THEORY AND APPLICATIONS OF
MICROSTRIP/NEGATIVE-REFRACTIVE-INDEX TRANSMISSION LINE
(MS/NRI-TL) COUPLED-LINE COUPLERS

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
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Abstract

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2011

The electromagnetic coupling of a microstrip transmission line (MS-TL) to a metamaterial backward wave Negative-Refractive-Index transmission line (NRI-TL) is the primary investigation of this dissertation. The coupling of forward waves in the MS-TL to the backward waves in the NRI-TL results in the formation of complex modes, characterized by simultaneous phase progression and attenuation along the lossless lines.

Through network-theoretic considerations, we investigate the properties of these modes in the complex-frequency plane of the Laplace domain to help unravel the confusion that has existed in the literature regarding the independent excitation of a pair of conjugate complex modes. We show that it is possible to arbitrarily suppress one of the modes over a finite bandwidth and completely eliminate it at a discrete set of frequencies using proper source and load impedances. Hence we use conjugate modes with independent amplitudes in our eigenmode expansion when we analyse various coupling configurations between the two types of lines (MS/NRI-TL coupler).

We derive approximate closed-form expression for the scattering parameters of the MS/NRI-TL coupler and these are complemented by design charts that allow the synthesis of a wide range of specifications. Moreover, these expressions reveal that such couplers allow for arbitrary backward coupling levels along with very high-isolation when they are made half a guided wavelength long. The MS/NRI-TL coupler offers some interesting applications which we highlight through the design and testing of a 3-dB power splitter,
a high-directivity signal monitor and a compact corporate power divider. We have included design, simulation and experimental data for the fabricated prototypes exhibiting good agreement and thereby justifying the theory that has been developed in this work to explain the coupling between a right-handed MS-TL and a left-handed NRI-TL.
Dedicated to my loving parents.
Acknowledgements

I would like to thank Prof. George Eleftheriades for his guidance, encouragement and patience in supervising this work. His willingness to be available at all times to share his deep insight and intuition was fundamental in the completion of this thesis and I wholeheartedly appreciate the wonderful learning experience in his company.

Over the years, I have had the fortune of learning under the excellent guidance of various faculty members in the electromagnetics group at the University of Toronto and in this regard, I extend a special note of appreciation to Prof. Sergei Dmitrevsky. The laboratory work that involved fabrication and testing of prototypes was only possible due to the company of knowledgeable and friendly lab managers and the guidance of my enthusiastic colleagues. In this regard, I express my gratitude for the help of Gerald and Tse. I thank Omar for his introduction to chemical etching, Marco for his tips on the fabrication of hybrid microwave circuits, Joshua for his tutorial on using the milling machine and M. Simcoe for the endless comedy.

The experience of being a graduate student for the last couple of years has been enjoyable by being in the company of the most lively group of knowledgeable colleagues. Among my colleagues, special thanks to Francis for pointing out the existence of complex modes in the MS/NRI-TL coupler. I would also like to thank Michael, Ashwin, Tony, Micah, Trevor, Mohamed, Alex, Roberto, Loic, Yan, Alam, Hassan, Wen, Abbas, Peter, Jackie and other members of our group whose pleasant acquaintances I will cherish.

I acknowledge the generosity of the National Sciences and Engineering Research Council (NSERC) of Canada and Nortel Networks for their financial assistance in conducting this work.

I thank my parents, brother Zubair, sister Shaima, aunts Ritu and Ratna and my family and friends both here and abroad for your continuous motivation and steady encouragement.

To my wife Helen, your love and affection, and your willingness to undertake most of the responsibilities in raising our son Ahmed, has been the anchor that has kept me steady and focussed through stressful times. Without your support and understanding, this task would have been impossible to undertake. I thank both of you for bearing with me and providing me with the motivation to complete this work.
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Chapter 1

Negative-Refractive-Index Metamaterials

1.1 Introduction to the concept of the MS/NRI-TL coupled-line coupler

The microwave directional coupler lies at the heart of microwave measurements and device characterization due to its ability to distinguish between a forward and a backward travelling wave in guided-wave structures. The performance of the coupler depends on the mechanism by which two of its adjacent waveguides or transmission lines exchange power and also on the electrical properties of the material out of which it is constructed. Nature offers limited flexibility in the choice of material parameters and hence the option of introducing artificial metamaterials into couplers to engineer their performance is very appealing. A major part of this work is concerned with the modelling of the power coupling mechanism between a regular microstrip transmission line (MS-TL) and a metamaterial Negative-Refractive-Index Transmission Line\(^1\) (NRI-TL) placed adjacent to it. We will refer to this configuration as a Microstrip/Negative-Refractive-Index Transmission-Line (MS/NRI-TL) coupled-line coupler.

\(^1\)The construction, characteristics and applications of the NRI-TL are discussed later in this chapter. The wave propagation in a grid constructed out of these metamaterial lines has been studied in the past to explore sub-diffraction imaging at microwave frequencies [1][2].
1.1.1 Motivation for developing the MS/NRI-TL coupler

The metamaterial MS/NRI-TL coupled-line coupler is the primary RF/microwave device investigated in this work. It was first introduced by R. Islam and G. V. Eleftheriades in 2003 [3] and its operation was later verified by C. Caloz and T. Itoh [4]. In 2004, R. Islam and G. V. Eleftheriades realized that the coupler supported complex modes (guided modes that exhibit simultaneous phase progression and attenuation, although the underlying structure is lossless) which played a fundamental role in describing the operation of the device [5]. Subsequently, closed-form scattering parameters for the coupler were developed in 2005 which revealed a possible coupler design with negligible power leakage into the isolated port [7]. A few applications of the MS/NRI-TL coupler were proposed such as a signal monitor [8] and a corporate power-dividing network [9].

The figure of merit relevant to any coupler includes the coupling level, isolation level and other attributes that are common to microwave components in general such as fabrication cost, losses, bandwidth, etc. Planar couplers are important as they are cost-effective and easy to integrate in hybrid and monolithic microwave systems. The coupling and isolation levels that can be achieved in practice from a conventional quarter-wave microstrip coupled-line coupler [10] are quite low. A large number of microstrip coupler topologies that address these issues have appeared in literature. For instance, the coupling level can be enhanced using the Lange coupler topology [11], mounting vertical conducting plates at the edges of the lines [12], using a multilayer broadside coupling configuration [13], etc. The directivity of the coupler can be improved by using phase velocity compensation techniques such as the introduction of lumped capacitors that bridge the lines at the terminal planes [14], ‘wiggly-lines’ [15], flared forward-coupling lines [16], dielectric overlays [17], etc. The MS/NRI-TL coupler that we have developed in this dissertation enhances both the coupling level and directivity of the coupled-line coupler using a planar topology that can be fabricated using standard PCB fabrication technology. This coupler can achieve a coupling level close to 0 dB with only a moderate spacing between the lines and an isolation level as low as 70 dB has been observed in an experimental prototype [8]. A few other examples of couplers that were constructed

\footnote{The existence of complex modes in this coupler was confirmed afterwards by H. Nguyen et. al. in 2007 [6].}

\footnote{For example, a 50 Ω coupler operating at 2.0 GHz on a 1.27 mm \( \epsilon_r = 9.6 \) substrate will offer a coupling level of less than 3.3 dB and a maximum isolation worse than 27 dB even when the lines are spaced 0.01 mm apart. A line spacing of 0.01 mm is difficult to obtain using standard PCB lithography and hence any coupling level larger than 3 dB is impractical for this simple topology.}
Chapter 1. Negative-Refractive-Index Metamaterials

Figure 1.1: Photographs of microstrip coupled-line couplers (a) A conventional quarter wave microstrip coupler (b) A metamaterial MS/NRI-TL coupler realized by replacing one of the lines with a NRI-TL (a line loaded periodically with series capacitors and shunt inductors to ground)

by the replacement of regular transmission-lines/waveguides with metamaterials can be found in [18, 19].

1.1.2 Construction of the MS/NRI-TL coupler

A conventional microstrip coupled-line coupler is realized by placing two parallel conducting strips above a grounded substrate. When one of these two strips is loaded periodically using series capacitors and shunt inductors to ground as depicted in Figure 1.1, the resulting structure is a MS/NRI-TL coupled-line coupler. The implementation depicted in Figure 1.1 uses chip capacitors placed in slits etched on one of the strips which is also loaded with chip inductors placed in holes drilled through the substrate. Monolithic implementation of this coupler is possible by using printed interdigital capacitors and grounded inductive stubs instead of chip components. If the unloaded strip of the MS/NRI-TL coupler is removed, then the resulting line is simply a metamaterial NRI-TL that supports a fundamental backward-wave mode (phase progression contra-directional to the power flow). The principle of coupling a forward-wave mode to a backward-wave mode was explored in the past by many authors and a few relevant examples can be found in [20]-[22].
1.1.3 Characteristics of the MS/NRI-TL coupler

When power is injected into the input port in Figure 1.1, it splits between the coupled port (which appears adjacent to the input) and the through port. In contradistinction to a regular backward microstrip coupled-line coupler, the coupled power of the MS/NRI-TL coupler can be increased to arbitrary levels (with corresponding drop in the power delivered to the through port) by increasing the total length of the structure. The power leakage into the isolated port (the isolation level) of the coupler is negligible when the device is made half a guided wavelength long. It should be noted that it is very difficult to construct regular microstrip couplers that can achieve coupling as large as $-3$ dB (let alone coupling levels that approach 0 dB) and one needs to employ cumbersome techniques to improve their isolation levels (discussed in Chapter 2).

The differences in the coupling and isolation levels of a regular microstrip coupled-line coupler and the MS/NRI-TL coupler can be explained by considering the peculiarities of their eigenmodes. A source placed at a port of a regular microstrip coupled-line coupler excites both the even and the odd modes strongly. The imbalance in the excitation amplitudes of these two modes determines the maximum coupling level and is dependent on the difference between the modal impedances. The isolation level of the regular coupler relies on destructive interference of the modes as they travel down the length of the coupler and is limited by the unequal phase velocity of quasi-TEM modes in microstrip. On the other hand, a source placed at a port of the MS/NRI-TL coupler strongly excites only a single complex mode whose profile features equal (in magnitude) and oppositely directed power flow on the two lines. The maximum coupling level is limited primarily by the reflection of the excited mode from the far end of the coupler. As these complex modes decay along the length of the coupler, a sufficient increase in its length can minimize reflections and facilitate large coupling levels. The isolation level of the MS/NRI-TL coupler is also dependent on destructive interference of the eigenmodes and in this case, the phase velocities of all modes are equal in the complex-mode band (they appear in complex conjugate pairs) and high isolation levels can be obtained.

1.1.4 Organization of this dissertation

In order to study the MS/NRI-TL coupler without resorting to exhaustive full-wave electromagnetic simulations, we model it as a chain of elementary blocks (unit cells) whose individual coupling characteristics are relatively straightforward to analyse. Standard
periodic analysis is then employed to obtain the eigenmodes of this system. In the
limit of electrically small unit cells, this periodic coupler can be described accurately
using an axially homogeneous model of coupled lines provided that the filling material is
substituted with an effective material parameter.

In the next step, we determine the scattering parameters of the coupler using a super-
position of its eigenmodes that satisfy source and impedance boundary conditions. Here
we are forced to make a slight detour to address a perplexing theoretical issue regarding
complex modes. The eigenmodes that exist in the MS/NRI-TL coupler are complex over
a finite frequency band. For the last few decades, it was believed that the modes in a
pair of conjugate complex modes cannot be excited independently to each other. This
would result in an incomplete modal set and incapacitate our efforts in determining the
scattering parameters through modal expansions. Hence before proceeding to analyse the
terminal response of the coupler, we outline the relevant network theory (in the Laplace
transform domain) associated with lossless and reciprocal coupled-line systems and then
show that these old beliefs on the excitability of complex modes are incomplete.

Analysis of the coupler using a superposition of multiply-reflected complex modes,
reveals that it is capable of achieving arbitrary backward coupling levels and high isolation.
Such characteristics are very desirable for a host of applications such as RF/microwave
power division and signal monitoring. In this regard, we present the design, fabrication
and testing of a 3-dB power divider, a signal-monitor and a compact corporate feed-
network for printed antenna arrays using MS/NRI-TL couplers. The justification for
adopting this new type of coupler for the mentioned applications over conventional ones,
is made through the fact that the benefits of large coupling levels and high isolation can
be obtained from a single compact and low-profile structure that can be fabricated in
a monolithic process. Moreover, the designs exhibit large usable bandwidths in which
the operation of the couplers can be predicted accurately using closed-form expressions
whose empirical parameters are relatively easy to extract. The good agreement between
the developed scattering-parameter equations and the corresponding microwave circuit
simulations supports our theory that mutually independent conjugate complex modes
(in time-harmonic analysis) are responsible for the coupling phenomenon in MS/NRI-TL
couplers.

The topics addressed in this dissertation fall under two distinct but related areas of
electromagnetics research. The design and applications of metamaterial RF/microwave
components comprise the first category, and in the present work, we focus on the MS/NRI-
The second category may be classified under the guided-wave theory of electromagnetics or network/state-space theory of guided waves. In this regard, the emphasis is on the analytic theory of the excitation of guided complex modes that are accompanied by potentially unstable branch-point singularities. Chapters 1, 2 and 7 contains background and summary material common to both areas. The network theory of the problem of complex mode excitation can be found in chapters 3 and 4 while the design equations and applications of the MS/NRI-TL coupler are confined to chapters 5 and 6. The ordering of the various chapters is based on their logical connection rather than chronological development and as they are self-contained, the reader will have no difficulty in focussing on the ones that are relevant to his/her interests. For example, the reader who is primarily interested in learning about the operation and design of the MS/NRI-TL coupler, will find all of the necessary background material, theory and design examples in chapters 2, 5 and 6.

Now we will briefly examine some of the work done on Negative-Index metamaterials and their applications to practical RF/microwave systems. We will concentrate on the properties of the NRI-TL to provide supplementary information as a guide in the modelling and analysis of its coupling to a standard transmission line.

1.2 Brief history of Negative-Refractive-Index metamaterials

The macroscopic description of electromagnetic phenomena in homogeneous media is feasible through the concepts of relative permittivity and permeability that describe their response to an applied field. The microscopic picture that underlies such a description is that of a dense and homogeneous arrangement of electric and magnetic multi-poles that align themselves to varying degrees when an external field is applied. Here the concepts of denseness and homogeneity depend on the resolution of our experimental apparatus that will determine if we can observe the finer details of the spatial variation of a field. This spatial resolution is dependent on the minimum wavelength and hence the maximum frequency that is of interest in any given experiment. Hence what qualifies as homogeneous at a given frequency range, will cease to do so at higher frequencies.

At radio frequencies, the wavelength of interest can be many centimetres long and hence we can construct bulk artificial structures with sub-wavelength intricate features.
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Figure 1.2: Orientation of the electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, wave propagation $\mathbf{k}$ and Poynting $\mathbf{S}$ vectors in a regular $\epsilon_r, \mu_r > 0$ material (shown on the left) and in a $\epsilon_r, \mu_r < 0$ metamaterial (shown on the right).

In this limited frequency range, the structure will appear homogeneous to the incident wave and its electromagnetic response can be tailored to provide a desired permittivity and permeability. Such artificially engineered dielectrics are referred to as Metamaterials of which the ones that are **Left-Handed** will be discussed due to their direct relevance to this work. A very brief introduction to this topic is presented below as the details can be found in a number of books including the one by G. V. Eleftheriades and K. G. Balmain [23].

In a left-handed medium, the propagation of plane electromagnetic waves is characterized by a left-handed orientation of the electric field, magnetic field and propagation vector triplet (see Figure 1.2). The electrodynamics of such media was first theoretically investigated by V. Veselago in the sixties [24] who identified the necessary conditions of negative permittivity and permittivity for this phenomenon to occur. This results in a unique propagation characteristic where the wave fronts move in a direction opposite to that of the power carried by the wave. When a wave is incident from vacuum onto its surface, the field continuity at the interface along with the contra-directional phase and power flow requirements, result in a refracted wave whose direction requires a negative index in Snell’s law. Hence left-handed media are described by a negative index of refraction and from simple geometrical optics considerations, it can be shown that a flat slab of this material functions likes a lens. Working out the detailed field theoretic wave transmission characteristics of such a slab, J. B. Pendry concluded that it was capable of beating the diffraction limit (and hence greatly improving the resolving power) of standard lenses [25].
To test these fascinating theoretical predictions, a large volume of research in left-handed material synthesis and fabrication has been conducted beginning with the initial experiments at the University of California in San Diego [26] involving split-ring-resonators and wires based metamaterials (see Figure 1.3a). This 3-dimensional topology, at least in its original form, is unsuitable for use in standard planar RF circuits and in this regard, the invention of the loaded transmission line based metamaterials (Figure 1.3b) is of significant importance [1]. Transmission line based NRI lenses have been successfully constructed to demonstrate sub-diffraction imaging in a 2-dimensional system [2] and focussing in 3-dimensions using a lens matched to free space [27].

The transmission line based Negative-Refractive-Index (NRI) metamaterial is constructed by loading a transmission-line grid with series capacitors and shunt inductors connected to the ground plane beneath it. The relationship between ordinary transmission line per-unit-length series reactance (usually inductive) and shunt susceptance (usually capacitive) with the permittivity and permeability respectively of the filling medium is well known. Hence the loading elements of the NRI grid alter the signs of the series and shunt immitances when operated below the corresponding branch resonances resulting in the emulation of negative permittivity and permeability respectively. The periodic Bloch-Floquet analysis can be used to rigorously compute the modes that are supported by the NRI grids, and the resulting propagation characteristics are those expected in a left-handed material.
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Figure 1.4: (a) Schematic of a metamaterial NRI-TL unit cell and (b) A sample monolithic implementation in microstrip consisting of an interdigital capacitor and spiral inductors grounded by vias.

At present, the investigation of metamaterials have branched off beyond the consideration of homogeneous NRI media to those that are inhomogeneous, anisotropic and non-linear with arbitrary refractive indexes and efforts are under way to push the successful attempts at microwave frequencies to the optical domain. Some interesting areas of research include resonance beams in anisotropic grids [28], the realization of nanocircuits at optical frequencies [29] and the electromagnetic cloaking of objects using transformation-optics metamaterials [30].

A major subject of this dissertation is the investigation of the coupling between a regular transmission line and a metamaterial NRI transmission line (NRI-TL). Hence in the following section we take a closer look at NRI-TLs, their construction, analysis and typical dispersion band diagrams.

1.3 Analysis of Negative-Refractive-Index Transmission-Line (NRI-TL) metamaterials

A metamaterial Negative-Refractive-Index Transmission Line (NRI-TL) is constructed using a periodic series capacitive and shunt inductive loading of a host transmission line such as a microstrip line, a strip line or a coplanar strip line [23]. They are also referred to as Composite Right/Left Handed Transmission Lines (CRLH-TL) [31] or simply as Left-handed lines in literature.
When operated in its first passband such that the electrical length of each unit cell is small compared to the guided wavelength, the NRI-TL can be characterized by effective homogeneous permittivity and permeability parameters with negative values [1]. In this passband, the NRI-TL supports backward wave propagation\(^4\) with phase increasing away from the source (in contradistinction to the decreasing phase of ordinary wave propagation away from the source under the assumption of \(e^{j\omega t}\) time dependence).

The schematic of a single unit cell of length \(d\) of this periodic structure is shown in Figure 1.4 along with a possible implementation that depicts a microstrip line hosting an interdigital capacitor and four spiral inductors grounded using vias. The characteristic impedance of the host microstrip line is \(Z_0 = \sqrt{L_0/C_0} = Y_0^{-1}\) with a propagation constant \(\beta = \omega \sqrt{L_0 C_0}\) and it is loaded with inductance \(L\) and capacitance \(C\). Let us define the following dimensionless parameters:

\[
L_n = \frac{L_0 d}{L} \\
C_n = \frac{C_0 d}{C} \\
\theta = \omega \sqrt{L_0 C_0} d
\]

The ABCD matrix\(^5\) for this unit cell is given by:

\[
\begin{pmatrix} V \\ I \end{pmatrix}_L = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}_R
\]

\[
A' = D' = 1 - 2 \left[ \sin \left( \frac{\theta}{2} \right) - \frac{L_n}{2\theta} \cos \left( \frac{\theta}{2} \right) \right] \left[ \sin \left( \frac{\theta}{2} \right) - \frac{C_n}{2\theta} \cos \left( \frac{\theta}{2} \right) \right]
\]

\[
B' = 2jZ_0 \cos \left( \frac{\theta}{2} \right) \left[ \sin \left( \frac{\theta}{2} \right) - \frac{C_n}{2\theta} \cos \left( \frac{\theta}{2} \right) \right]
\]

\[
C' = \frac{(A')^2 - 1}{B'}
\]

The subscripts \(R\) and \(L\) designate the right-side or left-side voltages and currents with the latter flowing towards the right at all ports of the unit cell depicted in Figure 1.4a. When we construct the NRI-TL through a periodic chain of these unit cells, the modes that are supported by it are those whose voltages and currents differ across each cell by a fixed complex constant (see Appendix G). These constants are functions of frequency.

---

\(^4\)Backward waves are characterized by contra-directional phase and power flow

\(^5\)Also known as the transfer or transmission matrix.
and are obtained as eigenvalues of the ABCD matrix of the system \[10][32\]. In this case, there are 2 eigenvalues and along with the associated eigenvectors, they form a complete set of functions that describe the voltages and currents at the unit cell terminals for arbitrary sources and terminations placed at the ends of the chain. We can obtain the eigenvalues, which we designate as \(e^{\gamma_{nri}d}\), and the scalar ratio of its voltage and current components \(Z_{nri}\) (that simply specifies the eigenvector) from the following expressions:

\[
sinh(\gamma_{nri}d) = A'
\]

\[
\gamma_{nri} = \frac{B'}{\sinh(\gamma_{nri}d)}
\]

It should be noted that these Bloch modes describe only the terminal voltages and currents of each unit cell from which one may deduce the waveforms inside them. The quantity \(A'\) is equation (1.8) is oscillatory and when its magnitude is larger than unity, the resulting mode is evanescent. On the other hand, when \(|A'| < 1\) the corresponding \(\gamma_{nri}\) is purely imaginary implying a propagating Bloch wave. The critical frequencies at which \(|A'| = 1\) correspond to band edges between propagating and evanescent modes. In the homogeneous limit, each unit cell is electrically short \(\theta \ll 1\) and the net phase incurred per unit cell is small as well \(|\gamma_{nri}d| \ll 1\) (which in turn implies that \(A' \approx 1\)). If we set \(L_n = C_n\) in equation (1.5), then two of the band edges corresponding to \(A' = 1\) merge, thereby causing the evanescent mode stop-band between them to disappear. This stop-band closing condition is equivalent to setting \(L/C = L_0/C_0\) and was reported in \[1\].

When the stop-band is not closed, the band edges corresponding to \(A' = 1\) are obtained as the solutions of the transcendental equations \(\tan(\theta/2) = L_n/(2\theta)\) and \(\tan(\theta/2) = C_n/(2\theta)\). Hence it is clear that when these band edges occur for small values of \(\theta\), then the terms \(L_n/(2\theta)\) and \(C_n/(2\theta)\) are small as well in the homogeneous limit. Hence when we expand (1.8) and (1.9) as a Taylor’s series in \(\theta\) upto the second order, the quantities \(L_n\) and \(C_n\) will be treated as small second-order terms. Ignoring terms of order three and higher in the series expansion of both sides of these equations, we obtain the following useful approximations of the NRI-TL characteristics:

\[
-(\gamma_{nri}d)^2 \approx \left[\theta - \frac{C_n}{\theta}\right] \left[\theta - \frac{L_n}{\theta}\right]
\]

\[
Z_{nri}^2 \approx Z_0^2 \left[\theta - \frac{C_n}{\theta}\right] / \left[\theta - \frac{L_n}{\theta}\right]
\]
These expressions are familiar from some of the early work in NRI-TLs [1] and the following observations can be made. First we notice that equations (1.10) and (1.11) suggests that the NRI-TL can be interpreted as a regular transmission line whole series inductance \(j\omega L_0d\) has been replaced by the resonator \(j\omega L_0d + \frac{1}{j\omega C}\) and whose shunt capacitance \(j\omega C_0d\) has been replaced by the resonator \(j\omega C_0d + \frac{1}{j\omega L}\). The left-handed behaviour in this NRI-TL is observed when both of these resonators are operated below resonance. This artificial transmission line is referred to as a NRI-TL even when it is not operated in the left-handed regime.

In Figure 1.5 we plot the dispersion characteristics (that is symmetric about the \(\theta\) axis) of the NRI-TL unit cell under the closed stop-band condition and we observe that there is good agreement between the Bloch/Floquet modes given by equations (1.8) and (1.9) and their second order approximation for values of \(\gamma_{nri}d\) close to 0. Hence in the homogeneous limit, the NRI-TL can be accurately characterized by the per unit length series and shunt resonators discussed earlier.

The backward wave behaviour of the NRI-TL is evident from the negative slope of the dispersion curve in Figure 1.5a below the \(\text{Im}(\gamma_{nri}d) = 0\) stop band. This slope is proportional to the group velocity and hence the wave \(\exp(-\gamma_{nri}z + j\omega t)\) that carries power in the \(+z\) axial direction of the NRI-TL must exhibit a \(\gamma_{nri}\) with a negative imaginary part. In other words, below the stop band, the NRI-TL dispersion curve,
reflected about the $\theta$ axis in Figure 1.5a, corresponds to the proper branch$^6$ that can be excited by a source placed at $z = 0$ in a line extending to $z \to +\infty$. Another way to verify this assertion is through the addition of a finite amount of dissipation into the system and then choosing the branch of the resulting dispersion plot with modes that decay to zero as $z \to +\infty$.

Adding a finite amount of dissipation is equivalent to the frequency transformation $\theta \to \theta - j\delta$ where $\delta \ll \theta$ is a positive real quantity. Applying this transformation to equation (1.10) and retaining up to first order terms in $\delta$:

$$(\gamma_{nri}d)^2 \approx -\left[\theta - \frac{C_n}{\theta}\right]\left[\theta - \frac{L_n}{\theta}\right] + 2j\delta\theta \left[1 - \frac{L_nC_n}{\theta^4}\right] \tag{1.12}$$

We recall that $\theta$, $L_n$, $C_n$ and $\delta$ are all positive and real. In the left-handed regime, if we set $\delta = 0$ in equation (1.12), then the right-hand side must evaluate to a negative real number so that $\gamma_{nri}$ is imaginary and corresponds to propagating modes. Below the stop-band resonance (given by $\theta^2 \approx L_n$ and $\theta^2 \approx C_n$) and when $\delta \neq 0$, the rightmost term in (1.12) enclosed in the square bracket, is negative. Consequently, the angle of the complex quantity $(\gamma_{nri}d)^2$ is slightly larger than $\pi$ and its two roots lie in the second and fourth quadrants of the complex plane. The root in the fourth quadrant has a positive real part and thereby corresponds to bounded modes as $z \to +\infty$ and its imaginary part is negative. Hence the branch of $\gamma_{nri}$ with a negative imaginary part corresponds to the proper modes below the stop-band of the NRI-TL.

### 1.4 RF/microwave applications of NRI-TL metamaterials

The synthesis of a network consisting of transmission lines and lumped loading elements to achieve a desired frequency response has been traditionally handled through the application of Richard’s transformation and Kuroda identities [10] to a vast repository of lumped element insertion-loss based prototypes. Although such techniques are powerful and systematic, at present they do not handle problems where the network to be

$^6$The terms ‘proper’ and ‘branch’ have been adopted from literature associated with leaky-wave antennas where it is common to separate the various modes of the structure based on their behaviour at infinity and place them on distinct Riemann sheets (sometimes referred to as ‘branches’) of a single multi-valued analytic function [33]. A ‘proper’ branch is an eigenmode that remains bounded (or carries power towards infinity) in the semi-infinite half-space problem under consideration.
synthesized needs to meet certain spatial as well as temporal specifications.

To elaborate on this, we will provide some examples. Filter synthesis can be used in the design of a distributed network with an optimal frequency response such as one with a maximally flat pass band. If we now insist that the total electrical length of the network should be a certain number of free space wavelengths, then it is unclear whether the Butterworth lumped element prototypes are necessarily optimal. Furthermore, it is possible for a loaded transmission line based network to radiate [34] and in this case, the existing network synthesis techniques cannot be relied upon to optimize the corresponding radiation pattern. Finally, if we require networks that are either 2 or 3 dimensional and use them to interact with an impinging electromagnetic wave, then this becomes a very demanding network analysis problem while the task of synthesis is practically impossible using the present state of filter theory.

Hence the idea of recognizing a periodically loaded structure such as a NRI-TL as an effective medium characterized by its Bloch/Floquet modes is a very useful one. It provides a new paradigm in tackling some of the design problems that are foreign to classical network synthesis [35]. This technique derives its strength through the homogenization principle where in the limit of long wavelength, the behaviour of a network is accurately modelled in terms of effective wave propagation parameters that hide the intricate circuit level details. This technique is not capable of simultaneous spatial and temporal (frequency domain) synthesis yet, but it has been successful in the realization of a large number of useful microwave structures. The strength of the effective medium theory lies in the use of principles borrowed from geometric and wave optics to understand the behaviour of complex networks instead of addressing it as a formidable multi-port network analysis problem [36]. We will outline a few applications in this section that were inspired by the NRI-TL.

**The backward end-fire leaky-wave antenna**

When we examine the typical dispersion plot for a NRI-TL (see Figure 1.5a), we notice that a portion of this plot is confined within the light line that corresponds to the propagation of a free-space wave. In this frequency range, where the magnitude of the propagation constant of the NRI-TL is less than that of free space, it is possible to couple energy into a radiating leaky-wave mode. A leaky-wave antenna operating on this principle was documented in [34] and exhibited a backward end-fire radiation at the design
frequency of 15 GHz. This particular design was implemented using coplanar waveguide technology by loading the central conductive strip with series flared gap capacitors and shunt inductive strips. The principle of its operation is illustrated in Figure 1.6a where the power flow (red arrow) and phase flow (grey arrow) are in opposite directions and phase matching at the antenna-air interface leads to two radiated beams in the backward direction (depending on the front-to-back ratio of the antenna). As one increases the frequency of operation, the main beam tilts from backward end-fire to broadside and then to the forward end-fire direction.

**Miniaturized microwave components**

Phase-shifting lines at low frequencies can be relatively large and hence various techniques have been employed in their miniaturization such as through meandering or low-pass loading [39]-[42]. The dispersion characteristics of the NRI-TL line reveals the possibility of obtaining large phase shifts at low frequencies and hence provides an alternative solution for achieving a large phase shift from a physically short line. Moreover, when the loading element values are small, the corresponding $L_n$ and $C_n$ in equation (1.10) are large leading to increased phase shift. This particular characteristic of the NRI-TL is very useful as small loading capacitances and inductances are practical to implement especially using a monolithic process.
Chapter 1. Negative-Refractive-Index Metamaterials

Figure 1.7: (a) A compact branchline coupler with 73% reduction in area (b) A wideband balun using NRI-TL without vias [38].

Figure 1.8: Phase-agile branchline couplers constructed by replacing the (a) Low impedance lines with NRI-TLs and (b) high impedance lines with NRI-TLs.

A NRI-TL based 1-to-4 series power divider was designed in [37] to obtain in-phase output and the topology is shown in Figure 1.6b. The alternative option that uses 1 wavelength long meandered line is also depicted and the latter occupies a larger circuit area and is narrowband compared to the NRI-TL implementation. This miniaturization technique was also applied to realize a compact branch-line coupler (shown in Figure 1.7a) that represents a 72% area reduction compared to a regular one operated at the same frequency and designed for the same system impedance [43]. This particular prototype used plated vias at the four corners to provide the required shunt inductances and series interdigital fingers for the loading capacitance.
Microwave devices incorporating NRI-TLs

A balun for converting a single ended input into a differential output [38] is shown in Figure 1.7b. This structure consists of a pair of microstrip lines that are periodically bridged by sections of transmission lines and are loaded in series with interdigital capacitors. Under the even mode excitation, each line is under cut-off due to its series and shunt capacitive loadings and hence this signal attenuates to a negligible level before reaching the output ports. On the other hand, the odd mode propagation is essentially along a pair of NRI-TLs operated in their pass-band (the bridged transmission-line segments act as shunt inductors due to the virtual short at their centres) and this signal reaches the output ports. Hence when one of the ports is excited, only the odd mode (that is out of phase on the two lines) voltage appears at the far end of the balun.

The two branch-line hybrids shown in Figure 1.8 were constructed by replacing either the low impedance or the high impedance branches with NRI-TLs [44]. The magnitude response of such a structure is very similar to a conventional hybrid but these substitutions allow the control of the phase at the output ports. In a regular branch-line coupler, the through phase is $-\pi/2$ while the phase at the coupled port is $\pi$. In these new prototypes, the coupled port phase is 0 and the through port phase is either $+\pi/2$ or $-\pi/2$ depending on whether the low impedance or the high impedance branch is substituted with a quarter-wave NRI-TL respectively. The ability to incorporate phase-shifting lines into power dividers can aid in the reduction of the overall size of a microwave feed network that possibly employs various phase shifting lines.

Other applications

The NRI-TL topology has been utilized in many microwave applications other than the ones highlighted above. The unique frequency dispersion characteristics of this line has been exploited in the construction of multi-band filters and components. Various coupled-line couplers, hybrids, antennas and diplexers have been investigated in recent years that employ NRI-TL lines to achieve characteristics such as broad bandwidth, multi-band operations and miniaturization.

A NRI-TL with open stop-band exhibits a bandpass response with slow roll-off at the band edges. This topology inspired a very compact and highly selective band-pass filter (shown in Figure 1.9a) by the replacement of the shunt inductor with a quarter wave open resonator to introduce a transmission zero into the magnitude response of the line [45]. A
A high-Q notch filter single unit cell of this highly selective band-pass filter was later attached to the ends of a conventional branch-line hybrid (see Figure 1.9b) to obtain a notch band-stop filter. The conversion of a band-pass to a band-stop notch using a hybrid can be understood in the following manner. In its pass-band, the filter surrounded by broken lines in Figure 1.9b is made extremely narrowband and hence its insertion loss due to substrate dissipation is very high. When connected to a hybrid, it presents matched loads to both output ports and allows minimal power transfer between them. As a result, there is no transfer of power from port 1 to the isolated port 2. On the other hand, in the stop-band of the filter attached to the hybrid, all power is reflected back and recombines at the previously isolated port in a manner analogous to the reflection phase-shifter described in section 2.8. This effectively channels the stop-band power flow through a lossy narrow-band filter and the pass-band power through the relatively low-loss branches of the hybrid coupler. The resulting notch filter has a quality factor of 50 and pass-band insertion loss of 1 dB over a 150% bandwidth [46].
Chapter 2

Theory of Coupling of Propagating Modes

2.1 Introduction

Coupled-mode theory deals with the analysis of energy exchange between two or more systems when their independent oscillations at infinite separation are affected due to proximity interaction. A classical example would be that of two swinging pendulums connected by a spring. The systems that can be analysed are not limited to physically distinct entities but can involve the interaction of forward and backward waves in periodic structures for instance. Coupled-mode formulations have been used in the investigation of phenomena ranging from the interaction of electron waves and electromagnetic fields inside vacuum tubes [47] to photo-elastic coupling of modes in dielectric waveguides using sound waves [49]. A selective historical account of the development and application of coupled-mode theory can be found in [50].

2.2 Pierce’s coupled-mode theory

Our primary interest is in the investigation of the coupling between two structures supporting guided electromagnetic waves. In this regard, John R. Pierce’s landmark paper titled “Coupling of Modes of Propagation” in 1953 is significant [47]. The importance of his work lies in the abstract formulation where the general features of coupling between two modes are investigated using the basic assumptions of linearity and energy conservation [48].
Following Pierce, let us designate by $P$ and $Q$ a pair of quantities (such as voltages or currents) which convey a measure of power flowing in transmission line 1 and line 2 respectively. Specifically, let $PP^* \pm QQ^*$ denote the total power flowing along the lines. A set of simultaneous linear ordinary differential equations in $P$ and $Q$ can be written as follows (see Appendix A for details):

\begin{align}
- \frac{dP}{dz} &= j(\kappa_{11} + \beta_p)P + j\kappa_{12}Q \\
- \frac{dQ}{dz} &= j\kappa_{21}P + j(\kappa_{22} + \beta_q)Q
\end{align} (2.1) (2.2)

In equations (2.1) and (2.2) above, $\kappa_{11}$, $\kappa_{12}$, $\kappa_{21}$ and $\kappa_{22}$ are complex constants that approach zero in the limit of infinite separation thereby resulting in two isolated modes with real propagation constants $\beta_p$ and $\beta_q$. The only differences between equations (2.1) and (2.2) and the ones in [47] are the $\kappa_{11}$ and $\kappa_{22}$ terms which were set to zero by Pierce. Assuming a $e^{-\gamma z}$ variation for both $P$ and $Q$ in (2.1) and (2.2), the eigenvalues and corresponding eigenvectors of the system are:

$$
\gamma_{1,2} = \frac{j}{2}[(\kappa_{11} + \beta_p) + (\kappa_{22} + \beta_q)] \pm \frac{1}{2} \sqrt{-4\kappa_{12}\kappa_{21} - [(\kappa_{11} + \beta_p) - (\kappa_{22} + \beta_q)]^2} 
$$ (2.3)

$$
\begin{pmatrix} P \\ Q \end{pmatrix}_{1,2} = \begin{pmatrix} 1 \\ R_{1,2} \end{pmatrix} = \begin{pmatrix} 1 \\ \gamma_{1,2} - j(\kappa_{11} + \beta_p) \\ j\kappa_{12} \end{pmatrix}
$$ (2.4)

The subscripts of $\gamma$ in equation (2.3) refer to the choice of sign in front of the radical. In equation (2.4), the $R_{1,2}$ parameters describe the complex amplitude ratios between the two lines for each eigenmode $\gamma_{1,2}$.

In a lossless system, conservation of energy reads \( \frac{d}{dz}(PP^* \pm QQ^*) = 0 \) with the upper sign corresponding to modes with co-directional power flow with respect to the choice of $\beta_p$ and $\beta_q$. Substituting (2.1) and (2.2) into the energy conservation expression, we obtain:

$$
PP^*(\kappa_{11} - \kappa_{11}^*) \pm QQ^*(\kappa_{22} - \kappa_{22}^*) \pm PQ^*(\kappa_{21} \mp \kappa_{12}^*) + P^*Q(\kappa_{12} \mp \kappa_{21}^*) = 0
$$ (2.5)

Equation (2.5) applies for all $z$ and for all physically realizable boundary conditions. Hence it is always possible to set either $P = 0$ or $Q = 0$ at some $z$ resulting in $\kappa_{22}$ and $\kappa_{11}$ respectively being real. The phase of $P$ and $Q$ are likewise arbitrary and therefore the factors in brackets appearing in the last two terms of (2.5) can be set to zero. The
restrictions on the coupling parameters due to energy conservation are summarized below.

\[ \kappa_{11} = \kappa_{11}^* \quad (2.6) \]
\[ \kappa_{22} = \kappa_{22}^* \quad (2.7) \]
\[ \kappa_{12} = \kappa_{21}^* \text{(co-directional coupling)} \quad (2.8) \]
\[ \kappa_{12} = -\kappa_{21}^* \text{(contra-directional coupling)} \quad (2.9) \]

If we denote \( \kappa_{11} + \beta_p \) and \( \kappa_{22} + \beta_q \) by \( \beta_{p*} \) and \( \beta_{q*} \) respectively, then for loss-less coupling, we can re-write \( \gamma_{1,2} \) and \( R_{1,2} \) in (2.3) and (2.4) as follows:

\[ \gamma_{1,2} = \frac{j}{2} (\beta_{p*} + \beta_{q*}) \pm \frac{j}{2} \sqrt{\pm 4|\kappa_{12}|^2 + (\beta_{p*} - \beta_{q*})^2} \quad (2.10) \]
\[ R_{1,2} = \frac{1}{2\kappa_{12}} (\beta_{q*} - \beta_{p*}) \pm \frac{1}{2\kappa_{12}} \sqrt{\pm 4|\kappa_{12}|^2 + (\beta_{p*} - \beta_{q*})^2} \quad (2.11) \]

The positive signs inside the radicals in (2.10) and (2.11) are chosen for co-directional coupling while the negative signs for contra-directional coupling. It can be seen that in the limit of no coupling (all parameters in equations (2.6) to (2.9) approaching zero), the propagation constants \( \gamma_{1,2} \) approach those of the isolated lines \( (\beta_p \text{ and } \beta_q) \). At the same time, the amplitude ratios \( R_{1,2} \) approach their extreme values (0 and \( \infty \)) indicating that the various modes consist of waves tightly bound to their respective lines. For both co-directional and contra-directional coupling, the strongest interaction between the lines occur when the difference \( \beta_{q*} - \beta_{p*} \) is small, resulting in \( |R| \) in (2.11) to be close to unity.

The modes corresponding to co-directional coupling propagate without attenuation when the difference between \( \beta_{p*} \) and \( \beta_{q*} \) is small in equation (2.10). An interesting observation about contra-directional coupling is that under the same condition, the propagation constants for both modes become complex-conjugate quantities. If we were to analyse coupling of forward waves and backward waves in a periodic structure such as a Bragg grating, the attenuation of these modes could be explained as the process by which the forward wave loses power to the reflected waves in the Bragg stop-band. On the other hand, when we consider the coupling of a forward-wave transmission line and a backward-wave transmission line, the modal attenuation is due to the continuous power leakage from one line to the other.

If strong coupling (with moderate values of coupling coefficient) is desired between two forward wave transmission lines, then they have to be operated in the frequency range where their isolated propagation constants are almost equal. An example would be the coupling between two identical microstrip lines with resulting even and odd modes.
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As the coupling is co-directional, it is impossible to excite just one of the modes by placing a single source at one end of a transmission line (forward power-flow in both lines require at least two sources to satisfy boundary conditions). Nevertheless, a single mode can indeed be strongly excited in the case of contra-directional coupling between a forward-wave and a backward-wave line with the additional benefit of decaying modes and hence minimal reflections off the end terminations.

Pierce’s formulation of coupled-mode theory, at least in its present form, is not well suited for analysing practical couplers at microwave frequencies. One of the difficulties is in the evaluation of the coupling coefficients (2.6) to (2.9) from the cross-sectional geometry of the transmission lines under consideration. For instance, in the case of dielectric optical waveguides, Yariv [49] derives expressions for these coefficients in terms of integrals over the complete modal set of each isolated guide comprising the coupler. Working out closed-form analytic expressions for the modal fields of periodically loaded microstrip transmission lines is impractical. On the other hand, the assumption of transverse electromagnetic (TEM) or quasi-TEM modal propagation allows one to model these transmission lines quite accurately in terms of distributed (per-unit-length) series, shunt and coupling immitances. Hence a coupled-mode formulation where the coupling coefficients can be directly related to the distributed immitance parameters is desirable.

Yet another shortcoming of the simple coupled-mode formulation outlined above is the failure to account for the impedances of the coupled modes. The formulation presented above is primarily concerned with the change in the propagation constants of the isolated waveguides/transmission lines due to proximity effects when they are brought near each other, and does not account for possible changes in the impedances of the original modes. A terminating impedance is a common boundary condition encountered in microwave circuits, and hence the change in the impedance level of the coupled-modes is crucial information that cannot be ignored. To overcome these difficulties we can adopt the coupled-mode formulation due to Schelkunoff [51].

2.3 Schelkunoff’s coupled-mode theory

The analysis of uniform multi-conductor transmission lines in homogeneous media can be traced back to the work of Lord Kelvin and Oliver Heaviside. Their approach relied on circuit theoretic principles based on the assumption that each conductor could be described in terms of distributed series inductance, shunt capacitance and their associated
dissipation parameters. In 1941, Louis Pipes analysed the same problem from a field-theoretic perspective showing the equivalence between Kelvin’s and Heaviside’s formulations and those obtained by solving Maxwell’s equations [52]. It was Sergei Schelkunoff at Bell Labs, who carried out the first thorough investigation of the coupling of modes of propagation in various waveguide geometries using Maxwell’s equations [51]. Unlike his predecessors, Schelkunoff’s formulation takes into account the interaction of higher-order non-TEM modes in generic non-uniform waveguides.

In Appendix B, we present the derivation of a simplified version of Schelkunoff’s coupled-mode differential equations\footnote{In his original work, Schelkunoff considered guides that could be axially non-uniform.} based on linear system theory of reciprocal and axially uniform networks which is in essence an extension of Pierce’s scalar coupled-mode equations. Although the general formulation of Schelkunoff considers an infinite number of modes, we will limit ourselves to the coupling between the dominant TEM or quasi-TEM modes between a pair of uniform (Bloch wavelength limit) transmission lines.

Using the notations in [97], the coupled-mode differential equations for two transmission lines are:

\[
\begin{pmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & Z_1 & Z_m \\
0 & 0 & Z_m & Z_2 \\
Y_1 & Y_m & 0 & 0 \\
Y_m & Y_2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2
\end{pmatrix}
\Rightarrow
\frac{d}{dz}
\begin{pmatrix}
V \\
I
\end{pmatrix}
= \begin{pmatrix}
N & Z \\
Y & N
\end{pmatrix}
\begin{pmatrix}
V \\
I
\end{pmatrix}
\tag{2.12}
\]

In equation (2.12), \(V_1\) and \(V_2\) are the voltages measured on line 1 and line 2 respectively relative to a common ground reference while \(I_1\) and \(I_2\) are the currents flowing in these lines. If the structure is lossless, \(\frac{d}{dz}(V^T I^* + I^T V^*) = 0\) and then using arguments similar to those in section 2.2, we can deduce that both matrices \(Z\) and \(Y\) are purely imaginary.

Modal solutions can be obtained by replacing the operator \(\frac{d}{dz}\) with \(-\gamma\) in (2.12). The squares of each eigenvalue of (2.12) are the eigenvalues of the following systems of equations:

\[
(ZY)V = \gamma^2 V
\tag{2.13}
\]
\[
(YZ)I = (ZY)^T I = \gamma^2 I
\tag{2.14}
\]
\[
ZY = \begin{pmatrix}
a_1 & b_1 \\
b_2 & a_2
\end{pmatrix} = \begin{pmatrix}
Z_1 Y_1 + Z_m Y_m & Z_1 Y_m + Z_m Y_2 \\
Z_2 Y_m + Z_m Y_1 & Z_2 Y_2 + Z_m Y_m
\end{pmatrix}
\] (2.15)

Let \(\gamma_c^2\) and \(\gamma_\pi^2\) denote the two eigenvalues of (2.13) and (2.14). The matrices \(P_V\) and \(P_I\) with column eigenvectors can be used to diagonalize \(ZY\) and \((ZY)^T\) according to:

\[
P_V^{-1} (ZY) P_V = P_I^{-1} (ZY)^T P_I = D = \begin{pmatrix}
\gamma_c^2 & 0 \\
0 & \gamma_\pi^2
\end{pmatrix}
\] (2.16)

Equation (2.16) results in \((P_I^T P_V) D = D (P_I^T P_V)\) from which it can be seen that the product \((P_I^T P_V)\) is diagonal. This will establish a relationship between the eigenvectors of (2.13) and (2.14) as follows:

\[
P_I^T P_V = \begin{pmatrix}
I_c & I_\pi
\end{pmatrix} \begin{pmatrix}
V_c & V_\pi
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
i_c & i_\pi
\end{pmatrix}^T \begin{pmatrix}
1 & 1 \\
R_c & R_\pi
\end{pmatrix} = \begin{pmatrix}
1 + i_c R_c & 1 + i_c R_\pi \\
1 + i_\pi R_c & 1 + i_\pi R_\pi
\end{pmatrix}
\] (2.17)

Enforcing (2.17) to be diagonal, results in the eigen-vectors:

\[
V_{c,\pi} = \begin{pmatrix}
1 \\
R_{c,\pi}
\end{pmatrix}, \quad I_{c,\pi} = \begin{pmatrix}
1 \\
-1/R_{\pi,c}
\end{pmatrix}
\] (2.18)

We can use the vectors in equation (2.18) to construct the eigenvectors of the original matrix problem in equation (2.12). Any constant multiple of the vectors above will satisfy (2.13) and (2.14) but the choice of the multipliers can be restricted using equation (2.12):

\[
\gamma (\mu V) = Z (\nu I) \quad \text{(2.19)}
\]

\[
\gamma (\nu I) = Y (\mu V) \quad \text{(2.20)}
\]

In (2.19) and (2.20) above, the vectors \(V\) and \(I\) are given by (2.18). Letting \(Z_{c,\pi} = \mu/\nu\), from equation (2.20) we obtain the following:

\[
\frac{\gamma_{c,\pi} \gamma_{c,\pi}}{Z_{c,\pi} \gamma_{c,\pi}} \begin{pmatrix}
1 \\
-1/R_{\pi,c}
\end{pmatrix} = \begin{pmatrix}
Y_1 & Y_m \\
Y_m & Y_2
\end{pmatrix} \begin{pmatrix}
1 \\
R_{c,\pi}
\end{pmatrix}
\] (2.21)

Now we can express the two modal impedances \(Z_{c,\pi}\) as:

\[
Z_{c,\pi} = \frac{\gamma_{c,\pi}}{Y_1 + Y_m R_{c,\pi}}
\] (2.22)

In equation (2.22), the four impedance term \(\pm Z_{c,\pi}\) are associated with their corresponding eigenvalues \(\pm \gamma_{c,\pi}\). Using the quantities defined in equation (2.15), the eigenvalues \(\gamma_{c,\pi}\)
and voltage ratio coefficients $R_{c,\pi}$ are given by:

$$\gamma^2_{c,\pi} = \frac{1}{2} \left( a_1 + a_2 \right) \pm \frac{1}{2} \sqrt{(a_2 - a_1)^2 + 4b_1b_2}$$  \hspace{1cm} (2.23)$$

$$R_{c,\pi} = \frac{1}{2b_1} \left[ (a_2 - a_1) \pm \sqrt{(a_2 - a_1)^2 + 4b_1b_2} \right]$$  \hspace{1cm} (2.24)$$

In the previous section, Pierce’s scalar coupled-mode theory was used to predict the existence of complex coupled-modes in the case of contra-directional coupling between two waves whose phase fronts travelled co-directionally with the same propagation constant. This is also the case under the present formulation and the details are presented later in this chapter.

