MSE-Based Linear Transceiver Designs for Multiuser MIMO Wireless Communications

by

Adam J. Tenenbaum

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto

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Abstract

This dissertation designs linear transceivers for the multiuser downlink in multiple-input multiple-output (MIMO) systems. The designs rely on an uplink/downlink duality for the mean squared error (MSE) of each individual data stream.

We first consider the design of transceivers assuming channel state information (CSI) at the transmitter. We consider minimization of the sum-MSE over all users subject to a sum power constraint on each transmission. Using MSE duality, we solve a computationally simpler convex problem in a virtual uplink. The transformation back to the downlink is simplified by our demonstrating the equality of the optimal power allocations in the uplink and downlink.

Our second set of designs maximize the sum throughput for all users. We establish a series of relationships linking MSE to the signal-to-interference-plus-noise ratios of individual data streams and the information theoretic channel capacity under linear minimum MSE decoding. We show that minimizing the product of MSE matrix determinants is equivalent to sum-rate maximization, but we demonstrate that this problem does not admit a computationally efficient solution. We simplify the problem by minimizing the product of mean squared errors (PMSE) and propose an iterative algorithm based on alternating optimization with near-optimal performance.
The remainder of the thesis considers the more practical case of imperfections in CSI. First, we consider the impact of delay and limited-rate feedback. We propose a system which employs Kalman prediction to mitigate delay; feedback rate is limited by employing adaptive delta modulation. Next, we consider the robust design of the sum-MSE and PMSE minimizing precoders with delay-free but imperfect estimates of the CSI. We extend the MSE duality to the case of imperfect CSI, and consider a new optimization problem which jointly optimizes the energy allocations for training and data stages along with the sum-MSE/PMSE minimizing transceivers. We prove the separability of these two problems when all users have equal estimation error variances, and propose several techniques to address the more challenging case of unequal estimation errors.
To Laura and Benjamin, with all my love.
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During my time spent at the University of Toronto, I have been fortunate to learn from and work with many outstanding individuals. I am most grateful to my advisor, Prof. Raviraj Adve, for his unwavering support throughout the formulation, research, and writing of this dissertation. His work ethic, devotion to his students, and dedication to his family have provided me with a role model that I hope to emulate.

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I thank my family and closest friends: my parents, for reinforcing the value of education; my brother, for helping me find my way along the path of academia; and Roni Gordon and Jonathan Lebi for their constant encouragement and friendship.

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advocated on my behalf when others questioned my choices, encouraged me when I felt defeated, and supported our family when a graduate student stipend would not suffice. She has given me the greatest gift of all in our beautiful son, Benjamin; despite having completed a doctoral dissertation, I view him today and the outstanding man that I know he will become, as my greatest accomplishment to date.

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<th>Description</th>
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<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>BC</td>
<td>Broadcast Channel</td>
</tr>
<tr>
<td>BD</td>
<td>Block Diagonalization</td>
</tr>
<tr>
<td>BS</td>
<td>Base Station</td>
</tr>
<tr>
<td>BER</td>
<td>Bit Error Rate</td>
</tr>
<tr>
<td>CSI</td>
<td>Channel State Information</td>
</tr>
<tr>
<td>DPC</td>
<td>Dirty Paper Coding</td>
</tr>
<tr>
<td>FDD</td>
<td>Frequency Division Duplexing</td>
</tr>
<tr>
<td>GGP</td>
<td>Generalized Geometric Program(ming)</td>
</tr>
<tr>
<td>GP</td>
<td>Geometric Program(ming)</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
</tr>
<tr>
<td>ISI</td>
<td>Intersymbol Interference</td>
</tr>
<tr>
<td>KKT</td>
<td>Karush-Kuhn-Tucker</td>
</tr>
<tr>
<td>LTE</td>
<td>Long Term Evolution</td>
</tr>
<tr>
<td>MAC</td>
<td>Multiple Access Channel</td>
</tr>
<tr>
<td>MISO</td>
<td>Multiple-Input Single-Output</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multiple-Input Multiple-Output</td>
</tr>
<tr>
<td>MMSE</td>
<td>Minimum Mean Squared Error</td>
</tr>
<tr>
<td>MS</td>
<td>Mobile Station</td>
</tr>
<tr>
<td>MSE</td>
<td>Mean Squared Error</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Full Form</td>
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<tr>
<td>MUI</td>
<td>Multi-User Interference</td>
</tr>
<tr>
<td>OFDM</td>
<td>Orthogonal Frequency Division Multiplexing</td>
</tr>
<tr>
<td>PDetMSE</td>
<td>Product of Mean Squared Error Matrix Determinants</td>
</tr>
<tr>
<td>PMSE</td>
<td>Product of Mean Squared Errors</td>
</tr>
<tr>
<td>PSK</td>
<td>Phase-Shift Keying</td>
</tr>
<tr>
<td>QAM</td>
<td>Quadrature Amplitude Modulation</td>
</tr>
<tr>
<td>QPSK</td>
<td>Quadrature-Phase-Shift Keying</td>
</tr>
<tr>
<td>QoS</td>
<td>Quality of Service</td>
</tr>
<tr>
<td>Rx</td>
<td>Receiver</td>
</tr>
<tr>
<td>SDP</td>
<td>Semidefinite Program(ming)</td>
</tr>
<tr>
<td>SQP</td>
<td>Sequential Quadratic Programming</td>
</tr>
<tr>
<td>SINR</td>
<td>Signal-to-Interference-plus-Noise Ratio</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
</tr>
<tr>
<td>Sum-MSE</td>
<td>Sum of Mean Squared Errors</td>
</tr>
<tr>
<td>SVD</td>
<td>Singular Value Decomposition</td>
</tr>
<tr>
<td>TDD</td>
<td>Time Division Duplexing</td>
</tr>
<tr>
<td>THP</td>
<td>Tomlinson-Harashima Precoding</td>
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<tr>
<td>Tx</td>
<td>Transmitter</td>
</tr>
<tr>
<td>ZF</td>
<td>Zero Forcing</td>
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# Notation

\(x\)  
Scalar

\(\mathbf{x}\)  
Vector

\(\mathbf{X}\)  
Matrix

\(\mathbf{x}_i\)  
Entry \(i\) in vector \(\mathbf{x}\)

\(\mathbf{X}_{i,j}\)  
Entry in row \(i\), column \(j\) of matrix \(\mathbf{X}\)

\(\mathbf{I}_n\)  
\(n \times n\) identity matrix

\(\mathbf{0}_n\)  
Zero vector of length \(n\)

\(\mathbf{0}_{m \times n}\)  
Zero matrix of dimension \(m \times n\)

\(\emptyset\)  
Empty set

\(\mathbf{1}_n\)  
Ones vector of length \(n\)

\(\mathbf{e}_i^n\)  
Standard basis vector of length \(n\) with a single one in the \(i^{th}\) position and zeroes elsewhere

\([x_1, x_2, \ldots, x_N]\)  
Concatenation of scalars into row vector

\([\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]\)  
Concatenation of column vectors into matrix

\([\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N]\)  
Concatenation of matrices

\(\text{diag}\left[\mathbf{x}\right]\)  
Concatenation of vector entries into diagonal matrix

\(\text{diag}\left[\mathbf{X}_1, \ldots, \mathbf{X}_n\right]\)  
Concatenation of matrices into block-diagonal matrix

\(\text{vec}\left[\mathbf{X}\right]\)  
Vectorization, stacks column vectors of \(\mathbf{X}\) into a single column vector

\(|\cdot|\)  
Magnitude of a complex number

\((\cdot)^*\)  
Complex conjugate (scalar)

\(\|\cdot\|_2\)  
Euclidean norm of a vector
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<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$| \cdot |_1$</td>
<td>1-norm of a vector (sum of entries)</td>
</tr>
<tr>
<td>$(\cdot)^T$</td>
<td>Transpose</td>
</tr>
<tr>
<td>$(\cdot)^H$</td>
<td>Conjugate transpose (Hermitian)</td>
</tr>
<tr>
<td>$\text{tr} [\cdot]$</td>
<td>Trace</td>
</tr>
<tr>
<td>$\text{det} [\cdot]$</td>
<td>Determinant</td>
</tr>
<tr>
<td>null [\cdot]</td>
<td>Null space</td>
</tr>
<tr>
<td>$\mathbb{E} [\cdot]$</td>
<td>Statistical expectation</td>
</tr>
<tr>
<td>$\Re [\cdot]$</td>
<td>Real part</td>
</tr>
<tr>
<td>$\Im [\cdot]$</td>
<td>Imaginary part</td>
</tr>
<tr>
<td>$\nabla_x [f(\mathbf{x})]$</td>
<td>Gradient of $f(\mathbf{x})$; $[\nabla_x [f(\mathbf{x})]]_i = \frac{\partial f(\mathbf{x})}{\partial x_i}$</td>
</tr>
<tr>
<td>$\mathcal{H}_x [f(\mathbf{x})]$</td>
<td>Hessian matrix of $f(\mathbf{x})$; $[\mathcal{H}<em>x [f(\mathbf{x})]]</em>{i,j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$</td>
</tr>
<tr>
<td>$\text{dom} [f(\mathbf{x})]$</td>
<td>The domain of $f(\mathbf{x})$; i.e., the values upon which it is defined</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>The set of dimension-$n$ vectors with real entries</td>
</tr>
<tr>
<td>$\mathbb{R}^n_+$</td>
<td>The set of dimension-$n$ vectors with non-negative real entries</td>
</tr>
<tr>
<td>$\mathbb{R}^n_{++}$</td>
<td>The set of dimension-$n$ vectors with positive real entries</td>
</tr>
<tr>
<td>$\mathbb{C}^n$</td>
<td>The set of dimension-$n$ vectors with complex entries</td>
</tr>
<tr>
<td>$\mathbb{C}^{m \times n}$</td>
<td>The set of $m \times n$ matrices with complex entries</td>
</tr>
<tr>
<td>$\mathbb{S}^n$</td>
<td>The set of symmetric $n \times n$ matrices</td>
</tr>
<tr>
<td>$\mathbb{S}^n_+$</td>
<td>The set of positive semidefinite $n \times n$ matrices</td>
</tr>
<tr>
<td>$\mathbb{S}^n_{++}$</td>
<td>The set of positive definite $n \times n$ matrices</td>
</tr>
<tr>
<td>$\mathbf{x} \succ 0$</td>
<td>$\mathbf{x}$ contains only positive entries</td>
</tr>
<tr>
<td>$\mathbf{x} \succeq 0$</td>
<td>$\mathbf{x}$ contains only non-negative entries</td>
</tr>
<tr>
<td>$\mathbf{X} \succ 0$</td>
<td>$\mathbf{X}$ is a positive definite matrix</td>
</tr>
<tr>
<td>$\mathbf{X} \succeq 0$</td>
<td>$\mathbf{X}$ is a positive semidefinite matrix</td>
</tr>
<tr>
<td>$\mathcal{CN}(\mu, \Sigma)$</td>
<td>Multivariate normal distribution with mean $\mu$ and covariance $\Sigma$</td>
</tr>
<tr>
<td>$\hat{e}_{\text{max}}(\mathbf{A}, \mathbf{B})$</td>
<td>Dominant generalized eigenvector of the pair of matrices $(\mathbf{A}, \mathbf{B})$</td>
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Chapter 1

Introduction

This thesis develops and analyzes techniques for a multiuser downlink in a cellular wireless network: the broadcast channel. To minimize complexity, we restrict our development to linear processing at the transmitter and receiver.

1.1 Motivation

The mobile Internet continues to experience incredibly rapid growth: nearly twenty percent of over five billion wireless subscribers worldwide have access to high-speed (3G or better) wireless service, and even faster 4G networks are beginning to appear in parts of Asia, Europe, and North America [1–4]. These growth trends are motivated by users’ desire for mobility, ubiquity, and demand for services such as social networking, high definition video-on-demand, and cloud computing, storage, and file-sharing.

In order to provide these services reliably, at high data rates, and to a large number of users, new technologies must be developed. Standards proposals for pre-4G broadband wireless systems (e.g., WiMAX [5] and 3GPP LTE [6]) encourage the use of multiple antennas at the transmitter and/or receiver. These multiple-input multiple-output (MIMO) systems use the additional spatial degrees of freedom provided by antenna arrays to address several challenges posed by the nature of the wireless channel.
1.1.1 Environmental Challenges

Mobility and Fading  The propagation environment around mobile receivers changes frequently due to their mobility. As a result, the wireless channel is a time-varying channel. Transmitter designs for mobile communications which depend on knowledge of the channels between the transmitter and mobile receivers must take this into account by keeping track of and adapting to the always-changing channel. A type of degradation known as fading is said to occur when changes in the propagation environment affect the condition of the channel. Fading occurs more often in urban environments where there are many scattering obstacles. If the propagation path from the transmitter to the receiver encounters a large obstruction, shadowing is said to occur and may possibly lead to deep fading, where the channel changes from ideal to unusable. Fading is a problem that fundamentally limits system performance, and special measures must be taken to design transmitters and receivers which mitigate fading in order to ensure consistency in wireless communications and to satisfy quality of service (QoS) requirements. This is accomplished by employing diversity techniques [7], whereby redundant copies of data are transmitted. Under diversity approaches, all copies of the data must be subject to a deep fade for communication to fail, which becomes decreasingly likely at an exponential rate as the number of redundant copies increases.

Limited capacity  Mobile receivers in the wireless environment typically encounter additive noise (which is most often modelled as Gaussian). As a result, the rate at which the transmitter can communicate with each receiver is fundamentally limited. However, MIMO techniques permit an alternative approach to diversity to be taken when multiple strong channel paths exist; this type of spatial multiplexing can vastly increase the transmission rate by transmitting multiple data streams to each receiver.
Multiuser interference  Information transmitted in the wireless domain is broadcast to all receivers, as illustrated in Figure 1.1. As a result, the information intended for each user can be received by other receivers; at users other than the desired one, this transmission acts as interference. This type of corruption is referred to as multiuser interference (MUI) for co-channel transmissions. Systems designed for the wireless downlink must be consider the avoidance or complete suppression of MUI.

Multipath propagation  The nature of the wireless environment leads to multipath propagation as reflected echoes of the transmitted signal arrive as delayed copies at the receiver. If the spread of these multiple paths is greater than a symbol interval, this phenomenon leads to intersymbol interference (ISI) which can be viewed in the frequency domain as frequency selective fading. Multipath is particularly problematic for systems with high data rate requirements, as the shorter symbol interval associated with increased data rates can cause multipath components to induce ISI that spans multiple symbols, necessitating the design of complex and costly equalizers. An approach of growing importance to solve this issue is the use of orthogonal frequency division multiplexing (OFDM), a multicarrier transmission technique which transforms the frequency-selective wideband
system into a set of parallel narrowband channels each experiencing flat fading [8]. In this thesis, we assume that channels are modelled as frequency-flat, as proposals for all 4G systems include a multicarrier modulation element that mitigates the effects of frequency selectivity. We thereby simplify our designs by restricting any approaches we take towards diversity and multiplexing to purely spatial processing.

1.1.2 Design Challenges

Channel State Information at the Transmitter and Receiver

Open loop approaches to diversity transmission and reception [7, 9] and spatial multiplexing [10] operate in the absence of channel state information (CSI) at the transmitter. However, while open loop techniques permit some improvements to diversity and rate, the greatest potential performance gains in MIMO systems occur when CSI is available at the transmitter. With complete knowledge of the channel, transmit diversity techniques can select the best antenna for transmission [11] or combine signals in an optimal manner using maximum ratio transmission [12]. Full CSI can also be used to diagonalize the channel, thus creating multiple parallel channels for increased data rates [13] or providing spatial multiplexing for simultaneous transmission to multiple users.

The primary focus of this dissertation is to design transmission and reception schemes for closed-loop multiuser MIMO downlink communications systems while exploiting the diversity and multiplexing benefits of MIMO. We begin by considering ideal designs under the assumption of perfect and instantaneous CSI for all channels at the transmitter and for each of the users’ individual channels at each respective receiver.

Designing precoders using perfect CSI provides insights into the limits on achievable performance in the system being considered; however, this assumption is unrealistic. In a system implementation of these precoders, the receivers must estimate the required channel (e.g. via training) and feed back the estimated CSI to the transmitter. Conse-
Chapter 1. Introduction

Consequently, CSI can take on imperfections in several forms, including delay, feedback error (i.e., quantization error), and estimation error. The second half of this thesis is devoted to examining the impact of these types of error and to proposing methods for their mitigation.

Computational Complexity, Power, and Energy Limitations

A major factor that influences the design of mobile communications systems is the computational complexity of the precoding and decoding algorithms. Mobile receivers containing small batteries are typically energy-limited; thus, computational resources must be used efficiently. Thus, we only consider receivers that perform relatively simple linear estimation and decoding.

While transmitters in the downlink are not typically battery-operated, lengthy computations may still be undesirable when data transmissions are sensitive to delay. Power limitations are also a concern as governmental regulations impose limits on average and peak powers that may be transmitted wirelessly. We take these limitations into consideration by proposing linear precoders that operate under a total instantaneous power constraint. We focus on designs based on optimization of tractable and efficiently computed functions of the mean squared errors (MSE) across all users and their data streams.

1.2 Organization

This dissertation is organized as follows. In Chapter 2, we outline the linear precoding system model in the downlink and introduce the virtual uplink as a useful tool to be used in solving MSE-based optimization problems. We derive expressions for the MSE matrices and the minimum-MSE receivers for both the downlink and virtual uplink configurations. Chapters 3 and 4 address the problem of linear precoder design to minimize the sum of mean squared errors (sum-MSE) and maximize the sum rate over all users,
respectively, in the presence of a sum power constraint on transmitted symbols and with perfect and instantaneous CSI. In Chapter 5, we consider the limited and delayed feedback of channel information and introduce a prediction mechanism to mitigate the impact of delay. Chapter 6 extends the designs of Chapters 3 and 4 to the case of imperfect CSI. We propose designs to jointly allocate energy to training and data symbols while designing the sum-MSE minimizing and sum-rate maximizing linear precoders. Chapter 7 summarizes the thesis and suggests future extensions of this work.

1.3 Contributions

The following contributions are made in this thesis:

- We extend the MSE and signal-to-interference-plus-noise (SINR) uplink/downlink dualities first developed for mobile users with single receive antennas in [14, 15] to the case of MIMO systems. Using this result, we propose an iterative algorithm employing convex optimization for the design of sum-MSE minimizing linear precoders and decoders in the multiuser MIMO downlink under a sum power constraint and in the presence of instantaneous and complete CSI. We present a previously unseen duality result for the case of sum-MSE minimization which demonstrates the equality of the optimal power allocations in the downlink and virtual uplink.

- We develop a series of relationships between the MSE/SINR for individual data streams and the information theoretical sum capacity under linear minimum MSE decoding. The resulting expressions are used to design linear precoders and decoders that perform sum-rate maximization for the multiuser MIMO downlink under a sum power constraint for the case of perfect CSI.

- We consider the impact of limited feedback and delayed CSI on the performance of the sum-MSE minimizing precoder and propose a prediction mechanism based on
linear filtering to mitigate its effect.

- We consider CSI that is stochastically corrupted by Gaussian noise, and design robust precoders to mitigate its impact. The relationship between the product of MSEs and the sum-rate are then extended to form a lower bound on sum-rate in the case of imperfect CSI. We address the problem of allocating energy optimally to training and data symbols, and find a closed form solution for the optimum energy allocation in the case of equal estimation error variances across users. We also analyze the problem for the case of unequal estimation error variances, and propose near-optimal and heuristic methods for precoder design in this scenario.
Chapter 2

System Model

In this chapter, we describe the linear precoding and decoding system under consideration in this thesis. The system models multiuser downlink communications, which involves a single base station broadcasting information to a number of mobile receivers (Figure 2.1). The transmitter and potentially the receivers are equipped with more than one antenna; this configuration with multiple users, multiple inputs, and multiple outputs is referred to as a multiuser (MU) MIMO system. While the transmitter is designed to perform joint processing for all users, cooperation is not permitted at the receivers. Each mobile receiver performs its own receive processing independently.

Figure 2.1: Transmit and receive processing for downlink communications
Chapter 2. System Model

2.1 Downlink

A block diagram for the downlink system model is depicted in Figure 2.2. The system comprises a base station (transmitter) with $M$ antennas communicating with $K$ decentralized mobile users. Each user $k$ has $N_k$ antennas, and the total number of receive antennas is $N = \sum_{k=1}^{K} N_k$. The channel between the transmitter and user $k$ is represented by the matrix $H_k^H \in \mathbb{C}^{N_k \times M}$, and the overall $N \times M$ channel matrix is $H^H$, where

$$H = [H_1, \ldots, H_K]. \quad (2.1)$$

2.1.1 Channel Model

In this thesis, we make a simplifying assumption by treating the channel $H$ as a Rayleigh channel experiencing slow, flat fading with independent coefficients for each antenna pair. The Rayleigh fading model is commonly used to model wireless systems in urban environments as it is the type of fading that typically arises due to phase interference caused by unresolvable multipath when no line-of-sight component is present. The algorithms designed in Chapters 3 and 4 do not depend on the fading rate, as the designs are based on the premise of the base station possessing perfect and instantaneous knowledge of the channel state information (CSI). In Chapter 5, we consider the impact of slow fading induced by movement of the mobile terminals, and in Chapter 6, we design robust precoders in the presence of imperfect channel estimates. We describe the required modifications to the channel model in each of these chapters.

The assumption of frequency-flat fading is a common one which vastly simplifies the
resulting model, as the channel response can be modelled by a matrix $H$ representing a single impulse/tap. Even though multipath propagation typically occurs in broadband communication systems (resulting in frequency-selective fading), we assume that in such a scenario, multicarrier techniques such as OFDM could be applied to transform the system into a set of parallel narrowband channels experiencing flat fading [8]. The algorithms and precoder designs presented in this thesis could then be applied on a per-subcarrier basis.

2.1.2 Linear Precoding and Decoding

The vector $\mathbf{x}_k$, defined as
\[
\mathbf{x}_k = [x_{k1}, \ldots, x_{kL_k}]^T \in \mathbb{C}^{L_k},
\]
contains $L_k$ data symbols to be transmitted to user $k$. A total of $L = \sum_{k=1}^{K} L_k$ source symbols are collected in the overall length-$L$ source symbol vector
\[
\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_K^T]^T.
\]
In the system described by Figure 2.2, we have divided the precoder into two stages. User $k$’s data streams are processed by the transmit filter
\[
\bar{U}_k = [\bar{u}_{k1}, \ldots, \bar{u}_{kL_k}] \in \mathbb{C}^{M \times L_k},
\]
where $\bar{u}_{kj}$ is the precoding beamformer for stream $j$ of user $k$ with $\|\bar{u}_{kj}\|_2 = 1$. These individual precoders are combined in the $M \times L$ global transmitter precoder matrix
\[
\bar{U} = [\bar{U}_1, \ldots, \bar{U}_K].
\]
Power is allocated to user $k$’s data streams in the vector
\[
p_k = [p_{k1}, \ldots, p_{kL_k}]^T \in \mathbb{R}^{L_k}
\]
and $\mathbf{P}_k = \text{diag}[p_k]$. We define the diagonal downlink power allocation matrix as $\mathbf{P} = \text{diag}[\mathbf{p}_1^T, \ldots, \mathbf{p}_K^T]$. 
Based on this model, user \( k \) receives a length-\( N_k \) vector

\[
y^{DL}_k = H_k^H \bar{U} \sqrt{P} x + n_k,
\]

where the superscript \( DL \) indicates the downlink, and \( n_k \) is an additive white complex Gaussian noise (AWGN) vector of dimension \( N_k \).

Each mobile receiver \( k \) applies the receive filter \( V^H_k \in \mathbb{C}^{L_k \times N_k} \), yielding estimates of its \( L_k \) source symbols \( x_k \), and denoted as

\[
\hat{x}^{DL}_k = V^H_k H_k^H \bar{U} \sqrt{P} x + V^H_k n_k.
\]

We also make use of the following block system representation that incorporates all users and their data substreams,

\[
\hat{x}^{DL} = V^H H^H \bar{U} \sqrt{P} x + V^H n,
\]

where

\[
\hat{x}^{DL} = \begin{bmatrix} \hat{x}^{DL}_1 \\ \vdots \\ \hat{x}^{DL}_K \end{bmatrix}, \quad n = \begin{bmatrix} n_1 \\ \vdots \\ n_K \end{bmatrix},
\]

and \( V^H = \text{diag} [V^H_1, \ldots, V^H_K] \) is a block diagonal decoder matrix of dimension \( L \times N \). It is important to emphasize the block diagonality of \( V \); that is, mobile users are not allowed to cooperate.
2.1.3 Mean Squared Error in the Downlink

We define the MSE matrix for user $k$ in the downlink for arbitrary transmit filter $\bar{U}$, downlink power allocation $P$, and receive filter $V_k^H$ as

$$
\varepsilon_{DL}^k = \mathbb{E} \left[ (\hat{x}_{DL}^k - x_k)(\hat{x}_{DL}^k - x_k)^H \right] = V_k^H H_k^H \bar{U} \sqrt{P} \mathbb{E} [xx^H] \sqrt{P} \bar{U} H_k V_k + V_k^H H_k^H \bar{U} \sqrt{P} \mathbb{E} [n_k^H] V_k \\
- V_k^H H_k^H \bar{U} \sqrt{P} \mathbb{E} [xx^H] V_k + V_k^H H_k^H \bar{U} \sqrt{P} \mathbb{E} [n_k^H] V_k \\
- \mathbb{E} [x_k x_k^H] \sqrt{P} \bar{U} H_k V_k + V_k^H H_k^H \bar{U} \sqrt{P} \mathbb{E} [n_k x_k^H] V_k + \mathbb{E} [x_k x_k^H] \\
= V_k^H \left( H_k^H \bar{U} \bar{P} \bar{U} H_k + \sigma_n^2 I_{N_k} \right) V_k - V_k^H H_k^H \bar{U} \sqrt{P} \bar{U} H_k V_k + I_{L_k} \\
= V_k^H R_k V_k - V_k^H H_k^H \bar{U} \sqrt{P} \bar{U} H_k V_k + I_{L_k},
$$

(2.11)

where we have defined the downlink signal-plus-interference-plus-noise covariance matrix for user $k$ as

$$
R_k = H_k^H \bar{U} \bar{P} \bar{U} H_k + \sigma_n^2 I_{N_k}.
$$

(2.12)

The evaluation of expectation terms follows from the following three assumptions:

1. $x$ are independent source symbols with unit average energy: $\mathbb{E} [xx^H] = I_L$.

2. $n_k$ consists of zero-mean, spatially white Gaussian noise: $n_k \sim \mathcal{CN}(0, \sigma_n^2 I_{N_k})$.

3. Source symbols and noise symbols are independent: $\mathbb{E} [xn_k^H] = 0_{L \times N_k}$.

2.1.4 Downlink MMSE Receivers

Since the objective functions being considered in this thesis are all functions of the MSE matrices $\varepsilon_{DL}^k$, we use the Wiener filter $V_k^H$ at each receiver $k$. This is the optimum linear receiver in the minimum mean squared error (MMSE) sense [16], and independently
minimizes each of the terms in $\varepsilon_{DL}^k$. It can found by complex differentiation of (2.11) with respect to $V_k$ as

$$V_k^{*H} = \sqrt{P_k} \bar{U}_k^H H_k R_k^{-1}. \quad (2.13)$$

Substitution of $V_k^*$ into (2.11) yields the MMSE matrix

$$\varepsilon_{DL}^k = I_{L_k} - \sqrt{P_k} \bar{U}_k^H H_k R_k^{-1} H_k^H \bar{U}_k \sqrt{P_k}. \quad (2.14)$$

In order for mobile user $k$ to determine its MMSE decoder $V_k^*$, it must possess knowledge of both the signal-plus-interference-plus-noise covariance matrix and the product of the channel matrix and precoder. In this thesis, we assume that these quantities are known to each receiver; in practice, they must each be estimated or communicated to the individual receivers. One possible approach is to perform blind and training-based sample estimation for $R_k$ and $H_k^H \bar{U}_k \sqrt{P_k}$, respectively, as considered in [17].

### 2.2 Virtual Uplink

While this thesis focuses on designing transceivers for communications in the multiuser downlink, we also frequently employ the use of the virtual uplink channel model, as illustrated in Figure 2.3. In this model, the mobile user $k$ communicates with the base station over the transpose channel $H_k$; that is, the communications channel is modelled as the conjugate transpose of the true channel as used in the downlink. The transmit and receive filters for user $k$ in the virtual uplink are $\tilde{V}_k$ and $U_k^H$ respectively, with

![Figure 2.3: Linear processing for user $k$ in the virtual uplink.](image-url)
normalized precoding beamformers; i.e., \( \| \bar{v}_{kj} \|_2 = 1 \). Power is allocated to user \( k \)'s data streams as

\[
q_k = [q_{k1}, \ldots, q_{kL_k}]^T \in \mathbb{R}^{L_k}_+,
\]

(2.15)

with \( Q_k = \text{diag} [q_k] \). As in the downlink, we introduce a notation that combines all users in a block form for convenience, with \( Q = \text{diag} [q_1^T, \ldots, q_K^T] \) and \( \bar{V} = \text{diag} [\bar{V}_1, \ldots, \bar{V}_K] \).

The received symbol vector at the base station is

\[
y_{UL} = \sum_{i=1}^K H_i \bar{V}_i \sqrt{Q_i} x_i + n
\]

(2.16)

the estimated symbol vector for user \( k \) is

\[
\hat{x}_{UL}^k = U_k^H \bar{V} \sqrt{Q} x + U_k^H n,
\]

(2.17)

and the estimated symbol vector for all users is

\[
\hat{x}_{UL} = U^H \bar{V} \sqrt{Q} x + U^H n.
\]

(2.18)

The assumptions made in Section 2.1.3 are applied to the virtual uplink as well, where the additive noise term in the virtual uplink is replaced by \( n \sim \mathcal{CN}(0, \sigma_n^2 I_M) \).

### 2.2.1 Mean Squared Error in the Virtual Uplink

As in the downlink, we define the MSE matrix for user \( k \) in the virtual uplink under arbitrary transmit filters \( \bar{V}_k \), virtual uplink power allocations \( Q_k \), and receive filters \( U_k^H \) as

\[
\varepsilon_{UL}^k = \mathbb{E} \left[ (\hat{x}_{UL}^k - x_k)(\hat{x}_{UL}^k - x_k)^H \right]
\]

\[
= U_k^H \left( \sum_{i=1}^K H_i \bar{V}_i Q_i \bar{V}_i^H H_i^H + \sigma_n^2 I_M \right) U_k - U_k^H \bar{V}_k \sqrt{Q_k} - \sqrt{Q_k} \bar{V}_k^H H_k^H U_k + I_{L_k}
\]

\[
= U_k^H RU_k - U_k^H \bar{V}_k \sqrt{Q_k} - \sqrt{Q_k} \bar{V}_k^H H_k^H U_k + I_{L_k},
\]

(2.19)
where we have defined the virtual uplink received signal-plus-noise covariance matrix

\[ R = \sum_{i=1}^{K} H_i V_i Q_i \bar{V}_i^H H_i^H + \sigma_n^2 I_M. \]  

(2.20)

Expectations in (2.19) have been evaluated using the same assumptions as in the downlink evaluation of (2.11).

Even though the MSE matrix expressions in (2.11) and (2.19) appear to be similar, there is no simple relationship between the two matrices that allows one to be expressed in terms of the other. We explore the relationship between the scalar MSE expressions for each data stream in the downlink and virtual uplink in Section 3.2, where we derive a duality relationship between the two quantities.

### 2.2.2 Virtual Uplink MMSE Receivers

The optimum linear receiver can once again be found as the Wiener filter, derived via complex differentiation of (2.19) as

\[ U^H_k = \sqrt{Q_k \bar{V}_k^H H_k^H R^{-1}}, \]  

(2.21)

with the associated MMSE matrix

\[ \varepsilon_k^{UL} = I_{L_k} - \sqrt{Q_k \bar{V}_k^H H_k^H R^{-1} H_k \bar{V}_k} \sqrt{Q_k}. \]  

(2.22)

### 2.3 Number of Data Streams and Resolvability

Since the system under consideration is limited to linear processing, we impose the following two constraints when choosing the number of data streams to transmit to each user:

\[ L \leq M \]  

(2.23)

\[ L_k \leq N_k \quad \forall k = 1, \ldots, K \]  

(2.24)
These can be thought of as “soft constraints”; i.e., it is possible to design MSE-based transceivers according to the framework proposed in this thesis while violating these constraints. However, such designs will likely exhibit poor performance due to unresolvable interference; as the signal-to-noise ratio (SNR) increases, the system will become interference-limited with an accompanying error floor.

Note that these resolvability constraints are a special case of the following general constraints:

\[ L \leq \text{rank}(H) \]
\[ L_k \leq \text{rank}(H_k) \quad \forall k = 1, \ldots, K. \]

In the case of i.i.d. channel coefficients (as specified in Section 2.1.1), these two sets of constraints are equivalent as the channel matrices are full-rank with probability 1.
Chapter 3

Sum-MSE Minimization with Perfect Channel State Information

In this chapter, we address the problem of designing a linear precoder that minimizes the sum of the mean squared errors over all users and each of their data streams, subject to a sum power constraint\(^1\).

The goal of this section is to optimize a measure of overall quality of service when a fixed number of data streams is required to be transmitted to each user. Minimizing the sum-MSE does not seem like the natural choice of objective function – an objective directly related to the communications application would be the minimization of the average bit error rate (BER). Expressions for the exact and approximate BER are well known functions of the signal-to-noise ratio (SNR) for a variety of modulation schemes, including phase-shift keying (PSK) and quadrature amplitude modulation (QAM), in the case of single-user communication \([20]\). These approximations can be applied to the multiuser case by substituting the SINR, and their accuracy depends on how close to Gaussian the noise and interference are.

\(^1\)The contributions presented in Sections 3.2 and 3.3 were developed in joint collaboration with Ali Khachan, a Master’s student from our research group, and they have been presented in part in his thesis \([18]\). These results were also presented at the IEEE International Conference on Communications (ICC 2006), and appear in its proceedings \([19]\).
Working with BER expressions is often difficult or intractable due to their dependence on the integral $Q$ function. In contrast, the MSE based expressions admit elegant solutions based on convex optimization and permit an uplink-downlink duality to be applied. Moreover, a relationship exists between the minimum sum-MSE and the minimum average BER using the approximation based on SINR substitution, following the equivalence relationship between the minimum MSE and maximum SINR \((3.25)\). MSE-based design is also an attractive option because of its independence from the choice of modulation scheme, subject to the source symbol constraint of uncorrelated symbols with unit average energy (Assumption 1 in Section 2.1.3).

