 < \gamma_i < \infty$) gain-scheduled in an appropriate fashion yields a hybrid VSP/finite gain system.

First, let us review the hybrid passive/finite gain systems framework originally presented in Sec. 5.5. To start, recall Eq. (5.16) on page 76:

$$A^\sim A + B^\sim B = 1.$$  \hfill (6.11)

This operator equation will be used quite frequently in the following gain-scheduling proof. Using $A^\sim A + B^\sim B = 1$, a function, such as $y$, can be written as

$$y(t) = (1y)(t) = (A^\sim Ay)(t) + (B^\sim By)(t).$$  \hfill (6.11)

Next, recall that if a hybrid system is ISP and has finite gain when passivity holds (i.e., over the passive bandwidth), and has finite gain when passivity has been violated (i.e., over the nonpassive but still finite gain bandwidth), then the system is hybrid VSP/finite gain. Let us explicitly characterize this fact. From Sec. 5.5, Definition 5.5.1 (page 76), we have that a system

$$y = (H u)(t), \quad H : L_{2e}(U) \to L_{2e}(Y), \quad u \in L_{2e}(U), \quad y \in L_{2e}(Y)$$

is ISP when passivity holds if there exists $\delta > 0$ such that

$$\langle Ay_T, Au_T \rangle \geq \delta \|Au_T\|_2^2, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+.$$  \hfill (6.12)

The same system $y = (H u)(t), \quad H : L_{2e}(U) \to L_{2e}(Y), \quad u \in L_{2e}(U), \quad y \in L_{2e}(Y)$ has passive gain that is finite (i.e., the gain of the system when passivity holds is finite) if there exists $0 < \kappa < \infty$ such that

$$\|Ay_T\|_2 \leq \kappa \|Au_T\|_2, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+$$  \hfill (6.13)

where $\kappa$ is the passive system gain. Also, the system $y = (H u)(t), \quad H : L_{2e}(U) \to L_{2e}(Y), \quad u \in L_{2e}(U), \quad y \in L_{2e}(Y)$ has finite gain when passivity has been violated if there exists $0 < \gamma < \infty$ such that

$$\|By_T\|_2 \leq \gamma \|Bu_T\|_2, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+$$  \hfill (6.14)

where $\gamma$ is the system gain when passivity has been violated. Thus, if the system $y = (H u)(t), \quad H : L_{2e}(U) \to L_{2e}(Y), \quad u \in L_{2e}(U), \quad y \in L_{2e}(Y)$ satisfies Eqs. (6.12), (6.13), and (6.14) for some $\delta > 0$, $0 < \kappa < \infty$, and $0 < \gamma < \infty$ then the system is ISP and has finite gain when passivity holds, and has finite gain when passivity is violated, which is to finally say that the system is a
hybrid VSP/finite gain system.

Let us now move on to the gain-scheduling architecture. Consider once again Fig. 6.2 on page 95. The scheduling signals, \( s_i \), are applied to both the input and output of the subsystems:

\[
y(t) = \sum_{i=1}^{N} s_i(t)y_i(t), \quad u_i(t) = s_i(t)u(t).
\]

The signals satisfy \( \sum_{i=1}^{N} s_i^2(t) \geq \nu > 0, \ s_i \in L^2 \cap L^\infty \). Using Eq. (6.11), Eq. (6.15) can be written as

\[
y(t) = \sum_{i=1}^{N} s_i(t)y_i(t) = \sum_{i=1}^{N} s_i(t) \left( (A^\sim A)y_i(t) + (B^\sim B)y_i(t) \right),
\]

\[
u_i(t) = s_i(t)u(t) = s_i(t) \left( (A^\sim A)u(t) + (B^\sim B)u(t) \right).
\]

(6.16a)

(6.16b)

As in Secs. 6.1 and 6.2, the scheduling signals can be a function of anything (e.g., time), and do not have to be linear functions of the scheduling variable(s).

We are now ready to show that a set of hybrid VSP/finite gain subsystems gain-scheduled as in Fig. 6.2 is overall a hybrid VSP/finite gain system as well.

**Theorem 6.3.1.** The map \( u \rightarrow y \) of Fig. 6.2 is hybrid VSP/finite gain with \( \delta > 0, 0 < \kappa < \infty \), and \( 0 < \gamma < \infty \) if the subsystems are scheduled via Eq. (6.16), and each of the subsystems \( y_i = (H_iu_i)(t) \) is hybrid VSP/finite gain with \( \delta_i > 0, 0 < \kappa_i < \infty \), and \( 0 < \gamma_i < \infty \).

**Proof.** The proof of Theorem 6.3.1 will be executed in two steps, in the same way Theorem 6.1.1 was. First it will be shown that a set of subsystems that are ISP when passivity holds and are scheduled via the architecture of Fig. 6.2 yields an overall gain-scheduled system that is also ISP when passivity holds. Then, it will be shown that, provided each hybrid subsystem possesses finite gain in both the passive and nonpassive bands, the overall hybrid system possesses finite gain in the passive and nonpassive bands as well (although the gain in the passive and nonpassive bands may be different). Combining the ISP and finite gain nature when passivity holds, plus the finite gain nature when passivity is violated, proves that the scheduled system is a hybrid VSP/finite gain system.

The ISP property when passivity holds will be considered first. To begin, note that

\[
\int_0^\infty (A^\sim A)u_T^T(t)(B^\sim B)y_T(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_T^T(-j\omega) A^T(-j\omega) A(j\omega) B^T(-j\omega) B(j\omega) y_T(j\omega)d\omega
\]

\[
= 0
\]

(6.17)

owing to the fact that the product \( \alpha(\omega)\beta(\omega) \) is zero (refer to Eqs. (5.16) and (5.17) on page 76, and the definition of \( \alpha \) in Sec. 5.5). Now, consider the following integral and subsequent
where zero has been added in the last line. Continuing,

\[ \sum_{i=1}^{N} \int_{0}^{\infty} s_i(t) (\mathcal{A}^\sim \mathbf{A}u_i)\mathbf{y}_i(t) \, dt = \sum_{i=1}^{N} \int_{0}^{\infty} s_i(t) (\mathcal{A}^\sim \mathbf{A}u_i)\mathbf{y}_i(t) \, dt \]

where the definition of \( \mathbf{u}_{i,T} \) provided in Eq. (6.15) has been employed. Knowing that each sub-system is ISP when passivity holds, and thus satisfies Eq. (6.12), it follows that

\[ \delta \int_{0}^{\infty} (\mathbf{A}u_i)^T(t)(\mathbf{A}u_i)(t) \, dt \]
where $\delta = v \min_{i=1 \ldots N} \delta_i$. Thus, a set of hybrid subsystems that are ISP when passivity holds scheduled appropriately is also ISP when passivity holds.

Next the finite gain nature of the scheduled hybrid subsystems in the passive band will be considered. Essentially, we want to show that Eq. (6.13) holds where $u$ and $y$ are the input and output of the gain-scheduled system, and each subsystem has finite gain when passivity holds (i.e., each hybrid subsystem satisfies Eq. (6.13)). To begin, consider the norm of $y_T$ filtered by $A$, and subsequent manipulation using Eqs. (6.16a) and (6.17):

$$
\|Ay_T\|_2 = \left[ \int_0^\infty (Ay_T)^T(t)(Ay_T)(t)dt \right]^{1/2} = \left[ \int_0^\infty (A^\sim Ay_T)^T(t) \left( \sum_{i=1}^N s_i(t) \left( (A^\sim Ay_{i,T})(t) + (B^\sim By_{i,T})(t) \right) \right) dt \right]^{1/2} \quad \text{(using Eq. (6.16a))}
$$

$$
= \left[ \int_0^\infty \sum_{i=1}^N s_i(t) \frac{(A^\sim Ay_T)^T(t)(Ay_T)(t)}{y_T(t)} \frac{(B^\sim By_T)^T(t)}{y_T(t)} dt \right]^{1/2} = \left[ \int_0^\infty \sum_{i=1}^N s_i(t)(A^\sim Ay_{i,T})^T(t)y_T(t)dt \right]^{1/2}.
$$

By substituting in Eq. (6.16a) for $y_T$ once again, then simplifying using Eq. (6.17) we have

$$
\|Ay_T\|_2 = \left\| \sum_{i=1}^N s_i (A^\sim Ay_i) \right\|_2.
$$

Using the triangle inequality, then manipulating further we have

$$
\|Ay_T\|_2 \leq \sum_{i=1}^N \|s_i (A^\sim Ay_i)\|_2 \leq \sum_{i=1}^N \|s_i\|_\infty \|Ay_{i,T}\|_2.
$$

Knowing that each hybrid subsystem satisfies Eq. (6.13), we can write

$$
\|Ay_T\|_2 \leq \sum_{i=1}^N \|s_i\|_\infty \kappa_i \|Au_{i,T}\|_2.
$$

By performing a few more tedious manipulations using Eqs. (6.16b) and (6.17) it can be shown that

$$
\|Ay_T\|_2 \leq \left\| \sum_{i=1}^N \|s_i\|_\infty^2 \kappa_i \right\|_{\kappa} \|Au_T\|_2.
$$

Thus, provided that $0 < \|s_i\|_\infty < \infty$ and $0 < \kappa_i < \infty$ the system has finite gain when passivity holds.

The finite gain nature of the scheduled subsystems when passivity has been violated is shown
in an identical fashion, resulting in

\[
\| B y_T \|_2 \leq \left[ \sum_{i=1}^{N} \| s_i \|_\infty^2 \gamma_i \right] \| B u_T \|_2
\]

where \( 0 < \gamma < \infty \).

Combining the above results shows that when a set of hybrid VSP/finite gain subsystems are gain-scheduled appropriately, the overall gain-scheduled system is also a hybrid VSP/finite gain system.

The above result is quite powerful; given a set of hybrid VSP/finite gain subsystems, the overall scheduled system is also a hybrid VSP/finite gain system. In the context of control of a nonlinear hybrid passive/finite gain system, a set of LTI hybrid VSP/finite gain controllers (designed in some fashion) scheduled as in Fig. 6.2 are guaranteed to stabilize the closed-loop via the Hybrid Passivity and Finite Gain Stability Theorem. How to design the controllers that make up the gain-scheduled controller is discussed in Chapter 9.

6.4 Summary

This chapter’s focus was the scheduling of various systems: passive systems, conic systems, and hybrid VSP/finite gain systems. Although the theorems pertaining to the scheduling of VSP and ISP systems were presented as review (as they are from Refs. 78 and 79), Theorems 6.2.1 and 6.3.1 are each new and novel results. Theorem 6.2.1 proves that when the subsystems are conic (in the traditional sense) and the subsystems are scheduled as in Fig. 6.3, the overall gain-scheduled system is also conic. Depending on the conic bounds of the subsystems (specifically, the cone centres and cone radii), the cone describing the scheduled input-output map may have been enlarged relative to the size of the individual subsystem cones (perhaps leading to a conservative result). Theorem 6.3.1 proves that a set of hybrid VSP/finite gain subsystems scheduled as in Fig. 6.2 also has hybrid VSP/finite gain properties. Theorem 6.3.1 will be explicitly used in Chapter 9, where the gain-scheduled control of a flexible manipulator will be considered. What is quite interesting about all the gain-scheduling results presented in this chapter is that the scheduling signals i) can be functions of any parameter (e.g., the plant output, the plant state, time, etc.), and ii) the scheduling signals do not have to be linear functions of the scheduling variable.
Part II

Applications to Aerospace Systems
Chapter 7

Linear Time-Varying Passivity-Based Attitude Control Using Magnetic and Mechanical Actuation

Spacecraft are usually equipped with some sort of attitude control system. Attitude control systems are usually required to enable pointing, slewing, or trajectory tracking (if not all), as well as disturbance rejection. Although spacecraft in geocentric orbits can exploit environmental disturbance torques such as aerodynamic, gravity gradient, or those associated with solar-radiation pressure for attitude control or actuator desaturation, critical pointing is usually delegated to mechanical actuators, such as reaction wheels. Another viable actuation scheme enabling attitude control is magnetic actuation whereby on-board magnetic dipole moments (created via current carrying coils) interact with the geomagnetic field thereby creating torques.\(^{80,81}\)

All attitude control schemes that rely solely on magnetic actuation suffer from the same fundamental problem: instantaneous underactuation. A general control torque cannot be realized instantaneously by magnetic actuation alone owing to the fact the magnetic torque vector is generated via the cross-product of the magnetic field vector and the dipole moment vector. Although the stability of spacecraft equipped with magnetic actuators can be guaranteed on average,\(^{82-84}\) improved performance in terms of pointing accuracy is desired.

As a spacecraft orbits the Earth the geomagnetic field properties relative to the spacecraft change in a (almost) periodic fashion. This periodic change has led many authors to investigate linear time-varying or linear periodic control schemes to be used in conjunction with magnetic actuation.\(^{85-91}\) For instance, Refs. 85–89 consider periodic state feedback control, while Refs. 90, 91 consider periodic output feedback control. Some authors explicitly consider disturbance rejection in the design of their periodic control law.\(^{85,88}\) Often Floquet analysis is used to assess stability of the closed-loop system;\(^{86,87,89}\) however, results that do not require a posteriori stability analysis are available.\(^{90,91}\)

Other authors have explored the use of tandem actuation, that is the collaborative use of magnetic and mechanical actuation, in order to, among other objectives, improve performance.
Refs. 92 and 93 consider spacecraft attitude control using reaction wheels and magnetic torque rods, while Ref. 94 utilizes thrusters and magnetic torque rods. Although both Refs. 92 and 93 use reaction wheels and magnetic torque rods in tandem, the control scheme of Ref. 92 allows for overlap of control torques, which is undesirable. The control architecture presented in Ref. 93 ensures torques from each actuator are orthogonal, disallowing the possibility of having control torques overlap and in effect cancelling each other over short periods of time.

Tandem magnetic and mechanical actuation has also been used after hardware failures have occurred to restore complete three axis control, thus saving the mission in question. For example, the primary and secondary pitch axis wheels of RADARSAT-1 failed on orbit, thus rendering the pitch axis uncontrollable. Similarly, the FUSE spacecraft experienced two reaction wheel failures (out of four), rendering three axis control impossible. In both scenarios, three axis control was restored via some form of tandem magnetic and mechanical actuation.

This chapter is concerned with passivity-based spacecraft attitude control subject to gravity-gradient disturbances. It is well known that the input-output map between body torques and the angular velocity of a spacecraft is passive. A passivity-based design approach is taken in order to guarantee robust closed-loop stability. The spacecraft attitude controller will be composed of quaternion-based proportional control and angular velocity based rate control. The rate controller is composed of a LTV ISP system, as defined in Theorem 4.3.2 of Chapter 4 (see page 49). Using a LTV system for control is motivated by the (almost) periodic nature of the linearized spacecraft model, where the periodicity is a result of the magnetic field imposing its periodic nature upon the system. Both magnetic and mechanical actuation will be used, where the desired control torque will be distributed based on the physical constraints of the magnetic actuators. This chapter can, in essence, be considered an applications chapter, where we are using some of the passivity LTV systems theory of Chapter 4 for control purposes.

7.1 Review of Passive, Input Strictly Passive, and Linear Time-Varying Systems

Before investigating the control formulation to be developed and employed, let us review passive systems, ISP systems, the Weak Passivity Theorem, and ISP LTV systems (see Sec. 3.4 starting on page 22, Sec. 3.7.2 starting on page 36, and Sec. 4.3 starting on page 47).