We can now express the solution to the system of coupled differential equations in (2.12) as a superposition of the individual eigenvectors multiplied by an exponential function of the corresponding eigenvalues:

$$\begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_c & V_c & V_\pi & V_\pi \\ I_c/Z_c & -I_c/Z_c & I_\pi/Z_\pi & -I_\pi/Z_\pi \end{bmatrix} \begin{bmatrix} V^+_c e^{-\gamma_c z} \\ V^-_c e^{+\gamma_c z} \\ V^+_\pi e^{-\gamma_\pi z} \\ V^-_\pi e^{+\gamma_\pi z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ R_c & R_c & R_\pi & R_\pi \\ 1/Z_c & -1/Z_c & 1/Z_\pi & -1/Z_\pi \\ -1/R_cZ_c & 1/R_cZ_c & -1/R_\piZ_\pi & 1/R_\piZ_\pi \end{bmatrix} \begin{bmatrix} V^+_c e^{-\gamma_c z} \\ V^-_c e^{+\gamma_c z} \\ V^+_\pi e^{-\gamma_\pi z} \\ V^-_\pi e^{+\gamma_\pi z} \end{bmatrix}$$  \hspace{1cm} (2.25)$$

Each column in (2.25) represents an eigenvector of (2.12) while the constants $V^+_c$, $V^-_c$, $V^+_\pi$ and $V^-_\pi$ are the amplitudes of each mode existing in the system and are determined from the boundary conditions.

The vector coupled-mode formulation outlined above can be used in the analysis of the coupling between an arbitrary pair of axially homogeneous transmission lines. We are interested in exploring the coupling phenomenon between artificial lines that may be realized through the periodic loading of a host medium. The formulation above applies in the homogeneous limit and hence we would expect to see deviations from theoretical predictions and actual experiments when this constraint is not satisfied. In later chapters it will be shown that this deviation can be ignored in the useful operating band of some practical microwave couplers. Nevertheless, we can further extend the vector coupled-mode formulation to accurately model the coupling between transmission lines that are periodically inhomogeneous in the longitudinal direction.
In the general case of the coupling between two longitudinally periodic transmission lines, we have to solve the inhomogeneous set of linear differential equations where the elements of the matrix in (2.12) are no longer constants but vary with distance $z$. If the periodic inhomogeneities occur in a discrete fashion such as the case of lumped-element loading on an otherwise homogeneous transmission line, we can apply the vector coupled-mode analysis to characterize those segments of a unit cell which are homogeneous. A transfer function description of one unit cell of the network can then be obtained by cascading the matrices which describe the homogeneous coupled transmission-line segments and those that describe the loadings. The modes which exist in such a system can then be determined using Bloch/Floquet analysis [53] and this will be the subject of the next section.

### 2.4 Coupling of periodically loaded lines

The schematic of two periodically loaded coupled-lines is depicted in Figure 2.1. The center block can be analysed using Schelkunoff’s system of linear and constant coefficient differential equations (2.12) and then represented using a $4 \times 4$ transmission matrix (in the case of two coupled lines). Similarly, a transmission matrix description of the loadings which appear at the outer segments in Figure 2.1 can be evaluated and the overall response from the left of the unit cell to its right may be stated as:

$$
\begin{pmatrix}
V_L \\
I_L
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix}
$$

(2.26)
The state vectors $V_L$ and $I_L$ in equation (2.26) comprise the left-hand side voltages and currents respectively appearing in Figure 2.1. We are after the modes which exist in a system consisting of an infinite periodic repetition of the typical unit cell depicted above. Using Bloch/Floquet analysis (see Appendix G), a modal solution will necessarily require that the left-hand side state vector be a constant multiple ($e^{\gamma d}$ where $d$ is the physical length of one unit) of the state vector on the right. Hence, the eigenmodes of the system can be computed from the relation:

$$\begin{pmatrix} A - e^{\gamma d} I & B \\ C & D - e^{\gamma d} I \end{pmatrix} \begin{pmatrix} V_R \\ I_R \end{pmatrix} = 0$$  \hspace{1cm} (2.27)$$

Taking the complex conjugate of equation (2.18) and using the properties of symmetric, reciprocal and loss-less networks (see Table C.2 in Appendix C):

$$\begin{pmatrix} A^* - e^{\gamma^* d} I & B^* \\ C^* & D^* - e^{\gamma^* d} I \end{pmatrix} \begin{pmatrix} V_R^* \\ I_R^* \end{pmatrix} = \begin{pmatrix} A - e^{\gamma^* d} I & -B \\ -C & D - e^{\gamma^* d} I \end{pmatrix} \begin{pmatrix} V_R^* \\ I_R^* \end{pmatrix} = 0$$  \hspace{1cm} (2.28)$$

or,

$$\begin{pmatrix} A - e^{\gamma^* d} I & B \\ C & D - e^{\gamma^* d} I \end{pmatrix} \begin{pmatrix} V_R^* \\ -I_R^* \end{pmatrix} = 0$$  \hspace{1cm} (2.29)$$

Hence it is evident that the eigenvalues $e^{\gamma d}$ of the coupled system appear in complex-conjugate pairs with the corresponding eigenvector shown in (2.29). It will be beneficial if the eigenvalues of the $2N \times 2N$ system shown in (2.27) (with $N = 2$) can be obtained by solving a $N \times N$ system instead. This reduction in computation can be achieved by considering 3 cases: (a) $V_R = 0$, $I_R \neq 0$; (b) $I_R = 0$, $V_R \neq 0$ and (c) $V_R \neq 0$, $I_R \neq 0$.

If we envision an infinite cascade of identical unit cells of the type depicted in Figure 2.1 where each cell is electrically small in the frequency band of interest, then the eigenmodes depict travelling electromagnetic waves in a longitudinally homogeneous media. In such instances, case (a) and case (b) would correspond to degenerate modes where either the voltage or current vectors are zero, while case (c) would be of primary importance.

Considering case (a) first, we set $V_R = 0$ in (2.27), and the existence of non-zero $I_R$ requires:

$$BI_R = 0$$  \hspace{1cm} (2.30)$$

$$(D - e^{\gamma d} I)I_R = 0$$  \hspace{1cm} (2.31)$$
Using similar arguments, for case (b) we obtain:

\[ CV_R = 0 \] (2.32)

\[ \left( A - e^{\gamma d} I \right) V_R = 0 \] (2.33)

Examining (2.31) and (2.33), we can see that in both cases the eigenvalues are the same \((A = D^T \text{ from Table C.2})\) and hence indeed these modes corresponding to case (a) and case (b) are degenerate. The remaining modal solutions are obtained from case (c) by multiplying the first row of (2.27) by \(C\):

\[ C \left( A - e^{\gamma d} I \right) V_R + CBI_R = 0 \] (2.34)

\[ CV_R + \left( D - e^{\gamma d} I \right) I_R = 0 \] (2.35)

Using the relations \(CA = DC\) and \(D^2 - CB = I\), we can now substitute (2.35) into (2.34) to obtain:

\[ \left( D - e^{\gamma d} I \right) CV_R + CBI_R = 0 \]

or, \( \left( CB - (D - e^{\gamma d} I)^2 \right) I_R = 0 \)

or, \( (\cosh(\gamma d)I - D) I_R = 0 \) (2.36)

In a similar fashion it can be shown that:

\[ (\cosh(\gamma d)I - A) V_R = 0 \] (2.37)

As both \(V_R\) and \(I_R\) are non-zero in (2.36) and (2.37), any one of these equations can be used to determine the eigenvalues of the original problem (2.27). As \(A = D^T\), these two vectors share exactly the same relationship which we observed in equation (2.18). The eigenvalues of the system can be obtained from the following two equations:

\[ \det \left( A - e^{\gamma d} I \right) = 0 \quad \text{(degenerate case)} \] (2.38)

\[ \det \left( A - \cosh(\gamma d) I \right) = 0 \] (2.39)

For a given eigenmode, the voltage and current vectors can be related using (2.27) and (2.36) in the following way:

\[ CV_R + \left( D - e^{\gamma d} I \right) I_R = 0 \]

or, \( (\cosh(\gamma d) - e^{\gamma d}) I_R = -CV_R \)

or, \( I_R = \frac{1}{\sinh(\gamma d)} CV_R = Y_\gamma V_R \) (2.40)
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We recognize that equation (2.40) is the counterpart to (2.22) which was derived in Section 2.3 for coupled-lines that are longitudinally homogeneous. Linear superposition of the various eigenmodes allows one to express the voltages and currents in this coupled-system for generic boundary conditions:

\[
\begin{pmatrix}
V \\
I
\end{pmatrix}_{z=Nd} = \sum_{i=1}^{n} \left\{ V_{i}^{+} e^{-\gamma_{i} z} \begin{pmatrix} V_{i} \\ Y_{i} V_{i} \end{pmatrix} + V_{i}^{-} e^{+\gamma_{i} z} \begin{pmatrix} V_{i} \\ -Y_{i} V_{i} \end{pmatrix} \right\}
\] (2.41)

In equation (2.41), \( N \) is an integer while \( V^{+}_{i} \) and \( V^{-}_{i} \) are complex scalars. The admittance matrices \( Y_{i} \) are obtained from (2.40) while vectors \( V_{i} \) are the \( V_{R} \) vectors defined in (2.37).

In our case, which involves two lines and two modes (and their corresponding reflections), we can express equation (2.41) as:

\[
\begin{pmatrix}
V \\
I
\end{pmatrix}_{z=Nd} = \begin{pmatrix}
V_{c} & V_{c} & V_{\pi} & V_{\pi} \\
Y_{c} V_{c} & -Y_{c} V_{c} & Y_{\pi} V_{\pi} & -Y_{\pi} V_{\pi}
\end{pmatrix} \begin{pmatrix}
V_{c}^{+} e^{-\gamma_{c} z} \\
V_{c}^{-} e^{+\gamma_{c} z} \\
V_{\pi}^{+} e^{-\gamma_{\pi} z} \\
V_{\pi}^{-} e^{+\gamma_{\pi} z}
\end{pmatrix}
\] (2.42)

The vectors \( V_{c} \) and \( V_{\pi} \) can be fixed using the definition in (2.18) without any loss of generality, and the corresponding current vectors are given by:

\[
I_{c,\pi} = \frac{1}{\sinh(\gamma_{c,\pi} d)} \begin{pmatrix} c_{11} & c_{12} \\
c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} 1 \\
R_{c,\pi} \end{pmatrix} = \frac{1}{\sinh(\gamma_{c,\pi} d)} \begin{pmatrix} c_{11} + c_{12} R_{c,\pi} \\
c_{21} + c_{22} R_{c,\pi} \end{pmatrix}
= \frac{c_{11} + c_{12} R_{c,\pi}}{\sinh(\gamma_{c,\pi} d)} \begin{pmatrix} 1 \\
-1/R_{\pi,c} \end{pmatrix}
\] (2.43)

In (2.43), \( c_{11}, c_{12} \ldots \) are the entries of the \( C \) matrix. We will define the scalar multiplier in (2.43) as the reciprocal of \( Z_{c,\pi} \) and this makes equation (2.42) identical in form to equation (2.25). Denoting the entries of the matrix \( A \) by \( a_{11}, a_{12} \ldots \), we will now summarize the formulae for evaluating the eigenmodes, the associated voltage ratio coefficients and impedances:

\[
cosh(\gamma_{c,\pi} d) = \frac{1}{2} (a_{11} + a_{22}) \pm \frac{1}{2} \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}}
\] (2.44)

\[
R_{c,\pi} = \frac{1}{2a_{12}} \left[ (a_{22} - a_{11}) \pm \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}} \right]
\] (2.45)

\[
Z_{c,\pi} = \frac{\sinh(\gamma_{c,\pi} d)}{c_{11} + c_{12} R_{c,\pi}}
\] (2.46)
For lossless, symmetric and reciprocal systems the entries of the \( A \) and \( C \) matrices are purely real and imaginary respectively. Hence equations (2.44) to (2.46) share the same properties of the expressions outlined in the context of Schelkunoff’s theory (equations (2.22) to (2.24)) differing only in their variation with frequency.

### 2.5 Complex modes from the coupling of a microstrip transmission line to a Negative-Refractive-Index transmission line

In this section, we will focus on the special case of complex modes in two coupled transmission lines in order to set the stage for the analysis of the coupling between a regular right-handed microstrip transmission line (MS-TL) and a left-handed metamaterial Negative-Refractive-Index transmission line (NRI-TL). We will begin by examining Schelkunoff’s coupled-mode theory for longitudinally homogeneous lines and in particular will focus on the expressions for \( \gamma_{c,\pi} \) and \( R_{c,\pi} \) in equations (2.23) and (2.24). It is evident that complex solutions are obtained in the frequency range defined by:

\[
(a_2 - a_1)^2 + 4b_1b_2 < 0 \quad (2.47)
\]

Noting that all quantities in (2.47) are real for a loss-less system, the product \( b_1b_2 < 0 \) is a necessary condition for the existence of these complex modes. Moreover, by examining equation (2.15), it is clear that if \( |Z_m|^2 \ll |Z_1Z_2| \) and \( |Y_m|^2 \ll |Y_1Y_2| \) (i.e. the mutual coupling immitances are much smaller than the self immitances of the lines), then the terms \( a_1 \) and \( a_2 \) dominate over the \( b_1 \) and \( b_2 \) terms. This implies that (2.47) can only be satisfied when \( a_1 \) is similar in value to \( a_2 \). In other words, these complex modes are expected to exist when the propagation constants of the individual isolated lines are similar in magnitude (\( \gamma^2_{c} = a_1 \) and \( \gamma^2_{\pi} = a_2 \) when \( Z_m = Y_m = 0 \)).

We now demonstrate that complex modes can form as a result of the coupling of a forward wave to a backward wave. Substituting the various per unit length impedance and admittance parameters from (2.15) into (2.47):

\[
(Z_2Y_2 - Z_1Y_1)^2 + 4 \left( Z_1Z_2Y_m^2 + Y_1Y_2Z_m^2 + Z_mY_m (Z_1Y_1 + Z_2Y_2) \right) < 0 \quad (2.48)
\]

Let \( \beta_i = \sqrt{-Z_iY_i} \) and \( \eta_i = \sqrt{Z_i/Y_i} \) define the propagation constant and impedances of the lines in absence of coupling. Using these relations we may denote \( Z_i = \pm j\beta_i\eta_i \).
Figure 2.2: A pair of conjugate complex modes supported by the MS/NRI-TL coupler. The red arrows indicate direction of power flow while the circles contain the voltage phasor along the length of the coupler. Note that the phase along both lines change in equal increments and the phase/power relationship exhibits the characteristics of the underlying transmission-lines (for instance, the phase always decreases along the direction of power flow on the forward-wave MS-TL). Also note that the \( c \)-mode depicted in Figure 2.2a is not the reflected counterpart of the \( \pi \)-mode in Figure 2.2b as such a mode will be identified with a change in the direction of the modal amplitude increase.

and \( Y_i = \pm j\beta_i/\eta_i \) where the positive sign is selected for propagating forward waves and the negative sign for propagating backward waves. Letting \( Z_m = jz_m \) and \( Y_m = jy_m \) represent the coupling terms in a loss-less system, then (2.48) takes the following form:

\[
(\beta_2 - \beta_1)^2 \left( (\beta_2 + \beta_1)^2 + 4z_my_m \right) + 4 \left( \pm \beta_1 \eta_1 \right) \left( \pm \beta_2 \eta_2 \right) \left( y_m + \frac{z_m}{\eta_1 \eta_2} \right)^2 < 0 \tag{2.49}
\]

In equation (2.49) all terms are real quantities when dealing with coupling of propagating modes in a loss-less system. Hence it is clear that the inequality is satisfied when \( \beta_1 \sim \beta_2 \) along with \( \beta_1 \beta_2 \eta_1 \eta_2 < 0 \) which is precisely the case when a forward wave is coupled to a backward wave.

From (2.23), it is clear that these modes exist in complex-conjugate pairs i.e. \( \gamma_c = \pm \gamma_\pi^* \). Moreover, the voltage ratio between the lines for each mode also appear in conjugate
pairs \((R_c = R_\pi^* \text{ in equation } (2.24))\). Having established the relationship between the complex-mode propagation constants and their relative voltage amplitudes on the two lines, we will now investigate the relationship between the modal impedances \(Z_c\) and \(Z_\pi\). From equation (2.22) it is seen that these impedances are complex when \(R_c\) and \(R_\pi\) are complex (see equations (2.18) and (2.22)). Replacing \(\alpha\) with \(1/Z_{c,\pi}\) in (2.19) we obtain:

\[
\gamma_{c,\pi} Z_c Z_\pi \left( \begin{array}{c}
1 \\
R_{c,\pi}
\end{array} \right) = \left( \begin{array}{cc}
Z_1 & Z_m \\
Z_m & Z_2
\end{array} \right) \left( \begin{array}{c}
1 \\
-1/R_{\pi,c}
\end{array} \right)
\] (2.50)

Examining the first row of (2.50) for each mode separately:

\[
\gamma_c Z_c = Z_1 - \frac{Z_m}{R_\pi}
\] (2.51)

\[
\gamma_\pi Z_\pi = Z_1 - \frac{Z_m}{R_c} = - (\gamma_c Z_c)^*
\] (2.52)

The last relation shown in (2.52) follows from the realization that for a loss-less system \(Z_1\) and \(Z_m\) are imaginary and for complex modes \(R_c = R_\pi^*\). This result enables us to pick the sign in (2.22) as soon as we agree on the sign convention for (2.23), or vice-versa.

By studying equation (2.25), we realize that for any given mode, the total power carried by it on one line is equal and in the opposite direction to that of the other line as depicted in Figure 2.2. Hence the net forward power carried by one of these complex modes is zero. This fact is demonstrated by setting \(V^-_c = V^+_\pi = V^-_\pi = 0\) in equation (2.25) and evaluating \(V_I I^*_1 + V_2 I^*_2\):

\[
V_I I^*_1 + V_2 I^*_2 = |V^+_c e^{-\gamma_c z}|^2 \left( \frac{1}{Z_c} - \frac{R_c}{R_\pi^* Z_c^*} \right) = 0
\] (2.53)

One can carry out similar exercises for all other modes and the conclusion is the same. For convenience, the various modes of the system and their characteristics are outlined in Table 2.1 and their primary features are shown pictorially in Figure 2.2.

### 2.6 Contra-directional coupling in guided-wave optics

At this point, it is important to distinguish between the nature of contra-directional coupling described in standard texts on guided-wave optics such as [54] and the complex modes which we have just discussed. To this end, we will draw attention to Pierce's
\[ \gamma_c = \alpha + j\beta \quad \alpha, \beta > 0 \]

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<th>Phase</th>
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<th>line 2</th>
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<td>$+\beta z$</td>
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<td>$-Z_c$</td>
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<thead>
<tr>
<th>Power flow</th>
<th>line 1</th>
<th>line 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{1}{Z_c} e^{-2\alpha z}$</td>
<td>$\frac{1}{Z_c} e^{+2\alpha z}$</td>
</tr>
</tbody>
</table>

\[ \gamma_c = \gamma_\pi^* \quad Z_c = -Z_\pi^* \quad R_c = R_\pi^* \]

Table 2.1: Properties of complex modes from Schelkunoff’s coupled-mode theory

coupled-mode equations (2.1) and (2.2) and assume weak coupling such that \( \kappa_{11} \) and \( \kappa_{22} \) are much smaller than \( \beta_p \) and \( \beta_q \) and can be neglected. The two equations can now be expressed in the following form:

\[
-\frac{dP'}{dz} = j\kappa_{12} e^{j(\beta_p - \beta_q)z} Q' \tag{2.54}
\]

\[
-\frac{dQ'}{dz} = j\kappa_{21} e^{j(\beta_p - \beta_q)z} P' \tag{2.55}
\]

\[
P' = Pe^{j\beta_p z} \tag{2.56}
\]

\[
Q' = Qe^{j\beta_q z} \tag{2.57}
\]

It is evident that the primed quantities \( P' \) and \( Q' \) represent the amplitude modulations of the waveforms \( P \) and \( Q \) respectively. Now we will explore contra-directional coupling between two regular guides assisted by a grating of spatial periodicity \( \Lambda \). As the coupling is between contra-directional waves in regular (positive index) media, \( \beta_p \) and \( \beta_q \) have opposite signs and \( \kappa_{12} = -\kappa_{21}^* \) (see equation (2.9)). Moreover, the effect of the grating will be considered as a perturbation which leaves \( \beta_p \) and \( \beta_q \) unchanged in equations (2.54) and (2.55) while the coupling coefficient \( \kappa_{12} \) is now a periodic function of the axial distance \( z \). Expressing \( \kappa_{12} \) as a Fourier series, we obtain the following coupled-mode
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In equations (2.58) and (2.59), \(\kappa_n\) is the \(n\)-th Fourier component of \(\kappa_{12}\). We will now integrate both sides of equations (2.58) and (2.59) over one spatial period. Using the same line of reasoning as Yariv [54], we assume slow variation of the amplitudes \(P'\) and \(Q'\) over one period of the grating such that they can be extracted out of the integrals to obtain:

\[
-\Delta P' \approx jQ' \sum_n \kappa_n \int_0^\Lambda e^{j\Delta \beta_n z} dz
\]  

(2.61)

\[
-\Delta Q' \approx -jP' \sum_n \kappa_n^* \int_0^\Lambda e^{-j\Delta \beta_n z} dz
\]  

(2.62)

The integrals in equations (2.61) and (2.62) are significant only when \(\Delta \beta_n \approx 0\) for some \(n\) and they are vanishingly small for all other values. This implies that the change in amplitude of the \(p\)-mode due to the \(q\)-mode (and vice-versa) is appreciable only when \(\Delta \beta_n \approx 0\) and that the other spatial harmonics of the grating do not play a significant role in describing the coupling between these modes.

Let us consider the case where the periodicity of the grating has been chosen such that one of its harmonics satisfies the phase matching condition \(\Delta \beta_n = 0\) between the \(\beta_p\) and \(\beta_q\) modes. In this case, we make negligible error in discarding all Fourier terms except the \(n\)-th one in equations (2.58) and (2.59) that now take the simpler form:

\[
-\frac{dP'}{dz} = j\kappa_n Q'
\]  

(2.63)

\[
-\frac{dQ'}{dz} = -j\kappa_n^* P'
\]  

(2.64)

Solving equations (2.63) and (2.64) for a semi-infinite system (where \(z\) ranges from 0 to \(+\infty\)), the amplitudes of the two original guided modes are given by:

\[
P(z) = P(0)e^{-|\kappa_n|z-j\beta_p z}
\]  

(2.65)

\[
Q(z) = -j\frac{\kappa_n^*}{|\kappa_n|} P(0)e^{-|\kappa_n|z-j\beta_q z}
\]  

(2.66)
Figure 2.3: Contra-directional coupling between two guides indicating the power flow (red arrows), phase (direction of blue arrows) and attenuation (length of blue arrows). (a) Backward coupling of power using a grating between two guides (b) Backward coupling due to a complex mode.

The amplitude profiles shown in equations (2.65) and (2.66) look deceptively similar to the complex modes we discussed in section 2.2. We know that $\beta_p$ and $\beta_q$ have opposite signs and that if the two guides were nearly identical, then $\beta_p \approx -\beta_q$ and we would nearly obtain complex-conjugate axial variation of the various amplitudes. We would like to warn the reader that these (see Figure 2.3a) are not considered as complex modes. The complex modes we discussed earlier (see Figure 2.3b) exhibited simultaneous attenuation and phase progression on both guides such that the field at one transverse plane could be obtained from another by a fixed multiplicative constant (say $e^{-(\alpha+j\beta)}$) raised to the distance $z$ (i.e. $e^{-(\alpha+j\beta)z}$) between the planes. In the present case, the phase of the field is increasing in one guide while it is decreasing in the other. Hence these are not the eigenmodes of the problem involving two guides coupled by a grating. The solutions in (2.65) and (2.66) only tell us how the original modes can be perturbed to account for the mutual interaction between the guides, and not what the eigenmodes of the newly created system are.

### 2.7 Common RF/microwave coupler topologies

A directional coupler is a four port device that splits the input power between two of its ports while leaving the fourth port isolated. A short history of the various forms of directional couplers in the RF/Microwave frequency spectrum that have emerged since the second world war can be found in [55]. We will briefly examine some of the more common ones whose details are available in standard texts on microwave engineering.
such as [10], [56] and [57]. They can be implemented using waveguides for applications requiring high performance and power handling, or using planar technologies such as microstrip.

**Planar couplers**

The simplest coupler is realized by printing two parallel microstrip transmission lines on a grounded dielectric substrate. The spacing between these lines depends on the desired level of coupling. The design depicted in Figure 2.4a is quarter-wave long at the operating frequency and usually most of the input power on one of the lines is delivered to the through port which occurs at the far end of the same line. A portion of it is coupled to the adjacent line in the backward direction\(^2\). In a properly designed coupler, the far end of the adjacent line (relative to the input) is perfectly isolated and receives no power.

The operation of this coupler is understood in terms of its two eigenmodes (the even and odd modes) of equal phase velocities, but unequal characteristic impedances. The even mode is characterized by equal (in-phase) voltages on the two lines and is of higher impedance than its odd mode counterpart characterized by out-of-phase voltages. A source placed on one of the lines excites the even mode more strongly than the odd mode (due to the difference in their impedances) resulting in a finite voltage across the coupled port. These two modes propagate down both lines and are always out of phase on the adjacent coupled line. Nevertheless, they do not cancel owing to unequal excitation amplitudes. At the far end of the coupler, these modes are mismatched with respect to the termination and undergo reflection such that the impinging even mode is decreased while the odd mode is increased. It can be shown that by choosing the system impedance to be the geometric mean of the even and odd mode impedances, it is possible to equalize the amplitudes of both modes upon reflection off the far end of the coupler. This results in perfect isolation at the far end of the coupled line. Furthermore, in a quarter wavelength coupler, the reflected waves interfere constructively at the coupled port resulting in maximum power transfer.

There are various shortcomings of this simple coupler. First, it is very difficult to achieve high coupling level using an edge coupled configuration. This drawback can be

---

\(^2\)Forward coupling microstrip coupled-line couplers can be realized as well, but these designs are more complicated (see for example [58]).
Chapter 2. Theory of Coupling of Propagating Modes

Figure 2.4: Various planar microstrip couplers.

Mitigated using the Lange coupler (to be discussed shortly) albeit at the expense of a complicated configuration which is not easy to model analytically. The second drawback is in the degradation of the isolation due to the unequal phase velocity of the two modes. This too can be remedied partially through the use of techniques such as wiggled lines, dielectric overlays or compensating capacitances across the two lines [14]. All of these techniques are cumbersome to implement and then are either imperfect or narrow band.

It is nearly impossible to achieve a practical line spacing that can provide a coupling level as high as 3 dB in a conventional\(^3\) microstrip coupled-line coupler. A popular solution to this problem was proposed by Julius Lange in 1969 [11] (see Figure 2.4b). The Lange coupler can be considered as a microstrip coupled-line coupler whose lines have been split into longitudinal segments and placed in a parallel but alternating fashion to increase the effective coupling capacitance between the lines. Electrical connectivity between segments belonging to the original line are achieved using bond wires as depicted in Figure 2.4b.

Although the Lange coupler design provides a certain amount of phase velocity compensation between the even and odd modes of the original coupled-line coupler and allows higher coupling levels (due to increased coupling capacitance), the main disadvantage is in the practicality of implementation. Bond wires are difficult to fabricate in a mono-

\(^3\)By conventional microstrip coupled-line coupler we mean designs that are quarter wave and fully planar. In recent years, various techniques have been proposed to increase the coupling level such as by using flanged edges [12] or multiple stages [60].
lithic single layer process and the required interdigitated lines can be quite thin and hence sensitive to fabrication tolerances and conductor losses. Moreover, for high frequency applications, the parasitics introduced by the bond wires are difficult to model and control.

The branch-line hybrid coupler provides a simple geometry that allows for large coupling levels and the implementation depicted in Figure 2.4c provides equal (3 dB) power split between the output ports at quadrature phase. Each line segment is a quarter wave long and the characteristic impedances of adjacent lines differ by a factor of $\sqrt{2}$ in the 3 dB implementation. The port adjacent to the input and connected to it by the higher impedance line segment, is the isolated port. The operation of the branch-line hybrid can also be understood in terms of even and odd mode excitations. It is inherently narrowband although multi-stage versions of this coupler exist for broader bandwidth [59]. This coupler occupies more space than either the coupled-line or the Lange couplers as all of its edges are quarter wavelengths long. One of the practical difficulties associated with this coupler is in the modelling of the junction effects due to the presence of three interconnecting transmission lines near each port. In the S-band, the ratio of the length of the each side to that of the junction is not very large and hence junction parasitics cannot be ignored.

Planar couplers are widely used due to their low cost of fabrication (compared to waveguide or coaxial line based ones) and the narrow bandwidth or imperfect isolation of these designs can often be tolerated for a variety of applications. We will highlight some of their uses in Section 2.8.

**Waveguide couplers**

Waveguide directional couplers can be made from the simple configuration of two rectangular guides that are made to exchange power by means of holes, slits or openings of other shapes on their common side walls. They can also assume more complex geometries such as the magic-T hybrid depicted in Figure 2.5a. They are more expensive to fabricate due to the precise three dimensional machining requirements but are widely used due to their low-loss and high-power handling capability compared to their planar counterparts.

The first coupler we consider is the magic-T depicted in Figure 2.5a which contains a horizontal H-plane T-junction and a vertical E-plane T-junction. When the port labelled as ‘In’ is excited with the dominant $\text{TE}_{10}$ mode, the input power splits equally but out
of phase between the two branches of the vertical T-junction into the ports labelled as ‘Through’ and ‘Coupled’. The fourth port remains isolated. Due to reciprocity, the input and isolated ports can interchange their roles but upon excitation of the port labelled as ‘Isolated’, the power splits equally (in the H-plane junction) and in-phase between the through and coupled ports. In a properly designed magic-T, the through and coupled ports are isolated from each other.

The operation of this coupler can be understood by considering the E-plane T-junction on its own and removing the isolated port by a metallic sheet. Examination of the the electric field lines of the TE_{10} mode in the input guide (oriented parallel to its narrower wall), lead us to the conclusion that in the junction, the fringing field lines should be reversed as they enter the through and coupled ports resulting in equal but out-of-phase power split. When we re-introduce the isolated port, we observe an electric field distribution with odd symmetry about its center (horizontally). The excitation amplitude of this port is governed by the integral across its cross-section of the equivalent currents of the incident field and their inner product with the field of its dominant TE_{10} mode. The modal field of the isolated port has even symmetry horizontally about its center while the equivalent currents have odd symmetry. This results in zero power leakage into the isolated port. Similar arguments can be made when the other ports are excited.

The ports labelled ‘Isolated’ and ‘In’ are known as sum and difference ports respectively. These designations can be understood by considering the waves which emerge
from them when the ports labelled as ‘Through’ and ‘Coupled’ are simultaneously excited. Due to reciprocity, we expect that the individual field contributions due to these two sources to be in phase in the sum port and out of phase in the difference port. Next we will consider the Bethe-hole coupler depicted in Figure 2.5b.

The Bethe-hole coupler achieves its directive property using two rectangular waveguides whose axes are slightly inclined and are coupled using a single hole placed at the center of the common broad wall of the guides. The operating principle of this coupler is understood by considering the TE$_{10}$ mode incident from the input port. The vertical electric field due to this mode induces a vertical electric dipole current at the center of the hole. The magnetic field of the incident wave likewise induces a horizontal magnetic current, and as the hole is located at the center of the guide, this current vector lies on the cross-sectional plane of the input guide. The excitation strengths of these current vectors depend on the incident field and shape of the hole. These currents radiate into the second waveguide which is shown to be on the top in Figure 2.5b.

In the top waveguide, the vertical electric dipole excites modes which move away from the hole with parallel electric fields and anti-parallel magnetic fields so that the boundary condition at the source is satisfied. This excites both ends of the top guide equally and in phase. On the other hand, the magnetic current excites anti-parallel electric fields resulting in equal but out-of-phase excitation of both ends of the top guide. The modal excitation strength due to the electric current, depends on its projection onto the electric field of the TE$_{10}$ mode in the top guide. In this case, it is independent of the angle between the axes of the two guides. The field excitation amplitude due to the magnetic current relies on its projection onto the magnetic field of the mode and depends on the angle between the guides. Hence, one can equate the modal excitation strengths due to the two currents by a judicious choice of angle between the guides resulting in constructive interference at one end and perfect cancellation of the modal fields at the other. This results in one of the ports in the top guide being isolated while the other receives finite amount of power.

Other waveguide couplers are possible which employ multiple holes to couple energy between them and uses the electrical spacing between them to achieve constructive and destructive interference to realize the coupled and isolated ports respectively. We will now discuss some of the applications of these couplers in RF/Microwave systems.
2.8 Applications

Directional couplers are key components in Network Analysers for the linear characterization in essentially all RF/Microwave systems. They can be used as power dividers and are found in some mixer applications. Couplers can be used in the realization of phase-shifters and in the construction of the Butler matrix used for feeding phased array antenna systems. The coupled-line coupler is one of the essential building blocks for planar filter synthesis. We will now provide a brief overview of some of the diverse applications of couplers to embrace the idea of exploring new topologies, such as the MS/NRI-TL coupled-line coupler, which is the subject of this dissertation.

Reflectometry

The measurement of the scattering or S-parameters of a microwave network requires the ability to distinguish between the forward and backward travelling waves in the main signal line. This necessitates a linear device with directional properties such as a coupler. The key parameter of the coupler in such an application is its directivity, which is the difference between its coupling level and isolation on a logarithmic scale. In Figure 2.6, the top line is the main signal line carrying power to the device under test and it is coupled to a second line with the assumption of infinite directivity. If the power is coupled backwards, then the measurements at port 1 and port 2 are proportional to the incident wave (solid line arrow) power and reflected wave (broken line arrow) power respectively. When the directivity is finite, some of the incident power appears in port 2.
and a portion of the reflected power in port 1 of Figure 2.6. Hence the directivity sets a lower limit to the reflected power which can be measured accurately.

**RF/Microwave power division**

Couplers can also be used in power distribution networks such as those designed to feed antenna arrays. A popular feed network known as the Butler matrix [61] is shown in Figure 2.7a. Each input port (labelled 1 to 4 in Figure 2.7a) excites the output ports (ports 5 to 8) with equal amplitudes but a different progressive phase-shift. Hence such a network can be connected to an antenna array and the input power can be switched between ports 1 to 4 to steer the main beam of the array.

We notice the use of 4 branch-line hybrid couplers to implement the $4 \times 4$ Butler matrix. The structure also depicts 2 microstrip cross-over junctions (inside boxes with broken lines in Figure 2.7a) each of which involves two juxtaposed branch-line couplers allowing the signal to cross past each other without the need for a multi-layer configuration.

A multiplexer is a device that can split a multi-band input into two or more bands and deliver them to physically separate output ports. They are commonly constructed by connecting two or more bandpass filters to a common junction [62] but another possible
topology is depicted in the diplexer [63] shown in Figure 2.7b. It consists of the two branch-line hybrid couplers connected using identical bandpass filters (surrounded by broken lines in Figure 2.7b). When a signal is input into port 1, one of the frequency bands is channelled into port 2 and the other into port 4. The operation of this diplexer is understood by considering excitations at frequencies inside and outside the pass-bands of the filter segments. In its pass-band, each filter acts like a piece of phase-shifting line which effectively realizes a microstrip cross-over junction involving two branch-line hybrids connected back-to-back. Hence power is transferred from port 1 to port 4. On the other hand, when the frequency of excitation corresponds to the stop-band of the filter, all power is reflected back into the first branch-line hybrid (assuming port 1 is excited). These reflected waves constructively interfere at port 2 and thereby the diplexer is able to separate the two frequency bands. This device can be used to connect a single antenna (say at port 1) simultaneously to a transmitter (at port 2) and a receiver (at port 4) operating at different frequency bands.

**Phase shifters**

A single matched transmission line can be used as a phase-shifter that provides a predetermined phase lead/lag between its input and output terminals. More sophisticated varieties that offer tunable phase shift can take the form of multiple sections of transmission lines that are switched using PIN-diodes. Some popular phase shifters employ couplers in their design as depicted in the examples in Figure 2.8. The first example that we will discuss is the Schiffman phase shifter (see Figure 2.8a) which is used due to its broad bandwidth [64].

This topology provides a differential out-of-phase signal between the ports 2 and 3 in Figure 2.8a when an input signal is applied to port 1. This can also be achieved using two transmission-line segments one of which is half wavelength longer than the other at the design frequency; but such an approach is narrowband. The differential phase in this case rapidly diverges from $\pi$ beyond the center frequency as the phase shift of the longer line increases faster with frequency compared to the shorter one.

The novelty of the Schiffman phase shifter is in the use of a coupled-line coupler with two of its ends connected by a short circuit. The resulting coupler provides a transmission phase of $\pi/2$ at the design frequency while the meandered line (connected between port 1 and 2 in Figure 2.8a) provides a phase shift of $3\pi/2$. In this case, the slope of the phase
change is equal for both paths to the output resulting in a broadband differential phase of $\pi$ between the output ports.

Couplers can also be used in the realization of tunable phase shifters and we depict a reflection based one using a branch-line hybrid coupler in Figure 2.8b. The phase shift between ports 1 and 2 in this device can be controlled by the bias voltage applied to the two identical varactor diodes attached to its output ports. When a signal is input into port 1 in Figure 2.8b, it splits equally between the ports terminated by the purely reactive varactor diodes. The waves are hence reflected back into the coupler and the phase of this reflection is determined by the bias condition of the diodes. As long as the diodes are identical and lossless, the waves upon re-entering, combine constructively at port 2. The phase shift measured at port 2 relative to port 1 depends on the reflection phase at the varactors.

Filter synthesis using couplers

Classical network synthesis [35] and insertion-loss filter design deals with the realization of a pole-zero constellation of a rational function (impedance or transfer function) using the basic lumped building blocks such as inductors, capacitors, resistors and transformers. At microwave frequencies, lumped components are unavailable for exact synthesis due to their finite electrical lengths. At these higher frequencies, the available building blocks are commensurate transmission-line segments and resonators such as open/short circuited
stubs. These elementary units can be realized using coupled strips which are widely used in the microwave frequency range [65]. As an example, Figure 2.9 depicts a coupled-line bandpass filter implementation in microstrip using 3 couplers of equal electrical length (assuming even and odd mode phase velocities are equal).

Systematic design of microwave filters from the lumped-element prototypes (derived for low frequencies), begins by the application of Richard’s frequency transformation [66] which converts lumped resonators into series/shunt stubs that can be either open or short circuited. These resonators are spaced by commensurate ideal transmission-line segments known as ‘Unit-Elements’. The 4-port coupled-line coupler can be converted into a 2-port lossless network by terminating two of its ports with a combination of short or open circuits. The resulting 2-port facilitates the realization of a wide variety of the basic filter building blocks at microwave frequencies [67].

In this chapter, we have discussed various approaches in the study of coupling between guided wave structures such as transmission lines. We have also outlined a few of the common coupler topologies that can be found in the microwave frequency range and have discussed some applications as well. Having concluded an overview of a diverse selection of topics such as NRI-TL metamaterials, coupled-mode theory, complex modes and microwave couplers, we will now address the two main topics of this dissertation. We will study the properties of complex modes in the context of electromagnetic guided-wave theory and examine their influence on the operation of the Microstrip/Negative-Refractive-Index Transmission Line (MS/NRI-TL) coupled-line coupler.
Chapter 3

Analytic Properties of Complex Modes

3.1 Introduction

Complex modes are proper (non-radiating) guided modes supported by lossless structures and they exhibit simultaneous exponential attenuation and phase progression along the direction of propagation [68]-[73]. A wide variety of structures supporting these modes has been catalogued since the early 1960’s such as dielectric-loaded circular waveguides [74], plasma-slabs [75], open dielectric guides [70], shielded printed lines [76] and more recently coupled Microstrip/Negative-Refractive-Index Transmission lines (MS/NRI-TL) [5]. When carrying out modal decomposition of fields, it is important to be aware of the presence of complex modes as they are required for completeness and the effects of ignoring them have been documented [77, 78]. In some structures, such as the MS/NRI-TL coupler, we shall see that these modes are primarily responsible for their operational peculiarities.

In the first half of this chapter, we will investigate the properties of complex modes supported by generic isotropic structures such as inhomogeneously filled waveguides, coupled uniform transmission lines and periodically loaded coupled lines. The second half will be concerned with the analytic behaviour of the eigenvalues $\gamma$ of a system of $N$ coupled lossless transmission lines when some of these modes are complex over a finite bandwidth.
3.2 Properties of complex modes at real frequencies

In this section, we will investigate the generic properties of complex modes in three classes of guided-wave problems. In the first case, we investigate complex modes in axially uniform waveguides that are filled inhomogeneously (in the transverse plane) with arbitrary isotropic dielectrics. The analysis carried out here is valid for both open and closed guides. Next we will study the case of $N$ coupled transmission lines which are axially uniform and this will be followed by the case when these coupled lines are periodically loaded in the axial direction. The discussion in this section will serve to highlight common features, such as mode orthogonality and power flow due to complex modes in isotropic structures. We will not pursue the analysis of complex modes in anisotropic guides in this work.

3.2.1 Complex modes in waveguides

Consider a generic inhomogeneously filled waveguide oriented along the $z$-axis with axial unit vector $a_z$ as shown in Fig. 3.1. Let the axial and temporal dependence of the the $n^{th}$ modal field be given by $e^{-\gamma_n z + j\omega t}$. The time-harmonic fields should satisfy the following source-free Maxwell’s equations:

\begin{align}
(\nabla_t - \gamma_n a_z) \times (E_{tn} + E_{zn} a_z) &= -j\omega \mu (H_{tn} + H_{zn} a_z) \\
(\nabla_t - \gamma_n a_z) \times (H_{tn} + H_{zn} a_z) &= j\omega \varepsilon (E_{tn} + E_{zn} a_z)
\end{align}

(3.1)  
(3.2)
In equations (3.1) and (3.2) above, the subscripts \( t \) and \( z \) refer to transverse and axial field components respectively, and the permittivity \( \epsilon \) and the permeability \( \mu \) are assumed isotropic, lossless, causal and dispersive (varies with frequency \( \omega \) which is a real quantity) functions of the transverse spatial coordinates. The following transformations leave the above equations unchanged:

\[
\begin{align*}
(\gamma_n, E_t, E_z, H_t, H_z) &\rightarrow (-\gamma_n, E_t, -E_z, -H_t, H_z) \quad (3.3) \\
(\gamma_n, E_t, E_z, H_t, H_z) &\rightarrow (\gamma_n^*, E_t^*, E_z^*, -H_t^*, -H_z^*) \quad (3.4) \\
(\gamma_n, E_t, E_z, H_t, H_z) &\rightarrow (-\gamma_n^*, E_t^*, -E_z^*, H_t^*, -H_z^*) \quad (3.5)
\end{align*}
\]

We can identify (3.3) as the reflected mode of the original solution. When \( \gamma_n \) is complex, (3.4) and (3.5) provide us with two additional independent solutions corresponding to a conjugate complex mode and its reflected wave. Hence these complex modes always appear in conjugate pairs.

The fact that a complex mode carries no net power across its transverse plane is well known (e.g. [75] and [68]). A simple but rigorous demonstration of this property, along with relevant mode orthogonality relations, is obtained by using the following reciprocity equation for two bounded modes \( \gamma_n \) and \( \gamma_m \) in an arbitrary reciprocal inhomogeneous guide [79]:

\[
(\gamma_m + \gamma_n) \int_S (E_{tm} \times H_{tn} - E_{tn} \times H_{tm}) \cdot a_z dS = 0 \quad (3.6)
\]

The integral in (3.6) can be carried out over any guide cross-section and this expression is also valid for waveguides with lossy walls as well as open guides with fields decaying faster than the inverse distance towards infinity [79]. We may establish three related expressions by substituting the field transformations (3.3) to (3.5) into (3.6) to obtain:

\[
\begin{align*}
(\gamma_m - \gamma_n) \int_S (E_{tm} \times -H_{tn} - E_{tn} \times H_{tm}) \cdot a_z dS &= 0 \quad (3.7) \\
(\gamma_m + \gamma_n^*) \int_S (E_{tm} \times -H_{tn}^* - E_{tn}^* \times H_{tm}) \cdot a_z dS &= 0 \quad (3.8) \\
(\gamma_m - \gamma_n^*) \int_S (E_{tm} \times H_{tn}^* - E_{tn}^* \times H_{tm}) \cdot a_z dS &= 0 \quad (3.9)
\end{align*}
\]

Two orthogonality relations can be obtained by adding and subtracting the equation pair...
(3.6) and (3.7), and performing the same operations on the pair (3.8) and (3.9) as well:

\[
(\gamma_m + \gamma_n) (\gamma_m - \gamma_n) \int \int_S (E_{tm} \times H_{tn}) \cdot a_z dS = 0 \quad (3.10)
\]

\[
(\gamma_m + \gamma_n^*) (\gamma_m - \gamma_n^*) \int \int_S (E_{tm} \times H_{tn}^*) \cdot a_z dS = 0 \quad (3.11)
\]

We make the following set of observations from equations (3.10) and (3.11):

- The surface integral in (3.10) is zero for an arbitrary pair of distinct modes \( \gamma_m \neq \pm \gamma_n \) including complex mode pairs \( \gamma_m = \pm \gamma_n^* \). Hence a pair of conjugate complex modes are orthogonal with respect to the integral in (3.10).

- The surface integral in (3.11) is zero for a pair of distinct propagating (\( \gamma \) imaginary) or evanescent (\( \gamma \) real) modes but non-zero for a pair of distinct conjugate modes \( \gamma_m = \pm \gamma_n^* \). Hence a pair of conjugate complex modes are non-orthogonal in terms of complex power flow.

- Setting \( m = n \) in (3.11) results in non-vanishing surface integrals for both propagating and evanescent modes. The integral is zero when \( \gamma_n \) is complex and hence each complex mode on its own carries no net axial power across the guide cross-section. Nevertheless, the local power flow (the integrand in (3.11)) can be non-zero for a complex mode.

Consider the transformation in equation (3.4) relating the field of a complex mode \( \gamma_n \) and its conjugate counterpart \( \gamma_n^* \) both of which are proper\(^1\). Under such a transformation, \( E_t \times H_t^* \rightarrow -E_t^* \times H_t \) which indicates that at every spatial point in the transverse plane, the real power flow is opposite for a pair of conjugate modes.

Let us investigate further the non-orthogonality of power carried by complex modes \( \gamma_c \) and \( \gamma_\pi \) such that \( \gamma_c = \gamma_\pi^* \). Let \( V_c^+, V_\pi^+, V_c^-, V_\pi^- \) denote the complex amplitudes of the two forward modes\(^2\) and their reflections. We can express two times the axial

\(^1\)Assuming an \( e^{-\gamma_n z} \) axial field variation, \( \gamma_n \) and \( \gamma_n^* \) can form a pair of proper modes. On the other hand, for an axial field variation \( e^{j\gamma_n z} \), \( \gamma_n \) and \( -\gamma_n^* \) would form a proper pair.

