GEOMETRIC ANALYSIS ON SOLUTIONS OF SOME DIFFERENTIAL INEQUALITIES AND
WITHIN RESTRICTED CLASSES OF HOLOMORPHIC FUNCTIONS

by

Damir Kinzebulatov

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University of Toronto

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Abstract

Geometric analysis on solutions of some differential inequalities and within restricted classes of holomorphic functions

Damir Kinzebulatov
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto
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Parts I and II are devoted to study of solutions of certain differential inequalities. Namely, in Part I we show that a germ of an analytic set (real or complex) admits a Gagliardo-Nirenberg type inequality with a certain exponent $s \geq 1$. At a regular point $s = 1$, and the inequality becomes classical. As our examples show, $s$ can be strictly greater than 1 even for an isolated singularity.

In Part II we prove the property of unique continuation for solutions of differential inequality $|\Delta u| \leq |Vu|$ for a large class of potentials $V$. This result can be applied to the problem of absence of positive eigenvalues for self-adjoint Schrödinger operator $-\Delta + V$ defined in the sense of the form sum. The results of Part II are joint with Leonid Shartser.

In Parts III and IV we derive the basic elements of complex function theory within some subalgebras of holomorphic functions (including extension from submanifolds, corona type theorem, properties of divisors, approximation property). Our key instruments and results are the analogues of Cartan theorems A and B for the ‘coherent sheaves’ on the maximal ideal spaces of these subalgebras, and of Oka-Cartan theorem on coherence of the sheaves of ideals of the corresponding complex analytic subsets.

More precisely, in Part III we consider the algebras of holomorphic functions on regular coverings of complex manifolds whose restrictions to each fiber belong to a translation-invariant Banach subalgebra of bounded functions endowed with sup-norm. The model examples of such subalgebras are Bohr’s holomorphic almost periodic functions on tube domains, and all fibrewise bounded holomorphic functions on regular coverings of complex manifolds.
In Part IV the primary object of study is the subalgebra of bounded holomorphic functions on the unit disk whose moduli can have only boundary discontinuities of the first kind. The results of Parts III and IV are joint with Alexander Brudnyi.
Dedication

To my father who would have been proud.

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Introduction

The characteristic Puiseux exponents of a germ of a complex irreducible curve provide a fairly complete measurement of the complexity of its singularity. However, a similar notion is not known in larger dimensions. A step in this direction is an invariant of a germ of a singular domain introduced by Bos-Milman in [BosM] via a Gagliardo-Nirenberg type inequality on this domain. Our attempt to extend these ideas to the case of singular analytic sets is the content of Part I of this thesis. More precisely, we show that for every (real or complex) analytic set $X$ its germ at $x \in X$ admits a Gagliardo-Nirenberg type inequality with some exponent $1 \leq s = s(x) < \infty$. If $x$ is a regular point of $X$, then $s(x) = 1$, and the inequality becomes classical; as our examples show, $s(x)$ can be strictly greater than 1 even for an isolated singularity at $x \in X$.

The results of Part I appeared in [K].

In Part II we obtain a sufficient condition for uniqueness of continuation for differential inequality $|\Delta u| \leq |Vu|$ for a large class of potentials $V$. This result can be applied to the problem of absence of positive eigenvalues for self-adjoint Schrödinger operator $-\Delta + V$ defined in the sense of the form sum (see, e.g., [Ka1]). More precisely, our class of potentials $V$ is a local analogue of the class of potential for which the operator $-\Delta + V$ is well defined; this result strengthens the classical results of Jerison-Koenig [JK] and E. Stein [St2]. Solutions of differential inequality $|\Delta u| \leq |Vu|$ for which we prove the unique continuation apriori contains the eigenfunctions corresponding to positive eigenvalues of $-\Delta + V$, but since these eigenfunctions must have compact support [Ka2], it follows that they must vanish identically due to the property of unique continuation. The results of Part II are contained in [KSh].

2. Towards complex function theory on the maximal ideal spaces of subalgebras of holomorphic functions (Parts III and IV).

In the 1950s K. Oka and H. Cartan (following Oka proving his crucial coherence lemma) laid down the foundations of modern complex function theory of several variables. They introduced the notion of a coherent sheaf, proved that the sheaf of ideals of a complex analytic subset is coherent ("Oka-Cartan theorem"), and that:
(A) Every germ of a coherent sheaf $\mathcal{A}$ on a Stein manifold $X$ is generated by its global sections ("Cartan theorem A").

(B) The sheaf cohomology groups $H^i(X, \mathcal{A})$ ($i \geq 1$) are trivial ("Cartan theorem B").

Recall that a sheaf of modules over the sheaf of germs of holomorphic functions on $X$ is called coherent if locally both this sheaf and its sheaf of relations are finitely generated. The class of coherent sheaves is closed under natural operations. Most sheaves that arise in complex analysis are coherent. Stein manifolds are complex manifolds that admit holomorphic embeddings into appropriate $\mathbb{C}^n$.

Cartan theorems A and B together with their numerous corollaries constitute the so-called Oka-Cartan theory of Stein manifolds. Consequences of the latter include solutions of numerous classical problems of the theory of several complex variables within the algebra $\mathcal{O}(X)$ of all holomorphic functions on $X$ (including solutions of the problem of extension from analytic subsets, corona problem, Cousin problems, Levi problem, Poincaré problem, and many others, e.g. as in the survey in [GR]).

The further development of complex function theory was driven, in part, by the problems that required study of behaviour of holomorphic functions (bounded or satisfying certain growth conditions) ‘at infinity’. As a consequence, the question of whether the problems of classical complex function theory can be solved within a proper subclass of $\mathcal{O}(X)$ (e.g., consisting of bounded or $L^p$-summable holomorphic functions) started to play an important role. However, trying to incorporate restrictions on holomorphic functions such as boundedness or $L^p$-summability, the immediate applications of the classical Oka-Cartan theory encounter considerable difficulties. Consequently, one has to complement the sheaf-theoretic approach of Oka-Cartan by various methods of hard analysis, e.g., by integral representation formulas on complex manifolds, estimates on solutions of $\bar{\partial}$-equation, etc (cf. [HL]).

We pursue the approach based on Oka-Cartan theory for the ‘coherent sheaves’ on the maximal ideal spaces of several subalgebras of holomorphic functions and derive the basic elements of complex functions theory within these subalgebras.

The ultimate goal for us is to extend our theory to the Hardy algebra of bounded holomorphic functions on a pseudoconvex domain in $\mathbb{C}^n$ in a hope that this may shed some light on the corona
problem for the latter, cf. [Br6].

To be more precise, in Part III we study the subalgebras of holomorphic functions on regular coverings of complex manifolds whose restrictions to each fiber belong to a translation-invariant Banach subalgebra of bounded functions endowed with sup-norm. Our key instruments and results are the analogues of Cartan theorems A and B for the ‘coherent sheaves’ on the maximal ideal spaces of these subalgebras.

The model examples of these subalgebras are:

1. Subalgebras of Bohr’s holomorphic almost periodic functions on tube domains in $\mathbb{C}^n$ (i.e. the uniform limits of exponential polynomials). The theory of almost periodic functions was created in 1920s. Its applications to various areas of mathematics include harmonic analysis, differential equations, etc. The results of the present thesis show that the general theory of holomorphic almost periodic functions of several variables (see, e.g., [Bo, Lev, FR] and references therein) to large extent can be reduced to solutions of $\bar{\partial}$-equation and applications of analytic sheaf theory.

2. Subalgebras of all fibrewise bounded holomorphic functions. This subalgebra arises, e.g., in study of holomorphic $L^2$-functions on coverings of pseudoconvex manifolds [GHS, Br2, Br5, La], in Caratheodory hyperbolicity (Liouville property) of $X$ [LS, Lin], in corona-type problems for bounded holomorphic functions on coverings of bordered Riemann surfaces, Hartogs-type theorems, integral representation of holomorphic functions of slow growth on coverings of Stein manifolds [Br1]-[Br4], etc.

Using our Cartan type theorems A and B, we study the problem of extension and the properties of their divisors within these subalgebras, we obtain corona and Hartogs type theorems, prove a holomorphic analogue of Peter-Weyl theorem and Runge type approximation, describe the uniqueness sets of the functions in these subalgebras, and derive other properties. Some of these results are new even for holomorphic almost periodic functions.

In our proofs we make use of some results and methods of the theory of coherent-type sheaves taking values in Banach or Frechet spaces that were pioneered by Bishop and Bungart [Bu1, Bu2], and developed further by Leiterer [Lt1], Douady, Lempert and Patyi [Lem], and others. The results of Part 3 were partly announced in [BrK1] and also appeared in [BrK2].
In Part IV the primary object of study is the subalgebra $A \subset H^\infty(\mathbb{D})$ of bounded holomorphic functions on the unit disk $\mathbb{D}$ whose moduli can have boundary discontinuities of only the first kind.

We elaborate the same approach as in Part 3 (albeit in a single variable case): we introduce the algebra $\mathcal{O}(M_A)$ of ‘holomorphic functions’ on the maximal ideal space of the subalgebra $A$, and show that the latter two algebras are isomorphic (our proof is via density arguments and solution of $\bar{\partial}$-equation). We then establish the connection between algebra $\mathcal{O}(M_A)$ and Bohr’s almost periodic functions (as well as with D. Sarason’s subalgebra of semi-almost periodic functions [Sar]).

Consequently, we combine the sheaf-theoretic methods with Bohr’s theory to study subalgebra $A$. In particular, we prove the Grothendieck’s approximation property. Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces that appear naturally in analysis, the approximation property was not so far established even for $H^\infty(\mathbb{D})$ [BR]). We obtain corona type theorem for subalgebra $A$, show that $A$ is projective free (and hence the corona-like problem of completion of matrices with entires in $A$ is solvable [C], see details in Section 13.2), describe the Šilov boundary of $A$, etc. We also describe the topological structure of $M_A$ (including its Čech cohomology groups).

We note that although our results were verified for a particular subalgebra of $H^\infty(\mathbb{D})$, our methods are rather abstract, and, in principle, can be applied to various other subalgebras of $H^\infty(\mathbb{D})$.

The results of Part IV are contained in [BrK3] and [BrK4].
Part I

Gagliardo-Nirenberg type inequalities on analytic sets

(2008-2009)
The classical Gagliardo-Nirenberg inequality is valid for smooth functions defined on a compact domain in $\mathbb{R}^n$ having Lipschitz regular boundary. As was shown in [BosM], in the case when the boundary of the domain has outward pointing cusps, a Gagliardo-Nirenberg type inequality holds but with a certain exponent $1 \leq s < \infty$ ($s = 1$ in the classical case of Lipschitz regular boundary). Below we show the validity of Gagliardo-Nirenberg type inequalities (i.e. in the quotient sup-norms) for smooth functions defined on a real or complex analytic set, also with a certain exponent $1 \leq s < \infty$, and give an example of an algebraic set with an isolated singularity which does not admit the Gagliardo-Nirenberg type inequality with any exponent smaller than a certain value $s > 1$.

0.0.1 Main result

Consider first the case of a real analytic set. Let $N$ be a relatively compact open subset of $\mathbb{R}^n$. Given $f \in C^\infty(\bar{N})$ and $m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, we define

$$|f|^N_m := \sum_{|\gamma| \leq m} |D^\gamma f|^N$$

(0.0.1)

where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n$, $D^\gamma = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$ and $|\cdot|^N$ is the ordinary sup-norm on $N$.

Let $X$ be a real analytic set in $\mathbb{R}^n$ (assume to be the closure in $\mathbb{R}^n$ of the set of its smooth points), $x \in X$. Given a relatively compact neighbourhood $U$ of $x \in \mathbb{R}^n$ and a function $f \in C^\infty(\bar{U})$, we define its quotient-norm

$$\|f\|_{U,m,X} := \inf_{(h = f)|_{U \cap X}} |h|^U_m,$$

(0.0.2)

where the infimum is taken over all $h \in C^\infty(\bar{U})$ such that $h = f$ on $X \cap U$.