A general square system with inputs $u \in L_{2e}(U)$ and outputs $y \in L_{2e}(Y)$ mapped through the operator $G : L_{2e}(U) \rightarrow L_{2e}(Y)$ is passive if there exists a constant $\beta$ such that

$$\int_0^T y^T(t)u(t)dt \geq \beta, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+$$  \hspace{1cm} (7.1)

and is ISP if there exist $\beta$ and $\delta > 0$ such that

$$\int_0^T y^T(t)u(t)dt \geq \delta \int_0^T u^T(t)u(t)dt + \beta, \quad \forall u \in L_{2e}(U), \quad \forall T \in \mathbb{R}^+.$$  \hspace{1cm} (7.2)
The weak version of the Passivity Theorem (i.e., the Weak Passivity Theorem; see Theorem 3.7.2 on page 36) states that the negative feedback interconnection of a passive system and an ISP system is $L_2$ stable.\(^{16}\)

As discussed in Sec. 3.2.2, in this chapter we will be interested in LTV systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (7.3a)$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad y \in \mathbb{R}^m, \quad (7.3b)$$

where $A : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n}$, $B : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times m}$, $C : \mathbb{R}^+ \rightarrow \mathbb{R}^{m \times n}$, and $D : \mathbb{R}^+ \rightarrow \mathbb{R}^{m \times m}$ are appropriately dimensioned real matrices that are continuous and bounded over the time interval of interest. It is assumed that the system is both controllable and observable. The nominal input-output equations are specified by Eqs. (7.3a) and (7.3b), while an alternate output is

$$z(t) = L(t)x(t) + W(t)u(t), \quad z \in \mathbb{R}^m$$

where $L : \mathbb{R}^+ \rightarrow \mathbb{R}^{m \times n}$, and $W : \mathbb{R}^+ \rightarrow \mathbb{R}^{m \times m}$. We will assume controllability of $(A(\cdot),B(\cdot))$, and observability of $(C(\cdot),A(\cdot))$ and $(L(\cdot),A(\cdot))$.

Recall Theorem 4.3.2 from Sec. 4.3 (see page 49), which states that a LTV system described by Eqs. (7.3a) and (7.3b) that is controllable and observable with $D(t) > 0$ is ISP if there exist continuous, bounded $P(t) = P^T(t) > 0$, $L(\cdot)$, $W(\cdot)$, and $\delta > 0$ such that

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -L^T(t)L(t), \quad (7.4a)$$

$$C^T(t) - P(t)B(t) = L^T(t)W(t), \quad (7.4b)$$

$$\ddot{D}(t) + \dot{D}^T(t) = W^T(t)W(t) \quad (7.4c)$$

where $D(t) = \dot{D}(t) + \delta 1 > 0$.

For the reminder of this chapter we will generally neglect writing the temporal argument of functions and time-varying matrices unless we deliberately want to emphasize a variable is time-varying.

### 7.2 Spacecraft Kinematics, Dynamics, Disturbances, and Actuation

The rotational dynamics of a generic rigid-body spacecraft in low Earth orbit are governed by Euler’s equation\(^{97}\)

$$I\dot{\omega} + \omega \times I\omega = \tau_d + \tau_w + \tau_m \quad (7.5)$$
where $\mathbf{I}$ is the moment of inertia matrix, $\boldsymbol{\omega}$ is the angular velocity of the spacecraft expressed in the body-fixed frame, $\tau_d$ is the disturbance torque, and

$$
\mathbf{a}^\times = \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
a_2 & a_1 & 0
\end{bmatrix}
$$

is a skew-symmetric matrix satisfying $\mathbf{a}^\times \mathbf{T} = -\mathbf{a}^\times$ where $\mathbf{a} = [a_1 \ a_2 \ a_3]^T$. The control torque, $\mathbf{u}$, will be distributed between reaction wheel torques, $\tau_w$, and magnetic torques, $\tau_m(t) = \mathbf{b}^\times \mathbf{T}(t) \mathbf{m}(t)$, where $\mathbf{b}$ is the Earth’s magnetic field vector expressed in the body-fixed frame, and $\mathbf{m}$ is the magnetic dipole moment. The vector $\mathbf{b}$ is not constant as a result of the spacecraft changing position and attitude while on orbit. The geomagnetic field vector expressed in the inertial frame is $\mathbf{b}_i$, and $\mathbf{b} = \mathbf{C}_{bi} \mathbf{b}_i$ where $\mathbf{C}_{bi}$ is the rotation matrix from the inertial frame to the body-fixed frame.

We elect to neglect disturbance torques that arise via aerodynamics (which dominate at low altitudes) and solar radiation pressure (which dominate at geostationary altitudes); the gravity-gradient torque will be considered the primary disturbance torque \(^{97}\)

$$
\tau_d = \frac{3\mu}{r^5} \mathbf{r} \times \mathbf{r}
$$

where $\mu = 3.98593 \times 10^{14} \text{ (m}^3/\text{s}^2\text{)}$ is the Earth’s gravitational constant, $\mathbf{r}$ is the position of the spacecraft relative to the Earth expressed in the spacecraft body frame, and $r = \sqrt{\mathbf{r}^T \mathbf{r}}$.

The spacecraft kinematics (i.e., the attitude of a spacecraft) can be described by the four parameter quaternion set $\mathbf{\epsilon} = [\epsilon_1 \ \epsilon_2 \ \epsilon_3]^T$ and $\eta$ that together satisfy $\mathbf{\epsilon}^T \mathbf{\epsilon} + \eta^2 = 1$. \(^{97}\) The quaternion rates and the angular velocity are related by

$$
\begin{bmatrix}
\dot{\mathbf{\epsilon}} \\
\dot{\eta}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\eta \mathbf{1} + \mathbf{\epsilon}^\times \\
-\mathbf{\epsilon}^T
\end{bmatrix} \boldsymbol{\omega} \quad \text{or} \quad \boldsymbol{\omega} = 2 \left[ (\eta \mathbf{1} - \mathbf{\epsilon}^\times) - \mathbf{\epsilon} \right] \begin{bmatrix}
\dot{\mathbf{\epsilon}} \\
\dot{\eta}
\end{bmatrix}.
\quad (7.7)
$$

### 7.3 Control Formulation

Neglecting the disturbance torque momentarily, consider a spacecraft endowed with three orthogonal reaction wheels and three orthogonal magnetic torque rods. The desired control torque, $\mathbf{u}$, will be composed of proportional control, $\mathbf{u}_p$, and rate control, $\mathbf{u}_r$:

$$
\mathbf{u} = \mathbf{u}_p + \mathbf{u}_r = \tau_w + \tau_m.
$$

The manner in which the desired control torque will be decomposed into wheel torques and magnetic torques will be discussed in Sec. 7.3.3.
7.3.1 Passivity Properties of a Spacecraft Compensated by Proportional Control

Proportional control of the form $u_p = -k \epsilon$ will be employed. In the following theorem we will interpret the plant compensated by proportional control in terms of a passive input-output map between $u_r$ and $\omega$.

**Theorem 7.3.1.** A spacecraft described by Eqs. (7.5) and (7.7) (with $\tau_d = 0$) compensated with proportional control of the form $u_p = -k \epsilon$ for $0 < k < \infty$ possesses a passive input-output map between $u_r$ and $\omega$.

**Proof.** Consider the following Lyapunov-like function, its temporal derivative, and subsequent simplification:

$$V = \frac{1}{2} \omega^T I \omega + k \left[ \epsilon^T \epsilon + (\eta - 1)^2 \right],$$
$$\dot{V} = \omega^T (-\omega \times I \omega + u_p + u_r) + 2k \left[ \epsilon^T \dot{\epsilon} + (\eta - 1)\dot{\eta} \right]$$
$$= -k \omega^T \epsilon + \omega^T u_r + k \left[ \epsilon^T (\eta I + \epsilon^x) \omega - (\eta - 1)\epsilon^T \omega \right]$$
$$= \omega^T u_r$$

where we have used Eq. (7.7) to simplify. Integrating the result between 0 and $T$ delivers

$$\int_0^T \omega^T u_r dt = \int_0^T \dot{V} dt = V(T) - V(0) \geq -\beta$$

which completes the proof. \qed

Notice that the spacecraft compensated with proportional control is passive for any inertia matrix $I$ and any positive but finite $k$; the passive nature of the system does not hinge on particular numeric values.

7.3.2 Input Strictly Passive Rate Control

Given that a spacecraft compensated by proportional control is passive, closed-loop stability of the system can be realized via the weak version of the Passivity Theorem provided the rate controller (to be connected in a negative feedback loop) is ISP. The controller input is $\omega$, while the output is $v_r = -u_r$. Consider the following rate control:

$$v_r = \delta b \hat{b}^T \omega + \hat{b}^T \mathcal{G} \left( \hat{b}^x \omega \right) \quad (7.8)$$

where $\hat{b}$ denotes a unit vector (i.e., $\hat{b} = b^{-1} b$), and $\mathcal{G}$ is an ISP operator satisfying Eq. (7.2). The $\delta$ in Eq. (7.8) is the same as the $\delta$ in Eq. (7.2) associated with $\mathcal{G}$. In the following theorem we will show that the map from $\omega$ to $v_r$ is ISP.

**Theorem 7.3.2.** The map between $\omega$ and $v_r$ is ISP, where $v_r$ is given in Eq. (7.8).

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Proof. Consider the following integral:

\[
\int_0^T \omega^T v_r \, dt = \int_0^T \omega^T \left[ \delta b b^T \omega + b^\times G (b^\times \omega) \right] \, dt \\
\geq \delta \int_0^T \omega^T (b b^T + b^\times b^\times) \omega \, dt + \beta \\
= \delta \int_0^T \omega^T \omega \, dt + \beta
\]

where we have used the identity \( b^\times b^\times = 1 - b b^T \). \qed

### 7.3.3 Distribution of Control Between Wheel Torques and Magnetic Torques

The control torques applied to the spacecraft will be distributed as follows:

\[
u = \tau_w + \tau_m = b b^T (-k \epsilon - \delta \omega) + b^\times \left[ -k b^\times \epsilon + G (b^\times \omega) \right].\] (7.9)

It is straightforward to have the reaction wheels apply the torque commanded to them, but the magnetic torques must be applied in the following way:

\[
\tau_m = b^\times m, \quad m = -|b|^{-1} \left[ k b^\times \epsilon + G (b^\times \omega) \right].
\]

The distribution presented in Eq. (7.9) can be interpreted in the following way using the results of Ref. 93. Recall that the torque created by magnetic actuation is restricted to lie in a plane orthogonal to the instantaneous magnetic field vector. The above distribution allocates control torques such that any torque that lies in \( \text{Ker} \{ b^\times \} \), that is parallel to \( b \), is applied by the reaction wheels. Torques that lie in \( \text{Im} \{ b^\times \} \), that is perpendicular to \( b \), are applied by the magnetic torque rods.

Control torques are distributed in this tandem manner in order to reduce reaction wheel load. The overall load experienced by the reaction wheels is less (because the wheels are applying lower torques to the spacecraft), leading to longer wheel life, or the use of smaller wheels. Depending on the spacecraft mission, ensuring longer wheel life may be more important (i.e., improving mission robustness and reliability). Alternatively, having smaller wheels may be desirable, especially in the context of micro- and nano-satellites. Smaller wheels may realize a reduced spacecraft mass, or the opportunity to increase the spacecraft payload.

### 7.4 Controller Design and Synthesis

Given the previously developed control formulation and decomposition, we are poised to design the ISP operator \( G \). Given that \( G \) must be ISP and the system to be controlled possesses periodic properties, \( G \) will be realized by an ISP LTV system, and must satisfy Theorem 4.3.2, which is presented on page 49. We will design our controller \( G \) based on a linearized model of the spacecraft.
to be controlled.

We will start by linearizing the spacecraft kinematics. We choose zero angular displacement and zero angular rate of the body-fixed frame relative to the inertial frame as our linearization point. Assuming small angles and rates we have \( \theta = 2\epsilon \), i.e., \( C_{bi} = 1 - \theta^\times \), and \( \dot{\theta} = \omega \). Next, we will linearize the spacecraft dynamics. To do so, we will write the system to be controlled in the following way:

\[
I\ddot{\theta} + \dot{\theta}^\times I\dot{\theta} = -k\epsilon - \delta \dot{\theta}^\times + \dot{\theta}^\times v \tag{7.10a}
\]

\[
= -k\epsilon - \delta C_{bi} \dot{b}_i^T C_{bi}^T \theta + C_{bi} \dot{b}_i^T C_{bi}^T v \tag{7.10b}
\]

where \( v = G_y \) and \( y = \dot{b}^\times \omega = C_{bi} \dot{b}_i^T C_{bi}^T \omega \) are the output and input of the ISP LTV controller, respectively, and \( \tau_d = 0 \). Notice that we used the identity \( (C_{bi} \dot{b}_i)^\times = C_{bi} \dot{b}_i^T C_{bi}^T \).

Substitution of the linearized kinematics into Eq. \( 7.10 \) gives

\[
I\ddot{\theta} + \dot{\theta}^\times I\dot{\theta} = -k\epsilon - \delta (1 - \theta^\times) \dot{b}_i^T (1 - \theta^\times)^T \theta + (1 - \theta^\times) \dot{b}_i^T (1 - \theta^\times)^T v. \tag{7.11}
\]

By neglecting terms that are of order greater than one Eq. \( 7.11 \) becomes

\[
I\ddot{\theta} = -k\epsilon - \delta \dot{b}_i^T \theta + \dot{b}_i^T v
\]

which can be written in conjunction with \( y = \dot{b}_i^\times \) in first order state-space form:

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-\frac{1}{2}I^{-1} & -\delta I^{-1} \dot{b}_i^T
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
I^{-1} \dot{b}_i^T
\end{bmatrix}
v(t),
\]

\[
y(t) = \begin{bmatrix}
0 & \dot{b}_i^\times
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix}
\tag{7.12a}
\]

Eq. \( 7.12 \) is the linearized plant model. We have written the temporal arguments in Eq. \( 7.12 \) to emphasize the linear model is time-varying.