### 3.1 Literature Review

#### 3.1.1 Minimum Sum-MSE Precoder Designs

The main challenge to effectively communicating information in a broadcast environment is the mitigation of multi-user interference and inter-stream interference. One straightforward way of addressing this issue is to employ a null-space approach \([21,22]\), whereby zero forcing (ZF) or block diagonalization (BD) is employed at the transmitter to orthogonalize users, eliminating MUI at the cost of noise enhancement. By designing the precoder matrices $\hat{U}_k$ to reside in the null space of all other users (i.e., $\mathbf{H}_j \hat{U}_k = \mathbf{0 \forall j \neq k}$), the $K$-user multiuser system is effectively transformed into a set of $K$ independent single-user channels. Such an approach suffers from a rigid constraint requiring that the number of transmit antennas must be greater than or equal to the total number of receive antennas.

A more generalized approach is to employ iterative algorithms that repeatedly cycle through power control, optimizing the transmit filter and optimizing the receive filter. This approach removes the previously stated constraint on the number of transmit antennas. The problem of transmitted power minimization with SINR constraints has been comprehensively investigated in literature. A solution to solve the multiuser downlink
problem with individual SINR constraints in a multiple-input single-output (MISO) system, i.e., a system where users have a single antenna, is proposed in [14]. The algorithm is based on the duality between the uplink and the downlink, and solves the problem iteratively in the uplink before switching to the downlink. This solution is extended in [15] to minimizing the sum-MSE. The downlink problem is also solved for single data stream MIMO systems in [23]; however, the algorithm diverges when some target SINR scenarios are infeasible. The scheme is extended in [24] by incorporating dirty paper precoding and results in a suboptimal non-linear solution. In our work, we focus on linear processing methods and generalize the scheme in [14] to MIMO systems with multiple data stream transmission. Our proposed algorithm does not have the drawback of diverging in some scenarios as in [23], which allows us to examine the problem of feasibility of a set of target SINRs.

In [25], the work of [26] on single user MIMO systems is extended to the multiuser domain using an iterative joint optimization algorithm based on sum-MSE minimization and a per-user power constraint. A numerical method employing sequential quadratic programming (SQP) is also proposed that solves the problem under a sum power constraint. In [27], the problem of joint transmit-receive optimization in order to minimize the sum-MSE is considered in the uplink with per-user power constraints. The proposed scheme allows for each user to transmit multiple data streams. In this chapter, we generalize the scheme in [15] to MIMO systems with sum power constraint by making use of the uplink solution in [27] and by exploiting duality.

3.1.2 Uplink-Downlink Duality for MISO Systems

In [14], the authors develop an uplink-downlink duality result for the signal-to-interference-plus-noise ratio (SINR) for the case of single-antenna receivers – the channel from the base station to each mobile user is thus a MISO system. In this section, we provide a reinterpretation of their duality under our system model, by replacing the channel matri-
ces $H_k^H$ with vector channels $h_k^H$, the precoding matrices $\bar{U}_k$ with transmit beamforming vectors $\bar{u}_k$, and by eliminating beamforming at the mobile users:

$$\hat{x}_k^{DL} = h_k^H \bar{U} \sqrt{P} x + n_k$$

$$\hat{x}_k^{UL} = u_k^H \left( \sum_{j=1}^{K} h_j \sqrt{q_j} x_j \right) + u_k^H n,$$

where $n_k \sim \mathcal{CN}(0, \sigma_n^2)$ is a scalar AWGN term.

We begin by noting that the decoded symbol estimates in the downlink and virtual uplink can be decomposed into desired signal components, $\hat{s}_k$, and additive interference-plus-noise components, $z_k$,

$$\hat{x}_k^{DL} = h_k^H \bar{u}_k \sqrt{p_k} x_k + h_k^H \sum_{j \neq k} \bar{u}_j \sqrt{p_j} x_j + n_k$$

$$\hat{x}_k^{UL} = u_k^H h_k \sqrt{q_k} x_k + u_k^H \sum_{j \neq k} h_j \sqrt{q_j} x_j + u_k^H n,$$

The receive SINR at user $k$ in the downlink can be defined as

$$\text{SINR}_{k}^{DL} = \frac{\mathbb{E} [ |\hat{s}_k^{DL}|^2 ]}{\mathbb{E} [ |z_k^{DL}|^2 ]} = \frac{p_k h_k^H \bar{u}_k h_k}{h_k^H \left( \sum_{j \neq k} p_j \bar{u}_j h_j^H \right) + \sigma_n^2}$$

and the post-decoding receive SINR for user $k$ in the virtual uplink can be defined as

$$\text{SINR}_{k}^{UL} = \frac{\mathbb{E} [ |\hat{s}_k^{UL}|^2 ]}{\mathbb{E} [ |z_k^{UL}|^2 ]} = \frac{q_k u_k^H h_k h_k^H u_k}{u_k^H \left( \sum_{j \neq k} q_j h_j h_j^H \right) u_k + \sigma_n^2 \|u_k\|^2},$$

Note that SINR$_{k}^{UL}$ is invariant to scaling of $u_k$; thus, the per-stream SINR values are the same if unit norm receive beamformers $\bar{u}_k = u_k / \|u_k\|_2$ are employed. Thus, we can rewrite (3.6) as
\[ \text{SINR}_{UL} = \frac{q_k |h_k^H \hat{u}_k|^2}{\sum_{j \neq k} q_k |h_j^H \hat{u}_k|^2 + \sigma_n^2}. \] (3.7)

Given this setup, we now interpret the SINR duality as developed in [13] in a form that will be relevant to our minimum sum-MSE design:

**Theorem 3.1 (Uplink-Downlink SINR Duality [13]).** Given an arbitrary set of beamforming vectors \( \hat{u}_1, \ldots, \hat{u}_K \), a set of target SINR values \( \gamma_1, \ldots, \gamma_K \), and the sum power constraints \( \|p\|_1 \leq P_{\text{max}} \) and \( \|q\|_1 \leq P_{\text{max}} \): if all \( \gamma_k \) are feasible SINR values, they can obtained in either the downlink or in the virtual uplink using the following power allocations:

\[ p = \sigma_n^2 (D^{-1} - \Psi)^{-1} 1_K \] (3.8)
\[ q = \sigma_n^2 (D^{-1} - \Psi^T)^{-1} 1_K, \]

where

\[ D = \text{diag} \left[ \frac{\gamma_1}{|h_1^H \hat{u}_1|^2}, \ldots, \frac{\gamma_K}{|h_K^H \hat{u}_K|^2} \right], \] (3.9)

and

\[ [\Psi]_{ij} = \begin{cases} |h_i^H \hat{u}_j|^2 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}. \] (3.10)

### 3.2 Extending Uplink-Downlink Duality to MIMO Systems

In order to extend the duality result described in Section 3.1.2 to the MIMO case (with \( L_k \) data streams per user), we treat each data substream as a *virtual user*; the overall system has \( L \) virtual users. We relabel the entries in the precoder matrix \( \hat{U} \), the downlink and virtual uplink power allocation matrices \( P \) and \( Q \), and the estimated received symbol.
vectors $\hat{x}^{DL}$ and $\hat{x}^{UL}$ to correspond to substream indices $l = 1, \ldots, L$: 

$$
\begin{align*}
\bar{U} &= [\bar{u}_1, \ldots, \bar{u}_L] & U &= [u_1, \ldots, u_L] \\
\bar{V}_k &= [v_{\tilde{L}_k+1}, \ldots, v_{\tilde{L}_k+L_k}] & \bar{V}_k &= [\bar{v}_{\tilde{L}_k+1}, \ldots, \bar{v}_{\tilde{L}_k+L_k}] \\
P &= \text{diag}[p_1, \ldots, p_L] & Q &= \text{diag}[q_1, \ldots, q_L] \\
x^{DL} &= [\hat{x}_1^{DL}, \ldots, \hat{x}_L^{DL}] & x^{UL} &= [\hat{x}_1^{UL}, \ldots, \hat{x}_L^{UL}],
\end{align*}
$$

(3.11)

where $\tilde{L}_k = \sum_{i=1}^{k-1} L_i$ is the total number of data stream indices for all users preceding user $k$. For convenience, we will sometimes refer to the block diagonal set of virtual uplink precoder matrices as $\bar{V} = \text{diag} [\bar{V}_1, \ldots, \bar{V}_K]$. We define the user index corresponding to data stream $l$ as $\kappa(l)$:

$$
\kappa(l) = \left\{ k : l \in [\tilde{L}_k + 1, \tilde{L}_k + L_k] \right\}.
$$

(3.12)

We also define the set of effective channels $\tilde{h}_l$, which combine the set of virtual uplink precoders $\bar{V}_k$ with the corresponding channel matrices $H_k$, in the overall effective channel matrix $\tilde{H}$,

$$
\tilde{H} = [H_1 \bar{V}_1, \ldots, H_K \bar{V}_K] = [H_{\kappa(1)} \bar{V}_1, \ldots, H_{\kappa(L)} \bar{V}_L],
$$

(3.13)

Under this remapping of indices, we can express the estimated downlink symbol for data stream $l$ from (2.8) as

$$
\begin{align*}
\hat{x}_l^{DL} &= v_l^H \bar{U} \sqrt{\text{P}} x + v_l^H \bar{n}
\end{align*}
$$

where $\hat{x}_l^{DL}$ is the signal component of the estimate $\hat{x}_l^{DL}$ and $z_l^{DL}$ is the interference-plus-
noise term. We define the post-decoding SINR of data stream $l$ in the downlink as

$$\text{SINR}_{DL}^l = \frac{\mathbb{E}[|\hat{s}^D_{DL}|^2]}{\mathbb{E}[|z_{DL}^D|^2]} = \frac{p_l v_l^H H_{\kappa(l)}^H \bar{u}_l \bar{u}_l^H v_{\kappa(l)} + \sigma_n^2}{v_l^H H_{\kappa(l)}^H \left( \sum_{j \neq l} p_j \bar{u}_j \bar{u}_j^H \right) H_{\kappa(l)} v_{\kappa(l)} + \sigma_n^2}.$$  

(3.15)

As before, $\text{SINR}_{DL}^l$ is invariant to scaling of the receive beamformer $v_l$; thus, the per-stream SINR values are the same if unit norm receive beamformers $\bar{v}_l = v_l / \|v_l\|^2$ are employed. We rewrite (3.15) using the effective channel notation in a more concise form:

$$\text{SINR}_{DL}^l = \frac{\mathbb{E}[|\hat{s}_l|^2]}{\mathbb{E}[|z_l|^2]} = \frac{p_l \bar{h}_l^H \bar{u}_l \bar{u}_l^H \bar{h}_l}{\bar{h}_l^H \left( \sum_{j \neq l} p_j \bar{u}_j \bar{u}_j^H \right) \bar{h}_l + \sigma_n^2}.$$  

(3.16)

Thus, for the purpose of extending the SINR duality from Section 3.1.2 to the MIMO scenario, the above definition of the effective channel $\bar{H}^H$ is valid for the downlink as well as the virtual uplink.

We rewrite the received symbol estimate (2.17) for the virtual uplink in a similar manner,

$$\hat{x}^L_{UL} = u_l^H H^H \sqrt{Q} x + u_l^H n$$

$$= u_l^H H_{\kappa(l)} \bar{v}_l \sqrt{q_l} + u_l^H \left( \sum_{j \neq l} H_{\kappa(j)} \bar{v}_j \sqrt{q_j} \bar{x}_j \right) + u_l^H n$$

(3.17)

$$= \underbrace{\bar{u}_l^H \bar{h}_l \sqrt{q_l} \bar{x}_l + u_l^H \left( \sum_{j \neq l} \bar{h}_j \sqrt{q_j} \bar{x}_j \right)}_{\hat{x}^L_{UL}} + u_l^H n,$$

where $\hat{x}^L_{UL}$ and $z^L_{UL}$ correspond to the signal and interference-plus-noise components of the estimated symbols in the virtual uplink. It follows that the post-decoding SINR of
data stream \( l \) in the virtual uplink can be expressed as

\[
\text{SINR}_{UL}^{UL} = \frac{\mathbb{E} \left[ |\hat{s}_l^{UL}|^2 \right]}{\mathbb{E} \left[ |z_l^{UL}|^2 \right]}
= \frac{q_l \bar{u}_l^H \hat{h}_l \hat{h}_l^H \bar{u}_l}{\bar{u}_l^H \left( \sum_{j \neq l} q_j \hat{h}_j \hat{h}_j^H \right) \bar{u}_l + \sigma_n^2 \| \bar{u}_l \|_2^2}
= \frac{q_l \| \hat{h}_l \|_2^2}{\sum_{j \neq l} q_j \| \hat{h}_j \|_2^2 + \sigma_n^2},
\]

which follows from setting \( \bar{u}_l = u_l / \| u_l \|_2 \) in accordance with the invariance of \( \text{SINR}_{UL}^{UL} \) to arbitrary scaling of \( u_l \).

By comparing (3.16) and (3.18) to (3.5) and (3.7), we see that the virtual user and effective channel models succeed in adapting a set of \( K \) MIMO users with \( L_k \) streams per user into an equivalent set of \( L \) MISO users. Thus, the SINR duality in Section 3.1.2 can be applied directly by replacing the MISO channels \( h_k \) with the effective channels \( \hat{h}_l \) and the user indices \( k = 1, \ldots, K \) with the substream indices \( l = 1, \ldots, L \).

### 3.2.1 MSE Duality under MMSE Reception

In [15], the authors employ MISO systems to demonstrate a duality result for the achievable MSE regions under a sum power constraint. Here, we extend this duality to MIMO systems, under the assumption that the linear receivers being used are the MMSE receivers \( V_k^* \) (2.13) and \( U_k^* \) (2.21).

The MSE terms for stream \( l \) in the downlink (2.14) and virtual uplink (2.22) can be rewritten using the virtual user notation as (3.19) and (3.20).

\[
\varepsilon_{DL}^l = 1 - p_l \bar{u}_l^H H_{\kappa(l)} \left( \sum_{j=1}^L p_j \bar{u}_j \bar{u}_j^H \right) H_{\kappa(l)} + \sigma_n^2 I_{N_{\kappa(l)}} \right)^{-1} H_{\kappa(l)}^H \bar{u}_l
= 1 - p_l \bar{u}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} H_{\kappa(l)}^H \bar{u}_l
\]

\[
\varepsilon_{UL}^l = 1 - q_l \bar{v}_l^H H_{\kappa(l)} \left( \sum_{j=1}^L q_j H_{\kappa(j)} \bar{v}_j \bar{v}_j^H H_{\kappa(j)} + \sigma_n^2 I_M \right)^{-1} H_{\kappa(l)} \bar{v}_l
= 1 - q_l \bar{v}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} H_{\kappa(l)} \bar{v}_l.
\]
We are able to rewrite the expressions for the MMSE receivers corresponding to data stream \( l \) from (2.13) and (2.21) as

\[ v_l^* \text{H} = \sqrt{p_l} u_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} \] (3.21)
\[ u_l^* \text{H} = \sqrt{q_l} v_l^H H_{\kappa(l)} R^{-1}. \] (3.22)

The downlink and virtual uplink SINR terms can be rewritten under this notation as

\[ \text{SINR}_{DL}^l = \frac{p_l v_l^H H_{\kappa(l)}^H \bar{u}_l \bar{u}_l^H H_{\kappa(l)} v_l}{v_l^H \left( R_{\kappa(l)} - p_l H_{\kappa(l)}^H \bar{u}_l \bar{u}_l^H H_{\kappa(l)} \right) v_l} \] (3.23)
\[ \text{SINR}_{UL}^l = \frac{q_l u_l^H H_{\kappa(l)}^H \bar{v}_l \bar{v}_l^H H_{\kappa(l)} u_l}{u_l^H \left( R - q_l H_{\kappa(l)} \bar{v}_l \bar{v}_l^H H_{\kappa(l)} \right) u_l} \] (3.24)

By manipulating (3.23) and substituting the MMSE receiver \( v_l^* \) from (3.21), we can show that

\[ \frac{1}{1 + \text{SINR}_{DL}^l} = \frac{v_l^* \left( R_{\kappa(l)} - p_l H_{\kappa(l)}^H \bar{u}_l \bar{u}_l^H H_{\kappa(l)} \right) v_l}{v_l^* R_{\kappa(l)} v_l} \]
\[ = 1 - \frac{v_l^* \left( p_l H_{\kappa(l)}^H \bar{u}_l \bar{u}_l^H H_{\kappa(l)} \right) v_l}{v_l^* R_{\kappa(l)} v_l} \]
\[ = 1 - \frac{p_l \bar{u}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} \bar{u}_l \bar{u}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} H_{\kappa(l)}^H \bar{u}_l}{p_l \bar{u}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} H_{\kappa(l)}^H \bar{u}_l} \]
\[ = 1 - \frac{\left( p_l \bar{u}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} H_{\kappa(l)}^H \bar{u}_l \right)^2}{p_l \bar{u}_l^H H_{\kappa(l)} R_{\kappa(l)}^{-1} H_{\kappa(l)}^H \bar{u}_l} \]
\[ = \varepsilon_{DL}^l. \]
Similarly,
\[
\frac{1}{1 + \text{SINR}_t^{UL}} = \frac{\mathbf{u}_l^H \left( \mathbf{R} - q_l \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \right) \mathbf{u}_l^*}{\mathbf{u}_l^H \mathbf{R} \mathbf{u}_l^*} = 1 - \frac{\mathbf{u}_l^H \left( q_l \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \right) \mathbf{u}_l^*}{\mathbf{u}_l^H \mathbf{R} \mathbf{u}_l^*} = 1 - \frac{q_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \mathbf{R}^{-1} \left( q_l \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \right) \mathbf{R}^{-1} \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l}{q_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \mathbf{R}^{-1} \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l} = 1 - \frac{\left( q_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \mathbf{R}^{-1} \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \right)^2}{q_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \mathbf{R}^{-1} \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l} = \varepsilon_l^{UL}.
\]

It thus holds true that any achievable MSE in the downlink can also be achieved in the virtual uplink, under MMSE decoding in both frameworks. This relationship applies to any choice of \(x\) satisfying the assumptions made in Section 2.1.3; this includes, but is not limited to, Gaussian source symbols with unit average power.

**Relationship between MSE Minimizing and SINR Maximizing Receivers**

If we examine the SINR expressions in (3.23) and (3.24), we observe that the SINR maximizing receive beamformers in the downlink and virtual uplink are the dominant generalized eigenvectors:

\[
\hat{\mathbf{v}}_l^{\text{SINR}_{\text{max}}} = \hat{\mathbf{e}}_{\text{max}} \left( p_l \mathbf{H}_\kappa(l) \bar{\mathbf{u}}_l \bar{\mathbf{u}}_l^H \mathbf{H}_\kappa(l), \mathbf{R}_\kappa(l) - p_l \mathbf{H}_\kappa(l) \bar{\mathbf{u}}_l \bar{\mathbf{u}}_l^H \mathbf{H}_\kappa(l) \right) \]  
\[
\hat{\mathbf{u}}_l^{\text{SINR}_{\text{max}}} = \hat{\mathbf{e}}_{\text{max}} \left( q_l \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l), \mathbf{R} - q_l \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l) \right).
\]

These solutions are in fact equivalent to the MMSE receiver expressions (3.21) and (3.22) since the matrices \(p_l \mathbf{H}_\kappa(l) \bar{\mathbf{u}}_l \bar{\mathbf{u}}_l^H \mathbf{H}_\kappa(l)\) and \(q_l \mathbf{H}_\kappa(l) \bar{\mathbf{v}}_l \bar{\mathbf{v}}_l^H \mathbf{H}_\kappa(l)\) are both rank-1.

The observation \(\text{MMSE} = 1/(1 + \text{SINR}_{\text{max}})\) has been seen in several other system scenarios as well (e.g. for MMSE detection in CDMA systems [28]).

\footnote{The set of generalized eigenvectors for any two square matrices \(A\) and \(B\) are the unit norm vectors \(x\) which satisfy \(Ax = \lambda Bx\). Consequently, the dominant generalized eigenvector \(x^* = \hat{\mathbf{e}}_{\text{max}} (A, B)\) is the eigenvector \(x\) corresponding to the maximum eigenvalue.}
3.2.2 Power Allocation for Equivalent SINR/MSE in MIMO Downlink and Virtual Uplink

By introducing the virtual user and effective channel notations in a modified version of Theorem 3.1, we are able to find equivalent power allocations for all achievable MSE/SINR regions in the downlink and virtual uplink. Given any set of unit-norm transmit beamformers, power allocation, and associated MMSE receive beamformers, we can find the set of corresponding feasible SINR values for each virtual user \((\gamma_1, \ldots, \gamma_L)\).

Recall that normalizing \(\bar{v}_i\) in the downlink or \(\bar{u}_i\) in the virtual uplink does not affect post-decoding SINR. We see that the power allocations which achieve the same SINR/MSE values for each data stream in the downlink and virtual uplink are:

\[
p = \sigma_n^2 \left( D^{-1} - \Psi \right)^{-1} 1_L
\]
\[
q = \sigma_n^2 \left( D^{-1} - \Psi^T \right)^{-1} 1_L,
\]

where

\[
D = \text{diag} \left[ \frac{\gamma_1}{|\bar{h}_1^H \bar{u}_1|^2}, \ldots, \frac{\gamma_L}{|\bar{h}_L^H \bar{u}_L|^2} \right],
\]

and

\[
[\Psi]_{ij} = \begin{cases} 
|\bar{h}_i^H \bar{u}_j|^2 & i \neq j \\
0 & i = j
\end{cases}
\]

3.3 Minimum Sum-MSE Precoder Design

With the available MMSE duality, we now approach the problem of designing a precoder that minimizes the sum-MSE in the downlink by exploiting convexity in the virtual uplink formulation.
3.3.1 Downlink Formulation

The minimum downlink sum-MSE (under linear MMSE receivers $V^*$) for any choice of $(\bar{U}, P)$ is found using $\epsilon_k^{DL}$ from (2.14):

$$\text{SMSE}^{DL} = \sum_{k=1}^{K} \text{tr} \left[ \epsilon_k^{*DL} \right]$$

$$= \text{tr} \left[ \sum_{k=1}^{K} I_{Lk} - \sqrt{P_k} \bar{U}_k^H H_k R_k^{-1} H_k^H \bar{U}_k \sqrt{P_k} \right]$$

$$= \text{tr} \left[ \sum_{k=1}^{K} I_{Lk} - \sqrt{P_k} \bar{U}_k^H H_k \left( H_k^H \bar{U} P \bar{U}^H H_k + \sigma_n^2 I_{N_k} \right)^{-1} H_k^H \bar{U}_k \sqrt{P_k} \right].$$  \hspace{1cm} (3.31)

The formal statement of the optimization problem under consideration, finding the sum-MSE minimizing precoders and power allocations in the downlink under a sum power constraint, is:

$$\begin{align*}
(\bar{U}, P) = & \arg \min_{\bar{U}, P} \text{SMSE}^{DL} \\
\text{s.t.} & \quad \text{tr}[P] \leq P_{\text{max}}, \\
& \quad \|\bar{u}_l\|_2 = 1 \quad l = 1, \ldots, L.
\end{align*}$$  \hspace{1cm} (3.32)

Close inspection of (3.31) reveals that in SMSE$^{DL}$ and all $\epsilon_k^{*DL}$, the precoder and power allocation matrices $\bar{U}_k$ and $P_k$ are coupled via channel matrices $H_k$ (introduced by the common term $\bar{U} P \bar{U}$ in each $R_k$). This complication makes direct optimization in the downlink a difficult problem; indeed, Problem (3.32) is non-convex in its direct formulation.\footnote{The authors of [29] demonstrate hidden convexity in the problem by optimizing receive matrices (using closed form MMSE precoders) and applying a modified cost function.} However, we now illustrate that a convex formulation is possible by transforming the problem to the virtual uplink.
### 3.3.2 Virtual Uplink Formulation

The sum-MSE in the virtual uplink can be expressed using $\mathbf{e}^*_{k,UL}$ from (2.22):

$$\text{SMSE}^{UL} = \sum_{k=1}^{K} \text{tr} \left[ \mathbf{e}^*_{k,UL} \right]$$

$$= \sum_{k=1}^{K} \text{tr} \left[ \mathbf{I}_{L_k} - \sqrt{Q_k} \bar{V}_k^H \bar{H}_k^H \mathbf{R}^{-1} \bar{H}_k \bar{V}_k \sqrt{Q_k} \right]$$

$$= L - \sum_{k=1}^{K} \text{tr} \left[ \sqrt{Q_k} \bar{V}_k^H \bar{H}_k^H \mathbf{R}^{-1} \bar{H}_k \bar{V}_k \sqrt{Q_k} \right],$$ \hspace{1cm} (3.33)

which follows from linearity of the trace operator. Employing linearity again along with the identity $\text{tr} [AB] = \text{tr} [BA]$ yields

$$\text{SMSE}^{UL} = L - \text{tr} \left[ \sum_{k=1}^{K} \mathbf{R}^{-1} \bar{H}_k \bar{V}_k Q_k \bar{V}_k^H \bar{H}_k^H \right]$$

$$= L - \text{tr} \left[ \mathbf{R}^{-1} \sum_{k=1}^{K} \bar{H}_k \bar{V}_k Q_k \bar{V}_k^H \bar{H}_k^H \right],$$ \hspace{1cm} (3.34)

which can be further simplified via substitution of $\mathbf{R}$,

$$\text{SMSE}^{UL} = L - \text{tr} \left[ \mathbf{R}^{-1} (\mathbf{R} - \sigma_n^2 \mathbf{I}_M) \right]$$

$$= L - M + \sigma_n^2 \text{tr} [\mathbf{R}^{-1}].$$ \hspace{1cm} (3.35)

We also consider rewriting (3.35) using the virtual user and effective channel notations introduced in Section 3.2 to be used later in this section and in Appendix A:

$$\text{SMSE}^{UL} = L - M + \sigma_n^2 \text{tr} \left[ \left( \bar{H} \bar{Q} \bar{V}^H \bar{H}^H + \sigma_n^2 \mathbf{I}_M \right)^{-1} \right]$$

$$= L - M + \sigma_n^2 \text{tr} \left[ \left( \sum_{k=1}^{K} \bar{H}_k \bar{V}_k Q_k \bar{V}_k^H \bar{H}_k^H + \sigma_n^2 \mathbf{I}_M \right)^{-1} \right]$$

$$= L - M + \sigma_n^2 \text{tr} \left[ \left( \sum_{l=1}^{L} \tilde{q}_l \tilde{h}_l^H \tilde{h}_l + \sigma_n^2 \mathbf{I}_M \right)^{-1} \right].$$ \hspace{1cm} (3.36)

We see that in the virtual uplink, the channel $\mathbf{H}_k$ and precoder pairs $(\bar{V}_k, Q_k)$ are decoupled. Minimizing the sum-MSE thus only requires minimization of $\text{tr} [\mathbf{R}^{-1}]$, which can easily be achieved due to convexity of the optimization problem.
Theorem 3.2. The sum-MSE objective function in (3.36) is jointly convex in each set of design variables; these include the collection of scalar power allocations $q_i$, the joint power allocation $Q$, the set of transmit direction (beamformer) covariance matrices $S_i = \bar{v}_i\bar{v}_i^H$, the collection of joint power allocation and transmit covariances $\tilde{S}_i = q_iS_i$, and the joint transmit covariance matrix $S = \bar{V}Q\bar{V}^H$ (as seen in [30, 31]).

Proof. See Appendix B.

The sum-MSE expression in (3.36) is a function of two variables; uplink power allocation $Q$ and uplink global transmit filter $\bar{V}$. In our proposed solution, we iterate between the design of each, assuming that the other is fixed.

Finding the optimum $Q$ for any fixed selection of $\bar{V}$ involves solving the simplified convex optimization problem

$$\begin{align*}
(q_1^*, \ldots, q_L^*) &= \arg\min_{q_1, \ldots, q_L} \text{tr} \left( \left( \sum_{l=1}^L q_l \tilde{h}_l \tilde{h}_l^H + \sigma^2 I_M \right)^{-1} \right) \\
\text{s.t.} \quad q_l &\geq 0 \quad l = 1, \ldots, L, \quad \sum_{l=1}^L q_l \leq P_{\text{max}},
\end{align*}$$

(3.38)

This optimization can be performed using generalized packages for solving convex problems (e.g., [32]); however, a more efficient solution is to design a customized algorithm suited to the specific problem based on primal-dual interior-point methods (see [33]). This is only possible when the gradient and Hessian of the objective function are available in closed form. We derive these expressions in Appendix A.

In [34], a detailed analysis of the convergence and computational complexity of interior-point algorithms for convex optimization is presented. There, it is shown that interior-point methods exhibit polynomial-time worst-case complexity. The number of

4Joint convexity is a stronger condition than individual convexity in each variable; for example, a function $f(x,y)$ that is jointly convex in $(x,y)$ requires that

$$f(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \leq \lambda f(x_1, y_1) + (1 - \lambda)f(x_2, y_2)$$

(3.37)

for all choices of $(x_1, y_1)$ and $(x_2, y_2)$ in the domain of $f(x, y)$ and $\lambda \in [0, 1]$. This condition implies individual convexity in $x$ and $y$ (i.e., by setting $y_1 = y_2$ or $x_1 = x_2$, respectively).
iterations required for convergence of a problem of dimension $n$ is $O(\sqrt{n}/\epsilon_{IP})$, where $\epsilon_{IP}$ is the required accuracy of the solution. The complexity of each inner iteration is dominated by the calculation of $R^{-1}$, which is $O(M^3)$. Since $L \leq M$ under the resolvability constraint (2.23), we can conclude that the calculating the convex solution to the power optimization subproblem is $O(M^{3.5}/\epsilon_{IP})$.

Next, we consider optimizing $\bar{V}$ for a fixed power allocation $Q$. We employ the method proposed in [27], where $\bar{V}$ was found for the case of per-user power constraints, and generalize it to our case. In order to optimize each $\bar{v}_i$, we first isolate the $\bar{v}_i$ terms from all other $\bar{v}_l, l \neq i$ in $R$:

$$R = \sum_{l=1}^{L} q_l \tilde{h}_l \tilde{h}_l^H + \sigma_n^2 I_M$$

$$= q_i \tilde{h}_i \tilde{h}_i^H + \sum_{l=1, l \neq i}^{L} q_l \tilde{h}_l \tilde{h}_l^H + \sigma_n^2 I_M$$

(3.39)

We employ the matrix inversion lemma\(^5\) to show that $\bar{v}_i$ can be isolated from $R^{-1}$ as well. Expansion of $R^{-1} = \left( Z_i + q_i \tilde{h}_i \tilde{h}_i^H \right)^{-1}$ using the matrix inversion lemma yields

$$R^{-1} = Z_i^{-1} - Z_i^{-1} \tilde{h}_i \left( \frac{1}{q_i} + \tilde{h}_i^H Z_i^{-1} \tilde{h}_i \right)^{-1} \tilde{h}_i^H Z_i^{-1}$$

$$= Z_i^{-1} - q_i \frac{Z_i^{-1} \tilde{h}_i \tilde{h}_i^H Z_i^{-1}}{1 + q_i \tilde{h}_i^H Z_i^{-1} \tilde{h}_i}$$

(3.41)

\(^5\)The matrix inversion lemma (Sherman-Morrison-Woodbury formula) \([35]\) states that when $A$ and $C$ are non-singular and $A, B, C, D$ are dimensionally consistent,\n
$$(A + BCD)^{-1} = A^{-1} - A^{-1}B \left( C^{-1} + DA^{-1}B \right)^{-1} DA^{-1}. \quad (3.40)$$
If we substitute (3.41) in (3.35), we get

$$SMSE_{UL} = L - M + \sigma_n^2 \text{tr} \left[ Z_i^{-1} - q_i \frac{Z_i^{-1} \bar{h}_i \bar{h}_i^H Z_i^{-1}}{1 + q_i \bar{h}_i^H Z_i^{-1} \bar{h}_i} \right]$$

$$= W_i - \sigma_n^2 q_i \frac{\bar{v}_i^H H_{\kappa(i)}^H Z_i^{-2} H_{\kappa(i)} \bar{v}_i}{\bar{v}_i^H \left( I_{N_i} + q_i H_{\kappa(i)}^H Z_i^{-1} H_{\kappa(i)} \right) \bar{v}_i},$$

where $W_i$ contains all the terms independent of the beamforming vector $\bar{v}_i$. Thus the optimum selection of $\bar{v}_i$, which minimizes the sum-MSE for a given power allocation when the beamforming vectors of all other streams are fixed, is the dominant generalized eigenvector

$$\bar{v}_i^* = \hat{e}_{\max} \left( H_{\kappa(i)}^H Z_i^{-2} H_{\kappa(i)}, I_{N_i} + q_i H_{\kappa(i)}^H Z_i^{-1} H_{\kappa(i)} \right)$$

(3.43)

Iteration between the optimization of $\bar{V}$ and $Q$ is a block coordinate descent algorithm. Since each of these two optimization steps finds an optimal solution in one set of variables while the other set of variables is fixed, each iteration is guaranteed to decrease unless a local minimum has been reached (i.e., each step is monotonically non-increasing in the sum-MSE). Combining this fact with the boundedness below of the sum-MSE function is sufficient to guarantee convergence to a local minimum [36]. A similar strategy based on iteration between optimization of powers across users and iterative single-user optimization of covariance matrices in the virtual uplink is shown to converge to the global minimum in [37].

With the optimum precoder designed in the virtual uplink, we could then exploit the SINR/MSE duality and associated power allocation described in Section 3.2 to find the optimum downlink precoder; however, the result presented in the following section obviates the need for finding a downlink power allocation.
Chapter 3. Sum-MSE Minimization with Perfect CSI

3.4 Equality of Optimal Power Allocations

In the previous section, we proposed an algorithm that employed the SINR/MSE duality and power allocation from Section 3.2 to converge to a globally optimum point in minimizing sum-MSE. In this section, we show that for the sum-MSE minimization problem, the step of transforming virtual uplink power allocations to the downlink is not needed, as the optimal power allocations are identical in the uplink and downlink.

**Theorem 3.3** (Equality of Optimal Power Allocations). Given an arbitrary set of virtual uplink precoders \( \bar{V}_1, \ldots, \bar{V}_K \), the associated optimum power allocation \( q^* = [q_1^*, \ldots, q_L^*] \) satisfying (3.38), and the corresponding MMSE receive beamformers \( u^*_l \) (3.22), the optimum downlink power allocations \( p^* \) are identical to \( q^* \).

**Proof.** Based on (3.28), we see that \( \Psi = \Psi^T \) is a sufficient condition for the equality of \( p \) and \( q \). We now proceed to prove that this transpose symmetry applies by analyzing the Karush-Kuhn-Tucker (KKT) conditions for the power allocation subproblem in the virtual uplink.