We will say that $X$ admits a Gagliardo-Nirenberg inequality at $x \in X$ with exponent $1 \leq s < \infty$ if there exist a neighbourhood $U$ of $x \in \mathbb{R}^n$ and constants $C_m > 0$ such that for all $f \in C^\infty(\bar{U})$ ($f \neq 0$ on $X \cap U$), all $m$ and all $1 \leq k < m/s$ we have

$$\|f\|_{k,X}^U \leq C_0 C_m^k \left( \frac{\|f\|_{0,X}^U}{\|f\|_{0,X}^U} \right)^{k/s}.$$  

(0.0.3)

It is well known that if $x$ is a regular point of $X$, then the quotient norms (in a neighbourhood of $x$) coincide with the ordinary norms, and $X$ admits a Gagliardo-Nirenberg inequality at $x$ with exponent $s = 1$, e.g., see [BosM].
Let us turn to the case when \( x \in X \) can be a singular point.

Let \( M \) be a smooth manifold and \( N \) be a relatively compact open subset of \( M \). There exist finitely many coordinate charts \( S_i \) on \( M \) with coordinate maps \( \eta_i : \{ x \in \mathbb{R}^n : |x| < 1 \} \to S_i \) \((l = \dim(M))\) which cover \( N \). Furthermore, we can find finitely many relatively compact open subsets \( T_j \subset M \) such that \( T_j \subset S_i \) for certain \( i = i(j) \) (in what follows, let us fix some choice of \( i \)), and \( N \subset \cup_j T_j \). Given \( f \in C^\infty(\bar{N}) \), define

\[
|f|^N_m = \max_j |\eta_i^* f|^{n_i^{-1}(T_j \cap N)}_m < \infty, 
\]

where \( n_i^{-1}(T_j \cap N) \) is, evidently, relatively compact for each \( j \) and \( i = i(j) \), and the norms in the right-hand side of (0.0.4) are defined by (0.0.1).

The norm defined by (0.0.4) is determined by our choice of \( S_i, \eta_i, T_j \) and \( i = i(j) \). Nevertheless, it is easy to see that any two of such norms are equivalent. The following estimate follows straightforwardly from [BM2, p. 774–775] and [ABM, p. 2].

**Proposition 0.0.1** ([ABM], [BM2]). Let \( X \subset \mathbb{R}^n \) be an analytic set, \( X \ni 0 \), and \( U \subset \mathbb{R}^n \) be a relatively compact neighbourhood of \( 0 \in \mathbb{R}^n \). Let \( M \) be an analytic manifold and let \( \varphi : M \to \mathbb{R}^n \) be a proper analytic map such that \( \varphi(M) = X \). Then there exists a constant \( c \in \mathbb{N} \) such that for all \( f \in C^\infty(\bar{U}) \) we have

\[
\|f\|_{k,X} \leq C_k \|\varphi^* f\|^{\varphi^{-1}(X \cap U)}_{c_k} 
\]

for a fixed \( c > 0 \) for all \( k \geq 1 \), for some \( C_k > 0 \).

**Theorem 0.0.2.** Given an analytic set \( X \), it admits a Gagliardo-Nirenberg inequality at any of its points \( x \) (with a certain exponent \( 1 \leq s = s(x) < \infty \)).

The proof of Theorem 0.0.2 actually provides an upper bound for \( s \).

**0.0.2 Examples**

**Example 0.0.3.** Let

\[
X = \{(x, y) \in \mathbb{R}^2 : y^q = x^p \},
\]

where \( q \) is even and \( \frac{p}{q} > 1 \).
Let us show that \( X \) does not admit a Gagliardo-Nirenberg inequality at 0 with any exponent smaller than \( s = \frac{p}{q} \). We employ, with slight modification, the family of functions that was used in [BosM] for the proof of an analogous statement for a compact domain in \( \mathbb{R}^n \).

(1) Suppose first that \( p \) is odd. Consider on \( U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \} \) the family of functions

\[
 f_k(x, y) = y\varphi(1 - kx),
\]

where

\[
 \varphi(x) = \begin{cases} 
  e^{-1/x}, & x > 0, \\
  0, & x \leq 0,
\end{cases}
\]

where \( g_k \in C^\infty(U) \) is any such that \( g_k|_{X \cap U} = f_k|_{X \cap U} \). Note that for each \( k \)

\[
e^{-1} = \left| \frac{\partial f_k}{\partial y}(0, 0) \right| \leq \inf_{g_k|_{X \cap U} = f_k|_{X \cap U}} \sup_{(x, y) \in U} \left\{ \left| \frac{\partial g_k}{\partial x}(x, y) \right| + \left| \frac{\partial g_k}{\partial y}(x, y) \right| \right\} \leq \|f_k\|_{1,X}^{U}. \tag{0.0.6}
\]

The last inequality follows straightforwardly from the definition of the quotient norm. The first inequality holds due to the following two facts: a) given any such extension \( g_k \) of \( f_k \), we have

\[
\frac{\partial g_k}{\partial x}(0, 0) = \lim_{x \to 0^+} \frac{f_k(x, x^{\frac{p}{q}}) - f_k(0, 0)}{x} = \frac{\lim_{x \to 0^+} \frac{\partial f_k}{\partial x}(0, 0)x + \frac{\partial f_k}{\partial y}(0, 0)x^{\frac{p}{q}} - o(|x|)}{x} = \frac{\partial f_k}{\partial x}(0, 0)
\]

and, similarly,

\[
\frac{\partial g_k}{\partial y}(0, 0) = \lim_{x \to 0^+} \frac{f_k(x, x^{\frac{p}{q}}) - f_k(x, -x^{\frac{p}{q}})}{2x^{\frac{p}{q}}} = \frac{\partial f_k}{\partial y}(0, 0),
\]

and b) for any such \( g \) we have

\[
\left| \frac{\partial f_k}{\partial y}(0, 0) \right| = \left| \frac{\partial g_k}{\partial y}(0, 0) \right| \leq \left| \frac{\partial g_k}{\partial x}(0, 0) \right| + \left| \frac{\partial g_k}{\partial y}(0, 0) \right| \leq \sup_{(x, y) \in U} \left\{ \left| \frac{\partial g_k}{\partial x}(x, y) \right| + \left| \frac{\partial g_k}{\partial y}(x, y) \right| \right\},
\]

so

\[
\left| \frac{\partial f_k(0, 0)}{\partial y} \right| \leq \inf_{g_k|_{X \cap U} = f_k|_{X \cap U}} \sup_{(x, y) \in U} \left\{ \left| \frac{\partial g_k}{\partial x}(x, y) \right| + \left| \frac{\partial g_k}{\partial y}(x, y) \right| \right\}.
\]

Thus, \( e^{-1} \leq \|f_k\|_{1,X}^{U} \) for all \( k \). Also, observe that we always have

\[
\|f_k\|_{0,X}^{U} = \sup_{(x, y) \in X \cap U} |f_k(x, y)|,
\]

9
and, as a result, for all $k$

$$
\|f_k\|_{0,X}^U = \sup_{(x,y) \in X \cap U} |f_k(x,y)| = \sup_{(x,y) \in X \cap U} |y\varphi(1 - kx)| = \sup_{0 \leq x < 1/k} |x^{\frac{p}{q}} \varphi(1 - kx)|
$$

since for any $x \geq 1/k$ we have $\varphi(1 - kx) = 0$. Furthermore,

$$
\sup_{0 \leq x < 1/k} |x^{\frac{p}{q}} \varphi(1 - kx)| \leq \sup_{0 \leq x < 1/k} |x^{\frac{p}{q}}| \sup_{0 \leq x < 1/k} |\varphi(1 - kx)| \leq k^{-\frac{p}{q}} e^{-1}.
$$

To estimate $\|f_k\|_{m,X}^U$ we take extension of $f_k|_{X \cap U}$ from $X \cap U$ to $U$, namely, $f_k$ itself. Then

$$
\|f_k\|_{m,X}^U \leq |f_k|_{m}^U \leq C_m k^m
$$

for a certain $C_m > 0$, as follows from the definition of $f_k$. Now it follows from the inequalities obtained above that for all $k$ and $m$

$$
e^{-1} \leq C_0 e^{-1} C(C_1)^m k^{-\frac{p}{q}} (1 - \frac{s}{m}) k^m \frac{p}{q},
$$

so fixing $m$ and taking $k \to \infty$ we obtain that

$$
-\frac{p}{q} \left(1 - \frac{s}{m}\right) + s \geq 0.
$$

Thus, $s \geq \frac{p}{q}$, otherwise taking $m$ sufficiently large and then letting $k \to \infty$ we arrive to contradiction $e^{-1} \leq 0$.

(2) In the case when $p$ is even, $X$ is symmetric with respect to $y$-axis, and we consider the family of functions

$$
f_k(x,y) = y\varphi(1 - kx)\varphi(1 + kx).
$$

We have

$$
\|f_k\|_{0,X}^U = \sup_{(x,y) \in X \cap U} |f_k(x,y)| = \sup_{(x,y) \in X \cap U} |y\varphi(1 - kx)\varphi(1 + kx)| = \\
= \sup_{-1/k \leq x < 1/k} |x^{\frac{p}{q}} \varphi(1 - kx)\varphi(1 + kx)| \leq \sup_{-1/k \leq x < 1/k} x^{\frac{p}{q}} \sup_{-1/k \leq y < 1/k} |\varphi(1 - kx)\varphi(1 + kx)| \leq \\
\leq k^{-\frac{p}{q}} e^{-1/2} e^{-1/2} = k^{-\frac{p}{q}} e^{-1}.
$$

The proof of the inequality $e^{-2} \leq \|f_k\|_{1,X}^U$, $k \in \mathbb{N}$, is analogous to the proof of (0.0.6). Similarly, we can show that $\|f_k\|_{m,X}^U \leq |f_k|_{m}^U \leq C'_m k^m$ for some choice of $C'_m$ for all $k \in \mathbb{N}$. This gives us the required result.
In the complex analytic case (in this case we change our definition of norm (0.0.1),
\[ \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n, \quad |\gamma| = \sum_{j=1}^{n} |\gamma_j|, \quad D^\gamma = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}, \]
\[ \frac{\partial^{\gamma_j}}{\partial x_j^{\gamma_j}} := \frac{\partial^{\gamma_j}}{\partial \text{Re} (x_j)} \quad \text{if } \gamma_j > 0, \quad \frac{\partial^{\gamma_j}}{\partial x_j^{\gamma_j}} := \frac{\partial^{\gamma_j}}{\partial \text{Im} (x_j)} \quad \text{if } \gamma_j < 0, \]
\[ |f|^N_m := \sum_{|\gamma| \leq m} |D^\gamma f|^N, \quad (0.0.7) \]
which gives us complex analogs of the norms (0.0.2) and (0.0.4)) our set \( X \subset \mathbb{C}^n \) can be viewed as a real analytic set embedded in \( \mathbb{R}^{2n} \), the algebras \( \mathbb{C}^\infty (U) \) and \( \mathbb{C}^\infty (\overline{U}) \), where \( U \) is an relatively compact domain, remain the same regardless of whether we consider \( U \) as a subdomain of \( \mathbb{C}^n \) or \( \mathbb{R}^{2n} \), so, in particular, Theorem 0.0.2 applies.

**Example 0.0.4.** Let \( X = \{(x, y) \in \mathbb{C}^2 : y^q = x^p\} \), where \( \frac{p}{q} > 1 \). Let us show that \( X \) does not admit Gagliardo-Nirenberg inequality at 0 with any exponent smaller than \( s = \frac{p}{q} \).

Indeed, let us consider the family of functions
\[ f_k(x, y) = \begin{cases} 
    ye^{-\frac{1}{2-k^p|x|^q}}, & |x| \leq \frac{1}{k}, \\
    0, & |x| > \frac{1}{k}.
\end{cases} \]
Let \( U = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 \leq 1\} \). First, let \( q \) be even. We have \( \text{Re} (y)^q = \text{Re} (x)^p \) for any \((x, y) \in X\). Then the same argument as in Example 1 gives us that
\[ e^{-1} = \left| \frac{\partial f_k}{\partial \text{Re} (y)}(0, 0) \right| \leq \|f_k\|_{U, 1, X}. \]
Similarly, \( \|f_k\|_{0, X} \leq k^{-\frac{1}{q}} e^{-1} \), \( \|f_k\|_{m, X} \leq \|f_k\|_{m, X} \leq C_m^r k^m \) for some \( C_m^r > 0 \) for all \( k \in \mathbb{N} \). The argument used in Example 1 gives us the required inequality \( s \geq \frac{p}{q} \).