The ISP controller \( G \) to be designed based on Eq. \( 7.12 \) can be written in state-space form as

\[
\dot{x}_c(t) = A_c(t)x_c(t) + B_c(t)y(t), \tag{7.13a}
\]

\[
v(t) = C_c(t)x_c(t) + D_c(t)y(t). \tag{7.13b}
\]

Rather than arbitrarily assigning the \( A_c(\cdot) \) and \( C_c(\cdot) \) matrices we will design them via the well known LQR formulation. Given the performance index\(^{59,100}\)

\[
J = x^T(T)Sx(T) + \int_0^T \left[ x^T(t)Mx(t) + v^T(t)Nv(t) \right] dt
\]
where \( S = S^T \geq 0, M = M^T \geq 0 \) and \( N = N^T > 0 \), one can derive an optimal state feedback \( C_c(t) = N^{-1}B^T(t)X(t) \). The matrix \( X(\cdot) = X^T(\cdot) \geq 0 \) can be found by solving the following time-varying matrix Riccati equation:

\[
-\dot{X}(t) = M + A^T(t)X(t) + X(t)A(t) - X(t)B(t)N^{-1}B^T(t)X(t), \quad X(T) = S. \tag{7.14}
\]

The matrix Riccati equation must be solved backward in time from \( t = T \) to \( t = 0 \) given the boundary condition \( X(T) = S \). Following the LQR formulation, we will let \( A_c(t) = A(t) - B(t)C_c(t) \). Given that we have specified \( A_c(\cdot) \) and \( C_c(\cdot) \), we must now design \( B_c(\cdot) \) and \( D_c(\cdot) \) such that the controller is ISP according to Theorem 4.3.2. By specifying \( L(\cdot) \) appropriately (i.e., so that \( (L(\cdot), A_c(\cdot)) \) is observable) and \( D_c(\cdot) \) to be positive definite (or alternatively \( D(\cdot) \), which in turn dictates \( W(\cdot) \)), the matrix \( B_c(\cdot) \) can be solved for via Eq. (7.4b):

\[
B_c(t) = P^{-1}(t) \left[ C_c^T(t) - L^T(t)W(t) \right]
\]

where \( P(\cdot) \) is found by solving Eq. (7.4a)

\[
\dot{P}(t) + P(t)A_c(t) + A_c^T(t)P(t) = -L^T(t)L(t) \tag{7.15}
\]

backwards in time from \( t = T \) to \( t = 0 \) given the boundary condition \( P(T) > 0 \). The resultant \( A_c(\cdot), B_c(\cdot), C_c(\cdot), \) and \( D_c(\cdot) \) compose an ISP controller of the form presented in Eq. (7.13).

The synthesis procedure presented above will always yield an ISP operator \( G \), even if the assumed spacecraft inertia matrix \( I \) and orbit (and hence \( b_i \) values) are incorrect.

### 7.5 Numerical Example

We will consider a spacecraft with moment of inertia matrix \( I = \text{diag} \{ 27, 17, 25 \} \) (kg \( \cdot \) m\(^2\)). The spacecraft is in a circular Keplerian orbit at an altitude of 450 (km), and inclination of 87\(^\circ\). The angle of the right ascension of the ascending node, argument of perigee, and time of perigee passage are equal to zero. We will use the magnetic field model described by Wertz\(^{98} \) Appendix H, and restated in Ref. 92.

The spacecraft is equipped with three reaction wheels and three torque rods. The controller is to regulate the quaternions to \( e = 0, \eta = 1 \), and the angular velocity to \( \omega = 0 \), while simultaneously rejecting the gravity-gradient disturbance torque of Eq. (7.6). Control will be computed using the design and synthesis method presented in Sec. 7.4. We will set \( k = 7.5 \times 10^{-4} \) (N \( \cdot \) m) and \( \delta = 5 \times 10^{-5} \) (N \( \cdot \) m \( \cdot \) s). The LQR weights will be \( M = \text{diag} \{ 1.5 \times 10^{-3}, 1.5 \times 10^{-3}, 1.5 \times 10^{-3}, 1, 1, 1 \} \) and \( N = \text{diag} \{ 1 \times 10^4, 1 \times 10^4, 1 \times 10^4 \} \), while \( X(T) = 1 \) where \( 1 \) is identity. The input strictly passive controller will be weighted by \( L = [(1 \times 10^{-8})I \ (10)I] \), while we will set \( D_c(t) = [2\delta + \frac{1}{80}(-\cos(t/T_0) + 1)]I \) where \( T_0 \) is the orbital period. Letting \( D(t) = [\delta + \frac{1}{80}(-\cos(t/T_0) + 1)]I \), it follows that

\[
W(t) = \sqrt{2 \left[ \delta + \frac{1}{80}(-\cos(t/T_0) + 1) \right]}I \text{ by Eq. (7.4c)}.
\]

Similarly, \( P(T) = 1 \) which will be required
to solve Eq. (7.15). The spacecraft initial conditions are \( \mathbf{\epsilon} = [-1/2 \ 1/2 \ 1/2]^T \), \( \eta = -1/2 \), and \( \omega = [0.02 \ -0.02 \ 0.02]^T \) (rad/s).

Fig. 7.1 shows the angular velocity and quaternion evolution versus time, while Fig. 7.2 shows the magnetic torques and reaction wheel torques versus time. Fig. 7.3 shows the combined torque \( \tau_w + \tau_m \) versus time and the magnetic dipole moments versus time, and Fig. 7.4 shows the maximum eigenvalues of the matrices \( \mathbf{X}(\cdot) \) and \( \mathbf{P}(\cdot) \) versus time; notice the nearly periodic nature of both indicating that the controller is not only time-varying, but close to periodic as well.
Recall the general structure of the rate control presented in Eq. (7.8). Let \( \mathbf{u}_{r,w} = -\delta \hat{\mathbf{b}} \hat{\mathbf{b}}^T \omega \) be the portion of the rate control applied by the reaction wheels and \( \mathbf{u}_{r,m} = -\hat{\mathbf{b}}^T \mathbf{G} \left( \hat{\mathbf{b}} \times \omega \right) \) be the portion applied by the magnetic torque rods. The instantaneous power of the wheels is then \( P_w = \omega^T \mathbf{u}_{r,w} \) while the instantaneous power of the magnetic torque rods is \( P_m = \omega^T \mathbf{u}_{r,m} \). Fig. 7.5 shows the total work done (i.e., the total energy dissipated) by each actuator as a function of time where the total work done is \( W_{\text{total}} = \int_0^t P(t') dt' \) and \( P \) is \( P_w \) or \( P_m \). The total work each actuator does after five orbits is \( W_{\text{total},w} = -9.9324 \times 10^{-5} \) (J) and \( W_{\text{total},m} = -0.0159 \) (J). Clearly the magnetic torque rods dissipate more energy than the reaction wheels do. This in turn means the wheels are not working as hard.

The advantage of passivity-based control is that robust stability can be achieved given model uncertainty. In our particular case, robust stability of a nonlinear plant is guaranteed to be stable via a linear time-varying input strictly passive controller. This is true given both spacecraft and magnetic field model perturbations. Previous studies where actuation is solely magnetic have only
been able to assess the stability of the true system via simulation. For example, Ref. 89 perturbs the assumed inertia matrix of the spacecraft by 22% and presents simulations that indicate “some signs of instability”. Ref. 87 perturbs the spacecraft inertia matrix (by 25% and 30%) and the orbit (by changing the altitude and eccentricity) and is able to show robust stability in simulation.

An example of robust closed-loop stability will be shown next. We will change both the spacecraft inertia properties, and orbit. The principal axes of the inertia matrix are reduced by 25% representing fuel loss, while the altitude, inclination, and eccentricity of the orbit have been changed to 500 (km), 67°, e = 0.05 respectively. Fig. 7.6 shows the spacecraft angular velocity and quaternion evolution versus time given these changes as controlled by the same controller developed for the assumed model and orbit. The system response is only moderately changed given these significant changes, highlighting the robust nature of the passivity-based control scheme employed.
Figure 7.4: Maximum eigenvalues of $X(\cdot)$ and $P(\cdot)$ versus orbit.

Figure 7.5: Total work done by reaction wheels and magnetic torque rods versus orbit.
Figure 7.6: Angular velocity and quaternion response when spacecraft inertia and orbit have been perturbed.

### 7.6 Practical Considerations and Additional Comments

Some remarks with respect to implementation of the proposed control architecture are in order. To start, recall that the steady-state solutions of $X(\cdot)$ and $P(\cdot)$ are acquired via backward integration of Eqs. (7.14) and (7.15), then $A_c(\cdot)$, $B_c(\cdot)$, $C_c(\cdot)$, and $D_c(\cdot)$ are computed given $X(\cdot)$ and $P(\cdot)$. Because the linearized system Eq. (7.12) is not perfectly periodic, the steady-state solutions of $X(\cdot)$ and $P(\cdot)$ are not perfectly periodic either. However, the solution of $X(\cdot)$ and $P(\cdot)$ are very close to periodic, and we can approximate them as periodic.

As previously mentioned (e.g., above, and in Sec. 4.2.1), $X(\cdot)$ and $P(\cdot)$ must be solved for numerically given $X(T)$ and $P(T)$. From a theoretical point of view, the choice of $T$ (the terminal time) is somewhat irrelevant as long as $X(T) = S \geq 0$ and $P(T) > 0$. From a practical point of view, one may simply pick $T$ to be 10 or 100 times the life of the spacecraft, thus ensuring $X(\cdot)$ and $P(\cdot)$ can be solved for over the active life of the spacecraft.

It is unreasonable to store the steady-state solutions of $X(\cdot)$ and $P(\cdot)$ (from $t = 0$ to $t = T$)
on-board the spacecraft. In practice, the upper (or lower) triangular parts of $X(\cdot)$ and $P(\cdot)$ would be approximated by a Fourier series. (Recall, $X(\cdot)$ and $P(\cdot)$ are symmetric, hence approximating every element of the matrices is unnecessary.) The Fourier coefficients associated with each Fourier series used to approximate the appropriate elements of $X(\cdot)$ and $P(\cdot)$ would then be stored on-board. Other approximations may be used; Ref. 85 suggests using a spline approximation.

### 7.7 Summary

In this chapter, the attitude control of a spacecraft equipped with both reaction wheels and magnetic torque rods has been considered. We have shown that a spacecraft compensated by proportional control possesses a passive input-output map, enabling the use of an ISP LTV controller to perform rate control. Closed-loop stability of the spacecraft angular velocity is guaranteed by the Passivity Theorem, resulting in robust closed-loop properties. Control torques are distributed between reaction wheels and magnetic torque rods in a natural way guided by the physical constraints imposed by magnetic actuation. In order to synthesize an input strictly passive controller, the spacecraft plus control distribution architecture is linearized; the linearized state-space model is time-varying. Theorem 4.3.2 from Sec. 4.3 is used along with the standard LQR formulation to design an ISP LTV feedback controller based on the linearized system. The resultant controller was found to be nearly periodic. Numerical simulation shows our scheme works well.
Chapter 8

Single-Link Flexible Manipulator Control Accommodating Passivity Violations: Theory and Experiments

Chapter 7 looked at the application of the theory presented in Chapter 4. The purpose of this chapter, and Chapter 9, is to apply the hybrid passive/finite gain systems framework of Chapter 5.

In this chapter we will investigate tip position and rate control of a single-link flexible manipulator. As we will show, a passive input-output map exists between a modified joint torque and a modified tip-rate, called the $\mu$-tip rate. The $\mu$-tip theory we employ was originally developed in Ref. 101, and is similar to the so-called “reflected-tip” theory investigated in Refs. 102 and 103.

Although the map between the modified joint torque and the $\mu$-tip is passive, as discussed in the motivation section of Chapter 5, a simple sensor, actuator, or filter will destroy the passive input-output map used as the basis for robust stabilization via the Passivity Theorem. As we will show, although the passive map between the modified joint torque and the $\mu$-tip is violated (specifically by a derivative filter), the system is then rendered hybrid possessing passive (PR) properties at low frequency, and finite gain (BR) properties at high frequency when passivity no longer holds. As such, the Hybrid Passivity/Finite Gain Stability Theorem can be used to guarantee closed-loop stability.

Given that the system has hybrid properties, we can guarantee closed-loop stability via Hybrid Passivity/Finite Gain Stability Theorem when the controller is hybrid VSP/finite gain. We consider the optimal design of a VSP/finite gain (that is a FF SPR/BR) controller. We pose a numerical optimization problem where the objective function attempts to have the FF SPR/BR controller “mimic” an unconstrained $H_2$ controller.

Having investigated the passive nature of the plant to be controlled, the form of the passivity violation experienced, and the synthesis of a FF SPR/BR controller, this chapter closes with experimental results. An optimal FF SPR/BR controller is used to control a single-link system carrying a large payload. We show that the FF SPR/BR controller synthesized using the opti-
mization formulation developed successfully stabilizes the single-link manipulator, demonstrating the success of not only the optimization procedure, but the hybrid systems framework as well.

8.1 Single-Link Flexible Manipulator Dynamics

8.1.1 Input-Output Model

In this chapter we will control the tip position and rate of a planar single-link flexible manipulator carrying a large payload described by

\[ M \ddot{q} + D \dot{q} + Kq = b \tau \]  

(8.1)

where \( M = M^T > 0 \), \( D = D^T \geq 0 \), and \( K = K^T \geq 0 \) are the mass, damping, and stiffness matrices, \( b = [1 \ 0]^T \), \( \tau \) is the joint torque, \( q = [\theta \ q_e]^T \), \( \theta \) is the joint angle of the hub, and \( q_e \) are the elastic coordinates associated with the flexible link discretization. The output we are interested in controlling is the \( \mu \)-tip rate

\[ y := \dot{\rho} = J_\theta \dot{\theta} + \mu J_e \dot{q}_e \]  

(8.2)

where \( J_\theta \) is the rigid Jacobian, \( J_e \) is the elastic Jacobian, and \( \mu \) is a fixed scalar parameter that will help us define a passive input-output map. When \( \mu = 1 \) the output is the true-tip rate, \( \dot{\rho} = \dot{\rho}_{\mu=1} \), when \( \mu = 0 \) the output is the rigid-tip rate, and when \( \mu = -1 \) the output is the reflected-tip rate as discussed in Ref. 102 and Ref. 103. Various authors have considered true-tip position and rate control; for example, one of the earliest works is Ref. 105. The non-collocated mapping between the joint torque and the true-tip rate is nonminimum phase, and hence not passive. However, if \( 0 \leq \mu < 1 \) and the manipulator is carrying a large payload, a passive input-output map can be defined.

Scaling \( J_e \) by \( \mu \) when \( 0 \leq \mu < 1 \) can be thought of as artificially retarding the natural flexibility of the link. From Eq. (8.2) we can equivalently write \( \dot{\rho}_\mu \) as

\[ \dot{\rho}_\mu = J_\theta \dot{\theta} + \mu J_e \dot{q}_e + \mu J_\theta \dot{\theta} - \mu J_\theta \dot{\theta} = (1 - \mu) J_\theta \dot{\theta} + \mu \left( J_\theta \dot{\theta} + J_e \dot{q}_e \right) = (1 - \mu) J_\theta \dot{\theta} + \mu \dot{\rho}. \]  

(8.3)

The above provides an expression for the \( \mu \)-tip position:

\[ \rho_\mu = (1 - \mu) J_\theta \theta + \mu \rho. \]  

(8.4)

By defining \( u = J_\theta^{-1} \tau \), which will be referred to as the modified joint torque, we can capture the
modified input-output dynamics in terms of a transfer function:

\[
y(s) = g(s)u(s),
\]

\[
g(s) = \frac{M_{\theta\theta}^{-1}J_{\theta}^2}{s} + \sum_{\alpha=1}^{N_e} \frac{s}{s^2 + 2\zeta_\alpha \omega_\alpha s + \omega_\alpha^2} c_\alpha b_\alpha,
\]

(8.5)

where \( g(s) = s\rho \mu(s), \) \( u(s) = J_{\theta}^{-1}\tau(s), \) \( M_{\theta\theta} \) is the rigid portion of the mass matrix, and \( \zeta_\alpha \) and \( \omega_\alpha \) are the damping ratios and natural frequencies associated with the \( N_e \) modes of the flexible link. The eigenvectors corresponding to the undamped, unforced form of Eq. (8.1) are \( q_\alpha = [\theta_\alpha \; q_{\alpha\alpha}^T]^T. \)

The transfer function \( g(s) \) follows from modal decomposition of Eqs. (8.1) and (8.2), followed by use of the Laplace transform. While carrying a large payload, the above transfer function is passive, that is PR, provided \( 0 \leq \mu < 1. \) Recall a PR transfer function has phase bounded by \( \pm 90^\circ, \) i.e., \(-\pi/2 \leq \arg g(j\omega) \leq \pi/2, \forall \omega \in \mathbb{R}. \) Because \( u(s) \rightarrow y(s) \) is a passive input-output map, robust closed-loop stability is guaranteed via the Passivity Theorem when a very strictly passive controller is employed. For example,

\[
-u(s) = \left[ \frac{K_p}{s} + g_{VSP}(s) \right] y(s)
\]

where \( 0 < K_p < \infty \) is the proportional control gain and \( g_{VSP}(s) \) is a very strictly passive transfer function would realize robust stabilization with respect to uncertain manipulator mass and stiffness. Unfortunately, the modified output \( y(s) \) is generally not directly available, as discussed in the next section.