From the objective and constraint functions in (3.38), the Lagrangian is

\[
L(q, \lambda) = \text{tr} \left[ \sum_{l=1}^{L} q_l \bar{h}_l \bar{h}_l^H + \sigma_n^2 I_M \right]^{-1} + \lambda_{\text{sum}} \left( \sum_{l=1}^{L} q_l - P_{\text{max}} \right) - \sum_{l=1}^{L} \lambda_l q_l, \quad (3.44)
\]

and the resulting KKT conditions are

\[
\nabla L = \left[ \begin{array}{c}
\bar{h}_1^H R^{-2} \bar{h}_1 \\
\vdots \\
\bar{h}_L^H R^{-2} \bar{h}_L
\end{array} \right] + \lambda_{\text{sum}} \mathbf{1}_L - \sum_{l=1}^{L} \lambda_l e^L_l = 0_L
\]

\[
\sum_{l=1}^{L} q_l \leq P_{\text{max}}, \quad q_l \geq 0
\]

\[
\lambda_{\text{sum}} \geq 0, \quad \lambda_l \geq 0
\]

\[
\lambda_{\text{sum}} \left( \sum_{l=1}^{L} q_l - P_{\text{max}} \right) = 0, \quad \lambda_l q_l = 0.
\]
The gradient in the stationarity condition is derived in Appendix A.

Having solved (3.38) for an arbitrary set of virtual uplink precoders $\mathbf{v}_l$, we then find the MMSE receive beamformers $\mathbf{u}_l^* = \mathbf{R}^{-1} \tilde{\mathbf{h}}_l \sqrt{q_l}$ (3.22). With the resulting virtual uplink stream MSEs $\varepsilon_l$ (and associated SINRs $\gamma_l$), we can then use (3.28) to find the downlink power allocation $\mathbf{p}$ that achieves the same MSEs for each data stream.

In the case where the optimal power allocation results in one or more inactive streams $\mathcal{S}_I = \{l \in (1, \ldots, L) \mid q_l^* = 0\}$, this algorithm fails since $\mathbf{u}_l^* = \mathbf{0}$ for $l \in \mathcal{S}_I$. However, the same MSEs can be achieved for these inactive streams in the downlink by setting $p_l = 0$.

The power allocation $\mathbf{p}$ for the set of active streams $\mathcal{S}_A = \{l \in (1, \ldots, L) \mid q_l^* > 0\}$ can then be found by following the specified procedure after deleting the rows and columns from $\mathbf{D}$ and $\mathbf{Ψ}$ corresponding to the inactive streams.

The coupling matrix $\mathbf{Ψ}$ is a real matrix whose off-diagonal entries $[\mathbf{Ψ}]_{ij}$ contain squared magnitudes of the end-to-end channel gains from transmitted symbol $x_j$ to the decoded symbol $\hat{x}_i$. We observe that $\mathbf{Ψ} = \mathbf{Ψ}^T$ is satisfied when

$$\frac{\tilde{\mathbf{h}}_i^H \mathbf{u}_j^*}{\|\mathbf{u}_j^*\|_2} = \frac{\mathbf{u}_i^* \tilde{\mathbf{h}}_j}{\|\mathbf{u}_i^*\|_2},$$

or equivalently,

$$\frac{\tilde{\mathbf{h}}_i^H \mathbf{R}^{-1} \tilde{\mathbf{h}}_j \sqrt{q_j^*}}{\sqrt{q_i^* \tilde{\mathbf{h}}_j^H \mathbf{R}^{-2} \tilde{\mathbf{h}}_j}} = \frac{\sqrt{q_i^* \tilde{\mathbf{h}}_i^H \mathbf{R}^{-1} \tilde{\mathbf{h}}_j}}{\sqrt{q_i^* \tilde{\mathbf{h}}_j^H \mathbf{R}^{-2} \tilde{\mathbf{h}}_i}}$$

(3.47)

The power allocation terms $q_i^*$ and $q_j^*$ cancel out, and numerators are equal; thus, an equivalent expression for the sufficient condition for $\mathbf{p} = \mathbf{q}$ is

$$\tilde{\mathbf{h}}_i^H \mathbf{R}^{-2} \tilde{\mathbf{h}}_i = \tilde{\mathbf{h}}_j^H \mathbf{R}^{-2} \tilde{\mathbf{h}}_j \quad \forall i, j \in \mathcal{S}_A.$$

(3.48)

We rewrite the individual terms in (3.45) as $\tilde{\mathbf{h}}_i^H \mathbf{R}^{-2} \tilde{\mathbf{h}}_i = (\lambda_{\text{sum}} - \lambda_l)$. Due to the complementary slackness condition ($\lambda_l q_l = 0$), the dual variables $\lambda_l$ are zero for all active streams $l \in \mathcal{S}_A$ with $q_l > 0$. Thus, it follows that

$$\tilde{\mathbf{h}}_l^H \mathbf{R}^{-2} \tilde{\mathbf{h}}_l = \lambda_{\text{sum}} \quad \forall l \in \mathcal{S}_A.$$
that is, (3.48) is satisfied, $\Psi = \Psi^T$, and the downlink and virtual uplink power allocations $p$ and $q$ that achieve the same minimum sum-MSE are identical.

It is worth emphasizing that this equality result applies for arbitrary beamformers $\bar{v}_l$, as long as the optimum power allocation and MMSE receivers are used. It follows that it also applies to the optimum covariance-based design, when covariance matrices for each stream are normalized as $S_l = q_l^* \bar{S}_l$ and $\bar{S}_l = \bar{v}_l \bar{v}_l^H$. This result implies that the virtual uplink to downlink transformation stage can be omitted from algorithms using both iterative and joint designs based on a virtual-uplink solution [19, 31, 38, 39], thus allowing for simplified implementations.

### 3.5 Sum-MSE Minimization Algorithm

The algorithm proposed in Table 3.1 finds the sum-MSE minimizing precoder and decoders for the downlink. The initialization step starts with the uplink power allocation $Q$ by distributing $P_{\text{max}}$ evenly among all data streams; however, any feasible initialization of $Q$ is acceptable. $\bar{V}_k = \text{SVD}(H_k)$ indicates that the $L_k$ dominant right singular vectors of $H_k$ (corresponding to the $L_k$ largest singular values) are used to initialize $\bar{V}_k$; the motivation for this choice is to approximate a maximum received SNR solution in the matched filters for each channel. The next step involves optimizing the set of $\bar{V}_k$ and $Q$ iteratively. Each iteration begins with optimizing $\bar{V}_k$ for the previous power allocation $Q$, where the optimum $\bar{v}_l$ of every virtual user is found using the dominant eigenvector method. After all of the beamforming vectors are found, power is allocated by solving the convex optimization problem (3.38). The iterative step is executed repeatedly for the virtual uplink until the sum-MSE converges to a global minimum (within a tolerance set by $\epsilon$). The MMSE downlink receiver $U^*$ is then found using (3.22) and each column (virtual user beamformer) is normalized. The power allocations $Q$ from the virtual uplink
are used as the downlink power allocations \( P \) and the associated downlink sum-MSE is identical to that found for the virtual uplink under the established MSE duality.

Computational complexity in the virtual uplink beamforming step is dominated by the calculations of \( Z_i^{-1}, Z_i^{-2} \) and by the generalized eigendecomposition (using the QR algorithm \([35]\)) for each virtual user; each of these operations has complexity \( O(M^3) \), although the calculation of \( Z_i^{-1} \) can be performed efficiently as a rank-1 update of \( R^{-1} \) as

\[
Z_i^{-1} = R^{-1} + \frac{h_i^H R^{-2} h_i}{q_i^{-1} - h_i^H R^{-1} h_i} I.
\] (3.50)

Since this step must be performed \( L \leq M \) times for each virtual user, the virtual uplink beamforming step has worst-case computational complexity \( O(M^4) \). Complexity of

<table>
<thead>
<tr>
<th>Table 3.1: Sum-MSE minimization algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Given:</strong> Sum-MSE tolerance ( \epsilon ) and sum power constraint ( P_{\text{max}} )</td>
</tr>
<tr>
<td><strong>Initialization:</strong> ( \bar{V}<em>k = \text{SVD}(H_k) ) and ( Q = \text{diag} \left[ (P</em>{\text{max}}/L) 1_L \right] )</td>
</tr>
<tr>
<td><strong>Iteration:</strong></td>
</tr>
<tr>
<td>1- Virtual Uplink Transmit Beamforming (for ( l = 1 : L ))</td>
</tr>
<tr>
<td>( \bar{v}<em>l = \bar{e}</em>{\text{max}} \left( H_{\kappa(l)}^H Z_l^{-2} H_{\kappa(l)} I_{N_l} + q_l H_{\kappa(l)}^H Z_l^{-1} H_{\kappa(l)} \right) )</td>
</tr>
<tr>
<td>where ( Z_l = \sum_{i=1, i \neq l}^L q_i \bar{h}_i \bar{h}_i^H + \sigma_n^2 I_M )</td>
</tr>
<tr>
<td>2- Virtual Uplink Power Allocation</td>
</tr>
<tr>
<td>( Q = \arg\min_Q \text{tr}[R^{-1}], \text{ subject to } q_l \geq 0, \text{ tr}[q] \leq P_{\text{max}} )</td>
</tr>
<tr>
<td>3- Repeat 1-2 until ( \left[ \text{SMSE}<em>{\text{old}}^{UL} - \text{SMSE}</em>{\text{new}}^{UL} \right] / \text{SMSE}_{\text{old}}^{UL} &lt; \epsilon )</td>
</tr>
<tr>
<td><strong>Update:</strong></td>
</tr>
<tr>
<td>4- Downlink Transmit Beamforming Directions (for ( l = 1 : L ))</td>
</tr>
<tr>
<td>( u_l^* = R^{-1} H_{\kappa(l)} \bar{v}_l \sqrt{q_l} )</td>
</tr>
<tr>
<td>( \bar{u}_l = \frac{u_l^<em>}{|u_l^</em>|_2} )</td>
</tr>
<tr>
<td>5- Downlink Power Allocation</td>
</tr>
<tr>
<td>( P = Q )</td>
</tr>
</tbody>
</table>
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Figure 3.1: BER vs. SNR for sum-MSE minimization.

the overall sum-MSE minimization algorithm described in Table 3.1 is $O(n_{\text{SMSE}}M^{3.5}/\epsilon)$, where $n_{\text{SMSE}}$ is the number of beamforming/power allocation iterations required for convergence within the specified threshold $\epsilon$, as the complexity of the $O(M^{3.5}/\epsilon)$ power allocation dominates the $O(M^4)$ beamforming step for all reasonable values of $M$ and $\epsilon^{-1}$. The required number of iterations $n_{\text{SMSE}}$ is difficult to characterize; we have observed in simulation that it is relatively low (e.g., 5–10 iterations for the problem illustrated in the following section), but that the number of required iterations increases with SNR.

3.5.1 Numerical Examples

In this section we illustrate the performance of the proposed algorithm. The total power is constrained as $P_{\text{max}} = 1$ and the noise variance $\sigma_n^2$ is varied to achieve a target transmit SNR, $P_{\text{max}}/\sigma_n^2$.

Figure 3.1 plots the average bit error rate as a function of transmitted signal-to-noise
ratio for the sum-MSE minimization algorithm, where $\text{SNR} = \frac{P_{\text{max}}}{\sigma_n^2}$. The figure also plots the BER when using the SQP approach of [25]. The simulated system has $K = 2$ users, receiving $L_1 = 2$ and $L_2 = 1$ data streams respectively. Both users are equipped with $N_k = 2$ antennas each. The figure shows that the proposed algorithm achieves an almost identical performance in BER to the SQP algorithm which suffers from being computationally intensive.

We can roughly estimate the difference in computational complexity between the two algorithms. Each iteration of SQP approximates the problem as a quadratic program, which is solved in turn using interior-point methods. These iterations thus exhibit worst-case polynomial time $O(M/\epsilon_IP)$, since the SQP-based algorithm jointly optimizes the precoder direction and power allocation matrices as $U = \bar{U}\sqrt{P} \in \mathbb{C}^{M \times L}$, which is an optimization problem with real dimension $2ML$.

As in the case of our proposed algorithm, calculating the number of iterations required by the SQP algorithm is quite difficult due to the nature of numerical optimization. We treat SQP as a “black-box” which can provide a near-optimal solution to the problem; thus, we are only able to compare relative complexity of the two algorithms by timing our simulations. In this particular case, the iterative approach appears to have a computational complexity that is an order-of-magnitude lower than the SQP algorithm.

### 3.6 Summary

In this chapter, we have generalized the SINR/MSE duality from the single receive antenna case to multiple antenna receivers with multiple data streams by introducing the effective channel notation and the notion of virtual users. We used this extended duality to propose a solution to the problem of joint beamforming and power allocation in the downlink of multiuser MIMO systems to minimize the sum of mean squared errors by solving convex optimization problems. A further extension of the duality, specific to
the case of sum-MSE minimization, shows that the optimum power allocation for the downlink is identical to the power allocation that minimizes sum-MSE in the virtual uplink.
Chapter 4

Sum-Rate Maximization with Perfect Channel State Information

In this chapter, we consider the problem of maximizing the sum data rate communicated from the base station to all mobile users, subject to a sum power constraint, under the MSE-based linear precoding framework established in Chapters 2 and 3.

4.1 Literature Review

4.1.1 Capacity of the Single-User Gaussian MIMO Channel

In the single-user case, linear precoding is capable of achieving the maximum communication rate in the Gaussian MIMO channel. This capacity-achieving solution is the waterfilling result first shown in [13]. Consider a single-user version of the system model described in Figure 2.2 with

\[ y = H^H U \sqrt{P} x + n \]

\[ = H^H s + n, \]  \hspace{1cm} (4.1)

where \( s = U \sqrt{P} x \) is the transmitted signal vector following precoding and power allocation, \( n \sim CN(0, K_n) \) is independent of the transmitted signal, and the channel matrix \( H \)
is assumed to be known. If the transmitted signal vector has zero mean, its covariance matrix can be defined as
\[ K_s = E[ss^H], \tag{4.2} \]
and the associated power constraint is
\[ \text{tr}[P] = \text{tr}[K_s] \leq P_{\text{max}}, \tag{4.3} \]
since \( \bar{U} \) is a unitary matrix.

Under the assumed model, \( y \) is zero-mean. Since it is a linear combination of \( s \) and \( n \), \( y \) is a complex Gaussian vector whose covariance matrix can be defined as
\[ K_y = E[yy^H] = H^H K_s H + K_n. \tag{4.4} \]

The differential entropy of a complex Gaussian vector \( v \sim \mathcal{CN}(0, K_v) \) is \[ h(v) = \log \det [\pi e K_v]; \tag{4.5} \]
thus, the mutual information \( I(s; y) \) between the transmitted and received symbol vectors can be expressed as
\[
I(s; y) = h(y) - h(y|s)
= h(y) - h(H^H s + n|s)
= h(y) - h(n|s)
= h(y) - h(n) \tag{4.6}
= \log (\det [\pi e K_y]) - \log (\det [\pi e K_n])
= \log \left( \frac{\det [K_y]}{\det [K_n]} \right)
= \log \left( \frac{\det [H^H K_s H + K_n]}{\det [K_n]} \right).
\]

It follows that the capacity of the MIMO channel with Gaussian noise is achieved by \( s \sim \mathcal{CN}(0, K_s) \), where the covariance matrix \( K_s \) is chosen to maximize the mutual
information,

\[
C = \arg \max_{K, \text{tr}[K] \leq P_{\text{max}}} \quad I(s; y)
\]

\[
= \arg \max_{K, \text{tr}[K] \leq P_{\text{max}}} \quad \det \left( H^H K_s H + K_n \right).
\]  

(4.7)

Without loss of generality, it is assumed that \( n \sim \mathcal{CN}(0, \sigma_n^2) \); in the case of coloured noise, an equivalent system can be found by whitening the noise via unitary rotation of the received symbols without altering the determinant in (4.7).

In this single-user case, the capacity achieving covariance matrix can be found by waterfilling over the singular values of the channel matrix \( H^H \), and transmitting in the directions of the channel’s right singular vectors.

**4.1.2 Capacity of the Multiuser MIMO Gaussian Broadcast Channel**

The optimal strategy for maximizing sum-rate in the multiuser MIMO downlink, also known as the broadcast channel (BC), was first proposed in [41]; the authors prove that Costa’s “writing on dirty paper” strategy [42] (also known as dirty paper coding, or DPC) is sum capacity achieving for a pair of single-antenna users. The sum-rate optimality of DPC was generalized to an arbitrary number of multi-antenna receivers using the notions of game theory [43] and uplink-downlink duality [44, 45]. This duality result states that the sum capacity of the MIMO BC is identical to that of the MIMO multiple access channel (MAC); however, the MIMO-MAC capacity admits an algorithmically easy solution by solving a concave optimization problem:

\[
R_{\text{sum}} = \max_{S_k} \log \det \left( I + \frac{1}{\sigma^2} \sum_{k=1}^{K} H_k S_k H_k^H \right)
\]

s.t. \( S_k \succeq 0, \quad k = 1, \ldots, K \)

\[
\sum_{k=1}^{K} \text{tr}[S_k] \leq P_{\text{max}}.
\]
Here, $S_k$ is the uplink transmit covariance matrix for mobile user $k$, and $P_{\text{max}}$ is the maximum sum power over all users.

In conjunction with this result and associated uplink-downlink duality, convex optimization is employed in [46,47] to derive iterative solutions that find the sum capacity. Furthermore, it has been shown that DPC is the optimal precoding strategy for the entire MIMO-BC capacity region [48].

### 4.1.3 Nonlinear Precoding for Rate Maximization

Finding a practical realization of the DPC precoding strategy has proven to be a difficult problem. Existing solutions are largely based on Tomlinson-Harashima precoding (THP) [49–52]. The fundamental concept in THP precoder design is analogous to the uplink-downlink duality for capacity [46,47]; the optimum feedback-based successive interference cancellation mechanism for the uplink is transformed to an equivalent feed-forward interference presubtraction component in the downlink. One downside of THP-based schemes is that their optimum design requires a combinatorial search over all possible user orders (as with DPC). THP-based schemes also suffer from rate loss when compared to the sum capacity due to modulo and shaping losses.

### 4.1.4 Orthogonalization-Based Linear Precoding for Rate Maximization

As in the case of sum-MSE minimization, linear precoding provides an alternative approach for transmission in the MIMO downlink, trading off a reduction in precoder complexity for possibly suboptimal performance. As described in Section 3.1.1, orthogonalization based schemes using zero forcing and block diagonalization transform the multiuser downlink into parallel single-user systems [21,22].

The optimum strategy for these schemes differs from the sum-MSE case in how power
is allocated to data streams – a waterfilling power allocation is used over the effective (block) diagonal channel to allocate powers to each of the users in order to maximize the sum-rate [53]. The simplicity of these approaches comes with several sacrifices. First, a penalty to achievable sum-rate may be incurred due to noise enhancement; however, this does not have an impact on the asymptotic performance of orthogonalization based schemes. A bigger detriment to these methods is that orthogonalization necessitates an antenna constraint requiring at least as many transmit antennas as the total number of receive antennas. These schemes, therefore, restrict the possibility of gains from additional receiver antennas. The constraint is relaxed under successive zero forcing [54], which requires only partial orthogonality but incurs higher complexity in finding an optimal user ordering (similar to DPC and THP). Coordinated beamforming [55] and generalized orthogonalization [56] are able to avoid the antenna constraint via iterative optimization of transmit and receive beamformers.

When the sum total number of receive antennas at all mobile users exceeds the number of transmit antennas, antenna selection may be introduced in order to design the sum-rate maximizing ZF precoder. The optimum precoder can be found by designing precoders for all possible subsets of available receive antennas and evaluating the resulting capacities with their corresponding waterfilling power allocations [41]. However, this strategy incurs combinatorial complexity on the order of the total number of receive antennas.

Another possible way to increase sum-rate is to exploit multiuser diversity by transmitting to a subset of a large number of users; this can be viewed as a form of antenna selection. Greedy and suboptimal strategies for user selection have been proposed with lower computational cost in [57–60]. We do not consider user selection in this thesis (despite the attractiveness of multiuser diversity), as its introduction would necessitate a separate scheduling mechanism that lies outside the scope of our purely spatial linear precoding designs.
4.1.5 Generalized Linear Precoding for Rate Maximization

Linear precoding approaches to sum-rate maximization had previously been proposed for the case of single-antenna receivers in [61, 62]. In [61], the authors suggest an iterative method for direct optimization of the sum-rate, while [62] exploits the SINR uplink-downlink duality of [14, 19, 38]. An iterative algorithm for the multiple antenna case is proposed in [63] which approximates a non-convex power allocation problem in the virtual uplink with a geometric program and transforms the resultant solution to the downlink using SINR duality. Each of these approaches yields potentially suboptimal solutions, as the proposed algorithms are only guaranteed to converge to a local optimum, if at all.

In [64], an equivalence relationship is developed between the single-user minimum MSE (MMSE) and mutual information (in nats),

$$\frac{\partial I(SNR)}{\partial SNR} = \frac{1}{2}\text{MMSE}(SNR).$$

(4.8)

Furthermore, [26] demonstrates that the single-user information rate maximizing design (i.e., waterfilling over the eigenmodes) has an equivalent weighted sum-MSE expression. The existence of these results motivates us to consider optimization of the sum-rate as a function of MSE-based criteria.

4.2 Scalar Processing and the Product of Mean Squared Errors

We begin by considering a scalarized approach to the sum-rate maximization problem, wherein the decoding of each of the $L$ data streams is performed as if they were $L$ separate virtual users in a single-user setting. It follows that under this formulation, when decoding user $k$’s $j$th data stream, all other data streams $l \neq j$ are considered as

\footnote{A nearly identical approach which optimizes sum-rate under linear precoding based upon the minimization of the PMSE was published independently and contemporaneously in [65].}
self-interference in addition to the multiuser interference.

### 4.2.1 Achievable Sum-Rate using Scalar Processing

Under the virtual user and effective channel notations, the achievable rate for virtual user \( l \) in the virtual uplink can be expressed as

\[
R_{LP}^{UL} = \log \left( 1 + \text{SINR}_{UL}^{UL} \right) = \log \left( 1 + \frac{q_l |\bar{h}^H \bar{u}_l|^2}{\sum_{j \neq l} q_j |\bar{h}^H \bar{u}_l|^2 + \sigma_n^2} \right),
\]

(4.9)

where \( \text{SINR}_{UL}^{UL} \) was derived in (3.18).

The scalar rate maximization problem with a sum power constraint under linear precoding in the virtual uplink can thus be written as

\[
(\bar{V}, Q) = \arg \max_{\bar{V}, Q} \sum_{l=1}^{L} \log \left( 1 + \text{SINR}_{UL}^{UL} \right)
\]

s.t. \( \|\bar{v}_l\|_2 = 1 \), \( l = 1, \ldots, L \)

\( q_l \geq 0 \), \( l = 1, \ldots, L \),

\( \|q\|_1 = \sum_{l=1}^{L} q_l \leq P_{\text{max}} \).

(4.10)

### 4.2.2 MSE Formulation: Product of Mean Squared Errors

With this scalar processing rate maximization problem in mind, we consider the MSE-equivalent formulation. Consider the following optimization problem, minimizing the product of mean squared errors (PMSE) under a sum power constraint,

\[
(\bar{V}, Q) = \arg \min_{\bar{V}, Q} \prod_{l=1}^{L} \varepsilon_{UL}^{UL}
\]

s.t. \( \|\bar{v}_l\|_2 = 1 \), \( l = 1, \ldots, L \)

\( q_l \geq 0 \), \( l = 1, \ldots, L \),

\( \|q\|_1 = \sum_{l=1}^{L} q_l \leq P_{\text{max}} \).

(4.11)
Theorem 4.1. When linear MMSE receivers $u_l^*$ (3.22) are employed at the base station, the optimization problems defined by (4.10) and (4.11) are equivalent.

Proof. As shown in Section 3.2.1, the post-decoding SINR optimizing linear receiver $u_l^{\text{SINR max}}$ is equivalent to the MMSE receiver $u_l^*$ (3.22). Furthermore, there is a simple relationship between SINR and MMSE under MMSE reception (3.26),

$$
\varepsilon_l^{UL} = \frac{1}{1 + \text{SINR}_l^{UL}}.
$$

It follows that the objective function in (4.10) can be rewritten as

$$
\sum_{l=1}^{L} \log \left(1 + \text{SINR}_l^{UL}\right) = -\sum_{l=1}^{L} \log \left(\frac{1}{1 + \text{SINR}_l^{UL}}\right)
$$

$$
= -\log \left(\prod_{l=1}^{L} \varepsilon_l^{UL}\right).
$$

Since the constraints on $\bar{v}_l$ and $q_l$ are identical in (4.10) and (4.11), the problem of maximizing sum-rate in (4.10) is therefore equivalent to minimizing the PMSE in (4.11).

4.2.3 Algorithm: PMSE Minimization

We now present an algorithm that minimizes the product of mean squared errors using the MSE duality from Section 3.2.1. In contrast to the sum-MSE minimizing precoder proposed in Chapter 3, it is not possible to find a closed-form expression for the optimum $\bar{v}_l$ given a fixed set of $q_l$. Thus, we suggest an algorithm which iteratively obtains the downlink precoder matrix $\bar{U}$ and power allocations $p$ and the virtual uplink precoder matrix $\bar{V}$ and power allocations $q$. Each step minimizes the PMSE objective function by modifying one of these four variables while leaving the remaining three fixed.

**Downlink Precoder**

For a fixed set of virtual uplink beamformers $\bar{v}_l$ and power allocation $q$, the optimum virtual uplink decoders $U^*$ are defined by (2.21). Each of the virtual users’ data stream
MSEs $\varepsilon_l$ is minimized individually by this MMSE receiver, thereby also minimizing the product of MSEs. This $U^*$ is then normalized and used as the downlink precoder.

**Downlink Power Allocation**

Given the set of $\tilde{v}_l$, $q$, and $\tilde{u}_l$, the achievable virtual-uplink SINR values $\gamma_l$ can be calculated as in (3.24). These values are used as the target SINRs to calculate the downlink power allocation $p$ according to (3.28).

**Virtual Uplink Precoder**

Given a fixed set of $\tilde{u}_l$ and $p$, the MMSE receivers $V^*_k$ are defined as in (2.13). Each column is normalized to find the virtual uplink transmit beamformers $\tilde{v}_l$.

**Virtual Uplink Power Allocation**

The power allocation problem on the virtual uplink solves (4.11) for a fixed set of beamformers $\tilde{v}_l$. While it is well accepted that the power allocation subproblem in PMSE minimization (or equivalently, in sum-rate maximization) is non-convex [62, 63, 66], recent work [65] has shown that the optimal power allocation can be found by formulating the subproblem as a geometric programming (GP) problem [33], which solves a specific type of non-convex problem via convex transformation (see Appendix C). We had not considered this possibility during the time of our initial research, and thus employ a numerical optimization technique based on sequential quadratic programming (SQP) [67] to solve the power allocation subproblem. SQP solves successive approximations of a constrained optimization problem and is guaranteed to converge to the optimum value for convex problems; however, in the case of non-convex optimization problems, SQP can only guarantee convergence to a local minimum.

\[\text{In fact, the power allocation subproblem for the weighted sum-rate also has a GP formulation, as we demonstrate in Section 6.3.}\]
In summary, the PMSE minimization algorithm keeps three of the four parameters \((\bar{U}, p, \bar{V}, q)\) fixed at each step and obtains the optimal value of the fourth. Convergence of the overall algorithm to a local minimum is guaranteed since the PMSE objective function is monotonically non-increasing at each of the four parameter update steps and bounded below. Termination of the algorithm is determined by the selection of a convergence threshold \(\epsilon\).

Since the overall minimization problem \((4.11)\) is not convex, the proposed method is only guaranteed to converge to a local minimum. Nonetheless, simulations suggest that the locally optimal value of the sum-rate is not overly sensitive to selection of an appropriate initialization point. It is important to ensure that the initial solution allocates power to all \(L\) substreams, as the iterative algorithm tends to not allocate power to streams with zero power. A reasonable initialization is to select random unit-norm precoder vectors in \(\bar{U}\) and uniform power allocated over all substreams. A summary of our proposed algorithm can be found in Table 4.1.

Theoretical complexity analysis for the PMSE minimization algorithm is similar to the analysis used for the sum-MSE minimization algorithm (Section 3.5) when the virtual uplink power allocation subproblem is solved as a GP, since the convex transformation of the GP can be solved by applying interior-point methods. In practice, however, the cost per iteration may be quite different. In performing simulations, we have observed that minimizing the PMSE via GP transformation is often slower than sum-MSE minimization; this is likely due to the additional steps of parsing the GP, transforming the GP to its equivalent convex form, and performing numerical estimation of the gradient and Hessian of the objective function.
Table 4.1: PMSE minimization algorithm

**Given:** PMSE tolerance $\epsilon$ and sum power constraint $P_{\text{max}}$

**Initialization:** $\tilde{V}_k = \text{SVD}(H_k)$ and $Q = \text{diag} \left[ \left( \frac{P_{\text{max}}}{L} \right)^{\frac{1}{L}} \right]$

**Iteration:**

1. **Downlink Precoder**
   
   $$\tilde{U}_k = J_k^{-1}H_k^{H}V_k\sqrt{Q_k}, \quad u_{kj} = \frac{\tilde{u}_{kj}}{\|\tilde{u}_{kj}\|_2}$$

2. **Downlink Power Allocation via MSE duality**
   
   $$p = \sigma^2(D^{-1} - \Psi)^{-1}1$$

3. **Virtual Uplink Precoder**
   
   $$\tilde{V}_k = J_k^{-1}H_k^{H}U_k\sqrt{P_k}, \quad v_{kj} = \frac{\tilde{v}_{kj}}{\|\tilde{v}_{kj}\|_2}$$

4. **Virtual Uplink Power Allocation**
   
   $$q = \arg \min_q \prod_{k=1}^{K} \prod_{j=1}^{L_k} \varepsilon_{kj}, \text{ s.t. } q_{kj} \geq 0, \|q\|_1 \leq P_{\text{max}}$$

5. **Repeat 1-4 until** $\left[ \text{PMSE}_{\text{old}} - \text{PMSE}_{\text{new}} \right] / \text{PMSE}_{\text{old}} < \epsilon$
4.3 Joint Optimization: Sum-Rate and the Product of MSE Matrix Determinants

In this section, we formulate the key theoretical contribution of this chapter: the sum-rate maximization problem under linear precoding when joint processing is allowed across data streams for each user. This formulation provides the theoretically optimum sum-rate under linear precoding; however, we will demonstrate that it does not admit practical implementation.

4.3.1 Achievable Sum-Rate under Linear Precoding

Consider the multiuser scenario where each user $k$ transmits a zero-mean Gaussian symbol vector with covariance matrix $\Sigma_k$, and receiver noise is also a zero-mean Gaussian vector with covariance matrix $\sigma_n^2I$. Under single-user decoding, where multi-user interference is treated as noise, user $k$ can achieve rate $R_k$ in the downlink as shown in (4.6):

$$R_k = \log \frac{\det \left( \sum_{j=1}^{K} H_k^H \Sigma_j H_k + \sigma_n^2 I \right)}{\det \left( \sum_{j \neq k} H_k^H \Sigma_j H_k + \sigma_n^2 I \right)}.$$  \hspace{1cm} (4.14)

Under the system model described in Chapter 2, the transmit covariance matrix $\Sigma_k = \bar{U}_k P_k \bar{U}_k^H$. The achievable rate for user $k$ under linear precoding is therefore

$$R_k^{LP} = \log \frac{\det \left( \sum_{j=1}^{K} H_k^H \bar{U}_j P_j \bar{U}_j^H H_k + \sigma_n^2 I \right)}{\det \left( \sum_{j \neq k} H_k^H \bar{U}_j P_j \bar{U}_j^H H_k + \sigma_n^2 I \right)}$$  \hspace{1cm} (4.15)

where $R_{N+I,k} = R_k - H_k^H \bar{U}_k P_k \bar{U}_k^H H_k$ is defined as the received noise-plus-interference covariance matrix at user $k$.

The rate maximization problem with a sum power constraint under linear precoding
can then be formulated as

\[
(\bar{U}, P) = \arg \max_{\bar{U}, P} \sum_{k=1}^{K} \log \frac{\det R_k}{\det R_{N+I,k}}
\]

subject to:

\[
\|\bar{u}_l\|_2 = 1, \quad l = 1, \ldots, L
\]
\[
p_l \geq 0, \quad l = 1, \ldots, L
\]
\[
\|p\|_1 = \sum_{l=1}^{L} p_l \leq P_{\text{max}}.
\]

(4.16)

4.3.2 MSE Formulation: Product of MSE Matrix Determinants

We now find an MSE-based formulation using joint processing of all streams (rather than treating each user’s own data streams as interference) which leads to an equivalent optimal formulation of the rate maximization problem under linear processing. We develop this relationship by using the MSE matrix determinants. The consideration of the determinant of the MSE matrix as a basis for selection of an optimization criterion is motivated by the single-user multicarrier case, wherein minimizing the PMSE is equivalent to minimizing the determinant of the MSE matrix and thus is also equivalent to maximizing the mutual information [68].

Consider the following optimization problem which minimizes the product of the determinants of the downlink MMSE matrices (2.14) under a sum power constraint:

\[
(\bar{U}, P) = \arg \min_{\bar{U}, P} \prod_{k=1}^{K} \det \epsilon_k^{DL}
\]

subject to:

\[
\|\bar{u}_l\|_2 = 1, \quad l = 1, \ldots, L
\]
\[
p_l \geq 0, \quad l = 1, \ldots, L
\]
\[
\|p\|_1 = \sum_{l=1}^{L} p_l \leq P_{\text{max}}.
\]

(4.17)

**Theorem 4.2.** When linear MMSE receivers \( V_k^* \) (2.13) are employed at the mobile terminals, the sum-rate maximization problem in (4.16) and the PDetMSE minimization problem in (4.17) are equivalent.
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Proof. The determinant of the downlink MSE matrix can be written as

\[
\det \varepsilon_k^{DL} = \det \left( I_L - H_k^H \bar{U}_k P_k \bar{U}_k^H H_k R_k^{-1} \right) 
\]

(4.18)

\[
= \det \left[ \left( R_k - H_k^H \bar{U}_k P_k \bar{U}_k^H H_k \right) R_k^{-1} \right] 
\]

(4.19)

\[
= \det \left[ R_{N+I,k} R_k^{-1} \right] 
\]

\[
= \frac{\det \left( R_{N+I,k} R_k^{-1} \right)}{\det R_k}.
\]

where (4.18) follows from (2.14) since \( \det(I + AB) = \det(I + BA) \) when \( A \) and \( B \) have appropriate dimensions. We then see the relationship to (4.15),

\[
\log \det \varepsilon_k^{DL} = - \log \frac{\det R_k}{\det R_{N+I,k}} 
\]

(4.19)

With this result, we can see that under MMSE reception, minimizing the determinant of the MSE matrix \( \varepsilon_k^{DL} \) is equivalent to maximizing the achievable rate for user \( k \). It follows that minimizing the product of MSE matrix determinants over all users is equivalent to sum-rate maximization,

\[
\min \prod_{k=1}^{K} \det \varepsilon_k^{DL} \equiv \min \log \left( \prod_{k=1}^{K} \det \varepsilon_k^{DL} \right) 
\]

(4.20)

\[
\equiv \min \sum_{k=1}^{K} \log \det \varepsilon_k^{DL} 
\]

\[
\equiv \max \sum_{k=1}^{K} R_k^{LP}.
\]

where (4.20) holds since \( \log(\cdot) \) is a monotonically increasing function of its argument.

\[\square\]

Note that this result represents an upper bound on the sum-rate of all linear precoding schemes in the broadcast channel.