Let \( q \) be odd. The estimations for \( \|f_k\|_{0, X}^U \) and \( \|f_k\|_{m, X}^U \) remain the same. Let us show that
\[ \left| \frac{\partial g_k(0, 0)}{\partial \text{Im} (y)} \right| = e^{-1} \quad (0.0.8) \]
for any \( C^\infty \)-extension \( g_k \) of \( f_k \) from \( X \cap U \) to \( U \), so that \( \|f_k\|_{1, X}^U \geq e^{-1} \). Then the argument identical to the one used in Example 0.0.3 will give us \( s \geq \frac{p}{q} \). Let \( x = re^{i\theta} \). Denote
\[ y_1 = e^{\frac{2\pi}{q} r^p e^{i\frac{p}{q} \theta}}, \quad y_2 = e^{-\frac{2\pi}{q} r^p e^{i\frac{p}{q} \theta}}. \]
Then \((x, y_1) \in X, (x, y_2) \in X\) for any value of \(\theta\), so we can put \(\theta = 0\). We have

\[
\frac{\partial g_k(0, 0)}{\partial \Im (y)} = \lim_{r \to 0^+} f_k(re^{i\theta}, r^{\frac{p}{q}} e^\frac{2\pi i}{q} e^{i\theta}) - f_k(re^{i\theta}, r^{\frac{p}{q}} e^\frac{2\pi i}{q} e^{i\theta}) |_{\theta = 0} = \lim_{r \to 0} \frac{f_k(r, 0, r^{\frac{p}{q}} \cos(2\pi/q), r^{\frac{p}{q}} \sin(2\pi/q)) - f_k(r, 0, r^{\frac{p}{q}} \cos(2\pi/q), -r^{\frac{p}{q}} \sin(2\pi/q))}{2r^{\frac{p}{q}} \sin(2\pi/q)} = e^{-1},
\]

which proves equality (0.0.8), as required.

### 0.0.3 Proof of Theorem 0.0.2

Without loss of generality \(0 \in X\), and we show that \(X\) admits a Gagliardo-Nirenberg type inequality at 0 with some exponent \(1 \leq s < \infty\). Let \(U = \{x \in \mathbb{R}^n : b_\varepsilon(x) < 0\}\), where \(b_\varepsilon = \sum_{k=1}^n x_k^2 - \varepsilon\) – an open ball of sufficiently small radius \(\varepsilon > 0\) centered at 0. There exists an analytic manifold \(M\) with \(\dim M = \dim X\) and a proper analytic function \(\phi : M \to \mathbb{R}^n\) such that \(\phi(M) = X\) (see, e.g., [BM3]). Define \(N = \phi^{-1}(X \cap U)\). Then \(N\) is a relatively compact subset of \(M\).

First we show that the sets \(T_j\) in the definition of the norm (0.0.4) can be chosen in such a way that \(\eta_i^{-1}(T_j \cap N)\) is a relatively compact and semianalytic subset of \(\mathbb{R}^l\) \((l = \dim M = \dim X)\). Let \(\iota\) be an (analytic) embedding of \(M\) in \(\mathbb{R}^q\) for some \(q\). Define

\[
T_j := \iota^{-1}(\iota(M) \cap B(x_j, \varepsilon_j))
\]

– a relatively compact subset of \(M\), where \(B(x_j, \varepsilon_j) \subset \mathbb{R}^q\) is the open ball of radius \(\varepsilon_j > 0\) centered at \(x_j \in \iota(M)\), and \(\varepsilon_j > 0\) and \(x_j\) are chosen in such way that the remaining conditions on \(T_j\) in (0.0.4) are satisfied. We have

\[
\eta_i^{-1}(T_j) = \{x \in \mathbb{R}^l : (\eta_i^* \iota^* b_{\varepsilon_j})(x) < 0\},
\]

where \(\eta_i^* \iota^* b_{\varepsilon_j}\) is an analytic function, and, by definition, \(\eta_i^{-1}(T_j)\) is semianalytic. Now, since \(\eta_i\) is a diffeomorphism,

\[
\eta_i^{-1}(T_j \cap N) = \eta_i^{-1}(T_j) \cap \eta_i^{-1}(N),
\]

where \(\eta_i^{-1}(N) = \{x \in \mathbb{R}^l : (\eta_i^* \phi^* b_c)(x) < 0\}\), \(b_c(y) = \sum_{k=1}^n y_k^2 - c\), and \(\eta_i^* \phi^* b_c\) is an analytic function. Thus, \(\eta_i^{-1}(N)\) is also semianalytic, and \(\eta_i^{-1}(T_j) \cap \eta_i^{-1}(N)\) is semianalytic as well.
Secondly we show that there exist $\hat{s} \geq 1$ and $\hat{C}_m > 0$ such that, given any $f \in C^\infty(\bar{N})$, we have for all $m$ and all $1 \leq k < m/\hat{s}$

$$\frac{\|\varphi^* f\|_k^N}{\|\varphi^* f\|_0^N} \leq \hat{C}_0 \hat{C}_m^k \left( \frac{\|\varphi^* f\|_m^N}{\|\varphi^* f\|_0^N} \right). \quad (0.0.9)$$

Indeed, since $\eta_i^{-1}(T_j \cap N)$ are semianalytic and relatively compact, we have, according to [BosM], that there exist $\hat{s}_j \geq 1$ and $\hat{C}_{mj} > 0$ such that for all $m$ and all $1 \leq k < m/\hat{s}_j$

$$\frac{\|\eta_i^* \varphi^* f\|_{\eta_i^{-1}(T_j \cap N)}^k}{\|\eta_i^* \varphi^* f\|_{\eta_i^{-1}(T_j \cap N)}^0} \leq \hat{C}_{0j} \hat{C}_{mj}^k \left( \frac{\|\eta_i^* \varphi^* f\|_{\eta_i^{-1}(T_j \cap N)}^m}{\|\eta_i^* \varphi^* f\|_{\eta_i^{-1}(T_j \cap N)}^0} \right).$$

Let us fix some $m$. Further, let $\hat{s} = \max_j \hat{s}_j$, $\hat{C}_m = \max_j \hat{C}_{mj}$. Taking into account that

$$\max_j \|\eta_i^* \varphi^* f\|_{\eta_i^{-1}(T_j \cap N)}^m = \|\varphi^* f\|_0^N,$$

we obtain, by definition of the norm (0.0.4), the required inequality. According to Proposition 0.0.1 there exists $c \in \mathbb{N}$ such that for all $f \in C^\infty(\bar{U})$ we have

$$\|f\|_{k,X}^U \leq C \|\varphi^* f\|_c^N \quad (0.0.10)$$

for some fixed $c > 0$, and $C := \max\{C_k : 1 \leq k \leq m\}$. Along with that, it follows from the definition of the norm (0.0.4) and the fact that the sets $\eta_i^{-1}(T_j \cap N)$ are relatively compact, that there exists $B > 0$ such that

$$|\varphi^* h|_m^N \leq B |h|_m^U \quad (0.0.11)$$

for any $h \in C^\infty(\bar{U})$. Since, given any $h \in C^\infty(\bar{U})$ such that $h|_X = f|_X$, we have $|\varphi^* h|_m^N = |\varphi^* f|_m^N$, we can take the infimum in (0.0.11) over all such $h$ to get

$$|\varphi^* f|_m^N \leq B \|f\|_{m,X}^U. \quad (0.0.12)$$

Now, as follows from the estimates (0.0.9), (0.0.10) and (0.0.12),

$$\frac{\|f\|_{k,X}^U}{\|f\|_{0,X}^U} \leq C \hat{C}_0 \hat{C}_m^c B^{\frac{c}{k}} m \left( \frac{\|f\|_{m,X}^U}{\|f\|_{0,X}^U} \right)^{\frac{c}{k}}.$$

for all $m$ and all $1 \leq k \leq \frac{m}{s_c}$. To complete our proof we put $C_0 := C \hat{C}_0$, $C_m := \hat{C}_m B^{\frac{c}{m}} (m \geq 1)$ and $s := \hat{s} c$. 

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Part II

Unique continuation for differential inequality $|\Delta u| \leq |Vu|$ (2009-2010)
Chapter 1

Introduction and main results

1.1 Introduction

Let $\Omega$ be an open set in $\mathbb{R}^d$ ($d \geq 3$). We set $X_p := L^p_{\text{loc}}(\Omega, dx)$ ($p \geq 1$), and denote by $H^{m,p}(\Omega)$ the standard Sobolev space. Let $\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ be the Laplace operator. We denote by $\mathcal{D}'(\Omega)$ the space of distributions on $\Omega$, and set

$$L^{2,1}_{\text{loc}}(\Omega) := \{ f \in X_1 : \Delta f \in \mathcal{D}'(\Omega) \cap X_1 \}.$$

From now on, we assume that $\Omega$ is connected.

Definition 1.1.1. For a given space of functions $Y_V \subset L^{2,1}_{\text{loc}}(\Omega)$ (depending on $V \in X_1$) we say that the differential inequality

$$|\Delta u(x)| \leq |V(x)||u(x)| \quad \text{a.e. in } \Omega$$  \hspace{1cm} \text{(1.1.1)}

has the property of weak unique continuation (WUC) in $Y_V$ ($=: Y_V^{\text{weak}}$) provided that whenever $u$ in $Y_V$ satisfies inequality (1.1.1) and vanishes in an open subset of $\Omega$ it follows that $u \equiv 0$ in $\Omega$.

Definition 1.1.2. We say that (1.1.1) has the property of strong unique continuation (SUC) in $Y_V$ ($=: Y_V^{\text{str}}$) if whenever $u$ in $Y_V$ satisfies (1.1.1) and vanishes to an infinite order at a point $x_0 \in \Omega$, i.e.,

$$\lim_{\rho \to 0} \frac{1}{\rho^k} \int_{|x-x_0|<\rho} |u(x)|^2 dx = 0, \text{ for all } k \in \mathbb{N},$$
it follows that \( u \equiv 0 \) in \( \Omega \).

The first result on unique continuation was obtained by T. Carleman [Ca]: differential inequality (1.1.1) has the WUC property in the case \( d = 2, V \in L^\infty_{\text{loc}}(\Omega) \). Since then, the properties of unique continuation were extensively studied by many authors (primarily following the original Carleman’s approach), with the best possible for \( L^p_{\text{loc}} \)-potentials SUC result obtained by D. Jerison and C. Kenig (\( p = \frac{d}{2}, Y^\text{str}_V = H^{2,\rho}_{\text{loc}}, \tilde{\rho} := \frac{2d}{d+2} \)) [JK], and its extension for \( L^{d/2,\infty}_{\text{loc}} \)-potentials obtained by E.M. Stein [St2]. Further improvements of Stein’s result were obtained in [CS, RV, Wo] where unique continuation is proved for potentials \( V \) locally in Campanato-Morrey class (see Section 1.3 for details), with \( Y^\text{str}_V = H^{2,2}_{\text{loc}}, H^{2,\rho}_{\text{loc}} \). Before that, in 1984, E.T. Sawyer proved the SUC property in the case \( d = 3 \) for potentials from Kato class (see Section 1.3).

Historically, the most important reason for establishing the WUC property is its application to the problem of absence of positive eigenvalues of the self-adjoint Schrödinger operators, discovered in 1959 by T. Kato [Ka2]. In what follows, we exploit this link and consider the class of potentials that is a local analogue of the potentials for which the self-adjoint Schrödinger operator is defined (in the sense of quadratic forms, see below), with the class of solutions containing all its eigenfunctions. The latter allows us to employ our WUC result to prove the absence of positive eigenvalues.

In contrast to [JK, St2, CS, RV, Wo] (and similarly to, e.g., [SS, F3H, Saw]), our class of potentials has an ‘abstract’ form, that is, is expressed in terms of the operators that are ‘specific to the problem’, i.e., which constitute the corresponding Schrödinger operator. The major novelty is, however, in the ‘abstract’ form of the class of solutions. Somewhat unexpectedly, it allows one to shorten and make more transparent the proofs of some of known results for ‘concrete’ classes (e.g., involving \( L^p_{\text{loc}} \)-conditions on \( V \) and \( H^{2,q}_{\text{loc}} \)-conditions on \( u \)).