### 8.1.2 Violation of Passivity

At any given time \( \theta \) and \( \rho \) are measured (using, for instance, an encoder and strain gauge), and \( \rho \mu \) can be calculated using Eq. (8.4). Using \( \rho \mu, \) proportional control of the form \( K_p\rho \mu(s) \) can be applied to the system without issue because proportional control does not alter the passivity of the system. Let \( \rho \mu(s) = g_p(s)u(s) \) be the plant with proportional control included. In order to implement rate control \( \dot{\rho} \mu \) is needed. By filtering \( \rho \mu \) with a derivative filter \( f(s), \) \( \dot{\rho} \mu \) can be approximated as \( y_1(s) = f(s)\rho \mu(s). \) If \( f(s) = s, \) that is perfect differentiation were possible, \( y_1(s) \equiv \dot{\rho} \mu(s) \) and use of the traditional Passivity Theorem would be appropriate. Unfortunately, a real derivative filter possesses dynamics and has a finite bandwidth; a realistic derivative filter is

\[
f(s) = s \left( \frac{\omega_f^2}{s^2 + 2\zeta_f \omega_f s + \omega_f^2} \right)
\]

(8.6)

and as a result the true plant output (which is the controller input) is really \( y_1(s) = g_1(s)u(s) \) where \( g_1(s) = f(s)g_p(s). \) The filter given in Eq. (8.6) differentiates at low frequency, but rolls off at high frequency. Although the nominal plant \( g(s) \) is passive, \( g_1(s) \) is not passive owing to the presence of the filter \( f(s). \) Fortunately, however, \( g_1(s) \) is still passive within a bandwidth; in a low
frequency region the passive characteristics of $g(s)$ are maintained because $f(s)$ essentially has no phase delay. As such, we would expect $g_1(s)$ to be phase bounded by $\pm 90^\circ$ at low frequency. At high frequency, however, $f(s)$ induces phase delay, thus destroying the positive realness of the plant; $g_1(s)$ is expected to have phase that exceeds $\pm 90^\circ$, that is to say passivity is violated at high frequency. However, in this high frequency region the plant will have finite gain due to the roll off of the both $g_p(s)$ and $f(s)$. The plant $g_1(s)$ can be segmented into two parts: a low frequency passive (i.e., PR) part, and a high frequency finite gain part. The hybrid systems framework is ideal for such systems.


In this section, we will review hybrid passive/finite gain systems theory in a LTI SISO context. The theory reviewed in this section is presented in Sec. 5.5.2 starting on page 81 in a LTI MIMO framework.

Consider two SISO hybrid passive/finite gain systems, $G_1 : L_{2e} \rightarrow L_{2e}$ and $G_2 : L_{2e} \rightarrow L_{2e}$, connected in a negative feedback loop as presented in Fig. 3.1 on page 18. The Hybrid Passivity and Finite Gain Stability Theorem (see Theorem 5.6.1 in Sec. 5.6 and Corollary 5.6.2 in Sec. 5.6.1) states that the negative feedback interconnection of $G_1$ and $G_2$ is $L_2$ stable if the variables $\delta_1$, $\epsilon_1$, $\gamma_1$, $\delta_2$, $\epsilon_2$, and $\gamma_2$ associated with $G_1$ and $G_2$ satisfy $\epsilon_1 + \delta_2 > 0$, $\epsilon_2 + \delta_1 > 0$, and $\gamma_1 \gamma_2 < 1$.

If the systems $G_1$ and $G_2$ are LTI, we can compute the parameters $\delta_1$, $\epsilon_1$ ($\kappa_1$), $\gamma_1$, $\delta_2$, $\epsilon_2$ ($\kappa_2$), and $\gamma_2$ by considering the frequency responses of the transfer functions $g_1(s)$ and $g_2(s)$ associated with the operators $G_1$ and $G_2$. As discussed in Sec. 5.5.2, each system with transfer function $g_i(s)$ (where $i = 1, 2$) has associated with it a $\delta_i$, $\epsilon_i$ ($\kappa_i$), and $\gamma_i$, where

\[
\delta_i = \inf_{-\omega_c < \omega < \omega_c} \text{Re} \{g_i(j\omega)\},
\]
\[
\kappa_i = \sup_{-\omega_c < \omega < \omega_c} |g_i(j\omega)|,
\]
\[
\gamma_i = \sup_{\omega \geq \omega_c} |g_i(j\omega)|
\]

and $\omega_c$ is the critical frequency.

In the context of this chapter, the LTI SISO plant to be controlled is a single-link flexible manipulator. As discussed in Sec. 8.1.2 the nominal plant is passive, but in reality passivity is violated as a result of rate signal approximation via the filter $f(s)$. The system has become hybrid possessing passive dynamics at low frequency (i.e., $\delta_1 = 0$, $\epsilon_1 = 0$), and finite gain dynamics above a critical frequency (i.e., $0 < \gamma_1 < \infty$). With the hybrid systems theory at our disposal, provided we design a controller to satisfy the hybrid stability criteria discussed above, closed-loop stability can be guaranteed. Because $\delta_1 = 0$ and $\epsilon_1 = 0$, for stability the controller must possess $\delta_2 > 0$, $0 < \kappa_2 < \infty$, and $0 < \gamma_2 < \infty$ such that $\gamma_1 \gamma_2 < 1$. (Recall that $\delta_2 > 0$, $0 < \kappa_2 < \infty$ implies $\delta_2 > 0$, $\epsilon_2 > 0$.) The optimal design of such a controller is the topic of the next section.
8.3 Controller Design Problem

The Hybrid Passivity and Finite Gain Stability Theorem provides sufficient conditions for closed-loop stability. The theorem does not address how to design an optimal hybrid VSP/finite gain controller, that is an optimal FF SPR/BR controller for a specific plant. The traditional $H_2$ formulation yields an optimal controller given certain weighting matrices on the states, control, disturbances, and noise. In theory, a $H_2$ controller should work exceptionally well in any situation, but in practice unmodeled plant dynamics and excessive noise often lead to poor closed-loop system characteristics, and in some cases an unstable closed-loop.

Knowing that a $H_2$ controller should yield a closed-loop with an optimal system response, we seek a controller that mimics the $H_2$ solution as closely as possible, but simultaneously satisfies the Hybrid Passivity/Finite Gain Stability Theorem in order to provide robustness. We seek an optimal, or a “as close to optimal as possible” FF SPR/BR controller. To find such a controller, we will pose a numerical optimization problem. Although the plant we wish to eventually control is SISO, for generality we will pose our numerical optimization problem in a MIMO context.

To properly pose our numerical optimization problem, let us review the traditional $H_2$ formulation. The nominal system (i.e., one that ignores sensors, actuators, etc. that induce passivity violations) to be controlled is

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{12}u \\
y &= C_2x + D_{21}w
\end{align*}
\]

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^nu$ is the control input, $y \in \mathbb{R}^ny$ is the measurement, $z \in \mathbb{R}^nz$ is the regulated output, the disturbances/noise are $w = [d^T v^T]^T \in \mathbb{R}^nw$, and all matrices are dimensioned appropriately. It is assumed that

1. $(A, B_1)$ is controllable and $(C_1, A)$ is observable,
2. $(A, B_2)$ is controllable and $(C_2, A)$ is observable,
3. $D_{12}^T C_1 = 0$ and $D_{12}^T D_{12} > 0$, and
4. $D_{21} B_1^T = 0$ and $D_{21} D_{21}^T > 0$.

The $H_2$ optimal controller takes the following form:

\[
\begin{align*}
\dot{x}_c &= (A - B_2 K_c - K_c C_2) x_c + K_c y_c \\
-u &= K_c x_c
\end{align*}
\]

\[\Leftrightarrow -u(s) = G^*_2(s) y(s) = K_c(sI - A_c)^{-1} K_c y(s) \quad (8.7)\]

where $K_c$ is the optimal feedback gain matrix and $K_c$ is the optimal observer gain matrix.

Now let us move on to formulating our numerical optimization problem. Consider a plant that is nominally passive, although passivity will be destroyed via sensors, actuators, etc. rendering the true plant hybrid (i.e., hybrid passive/finite gain, or FF PR/BR because the plant is LTI). Estimated values of the critical frequency, $\omega_c$, and the high frequency gain, $\gamma_1$, associated with
the FF PR/BR plant are assumed available (via approximate modeling, for example), for the true passivity violation is never fully known. To be conservative we assume \( \delta_1 = 0 \) and \( \kappa_1 \leq \infty \). Assuming \( \delta_1 = 0 \) is reasonable because the nominal plant to be controlled is PR. Assuming \( \kappa_1 \leq \infty \) is reasonable because, again, the nominal plant is PR, but also the system to be controlled most likely has finite gain, but it is difficult to characterize (such as in the context of flexible manipulator control). Therefore, a conservative assumption is \( \kappa_1 \leq \infty \).

Given our assumptions and an estimated passivity violation, for the closed-loop to be stable the controller \( G_2(s) \) must satisfy \( \delta_2 > 0, 0 < \kappa_2 < \infty, \) and \( \gamma_1 \gamma_2 < 1 \), that is the controller must be a hybrid VSP/finite gain (FF SPR/BR) controller where \( \gamma_2 \) satisfies \( \gamma_1 \gamma_2 < 1 \). A \( H_2 \) design provides the controller \( G_2^*(s) \). This controller will not necessarily be FF SPR/BR (it may by chance, however), but will perform well in simulation; it is optimal but not generally robust.

Let \( W(s) \) be a filter, \( G_{2,w}(s) = G_2(s)W(s) \), and \( G_{2,w}^*(s) = G_2^*(s)W(s) \) where

\[
W(s) = \begin{bmatrix} A_w & B_w \\ C_w & D_w \end{bmatrix}, \quad G_2(s) = \begin{bmatrix} A_c & K_e \\ K_o & 0 \end{bmatrix}, \quad G_{2,w}^*(s) = \begin{bmatrix} A_c & K_e \\ K_o & 0 \end{bmatrix}
\]

and

\[
G_{2,w}(s) = \begin{bmatrix} \tilde{A} & \tilde{B} \\ [K_o & 0] & 0 \end{bmatrix}, \quad G_{2,w}^*(s) = \begin{bmatrix} \tilde{A} & \tilde{B} \\ [K_c & 0] & 0 \end{bmatrix}.
\]

The matrices \( \tilde{A} \) and \( \tilde{B} \) depend on both the controller and filter state-space matrices:

\[
\tilde{A} = \begin{bmatrix} A_c & K_eC_w \\ 0 & A_w \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} K_eD_w \\ B_w \end{bmatrix}.
\]

Notice that the only difference between the \( H_2 \) controller, \( G_2^*(s) \), and the FF SPR/BR controller, \( G_2(s) \), to be optimally designed is the matrix \( K_o \). Let

\[
H(s) = \begin{bmatrix} \tilde{A} & \tilde{B} \\ C & 0 \end{bmatrix} = G_{2,w}^*(s) - G_{2,w}(s) = [G_2^*(s) - G_2(s)]W(s)
\]

which represents the filtered difference between the two controllers.

Consider the following objective function to be minimized: \(^{106}\)

\[
\mathcal{J}'(K_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \, H(j\omega)H^T(-j\omega) d\omega.
\]

Our objective function minimizes the filtered difference between \( G_{2,w}^*(s) \) and \( G_{2,w}(s) \) over all frequencies by changing the elements of \( K_o \) (i.e., the elements of \( K_o \) are the design variables). The filter adds flexibility into the design of \( G_2(s) \). Letting \( W(s) \) be lowpass allows \( G_2^*(s) \) and \( G_2(s) \) to differ at high frequency, while letting \( W(s) = 1 \) ensures \( G_2(s) \) is as close to \( G_2^*(s) \) as possible over all frequencies. The above objective function can be written as

\[
\mathcal{J}(K_o) = \text{tr} \, [(K_o - K_e)^T L (K_o - K_e)]
\]

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where $L$ is the upper $n \times n$ part of the matrix $L$ which is the solution of the Lyapunov equation $\dot{L}A + A^T L = -BB^T$. In a SISO context $\mathcal{J}(K_o) = (K_o - K_c)^T L (K_o - K_c)$ because $K_o, K_c \in \mathbb{R}^{1 \times n}$.

We are now poised to state our numerical optimization problem:

$$
\min_{K_o} \; \mathcal{J}(K_o) = \text{tr} \left[ (K_o - K_c)^T L (K_o - K_c) \right]
$$

w.r.t. \quad K_o

s.t. \quad \delta_2 > 0, \quad \kappa_2 < \infty, \quad \gamma_1 \gamma_2 < 1.

It is worth noting that we have chosen the above parameterization because it is simple; the SISO form of the objective function $\mathcal{J}(K_o) = (K_o - K_c)^T L (K_o - K_c)$ has a quadratic form ideal for numerical optimization. Although the objective function is quadratic (and hence convex), we cannot guarantee a global optimum can be found because the constraints are a nonlinear function of the design variables. As such, the optimum found is a local optimum.

The numerical algorithm employed to solve our optimization problem will be a Sequential Quadratic Programming (SQP) algorithm where constraints are enforced via Lagrange multipliers and derivative information is acquired via finite differencing. A SQP that employs finite differencing is used because it is simple to set up the optimization problem; only the objective function and the constraints must be provided to the optimizer. In particular, the optimization software we will use to solve our problem is `fmincon` within MATLAB’s optimization toolbox.

Before continuing, some comments on our controller design and synthesis method are in order. The way we go about designing a FF SPR/BR controller involves two approximations. First, it would be best if we could somehow synthesize a controller by directly minimizing the closed-loop $H_2$ norm subject to a set of constraints that ensure the controller is FF SPR/BR. Unfortunately, doing so is rather intractable; instead we are minimizing the difference (in a $H_2$ sense) between our FF SPR/BR controller and an unconstrained $H_2$ controller. This is our first approximation that is done so that our problem (that is, synthesizing a FF SPR/BR controller that is close to optimal) is tractable. Second, we parameterize our FF SPR/BR controller in terms of feedback gain matrix $K_o$; the dynamics matrix and input matrix of $G_2(s)$ are taken from the $H_2$ controller $G^*_2(s)$ (i.e., both $G_2(s)$ and $G^*_2(s)$ use $A_c$ and $K_e$; see Eq. (8.8)). This is our second approximation that leads to a tractable optimization problem easily solved by a numerical solver such as a SQP algorithm. To summarize, our controller design and synthesis procedure employs two approximations that lead to an optimization problem that can be easily solved.