The covariance matrices \( R_k \) and \( R_{N+I,k} \) in the MMSE matrix \( \varepsilon_k^{DL} \) are each functions of all precoder and power allocation matrices. Thus, the sum-rates \( R_k \) for each user \( k \)
(and the sum-rate for all users) are coupled across users. As such, finding $\bar{U}$ and $P$ jointly or finding only the power allocation $P$ for a fixed $\bar{U}$ are both non-convex problems and are just as difficult to solve as the rate maximization problem.

In the sum capacity and sum-MSE problems, the problem of non-convexity is addressed by solving a convex virtual uplink formulation and applying a duality-based transformation. Unfortunately, the sum-rate expression under linear precoding in the virtual uplink is nearly identical to that derived above for the downlink, and does not admit a cancellation or grouping of terms to decouple the problem across users.

Direct solution of the non-convex downlink problem for minimizing the product of MSE matrix determinants requires finding a complex $M \times L$ precoder matrix. Once again, we consider the application of SQP to solve this problem, as this computationally intensive approach is our only available option in the absence of a convex virtual uplink reformulation.

The numerical techniques used for solving most non-convex problems do not guarantee convergence to the global minimum. This is clearly not a desirable method for finding a practical precoder, especially when one of our major motivations for using linear precoding is reducing transmitter complexity. We do not suggest that this method be practically implemented; rather, we use it to illustrate the difference in performance between the solutions to the possibly optimal PDetMSE formulation and the more practical PMSE algorithm that we proposed in the previous section.

### 4.4 Numerical Examples

In this section, we present simulation results that compare the sum-rate achievable using linear precoding to the information theoretic capacity of the BC. That is, we consider the spectral efficiency (measured in bps/Hz) that could be achieved under ideal transmission by drawing transmit symbols from a Gaussian codebook. In Appendix D, we
present a simple scheme to illustrate performance of the PMSE algorithm using adaptive modulation for $M$-PSK constellations.

In each simulation, the fading channel is modelled as flat and Rayleigh, with i.i.d. channel coefficients distributed as $\mathcal{CN}(0,1)$. The examples use a maximum transmit power of $P_{\text{max}} = 1$; SNR is controlled by varying the receiver noise power $\sigma_n^2$. As stated earlier, the transmitter is assumed to have perfect knowledge of the channel matrix $H$. The PDetMSE results are determined by taking the best result from execution of the SQP-based algorithm using 10 random initializations.

Figure 4.1 illustrates how the proposed schemes perform when compared to the sum capacity for the broadcast channel (i.e., using dirty paper coding (DPC) [42]) and to linear precoding methods based on channel orthogonalization, i.e., block diagonalization and zero forcing [53]. The convergence threshold for the PMSE algorithm is set at $\epsilon = 10^{-6}$.

The simulations in Figure 4.1 model a $K = 2$ user system with $M = 4$ transmit antennas and $N_k = 2$ receive antennas per user. In simulation, we observe a negligible (but non-zero) difference in performance when comparing the PDetMSE algorithm to the PMSE solution (There is, however, a significant performance difference between BD and ZF.) This result is gratifying because it suggests that the marginal gains achieved by joint processing do not merit the greatly increased computational effort required to find a matrix in $2ML$ real variables using SQP; the computationally feasible PMSE solution can be used without a large penalty in performance. Both the PMSE and PDetMSE algorithms do demonstrate a divergence in performance from the theoretical DPC bound at higher SNR. This drop in spectral efficiency may reflect a fundamental gap between the (optimal) nonlinear DPC capacity and the rate achievable under linear precoding, but it may also be caused by the algorithms’ convergence to local minima due to the

---

3Simulation results for the DPC, BD, ZF, and NuSVD plots were obtained by using the cvx optimization package [32][69].

4Please see Section 4.5 for further discussion on this matter in the light of recent publications.
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Figure 4.1: Comparing PDetMSE, PMSE, DPC and orthogonalization-based methods, $K = 2$, $M = 4$, $N_k = 2$, $L_k = 2$

non-convexity of the optimization problems.

The PMSE algorithm outperforms the BD and ZF methods over the entire SNR range when the orthogonalization-based schemes are forced to use all $N$ receive antennas. However, this approach to orthogonalization is suboptimal; the optimal BD and ZF precoders may be found by selecting the best precoder from all $\sum_{k=1}^{\min(N,M)} \binom{N}{k}$ possible subsets of receive antennas. At high SNR, the PMSE and PDetMSE precoders perform equivalently to the BD precoder with selection; we have observed that the PMSE and PDetMSE precoders (in conjunction with the MMSE receivers) behave like the BD precoder in orthogonalizing the channel at high SNR. The biggest gain in performance over orthogonalization-based solutions occurs at low to mid-SNR values, where BD and ZF suffer due to noise enhancement.

Figure 4.2 presents simulation results for a similar system as Figure 4.1, but with $N_k = 4$ receive antennas per user. In this system, there are fewer transmit antennas than
Figure 4.2: Comparing PDetMSE, PMSE, DPC and orthogonalization-based methods, $K = 2$, $M = 4$, $N_k = 4$, $L_k = 2$

receive antennas ($M < N$), so BD/ZF cannot be employed without selection. We include simulation results for BD/ZF with selection, but note the large computational complexity required (selecting the best of 162 candidate precoders). We compare these results to a generalized orthogonalization based approach, referred to as nullspace-directed SVD (NuSVD) in [56], and observe a large difference in performance at high SNR. This gain in spectral efficiency can be attributed to NuSVD’s ability to use all $N = 8$ receive antennas, whereas BD and ZF are limited by an antenna constraint.

Once again, Figure 4.2 illustrates that the PMSE/PDetMSE approaches outperform orthogonalization, particularly at low to mid-SNR values. This improvement in performance comes at the expense of additional complexity. Even though NuSVD and PMSE are iterative algorithms, NuSVD requires only one (concave) waterfilling power allocation after convergence of precoder direction iterations, whereas the PMSE minimization method employs numerical optimization algorithms (SQP or GP) in each iteration.
Figure 4.3: Scaling of sum-rate with $K$, $M = 2K$, $N_k = L_k = 2$, SNR = 5dB

Figure 4.3 provides an alternative viewpoint to compare performance across schemes. The figure shows how the sum throughput scales with the number of users $K$, for $M = 2K$ transmit antennas and $N_k = 2$ receive antennas per user at 5 dB average SNR. The number of transmit antennas $M$ is chosen so that BD and ZF can be implemented without selection, as selection-based BD and ZF are exponentially complex with $4^K - 1$ possible precoders. This plot illustrates how the proposed scheme takes advantage of the available degrees of freedom at the transmitter and provides throughput significantly better than the orthogonalization based BD and ZF schemes.

The PMSE and PDetMSE algorithms do not require the explicit selection of $L_k$; rather, this parameter is determined implicitly by the power allocation. However, we can force the PMSE algorithm to allocate a maximum number of substreams $L_k$ to each user to gain further insight into its behaviour. In Figure 4.4 the number of streams in the $N_k = 4$ system described above is varied from $L_1 = L_2 = 2$ to $L_1 = 3$ and $L_2 = 1$. The achievable sum-rate in this system decreases in the latter case, as the asymmetric...
Figure 4.4: Data stream allocation in PMSE optimization, $K = 2$, $M = 4$, $N_k = 4$

stream allocation does not correspond to the symmetric (statistically identical) channel configuration. In this case, user 2 is restricted to only a single data stream, and thus can not take full advantage of good channel realizations. If the goal is always maximizing the sum rate, the users should be allocated the maximum number of data streams in as balanced a manner as possible. Note however that the PMSE algorithm can provide unbalanced allocations if desired for other reasons (e.g., quality of service provisioning).

4.5 Epilogue: Equivalence of Joint and Scalar Processing

In this chapter, we have described how the theoretically optimum formulation based on joint processing (PDetMSE minimization) can be closely approximated by the scalar PMSE minimization approach. Our original motivation for considering PMSE minimization stemmed from the equivalence between the two approaches in the single-user
multicarrier case [68]. There, diagonalization of the channel and MSE matrices results in the MSE matrix determinant minimization’s equivalence to the product of individual MSE terms. However, our work did not lead to such an equivalence in the multiuser case.

It has recently been demonstrated in [70] that the MSE matrices in the virtual uplink can also be diagonalized in the multiuser case. For any arbitrary set of virtual uplink precoders and power allocations, $V_k = \hat{V}_k \sqrt{Q_k}$, one can find the corresponding MMSE receivers as in (2.21),

\[
U_k^* = R_k^{-1} H_k V_k
= \left( \sum_{k=1}^{K} H_k V_k V_k^H H_k^H + \sigma_n^2 I_M \right)^{-1} H_k V_k,
\]

and the MMSE matrices as in (2.22):

\[
\varepsilon_{UL}^{\star} = I_{L_k} - V_k^H H_k^H R_k^{-1} H_k V_k
= I_{L_k} - V_k^H H_k^H \left( \sum_{k=1}^{K} H_k V_k V_k^H H_k^H + \sigma_n^2 I_M \right)^{-1} H_k V_k
= I_{L_k} - U_k^{\star H} H_k V_k
\]

Now, consider applying an arbitrary unitary transformation matrix $W_k$ to each precoder $V_k$ such that

\[
V_k' = V_k W_k,
\]

and let $\varepsilon_{UL}^{\star'}$ be the associated MMSE matrix (for the corresponding MMSE decoder, $U_k' = U_k W_k$). The determinant of the MMSE matrix is invariant to this arbitrary
unitary transformation, as shown below:

\[
\det \left[ \varepsilon^{UL}_k \right] = \det \left[ I_{L_k} - W_k^H V_k^H H_k^H \left( \sum_{k=1}^{K} H_k V_k W_k W_k^H V_k^H H_k^H + \sigma_n^2 I_M \right)^{-1} H_k V_k W_k \right]
\]

\[
= \det \left[ W_k^H W_k - W_k^H V_k^H H_k^H \left( \sum_{k=1}^{K} H_k V_k V_k^H H_k^H + \sigma_n^2 I_M \right)^{-1} H_k V_k W_k \right]
\]

\[
= \det \left[ W_k^H \left( I_{L_k} - V_k^H H_k^H \left( \sum_{k=1}^{K} H_k V_k V_k^H H_k^H + \sigma_n^2 I_M \right)^{-1} H_k V_k \right) W_k \right]
\]

\[
= \det \left[ \left( I_{L_k} - V_k^H H_k^H \left( \sum_{k=1}^{K} H_k V_k V_k^H H_k^H + \sigma_n^2 I_M \right)^{-1} H_k V_k \right) W_k W_k^H \right]
\]

\[
= \det \left[ \varepsilon^{UL}_k \right] \det \left[ W_k^H W_k \right]
\]

\[
= \det \left[ \varepsilon^{UL}_k \right],
\]

(4.24)
since \( W_k W_k^H = W_k^H W_k = I \) by definition.

Consider the eigendecomposition of the normal matrix

\[
U^*_H H_k V_k = W_k \Lambda_k W_k^H.
\]

(4.25)

\( W_k \) is also an eigenbasis for \( \varepsilon^{UL}_k \), as seen in (4.22). Thus, choosing this \( W_k \) as the transformation matrix diagonalizes both the end-to-end communication system for each user \( k \) and the corresponding MMSE matrix.

A similar result can be found in the downlink for precoder and power allocation matrices \( U_k = \tilde{U}_k \sqrt{P_k} \) by transforming \( U_k' = U_k W_k \) with the transformation matrices \( W_k \) selected as the eigenbases of

\[
V_k^* H_k^H U_k = U_k^H H_k \left( H_k^H U U^H H_k + \sigma_n^2 I_{N_k} \right)^{-1} H_k^H U_k.
\]

(4.26)

Since the MMSE matrix \( \varepsilon^*_k \) can be diagonalized to \( \varepsilon^{diag}_k \) via unitary transformation in both downlink and virtual uplink without changing its determinant, it follows that any sum rate that is achievable under joint precoding and decoding can be achieved using
the equivalent scalar formulation, since

$$\det \left[ \mathbf{e}_{k}^{\text{diag}} \right] = \prod_{l=1}^{L_k} \left[ \mathbf{e}_{k}^{\text{diag}} \right]_{l,l}. \quad (4.27)$$

### 4.5.1 Relative Performance of Joint and Scalar Approaches

In the light of this equivalence result, a question remains unanswered: Why does minimization of the PDetMSE yield a marginally higher rate in simulation when compared to minimizing the PMSE?

The algorithm proposed for PMSE minimization is only guaranteed to converge to a local optimum, since the problem in (4.11) is not convex. Thus, it is possible that the sum rate achieved by PMSE is suboptimal, and it is also possible that the numerical PDetMSE approach may achieve a higher sum rate. Due to the invariance of the MSE matrix determinant under unitary rotations, there are (infinitely) many possible precoders that achieve a given PDetMSE value; only one of these corresponds to a diagonal MSE matrix and an equivalent PMSE solution. We speculate that this extra rotational dimension in the PDetMSE solution space may make it easier for the numerical optimization techniques to find a solution near the global optimum. However, as shown in [70] and (4.27), the global optimum for PDetMSE and PMSE should coincide.

### 4.6 Summary

In this chapter, we have presented transceiver designs based on the product of stream MSEs and the product of MSE matrix determinants which maximize sum throughput in the multiuser MIMO downlink under a sum power constraint with perfect CSI. We have compared the maximum achievable sum rate performance of linear precoding schemes to the sum capacity in the general MIMO downlink, without imposing constraints on the number of users, base station antennas, or mobile antennas. Equivalent MSE-based formulations were derived for the joint processing solution (based on PDetMSE minimiza-
tion), which was shown to be theoretically optimal, but computationally infeasible. A near-optimal framework based on scalar (per-stream) processing was then proposed, and an implementation was provided based on PMSE minimization and employing a known uplink-downlink duality of MSEs. We evaluated the performance of these schemes in the context of orthogonalizing approaches, which suffer from noise enhancement, and have shown that the MSE based optimization schemes are able to achieve significant performance improvements. Furthermore, we have demonstrated that negligible performance losses occur when using the suboptimal PMSE criterion in comparison to the optimum PDetMSE criterion.
Chapter 5

Imperfect Channel State Information: Delay and Feedback

The precoder designs proposed in Chapters 3 and 4 rely on the transmitter and receivers possessing perfect knowledge of CSI estimates. In this chapter and in Chapter 6 we consider several potential sources of error in CSI, and propose methods to mitigate their adverse impact on system performance.

In practice, available estimates of channel coefficients are found using training in the presence of noise and are thus imperfect estimates. If channel reciprocity holds (i.e., the uplink and downlink channels are identical), these estimates can be provided by training in the uplink (e.g., using uplink sounding, as in the WiMAX standard [5]). However, in frequency division duplex (FDD) systems (and in some broadband time division duplex systems [7]), channel reciprocity does not apply. In the absence of channel reciprocity, CSI at the transmitter must be obtained via estimation of a training sequence by mobile terminals in the downlink and communicated back to the base station using an uplink feedback mechanism. Such a feedback system must provide the transmitter accurate CSI in a timely fashion while minimizing the overhead due to feedback. However, precoding performance may be greatly diminished when time-varying fading combines with prop-
agitation and computational delays, as the channel coefficients at the time of feedback reception may be drastically different from those used to design the precoder matrix.

In practice, a CSI feedback system requires three basic stages: channel estimation, channel prediction, and feedback quantization. This chapter focuses on prediction, while using basic approaches to scalar quantization and assuming perfect estimates of the channel at the mobile receivers.

### 5.1 Literature Review

In [72], a system for CSI prediction based on Kalman filtering is proposed for a system exploiting TDD reciprocity; as a result of this assumption, the authors are able to consider one-frame-ahead prediction as they only have to mitigate changes due to the fading rate of the channel. When the total delay is taken into account in our proposed system (i.e., block decoding of the pilot and feedback symbols, processing of the channel estimates, optimization of the new precoder, and propagation delays in both directions), a more realistic assumption is to consider a delay of three data frames or more\(^1\). We propose a generalized \(N\)-frames-ahead predictor based on the Kalman filter, and evaluate its performance in Section 5.4 for \(N = 3\).

The reduction of feedback overhead is also a significant challenge in multiuser systems. There is a fundamental tradeoff between the feedback rate and the accuracy of the CSI at the transmitter. Since the required feedback rate scales multiplicatively in the number of transmit antennas and in the number of users and receive antennas, feedback efficiency is a particularly important design criterion for MIMO systems. We propose the use of adaptive delta modulation (ADM) [74, 75] in Section 5.3.3 to reduce the feedback rate.

\(^1\)The choice of a minimum delay of three data frames is based on a suggestion arising from discussions with our industrial partner on this project (and co-author on the resulting work [73]), Young-Soo Yuk from LG Electronics, Inc. He noted that mobile receivers would likely give higher priority to decoding data symbols, and that pilot-based estimation following decoding of the entire block transmission might take span two or more transmission intervals. The three frame minimum results from the fact that feedback of the estimate takes place in the following uplink frame.
requirements by exploiting the temporal correlation of CSI.

### 5.2 Revised Channel Model

In this section, we consider the prediction and feedback of slow-fading Rayleigh channels. Rayleigh fading may not accurately model propagation in a real-world scenario (e.g., one which includes line-of-sight components or shadowing); however, assuming Rayleigh fading provides a tractable framework for analysis and helps to illustrate average performance in a dense urban environment where fading is dominated by small-scale fading. Coefficients in the MIMO channels are still assumed to be spatially uncorrelated; therefore, we are able to define the fading characteristic of the channel by considering a scalar channel model.

The scalar channels for each mobile user are defined using Jakes’ model \[7\], where the discrete time fading channel sequence \( h(n) \) models a channel with maximum Doppler frequency \( f_D \) and sampling period \( T_{fr} \), which has autocorrelation \( R(k) \) at lag \( k \),

\[
R(k) = \mathbb{E}[h(n)h^*(n-k)] = J_0(2\pi f_D k T_{fr}). \tag{5.1}
\]

Here, \( J_0(\cdot) \) is the order-zero Bessel function of the first kind.

We assume that the Rayleigh fading channels can be modeled by an order \( U \) autoregressive (AR) process; i.e.,

\[
h(n) = \sum_{k=1}^{U} a_k h(n-k) + v(n) \tag{5.2}
\]

and the process noise \( v(n) \sim \mathcal{CN}(0,\sigma_v^2) \); \( h(n) \) and \( v(n) \) are assumed independent. The AR coefficients \( a_k \) are determined by solving the set of \( U \times U \) Yule-Walker equations \[76\],

\[
\text{Yule-Walker equations:} \quad a_k = \frac{\mathbb{E}[h(n)h^*(n-k)]}{\mathbb{E}[h(n)^2]} \quad \forall k, 1 \leq k \leq U.
\]
which correspond to the set of autocorrelation equations for the first $U$ lags, $l = 1, \ldots, U$,

$$E[h(n)h^*(n - l)] = \sum_{k=1}^{U} a_k E[h(n - k)h^*(n - l)]$$

$$R(l) = \sum_{k=1}^{U} a_k R(l - k).$$

The solution to these $U$ equations in the AR coefficients $a_k$ is $a = R^{-1}r$, where

$$a = \begin{bmatrix} a_1 & a_2 & \ldots & a_U \end{bmatrix}^T,$$

$$R = \begin{bmatrix} R(0) & R(1) & \cdots & R(U - 1) \\ R(1) & R(0) & \cdots & R(U - 2) \\ \vdots & \vdots & \ddots & \vdots \\ R(U - 1) & R(U - 2) & \cdots & R(0) \end{bmatrix},$$

$$r = \begin{bmatrix} R(1) & R(2) & \cdots & R(U) \end{bmatrix}^T.$$

The transpose symmetry in $R$ follows from the even symmetry in the autocorrelation function $R(k)$. The process noise variance parameter can then be determined under the given AR model by substituting the coefficients $a_k$ into the equation for the zero-lag autocorrelation function,

$$E[h(n)h^*(n)] = \left( \sum_{k=1}^{U} \sum_{j=1}^{U} a_k a_j E[h(n - k)h^*(n - j)] \right) + E[v(n)v^*(n)]$$

$$\Rightarrow \sigma_p^2 = R(0) - \sum_{k=1}^{U} \sum_{j=1}^{U} a_k a_j R(j - k).$$

[76] notes that while low-order AR models can not perfectly model the irrational “bowl-shaped” Doppler power spectral density found as the Fourier transform of the autocorrelation function $(5.1)$, an arbitrarily accurate spectrum can be modelled with sufficiently high order. While our work focuses on low-complexity implementations with lower order AR models (e.g., $U < 6$), we refer the reader to [76] for a more detailed treatment of the tradeoff between AR model order and one-step prediction MSE.
5.3 Kalman Filtering and Channel Prediction

5.3.1 One-Step Kalman Filter

In order to predict the channel, we first formulate the one-step-ahead Kalman predictor \[77\]. The system uses the following model to describe the evolution of the state \( x_n \), representing the previous \( U \) values of a single scalar channel coefficient (as described in (5.2)), and the corresponding measurement \( z_n \),

\[
\begin{align*}
x_n &= A x_{n-1} + v_n \\
z_n &= C x_n + w_n,
\end{align*}
\]  

with state transition matrix \( A \) and measurement matrix \( C \):

\[
A = \begin{bmatrix}
a_1 & a_2 & \cdots & a_{U-1} & a_U \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]  

The temporally uncorrelated process and measurement noise are distributed as \( v_n \sim \mathcal{CN}(0, Q_p) \) and \( w_n \sim \mathcal{CN}(0, \sigma_o^2) \), respectively, with \( Q_p = \text{diag} \left( [\sigma_p^2 \ 0_{1 \times (U-1)}] \right) \). Here, due to the assumption of perfect channel estimation, \( \sigma_o^2 = 0 \). The time-update (prediction) equations and measurement (correction) equations are defined as follows, where the predicted and corrected forms of the state vector \( x \) and state covariance matrix \( P \) are
indicated by the subscripts $[\cdot]_{n|n-1}$ and $[\cdot]_{n|n}$, respectively:

\[
\begin{align*}
\mathbf{x}_{(n|n-1)} &= \mathbf{A}\mathbf{x}_{(n-1|n-1)} \\
\mathbf{P}_{(n|n-1)} &= \mathbf{A}\mathbf{P}_{(n-1|n-1)}\mathbf{A}^H + \mathbf{Q}_p \\
\mathbf{K}_n &= \mathbf{P}_{(n|n-1)}\mathbf{C}^H \left( \mathbf{C}\mathbf{P}_{(n|n-1)}\mathbf{C}^H + \sigma_n^2 \right)^{-1} \\
\mathbf{x}_{(n|n)} &= \mathbf{x}_{(n|n-1)} + \mathbf{K}_n \left( \mathbf{z}_n - \mathbf{C}\mathbf{x}_{(n|n-1)} \right) \\
\mathbf{P}_{(n|n)} &= (\mathbf{I} - \mathbf{K}_n\mathbf{C})\mathbf{P}_{(n|n-1)}.
\end{align*}
\]

We initialize the state vector and error covariance matrix as \( \mathbf{x}_{0|0} = 0_{U\times 1} \) and \( \mathbf{P}_{0|0} = \mathbf{I}_U \).

### 5.3.2 N-Step Ahead Prediction

We now generalize the one-step predictor to an arbitrary \( N \) steps.

#### Channel Prediction

By expanding the linear dynamic model in (5.6) for \( N \) subsequent time intervals, the state at time \( n + N \) can be written as

\[
\mathbf{x}_{n+N} = \mathbf{A}^N\mathbf{x}_n + \sum_{k=1}^{N} \mathbf{A}^{N-k}\mathbf{v}_{n+k}.
\]

We form our \( N \)-step-ahead prediction of the Kalman state vector \( \hat{\mathbf{x}}_{(n+N|n)} \) based on the measurement corrected state \( \mathbf{x}_{(n|n)} \) at time \( n \) as

\[
\hat{\mathbf{x}}_{(n+N|n)} = \mathbb{E} [\mathbf{x}_{n+N}] = \mathbf{A}^N\mathbf{x}_{(n|n)},
\]

and find the predicted channel coefficient \( \hat{h}(n+N) = \mathbf{C}\mathbf{x}_{(n+N|n)} \). Obtaining the estimate in this manner is equivalent to performing \( N \) iterations of the Kalman filter by using only the time-update (prediction) steps without performing any of the measurement update (correction) steps.
Prediction Error Analysis

Given the assumption of error-free observations $z_n$ of the channel at time $n$, it follows that $x_{(n|n)} = x_n$; that is, the measurement corrected state is the true state at time $n$. Thus, the $N$-step prediction error $e_N$ can be written as

$$e_N = x_{n+N} - \hat{x}_{(n+N|n)} = \sum_{k=1}^{N} A^{N-k}v_{n+k}.$$  \hspace{1cm} (5.11)

It follows that the $N$-step prediction error covariance matrix $P_{(n+N|n)}$ is

$$P_{(n+N|n)} = \mathbb{E}[e_N e_N^H] = \sum_{k=1}^{N} A^{N-k}Q_p (A^{N-k})^H,$$  \hspace{1cm} (5.12)

since the noise vectors $v_k$ are temporally uncorrelated. The $N$-step prediction MSE for the estimate $\hat{h}(n + N)$ is the first (top-left) entry in this matrix.

5.3.3 Adaptive Delta Modulation

Delta modulation can be employed effectively in systems using oversampling; in the slow-fading system under investigation, the oversampling rate is equivalent to the inverse of the normalized Doppler rate, $1/(f_D T_{fr})$. Slow fading occurs at pedestrian velocities (on the order of $v \simeq 3 \text{ km/h}$), resulting in an oversampling factor of approximately 30.

In the proposed system, we employ constant factor adaptive delta modulation (CF-ADM) with one-bit memory, as first described in [74] and recently used in [75] for channel feedback. This method of delta modulation uses an individual delta modulator to track each of the real and imaginary parts of each channel coefficient; thus, two bits of feedback per antenna path are transmitted per feedback period. The block diagram of this ADM-based quantizer is depicted in Figure 5.1 for a single (scalar) channel coefficient. For ease of illustration, we assume that $x_n$ is real. The error term $d_n$ is calculated as the difference between the previous quantizer output $y_{n-1}$ and the current value $x_n$. The sign of the

---

\footnote{This refers to the memory used in the quantizer for the real and imaginary components of each channel coefficient. The predictor in our implementation has a memory of $U$ complex floating-point channel coefficients.}
difference is determined by the $\text{sgn}(\cdot)$ operator, which in turn determines the bit-value $b_n$ corresponding to the sign of the correction term. The step size of the delta modulator is adapted according to a simple heuristic that depends only on the values of the current feedback bit $b_n$ and its previous value $b_{n-1}$. When both bits are identical, the step size is increased by a multiplicative factor $\alpha$; if they differ, the step size is decreased by the inverse factor $1/\alpha$. This method for scaling step sizes allows for both rapid tracking when the channel is changing relatively quickly and for convergence to the true value when the channel response is changing slowly. The quantizer output $y_n$ is calculated by adding the step size $s_n$ (multiplied by the $+1$ or $-1$ bit value, $b_n$) to a scaled version of the previous quantizer output, $\rho y_{n-1}$. When $\rho < 1$ is selected, the sum operator is referred to as a “leaky integrator”, which has the benefit of mitigating error propagation [75].

5.4 Numerical Examples

In this section, we present simulation results to illustrate the performance of the proposed prediction and quantization mechanisms for channel feedback in the system with an $N =$
Table 5.1: Typical WiMAX parameters based on ITU Pedestrian-A model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrier frequency ($f_c$)</td>
<td>2.3 GHz</td>
</tr>
<tr>
<td>Channel sampling rate ($f_s$)</td>
<td>200 Hz</td>
</tr>
<tr>
<td>Frame duration ($T_{fr}$)</td>
<td>5 ms</td>
</tr>
<tr>
<td>Max. feedback + processing delay</td>
<td>15 ms (3 frames)</td>
</tr>
<tr>
<td>Mobile velocity ($v$)</td>
<td>3 km/h</td>
</tr>
<tr>
<td>Maximum Doppler frequency ($f_D$)</td>
<td>6.4 Hz</td>
</tr>
</tbody>
</table>

3 frame feedback delay. We suggest some typical system parameters for a WiMAX/LTE system based on the IEEE 802.16e standard [5] and the ITU Pedestrian-A model) that result in slow-fading channels in Table 5.1.

Simulation results are averaged over 1000 realizations of a slow-fading Rayleigh channel, which is generated according to the modified Jakes’ model of [78]; this algorithm differs from the original Jakes simulators based on sum-of-sinusoids methods which generate wide-sense non-stationary channels. The introduction of an additional random initial phase term eliminates this problem, thereby permitting the generation of uncorrelated channels which correspond to our assumption of independent Rayleigh MIMO fading. We discard the first 100 predictions to allow the Kalman filter to converge to a steady-state solution. The simulated data points are then generated by averaging over the subsequent 400 predictions.

Figure 5.2 compares the simulated prediction MSE to the theoretical expression derived in (5.12). We are able to draw several conclusions from this figure. First, we see that the theoretical expression of (5.12) is not an exact match to the simulated prediction MSE; this difference may be attributed to the fact that the AR process does not accu-
Figure 5.2: Prediction MSE vs. velocity $v$ and AR order $U$

![Figure 5.2: Prediction MSE vs. velocity $v$ and AR order $U$](image)

...model the bandlimited power spectral density of the Rayleigh fading channel [79]. Nonetheless, the approximation is good enough to allow for effective prediction and the consistent difference between the theoretical and simulated MSE suggests that the theoretical MSE expression can be used for system design. It is evident that the use of an order $U = 2$ AR model (as proposed in [72]) may be insufficient when $N > 1$ and/or at higher velocities. Furthermore, this figure suggests that Kalman prediction is effective at the target ITU Pedestrian A velocity of 3 km/h, and demonstrates a tradeoff between prediction MSE and computational complexity (i.e., AR order). Finally, this figure illustrates that increasing the AR model order does not decrease prediction MSE above a threshold velocity (approximately 20 km/h, or a normalized Doppler rate of 0.21). At these higher velocities, such feedback schemes may not be feasible, and system designers may wish to revert to open loop techniques that are able to operate effectively in the absence of CSI.

Figure 5.3 illustrates the performance of the combined predictor/quantizer employing
delta modulation (1 bit / real channel coefficient). The MSE shown in Figure 5.3 is the total (prediction + quantization) MSE. Here, we have selected the value $\alpha = 1.5$ for the scaling factor by trial and error to minimize quantization error for the target velocity ($v = 3$); however, this value may be suboptimal for higher velocities. Similarly, we select $\rho = R(1)^{-1} \left( 1 - \sqrt{1 - R^2(1)} \right)$ as in [75], which is optimal for the case of $U = 1$, but not necessarily for higher order predictors. In comparing Figures 5.2 and 5.3, observe that the system performance under ADM is dominated by quantization error at low velocity, as the total MSE is several orders of magnitude larger than the prediction MSE for the corresponding values of $v$ and $U$. Furthermore, negligible gains are seen when increasing $U = 3$ to $U = 4$ for the ADM based system.

In order to determine the feedback savings enabled by ADM, we compare the proposed feedback scheme to one using direct scalar quantization of the channel coefficients (i.e., using the optimal Gaussian quantization codebook as determined by the Lloyd-
Max algorithm [80], with $B$ feedback bits per real channel coefficient, and the $U = 4$ AR model. At the target velocity of $v = 3$ km/h, the ADM-based system is able to reduce the number of feedback bits by a factor of 3; however, this advantage disappears as mobile velocity increases. Thus, while ADM can be used to reduce the required feedback rate at low velocity, higher rate feedback methods may be desired when lower total (prediction+quantization) MSE is needed. We also observe that with increasing mobile velocity, the total MSE becomes dominated by prediction error, and increasing the feedback rate no longer results in improved system performance.

In Figure 5.4 we illustrate the performance of the prediction-feedback scheme by examining average bit error rate (BER) performance under linear precoding (using the sum-MSE minimizing precoder described in Chapter 3). The proposed feedback scheme is employed with AR order $U = 4$. The system employs four transmit antennas to communicate one data stream to each of two users possessing two receive antennas each; data symbols are uncoded QPSK. We see that the Kalman predictor is sufficient to
provide nearly optimal performance (as compared to the genie-aided transmitter with perfect CSI). The delta modulated feedback scheme incurs only a small penalty in BER performance (approximately 1 dB and 2 dB at a target BER of $10^{-3}$ for $v = 3$ and $v = 6$ km/h, respectively). Finally, we observe an error floor that occurs at high SNR when using delta modulated feedback; this is evidence of the system becoming interference-limited due to the use of imperfect CSI.

5.5 Summary

In this chapter, we have proposed a scheme for predictive feedback using a Kalman-filter based predictor. We have extended the Kalman prediction framework to design an $N$-frames-ahead predictor. We implement low-rate feedback by employing adaptive delta modulation for quantization, and assess its performance as compared to the optimal scalar quantizers designed under the Lloyd-Max algorithm. Simulations based on typical WiMAX parameters suggest that the proposed scheme is quite effective at pedestrian velocities. The system is shown to be limited by high-speed mobility, as performance degradation occurs when the normalized Doppler rate grows large. However, at the designed pedestrian velocities, a tradeoff allows system designers to improve prediction quality at the expense of increased computation with increasing AR model order. Simulation results suggest that delta modulated feedback can be used to reduce feedback rate requirements for the target scenario, but that strict MSE requirements or higher velocity scenarios may require the use of alternate quantization schemes.
Chapter 6

Energy Allocation and Precoder Design for Systems with Imperfect Channel State Information

In Chapter 5, we assumed that receiver CSI estimates were perfect, and that the precoder and feedback design needed to compensate only for fading channel conditions, feedback error and delay. In contrast, in this chapter, we consider methods to adapt precoder designs to mitigate imperfect CSI estimation at the mobile receivers, but assume that the channel estimates are available immediately at the base station (via an error-free and delay-free feedback mechanism). The two schemes therefore take complementary viewpoints to the imperfect CSI problem.