1.1.1 Notation

Let \( 1_S \) denote the characteristic function of a set \( S \subset \mathbb{R}^d, B(x_0, \rho) := \{ x \in \mathbb{R}^d : |x - x_0| < \rho \} \) and \( B_S(x_0, \rho) := B(x_0, \rho) \cap S \) (also, set \( B(\rho) := B(0, \rho) \) and \( B_S(\rho) := B_S(0, \rho) \)), \( \|A\|_{p \rightarrow q} \) is the norm of operator \( A : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d), ( - \Delta )^{-\frac{z}{2}}, 0 < \Re(z) < d, \) stands for the Riesz operator.
whose action on a function \( f \in C_0^\infty(\mathbb{R}^d) \) is determined by the formula

\[
(-\Delta)^{-\frac{d}{2}} f(x) = c_z \int_{\mathbb{R}^d} (-\Delta)^{-\frac{d}{2}}(x, y) f(y) dy,
\]

where

\[
(-\Delta)^{-\frac{d}{2}}(x, y) := |x - y|^{z-d}, \quad c_z := \Gamma(\frac{d - z}{2}) \left( \frac{\pi^{d/2} \Gamma(\frac{z}{2})}{2} \right)^{-1}
\]

(see, e.g., [St1]).

Our main result is that differential inequality (1.1.1) has the WUC property in the space of solutions

\[
Y_V^{\text{weak}} := \left\{ f \in L^{2,1}_{\text{loc}} : |V|^\frac{1}{2} f \in X_2 \right\}
\]

and, respectively, the SUC property in

\[
Y_V^{\text{str}} := Y_V^{\text{weak}} \cap H^{1,\beta}_{\text{loc}}(\Omega),
\]

for potentials \( V \) in the class

\[
\mathcal{F}_{\beta, \text{loc}}^d := \left\{ W \in X_{d-1}^+ : \sup_{K} \lim_{R \to 0} \sup_{x \in K} \| 1_{B_K(x, \rho)} W \|_2^{d-1} |(-\Delta)^{-\frac{d}{2}} W|_2^{d-1} 1_{B_K(x, \rho)} \|_2 \leq \beta \right\},
\]

where \( K \) is a compact subset of \( \Omega \).

### 1.1.2 Motivation

In 1959 T. Kato proved that if \( V \) has a compact support, then all eigenfunctions corresponding to positive eigenvalues must vanish outside of a ball of finite radius, hence by WUC must be identically equal to zero. In what follows, we employ our WUC result for (1.1.1) to prove the absence of positive eigenvalues of the self-adjoint Schrödinger operator \( H \supset -\Delta + V \) in the complex Hilbert space \( \mathcal{H} := L^2(\mathbb{R}^d) \) defined in the sense of quadratic forms (see [Ka3, ReS]), namely:

\[
H := H_+ + (-V_-),
\]

where \( H_+ := H_0 + V_+ \), \( H_0 = (-\Delta|_{C_0^\infty(\mathbb{R}^d)})^* \), \( D(H_0) = H^{2,2}(\mathbb{R}^d) \), \( V = V_+ - V_- \), \( V_\pm \geq 0 \), \( V_\pm \in L^1(\mathbb{R}^d) \) and

\[
\inf_{\lambda > 0} \left\| V_\lambda^2 (\lambda + H_+)^{-1} V_\lambda^{1/2} \right\|_{2 \to 2} \leq \beta < 1.
\]

(1.1.3)
The latter inequality guarantees the existence of the form sum \((1.1.2)\), see [Ka3, Ch.VI]), and the inclusion \(D(H) \subset Y^\text{weak}_V\) (see Section 1.2).

The local nature of the problem of unique continuation and the form of differential inequality \((1.1.1)\) lead to the definition of the following ‘local analogue’ of potentials satisfying \((1.1.3)\):

\[
F_{\beta,\text{loc}} := \left\{ W \in X_1 : \sup_K \lim_{\rho \to 0} \sup_{x_0 \in K} \|1_{B_K(x_0,\rho)}|W|^\frac{1}{2}(-\Delta)^{-\frac{1}{2}}1_{B_K(x_0,\rho)}\|_{2 \to 2} \leq \beta \right\}, \quad (1.1.4)
\]

where \(K\) is a compact subset of \(\Omega\). This class coincides with \(F_{\beta,\text{loc}}^d\) if \(d = 3\), and contains \(F_{\beta,\text{loc}}^d\) as a proper subclass if \(d \geq 4\) (the latter easily follows from Heinz-Kato inequality, see, e.g., [Ka1]). Arguments of this article do not apply to the larger class of potentials \(F_{\beta,\text{loc}}^d\) for \(d \geq 4\).

Class \(F_{\beta,\text{loc}}^d\) contains the potentials considered in [JK, St2, Saw, CS, Wo] as proper subclasses. Previously WUC and SUC properties were derived only for \(Y_V = H^2_{\text{loc}}(\Omega)\). We note that though the dependence of \(Y_V\) on \(V\) (i.e., \(u \in Y_V\) implies \(|V|^\frac{1}{2}u \in X_2\)) does not appear explicitly in the papers cited above, it is implicit, see Section 1.3.

Following Carleman, most proofs of unique continuation rely on Carleman type estimates on the norms of the appropriate operators acting from \(L^p\) to \(L^q\), for certain \(p\) and \(q\) (e.g., Theorem 2.1 in [JK], Theorem 1 in [St2]). Our method is based on an \(L^2 \mapsto L^2\) estimate of Proposition 2.1.1 and Lemma 2.1.2, proved in [Saw]. In the case \(d = 3\) we derive Proposition 2.1.1 using only Lemma 2.1.2. The case \(d \geq 4\) is reduced to the case \(d = 3\) at the cost of a more restrictive class of potentials: the proof uses Stein’s interpolation theorem for analytic families of operators [SW], and relies on Lemma 2.1.3 of [JK] and Lemma 2.1.4 – an extension of pointwise inequalities considered in [Saw] and [St2] (cf. Lemma 1 in [Saw], Lemma 5 in [St2]).

### 1.2 Main results

Our main results state that \((1.1.1)\) has the WUC and SUC properties with potentials from \(F_{\beta,\text{loc}}^d\). The difference between the results is in the classes \(Y_V\) within which we look for solutions to \((1.1.1)\).
**Theorem 1.2.1.** There exists a sufficiently small constant $\beta < 1$ such that if $V \in F_{\beta,\text{loc}}^d$, then \((1.1.1)\) has the WUC property in $Y_V^{\text{weak}}$.

**Theorem 1.2.2.** There exists a sufficiently small constant $\beta < 1$ such that if $V \in F_{\beta,\text{loc}}^d$, then \((1.1.1)\) has the SUC property in $Y_V^{\text{str}}$.

The proofs of Theorems 1.2.1 and 1.2.2 are given in Section 2.1. Concerning the eigenvalue problem, we have the following result.

**Theorem 1.2.3.** Suppose that $H$ is defined by \((1.1.2)\) in assumption that \((1.1.3)\) holds. Let us also assume that $V \in F_{\beta,\text{loc}}^d$ for $\beta < 1$ sufficiently small, and $\text{supp}(V)$ is compact in $\mathbb{R}^d$. Then the only solution to the eigenvalue problem

\[
Hu = \lambda u, \quad u \in D(H), \quad \lambda > 0
\]

is zero.

**Proof.** The following inclusions are immediate from the definition of operator $H$:

\[
D(H) \subset H^{1,2}(\mathbb{R}^d) \cap D(V_1^{1/2}) \cap D(V_2^{1/2}) \cap D(H_{\text{max}}),
\]

where

\[
D(H_{\text{max}}) := \{ f \in \mathcal{H} : \Delta f \in \mathcal{D}'(\mathbb{R}^d) \cap L^1_{\text{loc}}(\mathbb{R}^d), Vf \in L^1_{\text{loc}}(\mathbb{R}^d), -\Delta f + Vf \in \mathcal{H} \}.
\]

Therefore, $D(H) \subset Y_V^{\text{weak}}$ and if $u \in D(H)$ is a solution to \((1.2.5)\), then

\[
|\Delta u| = |(V - \lambda)u| \quad \text{a.e. in } \mathbb{R}^d.
\]

By Kato’s theorem [Ka2] $u$ has compact support. Now Theorem 1.2.3 follows from Theorem 1.2.1.

**1.3 Comparison with the classical results**

(1) D. Jerison and C. Keing [JK] and E. M. Stein [St2] proved the validity of the SUC property for potentials from classes $L_{\text{loc}}^{d/2}(\Omega)$ and $L_{\text{loc}}^{d,\infty}(\Omega)$ (weak type $d/2$ Lorentz space), respectively.
Below \( \| \cdot \|_{p,\infty} \) denotes weak type \( p \) norm. One has
\[
L^d_{\text{loc}}(\Omega) \subset \bigcap_{\beta > 0} F^d_{\beta,\text{loc}}, \quad (1.3.6)
\]
\[
L^{\frac{d}{2},\infty}_{\text{loc}}(\Omega) \subset \bigcup_{\beta > 0} F^d_{\beta,\text{loc}}. \quad (1.3.7)
\]
The first inclusion follows straightforwardly from the Sobolev embedding theorem. For the following proof of the second inclusion let us note first that
\[
\|1_{B(x_0,\rho)}|W|^{\frac{d-1}{4}}(-\Delta)^{-\frac{d-1}{4}}1_{B(x_0,\rho)}\|_{2\rightarrow 2} = \|1_{B(x_0,\rho)}|V|^{\frac{d-1}{4}}(-\Delta)^{-\frac{d-1}{4}}\|_{2\rightarrow 2}.
\]
Next, if \( V \in L^{d/2,\infty}_{\text{loc}} \), then
\[
\|1_{B(x_0,\rho)}|V|^{\frac{d-1}{4}}(-\Delta)^{-\frac{d-1}{4}}\|_{2\rightarrow 2} \leq \left( \frac{2d^{-1} \pi^{\frac{d}{2}} c_1^2}{\Gamma \left( \frac{d}{2} \right) c_d^2} \right) \|1_{B(x_0,\rho)}|V|^{\frac{d-1}{4}}\|_{\frac{d}{2},\infty}, \quad (1.3.8)
\]
which is a special case of Strichartz inequality with sharp constants, proved in [KPS]. Inclusion (1.3.7) follows.

To see that the latter inclusion is strict we introduce a family of potentials
\[
V(x) := C \left( 1_{B(1+\delta)}(x) - 1_{B(1-\delta)}(x) \right) \left( \frac{|x| - 1}{\pi^2} \right)^{\frac{d}{2}} \left( -\ln |x| - 1 \right)^b, \quad \text{where } b > \frac{2}{d-1}, \quad 0 < \delta < 1. \quad (1.3.9)
\]
A straightforward computation shows that \( V \in F^d_{\beta,\text{loc}} \), as well as \( V \in L^{\frac{d-1}{2}}_{\text{loc}}(\Omega) \setminus L^{\frac{d-1}{2}+\varepsilon}_{\text{loc}}(\Omega) \) for any \( \varepsilon > 0 \), so that \( V \not\in L^{\frac{d}{2},\infty}_{\text{loc}}(\Omega) \).

The result in [St2] can be formulated as follows. Suppose that \( d \geq 3 \) and \( V \in L^{\frac{d}{2},\infty}_{\text{loc}}(\Omega) \).

There exists a sufficiently small constant \( \beta \) such that if
\[
\sup_{x_0 \in \Omega} \lim_{\rho \to 0} \|1_{B(x_0,\rho)}|V|\|_{\frac{d}{2},\infty} \leq \beta,
\]
then (1.1.1) has the SUC property in \( Y_V := H^{2,\beta}_{\text{loc}}(\Omega) \), where \( \bar{p} := \frac{2d}{d+2} \). (It is known that the assumption of \( \beta \) being sufficiently small can not be omitted, see [KoT].)