\(^2\)Forward waves are identified by either real power-flow (for normal propagating modes) or attenuation (for evanescent and complex modes) towards \( z \rightarrow +\infty \).
Poynting vector at any given point on the transverse plane as:

\[
E \times H^* = [(V_c^+ + V_c^-)E_t + (V_\pi^+ + V_\pi^-)E_t^*] \times \left[ (V_c^+ - V_c^-)H_t + (-V_\pi^+ + V_\pi^-)H_t^* \right]^* \\
= [(V_c^+)^2 - (V_c^-)^2 + V_c^{+*}V_c^- - V_c^{+*}V_c^-]E_t \times H_t^* \\
+ [(V_\pi^+)^2 - (V_\pi^-)^2 + V_\pi^{+*}V_\pi^- - V_\pi^{+*}V_\pi^-]E_t^* \times H_t \\
+ 2\text{Re}((V_c^+V_\pi^{+*} - V_c^-V_\pi^{+*})E_t \times H_t] \\
- 2j\text{Im}[(V_c^+V_\pi^{+*} - V_c^-V_\pi^{+*})E_t \times H_t] \\
\]

The various terms in (3.12) were obtained by utilizing the transformations in equations (3.3) to (3.5) and we can deduce the following:

- If we set either the forward (\(+\)) waves or the reflected (\(-\)) waves to zero in (3.12), then \(\text{Re}(E \times H^*)\) is equal to the sum of the real power carried by the individual modes. This is the case in a semi-infinite guide where the real part of the complex power is orthogonal between a pair of conjugate complex modes. Also note that if the amplitudes of both modes are equal, then the power is purely reactive at each point in the transverse plane.

- If we set the forward wave of the \(c\)-mode to zero and the reflected wave of the \(\pi\)-mode to zero, or vice-versa, then we observe that the \(\text{Im}(E \times H^*)\) equals to the sum of the reactive power carried by each mode on its own. In this case, if the two modes are excited with equal amplitude, the power flow is purely real.

### 3.2.2 Complex modes in uniform coupled lines

The time-harmonic linear coupling between \(N\) non-identical axially-homogeneous transmission lines (Figure 3.2) can be described by the following system of coupled differential equations which we discussed in section 2.3 for the special case of \(N = 2\) lines:

\[
-\frac{d}{dz} \begin{pmatrix} V_n \\ I_n \end{pmatrix} = \begin{pmatrix} 0 & Z \\ Y & 0 \end{pmatrix} \begin{pmatrix} V_n \\ I_n \end{pmatrix} 
\]

In the following discussion, we will consider coupling between \(N\) lines that are lossless and reciprocal. The relevant eigenvalue problem that needs to be addressed is obtained by replacing \(\frac{d}{dz}\) in equation (3.13) by \(-\gamma_n\). In equation (3.13), \(V_n\) and \(I_n\) are \(N \times 1\) column vectors containing transmission-line voltages and currents. The \(N \times N\) paraske...
Hermitian matrices $^3 Z$ and $Y$ represent the per-unit-length self and mutual impedances and admittances of the coupled lines, and $0$ is the null-matrix. Just like in the case of guided wave theory, the following transformations leave (3.13) unchanged:

\[
\begin{align*}
(\gamma_n, V_n, I_n) & \rightarrow (-\gamma_n, V_n, -I_n) \quad (3.14) \\
(\gamma_n, V_n, I_n) & \rightarrow (\gamma_n^*, V_n^*, -I_n^*) \quad (3.15) \\
(\gamma_n, V_n, I_n) & \rightarrow (-\gamma_n^*, V_n^*, I_n^*) \quad (3.16)
\end{align*}
\]

The fact that the complex modes should appear in conjugate pairs is again confirmed using equations (3.15) and (3.16). Moreover, we observe that the eigenvectors of a complex mode and its conjugate counterpart are linearly independent (3.15). The relevant orthogonality relation applicable to the paraskew Hermitian matrix in (3.13) is:

\[
(\gamma_m + \gamma_n) (V_m^T I_n - I_m^T V_n) = 0 
\]

(3.17)

Substituting into (3.17) the various transformations outlined in equations (3.14) to (3.16), the following useful orthogonality relations are obtained:

\[
\begin{align*}
(\gamma_m + \gamma_n) (\gamma_m - \gamma_n) V_m^T I_n &= 0 \quad (3.18) \\
(\gamma_m^* + \gamma_n) (\gamma_m^* - \gamma_n) V_m^\dagger I_n &= 0 \quad (3.19)
\end{align*}
\]

$^3$The lines are assumed to be reciprocal and lossless and hence $Z^\dagger = -Z$ with $\dagger$ representing the conjugate-transpose operation.
In agreement with field theory, equation (3.19) confirms that complex modes carry no net power. Now consider the following Schur-Hadamard (element by element multiplication of two matrices of the same dimension) products $V_n^* \odot I_n$ and $V_m^* \odot I_m$ whose entries are the power flows on the various lines. Under transformation (3.15) we see that the real part of the complex power carried on each line by a mode is equal and in the opposite direction to that carried by its conjugate counterpart.

In a way analogous to the waveguide problem, we can investigate the non-orthogonality of power flow between the four complex modes ($\gamma_c, \gamma_\pi$ and their reflections) in the coupled transmission-line problem as follows:

$$[(V_c^+ + V_c^-)V + (V_\pi^+ + V_\pi^-)V^*] \odot [(V_c^+ - V_c^-)I + (-V_\pi^+ + V_\pi^-)I^*]^*$$
$$= [\lvert V_c^+ \rvert^2 - \lvert V_c^- \rvert^2 + V_c^{+*}V_c^- - V_c^{+*}V_c^-]V \odot I^*$$
$$+ [\lvert V_\pi^- \rvert^2 - \lvert V_\pi^+ \rvert^2 + V_\pi^{+*}V_\pi^- - V_\pi^{+*}V_\pi^-]V^* \odot I$$
$$+ 2\Re[(V_c^+V_\pi^{-*} - V_c^{-*}V_\pi^+)V \odot I]$$
$$- 2j\Im[(V_c^+V_\pi^{+*} - V_c^{-*}V_\pi^{-*})V \odot I]$$

(3.20)

From (3.20), we confirm that when two forward (or backward) waves are present on the lines, the real power flow on each line is the sum of the contributions due to the individual modes on their own.

### 3.2.3 Complex modes in axially periodic coupled lines

In this section we will demonstrate that the generic properties of complex modes which became evident by examination of guided waves in axially uniform inhomogeneous waveguides and coupled transmission lines can be extended to axially periodic structures such as periodically loaded coupled transmission lines. We will assume that the $N$ non-identical periodic lines are symmetric (about the midpoint of each periodic block), isotropic and lossless and is described by the following $2N$-port transmission matrix whose properties are described in Table C.2 of Appendix C:

$$
\begin{pmatrix}
V_L \\
I_L
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix}
$$

(3.21)

In (3.21) above, $V$ and $I$ are $N$ dimensional voltage and current vectors respectively with subscripts $L$ and $R$ denoting the left-end or the right-end of the coupling block depicted in Figure 3.3. In equation (3.21), four $N \times N$ block matrices characterize the linear coupling
between $N$ lines with real matrices $A^T = D$ and paraskew Hermitian matrices $B$ and $C$. In section 2.4 it is shown that if one assumes $(V_R \ I_R)^T = e^{-\gamma d}(V_L \ I_L)^T$, then the corresponding eigenvalues and eigenvectors are obtained from the following equations:

$$\det (A - \cosh(\gamma_n d)I) = 0 \quad (3.22)$$

$$(A - \cosh(\gamma_n d)I) V_n = 0 \quad (3.23)$$

$$\sinh(\gamma_n d)I_n = CV_n \quad (3.24)$$

It is to be noted that in equations (3.22) to (3.24), $I$ is the $N \times N$ identity matrix while $V_n$ and $I_n$ are the voltage and current portions of the eigenvector corresponding to the eigenvalue $\gamma_n$. Hence we again establish that the if $\gamma$ is an eigenvalue of the system, then so are $-\gamma$ and $\pm \gamma^*$. Moreover, the transformations in equations (3.14) to (3.16) which were developed for the axially uniform coupled line system, are also applicable in this case and can be verified by direct substitution into equations (3.22) to (3.24). These transformations allowed us to deduce that a complex mode carries no net power axially. Moreover, it was also determined that for any pair of conjugate complex modes, the corresponding power flow on each line was in the opposite direction to each other. The fact that the periodically loaded lines share the same transformations allows us to extrapolate the properties of complex modes observed in the cases of homogeneous systems to those that are axially periodic.
3.3 Analytic study of the eigenmodes of coupled transmission lines

In this section, we will examine the analytic characteristics of the eigenvalues of the uniform coupled transmission-line problem with particular attention to the cases when some of these modes are complex in a finite bandwidth. The investigation in this section will assume that the coupled lines are lossless and characterized by the Laplace domain equation (3.13) with a finite number of poles and zeroes in the entire complex-frequency plane. We will rely on Network Theoretic principles [35][80] to deduce the behaviour of the eigenvalues $\gamma$ of the system at complex frequencies defined over multiple Riemann sheets in the most general case. The study of the analytic properties of $\gamma$ in this chapter was motivated in part by the interesting ideas presented by Rhodes in [81] and those by Hanson and Yakovlev in [82]. The results deduced in this section will provide the required framework when we answer the question about the independent excitation of complex modes in Chapter 4. Moreover, the class of problems we consider here, namely lossless finite pole/zero coupled transmission lines, contains the case of the MS/NRI-TL coupled-line coupler in the homogeneous limit.

3.3.1 Poles and zeroes of lossless coupled transmission lines

Physical Constraints

Our analysis will be based on the assumption that the $N$ coupled transmission lines can be modelled using a finite number of poles and zeroes in equation (3.13). In other words, all entries of the impedance and admittance matrices which characterize these lines can be represented by rational functions. These matrices are functions of the complex frequency $s$ and their allowable mathematical form will be restricted to model causal, realizable, real, lossless, reciprocal and stable systems. The causality requirement will be established by requiring that the impulse response of the system is zero for time $t < 0$. By realizability, we refer to the fact that the inductors, capacitors, transformers . . . etc. that characterize a small differential element are all real — i.e. in all impedance expressions of the form $sL$ and $1/sC$ the inductance $L$ and capacitance $C$ are real\(^4\) quantities. The

\(^4\)In general passive systems, these quantities can be either positive or negative. In the cases of negative parameters, such as the negative inductors that appear in the T-junction models of some coupled coils, the values of these components will have to be constrained such that passivity is not violated. Such
condition of reality implies that the time domain output is real when the input is real. We will also restrict the parameters of the coupled-mode differential equation (3.13) such that the system in BIBO (Bounded Input Bounded Output) stable.

It should be pointed out that there is a fundamental difference between finite pole/zero electrical filters and the coupled-lines we consider here. In the case of filters, the terminal responses such as the port impedances and transmission functions contain a finite number of poles and zeroes. This is not the case for the coupled-lines as the finite pole/zero response is only applicable to a small differential element.

**Analytic properties of** $Z$ and $Y$

The eigenvalues $\gamma$ of the lossless $N$ coupled uniform transmission-line system are obtained from (see equation (3.13)):

$$\det \begin{pmatrix} -\gamma(s)I & Z(s) \\ Y(s) & -\gamma(s)I \end{pmatrix} = 0$$

In equation (3.25), $I$ is the identity matrix while $Z$ and $Y$ are the $N \times N$ impedance and admittance matrices whose Laplace domain properties will be outlined. constraints are of no concern in the following discussion as we are dealing with the analysis rather than the synthesis of passive networks.
These matrices share the same properties as those of the impedance/admittance matrices which describe lossless finite pole/zero electrical networks and the circuit level interpretation for the case of \( N = 2 \) lines is depicted in Figure 3.4.

As the system is lossless and reciprocal, the matrices \( Z \) and \( Y \) are symmetric and purely imaginary when \( s = j\omega \) i.e. along the imaginary axis of the complex-frequency plane, these matrices are paraskew Hermitian. A small differential segment of the \( N \) coupled lines consists of a finite number of poles and zeroes, and hence each element of these matrices is a rational function of \( s \). Due to the realizability constraint, these rational functions are obtained by performing a finite number of arithmetic operations on terms of the form \( sL, 1/sC \) etc. Hence the coefficients of the numerator and denominator polynomials of these rational functions are real.

Along the real axis of the complex-frequency plane, \( Z \) and \( Y \) are Hermitian as they are symmetric and real — each rational function with real coefficients is real when \( s \) is real. The real part of each of these matrices \( Z + Z^\dagger \) and \( Y + Y^\dagger \) must be positive-definite\(^5\) (due to passivity) and devoid of poles (due to stability) in the right-half frequency plane [35]. In fact, the positive-definite, analytic (no poles) along with conjugate reflection symmetry (about the real axis) properties in the right-half plane are sufficient to guarantee passive realizability of a network represented by a given immitance matrix. Hence, if we apply the frequency transformation \( s \rightarrow s + \sigma \) where \( \sigma \) is a positive constant to a realizable matrix \( Z(s) \), the resulting matrix is also passive realizable as we preserve its properties in the right-half plane.

\section*{Symmetries of the characteristic polynomial of the system}

The eigenvalues \( \gamma \) of the \( N \) coupled-lines can also be obtained from the following simplified determinant equation (see equation (2.13)):

\[
\det \begin{vmatrix} Z(s) & Y(s) \\ \gamma^2 \end{vmatrix} = 0 \tag{3.26}
\]

The characteristic polynomial in \( \gamma \) that we obtain after evaluating (3.26) is hence even in \( \gamma \) and is of the form:

\[
\gamma^{2N} + \gamma^{2(N-1)}a_{N-1}(s) + \gamma^{2(N-2)}a_{N-2}(s) + \ldots + a_0(s) = 0 \tag{3.27}
\]

Each coefficient function \( a_n(s) \) in equation (3.27) is obtained by adding and multiplying the entries of the impedance and admittance matrices, and as a result, they are

\(^5\)A matrix is positive definite if the quadratic form \( X^\dagger ZX > 0 \) for arbitrary complex vectors \( X \).
also rational with real coefficients.

The matrix product $ZY$ in equation (3.26) is purely real for all $s = j\omega$ due to the lossless nature of the system. Hence the rational functions $a_n(s)$ must be real when $s = j\omega$ which implies that they are even functions of $s$. To see this, consider the general form of $a_n(s)$:

$$a_n(s) = A_n s^p + N_{p-1}s^{p-1} + N_{p-2}s^{p-2} + \ldots + N_0$$  \hspace{1cm} (3.28)

In equation (3.28), the factor $A_n$, the coefficients of the numerator polynomial of order $p$ and those of the denominator polynomial of order $q$ are all real constants. If we substitute $s = j\omega$, the even terms are real while the odd term are imaginary both in the numerator and denominator. Hence, if $a_n(s)$ is to be real for all points on the imaginary axis of the complex-frequency plane, we will have to set the odd coefficients in (3.28) to zero. Therefore the rational functions $a_n(s)$ in the eigenmode equation (3.27), are even functions of $s$. We also observe that these rational functions are real on the real axis of the complex plane.

The real coefficients of the $a_n(s)$ imply that the complex roots of the numerator and denominator always occur in conjugate pairs. The even nature of these functions guarantees reflection symmetry of their roots through the origin in the complex-frequency plane. Hence we will find the poles and zeroes of $a_n(s)$ in reflection symmetry pairs when they lie on the real or imaginary axis, and in reflection symmetry quartets for all other cases.

**Analytic properties of $\gamma$**

The $2N$ roots of equation (3.27) can be represented as analytic functions of $s$ in $2N$ Riemann sheets. Some of them are connected by common branch-points and they form mode sets [81] which are disjoint from the remaining sheets. The details of the branch-points and mode sets will presented later. We will classify each sheet as being proper or improper depending on whether $\gamma$ remains finite or not (as $z \to +\infty$) when evaluated along the imaginary axis (in the presence of infinitesimal loss).

In a semi-infinite system of coupled-lines located between $z = 0$ and $z = +\infty$, any physical source will excite only the proper modes. Let us consider the time domain voltages and currents at $z = 0$ as the input and those at $z = d$ as the output given by:

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix}_{z=d} = \frac{1}{2\pi j} \int_{Br} \sum_n b_n(s) e^{-\gamma_n(s)d+st} \begin{pmatrix} V(s) \\ I(s) \end{pmatrix}_n ds \hspace{1cm} (3.29)$$
In equation (3.29) we sum over the proper modes where \((V^T \ I)_n\) is the eigenvector corresponding to \(\gamma_n\) with complex excitation amplitude \(b_n\). The inverse Laplace integral will be carried out over a straight line parallel to the imaginary axis in the complex-frequency plane (denoted by ‘Br’ — the Bromwich contour) and we will place it to the right of all singularities which are poles (and not branch-points). This choice of the Bromwich contour will enforce a causal response from the system. This is evident by recognising that the integral in equation (3.29) can be closed in the right-half complex plane when \(t < 0\) and \(d > 0\). From Cauchy’s theorem, the resulting closed contour integral will be zero as it contains no poles of the integrand\(^6\) and this ensures that the output signal does not precede the input.

In order for the response in (3.29) to be stable, there are no poles (of \(\gamma\)) in the right-half plane\(^7\) and we will demonstrate shortly, that the zeroes of \(\gamma\) are not allowed in this half of the plane as well. From our earlier discussion, we know that \(\gamma\) enjoys reflection symmetry through the origin of the complex-frequency plane. Hence all poles and zeroes must be limited to the imaginary axis. If there is any pole/zero in the left-half plane at some frequency \(s\), we would find its counter-part at \(-s\) in the right-half plane which is clearly not possible. Hence the Bromwich integral in equation (3.29) will be carried out along the imaginary axis with small semi-circular indentations to the right around all singularities.

Now we will demonstrate that the imaginary axis poles and zeroes of \(\gamma(s)\) in any Riemann sheet must be simple. In the vicinity of an imaginary axis pole of order \(m\) (say at \(s_0\)), we can approximate \(\gamma_n(s)\) in the \(n\)-th proper Riemann sheet by:

\[
\gamma_n(s) \approx \frac{R_0}{(s - s_0)^m}
\]

Let us introduce a small amount of dissipation into the system by the frequency transformation \(s \to s + \sigma\) where \(\sigma\) is real and positive. This is always possible by transforming the lossless inductive impedances in the system \(sL \to sL + R_L = (s + \sigma)L\) and the lossless capacitive impedances \(1/sC \to 1/(sC + G_C) = 1/(s + \sigma)C\) where \(\sigma L\) and \(\sigma C\) are lossy series resistance and parallel conductance respectively. The introduction of loss is hence equivalent to a slight shift of the imaginary axis pole at \(s_0\) to the left-half

\(^6\)We will allow right-half plane branch-point singularities which are bounded and show in Chapter 4 that they do not violate causality or stability requirements.

\(^7\)The product of the amplitude \(b_n(s)\) and the modal vector \((V^T \ I)_n\) in equation (3.29) has no poles in the right half plane. This can be seen by taking the limit \(d \to 0\) and observing that it is simply a vector of inverse Laplace transform coefficients corresponding to the causal sources that feed the lines.
The points on the imaginary axis correspond to real frequencies of harmonic analysis and as we sweep from a point slightly below the pole (at \( s_0^- \)) to a point above it (at \( s_0^+ \)), we change the argument of \((s - s_0)\) in equation (3.30) by an amount slightly smaller than \(+\pi\). As \( \gamma_n \) is a proper mode, in the presence of small loss, we require it to have a positive real part at both \( s_0^- \) and \( s_0^+ \). If we start off (at \( z = 0 \)) with a voltage/current excitation at the frequency \( s_0^- \) that decays to zero along the lines towards \( z \to +\infty \), then a small change in frequency should not change the nature of the mode from one that is decaying to one that is exponentially increasing. Hence the pole must be simple (\( m = 1 \)) as otherwise the argument of \((s - s_0)^m\) will change by a factor greater than \(+\pi\) when we move from \( s_0^- \) to \( s_0^+ \). This also implies that the pole residue \( R_0 \) in equation (3.30) is a positive real constant. We can use exactly the same line of reasoning to deduce that the zeroes on the imaginary axis must be simple as well.

We are not obliged to introduce only a small amount of loss into the system. In fact we can make the system arbitrarily lossy by choosing a large positive \( \sigma \) in the transformation \( s \to s + \sigma \) and the resulting impedance and admittance matrices which describe the system will still be physically realizable. But in such a lossy scenario, the real part of \( \gamma \) (and hence \( \gamma \) itself) cannot be zero on the imaginary axis (i.e. the modes must attenuate if they are proper and grow if they are improper) of the transformed complex-frequency plane. In other words, we have just verified that there can be no zeroes of \( \gamma \) in the right-half plane. This fact can be rigorously confirmed from the locations of the natural resonances of the system.

From equation (3.26), it is clear that \( \gamma = 0 \) implies that \( \det(Z)\det(Y) = 0 \). When the terminals of a passive network described by an impedance matrix are shorted, the frequencies corresponding to the zeroes of the determinant are the ones at which non-zero currents can exist at the shorted ports. Similarly, the zeroes of the determinant of an admittance matrix correspond to source-free voltage oscillations at the open circuited ports of the network. The frequencies of oscillation, and hence the zeroes of the determinant, cannot be in the right-half complex-frequency plane due to the stability requirement. Hence the zeroes of \( \gamma \) are restricted to the imaginary axis.

At the origin and the point at infinity, \( \gamma \) will usually have a pole or a zero and is some cases can be purely real\(^8\). At these extreme frequencies the inductors and capacitors

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\(^8\)When \( \gamma \) has a simple pole at infinity, this implies that the phase of the voltage transfer function
which model a small differential length of the \( N \) coupled-lines are either open or short circuits and hence any non-zero finite phase shift or attenuation between the input and output planes of a lossless network is meaningless. Nevertheless, since ideal transformers are allowable lossless circuit elements, a real \( \gamma \) at the origin or infinity simply represents an impedance level shift.

When we take the complex conjugate of equation (3.27) in which the \( a_n(s) \) are rational functions with real coefficients, we observe that \( \gamma \) being a solution at \( s \), implies that \( \gamma^* \) is a solution at \( s^* \). We also observe that for equation (3.29) to be real, we require \( \gamma_n(s) = \gamma_n^*(s^*) \) in each Riemann sheet. Hence the solution pair \( \gamma \) and \( \gamma^* \) of equation (3.27) must be located on the same Riemann sheet. The question which naturally arises in this case is whether the solution pair of \( \gamma_n(s) \) and \( \gamma_n^*(s^*) \) constitute a smooth analytic continuation from the top half plane to the bottom half plane?

To answer this question, we refer back to the eigenvalue matrix equation (3.26) which can be cast in the form of the generalized eigenvalue problem \( ZV = \gamma^2 Y^{-1}V \) where \( V \) is the voltage eigenvector. The eigenvalues of this problem are real when the matrices involved are positive/negative definite [83] which happens to be the case on the real axis \((Z + Z^T = 2Z\) when \( s \) is real) from our previous discussion on the realizability criteria of impedance/admittance matrices. Hence \( \gamma_n(s) \) in the upper half plane will continuously merge into \( \gamma_n^*(s^*) \) through the real axis and the resulting function will be analytic due to the Schwartz reflection principle\(^9\).

Now we conclude that the pole or zero at the origin must be simple. To see this, we can add a finite amount of dissipation into the system like before, which will cause the pole/zero at the origin to shift to the left of the imaginary axis. Due to conjugate symmetry, we require that the small real part of a proper mode to be positive both above and below the real axis. This implies this pole/zero must of the first order.

In order to see that the pole/zero at infinity must be simple as well, we apply the frequency transformation \( s \rightarrow 1/s \) to \( \gamma_n(s) \) in one of the Riemann sheets. This transformation maps the imaginary axis on to itself and the point at infinity to the origin. If the

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\(^9\) The Schwartz reflection principle states that the analytic continuation of a function defined in region lying on one side of the real axis with real values on the axis, is the conjugate of the function reflected in the region about this axis [84].
pole/zero is not simple at the transformed origin, we run into exactly the same problem with the real part of $\gamma_n$ on the imaginary axis in the presence of small dissipation.

At any analytic point on the imaginary axis, $\gamma = \alpha + j\beta$ satisfies the Cauchy-Riemann conditions $\frac{\partial \alpha}{\partial \sigma} = \frac{\partial \beta}{\partial \omega}$ and $\frac{\partial \alpha}{\partial \omega} = -\frac{\partial \beta}{\partial \sigma}$ where we designate the complex frequency $s = \sigma + j\omega$. If $\gamma$ is imaginary and corresponds to a proper mode, then $\frac{\partial \beta}{\partial \omega} > 0$ such that when we add a little dissipation into the system, the resulting mode attenuates $\frac{\partial \alpha}{\partial \sigma} > 0$ towards $+\infty$. This is equivalent to stating that the group velocity of the proper propagating mode must be positive at all frequencies. It should be noted that this positive group velocity argument holds only for purely propagating modes and not necessarily for complex modes as their real part is non-zero on the imaginary axis (i.e. if $\frac{\partial \alpha}{\partial \sigma} \leq 0$, then the addition of loss will not necessarily change a proper mode into one that is improper).

We will summarize the analytic properties of the eigenvalues $\gamma$ of the realizable, real, lossless, reciprocal, causal and stable coupled transmission-line system:

- The poles and zeroes of $\gamma$ are simple and located exclusively on the imaginary axis with reflection symmetry about the origin.

- The origin and the point at infinity in the complex plane can be a simple pole, a simple zero or $\gamma$ can be real at these points due to the presence of an ideal transformer.

- The $2N$ eigenvalues of the $N$ coupled transmission-line problem are defined on $2N$ Riemann sheets where they are analytic, real on the real axis and exhibit conjugate reflection symmetry about this axis.

- Some of these sheets are connected by branch-points and a set of connected sheets form a mode-set\textsuperscript{10}. Each mode set contains both $\gamma(s)$ and $-\gamma(s)$ which are simply different branches of the same analytic function.

- Each Riemann sheet is classified as either proper or improper depending on whether $e^{-\gamma_n(s)z}$ is bounded or unbounded as $z \to +\infty$ at frequencies slightly to the right of the imaginary axis.

\textsuperscript{10}We borrow the term ‘mode-set’ from [81] and generalize it to refer to a collection of Riemann sheets that share common branch-points and the associated cuts throughout the complex-frequency plane (instead of just the right-half plane in [81]).
• When $\gamma$ is purely imaginary and proper, it is an increasing function of frequency along the imaginary axis. Consequently, any two poles of $\gamma$ are separated by at least one zero.

### 3.3.2 Riemann surface, branch-points and complex modes

The $2N$ solutions of equation (3.27) at every complex frequency $s$ defines a multivalued analytic function $\gamma$ that can be conveniently described on $2N$ complex-frequency planes. This extended multi-sheeted domain is referred to as the Riemann surface on which $\gamma$ is single valued. Some of these sheets can be clustered together when they are connected by branch-points and we will refer to them as mode-sets. The topography of the Riemann surface corresponding to equation (3.27) can consist of a collection of disjoint mode-sets.

Let us assume that equation (3.27) can be expressed as the product of $M$ factors ($1 \leq M \leq N$) such that:

\[
\prod_{m=1}^{M} \left[ \gamma^{2N_m} + \gamma^{2(N_m-1)}a_{m,N_m-1}(s) + \ldots + a_{m,0} \right] = 0
\] (3.31)

where $\sum N_m = N$ and each $a_{m,N_m}(s)$ is a rational function of frequency with the additional condition that each term of the product cannot be factorized into simpler functions with rational coefficients. In this case, each square bracket above produces a mode-set that is completely independent of the other solutions of the system.

The Riemann sheets that constitute a mode-set are connected smoothly to ensure that $\gamma$ is continuous when it is analytically continued around a common branch-point. There are no natural boundaries of this multi-dimensional surface but it is common to use the artificial concept of the branch-cut to separate out the various sheets according to some peculiar behaviour of $\gamma$. The criteria commonly used to define these branch-cuts in guided-wave problems is the behaviour at infinity. Hence cuts are introduced to separate those mode that are bounded as $z \to +\infty$ (the proper modes) from those that are not (the improper modes).

The first order branch-points of $\gamma$ that lie on the imaginary axis denote the transition frequencies about which the characteristics of the mode changes between propagating, evanescent and complex. Branch-points represent modal degeneracy where two or more modes share the same propagation constant. For example, the proper and improper modes have the same propagation constant $\gamma = 0$ at the cut-off frequency of a given mode in a rectangular waveguide. On the other hand, not all modal degeneracies are
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associated with branch-points (e.g. the $\gamma = 0$ point of a NRI-TL with closed stop-band is not a branch-point).

When a mode transforms from one that is evanescent to propagating (or vice-versa) at the point $s_0$, the corresponding phase of $\gamma$ changes by $\pi/2$. Hence if $\gamma \approx \sqrt{jB(s - s_0)}$ in the vicinity of this transition point (where $B$ is some real constant), the angle of $\gamma$ changes by $\pi/2$ when one sweeps the frequency $s$ past $s_0$ about a semi-circular arc in the right half-plane. The sign of $B$ determines if the transition is from evanescent to propagating or vice-versa. On the other hand, if $\gamma \approx \gamma_0 + \sqrt{jB(s - s_0)}$ where $\gamma_0$ is some non-zero purely real or imaginary constant, then crossing this branch-point converts an evanescent or propagating mode to one that is complex (depending on the sign of $B$).

In the limit when the $N$ lines are separated from each other by infinite distance, their modes are independent and the Riemann surface consists of $N$ mode-sets comprising two joint sheets each and containing a pair of proper and improper modes (in some cases, even these two sheets might be disjoint). It follows from the fact that in this limit, the matrices $Z$ and $Y$ in equation (3.26) are diagonal and the eigenvalue equations are of the form $\gamma^2 - Z_{nn}(s)Y_{nn}(s) = 0$ where $Z_{nn}$ and $Y_{nn}$ are the rational functions corresponding to the $n$-th diagonal entry.

When the $p$-th line is coupled to the $q$-th line, the terms $Z_{pq} = Z_{qp}$ and $Y_{pq} = Y_{qp}$ are non zero and the eigenvalue equations are $\gamma^2 - Z_{nn}(s)Y_{nn}(s) = 0$ for all $n \neq p, q$. The remaining eigenmodes are given by a 4-th order algebraic function that connects the $p$-th and $q$-th mode-sets into a 4-sheeted surface. Hence the process of coupling manifests itself analytically through the formation of new branch-points that connect the previously disjoint mode-sets. These new branch-points move continuously through the complex-frequency plane when the coupling parameters are smoothly varied. As mentioned earlier, branch-points are associated with modal degeneracy and hence when the lines are slowly decoupled, they must vanish into degeneracies of the uncoupled system. In other words, the process of coupling creates branch-points that move out of locations in the Riemann surface where the uncoupled modes have the same value of $\gamma$ (for example a common pole at infinity or simply a frequency point where the two isolated modes have the same finite propagation constant).

If we write the equations for the uncoupled eigenmodes as $\gamma = \sqrt{Z_{nn}(s)Y_{nn}(s)}$, then it is evident that the branch-points of these modes correspond to the simple zeroes of $Z_{nn}(s)Y_{nn}(s)$ at which $\gamma = 0$. Hence complex eigenmodes are not possible in this case and the minimum requirement for their existence is a system that contains at least 2
coupled lines.

From here on, we focus on the case of 2 coupled lines whose eigenmodes are obtained from an algebraic equation of the general form:

$$\gamma^4 + \gamma^2 a_1(s) + a_0(s) = 0 \quad (3.32)$$

where $a_1(s)$ and $a_0(s)$ are real rational functions on the imaginary axis of the frequency plane. One can show that $a_1 = -\text{tr}(Z Y)$ and that $a_0 = \det(Z Y)$ where ‘tr’ is the trace. We can designate the four modal functions of equation (3.32) as $\gamma_+^c(s)$, $\gamma_-^c(s)$, $\gamma_+^\pi(s)$ and $\gamma_-^\pi(s)$ each of which is defined on a separate Riemann sheet. In the neighbourhood of a regular frequency point, each of these modes can be represented by a distinct functional element in the form of a Taylor series with a finite radius of convergence. If we analytically continue one of the functional elements along a closed path that encircles a branch-point of equation (3.32), then it is either converted to one of the remaining three functional elements or remains unchanged\footnote{Two of the functional elements can be connected by a branch-point at $s_0$ while the other two remains regular at this point.}. It is impossible for two distinct functional elements to transformed into the same element when this procedure is carried out [85].

Hence if we start with an ordered set of functional elements \{$\gamma_+^c, \gamma_-^c, \gamma_+^\pi, \gamma_-^\pi$\}, analytic continuation around any branch-point gives us a new ordered set of elements that is merely a permutation of the initial set (for example \{$\gamma_-^\pi, \gamma_+^\pi, \gamma_-^c, \gamma_+^c$\}). This property enables the use of the set of functions above to construct new ones that are free of branch-points in the entire complex-frequency plane. For instance the symmetric functions $\gamma_+^c + \gamma_-^c + \gamma_+^\pi + \gamma_-^\pi$ or $\gamma_+^c \gamma_-^c \gamma_+^\pi \gamma_-^\pi$ do not exhibit any of the branch-points of the original elements. Analytic continuation around the branch-point $s_0$ of the individual elements leaves these newly formed functions unchanged as their subscripts/superscripts are merely shuffled in the process (and hence by the Riemann Monodromy theorem, they are regular at $s_0$).

The symmetric functions of these eigenmodes are helpful in developing a deeper understanding of the analytic structure that these modes exhibit. This is the case as the non-rational functional elements are difficult to analyse in the entire complex plane compared to the rational symmetric functions. As $\gamma$ is even in equation (3.32), we let...
\[ \gamma_c = \gamma_c^+ = -\gamma_c^- \quad \text{and} \quad \gamma_\pi = \gamma_\pi^+ = -\gamma_\pi^- . \]

Now we define the following functions:

\[
F_1(s) = -\left( \gamma_c^2 + \gamma_\pi^2 \right) = a_1(s) \tag{3.33}
\]
\[
F_2(s) = \gamma_c^2 \gamma_\pi^2 = a_0(s) \tag{3.34}
\]
\[
F_3(s) = \left[ \gamma_c^2 - \gamma_\pi^2 \right]^2 = a_1^2(s) - 4a_0(s) \tag{3.35}
\]

The right most expressions in equations (3.33) to (3.35) above can be obtained from the quadratic formula for the roots of equation (3.32) and it is readily seen that these symmetric functions are rational and hence devoid of any branch-points. The ability to construct rational functions out of half of the total number of modal elements is possible only due to the even nature of \( \gamma_{12} \).

The usefulness of these functions can be appreciated as follows. We observe that the poles and zeroes of the modes can be located from those of the function \( a_0(s) \) in equation (3.34) which is the determinant of the matrix \( ZY \). The symmetric function \( F_3(s) \) defined in equation (3.35) will be used later in this chapter to understand the origin of the peculiar phenomenon of right-half plane branch-points associated with complex eigenmodes.

There are 4! possible permutations of the set \( \{ \gamma_c^+, \gamma_c^-, \gamma_\pi^+, \gamma_\pi^- \} \) but not all of them are obtainable through the analytic continuation around a single branch-point. Hence we list below the ones that are allowed as well as the ones that are not, and also describe the behaviour of the functional element in the vicinity of the branch-point that is responsible for the specific permutation. In the interest of conciseness, we use standard notations such as \( (\gamma_c^+ \gamma_\pi^- \gamma_c^-) \) to indicate cyclic replacement of the object \( \gamma_c^+ \) by \( \gamma_\pi^+ \), \( \gamma_\pi^- \) by \( \gamma_c^- \) and \( \gamma_c^- \) by \( \gamma_\pi^+ \). A product of two cycles imply successive application of each cycle to the set of modes.

- Transpositions with the same subscript \( (\gamma_c^+ \gamma_c^-) \) and \( (\gamma_\pi^+ \gamma_\pi^-) \) are associated with analytic continuation around branch-points at which \( \gamma \) can be either 0 or infinity. This ensures that the associated functional elements are identical except for a negative sign. As the poles and zeroes of \( \gamma \) are confined to the imaginary axis of the complex-frequency plane (due to linearity, passivity . . . etc.), such branch-points can only occur on this axis. Any one of these transpositions or their product can be associated with a single branch-point.

\footnote{If for instance \( \gamma_c^+ \to \gamma_\pi^- \) when continued around a branch-point, then this automatically requires that \( \gamma_c^- \to \gamma_\pi^+ \) as the the starting Taylor series for \( \gamma_c^+ \) is simply the negative of that of \( \gamma_c^- \).}
Chapter 3. Analytic Properties of Complex Modes

• Transpositions with different subscripts such as \((\gamma_c^+ \gamma_c^-)\) cannot be realized on their own from a single branch-point. However, the disjoint products such as \((\gamma_c^+ \gamma_c^-)(\gamma_c^- \gamma_c^+)\) can be associated with a single first order branch-point that can be anywhere in the complex-frequency plane. If this branch-point is located at \(s_0\), then in its vicinity, the four functional elements can be obtained from the expression \(\gamma \approx \pm (\gamma_0 + C\sqrt{s - s_0})\). When such a point is located on the imaginary axis, \(\gamma_0\) must be purely real or imaginary and \(s_0\) must correspond to the transition between two evanescent or two propagating modes to a pair of complex modes (or vice-versa). This can be demonstrated from the requirements that the eigenmodes appear in conjugate pairs on this axis and that the proper modes remain bounded in the right vicinity of the branch-point.

• Three cycles such as \((\gamma_c^+ \gamma_c^- \gamma_c^- \gamma_c^+)\) cannot be obtained by analytic continuation around any branch-point. If \(\gamma_c^+\) transforms to \(\gamma_c^-\) then \(\gamma_c^-\) cannot transform to \(\gamma_c^+\) at the end of the cycle. This is the case because \(\gamma_c^+\) and \(\gamma_c^-\) are linearly dependent functional elements (one is the negative of the other) and \(\gamma_c^+\) is an element independent of the former. Consequently, second order branch-points do not exist due to the even nature of the eigenvalue characteristic equation.

• The only four cycles that may be found in this system are \((\gamma_c^+ \gamma_c^- \gamma_c^- \gamma_c^+)\) and \((\gamma_c^+ \gamma_c^- \gamma_c^- \gamma_c^-)\). Such transformations are affected by analytic continuation around a branch-point of the third order. All functional elements have the same value at the branch-point frequency and hence the corresponding \(\gamma\) must be zero or infinity at this degenerate point to ensure that \(\gamma_c^+\) is the negative of \(\gamma_c^-\). Hence third order branch-points, if they exist, must be located on the imaginary axis of the complex-frequency plane. They describe the transition from one propagating and one evanescent mode to a complex-mode pair (or vice-versa).

3.3.3 Group velocity in the complex-mode band

In this section we will examine the dispersive properties of \(\gamma\) in a system consisting of two \((N = 2)\) lossless coupled-lines in the particular case when the modes become complex in a finite segment of the imaginary axis and are ordinary (propagating or evanescent) elsewhere. A few representative examples of the frequency dispersion of complex modes are shown in the plots in Figure 3.5.
Figure 3.5: (a) Typical dispersion characteristics (where $p$ is the normalized complex frequency defined in section 3.3.4) of the modes of a MS/NRI-TL coupler depicting a single complex-mode band between the horizontal black lines. The solid lines represent the eigenmodes of the coupler with finite line separation. The broken lines represent the modal dispersion in the limit of infinite line separation corresponding to those of an isolated microstrip line (red) and an isolated NRI-TL (blue). Note that the real parts of the modes are not shown in this plot. (b) Complex modes in circular waveguides with dielectric inserts are taken from [81]. Note that in these plots, the frequency axes are horizontal, $\alpha$ and $\beta$ correspond to the real and imaginary parts of $\gamma$ and the complex-mode bands are marked by vertical broken lines.
In the case of \( N = 2 \) coupled-lines, the eigenvalues \( \gamma \) are obtained from equation (3.32) which can also be expressed as:

\[
2\gamma^2 = -a_1(s) + \sqrt{a_1^2(s) - 4a_0(s)}
\]  
(3.36)

We notice that the eigenmodes can be complex in the lossless system if \( a_1^2(s) - 4a_0(s) \) in equation (3.36) becomes negative and that they are ordinary when this term is positive. From equation (3.35), the term under the radical is equal to \( [\gamma_c^2 - \gamma_\pi^2]^2 \). In order to gain physical insight, we consider the limit of infinite spacing between the lines such that \( \gamma_c \rightarrow \gamma_1 \) and \( \gamma_\pi \rightarrow \gamma_2 \) where \( \gamma_1 \) and \( \gamma_2 \) are the propagation constants of the isolated lines. Hence we can expect complex modes in the phase-matched coupling (crossing of the isolated dispersion curves) of a forward and a backward wave line (see Figure 3.5a).

Let \( s = j\omega_c \) denote the critical frequency (where \( \omega_c \in \mathbb{R}^+ \)) that marks the transition between regular propagating modes of the system and the complex proper modes — i.e. \( \gamma \) is purely imaginary for \( \omega \) lower than \( \omega_c \) and complex above it (this would correspond approximately to the frequency point \( \text{Im}(\gamma) = 0.5 \) in Figure 3.5a). Hence the quantity under the radical in equation (3.36) changes its sign about this frequency point implying that \( a_1^2(j\omega_c) - 4a_0(j\omega_c) = 0 \). One can show that this zero must be simple to bring about the transition from propagating modes to complex modes in a passive system and thereby we conclude that \( s = j\omega_c \) is a branch-point of \( \gamma \) in the complex plane.

Let \( \gamma = j\beta_c \) (where \( \beta_c \in \mathbb{R}^+ \)) be the eigenvalue of one of the proper modes at \( s = j\omega_c \). On this particular Riemann sheet and in the vicinity of the branch-point, we can expand equation (3.36) as:

\[
2\gamma^2 = -\left[2\beta_c^2 + jA(s - j\omega_c)\right] + \sqrt{jB(s - j\omega_c)}
\]  
(3.37)

In equation (3.37), the quantity inside the square bracket is the Taylor series expansion of \( a_1(s) \) about \( j\omega_c \). The slope of \( a_1(s) \) is taken to be \( jA \) (where \( A \) is real) to ensure that the function is real on the imaginary axis and this derivative can be positive or negative. On the other hand, the constant \( B \) is real and positive so that when the angle of \( (s - j\omega_c) \) changes from \(-\pi/2\) to \(+\pi/2\), the eigenmode \( \gamma \) transforms from being purely imaginary to one which is complex with a positive real part.

We can now differentiate (3.37) at a point \( j\omega_c^+ = j(\omega_c + \delta) \) slightly above the branch-
point frequency (where \( \delta \in \mathbb{R}^+ \)) to obtain:

\[
4 \gamma \frac{d\gamma}{ds} = 0.5 \sqrt{B/\delta} - j A
\]

or,

\[
8 \delta \frac{d\gamma}{ds} = \frac{\sqrt{B \delta - 2 j A \delta}}{\gamma(j \omega_c^+)} = \frac{\sqrt{B \delta - 2 j A \delta}}{\sqrt{-[\beta_c^2 - A \delta/2] + j \sqrt{B \delta/2}}}
\]

(3.38)

At \( s = j \omega_c^+ \), \( \gamma \) must have positive real and imaginary parts when it is analytically continued on the imaginary axis through an arc in the right-half plane which circumvents the branch-point. When \( A \) is positive, it is likely that the angle of the right-hand side of equation (3.38) will be less than \(-\pi/2\). To see this, make \( \delta \) so small that setting \( \gamma = j \beta_c \) in the denominator of equation (3.38) will result in negligible error.

We can write \( d\gamma/ds = -j d\alpha/d\omega + d\beta/d\omega \) where \( \alpha \) and \( \beta \) are the real and imaginary parts of \( \gamma \). If the angle of \( d\gamma/ds \) is less than \(-\pi/2\), then \( d\beta/d\omega \) is negative implying that the group velocity \( d\omega/d\beta \) is negative as well.

Whether \( A \) can be made positive in practice and if its magnitude can be controlled are questions whose answers are still pending. In this regard, recall that \( a_1(s) = - (\gamma_1^2(s) + \gamma_2^2(s)) \) in the limit of infinite separation between the lines. Letting \( \gamma_1(s) = j \beta_1(s) \) and \( \gamma_2(s) = j \beta_2(s) \), the frequency derivative of \( a_1(j \omega) \) is:

\[
A = - \frac{da_1(j \omega)}{d\omega} = -2 \left( \beta_1 \frac{d\beta_1}{d\omega} + \beta_2 \frac{d\beta_2}{d\omega} \right)
\]

(3.39)

When we consider the coupling between a forward-wave (\( \beta_1 \frac{d\beta_1}{d\omega} > 0 \)) and a backward-wave (\( \beta_2 \frac{d\beta_2}{d\omega} < 0 \)) such that \( |\beta_2 \frac{d\beta_2}{d\omega}| > |\beta_1 \frac{d\beta_1}{d\omega}| \), then \( A \) is positive. This is always possible in practice and in fact \( |\beta_2 \frac{d\beta_2}{d\omega}| \) can be made quite large compared to \( |\beta_1 \frac{d\beta_1}{d\omega}| \). An example would be the coupling between a forward-wave MS-TL and a backward-wave NRI-TL whose absolute group velocity \( \frac{d\omega}{d\beta} \) can be made very small.

Associating a negative group velocity with a complex mode (that carries no net power) is meaningful as the power flow on each individual line is non-zero. Imagine that a band-limited pulse is fed into a single line of the coupled system while simultaneously ensuring that only the peculiar mode associated with the negative group velocity phenomenon is predominantly excited. In this case, the resulting peak of the output pulse would appear to precede the input on the line which is excited. This doesn’t violate causality as the front edge of the pulse still travels at a velocity bounded by the speed of light in vacuum. Moreover, this phenomenon can only be observed for short distances away from the source due to strong modal attenuation causing significant pulse-shape distortion. Nevertheless,
this peculiar dispersion characteristic can potentially be exploited for pulse-shaping and phase-shifting applications.

### 3.3.4 Singularities of the MS/NRI-TL coupler eigenmodes

The eigenmodes $\gamma$ that result from the coupling of a microstrip to a NRI-TL (see Figure 3.5a) in the homogeneous limit can be obtained from the 4-th order equation (see Chapter 5 for details):

$$2 (p \gamma d)^2 = [2(1 + \kappa_L \kappa_C)p^4 + (\rho_L + \rho_C)p^2 + \rho_L \rho_C]$$

$$\pm \sqrt{[(\rho_L + \rho_C)p^2 + \rho_L \rho_C]^2 + 4p^4[(\kappa_L + \kappa_C)p^2 + \kappa_L \rho_L][(\kappa_L + \kappa_C)p^2 + \kappa_C \rho_C]}$$

(3.40)

In equation (3.40), the dimensionless parameters $\kappa_L$ and $\kappa_C$ are real and represent the inductive and capacitive coupling between the lines; they tend to zero when the line separation approaches infinity. The loading parameters $\rho_L$ and $\rho_C$ and the unit cell size $d$ are real positive quantities. The numerical value of the normalized complex frequency variable $p = s\sqrt{\varepsilon_0 \mu_0}d$ is approximately equal to the insertion phase across the microstrip line of length $d$. We will now verify that the eigenvalues obtained from this homogeneous model satisfy all the requirements of a lossless, reciprocal, stable and real system.

In equation (3.40), $\gamma$ is an even functions of the complex frequency $p$ whose coefficients are all real. This is the characteristic we expect from of a real lossless system. It is also evident that if $\gamma$ is a modal solution, then so is $-\gamma$ implying that the underlying system is that of a reciprocal guide that allows identical wave propagation in both directions. In the limit $p \to 0$:

$$2 (p \gamma d)^2 \to \rho_L \rho_C \pm \rho_L \rho_C$$

(3.41)

When we choose the + sign in equation (3.41), we obtain two modes that exhibit simple poles at the origin with real residues of which only one is proper. When the − sign is chosen instead, the resulting two modes exhibit zeroes at the origin. This is seen in Figure 3.6a where near the origin, two of the modes approach zero while the others diverge towards infinity. To demonstrate that these zeroes are simple with real coefficients we will need to obtain a Taylor series expansion of the right hand side of equation (3.40). Differentiating this right-hand side expression 4 times it can be shown that the first non-zero coefficient (that is also real and positive) is that corresponding to the $p^4$ term. Hence the $\gamma$ corresponding to these modes has simple zeroes.
At infinity $p \to \infty$, all modes possess simple poles which is evident by retaining the highest order polynomial terms in equation (3.40):

$$(p\gamma d)^2 \to (1 + \kappa_L\kappa_C \pm |\kappa_L + \kappa_C|) p^4$$

All possible values of the coefficient of $p^4$ in equation (3.42) can be obtained from the four possible products $(1 \pm \kappa_L)(1 \pm \kappa_C)$ which are all real and positive. This is the case as $\kappa_L = L_m/L_0$ and $\kappa_C = C_m/C_0$ represent the ratio between the mutual and self inductance/capacitance of a pair of regular coupled microstrip lines and this must be less than unity in magnitude. Therefore the pole residue at infinity is real as required. There are no other poles and hence the modes that we obtain from the homogeneous approximation represent stable and causal solutions.