6.1 Introduction

6.1.1 Block Transmission Model

In Chapters 3 and 4 precoders were designed using instantaneous knowledge of each channel realization. Within this context, it was sensible to consider only a sum-power
constraint for each data transmission. In this chapter, we assume that the communications channel is fixed for a coherence interval of $n$ consecutive symbol periods, which we refer to as a transmission block. We introduce a total energy constraint of $E_{\text{max}}$ for each transmission block, which is divided into a training phase and a data transmission phase, as illustrated in Figure 6.1. If we scale the total energy $E_{\text{max}}$ to be proportional to the block-length $n$ (as in Section 6.6.1), this constraint can be viewed as an average power constraint. Each transmission block consists of $n_T$ training symbols and $n_D$ data symbols, respectively.

At the start of each transmission block, $n_T$ training symbols are broadcast (using total training energy $E_T$) which the mobile receivers use to form estimates of the downlink channel. The remaining $n_D = n - n_T$ symbols in the transmission block are used for data transmission using all remaining energy $E_D = E_{\text{max}} - E_T$, following the system model as previously described in Chapter 2.

Due to the block fading assumption, the channel $H$ does not vary during the $n_T$ training symbols and $n_D$ data transmissions. We are able to simplify our design by choosing a single precoder/decoder pair to be used for all data transmissions in the block. Furthermore, the available energy should be divided equally over the $n_D$ data transmissions, resulting in a maximum transmit power per data time slot $P_D = \frac{E_{\text{max}} - E_T}{n_D}$.

---

1 Under linear estimation, $n_T \geq M$ training symbol vectors must be sent to guarantee resolvability of the individual channel coefficients. [81]

2 Both the minimum sum-MSE and PMSE are optimized by using all available transmit data power; the proof can be found in Appendix E.
6.1.2 Chapter Overview

In this chapter, we introduce channel estimation by broadcasting pilot/training symbols during a training phase, as illustrated in Figure 6.1. As such, a new aspect of optimization is introduced: the MSE-optimal allocation of energy to training and data symbols within a block transmission. We address the problem of jointly designing a training sequence for MMSE CSI estimation, designing linear transceivers for minimum sum-MSE and minimum PMSE communication, and optimizing the allocation of the limited available energy between the training and data communication phases. Section 6.2 presents a survey of recent literature on robust precoding in the presence of imperfect CSI and MMSE channel estimation. In Section 6.3, we extend the MIMO MSE duality of Chapter 5 to the case of imperfect CSI, and in Section 6.4, we reformulate the MSE expressions in the virtual uplink as a geometric program. Section 6.5 formulates a lower bound on sum-rate under scalar and joint decoding as an extension of the results of Chapter 4. Section 6.6 formulates the problem of jointly optimizing the training/data power allocation and the precoder design for sum-MSE and PMSE minimization under equal estimation error variances for each user, demonstrates separability of the two problems, and presents an elegant closed form solution. Section 6.7 addresses the same problem in the presence of unequal estimation error variances, and proposes several algorithmic solutions which include a theoretically near-optimal but practically infeasible formulation and a practical heuristic implementation.

6.2 Literature Review

The available literature in the area of robust precoder design is fairly limited. Recent studies [82–85] extend the sum-MSE minimizing precoder designs (under a sum-power constraint) like the ones proposed in Chapter 3 and in [19, 31, 38, 39] to the case of imperfect CSI, by adapting the uplink-downlink MSE duality described in Section 3.2.
to account for channel estimation errors (see Section 6.3). This extended duality is used to design robust transceivers, under the assumption that fixed and known channel estimation error variances $\sigma_k^2$ are provided by a predetermined estimation mechanism.

### 6.2.1 System Model with Estimation Error

In [82–85], a modified channel model is considered (as compared to the model described in Chapter 2) based on stochastic errors. User $k$’s channel is modelled as Rayleigh fading, with i.i.d. complex coefficients having variance $\sigma_{H_k}^2$. The channel is represented as a sum of the channel estimate $\hat{H}_k$ and an independent additive zero-mean white Gaussian error term $E_k$,

$$H_k = \hat{H}_k + E_k,$$

(6.1)

where $\text{vec}[E_k] \sim \mathcal{CN}(0, \sigma_k^2 I_{MN_k})$. We describe a training-based mechanism for MMSE channel estimation in Section 6.2.2, where we calculate the estimation error variance $\sigma_k^2$. These imperfect channel estimates are assumed to be available at the base station via an error-free and delay-free feedback mechanism and are used to design the precoders for data transmission.

### 6.2.2 MMSE Channel Estimation and Training

A study of various training-based MIMO estimation techniques is presented in [81]; in this chapter, with MSE-based criteria as our objective, we focus on MMSE channel estimation.

Training sequence and estimator design can be simplified under the assumption of uncorrelated channel coefficients by considering training for vector channels from the $M$ transmit antennas to each individual receive antenna. Without loss of generality, we simplify notation in this section by considering training for a single (MISO) vector channel $h^H$, whose complex entries are zero mean and have magnitudes with variance
\( \mathbb{E}[|h_i|^2] = \sigma_H^2 \).

Channel estimation is performed by broadcasting a set of \( n_T \) length-\( M \) training signal vectors, \( \mathbf{x}_{T,1}, \ldots, \mathbf{x}_{T,n_T} \), from the \( M \) transmit antennas without any precoding. (This can be represented using the system model in Figure 2.2 with \( \bar{U} = P = I_M \)). For convenience, we concatenate the training vectors as

\[
\mathbf{X}_T = [\mathbf{x}_{T,1}, \ldots, \mathbf{x}_{T,n_T}],
\]

with

\[
\text{tr} [\mathbf{X}_T^H \mathbf{X}_T] \leq E_T. \tag{6.3}
\]

The received length-\( n_T \) training signal vector is modelled as

\[
\mathbf{y}_T = \mathbf{h}^H \mathbf{X}_T + \mathbf{z} \tag{6.4}
\]

where \( \mathbf{z} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_{n_T}) \), and the MMSE channel estimate

\[
\hat{\mathbf{h}}_{\text{MMSE}}^H = \mathbf{y}_T \mathbf{A}_0 \tag{6.5}
\]

is found using the linear MMSE estimator

\[
\mathbf{A}_0 = \left( \mathbf{X}_T^H \mathbf{X}_T + \frac{\sigma_n^2}{\sigma_H^2} \mathbf{I}_{n_T} \right)^{-1} \mathbf{X}_T^H. \tag{6.6}
\]

Under the training energy constraint and the assumption of independent channel coefficients, a sufficient condition for optimality of the training matrix is derived in [81][86]:

\[
\mathbf{X}_T \mathbf{X}_T^H = \frac{E_T}{M} \mathbf{I}_M = P_T \mathbf{I}_M; \tag{6.7}
\]

where we have defined \( P_T = \frac{E_T}{M} \) as the average power per training symbol.

In other words, we are free to choose any training matrix with orthogonal rows, assign equal training power \( P_T \) to each transmitted training symbol, and maintain optimality in the MMSE sense. When no other constraints are present, there is no benefit (in the sense of reducing the estimation MSE) to using any more than \( n_T = M \) training symbols.
under MMSE channel estimation. For the rest of this chapter, we use \( n_T = M \). For algorithmic simplicity, we use a scaled identity matrix as the set of training vectors (i.e., training each transmit antenna in sequence), as 

\[
X_T = \sqrt{\frac{E_T}{M}} I_M. \tag{6.8}
\]

### MMSE Channel Estimation Error

Using (6.8), the minimum MSE matrix for the estimation of \( h \) can be written by taking the expectation over the true channel vector \( h \) and AWGN vector \( z \),

\[
\begin{align*}
\varepsilon_{\text{MMSE,est}} & = \mathbb{E}_{h,z} \left[ (\hat{h}_{\text{MMSE}} - h) (\hat{h}_{\text{MMSE}} - h)^H \right] \\
& = \sigma_H^2 \left[ A_0^H \left( X_T^H X_T + \frac{\sigma_n^2}{\sigma_H^2} I_{n_T} \right) A_0 - \left( A_0^H X_T^H + X_T A_0 \right) + I_{n_T} \right] \\
& = \sigma_H^2 \left( I_{n_T} - X_T^H \left( X_T^H X_T + \frac{\sigma_n^2}{\sigma_H^2} I_{n_T} \right)^{-1} X_T \right) \\
& = \sigma_H^2 \left( I_{n_T} + \frac{\sigma_n^2}{\sigma_H^2} X_T^H X_T \right)^{-1} \\
& = \left( \sigma_H^{-2} + \frac{1}{\sigma_n^2 M} \right)^{-1} I_{n_T},
\end{align*}
\]

where we have assumed that \( h \) and \( z \) are independent; this MMSE matrix is diagonal due to the assumption of i.i.d. channel coefficients. The fourth equality follows from application of the matrix inversion lemma \((3.40)\), with \( A = I_{n_T}, B = X_T, C = \frac{\sigma_n^2}{\sigma_H^2} I_{n_T}, \) and \( D = X_T^H \). Since the estimation error \( \hat{h}_{\text{MMSE}} - h \) is a linear combination of random vectors from a multivariate Gaussian distribution with uncorrelated components, it follows that the estimation errors are also independent Gaussian random variables.

We see in (6.9) that for i.i.d. channels whose coefficients have variance \( \sigma_H^2 \), the estimation errors of each channel coefficient are equal under the assumption of i.i.d. channels.

---

3The result is quite different, however, when training and data powers are constrained to be equal. The authors of [86] optimize the training sequence, length, and training/data power allocations in order to maximize a lower bound on the single-user capacity of a fading channel. They show that under an equal power constraint for training and data symbols, the optimal training sequence is a function of both the total SNR and the block-length \( n \); as SNR decreases, the optimal training interval converges to \( n_T = n/2 \).
with variance $\sigma_H^2$, taking the value

$$\sigma_k^2 = \left(\sigma_H^{-2} + \frac{1}{\sigma_n^2} \frac{E_T}{M}\right)^{-1}. \quad (6.10)$$

### 6.2.3 Mean Squared Error and Robust Receiver Design

#### Downlink

The MSE matrix for the source symbols $x_k$ in the downlink can be found in an analogous fashion to (2.11):

$$\varepsilon_{DL}^k = \mathbb{E}_{E_k, \mathbf{n}_k} \left[ (\hat{x}_{DL}^k - x_k) (\hat{x}_{DL}^k - x_k)^H \right]$$

$$= \mathbb{E}_{E_k} \left[ V_k^H \left( \mathbf{H}_k^H \mathbf{U} \mathbf{P}_k^H \mathbf{H}_k + \sigma_n^2 \mathbf{I}_{N_k} \right) V_k - V_k^H \mathbf{H}_k^H \hat{U}_k \sqrt{\mathbf{P}_k} - \sqrt{\mathbf{P}_k} \mathbf{H}_k^H \mathbf{V}_k + \mathbf{I}_{L_k} \right]$$

$$= \mathbb{E}_{E_k} \left[ V_k^H \left( \left( \mathbf{H}_k^H + \mathbf{E}_k^H \right) \mathbf{U} \mathbf{P}_k^H \left( \hat{\mathbf{H}}_k + \mathbf{E}_k \right) \mathbf{V}_k - V_k^H \left( \mathbf{H}_k^H + \mathbf{E}_k^H \right) \hat{U}_k \sqrt{\mathbf{P}_k} \right) \mathbf{V}_k + \mathbf{I}_{L_k} \right]$$

$$= V_k^H \tilde{R}_k V_k - V_k^H \hat{\mathbf{H}}_k^H \hat{U}_k \sqrt{\mathbf{P}_k} - \sqrt{\mathbf{P}_k} \mathbf{H}_k^H \mathbf{V}_k + \mathbf{I}_{L_k}, \quad (6.11)$$

where

$$\tilde{R}_k = \hat{\mathbf{H}}_k^H \mathbf{U} \mathbf{P}_k^H \hat{\mathbf{H}}_k + \sigma_n^2 \mathbf{I}_{N_k} + \mathbb{E}_{E_k} \left[ \mathbf{E}_k^H \mathbf{U} \mathbf{P}_k^H \mathbf{E}_k \right]$$

$$= \hat{\mathbf{H}}_k^H \mathbf{U} \mathbf{P}_k^H \hat{\mathbf{H}}_k + \sigma_n^2 \mathbf{I}_{N_k} + \sigma_k^2 \text{tr} \left[ \mathbf{U} \mathbf{P}_k^H \right] \mathbf{I}_{N_k} \quad (6.12)$$

$$= \hat{\mathbf{H}}_k^H \mathbf{U} \mathbf{P}_k^H \hat{\mathbf{H}}_k + \left( \sigma_n^2 + \sigma_k^2 \|p\|_1 \right) \mathbf{I}_{N_k}$$

is the received signal-plus-interference-plus-effective-noise covariance matrix at user $k$, incorporating the effects of thermal noise (with variance $\sigma_n^2$) as well as the effective noise due to estimation error with variance $\sigma_k^2$. Here, $\|p\|_1$ is the sum-power allocated to all data streams. We have assumed the independence of data symbols, noise, and estimation errors.

The MSE matrix resulting from using the linear MMSE decoder $\mathbf{V}_k^* = \tilde{R}_k^{-1} \hat{\mathbf{H}}_k^H \hat{\mathbf{U}}_k \sqrt{\mathbf{P}_k}$

---

4\(^\text{Note that the linear MMSE decoder is not (necessarily) the MMSE estimator in the case of imperfect CSI, since the inputs and outputs are not jointly Gaussian. The terms $\mathbf{E}_k^H \hat{\mathbf{U}}_j \sqrt{\mathbf{P}_j x_j}$, $j \neq k$ in the received signal expression $y_k$ follow a normal product distribution.}\)
\[ \varepsilon_k^{DL} = I_{L_k} - \sqrt{P_k} \tilde{U}_k^H \tilde{H}_k \tilde{R}_k \tilde{H}_k^{-1} \tilde{H}_k^H \tilde{U}_k \sqrt{P_k}. \] (6.13)

In order to determine its linear MMSE decoder \( V_k^* \), mobile user \( k \) requires knowledge of the received signal-plus-interference-plus-noise covariance matrix \( \tilde{R}_k \) and the product of the estimated channel with its precoder, \( \tilde{H}_k^H \tilde{U}_k \sqrt{P_k} \). We assume that the mobile user has perfect knowledge of these matrices. As in the case of perfect CSI (Section 2.1.4), practical implementations require these quantities to either be communicated to the receivers or estimated; the latter would require an additional dedicated training stage following channel estimation and feedback. A detailed treatment of this topic has been addressed in [87, 88] for zero-forcing based receive beamforming.

**Virtual Uplink**

The MSE matrix for user \( k \) in the virtual uplink can be derived in a similar manner as (2.19) (i.e., for the perfect CSI case), while taking estimation error into account:

\[ \varepsilon_k^{UL} = E_{E,x,n} \left[ \left( x_k^{UL} - x_k \right) \left( x_k^{UL} - x_k \right)^H \right] \]
\[ = E_{E} \left[ U_k^H \left( \tilde{H}^H V Q \tilde{V}^H \tilde{H}^H + \sigma_n^2 I \right) U_k - U_k^H \tilde{H}_k \tilde{V}_k \sqrt{Q_k} - \sqrt{Q_k} \tilde{V}_k^H \tilde{H}_k^H U_k + I_{L_k} \right] \]
\[ = E_{E} \left[ U_k^H \left( \left( \tilde{H}^H + E^H \right) V Q \tilde{V}^H \left( \tilde{H}^H + E^H \right) + \sigma_n^2 I \right) U_k - U_k^H \left( \tilde{H}_k + E_k \right) \tilde{V}_k \sqrt{Q_k} - \sqrt{Q_k} \tilde{V}_k^H \left( \tilde{H}_k^H + E_k^H \right) U_k + I_{L_k} \right] \]
\[ = U_k^H \tilde{R} U_k - U_k^H \tilde{H}_k \tilde{V}_k \sqrt{Q_k} - \sqrt{Q_k} \tilde{V}_k^H \tilde{H}_k^H U_k + I_{L_k}, \]

(6.14)

where \( \tilde{R} = \tilde{H} V Q \tilde{V}^H \tilde{H} + \sigma_{\text{eff}}^2 I_M \) is the received signal-plus-interference-plus-effective-noise covariance matrix. We have defined the effective noise power under the general
model with (possibly) different estimation error variances $\sigma_k^2$ for each user $k$ as

$$\sigma_{\text{eff}}^2 = \sigma_n^2 + \sum_{k=1}^{K} \sigma_k^2 \text{tr} \left[ \hat{V}_k Q_k \hat{V}_k^H \right]$$

$$= \sigma_n^2 + \sum_{k=1}^{K} \sigma_k^2 \left( \sum_{l=1}^{L} q_{kl} \right)$$

$$= \sigma_n^2 + \sum_{k=1}^{K} \sigma_k^2 \|q_k\|_1,$$  

(6.15)

which follows from the fact that the beamforming vectors $\bar{v}_l$ have unit norm. Here, $\|q_k\|_1$ is the sum of powers allocated to user $k$’s data streams.

As in the downlink, this expression contains contributions from thermal noise (with variance $\sigma_n^2$) and terms originating from estimation error terms with variances $\sigma_k^2$; it is derived under the same assumptions of independent data symbols, noise, and estimation errors. The optimum robust virtual uplink receiver for user $k$ is found using the MMSE filter,

$$U_k^* = \sqrt{Q_k} \bar{V}_k^H \hat{H}_k^H \hat{R}^{-1} \hat{H}_k \bar{V}_k Q_k \bar{V}_k^H = U_k^H = \sqrt{Q_k} \bar{V}_k^H \hat{H}_k^H \hat{R}^{-1} \hat{H}_k \bar{V}_k Q_k \bar{V}_k^H.$$  

(6.16)

The virtual uplink MMSE matrix (6.14) is thus

$$\varepsilon_{k, UL} = \sqrt{Q_k} \bar{V}_k^H \hat{H}_k^H \hat{R}^{-1} \hat{H}_k \bar{V}_k Q_k \bar{V}_k^H = \sqrt{Q_k} \bar{V}_k^H \hat{H}_k^H \hat{R}^{-1} \hat{H}_k \bar{V}_k Q_k \bar{V}_k^H.$$  

(6.17)

### 6.2.4 Robust Minimum Sum-MSE Precoder Design

With the MMSE matrix (6.17), the resulting minimum sum-MSE is

$$\text{SMSE}^{UL} = \sum_{k=1}^{K} \text{tr} \left[ \varepsilon_{k, UL} \right] = \sum_{k=1}^{K} L_k - \text{tr} \left[ \hat{R}^{-1} \sum_{k=1}^{K} \hat{H}_k \bar{V}_k Q_k \bar{V}_k^H \hat{H}_k^H \hat{R}^{-1} \hat{H}_k \bar{V}_k Q_k \bar{V}_k^H \right]$$

$$= L - M + \sigma_{\text{eff}}^2 \text{tr} \left[ \hat{R}^{-1} \right]$$

(6.18)

which follows from $\text{tr} \left[ AB \right] = \text{tr} \left[ BA \right]$, linearity of the trace operator, and the definition of $\hat{R}$. Under a sum-power constraint with a maximum transmit power of $P_D$, the (possibly) non-convex virtual uplink sum-MSE minimization power allocation subproblem can be
formally defined as

\[ Q^* = \arg \min_Q \left( \sigma_n^2 + \sum_{k=1}^{K} \sigma_k^2 \|q_k\|_1 \right) \text{tr} \left[ \hat{R}^{-1} \right] \]

s.t. \[ q_{kl} \geq 0 \quad k = 1, \ldots, K; \quad l = 1, \ldots, L_k, \]
\[ \text{tr} \left[ Q \right] \leq P_D. \] (6.19)

### 6.3 Extending MSE Duality to Imperfect CSI

In this section, we develop a per-stream MSE duality for the case of MIMO multi-stream communications. This duality result was derived concurrently to, and independently from, [85]; they arrive at the same result using a slightly different approach.

We see from (6.17) that the form of the virtual uplink MMSE with estimation error is nearly identical to (2.22) as found under perfect CSI. The algorithms proposed in this chapter make use of the virtual uplink to extend the sum-MSE minimizing and PMSE maximizing designs of Chapters 3 and 4 and to thus solve simpler problems than the original downlink formulations. In order to apply the resulting designs to the MIMO downlink, we develop the MSE uplink-downlink duality for the case of imperfect CSI.

First, we introduce three lemmas that will be used in the proof:

**Lemma 6.1** (Invertibility of strictly diagonally dominant matrices). A strictly diagonally dominant matrix \( X \), i.e., one that satisfies the property

\[ |[X]_{ii}| - \sum_{j \neq i} |[X]_{ij}| > 0 \quad \forall i, \] (6.20)

is non-singular.

**Proof.** See [35].

**Lemma 6.2** (Product of non-singular matrices). The product of two non-singular matrices \( A \in \mathbb{C}^{L \times L} \) and \( B \in \mathbb{C}^{L \times L} \) is also non-singular.
Proof. If $A$ and $B$ are non-singular, then $\det [A] \neq 0$ and $\det [B] \neq 0$. Consequently,

$$\det [AB] = \det [A] \det [B] \neq 0.$$ (6.21)

In other words, Lemma 6.2 claims that the set of non-singular matrices in $\mathbb{C}^{L \times L}$ forms a group under matrix multiplication. The group axioms are trivially satisfied: matrix multiplication is associative, the identity element is $I_L$, and the inverse element is the matrix inverse which exists by definition for non-singular matrices.

Lemma 6.3 (Inverse of the transpose). The inverse of the transpose of an invertible matrix $A$ is the transpose of its inverse; i.e., $[A^T]^{-1} = [A^{-1}]^T$.

Proof.

$$I = [AA^{-1}]^T = [A^{-1}]^T A^T = I.$$ (6.22)

Theorem 6.1 (Uplink-Downlink MSE Duality for Imperfect CSI). Given an arbitrary set of normalized transmit beamforming vectors $\bar{v}_1, \ldots, \bar{v}_L$, power allocations $q_1, \ldots, q_L$, and MMSE receive beamformers $u\star_1, \ldots, u\star_L$ resulting in virtual uplink MSE values $\varepsilon_{UL}^l$, the same MSE values $\varepsilon_{DL}^l = \varepsilon_{UL}^l$ can be achieved in the downlink using normalized transmit beamformers $\bar{u}_l = u\star_l / \|u\star_l\|$ and power allocations

$$p = \sigma_n^2 \text{diag} [\Phi^{-1} 1_L] q.$$ (6.23)

where

$$[\Phi]_{lj} = \begin{cases} \sigma_n^2 + \sum_{i \neq l} q_i u\star_i^H (\bar{h}_i \bar{h}_i^H + \sigma_{k(l)}^2 I_M) u\star_l & l = j \\ -q_j u\star_j^H (\bar{h}_j \bar{h}_j^H + \sigma_{k(l)}^2 I_M) u\star_l & j \neq l \end{cases}.$$ (6.24)

Proof. In [31], the authors derive the MSE duality for each data stream in the MIMO downlink and virtual uplink by direct manipulation of the MSE expressions (rather than
using the SINR expressions as in Chapter 3. In this proof, we extend the MIMO MSE duality using the approach of [31] to the case of imperfect CSI.

In [31], the system model is modified to normalize the receiver decoder matrices in both the downlink and the virtual uplink by factoring them as

\[ U_k^H = Q_k^{-1/2} \beta_k \bar{U}_k^H \]
\[ V_k^H = P_k^{-1/2} \beta_k \bar{V}_k^H, \]

where \( \beta_k, Q_k^{-1/2}, \) and \( P_k^{-1/2} \) are diagonal matrices containing real scalars \( \beta_l, q_l^{-1/2}, \) and \( p_l^{-1/2} \) for each data stream. These modified system models are illustrated in Figures 6.2 and 6.3 for the virtual uplink and downlink.

The changes to the system model result in the following expression for the estimated symbol for virtual user \( l \):

\[
\hat{x}_{UL}^L = q_l^{-1/2} \beta_l \hat{u}_l^H \left( \sum_{k=1}^K H_k \bar{V}_k \sqrt{Q_k} x_k + n \right) = \beta_l \hat{u}_l^H H_{\kappa(l)} \tilde{v}_l x_l + \sum_{i \neq l} \sqrt{q_i q_l^{-1}} \beta_i \hat{u}_i^H H_{\kappa(i)} \tilde{v}_i x_i + q_l^{-1/2} \beta_l \hat{u}_l^H n. \]
The corresponding MSE for data stream $l$ can be expressed using the effective channel notation from Chapter 3:

\[
\varepsilon_{UL}^l = \mathbb{E}_{E, x, n} \left[ (\hat{x}_{UL}^l - x_i) (\hat{x}_{UL}^l - x_i)^* \right] = \beta_l^2 q_l^{-1} \bar{u}^H \tilde{R} \bar{u} - \beta_l \bar{u}^H \bar{H}_{\kappa(l)} \bar{v}_l - \beta_l \bar{v}_l^H \bar{H}_{\kappa(l)} \bar{u}_l + 1
\]

\[
= \beta_l^2 q_l^{-1} \bar{u}^H \left[ \sigma_n^2 I + \sum_{i=1}^L \left( \bar{h}_i \bar{h}_i^H q_i + \sigma_{\kappa(l)}^2 q_i I \right) \right] \bar{u}_l - 2 \beta_l \Re \left[ \bar{u}^H \bar{h}_l \right] + 1
\]

\[
= \beta_l^2 \bar{u}^H \bar{h}_l \bar{h}_l^H \bar{u}_l - 2 \beta_l \Re \left[ \bar{u}^H \bar{h}_l \right] + 1 + \beta_l^2 q_l^{-1} \left( \sum_{i \neq l} \bar{h}_i \bar{h}_i^H q_i \right) \bar{u}_l
\]

\[
+ \beta_l^2 q_l^{-1} \left( \sigma_n^2 + \sum_{i=1}^L \sigma_{\kappa(l)}^2 q_i \right).
\]

(6.27)

The virtual uplink MSE for virtual user $l$ in (6.27) can be rewritten as:

\[
\varepsilon_{UL}^l = \sum_{i=1}^L q_l^{-1} \beta_l^2 \bar{u}^H q_i \bar{h}_l \bar{h}_l^H \bar{u}_l + q_l^{-1} \beta_l^2 \left( \sigma_n^2 + \sum_{i=1}^L \sigma_{\kappa(l)}^2 q_i \right) - 2 \beta_l \Re \left[ \bar{u}^H \bar{h}_l \right] + 1. \quad (6.28)
\]

Following a similar derivation, we find the downlink MSE for virtual user $l$ using the factored receivers $\bar{v}_l^H = p_l^{-1/2} \beta_l \bar{v}_l^H$ as introduced in Figure 6.3. The estimated symbol for virtual user $l$ can be written as

\[
\hat{x}_{DL}^l = p_l^{-1/2} \beta_l \bar{v}_l^H \left( \bar{H}_{\kappa(l)}^H \bar{U} \sqrt{\bar{P}} \bar{x} + \bar{n}_{\kappa(l)} \right)
\]

\[
= \beta_l \bar{v}_l^H \bar{H}_{\kappa(l)} \bar{u}_l x_i + \sum_{i \neq l} \sqrt{\frac{p_l}{p_i}} \beta_l \bar{v}_l^H \bar{H}_{\kappa(l)} \bar{u}_i x_i + p_l^{-1/2} \beta_l \bar{v}_l^H \bar{n}_{\kappa(l)}. \quad (6.29)
\]

Modifying (6.11) to include the $\beta_l$ terms results in the following set of MSEs for each data stream $l$:

\[
\varepsilon_{DL}^l = \mathbb{E}_{E_{\kappa(l)}, x, n_{\kappa(l)}} \left[ (\hat{x}_{DL}^l - x_i) (\hat{x}_{DL}^l - x_i)^* \right] = \beta_l^2 p_l^{-1} \bar{v}_l^H \bar{H}_{\kappa(l)} \bar{v}_l - \beta_l \bar{v}_l^H \bar{H}_{\kappa(l)} \bar{v}_l + 1
\]

\[
= \beta_l^2 p_l^{-1} \bar{v}_l^H \left( \bar{H}_{\kappa(l)}^H \bar{U} \bar{P} \bar{U}^H \bar{H}_{\kappa(l)} + \left( \sigma_n^2 + \sum_{i=1}^L p_i \right) I \right) \bar{v}_l - 2 \beta_l \Re \left[ \bar{u}^H \bar{h}_l \right] + 1
\]

\[
= \sum_{i=1}^L \beta_l^2 p_l^{-1} \bar{h}_i \bar{h}_i^H p_i \bar{u}_i \bar{u}_i^H + p_l^{-1} \beta_l^2 \left( \sigma_n^2 + \sum_{i=1}^L \sigma_{\kappa(l)}^2 p_i \right) - 2 \beta_l \Re \left[ \bar{u}^H \bar{h}_l \right] + 1.
\]

(6.30)
In order to derive a duality based purely on the equivalence of the MSE terms $\varepsilon_i^{DL}$ and $\varepsilon_i^{UL}$, we let $p_l = \omega_l q_l$ and equate (6.28) and (6.30), yielding

$$q_l^{-1} \beta^2_l \left[ \sigma_n^2 + \sum_{i=1}^{L} q_i \left( \bar{u}_i^H \tilde{h}_i \tilde{h}_i^H \bar{u}_i + \sigma_n^2 \right) \right] = \omega_l^{-1} q_l^{-1} \beta^2_l \left[ \sigma_n^2 + \sum_{i=1}^{L} \omega_i q_i \left( \bar{u}_i^H \tilde{h}_i \tilde{h}_i^H \bar{u}_i + \sigma_n^2 \right) \right]$$

or equivalently,

$$\omega_l \left[ \sum_{i \neq l} q_i \left( \bar{u}_i^H \tilde{h}_i \tilde{h}_i^H \bar{u}_l + \sigma_n^2 \right) + \sigma_n^2 \right] - \sum_{i \neq l} \omega_i q_i \left( \bar{u}_i^H \tilde{h}_i \tilde{h}_i^H \bar{u}_i + \sigma_n^2 \right) = \sigma_n^2. \quad (6.32)$$

Combining the set of $L$ equalities for each of the virtual users' MSE terms forms the set of linear equations $\Phi \omega = \sigma_n^2 1_L$, where $\omega = [\omega_1, \omega_2, \ldots, \omega_L]^T$, and

$$[\Phi]_{lj} = \begin{cases} 
\sigma_n^2 + \sum_{i \neq l} q_i \bar{u}_i^H \left( \tilde{h}_i \tilde{h}_i^H + \sigma_n^2 I_M \right) \bar{u}_l & l = j \\
- q_j \bar{u}_j^H \left( \tilde{h}_j \tilde{h}_j^H + \sigma_n^2 I_M \right) \bar{u}_j & l \neq j 
\end{cases}. \quad (6.33)$$

We are able to find $\omega$ that satisfies this set of equalities (and thus provide an equivalent set of downlink powers achieving the same MSEs $\varepsilon_i^{DL}$) as long as $\Phi$ is non-singular.

For convenience, we define the terms

$$G_{jl} = \bar{u}_j^H \left( \tilde{h}_j \tilde{h}_j^H + \sigma_n^2 I_M \right) \bar{u}_j, \quad (6.34)$$

and rewrite (6.33) as

$$[\Phi]_{lj} = \begin{cases} 
\sigma_n^2 + \sum_{i \neq l} q_i G_{li} & l = j \\
-q_j G_{jl} & l \neq j 
\end{cases}. \quad (6.35)$$

Now, consider the matrix $\Psi = \Phi^T$,

$$[\Psi]_{lj} = \begin{cases} 
q \sigma_n^2 + \sum_{i \neq l} q_i G_{li} & l = j \\
-q_j G_{lj} & l \neq j 
\end{cases}. \quad (6.36)$$

Evaluating the expression in Lemma 6.1 yields

$$|[\Psi]_{ll}| - \sum_{j \neq l} |[\Psi]_{lj}| = q \sigma_n^2 + \sum_{i \neq l} q_i G_{li} - \sum_{j \neq l} q_j G_{lj}$$

$$= q \sigma_n^2 > 0. \quad (6.37)$$
Under Lemma 6.1, we can conclude that $\Psi$ is non-singular. Under the assumption of non-zero powers $q_l$, $Q$ is invertible, and we can express $\Phi$ as $\Phi = \Psi^T Q^{-1}$. Since $\Psi$ and its transpose $\Psi^T$ are non-singular, $\Phi$ is invertible under Lemma 6.2.

To complete the proof of duality, we show that the sum-powers in the downlink and virtual uplink are identical (i.e., $\|p\|_1 = \|q\|_1$), using the techniques applied in [31]. By defining matrices $D$, $\Gamma$, and $\beta$ as

$$
[D]_{lj} = \begin{cases} 
\beta_l^2 G_{ll} - 2 \beta_l \Re \left[ \bar{u}_l^H \tilde{h}_l \right] + 1 & l = j \\
0 & l \neq j 
\end{cases},
$$

$$
[\Gamma]_{lj} = \begin{cases} 
0 & l = j \\
G_{lj} & l \neq j 
\end{cases},
$$

$$
\beta = \text{diag}[\beta_1, \ldots, \beta_L],
$$

we can solve the set of equations (6.28) and (6.30) to find the downlink and virtual uplink power allocations $p$ and $q$ corresponding to any set of achievable MSEs $\varepsilon = \text{diag}[\varepsilon_1, \ldots, \varepsilon_L]$ as

$$
p = \sigma_n^2 (\varepsilon - D - \beta^2 \Gamma)^{-1} \beta^2 1_L \\
q = \sigma_n^2 (\varepsilon - D - \beta^2 \Gamma^T)^{-1} \beta^2 1_L.
$$

(6.39)

It follows that the sum-power used in the downlink can be expressed as

$$
\|p\|_1 = p^T 1_L \\
= \left[ \sigma_n^2 (\varepsilon - D - \beta^2 \Gamma)^{-1} \beta^2 1_L \right]^T 1_L \\
= \left[ \sigma_n^2 (\beta^{-2} \varepsilon - \beta^{-2} D - \Gamma)^{-1} 1_L \right]^T 1_L \\
= \sigma_n^2 1_L^T \left[ (\beta^{-2} \varepsilon - \beta^{-2} D - \Gamma)^{-1} \right]^T 1_L \\
= \sigma_n^2 1_L^T (\beta^{-2} \varepsilon - \beta^{-2} D - \Gamma^T)^{-1} 1_L \\
= 1_L^T \sigma_n^2 (\varepsilon - D - \beta^2 \Gamma^T)^{-1} \beta^2 1_L \\
= 1_L^T q = \|q\|_1,
$$

(6.40)
which follows from the application of Lemma 6.3. The sum-powers allocated in the downlink and virtual uplink for the same set of achievable MSEs $\varepsilon$ are thus equal, concluding the proof of Theorem 6.1.