In view of (1.3.6), (1.3.7), the results in [St2] and in [JK] follow from Theorem 1.2.2 provided that we show \( |V|^\frac{d}{2} u \in X_2 \). Indeed, let \( L^{q,p} \) be the \((q,p)\) Lorentz space (see [SW]). By Sobolev embedding theorem for Lorentz spaces (see [SW])
\[
H^{2,\beta}_{\text{loc}}(\Omega) \hookrightarrow L^{\frac{q}{d},\frac{\bar{p}}{d}}_{\text{loc}}(\Omega), \quad \bar{q} := \frac{2d}{d-2},
\]

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Hence, by Hölder inequality in Lorentz spaces $|V|^{\frac{1}{2}} u \in X_2$ whenever $u \in L^{\frac{d}{2}, \infty}_{\text{loc}}(\Omega)$ and $V \in L^{d/2, \infty}_{\text{loc}}$. Also, $H^{2, \frac{d}{2}}_{\text{loc}}(\Omega) \hookrightarrow H^{1, \frac{d}{2}}_{\text{loc}}(\Omega)$, so $H^{2, \frac{d}{2}}_{\text{loc}}(\Omega) \subset Y^{\text{str}}_{V}$, as required.

(2) E.T. Sawyer [Saw] proved uniqueness of continuation for the case $d = 3$ and potential $V$ from the local Kato-class

\[ K_{\beta, \text{loc}} := \{ W \in L^1_{\text{loc}}(\Omega) : \sup_{K} \lim_{\rho \to 0} \sup_{x_0 \in K} \| (-\Delta)^{-1} 1_{B_K(x_0, \rho)} W \|_{\infty} \leq \beta \}, \]

where $K$ is a compact subset of $\Omega$. It is easy to see that $K_{\beta, \text{loc}} \subsetneq F_{\beta, \text{loc}}$.

To see that the latter inclusion is strict consider, for instance, potential $V_{\beta}(x) := \beta v_0$, $v_0 := \left( \frac{d-2}{2} \right)^2 |x|^{-2}$.

By Hardy’s inequality, $V_{\beta} \in F_{\beta, \text{loc}}$. At the same time, $\| (-\Delta)^{-1} v_0 1_{B(\rho)} \|_{\infty} = \infty$ for all $\rho > 0$, hence $V_{\beta} \notin K_{\beta, \text{loc}}$ for all $\beta \neq 0$.

The next statement is essentially due to E.T. Sawyer [Saw].

**Theorem 1.3.1.** Let $d = 3$. There exists a constant $\beta < 1$ such that if $V \in K_{\beta, \text{loc}}$ then (1.1.1) has the WUC property in $Y^K_V := \{ f \in X_1 : \Delta f \in X_1, Vf \in X_1 \}$.

The proof of Theorem 1.3.1 is provided in Section 2.2.

Despite the embedding $K_{\beta, \text{loc}} \hookrightarrow F_{\beta, \text{loc}}$, Theorem 1.2.1 does not imply Theorem 1.3.1. The reason is simple: $Y^K_V \not\subset Y^\text{weak}_V$.

(3) S. Chanillo and E.T. Sawyer showed in [CS] the validity of the SUC property for (1.1.1) in $Y_V = H^{2, \frac{d}{2}}_{\text{loc}}(\Omega)$ ($d \geq 3$) for potentials $V$ locally small in Campanato-Morrey class $M^p$ ($p > \frac{d-1}{2}$),

\[ M^p := \{ W \in L^p : \| W \|_{M^p} := \sup_{x \in \Omega, \rho > 0} r^{2-\frac{d}{p}} \| 1_{B(x, r)} W \|_p < \infty \}. \]

Note that for $p > \frac{d-1}{2}$

\[ M^p_{\text{loc}} \subsetneq \bigcup_{\beta > 0} F^{d}_{\beta, \text{loc}} \]

(see [CS, F, KeS]). To see that the above inclusion is strict one may consider, for instance, potential defined in (1.3.9).

It is easy to see, using Hölder inequality, that if $u \in H^{2, \frac{d}{2}}_{\text{loc}}(\Omega)$ and $V \in M^p_{\text{loc}}$ ($p > \frac{d-1}{2}$), then $|V|^{\frac{1}{2}} u \in X_2$, i.e., $u \in Y^\text{weak}_V$.  

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Chapter 2

Proofs

2.1 Proofs of Theorems 1.2.1 and 1.2.2

Let us introduce some notations. In what follows, we omit index $K$ in $B_K(x_0, \rho)$, and write simply $B(x_0, \rho)$.

Let $W \in X_{d-1}$, $x_0 \in \Omega$, $\rho > 0$, $d \geq 3$, define

$$
\tau(W, x_0, \rho) := \| 1_{B(x_0, \rho)} |W|^{\frac{d-1}{2}} (-\Delta)^{-\frac{d-1}{2}} |W|^{\frac{d-1}{4}} 1_{B(x_0, \rho)} \|_{2 \to 2}.
$$

(2.1.1)

Note that if $V$ is a potential from $\mathcal{F}_{d, \beta, \text{loc}}$, and $V_1 := |V| + 1$, then

$$
\tau(V_1, x_0, \rho) \leq \tau(V, x_0, \rho) + \varepsilon(\rho),
$$

(2.1.2)

where $\varepsilon(\rho) \to 0$ as $\rho \to 0$.

Let $1_{B(\rho \setminus a)}$ be the characteristic function of set $B(0, \rho) \setminus B(0, a)$, where $0 < a < \rho$. We define integral operator

$$
\left[ (-\Delta)^{-\frac{d}{2}} \right]_N f(x) := \int_{\mathbb{R}^d} \left[ (-\Delta)^{-\frac{d}{2}} \right]_N (x, y) f(y) dy,
$$

where $\Re(z) \leq d - 1$ and

$$
\left[ (-\Delta)^{-\frac{d}{2}} \right]_N (x, y) := c_Z \left( |x - y|^{z-d} - \sum_{k=0}^{N-1} \frac{(x \cdot \nabla)^k}{k!} |0 - y|^{z-d} \right),
$$
where \((x \cdot \nabla)^k := \sum_{|\alpha|=k} k! \frac{\partial^k}{\partial x_1^{\alpha_1} \cdots x_d^{\alpha_d}} x_1^{\alpha_1} \cdots x_d^{\alpha_d}\) is the multinomial expansion of \((x \cdot \nabla)\). Define, further,

\[\left( (\nabla)^{-\frac{d}{2}} \right)^N_{N,t} := \varphi_t \left( (\nabla)^{-\frac{d}{2}} \right)^N_{N,t} \varphi_t^{-1},\]

where \(\varphi_t(x) := |x|^{-t}\).

### 2.1.1 Proof of Theorem 1.2.1

Our proof is based on the inequalities of Proposition 2.1.1 and Lemma 2.1.2.

**Proposition 2.1.1.** If \(\tau(V,0,\rho) < \infty\), then there exists a constant \(C = C(\rho, \delta, d) > 0\) such that

\[\|1_{B(\rho, a)}[V]^\frac{1}{2} \left( (\nabla)^{-1} \right)^N_{N,N_d} [V]^\frac{1}{2} 1_{B(\rho, a)} \|_{2 \rightarrow 2} \leq C\tau(V,0,\rho) \frac{2^d}{\pi} ,\]

for all positive integers \(N\), where \(0 < \delta < 1/2\) and

\[N_d^d := N + \left( \frac{d}{2} - \delta \right) \frac{d - 3}{d - 1} .\]

**Lemma 2.1.2.** There exists a constant \(C = C(d)\) such that

\[\left| \left( (\nabla)^{-1} \right)^N_{N} (x,y) \right| \leq C N^{d-3} \left( \frac{|x|}{|y|} \right)^N \left( (\nabla)^{-1} \right)^{(x,y)}\]

for all \(x, y \in \mathbb{R}^d\) and all positive integers \(N\).

Lemma 2.1.2 is a simple consequence of Lemma 2.1.4 below for \(\gamma = 0\). Lemmas 2.1.3 and 2.1.4 are required for analytic interpolation procedure used in the proof of Proposition 2.1.1 when \(d \geq 4\).

**Lemma 2.1.3 ([JK]).** There exist \(C_2 = C_2(\rho_1, \rho_2, \delta, d)\) and \(c_2 = c_2(\rho_1, \rho_2, \delta, d) > 0\) such that

\[\|1_{B(\rho_1 \setminus a)} \left[ (\nabla)^{-1} \right]_{N,N_d + \frac{d}{2} - \delta} 1_{B(\rho_2 \setminus a)} \|_{2 \rightarrow 2} \leq C_2 e^{c_2|\gamma|} ,\]

where \(0 < \delta < 1/2\), for all \(\gamma \in \mathbb{R}\) and all positive integers \(N\).

**Lemma 2.1.4.** There exist constants \(C_1 = C_1(d)\) and \(c_1 = c_1(d) > 0\) such that

\[\left| \left[ (\nabla)^{-\frac{d-1+2\gamma}{2}} \left( (\nabla)^{-\frac{d-1}{2}} (x,y) \right) \right]_{N} \right| \leq C_1 e^{c_1\gamma^2} \left( \frac{|x|}{|y|} \right)^N \left( (\nabla)^{-\frac{d-1}{2}} (x,y) \right)\]

for all \(x, y \in \mathbb{R}^d\), all \(\gamma \in \mathbb{R}\) and all positive integers \(N\).
We prove Lemma 2.1.4 at the end of this section.

Proof of Proposition 2.1.1. If \( d = 3 \), then Proposition 2.1.1 follows immediately from Lemma 2.1.2. Suppose that \( d \geq 4 \). Consider the operator-valued function

\[
F(z) := 1_{B(\rho,a)}|V|^\frac{d-1}{4} \varphi_{N+(\frac{d}{2}-\delta)(1-z)} \left[ (-\Delta)^{-\frac{d-1}{2}} \right] N \varphi_{N+(\frac{d}{2}-\delta)(1-z)}|V|^\frac{d-1}{4} 1_{B(\rho,a)}
\]

defined on the strip \( \{ z \in \mathbb{C} : 0 \leq \text{Re} \, (z) \leq 1 \} \) and acting on \( L^2 \). By Lemma 2.1.3,

\[
\|F(i\gamma)\|_{2 \to 2} \leq C_2 e^{\epsilon_2 |\gamma|}, \quad \gamma \in \mathbb{R},
\]

and by Lemma 2.1.4,

\[
\|F(1+i\gamma)\|_{2 \to 2} \leq \tau(V,0,\rho)C_1 e^{\epsilon_1 \gamma^2}, \quad \gamma \in \mathbb{R}.
\]

Together with obvious observations about analyticity of \( F \) this implies that \( F \) satisfies all conditions of Stein’s interpolation theorem. In particular, \( F\left(\frac{\gamma}{\tau(V,0,\rho)}\right) : L^2 \mapsto L^2 \) is bounded, which completes the proof.

Proof of Theorem 1.2.1. Let \( u \in Y^\text{weak}_1 \). Without loss of generality we may assume \( u \equiv 0 \) on \( B(0,a) \) for \( a > 0 \) sufficiently small, such that there exists \( \rho > a \) with the properties \( \rho < 1 \) and \( \bar{B}(0,3\rho) \subset \Omega \). In order to prove that \( u \) vanishes on \( \Omega \) it suffices to show that \( u \equiv 0 \) on \( B(0,\rho) \) for any such \( \rho \).