### 8.4 Experimental Implementation and Results

#### 8.4.1 Experimental Apparatus

The theoretical developments discussed in the previous sections will be tested on an experimental apparatus, specifically, the single-link flexible manipulator shown in Fig. 8.1. This test-bed is manufactured by Quanser Consulting Inc., and is in fact a two-link apparatus. We have removed the second link thus creating a single-link test bed. (Note, control of the two-link system
will be considered in Chapter 9.) The motor and gearbox usually used to move the second link remain affixed to the end of the first link thus acting as a large payload. When we say large, we are referring to the fact the payload is much more massive than the slender flexible link.

The flexible link is made of steel, is 210.00 (mm) long, 1.27 (mm) thick, and 76.20 (mm) high. The payload mass is 0.6 (kg), while the hub inertia (including the gearbox) is $6.4 \times 10^{-2}$ (kg · m²). The flexible link’s first natural frequency is 19.5 (rad/s). Affixed to the base of the link is a strain gauge, while a digital encoder is mounted to the output shaft of the motor. The encoder provides a measurement of $\theta$, and the strain gauge and encoder together can be used to calculate the true tip position, $\rho$. Thus, as mentioned in Sec. 8.1.2, $\rho_\mu$ can be calculated via Eq. (8.4) and proportional control can be implemented in a straightforward manner. In practice, the proportional control gain is set to $K_p = 20$ (N/m). Note that $\dot{\theta}$ and $\dot{\rho}$ are not directly measured, and as mentioned in Sec. 8.1.2, $\dot{\rho}_\mu$ will be acquired by filtering $\rho_\mu(s)$ using the derivative filter $f(s)$ shown in Eq. (8.6) where $\omega_f = 45$ (rad/s) and $\zeta_f = 1.15$. Readers interested in other specific details of the apparatus are referred to Ref. 107.

Guided by the theoretical developments of Ref. 108, $\mu$ was found to be approximately 0.4. However, Ref. 108 only provides an estimate of $\mu$; in practice $\mu = 0.6$ is used, which satisfies $0 \leq \mu < 1$ and, after experimental testing, was found to provide better closed-loop performance.
8.4.2 Passivity Violation Approximation

In order to design and implement a FF SPR/BR controller, we must estimate various parameters associated with the Hybrid Passivity/Finite Gain Stability Theorem such as the critical frequency $\omega_c$ and $\gamma_1$. Recall from Sec. 8.3 that $\delta_1 = 0$ and $\kappa_1 = \infty$. It is reasonable to assume $\delta_1 = 0$ because the ideal plant is passive, but also because at the point passivity is violated $\delta_1 = 0$ by definition. With respect to $\kappa_1$, the flexible system will have some natural damping (both rigid body and modal damping), therefore $\kappa_1$ is realistically large, but not infinite. We assume so mainly because it is difficult to estimate the damping of the system, and hence estimate $\kappa_1$.

To estimate $\omega_c$ and $\gamma_1$ we will consider our ideal plant and the effect of $f(s)$, as discussed in Sec. 8.1.2. Recall from Sec. 8.1.2 that $g_p(s) = \rho_{\mu}(s)/u(s)$ is the plant compensated with proportional control. As such, $g_1^*(s) = sg_p(s) = \dot{\rho}_{\mu}(s)/u(s)$ represents the ideal PR plant, while $g_1(s) = f(s)g_p(s) = y_1(s)/u(s)$ is the perturbed plant that uses the derivative filter $f(s)$ to estimate $\dot{\rho}_{\mu}$. Both $g_1^*(s)$ and $g_1(s)$ are shown in Fig. 8.2; clearly the ideal plant $g_1^*(s)$ is PR (i.e., has phase bounded by $\pm 90^\circ$). The system $g_1(s)$ is PR up until approximately 8.25 (rad/s). Above this frequency $g_1(s)$ has finite gain, but does not behave passively. The system $g_1(s)$ is clearly hybrid possessing a passive (PR) region below 8.25 (rad/s), and a non-passive but finite gain (BR) region above 8.25 (rad/s).

![Figure 8.2: Frequency response of ideal and perturbed plant. The ideal plant includes a pure derivative operator $s$, while the perturbed plant includes the derivative filter $f(s)$.](image)

Given the assumed frequency response of $g_1(s)$ in Fig. 8.2, $\omega_c$ and $\gamma_1$ of $g_1(s)$ are chosen to be 8.25 (rad/s) and 0.85 (m/(N·s)). It should be stressed that we can only estimate $\omega_c$ and $\gamma_1$; given our original plant model and an idea of how the sensors behave, we are estimating the value of $\omega_c$. In turn, we are estimating the range where the plant behaves passively, and above such range we are confident the plant has finite gain, but again, we are estimating the gain $\gamma_1$ as well.
8.4.3 Controller Optimization Results

The numerical optimization formulation of Sec. 8.3 will be used to design a FF SPR/BR controller for the single-link manipulator under consideration. The nominal plant used as the basis for optimization is \( g_1^*(s) \) (i.e., the “perfect” plant augmented with proportional control). The weights \( D_{12}^T D_{12} = 1 \times 10^{-2} \) and \( D_{21}^T D_{21} = 1 \times 10^{-4} \) are used for controller synthesis. Recall the use of the transfer function \( W(s) \) within the general MIMO optimization formulation; in the SISO context this filter is just the transfer function \( w(s) \), which we will specify to be a fourth order lowpass Butterworth transfer function with a bandwidth of \( \omega_0 = 100 \) (rad/s). By choosing this bandwidth the optimization algorithm neglects differences in the magnitude responses of the \( H_2 \) controller and the FF SPR/BR controller being optimized above \( \omega_0 \). We do this because we expect the FF SPR/BR controller to roll-off owing to the high frequency gain constraint \( \gamma_1 \gamma_2 < 1 \). At the same time, we desire our FF SPR/BR controller to perform well at low frequency, and should mimic the \( H_2 \) controller as closely as possible.

During optimization, \( \delta_2 \), \( \kappa_2 \), and \( \gamma_2 \) can be calculated in order to enforce the required optimization constraints. Specifically, for a SISO controller \( g_2(s) \) we have \( \delta_2 = \inf_{-\omega_c < \omega < \omega_c} \text{Re} \{ g_2(j\omega) \} \), \( \kappa_2 = \sup_{-\omega_c < \omega < \omega_c} |g_2(j\omega)| \), and \( \gamma_2 = \sup_{\omega > \omega_c} |g_2(j\omega)| \) as presented in Sec. 5.5.2.

The frequency response of the numerically optimized FF SPR/BR controller, \( g_2(s) \), as well as the traditional \( H_2 \) controller, \( g_2^*(s) \), used within the optimization formulation are shown in Fig. 8.3. The gain of \( g_2(s) \) is less than the gain of \( g_2^*(s) \) over all frequencies. Because \( \gamma_1 \gamma_2 < 1 \) must be satisfied above \( \omega_c \), the gain of \( g_2(s) \) is reduced over all frequencies (including DC). Notice that at approximately 19.6 (rad/s) and 66.0 (rad/s) the magnitude of the \( H_2 \) controller increases. The FF SPR/BR controller also has a slight increase in gain at these two frequencies. Similarly, at 19.6 (rad/s) the phase of \( g_2^*(s) \) dips below \(-135^\circ\), and \( g_2(s) \) attempts to mimic the phase response by dipping to \(-65^\circ\). Clearly the FF SPR/BR controller is attempting to mimic the \( H_2 \) controller as best it can while satisfying a low frequency phase constraint and high frequency gain constraint.

![Frequency response of \( H_2 \) controller and hybrid VSP/finite gain controller.](image)
Referring to Fig. 8.2, notice the flexible link’s lower two modes are excited at approximately 19.6 (rad/s) and 66.0 (rad/s). The gain increase in both $g_2(s)$ and $g_2^*(s)$ at 19.6 (rad/s) and 66.0 (rad/s) can be attributed to damping these two modes.

Consider the open-loop frequency response of $g_1^*(s)$ and $g_2^*(s)$, as well as $g_1^*(s)$ and $g_2(s)$ shown in Fig. 8.4. Notice that the overall shape of the two frequency responses is similar. Because the FF SPR/BR controller $g_2(s)$ was designed to mimic the $H_2$ controller $g_2^*(s)$, it is logical that the open-loop frequency response $g_1^*(s)g_2(s)$ mimics the open-loop frequency response of $g_1^*(s)g_2^*(s)$. When controlled by the $H_2$ controller $g_2(j\omega)$, the ideal system can tolerate a gain decrease of 5.1 dB (at 25.0 (rad/s)) before instability, while the phase margin is 49.6° (at 131 (rad/s)). When the FF SPR/BR controller $g_2(j\omega)$ is used to control $g_1^*(j\omega)$, the closed-loop system is significantly more robust; the gain margin is infinite, while the phase margin is 86.9° (at 8.16 (rad/s)). The gain margin is infinite because, by chance (that is to say, not by design on our part), the optimal FF SPR/BR controller is actually SPR over all frequencies, not just at low frequency. It is well known the gain margin associated with a positive real plant being controlled by a SPR controller is infinite.

![Figure 8.4: Frequency response of open-loop systems $g_1^*(j\omega)g_2^*(j\omega)$ and $g_1^*(j\omega)g_2(j\omega)$](image)

Now let us investigate how the robustness properties of the system change (in terms of gain margin and phase margin) when $g_2^*(s)$ and $g_2(s)$ are used to control $g_1(s)$, the plant that uses $f(s)$ to acquire rate information. Interestingly, when the $H_2$ controller $g_2^*(s)$ is used to control $g_1(s)$, the closed-loop is unstable. As such, plotting the open-loop frequency response $g_1(j\omega)g_2^*(j\omega)$ is meaningless, as is stating gain and phase margins. When $g_2(s)$ is used to control $g_1(s)$ the closed-loop system is stable (as we would expect), but the gain and phase margins are quite good as well. The open-loop frequency response $g_1(s)g_2(s)$ is shown in Fig. 8.5. The gain margin is 14.3 dB ($\phi_{gm} = 66.0$ (rad/s)) while the phase margin is 64.3° ($\phi_{pm} = 8.1$ (rad/s)). These margins indicate the closed-loop system is quite robust in that the system can tolerate additional perturbations before instability ensues. Compared to the perfect plant being controlled by $g_2(s)$, although the gain and phase margins are deteriorated when $f(s)$ is present, they have not been
destroyed all together.

![Image of frequency response graph]

Figure 8.5: Frequency response of open-loop system, \( g_1(j\omega)g_2(j\omega) \).

### 8.4.4 Experimental Results

The controllers \( g_2(s) \) and \( g_2^*(s) \) depicted in Fig. 8.3 have been used to control the single-link flexible manipulator apparatus depicted in Fig. 8.1. The manipulator is to follow a desired trajectory, \( \rho_d = J_\theta \theta_d \), starting at \( \rho_a = J_\theta \theta_a \), moving to \( \rho_b = J_\theta \theta_b \), then moving back to \( \rho_a \). Specifically, \( \theta_a = -\pi/4 \) (rad), while \( \theta_b = \pi/4 \) (rad). The desired trajectory between set points is

\[
\theta_d = \left[ 10 \left( \frac{t}{t_f} \right)^3 - 15 \left( \frac{t}{t_f} \right)^4 + 6 \left( \frac{t}{t_f} \right)^5 \right] (\theta_f - \theta_i) + \theta_i
\]

where \( t_f = 2 \) (s), \( \theta_f \) is the final angular position, and \( \theta_i \) is the initial angular position. Between maneuvers there is a 2 (s) dwell.

Fig. 8.6 shows the system response of the manipulator as controlled by the FF SPR/BR controller and \( H_2 \) controller, while Fig. 8.7 shows the system response error where \( e = \rho - \rho_d \). The rms error is shown in Table 8.1; the FF SPR/BR control outperforms the traditional \( H_2 \) controller. In Fig. 8.6 one can see, especially around 4 (s), that the \( H_2 \) controller does not suppress the tip-rate as quickly as the FF SPR/BR controller does. The FF SPR/BR controller is a better rate controller. With respect to tip position errors, the system controlled by \( g_2^*(s) \) never reaches a steady-state tip position error of zero, as shown in the tip position plot of Fig. 8.6 and the tip position error plot of Fig. 8.7. This can be attributed to the fact that the gearbox stiction (i.e., static friction) is not perfectly compensated, and there is no integral term (with respect to the \( \mu \)-tip position) in the feedback control loop that would force zero steady-state tip position error. The system controlled by \( g_2(s) \) has basically zero steady-state tip position error because both the rate and position tracking are quite good as the manipulator slews to and from each set-point, and hence the error upon completion of the maneuver is very small.
The FF SPR/BR controller performs well for a variety of reasons. Recall that the FF SPR/BR controller is optimally designed assuming an accurate plant model within a low frequency range (i.e., $\omega \in (-\omega_c, \omega_c)$). Therefore, the FF SPR/BR controller approximates the optimal $H_2$ controller as best it can at low frequency while satisfying the Hybrid Passivity/Finite Gain stability requirements. Particularly, the FF SPR/BR controller has higher gain at low frequency, gain increases at certain frequencies (i.e., 19.6 (rad/s) and 66.0 (rad/s)), and a phase response that tries to approximate the $H_2$ solution. However, above $\omega_c$, where passivity has been violated and it is assumed the nominal plant model is not representative of the true system, the controller rolls off, essentially ignoring high frequency information contained in the feedback measurement. Additionally, the high frequency gain constraint forces the controller to have gain that is less than the $H_2$ controller in the passive band. This rejection of high frequency data, and noise, but retention of “good” low frequency data yields the performance observed.

On the other hand, the $H_2$ controller assumes the nominal plant is accurate over all frequen-
cies, including the region where passivity has been violated. The gain of the $\mathcal{H}_2$ controller is (relatively) high above $\omega_c$, where the nominal plant model does not accurately represent the true plant. Additionally, the gain profile of the $\mathcal{H}_2$ controller does not reject high frequency noise, which is amplified, as shown in the error signals of Fig. 8.7.

It is interesting to see that although that closed-loop involving $g_1(s)$ (the plant with $f(s)$) and $g_2^*(s)$ is theoretically unstable, the $\mathcal{H}_2$ controller stabilizes the physical manipulator test-bed. This fact highlights the reality that our plant and derivative filter model representing $g_1(s)$ do not capture the true dynamics of the system.

### 8.4.5 Attempted Modifications to the Controller Optimization Scheme

It is unfortunate that the optimal FF SPR/BR controller $g_2(s)$ has gain that is less than that of $g_2^*(s)$ at low frequency (see Fig. 8.3). It would be ideal if the low frequency gain of $g_2(s)$ was closer to that of $g_2^*(s)$, thus realizing better performance. An attempt was made to do so; within the optimization formulation, another constraint was added in which the DC gain of $g_2(s)$ had to be greater than or equal to the DC gain of $g_2^*(s)$, i.e., $|g_2(j0)| \geq |g_2^*(j0)|$. It was found that this constraint severely conflicted with the constraint $\gamma_{12} < 1$, and as such the optimization procedure could not converge (i.e., no solution was found). The constraint $|g_2(j0)| \geq |g_2^*(j0)|$ pushes the low frequency gain of $g_2(s)$ up, while the constraint $\gamma_{12} < 1$ pulls down the gain of $g_2(s)$ around and above $\omega_c$. The controller can not satisfy $|g_2(j0)| \geq |g_2^*(j0)|$, roll off fast enough to satisfy $\gamma_{12} < 1$, and satisfy the required low frequency phase constraint $\delta_2 > 0$. However, if $\omega_c$ were much larger (e.g., 100 (rad/s) or 1000 (rad/s)), then including $|g_2(j0)| \geq |g_2^*(j0)|$ may be tolerable because $g_2(s)$ would not have to roll off as fast in the low frequency passive/positive real frequency band.