Note that in this proof, we assume that all virtual users have non-zero power allocations; in the case where any virtual uplink power allocations $q_l = 0$, the same result applies by omitting those stream indices from the formulation and assigning the trivial downlink power allocation $p_l = 0$ to the corresponding data streams.

With the MSE duality, we are able to design precoders using MSE-based optimization algorithms based in the virtual uplink. The main challenge, as seen in Chapters 3 and 4, is finding a tractable formulation for the power allocation subproblem in the virtual uplink. In the following section, we proceed to demonstrate that even in the case of unequal estimation error variances $\sigma_k^2$, the power allocation subproblems for both the sum-MSE and the PMSE minimization problems are geometric programs.

6.4 Precoder Design and Geometric Programming

As discussed in Section 4.2.3, the power allocation subproblem in the virtual uplink for the perfect CSI case is a geometric program. In this section, we discuss the extension of a GP solution for the case of imperfect CSI, and describe how it is also applicable to sum-MSE optimization. A brief explanation on the formulation and solution of GPs can be found in Appendix C.

In order to transform the problem of PMSE minimization to GP form, we take advantage of the virtual uplink MSE expression (6.27), which we repeat here for clarity:

$$\varepsilon_{UL} = \beta_i^2 \bar{u}_l^H \hat{h}_l \hat{h}_l^H \bar{u}_l - 2\beta_i \Re \left[ \bar{u}_l^H \bar{h}_l \right] + 1 + \beta_i q_l^{-1} \bar{u}_l^H \left( \sum_{i \neq l} \hat{h}_i \hat{h}_i^H q_i \right) \bar{u}_l$$

$$+ \beta_i q_l^{-1} \bar{u}_l^H \left( \sigma_n^2 + \sum_{i=1}^L \sigma_{\kappa(i)}^2 q_i \right) \bar{u}_l. \quad (6.41)$$
Let $z = \beta_l \tilde{u}_l^H \tilde{h}_l$. We observe that the sum of the first three terms in (6.41) can be expressed as

$$zz^* - 2\Re\{z\} + 1 = zz^* - (z + z^*) + 1 = |z - 1|^2. \quad (6.42)$$

Thus, we can rewrite (6.41) as

$$\varepsilon_l = |\beta_l \tilde{u}_l^H \tilde{h}_l - 1|^2 + \beta_l^2 q_l^{-1} \tilde{u}_l^H \left( \left( \sigma_n^2 + \sigma_{\kappa(l)}^2 q_l \right) \mathbf{I}_M + \sum_{i \neq l} q_i \left( \tilde{h}_i \tilde{h}_i^H + \sigma_{\kappa(i)}^2 \mathbf{I}_M \right) \right) \tilde{u}_l. \quad (6.43)$$

This expression is a posynomial function (see Appendix C) of the set of virtual uplink power allocations $q_i$. As a result, all additive and multiplicative positive scaled combinations of $\varepsilon_l$ are posynomials; of particular interest in this class of functions are both the (weighted) sum-MSE and PMSE. Maximizing the weighted sum-rate,

$$\text{WSR} = \sum_{l=1}^{L} w_l \log (1 + \text{SINR}_l) = -\log \left( \prod_{l=1}^{L} \varepsilon_l^{w_l} \right), \quad (6.44)$$

can be achieved by minimizing the product of exponentially weighted posynomials

$$f(\varepsilon_l) = \prod_{l=1}^{L} \varepsilon_l^{w_l}. \quad (6.45)$$

While $f(\varepsilon_l)$ is not a posynomial for fractional weights $w_l$, it belongs to the class of generalized posynomials that give rise to generalized geometric programs (GGP) which have GP-equivalent formulations (see example in Appendix C.4).

It follows that the optimal power allocations that minimize the GP-based problems listed above can be found efficiently by solving the associated geometric programs.

### 6.5 Imperfect CSI and Minimum PMSE Design

We now consider a solution for the problem of rate maximization in the presence of imperfect CSI estimates. In this section, we extend the PMSE minimizing precoder design of Chapter 4 to the case of imperfect CSI by finding a lower bound on achievable
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sum rate as a function of the MMSE. We use the fact that in the single-user Gaussian MIMO channel

\[ y = Hx + n, \]  

(6.46)

the following lower bound for achievable rate is derived [89]:

\[
I(x; y|\hat{H}) \geq \log \det \left[ I + \frac{1}{1 + \sigma_e^2 P \hat{H}^H \hat{H} S_x} \right].
\]  

(6.47)

Here, \( \hat{H} \) is the channel estimate, \( \sigma_e^2 \) is the channel estimation error variance, \( S_x = \mathbb{E}[xx^H] \) is the channel input covariance matrix, and \( P = \text{tr}[S_x] \) is the average input power constraint.

Consider the transmitted symbol vector for user \( k \) defined as

\[ s_k = \bar{U}_k \sqrt{P_k} x_k \]  

(6.48)

which is a zero-mean Gaussian random vector with covariance matrix

\[ S_k = \mathbb{E}[s_k s_k^H] = \bar{U}_k P_k \bar{U}_k^H \]  

(6.49)

when \( x_k \sim \mathcal{CN}(0_{L_k}, I_{L_k}) \). The mutual information between the transmitted symbols \( s_k \) and the received symbols \( y_k \) at user \( k \), conditioned on knowledge of the channel estimate \( \hat{H}_k \) is

\[
I(s_k; y_k|\hat{H}_k) = h(s_k|\hat{H}_k) - h(s_k|y_k, \hat{H}_k).
\]  

(6.50)

Choosing \( s_k|\hat{H}_k \) as zero-mean Gaussian with covariance matrix \( S_k \) is not necessarily optimal when CSI estimates are imperfect [89], but it allows us to find a tractable lower bound on mutual information. The differential entropy of the first term is

\[
h(s_k|\hat{H}_k) = \log \det [\pi e S_k].
\]  

(6.51)

An upper bound on the second term in (6.50) can be found by extending the procedure applied to the single-user case in [86]; there, it is shown that when signal and noise
are uncorrelated, the worst-case additive noise is zero-mean Gaussian distributed with covariance matrix equal to the MSE matrix of the MMSE estimate of the signal term. Consider rewriting the received signal \( y_k \) as

\[
y_k = H_k^H \sum_{i=1}^{K} s_i + n_k
= \hat{H}_k^H s_k + E_k s_k + H_k \sum_{i \neq k} s_i + n_k.
\]

(6.52)

The signal term \( s_k \) and “noise” term \( z_k \) (consisting of the signal \( s_k \) projected along the direction of the estimation error matrix \( E_k \), interference, and thermal noise) are uncorrelated when the MMSE estimator \( \hat{H}_k \) is used, since the MMSE estimator satisfies the orthogonality principle,

\[
E \left[ \hat{H}_k E_k^H \right] = 0.
\]

(6.53)

The covariance matrix of \( z_k \) is the MSE matrix of the linear MMSE estimate of \( s_k \) given knowledge of the channel estimate \( \hat{H}_k \). The MSE matrix derived in (6.13) is for the estimate of the source symbols \( x_k \); the corresponding MMSE matrix for estimated symbols \( \hat{s}_k^{DL} = \hat{U}_k \sqrt{P_k} \hat{x}_k^{DL} \) is

\[
\hat{\epsilon}_k^{*DL} = E_{E_k, x, n_k} \left[ \left( \hat{s}_k^{DL} - s_k \right) \left( \hat{s}_k^{DL} - s_k \right)^H \right]
= \hat{U}_k \sqrt{P_k} \epsilon_k^{*DL} \sqrt{P_k} \hat{U}_k^H
= S_k - S_k \hat{H}_k \hat{R}_k^{-1} \hat{H}_k^H S_k.
\]

(6.54)

It follows from the worst-case noise theorem of [86] that

\[
h(s_k | y_k, \hat{H}_k) \leq \log \det \left[ \pi e S_k \left( I_{L_k} - \hat{H}_k \hat{R}_k^{-1} \hat{H}_k^H S_k \right) \right],
\]

(6.55)

\footnote{This was a new result presented in [86], in contrast to the traditional literature which considered independent signal and noise terms.}
Chapter 6. Imperfect CSI: Energy Allocation and Precoder Design

and thus

\[ I(s_k; y_k | \hat{H}_k) \geq \log \det [\pi e S_k] - \log \det \left[ \pi e S_k \left( I_{L_k} - \hat{H}_k \hat{R}_k^{-1} \hat{H}_k^H S_k \right) \right] \]

\[ = - \log \det \left[ I_{L_k} - \hat{H}_k \hat{R}_k^{-1} \hat{H}_k^H S_k \right] \]

\[ = - \log \det \left[ I_{L_k} - \sqrt{P_k} \tilde{U}_k^H \hat{H}_k \hat{R}_k^{-1} \hat{H}_k^H \tilde{U}_k \sqrt{P_k} \right] \]

\[ = - \log \det \varepsilon_{DL}^* \]. (6.56)

Since the determinant of the MSE matrix provides a lower bound on mutual information in the downlink even with imperfect CSI, we could again consider minimizing the product of MSE matrix determinants in order to increase sum-rate. However, as suggested in Chapter 4, minimization of the product of MSE matrix determinants is infeasible from a computational/algorithmic point of view. The lower bound derived above applies equally to the case of scalar processing by considering each data stream as a virtual user \( l \), in which case the mutual information becomes

\[ I(s_l; y_{\kappa(l)} | \hat{H}_{\kappa(l)}) \geq - \log \left( 1 - p_l \hat{u}_l^H \hat{H}_{\kappa(l)} \hat{R}_{\kappa(l)}^{-1} \hat{H}_{\kappa(l)}^H \hat{u}_l \right) \]

\[ = - \log \varepsilon_{DL}^* \]. (6.57)

It follows that a lower bound on the sum-rate can be maximized by minimizing the product of individual stream MSEs; as in Chapter 4 we can perform this in the virtual uplink and transform the resulting precoder/decoder to the downlink using the MIMO extension of the duality derived in Section 6.3. Minimization of the PMSE with channel estimation errors under a sum power constraint can be stated formally as the non-convex optimization problem:

\[
(\hat{V}, Q) = \arg \min_{\hat{V}, Q} \prod_{l=1}^{L} \xi_{UL}^{UL} \\
\text{s.t. } \|\hat{v}_l\|_2 = 1, \quad l = 1, \ldots, L \\
q_l \geq 0, \quad l = 1, \ldots, L, \\
\|q\|_1 = \sum_{l=1}^{L} q_l \leq P_D. \quad \text{(6.58)}
\]

Effective approaches to solve this problem are presented in the next two sections.
6.6 Joint Optimization of Energy and Precoder Design: Equal Estimation Error

The formulations for robust minimum sum-MSE and minimum PMSE optimization presented in Sections 6.2.4 and 6.5 find precoders for an arbitrary choice of $P_D$, $P_T$, and the associated estimation error variances $\sigma^2_k$. We now consider the joint allocation of energy/power to training and data transmission phases in conjunction with precoder design.

When the channel estimation error variances are equal; that is, $\sigma_k^2 = \sigma_e^2 \forall k = 1, \ldots, K$, the effective noise becomes

$$\sigma_{\text{eff}}^2 = \sigma_n^2 + \sigma_e^2 \sum_k \|q_k\|_1$$

(6.59)

The second equality follows from the fact that sum-MSE and PMSE are non-increasing functions of total available power for data transmission (Appendix E); thus, all available power allocated to data transmission will be used in the optimum precoder design. Since the effective noise power is no longer a function of the uplink power allocations $q_{kl}$, the optimization problem (6.19) for sum-MSE minimization becomes convex (the minimization of $\text{tr} \left[ \tilde{R}^{-1} \right]$ under a sum power constraint, as in Chapter 3), and can thus be solved using the algorithm designed for the perfect CSI case in Section 3.3 by substituting the effective noise $\sigma_{\text{eff}}^2$ for the noise term $\sigma_n^2$ in the original design. Similarly, the GP for the PMSE case can be solved as described in Chapter 4 using the result of [65]. Essentially, for the case with equal average estimation errors, the algorithms of Chapters 3 and 4 do not change. In the remainder of this section, we will refer to either of these functions as $f(\varepsilon_{UL}^I)$.

As explained in Section 6.2.1, the optimum strategy for sharing the available data energy $E_D$ over $n_D$ transmitted symbols (for a quasi-static channel which does not change during each transmission block) is to equally divide the available energy over each trans-
mission. Using this strategy, and substituting the estimation error variance from (6.10) into the effective noise variance, the joint optimization problem for the virtual uplink power allocation and training/data energy allocation is

\[
(Q^\star, E_{T}^\star) = \arg \min_{Q, E_{T}} f(\varepsilon_{i}^{UL})
\]

s.t. \[q_{kl} \geq 0 \quad k = 1, \ldots, K; \quad l = 1, \ldots, L_{k},\]

\[\text{tr}[Q] = P_{D}, \quad P_{D} = \frac{E_{\text{max}} - E_{T}}{n_{D}},\]

\[\sigma_{\text{eff}}^{2} = \sigma_{n}^{2} + \frac{P_{D}}{(\sigma_{H}^{-2} + \frac{1}{\sigma_{n}^{2}}) E_{T}}.\] (6.60)

Theorem 6.2. The optimum training energy \(E_{T}^\star\) that minimizes both sum-MSE and PMSE in data symbol transmission under equal MMSE channel estimation error variances can be expressed independently from the choice of precoder as

\[
E_{T}^\star = \begin{cases} 
\frac{E_{\text{max}} - \left(\frac{\sigma_{n}^{2}}{\sigma_{H}^{2}}\right)^{\frac{1}{2}}}{\sqrt{Mn_{D} + M}} & E_{T} > \frac{\sigma_{n}^{2}}{\sigma_{H}^{2}}\sqrt{Mn_{D}} \\
0 & \text{otherwise}
\end{cases}
\] (6.61)

Proof. We perform the optimization in terms of the training power \(P_{T} = E_{T}/M\). Using the virtual uplink sum-MSE or PMSE as the objective function, and the energy constraints \(E_{T} \geq 0\) and \(E_{T} \leq E_{\text{max}}\), we derive the Karush-Kuhn-Tucker (KKT) conditions for Problem (6.60) as

\[
\frac{\partial f(\varepsilon_{i}^{UL})}{\partial P_{T}} + \lambda_{\text{max}}M - \lambda_{+} = 0
\] (6.62)

\[P_{T}M \geq 0, \quad P_{T}M \leq E_{\text{max}}\] (6.63)

\[\lambda_{+} \geq 0, \quad \lambda_{\text{max}} \geq 0\] (6.64)

\[\lambda_{+}P_{T}M = 0, \quad \lambda_{\text{max}}(P_{T}M - E_{\text{max}}) = 0,\] (6.65)

where \(\lambda_{+}\) and \(\lambda_{\text{max}}\) are the Lagrange multipliers (dual variables) corresponding to the positivity constraint \(E_{T} \geq 0\) and maximum energy constraint \(E_{T} \leq E_{\text{max}}\), respectively.
We consider only the solutions where the constraints are not binding, as allowing either constraint to hold with equality prevents us from reaching a global minimum for the optimization problem. When $P_T M = 0$, no training symbols are sent, and the resulting channel estimate is $\hat{\mathbf{H}}^H = \mathbf{0}$. If $P_T M = E_{\text{max}}$, zero energy remains for data transmission. In either of these cases, the resulting data symbol estimates are $\hat{x}_k^{UL} = \mathbf{0}$, and no information can be communicated. Since neither constraint is binding, complementary slackness (6.65) requires that $\lambda_{\text{max}} = \lambda_+ = 0$; thus, any minimizer can be found by considering the unconstrained minimization of $f(\varepsilon_i^{UL})$ and checking feasibility of the resulting solutions. We begin by rewriting the effective noise power,

$$\sigma_{\text{eff}}^2 = \sigma_n^2 + \frac{\sigma_n^2}{n_D} \left( \frac{E_{\text{max}} - P_T M}{\rho + P_T} \right),$$

with $\rho = \sigma_n^2/\sigma_H^2$. Define the derivative

$$D_\sigma = \frac{\partial \sigma_{\text{eff}}^2}{\partial P_T} = \frac{-\sigma_n^2 (E_{\text{max}} + \rho M)}{n_D (\rho + P_T)^2}.$$  

(6.67)

We then separate the data power $P_D$ from the uplink power allocation by rewriting $\mathbf{Q} = P_D \bar{\mathbf{Q}}$, with associated normalized sum-power constraint $\text{tr} \left[ \mathbf{Q} \right] \leq 1$. It follows that

$$\varepsilon_k^{UL} = \mathbf{I}_{L_k} - \left( \frac{E_{\text{max}} - P_T M}{n_D} \right) \sqrt{\bar{Q}_k} \hat{\mathbf{V}}_k^H \hat{\mathbf{H}}_k^H \hat{\mathbf{R}}^{-1} \hat{\mathbf{H}}_k \hat{\mathbf{V}}_k \sqrt{\bar{Q}_k},$$

(6.68)

and that

$$\hat{\mathbf{R}} = \left( \frac{E_{\text{max}} - P_T M}{n_D} \right) \hat{\mathbf{H}} \hat{\mathbf{V}} \bar{\mathbf{Q}} \hat{\mathbf{V}}^H \hat{\mathbf{H}}^H + \sigma_{\text{eff}}^2 \mathbf{I}.$$  

(6.69)

The derivative of $\hat{\mathbf{R}}$ with respect to $P_T$ is simply

$$\frac{\partial \hat{\mathbf{R}}}{\partial P_T} = \frac{-M}{n_D} \hat{\mathbf{H}} \hat{\mathbf{V}} \bar{\mathbf{Q}} \hat{\mathbf{V}}^H \hat{\mathbf{H}}^H + D_\sigma \mathbf{I}.$$  

(6.70)

We differentiate each element of the virtual uplink MMSE matrix $\varepsilon_k^{UL}$ with respect
to the training power $P_T$:

$$
\frac{\partial \varepsilon_{UL}^k}{\partial P_T} = \frac{M}{n_D} \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k V_k \sqrt{\hat{Q}_k}} - P_D \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \frac{\partial \hat{R}_k^{-1}}{\partial P_T} \hat{H}_k V_k \sqrt{\hat{Q}_k}} \\
= \frac{M}{n_D} \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k^{-1} \hat{H}_k V_k \sqrt{\hat{Q}_k}} \\
- P_D \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k^{-1} \left( \frac{M}{n_D} \hat{H}_k \tilde{V}_k \hat{V}_k^{H^H} - D_s \mathbf{I} \right) \hat{R}_k^{-1} \hat{H}_k V_k \sqrt{\hat{Q}_k}} \\
= \frac{M}{n_D} \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k^{-1} \left( \hat{R}_k - P_D \hat{H}_k \tilde{V}_k \hat{V}_k^{H^H} \right) \hat{R}_k^{-1} \hat{H}_k V_k \sqrt{\hat{Q}_k}} \\
+ P_D D_s \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k^{-2} \hat{H}_k V_k \sqrt{\hat{Q}_k}} \\
= \left( \frac{M}{n_D} \sigma^2_{\text{eff}} + P_D D_s \right) \sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k^{-2} \hat{H}_k V_k \sqrt{\hat{Q}_k}},
$$

(6.71)

where we have employed the differential identity $\frac{\partial \hat{R}_k^{-1}}{\partial P_T} = -\hat{R}_k^{-1} \left( \frac{\partial \hat{R}_k}{\partial P_T} \right) \hat{R}_k^{-1}$ as in (A.1). The matrix term $\sqrt{\hat{Q}_k V_k^H \hat{H}_k^H \hat{R}_k^{-2} \hat{H}_k V_k \sqrt{\hat{Q}_k}}$ is positive definite, so the only possible candidate values for $P_T^*$ corresponding to stationary points arise from solving the scalar equation

$$
\frac{M}{n_D} \sigma^2_{\text{eff}} + P_D D_s = 0.
$$

(6.72)

Substituting the definitions of (6.66) and (6.67) result in the following equation,

$$
\frac{M}{n_D} \left[ \sigma^2_n + \sigma^2_{\text{eff}} \left( \frac{E_{\text{max}} - P_T M}{\rho + P_T} \right) \right] = \frac{E_{\text{max}} - P_T M \sigma^2_n (E_{\text{max}} + \rho M)}{n_D (\rho + P_T)^2},
$$

(6.73)

which can be reduced to the following quadratic equation in $P_T$,

$$
P_T^2 (n_D - M) + 2P_T (E_{\text{max}} + \rho n_D) = \frac{E_{\text{max}}^2}{M} - \rho^2 n_D.
$$

(6.74)

The two roots of this quadratic equation are

$$
P_T = \frac{1}{n_D - M} (-E_{\text{max}} - \rho n_D \pm \gamma),
$$

(6.75)

with

$$
\gamma = \sqrt{n_D \left( \rho^2 M + 2\rho E_{\text{max}} + \frac{E_{\text{max}}^2}{M} \right)}
$$

(6.76)
For the most common case of “long” blocks \((n \gg M)\), the number of data symbols is greater than the number of training symbols \((n_D > M)\). In this case, the negative root \((-\gamma)\) results in an infeasible solution with \(P_T < 0\). For the case of short blocks with \(n_D < M\), we can rewrite (6.75) as

\[
P_T = \frac{1}{M - n_D} (E_{\text{max}} + \rho n_D \mp \gamma),
\]

and observe that selecting the \(+\gamma\) term also leads to infeasibility, since

\[
E_T = P_T M = \frac{M}{M - n_D} \left( E_{\text{max}} \left[ 1 + \sqrt{n_D M} \right] + \rho \left( n_D + \sqrt{n_D M} \right) \right) > E_{\text{max}}.
\]

Choosing the positive root in (6.75) results in

\[
P_T^* = E_{\text{max}} \frac{\left( \sqrt{\frac{n_D}{M}} - 1 \right) - \rho n_D \left( 1 - \sqrt{\frac{M}{n_D}} \right)}{n_D - M} \frac{E_{\text{max}} \left( \sqrt{n_D - \sqrt{M}} \right) - \rho n_D \left( \sqrt{n_D - \sqrt{M}} \right)}{\sqrt{n_D} - \sqrt{M}} \left( \sqrt{n_D} + \sqrt{M} \right) = \frac{E_{\text{max}}}{\sqrt{M}} - \frac{\rho \sqrt{n_D}}{\sqrt{n_D} + \sqrt{M}} = \frac{E_{\text{max}} - \rho \sqrt{M n_D}}{\sqrt{M n_D} + M}.
\]

This solution always satisfies \(P_T^* M < E_{\text{max}}\), and is only infeasible (with \(P_T^* < 0\)) if \(E_{\text{max}} < \rho \sqrt{n_D M}\).

For completeness, we consider the case where \(n_D = M\). In this case, the quadratic expression in (6.75) becomes a linear equation with the solution

\[
P_T^* = \frac{E_{\text{max}}^2 - \rho^2 n_D}{2 \left( E_{\text{max}} + \rho n_D \right)} = \frac{E_{\text{max}}^2 - \rho^2 M^2}{2 M \left( E_{\text{max}} + \rho M \right)} = \frac{E_{\text{max}} - \rho M}{2 M},
\]
which is a special case of (6.79) for \( n_D = M \).

We finally prove that the stationary point \( P^*_T \) is indeed a global minimum. We observe that the second derivative of the matrix \( \varepsilon_{UL}^k \) can be written as

\[
\frac{\partial^2 \varepsilon_{UL}^k}{\partial P_T^2} = \left( \frac{M}{n_D} D_\sigma + P_D \frac{\partial D_\sigma}{\partial P_T} - \frac{M}{n_D} D_\sigma \right) \sqrt{Q_k V_k^H H_k^H R_k^{-2} H_k V_k} \sqrt{Q_k},
\]

\[
+ \left( \frac{M}{n_D} \sigma_{\text{eff}}^2 + P_D D_\sigma \right) \frac{\partial}{\partial P_T} \left[ \sqrt{Q_k V_k^H H_k^H R_k^{-2} H_k V_k} \sqrt{Q_k} \right],
\]

Evaluating this expression at the stationary point \( P^*_T \) results in the second term reducing to zero. Thus, we see that

\[
\frac{\partial^2 \varepsilon_{UL}^k}{\partial P_T^2} \bigg|_{P_T = P^*_T} = \left( P_D^* \left. \frac{\partial D_\sigma}{\partial P_T} \right|_{P_T = P^*_T} \right) \sqrt{Q_k V_k^H H_k^H R_k^{-2} H_k V_k} \sqrt{Q_k}
\]

\[
= \frac{2 \sigma_n^2 (E_{\text{max}} + \rho M) P_D^*}{n_D (\rho + P_T^*)^3} \sqrt{Q_k V_k^H H_k^H R_k^{-2} H_k V_k} \sqrt{Q_k}
\]

\[
> 0,
\]

since the first (scalar) expression is positive for all feasible solutions (corresponding to \( P_D^* \neq 0 \) and \( P_T^* \neq 0 \), and since the matrix \( \sqrt{Q_k V_k^H H_k^H R_k^{-2} H_k V_k} \sqrt{Q_k} \) is itself positive definite. Consequently, it follows from linearity of the trace and differential operators that \( P_T^* \) corresponding to (6.61) is a global minimizer for the sum-MSE. It also follows that this choice of \( P_T^* \) minimizes the PMSE \( \prod_{l=1}^L [\varepsilon_{UL}^k]_{l,l} \), since a positive definite matrix has strictly positive entries along its main diagonal; that is, this value of \( P_T^* \) minimizes each of the individual (scalar) MSE terms along the diagonal of the MSE matrix.

\[\square\]

**Corollary 6.1.** The optimization of training/data energy allocation and the optimum precoder design in problem (6.60) are separable problems when estimation error variances are equal. This result can be seen directly in (6.61), as the optimum value, \( E^*_T \), is neither a function of \( \mathbf{V} \) nor \( \mathbf{Q} \).

**Corollary 6.2.** Since optimization of the available energy is separable from precoder design, the sum-MSE minimizing precoder and PMSE minimizing precoder can be designed...
using the algorithms proposed in Table 3.1 and Table 4.1, respectively, by modifying the sum power constraint to \( P_{\text{max}} = P_D^* = (E_{\text{max}} - E_T^*)/n_D \) and by replacing the noise power term \( \sigma_n^2 \) and channel matrices \( H_k \) with the effective noise power \( \sigma_{\text{eff}}^2 = \sigma_n^2 + \sigma_e^2 P_D^* \) and channel estimates \( \hat{H}_k \), respectively. The resulting algorithm is summarized in Table 6.1.

Table 6.1: Sum-MSE/PMSE minimization algorithm with joint training/data energy allocation and precoder design for equal estimation error

<table>
<thead>
<tr>
<th>Training Energy Optimization:</th>
<th>Find ( E_T^* ) via (6.61)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel Estimation:</td>
<td>Broadcast training vectors as in (6.8) and form MMSE channel estimates ( \hat{H}_k ) as in (6.5) and (6.6)</td>
</tr>
<tr>
<td>Precoder Design:</td>
<td>Substitute ( P_{\text{max}} \rightarrow P_D^* ), ( \sigma_n^2 \rightarrow \sigma_{\text{eff}}^2 )</td>
</tr>
<tr>
<td></td>
<td>Find ( (\bar{V}, Q) ) using the sum-MSE minimization algorithm from Table 3.1 or the PMSE minimization algorithm from Table 4.1</td>
</tr>
</tbody>
</table>

Corollary 6.3. The result presented in Section 3.4 on the equality of optimum downlink and virtual uplink power allocations in the case of sum-MSE minimization can also be (trivially) extended to the imperfect CSI case when estimation error variances are equal by substituting \( \sigma_{\text{eff}}^2 \) for \( \sigma_n^2 \) and \( \hat{H}_k \) for \( H_k \).

Threshold Behaviour

We observe from (6.61) that no information can be communicated using the proposed algorithm in the case where \( E_{\text{max}} \leq \frac{\sigma_n^2}{\sigma_H^2} \sqrt{M n_D} \). If the total available energy fails to exceed this threshold, there is zero energy allocated to training; as a result, the estimated channel is \( \hat{H} = 0 \) and the resulting symbol estimates are \( \hat{x}^{DL} = 0 \) as well. It is difficult to provide an intuitive understanding of this result (in the case of sum-MSE minimization) without a closed-form expression for the minimum sum-MSE as a function of \( E_T \); however, we have
observed in simulations that when $E_{\text{max}}$ falls below the threshold, the resulting minimum sum-MSE is an increasing function of $E_T$. It follows that the “best” strategy is to avoid training. Note that this may also be an artifact of choosing MSE as the optimization objective, rather than BER.

This result can be interpreted in the information theoretic sense by considering the lower bound on achievable rate derived in Section 6.5. When $\hat{H} = 0$, the corresponding MMSE matrices are $\epsilon_k^{*DL} = I_{L_k}$; consequently, the lower bound on achievable rate (6.56) is

$$I(s_k; y_k|\hat{H}_k) \geq -\log \det I_{L_k} = 0.$$ (6.83)

In other words, in the case of Gaussian signaling and (worst-case) Gaussian noise-plus-interference, if the threshold constraint in (6.61) is violated, there exists no coding technique (under the proposed linear precoding mechanism on a per-block basis) which permits communication with arbitrarily low probability of error at non-zero rates. [90] proposes an open-loop strategy to apply channel coding across multiple blocks in the non-coherent case (i.e., where channel coefficients are not known); this may be an appropriate technique to consider when the total available energy is below the threshold, as the channel coefficients can not be effectively estimated.

We can reinterpret this threshold result in the context of average received SNR. If we define the average transmitted power as $P_{\text{avg}} \doteq E_{\text{max}}/n$, we can rewrite the constraint as

$$SNR_{\text{rx}} \doteq \frac{P_{\text{avg}} \sigma_H^2}{\sigma_n^2} \leq \frac{\sqrt{Mn_D}}{n_D + M}.$$ (6.84)

It follows that as $n \to \infty$, a strictly positive optimum training power allocation is always feasible. Furthermore, the largest average received SNR value that the threshold can take on is $SNR_{\text{rx}} = -3\text{dB}$, corresponding to the maximum value of the right hand side of (6.84) when $n_D = M$. 
6.6.1 Numerical Examples

We now present both analytical and simulation results to illustrate the behaviour and performance of the proposed algorithm. In these results, the flat Rayleigh fading channels are modelled with $\sigma_H^2 = 1$. We scale the total energy $E_{\text{max}}$ proportionally to the block-length $n$ to reflect a realistic average power constraint, $P_{\text{avg}} = E_{\text{max}}/n$; in these simulations, we illustrate the case of $E_{\text{max}}/n = 1$. As such, we define the average transmit SNR as $P_{\text{avg}}/\sigma_n^2$, and find different SNR values by varying the noise power $\sigma_n^2$. These results illustrate performance in a system with $K = 2$ users, $M = 4$ base station antennas, and $N_1 = N_2 = L_1 = L_2 = 2$ receive antennas and data streams per user.

Figure 6.4 illustrates how the optimum power allocated to training, $P_T^*$, grows with average SNR and with block length $n$. We observe that as $n$ grows, the optimum power allocated to training becomes significantly larger than equal power allocation $P_T = 1$; however, $P_T^*$ converges fairly rapidly with increasing SNR. We also observe the threshold behaviour described in the discussion at the end of the previous section.
Figure 6.5: Asymptotic behaviour of fractional training energy allocation

Figure 6.5 illustrates a different view of this result by examining the behaviour of the fractional energy allocated to training \( \frac{E^*_T}{E_{\text{max}}} \) as block length \( n \) increases asymptotically for three different values of SNR\(_{\text{avg}}\). Since \( P^*_T \) converges to a finite value, it follows that the fractional training energy allocation decreases to zero with increasing \( n \); the rate of descent appears to follow a power-law for large block length \( n \).

Figure 6.6 illustrates the “no-communication” threshold behaviour described in the previous section. This figure depicts the sum-MSE performance for a single channel realization for block length \( n = 100 \) and SNR\(_{\text{avg}}\) = \(-10\) dB, which clearly falls below the threshold depicted in Figure 6.4. This illustration suggests that when \( E_{\text{max}} \) falls below the training threshold in (6.61), transferring energy from training symbols to data symbols increases the sum-MSE objective.

Figures 6.7 and 6.8 illustrate the sum-MSE and average BER performance of the proposed algorithm. Results in each of these plots are generated using 5000 channel realizations per average SNR value, and data symbols are generated as uncoded QPSK.
Here, we compare performance of the proposed algorithm to the case where equal power is allocated to both training and data symbols (i.e., $P_T = P_D = 1$). This offers a reasonable point of comparison in the absence of alternative algorithms for joint energy optimization and multiuser linear precoding. We observe notable performance improvements for large block lengths ($n \gg M$), with approximately 3 dB of SNR gain for $n = 1000$.

Figure 6.6 illustrates the threshold behaviour and optimality of $E_T^* = P_T^* = 0$

Figure 6.9 illustrates the sum-rate performance of the modified PMSE minimization algorithm described in Section 6.4. In this figure, the lower bounds on achievable rate are calculated based on simulation over 300 channel realizations per average SNR value. For comparison, we provide results for the PMSE-minimizing design under equal power allocation for training and data ($P_T = P_D = 1$), and also include the PMSE-minimizing precoder design under perfect CSI with $P_D = 1$. We observe that the proposed scheme achieves significant gains over equal power allocation as block length $n$ increases; we note an SNR gain of approximately 2.5 dB for block length $n = 1000$. 
Figure 6.7: Sum-MSE performance for equal and optimal energy allocations

Figure 6.8: Average BER performance for equal and optimal energy allocations
Figure 6.9: Lower bound on spectral efficiency for equal and optimal energy allocations

6.7 Joint Optimization of Energy and Precoder Design: Unequal Estimation Error

We now consider the problem of jointly allocating energy to training and data symbols and designing sum-MSE and PMSE minimizing precoders when the estimation error variances $\sigma_k^2$ differ for each user.

In the previous section, we saw that the case where all users have equal estimation errors ($\sigma_k^2 = \sigma_e^2$) can be treated in the same manner as the the perfect CSI case, due to the elimination of the power allocation terms $\tilde{q}_l$ from $\sigma_{\text{eff}}^2$ in (6.15) via the grouping $\sum_l \sigma_{\kappa(l)}^2 q_l = P_D \sigma_e^2$. This result enables the sum-MSE minimizing design to be found via convex optimization in the virtual uplink as in Chapter 3. Furthermore, the problem of optimizing the training energy allocation $E_T$ (and thus implicitly optimizing $P_T$ and $P_D$) and precoder design are separable as (6.72) is not a function of $\tilde{Q}$.