Let \( \eta \in C_0^\infty(\Omega) \) be such that \( 0 \leq \eta \leq 1, \eta \equiv 1 \) on \( B(0,2\rho) \), \( \eta \equiv 0 \) on \( \Omega \setminus B(0,3\rho) \), \( |\nabla \eta| \leq \frac{\epsilon}{\rho} \), \( |\Delta \eta| \leq \frac{\epsilon^2}{\rho^2} \). Let \( E_\eta(u) := 2\nabla \eta \nabla u + u \Delta \eta \in X_1 \). Denote \( u_\eta := u \eta \). Since \( L^{\frac{d+1}{d-1}}_\text{loc}(\Omega) \subset H^{\frac{d+1}{2}}_\text{loc}(\Omega) \), \( p < \frac{d}{d-1} \), we have \( E_\eta(u) \in L^1_\text{com}(\Omega) \) and hence

\[
\Delta u_\eta = \eta \Delta u + E_\eta(u)
\]

implies \( \Delta u_\eta \in L^1_\text{com}(\Omega) \). Thus, we can write

\[
u_\eta = (-\Delta)^{-1}(\Delta u_\eta).
\]

The standard limiting argument (involving consideration of \( C_0^\infty \) -mollifiers, subtraction of Taylor polynomial of degree \( N-1 \) at 0 of function \( u_\eta \) and interchanging the signs of differentiation and integration) allows us to conclude further

\[
u_\eta = [(-\Delta)^{-1}]_N(\Delta u_\eta).
\]

(2.1.3)
Let us denote $1^c_{B(\rho)} := 1 - 1_{B(\rho)}$, so that $\Delta u_\eta = (1_{B(\rho,a)} + 1^c_{B(\rho)}) \Delta u_\eta$. Observe that

$$\supp \eta \Delta u \subset \bar{B}(0,3\rho) \setminus B(0,a), \quad \sup E_\eta(u) \subset \bar{B}(0,3\rho) \setminus B(0,2\rho)$$

and, thus, $1^c_{B(\rho)} \eta \Delta u = 1_{B(3\rho,\rho)} \Delta u, \quad 1^c_{B(\rho)} E_\eta(u) = 1_{B(3\rho,2\rho)} E_\eta(u)$. Identity (2.1.3) implies then

$$1^c_{B(\rho)} V_{1}^\frac{1}{2} \varphi_{N_d}^\delta u = 1_{B(\rho)} V_{1}^\frac{1}{2} \left[ (-\Delta)^{-1} \right]_{N_d} V_{1}^\frac{1}{2} 1^c_{B(\rho,a)} \varphi_{N_d}^\delta \frac{-\Delta u}{V_{1}^\frac{1}{2}} +$$

$$+ 1_{B(\rho)} V_{1}^\frac{1}{2} \left[ (-\Delta)^{-1} \right]_{N_d} V_{1}^\frac{1}{2} 1^c_{B(\rho)} \varphi_{N_d}^\delta \frac{-\eta \Delta u}{V_{1}^\frac{1}{2}} +$$

$$+ 1_{B(\rho)} V_{1}^\frac{1}{2} \left[ (-\Delta)^{-1} \right]_{N_d} 1_{B(3\rho,2\rho)} \varphi_{N_d}^\delta (-E_\eta(u))$$

(we assume that $0 < \delta < 1/2$ is fixed throughout the proof) or, letting $I$ to denote the left hand side and, respectively, $I_1$, $I_1^c$ and $I_2$ the three summands of the right hand side of the last equality, we rewrite the latter as

$$I = I_1 + I_1^c + I_2.$$  

We would like to emphasize that a priori $I \not\in L^2$, but only $I \in L^s$, $s < d/(d-2)$. Hence, we must first prove that $I_1$, $I_1^c$ and $I_2$ are in $L^2$, so that $I \in L^2$ as well. After this done, we obtain the estimates $\|I_1\|_2 \leq c_1 \varphi_{N_d}^\delta(\rho)$, $\|I_2\|_2 \leq c_2 \varphi_{N_d}^\delta(\rho)$ and $\|I_1\|_2 \leq \alpha \|I\|_2$, $\alpha < 1$, and conclude that $(1-\alpha)\|I\|_2 \leq (c_1 + c_2) \varphi_{N_d}^\delta(\rho)$, and therefore that

$$\left\| \frac{1_{B(\rho,a)} \varphi_{N_d}^\delta(\rho)}{1_{B(\rho,a)} \varphi_{N_d}^\delta(\rho)} \right\|_2 \leq \frac{c_1 + c_2}{1 - \alpha} .$$

Letting $N \to \infty$, we derive identity $u \equiv 0$ in $B(0,\rho)$.

1) Proof of $I_1 \in L^2$ and $\|I_1\|_2 \leq \alpha \|I\|_2$, $\alpha < 1$. Observe that

$$1_{B(\rho,a)} \frac{|\Delta u|}{V_{1}^\frac{1}{2}} \leq 1_{B(\rho)} \frac{|V||u|}{V_{1}^\frac{1}{2}} \leq 1_{B(\rho)} |V|^{1/2} |u| \in X_2 \quad \text{(since } u \in Y^\text{weak}_V),$$

and hence, according to Proposition 2.1.1,

$$\|I_1\|_2 \leq \left\| 1_{B(\rho,a)} V_{1}^\frac{1}{2} \left[ (-\Delta)^{-1} \right]_{N_d} V_{1}^\frac{1}{2} 1_{B(\rho,a)} \right\|_{2 \to 2} \left\| 1_{B(\rho)} \varphi_{N_d}^\delta |V|^{1/2} u \right\|_2 \leq$$

$$\beta_1 \|1_{B(\rho)} \varphi_{N_d}^\delta |V|^{1/2} u\|_2 .$$

Here $\beta_1 := C \tau(V_1,0,\rho)^\frac{1}{1-t}$, where $C$ is the constant in formulation of Proposition 2.1.1. We may assume that $\beta_1 < 1$ (see (2.1.2)).
2) **Proof of** $\|I_1\|_2 \leq c_1 \varphi_{N_d}^\phi(\rho)$. By Proposition 2.1.1,

$$\|I_1\|_2 \leq \left\|1_B(\rho) V_1^1 [(-\Delta)^{-1}]_{N,N_d} V_1^2 1_B(3\rho;2\rho)\right\|_{2 \to 2} \left\|1_B(3\rho;2\rho) \varphi_{N_d}^\phi V_1^2 u\right\|_2 \leq \beta_2 \varphi_{N_d}^\phi(\rho) \|1_B(3\rho) V|^{1/2} u\|,$$

where $\beta_2 := C\tau(V_1,0,3\rho)^{\frac{1}{d+1}} < \infty$.

3) **Proof of** $\|I_2\|_2 \leq c_2 \varphi_{N_d}^\phi(\rho)$. We need to derive an estimate of the form

$$\|I_2\|_2 \leq C \varphi_{N_d}^\phi(\rho) \|E(u)\|_1,$$

where $C$ can depend on $d$, $\delta$, $\alpha$, $\rho$, $\|1_B(\rho) V\|_1$, but not on $N$. We have

$$\|I_2\|_2 \leq \left\|1_B(\rho) V_1^{1/2} [(-\Delta)^{-1}]_{N,N_d} 1_B(3\rho;2\rho)\right\|_{1 \to 2} \left\|1_B(3\rho;2\rho) \varphi_{N_d}^\phi E(u)\right\|_1 \leq \left\|1_B(\rho) V_1^{1/2} [(-\Delta)^{-1}]_{N,N_d} 1_B(3\rho;2\rho)\right\|_{1 \to 2} 2^{-\delta} \varphi_{N_d}^\phi(\rho) \|E(u)\|_1.$$

Now for $h \in L^1(\mathbb{R}^d)$, in virtue of Lemma 2.1.2,

$$\|1_B(\rho) V_1^{1/2} [(-\Delta)^{-1}]_{N,N_d} 1_B(3\rho;2\rho) h\|_2 \leq \|1_B(\rho) V_1^{1/2}\|_2 \|1_B(\rho) [(-\Delta)^{-1}]_{N,N_d} 1_B(3\rho;2\rho) h\|_\infty \leq \|1_B(\rho) V_1^{1/2}\|_2 C N^{d-3} \varphi(\frac{d}{2} - \delta) \frac{d-3}{d+1} (3\rho) \|1_B(\rho) (-\Delta)^{-1} 1_B(3\rho;2\rho) h\|_\infty \leq \left(\|1_B(\rho)\|_1 + \|1_B(\rho) V\|_1\right)^{1/2} C N^{d-3} \left(\frac{3\rho}{a}\right)^{\frac{d}{2} - \delta} \frac{d-3}{d+1} M_\rho,$$

where

$$M_\rho := C_2 \text{esssup}_{x \in B(0,\rho)} \int_{2\rho \leq |y| \leq 3\rho} |x - y|^{2-d} |h(y)| dy \leq C_2 \rho^{2-d} \|h\|_1.$$

Therefore

$$\|1_B(\rho;\alpha) V_1^{1/2} [(-\Delta)^{-1}]_{N,N_d} 1_B(3\rho;2\rho)\|_{1 \to 2} \leq \left(\|1_B(\rho)\|_1 + \|1_B(\rho) V\|_1\right)^{1/2} C_2 N^{d-3} \left(\frac{3\rho}{a}\right)^{\frac{d}{2} - \delta} \frac{d-3}{d+1} \rho^{2-d}$$

Hence, there exists a constant $\hat{C} = \hat{C}(d,\delta,\alpha,\rho,\|1_B(\rho) V\|_1)$ such that

$$\|I_2\|_2 \leq \hat{C} N^{d-3} 2^{-\delta} \varphi_{N_d}^\phi(\rho) \|E(u)\|_1,$$

which implies the required estimate. 

\[\square\]
Proof of Lemma 2.1.4. The proof essentially follows the argument in [Saw]. Put
\[
\begin{bmatrix}
-\frac{1}{2} + \frac{i\gamma}{2}
\end{bmatrix}
\frac{k}{k} := \prod_{j=1}^{k} \left( 1 + \frac{-\frac{1}{2} + \frac{i\gamma}{2}}{j} \right).
\]
Then
\[
\left| \begin{bmatrix}
-\frac{1}{2} + \frac{i\gamma}{2}
\end{bmatrix}
\frac{k}{k} \right| = \prod_{j=1}^{k} \left( 1 - \frac{1}{2j} \right) \prod_{j=1}^{k} \sqrt{1 + \frac{\gamma^2}{(2j-1)^2}} \leq \prod_{j=1}^{k} \left( 1 - \frac{1}{2j} \right) e^{\gamma^2 c}, \quad c = \frac{\pi^2}{48}. \tag{2.1.4}
\]
We may assume, after a dilation and rotation, that
\[x = (x_1, x_2, 0, \ldots, 0), \quad y = (1, 0, \ldots, 0).\]
Thus, passing to polar coordinates \((x_1, x_2) = te^{i\theta}\), we reduce our inequality to inequality
\[|1 - te^{i\theta}|^{-1} - i\gamma - P_{N-1}(t, \theta) | \leq Ce^{\gamma^2 t^N}|1 - te^{i\theta}|^{-1}, \quad \text{for all } \gamma \in \mathbb{R}\]
and for appropriate \(C > 0, c > 0\). Here \(P_{N-1}(t, \theta)\) denotes the Taylor polynomial of degree \(N - 1\) at point \(z = 0\) of function \(z = te^{i\theta} \mapsto |1 - z|^{-1}\). Similarly to [Saw], via summation of geometric series we obtain a representation
\[P_{N-1}(t, \theta) = \sum_{m=0}^{N-1} a_m^\gamma(\theta) t^m, \]
where
\[a_m^\gamma(\theta) := \sum_{k+l=m} \begin{bmatrix}
-\frac{1}{2} + \frac{i\gamma}{2}
\end{bmatrix}
\frac{k}{l} \begin{bmatrix}
-\frac{1}{2} + \frac{i\gamma}{2}
\end{bmatrix}
e^{i(k-l)\theta}.
\]
Note that
\[a_m^0(0) = \sum_{k+l=m} \begin{bmatrix}
-\frac{1}{2}
\end{bmatrix}
\frac{k}{l} \begin{bmatrix}
-\frac{1}{2}
\end{bmatrix} = 1
\] since
\[\sum_{m=0}^{\infty} a_m^0(0) t^m = (1 - t)^{-1} = \sum_{m=0}^{\infty} t^m.
\] Now estimate (2.1.4) and identity \(a_m^0(0) = 1\) yield
\[|a_m^\gamma(\theta)| \leq \sum_{k+l=m} \begin{bmatrix}
-\frac{1}{2}
\end{bmatrix}
\frac{k}{l} \begin{bmatrix}
-\frac{1}{2}
\end{bmatrix} e^{2c\gamma^2} = e^{2c\gamma^2}. \]
We have to distinguish between four cases $t \geq 2$, $1 < t < 2$, $0 \leq t \leq \frac{1}{2}$ and $\frac{1}{2} < t < 1$. Below we consider only the cases $t \geq 2$ and $1 < t < 2$ (proofs in two other cases are similar).