One set of branch-points of the $\gamma$ can be determined from the simple roots of the 8-th order polynomial under the radical in equation (3.40). A second set is determined from the zeroes of $\gamma^2$ obtained by setting the right-hand side of equation (3.40) to zero. The following equations in $p$ can be solved to obtain all branch-points of the system:

$$4(\kappa_L + \kappa_C)^2 p^8 + 4(\kappa_L + \kappa_C)(\kappa_L\rho_L + \kappa_C\rho_C)p^6 + ((\rho_L + \rho_C)^2 + 4\kappa_L\kappa_C\rho_L\rho_C) p^4 + 2\rho_L\rho_C(\rho_L + \rho_C)p^2 + (\rho_L\rho_C)^2 = 0$$

$$[(1 - \kappa_L^2)p^2 + \rho_C] [(1 - \kappa_C^2)p^2 + \rho_L] = 0$$

The branch-points of the eigenmodes of a typical MS/NRI-TL coupler are depicted in Figure 3.6b which were obtained using the same parameters as those used to generate the dispersion plot in Figure 3.6b. We observe that these singularities are located symmetrically about both the real and imaginary axes as expected from the characteristics of a lossless and reciprocal system. A very interesting observation to make is that there are branch-points in the right-half plane. The presence of such singularities has been associated with the existence of complex modes and have led to a long standing belief that a pair of conjugate modes cannot be excited independently [81]. We will show that this assertion is not entirely correct and that the amplitudes of the pair can be controlled independently (and a single mode can be eliminated at a set of discrete frequencies) without resulting in any unstable or non-causal response from the system (see Chapter 4).
Figure 3.6: (a) Typical eigenmodes of a MS/NRI-TL coupler with solid lines depicting proper modes that are bounded as $z \to +\infty$. (b) Branch-points of $\gamma$ on the top-half of the complex frequency $p$-plane that are also present in the bottom-half plane with reflection symmetry.
3.3.5 Proper and improper branches of the eigenmodes

We present a sample plot of the modal dispersion along the imaginary axis of the normalized frequency plane in Figure 3.6a using the parameters\(^{13}\) \(\rho_L = 0.56\), \(\rho_C = 0.75\), \(\kappa_L = 0.41\) and \(\kappa_C = -0.31\). The solid curves in this plot are proper modes bounded as \(z \to +\infty\) while the broken lines are color-coordinated to indicate the corresponding improper modes. We observe a complex-mode band between points \(a\) and \(c\) (real part of \(\gamma\) is not shown but is non-zero and positive for the proper modes).

All points in Figure 3.6a that are annotated represent imaginary axis branch-points of equation (3.40). The systematic procedure for selecting segments of the dispersion curve that lie on the same Riemann sheet is through the addition of a small positive real part to the frequency variable \(p\). These modes are then assigned to the various sheets based on local continuity and the sign of the real part of \(\gamma\).

To demonstrate that each solid curve depicted in the dispersion plot lies on a single proper Riemann sheet, we examine the two red curves in Figure 3.6a. These modes exhibit poles at \(p = 0\) with a frequency variation (see equation (3.41)) of the form:

\[
\gamma d = \pm \frac{\sqrt{\rho_L \rho_C}}{p} \tag{3.45}
\]

Letting \(p = j\omega' + \sigma'\) where \(\sigma'\) is a real positive quantity, it is evident that the real part of \(\gamma\) is positive (and its imaginary part is negative) in the right-half frequency plane when we choose the + sign in equation (3.45). Hence the tail of the solid red curve (below the point marked ‘\(a\)’) is proper while the one plotted using red broken lines is improper.

We increase the frequency and approach the branch-point at ‘\(a\)’ in Figure 3.6a and analytically continue the red curve through a small indentation in the right-half of the complex-frequency plane. Let us imagine plotting dispersion diagrams of the type shown in Figure 3.6a by fixing the real part of \(p\) to a non-zero value and then stacking these plots in order of increasing real value of the complex frequency. Using this arrangement, the required continuation is affected through a semi-circular loop lying on the plane perpendicular to the plot in Figure 3.6a that circumvents the point ‘\(a\)’. There are 4 possible branches\(^{14}\) that are possible continuations of the curve below. We select the

\(^{13}\) These dimensionless quantities can be obtained in practice from a coupler whose parameters are those given in Table 5.1 with the exception of a loading inductance of 5 nH and loading capacitance of 2 pF.

\(^{14}\) There are two segments that join points ‘\(a\)’ and ‘\(c\)’ in Figure 3.6a both of which have the same imaginary part but oppositely signed real parts. The same applies to the curve segment joining points ‘\(b\)’ and ‘\(d\)’.
branch with the negative imaginary part and positive real part to satisfy the requirements of continuity and bounded behaviour at infinite distance away from the source.

We continue tracking the red curve until it reaches the branch-point at ‘c’ above which there are two distinct purely propagating modes that can be reached continuously. One of these modes approach infinity as the frequency is increased while the other intersects the \( \gamma = 0 \) axis at the point labelled ‘e’. In order to verify that the arc joining points ‘c’ and ‘e’ in Figure 3.6a is indeed proper, we write out the first few terms of the series expansion about the first order algebraic branch-point at ‘c’. We divide both sides of equation (3.40) by \( 2(pd)^2 \) and then form the first order Taylor series of the terms inside and outside the radical. Keeping in mind that the term outside the radical is real along the imaginary axis of the \( p \) plane and that the term inside is a simple zero at the branch-point ‘c’, this series expansion is of the form:

\[
\gamma^2 \approx [-\beta_c^2 + jA(p - j\omega_c)] + \sqrt{-jB(p - j\omega_c)}
\]  

(3.46)

In equation (3.46) above, the constants \( \beta_c, \omega_c, A \) and \( B \) are all real of which only \( A \) is allowed to be negative. The branch-point at ‘c’ is indicated by the frequency \( \omega_c \) with the corresponding value of \( \gamma(j\omega_c) = -j\beta_c \). At a frequency point \( p = j\omega^-_c \) just below the branch-point, the term under the radical is negative while at the point \( p = j\omega^+_c \) just above, this term is positive. Hence we obtain complex modal solutions from equation (3.46) below the branch-point and regular propagating modes above it as required. If we set the angle of the quantity \(-jB(j\omega^-_c - j\omega_c)\) to \(-\pi\), then the dispersion curve traced by equation (3.46) below the branch-point corresponds to the proper complex mode indicated by the segment between ‘a’ and ‘c’ in Figure 3.6a.

Now consider the analytic continuation from \( j\omega^-_c \) to \( j\omega^+_c \) through a semi-circular arc in the right-half of the \( p \)-plane that results in the argument of \(-jB(j\omega^-_c - j\omega_c)\) to increase by \(+\pi\). The radical in equation (3.46) is now a positive real quantity and the magnitude of \( \gamma \) is definitely less than that of \(-\beta_c^2 - A(\omega^+_c - \omega_c)\). Hence the segment that joins ‘a’ and ‘c’ with decreasing magnitude of \( \gamma \) is the proper branch lying on the same Riemann sheet.

We extend the red curve to the point ‘e’ and continue along the \( \gamma = 0 \) axis to ‘f’ such that the mode between them has a real positive part. At the branch-point ‘f’, the eigenmode \( \gamma \) is zero but the radical in (3.40) is not. This is is evident by observing that there are 3 distinct modes at this frequency (see Figure 3.6a) and hence the entire
right-hand side of equation (3.40) is analytic and $\gamma$ can be expressed as:

$$\gamma^2 \approx jB(p - j\omega_f)$$  (3.47)

where $B$ is some positive number. In this case we choose the angle of $jB(j\omega_f - j\omega_f)$ to be 0 at a point $j\omega_f^-$ slightly below the branch-point to obtain the proper evanescent mode on the segment between 'e' and 'f'. Analytic continuation around this branch-point through the right-half plane increases the angle by $\pi$ and the resulting $\gamma$ has a positive imaginary part. This line of reasoning can be applied to all other modes shown in Figure 3.6a to deduce that they lie on the same proper or improper Riemann sheet.

### 3.4 Right-half plane branch-points and Nyquist theory

Some structures that support complex modes such as the MS/NRI-TL coupler are associated with right-half plane (R.H.P.) branch-points as pointed out earlier. Yet there are cases when such singularities are absent even when the structure supports complex modes\(^{15}\) [86]. The existence of these singularities can be confirmed easily if one is provided with an analytic eigenmode equation that is valid in the entire complex plane. If the available data is limited to the eigenmode dispersion plot along the imaginary axis alone, then the question is whether we can predict the existence of such R.H.P. branch-

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\(^{15}\)An example of the coupling between the lowest order TE and TM modes of a biaxial waveguide is given in [86] that exhibits a complex-mode band without right-half plane branch-point singularities.
points without exhaustive analytic continuation of the function to the entire complex plane.

We answer this question for the case of 2 lines by adopting the technique due to Harry Nyquist [80] to study the stability of rational transfer functions. Our objective here is to detect R.H.P. branch-points rather than poles and zeroes which was the aim of the original Nyquist technique. Moreover, our eigenmodes are not rational functions. The strategy here is to study the symmetric function $F_3(s)$ defined in equation (3.35) which is rational instead of $\gamma$.

The main concept used here is that of the ‘winding number’ of a curve from complex analysis. Given a closed curve $c.c$ in the complex $z$-plane, the total number of times $N_{c.c}$ that the curve winds itself around the origin (where a positive sign is assigned to a counter-clockwise loop) can be obtained by performing the complex integration:

$$N_{c.c} = \frac{1}{2\pi j} \oint_{c.c} \frac{dz}{z}$$  \hspace{1cm} (3.48)

For equation (3.48) to be meaningful, the curve $c.c$ cannot pass through the origin. As $d(ln(z)) = dz/z$, a closed loop integration of $1/z$ is equal to the change in the imaginary part of the natural logarithm that increases by $2\pi$ every time the curve circumvents the origin in the counter-clockwise direction. If we now map the normalized frequency $p$-plane to the $z$-plane using the function $z = F_3(p)$, then for any Jordan curve $^{16}j.c.$ in the $p$-plane, the corresponding winding number of the image curve in the $z$-plane is given by:

$$N_{j.c} = \frac{1}{2\pi j} \oint_{j.c} \frac{1}{F_3} \frac{dF_3}{dp} dp = \frac{1}{2\pi j} \oint_{j.c} d(ln F_3)$$  \hspace{1cm} (3.49)

If $F_3(p) = \frac{N_3(p)}{D_3(p)}$ is a rational function constructed out of polynomials $N_3(p)$ and $D_3(p)$, then $\frac{1}{F_3} \frac{dF_3}{dp} = \frac{1}{N_3} \frac{dN_3}{dp} - \frac{1}{D_3} \frac{dD_3}{dp}$ whose poles are the zeroes of the numerator and denominator polynomials. These poles have residues equal to the multiplicity of the roots and the sign depends on whether the zero belongs to $N_3$ or $D_3$. As the right-hand side of equation (3.49) measures the number of times $F_3$ circumvents the origin, this number is numerically equal to the difference between the poles and zeroes (weighted by their multiplicity) that are enclosed by the Jordan curve $j.c.$

From our previous discussion, $F_3 = [\gamma_1^2 - \gamma_2^2]^2$ is a rational function and we know that all poles and zeroes of $\gamma$ are confined to the imaginary axis. All finite degeneracies (including branch-points) of $\gamma$ and its poles are the zeroes and poles of $F_3$ respectively.

\(^{16}\)A simple closed curve is referred to as a Jordan curve (i.e. a curve without overlaps).
Chapter 3. Analytic Properties of Complex Modes

Figure 3.8: (a) Frequency contour on the normalized frequency \( p \) plane along which the symmetric function \( F_3 \) defined in equation (3.35) is evaluated. Indentations around imaginary axis branch-points and pole at the origin are shown. (b) Nyquist plot of \( F_3 \) depicting two large loops that encircle the origin in the counter-clockwise direction and a small loop (inset) that encircles the origin in the clockwise direction due to the complex-mode band.

Hence we construct a Jordan curve that runs along the imaginary axis with indentations to the R.H.P. around the poles and zeroes of \( F_3 \) and we close this curve using a large semi-circle that lies in the R.H.P. The number of times \( F_3 \) encircles the origin is now directly a measure of the number of degeneracies of \( \gamma \) in the R.H.P. (we already know that there are no poles in this region). In most cases, it is possible to determine the general characteristics of \( F_3 \) by simply examining the frequency dispersion plot of the two modes (see Appendix E for details).

In Figure 3.8 we present the portion of the Nyquist plot that corresponds the Jordan curve in to the top half \( p \)-plane. The rest of the plot will be the mirror image of the one shown in Figure 3.8b about the real axis. The Nyquist plot shown in this figure corresponds to the dispersion diagram and the associated branch-points shown in Figure 3.6. Hence the Jordan curve is indented around all the singular points of \( \gamma \) that are shown in Figure 3.6.

We begin plotting \( F_3 \) by sweeping \( p \) from a point on the real axis near the origin.
In Figure 3.8a, observe a quarter circular loop that avoids the pole at the origin. From the dispersion diagram in Figure 3.6a it is clear that only one of the modes approaches infinity as \( p \to 0 \). Hence \( F_3 \) exhibits a 4-th order pole thereby mapping the counter-clockwise quarter circular arc to a loop that encircles the origin of the \( F_3 \) plane in the clockwise direction. This portion of the mapping is depicted in Figure 3.8b by the curved segment that starts from the right-most portion of the real axis and swirls to the point close to the origin.

As the Jordan curve on the \( p \) plane is swept along the imaginary axis towards the first branch-point, the corresponding image on the \( F_3 \) plane, moves along the real axis towards the origin (see inset of Figure 3.8b). The first branch-point is encountered at around \( p = 0.5 \), and the loop taken to avoid it is mapped to the large arc that is shown in the inset of Figure 3.8b. This arc bridges from the positive real axis to the negative real axis of the \( F_3 \) plane and this is to be expected from the behaviour of the mapping near a transition from regular modes to complex modes (see Appendix E).

When \( p \) is swept along the complex-mode band (between \( p = 0.5 \) and \( p = 0.7 \)), these points are mapped to the negative real axis. The loop around the branch-point at \( p = 0.7 \) appears as the small arc directly below the largest one inset of Figure 3.8b. Hence the process of traversing past a complex-mode band results in a loop that encircles the origin once in the Nyquist diagram.

The rest of the branch-points (near \( p = 0.8 \) and \( p = 0.9 \)) are neither poles nor zeroes of \( F_3 \) because \( |\gamma_c| \neq |\gamma_\pi| \) (see Figure 3.6a) and hence the loops around these points do not circumvent the origin of the \( F_3 \) plane. Finally, the large quarter circular loop from the imaginary axis to the real axis of the \( p \)-plane maps to the circle that encloses the origin once in the \( F_3 \) plane.

The two large loops that encircle the origin in Figure 3.8b are in the clockwise direction while the small loop inset is in the counter-clockwise direction. Hence if we were to include the entire Jordan curve in the Nyquist plot, the net winding of the curve would be twice in the clockwise sense. This indicates the presence of two degenerate points one of which is shown as the R.H.P. branch-point in Figure 3.6b.

If a structure supports complex modes, then each portion of the Jordan curve in the vicinity of the complex band, gets mapped to a loop that encircles the origin. The pole at infinity contributes to at most two loops that circulate the origin. If the pole at the origin was missing in our case, the winding number of the Nyquist plot would be zero. This is precisely the case in the complex mode examples presented in [86] where the
structures lacked R.H.P. branch-point singularities. Hence we have outlined a procedure that allows one to examine the dispersion characteristics of a coupled system to determine the presence of degeneracies in the R.H.P. The presence of degeneracies that are branch-points in the R.H.P. have subtle implications on the excitability of modes which possess them. This will be the subject of the next chapter.
Chapter 4

Independent Excitation of Complex Modes

4.1 Historical perspective

The discovery of the existence of proper complex modes in guided wave structures since the early 60’s have been accompanied by the belief that a single mode out of a pair of conjugate modes can never be excited independently, even at a single frequency. This long standing belief has been championed by many authors despite being at odds with the well known fact that such a pair of conjugate modes are orthogonal [68] and hence one should be able to formulate a current distribution across a guide cross-section which excites only one of them [79].

A serious objection against the independence of a pair of conjugate modes was raised by J. D. Rhodes [81] due to the presence of temporal right-half complex-plane branch points who stated that these complex modes are not only dependent but that they must produce fields with equal amplitudes. In the example of an open plasma slab, it was believed that a single complex-mode on its own violated power conservation in the transverse plane [75]. Integral formulation of the excitation of structures supporting complex modes seemed to indicate that for any source configuration, the complex wavenumber poles corresponding to both modes contributed to the discrete spectral fields [75][76]. The fact that a complex mode on its own carried neither real nor reactive power was also considered as a possible reason why they couldn’t be excited on their own [77].

In the time-harmonic analysis of the excitation of reactive surfaces with impressed current sources, the poles corresponding to the guided complex-waves and those due to
the improper leaky waves appear on different Riemann sheets of the complex spatial
wavenumber and are connected by Sommerfeld branch-cuts (see for example the chapter
by A. Hessel in [33]). Their allocation to the appropriate Riemann sheet is governed by
the behaviour of the field at infinite distance corresponding to the pole. Although one
cannot strictly discuss the excitation of a leaky-wave pole in the same manner as the
excitation of a complex-mode pole, the former can play a significant role in determining
the far-field radiation pattern. This is evident from the evaluation of the radiation integral
along the steepest-descent contour in the complex wavenumber plane. The deformed
contour is moved on this plane based on the observation angle of the far-field radiation,
and hence one can correlate peaks in the radiation pattern with the proximity of the leaky
poles to the integration contour. It is a known fact that only a single leaky pole out of a
conjugate pair\(^1\), at least for the case of the plasma-slab mentioned earlier, influences the
far-field pattern [75]. On a fundamental level, the strength of the excitation of proper
complex waves or improper leaky waves is gauged by deforming the field integration
contour on either the proper or the improper Riemann sheet and evaluating the residues
of the captured poles. It is not obvious why we should believe that an individual mode
in a pair of conjugate complex waves cannot exist on its own, whereas a leaky-wave pole
can.

We will address all of these concerns and show how to independently control the exci-
tation amplitudes of a pair of conjugate modes in practice in lossless isotropic (reciprocal)
guides. Specifically, we will show how one of the modes in a pair can be suppressed over
a finite bandwidth and completely eliminated at a discrete set of frequency points.

In order to investigate the excitation of complex modes, we will use the example of
the coupling between a regular forward-wave microstrip Transmission Line (MS-TL) and
a backward-wave metamaterial Negative-Refractive-Index Transmission Line (NRI-TL).
The fact that such a coupler supports complex modes was first pointed out in [5] and
it is probably the simplest and most convenient platform to study their existence and
excitation as the total number of modes are limited [87]. The excitation amplitudes
of each mode under generic source and boundary conditions can then be investigated
with ease, devoid of unnecessary cluttering by infinite number of modes in waveguide
problems.

\(^1\)By conjugate leaky poles, we are referring to those field solutions whose phase increases or de-
creases at the same rate away from the excitation surface but with increasing amplitude towards infinity
(improper waves).
In this chapter we develop the necessary theoretical framework to show that the excitation amplitude of each complex mode in a conjugate pair can, for all practical purposes, be independently controlled. This result is of fundamental importance when we develop the scattering parameters of the MS/NRI-TL coupled-line coupler in its complex-mode band. Hence the theory presented in this section will justify the independent linear superposition of its four conjugate complex modes to satisfy four arbitrary boundary conditions (sources and impedance terminations) in Chapter 5.

### 4.2 Complex modes and temporal frequency branch-points

The linear independence/orthogonality of complex-conjugate modes as expressed through equations (3.10) and (3.18) should be sufficient to guarantee independent excitation of each constituent mode within a pair, at least at a single frequency. A compelling objection against this modal independence was put forward by Rhodes [81] who showed that when these eigenmodes are analytically continued from the \( s = j\omega \) axis into the right-half complex-frequency plane, one encounters branch-point singularities which require that the two modes must be excited equally. In this section, we will discuss the implications of these right-half plane (R.H.P.) branch-point singularities.

Consider a simple example involving only two scalar modes \( \gamma_c \) and \( \gamma_\pi \) which are complex-conjugate quantities over a finite portion of the imaginary axis and share first-order R.H.P. branch-points along with the associated branch-cuts. The time domain ‘field’ due to these two modes is obtained by inverting their Laplace-domain description using the Bromwich integral with an appropriate choice of the region-of-convergence (R.O.C.)[88]:

\[
E^+(t) = \frac{1}{2\pi j} \int_{Br} a_c(s)e^{-\gamma_c(s)z}e^{st}ds + \frac{1}{2\pi j} \int_{Br} a_\pi(s)e^{-\gamma_\pi(s)z}e^{st}ds \tag{4.1}
\]

In (4.1), \( a_c(s) \) and \( a_\pi(s) \) are single-valued analytic excitation amplitudes for each mode. The + superscript on \( E \) indicates that we are considering only the region \( z > 0 \). The Bromwich contour (denoted as \( Br \)-path in Figure 4.1) runs parallel to the imaginary axis and its location is dictated by physical considerations such as causality and stability. Assuming that the allowable time-domain sources are sufficiently restricted\(^2\) to model...
Figure 4.1: The Bromwich inversion integral (a) Region of convergence defined to the left of the R.H.P. branch-points. (b) Region of convergence defined to the right of all integrand singularities.

realistic scenarios, the original integral maybe deformed as depicted in Figure 4.1.

In Figure 4.1(a), the R.O.C. is assumed to be on the left of the depicted branch-points and the $Br$-path integral is taken over the $j\omega$ axis. In this case, for $t < 0$ it is possible to deform the contour in the R.H.P. such that the contribution from the depicted large arc approaches zero. If the integrand is multi-valued with branch-points in the R.H.P., then the original integral is numerically equal to the non-zero branch-cut integrals (assuming no other singularities in the R.H.P.). This would indicate non-zero fields for $t < 0$ during which the excitation sources are turned-off (i.e. a non-causal response).

On the other hand, if the R.O.C. (accompanied by the $Br$-path) is taken to the right of all singularities in the R.H.P., the signal is causal although unstable for $t > 0$. This can be seen from Figure 4.1(b) where the $Br$-path is closed in the left-half plane ($t > 0$) and the branch-cut integral contribution in the R.H.P. is divergent.

A simple solution to this R.H.P. branch-point dilemma was presented by Rhodes requiring that a set of complex modes which are connected through branch-cuts, should all be excited with equal amplitudes [81]. An arbitrary guide in general can have multiple sets of complex modes; each set is distinguished by a lack of branch-cut connectivity.

When the original Bromwich contour (assumed to be on the $s = j\omega$ axis) is deformed into the R.H.P. as shown in Figure 4.1(a), each integrand in (4.1) is analytically continued
Figure 4.2: Two Riemann sheets of first-order branch-points. Width of lower half-plane cuts on both sheets has been exaggerated to visualize continuity of a multi-valued integrand across a branch-cut.

on the Riemann sheet where it has been ‘uniformized’ [85]. Hence the original integral can be replaced by branch-cut integrals on different Riemann sheets as depicted in Figure 4.2. In our example, the \( c \)-mode integral is evaluated as the branch-cut integral on sheet 1 and the \( \pi \)-mode integral is evaluated on sheet 2.

When we are dealing with modes that look quite distinct along the imaginary axis, but are connected by branch-points, then from an analytic point of view, the notion of considering them as distinct entities is quite superficial. Hence we may replace \( \gamma_c(s) \) and \( \gamma_\pi(s) \) in equation (4.1) by simply \( \gamma(s) \) as long as we indicate on which Riemann sheet each integral should be evaluated. The topography of the two Riemann sheets depicted in Figure 4.2 should be interpreted in the following manner. The mapping \( s \rightarrow \gamma \) is continuous along any smooth curve joining two points that may or may not lie on the same sheet. Hence if we start at a point on the imaginary axis on sheet 1 and follow a winding path through the branch cuts into the corresponding point on sheet 2, the function \( \gamma(s) \) varies continuously. Using this construct, there is no reason for \( \gamma \) to yield identical values along the imaginary axes of the two sheets. The mapping \( \gamma(j\omega) \) on sheet 1 and sheet 2 were distinguished previously by functions \( \gamma_c(j\omega) \) and \( \gamma_\pi(j\omega) \) respectively before becoming aware of the bigger picture in which they are simply different branches of the same multivalued function.

When the function \( e^{-\gamma(s)z} \) is evaluated along the branch-cut lip marked as \( a_1 \) on sheet 1 and \( b_2 \) on sheet 2 (see Figure 4.2), the resulting values are identical. Similarly, the two branches are also equal along lips \( b_1 \) and \( a_2 \). If we let the single-valued analytic
amplitude $a_c(s = j\omega)$ equal to $a_x(s = j\omega)$, analytic continuation into the R.H.P. results in an amplitude function which is equal on both sides of the branch-cut (continuity of single-valued analytic functions). Hence the two branch-cut integrals in (4.1) cancel out owing to the reversal of the integration direction when going from sheet 1 to sheet 2.

This is the essence of the argument proposed by Rhodes explaining why conjugate complex mode pairs must be excited with equal amplitudes. The issues of instability or violation of causality is tied to the integration around the R.H.P. singularities and in the present case, by equal excitation of the two modes, this integral is nullified.

4.3 Limitations placed by R.H.P. branch-points towards modal excitation by realizable sources

In this section, we will demonstrate that the ideas proposed in [81] to resolve the issues associated with R.H.P. branch-point singularities are too restrictive and unnecessary. To accomplish this, we present two main results here. First, we prove that in a problem that involves expanding a realizable source function in terms of the eigenmodes of a system (that are multivalued in general), the necessary and sufficient condition to ensure consistency of the process is that the modal amplitudes must contain the branch-points that are present in the eigenmodes but not in the source function. Next we prove that even though it is impossible to completely eliminate an eigenmode (over a non-zero bandwidth) which contains branch-point singularities that are not shared by a realizable source, it is always possible to design such a source that can arbitrarily suppress the mode over a given bandwidth.

The theorems in this section are original results to the best of the author’s knowledge. The author is aware of a result similar to Theorem 2 involving Padé approximants [89] without the restrictions that have been placed in this work on the locations and residues of the poles such that the inverse Laplace transform of the resulting complex function corresponds to a physically realizable source function.

Let $X$ denote the Laplace-domain state vector of a system, $\Theta$ a square matrix and $G$ the source vector located at a point $z = 0$ along the axis of the guided-wave structure modelled by the first-order linear differential equation:

$$ \frac{dX}{dz} + \Theta X = \delta(z)G $$

(4.2)
Proposition 1. We will assume that $\Theta$ is a single-valued analytic function of the complex-frequency variable $s$ and that all of its eigenvalues are distinct in the neighbourhood of some frequency $s_0$.

Definition 1. A realizable source is a function of time $t$ that is zero for $t < 0$, and is bounded and Lipschitz continuous for all $t$.

Let $G$ be a realizable source. We can expand the state vector in terms of the eigenvalues $\gamma_m$ (of $\Theta$) and their associated eigenvectors $X_m$ in the following manner:

$$X(z, s) = \sum_m a_m^+ X_m^+ e^{-\gamma_m z} \quad z > 0 \quad (4.3)$$

$$X(z, s) = \sum_m a_m^- X_m^- e^{\gamma_m z} \quad z < 0 \quad (4.4)$$

The complex expansion coefficients $a_m$, the eigenvalues and the eigenvectors above are functions of $s$ only and the positive and negative signs in the superscripts refer to proper and improper modes respectively. We can associate the state vector $X$ with an array of line-voltages followed by line-currents in a coupled transmission-line problem and the source function can be imagined as a $T$-network whose series branches contain ideal voltage sources and the shunt branch is an ideal current source. It is also possible to relate the state vector to the field expansion coefficients in an arbitrary inhomogeneously-filled waveguide problem as discussed by Omar and Schunemann in [68]. The discussion that follows may be extended to open-waveguides although the associated modal expansion will involve both the discrete spectrum and a continuous one$^3$. The excitation amplitudes are related to the source vector and its subsequent multiplication by the matrix $\Theta$ by:

$$\Theta^n G = \sum_m \gamma_m^n (a_m^+ X_m^+ + (-1)^{n+1} a_m^- X_m^-) \quad (4.5)$$

We observe that the left-side of equation (4.5) is single-valued and analytic in the right-half plane while the $\gamma_m$ terms in the right-side of the equation may contain branch-point singularities in this portion of the frequency plane. The trivial resolution to this dilemma is the assertion that the amplitudes of the modes associated with such singularities are zero. If on the other hand these multivalued eigenmodes are combined like the symmetric functions discussed in Section 3.3.2, then the unwanted singularities disappear. Let the

$^3$As we are only interested in the problem of the excitation of discrete modes associated with branch-point singularities, the continuous spectrum of an open structure is irrelevant to the discussion that follows.
Chapter 4. Independent Excitation of Complex Modes

Figure 4.3: Illustration of the analytic continuation around a second order branch-point $s_b$ that results in a permutation of the germs of the eigenmodes $\gamma$ and their associated amplitudes $a$ such that this singularity is absent from the source $G$. Note that the shaded regions in the $\gamma$ and $a$ planes are the mappings of the circle shown in the $s$ plane.
germs\textsuperscript{4} \gamma_m \) for \( m = 1, 2, \ldots, k \) be connected through a \((k - 1)\)-th order branch-point at \( s = s_b \) in the right-half plane. Let us define the permutations \( P = (\gamma_1 \gamma_2 \ldots \gamma_k) \), \( Q = (a_1^+ X_1^+ a_2^+ X_2^+ \ldots a_k^+ X_k^+) \) and \( R = (a_1^- X_1^- a_2^- X_2^- \ldots a_k^- X_k^-) \) that cyclically replaces the germs in the order they appear inside the brackets. The duplicate association of symbols such as \( \gamma_m, X_m \ldots \) etc. to both points and germs is to be noted and the context will determine the intended use of the symbols. The definition of \( P \) is intended to convey the sequence of conversions between the germs \( \gamma_m \) \(( m = 1, 2, \ldots, k \) upon a single closed-loop analytic continuation around \( \omega_b \). If this closed loop analytic continuation also results in the permutations \( Q \) and \( R \), then the right-side of (4.5) remains unchanged — the various terms inside the summation are merely shuffled in their ordering (see Figure 4.3 for an illustration involving a second order branch-point). Verification of this observation will be provided in later sections when we consider actual examples.

The simultaneous applications of permutations \( P, Q \) and \( R \) is a sufficient condition to preserve the single-valued nature of equation (4.5). Now we will prove that this condition is necessary as well. For convenience of notation, we drop the positive and negative superscripts in equation (4.5) and assume that the summation now runs over both the proper and the improper eigenvalues:

\[
\Theta^n G = \sum_{m=1}^{N} \gamma^m a_m X_m
\]

**Theorem 1.** Let \( s_b \) be a branch-point that is common to a subset \( S \) of \( \gamma_m \) in equation (4.6). The necessary and sufficient condition that this singularity is absent from the sum in equation (4.6), is that each factor that multiplies \( \gamma_m \) in \( S \) must permute in the same order as the elements of \( S \) upon closed-loop analytic continuation around the branch-point \( s_b \). The factors that multiply the complement of \( S \) cannot contain this branch-point.

**Proof.** The proof of sufficiency was given at the beginning of this section. To prove that the condition above is also necessary, define \( P(m) \) to be the \( m \)-th element when the
permutation \( P \) is applied to the set \( \{1, 2, \ldots, N\} \). Hence the set \( \{\gamma_{P(m)}\} \) is a rearrangement of the elements in the set of eigenvalues \( \{\gamma_m\} \) (note that this set now contains both proper and improper eigenvalues) that results from a single closed-loop analytic continuation around the branch-point \( s_b \). The locations of the eigenvalues corresponding to those germs that aren’t connected through \( s_b \) remain fixed in the mentioned sets. Now let \( \sum_r b_{rm} X_r \) represent the transformation of the vectors \( a_m X_m \) under the same continuation. This continuation leaves the left-side of equation (4.6) unchanged and hence the following must hold:

\[
\sum_m \gamma_m^n a_m X_m = \sum_m \gamma_{P(m)}^n \sum_r (b_{rm} X_r) \tag{4.7}
\]

Comparing the coefficients of \( X_m \) on both sides of equation (4.7):

\[
\gamma_m^n a_m = \sum_r \gamma_{P(r)}^n b_{mr} = \sum_r \gamma_r^n b_{mP^{-1}(r)} \tag{4.8}
\]

\[
\sum_{r \neq m} \gamma_r^n b_{mP^{-1}(r)} + \gamma_m^n (b_{mP^{-1}(m)} - a_m) = 0 \tag{4.9}
\]

The expression in equation (4.9) is valid for all \( n \) and hence the coefficients that accompany \( \gamma_r^n \) must be zero (this fact is elaborated upon at the end of this proof):

\[
b_{mP^{-1}(m)} = a_m \quad \forall \ m
\]

\[
b_{mP^{-1}(r)} = 0 \quad r \neq m \tag{4.10}
\]

The results expressed in equation (4.10) can be restated as \( b_{P(m)m} = a_{P(m)} \) and that \( b_{mr} = 0 \) if \( P(r) \neq m \). Hence when \( \gamma_m \) transforms to \( \gamma_{P(m)} \), the term \( a_m X_m \) transforms to \( \sum_r b_{rm} X_r = a_{P(m)} X_{P(m)} \).

A final detail that remains is to show that the coefficients of \( \gamma \) in equation (4.9) must be zero. As this equation is valid for all \( n \), one can list it for \( n = 0, 1, 2, \ldots, (N - 1) \) and express the system of equations in a matrix form \( \Gamma Y = 0 \), where the rows of the matrix \( \Gamma \) are of the form \( [\gamma_1^n \gamma_2^n \ldots \gamma_N^n] \) and the column vector \( Y \) contains the amplitude coefficients \( b_{mn} \) and \( a_m \). If the determinant of the matrix \( \Gamma \) is zero, then coefficients \( c_n \) can be found such that \( \sum_{n=0}^{N-1} c_n [\gamma_1^n \gamma_2^n \ldots \gamma_N^n] = 0 \). This implies that all eigenmode \( \gamma_m \) are obtained as roots of the polynomial \( c_0 + c_1 x + \ldots + c_{N-1} x^{N-1} = 0 \). This violates the assumption made in Proposition 1 that all \( N \) eigenmodes are distinct at \( s_0 \) (the frequency point about which the expansion in equation (4.6) is evaluated). Hence the matrix \( \Gamma \) is invertible and setting the amplitude coefficients to zero is justified.
We have established that the amplitudes of the eigenmodes that are connected by a R.H.P. branch-point must be different branches of a multi-valued function sharing the same branch-point as the eigenmodes. At a fixed point $s_0$ on the imaginary axis of the complex-frequency plane, these amplitudes can be independently set to arbitrary values (including zero) by a proper choice of the source function as seen from equation (4.5) (by setting $n = 0$). A problem of significant interest is whether any given mode that exhibits such singularities can be suppressed to arbitrarily small values over a finite neighbourhood of $s_0$ using realizable sources. It should be noted that it is impossible to set one such modal amplitude identically to zero throughout a finite disk (or line segment) in the complex-frequency plane surrounding $s_0$ without causing all other connected branches to be zero as well. This is a direct consequence of the result that the zeros of an analytic function are isolated points. Now we will show that it is possible to synthesize a realizable source that can suppress any given mode (containing a R.H.P. branch-point) in a finite sized disk surrounding a frequency $s_0$.

**Lemma 1.** Consider a real square matrix $M = [F(x_1) \ F(x_2) \ F(x_3) \ldots]$ where $F(x)$ is a column vector that is a function of the real scalar $x$. If $F$ is differentiable over a finite interval and all of its components are non-zero linearly independent functions, then it is always possible to choose a set of points $x_1, x_2, x_3 \ldots$ in that interval that make $M$ invertible.

**Proof.** Let the vector $F$ be of dimension $n + 1$. The lemma above can be proven using induction. A point $x_1$ can be found such that the first component of $F(x_1)$ is non-zero and hence constitutes a trivial $1 \times 1$ invertible matrix. Let $M'$ be an invertible $n \times n$ matrix obtained from the first $n$ components of $F(x)$ evaluated at the points $x_1, x_2, \ldots, x_n$ all of which will be held fixed. We need to show that a point $x_{n+1}$ can be found such that $M = [F(x_1) \ F(x_2) \ldots F(x_{n+1})]$ is invertible. Assume at first that no such point exists, and hence for all $x$, $\det[F(x_1) \ F(x_2) \ldots F(x)] = 0$. This would imply that the column vectors are linearly dependent allowing the linear combination $F(x) = \lambda_1(x)F(x_1) + \lambda_2(x)F(x_2) + \ldots + \lambda_n(x)F(x_n)$ where at least one of the coefficients $\lambda_i(x)$ is non-zero. The matrix $[F(x_1) \ F(x_2) \ldots F(x_n)]$ is of rank $n$, and hence Gaussian elimination can be employed to set one of its rows to zero. Referring to the equation $F(x) = [F(x_1) \ F(x_2) \ldots F(x_n)][\lambda_1(x) \ \lambda_2(x) \ldots \ \lambda_n(x)]^T$, it is seen that a linear combination of the components of $F(x)$ can be set to zero thereby contradicting the assumption that its components are linearly independent functions. \hfill $\blacksquare$
Definition 2. A function $F_a$ will be defined as an $n$-th order approximant to another function $F$ over an open disk centred at a point $j\omega_0$ on the imaginary axis of the complex plane if the first $n+1$ terms of their Taylor series are identical and convergent on the disk. Moreover, $F_a$ will be defined as a realizable approximant if it is the Laplace transform of a real, bounded and Lipschitz continuous function of $t$ and is zero for $t < 0$.

Theorem 2. There exists a realizable $n$-th order approximant to any analytic function at a regular point $j\omega_0$ on the imaginary axis of the complex plane for an arbitrary natural number $n$.

Proof. We begin by considering the following rational function whose simple poles are all restricted to the imaginary axis:

$$F_a(s) = \sum_m \left[ \frac{a_m}{s - j p_m} + \frac{a_m^*}{s + j p_m} \right]$$ (4.11)

In equation (4.11), $p_m \in \mathbb{R}^+$ and $a_m \in \mathbb{C}$ and if we set $\sum_m a_m = 0$, then $\sum_m \text{Re}(a_m)$ will also vanish, and the inverse Laplace transform of $F_a$ will be continuous at $t = 0$. It is clear that if $F_a$ is an approximant to a function $F$, then it will also be a realizable approximant.

A subtle difference between the rational approximants to impedance functions in electrical network synthesis and the one being considered here is to be noted—the residues of the imaginary axis poles $a_m$ above are allowed to be complex. Differentiating equation (4.11) $n$ times at $s = j\omega_0$ and equating each to the corresponding derivative of $F$:

$$F^{(n)}(j\omega_0) = \sum_m -n!(j)^{n+1} \left[ \frac{a_m}{(\omega_0 - p_m)^{n+1}} + \frac{a_m^*}{(\omega_0 + p_m)^{n+1}} \right]$$ (4.12)

Noting that $\omega_0 \in \mathbb{R}^+$, we can equate the real and imaginary parts of both sides of equation (4.12) and the following systems of equations are obtained:

$$\sum_m \text{Re}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^{1}} + \frac{1}{(\omega_0 + p_m)^{1}} \right] = -\frac{1}{0!} \text{Im} [F^{(0)}(j\omega_0)]$$

$$\sum_m \text{Re}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^{2}} + \frac{1}{(\omega_0 + p_m)^{2}} \right] = +\frac{1}{1!} \text{Re} [F^{(1)}(j\omega_0)]$$

$$\sum_m \text{Re}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^{3}} + \frac{1}{(\omega_0 + p_m)^{3}} \right] = +\frac{1}{2!} \text{Im} [F^{(2)}(j\omega_0)]$$

$$\sum_m \text{Re}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^{4}} + \frac{1}{(\omega_0 + p_m)^{4}} \right] = -\frac{1}{3!} \text{Re} [F^{(3)}(j\omega_0)]$$

$$\vdots$$

$$\sum_m \text{Re}(a_m) = 0$$ (4.13)
\[ \sum_m \text{Im}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^1} - \frac{1}{(\omega_0 + p_m)^1} \right] = + \frac{1}{0!} \text{Re} \left[ F^{(0)}(j\omega_0) \right] \]
\[ \sum_m \text{Im}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^2} - \frac{1}{(\omega_0 + p_m)^2} \right] = + \frac{1}{1!} \text{Im} \left[ F^{(1)}(j\omega_0) \right] \]
\[ \sum_m \text{Im}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^3} - \frac{1}{(\omega_0 + p_m)^3} \right] = - \frac{1}{2!} \text{Re} \left[ F^{(2)}(j\omega_0) \right] \]
\[ \sum_m \text{Im}(a_m) \left[ \frac{1}{(\omega_0 - p_m)^4} - \frac{1}{(\omega_0 + p_m)^4} \right] = - \frac{1}{3!} \text{Im} \left[ F^{(3)}(j\omega_0) \right] \]
\[ \vdots \]
\[ \sum_m \text{Im}(a_m) = 0 \quad (4.14) \]

If we can find \( p_m \)'s such that the matrices corresponding to equations (4.13) and (4.14) are invertible, then we can solve for the residues and the proof is complete. The pole closest to \( j\omega_0 \) should be placed (if possible) at a distance equal to or greater than the radius of convergence of the Taylor series of \( F \) at this point. We observe that the matrices involved in this problem are invertible since the associated column vectors (with linearly independent components of the form \( \frac{1}{(\omega_0 - p)^n} \pm \frac{1}{(\omega_0 + p)^n} \)) are of the type described in Lemma 1 as long as \( \omega_0 \neq 0 \). Thus we can compute all parameters in equation (4.11) to obtain a realizable approximant to any analytic function.

**Definition 3.** We define the arbitrary suppression of an eigenmode \( \gamma_m \) in a closed disk centred at \( s = j\omega_0 \) as the condition that \( |a_m| < \epsilon \) in equation (4.6) (for all \( s \) in this disk) for any real \( \epsilon > 0 \).

**Theorem 3.** It is possible to excite a single eigenmode in a finite closed disk (not containing any singularities) centred at a point \( j\omega_0 \) on the imaginary axis of the complex frequency plane, and arbitrarily suppress all others (in equation (4.6)) using a realizable source \( G \).

**Proof.** Set \( n = 0 \) in equation (4.6) and let \( G = X_p \) for some integer \( p \). As the eigenvectors in the expansion are all linearly independent, the solution to this system is \( a_m = 0 \) for all \( m \neq p \). This is also the case when one substitutes in \( G = X_p \) in the corresponding matrix equation:
\[ \langle G, X_\mu \rangle = \sum_m a_m \langle X_m, X_\mu \rangle \quad (4.15) \]

The inner-product used in this context is of the Hermitian type. Equation (4.15) defines a linear, continuous and invertible map between the vector \( \{ \langle G, X_\mu \rangle \} \) and \( \{ a_m \} \) in an
open disk containing no singularities and centred at the point \( j\omega_0 \) on the imaginary axis. The fact that the mapping is continuous and invertible is obvious from the boundedness\(^5\) of \( \langle X_m, X_\mu \rangle \) and the linear independence of the eigenvectors. Equation (4.15) can be modified to:

\[
\langle G - X_p, X_\mu \rangle = \sum_m (a_m - b_m) \langle X_m, X_\mu \rangle
\]  

(4.16)

where \( b_m = 0 \) for all \( m \neq p \). The continuity of the map ensures that it is possible to make \( \| \{a_m - b_m\} \| < \epsilon \) for any \( \epsilon > 0 \) when \( \| \{\langle G - X_p, X_\mu \rangle\} \| < \delta \) for some \( \delta > 0 \) (here \( p \) is fixed and \( m, \mu = 1, 2, \ldots, N \)). The condition \( \| \{\langle G - X_p, X_\mu \rangle\} \| < \delta \) can be met by setting \( \|G - X_p\| < \delta / \max ||X_\mu||/\sqrt{N} \) and this follows from the application of the Cauchy-Schwarz inequality (here \( N \) is the dimension of the vector space). Hence, it is sufficient to set the magnitude of each component of \( G - X_p \) to be less that \( \delta / \max ||X_\mu||/\sqrt{N} \) and from Theorem 2 we know that there exists an \( n \)-th order realizable approximant (of the components of the vector \( G \)) to each component of \( X_p \) in an open disk centred at \( j\omega_0 \). Inside this disk, the Taylor series of each component of \( X_p \) is convergent, and hence the residual series corresponding to each component of \( G - X_p \) can be made arbitrarily small (in a closed disk contained inside the open one where the series converges) using sufficiently large \( n \). When this is accomplished, \( \| \{a_m - b_m\} \| < \epsilon \) and \( |a_m| < \epsilon \) for all \( m \neq p \).

\[\blacksquare\]

A few observations about the results obtained in this section are as follows. The result that the amplitudes of the undesired modes can be arbitrarily suppressed in a finite bandwidth doesn’t imply that they can be identically set to zero. In order to decrease an unwanted modal amplitude to an arbitrarily small level, a large number of terms in the Taylor series of the desired mode function and its approximant have to agree which in turn will require an increase in the number of poles in the approximant. As the realizable source \( G \) is assumed to be analytic and Lipschitz continuous (i.e. there is an upper limit on how fast the source can change with time), there is an upper-bound on the distance of the imaginary-axis poles from the origin. If an infinite number of terms are used in the approximant, the Bolzano-Weierstrass theorem states that there must be an accumulation point for these poles thereby violating the analyticity condition of the source. This can be intuitively understood by considering the Fourier series expansion of a periodic square wave that oscillates between zero and one. The truncated series can

---

\(^5\)Continuity is equivalent to boundedness for any linear operator [91].
approximate the zero region of the square wave to an arbitrary level of accuracy and is an entire analytic function. In the limit, such a series converges to a function which is discontinuous and doesn’t meet the realizability criteria in Definition 1.

There are techniques in literature for approximating a given analytic function using rational functions such as through the use of the Padé approximants [89]. A few others, including time-domain ones are outlined in network synthesis texts such as the one by Guillemin [35]. The approach used in Theorem 2 is more restrictive than Padé approximants since the latter requires no constrains on the symmetry of the pole distribution and their residues. On the other hand, this technique is more general than most commonly encountered synthesis techniques in network theory literature where the problem of simultaneously approximating the magnitude and phase of a function is addressed infrequently.

There are claims in literature that a complex mode cannot be excited on its own due to the presence of right-half plane branch-points by any realizable source. Moreover, it has been frequently suggested that these modes must be excited with equal amplitudes [75][76][81]. Also, it has been suggested that a backward mode cannot be excited with the presence of a forward mode. In other words a negative index has to be always associated with a positive index solution over the same frequency band [92].

In light of the results obtained in this section, a revision of these old assertions is warranted. Any analytic eigenmode, complex or otherwise, will possess a region where it can be defined using a Taylor series and in this region, a realizable source exists according to Theorem 3 that can suppress all of the undesired modes. In other words, it is practically possible to independently control the amplitudes of each individual complex mode in a conjugate pair.

Most of the essential results in this section, may be extended heuristically to the case of infinite dimensional operators that allow an eigenmode expansion of the type in equations (4.3) and (4.4). For instance, such an expansion is common in closed waveguide problems and it is obvious that the sufficiency condition regarding the permutation of the modal amplitudes applies here as well. To address the question on whether a certain mode with the undesirable singularities can be suppressed over a finite bandwidth using a realizable source, one will first have to define a realizable source for the waveguide problem. Let us define it as an arbitrary current distribution across the guide cross-section such that the time-domain behaviour at each point follows definition 1. At each frequency, one can formulate a cross-sectional current distribution that excites a single mode [79] and we
may assume that the transverse profile of these modes change in a continuous manner with frequency under the condition that the operator is an analytic function of $s$. Hence to suppress all undesired modes, we need to formulate scalar functions of time at each spatial point on the transverse plane, such that it provides the desired approximant to the Laplace domain variation of the eigenfunction at the same point. A rigorous exposition of the continuity assumption (with respect to $s$) regarding the change in each eigenmode’s transverse profile is beyond the scope of this dissertation, but this assumption seems like a reasonable one.

At this point, it is worthwhile to compare and contrast the results of the derivations above to the claims made in [81]. The assertions that complex conjugate modes are not independent and that they must be excited with equal amplitudes are merely conjectures in [81] extrapolated from observations of particular cases involving the numerical simulation of inhomogeneous waveguides that support such modes. These conjectures were stated by only investigating the analytic properties of the source-free modes without considering the importance of introducing the source into the system. Hence by showing that there is indeed a class of problems where the claims in [81] are inapplicable, we have disproved the universality of the argument that a pair of conjugate complex modes must be excited equally.

4.4 Complex-mode amplitudes in waveguides and transmission lines

In this section, we will show how the general results of section 4.3 apply to practical problems such as semi-infinite waveguides and coupled transmission-lines excited by current/voltage sources and terminated by impedances (in the latter case).