Picking different $\omega_o$ values during the optimization process was also investigated. Recall that $\omega_o$ is the frequency where $w(s)$ rolls off. For example, when $\omega_o$ is set to 30 (rad/s), the optimization scheme converges, however, it was found the resultant controller did not perform as well $g_2(s)$, the controller shown in Fig. 8.3 which was optimized using $\omega_o = 100$ (rad/s). The reason is related to having the controller suppress the second vibration mode of the link. The $\mathcal{H}_2$ controller gain is high at 66.0 (rad/s), and hence $g_2^*(s)$ suppresses the second vibration mode of the link. When the cut-off of $w(s)$ is set to 100 (rad/s), the FF SPR/BR controller is forced to mimic the $\mathcal{H}_2$ controller (or, said another way, penalized if it does not mimic $\mathcal{H}_2$ controller), and hence the FF SPR/BR controller tries to suppress the vibration mode at 66.0 (rad/s) just as the $\mathcal{H}_2$ controller does. However, when the cut-off of $w(s)$ is set to 30 (rad/s) the FF SPR/BR controller synthesized during optimization is not penalized for not mimicking the $\mathcal{H}_2$ controller above 30 (rad/s); the resultant FF SPR/BR controller essentially ignores the vibration mode.

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|}
\hline
Controller & $\varepsilon_{rms}$ (m) & $\dot{e}_{rms}$ (m/s) \\
\hline
FF SPR/BR Control & $2.664 \times 10^{-3}$ & $2.258 \times 10^{-2}$ \\
$\mathcal{H}_2$ Control & $13.058 \times 10^{-3}$ & $10.7074 \times 10^{-2}$ \\
\hline
\end{tabular}
\caption{System response rms error.}
\end{table}
at 66.0 (rad/s) that should be suppressed, which was found to lead to a decrease in closed-loop performance.

8.5 Summary

In this chapter we considered the design and optimization of a SISO hybrid VSP/finite gain controller to control a single-link flexible manipulator that has experienced a passivity violation. After discussing the single-link flexible manipulator model and the relevant passive input-output mapping, hybrid passive/finite gain systems theory was reviewed in a LTI, SISO context. We discussed $\delta$, $\epsilon$, and $\gamma$ in terms of phase and gain in different frequency bands. The Hybrid Passivity/Finite Gain Stability Theorem was also reviewed. Next, a numerical optimization problem was posed whereby a FF SPR/BR controller was designed to mimic a traditional $\mathcal{H}_2$ controller subject to constraints dictated by the Hybrid Passivity/Finite Gain Stability Theorem. Experimental results confirmed the success of the optimal FF SPR/BR controller found relative to a traditional $\mathcal{H}_2$ controller.

In the next chapter we will investigate the control of a two-link flexible manipulator using the hybrid passivity/finite gain systems theory.
Chapter 9

Two-Link Flexible Manipulator Control Accommodating Passivity Violations: Theory and Experiments

In this chapter, we will consider joint position and rate control of a two-link flexible manipulator. The control of such a flexible manipulator (and in general a system with \( N_r \) links) is quite challenging, and has been investigated previously by other authors; control methods include passivity-based control, gain-scheduled control, and LPV control\(^{24,77,109–111}\).

The map between the manipulator joint torques and joint rates is passive\(^{24}\). However, as we saw in Chapter 8, when a filter is used to acquire rate information, the passive input-output map is destroyed: passivity is violated. In what follows, we will investigate the hybrid passive/finite gain mapping of the two-link manipulator system in question, and the synthesis of a hybrid VSP/finite gain controller.

Previously in Chapter 8, the synthesis of a hybrid VSP/finite gain, that is, a FF SPR/BR controller was considered. Although the synthesis method was sound, the low frequency PR and high frequency BR constraints were enforced in a “brute force” manner. In this chapter, we will use the GKYP Lemma to enforce a FF SPR/BR structure, and pose a convex optimization problem where our objective function is convex, and the constraints are LMIs. Such an optimization problem is nice in that a solution can be found quickly and efficiently using numerical algorithms that exploit the convex nature of the problem. Additionally, we will employ the theory presented in Chapter 6; we will synthesize two FF SPR/BR controllers using the convex optimization formulation just mentioned, and use the FF SPR/BR controllers within a gain-scheduling algorithm. As such, the overall controller we will use to control the two-link system is a gain-scheduled hybrid VSP/finite gain controller, where the subcontrollers are optimal FF SPR/BR controllers.
9.1 Multi-Link Flexible Manipulator Dynamics

9.1.1 Ideal Input-Output Model

A flexible robotic manipulator with \( N_r \) joints is described by

\[
M(q)\ddot{q} + D\dot{q} + Kq = \hat{B}\tau + f_n(q, \dot{q})
\]

(9.1)

where \( M > 0 \), \( D \geq 0 \), and \( K \geq 0 \) are the mass, damping, and stiffness matrices (each of which is symmetric), \( f_n \) are the nonlinear inertial forces, \( \hat{B} = [1 \ 0]^T \), \( \tau = [\tau_1 \cdots \tau_{N_r}]^T \) are the joint torques, \( q = [\theta^T \ q_e^T]^T \), \( \theta = [\theta_1 \cdots \theta_{N_r}]^T \) are the joint angles, and \( q_e \) are the elastic coordinates associated with discretization of the flexible links.

It is well known that in the context of flexible mechanical systems, collocated force/torque actuators and velocity/angular velocity rate sensor yield a passive input-output map\(^{24,112} \). In particular, the map \( \tau \rightarrow \dot{\theta} \) associated with flexible robotic manipulators is a passive one. The map remains passive regardless of the assumed modes or discretization method, mass distribution, etc. To robustly stabilize via the Passivity Theorem, an appropriate control law would be of the following form:

\[
\tau(s) = \tau_f(s) - \left[ \frac{1}{s}K_p + GVSP(s) \right] \dot{\theta}(s)
\]

where \( \tau_f \) is a feedforward control that effectively negates a portion of the nonlinear manipulator dynamics, \( K_p = K_p^T > 0 \) realizes proportional control (note, \( \frac{1}{s}K_p \) is passive), and \( GVSP(s) \) is a very strictly passive (VSP) transfer matrix. In instances where \( \dot{\theta} \) can be directly measured, such a control law may be employed. In practice, however, \( \dot{\theta} \) is not directly measured, but estimated by filtering or finite differencing of \( \theta \) measurements.

9.1.2 Violation of Passivity

As just mentioned, in many practical situations \( \theta \) will be measured directly, but \( \dot{\theta} \) will not be. With \( \theta \) directly available proportional control may be implemented without difficulty, and the result (i.e., the flexible system compensated by proportional control) is a passive plant with no rigid body modes.

For rate control to be implemented \( \dot{\theta} \) must be estimated. A simple method to estimate \( \dot{\theta} \) is by filtering \( \theta \) as follows:

\[
y(s) = F(s)\theta(s) = \text{diag} \{f(s)\} \theta(s), \quad f(s) = \frac{\omega_f^2 s}{s^2 + 2\zeta_f \omega_f s + \omega_f^2}
\]

(9.2)

where \( \omega_f \) and \( \zeta_f \) are the natural frequency and damping ratio of the derivative filter \( f(s) \). Although the flexible manipulator to be controlled is nonlinear, to facilitate a simpler discussion, the system’s frequency response will be referred to (which, if it is assumed the system has been linearized about a specific joint configuration, is acceptable). At low frequency \( F(s) \) accurately approximates \( \theta \) and as a result the input-output map \( \tau \rightarrow y \) behaves as if \( \dot{\theta} \) were measured di-
rectly, i.e., passively. At higher frequencies, however, $F(s)$ induces phase lag in the measurement signal to be used for control. As a result, what is fed back to the controller does not represent $\dot{\theta}$ closely, and the input-output map $\tau \rightarrow y$ does not behave passively; *passivity has been violated* above a certain frequency. The true input-output map $\tau \rightarrow y$ can effectively be divided into two parts: a low frequency part where the input-output map possesses passive characteristics, and a high frequency part where passivity is violated. Although passivity has been violated at high frequency, the input-output map maintains finite gain characteristics owing to the small amount of natural damping in the structure, and the natural roll-off of $F(s)$. The system is hybrid, possessing both passive and finite gain properties.


In this section, we will review hybrid passive/finite gain systems theory that was first presented in Sec. 5.5.2, starting on page 81.

Consider the negative feedback interconnection of two hybrid passive/finite gain systems, $G_1 : L_{2e} \rightarrow L_{2e}$ and $G_2 : L_{2e} \rightarrow L_{2e}$, presented in Fig. 3.1 on page 18. The hybrid parameters associated with each system are $\delta_1$, $\epsilon_1$, and $\gamma_1$ and $\delta_2$, $\epsilon_2$, and $\gamma_2$ respectively. The Hybrid Passivity/Finite Gain Stability Theorem (see Theorem 5.6.1 in Sec. 5.6 and Corollary 5.6.2 in Sec. 5.6.1) states that the negative feedback interconnection presented in Fig. 3.1 is $L_2$ stable if the variables $\delta_1$, $\epsilon_1$, $\gamma_1$, $\delta_2$, $\epsilon_2$, and $\gamma_2$ satisfy $\epsilon_1 + \delta_2 > 0$, $\epsilon_2 + \delta_1 > 0$, and $\gamma_1 \gamma_2 < 1$.

In a LTI context, both $G_1$ and $G_2$ can be expressed in terms of transfer matrices, $G_1(s)$ and $G_2(s)$. In a LTI context the parameters $\delta_1$, $\epsilon_1$, $\gamma_1$, $\delta_2$, $\epsilon_2$, and $\gamma_2$ can be calculated via frequency domain methods or using the GKYP, as discussed in Sec. 5.5.2. In the last chapter we relied on frequency domain methods. In this chapter, we will used various forms of the GKYP Lemma. In particular, we will make use of Lemmas 5.5.1, 5.5.2, and 5.5.3 that, when satisfied, ensure a system with minimal state-space realization ($A$, $B$, $C$, $D$) is FF PR over a low frequency region, FF SPR over a low frequency region, and FF BR over a high frequency region.

A specific form of the Hybrid Passivity/Finite Gain Stability Theorem that we will make use of is the negative feedback interconnection of a hybrid passive/finite gain plant and a hybrid VSP/finite gain controller. A hybrid passive/finite gain plant (i.e., a FF PR/BR plant in a LTI context) will have $\delta_1 = \epsilon_1 = 0$ and $0 < \gamma_1 < \infty$. A hybrid VSP/finite gain controller (i.e., a FF SPR/BR controller in a LTI context) will have $\delta_2 > 0$, $\epsilon_2 > 0$ ($0 < \kappa_2 < \infty$) and $0 < \gamma_2 < \infty$. In this particular situation, if $\gamma_1 \gamma_2 < 1$ the closed-loop system will be stable. This form of the theorem is particularly useful for stabilizing plants that are nominally passive in the traditional sense (i.e., in a LTI context are PR over all frequencies), but have their passive input-output map destroyed in some way.
9.3 Control Design Problem

Our work pertains to systems that are ideally PR over all frequencies, but have their PR nature destroyed above $\omega_c$ and are rendered hybrid having PR properties below $\omega_c$ and BR properties above $\omega_c$. In terms of the passivity and finite gain parameters, a LTI plant $G_1(s)$ that is FF PR/BR will have $\delta_1 = 0$, $\epsilon_1 = 0$, and $0 < \gamma_1 < \infty$. In order to stabilize such a system via the Hybrid Passivity/Finite Gain Stability Theorem a controller $G_2(s)$ must be synthesized so that $\delta_2 > 0$, $\epsilon_2 > 0$ (i.e., $0 < \kappa_2 < \infty$), and $0 < \gamma_2 < \infty$ where $\gamma_1 \gamma_2 < 1$. Such a controller takes the form of a FF SPR/BR transfer matrix.

The purpose of this section is to formulate a convex optimization problem that yields a FF SPR/BR system given $\omega_c$ and $\gamma_1$ to act as a controller. In particular, our approach will be to mimic a classic $\mathcal{H}_2$ controller as closely as possible. We will first review the standard $\mathcal{H}_2$ formulation.

The nominal system (i.e., one that ignores sensors, actuators, etc. that induce passivity violations) to be controlled is

$$\dot{x} = Ax + B_1w + B_2u$$
$$z = C_1x + D_{12}u$$
$$y = C_2x + D_{21}w$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measurement, $z \in \mathbb{R}^{n_z}$ is the regulated output, the disturbances/noise are $w = [d^T v^T]^T \in \mathbb{R}^{n_w}$, and all matrices are dimensioned appropriately. It is assumed that

1. $(A, B_1)$ is controllable and $(C_1, A)$ is observable,
2. $(A, B_2)$ is controllable and $(C_2, A)$ is observable,
3. $D_{12}^T C_1 = 0$ and $D_{12}^T D_{12} > 0$, and
4. $D_{21}^T B_1 = 0$ and $D_{21}^T D_{21} > 0$.

The $\mathcal{H}_2$ optimal controller takes the following form:

$$\begin{aligned}
\dot{x}_c &= \frac{A_c}{(A - B_2 K_c - K_c C_2)} x_c + K_c y \\
-u &= K_c x_c
\end{aligned}$$

where $u(s) = G_2^*(s)y(s) = K_c(sI - A_c)^{-1}K_c y(s)$ (9.3)

where $K_c$ is the optimal state-feedback gain matrix and $K_e$ is the optimal estimator gain matrix. Both $K_c$ and $K_e$ are found by solving two different Riccati equations, or solving two different convex optimization problems constrained by LMIs. Our approach to designing an optimal FF SPR/BR controller is to keep the controller dynamic matrix $A_c$ and input matrix $K_e$ the same as the standard $\mathcal{H}_2$ controller, then find a state-feedback gain matrix $K_c$ that renders $G_2(s) = K_c(sI - A_c)^{-1}K_c$ FF SPR/BR. This approach is similar to the approaches proposed in Ref. 106 and Sec. 8.3 of Chapter 8, but here we will form a convex optimization problem using LMIs, much like the work of Refs. 113 and 114.