If $t \geq 2$, then

$$|P_{N-1}(t, \theta)| \leq \sum_{m=0}^{N-1} |a_m(\theta)| t^m \leq e^{2c\gamma^2} t^N \leq \frac{3}{2} e^{2c\gamma^2} t^N |1 - te^{i\theta}|^{-1}$$

since $1 \leq \frac{3}{2} |1 - te^{i\theta}|^{-1}$. Hence, using $|1 - te^{i\theta}|^{-1-i\gamma} \leq t^N |1 - te^{i\theta}|^{-1}$, we obtain

$$|1 - te^{i\theta}|^{-1-i\gamma} - P_{N-1}(t, \theta) \leq t^N |1 - te^{i\theta}|^{-1} + \frac{3}{2} e^{2c\gamma^2} t^N |1 - te^{i\theta}|^{-1} \leq C e^{2c\gamma^2} t^N |1 - te^{i\theta}|^{-1}$$

for an appropriate $C > 0$, as required.

If $1 < t < 2$, then, after two summations by parts, we derive

$$P_{N-1}(t, \theta) = \sum_{l=0}^{N-3} S \left[ -\frac{1}{2} + \frac{i\gamma}{2} \right] \frac{1}{l} D_l(\bar{z}) \sum_{k=0}^{N-1-l} S \left[ -\frac{1}{2} + \frac{i\gamma}{2} \right] \frac{1}{k} D_k(z) +$$

$$+ \sum_{l=0}^{N-2} S \left[ -\frac{1}{2} + \frac{i\gamma}{2} \right] \frac{1}{l} D_l(\bar{z}) D_{N-l-2}(z) +$$

$$+ \sum_{k=0}^{N-1} S \left[ -\frac{1}{2} + \frac{i\gamma}{2} \right] \frac{1}{k} D_{N-k-1}(z) = J_1 + J_2 + J_3,$$

where

$$S \left[ \begin{array}{c} \delta \\ k \end{array} \right] := \delta - \delta \left[ \begin{array}{c} \delta \\ k+1 \end{array} \right], \quad D_k(z) := \sum_{j=0}^{k} z^j.$$

We use estimate

$$\left| S \left[ -\frac{1}{2} + \frac{i\gamma}{2} \right] \frac{1}{k} \right| \leq \left| \left[ -\frac{1}{2} + \frac{i\gamma}{2} \right] \left( \frac{-\frac{1}{2} + \frac{i\gamma}{2}}{1+k} \right) \right| \leq C(k+1)^{-\frac{1}{2}} e^{c\gamma^2}$$

to obtain, following an argument in [Saw], that each $J_i$ $(i = 1, 2, 3)$ is majorized by $C e^{c\gamma^2} t^N |1 - te^{i\theta}|^{-1}$ for some $C > 0$. Since $|1 - te^{i\theta}|^{-1-i\gamma} \leq t^N |1 - te^{i\theta}|^{-1}$, Lemma 2.1.4 follows.

**2.1.2 Proof of Theorem 1.2.2**

Choose $\Psi_j \in C^\infty(\Omega)$ in such a way that $0 \leq \Psi_j \leq 1$, $\Psi_j(x) = 1$ for $|x| > \frac{2}{7}$, $\Psi_j(x) = 0$ for $|x| < \frac{1}{7}$, $|\nabla \Psi_j(x)| \leq c' j$, $|\Delta \Psi_j(x)| \leq c' j^2$. 

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Proposition 2.1.5. Let $\tau(V,0,\rho) < \infty$. There exists a constant $C = C(\rho,\delta,d) > 0$ such that for all positive integers $N$ and $j$

\begin{align}
(E1) & \quad \| 1_{B(\rho)} \psi_j V^{\frac{1}{2}} [(-\Delta)^{-1}]_{N,N_d} V^{\frac{1}{2}} \psi_j 1_{B(\rho)} \|_{2 \to 2} \leq C \tau(V,0,\rho)^{\frac{2}{p+1}}, \\
(E2) & \quad \| 1_{B(\rho)} \psi_j V^{\frac{1}{2}} [(-\Delta)^{-1}]_{N,N_d} V^{\frac{1}{2}} 1_{B(3\rho \setminus \rho)} \|_{2 \to 2} \leq C \tau(V,0,\rho)^{\frac{2}{p+1}}, \\
(E3) & \quad \| 1_{B(\rho)} \psi_j V^{\frac{1}{2}} [(-\Delta)^{-1}]_{N,N_d} 1_{B(\frac{3}{2} \rho \setminus \rho)} \|_{p \to 2} \leq C \tau(V,0,\rho)^{\frac{1}{p-1}}, \\
(E4) & \quad \| 1_{B(\rho)} \psi_j V^{\frac{1}{2}} [(-\Delta)^{-1}]_{N,N_d} 1_{B(3\rho \setminus 2\rho)} \|_{p \to 2} \leq C \tau(V,0,\rho)^{\frac{1}{p-1}},
\end{align}

where $p = \frac{2d}{d+2}$.

We prove Proposition 2.1.5 at the end of this section.

Proof of Theorem 1.2.2. We use the same notations as in the proof of Theorem 1.2.1. Suppose that $u \in Y_{\text{str}}^V$ satisfies (1.1.1) and vanishes to an infinite order at $0 \in \Omega$. We wish to obtain an estimate of the form

$$
\| 1_{B(\rho)} \frac{\varphi_{N_d}^j}{\varphi_{N_d}^{\rho}} u \|_2 \leq C. \tag{2.1.5}
$$

Then, letting $N \to \infty$, we would derive the required identity: $u \equiv 0$ in $B(0,\rho)$.

The same argument as in the proof of Theorem 1.2.1 leads us to an identity

$$
u_{\eta_j} = (-\Delta)^{-1}(-\Delta u_{\eta_j}), \quad \eta_j = \eta \psi_j,$$

which, in turn, implies

$$
1_{B(\rho)} \psi_j V^{\frac{1}{2}} \varphi_{N_d}^j u = \frac{\eta_j \Delta u}{V^{\frac{3}{2}}_1} = 1_{B(\rho)} \psi_j V^{\frac{1}{2}} [(-\Delta)^{-1}]_{N,N_d} V^{\frac{1}{2}}_1 \varphi_{N_d}^j E_j(u) + 1_{B(\rho)} \psi_j V^{\frac{1}{2}} [(-\Delta)^{-1}]_{N,N_d} \varphi_{N_d}^j E_j(u).
$$

Letting $I$ to denote the left hand side of the previous identity, and, respectively, $I_1$ and $I_2$ the two summands of the right hand side, we rewrite the latter as

$$I = I_1 + I_2.$$
Here $0 < \delta < 1/2$ is fixed, $2/j \leq \rho$, $\Delta u_{\eta_j} = \eta_j \Delta u + E_j(u)$ and

$$E_j(u) := 2\nabla \eta_j \nabla u + (\Delta \eta_j)u.$$ 

Note that $I \in L^2$, since $H^1_{\text{loc}}(\Omega) \subset X_2$ by Sobolev embedding theorem, and $|V|^{\frac{1}{2}}u \in X_2$ by the definition of $Y^\text{str}_V$.

Next, we expand $I_1$ as a sum $I_{11} + I_{12}^c$, where

$$I_{11} := 1_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}[(-\Delta)^{-1}]_{N,N_d} V_1^{\frac{1}{2}} 1_{B(\rho)}\varphi_{N_d} \frac{-\Psi_j \Delta u}{V_1^{\frac{1}{2}}}$$

and

$$I_{12}^c := 1_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}[(-\Delta)^{-1}]_{N,N_d} V_1^{\frac{1}{2}} 1_{B(\rho)}\varphi_{N_d} \frac{-\eta \Delta u}{V_1^{\frac{1}{2}}}.$$ 

Proposition 2.1.5 and inequalities (E1) and (E2) imply the required estimates:

$$\|I_{11}\|_2 \leq C\tau(V_1,0,\rho)^{\frac{2}{d-1}} \|I\|_2$$

and

$$\|I_{12}^c\|_2 \leq C\varphi_{N_d}(\rho)\tau(V_1,0,3\rho)^{\frac{2}{d-1}} \|1_{B(3\rho)}|V|^{\frac{1}{2}}u\|_2.$$ 

Finally, we represent $I_2$ as a sum $I_{21} + I_{22}$, where

$$I_{21} := 1_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}[(-\Delta)^{-1}]_{N,N_d} 1_{B(3\rho)} \varphi_{N_d} E_j^{(1)}(u)$$

and

$$I_{22} := 1_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}[(-\Delta)^{-1}]_{N,N_d} 1_{B(3\rho)} \varphi_{N_d} E_j^{(2)}(u).$$

Here

$$E_j^{(1)}(u) := -2\nabla \Psi_j \nabla u - (\Delta \Psi_j)u, \quad E_j^{(2)}(u) := -2\nabla \eta \nabla u - (\Delta \eta)u.$$ 

In order to derive an estimate on $\|I_{21}\|_2$, we expand

$$I_{21} = I_{21}' + I_{21}''.$$ 

where

$$I_{21}' := 1_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}[(-\Delta)^{-1}]_{N,N_d} 1_{B(3\rho)} \varphi_{N_d} (-\Delta \Psi_j)u,$$

$$I_{21}'' := 1_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}[(-\Delta)^{-1}]_{N,N_d} 1_{B(3\rho)} \varphi_{N_d} (-2\nabla \eta \nabla u).$$
1) Term $I'_{21}$ presents no problem: by (E3),

$$\|I'_{21}\|_2 \leq \left\|1_{B(\rho)} \Psi_j V_1^{\frac{1}{2}} (-\Delta)^{-1} \right\|_{N,N_d} \right\|_2 \leq C\tau(V_1, 0, \rho)^{\frac{1}{\gamma}} \left\|1_{B(\rho)} \Psi_j \varphi_{N_d}^\delta (\Delta \Psi_j) u \right\|_2,$$

where

$$\left\|1_{B(\rho)} \Psi_j \varphi_{N_d}^\delta (\Delta \Psi_j) u \right\|_2 \leq Cj^{N_d+1} \left\|1_{B(\rho)} u \right\|_2 \rightarrow 0 \text{ as } j \rightarrow \infty$$

by the definition of the SUC property.

2) In order to derive an estimate on $I''_{21}$, we once again use inequality (E3):

$$\|I''_{21}\|_2 \leq \left\|1_{B(\rho)} \Psi_j V_1^{\frac{1}{2}} (-\Delta)^{-1} \right\|_{N,N_d} \left\|1_{B(\rho)} \Psi_j \nabla \Psi_j \nabla u \right\|_p \leq \leq C\tau(V_1, 0, \rho)^{\frac{1}{\gamma}} \left\|1_{B(\rho)} \Psi_j \nabla \Psi_j \nabla u \right\|_p \leq \tilde{C}j^{N_d+1} \left\|1_{B(\rho)} \nabla u \right\|_p,$$

where $p := \frac{2d}{d+2}$. We must estimate $\|1_{B(\rho)} \nabla u \|_2$ by $\|1_{B(\rho)} \nabla u \|_2$ in order to apply the SUC property. For this purpose, we make use of the following well known interpolation inequality

$$\left\|1_{B(\rho)} \nabla u \right\|_p \leq Cj^d \left( C'j^{\frac{d}{2}-1} \left\|1_{B(\rho)} u \right\|_2 + j^{\frac{d+6}{2}} \left\|1_{B(\rho)} \Delta u \right\|_p \right),$$

where $r := \frac{2d}{d+4}$ (see [M]). Using differential inequality (1.1.1), we reduce the problem to the problem of finding an estimate on $\|1_{B(\rho)} V u \|_r$ in terms of $\|1_{B(\rho)} u \|_r$, $\mu > 0$. By Hölder inequality,

$$\left\|1_{B(\rho)} V u \right\|_r \leq \left\|1_{B(\rho)} V \right\|_2 \left\|1_{B(\rho)} V \right\|_2 \left\|1_{B(\rho)} u \right\|_2,$$

as required.

As the last step of the proof, we use inequality (E4) to derive an estimate on term $I_{22}$:

$$\|I_{22}\|_2 \leq C\tau(V_1, 0, 3\rho)^{\frac{1}{\gamma}} \varphi_{N_d}^\delta (\rho) \left\|E^{(2)}_j (u) \right\|_p.$$

This estimate and the estimates obtained above imply (2.1.5).