In the following discussion, we will use the notation used in sections 3.2.1 and 3.2.2 to describe the modal expansion in uniform inhomogeneous waveguides and coupled transmission lines. Let us denote by $a_n(\omega) F_n(\omega)e^{-\gamma_n(\omega)z}$ the $n$-th modal expansion term where the amplitudes $a_n$, the modal function$^6 F_n$ and the modes $\gamma_n$ are all functions of harmonic frequency $\omega$ measured along the imaginary axis of the complex-frequency plane.

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$^6$This modal function can be the transverse electric and magnetic field function for the waveguide problem or a vector of voltages and currents for the coupled transmission-line problem.
We will now demonstrate that if two non-degenerate modes \( n = p \) and \( n = q \) are connected by branch-cuts, then their corresponding amplitude functions \( a_p \) and \( a_q \) are simply two branches of the same multivalued analytic function that shares the same branch-points and cuts as \( \gamma_p \) and \( \gamma_q \). This multivalued nature of the amplitudes must be recognized even though the voltage sources, current sources and impedance terminations involved in the problem are all considered to be single-valued analytic functions of frequency.

Analytically continuing along a closed loop starting from a point \( s = j\omega \) and encircling one branch-point carries \( \gamma_p \) to \( \gamma_q \) and vice-versa. Similarly the associated field \((E_{tp}, H_{tp})\) or the voltage-current eigenvector \((V_p, I_p)\) transforms to \((\kappa E_{tq}, \kappa H_{tq})\) and \((\sigma V_q, \sigma I_q)\) respectively \((\kappa \text{ and } \sigma \text{ are two complex scalars})\). This must be the case as otherwise we end up with two linearly independent field functions or voltage-current eigenvectors corresponding to the same eigenvalue \(\text{degenerate-mode}\). By analytically continuing the surface integral in \((3.10)\) \((m = n = p \text{ mode to the } m = n = q \text{ mode})\) which has been normalized to unity when \(m = n\), we can reason that under such a field normalization, \(\kappa = 1\). Similar arguments also hold for \(\sigma\) which can be set to unity under the normalization \(V_n^T I_n = 1\).

The time domain electric field in the semi-infinite guided-wave problem \(\text{(see Figure 3.1)}\) or the voltage-current waveforms in the semi-infinite coupled transmission-line problem \(\text{(see Figure 3.2)}\) in the region \(z > 0\) \(\text{(indicated by the} + \text{subscript in equations (4.17) and (4.18)) are obtained from the following Laplace inversion formulae, respectively:}\)

\[
\mathcal{E}_t = \frac{1}{2\pi j} \sum_{n=1}^{N} \int_{Br} a_n(s) E_{tn}^+(s) e^{-\gamma_n(s)z} e^{st} ds \tag{4.17}
\]

\[
\begin{pmatrix} V \\ I \end{pmatrix} = \frac{1}{2\pi j} \sum_{n=1}^{N} \int_{Br} b_n(s) \begin{pmatrix} V_n^+(s) \\ I_n^+(s) \end{pmatrix} e^{-\gamma_n(s)z} e^{st} ds \tag{4.18}
\]

The expansion coefficients \(a_n(s)\) in the waveguide problem are obtained by computing the reaction of each modal field propagating in the negative \(z\) direction to the source current sheet \(J_s\) \([79]\) located at \(z = 0\) in Figure 3.1. The coefficients \(b_n(s)\) in the coupled transmission-line problem are obtained by solving the following matrix problem obtained by applying the source and impedance boundary conditions at \(z = 0\) in Figure 3.2:

\[
V_s = \sum_{n=1}^{N} b_n \left( V_n^+ + Z_s I_n^+ \right) \tag{4.19}
\]

In equation (4.19), \(V_s\) is a \(1 \times N\) column vector of line voltage sources and \(Z_s\) is a \(N \times N\)
diagonal matrix containing the source impedances depicted in Figure 3.2. We will now list the expressions which will allow us to compute the expansion coefficients for both the waveguide and coupled transmission-line problems:

\[ a_n(s) = -\int_S J_s \cdot E^\infty_{in} dS \quad (4.20) \]

\[ I^{T+}_m V_s = \sum_{n=1}^{N} \left( I^{T+}_m V^+_n + I^{T+}_m Z_s I^+_n \right) b_n(s) \quad (4.21) \]

Examining the guided-wave amplitude in (4.20) we see that \(a_p(s)\) and \(a_q(s)\) share the same branch-points as \(\gamma_p(s)\) and \(\gamma_q(s)\) for all single-valued analytic source function \(J_s(s)\).

For the coupled transmission-line case, a closed loop analytic continuation around a branch-point of \(\gamma_p(s)\) transforms every voltage and current vector in equation (4.21) with \(p\) indices to \(q\) indices and vice-versa. This corresponds to switching the \(p\)-th and \(q\)-th rows and columns of the matrix \(A_{mn} = I^{T+}_m V^+_n + I^{T+}_m Z_s I^+_n\) and vector \(x_m = I^{T+}_m V_s\). Hence swapping \(b_p(s)\) and \(b_q(s)\) satisfies equation (4.21). This indicates that the amplitudes \(b_p(s)\) and \(b_q(s)\) are just different branches of the same multi-valued analytic function that share the same branch-points as \(\gamma_p(s)\) for all single-valued analytic sources \(V_s\) and terminating impedances \(Z_s\).

As the amplitudes of a pair of connected complex modes share the same branch-cuts as the modes themselves, the R.H.P. branch-cut integrals in equations (4.17) and (4.18) also vanish. The two amplitude functions smoothly merge at the branch-cuts but are quite distinct when one considers their behaviour along analytic line segments on the imaginary \(s = j\omega\) axis. Using appropriate sources and impedance terminations\(^7\), it is possible to selectively set the amplitude of one of a pair of complex modes to zero over a set of isolated \(j\omega\) frequency points\(^8\). Although a complex mode can never be eliminated over a continuous frequency range (without eliminating its conjugate counterpart as well), it can nevertheless be limited in magnitude over a finite bandwidth. We will discuss these assertions in the context of a simple example using a MS/NRI-TL coupled-line coupler excited by a source and terminated by various impedances.

\(^7\)All sources and impedances are restricted to the \(z = 0\) plane and the guide is assumed to be semi-infinite, i.e. extends to \(+\infty\).

\(^8\)Due to uniqueness of analytic functions, setting \(a_p(s)\) to zero in a non-isolated set (an accumulation point) such as a segment of an arc or an open disk, will imply that the function is identically zero [85].
Chapter 4. Independent Excitation of Complex Modes

4.5 Resolving further objections against the independence of complex modes

We will now clarify several other objections in literature, other than the R.H.P. branch-point problem above, against the independence of a complex mode in a conjugate pair. These are claims based on harmonic analysis of the spatial variation of a complex mode. In analysing modes guided by plasma slabs [75], it was suggested that a single complex-mode on its own causes violation of power conservation in the transverse direction. Yet another objection has been cast in terms of simultaneous capture of both complex-mode poles by contour deformation in the complex spatial wave-number plane [75][76]. Lastly we will resolve the conceptual difficulty in exciting a mode which carries no power in the axial direction [77].

4.5.1 Power conservation in the transverse direction of a complex mode

A complex mode carries no net power in the axial direction and this is a direct consequence of power conservation. If in a lossless structure, such a mode carried net power past a transverse plane at \( z = 0 \), then on any other transverse plane \( z > 0 \), the net power flow would be smaller thereby violating power conservation. Nevertheless, complex modes do carry finite power in the transverse direction as shown in the case of an infinite plasma-slab [75]. As these modes exponentially decay in the transverse direction, it would seem that power conservation is being violated, thus requiring the simultaneous excitation of a pair of conjugate modes. This is not the case, as the simple demonstration below shows.

Consider a proper TM complex mode supported by a plasma slab oriented perpendicular to the \( a_x \) direction as shown in Figure 4.4. Let the axial and transverse (no \( y \) variation) dependence of fields in the air-region above the slab be of the form \( e^{-\lambda_xx - \gamma_z z} \) with \( \lambda_x = \lambda_{xr} + j\lambda_{xi} \) and \( \gamma_z = \gamma_{zr} + j\gamma_{zi} \). This being a proper complex-mode, the real parts \( \lambda_{xr} > 0 \) and \( \gamma_{zr} > 0 \). The condition \( k_0^2 + \lambda_x^2 + \gamma_z^2 = 0 \) where \( k_0 \) is the free-space wave number, implies that \( \lambda_{xr}\lambda_{zi} + \gamma_{zr}\gamma_{xi} = 0 \). Hence either \( \lambda_{xi} \) or \( \gamma_{zi} \) is negative; in the example depicted in Figure 4.4 the former is negative (\( \gamma_{zi} \) must be positive to carry power forward in the free space region above the plasma slab). The modal impedances in the positive \( z \)-direction and positive \( x \)-direction are \( \gamma_z/(j\omega\epsilon_0) \) and \( \lambda_x/(j\omega\epsilon_0) \) respectively.
Figure 4.4: TM complex modes supported by an infinite plasma slab in free space. Black/white arrows depict normal component of power flow across various cross-sectional surfaces. The Poynting vector (red arrows) profiles for a pair of conjugate modes (labelled as $c$ and $\pi$ arbitrarily) are shown on the right. The depicted mode profiles are approximate and not to scale.

The total complex power flowing into the surface $S$ in Figure 4.4 can be evaluated as:

$$\int_S E \times H^* \cdot n \, dS = \frac{|H_y|^2}{2j\omega\varepsilon_0} e^{-2\gamma_x x_1} \left( e^{-2\lambda_x x_1} - e^{-2\lambda_x x_2} \right) \left( \frac{\lambda_x}{\gamma_x} + \frac{\gamma_x}{\lambda_x} \right)$$

(4.22)

In equation (4.22), $|H_y|$ is the amplitude of the magnetic field, the unit normal vector $n$ points into the surface and it is seen that the total real power flowing into the volume of space enclosed by $S$ is zero. The real power flowing into the top surface ($x = x_2$, $z \geq z_1$) is smaller than the power leaving the surface ($x = x_1$, $z \geq z_1$) but a compensating power flow from the side ($z = z_1$, $x_1 \leq x \leq x_2$) leads to power conservation. This is simply a consequence of the fact that the total divergence of the Poynting vector $\nabla \cdot S$ can be zero with non-zero divergence in orthogonal directions. This reassures us that power is indeed conserved by a single complex-mode.
4.5.2 Complex modes and spatial wave-number poles

A common procedure involved in solving for the time-harmonic fields excited by a source in an infinite guided wave structure involves computation of the relevant Green’s function for the problem. Integration in the complex wave-number plane is quite often involved in determining the Green’s function and poles of the integrand are recognized as proper guided waves or improper leaky waves. Quite often the integrand is well-behaved to allow the closure of the path of integration at infinity and determine the amplitude of the excited modes in terms of their residues.

Such techniques have been cited in the past to argue about the impossibility of exciting a complex mode on its own [75][76]. It involves the argument that these complex-mode poles appear symmetrically in all four quadrants of the complex wave-number plane, and hence any contour deformation will capture at least two poles with residues proportional to their amplitudes. In the discussion below we will show that, although both poles are involved, by a judicious choice of source configuration, we can selectively suppress one of these modes by setting its amplitude to zero. We will use the example of the TM modes excited in the upper-right spatial quadrant \((x > 0, z > 0)\) above a plasma-slab [75] as the integrand for the Green’s function is conveniently available in closed form.

\[
G(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\kappa, z, h) e^{j\zeta x} d\zeta \tag{4.23}
\]

In equation (4.23), \(\zeta\) is the propagation constant of a mode in a guide oriented along the \(x\) direction with \(\kappa\) the transverse wave-number (in air) in the \(z\) direction perpendicular to the plasma slab (of real relative permittivity \(\epsilon_p\)). The source is situated at \(z = h\) units above the plasma slab. Let it be noted that co-ordinates shown in Figure 4.4 are different from the one in equation (4.23) above. The integrand \(g(\kappa, z, h)\) inside the plasma-slab (\(-\) superscript) and above it (\(+\) superscript) are given by:

\[
g^-(\kappa, z, h) = \frac{j\epsilon_p e^{j\kappa(h-1)}}{T_D(\kappa)} \psi(\kappa, z) \tag{4.24}
\]

\[
g^+(\kappa, z, h) = \frac{j}{2\kappa} e^{j\kappa(z_l-z_s)} + \frac{T_N(\kappa)}{T_D(\kappa)} e^{j\kappa(z+h-2)} \tag{4.25}
\]

The various functions \(\psi(\kappa, z), T_N(\kappa)\) and \(T_D(\kappa)\) in equations (4.24) and (4.25) are defined in [75] and their exact form in not required for our discussion. Nevertheless, what we do need are the following symmetry operations for any complex \(\kappa\): \(T_N(-\kappa^*) = -T_N^*(\kappa)\), \(T_D(-\kappa^*) = -T_D^*(\kappa)\) and \(\psi(-\kappa^*, z) = \psi^*(\kappa, z)\). Another important point to note is that
the zeroes of $T_D(\kappa)$ provide all the poles including those for the complex modes and at these pole wave-numbers, the other functions remain bounded.

The outlined symmetry relations let us conclude that $g(-\kappa^*, z, h) = g^*(\kappa, z, h)$. Hence if $\kappa_0$ corresponds to a complex-wave pole, then so does the wave-number $-\kappa_0^*$. In the vicinity of these two simple poles, $g(\kappa, z, h) = a/(\kappa - \kappa_0)$ and $g(\kappa, z, h) = b/(\kappa + \kappa_0^*)$ respectively, where $a$ and $b$ are the pole residues in the $\kappa$ plane. Using the symmetry relation of $g(\kappa, z, h)$, we can conclude that $a = -b^*$ and the same relation holds for the corresponding poles in the complex $\zeta$ plane\(^9\). Consider two vertically stacked point sources of complex amplitudes $J_1$ and $J_2$ situated at $z = h_1$ and $z = h_2$ respectively. We may now express the portion of the magnetic field $H_{y0}$ attributed to a proper complex-mode pair $(\zeta_0, \kappa_0)$ and $(-\zeta_0^*, -\kappa_0^*)$ excited by these two sources as:

$$H_{y0} = J_1 \left[ a(z)e^{j\kappa_0 h_1}e^{j\zeta_0 x} - a^*(z)e^{-j\kappa_0^* h_1}e^{-j\zeta_0^* x} \right] + J_2 \left[ a(z)e^{j\kappa_0 h_2}e^{j\zeta_0 x} - a^*(z)e^{-j\kappa_0^* h_2}e^{-j\zeta_0^* x} \right]$$

(4.26)

In equation (4.26), we have split the residue of each pole to highlight their dependence in the transverse direction, axial direction and source height. Hence we immediately see that by setting $J_1 = -J_2 e^{j\kappa_0(h_2-h_1)}$ we can cause the $(\zeta_0, \kappa_0)$ mode in equation (4.26) to vanish without eliminating its conjugate counterpart $(-\zeta_0^*, -\kappa_0^*)$. This result can be intuitively understood by examining the spatial profile of the two conjugate modes. In the transverse $z$ direction, the phase of one of the modes increases with distance while it decreases for the other, while the amplitude profile remains identical for both modes. Hence by coinciding the phase and amplitude of two point sources with one of the modes, we can completely eliminate the other.

### 4.5.3 Excitation of a mode which carries no net forward power

A complex mode on its own carries no net power axially although in combination with its conjugate counterpart, the pair can carry reactive power resembling cut-off evanescent modes in waveguides [77][68]. This feature of a pair of complex modes might tempt the conclusion that in practical excitation/discontinuity problems, one needs both modes to bring about finite energy transfer. This energy transfer dilemma is again resolved by

\(^9\)The $\zeta$-plane residue differs from the $\kappa$ plane residue only by a factor of $\frac{d\zeta}{d\kappa}$ which enjoys the same conjugate symmetry (with respect to a pair of proper complex modes) as the integrand in equation (4.23).
close examination of the spatial power variation of a single complex-mode in its transverse plane.

In the transverse plane, there are regions where a complex mode carries forward power and others with backward power resulting in a net zero axial power flow. Moreover, as discussed in Chapter 3, the power flow profile across the transverse plane is reversed in direction when comparing a complex mode to its conjugate counterpart (see Figure 4.4). Hence, by selectively illuminating only those spatial regions of a mode which correspond to forward power flow, we can potentially exclude its conjugate pair provided that we have matched absorbing sheet/load for the regions corresponding to backward power flow. We will illuminate this principle in our discussion on the excitation of a single complex-mode in a MS/NRI-TL coupled-line coupler.

### 4.6 Independence of modes in MS/NRI-TL couplers

In this section we will explore the particular case of complex modes in a MS/NRI-TL coupled-line coupler and show the independence of its conjugate modes which are connected by R.H.P. branch-points.

The typical dispersion relation for a MS/NRI-TL coupler is given in Figure 4.5 which shows the phase $\gamma d$ vs. frequency characteristics in the irreducible Brillouin zone\(^\text{10}\). It depicts a lower frequency band in which the structure supports both forward and backward waves, followed by a complex-mode band (from 1.58 GHz to 2.55 GHz approximately) and at higher frequencies, a band with two forward-wave modes.

We can evaluate the amplitudes of the two modes in a semi-infinite MS/NRI-TL coupler using equation (4.21).

\[
\begin{pmatrix}
I_c^T V_s \\
I_\pi^T V_s
\end{pmatrix} =
\begin{pmatrix}
I_c^T V_c + I_c^T Z_s I_c & I_c^T V_\pi + I_c^T Z_s I_\pi \\
I_\pi^T V_c + I_\pi^T Z_s I_c & I_\pi^T V_\pi + I_\pi^T Z_s I_\pi
\end{pmatrix}
\begin{pmatrix}
b_c \\
b_\pi
\end{pmatrix}
\]

In equation (4.27), $V_s$ is a column vector containing sources $V_{s1}$ and $V_{s2}$ attached to line 1 and line 2 respectively, while $Z_s$ is a $2 \times 2$ diagonal matrix containing the source impedances $Z_{s1}$ and $Z_{s2}$. The modal vectors $V_{c,\pi}$ and $I_{c,\pi}$ are obtained from the columns

\(^\text{10}\)The dispersion relation exhibits mirror symmetry about the frequency axis and conjugate symmetry about the phase axis.
Figure 4.5: Modes in a MS/NRI-TL coupled-line coupler obtained by using with a complex-mode band in this case is located between 1.58 GHz and 2.55 GHz. Time domain circuit simulation carried out in Agilent ADS — details in Section 4.7 of the matrix in (2.25):

\[
V_{c,\pi} = \begin{pmatrix} 1 \\ R_{c,\pi} \end{pmatrix}, \quad I_{c,\pi} = \begin{pmatrix} 1/Z_{c,\pi} \\ -1/(R_{\pi,c}Z_{c,\pi}) \end{pmatrix}
\]

(4.28)

Solving for \(b_c\) and \(b_\pi\) in equation (4.27) we obtain:

\[
b_c = \frac{V_{s1}}{\Delta} \left[ R_\pi - \frac{Z_{s2}}{R_c Z_\pi} \right] - \frac{V_{s2}}{\Delta} \left[ 1 + \frac{Z_{s1}}{Z_\pi} \right]
\]

(4.29)

\[
b_\pi = \frac{V_{s1}}{\Delta} \left[ \frac{Z_{s2}}{R_\pi Z_c} - R_c \right] + \frac{V_{s2}}{\Delta} \left[ 1 + \frac{Z_{s1}}{Z_c} \right]
\]

(4.30)

where,

\[
\Delta = \left[ 1 + \frac{Z_{s1}}{Z_c} \right] \left[ R_\pi - \frac{Z_{s2}}{R_c Z_\pi} \right] - \left[ 1 + \frac{Z_{s1}}{Z_\pi} \right] \left[ R_c - \frac{Z_{s2}}{R_\pi Z_c} \right]
\]

(4.31)

At any given frequency in the complex-mode band, one of the the amplitudes, say \(b_\pi\) in equation (4.30). These complex modes carry equal but oppositely directed power on
the two lines and hence by exciting one of the lines with a source and terminating the other with a matched load, we obtain a single mode. This observation was alluded to at the end of Section 4.5.3.

From a time-harmonic perspective, it is evident that there is no restriction on the relative amplitudes of the two modes that can be excited on a semi-infinite structure. We will now demonstrate how the arguments regarding \( N \) coupled lines presented in Section 4.4, applies to our case (\( N = 2 \)). We need to show that the amplitudes \( b_c \) and \( b_\pi \) appearing in equations (4.29) and (4.30) are simply two branches of the same multi-valued function and share the same branch-points as \( \gamma_c \) and \( \gamma_\pi \). Once this fact is established, the eigenvectors of each mode may be linearly combined and the resulting function converted to its time domain representation using the Bromwich integral.

The possible branch points in equation (5.20) are obtained by setting \( \gamma_{c,\pi}^2 = 0 \) or by setting \((a_2 - a_1)^2 + 4b_1b_2 = 0\). The roots of \( \gamma_{c,\pi}^2 = 0 \) are located on the imaginary axis and hence the R.H.P. branch points are generated by the roots of the term under the radical in (5.20). This radical also appears in the defining equations of \( R_c \) and \( R_\pi \) in (2.18) with the implication that these two quantities share the same R.H.P. branch-points as \( \gamma_{c,\pi} \). Similar arguments can also be applied to \( Z_c \) and \( Z_\pi \) in equation (5.29). A closed loop analytic continuation encircling one of these R.H.P. branch-points will result in the transformation: \( \gamma_c \to \gamma_\pi, \ R_c \to R_\pi, \ Z_c \to Z_\pi \) and vice-versa. Consequently, with the assumption that the source voltages and impedances are single-valued analytic functions, it can be seen from equation (4.31) that \( \Delta \) undergoes a sign change under this continuation. This establishes that \( b_c \) and \( b_\pi \) are two branches of the same function. The total Laplace domain function which we are interested in inverting is obtained by a linear combination of the eigenvectors:

\[
\begin{pmatrix} V \\ I \end{pmatrix} = b_c \begin{pmatrix} 1 \\ R_c \\ 1/Z_c \\ -1/R_\pi Z_c \end{pmatrix} e^{-\gamma_c z} + b_\pi \begin{pmatrix} 1 \\ R_\pi \\ 1/Z_\pi \\ -1/R_c Z_\pi \end{pmatrix} e^{-\gamma_\pi z}
\]

The two terms appearing in (4.32) are different branches of a multivalued analytic function which smoothly merges from one sheet to the next across R.H.P. branch-cuts. Hence, from our discussion in Section 4.2, the R.H.P. branch-cut integrals will no longer pose a causality/stability problem and will vanish. The independent control of the two modal amplitudes is hence established.
4.7 Verification of independent modal excitation

In order to verify our assertion that complex modes can indeed be controlled independently in practice in a MS/NRI-TL coupled-line coupler using realistic sources and terminations, we carry out a few time domain simulations using Agilent’s Advanced Design System (ADS) microwave circuit simulator. We analyse the voltages and currents excited in a long MS/NRI-TL coupled-line coupler under various pulsed RF source configurations and impedance terminations.

4.7.1 The simulation setup

In our design, we employ 2 mm long MS/NRI-TL unit cells with 67.5 Ω and 25.0 Ω even and odd mode characteristic impedances for the coupled segments. These coupled segments can be realized on a 50 mils Rogers RO2010® substrate ($\epsilon_r = 11.05$) using 1.09 mm wide lines spaced 0.1 mm apart. The loading inductors $L$ and capacitors $C$ are 8.75 nH and 3.5 pF respectively.

We excite the structure ($V_{s1}$ and $V_{s2}$ on line 1 and line 2 respectively) using Gaussian edged pulses of 0.4 ns duration modulated by a 2.5 GHz carrier. Over 1500 cascaded unit cells are used in the simulations and the excited voltage and current waveforms in the structure are sampled at 24 consecutive unit cells at 50 ps intervals for a total duration of 10 ns. Such a setup guarantees sufficiently large source amplitude from 0.5 GHz to 5 GHz (to accurately resolve the eigenmodes in the structure) with a Fast Fourier Transform (FFT) bandwidth of 10 GHz and 0.1 GHz frequency resolution. The usage of at least 1500 unit cells ensures that any reflections from the far end of the structure are minimized in the observation window of 10 ns thereby emulating semi-infinite lines.

4.7.2 Data analysis and verification of the numerical algorithm

The simulations are carried out in the time domain to emulate realistic excitation scenarios but in order to examine the excitation amplitudes of each mode, we still require frequency domain data. Hence, at each sampling point in the MS/NRI-TL coupler, we perform a FFT on the time domain waveforms to obtain the corresponding frequency domain spectrum. This provides us with both the amplitude and phase of the voltages and currents on the lines at 0.1 GHz intervals from DC to 10 GHz.

At each frequency point, we record 24 voltage and current samples in each line from
which both the eigenmodes and their corresponding amplitudes are extracted. For this task we use Prony’s method \[93\] with a two-mode extraction. Although a total of 4 sample points would have been sufficient for the procedure, we use 24 along with the Method of Least Squares to compensate for truncation and interpolation errors.

Before examining the relative amplitudes of the two modes in the coupler under various source and impedance configurations, we verify the accuracy of our numerical script by extracting the dispersion curve. In Figure 4.5, we plot the real and imaginary phase-shift per unit cell as a function of frequency using the frequency domain equation (5.20). Next we use the time domain sampled waveforms and Prony’s method to extract the eigenmodes and plotted the phase shifts of both modes. These extracted modes are shown by dots in Figure 4.5 where we have mirrored the backward-wave modes about the imaginary axis for convenience of displaying the dispersion profile.

As seen from Figure 4.5 there is close agreement between the theoretical dispersion plot and the numerically extracted modes. We do see deviations at higher frequencies and this is to be expected considering the various limitations of our setup. The coupled transmission-line differential equations assume longitudinally homogeneous or infinitesimally small unit cells which is not entirely true for the realistic cell size of 2 mm used in the simulation. The theoretical dispersion curves can be improved by employing a more rigorous periodic Bloch-Floquet type analysis \[32\]. The width of each stop band is sensitive to the mutual coupling inductance and capacitance values used in the theoretical model, and any extracted values for these parameters, approximately model the even and odd mode impedances of the two lines.

Nevertheless, the results in Figure 4.5 verify two crucial points that provide us with confidence over the results to be presented next. Firstly, the MS/NRI-TL structure does indeed support a complex-mode band and secondly, our coupled transmission-line equations, on which we have based a lot of our theoretical discussions, and numerically extracted modes are in good agreement. This will allow meaningful comparison between simulation results and theoretical predictions regarding amplitude control of individual complex modes in a pair.
4.7.3 Complex-mode amplitude control using various impedance terminations

In order to demonstrate the process of controlling the amplitude of individual complex modes, we excite Line 1 (MS-TL) of a cascade of 1500 unit cells using the pulsed source described earlier. We apply four different impedance terminations to that end of Line 2 (NRI-TL) which is adjacent to the source. The far ends of the MS/NRI-TL coupler are simply terminated with 50 Ω resistors.

From our initial modal dispersion simulations, we deduce that the value of $Z_{s2} = R_c R_{\pi} Z_{\pi}$ (see Section 2.5) which will nullify the amplitude of the $\gamma_{\pi}$ mode\textsuperscript{11} at 2.0 GHz, can be obtained using the series combination of a 62.6 Ω resistor and a 0.58 nH inductor. We utilize this impedance, along with a short circuit (0 Ω), 10 Ω and 30 Ω terminations on Line 2.

As discussed in Chapter 3, in the complex-mode band, the two modes in the MS/NRI-TL coupler exhibit orthogonality in terms of the real power they carry. Moreover, in this frequency band, the two modal impedances are negative conjugates of each other. Hence the square of the absolute value of the voltage amplitude of each mode measured in Line 1, provides comparable information on the power it carries. If the real part of the impedances of these modes were different in magnitude, comparing only the voltage amplitude coefficients might have caused one to doubt the validity of our conclusions about the suppression of one mode relative to the other. Moreover, the real power orthogonality relation, legitimize any discussion which involves comparison of the total real power in a line with the real power carried by individual modes.

In Figure 4.6 we plot the normalized difference between the amplitude squares of the two modes computed by the expression $(|b_c|^2 - |b_{\pi}|^2) / (|b_c|^2 + |b_{\pi}|^2)$. In the complex-mode band, highlighted by the two vertical lines in Figure 4.6, this expression provides a valid description of the difference in the real power carried by the two modes. A value close to zero is indicative of equal excitation of the two modes, whereas at the vicinity of unity, the normalized difference denotes that essentially only a single mode is excited.

As expected, when Line 2 is terminated with a short circuit, the input impedance looking into Line 1 is purely reactive and both modes are equally excited as seen from

\textsuperscript{11}The $\gamma_{\pi}$ mode displays attenuation and phase lead with distance away from the source along with backward power flow on Line 1 and forward power flow on Line 2. On the other hand, the $\gamma_c$ mode exhibits attenuation and phase lead away from the source, with forward power on Line 1 and backward power on Line 2.
Figure 4.6: Normalized difference in real power between a pair of complex-conjugate modes in a MS/NRI-TL coupler with a pulsed source attached to the MS-TL line and various impedance terminations attached to the NRI-TL line.

The values near zero in Figure 4.6. As we gradually change the termination to the matching condition at 2.0 GHz, we notice that the difference in the amplitudes of and the power carried by the two modes increase until the $\gamma_c$ modes completely dominates. We also notice in Figure 4.6 that this mode is dominant even outside the complex-mode band under the simple matched termination. One may envision a process whereby a load impedance $Z_{s2}$ is synthesized such that $Z_{s2} = R_c R_\pi Z_\pi$ for discrete set of frequency points thereby allowing one to control the relative amplitudes of each mode over a large bandwidth.

We have also plotted the ratio of the squares of the amplitudes (see Figure 4.7) of the two modes $|b_\pi|^2/|b_c|^2$ as a function of frequency under various terminating impedances to demonstrate the independent control of the excitation of each mode in the MS/NRI-TL coupler. In this case, a value near unity is indicative of equal excitation of the modes whereas near zero indicates dominating presence of the $\gamma_c$ mode.
Figure 4.7: Ratio of real power between two complex conjugate modes in a MS/NRI-TL coupler with a pulsed source attached to the MS-TL line and various impedance terminations attached to the NRI-TL line.
Chapter 5

Coupling Between a Right-handed Microstrip-TL and a Left-handed NRI-TL

5.1 Introduction

In this chapter, we carry out a detailed study of the coupling between a regular right-handed Microstrip Transmission Line (MS-TL) and a metamaterial Negative-Refractive-Index Transmission Line (NRI-TL). We compare its eigenmodes obtained from periodic Floquet analysis and the homogeneous Schelkunoff’s coupled-mode equations to justify the use of the latter in modelling the MS/NRI-TL coupler. The scattering parameters of this coupler are then obtained by solving for the modal amplitudes in the complex-mode band. It is shown that such a coupler is capable of achieving arbitrary backward coupling levels. Moreover, when the structure is made half a guided wavelength long, it exhibits perfect isolation.

5.2 Coupling between two identical lines

In this section we will briefly consider the coupling between two identical lines using the coupled mode formulation introduced in Chapter 2. In the process, we will equip ourselves with parameters which will help in the characterization of the more complicated case of a MS/NRI-TL coupled line coupler.
In the case of two identical lines, we set $a_1 = a_2 = (Z_1 Y_1 + Z_m Y_m)$ and $b_1 = b_2 = (Z_1 Y_m + Z_m Y_1)$ in equation (2.15) and the propagation constants $\gamma$, line voltage ratio $R$ and modal impedance $Z$ from equations (2.22) to (2.24) are given by:

$$\gamma_{c,\pi}^2 = (Z_1 \pm Z_m) (Y_1 \pm Y_m) \quad (5.1)$$

$$R_{c,\pi} = \pm 1 \quad (5.2)$$

$$Z_{c,\pi} = \frac{\gamma_{c,\pi}}{Y_1 \pm Y_m} \quad (5.3)$$

We can now solve for the four parameters $Z_1$, $Z_m$, $Y_1$ and $Y_m$ in terms of the coupled propagation constants and impedances to obtain the following useful relations:

$$Z_1 = \frac{1}{2} \left( \gamma_c Z_c + \gamma_{\pi} Z_{\pi} \right) \quad (5.4)$$

$$Z_m = \frac{1}{2} \left( \gamma_c Z_c - \gamma_{\pi} Z_{\pi} \right) \quad (5.5)$$

$$Y_1 = \frac{1}{2} \left( \gamma_c / Z_c + \gamma_{\pi} / Z_{\pi} \right) \quad (5.6)$$

$$Y_m = \frac{1}{2} \left( \gamma_c / Z_c - \gamma_{\pi} / Z_{\pi} \right) \quad (5.7)$$

For two identical lines, the $c$-mode and $\pi$-mode are normally referred to as the even mode and the odd mode respectively, and their corresponding propagation constants and associated impedances have been tabulated for a variety of transmission-line technologies such as strip-lines and microstrip-lines [10] or are available from commercial microwave circuit simulators. Hence equations (5.4) to (5.6) are useful in empirically determining the coupling parameters of the MS/NRI-TL coupler comprised of segments of coupled microstrips of equal width of which one is loaded with lumped components. In the event that one chooses to utilize microstrip lines of unequal width, equations (2.22) to (2.24) provide six relations for the six unknown terms in equation (2.15), and the system can be solved numerically.

### 5.3 Coupled modes in a MS/NRI-TL coupler

The geometry of the proposed MS/NRI-TL coupler is shown in Figure 5.1. The use of even-odd mode analysis used in studying conventional microstrip coupled-line couplers does not apply here as the two lines are non-identical. Hence the coupled-mode formulation described in Chapter 2 will be used and the justification of using Schelkunoff’s ‘simpler’ formulation (section 2.3) over the more accurate Bloch/Floquet coupled-mode
Chapter 5. Coupling Between MS-TL and NRI-TL

5.3.1 Modes using Schelkunoff’s formulation

A single microstrip transmission line can be characterized in terms of its per-unit-length series reactance \( Z_1 = sL_0 \) and shunt susceptance \( Y_1 = sC_0 \) where \( s \) is the complex-frequency variable. When such a line is loaded with series capacitors and shunt inductors, we construct a metamaterial NRI-TL. In the homogeneous limit, the NRI-TL is characterized using resonant series reactance \( Z_2 = sL_0 + 1/sCd \) and shunt susceptance \( Y_2 = sC_0 + 1/sLd \) where \( L \) and \( C \) are the values of the lumped inductive and capacitive loadings and \( d \) is the length of one unit cell. In the characterization above, it has been assumed that the width of the host microstrip medium is the same for both lines.

Let us denote the MS-TL as line 1 and NRI-TL as line 2 in the MS/NRI-TL coupler depicted in Figure 5.1 and assume that they are both of equal width. In the homogeneous limit, such a system is characterized by equation (2.12) where the matrix elements are
functions of frequency only and we include the equation here for convenience:

\[
-\frac{d}{dz} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & Z_1 & Z_m \\ 0 & 0 & Z_m & Z_2 \\ Y_1 & Y_m & 0 & 0 \\ Y_m & Y_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix}
\]

(5.8)

Our previous discussion regarding the homogeneous characterization of a NRI-TL will guide us in computing the various elements of the matrix in equation (5.8). Hence for the self impedance and admittance terms of line 1 (\(Z_1\) and \(Y_1\) respectively), we use the values computed in equations (5.4) and (5.6) corresponding to the case of coupling between two identical lines. The self impedance/admittance terms for line 2 are obtained by adding the immitances of the loading elements to those of the MS-TL (line 1) such that \(Z_2 = Z_1 + 1/sCd\) and \(Y_2 = Y_1 + 1/sLd\). We retain the coupling terms \(Z_m\) and \(Y_m\) as the ones obtained for the case of identical coupled lines (equations (5.5) and (5.6)). For future reference, we will summarize the formulae for computing the various parameters of the coupler:

\[
Z_1 = sL_0 = \frac{1}{2} (\gamma_e Z_e + \gamma_o Z_o)
\]

(5.9)

\[
Y_1 = sC_0 = \frac{1}{2} (\gamma_e/Z_e + \gamma_o/Z_o)
\]

(5.10)

\[
Z_2 = sL_0 + \frac{1}{sCd}
\]

(5.11)

\[
Y_2 = sC_0 + \frac{1}{sLd}
\]

(5.12)

\[
Z_m = sL_m = \frac{1}{2} (\gamma_e Z_e - \gamma_o Z_o)
\]

(5.13)

\[
Y_m = sC_m = \frac{1}{2} (\gamma_e/Z_e - \gamma_o/Z_o)
\]

(5.14)

In equations (5.9) to (5.14) above, the terms with subscripts \(e\) and \(o\) denote the even and odd mode parameters of the coupled microstrip lines. The validity of the proposed model will be verified from simulations and their choice is reasonable when one considers two extreme cases. In the first case, if we set the loading immitances to zero, i.e. infinite series capacitance and shunt inductance, the modes reduce to those of a pair of identical coupled transmission lines as expected. In the second case, if the lines are separated by infinite distance, the even and odd mode impedance and propagation constant parameters in equations (5.4) to (5.7) converge to a single value, namely those of each isolated microstrip line. Hence in this case, \(Z_m = Y_m = 0\) and the two eigenmodes
of equation (5.8) correspond to those of the isolated MS-TL and the isolated NRI-TL in the homogeneous limit.

From Chapter 2, the eigenmodes of the MS/NRI-TL coupler are completely characterized by their propagation constants $\gamma_{c,\pi}$, voltage ratio coefficients $R_{c,\pi}$ and impedances $Z_{c,\pi}$. For convenience, we define $R_0 = \sqrt{L_0/C_0}$ and the following dimensionless parameters:

$$p^2 = s^2 L_0 C_0 d^2$$

(5.15)

$$\rho_L = L_0 d/L$$

(5.16)

$$\rho_C = C_0 d/C$$

(5.17)

$$\kappa_L = L_m/L_0$$

(5.18)

$$\kappa_C = C_m/C_0$$

(5.19)

Now we rewrite equations (2.22) to (2.24) in the following form:

$$\begin{align*}
(\gamma_{c,\pi} d)^2 &= \frac{d^2}{2} (a_1 + a_2) \pm \frac{d^2}{2} \sqrt{(a_2 - a_1)^2 + 4b_1 b_2} \\
R_{c,\pi} &= \frac{1}{2b_1} \left[ (a_2 - a_1) \pm \sqrt{(a_2 - a_1)^2 + 4b_1 b_2} \right] \\
Z_{c,\pi}/R_0 &= \frac{\gamma_{c,\pi} d}{(1 + \kappa_C R_{c,\pi}) p} \\
a_1 d^2 &= (1 + \kappa_L \kappa_C) p^2 \\
a_2 d^2 &= (1 + \kappa_L \kappa_C) p^2 + \frac{\rho_L \rho_C + (\rho_L + \rho_C) p^2}{p^2} \\
b_1 d^2 &= (\kappa_L + \kappa_C) p^2 + \kappa_L \rho_L \\
b_2 d^2 &= (\kappa_L + \kappa_C) p^2 + \kappa_C \rho_C
\end{align*}$$

(5.20) (5.21) (5.22) (5.23) (5.24) (5.25) (5.26)

In the limit of infinite separation between the lines, the coupling parameters $\kappa_L$ and $\kappa_C$ approach zero and the eigenvalues are given by $\gamma^2 = a_1$ and $\gamma^2 = a_2$ that correspond to the propagation constants of an isolated MS-TL and an isolated NRI-TL. When the square of the normalized frequency is set to:

$$p^2 = s_0^2 L_0 C_0 d^2 = -\frac{\rho_L \rho_C}{\rho_L + \rho_C}$$

(5.27)

the expressions for $a_1$ and $a_2$ given in equations (5.23) and (5.24) are equal. At this the frequency $s_0$ the two lines are tuned to the same propagation constant leading to a strong
coupling effect. Substituting the expression for $s_0$ into equations (5.25) and (5.26):

\begin{align*}
 b_1d^2 &= \rho_L (\kappa_L \rho_L - \kappa_C \rho_C) \\
 b_2d^2 &= \rho_C (\kappa_C \rho_C - \kappa_L \rho_L)
\end{align*}

(5.28) \quad (5.29)

The loading parameters $\rho_L$ and $\rho_C$ are both positive and since $a_1 = a_2$, we can see that the eigenmodes in equation (5.20) are complex at this frequency. The MS/NRI-TL coupler will be operated around this center frequency. The loading capacitors and inductors of the coupler in our design will be chosen by setting $\rho_L = \rho_C$ which is equivalent to the condition $L_0/C_0 = L/C$. This choice of loading parameters is not mandatory but will simplify the closed form expressions for the scattering parameters derived later in this chapter.

### 5.3.2 Modes using Bloch/Floquet formulation

The unit cell depicted in Figure 5.1 shows a cascade of 5 elementary building blocks. The first block is a shunt inductor connected to one of the lines. It will be characterized using the 4 port transmission matrix $M_L$ that models two pairs of ports connected back-to-back with one of them containing a shunt inductor. The second block is a segment of coupled microstrip-lines of length $d/2$ and is described by the matrix $M_{TL}$. The third block $M_C$ (which also represents a 4-port network) contains a series capacitance placed between two of its ports while the other two are connected via ideal shorts. The resulting $4 \times 4$ transmission matrix (equation (2.26)) for the unit cell depicted in Figure 5.1 is given by the matrix product $M_LM_{TL}M_CM_{TL}M_L$ where:

\[
 M_L = \begin{pmatrix}
 I & N \\
 Y & I
\end{pmatrix}, \quad M_C = \begin{pmatrix}
 I & Z \\
 N & I
\end{pmatrix}, \quad M_{TL} = \begin{pmatrix}
 I' & Z' \\
 Y' & I'
\end{pmatrix}
\]

(5.30)
In equation (5.30), $I$ and $N$ are $2 \times 2$ identity and null matrices respectively, while the rest of the block matrices are defined below:

$$\left(\gamma_{e,o}d\right)^2 = p^2(1 \pm \kappa_L)(1 \pm \kappa_C)$$  \hspace{1cm} (5.31)

$$\left(\begin{array}{c} Z_{e,o} \\ R_0 \end{array}\right)^2 = \frac{1 \pm \kappa_L}{1 \pm \kappa_C}$$  \hspace{1cm} (5.32)

$$R_0^{-1}Z = \left(\begin{array}{cc} 0 & 0 \\ 0 & \frac{p_R}{p} \end{array}\right)$$  \hspace{1cm} (5.33)

$$R_0Y = \left(\begin{array}{cc} 0 & 0 \\ 0 & \frac{p_L}{p} \end{array}\right)$$  \hspace{1cm} (5.34)

$$I' = \frac{1}{2} \left(\begin{array}{cc} \cosh(\frac{\gamma_{e}}{2}) + \cosh(\frac{\gamma_{o}}{2}) & \cosh(\frac{\gamma_{e}}{2}) - \cosh(\frac{\gamma_{o}}{2}) \\ \cosh(\frac{\gamma_{o}}{2}) - \cosh(\frac{\gamma_{e}}{2}) & \cosh(\frac{\gamma_{o}}{2}) + \cosh(\frac{\gamma_{e}}{2}) \end{array}\right)$$  \hspace{1cm} (5.35)

$$Z' = \frac{1}{2} \left(\begin{array}{cc} Z_e \sinh(\frac{\gamma_{e}}{2}) + Z_o \sinh(\frac{\gamma_{o}}{2}) & Z_e \sinh(\frac{\gamma_{e}}{2}) - Z_o \sinh(\frac{\gamma_{o}}{2}) \\ Z_e \sinh(\frac{\gamma_{e}}{2}) - Z_o \sinh(\frac{\gamma_{o}}{2}) & Z_e \sinh(\frac{\gamma_{e}}{2}) + Z_o \sinh(\frac{\gamma_{o}}{2}) \end{array}\right)$$  \hspace{1cm} (5.36)

$$Y' = \frac{1}{2} \left(\begin{array}{cc} Y_e \sinh(\frac{\gamma_{e}}{2}) + Y_o \sinh(\frac{\gamma_{o}}{2}) & Y_e \sinh(\frac{\gamma_{e}}{2}) - Y_o \sinh(\frac{\gamma_{o}}{2}) \\ Y_e \sinh(\frac{\gamma_{e}}{2}) - Y_o \sinh(\frac{\gamma_{o}}{2}) & Y_e \sinh(\frac{\gamma_{e}}{2}) + Y_o \sinh(\frac{\gamma_{o}}{2}) \end{array}\right)$$  \hspace{1cm} (5.37)

Noting that the symmetric matrices $I'$, $Z'$ and $Y'$ commute and that $I'^2 - Z'Y' = I$ (from the properties of symmetric and reciprocal networks), we can now evaluate the elements of the transmission matrix in equation (5.30):

$$A = D^T = I'(2I' + 2Z'Y + ZY' + ZI'Y) - I$$  \hspace{1cm} (5.38)

$$B = I'(2Z' + ZI')$$  \hspace{1cm} (5.39)

$$C = (Y' + YI')(2Z' + ZI')Y + Y(2Z' + I'Z)Y' + (2I' + Y'Z)Y' + 2Y$$  \hspace{1cm} (5.40)

For convenience, we will now list the expressions for the eigenmodes, voltage ratios and impedances of the coupled system given in equations (2.44) to (2.46) with $a$ and $c$ representing the elements of the $\mathbf{A}$ and $\mathbf{C}$ matrices respectively:

$$\cosh(\gamma_{e,o}d) = \frac{1}{2} (a_{11} + a_{22}) \pm \frac{1}{2} \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}}$$  \hspace{1cm} (5.41)

$$R_{e,o} = \frac{1}{2a_{12}} \left[ (a_{22} - a_{11}) \pm \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}} \right]$$  \hspace{1cm} (5.42)

$$Z_{e,o} = \frac{\sinh(\gamma_{e,o}d)}{c_{11} + c_{12}R_{e,o}}$$  \hspace{1cm} (5.43)

It is shown in Appendix D that in the limit of electrically small cell size and small Bloch phase shift per unit cell, the modes obtained through the periodic analysis approach
Table 5.1: Simulation parameters to compare the periodic analysis of the MS/NRI-TL coupler eigenmodes with the homogeneous approximation.

We expect the homogeneous approximation to fail when the cell size becomes electrically large (at higher frequencies) and at the edges of the Brillouin zone (imaginary part of $\gamma_{c,\pi}d$ equals $\pi$). We plot the eigenmodes $\gamma_{c,\pi}$, the associated voltage ratio of each mode between the two lines $R_{c,\pi}$ and the modal impedances $Z_{c,\pi}$ in Figure 5.2 where the real and imaginary parts are plotted in broken and solid lines respectively. The red lines correspond to the homogeneous Schelkunoff’s theory (equations (5.20) to (5.22)) and the blue lines were obtained using the more accurate periodic analysis (equations (5.41) to (5.43)).

In Figures 5.2a, 5.2c and 5.2e we see good agreement between the two theories as these plots correspond to unit cells whose electrical length $p \leq 2$. A complex-mode band is observed around $p = 0.5j$ and we notice that even when the two lines are closely spaced (by a separation of 0.1 mm) the imaginary part of the impedance in the complex-mode regime (see Figure 5.2e) is much smaller than its real part. As expected we observe deviation between the periodic analysis and the homogeneous approximation for larger electrical cell size (see Figures 5.2b, 5.2d and 5.2f). An interesting feature to note is that of a higher order complex-mode band which appears just below $p = 3.5j$ in Figure 5.2b. Complex modes in this region are formed as a result of the coupling of the second backward-wave band of the NRI-TL to the forward-wave mode of the MS-TL. In the homogeneous approximation, the NRI-TL line dispersion contains a single fundamental
backward-wave band and a higher forward wave band that extends to infinity and hence this model is unable to capture the additional complex-mode bands.

We are going to be primarily interested in the operation of the MS/NRI-TL coupler in and around its first complex-mode band and hence Schelkunoff’s theory will be adequate. The more accurate Bloch/Floquet formulation is desirable if we are interested in modelling the wide band response of the coupler. Nevertheless, it should be borne in mind that the characterization of the coupled microstrip segments using frequency independent even/odd mode impedances and phase velocities will have to be replaced by frequency dependent ones obtained empirically (or from standard RF/Microwave simulators). In this case, the analytic study of equations (5.41) to (5.43) will be difficult. Hence for wideband designs, it is convenient to use commercial RF/Microwave simulators over numerical evaluation of the Bloch/Floquet equations. On the other hand, for analytic study in the neighbourhood of the complex-mode band and in the limit of small cell dimensions, Schelkunoff’s theory will be of great utility.