In this section, we will demonstrate that in the presence of unequal estimation error
variances $\sigma_k^2$, separability no longer exists, since the $\sigma_{\text{eff}}^2$ and $D_\sigma$ terms in (6.72) are both functions of $\tilde{Q}$. Furthermore, the convex formulation for the sum-MSE minimizing power allocation subproblem can no longer be applied, and we must design an algorithm using MSE duality (and its associated power transformation) and the GP formulation developed in Sections 6.3 and 6.4 to solve the design problem using alternating optimization, as in the PMSE minimizing design of Chapter 4.

We consider three approaches to this alternating optimization, organized from highest to least complexity. First, we develop an approach based on geometric programming that jointly optimizes the training/data and per-stream power allocation problems; the resulting algorithm is, unfortunately, infeasible for practical implementation. We then consider the problem as an extension of the results of Section 6.6 whereby we alternate between finding the optimum precoder pair $(\bar{V}, Q)$ and training/data power $(P_T, P_D)$ while fixing the other set of terms. Finally, we propose a heuristic iterative method with far lower computational complexity.

### 6.7.1 Geometric Programming Approach

We begin by considering an extension of the iterative approach of Chapter 4 which alternates between finding virtual uplink precoders $\tilde{V}$, power allocations $Q$, virtual uplink receivers $U$ (and their associated normalized downlink precoders $\tilde{U}$), and equivalent downlink power allocations $P$, to include optimization of the training and data power allocations $P_T$ and $P_D$.

In place of the power allocation subproblem, we perform the optimization of the normalized power allocations $\tilde{q}_l$ as well as the data and training symbol power allocations $P_D$ and $P_T$ in a single geometric program; these two are related by the standard form posynomial inequality constraint

$$\frac{P_T M + P_D n_D}{E_{\text{max}}} \leq 1.$$ 

(6.85)
We extend the GP formulation (6.43) to include $P_T$ and $P_D$ as follows, and begin by factoring $Q = P_D \tilde{Q}$:

\[
\varepsilon_{UL}^l = |\beta_l \bar{u}_l^H \bar{h}_l - 1|^2 + \beta_l^2 P_D^{-1} \tilde{q}_l^{-1} \bar{u}_l^H \left[ \sigma_n^2 + \sigma_n^2 P_D \tilde{q}_l + \sum_{i \neq l} P_D \tilde{q}_i \left( \tilde{h}_i^H \tilde{h}_l^H + \sigma_n^2 \rho \right) \right] \bar{u}_l \\
= |\beta_l \bar{u}_l^H \bar{h}_l - 1|^2 + \beta_l^2 P_D^{-1} \tilde{q}_l^{-1} \sigma_n^2 \bar{u}_l^H \bar{u}_l + \beta_l^2 \tilde{q}_l^{-1} \bar{u}_l^H \left( \sum_{i \neq l} \tilde{q}_i \tilde{h}_i \tilde{h}_l^H \right) \bar{u}_l \\
+ \beta_l^2 \tilde{q}_l^{-1} \left( \sum_{i=1}^L \tilde{q}_i \frac{\sigma_n^2}{\rho + P_T} \right),
\]

(6.86)

where

\[
\sigma_{\kappa(i)}^2 = \left( \sigma_{H_{\kappa(i)}}^{-2} + \frac{1}{\sigma_n^2 M} \right)^{-1} = \frac{\sigma_n^2}{\rho_i + P_T}
\]

(6.87)
as in (6.10), with $\rho_i = \sigma_n^2 / \sigma_{H_{\kappa(i)}}^2$. We formally state the optimization problem under consideration as

\[
\left( \tilde{Q}^*, P_T^*, P_D^* \right) = \arg \min_{Q, P_T, P_D} f(\varepsilon_{UL}^l) \\
\text{s.t.} \quad \sum_{i=1}^L \tilde{q}_i \leq 1 \\
\frac{P_D n_D + P_T M}{E_{\max}} \leq 1,
\]

(6.88)

and note that this problem does not correspond to a geometric program. While the first three terms in (6.86) comprise a posynomial in $P_D$ and $\tilde{q}_i$ for any fixed choice of $\bar{u}_l$ and $\bar{v}_l$, $\varepsilon_{UL}^l$ is itself not a posynomial due to the term $(\rho_i + P_T)^{-1}$. Consequently, composing the sum-MSE or PMSE functions from $\varepsilon_{UL}^l$ does not result in a posynomial function either.

In order to yield a somewhat efficient solution, we transform (6.86) to a posynomial form by introducing the substitution of variables $t_i, i = 1, \ldots, L$ as

\[
t_i = \rho_i + P_T.
\]

(6.89)
The modified form for the virtual uplink MSE following substitution is

\[ \varepsilon_l = |\beta_l \hat{u}_l^H \hat{h}_l - 1|^2 + \beta_l^2 P_D^{-1} \tilde{q}_l^{-1} \sigma_n^2 + \beta_l^2 \tilde{q}_l^{-1} \tilde{u}_l^H \left( \sum_{i \neq l} \tilde{q}_i \tilde{h}_i \tilde{h}_i^H \right) \tilde{u}_l \]

(6.90)

which is a posynomial expression in \( P_D, \tilde{q}_i \), and \( t_i \).

Even with a posynomial objective function, the new problem is not a standard-form GP, since (6.89) can only be expressed as a set of posynomial (and not standard-form monomial) equality constraints:

\[ t_i + \frac{P_D n_D}{\rho_i + \frac{E_{\text{max}}}{M}} = 1. \]

(6.91)

In summary, we have shown that the sum-MSE and PMSE problems with unequal channel estimation errors can be formulated as a posynomial objective function with posynomial equality and inequality constraints. However, this is not a standard form GP and hence cannot be solved efficiently via convex transformation.

Thus, we are motivated to relax (6.91) to a set of inequality constraints, thereby solving a relaxed version of the original optimization problem,

\[ \left( \tilde{Q}^*, P_T^*, P_D^*, t_i^* \right) = \arg \min_{\tilde{Q}, P_T, P_D, t_i} f(\varepsilon_l^{UL}) \]

s.t. \( \sum_{i=1}^{L} \tilde{q}_i \leq 1 \)

\( \frac{P_D n_D + P_T M}{E_{\text{max}}} \leq 1 \)

\( \frac{t_i + \frac{P_D n_D}{\rho_i + \frac{E_{\text{max}}}{M}}}{M} \leq 1, \quad i = 1, \ldots, L. \)

(6.92)

Let the optimum value of this GP be \( f^* \). If the relaxed inequalities are satisfied with equality at \( f^* \) (i.e., if (6.89) is satisfied) then the resulting optimal solution is a feasible solution for the the original problem (6.88). If not, we can follow a strategy proposed in [91, Section 7.4] to find an optimal solution: a tightening of the variables \( t_i \) and \( P_D \) that are involved in the relaxed inequalities can be performed until (6.89) is satisfied.

This is possible in this particular case due to the following conditions being satisfied:
1. The objective function $f$ is monotonic decreasing in $P_D$ and $t_i$.

2. The relaxed inequality constraint functions are monotonic strictly increasing in $P_D$ and $t_i$.

3. The other posynomial inequality constraint functions are monotonic decreasing in $P_D$ and $t_i$.

Consequently, it is possible to increase one or more of the variables $P_D$ and $t_i$ until (6.89) is satisfied without violating any of the inequality constraints or altering the objective function from $f^*$.

The associated values of $P_D$ and $t_i$ that satisfy the set of constraints with equality can be found by solving an auxiliary GP:

$$
(t^*_i, P^*_D) = \arg \min_{t_i, P_D} \frac{1}{P_D} + \sum_{i=1}^{L} \frac{1}{t_i}
$$

s.t.  
$$
\sum_{i=1}^{L} \tilde{q}_i \leq 1
$$

$$
\frac{P_D n_D + P_T M}{E_{\text{max}}} \leq 1
$$

$$
\frac{t_i + P_D n_D}{\rho_i + \frac{E_{\text{max}}}{M}} \leq 1, \quad i = 1, \ldots, L
$$

$$
f(\varepsilon^{UL}) \leq f^*.
$$

(6.93)

The objective in (6.92) is chosen somewhat arbitrarily; the tutorial in [91] suggests that any function that “puts pressure on the optimization variables to increase” suffices.

In practice, we have seen that solving the auxiliary GP (6.93) is unnecessary. In all simulations, we have observed that training does not hurt; i.e., the optimal values from (6.92) are

$$
t^*_i = \rho_i + \frac{E_{\text{max}} - P^*_D n_D}{M},
$$

(6.94)

which corresponds to solving (6.89) with equality for all $t_i$ when all remaining energy not used for data symbols is allocated to training; that is,

$$
P^*_T = \frac{E_{\text{max}} - P^*_D n_D}{M}.
$$

(6.95)
Table 6.2: Sum-MSE/PMSE minimization algorithm with joint training/data energy allocation and precoder design for unequal estimation error $\sigma_k^2$

**Training Energy Initialization:** Select feasible $E_T$, e.g., $E_T = E_{\text{max}}/n$

**Channel Estimation:** Broadcast training vectors as in (6.8) and form MMSE channel estimates $\hat{H}_k$ as in (6.5) and (6.6)

**Precoder Initialization:** $\bar{V}_k = \text{SVD}(\hat{H}_k)$, $P_D = \frac{E_{\text{max}} - E_T}{n_D}$, $Q = \text{diag} \left[ \frac{(P_D/L)}{L} \right]$

**Iteration:**

1- **Downlink Precoder**
   
   $U_k^\star = \tilde{R}^{-1} \hat{H}_k^H \bar{V}_k \sqrt{Q_k}, \quad \bar{u}_l = \frac{u_l^\star}{\|u_l^\star\|_2}$

2- **Equivalent Downlink Power Allocation** $P$ (6.23)

3- **Virtual Uplink Precoder**
   
   $V_k^\star = \tilde{R}^{-1} \hat{H}_k^H \bar{U}_k \sqrt{P_k}, \quad \bar{v}_l = \frac{v_l^\star}{\|v_l^\star\|_2}$

4- **Virtual Uplink Power and Training/Data Power Allocation** $(\tilde{Q}^\star, P_T^\star, P_D)$ using GP Relaxation (6.92)

5- **New Channel Estimation using** $E_T = P_T^\star M$

6- **Repeat 1–5 until** $\left[ f(\varepsilon^{UL})_{\text{old}} - f(\varepsilon^{UL})_{\text{new}} \right] / f(\varepsilon^{UL})_{\text{old}} < \epsilon$

The structure of the resulting iterative algorithm is described in Table 6.2. It is similar to the PMSE minimizing algorithm of Table 4.1 with an important difference: channel estimation must be performed in each iteration of the precoder design.
6.7.2 Iterating between Energy Allocation and Precoder Optimization

The approach employing alternative optimization proposed in the previous section is not practical to implement, as it requires a new channel estimation in each iteration of the alternating procedure. We thus propose to solve the problem by alternating between finding the set of optimal precoder/decoder and power allocations \( \tilde{q}^*, \tilde{V}^*, U^* \) for a given \( P_T \), and solving for \( P_T^* \) for a fixed set of precoder, power allocation, and MMSE decoder. The first step presented here is finding \( P_T^* \), the optimal training power given \( \tilde{q}^*, \tilde{V}^* \).

We express the effective noise variance using the virtual user notation and (6.87):

\[
\sigma^2_{\text{eff}} = \sigma^2_n + P_D \sum_{l=1}^L \tilde{q}_l \sigma^2_{\kappa(l)}
\]

(6.96)

As shown in Section 6.6, solving (6.72) finds candidate values of \( P_T^* \) for optimality (i.e., stationary points for functions of the MMSE channel estimate) for all choices of \( \sigma^2_k \).

We define the function

\[
g(P_T) = \frac{M}{n_D} \sigma^2_{\text{eff}} + P_D D_\sigma;
\]

(6.97)

finding the zeroes of \( g(P_T) \) will allow us to identify candidates for optimality.

With different estimation error variances, we rederive \( D_\sigma \) by differentiating (6.96) with respect to \( P_T \):

\[
D_\sigma = \sigma^2_n \frac{\partial}{\partial P_T} \left[ 1 + \left( \frac{E_{\text{max}} - P_T M}{n_D} \right) \sum_{l=1}^L \frac{\tilde{q}_l}{\rho_l + P_T} \right]
= -\sigma^2_n \left[ \frac{M}{n_D} \sum_{l=1}^L \frac{\tilde{q}_l}{\rho_l + P_T} + \frac{E_{\text{max}} - P_T M}{n_D} \sum_{l=1}^L \frac{\tilde{q}_l}{(\rho_l + P_T)^2} \right]
= -\sigma^2_n \left[ \sum_{l=1}^L \frac{\tilde{q}_l (E_{\text{max}} + \rho_l M)}{(\rho_l + P_T)^2} \right]
\]

(6.98)

We then find the derivative

\[
\frac{\partial D_\sigma}{\partial P_T} = 2 \frac{\sigma^2_n}{n_D} \left[ \sum_{l=1}^L \frac{\tilde{q}_l (E_{\text{max}} + \rho_l M)}{(\rho_l + P_T)^3} \right] > 0.
\]

(6.99)
We observe that $g(P_T)$ is an increasing function in $P_T$, since

$$\frac{\partial g(P_T)}{\partial P_T} = \frac{M}{N_D}D_\sigma + P_D\frac{\partial D_\sigma}{\partial P_T} - \frac{M}{N_D}D_\sigma = P_D\frac{\partial D_\sigma}{\partial P_T} > 0; \quad (6.100)$$

as such, there can only be one potential stationary point when choosing $P_T$ for MSE-based optimization under MMSE channel estimation corresponding to a zero of $g(P_T)$ when $g(0) < 0$ and $g(E_{\text{max}}/M) > 0$. If this stationary point does exist, the positivity of the second derivative means that it must correspond to a global minimum for the sum-MSE and PMSE functions, as seen in Section 6.6.

Now, we provide a condition for existence of the stationary point $P^*_T$. We begin by rewriting $g(P_T)$ as

$$g(P_T) = \frac{M}{n_D}\sigma_{\text{eff}}^2 + P_D D_\sigma$$

$$= \frac{M}{n_D}\sigma_n^2 \left(1 + P_D \sum_{l=1}^L \frac{\bar{q}_l}{\rho_l + P_T} \right) - \sigma_n^2 P_D \left[ \sum_{l=1}^L \frac{\bar{q}_l (E_{\text{max}} + \rho_l M)}{(\rho_l + P_T)^2} \right]$$

$$= \frac{M}{n_D}\sigma_n^2 \left(1 + P_D \sum_{l=1}^L \frac{\bar{q}_l}{\rho_l + P_T} - P_D \sum_{l=1}^L \frac{\bar{q}_l (E_{\text{max}} + \rho_l M)}{M (\rho_l + P_T)^2} \right)$$

$$= \frac{M}{n_D}\sigma_n^2 \left(1 - P_D \sum_{l=1}^L \frac{\bar{q}_l (E_{\text{max}} - \rho_l M)}{M (\rho_l + P_T)^2} \right)$$

$$= \frac{M}{n_D}\sigma_n^2 - \sigma_n^2 P_D^2 \sum_{l=1}^L \frac{\bar{q}_l}{(\rho_l + P_T)^2}. \quad (6.101)$$

At $P_T = E_{\text{max}}/M$, $P_D = 0$; thus,

$$g(E_{\text{max}}/M) = \frac{M}{n_D}\sigma_n^2 > 0. \quad (6.102)$$

Since $g(P_T)$ is strictly increasing in $P_T$, a solution to $g(P_T) = 0$ in $(0, E_{\text{max}}/M)$ exists iff $g(0) < 0$; that is,

$$\frac{M}{n_D} < \frac{E_{\text{max}}^2}{n_D^2} \sum_{l=1}^L \frac{\bar{q}_l}{\rho_l^2}. \quad (6.103)$$
We can express this condition for the existence of an optimal $P_T^\star$ as

$$E_{\text{max}} > \sigma_n^2 \sqrt{\frac{Mn_D}{\sum_{l=1}^{L} \tilde{q}_l \sigma_{H_{k(l)}}^4}},$$

(6.104)

which is equivalent to (6.61) and reduces to the feasibility condition stated therein for the case of equal $\sigma_{H_k}^2$.

We observe from both the feasibility condition and from $g(P_T)$ that, in contrast to the case of equal $\sigma_{H_k}^2$, the problems of allocating energy for training/data symbols and designing the optimum precoders are no longer decoupled; rather, both are interdependent on $P_T$, $P_D$, and the virtual uplink power allocations $\tilde{q}_l$. Consequently, joint optimization of the training/data energy allocation and precoder design is a difficult problem to solve.

Since there is at most one possible solution to $g(P_T) = 0$ in (6.101), the latter problem can be solved efficiently by employing numerical root-finding tools as long as the feasibility condition (6.104) is satisfied.

We use this result to design our second algorithm for iterative optimization of $(P_T, P_D)$ and $(\bar{V}, \bar{Q})$ in Table 6.3. While the algorithm described by Table 6.3 seems to be more complicated than that of Table 6.2 due to the additional layer of iteration, it is important to note that the inner iteration is the computationally expensive segment which takes far longer to iterate than the outer loop (as will be shown in Section 6.7.4 via Figure 6.16).

By moving the channel estimation step out of the inner loop, we can dramatically reduce the number of estimations required, making the iterative algorithm proposed in Table 6.3 more feasible than the joint algorithm in Table 6.2.

Even though we are able to optimize $P_T$ for an arbitrary power allocation $\bar{q}$, examining (6.104) reveals that a poorly selected power allocation $\bar{q}$ may yield an infeasible solution from the energy optimization subproblem. As a result, the alternating optimization algorithm may not converge if

$$E_{\text{max}} \leq \frac{\sigma_n^2}{\sigma_{H_k}^2} \sqrt{Mn_D}$$

(6.105)
Table 6.3: Sum-MSE/PMSE minimization algorithm with iterative training/data energy allocation and precoder design for unequal estimation error $\sigma^2_k$

**Training Energy Initialization:** Select feasible $E_T$, e.g., $E_T = E_{\text{max}}/n$

**Outer Iteration:**
1. **Channel Estimation:** Broadcast training vectors as in (6.8) and form MMSE channel estimates $\hat{H}_k$ as in (6.5) and (6.6)
2. **Precoder Initialization:** $V_k = \text{SVD}(\hat{H}_k)$, $P_D = \frac{E_{\text{max}} - E_T}{n_D}$
3. **Inner Iteration – Precoder Design:**
   i) **Downlink Precoder**
   $$U^*_k = \tilde{\mathbf{R}}^{-1}\hat{H}_k^H\tilde{\mathbf{V}}_k\sqrt{\tilde{Q}_k}, \quad \tilde{u}_l = \frac{u^*_l}{\|u^*_l\|^2}$$
   ii) **Equivalent Downlink Power Allocation** $P$ (6.23)
   iii) **Virtual Uplink Precoder**
   $$V^*_k = \tilde{\mathbf{R}}^{-1}k\hat{H}_k^H\tilde{U}_k\sqrt{\tilde{P}_k}, \quad \tilde{v}_l = \frac{v^*_l}{\|v^*_l\|^2}$$
   iv) **Virtual Uplink Power Allocation** ($\tilde{Q}^*$) via GP (6.43)
   v) **Repeat i)–iv)** until $[f(\varepsilon^{UL})_{\text{old}} - f(\varepsilon^{UL})_{\text{new}}]/f(\varepsilon^{UL})_{\text{old}} < \epsilon$
4. **Training Energy Optimization:** Find $E_T^*$ by solving (6.101) via root-finding
5. **Repeat 1–4** until convergence

for any user $k$. In contrast, if the following (stricter) sufficient condition is satisfied for existence of an optimal $P_T^*$,

$$E_{\text{max}} > \frac{\sigma^2_n}{\sigma^2_{H_{\text{min}}}} \sqrt{Mn_D},$$

(6.106)

the alternating optimization algorithm is guaranteed to converge to a local optimum as each step of the algorithm is monotonic non-increasing in per-stream MSEs and because each of the objective functions is bounded from below. This condition corresponds to feasibility for the single-user / equal estimation error case (6.61) for users with the worst-
case channel variance $\sigma^2_{H,\text{min}}$. If this condition is satisfied, then all power allocations $\tilde{q}$ admit feasible training/data energy allocations. If not, one possible approach for finding a convergent solution is to drop the user(s) with the worst channels until (6.106) is satisfied; however, such an approach is potentially suboptimal for rate maximization, and may be unacceptable under sum-MSE minimization with a fixed set of operating data streams.

### 6.7.3 Heuristic Approaches

The optimization algorithms suggested in Sections 6.7.1 and 6.7.2 lead to a theoretically optimal or near-optimal solution to the problem of jointly optimizing the training/data energy allocation and the sum-MSE / PMSE minimizing precoder design; however, a fundamental problem exists which prevent these algorithms from being practically implemented. The main difficulty is that the process of training and channel estimation must be performed in each iteration when the training power $P_T$ is altered. This is impractical to implement for realistic fading rates because of the large amount of overhead that would be incurred with each iteration. On the other hand, the equal estimation error case allows for efficient solutions.

We propose a heuristic approach to solving the joint optimization problem, whereby we apply the algorithm designed for equal estimation errors $\sigma^2_k$ in Section 6.6 by assigning a value to $\sigma^2_H$ according to one of three criteria:

1. $\sigma^2_H = \text{mean} (\sigma^2_{H_k})$ (averaging)
2. $\sigma^2_H = \text{min} (\sigma^2_{H_k})$ (making an optimistic choice)
3. $\sigma^2_H = \text{max} (\sigma^2_{H_k})$ (making a ‘worst-case’ choice)

### 6.7.4 Numerical Examples

We compare the performance of the proposed iterative and heuristic approaches to the iterative algorithm based on KKT conditions proposed in [82, Section 5.3]. The KKT-based
algorithm alternates between solving precoder and decoder matrices and the associated Lagrange multiplier until convergence following the strategy proposed for the perfect CSI case in [27]. Since this KKT-based design does not account for the optimization of energy allocated to training and data symbols, we consider the case of equal power allocation (as in Section 6.6.1) as well as the mean/min/max heuristic criteria proposed in Section 6.7.3.

Example 1: Figures 6.10 and 6.11 illustrate the average sum-MSE and average BER performance of the proposed algorithms for a similar system configuration to that presented in Section 6.6.1. There are $K = 2$ mobile users, $M = 4$ base station antennas, and $N_1 = N_2 = L_1 = L_2 = 2$ receive antennas and data streams per user; transmission blocks are of length $n = 200$. Here, in contrast to the previous examples, users 1 and 2 have complex Rayleigh modelled channel coefficients with different variances $\sigma_{H_1}^2 = 10$ and $\sigma_{H_2}^2 = 1$. 

Figure 6.10: Average sum-MSE performance for KKT-based, heuristic, and iterative algorithms with $\sigma_{H_1}^2 = 10, \sigma_{H_2}^2 = 1$.
Figure 6.11: Average BER performance for heuristic and iterative algorithms with $\sigma_{H_1}^2 = 10, \sigma_{H_2}^2 = 1$, respectively. As in Section 6.6.1, $E_{\text{max}}$ is set equal to the block-length $n$ to reflect an average power constraint, $P_{\text{avg}} = 1$. The data streams in Figure 6.11 are grouped by user. Nearly identical average sum-MSE and BER performance are achieved by each of the heuristic and iterative algorithms under consideration. Slight improvements in average BER performance can be seen at the higher end of average transmit SNR (i.e., above 15 dB), which may possibly be related to convergence of the KKT-based scheme to local minima.

The only true decline in performance can be seen in the case of equal power allocation ($P_T = P_D = 1$). Figure 6.12 illustrates the “optimum” training powers that would be allocated in the case of equal estimation error under the max/min/mean heuristic designs, as well as the locally optimal training power found by the iterative algorithm. Here, we see that the range of training power allocations in the average transmit SNR regime under consideration is $P_T^* \in [5.4, 6.3]$ – this accounts for the marked improvement over the equal
Figure 6.12: Training power $P^*_T$ from [6.61] for maximum $\sigma_{H_1}^2 = 10$, minimum $\sigma_{H_2}^2 = 1$, mean $\sigma_{H}^2 = 5.5$, and average power allocation under the proposed iterative algorithm.

Power allocation case with $P_T = 1$. For this system configuration, the range of candidate training powers according to the proposed schemes is not particularly large; combining this result with Figures 6.10 and 6.11 suggests that the KKT-based and heuristic schemes are not particularly sensitive to which power allocation is used within this range. Finally, we note that in this system scenario (with one strong and one weak user), the average training power allocated by the iterative scheme at lower SNR values falls in the middle of the min and max heuristic approaches; at higher SNRs, it appears to converge to the minimum optimum training power.

**Example 2:** Figures [6.13] [6.15] depict a system with a similar configuration but where the users’ channel variances are much closer than in the first example ($\sigma_{H_1}^2 = 0.1$ and $\sigma_{H_2}^2 = 0.05$). In this system configuration, the performance of the heuristic and iterative
Figure 6.13: Average sum-MSE performance for KKT-based, heuristic, and iterative algorithms with $\sigma^2_{H_1} = 0.1, \sigma^2_{H_2} = 0.05$

Figure 6.14: Average BER performance for heuristic and iterative algorithms with $\sigma^2_{H_1} = 0.1, \sigma^2_{H_2} = 0.05$
Figure 6.15: Training power $P^*_T$ from $[6.61]$ for maximum $\sigma^2_{H_1} = 0.1$, minimum $\sigma^2_{H_2} = 0.05$, mean $\sigma^2_H = 0.075$, and average power allocation under the proposed iterative algorithm.

schemes once again appears approximately equal; only minor performance improvements can be achieved by employing the iterative approach.

Figure 6.15 demonstrates a different behaviour for the average training power allocated by the iterative scheme than that seen in Figure 6.12. Here, the average training power under the iterative scheme is close to the maximum optimal training power at lower SNR values, and appears to converge to the mean heuristic training power at high SNR.

Figure 6.16 illustrates the convergence of the average training power found under the iterative algorithm for the case depicted in the second example. Even though the algorithm is initialized with a random training power (which is potentially far from the optimum), we see that the algorithm demonstrates stable and rapid convergence in this case and in other cases that we have simulated.
Figure 6.16: Convergence of training power $P_T$ for $\sigma_{H_1}^2 = 0.1$, $\sigma_{H_2}^2 = 0.05$ under the proposed iterative algorithm.

Discussion

We are able to draw some conclusions from the two examples presented. First, we have observed that sum-MSE and BER performance do not appear to be overly sensitive to the choice of training power allocation, as long as the selected power falls within the feasible range. Furthermore, the heuristic algorithm (which assumes equal estimation errors) performs nearly identically to the iterative KKT-based and iterative training/data power allocation/precoder design algorithms. The heuristic design requires only a single (relatively computationally inexpensive) convex optimization, making it a more attractive option for implementation. Finally, we note that while the results on joint optimization for the case of unequal estimation error are theoretically interesting, a real systems-level implementation could take advantage of servicing groups of users with equal (or near-equal) estimation errors using the optimal energy allocation strategy.
6.8 Summary

In this chapter, we have introduced a new challenge to the problem of transceiver design by implementing training and MMSE channel estimation elements in the system. Consequently, we have extended the existing transceiver designs to make them robust to CSI imperfections. We began by extending the MIMO MSE duality result to the case of imperfect CSI, and then demonstrated that both the sum-MSE and PMSE power allocation subproblems could be formulated as geometric programs.

The separation of transmission blocks into training and data stages necessitated the formulation of a new optimization problem, whereby we consider the optimization of energy allocated to either stage. We have thus considered the joint optimization of the training/data energy allocation in conjunction with the sum-MSE and PMSE minimizing transceiver designs.

In the case where all users have equal estimation error variances, we have demonstrated the separability of the training/data energy allocation and transceiver design problems, and have derived the optimum closed-form energy allocation for the case of MMSE channel estimation when all users have statistically identical channels. As a result of separability, we see that existing algorithms for minimum sum-MSE and PMSE precoding can be applied following energy optimization.

The case of unequal estimation error variances proves to be a great deal more challenging, as the separability and equivalence results no longer apply. We have derived a near-optimal formulation based on geometric programming which proves to be infeasible for practical implementation. Next, we have proposed an approach based on root-finding to solve the energy optimization problem for a given virtual-uplink power allocation; we employ this approach in an iterative manner to jointly find the training/data energy allocation and precoder design. Finally, we have proposed several heuristics under which we apply the design for the equal estimation error case, and demonstrate that these computationally simpler methods can be applied with little loss in overall performance.
Chapter 7

Conclusions

This dissertation addresses the problem of optimizing linear transceivers for use in the multiuser MIMO downlink, when both the transmitter and the receivers may be equipped with multiple antennas and each mobile user may receive multiple data streams.

We first considered the problem of designing transceivers when perfect CSI was available at both ends of the communications link. The key to solving the proposed design problems was using a duality result stating the equivalence of achievable MSEs in the downlink and virtual uplink. We extended this duality to the multiuser MIMO case by treating each data stream as a virtual user, and found the equivalent uplink-to-downlink transformation for arbitrary precoder/decoder pairs.

The first design that we considered was minimization of the sum-MSE over all users. We found a computationally feasible design for the downlink by solving an equivalent convex problem in the virtual uplink, and applied the MSE duality and associated uplink-to-downlink transformation. The transformation was further simplified by introducing a new extension to the known duality proving the equality of sum-MSE minimizing uplink and downlink power allocations.

The second set of designs we considered aim to maximize the sum throughput of all users. We introduced several MSE-based criteria for optimization and established
a series of relationships linking these criteria to the signal-to-interference-plus-noise ratios of individual data streams and the information theoretic sum capacity under linear minimum MSE decoding. The sum-rate maximizing design was shown to be equivalent to minimizing the product of MSE matrix determinants, but we demonstrated that this problem does not admit a computationally efficient solution. A simplified version of the problem was considered that minimizes the product of mean squared errors (PMSE), and an iterative algorithm based on alternating optimization in the downlink and virtual uplink was proposed to solve the problem with near-optimal performance.

Next, we considered the impact of two different forms of imperfect CSI on performance: limited-rate feedback (with delay), and channel estimation error. We proposed a flexible system to mitigate delay by employing Kalman prediction, and applied adaptive delta modulation to feedback bits to yield a low-rate feedback scheme. We extended the MSE duality to the case of imperfect CSI resulting from channel estimation, and demonstrated that the sum-MSE and PMSE power allocation subproblems have equivalent GP formulations. We thus considered the robust extensions of the sum-MSE and PMSE minimizing precoders when transmitter and receivers possess identical, delay-free, but imperfect estimates of the CSI.

We introduced a modified block-transmission framework which incorporated training/estimation and data transmission stages. We thus considered a new optimization problem of joint optimization of the energy allocations to training and data symbols in conjunction with the sum-MSE/PMSE minimizing transceiver designs. We proved the separability of the training/data energy allocation and precoder design problems in the case of equal estimation error variances, and demonstrate the equivalence of the precoder designs to the perfect CSI case. The case of unequal estimation error variances proved to be quite a bit more challenging as separability no longer applied. We derived a near-optimal formulation based on solving a GP which proved to be infeasible to implement, and considered an alternative approach based on iteration between precoder design and
energy optimization using root-finding. Performance of the iterative approach was compared to several computationally simpler heuristic approaches (employing the equal error variance solution), with little to no degradation in overall performance.

**Topics for Future Research**

The contributions of this dissertation could be extended in several ways that are described here. Many of the contributions made in this thesis were primarily theoretical; as such, the suggestions for future work focus largely upon further exploration of system design via simulation.

- In Chapter 1 we described how multicarrier techniques (i.e., OFDM) could be applied to transform frequency-selective broadband channels to a set of frequency-flat subcarriers; thus, the designs we propose in this thesis can be easily applied to each subcarrier in an OFDM system. Direct application may be infeasible in practice, since the complexity of the resulting system would scale linearly in the number of subcarriers. In [92], our designs for sum-MSE minimization were extended to the case of OFDM using complexity-saving tools such as interpolation and clustering. These tools could also be applied in the context of the rate maximizing designs of Chapter 4 and to the cases of imperfect CSI cases as discussed in Chapters 5 and 6.

- The prediction and feedback mechanism proposed in Chapter 5 assumes that the channel fading model is Rayleigh with i.i.d. channel coefficients. While the Rayleigh fading model may offer insights into the average performance of the proposed algorithm in dense urban environments (i.e., where small-scale fading has the biggest impact on channel condition), it does not take into account the impact of line-of-sight components (i.e., Rician fading), shadowing, or path loss. As well, techniques that exploit spatial redundancy could be employed in the case of correlated channel coefficients to reduce the total cost of feedback. More realistic channel models
should be considered in order to make the proposed system and subsequent analysis better reflect performance under real-world implementation.

- In this thesis, we only considered the unweighted sum-MSE minimization and sum-rate maximization problems; another important problem to consider is optimization of the weighted sum-MSE and weighted sum-rate. Several existing works have investigated these problems in the case of perfect CSI (e.g., [63, 65]), and connections between weighted sum-MSE minimization and weighted sum-rate maximization have been developed in [26, 93]. One possible concern about using the unweighted versions of the problems as a measure of overall system performance is that the framework does not impose fairness across users; introducing weights (which may be changed dynamically) might be a good method for introducing fairness / QoS constraints in practical system implementations. An interesting and useful extension of this dissertation would be the extension of the weighted sum-MSE and weighted sum-rate optimization problems to the case of imperfect CSI.

- In Section 4.1.4, we suggested that one possible method for increasing sum-rate would be to introduce multiuser diversity by transmitting to a subset of a large group of users. In practice, this would not only be desirable but also necessary, as base stations servicing a large number of users must implement a scheduling mechanism to communicate with all users within their cell / sector. Scheduling could also help address the issue of fairness by preferentially selecting users in round robin fashion who did not meet their QoS requirements in previous transmission blocks. One important concern that must be considered in such a scenario is the uplink resource usage required for communication of the channel estimates of many mobile users. Introducing scheduling for a large number of users into the existing algorithms may lead to some interesting and challenging research problems.

- The contributions made in Chapter 5 on prediction and limited feedback and in
Chapter 6 on MMSE channel estimation and error are treated as two separate problems; however, in practice, these two sources of CSI imperfection are likely not separable. While combining the impact of quantization and channel estimation error may result in complicated formulations for the purpose of analysis, both aspects could be included in a combined simulation.

- We did not simulate the performance of the rate maximizing designs of Chapter 4 under delayed feedback with quantization error; it would be interesting to see if the same conclusions apply regarding the suitability of the design with respect to mobile velocity and number of feedback bits as was demonstrated for the sum-MSE minimizing precoder design.