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Let us first discuss our controller constraints. As previously mentioned, given $\omega_c$ and $\gamma_1$ we must design our controller $G_2(s)$ to be FF SPR/BR. We assume that $G_2(s) = K_o(sI - A_c)^{-1}K_e$ is Hurwitz for any $K_o$ by assuming that the nominal $H_2$ solution renders $A_c$ Hurwitz. Now, using Lemma 5.5.2 (page 83) the controller $G_2(s)$ will be SPR $\forall \omega \in \Omega$ if $\exists P_p, Q_p \in \mathbb{R}^{n \times n}$ where $P_p = P_p^T$ and $Q_p = Q_p^T > 0$ such that

$$
\begin{bmatrix}
A_c & K_e \\
1 & 0
\end{bmatrix}^T \begin{bmatrix}
-Q_p & P_p \\
P_p & (\omega_c - \omega_c)^2 Q_p
\end{bmatrix} \begin{bmatrix}
A_c & K_e \\
1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & -K_o^T \\
-K_o & 0
\end{bmatrix} < 0. \quad (9.4)
$$

Notice that this is a LMI in $K_o, P_p, Q_p$. Next, using Lemma 5.5.3 (page 83) our controller will be BR $\forall \omega \in \Omega_h$ with gain $\gamma_2 < 1/\gamma_1$ if $\exists P_b, Q_b \in \mathbb{R}^{n \times n}$ where $P_b = P_b^T$ and $Q_b = Q_b^T \geq 0$ such that

$$
\begin{bmatrix}
A_c & K_e \\
1 & 0
\end{bmatrix}^T \begin{bmatrix}
Q_b & P_b \\
P_b & -\omega_c^2 Q_b
\end{bmatrix} \begin{bmatrix}
A_c & K_e \\
1 & 0
\end{bmatrix} + \begin{bmatrix}
K_o^T & 0 \\
0 & -\gamma_2^2 I
\end{bmatrix} \leq 0.
$$

This matrix inequality is linear in $P_b$ and $Q_b$, but not linear in $K_o$. By using the Schur complement\(^{31}\) we can transform it into a LMI:

$$
\begin{bmatrix}
A_c & K_e \\
1 & 0
\end{bmatrix}^T \begin{bmatrix}
Q_b & P_b \\
P_b & -\omega_c^2 Q_b
\end{bmatrix} \begin{bmatrix}
A_c & K_e \\
1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & -\gamma_2^2 I
\end{bmatrix} \begin{bmatrix}
K_o \\
0
\end{bmatrix} \leq 0. \quad (9.5)
$$

This is now a LMI in terms of $K_o, P_b, Q_b$. Therefore, the controller $G_2(s) = K_o(sI - A_c)^{-1}K_e$ will be FF SPR/BR if $\exists P_p, Q_p, P_b, Q_b \in \mathbb{R}^{n \times n}$ where $P_p = P_p^T, Q_p = Q_p^T > 0, P_b = P_b^T$, and $Q_b = Q_b^T \geq 0$ such that both Eqs. (9.4) and (9.5) are satisfied.

Although any $G_2(s)$ that is FF SPR/BR will stabilize a FF PR/BR plant via the Hybrid Passivity/Finite Gain Stability Theorem, we want a $G_2(s)$ that is optimal in some sense. As mentioned at the beginning of this section, we will formulate our convex optimization problem so that $G_2(s)$ mimics $G_2^*(s)$ as best is can while satisfying the FF SPR/BR constraints. As such, consider the following objective function to be minimized:

$$
J = \text{tr} \left[ (K_o - K_c)(K_o - K_c)^T \right].
$$

By using “slack” variable $g \in \mathbb{R}^+$ and the symmetric positive semidefinite “slack” matrix $Z \in \mathbb{R}^{nu \times nu}$ the above objective function can be equivalently written

$$
J = g \quad (9.6)
$$
subject to

\[ \text{tr} Z \leq g \quad (9.7a) \]
\[ (K_o - K_c)(K_o - K_c)^T \leq Z \quad (9.7b) \]

As \( g \) is minimized \( Z \) is minimized, and as \( Z \) is minimized \((K_o - K_c)(K_o - K_c)^T \leq Z\) is minimized. By using the Schur complement, the constraint \((K_o - K_c)(K_o - K_c)^T \leq Z\) can be equivalently written

\[ \begin{bmatrix} Z & (K_o - K_c) \\ (K_o - K_c)^T & 1 \end{bmatrix} \geq 0. \quad (9.8) \]

If the objective function in Eq. (9.6) is minimized subject to the constraints given in Eqs. (9.4), (9.5), (9.7), and (9.8) the resultant controller will be FF SPR/BR, but also in effect be the closest approximation to an unconstrained \( \mathcal{H}_2 \) controller.

Our optimization problem can be summarized as follows:

\[
\begin{align*}
\min_{K_o, \ P_p, \ Q_p, \ P_b, \ Q_b, \ g, \ Z} & \quad \mathcal{J}(K_o, P_p, Q_p, P_b, Q_b, g, Z) = g \\
\text{w.r.t.} & \quad K_o, \ P_p, \ Q_p, \ P_b, \ Q_b, \ g, \ Z \\
\text{s.t.} & \quad Q_p = Q_p^T > 0, \text{ Eq. (9.4)}, \quad Q_b = Q_b^T \geq 0, \text{ Eq. (9.5)}, \quad Z = Z^T \geq 0, \text{ Eq. (9.7)}, \text{ Eq. (9.8)}.
\end{align*}
\]

This optimization problem is convex; in fact, this optimization problem is a semidefinite programme (SDP) easily solve by a numerical algorithm such as an interior point method. In particular, we will use the MATLAB interface \textsc{Yalmip} and the solver \textsc{SeDuMi}. Note that, just like \( g \) and \( Z \), the symmetric matrices \( P_p, Q_p, P_b, Q_b \) are slack matrices.

### 9.4 Experimental Implementation and Results

#### 9.4.1 Experimental Apparatus

Control of the two-link flexible manipulator test-bed shown in Fig. 9.1 will now be considered. The apparatus is manufactured by Quanser Consulting Inc. The links are made of steel; the first link is 210.00 mm long, 1.27 (mm) thick, and 76.20 (mm) high, while the second link is 210.00 (mm) long, 0.89 (mm) thick, and 38.1 (mm) high. Affixed to the base of each link is a strain gauge, while a digital encoder is mounted to the output shaft of each motor. The encoders provide \( \theta = [\theta_1 \ \theta_2]^T \), the base and elbow joint angles, and the strain gauges are used to determine the deflection in each link. Readers interested in other specific details of the apparatus are referred to Ref. 107.

Because \( \theta \) is directly measured, proportional control can be implemented with ease; the proportional control gain will be set to \( K_p = \text{diag} \{40, 40\} \) (N \cdot m/rad). The velocity \( \dot{\theta} \) is not directly measured, and will be acquired by filtering \( \theta \) with \( F(s) \) (as shown in Eq. (9.2)) thus destroying the passive nature of the nominal plant. Knowing that \( F(s) \) destroys passivity, we will

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DC motors and harmonic drives.


Figure 9.1: Two-link flexible robot experiment manufactured by Quanser Consulting Inc.

control the system via the Hybrid Passivity/Finite Gain Stability Theorem. To do so, we must be able to estimate the nature of the passivity violation. In particular, we will linearize the two-link system and investigate the FF PR/BR nature of the linearized system, as discussed next.

### 9.4.2 Passivity Violation Approximation via Linearization

Consider the manipulator dynamics presented in Eq. (9.1) augmented with proportional control and linearized about a specific joint configuration $\theta_d$. The linearized, unforced/undamped, augmented system can be written as

$$
\begin{align*}
M\ddot{\delta q} + K_a \delta q &= 0
\end{align*}
$$

where $K_a = K_a^T > 0$ is the augmented stiffness matrix, and $\delta q = q - q_d$ where $q_d = [\theta_d^T \ 0]^T$. By solving the eigenproblem associated with this simplified system we can define a set of modal coordinates, $\delta q = Q_e \eta$, where $\eta$ are the modal coordinates and $Q_e = \text{row} \{q_\alpha\}$ where $q_\alpha$ are the eigenvectors normalized with respect to the mass matrix (i.e., $q_\alpha^T M q_\beta = \delta_{\alpha\beta}$). Additionally, we can define $\Omega = \text{diag} \{\omega_\alpha\}$ where $\omega_\alpha$ are the natural frequencies of each mode corresponding to the eigenvalues associated with the original eigenproblem.

We can now write the linearized equations in the following first order state-space form:

$$
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & \Omega \\ -\Omega & -2\Omega \end{bmatrix} \begin{bmatrix} \Omega \eta \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ Q_e^T \hat{B} \end{bmatrix} \tau
\end{align*}
$$
where $Z = \text{diag} \{ \zeta \}$ and $\zeta$ is the damping ratio associated with each mode. Note that we are deliberately writing the linearized motion equations in this form because numerical computations tend to be much more stable. Given the above state-space form, the relation between $x$, $\theta$, and $\dot{\theta}$ is

$$
\theta = \begin{bmatrix}
\hat{B}^T Q_c \Omega^{-1} \\
C_p
\end{bmatrix} x, \quad \dot{\theta} = \begin{bmatrix}
0 \\
\hat{B}^T Q_c
\end{bmatrix} x.
$$

Let $\theta(s) = G_p(s) \tau(s) = C_p(s1 - A)^{-1} B \tau(s)$ and $\dot{\theta}(s) = G(s) \tau(s) = C(s1 - A)^{-1} B \tau(s)$ where $G(s) := s G_p(s)$.

Figure 9.2: Frequency response of ideal and perturbed plant. The ideal plant includes a pure derivative operator $s1$, while the perturbed plant includes the derivative filter $F(s)$.

The frequency response of the ideal (linearized) system $G(s)$ is shown in Fig. 9.2. The linearization is performed about $\theta_d = [-\pi/4 \ 0]^T$ (rad). Within Fig. 9.2 is plotted the maximum singular value of $G(s)$ and the minimum hermitian part as a function of frequency. The maximum singular value of $G(j\omega)$ is $\bar{\sigma}(G(j\omega)) = \sqrt{\lambda[G^H(j\omega)G(j\omega)]}$, while the minimum hermitian part is

$$
\frac{1}{2} \lambda[\hat{G}(j\omega) + G^H(j\omega)].
$$

Clearly the linearized system is PR over all frequencies owing to the fact the hermitian part is positive over all frequencies.

Now, consider the input-output mapping where $\dot{\theta}$ is not directly measured, but acquired via differentiation using $F(s)$, i.e., $y(s) = G_1(s) \tau(s)$ where $F(s) G_p(s)$. The frequency response of this transfer matrix is also plotted in Fig. 9.2. The system $G_1(s)$ is PR over a specific frequency range; below approximately 100 (rad/s) the transfer matrix has a hermitian part that is positive, and hence PR. Above 100 (rad/s) the system is no longer PR (i.e., the hermitian part is negative) but is BR. The system is clearly hybrid possessing a frequency response that is FF PR/BR. Assuming our linearized model accurately approximates the nonlinear system, by using the Hybrid Passivity/Finite Gain Stability Theorem this system can be stabilized by a FF SPR/BR controller.
9.4.3 Controller Optimization Results

Rather than using one FF SPR/BR controller, we will synthesize and use two FF SPR/BR controllers within the gain-scheduling architecture presented in Chapter 6, Sec. 6.3, page 103. Because we are scheduling two FF SPR/BR controllers, the scheduling architecture originally presented in Fig. 6.2 simplifies, as shown in Fig. 9.3(a). Notice that the two scheduling signals, \( s_1 \) and \( s_2 \), are still applied to the input and the output of the FF SPR/BR rate controllers they schedule (while the proportional control is not scheduled). From Chapter 6, Sec. 6.3 we know that this particular scheduling architecture ensures that the overall gain-scheduling controller maintains a hybrid VSP/finite gain character.

The scheduling signals may be a function of \( \theta_2 \) or time. We elect to specify the scheduling signals to be a explicit function of time only. The scheduling signal profiles are shown in Fig. 9.3(b) where \( t_f = 2.5 \) (s). This form of scheduling is simple to implement, and essentially is scheduling the controllers based on the assumed position of the manipulator.

![Figure 9.3: Scheduling architecture and scheduling signals.](image)

The two controllers within the scheduling algorithm, \( H_1(s) \) and \( H_2(s) \), will each be designed about a specific linearization point; \( H_1(s) \) about set-point one, and \( H_2(s) \) about set-point two. Set-point one corresponds to \( [-\pi/4 \quad 0]^T \) (rad) while set-point two corresponds to \( [\pi/4 \quad \pi/3]^T \) (rad).

Similar to the single-link case, to design the FF SPR/BR controller both \( \omega_c \) and \( \gamma_1 \) must be estimated. From Fig. 8.2 \( \omega_c = 100 \) (rad/s) while \( \gamma_1 = 1.25 \) (rad/(N·m·s)). Note that these values are an estimate based on a linearization and an assumed filter \( F(s) \); the true nonlinear high frequency gain may not be the \( \gamma_1 \) we have chosen. However, with no way to calculate a true nonlinear gain we resort to estimating the high frequency gain in this way.

The frequency response of the FF SPR/BR controllers synthesize about set-points one and two using the scheme presented in Sec. 9.3 are shown in Figs. 9.4(a) and 9.4(b). The frequency response of \( H_2 \) controllers designed about each set-point are shown in Figs. 9.4(a) and 9.4(b) as well. The singular value and hermitian profiles of each FF SPR/BR controller mimics the \( H_2 \) controller frequency responses as close as possible without violating the FF SPR and FF BR constraints. In particular, notice the hermitian part of the FF SPR/BR controllers aggressively
trace the hermitian part of the $\mathcal{H}_2$ controllers, but each always remain positive in the frequency range below $\omega_c$, thus adhering to the FF SPR low frequency constraint. The gain profile of each FF SPR/BR controller is reduced below the gain profile of the $\mathcal{H}_2$ controller so that the constraint $\gamma_1 \gamma_2 < 1$ is satisfied as well.

![Graphs](image)

(a) Controller designed about set-point one, $\mathbf{H}_1(s)$.  
(b) Controller designed about set-point two, $\mathbf{H}_2(s)$.

Figure 9.4: Frequency response of FF SPR/BR controllers designed about set-points one and two.

The closed-loop (that is, the map from $\mathbf{w}^T = [\mathbf{d}^T \mathbf{v}^T] \rightarrow [\dot{\theta}^T \tau^T]$) maximum singular values versus frequency using the FF SPR/BR controllers designed about set-points one and two are shown in Figs. 9.5(a) and 9.5(b). The closed-loop maximum singular values versus frequency using the $\mathcal{H}_2$ controllers are also shown in Figs. 9.5(a) and 9.5(b) as well. Notice, the overall gain (i.e., maximum singular values) of the closed-loop is reduced when the FF SPR/BR controllers are used as compared to when the $\mathcal{H}_2$ controllers are used.

### 9.4.4 Experimental Results

The FF SPR/BR controllers of Figs. 9.4(a) and 9.4(b) have been used within the scheduling architecture described above to control the two-link manipulator. The manipulator is to follow a desired trajectory starting at set-point one, moving to set-point two, then moving back to set-point one. The desired trajectory between set-points is

$$\theta_d = \left[ 10 \left( \frac{t}{t_f} \right)^3 - 15 \left( \frac{t}{t_f} \right)^4 + 6 \left( \frac{t}{t_f} \right)^5 \right] (\theta_f - \theta_i) + \theta_i$$

where $t_f$ is 2 (s), $\theta_f$ is the final angular position, and $\theta_i$ is the initial angular position. Between maneuvers there is a 2.5 (s) dwell.

Fig. 9.6 shows the position and rate response of the system controlled by set-point one controller alone (i.e., there is no controller scheduling, and only $\mathbf{H}_1(s)$ is used) and the gain-scheduled controller. Fig. 9.7 shows the position and rate error of the system controlled by each scheme.
were $e(t) = \theta(t) - \theta_d(t)$. The rms errors are presented in Table 9.1.