5.4 S-parameters of a MS/NRI-TL coupler

In this section we will outline the procedure involved in computing the scattering parameters (s-parameters) of the MS/NRI-TL coupler once we have determined its eigenmodes using either the Schelkunoff or Bloch/Floquet formulation. We need to solve for the voltages and currents in a coupler consisting of \( N \) unit cells of length \( d \) each under the following generic boundary conditions:

\[
\begin{align*}
V_{s1} &= V_1(0) + Z_{01}I_1(0) \\
V_{s2} &= V_2(0) + Z_{02}I_2(0) \\
V_{s3} &= V_1(Nd) - Z_{03}I_1(Nd) \\
V_{s4} &= V_2(Nd) - Z_{04}I_2(Nd)
\end{align*}
\]

The voltage source \( V_{si} \) and associated impedance \( Z_{0i} \) in equations (5.44) to (5.47) are attached to the \( i \)-th port of the coupler (see Figure 5.3) and due to the reciprocal nature of the device, only two of these sources (attached to different lines) are needed to determine all 16 scattering parameters. Ordinarily all \( Z_{0i} \)’s will be set equal to the system impedance \( Z_0 \) but we have allowed these impedances to be distinct so that we may
Figure 5.2: Eigenmodes $\gamma_{c,\pi}$, voltage ratios $R_{c,\pi}$ and impedances $Z_{c,\pi}$ using Schelkunoff’s coupled-mode theory (red lines) and Bloch/Floquet theory (blue lines). The real and imaginary parts are plotted using broken and solid lines respectively.
explore the possibility of adding reactive loadings to the input terminals of the coupler. We now utilize the modal vectors in equation (2.25) and apply the boundary conditions above to solve for the amplitudes $V_{c}^{+}$, $V_{c}^{-}$, $V_{\pi}^{+}$ and $V_{\pi}^{-}$:

$$
\begin{bmatrix}
V_{s1} & V_{s2} & V_{s3} & V_{s4}
\end{bmatrix}^T =
\begin{bmatrix}
(1 + \frac{Z_{01}}{Z_{c}}) & (1 - \frac{Z_{01}}{Z_{\pi}}) & (1 + \frac{Z_{01}}{Z_{\pi}}) & (1 - \frac{Z_{01}}{Z_{\pi}})
(R_{c} - \frac{Z_{02}}{R_{\pi}Z_{c}}) & (R_{c} + \frac{Z_{02}}{R_{\pi}Z_{c}}) & (R_{\pi} - \frac{Z_{02}}{R_{c}Z_{\pi}}) & (R_{\pi} + \frac{Z_{02}}{R_{c}Z_{\pi}})
(1 - \frac{Z_{03}}{Z_{c}})e^{-\gamma_{c}Nd} & (1 + \frac{Z_{03}}{Z_{c}})e^{\gamma_{c}Nd} & (1 - \frac{Z_{03}}{Z_{\pi}})e^{-\gamma_{\pi}Nd} & (1 + \frac{Z_{03}}{Z_{\pi}})e^{\gamma_{\pi}Nd}
(R_{c} + \frac{Z_{04}}{R_{\pi}Z_{c}})e^{-\gamma_{c}Nd} & (R_{c} - \frac{Z_{04}}{R_{\pi}Z_{c}})e^{\gamma_{c}Nd} & (R_{\pi} - \frac{Z_{04}}{R_{c}Z_{\pi}})e^{-\gamma_{\pi}Nd} & (R_{\pi} + \frac{Z_{04}}{R_{c}Z_{\pi}})e^{\gamma_{\pi}Nd}
\end{bmatrix}
\begin{bmatrix}
V_{c}^{+}
V_{c}^{-}
V_{\pi}^{+}
V_{\pi}^{-}
\end{bmatrix}
$$

Equation (5.48) can be solved numerically in the most general case for all frequencies of interest. Once the modal amplitudes are known, the voltages and current on both lines at the end points can be determined. From these, the scattering parameters (with respect to port impedances $Z_{0i}$) can be computed by the following formula:

$$
S_{ij} = \sqrt{\frac{Z_{0j}}{Z_{0i}}} \left| \frac{V_{i}^{+}}{V_{j}^{+}} \right|_{V_{m}^{+} = 0 (m \neq i)} = \sqrt{\frac{Z_{0j}}{Z_{0i}}} \left| \frac{V_{i}^{+} - Z_{0i}I_{pi}}{V_{j}^{+} + Z_{0j}I_{pj}} \right|_{V_{m}^{+} = 0 (m \neq i)}
$$

In equation (5.49), the voltages $V_{i}$ and currents $I_{pi}$ (directed into the ports) are computed at the $i$-th port of the MS/NRI-TL coupler.
If we are interested in the performance of the coupler in the complex-mode stop-band, then we let $\gamma_c = \gamma_\pi^*", R_c = R_\pi^*$ and $Z_c = -Z_\pi^*$ (see Table 2.1). Even if we restrict our attention to the complex-mode band, equation (5.48) is too cumbersome to handle algebraically unless we make weak/moderate coupling assumptions to further simplify the problem. Such a task was undertaken (at a single frequency point) in [7] where it was assumed that the imaginary part of the complex-mode impedance is much smaller than its real part in a practical device (see Fig. 5.2e). Although laborious, such an analysis revealed an interesting fact about the MS/NRI-TL coupler – namely, it is possible to achieve very high directivity by making the structure half a guided wavelength long. The process of algebraically inverting the matrix in equation (5.48), even in the case of weak/moderate coupling, doesn’t offer much insight into the role played by the individual modes.

Hence, for the sake of analytic study, we will resort to a different approach in studying generic finite length MS/NRI-TL couplers. We will examine how the various complex modes are excited in such couplers and the mechanics of their reflection off terminations. By carrying out an infinite sum of the bouncing modes between the two ends of the coupler, we can solve for the port voltages and currents and determine its s-parameters.

### 5.5 Excitation and reflection of modes in a MS/NRI-TL coupler

In this section we will explore the dynamics of mode excitation and reflections in a finite length MS/NRI-TL coupler with sources $V_s1$ and $V_s2$ at $z = 0$ and passive terminations at $z = Nd$. Such a setup is sufficient for obtaining all s-parameters of this reciprocal device.

Let us first consider the excitation of modes in a semi-infinite MS/NRI-TL coupled-line coupler which extends from $z = 0$ to $z \to +\infty$. We define the positive travelling (+), negative travelling (−) wave vectors and voltage ratio matrix $R$ by:

$$
V_0^+ = \begin{pmatrix} V_c^+ \\ V_\pi^+ \end{pmatrix}, \quad V_0^- = \begin{pmatrix} V_c^- \\ V_\pi^- \end{pmatrix}, \quad R = \begin{pmatrix} R_c & 0 \\ 0 & R_\pi \end{pmatrix}
$$

(5.50)

In equation (5.50), we associate with the + voltage vector with those wave fronts which either carry net positive power or attenuate (in the case of complex and evanescent modes) in the +z direction. We make the observation from equation (2.25) that the total voltage
at \( z = 0 \) on line 1 is simply the sum of the elements in the vector \((V_0^+ + V_0^-)\) while for line 2 it is given by the sum of the entries in \(R(V_0^+ + V_0^-)\).

Now assume that line 1 (MS-TL) and line 2 (NRI-TL) are attached to voltage sources \(V_{s1}\) and \(V_{s2}\) with associated series impedances \(Z_{01}\) and \(Z_{02}\) respectively at \( z = 0 \). Such a setup will excite the \(e^{-\gamma c z}\) and \(e^{-\gamma \pi z}\) modes corresponding to the \(V_0^+\) mode vector. The resulting boundary conditions for the two lines are \(V_{s1} = V_1(0) + Z_{01}I_1(0)\) and \(V_{s2} = V_2(0) + Z_{02}I_2(0)\). Applying these conditions to equation (2.25) by setting \(V_0^- = 0\), gives us the following equation:

\[
\begin{pmatrix}
    1 + \frac{Z_{01}}{Z_c} & 1 + \frac{Z_{01}}{Z_c} \\
    R_c - \frac{Z_{02}}{R_c Z_c} & R_\pi - \frac{Z_{02}}{R_c Z_c}
\end{pmatrix}
\begin{pmatrix}
    V_0^+ \\
    V_0^-
\end{pmatrix}
= \begin{pmatrix}
    V_{s1} \\
    V_{s2}
\end{pmatrix}
\]

Solving equation (2.41) for \(V_0^+ = V_{\text{init}}^+\) which contains the initial excitation amplitudes of the two modes:

\[
V_{\text{init}}^+ = \frac{1}{\Delta} \begin{pmatrix}
    R_\pi - \frac{Z_{02}}{R_c Z_c} & -1 - \frac{Z_{01}}{Z_c} \\
    -R_c + \frac{Z_{02}}{R_c Z_c} & 1 + \frac{Z_{01}}{Z_c}
\end{pmatrix}
\begin{pmatrix}
    V_{s1} \\
    V_{s2}
\end{pmatrix}
\]

\[
\Delta = \left(1 + \frac{Z_{01}}{Z_c}\right)\left(R_\pi - \frac{Z_{02}}{R_c Z_c}\right) - \left(1 + \frac{Z_{01}}{Z_\pi}\right)\left(R_c - \frac{Z_{02}}{R_c Z_\pi}\right)
\]

The amplitude and phase of these two modes at \( z = Nd \) relative to \( z = 0 \) is given by:

\[
V_{N d}^+ = \begin{pmatrix}
    e^{-\gamma c N d} & 0 \\
    0 & e^{-\gamma \pi N d}
\end{pmatrix} V_0^+ = TV_0^+
\]

The matrix \(T^{-1}\) relates the mode vector \(V_0^-\) at \( z = 0 \) to its value at \( z = Nd\):

\[
V_{N d}^-= T^{-1}V_0^-
\]

The \(V_0^+\) modes which are created at \( z = 0\), generate secondary reflected waves \(V_{N d}^-\) at the \( z = Nd \) impedance terminations. The relationship between the incident and reflected waves at \( z = Nd\) can be determined from the last two rows of equation (5.48) by setting \(V_{s3} = V_{s4} = 0\):

\[
\begin{pmatrix}
    1 - \frac{Z_{03}}{Z_c} & 1 - \frac{Z_{04}}{Z_c} \\
    R_c + \frac{Z_{04}}{R_c Z_c} & R_\pi + \frac{Z_{04}}{R_c Z_\pi}
\end{pmatrix} TV_0^+ = - \begin{pmatrix}
    1 + \frac{Z_{03}}{Z_c} & 1 + \frac{Z_{04}}{Z_c} \\
    R_c - \frac{Z_{04}}{R_c Z_c} & R_\pi - \frac{Z_{04}}{R_c Z_\pi}
\end{pmatrix} T^{-1} V_0^-
\]
Similarly, the \( V_{Nd}^{-} \) wave which travels from \( z = Nd \) and impinges upon the terminations at \( z = 0 \) produces the \( V_{0}^{+} \) wave whose amplitude and phase can be derived from the first two rows in (5.48) by setting \( V_{s1} = V_{s2} = 0 \):

\[
\begin{pmatrix}
1 + \frac{Z_{0}}{Z_{c}} & 1 + \frac{Z_{0}}{Z_{e}} \\
R_{c} - \frac{Z_{0}}{R_{c}Z_{c}} & R_{e} - \frac{Z_{0}}{R_{e}Z_{e}}
\end{pmatrix}
\begin{pmatrix}
V_{0}^{+} \\
V_{0}^{-}
\end{pmatrix}
= - \begin{pmatrix}
1 - \frac{Z_{0}}{Z_{c}} & 1 - \frac{Z_{0}}{Z_{e}} \\
R_{c} + \frac{Z_{0}}{R_{c}Z_{c}} & R_{e} + \frac{Z_{0}}{R_{e}Z_{e}}
\end{pmatrix}
\begin{pmatrix}
V_{0}^{-} \\
V_{0}^{+}
\end{pmatrix} \quad (5.57)
\]

Now we define the reflection coefficient matrices \( \Gamma_{0} \) and \( \Gamma_{Nd} \) by:

\[
\Gamma_{0} = - \begin{pmatrix}
1 + \frac{Z_{0}}{Z_{c}} & 1 + \frac{Z_{0}}{Z_{e}} \\
R_{c} - \frac{Z_{0}}{R_{c}Z_{c}} & R_{e} - \frac{Z_{0}}{R_{e}Z_{e}}
\end{pmatrix}^{-1} \begin{pmatrix}
1 - \frac{Z_{0}}{Z_{c}} & 1 - \frac{Z_{0}}{Z_{e}} \\
R_{c} + \frac{Z_{0}}{R_{c}Z_{c}} & R_{e} + \frac{Z_{0}}{R_{e}Z_{e}}
\end{pmatrix} \quad (5.58)
\]

\[
\Gamma_{Nd} = - \begin{pmatrix}
1 + \frac{Z_{0}}{Z_{c}} & 1 + \frac{Z_{0}}{Z_{e}} \\
R_{c} - \frac{Z_{0}}{R_{c}Z_{c}} & R_{e} - \frac{Z_{0}}{R_{e}Z_{e}}
\end{pmatrix}^{-1} \begin{pmatrix}
1 - \frac{Z_{0}}{Z_{c}} & 1 - \frac{Z_{0}}{Z_{e}} \\
R_{c} + \frac{Z_{0}}{R_{c}Z_{c}} & R_{e} + \frac{Z_{0}}{R_{e}Z_{e}}
\end{pmatrix} \quad (5.59)
\]

Using these reflection matrices, equations (5.56) and (5.57) can be expressed simply as \( V_{Nd}^{-} = \Gamma_{Nd}V_{Nd}^{+} \) and \( V_{0}^{+} = \Gamma_{0}V_{0}^{-} \) respectively. The total voltage vectors at \( z = 0 \) and \( z = Nd \) due to the infinite reflections of the positive and negative travelling waves are:

\[
V_{0} = (I + TT_{Nd}T) (I + [\Gamma_{0}TT_{Nd}T] + [\Gamma_{0}TT_{Nd}T]^{2} + \ldots )V_{init}^{+} \quad (5.60)
\]

\[
V_{Nd} = (T + \Gamma_{Nd}T) (I + [\Gamma_{0}TT_{Nd}T] + [\Gamma_{0}TT_{Nd}T]^{2} + \ldots )V_{init}^{+} \quad (5.61)
\]

We will now assume that the infinite sums in equations (5.60) and (5.61) are convergent and hence can be summed as a geometric series of matrices. The total voltage at all four ports of the coupler are hence given by:

\[
V_{1} = \begin{pmatrix} 1 & 1 \end{pmatrix} (I + TT_{Nd}T) (I - \Gamma_{0}TT_{Nd}T)^{-1} V_{init}^{+} \quad (5.62)
\]

\[
V_{2} = \begin{pmatrix} 1 & 1 \end{pmatrix} R (I + TT_{Nd}T) (I - \Gamma_{0}TT_{Nd}T)^{-1} V_{init}^{+} \quad (5.63)
\]

\[
V_{3} = \begin{pmatrix} 1 & 1 \end{pmatrix} (T + \Gamma_{Nd}T) (I - \Gamma_{0}TT_{Nd}T)^{-1} V_{init}^{+} \quad (5.64)
\]

\[
V_{4} = \begin{pmatrix} 1 & 1 \end{pmatrix} R (T + \Gamma_{Nd}T) (I - \Gamma_{0}TT_{Nd}T)^{-1} V_{init}^{+} \quad (5.65)
\]

The expressions in equations (5.62) to (5.65) involve the evaluation of \( 2 \times 2 \) matrices rather than the \( 4 \times 4 \) system in equation (2.25) and hence are more suitable for analytical study. In the next few sections, we will use these equations to evaluate the s-parameters of a MS/NRI-TL coupled-line coupler under various terminations and weak coupling approximation. Before we can proceed with such a study, it is imperative that we examine the convergence of the series in equations (5.60) and (5.61).
5.5.1 Convergence of the infinite sum of reflected modes

In order for each equation from (5.62) to (5.65) to be meaningful, we need to ensure the convergence of the associated infinite series and to this end we closely examine the matrix product $\Gamma_0 T \Gamma_{Nd} T$. We assume that these matrices are invertible and diagonalizable at all but a finite number of discrete frequency points (we will not investigate critical points such as band edges, zero and infinite frequency). Let $P$ be a matrix of column eigenvectors and $D$ be a diagonal matrix of eigenvalues such that:

$$\Gamma_0 T \Gamma_{Nd} T P = PD$$ \hspace{1cm} (5.66)

The geometric series in equations (5.60) and (5.61) can be expressed as:

$$I + [\Gamma_0 T \Gamma_{Nd} T] + [\Gamma_0 T \Gamma_{Nd} T]^2 + \ldots = P \left( I + \sum_{n=1}^{\infty} D^n \right) P^{-1}$$ \hspace{1cm} (5.67)

The geometric sum of the diagonal matrix in (5.67) is simply a matrix formed by taking the geometric sum of each component. Hence we require the magnitude of each component of $D$, or the magnitude of each eigenvalue of $\Gamma_0 T \Gamma_{Nd} T$, to be less than unity. This also implies that the magnitude of the determinant of $D$ must be less than one. As the determinant of a matrix and its diagonal form are equal, for convergence we require that:

$$|\det \Gamma_0| |\det \Gamma_{Nd}| |\det T|^2 < 1$$ \hspace{1cm} (5.68)

For propagating modes, $|\det T| = 1$ while for evanescent and complex modes $|\det T| < 1$. From equation (5.58), the determinant of $\Gamma_0$ is:

$$\det \Gamma_0 = -\left( 1 - \frac{Z_{01}}{Z_c} \right) \left( R_\pi + \frac{Z_{02}}{R_c Z_c} \right) - \left( 1 - \frac{Z_{01}}{Z_c} \right) \left( R_c + \frac{Z_{02}}{R_c Z_c} \right)$$

$$= -\left( R_\pi - R_c \right) \left[ 1 - \frac{Z_{01} Z_{02}}{(R_c Z_c) \left( R_c Z_c \right)} \right] - \left[ Z_{01} \left( \frac{R_c}{Z_c} - \frac{R}{Z_\pi} \right) + Z_{02} \left( \frac{1}{R_c Z_c} - \frac{1}{R_c Z_\pi} \right) \right]$$

$$\left( R_\pi - R_c \right) \left[ 1 - \frac{Z_{01} Z_{02}}{(R_c Z_c) \left( R_c Z_c \right)} \right] + \left[ Z_{01} \left( \frac{R_c}{Z_c} - \frac{R}{Z_\pi} \right) + Z_{02} \left( \frac{1}{R_c Z_c} - \frac{1}{R_c Z_\pi} \right) \right]$$

$$= -\left( R_\pi - R_c \right) \left[ 1 - \frac{Z_{01} Z_{02}}{(R_c Z_c) \left( R_c Z_c \right)} \right] - \left[ Z_{01} \left( \frac{R_c}{Z_c} - \frac{R}{Z_\pi} \right) + Z_{02} \left( \frac{1}{R_c Z_c} - \frac{1}{R_c Z_\pi} \right) \right]$$

$$= -\left( R_\pi - R_c \right) \left[ 1 - \frac{Z_{01} Z_{02}}{(R_c Z_c) \left( R_c Z_c \right)} \right] - \left[ Z_{01} \left( \frac{R_c}{Z_c} - \frac{R}{Z_\pi} \right) + Z_{02} \left( \frac{1}{R_c Z_c} - \frac{1}{R_c Z_\pi} \right) \right]$$

$$\left( R_\pi - R_c \right) \left[ 1 - \frac{Z_{01} Z_{02}}{(R_c Z_c) \left( R_c Z_c \right)} \right] + \left[ Z_{01} \left( \frac{R_c}{Z_c} - \frac{R}{Z_\pi} \right) + Z_{02} \left( \frac{1}{R_c Z_c} - \frac{1}{R_c Z_\pi} \right) \right]$$

$$= -\left( R_\pi - R_c \right) \left[ 1 - \frac{Z_{01} Z_{02}}{(R_c Z_c) \left( R_c Z_c \right)} \right] - \left[ Z_{01} \left( \frac{R_c}{Z_c} - \frac{R}{Z_\pi} \right) + Z_{02} \left( \frac{1}{R_c Z_c} - \frac{1}{R_c Z_\pi} \right) \right]$$

If $Z_{01}$ and $Z_{02}$ are purely resistive, then in the complex-mode band ($R_c = R_\pi^*$ and $Z_c = -Z_\pi$), the first term in the numerator of equation (5.69) is imaginary while the second term is real. Hence the numerator and denominator terms are simply negative conjugate quantities implying that, with resistive terminations in the complex-mode
band, $|\det \Gamma_0| = 1$. Similar arguments show that $|\det \Gamma_N d| = 1$ and this satisfies the convergence condition in equation (5.68).

We also note that in our definition of the $+$ voltage vector in equation (5.50), we demanded that the total real power carried by the mode associated with this vector should be greater than zero (for propagating modes) in the $+z$ direction. This implies, with reference to equation (2.25), that:

$$\text{Re} \left( \frac{1}{Z_c} - \frac{R_c}{R_{\pi} Z_c} \right) > 0 \quad (5.70)$$

$$\text{Re} \left( \frac{1}{Z_\pi} - \frac{R_\pi}{R_c Z_\pi} \right) > 0 \quad (5.71)$$

Hence for propagating modes ($R_c, R_\pi, Z_c$ and $Z_\pi$ are all real), we can satisfy equations (5.70) and (5.71) simultaneously by requiring that $R_c > R_\pi$, $(R_c Z_\pi) > 0$ and $(R_\pi Z_c) < 0$. Examining equation (5.69) with real impedances $Z_{01}$ and $Z_{02}$ in the propagating bands, we conclude that both terms in the numerator are negative real quantities. Hence in this case $|\det \Gamma_0| < 1$ and using similar arguments, $|\det \Gamma_N d| < 1$. The convergence of the infinite series of bouncing modes is hence assured outside the complex-mode band as well.

### 5.5.2 Weak coupling assumption in the complex-mode band

Before we undertake the laborious task of deriving the $s$-parameters of the MS/NRI-TL coupler in the complex-mode band, we will formally define the weak/moderate coupling assumption. Referring to the dimensionless parameters described in equations (5.15) to (5.19), we will assume that:

$$|\kappa_L|, |\kappa_C| \ll 1 \quad (5.72)$$

The loading elements used in the coupler are selected such that they are related by same form of expression as the closed stop-band condition for the isolated NRI-TL:

$$\rho_L = \rho_C \quad (5.73)$$

When equation (5.72) is valid, we expect the coupled modes to resemble the isolated modes of the MS-TL and NRI-TL and if equation (5.73) is satisfied as well, the two isolated lines will have the same characteristic impedance (away from the edges of the Brillouin zone). In the complex-mode band we know that the net power carried by each mode is zero and hence we expect that the corresponding voltage amplitudes on the two
lines to be equal. In other words, the magnitude of the voltage ratio coefficients $R_c = R_\pi^*$ should be close to unity. Hence we let:

$$R_c = e^{j\phi}$$  \hspace{1cm} (5.74)

In equation (5.74), $\phi$ is real function of frequency in the complex-mode band. We justify this approximation by drawing attention to equation (5.21) which is also complex in this band and at the frequency $s_0$ is given by:

$$R_{c,\pi}(s_0) = \pm j \sqrt{\frac{b_2}{b_1}} = \pm j \sqrt{\frac{\rho_c}{\rho_L}}$$  \hspace{1cm} (5.75)

We have substituted the expressions for $b_1$ and $b_2$ from equations (5.28) and (5.29) and from the closed stop-band constraint, we reason that $R_{c,\pi} = \pm j$.

It is also evident that under the weak coupling assumption, the imaginary part of the modal impedance $Z_c = -Z_\pi^*$ is small compared to its real part (see Figure 5.2e). Hence we will set:

$$Z_c = Z(1 + j\delta)$$  \hspace{1cm} (5.76)

In equation (5.76), it is understood that $|\delta| \ll 1$ and in the derivations to follow, the order of the approximation will refer to number of terms we retain in the Maclaurin series expansion of functions of $Z_{c,\pi}$.

### 5.6 S-parameters of a weakly coupled MS-TL and NRI-TL

In this section, we present the scattering parameters of a weakly coupled MS/NRI-TL coupled-line coupler in the complex-mode band and consider the case where all ports are terminated by purely resistive impedances. We will use the derivations in this section to obtain design insight in the application of this coupler to practical RF/microwave components.

#### 5.6.1 S-parameters using 0-th order approximation

In the following derivation we will ignore the small imaginary part of the modal impedance $Z_c$ in equation (5.76) and will examine the case when all port impedances are real and equal to $Z_0$. We let the port and modal impedances be equal ($Z_0 = Z_c = -Z_\pi$) and
evaluate the initial excitation vector $V_{\text{init}}^\pm$ and the reflection matrix $\Gamma_0 = \Gamma_{Nd}$ from equations (5.52) and (5.58) respectively:

$$V_{\text{init}}^\pm = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & e^{j\phi} \end{pmatrix} \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix}$$ (5.77)

$$\Gamma_0 = - \begin{pmatrix} 0 & 1 \\ e^{2j\phi} & 0 \end{pmatrix}$$ (5.78)

We make the following observations from the results above. When the port impedances are set to the real part of $Z_c$ in the 0-th order approximation, a single mode is initially excited if we set one of the sources ($V_{s1}$ or $V_{s2}$) to zero in equation (5.77). More interestingly, from equation (5.78) we see mode conversion taking place as a result of reflection from the ports. Hence when line 1 is attached to source $V_{s1}$, the $c$-mode which carries power in the $+z$ direction in line 1 (see Table. 2.1) is excited. The $c$-mode exponentially decays and upon reflection at $z = Nd$ is converted to the $\pi$-mode. This $\pi$-mode decays as it travels back to the source end and the mode conversion process repeats itself. In short, when line 1 is excited, to the 0-th order approximation it is valid to state that only the $e^{-\gamma_c z}$ and $e^{+\gamma_z z}$ modes are involved. On the other hand, if line 2 is excited, we can determine the voltages and currents throughout the coupler by using only the $e^{+\gamma_c z}$ and $e^{-\gamma_z z}$ modes. Letting $\gamma_c = \gamma = \alpha + j\beta$, we now compute the port voltages using equations (5.62) to (5.65).

$$V_1 = \frac{1}{1 - e^{2(j\phi - \alpha Nd)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 & -e^{-2\alpha Nd} \\ -e^{2(j\phi - \alpha Nd)} & 1 \end{pmatrix} \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix}$$ (5.79)

$$V_2 = \frac{1}{1 - e^{2(j\phi - \alpha Nd)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} e^{j\phi} & -e^{j\phi - 2\alpha Nd} \\ -e^{-j\phi - 2\alpha Nd} & e^{-j\phi} \end{pmatrix} \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix}$$ (5.80)

$$V_3 = \frac{1}{1 - e^{2(j\phi - \alpha Nd)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} e^{-\gamma Nd} & -e^{-\gamma^* Nd} \\ -e^{2j\phi - \gamma Nd} & e^{-\gamma^* Nd} \end{pmatrix} \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix}$$ (5.81)

$$V_4 = \frac{1}{1 - e^{2(j\phi - \alpha Nd)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} e^{j\phi - \gamma Nd} & -e^{j\phi - \gamma^* Nd} \\ -e^{j\phi - \gamma^* Nd} & e^{j\phi - \gamma Nd} \end{pmatrix} \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix}$$ (5.82)
We can use equation (5.49) to compute the s-parameters of the coupler by setting \( Z_0i = Z_0 \) for all ports; \( V_{si} = I_{pi}Z_0 + V_i \) at ports 1 and 2 and \( V_i = -Z_0I_{pi} \) at ports 3 and 4.

\[
S_{11} = \left. \frac{2V_i - V_{s1}}{V_{s1}} \right|_{V_{s2}=0} = 0 \tag{5.83}
\]

\[
S_{22} = \left. \frac{2V_2 - V_{s2}}{V_{s2}} \right|_{V_{s1}=0} = 0 \tag{5.84}
\]

\[
S_{21} = \left. \frac{2V_2}{V_{s1}} \right|_{V_{s2}=0} = \frac{1}{\cos(\phi) - j\sin(\phi) \coth(\alpha Nd)} \tag{5.85}
\]

\[
S_{31} = \left. \frac{2V_3}{V_{s1}} \right|_{V_{s2}=0} = e^{-j\beta Nd} \cosh(\alpha Nd) + j\sinh(\alpha Nd) \cot(\phi) \tag{5.86}
\]

\[
S_{42} = \left. \frac{2V_4}{V_{s2}} \right|_{V_{s1}=0} = e^{j\beta Nd} \cosh(\alpha Nd) + j\sinh(\alpha Nd) \cot(\phi) \tag{5.87}
\]

\[
S_{41} = \left. \frac{2V_4}{V_{s1}} \right|_{V_{s2}=0} = 0 \tag{5.88}
\]

In equations (5.83) to (5.88), \( \phi \) is the transverse voltage phase difference between the two lines for the c-mode in a MS/NRI-TL coupler (operated in the complex-mode band) which has \( N \) sections of length \( d \) each. Examining the s-parameters given above, we conclude that the coupler is matched at all ports and is isolated at port 4 (located diagonally opposite to the excited port) in the 0-th order approximation. Moreover, the backward coupling from port 1 into adjacent port 2 increases to unity when the length of the coupler \( Nd \) is made sufficiently large (see equation (5.85)). As expected, the through port (port 3) exhibits a simple \( e^{-j\beta Nd} \) phase shift (with respect to the input at port 1) for short coupler length (see equation (5.86)), and the power delivered to this port drops to zero as the length of the coupler is increased.

We can improve upon these ideal results (frequency independent perfect match at all ports and perfect isolation) by retaining first order terms which contain the small imaginary component of the modal impedance in equation (5.76). The relatively simple derivation carried out in this section captures most of the essential features of the MS/NRI-TL coupler which is operated in the complex-mode band under weak coupling, closed stop-band and matched resistive termination (\( Z_0 = \text{Re}(Z_c) \)) constraints.

### 5.6.2 S-parameters using 1-st order approximation

In the following discussion, we will now retain only the 1-st order terms in \( \delta \) which is defined in equation (5.76). We will again consider the case where all port impedances
are real and equal to \( \text{Re}(Z_c) \). The initial excitation vector at \( z = 0 \) and the reflection matrix in this case are:

\[
V_{\text{init}}^{+} = \frac{1}{4} \begin{pmatrix}
(2 + j\delta) & j\delta e^{j\phi} \\
-j\delta e^{2j\phi} & (2 - j\delta) e^{j\phi}
\end{pmatrix} \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix}
\]

\( \Gamma_0 = \begin{pmatrix}
-j\delta \cos(\phi) e^{j\phi} & -(1 + j\delta) \\
-(1 - j\delta) e^{2j\phi} & j\delta \cos(\phi) e^{j\phi}
\end{pmatrix}
\]

(5.89)

(5.90)

Using the same procedure employed in the previous section, we can now evaluate the s-parameters of the coupler:

\[
S_{11} = \frac{\delta \sin(\phi) (\cosh(2\alpha N d - j\phi) - \cos(\phi)e^{-2j\beta N d})}{\sinh^2(\alpha N d - j\phi)}
\]

(5.91)

\[
S_{22} = -\frac{\delta \sin(\phi) (\cosh(2\alpha N d - j\phi) - \cos(\phi)e^{2j\beta N d})}{\sinh^2(\alpha N d - j\phi)}
\]

(5.92)

\[
S_{21} = \frac{1}{\cos(\phi) - j \sin(\phi) \coth(\alpha N d)} e^{-j\beta N d}
\]

(5.93)

\[
S_{31} = \frac{\cosh(\alpha N d) + j \sinh(\alpha N d) \cot(\phi)}{e^{j\beta N d}}
\]

(5.94)

\[
S_{42} = \frac{\cosh(\alpha N d) + j \sinh(\alpha N d) \cot(\phi)}{e^{j\beta N d}}
\]

(5.95)

\[
S_{41} = \frac{j\delta \csc(\phi) \sech(\alpha N d) \sin(\beta N d)}{(1 + j \cot(\phi) \tanh(\alpha N d))^2}
\]

(5.96)

From equations (5.91) to (5.96), we infer that the coupling (\( S_{21} \)) and insertion loss of the lines (\( S_{31} \) and \( S_{42} \)) are identical to the ones obtained from the 0-th order analysis. We observe a first order degradation in matching at the various ports (\( S_{11} \) and \( S_{22} \)) and deviation from perfect isolation (\( S_{41} \)). Nevertheless the mismatch is small and the isolation can be made perfect by setting \( \beta N d = \pi \) in equation (5.96) at the design frequency. At the frequency \( s_0 \) defined in equation (5.27), \( R_c = j \) (see discussion in section 5.5.2) and hence \( \phi = \pi/2 \). If we substitute this value of \( \phi \) into the s-parameter equations derived under the 1-st order approximation, we obtain:

\[
S_{11} = j\delta \tanh(\alpha N d)
\]

(5.97)

\[
S_{22} = -j\delta \tanh(\alpha N d)
\]

(5.98)

\[
S_{21} = j \tanh(\alpha N d)
\]

(5.99)

\[
S_{31} = \sech(\alpha N d) e^{-j\beta N d}
\]

(5.100)

\[
S_{42} = \sech(\alpha N d) e^{j\beta N d}
\]

(5.101)

\[
S_{41} = j\delta \sech(\alpha N d) \sin(\beta N d)
\]

(5.102)
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The s-parameters listed above in equations (5.97) to (5.102) are identical to the ones given in [7] where they were derived by directly inverting the matrix equation in (5.48) at the single frequency point $s_0$. Examining equations (5.97) and (5.98), we see that the input impedances looking into ports 1 and 2 of the coupler are of the form $Z(1 \pm 2j\delta \tanh(\alpha N d))$. This is to be expected from our consideration of the 0-th order analysis where we witnessed dominant excitation of the $c$-mode or the $\pi$-mode depending on whether the source was attached to port 1 or port 2 respectively. Hence we may expect to improve the port match of the coupler by adding small reactive elements in series to the ends of the MS/NRI-TL coupler.

5.6.3 MS/NRI-TL coupler with perfect match and isolation

The impedances of the complex modes that are excited in the MS/NRI-TL coupler are also complex and result in non-zero return-loss at the output ports (see equations (5.97) and (5.98)). In many designs, this port mismatch is quite low as is evident from the examples of the 3-dB and high-directivity prototypes discussed in Chapter 6. If the application requires an extremely well-matched coupler, then this is possible by adding compensating reactances to the various output ports. The technique to compensate for the imaginary part of the complex modal impedance and achieve zero return-loss designs using the MS/NRI-TL coupler will be addressed in this section. Now we will derive the S-parameters of an MS/NRI-TL coupler under 1-st order approximation in the complex-mode band when small series reactances $\pm j\epsilon Z$ (with $\epsilon$ real) are attached to its ends. The application of equations (5.62) to (5.65) will require the convergence of the infinite sum of reflected modes when the coupler is terminated with complex impedances. In section 5.5.1, we verified convergence only for real terminations and will assume that it also holds for the present case.

To state the problem precisely, we will first evaluate equations (5.62) to (5.65) by setting $Z_{01} = Z_{03} = Z(1 + j\epsilon)$ and $Z_{02} = Z_{04} = Z(1 - j\epsilon)$ along with the modal impedance $Z_c = Z(1 + j\delta)$. We will ignore second order and higher terms in $\delta$ and $\epsilon$ and evaluate the s-parameters of the coupler with respect to the system impedance $Z$ (the real part of the modal impedance).
The initial mode excitation vector and reflection matrix in this case are:

\[
V_{\text{init}}^+ = \frac{1}{4} \begin{pmatrix}
2 + j(\delta - \epsilon) & je^{j\phi}(\delta + \epsilon) \\
-je^{2j\phi}(\delta + \epsilon) & e^{j\phi}(2 - j(\delta - \epsilon))
\end{pmatrix}
\begin{pmatrix}
V_{s1} \\
V_{s2}
\end{pmatrix}
\] (5.103)

\[
\Gamma_0 = e^{i\phi} \begin{pmatrix}
\epsilon \sin(\phi) - j\delta \cos(\phi) & -e^{-j\phi}(1 + j\delta) \\
-e^{j\phi}(1 - j\delta) & -\epsilon \sin(\phi) + j\delta \cos(\phi)
\end{pmatrix}
\] (5.104)

We need to modify the expression for the scattering parameters in (5.49) to account for the fact that the line voltages in equations (5.62) to (5.65) are not equal to the port voltages. In the present scenario, we have to de-embed the series reactances \(\pm j\epsilon Z\) between the MS/NRI-TL coupler and the ports (of impedance \(Z_0 = Z\)). The de-embedded s-parameters are obtained by adding the voltage drop across the reactive loadings to the line voltages in (5.62):

\[
S_{mn} = \left. \frac{V_n + j\epsilon Z I_{pm}(-1)^{m+1}}{V_n + j\epsilon Z I_{pm}(-1)^{n+1} + Z I_{pm}} \right|_{V_{sk} = 0, (k \neq n)}
\] (5.105)

\[
I_{pk} = \frac{V_{sk} - V_k}{Z(1 + j\epsilon(-1)^{k+1})}
\] (5.106)

We will choose \(\epsilon\) such that \(S_{11}\) and \(S_{22}\) vanish at the frequency \(s_0\). Setting \(\phi = \pi/2\) along with \(S_{11} = 0\) and \(S_{22} = 0\) results in the following two relations between \(\epsilon\) and \(\delta\):

\[
\frac{\epsilon}{\delta} = -\frac{\sinh(2\alpha N d)}{\cosh(2\alpha N d) + e^{-2j\beta N d}} \quad (S_{11} = 0) \quad (5.107)
\]

\[
\frac{\epsilon}{\delta} = -\frac{\sinh(2\alpha N d)}{\cosh(2\alpha N d) + e^{+2j\beta N d}} \quad (S_{22} = 0) \quad (5.108)
\]

We observe that in order to satisfy both equations (5.107) and (5.108) simultaneously with real values of \(\epsilon\) and \(\delta\), we require \(\beta N d\) to be an integer multiple of \(\pi/2\). From the results in section 5.6.2, we know that the MS/NRI-TL coupler exhibits perfect isolation when \(\beta N d = \pi\) and for this electrical length:

\[
\epsilon = -\tanh(\alpha N d)\delta
\] (5.109)

Hence at the frequency \(s_0\), we expect to obtain perfect matching at all ports of the coupler by attaching in series \(-j\delta Z \tanh(\alpha N d)\) reactances to ports 1 and 3 and their negative values to ports 2 and 4 if the length of the coupler is half its guided wavelength \((\beta N d = \pi)\). This result is plausible from our discussion at the end of section 5.6.2 where we noticed an input impedance mismatch of \(2j\delta Z \tanh(\alpha N d)\) without reactive loading.
The condition in (5.109) conjugately matches all ports of the coupler and we will now evaluate the rest of its s-parameters at the frequency $s_0$ (i.e. we fix $\phi = \pi/2$).

\[
\begin{align*}
S_{11} &= -\delta \sin(\beta N d) \tanh(\alpha N d) \sech^2(\alpha N d) e^{-j\beta N d} \\
S_{22} &= -\delta \sin(\beta N d) \tanh(\alpha N d) \sech^2(\alpha N d) e^{j\beta N d} \\
S_{21} &= j \tanh(\alpha N d) \\
S_{31} &= e^{-j\beta N d} \sech(\alpha N d) (1 + j\delta \tanh(\alpha N d)) \\
S_{42} &= e^{j\beta N d} \sech(\alpha N d) (1 - j\delta \tanh(\alpha N d)) \\
S_{41} &= j\delta \sin(\beta N d) \sech^3(\alpha N d)
\end{align*}
\] (5.110-5.115)

From equations (5.110) to (5.115) we observe that when $\beta N d = \pi$, the coupler exhibits perfect isolation and match at all ports. Before we conclude our discussion in this section, we need to investigate the issue of convergence of the infinite sum of reflected coupled modes when the coupler is terminated with complex-conjugate impedances.

We explore convergence in the complex-mode band at the frequency $s_0$ for the present case by setting $Z_{01} = Z_{02}^* = Z(1 + j\epsilon)$, $Z_c = -Z_{\pi}^* = Z(1 + j\delta)$ and $R_c = R_{\pi}^* = j$ in equation (5.69) to obtain:

\[
\det(\Gamma_0) = -\frac{2 + (\epsilon - \delta)^2}{2 + (\epsilon + \delta)^2}
\] (5.116)

In the complex-mode band $\det(T) = e^{-2\alpha N d} < 1$. Let the ratio $\epsilon/\delta = -x$ where $x$ is some real number, then the convergence condition $|\det(\Gamma_0 T)| < 1$ can be expressed as:

\[
|x - \tanh(\alpha N d/2)| [x - \coth(\alpha N d/2)] > -\frac{2}{\delta^2}
\] (5.117)

If we substitute $x = \tanh(\alpha N d)$ (see equation (5.109)), the left hand side of equation (5.117) reads $(-1) [1 - \tanh^2(\alpha N d/2)]^2 / [1 + \tanh^2(\alpha N d/2)]^2$, whose magnitude is clearly less than unity. As $\delta \ll 1$, the inequality in equation (5.117) is fulfilled in our case and convergence of the infinite sum of reflected modes is hence established. This confirms our assumption at the beginning of this section and mathematically validates the formulae which describe the operation of a perfectly matched and isolated MS/NRI-TL coupled-line coupler.

## 5.7 Verification of theoretical results

The theory presented in this chapter is based on the idea that 2 independent eigenmodes (and their reflections) form a complete modal set which describes every possible state
of the MS/NRI-TL coupler and that the amplitudes of these modes are controlled by the source and terminations attached to it. Modal analysis showed us the existence of complex-conjugate modes when the two lines in their isolated state had equal wavenumbers.

We then adopted the homogeneous approximation (Schelkunoff’s coupled-mode theory) over the more accurate periodic Bloch/Floquet analysis based on the argument that the periodicity of the NRI-TL line in the coupler is usually much smaller than the guided wavelength. We established that the operation of the coupler, can be described by an infinite set of bouncing waves similar to those in a single transmission line. The algebraic difficulty in presenting closed-form expressions for the s-parameters of the coupler in the general case, led us to develop design equations which are valid for moderate spacing between the two lines and we focussed on the operation inside the complex-mode band.

We will now test the validity and usefulness of our formulation by comparing the predictions of equations (5.110) to (5.115) with simulation results from a commercial microwave circuit simulator (Agilent’s Advanced Design System). We will fix the frequency of operation at $s_0$ and test how the s-parameter equations hold up as a function of the coupler length, the line separation and the periodicity of the loading elements. The effects of dissipation will also be investigated.

5.7.1 Variation of s-parameters with coupler length

We examine the variation of the s-parameters of the MS/NRI-TL coupled-line coupler as a function of the number of unit cells at the single frequency point $s_0$ in equation (5.27). We will utilize the characteristics of the Rogers TMM4 substrate ($\varepsilon_r = 4.5$) of height 1.27 mm and will neglect substrate and conductor dissipation for the simulation results presented in this section.

The schematic of the simulation setup is depicted in Figure 5.4 where the contents of the unit cell, the impedances of the ports (50.092Ω) and the frequency (2.0285 GHz) remains fixed. The reactive loadings seen at the ends of the coupler in Figure 5.4a provide impedances of $\pm jZ\varepsilon$ (see equation (5.109)) and hence depends on the number of unit cells $N$ used in the simulation. We simulate couplers ranging from 1 to 20 unit cells in length with the physical and electrical parameters given in Table 5.2. The loading elements for the NRI-TL line used in the coupler were selected to satisfy equation (5.73). Note that the shunt inductors in Figure 5.4b are twice the value listed in Table 5.2 such
Figure 5.4: (a) Simulation set-up of the MS/NRI-TL coupler in Agilent ADS with 8 unit cell blocks (b) Schematic of each unit cell of the coupler where the upper line is MS-TL and the lower is NRI-TL.
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| Physical parameters (on TMM4 1.27 mm substrate) | Microstrip width: 2.369 mm  
Line spacing: 2.006 mm  
Unit-cell size: 5.0 mm |
|------------------------------------------------|----------------------------------------------------------|
| Electrical parameters                            | $Z_e = 54 \, \Omega$  
$Z_0 = 46 \, \Omega$  
$\epsilon_e = 3.638$  
$\epsilon_o = 3.153$ |
| Coupled-mode parameters ($s_0/2\pi = 2.029 \, \text{GHz}$) | $L_0 = 307.8 \, \text{nH/m}$  
$L_m = 35.53 \, \text{nH/m}$  
$L = 4.996 \, \text{nH}$  
$C_0 = 123.2 \, \text{pF/m}$  
$C_m = -5.467 \, \text{pF/m}$  
$C = 2.0 \, \text{pF}$  
$\gamma_c = 6.267 + j78.537 \, \text{m}^{-1}$  
$Z_c = 50.092 - j1.768 \, \Omega$ |

Table 5.2: Simulation parameters of the MS/NRI-TL coupler

that when these cells are periodically cascaded, adjacent inductors combine in parallel to half their individual values.

The results of this simulation setup along with theoretical predictions from the s-parameter equations (5.110) to (5.115) are shown in Figure 5.5. Examining the magnitude responses of the reflection $S_{11}$, coupling $S_{21}$, through $S_{31}$ and isolation $S_{41}$ in plots 5.5a to 5.5d, we observe excellent agreement between simulation and theory. We observe deviations at the nulls of $S_{11}$ and $S_{41}$ where second order effects have to be considered. Nevertheless, at these points the reflection and isolation are below $-50 \, \text{dB}$ and as such, the absolute differences between simulation and theory are vanishingly small.

The phase plots in 5.5e to 5.5h show good agreement as well although small deviations are expected. At very low return-loss levels, second-order effects will play a significant role in determining the phase. Nevertheless we see surprisingly good agreement in the phase plot of the isolation level (see Figure 5.5h). These results validate the theory which we developed in this section and the derived s-parameter equations provide us with valuable tools in the design of MS/NRI-TL couplers.

5.7.2 Variation of s-parameters with line spacing

The closed form s-parameters derived in this chapter were based on moderate separation between the lines which allows us to assume that the imaginary part of the modal impedance is much smaller than its real part in the complex-mode band. We observed excellent agreement between simulation and theory with a spacing of 2 mm and expect
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Figure 5.5: Comparison of the magnitude (in plots (a) to (d)) and phase (in plots (e) to (h)) of the theoretical (see equations (5.110) to (5.115)) and simulated s-parameters of MS/NRI-TL couplers as a function of the number of unit cells (of approximate electrical length $\pi/8$ radians each) and operated at 2.03 GHz. Theoretical results are in red and ADS simulation results are in black.
this to remain true if the spacing is increased. Now we will decrease the line spacing from 2 mm to 0.05 mm for a \( N = 10 \) unit cells long coupler operated at the frequency \( s_0 \). The unit cell size and loading capacitor for the NRI-TL line will be maintained at 5 mm and 2 pF respectively as in the previous case. The substrate is the same TMM4 as before and the width of the lines will be held fixed at 2.4 mm. The operating frequency is around 2 GHz in this case.

The variation of line spacing results in the change of the four parameters \( (L_0, C_0, L_m, C_m) \) that describe the coupled microstrip-line segments of the coupler. As the loading capacitor is fixed, for each line spacing we choose a different loading inductor according to equation (5.73). We find out that this shunt inductance varies from 4.918 nH to 3.444 nH when the spacing is swept from 2 mm to 0.05 mm. The frequency \( s_0 \) changes as well, and we carry out the simulations at different frequency points based on the spacing. The simulation frequency varies from 2.036 GHz to 2.151 GHz with decreasing spacing.

With different \( \gamma_c \) and \( Z_e \) for each line spacing, we adjust the matching reactances and port impedances such that they sum to \( Z(1 \pm j\epsilon) \). The simulation results in this case are presented in Fig. 5.6 along with predictions from equations (5.110) and (5.115).

We observe good agreement between theory and simulation from the plots in Figure 5.6 when the line spacing is maintained above 0.2 mm corresponding to coupled transmission lines with \( Z_e = 61.4 \, \Omega \) and \( Z_o = 32.1 \, \Omega \). The error between simulation and theory for line spacing less than 0.2 mm is greatly magnified in the \( S_{11} \) and \( S_{41} \) plots owing to additive second order effects which are more pronounced when the magnitude of the s-parameter is very small. We observe negligible deterioration in the accuracy of the theory in the coupling and transmission plots in Figures 5.6b and 5.6c. Hence these equations still serve as useful tools in the estimation of the coupling level for line spacings as small as 0.05 mm\(^1\).

### 5.7.3 Variation of s-parameters with cell size

Having examined the agreement between theory and simulation as functions of the number of unit cells and line spacing, in the next set of simulations we study the effect of cell size \( d \) on the s-parameters. We expect good agreement for small cell sizes compared

\[ ^1 \text{A line spacing of 0.05 mm is very challenging to achieve using standard PCB photo-lithographic techniques.} \]
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Figure 5.6: Comparison of the magnitude (in plots (a) to (d)) and phase (in plots (e) to (h)) of the theoretical and simulated s-parameters of MS/NRI-TL couplers as a function of the line spacing. Ten unit cells of length 5 mm each were used in each simulation along with constant line width. Theoretical results are in red and ADS simulation results are in black.
to the guided wavelength as under this condition, the assumption of axial homogeneity (implicit in Schelkunoff’s coupled differential equations) is valid.