- We were only able to illustrate a few basic configurations for the designs implemented in Chapter 6. A more comprehensive set of simulations is merited for different system parameters (i.e., number of users, unequal number of receiver antennas, etc). One particularly important parameter to investigate is the impact of different estimation error variances; i.e., how do the designs perform as the users’ channels become drastically different?
Appendix A

Gradient and Hessian of Sum-MSE Function

In order to differentiate the sum-MSE objective function in (3.38) with respect to each of the optimization variables \( q_i, i = 1, \ldots, L \), we make use of the following differential identity for the matrix inverse [94]:

\[
\partial (X^{-1}) = -X^{-1} (\partial X) X^{-1}.
\]

It follows that the derivative of the objective function with respect to the \( i \)th virtual uplink power \( q_i \) is

\[
\frac{\partial \text{tr} [R^{-1}]}{\partial q_i} = \text{tr} \left[ \frac{\partial (R^{-1})}{\partial q_i} \right] \\
= -\text{tr} \left[ R^{-1} \left( \frac{\partial R}{\partial q_i} \right) R^{-1} \right] \\
= -\text{tr} \left[ R^{-1} D_i(R) R^{-1} \right],
\]

where we have made use of linearity of the trace operator, and we have defined the notation

\[
D_i(R) = \frac{\partial R}{\partial q_i}.
\]
Using the expression for $R$ from (3.39),

$$R = \sum_{j=1}^{L} q_j \tilde{h}_j \tilde{h}_j^H + \sigma_n^2 I_M,$$

we observe that

$$D_i(R) = \tilde{h}_i \tilde{h}_i^H,$$

and that

$$\frac{\partial \text{tr} [R^{-1}]}{\partial q_i} = -\text{tr} \left[ R^{-1} \tilde{h}_i \tilde{h}_i^H R^{-1} \right]\$$

$$= -\tilde{h}_i^H R^{-2} \tilde{h}_i.$$

The gradient of the objective function in (3.38) with respect to the virtual uplink power allocation $q$ is thus

$$\nabla_q \left[ \text{tr} [R^{-1}] \right] = -\left[ \begin{array}{c} \tilde{h}_i^H R^{-2} \tilde{h}_i \\ \vdots \\ \tilde{h}_L^H R^{-2} \tilde{h}_L \end{array} \right].$$

We can find the elements of the Hessian matrix by applying a matrix version of the product rule for differential operators [94],

$$\partial(XY) = (\partial X) Y + X (\partial Y).$$

We apply linearity again to rewrite the second derivative with respect to optimization variables $q_i$ and $q_j$ as

$$\frac{\partial^2 \text{tr} [R^{-1}]}{\partial q_i \partial q_j} = -\text{tr} \left[ \tilde{h}_i \tilde{h}_j^H \frac{\partial (R^{-2})}{\partial q_j} \right]$$

$$= -\text{tr} \left[ \tilde{h}_i \tilde{h}_j^H \left\{ ( -R^{-1} D_j (R) R^{-1} ) R^{-1} + R^{-1} ( -R^{-1} D_j (R) R^{-1} ) \right\} \right]$$

$$= \text{tr} \left[ \tilde{h}_i \tilde{h}_j^H \left\{ R^{-1} \tilde{h}_j \tilde{h}_j^H R^{-2} + R^{-2} \tilde{h}_j \tilde{h}_j^H R^{-1} \right\} \right]$$

$$= \tilde{h}_i^H R^{-1} \tilde{h}_j \tilde{h}_j^H R^{-2} \tilde{h}_i + \tilde{h}_j^H R^{-2} \tilde{h}_i \tilde{h}_j^H R^{-1} \tilde{h}_i$$

$$= 2 \text{Re} \left[ \tilde{h}_i^H R^{-1} \tilde{h}_j \tilde{h}_j^H R^{-2} \tilde{h}_i \right].$$

The entries in the Hessian matrix $\mathcal{H}_q [\text{tr} [R^{-1}]]$ are the second derivatives found in (A.9).
Appendix B

Convexity of Sum-MSE Function

This proof is based heavily on [33, Exercise 3.18a and Example 3.1.5]. An alternate proof follows from the classification of \( \text{tr} [X^{-1}] \) as a spectral function (i.e., a function of the eigenvalues of a Hermitian matrix which is invariant to permutation of their order, see [95]).

First, we verify convexity of the function

\[
 f(X) = \text{tr} [X^{-1}] \tag{B.1}
\]

for \( X \in S_{++}^n \) by considering an arbitrary line,

\[
 X = Z + tV, \tag{B.2}
\]

where \( Z \in S_{++}^n \) and \( V \in S_+^n \). We introduce the function

\[
 g(t) = f(Z + tV) \tag{B.3}
\]

and restrict \( g \) to the interval of values for which \( Z + tV \succ 0 \) (i.e., so that \( X \in S_{++}^n \) still holds).

We form the eigendecomposition

\[
 Z = ADA^H, \tag{B.4}
\]
Appendix B. Convexity of Sum-MSE Function

where \([D]_{i,i} \in \mathbb{R}^{++}\) due to \(Z \in \mathbb{S}^n_{++}\); it follows that a matrix square root for \(Z\) can be defined as

\[
Z^{1/2} = AD^{1/2}A^H. \tag{B.5}
\]

with \([D^{1/2}]_{i,i} \in \mathbb{R}^{++}\) as well. Similarly, we define

\[
Z^{-1/2} = AD^{-1/2}A^H. \tag{B.6}
\]

and note that \(Z^{-1/2} \in \mathbb{S}^n_{++}\) and \(Z^{-1/2} = (Z^{1/2})^{-1}\).

With this result, we define \(W = Z^{-1/2}VZ^{-1/2}\) and its eigendecomposition,

\[
W = BAB^H. \tag{B.7}
\]

Since \(Z^{-1/2}\) is Hermitian and \(V \in \mathbb{S}^n_+\), it follows that \(W \in \mathbb{S}^n_+\) and \([\Lambda]_{i,i} \in \mathbb{R}_+\).

We rewrite

\[
g(t) = \text{tr} \left[ (Z + tV)^{-1} \right] \\
= \text{tr} \left[ Z^{-1/2} \left( I + tZ^{-1/2}VZ^{-1/2} \right)^{-1} Z^{-1/2} \right] \\
= \text{tr} \left[ Z^{-1/2} \left( I + tBAB^H \right)^{-1} Z^{-1/2} \right] \\
= \text{tr} \left[ Z^{-1/2}B \left( I + t\Lambda \right)^{-1} B^HZ^{-1/2} \right] \\
= \text{tr} \left[ B^HZ^{-1}B \left( I + t\Lambda \right)^{-1} \right] \\
= \sum_{i=1}^{n} \frac{[B^HZ^{-1}B]_{i,i}}{1+t\lambda_i} \\
= \sum_{i=1}^{n} \frac{c_i}{1+t\lambda_i}, \tag{B.8}
\]

where we have defined \(c_i = [B^HZ^{-1}B]_{i,i}\). Note that \(c_i \in \mathbb{R}^{++}\) since \(B^HZ^{-1}B \succ 0\).

Differentiating \(g(t)\) twice with respect to \(t\) yields

\[
g'(t) = -\sum_{i=1}^{n} \frac{c_i\lambda_i}{(1+t\lambda_i)^2} \tag{B.9}
\]

and

\[
g''(t) = \sum_{i=1}^{n} \frac{2c_i\lambda_i^2}{(1+t\lambda_i)^3}. \tag{B.10}
\]
Earlier in the proof, we had restricted $t$ to the interval that kept $X = Z + tV$ positive definite; we see from (B.8) that

$$Z^{-1/2}B(I + tA)^{-1}B^HZ^{-1/2} > 0$$

(B.11)

when $t > -\lambda_i^{-1}, i = \{1, 2, \ldots, n\}$. It follows that $1 + t\lambda_i > 0$ and $g''(t)$ is always positive; thus, $\text{tr}[X^{-1}]$ is convex on $S_{++}^n$.

The objective function $\text{SMSE}^{UL} = L - M + \sigma_n^2 \text{tr}
\left[(H\bar{V}Q\bar{V}^H H^H + \sigma_n^2 I_M)^{-1}\right]$ is thus convex in $Q$, since convexity is preserved by composition with the affine mapping of the variable $Q$, $H\bar{V}Q\bar{V}^H H^H + \sigma_n^2 I_M$. Similarly, the sum-MSE function is convex in $S = VQ\bar{V}^H$, since $HSH^H + \sigma_n^2 I_M$ is an affine mapping of $S$.

Convexity in each of the scalar variables $q_i, i = 1, \ldots, L$, follows from joint convexity in $Q$. Finally, if we rewrite the sum-MSE as

$$\text{SMSE}^{UL} = L - M + \sigma_n^2 \text{tr}
\left[\sum_{l=1}^{L} \left(q_lH_{\kappa(l)}\bar{v}_l\bar{v}_l^H H_{\kappa(l)}^H + \sigma_n^2 I_M\right)^{-1}\right]$$

$$= L - M + \sigma_n^2 \text{tr}
\left[\left(q_lH_{\kappa(i)}\bar{v}_i\bar{v}_i^H H_{\kappa(i)}^H + \sum_{l\neq i} q_lH_{\kappa(l)}\bar{v}_l\bar{v}_l^H H_{\kappa(l)}^H + \sigma_n^2 I_M\right)^{-1}\right]$$

$$= L - M + \sigma_n^2 \text{tr}
\left[\left(q_lH_{\kappa(i)}S_i H_{\kappa(i)}^H + \sum_{l\neq i} q_lH_{\kappa(l)}S_l H_{\kappa(l)}^H + \sigma_n^2 I_M\right)^{-1}\right],$$

(B.12)

we can observe that the sum-MSE function is both individually and jointly convex in the transmit direction covariance matrices $S_i = \bar{v}_i\bar{v}_i^H$. 
Appendix C

Geometric Programming

This brief synopsis of geometric programming is a summary of the topic as discussed in [91].

C.1 Monomial and Posynomial Functions

A monomial function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) with \( \text{dom}[f] = \mathbb{R}^n_{++} \) is defined as:

\[
f(x) = cx_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}.
\]

(C.1)

where \( c \in \mathbb{R}_{++} \) and \( a_i \in \mathbb{R} \) are arbitrary non-negative and real values, respectively.

A posynomial function is a sum of any number of monomial functions; i.e.,

\[
f_i(x) = \sum_{j=1}^{N} c_j x_1^{a_{1j}}x_2^{a_{2j}} \cdots x_n^{a_{nj}}
\]

(C.2)

with \( c_j \in \mathbb{R}_{++} \) and \( a_{ij} \in \mathbb{R} \).
C.2 Standard Form

The standard form geometric program with $p$ inequality constraints and $m$ equality constraints can be expressed as

\[
\begin{align*}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 1, \quad i = 1, \ldots, p \\
& h_i(x) = 1, \quad i = p + 1, \ldots, p + m,
\end{align*}
\]  

\tag{C.3}

where

\[
f_i(x) = \sum_{j=1}^{N_i} c_{ij} x_1^{a_{1ij}} x_2^{a_{2ij}} \cdots x_n^{a_{nij}}
\]  

\tag{C.4}

are posynomial functions and

\[
h_i(x) = c_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}
\]  

\tag{C.5}

The domain of the problem (i.e. both the objective and constraint functions) is $\mathbb{R}_{++}^n$; the positivity of the optimization variables is implicit (i.e., $x \succ 0_n$).

C.3 Log-Transformation to Convex Form

While (C.3) is not a convex program, substitution of variables allows one to find an equivalent convex program. Due to the positivity of both the variables and functions, one may apply a log transformation to each as follows. By substituting

\[
y_i = \log(x_i) \\
\mathbf{a}_i = [a_{1i}, a_{2i}, \cdots, a_{ni}], \quad \mathbf{a}_{ij} = [a_{1ij}, a_{2ij}, \cdots, a_{nij}] \\
b_i = \log(c_i), \quad b_{ij} = \log(c_{ij})
\]  

\tag{C.6}
into (C.3), we see that the GP can be expressed as

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{N_0} e^{a_{ij}^T y + b_{ij}} \\
\text{subject to} & \quad \sum_{j=1}^{N_i} e^{a_{ij}^T y + b_{ij}} \leq 1, \quad i = 1, \ldots, p \\
& \quad e^{a_i^T y + b_i} = 1, \quad i = p + 1, \ldots, p + m.
\end{align*}
\]

While this form is not itself convex, taking a log transformation of the objective and constraint functions yields an equivalent convex problem:

\[
\begin{align*}
\text{minimize} & \quad \log \left( \sum_{j=1}^{N_0} e^{a_{ij}^T y + b_{ij}} \right) \\
\text{subject to} & \quad \log \left( \sum_{j=1}^{N_i} e^{a_{ij}^T y + b_{ij}} \right) \leq 0, \quad i = 1, \ldots, p \\
& \quad a_i^T y + b_i = 0, \quad i = p + 1, \ldots, p + m.
\end{align*}
\]

Here, the objective and inequality constraints are convex functions in \(y_i\) due to the convexity of log-sum-exp and since convexity is preserved by affine mappings [33]; the equality constraints are affine functions of \(y_i\) as required for convex programming. It follows that (C.8) is a convex program, and the equivalent solution in the original space can be found via inversion of the mapping \(x_i = e^{y_i}\).

### C.4 Fractional Coefficients and Generalized Geometric Programming

If \(f_i(x)\) is a posynomial, then \(g_i(x) = f_i^a(x)\) is clearly a posynomial for positive integers \(a\) following polynomial expansion. However, this can not be generalized to all \(a \in \mathbb{R}_+\). As a result, objective functions and inequality constraints containing fractional exponents of posynomials can not be solved directly by geometric programming.
By introducing a new set of variables \( t_i \) for each function \( f_i \), and a new set of (posynomial) inequality constraints

\[
f_i(x) \leq t_i,
\]

the optimization problem can be transformed to an equivalent GP. This type of problem belongs to the class of \textit{generalized geometric programs} (GGP).

**Example:** Consider the problem of minimizing a product of posynomials with fractional positive coefficients, subject to posynomial inequality constraints and monomial equality constraints:

\[
\begin{align*}
\text{minimize} & \quad \prod_{i=1}^{N} f_i^{a_i}(x), \quad a_i \in \mathbb{R}_+ \\
\text{subject to} & \quad f_i(x) \leq 1, \quad i = N + 1, \ldots, N + p \\
& \quad h_i(x) = 1, \quad i = N + p + 1, \ldots, N + p + m.
\end{align*}
\]

The equivalent GP formulation is

\[
\begin{align*}
\text{minimize} & \quad \prod_{i=1}^{N} t_i^{a_i} \\
\text{subject to} & \quad f_i(x) \leq t_i, \quad i = 1, \ldots, N \\
& \quad f_i(x) \leq 1, \quad i = N + 1, \ldots, N + p \\
& \quad h_i(x) = 1, \quad i = N + p + 1, \ldots, N + p + m,
\end{align*}
\]

where the objective function has been transformed to a monomial in \( t_i \). The inequality constraints \( f_i(x) \leq t_i \) of (C.10) are satisfied with equality at its optimum (i.e., \( f_i(x^*) = t_i^* \)). This can be easily proven by contradiction: At the optimum, assume that one or more \( \hat{t}_i > f_i(x^*) \). These values of \( \hat{t}_i \) are not minimizers of the objective function, as a product of positive terms \( t_i \) with positive coefficients is monotonic increasing in all \( t_i \). Their values can be reduced to \( t_i^* = f_i(x^*) < \hat{t}_i \) without violating any of the inequality constraints, and the objective function can be reduced accordingly. Consequently, we are able to directly solve the GGP (C.10) by solving the equivalent GP (C.11).
Appendix D

A Simple Scheme for Adaptive Modulation

Section 4.4 contains simulation results which illustrate theoretically achievable rates when users employ Gaussian codebooks. In contrast to these previous results, we now consider the selection of constellations for modulation to achieve a maximum throughput for a specified bit error rate (BER) target of $\beta_{kj}$ on user $k$’s $j$th substream. The only assumption relating to choice of constellation in the PMSE minimization algorithm is that transmitted symbols must possess unit energy on average; the approach suggested in this section can thus be applied to any appropriately scaled constellation with a suitably accurate functional form relating BER and $M$. In this implementation, we employ $M$-PSK constellations for simplicity of analysis.

The precoder designs in Sections 4.2 and 4.3 assume that the noise-plus-interference at each mobile receiver is Gaussian; that assumption may become invalid under fixed modulation schemes such as $M$-PSK. It may remain valid when the number of interferers is large (due to the central limit theorem), but this is not necessarily true for systems with a small number of users (e.g., with $K = 2$ as illustrated in this appendix). In our results, we present an ensemble average over a number of realizations of Rayleigh fading channels.
(i.e., with complex Gaussian coefficients), and the average noise-plus-interference can be viewed as Gaussian; in channels with line-of-sight components (e.g., Rician fading), the assumption may not hold.

We propose two approaches for selecting the modulation scheme for each substream.

**D.1 Naive Approach**

This approach selects the largest PSK constellation of $b_{kj}$ bits per stream that satisfies the required BER constraint. The constraint is satisfied using a closed form BER approximation [96],

$$\text{BER}_\text{PSK}(\gamma) \approx c_1 \exp\left(\frac{-c_2 \gamma}{2c_3 b - c_4}\right),$$

where $M = 2^b$ is the size of the PSK constellation. We apply the least aggressive of the bounds proposed in [96] by using the values $c_1 = 0.25, c_2 = 8, c_3 = 1.94, \text{ and } c_4 = 0$. We note that this approximation only holds for $b \geq 2$; as such, for $b = 1$, one can use the exact expression for BPSK:

$$\text{BER}_\text{BPSK}(\gamma) = \frac{1}{2} \text{erfc}\left(\sqrt{\gamma}\right).$$

The BPSK expression can be used as a test of feasibility for the specified BER target; if the resulting BER under BPSK modulation is higher than $\beta_{kj}$, then we have two options: either declare the BER target infeasible, or transmit using the lowest modulation depth available (i.e., BPSK). In this work, we have elected to transmit using BPSK whenever the PMSE stage has allocated power to the data stream.

**D.2 Probabilistic Approach**

The naive approach is quite conservative in that there may be a large gap between the BER requirement and $\beta_{kj}$, the BER achieved for each channel realization. We suggest
A probabilistic bit allocation scheme that switches between $b_{kj}$ bits (as determined by the naive approach) and $b_{kj} + 1$ bits with probability

$$p_{kj} = \frac{\beta_{kj} - \text{BER}_{b_{kj}}}{\text{BER}_{b_{kj}+1} - \text{BER}_{b_{kj}}}.$$  

This modulation strategy may not be appropriate for systems requiring instantaneous satisfaction of BER constraints; however, the probabilistic method will still achieve the desired BER in the long-term over multiple channel realizations.

Figure D.1 shows the sum-rate achieved in the same system configuration as in Figure 4.1 ($K = 2$, $M = 4$, $N_k = L_k = 2$) under the $M$-PSK modulation scheme described above. The simulations use two data streams per user and a target bit error rate of $\beta_{kj} = 10^{-2}$; 5000 data and noise realizations are used for each channel realization. The plot illustrates the average number of bits per transmission for user 1; due to symmetry, the corresponding plot for user 2 is identical. Note that in contrast to the previous
Figure D.2: BER vs. SNR for user 1 with M-PSK modulation, $K = 2$, $M = 4$, $N_k = L_k = 2$

results based on Gaussian coding using spectral efficiency, the sum-rate in Figure D.1 is the average number of bits transmitted per realization using symbols from a PSK constellation.

In Figure D.1 we consider using the PSK modulation scheme for the PMSE precoder and the sum-MSE precoder designed in [19]. Examining this plot reveals that using the PMSE criterion is justified at practical SNR values with improvements of approximately one bit per transmission near 15 dB. Furthermore, using the probabilistic modulation scheme (designated “PMSE-P”) yields an additional improvement of more than half a bit per transmission across all SNR values.

In Figure D.2, we plot average BER versus SNR for the same system configuration as in Figure D.1. This plot illustrates how the naive bit allocation algorithm attempts to achieve the target BER of $10^{-2}$ for all data streams under PMSE, but also overshoots the target, converging to a BER of approximately $5 \times 10^{-4}$. This can be attributed
to the looseness of the BER bound mentioned above. In contrast, the probabilistic rate allocation algorithm not only increases the rate, as shown in Figure [D.1] but also converges to a BER that is much closer to the desired target BER. The remaining gap between the actual BER achieved and the target BER can again be attributed to looseness in the approximations of (D.1) and (D.2).
Appendix E

MSE-Based Functions Use All Available Data Power

This appendix demonstrates that the precoder designs which minimize MSE-based objective functions as considered in this thesis are optimized by using all available data power. This claim is made in [82] for equal estimation error variances $\sigma_k^2$ in the sum-MSE case; here, we offer a stronger proof which is generalized to arbitrary $\sigma_k^2$ and to the PMSE function.

E.1 KKT Based Proof for Sum-MSE

For the purposes of this proof, we employ the virtual user notation from Section 3.2, where the entire group of data substreams across all users are relabelled as $l = 1, \ldots, L$. Expanding the $\sigma_{\text{eff}}^2$ and $\tilde{\mathbf{R}}^{-1}$ terms in (6.18) results in the following expression:

$$
\text{SMSE}^{UL} = L - M + \left( \sigma_n^2 + \sum_{l=1}^{L} \sigma_{\kappa(l)}^2 q_l \right) \text{tr} \left[ \left\{ \sum_{l=1}^{L} q_l \left( \tilde{\mathbf{h}}_l \tilde{\mathbf{h}}_l^H + \sigma_{\kappa(l)}^2 \mathbf{I} \right) + \sigma_n^2 \mathbf{I} \right\}^{-1} \right]. \quad (E.1)
$$

In [82] p. 131–132, Section 5.8.6], it is demonstrated that the sum-MSE minimization problem may be solved by replacing the sum power inequality constraint with an equality constraint, as the sum-MSE objective function is non-increasing in $P_{\text{max}}$ (when all $\sigma_k^2$ are
equal). Here, we offer a stronger claim, by generalizing to the case of arbitrary $\sigma_k^2$, and by proving that the power constraint must be tight at any optimum.

The form of the power allocation optimization subproblem is similar to (3.38):

$$ (q_1^*, \ldots, q_L^*) = \arg \min_{q_1, \ldots, q_L} \text{SMSE}^{UL} $$

$$ \text{s.t. } q_l \geq 0 \quad l = 1, \ldots, L, \quad \sum_{l=1}^L q_l \leq P_{\text{max}}, $$

(E.2)

The KKT conditions for optimization of the sum-MSE with respect to power allocations $q_l$ under a sum power constraint can be expressed as

$$ \frac{\partial \text{SMSE}^{UL}}{\partial q_i} + \lambda_{\text{sum}} - \lambda_i = 0 \quad \forall i = 1, \ldots, L, $$

(E.3)

$$ q_i \geq 0 \quad \forall i = 1, \ldots, L, \quad \sum_{l=1}^L q_l \leq P_{\text{max}} $$

(E.4)

$$ \lambda_i \geq 0 \quad \forall i = 1, \ldots, L, \quad \lambda_{\text{sum}} \geq 0 $$

(E.5)

$$ \lambda_i q_i = 0 \quad \forall i = 1, \ldots, L, \quad \lambda_{\text{sum}} \left( \sum_{l=1}^L q_l - P_{\text{max}} \right) = 0. $$

(E.6)

Differentiating with respect to $q_i$ follows using the differential identity for the matrix inverse employed in (A.1):

$$ \frac{\partial \text{SMSE}^{UL}}{\partial q_i} = \sigma^2_{\kappa(i)} \text{tr} \left[ \tilde{R}^{-1} \right] - \left( \sigma^2_n + \sum_{l=1}^L \sigma^2_{\kappa(l)} q_l \right) \text{tr} \left[ \tilde{R}^{-1} \left( \tilde{h}_i \tilde{h}_i^H + \sigma^2_{\kappa(i)} I \right) \tilde{R}^{-1} \right] $$

$$ = \sigma^2_{\kappa(i)} \text{tr} \left[ \tilde{R}^{-1} \right] - \sigma^2_{\text{eff}} \sigma^2_{\kappa(i)} \text{tr} \left[ \tilde{R}^{-2} \right] - \sigma^2_{\text{eff}} \tilde{h}_i \tilde{R}^{-2} \tilde{h}_i $$

$$ = \sigma^2_{\kappa(i)} \text{tr} \left[ \tilde{R}^{-1} \right] - \left( \sigma^2_{\text{eff}} I \right) \tilde{R}^{-2} - \sigma^2_{\text{eff}} \tilde{h}_i \tilde{R}^{-2} \tilde{h}_i $$

$$ = \sigma^2_{\kappa(i)} \text{tr} \left[ \tilde{R}^{-1} - \left( \tilde{R} - \sum_{l=1}^L q_l \tilde{h}_l \tilde{h}_l^H \right) \tilde{R}^{-2} \right] - \sigma^2_{\text{eff}} \tilde{h}_i \tilde{R}^{-2} \tilde{h}_i $$

$$ = \sigma^2_{\kappa(i)} \sum_{l=1}^L q_l \tilde{h}_l \tilde{h}_l^H \tilde{R}^{-2} \tilde{h}_l - \sigma^2_{\kappa(i)} \sum_{l=1}^L q_l \tilde{h}_l \tilde{h}_l^H \tilde{R}^{-2} \tilde{h}_l. $$

(E.7)

Substituting this expression into the stationarity condition (E.3) yields the following set of equations:

$$ \lambda_{\text{sum}} - \lambda_i = \sigma^2_{\text{eff}} \tilde{h}_i \tilde{R}^{-2} \tilde{h}_i - \sigma^2_{\kappa(i)} \sum_{l=1}^L q_l \tilde{h}_l \tilde{h}_l^H \tilde{R}^{-2} \tilde{h}_l. $$

(E.8)
Next, we expand the set of equations in (E.8) as

\[
\begin{bmatrix}
\sigma_{\text{eff}}^2 - q_1 \sigma_{\kappa(1)}^2 & -q_2 \sigma_{\kappa(1)}^2 & \ldots & -q_L \sigma_{\kappa(1)}^2 \\
-q_1 \sigma_{\kappa(2)}^2 & \sigma_{\text{eff}}^2 - q_2 \sigma_{\kappa(2)}^2 & \ldots & -q_L \sigma_{\kappa(2)}^2 \\
\vdots & \vdots & \ddots & \vdots \\
-q_1 \sigma_{\kappa(L)}^2 & -q_2 \sigma_{\kappa(L)}^2 & \ldots & \sigma_{\text{eff}}^2 - q_L \sigma_{\kappa(L)}^2
\end{bmatrix}
\begin{bmatrix}
\tilde{h}_1^H \tilde{R}^{-2} \tilde{h}_1 \\
\tilde{h}_2^H \tilde{R}^{-2} \tilde{h}_2 \\
\vdots \\
\tilde{h}_L^H \tilde{R}^{-2} \tilde{h}_L
\end{bmatrix}
= \begin{bmatrix}
\lambda_{\text{sum}} - \lambda_1 \\
\lambda_{\text{sum}} - \lambda_2 \\
\vdots \\
\lambda_{\text{sum}} - \lambda_L
\end{bmatrix},
\tag{E.9}
\]

It is possible that the optimum power allocation contains one or more inactive streams \(i \in S_I\); that is, with \(q_i = 0\) (see Section 3.4). Without loss of generality, we assume in this proof that the first \(a\) data streams are active, and the remaining \(L - a\) data streams are inactive; that is,

\[
S_A = \{1, 2, \ldots, a\} \\
S_I = \{a + 1, a + 2, \ldots, L\}.
\tag{E.10}
\]

In the first \(a\) rows of (E.9), the rightmost \(L - a\) columns in the first matrix have zero value (corresponding to \(q_i = 0\) for \(i \in S_I\)); moreover, since the first \(a\) data streams are active, the corresponding Lagrange multipliers \(\lambda_i = 0\) for \(i = \{1, \ldots, a\}\) due to complementary slackness. Examining these first \(a\) rows provides a new system of linear equations

\[
\begin{bmatrix}
\sigma_{\text{eff}}^2 - q_1 \sigma_{\kappa(1)}^2 & -q_2 \sigma_{\kappa(1)}^2 & \ldots & -q_a \sigma_{\kappa(1)}^2 \\
-q_1 \sigma_{\kappa(2)}^2 & \sigma_{\text{eff}}^2 - q_2 \sigma_{\kappa(2)}^2 & \ldots & -q_a \sigma_{\kappa(2)}^2 \\
\vdots & \vdots & \ddots & \vdots \\
-q_1 \sigma_{\kappa(a)}^2 & -q_2 \sigma_{\kappa(a)}^2 & \ldots & \sigma_{\text{eff}}^2 - q_a \sigma_{\kappa(a)}^2
\end{bmatrix}
\begin{bmatrix}
\tilde{h}_1^H \tilde{R}^{-2} \tilde{h}_1 \\
\tilde{h}_2^H \tilde{R}^{-2} \tilde{h}_2 \\
\vdots \\
\tilde{h}_L^H \tilde{R}^{-2} \tilde{h}_L
\end{bmatrix}
= \begin{bmatrix}
\lambda_{\text{sum}} \\
\lambda_{\text{sum}} \\
\vdots \\
\lambda_{\text{sum}}
\end{bmatrix},
\tag{E.11}
\]
which can be further simplified as

\[
\begin{bmatrix}
\sigma_{\text{eff}}^2 - q_1 \sigma_{\kappa(1)}^2 & -q_2 \sigma_{\kappa(1)}^2 & \ldots & -q_a \sigma_{\kappa(1)}^2 \\
-q_1 \sigma_{\kappa(2)}^2 & \sigma_{\text{eff}}^2 - q_2 \sigma_{\kappa(2)}^2 & \ldots & -q_a \sigma_{\kappa(2)}^2 \\
\vdots & \vdots & \ddots & \vdots \\
-q_1 \sigma_{\kappa(a)}^2 & -q_2 \sigma_{\kappa(a)}^2 & \ldots & \sigma_{\text{eff}}^2 - q_a \sigma_{\kappa(a)}^2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{h}_1^H \tilde{R}^{-2} \tilde{h}_1 \\
\tilde{h}_2^H \tilde{R}^{-2} \tilde{h}_2 \\
\vdots \\
\tilde{h}_a^H \tilde{R}^{-2} \tilde{h}_a \\
\end{bmatrix} = \lambda_{\text{sum}} \begin{bmatrix} 1 \\
1 \\
\vdots \\
1 \\
\end{bmatrix}.
\]

(E.12)

The matrix \(A\) can be decomposed as

\[
A = \sigma_{\text{eff}}^2 I_a - s_A q_A^T,
\]

where \(s_A\) and \(q_A\) are the vectors of channel estimation error variances and virtual uplink power allocations for all active streams,

\[
s_A = \begin{bmatrix}
\sigma_{\kappa(1)}^2 \\
\sigma_{\kappa(2)}^2 \\
\vdots \\
\sigma_{\kappa(a)}^2
\end{bmatrix},
q_A = \begin{bmatrix} q_1 \\
q_2 \\
\vdots \\
q_a \end{bmatrix}.
\]

(E.14)

Since \(s_A q_A^T\) is an outer product of two vectors, it has one non-zero eigenvalue,

\[
q_A^T s_A = \sum_{i=1}^a q_i \sigma_{\kappa(i)}^2.
\]

(E.15)

It follows that the eigenvalues of \(A\) are \(\sigma_{\text{eff}}^2\) (with multiplicity \(a-1\)) and a single eigenvalue

\[
\sigma_{\text{eff}}^2 - \sum_{i=1}^a q_i \sigma_{\kappa(i)}^2 = \sigma_n^2.
\]

(E.16)

Since none of the eigenvalues of \(A\) is zero, it follows that \(\text{null } [A] = \emptyset\). Since \(b \neq 0_a\), we can conclude that \(\lambda_{\text{sum}} \neq 0\). Complementary slackness in the sum power constraint thus dictates that the constraint is tight and all available transmit power is used; i.e.,

\[
\sum_{l=1}^L q_l = P_{\text{max}}.
\]
E.2 Generalized Proof for Sum-MSE and PMSE

Consider a set of fixed estimation error variances $\sigma_k^2$ corresponding to fixed values of $\sigma_{\hat{H}_k}^2$ and $P_T$; we can rewrite (6.69) as

$$\tilde{R} = P_D \left( \hat{H} \tilde{V} \hat{Q} \tilde{V}^H \hat{H}^H + \sum_{l=1}^{L} \tilde{q}_l \sigma_{\kappa(l)}^2 I \right) + \sigma_n^2 I; \quad (E.17)$$

thus, the element-wise matrix derivative of $\tilde{R}$ with respect to $P_D$ is

$$\frac{\partial \tilde{R}}{\partial P_D} = \hat{H} \tilde{V} \hat{Q} \tilde{V}^H \hat{H}^H + \sum_{l=1}^{L} \tilde{q}_l \sigma_{\kappa(l)}^2 I = \frac{1}{P_D} \left( \tilde{R} - \sigma_n^2 I \right). \quad (E.18)$$

Differentiation of the virtual uplink MMSE matrix (6.17) with respect to $P_D$ yields the following result,

$$\frac{\partial \varepsilon^{UL}_k}{\partial P_D} = -\sqrt{\tilde{Q}_k} \tilde{V}_k^H \hat{H}_k^R \tilde{H}_k \tilde{V}_k \sqrt{\tilde{Q}_k} - P_D \sqrt{\tilde{Q}_k} \tilde{V}_k^H \hat{H}_k^R \frac{\partial \tilde{R}^{-1}}{\partial P_D} \tilde{H}_k \tilde{V}_k \sqrt{\tilde{Q}_k}$$

$$= -\sigma_n^2 \sqrt{\tilde{Q}_k} \tilde{V}_k^H \hat{H}_k^R \tilde{R}^{-2} \tilde{H}_k \tilde{V}_k \sqrt{\tilde{Q}_k} \quad (E.19)$$

that is, the matrix derivative with respect to $P_D$ (for fixed $P_T$ and $\sigma_k^2$) is negative definite; thus, its diagonal elements (corresponding to individual data stream MSEs) are negative, and $[\varepsilon^{UL}_k]_{l,l}$ are decreasing in $P_D$. Linearity dictates that $\text{tr} [\varepsilon^{UL}_k]$ also decreases in $P_D$ for all $k$. Consequently, the sum-MSE and PMSE functions are decreasing functions of $P_D$, and optimization will always satisfy data power constraints with equality.
Appendix F

List of Publications

The following publications are based upon the contributions made in this dissertation:

Journals


Conferences


MIMO-OFDM systems,” *IEEE International Conference on Communications*, Glasgow, Scotland, June 2007.


Bibliography