Figure 9.6: Two-link manipulator system response using hybrid control and gain-scheduled hybrid control.

Although it is perhaps hard to visually discern the quality of the controlled responses, Table 9.1 clearly shows that the scheduled FF SPR/BR control scheme realizes lower position and rate error as compared to control that uses $H_1(s)$ alone. Although errors associated with the first joint angle do not change significantly (there is only a modest improvement, that is, the rms error in $\theta_1$ is improved 3.285%, while the rms error in $\dot{\theta}_1$ is improved 0.232%), the errors associated with the second joint are improved greatly (the rms error in $\theta_2$ is improved 21.497%, and the rms error in $\dot{\theta}_2$ is improved 33.530%) when the scheduled controller is used.
In this chapter, we considered the synthesis of a MIMO hybrid VSP/finite gain controller to control a two-link flexible manipulator. Specifically, the hybrid VSP/finite gain controller employed is a gain-scheduled controller where the subcontrollers are FF SPR/BR controllers. As discussed, a generic flexible manipulator has a passive input-output map that is destroyed when a derivative filter is used to acquire rate information. We are able to estimate the passive and finite gain regions by using a linearized model of the nonlinear plant. To design the FF SPR/BR controllers to be used within the gain-scheduling algorithm, we pose a convex optimization problem. The optimization objective function seeks to mimic a $\mathcal{H}_2$ controller. The low frequency FF SPR and high frequency FF BR constraints are enforced using the GKYP Lemma. Given that the optimization problem has a convex objective function and constraints in the form of LMIs, the problem can be solved very easily and efficiently. We use two FF SPR/BR controllers (each optimally designed about a linearization point) within a gain-scheduling algorithm to control a two-link manipulator experimental apparatus. Results show that both unscheduled and scheduled control is good, but that better tracking can be achieved using the gain-scheduled controller.

In Chapter 8, an unconstrained $\mathcal{H}_2$ controller was used to control the single-link manipulator along with the FF SPR/BR controller synthesized. We also tried to use an unconstrained $\mathcal{H}_2$ controller (that is, the $\mathcal{H}_2$ controller used as the basis for FF SPR/BR controller synthesis) to control the two-link system. Unfortunately, when implemented on the experimental test bed, the
unconstrained $\mathcal{H}_2$ controller was not able to stabilize the system. If we changed the performance index (that is, changed the weighting matrices $C_1, D_{12}, B_1, D_{21}$) so that less emphasis was placed on minimization of the states to be controlled, an unconstrained $\mathcal{H}_2$ controller that could stabilize the system was found, but the performance was rather poor. What this shows is that given an aggressive performance index, if the controller is also constrained to be FF SPR/BR, robust yet good performance can be attained over a traditional unconstrained $\mathcal{H}_2$ controller.

Note that a low fidelity model was used for controller synthesis; only two flexible modes were modeled, and the mass distribution of the links used in the model are assumed to be close to the real values, but not exact. Additionally, we did not perform any sort of system identification, which can often be very time consuming. This, in our opinion, is exactly how controller design should be done. When designing a control system, an engineer should try to model as little as possible, and avoid system identification is possible. If the control does not work on the real system given the low fidelity model used for design purposes, then the engineer has succeeded. If the control does not work on the real system, the engineer must revisit the model used for design, increasing the model fidelity. (Recall from Chapter 8 that an unconstrained $\mathcal{H}_2$ controller was able to stabilize the single-link system in practice; evidently the model fidelity was “good enough”.) We did not attempt to do so simply because our hybrid VSP/finite gain gain-scheduled controller worked very well. Our stability and control framework is robust to errors not only in the mass distribution and stiffness parameters of the manipulator, but also to all sorts of dynamics we did not consider (e.g., motor dynamics, computer processing delay, etc.).
Chapter 10

Conclusions

In this thesis we consider extensions and modifications to the traditional input-output systems framework, and their application to aerospace systems. We focus on three theoretical areas and two applications. We first consider sufficient conditions that ensure a LTV system has finite gain, is passive, or is conic. Second, we investigate passivity violations, and how to overcome them. Our solution, when generalized, is the hybrid input-output framework where a system’s input-output map is described in terms of multiple properties such as, for example, passive properties and nonpassive but finite gain properties when passivity has been violated. Third, the input-output properties (specifically, the conic and hybrid VSP/finite gain properties) of gain-scheduled systems are analyzed.

Each of the contributions presented in this thesis were originally motivated by practical engineering problems. Our LTV input-output results were originally motivated by the design of a LTV ISP controller to control the attitude of a spacecraft endowed with magnetic torque rods and reaction wheels. The hybrid input-output framework was originally motivated by passivity violations, which in reality will always occur (to some degree at least). The input-output properties of gain-scheduled systems were investigated in order to guarantee better closed-loop control. Overall, we believe that our contributions are not only novel and theoretically pleasing, but practically relevant.

10.1 Summary of Major Contributions

The major contributions of this thesis are summarized below.

1. In Chapter 4 we consider the input-output properties of linear LTV systems. We pay special attention to
   1.1. sufficient conditions that ensure a LTV system has finite gain (Sec. 4.2, page 45), is passive (either VSP, ISP, OSP, or purely passive; see Sec. 4.3, page 47), or is conic (Sec. 4.4, page 52), and
   1.2. the stability of various feedback interconnections involving LTV passive systems and sector bounded memoryless nonlinearities (Sec. 4.5, page 54).
2. In Chapter 5, we develop the hybrid input-output systems framework. In particular we
2.1. define hybrid conic systems (Sec. 5.4, page 69), and discuss how to determine if a LTI
hybrid conic system is bounded by a particular cone using a frequency domain method
(Sec. 5.4.1, page 70) or the GKYP Lemma (Sec. 5.4.1, page 71),
2.2. show how the variable cones developed by Safonov\textsuperscript{18} can be approximated using hybrid
cones (Sec. 5.4.2, page 73),
2.3. provide a means to overcome passivity violations by developing the hybrid passive/finite
gain systems framework (Sec. 5.5, page 76),
2.4. show how to calculate the hybrid passive/finite gain parameters in the frequency do-
main (Sec. 5.5.2, page 81), and show how to determine if a system is PR, SPR, or BR
in a low or high frequency region using the GKYP Lemma (Sec. 5.5.2, page 82), and
2.5. derive sufficient conditions that ensure the negative feedback interconnection of two
hybrid systems is $L_2$ stable (Sec. 5.6, page 84 and Sec. 5.6.1, page 88).

3. In Chapter 6 we consider gain-scheduling. Specifically,
3.1. we show that a set of conic subsystems gain-scheduled appropriately has specific conic
bounds (Sec. 6.2, page 98), and
3.2. we show that a set of hybrid VSP/finite gain subsystems gain-scheduled appropriately
also has hybrid VSP/finite gain properties (Sec. 6.3, page 103).

4. Chapter 7 considers spacecraft attitude control using magnetic torque rods and reaction
wheels. Our main contributions are
4.1. applying the theory of Chapter 4, specifically Theorem 4.3.2, to a practical control
problem, and
4.2. outlining a controller synthesis method based on the LQR formulation and Theorem
4.3.2.

5. Chapter 8 uses the hybrid passive/finite gain systems framework to control a single-link
flexible manipulator. The contributions of this chapter are
5.1. characterizing how passivity is violated by a rate filter in a SISO context, rendering
the nominally PR plant FF PR/BR (Sec. 8.1.2, page 125 and Sec. 8.4.2, page 131),
5.2. developing a simple controller synthesis technique using (nonlinear) numerical opti-
mization (Sec. 8.3, page 127), and
5.3. experimentally demonstrating closed-loop control of a LTI system using the hybrid
systems framework (Sec. 8.4, page 129).

6. Chapter 9 considers the control of a two-link flexible manipulator within the hybrid pas-
sive/finite gain systems framework. Our major contributions are
6.1. identifying passivity violations in a MIMO context (Sec. 9.1.2, page 139 and Sec. 9.4.2,
page 144),
6.2. synthesizing FF SPR/BR controllers by posing a convex optimization problem where
constraints are in the form of LMIs (owing to the fact the GKYP Lemma is used) (Sec.
9.3, page 141),
6.3. employing the hybrid VSP/finite gain-scheduling architecture of Chapter 6 (specifically, Theorem 6.3.1 on page 104) (Sec. 9.4.3, page 146), and
6.4. experimentally validating the hybrid passive/finite gain systems framework in a non-linear MIMO context (Sec. 9.4.4, page 147).

Without a doubt, the most significant contribution of this thesis is the hybrid input-output framework. Additionally, the connection between the hybrid input-output systems framework and a) Safonov’s variable cones, and b) Iwasaki and Hara’s GKYP Lemma are also quite significant. In particular, the relation between the hybrid input-output framework and the GKYP Lemma is perhaps just as important as hybrid input-output theory itself. The hybrid input-output framework provides a stability result, while the GKYP Lemma provides a means to assess if a LTI system is PR, SPR, BR, or conic in a finite frequency band. Together the hybrid input-output framework and the GYKP Lemma create a unified control architecture.

10.2 Future Work

There are many research avenues on which we may embark in the future.

Linear Time-Varying Systems Although the results of Chapter 4 are novel and interesting, there is additional research to be done. To start, only sufficient conditions are provided in the various theorems of Chapter 4. Naturally, the next step is to derive necessary conditions.

With respect to application of the theory presented in Chapter 4, only the ISP LTV results were used. In the future, we hope to investigate the control of LTV systems that are finite gain or conic, and design finite gain and conic LTV controllers.

At the heart of the $\mathcal{H}_\infty$ control problem is the Small Gain Theorem. In the $\mathcal{H}_\infty$ control framework, there is an uncertainty block $\Delta(s)$ with gain $\gamma_1$, a plant, and a controller. The controller is designed so that the gain of the closed-loop is $\gamma_2$, and provided that $\gamma_1 \gamma_2 < 1$, the closed-loop system is robust to the bounded real (BR) uncertainty expressed by the uncertainty block $\Delta(s)$. A related problem is the Positive Real (PR) control problem.118

Again, there is an uncertainty block $\Delta(s)$, a plant, and a controller; however the block $\Delta(s)$ is PR (not BR). In the PR control problem, the controller is designed so that the closed-loop is SPR, and as such the closed-loop system is robust to the PR uncertainty expressed by the uncertainty block $\Delta(s)$. Future work will investigate a LTV version of the PR control problem, that being the design of a LTV controller to control a LTV plant such that the closed-loop is ISP but still LTV. As such, the closed-loop will be robust to passive uncertainty.

Hybrid Input-Output Systems Recall the $\mathcal{H}_\infty$ and PR control problems just mentioned. In each problem, the uncertainty block $\Delta(s)$ represents either BR or PR uncertainty. In the $\mathcal{H}_\infty$ control problem only the gain of $\Delta(s)$ is relevant, while phase information is discarded.
Similarly, in the PR control problem only the phase of $\Delta(s)$ is relevant, while gain information is discarded. Now consider a problem that is a "hybrid" or "mix" of the $H_\infty$ and Positive Real control problem where the uncertainty $\Delta(s)$ captures PR uncertainty up until a critical frequency, and BR uncertainty past the critical frequency. Such a problem, which we call the Hybrid PR/$H_\infty$ control problem, can now be solved with the hybrid passivity/finite gain systems framework at our disposal. In fact, given the more general hybrid conic systems framework, we are posed to solve similar problems where the uncertainty block $\Delta(s)$ captures different kinds of uncertainty over different frequency bands. For example, consider uncertainty that is BR with gain $\gamma_{11}$ up to $\omega_1$, BR with gain $\gamma_{12}$ between $\omega_1$ and $\omega_2$, and BR again with gain $\gamma_{13}$ above $\omega_2$. Provided that the closed-loop system has gain $\gamma_{21}$ below $\omega_1$ such that $\gamma_{11}\gamma_{12} < 1$, gain $\gamma_{22}$ between $\omega_1$ and $\omega_2$ such that $\gamma_{12}\gamma_{22} < 1$, and gain $\gamma_{23}$ above $\omega_2$ such that $\gamma_{13}\gamma_{23} < 1$, by the Hybrid Conic Sector Theorem developed in Chapter 5 (see Theorem 5.6.1 and Corollary 5.6.1), the system will be robust to uncertainty block $\Delta(s)$. Interestingly, Iwasaki and Hara have already investigated the design of static and dynamic controllers that render a closed-loop hybrid in the sense that the closed-loop system has a certain gain over a specific frequency range.\(^\text{119–121}\) Future work will be identifying how the hybrid input-output systems framework dovetails with Iwasaki and Hara’s results, as well as applications.

Because our original motivation was to overcome passivity violations, we used the hybrid passivity/finite gain framework to control hybrid passive/finite gain systems. Future work will involve the control of systems that are hybrid conic in nature. In particular, how can we design a hybrid conic controller using the GKYP Lemma will be investigated.

Speaking of controller design, in Chapter 9 we showed how to use the GKYP Lemma to constrain a controller to be FF SPR/BR while it is forced (through an optimization objective function) to mimic a standard $H_2$ controller. The optimization problem we posed was rather conservative in that we only parameterized the problem in terms of $K_c$, the state-feedback gain matrix of the controller. We did so because of the “bilinear” terms involving the controller $A_c$ and $K_e$ matrices. In Refs. 122–124 a new “dilated” LMI is presented where the matrix $A_c$ does not have any bilinear terms associated with it. As such, alternative synthesis methods involving $A_c$ as a design variable are possible. In the future, we will investigate how to optimally design FF SPR/BR controllers where constraints are imposed using the GKYP Lemma again, but hopefully we can parameterize our problem in terms of $K_c$, $A_c$, and $K_e$, and convert the resultant bilinear matrix inequality back into a linear matrix inequality using the dilated LMI results of Refs. 122–124.

The passivity violation we considered in this thesis (in both the single-link and two-link work of Chapters 8 and 9) arose from rate approximation using a derivative filter (either $f(s)$ or $F(s)$). The filters used were rather crude; $f(s)$ was a simple second order transfer functions, while $F(s)$ was a diagonal transfer matrix composed of second order transfer functions. Perhaps better rate estimates can be acquired by using an advanced filtering technique, such as a Kalman filter or a sigma point filter (also called an unscented Kalman
filter). Naturally, what the passivity violations look like when an advanced filter is used will be investigated in the future.

Adaptive control design is quite interesting, with great potential. In particular, passivity-based adaptive control has been successfully developed for robotic manipulators and spacecraft. A future research endeavor would be to use the hybrid passivity/finite gain systems framework within an adaptive control architecture.

Gain-Scheduling In Chapter 6, the passivity-based gain-scheduling schemes of Refs. 78 and 79 were reviewed. In both Refs. 78 and 79, the passivity-based gain-scheduling scheme was used to control flexible robotic manipulators. In the future, we hope to utilize the conic gain-scheduling results presented in this thesis to control a system. Identifying a systems that are conic (and nonlinear) will also be investigated in the future.

As noted previously, one of the contributions of Ref. 79 was to optimally design the scheduling signals. The resultant time-dependent scheduling signals were found to be nonlinear when excellent performance was required. In the future we hope to investigate the optimal design of scheduling signals that are a function of more than one variable (such as the outputs of a nonlinear system). Such a project may involve the parameterization of a scheduling “surface”.

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Bibliography


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