For large cell sizes, the derivations in section 5.6 can be used provided that we evaluate $\gamma_{c,\pi}, R_{c,\pi}$ and $Z_{c,\pi}$ using the Floquet modal formulation in equations (5.41) to (5.43). This also implies that for large cell sizes, equations (5.27) and (5.73) used to determine the values of $s_0$ and the choice of loading elements which set $R_{c,\pi} = \pm j$ will have to be generalized to the periodic Floquet case.

The simulation setup in the present case is as follows. On a TMM4 substrate we consider a coupler with $N = 10$ unit cells on which the microstrip lines are 2.4 mm wide and separated by a distance of 2 mm. The loading capacitance of the NRI-TL is taken to be 1 pF and the cell size is varied from 3 mm to 39 mm spanning the electrical length from $0.137\pi$ to $0.497\pi$ radians (with respect to the phase shift of a 50 Ω microstrip line printed on the same substrate and operated at the same frequency).

From equation (5.27), the center frequency $s_0$ depends on the cell size and varies from 3.705 GHz to 1.033 GHz with increasing $d$. Although the width and spacing between the lines are held fixed, being slightly dispersive, the coupled microstrip-lines exhibit changes in $L_0, C_0, L_m$ and $C_m$ with frequency. Accordingly the shunt inductive loading of the NRI-TL line varies from 2.465 nH to 2.458 nH with increasing cell size such that the constraint in equation (5.73) is maintained. We apply similar changes as before to the port impedances and the associated matching reactances for each cell size. The comparison between theory and simulation is depicted in Figure 5.7.
Figure 5.7: Comparison of the magnitude and phase of the theoretical and simulated s-parameters of MS/NRI-TL couplers (10 unit cells long) as a function of the cell size measured in terms of the guided wavelength. Theoretical results are in red and ADS simulation results are in black.

We observe good agreement between theory and simulation up to a cell size of \(0.3\pi\) radians corresponding to a physical length of approximately 15 mm at 1.66 GHz on the TMM4 substrate of relative permittivity of 4.5. We have excluded the isolation and reflection plots in this case due to their inconclusive nature. The deviation between theory and simulation in \(S_{11}\) and \(S_{41}\) is a strong function of the total length (with largest deviation at their respective nulls) of the coupler and hence masks the effect of inhomogeneity which we are after.

If a large cell size is unavoidable\(^2\), we have to resort to equations (5.41) to (5.43) to obtain accurate propagation constants and impedances for use in the s-parameter equations found in section 5.6.3. In this case, the equations for choosing the design frequency \(s_0\) and loading elements based on Schelkunoff’s formulation will have to be abandoned for the more general approach. This will involve a pair of simultaneous

\(^2\)For instance, a high frequency design or one involving monolithic printed loading elements of large values may require an electrically large unit cell.
equations where we set $a_{11} = a_{22}$ in equation (5.41) to obtain an expression for $s_0$. Next we will have to set $a_{12} = -a_{21}$ in (5.42) at the frequency $s_0$ to obtain an expression relating the loading elements to the coupled-line parameters $Z_e$, $Z_o$, $\gamma_e$ and $\gamma_o$. We expect this expression to approach equation (5.73) in the limit of small cell size. These ideas won’t be pursued any further in this work and we will limit ourselves to electrically small cell sizes which allow the definition of a homogeneous negative-index 'material' in the first place.

### 5.7.4 Variation of s-parameters with loss

We will now repeat the simulations carried out in section 5.7.1 but will add substrate and conductor losses to investigate expected deviations from theoretical results in the presence of finite dissipation. We will assume the capacitors used in the design are lossless and that inductors have finite quality factors.

We set the dielectric loss tangent to 0.0023 and copper conductivity to 60 MS m$^{-1}$ and assume a copper thickness of 17 $\mu$m on a 50 mils Rogers TMM4 substrate. We also assume a quality factor of 75 for the inductors in the coupler\(^3\). The rest of the parameters for the lossy simulations are the same as the ones outlined in Table 5.2.

We observe (see Figure 5.8) that the losses primarily affect the return loss of the coupler and in particular, the predicted phase deviates the most from those seen in the simulation. This is to be expected considering the magnitude of $S_{11}$ is quite small for all simulation points. Interestingly we observe that the isolation $S_{41}$ suffers negligible deviation due to the presence of loss.

### 5.8 Guidelines for designing a MS/NRI-TL coupler

We conclude this chapter by providing a brief strategy for the realization of MS/NRI-TL couplers to meet design specifications. First we have to avoid very closely spaced lines such that the difference between the even and odd mode impedances do not exceed $Z_e - Z_o = (61.4 - 32.1) \, \Omega$ (see discussion at the end of section 5.7.2). To be able to use the homogeneous approximation, the unit cell size should be maintained below three twentieths (or about a seventh) of the guided wavelength. These conditions will enable

\(^3\)Surface mount chip inductors of 5 nH inductance with Q-factors around 100 at 2.0 GHz are common in the market.
Figure 5.8: Comparison of the magnitude (in plots (a) to (d)) and phase (in plots (e) to (h)) of the theoretical and simulated s-parameters of lossy MS/NRI-TL couplers as a function of the number of unit cells. Theoretical results are in red and ADS simulation results are in black.
the proper usage of design equations (5.110) to (5.115).

Some applications of the coupler will require very high isolation (for instance, in signal monitoring using reflectometry) and we will consider this case first. The design frequency \( s_0 = j\omega_0 \), system impedance \( Z_0 \) and coupling level \(|S_{21}|\) are usually fixed by the intended application. The parameters which we have control over are the width of the lines, the line spacing, the ratio of loading elements \( L/C \) and the product of cell size and number of cells \( Nd \). We list a set of design steps below:

- **System impedance**: The line width primarily influences the system impedance of the coupler and as a first iterative step, one can choose this width to be that of an isolated microstrip line of the desired system impedance (i.e \( Z_{MS-TL} \approx Z_0 = \text{Re}(Z_c(s_0)) \)). Given the design frequency \( \omega_0 \), most microwave circuit simulators can provide its guided wavenumber \( \beta_{MS-TL} \) as well. This impedance and wavenumber will be the first estimate of the coupler’s \( L_0 \approx \beta_{MS-TL} Z_0/\omega_0 \) and \( C_0 \approx \beta_{MS-TL}/Z_0/\omega_0 \).

- **Isolation**: In the first iteration, we can assume that \( \beta_{MS-TL} \) is equal to the imaginary part of the complex propagation constant at the design frequency. This implies that for perfect isolation, we need \( \beta_{MS-TL} Nd \approx \pi \). Now we let \( N = 4 \) such that we satisfy the homogeneity requirement of at least 7 cells per wavelength. An estimate of the required size of each unit cell \( d \) is now available.

- **Design frequency**: At \( s_0 \), the magnitudes of the wavenumbers of the isolated MS-TL and NRI-TL (that make up the coupler) are approximately equal. Having fixed the cell size and line width at \( s_0 \) in this iterative step, we can only tune the ratio of the loading elements \( L/C \) such that the NRI-TL wavenumber equals that of its MS-TL counterpart. In the homogeneous limit, setting \( \gamma_{nri} d = j\theta \) in equation (1.10) along with the constraint in equation (5.73) can be used to determine the shunt \( L \) and series \( C \) components which load each unit cell of the coupler.

- **Coupling level**: Substituting the desired coupling level into equation (5.112), we obtain the imaginary part \( \alpha \) of the complex wavenumber at \( s_0 \). This parameter is primarily influenced by the spacing between the two lines. As a first approximation, we set \( \gamma_e = \gamma_o \) in equations (5.9) to (5.14) to obtain \( Z_0^2 \approx L_0/C_0 \approx Z_c Z_0 \) and \( L_m/C_m \approx -Z_c Z_o \). Hence for the first iteration, we set the ratio of the coupling inductance to capacitance equal to the negative of the square of the system
impedance. The mutual capacitance term $C_m$ is negative for coupled microstrip lines. We can obtain an equation for $\gamma_{c,\pi}$ (see equation (5.20)) in terms of $L_0$, $C_0$, $L$, $C$ and $C_m$ and a root finder can be used to solve this equation for $C_m$ with $\gamma_c = \alpha + j\beta_{MS-TL}$. We can now compute the required even and odd mode impedances of coupled microstrip lines and determine their spacing either from design curves or a microwave circuit simulator.

- **Second iteration:** The geometric parameters obtained in the first iteration can be used to determine accurate values for $L_0$, $C_0$, $L_m$ and $C_m$. These quantities along with $L$, $C$, $d$ and $N = 4$ can be used in equations (5.20) to (5.27) and equation (5.112) to compare the design frequency, system impedance and coupling level with the desired specifications. The design frequency is primarily influenced by the loading elements and length of each unit cell. Increasing $L$ (while maintaining the ratio in equation (5.73)) and $d$ lowers the value of $s_0$. The system impedance and the coupling level can be tweaked by the line width and spacing respectively.

- **Final design:** Higher iterations can be carried out to meet the specifications to any desired level of accuracy. It should be borne in mind, that excessive number of iterations is unnecessary at the circuit level design of a MS/NRI-TL coupler. Tuning of the coupler layout in a full-wave EM solver is unavoidable to account for losses, mutual coupling between adjacent components, finite extension of the idealized lumped component, axial inhomogeneity, uneven current flow on the microstrip lines\(^4\)...etc. The final step in the circuit design will involve the choice of the matching reactances according to equation (5.109).

In applications where the coupler will be used simply as a signal sampler or a power divider, high isolation might not be necessary. This will allow us to design even more compact sized couplers which are half as long as the ones with perfect isolation. When we re-examine equations (5.107) and (5.108), we notice that perfect matching at all ports can also be achieved by setting $\beta Nd = \pi/2$ albeit with different matching reactance than the one in (5.109). The design procedure in this case is again similar to that outlined above for the case with high isolation.

We can introduce yet one more degree of freedom into our design if perfect matching

\(^4\)The current flow across the width of each microstrip is not uniform as a function of axial distance due to the loading elements which alters the effective coupling parameters of the lines.
is unnecessary. We can achieve better than 20 dB of return loss without the additional matching reactances and this can be sufficient for most applications. Moreover, in the process we avoid the use of additional lumped components in our design. In this case, the electrical length of the coupler can be chosen arbitrarily and the main difference in the design procedure is in the use of s-parameter equations given in (5.91) to (5.96).

Having presented the theory of operation of the MS/NRI-TL coupler, its s-parameters and design guidelines, we will show actual realizations, experimental results and a few representative applications in the next chapter.
Chapter 6

Design and Applications of MS/NRI-TL Couplers

6.1 Introduction

In Chapter 2 we outlined some of the RF/microwave applications of couplers and compared both the benefits and limitations of standard topologies. The choice of the appropriate coupler for a given application is determined by a number of factors such as cost, bandwidth, coupling level, return loss, insertion loss and isolation. Planar couplers are cost-effective as they can be manufactured using standard photo-lithographic processes. In this chapter we will present simulation and experimental results of a few devices which employ the planar MS/NRI-TL coupler.

The analysis of the MS/NRI-TL coupled-line coupler carried out in previous chapters revealed that it was capable of providing arbitrary coupling levels, good isolation and low return loss. First we will outline a systematic design methodology which will furnish the engineer with starting parameters of the MS/NRI-TL coupler to meet the application specifications. Then we will document three possible applications of this coupler. We will describe the design and construction of a 3-dB prototype which splits power equally between the through and coupled ports. Next we present a high-directivity implementation of the coupler for reflectometry. Finally we describe a compact corporate 1 : 4 power divider for feeding printed antennas such us dipole or patch arrays. All prototypes considered in this chapter were designed, simulated, fabricated and tested in-house.
6.2 Synthesis of MS/NRI-TL couplers

The choice of a commercial substrate of a given thickness depends on a number of factors such as availability, cost, dissipation, compactness of the design, machining requirements (such as milling, chemical etching, drilling and through-hole plating), chip component placement (if they are used in the design), etc. Once the substrate is chosen, the specifications that have to be met include the system impedance $Z_0$, the coupling level $S_{21}$, operating frequency $\omega_0$ and the isolation level. Meeting the bandwidth and insertion loss specifications is beyond the scope of this work and we will only address these issues qualitatively.

The physical design parameters that we have to optimize are $w/h$ (normalized width of each microstrip line where $h$ is the substrate height), $s/h$ (normalized spacing$^1$), $L$ (loading shunt inductance), $C$ (loading series capacitance), $N$ (number of unit cells) and $d$ (length of each unit cell). Typically, a three step synthesis process which involves the numerical evaluation of the dispersion and scattering parameters of the coupler using equations in Chapter 5, followed by microwave circuit simulation and full-wave electromagnetic simulation have to be carried out.

Stage 1: Ideal coupler synthesis using design equations

All components of the the ideal MS/NRI-TL coupler are lossless and non-radiating and in addition, the coupled microstrip lines are assumed to be non-dispersive and the loading elements are lumped. Before addressing the synthesis problem, we will list the equations from Chapter 5 that can be used to evaluate the scattering parameters of the coupler very accurately compared to microwave circuit simulations.

- The MS/NRI-TL coupler dispersion parameters $\gamma_{c,\pi}$, $R_{c,\pi}$ and $Z_{c,\pi}$ can be obtained using equations (5.30) to (5.43).

- The mode excitation amplitudes $V_{c}^+$, $V_{c}^-$, $V_{\pi}^+$ and $V_{\pi}^-$ can be obtained from equation (5.48).

- The scattering parameters of the chosen coupler geometry can be obtained using equation (5.49).

$^1$Note that $s$ here refers to the line spacing and not the complex frequency variable.
The availability of cheap computing power and standard optimization algorithms means that the steps outlined above can potentially be used to numerically synthesize couplers which satisfy design specifications. This is often unnecessary, especially when the lumped loadings have to be implemented using distributed components such as transmission-line stubs and interdigital capacitors, and the final design will have to be tuned to compensate for parasitic coupling between components, discontinuity effects, losses ... etc. Commercial simulators are usually equipped with tuning and optimization capabilities and hence it will be useful to have design charts that provide the engineer with physical parameters that yield performance close to the specifications. We have included sample design charts for the coupler on a substrate with dielectric constant $\varepsilon_r = 4.5$ in Appendix H and will now list the relevant equations that can generate such charts for any arbitrary dielectric constant.

Examining equations (5.20) to (5.27) at the design frequency $\omega_0 = j\omega_0$ with the additional constraint that $\rho_L = \rho_C$, the modal propagation constants and impedances of the MS/NRI-TL complex modes can be expressed as:

\[
\left( \frac{\gamma}{\omega_0} \right)^2 = \left( \frac{\alpha}{\omega_0} + j \frac{\beta}{\omega_0} \right)^2 = -(L_0 \mp jL_m)(C_0 \pm jC_m) \quad (6.1)
\]

\[
Z^2 = \frac{L_0 \mp jL_m}{C_0 \pm jC_m} \quad (6.2)
\]

\[
\omega_0^2 = \frac{1}{2L_0Cd} = \frac{1}{2C_0Ld} \quad (6.3)
\]

The absolute value of the real part of equation (6.2) is taken as the system impedance. Let $\beta Nd = \theta_0$ be the electrical length of the coupler where $\beta$ is the imaginary part of its propagation constant at $\omega_0$. With $\theta_0$ equal to either $\pi/2$ or $\pi$, we obtained closed form expressions for the reactances that simultaneously match all ports of the coupler. Moreover for $\theta_0 = \pi$, the coupler exhibited perfect isolation. Hence these two electrical lengths are of primary interest to the designer. We will utilize the formula $S_{21} = j \tanh(\alpha Nd)$ in equation (5.99) to approximate the coupling level at the design frequency. From equation (6.3) above and the definition $\beta Nd = \theta_0$, we see that $\beta/(2\omega_0L_0) = \theta_0\omega_0C/N$ and $\beta/(2\omega_0C_0) = \theta_0\omega_0L/N$. Hence the coupling level, loading components of the NRI-TL
line of the coupler and its physical length can be obtained from:

\[ S_{21} = j \tanh \left( \frac{\theta_0 \alpha/\omega_0}{\beta/\omega_0} \right) \]  
\[ \frac{\omega_0 C}{N} = \frac{1}{2 \theta_0 L_0} \left( \frac{\beta}{\omega_0} \right) \]  
\[ \frac{\omega_0 L}{N} = \frac{1}{2 \theta_0 C_0} \left( \frac{\beta}{\omega_0} \right) \]  
\[ \frac{N d}{\lambda_0} = \frac{\theta_0}{2 \pi c} \left( \frac{\beta}{\omega_0} \right)^{-1} \]

In equation (6.7) above, \( \lambda_0 \) is the free space wavelength and \( c \) is the speed of light in vacuum. Finally, for the cases of quarter and half wavelengths long couplers, the reactances that perfectly match the coupler when attached in series to the terminations are given by (see equation (5.109)):

\[ X_m = -2 \text{Im}(Z) \tanh \left( \frac{\pi \alpha/\omega_0}{2 \beta/\omega_0} \right) \quad (\theta_0 = \pi/2) \]  
\[ X_m = -\text{Im}(Z) \tanh \left( \frac{\pi \alpha/\omega_0}{\beta/\omega_0} \right) \quad (\theta_0 = \pi) \]

The sign of the imaginary part of \( Z \) is determined in the following manner. From the two possible combinations of \( \pm \) in equation (6.1) select the one that results in a \( \gamma^2 \) with a positive imaginary part (the two roots of this quantity are the \( c \)-mode and its reflection). Use the same choice of \( \pm \) in equation (6.2) and select the root with a positive real part. The imaginary part of this root is used in equations (6.8) and (6.9). A matching reactance of \( j X_m \) attached in series to the terminations of the MS-TL line of the coupler and \( -j X_m \) attached to the NRI-TL line of the coupler will result in proper match at the design frequency.

Once we choose the desired electrical length of the coupler \( \theta_0 \), the coupling level and the system impedance can be expressed in terms of \( L_0, C_0, L_m \) and \( C_m \) (or equivalently in terms of the even and odd mode impedances and effective dielectric constants given in equations (5.9) to (5.14)). These quantities characterize the coupled microstrip segments and are functions of \( w/h \) and \( s/h \) for a given dielectric constant. Hence one can use coupled microstrip formulae in literature, such as the ones due to E. Hammerstad and O. Jensen [94], to plot the coupling level and impedance as a function of normalized microstrip width and line spacing (see Figure H.1 in Appendix H). The same can be done for the loading capacitance, inductance, coupler length and matching reactance given by equations (6.5) to (6.9) (see Figures H.2 to H.5 in Appendix H). The Hammerstad/Jensen
formulae depend on the dielectric constant and hence these plots can be generated for any desired substrate. One can use the more accurate Kirschning/Jansen formulation [95] to account for microstrip coupled-line dispersion, but this is unnecessary in obtaining approximate coupler parameters for optimization in the next design stage.

The final step is in the choice of the number of unit cells $N$ for the design. This choice is dependent on the electrical length of the coupler and one should ensure that each unit cell is electrically small. Hence $N = 4$ is a minimum recommendation for $\theta = \pi$ coupler such that each cell has a maximum electrical length which is an eighth of a guided wavelength. Extremely small cell sizes are undesirable from practical considerations such as increased losses due to the loading components and fabrication tolerances.

We have included sample charts in Appendix H for designing half-wave and quarter-wave long MS/NRI-TL couplers on a substrate with dielectric constant $\varepsilon_r = 4.5$ (e.g. the TMM4 substrate from Rogers Corp.). The equations in this section along with the Hammerstad/Jensen formulae can be used to generate such charts for arbitrary substrates.

Stage 2: Microwave circuit simulation

The coupler parameters that we obtained from the design equations above ignored loss, dispersion and bandwidth considerations. To take these factors into account, we will need to tweak the design using a microwave circuit simulator. In this section, we will present results to indicate that the values obtained from the design charts are quite close to the ones obtained through optimization using commercial simulators.

We present 5 designs on a 1.27 mm thick substrate of dielectric constant $\varepsilon_r = 4.5$ for an operating frequency of 2.0 GHz. The couplers were designed to be one-half wavelength long with $N = 6$ unit cells using the design curves and these values were tweaked using the circuit optimizer in Agilent ADS to meet the required goals of port matching, coupling level and center frequency. The results are tabulated in Table 6.1 below.

The first four designs outlined in Table 6.1 consider extreme cases of impedance and coupling level to test the worst case scenarios in using the charts in Appendix H. Nevertheless, we observe that most of the optimized design parameters using the circuit simulator are within 15% of the values obtained from the charts. There are two line spacing values that deviate by more than 20% and this is probably due to the non-uniqueness of the optimal design parameters that satisfy the optimization constraints (observe that the initial simulation coupling level is very close to the design objective.
<table>
<thead>
<tr>
<th>Goal</th>
<th>Design curve</th>
<th>Initial simulation</th>
<th>Optimized values</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design 1</td>
<td>$w = 0.16 \text{ mm}$</td>
<td>Coupling: 1.1 dB</td>
<td>$w = 0.14 \text{ mm}$</td>
<td>14 %</td>
</tr>
<tr>
<td>$Z_0 = 130 \Omega$</td>
<td>$s = 0.24 \text{ mm}$</td>
<td>Return loss: 32 dB</td>
<td>$s = 0.23 \text{ mm}$</td>
<td>4 %</td>
</tr>
<tr>
<td>$S_{21} = -1 \text{ dB}$</td>
<td>$d = 6.51 \text{ mm}$</td>
<td>Resonance: 1.96 GHz</td>
<td>$d = 6.36 \text{ mm}$</td>
<td>2 %</td>
</tr>
<tr>
<td></td>
<td>$L = 9.80 \text{ nH}$</td>
<td>Isolation: 76 dB</td>
<td>$L = 9.67 \text{ nH}$</td>
<td>1 %</td>
</tr>
<tr>
<td></td>
<td>$C = 0.59 \text{ pF}$</td>
<td></td>
<td>$C = 0.59 \text{ pF}$</td>
<td>0 %</td>
</tr>
<tr>
<td>Design 2</td>
<td>$w = 7.78 \text{ mm}$</td>
<td>Coupling: 5.2 dB</td>
<td>$w = 7.20 \text{ mm}$</td>
<td>8 %</td>
</tr>
<tr>
<td>$Z_0 = 20 \Omega$</td>
<td>$s = 0.11 \text{ mm}$</td>
<td>Return loss: 46 dB</td>
<td>$s = 0.09 \text{ mm}$</td>
<td>22 %</td>
</tr>
<tr>
<td>$S_{21} = -4 \text{ dB}$</td>
<td>$d = 6.35 \text{ mm}$</td>
<td>Resonance: 1.94 GHz</td>
<td>$d = 6.33 \text{ mm}$</td>
<td>0 %</td>
</tr>
<tr>
<td></td>
<td>$L = 1.51 \text{ nH}$</td>
<td>Isolation: 69 dB</td>
<td>$L = 1.52 \text{ nH}$</td>
<td>1 %</td>
</tr>
<tr>
<td></td>
<td>$C = 3.84 \text{ pF}$</td>
<td></td>
<td>$C = 3.72 \text{ pF}$</td>
<td>3 %</td>
</tr>
<tr>
<td>Design 3</td>
<td>$w = 8.67 \text{ mm}$</td>
<td>Coupling: 26 dB</td>
<td>$w = 8.72 \text{ mm}$</td>
<td>1 %</td>
</tr>
<tr>
<td>$Z_0 = 20 \Omega$</td>
<td>$s = 9.29 \text{ mm}$</td>
<td>Return loss: 87 dB</td>
<td>$s = 8.77 \text{ mm}$</td>
<td>6 %</td>
</tr>
<tr>
<td>$S_{21} = -25 \text{ dB}$</td>
<td>$d = 6.41 \text{ mm}$</td>
<td>Resonance: 1.95 GHz</td>
<td>$d = 6.09 \text{ mm}$</td>
<td>5 %</td>
</tr>
<tr>
<td></td>
<td>$L = 1.52 \text{ nH}$</td>
<td>Isolation: 90 dB</td>
<td>$L = 1.55 \text{ nH}$</td>
<td>2 %</td>
</tr>
<tr>
<td></td>
<td>$C = 3.80 \text{ pF}$</td>
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<td>$C = 3.82 \text{ pF}$</td>
<td>1 %</td>
</tr>
<tr>
<td>Design 4</td>
<td>$w = 0.18 \text{ mm}$</td>
<td>Coupling: 32 dB</td>
<td>$w = 0.17 \text{ mm}$</td>
<td>6 %</td>
</tr>
<tr>
<td>$Z_0 = 140 \Omega$</td>
<td>$s = 8.46 \text{ mm}$</td>
<td>Return loss: 79 dB</td>
<td>$s = 6.83 \text{ mm}$</td>
<td>24 %</td>
</tr>
<tr>
<td>$S_{21} = -30 \text{ dB}$</td>
<td>$d = 7.24 \text{ mm}$</td>
<td>Resonance: 1.94 GHz</td>
<td>$d = 6.88 \text{ mm}$</td>
<td>5 %</td>
</tr>
<tr>
<td></td>
<td>$L = 10.64 \text{ nH}$</td>
<td>Isolation: 104 dB</td>
<td>$L = 10.8 \text{ nH}$</td>
<td>1 %</td>
</tr>
<tr>
<td></td>
<td>$C = 0.54 \text{ pF}$</td>
<td></td>
<td>$C = 0.54 \text{ pF}$</td>
<td>0 %</td>
</tr>
<tr>
<td>Design 5</td>
<td>$w = 2.20 \text{ mm}$</td>
<td>Return loss: 42 dB</td>
<td>$w = 2.12 \text{ mm}$</td>
<td>4 %</td>
</tr>
<tr>
<td>$Z_0 = 50 \Omega$</td>
<td>$s = 0.47 \text{ mm}$</td>
<td>Coupling: 3.9 dB</td>
<td>$s = 0.42 \text{ mm}$</td>
<td>12 %</td>
</tr>
<tr>
<td>$S_{21} = -3 \text{ dB}$</td>
<td>$d = 6.62 \text{ mm}$</td>
<td></td>
<td>$d = 7.28 \text{ mm}$</td>
<td>9 %</td>
</tr>
<tr>
<td></td>
<td>$L = 3.76 \text{ nH}$</td>
<td></td>
<td>$L = 3.64 \text{ nH}$</td>
<td>3 %</td>
</tr>
<tr>
<td></td>
<td>$C = 1.54 \text{ pF}$</td>
<td></td>
<td>$C = 1.51 \text{ pF}$</td>
<td>2 %</td>
</tr>
</tbody>
</table>

Table 6.1: Comparison of MS/NRI-TL coupler designs at 2.0 GHz using plots in Appendix H and their optimized values using Agilent’s ADS. The initial simulation column contains circuit simulation results using parameters from the design graphs and the term ‘resonance’ refers to the minima of the isolation level. The deviation column compares the physical dimensions read off the plots (column 2) and those obtained by optimization (in column 4 simulated using 20,000 iterations) in ADS.
before optimization for the design 2 and 4 in Table 6.1). This is reasonable considering that these graphs were generated using approximate equations for the s-parameters of the coupler and that the Hammerstad/Jansen formulae for the coupled lines ignore microstrip dispersion. The difference between the circuit optimized and chart values is acceptable in anticipation of the full-wave design step to account for parasitics not captured by the circuit simulator.

If one chooses to use printed components such as interdigital lines, microstrip gaps, short circuit stubs, plated vias, spirals ... etc. to realize the loading components of the NRI-TL line of the coupler, then the geometry of these components can be approximated at this stage by matching their inbuilt circuit simulator model to the desired reactances of the loading elements.

**Stage 3: Layout and full-wave simulation**

Full-wave simulations are carried out to account for non-idealities such as lumped component parasitics, mutual coupling, discontinuity effects, non-uniform current flow on the microstrip lines, radiation losses, etc.

The shunt inductors that load one of the microstrip lines of the coupler are preferably implemented using short-circuited stubs of high impedance. High impedance lines are preferred due to their narrow width which enables them to act as ‘lumped’ reactive loadings. If microwave chip inductors are used, then they may be placed in periodic holes drilled through the substrate at the center of one of the strips or soldered to open stubs and then grounded using plated vias. The former approach using chip components is more difficult to implement as it requires manual fabrication and the height of the substrate that can be used will depend on the physical length of the chip inductor.

Realization of the series capacitors that load the NRI-TL lines requires additional effort compared to the shunt inductors as they are usually electrically large. For instance, an interdigital capacitor may be modelled as a lumped capacitor at the center of transmission-line segments of finite length. This extra length must be compensated for and incorporated into the unit cell size. A reasonable starting layout for the full-wave optimization will involve coupler cells whose total physical length matches those of the optimized circuit simulation.
6.3 3-dB Power divider

In this section we will describe the design of a planar 3-dB MS/NRI-TL coupler operating at 3 GHz (see Figure 6.1). Couplers with such high coupling level, isolation and bandwidth are usually very difficult to realize in a uni-planar printed topology. Such an implementation will be useful in applications such as RF power division to feed antenna arrays and power combining from distributed amplifiers.

Design parameters and fabrication

The physical and electrical parameters of a 3-dB coupler for a 50 Ω system on a 50 mils Rogers TMM4 ceramic substrate of dielectric constant 4.6 and loss tangent of 0.002 are given in Table 6.2. The eigenmodes corresponding to this design are plotted in Figure 6.1b where we observe that the operating frequency of 3.0 GHz lies inside the complex-mode band.

This design does not provide optimal performance (in terms of return loss and isolation) and this is partly because proper design equations and charts were unavailable to the author at the time of its fabrication. Erroneous parameter extraction of the self and mutual impedances/admittances of the coupled-lines resulted in a design which would have worked better at 2.73 GHz (corresponding to the condition in equation (5.27)) instead of 3.0 GHz.

This prototype was chemically etched and chip components were used to load the NRI-TL line. Panasonic 0402 chip capacitors were soldered across 0.5 mm gaps on
### Table 6.2: Design parameters of the 3-dB MS/NRI-TL coupler. The coupled transmission line equivalent parameters were obtained using the LineCalc utility in ADS and the eigenmode data were computed using equations (5.20) to (5.22)

<table>
<thead>
<tr>
<th>Layout</th>
<th>Rogers TMM4 substrate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\epsilon_r = 4.6$ 1.27 mm tan $\delta = 0.002$</td>
</tr>
<tr>
<td>Microstrip trace</td>
<td>$w = 2.343$ mm $s = 0.2$ mm $d = 4.0$ mm</td>
</tr>
<tr>
<td>Equivalent</td>
<td>$Z_e = 61.9$ Ω $\epsilon_e = 3.782$</td>
</tr>
<tr>
<td></td>
<td>$Z_0 = 32.1$ Ω $\epsilon_o = 2.972$</td>
</tr>
<tr>
<td>Coupled-mode</td>
<td>$L_0 = 293$ nH/m $L_m = 108$ nH/m $L = 3.3$ nH</td>
</tr>
<tr>
<td></td>
<td>$C_0 = 142$ pF/m $C_m = -37$ pF/m $C = 1.3$ pF</td>
</tr>
<tr>
<td>Eigenmode</td>
<td>$f_0 = 2.73$ GHz $\gamma_c = 34 + j111$ m$^{-1}$ $Z_c = 45.7 - j0.9$ Ω</td>
</tr>
</tbody>
</table>

The NRI-TL strip and holes were drilled between each capacitor to introduce 0603 chip inductors in shunt.

### Simulated and measured results

We simulate the response of this non-optimal design using Agilent ADS and compare it with that predicted from the analytic expressions in Chapter 5 and we observe very good agreement between the two (see Figure 6.2). If we had used the eigenmodes from the homogeneous approximation (Schelkunoff’s theory), the solid and dotted curves in Figure 6.2 would agree well in the complex-mode band (indicated by vertical red lines) but would deviate outside it. The Floquet eigenmodes result in surprisingly good agreement over a large bandwidth considering that we ignore frequency dispersion of the coupled-strips (we used frequency independent even and odd mode impedance and propagation constants extracted at 3.0 GHz to model the coupled-lines in our equations).

The measured scattering parameters of the coupler are given in Figure 6.3. We observe near equal power split between the output ports with a 3 dB bandwidth larger than 1 GHz. Although the port match and isolation are poor compared to those in the simulation, the return loss is less than 10 dB over the entire band and this is can be sufficient for some applications. This particular prototype being one of the very first
Figure 6.2: Simulated s-parameters of the 3-dB MS/NRI-TL coupler. The solid lines were obtained from microwave circuit simulations in ADS. The broken lines were obtained by solving equation (5.48) using eigenmodes computed from Floquet theory.

Figure 6.3: Measured (solid lines) and circuit simulation (broken lines) s-parameters of the 3-dB MS/NRI-TL coupler.
ones, suffers from poor performance due to numerous construction inaccuracies which were fixed in later designs. The stray capacitance due to soldering of the series chip capacitors is problematic when using low values (a couple of pico Farads) in the design and one can either use silver epoxy for accuracy or rely on fully printed interdigital capacitors. The other option is to measure the capacitance of a soldered chip in-situ and apply appropriate compensation to the design.

The use of shunt chip inductors placed in holes drilled into the substrate suffers from a number of drawbacks. Their placement was done manually and hence not centred at each unit cell location although this can easily be remedied by the use of computer-controlled PCB milling/drilling machines. Moreover the process of drilling into the strips changes their effective dielectric constant and their impedance. A better option would be to connect the shunt inductors to the edge of one of the strips and ground them using vias.

Furthermore, we have not accounted for the tolerances in the chip components and mutual coupling between them in the circuit simulation. The design considered in the next section is fully printed and have been optimized through full-wave simulation. It exhibits much better agreement between simulation and measurement.

### 6.4 High-directivity coupler

In this section, we describe the construction of a low-coupling but high-isolation coupler which is useful for signal monitoring and reflectometry applications [8]. The objective of this design is to provide a $-30$ dB sample of a signal carried by the main line (the MS-TL) while maintaining the isolated port power below $-65$ dB at the design frequency of 2.0 GHz. This correspond to a directivity of 35 dB which is much better than that of a conventional microstrip coupled-line coupler of low-coupling level (a directivity value of less than 10 dB for the conventional case has been reported by many authors [14] for coupling levels below 15 dB). The fabricated prototype of this coupler is shown in Figure 6.4.

#### Design parameters and fabrication

This particular prototype was realized by monolithic fabrication on a 50 mils Rogers TMM4 substrate using interdigitated fingers for the series capacitors and shorted stubs
Figure 6.4: (a) Photograph of the fabricated high-directivity MS/NRI-TL coupled-line coupler (b) Layout parameters of the coupler with values given in Table 6.3 for the shunt inductors that load the NRI-TL of the coupler. The design parameters for this coupler are listed in Table 6.3 including the dimensions of the quarter-wave impedance transformers to convert the 30 Ω impedance of the coupled lines to the desired 50 Ω system impedance. This prototype was milled using the *LPKF ProtoMat® H100* mechanical PCB router and the vias were copper plated using the *LPKF* through-hole plating system.

To fabricate this prototype, we drilled all of the via holes of diameter 0.2 mm into the TMM4 substrate with copper cladding on both sides. These holes were mechanically cleaned under the microscope to prevent clogging during the plating process. Next we copper plated these vias using the Reverse-Pulse-Plating technique where the polarity of the electrodes used in the process was alternated to ensure uniform copper deposition without clogging. Alignment holes were used as reference to mill out the coupler at the proper location with respect to the plated vias.

The low impedance of 30 Ω for the coupled sections was chosen to provide sufficient width to accommodate the 1.6 pF series capacitors given that the finest track that we could reliably mill on a PCB was around 0.15 mm (finer feature size can be obtained
### Layout

<table>
<thead>
<tr>
<th>Component</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rogers TMM4 substrate</td>
<td>(\varepsilon_r = 4.6) (h = 1.27) mm (\tan\delta = 0.002)</td>
</tr>
<tr>
<td>Microstrip trace</td>
<td>(w = 5.185) mm (s = 9.0) mm (d = 9.76) mm</td>
</tr>
<tr>
<td>Interdigital fingers ((N_f = 12))</td>
<td>(w_f = 0.295) mm (s_f = 0.155) mm (d_f = 4.643) mm</td>
</tr>
<tr>
<td>Exterior stubs (shorted with 0.1 mm radii vias)</td>
<td>(w_s = 0.2) mm (d_s = 2.5) mm</td>
</tr>
<tr>
<td>Interior stubs (shorted with 0.1 mm radii vias)</td>
<td>(w_s = 0.2) mm (d_s = 0.7) mm</td>
</tr>
<tr>
<td>Curved 90° bends</td>
<td>(w_b = 4.93) mm (r_b = 10) mm</td>
</tr>
<tr>
<td>Quarter-wave transformer (30 (\Omega) to 50 (\Omega))</td>
<td>(w_t = 3.6) mm (d_t = 19.5) mm</td>
</tr>
</tbody>
</table>

### Equivalent

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z_e)</td>
<td>30.40 (\Omega)</td>
</tr>
<tr>
<td>(\varepsilon_e)</td>
<td>3.659</td>
</tr>
<tr>
<td>(Z_0)</td>
<td>29.54 (\Omega)</td>
</tr>
<tr>
<td>(\varepsilon_o)</td>
<td>3.560</td>
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### Coupled-mode

<table>
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<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_0)</td>
<td>190 nH/m</td>
</tr>
<tr>
<td>(L_m)</td>
<td>4.03 nH/m</td>
</tr>
<tr>
<td>(L)</td>
<td>1.45 nH</td>
</tr>
<tr>
<td>(C_0)</td>
<td>211 pF/m</td>
</tr>
<tr>
<td>(C_m)</td>
<td>-1.59 pF/m</td>
</tr>
<tr>
<td>(C)</td>
<td>1.60 pF</td>
</tr>
</tbody>
</table>

### Eigenmode

<table>
<thead>
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<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_0)</td>
<td>2.06 GHz</td>
</tr>
<tr>
<td>(\gamma_c)</td>
<td>(1.18 + j82.1) m(^{-1})</td>
</tr>
<tr>
<td>(Z_c)</td>
<td>30.0 - (j0.20) (\Omega)</td>
</tr>
</tbody>
</table>

Table 6.3: Design parameters of the high-directivity MS/NRI-TL coupler. The subscripts ‘f’, ‘s’, ‘b’ and ‘t’ refer to finger, stub, bend and transformer dimensions respectively (see Figure 6.4b). The length of each unit cell \(d\) incorporates the length of the interdigital capacitive fingers \(d_f\). The parameter \(r_b\) of the curved bend refers to its radius of curvature.
Figure 6.5: Simulated s-parameters of the high directivity MS/NRI-TL coupler for operation at 2.01 GHz. The solid lines were obtained from microwave circuit simulations in ADS while the broken lines were computed using equation (5.48) and the eigenmodes from Floquet analysis.

Using standard laser PCB prototyping systems).

Simulated and measured results

As a first step, we compare the results from ADS circuit simulations with those obtained from the Floquet theory based s-parameters using the values listed in Table 6.3. These ideal coupler simulation results are shown in Figure 6.5 where we observe good agreement between the two except in the region below $-60$ dB where we believe that the subtleties of microstrip dispersion come into effect and are not captured by our simplistic model of ideal coupled transmission lines. We observe a sharp dip in the isolation $S_{41}$ of the coupler operating at 2.01 GHz. At this frequency, all ports are well matched to 30 $\Omega$ and the coupling level is equal to 26.7 dB while the through power is practically 0 dB.

Next we compare the performance of our MS/NRI-TL coupler optimized in full-wave simulation using Agilent Momentum® with the measured s-parameters of the fabricated

$^2$The slight discrepancy between the frequency value of 2.06 GHz listed in Table 6.3 and the optimal value of 2.01 GHz observed from the simulation plot is due to the use of the Schelkunoff and Floquet coupled-mode formulations respectively for the two cases.
Figure 6.6: Simulated (full-wave) and measured s-parameters of the high-directivity MS/NRI-TL coupler prototype indicated by broken and solid lines respectively. Measured data for $S_{31}$ is not available but was observed to be nearly identical to the simulated results.

The measured coupling level is seen to be 27.3 dB at 2.04 GHz where we see the resonant dip in the isolation level of the coupler to 71 dB representing a directivity of over 40 dB. The shift in the operating frequency from the intended design is only 1.5%. The insertion loss of the coupler is very low as the lines are widely spaced and $S_{31}$ mainly represents the transmission through an impedance transformed microstrip line.
6.5 Antenna feed network

A systematic way of feeding a large array of antenna elements or amplifiers is to use a binary tree of power dividers. Such a corporate feed network scales in complexity very rapidly as the number of elements increase and will occupy a large portion of the limited housing available for front-end components in any RF/Microwave system. Ideally each power splitting node must be impedance matched at all ends over a reasonable bandwidth and its output ports should be well isolated. In this section, we will describe an arrangement of 3-dB MS/NRI-TL coupled-line couplers that exhibits these desirable characteristics and can be arranged in a manner which is laterally compact. This design was realized on a 1.27 mm Rogers RO3010 substrate for operation at 2.1 GHz. The intended application in this case is in the feeding of a patch antenna array in phase which would require that the output ports be spaced apart by at least half a guided wavelength. A schematic of the proposed structure is shown in Figure 6.7.

Design parameters and fabrication

A 1-to-4 corporate power divider was designed using three half-wave 3-dB MS/NRI-TL coupled-line couplers and 6 phase-shifting lines. This structure is laterally compact, broad-band and matched at all ports. The central coupler in Figure 6.7 attached to port 1 (labelled as P1) splits the input power in half between its adjacent port (backward-coupling) and its through port. These are fed to two more couplers attached to the four output ports.

The phase-shifting lines are designed such that all output ports (labelled P2 to P5) are fed in phase. We recall that the half-wave MS/NRI-TL coupler operated at its design frequency given by equation (5.27), exhibits a phase of $\pi/2$ at the coupled-port and $\pi$ at the through port. We can now follow the input signal phase through the divider and establish that the phase-shift at all of the output ports are:

$$\angle S_{21} = \frac{3\pi}{2} + \phi_2 + \phi_3$$  \hspace{1cm} (6.10)

$$\angle S_{31} = \pi + \phi_2 + \phi_4$$  \hspace{1cm} (6.11)

$$\angle S_{41} = \frac{3\pi}{2} + \phi_1 + \phi_4$$  \hspace{1cm} (6.12)

$$\angle S_{51} = \phi_1 + \phi_3$$  \hspace{1cm} (6.13)

The various $\phi$ terms in equation (6.10) to (6.13) refer to the phase shift provided by
Figure 6.7: Schematic of the 1-to-4 corporate feed-network realized using 3 MS/NRI-TL couplers (surrounded by dotted lines with the grey rectangle representing the NRI-TL line). The 6 phase-shifting lines with phase shifts $\phi_i$ are required to obtain outputs with equal phase. The darkened nodes in this figure indicate 50 $\Omega$ shunt impedances to ground.

In our design we utilize a meandered quarter wave MS-TL for $\phi_1$ and a single unit cell NRI-TL of $+\pi/2$ phase shift for $\phi_2$ (see Figure 6.8) to achieve the required differential phase given by equation (6.14). Meandered MS-TL lines of different lengths connecting the output ports provide the required differential phase set by equation (6.15). This choice of phase-shifting lines is not optimal in terms of bandwidth, but we have used them for simplicity. The usage of Schiffman phase-shifting lines [64] to provide the differential $\pi/2$ phase shift is always an option to improve the phase balance bandwidth of this power divider. We have used a matching stub at the input port to improve the input-match of the divider. The full-wave optimized layout dimensions of the proposed divider are listed in Table 6.4.

We utilized chemical etching to fabricate this design as mechanical milling was found to be incompatible with the softness of the RO3010 substrate and the stubs used in this design were too long compared to their width to withstand the milling process. The 0.2
### Substrate

Rogers RO3010 substrate  

- \( \epsilon_r = 11.05 \)  
- \( h = 1.27 \text{ mm} \)  
- \( \tan \delta = 0.0023 \)

### 3-dB Coupler

- Left coupler (MS-TL left aligned to NRI-TL)  
  - \( s = 0.53 \text{ mm} \)  
  - MS-TL total length: 25.6 mm

- Centre coupler (MS-TL offset 0.77 mm to left of NRI-TL)  
  - \( s = 0.62 \text{ mm} \)  
  - MS-TL total length: 25.3 mm

- Right coupler (MS-TL right aligned to NRI-TL)  
  - \( s = 0.53 \text{ mm} \)  
  - MS-TL total length: 24.0 mm

### Common dimensions of each unit cell

- \( w = 1.2 \text{ mm} \)  
- \( d = 7.125 \text{ mm} \)

### Interdigital fingers \((N_f = 4)\)

- \( w_f = 0.185 \text{ mm} \)  
- \( s_f = 0.151 \text{ mm} \)  
- \( d_f = 5.323 \text{ mm} \)

### Exterior stubs (shorted with 0.1 mm radii vias)

- \( w_s = 0.1 \text{ mm} \)  
- \( d_s = 4.4 \text{ mm} \)

### Interior stubs (shorted with 0.1 mm radii vias)

- \( w_s = 0.1 \text{ mm} \)  
- \( d_s = 2.15 \text{ mm} \)

### Miscellaneous

- Curved 90° bend  
  - \( w_b = 1.2 \text{ mm} \)  
  - \( r_b = 2 \text{ mm} \)

- Input matching short-circuit stub  
  - \( w_s = 0.15 \text{ mm} \)  
  - \( d_s = 7.5 \text{ mm} \)

- Input feed line width and length  
  - \( w_{in} = 1.10 \text{ mm} \)  
  - \( d_{in} = 13.18 \text{ mm} \)

- Output lines width and spacing  
  - \( w_{out} = 1.14 \text{ mm} \)  
  - \( s_{out} = 38 \text{ mm} \)

### Table 6.4

| Substrate | Rogers RO3010 substrate  
- \( \epsilon_r = 11.05 \)  
- \( h = 1.27 \text{ mm} \)  
- \( \tan \delta = 0.0023 \) |
| --- | --- |
| 3-dB Coupler | Left coupler (MS-TL left aligned to NRI-TL)  
  \( s = 0.53 \text{ mm} \)  
  MS-TL total length: 25.6 mm  
Centre coupler (MS-TL offset 0.77 mm to left of NRI-TL)  
  \( s = 0.62 \text{ mm} \)  
  MS-TL total length: 25.3 mm  
Right coupler (MS-TL right aligned to NRI-TL)  
  \( s = 0.53 \text{ mm} \)  
  MS-TL total length: 24.0 mm |
| Common dimensions of each unit cell | \( w = 1.2 \text{ mm} \)  
  \( d = 7.125 \text{ mm} \) |
| Interdigital fingers \((N_f = 4)\) | \( w_f = 0.185 \text{ mm} \)  
  \( s_f = 0.151 \text{ mm} \)  
  \( d_f = 5.323 \text{ mm} \) |
| Exterior stubs (shorted with 0.1 mm radii vias) | \( w_s = 0.1 \text{ mm} \)  
  \( d_s = 4.4 \text{ mm} \) |
| Interior stubs (shorted with 0.1 mm radii vias) | \( w_s = 0.1 \text{ mm} \)  
  \( d_s = 2.15 \text{ mm} \) |
| Miscellaneous | Curved 90° bend  
  \( w_b = 1.2 \text{ mm} \)  
  \( r_b = 2 \text{ mm} \) |
| Input matching short-circuit stub | \( w_s = 0.15 \text{ mm} \)  
  \( d_s = 7.5 \text{ mm} \) |
| Input feed line width and length | \( w_{in} = 1.10 \text{ mm} \)  
  \( d_{in} = 13.18 \text{ mm} \) |
| Output lines width and spacing | \( w_{out} = 1.14 \text{ mm} \)  
  \( s_{out} = 38 \text{ mm} \) |

Table 6.4: Layout dimensions of the 1-to-4 corporate power divider utilizing three MS/NRI-TL coupled-line couplers with 3 dB coupling level. The network is designed for a 50 Ω system operating at 2.1 GHz.
mm wide via holes were first drilled into the substrate (with copper cladding on both sides) using the LPKF milling machine to avoid drilling into the tips of very thin stubs after the etching process. Next we aligned the mask using the drilled vias and phototetched the divider pattern. The final step of the process involved threading the vias with copper wires of slightly smaller diameter and soldering them to both sides of the board. We measured the performance of the prototype both with and without the shunt resistive terminations indicated by darkened nodes in Figure 6.7. These were implemented using 50 Ω chip resistors that were soldered into holes drilled into the substrate. The power level at each port of the fabricated prototype was manually tuned to achieve balance by widening the spacing between the lines using a very fine tipped cutter. The fabricated prototype is shown in Figure 6.8.

**Simulated and measured results**

Each MS/NRI-TL coupler used in the design is half-wavelength long and hence when port 1 in Figure 6.7 is excited, the darkened nodes ideally would have negligible incident power. If we do not terminate them, the divider will still be functional in terms of equal and in-phase power split at the output ports although its return loss and the isolation will be poor. In Figure 6.9 we present the full-wave simulation (with losses) and measured s-parameters of the divider.

We observe good agreement between simulation and measurement with a nearly equal
power level of $-7$ dB at all output ports at the design frequency of 2.1 GHz. This represents a net insertion loss (due to material dissipation) of about 1.2 dB while the return loss is better than 30 dB. In Figure 6.9b we notice that the phase difference between the various ports is very small at the operating frequency although their absolute value is different from simulation. This is due to the fact that we did not perform a reference plane calibration\(^3\) resulting in a constant offset for all phase measurements.

Now we attach the terminating 50 Ω chip resistors to the open ends of each MS/NRI-TL coupler to improve the output port match and isolation. The measured results for the modified feed network prototype are given in Figures 6.10 and 6.11. We observe that the power balance between the output ports has now deteriorated. This is indicative of the fact that the couplers that were optimized for our initial design without the terminating resistors, did not exhibit perfect isolation. The imbalance is reflected both in simulation and measurements (see Figure 6.10a) and can be fixed by the full-wave optimization and fabrication of a new prototype. Such an endeavour is straightforward but of very little academic interest. The phase balance between the output ports shown in Figure 6.10b is very similar to the case without shunt chip resistors.

We present the output port match and isolation data in Figure 6.11. Here we observe that the measured isolation between the output ports is better than 15 dB and the return loss is better than 10 dB over a wide frequency band. The discrepancy between simulation and measurement is probably due to the fact that we mechanically tuned (by increasing line spacing) the divider for the initial set of measurements (on the low-isolation version) before adding in the shunt resistors that terminated the unused ports of the 3 dB couplers. These changes in the physical dimensions are not reflected in the full-wave simulations. Moreover, the substrate suffered mechanical damage from multiple measurements due to its softness and flexibility and it should have been housed in a metallic enclosure.

### 6.6 Other applications

We have considered only a few of the many interesting applications that can be explored using the MS/NRI-TL coupler. The ordinary coupled-line coupler is widely used in the realization of narrow-band filters and is an important building block for microwave filter synthesis. The usage of MS/NRI-TL couplers in filters is yet to be investigated.

---

\(^3\)The Network analyser was calibrated using the simple SOLT (Short-Open-Load-Through) standards.
Figure 6.9: Full-wave simulation (broken line) and measured (solid line) s-parameter (a) magnitude and (b) transmission phase between input and output ports of the fabricated feed network prototype without the shunt resistors shown as filled dots in the schematic of Figure 6.7.
Figure 6.10: Full-wave simulation (broken line) and measured (solid line) s-parameter (a) Magnitude and (b) Transmission phase between input and output ports of the fabricated feed network prototype with shunt resistors shown as filled dots in the schematic of Figure 6.7.
Figure 6.11: Full-wave simulation (broken line) and measured (solid line) $s$-parameter (a) isolation and (b) output port match of the fabricated feed network prototype with shunt resistive terminations.
Figure 6.12: (a) Schematic of a multiplexer using MS/NRI-TL couplers. Power flowing in a single microstrip line is coupled-out using NRI-TL lines that operate at different frequency bands (b) Circuit simulation results for a prototype operating between 1 GHz and 3 GHz.

The ability to achieve near 0 dB coupling level implies that these couplers can be used as multiplexers for separating frequency bands in a communication system (see Figure 6.12). For example, consider a linear arrangement of 0 dB couplers each tuned to operate in a different frequency band, such that the MS-TLs of each coupler are connected back-to-back. The coupled port on the NRI-TL line of each device acts as the multiplexing port. When a signal is input into one end of the MS-TL it passes through the coupler chain unaffected until it reaches the one that is tuned to its frequency band. At this stage, the adjacent NRI-TL line taps out the desired signal.

We observed interesting dispersion characteristics of this coupler in Chapter 3. Just below the edge of the complex-mode band, there is the possibility of achieving slow-wave propagation as the group velocity \( \frac{d\omega}{d\beta} \) becomes very small in this region. Moreover, at the center of the complex-mode band, the group velocity is negative for one of the modes. Hence the pulse shaping effect in this band will result in the output pulse to appear before the input. Such interesting dispersive effects can be used for pulse shaping to compensate for dispersion in communication links.

The MS/NRI-TL coupler has a closely related counterpart at optical frequencies [96] that uses the coupling of surface plasmon/polariton waves in metals. Such structures support complex modes with characteristics similar to their microwave counterparts and potential applications to optical filtering and multiplexing are yet to be explored.
Chapter 7

Concluding Remarks

7.1 Summary of the completed work

The main focus of this dissertation is in the introduction and development of the the MS/NRI-TL coupler that is constructed by edge coupling a regular microstrip transmission-line (MS-TL) to a metamaterial negative-refractive-index transmission-line (NRI-TL). It was discovered that the eigenmodes of this coupler are complex (i.e. they exhibit simultaneous phase progression and attenuation along the axis of a lossless guide) in its primary frequency band of operation. These complex modes are responsible for the important attributes of the coupler and exhibit equal and oppositely directed power flow on the two lines. A misconception about complex modes that has existed in literature since their discovery, was the belief that it is impossible to excite such a mode independently of its related conjugate counterpart. Hence we have included an extensive discussion on the topic of complex mode excitation before presenting the analysis and applications of MS/NRI-TL couplers.

Starting from the fundamental realizability constraints of linear, reciprocal and lossless networks, we determined the constraints placed on the analytic characteristics, such as the location and order of the poles, zeros and branch-points, of the eigenmodes that describe coupled transmission-line systems. This equipped us with the necessary theoretical framework to address the complex mode excitation problem. We have shown that the dilemma associated with the potentially unstable right-half-plane branch-points of complex modes can be resolved without requiring equal amplitude excitation of a pair of conjugate modes.

The resolution of this five decade old misconception emerges from the recognition that
Chapter 7. Concluding Remarks

multi-valued modal amplitude functions (that share the same right-half-plane branch-points as the eigenmodes) are legitimate responses from a system excited by single-valued analytic sources. We have used the well known symmetric functions (in the theory of analytic functions) to construct eigenmode expansions in coupled transmission-line problems that allow the independent selection of modal amplitudes at any frequency while simultaneously eliminating the branch-point singularities in the right-half of the complex frequency plane. Moreover, we have shown that a single complex mode can be excited strongly in a finite bandwidth while its conjugate counterpart can be arbitrarily suppressed. We examined a host of other ideas related to complex modes in literature that have been used in the past to support the equal-strength excitation hypothesis. We have shown that they were mathematically inconsistent as well. For example, we have carefully explained why the idea of unequal excitation amplitudes of a pair of complex modes does not result in violation of power conservation in the transverse plane of the guide, and that it is possible to formulate a source distribution that excites a single mode predominantly, even though the Green’s function of the problem indicates equal amplitude excitation of the modal pair. The assertion that the amplitudes of a pair of conjugate complex modes can indeed be controlled independently was verified directly through simulation and indirectly through the good agreement between the theory and measurements of MS/NRI-TL couplers. This result is of importance in problems that involve the field expansion in a structure that supports complex modes. In the context of negative-refractive-index metamaterials, these results justify the assumption of dominant backward-wave excitation without requiring the equal strength excitation of any associated forward-waves. This is significant because it negates the argument that it would be impossible/erroneous to characterize some of these novel metamaterials with effective material parameters without being able to suppress the parasitic modes (in this case the forward-wave mode in the backward-wave band) [92].

Next we compared the eigenmodes of the MS/NRI-TL coupler obtained from a Bloch-Floquet type analysis to those from a simpler homogeneous coupled-mode formulation to establish the practical considerations under which this homogenization was justified. Then we investigated how these modes were excited and reflected in semi-infinite structures. This enabled us to develop the modal amplitudes in a finite length coupler (excited by sources and terminated with impedances) by considering an infinite series of modes bouncing between its terminal planes. Closed-form expressions for the scattering parameters of the MS/NRI-TL coupler were obtained for the first time and these revealed
interesting characteristics of the device. We observed that in contrast to regular couplers consisting of two adjacent microstrip lines, the MS/NRI-TL coupler was able to achieve arbitrary backward coupling levels using only a moderate spacing between the lines by merely increasing their lengths. We realized that it was possible to achieve very large isolation (and directivity) level by making the length of the coupler equal to half of the guided wavelength. We observed good agreement between simulation results and our closed-form scattering parameter expressions and hence we used the latter to develop useful nomographs. These nomographs facilitate the selection of parameters such as line width, spacing, length and loading elements to design MS/NRI-TL couplers of arbitrary coupling level and system impedance. We have designed, fabricated and tested a few sample prototypes of the coupler to illustrate possible applications such as a 3-dB power divider, a high-directivity signal monitor and a compact corporate feed network for exciting antenna arrays. Good agreement between theory, simulation and measurements validates our work (limited only by uncontrollable parasitics such as losses, component and fabrication tolerances and transmission-line dispersion).

7.2 Future Work

Several aspects of the MS/NRI-TL coupled-line system that need further studies include the effects of loss, dispersion and the modelling of electrically large loading elements. The effect of dissipation has been examined only superficially through full-wave and circuit simulation and needs further study to determine their effects on the complex-mode band dispersion, line impedance and insertion loss of the system. Losses along with the loading reactances will limit the power handling capability of this coupler and this is yet to be investigated.

The accuracy of the circuit model developed in this work to analyse the MS/NRI-TL coupler is limited primarily by the dispersive nature of the underlying microstrip line and those of the electrically large ‘lumped’ loading. We compensated for dispersion by extracting the per-unit-length parameters at the design frequency and this provided us with good agreement between simulation and theory over a bandwidth less than 10 GHz. If such couplers are to be incorporated in systems with much larger bandwidth requirements such as microwave measurement systems, then the effects of dispersion merits further study.

We maintained good agreement between the circuit model (that contains lumped com-
ponents between ideal transmission lines) and the actual coupler layout (that features interdigital capacitors and shorted stubs) by imposing constraints on their geometry. Through various simulations we observed that these interdigital capacitors can be accurately modelled over a small fractional bandwidth by the combination of short segments of microstrip lines flanking an ideal lumped capacitor. It may be worthwhile to model the enhanced interaction between the interdigital fingers and the adjacent line as a multi-conductor microstrip coupled-line component consisting of a wide MS-TL edge coupled to the thin interdigital fingers on the NRI-TL. We were able to absorb these extra segments into the unit-cell length as long as the widths of the microstrip line and the capacitor were equal. We used shorted stubs with the smallest width that we could fabricate so that they emulated the lumped inductors in our circuit model. In the process we have ignored the self-resonance and quality factor of these components and these must be investigated for any systematic wideband synthesis.

The issue of possible radiation from this coupler was not addressed in this work. This might happen due to the discontinuities on the NRI-TL line of the coupler and will need further study to anticipate the effect of a boxed enclosure in a commercial application. In order to be able to use this coupler for filter synthesis, the frequency dependence of the modes has to be incorporated explicitly into the analytical s-parameter expressions that we derived. A few interesting applications of the MS/NRI-TL coupler that can be investigated include multiplexer/directional filters, and possible control/tapering of the leakage constant from a NRI-TL based leaky-wave antenna. Investigation of the coupling between multiple MS-TLs and NRI-TLs in various arrangements may lead to the realization of interesting microwave devices such as analog spectrum analysers and multiplexers. One may also examine the possibility of tailoring the anomalous dispersion in the complex-mode band for pulse-shaping and broadband phase-shifting applications.

7.3 Contributions

The following conference and journal papers provide a summary of the research projects pursued and completed as a part of this degree.
Journal Papers


**Conference Papers**


Appendix A

Scalar Coupled-mode Differential Equations

Let $P$ and $Q$ designate two propagating waves (see Fig. A.1) such that $PP^*$ and $QQ^*$ quantify the power they carry respectively. The $ABCD$ transfer matrix in Fig. A.1 allows linear power exchange between $P$ and $Q$ after undergoing phase shifts $\beta_p \Delta z$ and $\beta_q \Delta z$ respectively. A pair of simultaneous equations for these modes identical to those of Pierce’s [47], are given below:

\begin{align*}
    P_{n+1} &= Ae^{-j\beta_p \Delta z}P_n + Be^{-j\beta_q \Delta z}Q_n \\
    Q_{n+1} &= Ce^{-j\beta_p \Delta z}P_n + De^{-j\beta_q \Delta z}Q_n
\end{align*}

Figure A.1: Linear Coupling between two Transmission Lines

In order to extend Pierce’s discrete formulation to the continuous case, we expand $A,B,C,D$ and the arguments of the exponentials in a Maclaurin’s series with respect to
\[ \Delta z: \]

\[ \frac{dP}{dz} = \frac{1}{\Delta z} \left\{ \sum_{l=0}^{\infty} A_l \Delta z^l \sum_{m=0}^{\infty} \frac{(-j \beta_p \Delta z)^m}{m!} - 1 \right\} P \]

\[ + \frac{1}{\Delta z} \left\{ \sum_{l=0}^{\infty} B_l \Delta z^l \sum_{m=0}^{\infty} \frac{(-j \beta_q \Delta z)^m}{m!} \right\} Q \quad (A.3) \]

\[ \frac{dQ}{dz} = \frac{1}{\Delta z} \left\{ \sum_{l=0}^{\infty} C_l \Delta z^l \sum_{m=0}^{\infty} \frac{(-j \beta_p \Delta z)^m}{m!} \right\} P \]

\[ + \frac{1}{\Delta z} \left\{ \sum_{l=0}^{\infty} D_l \Delta z^l \sum_{m=0}^{\infty} \frac{(-j \beta_q \Delta z)^m}{m!} - 1 \right\} Q \quad (A.4) \]

In the limit \( \Delta z \to 0 \), the first few terms in the series expansions of equations (A.3) and (A.4) can be written out as follows:

\[ \frac{dP}{dz} = \left\{ \frac{A_0 - 1}{\Delta z} + A_1 - j \beta_p A_0 + \left( A_2 - \beta_p^2 - j \beta_p A_1 \right) \Delta z + \ldots \right\} P \]

\[ + \left\{ \frac{B_0}{\Delta z} + B_1 - j \beta_q B_0 + \left( B_2 - \beta_q^2 - j \beta_q B_1 \right) \Delta z + \ldots \right\} Q \quad (A.5) \]

\[ \frac{dQ}{dz} = \left\{ \frac{C_0}{\Delta z} + C_1 - j \beta_p C_0 + \left( C_2 - \beta_p^2 - j \beta_p C_1 \right) \Delta z + \ldots \right\} P \]

\[ + \left\{ \frac{D_0 - 1}{\Delta z} + D_1 - j \beta_q D_0 + \left( D_2 - \beta_q^2 - j \beta_q D_1 \right) \Delta z + \ldots \right\} Q \quad (A.6) \]

Now we let \( A_0 = D_0 = 1 \) and \( B_0 = C_0 = 0 \) in (A.5) and (A.6) to ensure that all terms remain finite in the limit. This is reasonable, since as \( \Delta z \to 0 \), we expect that to the zeroth order, the transfer matrix simply transforms to the identity matrix. Now we can relabel \( A_1 = -j \kappa_{11}, \ B_1 = -j \kappa_{12}, \ B_1 = -j \kappa_{12} \) and \( C_1 = -j \kappa_{21} \) to obtain a pair of simultaneous linear ordinary differential equations in \( P \) and \( Q \):

\[ -\frac{dP}{dz} = j (\kappa_{11} + \beta_p) P + j \kappa_{12} Q \quad (A.7) \]

\[ -\frac{dQ}{dz} = j \kappa_{21} P + j (\kappa_{22} + \beta_q) Q \quad (A.8) \]
Appendix B

Vector Coupled-mode Differential Equations

Figure B.1: Linear Coupling between two Transmission Lines

Referring to Fig. B.1, we can define the linear relationship between voltages and currents measured at the various ports with the aid of a $4 \times 4$ transmission matrix as shown below:

\[
\begin{pmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2 \\
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & b_{11} & b_{12} \\
a_{21} & a_{22} & b_{21} & b_{22} \\
c_{11} & c_{12} & d_{11} & d_{12} \\
c_{21} & c_{22} & d_{21} & d_{22} \\
\end{pmatrix}
\begin{pmatrix}
V_1 + \frac{dV_1}{dz} \Delta z \\
V_2 + \frac{dV_2}{dz} \Delta z \\
I_1 + \frac{dI_1}{dz} \Delta z \\
I_2 + \frac{dI_2}{dz} \Delta z \\
\end{pmatrix}
\]

(B.1)
\[
\begin{pmatrix}
V \\
I
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V \\
I
\end{pmatrix} + \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\frac{d}{dz}
\begin{pmatrix}
V \\
I
\end{pmatrix} \Delta z
\]

(B.2)

In equation (B.2), \(V\) and \(I\) are column vectors of line voltages and currents respectively, while \(A, B, C\) and \(D\) are \(2 \times 2\) square matrices which are functions of length \(\Delta z\). In the limit \(\Delta z \to 0\), the matrices \(A, D \to I\) and \(B, C \to N\), where \(I\) and \(N\) are \(2 \times 2\) identity and null matrices respectively. Equation (B.2) can be rewritten in the form:

\[
- \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\frac{d}{dz}
\begin{pmatrix}
V \\
I
\end{pmatrix} = \begin{pmatrix}
(A - I) / \Delta z & B / \Delta z \\
C / \Delta z & (D - I) / \Delta z
\end{pmatrix}
\begin{pmatrix}
V \\
I
\end{pmatrix}
\]

(B.3)

Expanding each \(2 \times 2\) matrix (keeping in mind their limiting forms when the length of the sections vanish) in (B.3) and applying the limit \(\Delta z \to 0\) results in a system of coupled differential equations:

\[
- \frac{d}{dz}
\begin{pmatrix}
V(z) \\
I(z)
\end{pmatrix} = \begin{pmatrix}
A' & B' \\
C' & D'
\end{pmatrix}
\begin{pmatrix}
V(z) \\
I(z)
\end{pmatrix}
\]

(B.4)

In equation (B.4) above, each primed constant matrix involves the coefficients of the linear term in a Maclaurin series expansion with respect to length. We can further simplify the form of this system by invoking homogeneity and reciprocity. The two lines in general are not identical, but homogeneity results in symmetric transmission properties from left-to-right or vice-versa in Fig. B.1. In Appendix C it is shown that the assumption of reciprocity causes the matrices \(B\) and \(C\) to be symmetric. Furthermore, reciprocity and lateral symmetry implies that if the pair \(V(z), I(z)\) is a solution to (B.4), then \(V(-z), -I(-z)\) must also be a solution pair. This is required to allow identical propagation of fields in both directions with the negative sign infront of the current vector to enforce reversal of power flow. Under these conditions, (B.4) can be rewritten as:

\[
- \frac{d}{dz}
\begin{pmatrix}
V(-z) \\
-I(-z)
\end{pmatrix} = \begin{pmatrix}
A' & B' \\
C' & D'
\end{pmatrix}
\begin{pmatrix}
V(-z) \\
-I(-z)
\end{pmatrix}
\]

(B.5)

Applying the transformation \(z' = -z\), extracting the negative signs from \(I\) and absorbing them back into the primed block matrices, equation (B.5) can be rewritten as:

\[
- \frac{d}{dz'}
\begin{pmatrix}
V(z') \\
I(z')
\end{pmatrix} = \begin{pmatrix}
-A' & B' \\
C' & -D'
\end{pmatrix}
\begin{pmatrix}
V(z') \\
I(z')
\end{pmatrix}
\]

(B.6)

Comparing (B.4) to (B.6), they describe the physical laws of the same system albeit with different variable names. Hence both \(A'\) and \(D'\) must be the null matrix. Relabelling
the symmetric matrices $B'$ and $C'$ as $Z$ and $Y$ respectively, the final form of the vector coupled-mode differential equations is:

$$-\frac{d}{dz}V = ZI$$  
(B.7)

$$-\frac{d}{dz}I = YV$$  
(B.8)

The system of coupled-mode differential equations shown above can be easily expanded to the case of an arbitrary number of lines and is valid for all linear, reciprocal and homogenous lines supporting TEM or quasi-TEM modes. Although these equations can be applied to arbitrary waveguide modes, the evaluation of matrices $Z$ and $Y$ are relatively simple for TEM-type modes. Finally, (B.7) and (B.8) in their expanded form can be expressed as [97]:

$$-\frac{dV_1}{dz} = Z_1I_1 + Z_m I_2$$  
(B.9)

$$-\frac{dV_2}{dz} = Z_m I_1 + Z_2 I_2$$  
(B.10)

$$-\frac{dI_1}{dz} = Y_1 V_1 + Y_m V_2$$  
(B.11)

$$-\frac{dI_2}{dz} = Y_m V_1 + Y_2 V_2$$  
(B.12)
Appendix C

2N-port Block Transmission Matrices

Figure C.1: 2N-Port Linear Network

The linear 2n-port structure shown in Fig. C.1 may be characterized by a transfer
Appendix C. 2N-port Block Transmission Matrices

matrix of the form:

\[
\begin{pmatrix}
V_L \\
I_L
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
a_{11} & \ldots & a_{1n} \\
b_{11} & \ldots & b_{1n}
\end{pmatrix} & \begin{pmatrix}
a_{n1} & \ldots & a_{nn} \\
b_{n1} & \ldots & b_{nn}
\end{pmatrix} \\
\begin{pmatrix}
c_{11} & \ldots & c_{1n} \\
d_{11} & \ldots & d_{1n}
\end{pmatrix} & \begin{pmatrix}
c_{n1} & \ldots & c_{nn} \\
d_{n1} & \ldots & d_{nn}
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix}
\]

(C.1)

In equation (C.1) above, the subscripts \( R \) and \( L \) are used to indicate the right-side or the left-side of the device. We are interested in the properties of the \( n \times n \) square matrices \( A, B, C \) and \( D \) in the case of reciprocal, symmetric (left-to-right) and loss-less structures. The final relations obtained in this appendix can be found in [99], [100] and [101]. However, some of the properties of the transmission matrix (e.g. reciprocity relations) have been derived directly without borrowing properties known from impedance and admittance matrices. Moreover, no assumption is made on the invertibility of the constituent block matrices which make up the transmission matrix. Although the final results are the same as in literature, it is comforting to know that the properties listed in Table C.2 are applicable even in cases when the corresponding impedance/admittance matrices do not exist or when a block matrix do not possess an inverse.

### Symmetrical\(^1\) and Reciprocal Devices

If equation (C.1) describes a 2n-port circuit which is symmetrical from left to right, then the inverse transmission matrix should look exactly like the original after negating the directions of currents in the original expression:

\[
\begin{pmatrix}
V_L \\
I_L
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix} \Longleftrightarrow \begin{pmatrix}
V_R \\
-I_R
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V_L \\
-I_L
\end{pmatrix}
\]

(C.2)

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
A & -B \\
-C & D
\end{pmatrix} = \begin{pmatrix}
I & N \\
N & I
\end{pmatrix}
\]

(C.3)

\(^1\)Left-right symmetry as defined by (C.2) is implied here with the freedom of choice of non-symmetric top-bottom configuration of the ports.
Hence from equation (C.3), a symmetrical network will result in the following relations between the block matrices:

\[ A^2 - BC = I \]  \hspace{1cm} (C.4)
\[ D^2 - CB = I \]  \hspace{1cm} (C.5)
\[ AB = BD \]  \hspace{1cm} (C.6)
\[ CA = DC \]  \hspace{1cm} (C.7)

From equation (C.3) we can also infer that the magnitude of the determinant of the transmission matrix is unity because:

\[ \det \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} = \det \begin{pmatrix} -I & N \\ N & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -I & N \\ N & I \end{pmatrix} = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]  \hspace{1cm} (C.8)

Let \( J^a \) and \( J^b \) be two electric-current sources and let \( M^a \) and \( M^b \) be two magnetic-current sources. If these sources are time-harmonic and of finite extent, then field reciprocity requires that \[ \int \int \int (E^a \cdot J^b - H^a \cdot M^b) \, dV = \int \int \int (E^b \cdot J^a - H^b \cdot M^a) \, dV \]  \hspace{1cm} (C.9)

In (C.9), \( E^a \) and \( H^a \) are the total electric and magnetic fields produced by ‘a’ sources \( J^a \) and \( M^a \) in the absence of the ‘b’ sources. In the circuit formulation of reciprocity, \( J \) and \( M \) are replaced by ideal current and voltage sources respectively. Moreover, when applying (C.9) to circuits, the polarity/direction of voltage/current impressed by one source on the other obeys the Ohm’s law convention. We can define sources ‘a’ and ‘b’ in four different ways to study the reciprocal properties of the transmission matrix as listed in Table C.1.

Application of equation (C.9) using the ‘case 1’ source configuration of Table C.1 yields:

\[ (I^b)^T \cdot V_s^a = (I^a)^T \cdot V_s^b \]  \hspace{1cm} (C.10)
\[ \begin{pmatrix} 0 \\ -I^b \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_s^b \\ I' \end{pmatrix} \]  \hspace{1cm} (C.11)
\[ \begin{pmatrix} V_s^a \\ I'' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 \\ I^a \end{pmatrix} \]  \hspace{1cm} (C.12)
### Appendix C. 2N-port Block Transmission Matrices

#### Table C.1: Voltage and current sources to test the constraints on the network transmission matrices due to reciprocity.

<table>
<thead>
<tr>
<th>Source 'a'</th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
<th>case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} V_L \ I_L \end{pmatrix}$ =</td>
<td>$\begin{pmatrix} V_s^a \ 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 \ I_s^a \end{pmatrix}$</td>
<td>$\begin{pmatrix} V_s^a \ 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 \ I_s^a \end{pmatrix}$</td>
</tr>
<tr>
<td>Source 'b'</td>
<td>$\begin{pmatrix} V_R \ I_R \end{pmatrix}$ =</td>
<td>$\begin{pmatrix} V_s^b \ 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 \ I_s^b \end{pmatrix}$</td>
<td>$\begin{pmatrix} V_s^b \ 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

In equation (C.10), $I^b$ is the current measured at the $I_L$ with source $V_R = V_s^b$ and source $V_L = V_s^a = 0$. This is indicated in (C.11) while a complementary description of $I^a$ is given in (C.12). The negative sign front of $I^b$ is required to reflect that (C.10) utilizes Ohm’s law convention whereas $I_L$ in the definition of the transmission matrix (C.1) doesn’t. We can now solve the systems (C.11) and (C.12) for $V_s^a$ and $V_s^b$ in terms of $I^a$ and $I^b$ so that we may substitute the resulting expressions into the reciprocity relation (C.10).

\[
0 = A \left( AV_s^b + BI' \right) \quad (C.13)
\]
\[
-BI^b = B \left( CV_s^b + DI' \right) \quad (C.14)
\]
\[
V_s^a = BI^a \quad (C.15)
\]
\[
I'' = DI^a \quad (C.16)
\]

Symmetry relations (C.4) and (C.6) enable one to simplify equations (C.13) and (C.14) above to yield $BI^b = V_s^b$. Applying reciprocity, one obtains:

\[
(I^b)^T BI^a = (I^a)^T BI^b = (I^b)^T B^T I^a \quad (C.17)
\]

The last expression in (C.17) follows by tranposition of the previous term which is a scalar. Hence $(I^b)^T \left( B - B^T \right) I^a = 0$ for arbitrary choices of excitations of the form shown as ‘case 1’ in Table C.1, and consequently, one concludes that $B$ is a symmetric matrix for a symmetric reciprocal network. In a similar fashion, the other choices of
Excitations maybe used to deduce further properties of a symmetric reciprocal network and these are summarized in Table C.2.

**Loss-less Devices**

If the network under consideration is loss-less, then it must satisfy:

\[
\begin{pmatrix}
V_L^H & I_L^H \\
I_L & V_L
\end{pmatrix}
\begin{pmatrix}
I_L \\
V_L
\end{pmatrix} =
\begin{pmatrix}
V_R^H & I_R^H \\
I_R & V_R
\end{pmatrix}
\begin{pmatrix}
I_R \\
V_R
\end{pmatrix}
\]

(C.18)

or,

\[
\begin{pmatrix}
V_L^H \\
I_L \\
V_L
\end{pmatrix} =
\begin{pmatrix}
V_R^H \\
I_R \\
V_R
\end{pmatrix}
\begin{pmatrix}
A^H & C^H \\
B^H & D^H
\end{pmatrix}
\begin{pmatrix}
C & D \\
A & B
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R \\
V_R
\end{pmatrix} =
\begin{pmatrix}
V_R^H \\
I_R \\
V_R
\end{pmatrix}^H
\begin{pmatrix}
I_R \\
V_R
\end{pmatrix}
\]

(C.19)

or,

\[
\begin{pmatrix}
V_R^H \\
I_R \\
V_R
\end{pmatrix} =
\begin{pmatrix}
V_R^H \\
I_R \\
V_R
\end{pmatrix}
\begin{pmatrix}
A^H C + C^H A \\
B^H C + D^H A - I \\
B^H D + D^H B
\end{pmatrix}
\begin{pmatrix}
V_R \\
I_R \\
V_R
\end{pmatrix} = 0
\]

(C.20)

In equations (C.18) to (C.20), the superscript \( H \) refers to the ‘Hermitian Transpose’ operation. The complex vectors \( V_R \) and \( I_R \) are completely arbitrary and hence each element of the matrix in (C.20) must be zero, and the resulting constraints imposed by the loss-less nature of the network can be expressed as:

\[
A^H C = -C^H A \quad (C.21)
\]

\[
B^H D = -D^H B \quad (C.22)
\]

\[
(A^H D + C^H B) D = D \quad (C.23)
\]

\[
(B^H C + D^H A) A = A \quad (C.24)
\]

Applying the symmetry conditions (C.5) and (C.6), and equation (C.21) to the expression in (C.23), we conclude that \( A^H = D \). Comparing with the reciprocity constraint \( A^T = D \) (see Table C.2), we can conclude that for a loss-less, reciprocal and symmetric network, the matrices \( A \) and \( D \) must be real. Inserting these results back into (C.21) and (C.22), one can infer that the matrices \( B \) and \( C \) are skew-Hermitian.
### Symmetry

| Symmetry & Reciprocity | \( A^2 - BC = I \)  
|                       | \( D^2 - CB = I \)  
|                       | \( AB = BD \)  
|                       | \( CA = DC \)  
|                       | \( \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \pm 1 \)  

### Symmetry & Reciprocity

| Symmetry, Reciprocity & Loss-less | \( A^T = D \)  
|                                | \( B^T = B \)  
|                                | \( C^T = C \)  

### Symmetry, Reciprocity & Loss-less

| Symmetry, Reciprocity & Loss-less | \( A^* = A \)  
|                                  | \( D^* = D \)  
|                                  | \( B^* = -B \)  
|                                  | \( C^* = -C \)  

Table C.2: Symmetry, Reciprocity and Loss-less constraints on Transmission Matrices
Appendix D

Homogenization of the MS/NRI-TL Coupler

In a manner similar to the homogenization of the metamaterial NRI-TL, we keep the product of the lumped loading elements \( Ld = L' \) and \( Cd = C' \) constant in the limit \( d \rightarrow 0 \). Carrying out the first order expansion in \( d = \Delta z \) of the matrices in equations (5.33) to (5.37), we obtain:

\[
Z = \Delta z \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{sC'} \end{pmatrix} \tag{D.1}
\]

\[
 Y = \Delta z \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2sL'} \end{pmatrix} \tag{D.2}
\]

\[
 I' = I \tag{D.3}
\]

\[
 Z' = \frac{\Delta z}{2} \begin{pmatrix} sL_0 & sL_m \\ sL_m & sL_0 \end{pmatrix} \tag{D.4}
\]

\[
 Y' = \frac{\Delta z}{2} \begin{pmatrix} sC_0 & sC'_m \\ sC'_m & sC_0 \end{pmatrix} \tag{D.5}
\]

In the equations above, \( L_0, C_0, L_m \) and \( C_m \) are the per unit length inductances and capacitances that describe a uniform coupled microstrip line and \( I \) is the \( 2 \times 2 \) identity matrix. The transmission matrix of the system can now be evaluated using equations
Appendix D. Homogenization of the MS/NRI-TL Coupler

(5.38) to (5.40) and we ignore all terms higher than first order in $\Delta z$:

\[ A = D = I \quad (D.6) \]
\[ B = \Delta z \begin{pmatrix} sL_0 & sL_m \\ sL_m & sL_0 + \frac{1}{sC_m} \end{pmatrix} \quad (D.7) \]
\[ C = \Delta z \begin{pmatrix} sC_0 & sC_m \\ sC_m & sC_0 + \frac{1}{sL_0} \end{pmatrix} \quad (D.8) \]

This transmission matrix relates the voltage-current state vector at the left-side of the network to those at its right, and using a first order Taylor expansion we can say that:

\[
\begin{pmatrix} V \\ I \end{pmatrix}_L = \begin{pmatrix} V \\ I \end{pmatrix}_R - \Delta z \frac{d}{dz} \begin{pmatrix} V \\ I \end{pmatrix}_R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}_R \quad (D.9)
\]

We have used the subscripts $R$ and $L$ to designate right-hand side and left-hand side voltage ($V = (V_1 \ V_2)^T$) and current ($I = (I_1 \ I_2)^T$) vectors measured at the ports of the coupler unit cell. Rearranging the terms in equation (D.9) and dropping the subscripts we obtain:

\[
-\frac{d}{dz} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & sL_0 & sL_m \\ 0 & 0 & sL_m & sL_0 + \frac{1}{sC_m} \\ sC_0 & sC_m & 0 & 0 \\ sC_m & sC_0 + \frac{1}{sL_0} & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} \quad (D.10)
\]

The expression in equation (D.10) is identical to the one we deduced intuitively in section 5.3.1 and is valid in the homogeneous limit when the electrical length of each unit cell is small and also when the Bloch phase shift per unit cell $\gamma d$ is small as well. We have not explicitly invoked the condition that $|\gamma d| \ll 1$ in our derivation but it was implicit in equation (D.9) above when we carried out a first order Taylor expansion of the voltages and currents on the two lines.
Appendix E

Nyquist Analysis of Coupled Modes
\((N = 2)\)

In this Appendix, we outline the qualitative features of the Nyquist plot of the symmetric function:

\[ F_3(p) = \left[ \gamma_c^2 - \gamma_\pi^2 \right]^2 \]  

(E.1)

that can be used to locate degeneracies (and hence potential branch-points) of a pair of coupled eigenmodes in the right-half complex-frequency plane. In equation (E.1), \(\gamma_c\) and \(\gamma_\pi\) are the two proper eigenmodes of a system consisting of 2 lossless and reciprocal coupled-lines and \(p\) is the complex-frequency variable. From our discussion in section 3.3.2, we know that such a function is rational even though the individual modes aren’t necessarily so.

The Nyquist plot of \(F_3\) is the image curve corresponding to a semi-circle of infinite radius on the right-half of the \(p\) plane whose straight edge is aligned with the imaginary axis and is indented to its right at all poles and zeroes of \(F_3\). We will refer to this curve as \(j.c.\).

**Mapping of regular points on the imaginary \(p\) axis**

At all regular points where the modes are propagating or evanescent and when \(\gamma_c^2 \neq \gamma_\pi^2\), \(F_3(p)\) maps the imaginary \(p\) axis to the positive real axis of the \(F_3\) plane. Hence, when \(p\) is swept along the imaginary axis, \(F_3(p)\) moves along the real axis either towards the origin or away from it depending on whether the dispersion diagram of the two modes
approach or diverge from each other. The complex modes of the system always appear in conjugate pairs. Hence when $\gamma_c^2$ is complex, $\gamma_\pi^2$ is its conjugate. This implies that $\gamma_c^2 - \gamma_\pi^2$ purely imaginary and hence $F_3(p)$ maps this portion of the imaginary $p$ axis to the negative real axis of the $F_3$ plane.

**Mapping of the region around a pole**

If one of the eigenmodes has a simple pole at $p$ while the other is regular, then the corresponding $F_3(p)$ exhibits a 4-th order pole. From the Laurent series expansion of a function in the vicinity of a 4-th order pole, it is a simple matter to demonstrate that a small clockwise arc of angle $\theta$ that surrounds it, is mapped to a large circular arc of angle $4\theta$ in the counter-clockwise direction. The only exception is the pole at infinity. In this case, the large circular arc of angle $\theta$ on the $p$ plane is mapped to one whose angle is also four times larger but is oriented in the same direction. Hence the curved portion of $j.c.$ is mapped to a loop that encircles the origin twice in the same sense. On the other hand, the semi-circular indentation around an imaginary axis pole is mapped to a large loop that encircles the origin of the $F_3$ plane twice in the opposite sense.

If both eigenmodes possess simple poles at a point $p_0$, the situation gets complicated. Expanding $\gamma_c$ and $\gamma_\pi$ about this simple pole in a Laurent series:

$$\gamma_c^2 - \gamma_\pi^2 = \left[ \frac{a_{-1}}{p - p_0} + a_0 + a_1(p - p_0) + \ldots \right]^2 - \left[ \frac{b_{-1}}{p - p_0} + b_0 + b_1(p - p_0) + \ldots \right]^2$$

$$= \frac{a_{-1}^2 - b_{-1}^2}{(p - p_0)^2} + 2\frac{a_{-1}a_0 - b_{-1}b_0}{p - p_0} + (a_0^2 + 2a_{-1}a_1 - b_0^2 - 2b_{-1}b_1) + \ldots$$  \hspace{1cm} (E.2)

Hence in this case, $F_3$ exhibits either a pole of the 4-th order (if $a_{-1} \neq b_{-1}$ in equation (E.2)), or a 2-nd order pole, or approaches a constant . . . etc. The specific case requires knowledge of the constants $a_i$ and $b_i$ which are the Laurent expansion coefficients of $\gamma_c$ and $\gamma_\pi$ respectively at $p_0$. The simultaneous occurrence of poles for both modes is commonly seen at infinity and in such cases, one can examine the relative slope of the dispersion curves for each mode for large values of frequency along the imaginary axis (i.e. when the modes show a linear trend). If these slopes are different, then $F_3$ is definitely a 4-th order pole. If such an observation cannot be made, then the portion of the Nyquist plot at the vicinity of the pole must be generated numerically.
Mapping of the region around a zero

If a single mode is zero at a given frequency point while the other is propagating or evanescent, then the corresponding point maps to the positive real axis of the $F_3$ plane and there is no need to indent the j.c. curve in the $p$ plane. On the other hand, if both modes have simple zeroes at the point $p_0$, then using arguments similar to the case above, the function $F_3$ will exhibit either a 4-th order zero, or a higher even order zero. If the slopes of $\gamma_c$ and $\gamma_\pi$ are different as they approach the common zero (and is continuous about the critical frequency), then $F_3$ will definitely possess a 4-th order zero at this location.

Mapping of the region around a first order branch-point

The branch-point degeneracies that mark the edges of the complex band are mapped into the origin of the $F_3$ plane. If the branch-point on the imaginary axis is of the first order, then a semicircular counter-clockwise indentation around it (in the right half of the $p$-plane) gets mapped to a counter-clockwise arc. The arc in the $F_3$ plane circumvents the origin from the positive to the negative real axis (or vice-versa) depending on whether sweeping past the branch-point in question transforms the mode from propagating/evanescent to complex (or vice-versa).

Symmetry of the mapping about the real axis

Lastly, due to the symmetry $\gamma(p) = \gamma(-p)^*$ about the real axis of the $p$-plane, it is unnecessary to determine the Nyquist plot for the entire Jordan curve that stretches from negative to positive infinity along the imaginary axis and then closes with a larger semi-circular loop in the R.H.P. We only need to sketch the Nyquist loops corresponding to $F_3(p)$ for frequencies that lie above the real axis of the $p$-plane. Once we have this plot, we simply mirror the curves about the real axis in the $F_3$ plane to obtain the diagram for the entire Jordan curve.
Other singularities

We have not considered a couple of other scenarios that involve both modes possessing a zero at the frequency $p_0$. These include for instance the case where $\gamma_c$ has a simple zero at $p_0$ and $\gamma_\pi$ has a branch-point with $\gamma_\pi(p_0) = 0$. The other possibility is that of two modes with first order branch-points and a zero value at $p_0$ (e.g. two identical uncoupled guides with a cut-off frequency at $p_0$). Lastly, a third order branch-point at which $\gamma$ is zero has been left out of the discussion in this section. Moreover, all these cases have equivalent counterparts when the point in question is a pole instead of a zero. Such scenarios can be laboriously investigated through series expansion around the singularity in question, but is beyond the scope of this brief Appendix.
A powerful technique to fit a finite set of discrete points to a linear sum of complex exponentials is one due to Gaspard de Prony [93]. Prony’s method is very useful for the analysis of discrete signals when one anticipates that its constituent oscillations are in general damped and limited to a finite set of modes. In our case, we apply this technique to extract spatial propagation constants of the eigen-modes in a MS/NRI-TL coupler (which can be complex) and their relative amplitudes of excitation.

Let $f(z)$ be a function of $z$ which we would like to expand as a sum of $N$ complex exponential functions $/mu_{n}^{z}$ where $n$ is an integer between 0 and $N - 1$. Hence we can express $f(z)$ as:

$$f(z) = \sum_{n=0}^{N-1} C_{n}\mu_{n}^{z} \quad (F.1)$$

Now let $f(m)$ be the $m$-th sampling of $f$ out of a total of $M$ sample points. We can rewrite equation (F.1) as:

$$f(m) = \sum_{n=0}^{N-1} C_{n}\mu_{n}^{m} \quad m = 0, 1, 2, \ldots, M - 1 \quad (F.2)$$

We now have $M$ equations and $2N$ unknowns - where the $f(m)$ are the known quantities while $C_{n}$ and $\mu_{n}$ have to be determined. Hence it is clear that $M \geq 2N$ for the system to be solvable. In special cases, the $\mu_{n}$ might be known, for instance, the various modal propagation constants may be available from full-wave simulations. In such cases, (F.2) can be solved directly if $M = N$ or by the method of Least Squares if $M > N$. But in
general, we need to first determine the modes $\mu_n$ before we establish their corresponding amplitudes $C_n$. Let us now assume that the $N$ modes ($\mu_n$) are the roots of a $N$-th order polynomial in $\mu$:

$$\prod_{n=0}^{N-1} (\mu - \mu_n) = \sum_{k=0}^{N-1} \alpha_k \mu^k + \mu^N = 0 \quad (F.3)$$

In equation (F.3), we show both the factorized and expanded versions of the polynomials in $\mu$ with coefficients $\alpha_k$. The problem of determining the $\mu_n$ modes has been transformed into finding the $\alpha_k$ coefficients. Now consider the following summation involving the polynomial coefficients in (F.3) and the first $N - 1$ samples of $f$:

$$\sum_{k=0}^{N-1} \alpha_k f(k) = \sum_{k=0}^{N-1} \alpha_k \sum_{n=0}^{N-1} C_n \mu_n^k = \sum_{n=0}^{N-1} C_n \sum_{k=0}^{N-1} \alpha_k \mu_n^k \quad (F.4)$$

Here, $\mu_n$ is a root of the polynomial in (F.3), and hence we can replace the inner summation in (F.4) by $-\mu_n^N$:

$$\sum_{k=0}^{N-1} \alpha_k f(k) = - \sum_{n=0}^{N-1} C_n \mu_n^N = -f(N) \quad (F.5)$$

Instead of utilizing the first $N - 1$ samples of $f$ (recalling that $f$ has $M$ samples where $M \geq 2N$) to form the summation in (F.4), we can generalize the procedure above the the $p$-th sample set comprising $N - 1$ terms as shown below:

$$\sum_{k=p}^{N-1+p} \alpha_{k-p} f(k) = \sum_{n=0}^{N-1} C_n \sum_{k=p}^{N-1+p} \alpha_{k-p} \mu_n^k = \sum_{n=0}^{N-1} C_n \sum_{k=0}^{N-1} \alpha_k \mu_n^{k+p} = \sum_{n=0}^{N-1} C_n \mu_n^p \sum_{k=0}^{N-1} \alpha_k \mu_n^k = - \sum_{n=0}^{N-1} C_n \mu_n^{p+N} = -f(N+p) \quad (F.6)$$

In equation (F.6) we can choose $p$ to be any integer from 0 upto $M - N - 1$. The polynomial coefficients $\alpha_k$ can be evaluated from the following matrix equation:

$$\sum_{k=0}^{N-1} f(k+p) \alpha_k = -f(N+p) \quad p = 0, 1, 2, \ldots, M - N - 1 \quad (F.7)$$
If $M = 2N$, equation (F.7) can be solved immediately and the $\alpha_k$ coefficients can be used in (F.3) to solve for the roots $\mu_n$. These $\mu_n$ modes can then be substituted into (F.2) and any $N$ set of equations (out of the total $M$) can be used to obtain the excitation amplitudes $C_n$. In order to improve the accuracy of this technique in the presence of noise and sampling round-off errors, one can utilize the technique of least squares where all $M - N$ equation in (F.7) are utilized in solving for $\alpha_k$ and all $M$ equations are used in (F.2) in solving for $C_n$. 
Appendix G

Bloch/Floquet Modes in a Periodic Lattice

Let $f(z)$ be a function of periodicity $d$ (i.e. $f(z + d) = f(z)$) associated with the following second order differential equation:

$$\frac{d^2\Psi}{dz^2} + f(z)\Psi = 0 \quad (G.1)$$

A well known result in quantum mechanics is that the general solution of equation (G.1) is of the form $\Psi(z) = \phi(z)e^{-j\beta z}$ where the function $\phi(z + d) = \phi(z)$ has the same period as $f(z)$ [53]. A solution of this form is referred to as a Bloch or Floquet mode and its distinguishing feature is that of a travelling wave solution $e^{-j\beta z}$ modulated by a periodic function.

We can obtain a similar result for a periodic electrical network [32] and the resulting Bloch modes will provide a complete set of functions to represent the voltages and currents at the various nodes under arbitrary excitations and terminations. To see that this is indeed the case, let us consider a linear network formed by back-to-back connections of identical unit cells characterized by the $2N \times 2N$ transmission matrix $T$. We define the state vector $X[m] = (V[m] \ I[m])^T$ which contains a column of $N$ voltages and $N$ currents (flowing in the $+z$ direction) at the $m$-th plane between adjacent unit cells:

$$X[m] = TX[m + 1] \quad (G.2)$$

In equation (G.2) the terminal planes are numbered such that $m$ increases in the $+z$ direction. Let the eigenvalues of $T$ be denoted by $\lambda_n$ with the corresponding eigenvectors $E_n$ such that $TE_n = \lambda_n E_n$. We will assume that these eigenvectors span the $2N$
dimensional vector space. This assumption might not hold for certain forms of the transmission matrix $T$ at a discrete set of frequency points or for pathological cases such as unit cells that contain a single series or shunt element. Excluding the special cases mentioned above, we can expand the voltages and currents at each terminal plane of the network chain using the eigenvectors of $T$:

$$X[m] = \sum_{n} a_{m,n} E_n$$  \hspace{1cm} (G.3)

Multiplying both sides of equation (G.3) by $T$:

$$X[m-1] = TX[m] = \sum_{n} a_{m,n} \lambda_n E_n$$

$$\therefore a_{m-1,n} = \lambda_n a_{m,n}$$  \hspace{1cm} (G.4)

It is evident from equation (G.4) that if we designate $a_{0,n} = a_n$, then all other expansion coefficients are obtained by $a_{m,n} = a_n \lambda_n^{-m}$. It is customary to express the eigenvalues as $\lambda_n = e^{\gamma_n d}$ for some complex $\gamma_n$ where $d$ is the physical length of each unit cell. With this notation, we can rewrite equation (G.3) as a function of axial distance $z = md$:

$$X[m] = \sum_{n} a_n e^{-\gamma_n md} E_n = \sum_{n} a_n e^{-\gamma_n z} E_n$$  \hspace{1cm} (G.5)

The $2N$ unknown amplitudes $a_n$ are obtained from the source and impedance conditions at both ends of a finite length chain.
Appendix H

MS/NRI-TL Coupler Design Graphs
Figure H.1: Normalized system impedance and coupling level on a $\varepsilon_r = 4.5$ substrate as a function of microstrip width and line spacing normalized to the substrate height for (a) Quarter-wave long MS/NRI-TL coupler and (b) Half-wave long MS/NRI-TL coupler.
Figure H.2: Normalized loading lumped capacitance on a $\varepsilon_r = 4.5$ substrate as a function of microstrip width and line spacing for (a) Quarter-wave long MS/NRI-TL coupler and (b) Half-wave long MS/NRI-TL coupler.
Figure H.3: Normalized loading lumped inductance on a $\varepsilon_r = 4.5$ substrate as a function of microstrip width and line spacing for (a) Quarter-wave long MS/NRI-TL coupler and (b) Half-wave long MS/NRI-TL coupler.
Figure H.4: Coupler length normalized with respect to free space wavelength on a \( \varepsilon_r = 4.5 \) substrate as a function of microstrip width and line spacing for (a) Quarter-wave long MS/NRI-TL coupler and (b) Half-wave long MS/NRI-TL coupler.
Figure H.5: Normalized matching reactance for the MS-TL line of the coupler on a $\varepsilon_r = 4.5$ substrate as a function of microstrip width and line spacing for (a) Quarter-wave long MS/NRI-TL coupler and (b) Half-wave long MS/NRI-TL coupler. The NRI-TL matching reactance is the negative of values shown in the plot.
Bibliography