Proof of Proposition 2.1.5. Estimates (E1) and (E2) follow straightforwardly from Proposition 2.1.1. In order to prove estimate (E3), we introduce the following interpolation function:

$$F_1(z) := 1_{B(\rho)} \Psi_j |V|^\frac{d-1}{4} z \varphi_{N+\frac{d}{4} - \delta (1-z)} \left[ (-\Delta)^{-\frac{d+1}{2}} z \right] N \varphi^{-1}_{N+\frac{d}{4} - \delta (1-z)} 1_{B(\rho)}.$$
where \(0 \leq \text{Re} \, (z) \leq 1\). According to Lemma 2.1.3, \(\|F_1(i\gamma)\|_{2 \rightarrow 2} \leq C_1 e^{c_1|\gamma|}\) for appropriate \(C_1, c_1 > 0\). Further, according to Lemma 2.1.4,

\[
\|F_1(1 + i\gamma)\|_{2 \rightarrow 2} \leq C_2 e^{c_2 \gamma^2} \|1_{B(\rho)} |V|^{-\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{4}}\|_{2 \rightarrow 2} \leq C_2 e^{c_2 \gamma^2} \|1_{B(\rho)} |V|^{-\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{4}}\|_{2 \rightarrow 2} \leq C_2 e^{c_2 \gamma^2} \tau(V, x_0, \rho)^\frac{1}{2} \|(-\Delta)^{-\frac{d-1}{4}}\|_{2 \rightarrow 2}
\]

for appropriate \(C_2, c_2 > 0\), where, clearly, \(\|(-\Delta)^{-\frac{d-1}{4}}\|_{2 \rightarrow 2} < \infty\). Therefore, by Stein’s interpolation theorem,

\[
\|F_1 \left(\frac{2}{d-1}\right)\|_{p \rightarrow 2} \leq C\tau(V, x_0, \rho)^\frac{1}{d-1}.
\]

The latter inequality implies (E3).

The proof of estimate (E4) is similar: it suffices to consider interpolation function

\[
F_2(z) := 1_{B(\rho)} \Psi_j |V|^{-\frac{d+1}{2}} N_{\frac{d}{2} - \delta} (1-z) \left( (-\Delta)^{-\frac{d+1}{2}} \right)^{-1} 1_{B(3\rho \setminus 2\rho)}
\]

for \(0 \leq \text{Re} \, (z) \leq 1\).

### 2.2 Proof of Theorem 1.3.1

Let \(u \in Y_V^K\). Suppose that \(u \equiv 0\) in some neighbourhood of 0. Assume that \(\rho > 0\) is sufficiently small, so that \(B(0, 2\rho) \subset \Omega\), and let \(\eta \in C^\infty(\Omega)\) be such that \(\eta \equiv 1\) on \(B(0, \rho)\), \(\eta \equiv 0\) on \(\Omega \setminus B(0, 2\rho)\). We may assume, without loss of generality, that \(V \geq 1\). The standard limiting argument implies the following identity:

\[
1_{B(\rho)} u = 1_{B(\rho)} [(-\Delta)^{-1}] N(-\Delta \eta).
\]

Therefore, we can write

\[
1_{B(\rho)} \varphi_N V u = 1_{B(\rho)} \varphi_N V [(-\Delta)^{-1}] N^{-1} 1_{B(\rho)} \varphi_N (-\Delta u) + 1_{B(\rho)} \varphi_N V [(-\Delta)^{-1}] N^{-1} 1_{B(\rho)} \varphi (-\Delta \eta),
\]

or, letting \(K\) to denote the left hand side and, respectively, \(K_1\) and \(K_2\) the two summands of the right hand side of the last equality, we rewrite the latter as

\[
K = K_1 + K_2.
\]
Note that $K \in L^1(\mathbb{R}^d)$, as follows from definition of space $Y^K_V$. Lemma 2.1.2 implies that

$$\|1_{B(\rho)}\varphi_NV[(-\Delta)^{-1}]N\varphi^{-1}_N f\|_1 \leq C\|1_{B(\rho)}V(-\Delta)^{-1}f\|_1 \leq C\beta\|f\|_1,$$

for all $f \in L^1(\Omega)$, which implies an estimate on $K_1$:

$$\|K_1\|_1 \leq C\beta\|K\|_1.$$

In order to estimate $K_2$, we first note that $1_{B(\rho)}^c(-\Delta u_\eta) = 1_{B(2\rho,\rho)}(-\Delta u_\eta)$. According to Lemma 2.1.2 there exists a constant $\hat{C} > 0$ such that

$$\|1_{B(2\rho)}\varphi_NV[(-\Delta)^{-1}]N\varphi^{-1}_N\|_1 \leq \hat{C}.$$

Hence,

$$\|K_2\|_1 \leq \hat{C}\|1_{B(2\rho,\rho)}\varphi_NV(-\Delta u_\eta)\|_1 \leq \hat{C}\rho^{-N}\|\Delta u_\eta\|_1.$$

Let us choose $\beta > 0$ such that $C\beta < 1$. Then the estimates above imply

$$(1 - C\beta)\|1_{B(\rho)}\rho^N\varphi_N u\|_1 \leq (1 - C\beta)\|\rho^NK\|_1 \leq \|\rho^NK_2\|_1 \leq \hat{C}\|\Delta u_\eta\|_1.$$

Letting $N \to \infty$, we obtain $u \equiv 0$ in $B(0, \rho)$. 
Part III

Oka-Cartan type theory for subalgebras of holomorphic functions on coverings of complex manifolds (2009-2011)
Chapter 3

Introduction

Here we develop some basic elements of complex function theory within Frechet (sub) algebras $O_a(X)$ of holomorphic functions defined on a regular covering $p : X \to X_0$ of a complex manifold $X_0$ whose restrictions to each fiber $p^{-1}(x)$, $x \in X_0$, belong to a translation-invariant Banach algebra $a$ endowed with sup-norm.

Our approach is based on analysis of algebra $O(c_aX)$ of ‘holomorphic functions’ on the topological space $c_aX$ of maximal ideals of algebra $O_a(X)$ (the latter two algebras are isomorphic under the assumption that $a$ is closed with respect to the complex conjugation of its elements). More precisely, we prove the analogues of Cartan theorems A and B for ‘analytic sheaves’ on $c_aX$ that we define in Section 6.1.3. Consequently, we obtain results on extension from submanifolds, divisors, corona theorem, approximation within algebra $O_a(X)$ etc, that extend the classical results on Stein manifolds.

The main difficulty in the proof of Cartan type theorems A and B on $c_aX$ comes from The fact that one can no longer use the properties of the local rings of holomorphic germs, crucial for the classical proofs on Stein manifolds (cf. [GR, GrR]), is the main obstruction in proving our extension of Cartan theorems. To overcome this obstruction we exploit some methods from [Lem] for cohesive sheaves (on Banach complex manifolds).

The model examples of algebras $O_a(X)$ are:

**Example 1: Holomorphic almost periodic functions.** A function $f \in O(T)$ on a tube domain $T = \mathbb{R}^n + i \Omega \subset \mathbb{C}^n$, $\Omega \subset \mathbb{R}^n$ is open and convex, is called *holomorphic almost periodic*...
if the family of translates
\[ \{ z \to f(z + s), \ z \in T \}_{s \in \mathbb{R}^n} \]
is relatively compact in the topology of uniform convergence on tube subdomains \( T' = \mathbb{R}^n + i\Omega' \), \( \Omega' \subseteq \Omega \). The cornerstone of Bohr’s theory (see [Bo]) is his approximation theorem, which states that every holomorphic almost periodic function is the uniform limit (on tube subdomains \( T' \) of \( T \)) of exponential polynomials
\[ z \to \sum_{k=1}^{m} c_k e^{i(z, \lambda_k)}, \quad z \in T, \quad c_k \in \mathbb{C}, \quad \lambda_k \in \mathbb{R}^n \]
where \((z, \lambda_k)\) is the Hermitian scalar product on \( \mathbb{C}^n \).

The classical approach to study of holomorphic almost periodic functions exploits the fact that \( T \) is the trivial bundle with base \( \Omega \) and fibre \( \mathbb{R}^n \) (for example as in the characterization of almost periodic functions in terms of their Jessen functions defined on \( \Omega \), see, e.g., [Lev], [JT], [Ron], [FR], etc). By considering \( T \) as a regular covering \( p : T \to T_0 := p(T) \subset \mathbb{C}^n \) with the deck transformation group \( \mathbb{Z}^n \), and
\[ p(z) := (e^{iz_1}, \ldots, e^{iz_n}), \quad z = (z_1, \ldots, z_n) \in T \]
(a complex strip covering an annulus if \( n = 1 \)), we obtain (Theorem 5.2.5) that holomorphic almost periodic functions on \( T \) are precisely the functions \( f \in \mathcal{O}(T) \) bounded on subsets \( p^{-1}(U_0), \ U_0 \subseteq T_0 \), and such that for each \( x \in T \) the functions \( f_x(t) := f(x + t), \ t \in \mathbb{Z}^n \), belong to algebra \( AP(\mathbb{Z}^n) \) which is generated by exponential polynomials \( t \to \sum_{k=1}^{m} c_k e^{i(t, \lambda_k)}, \ t \in \mathbb{Z}^n, \lambda \in \mathbb{R}^n \) (von Neumann almost periodic functions on group \( \mathbb{Z}^n \)). This result enables us to regard holomorphic almost periodic functions on \( T \) as:

(a) holomorphic sections of a certain holomorphic Banach vector bundle on \( T_0 \);

(b) ‘holomorphic’ functions on the fibrewise Bohr compactification of the covering \( p : T \to T_0 \), a topological space having some properties of a complex manifold.

Using these identifications, we employ the methods of analytic sheaf theory and the results of Banach-valued complex analysis to study holomorphic almost periodic functions (in addition
to the variety of techniques already used in the theory of holomorphic almost periodic functions, such as Fourier analysis-type arguments, cf. [Bo], [L], [Lev], or arguments based on the properties of Monge-Amperé currents, cf. [FRR2], etc). We obtain new results on holomorphic almost periodic extensions from almost periodic complex submanifolds, Hartogs-type theorems, recovery of almost periodicity of a holomorphic function from that for its trace to a real periodic hypersurfaces, etc. We also show that many results on holomorphic almost periodic functions on tube domains (e.g., Bohr’s approximation theorem, on properties of almost periodic divisors) are valid, in the corresponding setting, for an arbitrary algebra \( \mathcal{O}_a(X) \).

It is interesting to note that already in his monograph [Bo] H. Bohr uses equally often the aforementioned ”trivial fibre bundle” and ”regular covering” points of view on a complex strip. Note also that the Bohr compactification \( b\mathbb{R}^n + i\Omega \) of a tube domain \( \mathbb{R}^n + i\Omega \) was used earlier in [Fav1], [Fav2] in the context of the problem of a characterization of zero sets of holomorphic almost periodic functions among all almost periodic divisors.

**Example 2: Fibrewise bounded holomorphic functions.** Let \( \mathfrak{a} \) be the algebra \( \ell_\infty(G) \) of all bounded functions on the deck transformation group \( G \cong p^{-1}(x), x \in X_0 \), of covering \( p : X \to X_0 \). The algebra \( \mathcal{O}_{\ell_\infty}(X) \) arises, e.g., in study of holomorphic \( L^2 \)-functions on coverings of pseudoconvex manifolds [GHS, Br2, Br5, La], Caratheodory hyperbolicity (Liouville property) of \( X \) [LS, Lin], corona-type problems for bounded holomorphic functions on \( X \) [Br1]. Earlier, some methods similar to the ones developed in the thesis were elaborated for algebra \( \mathcal{O}_{\ell_\infty}(X) \) in [Br1]-[Br4] in connection with corona-type problems for some algebras of bounded holomorphic functions on coverings of bordered Riemann surfaces, Hartogs-type theorems, integral representation of holomorphic functions of slow growth on coverings of Stein manifolds.

**Outline.** In Chapter 4 we recall basic facts on Stein manifolds, pseudoconvex domains, fibre bundles and (unbranched) coverings.

In Chapter 5 we define algebras of continuous and holomorphic functions needed to study \( \mathcal{O}_a(X) \). We also give some other examples of algebras \( \mathfrak{a} \) and \( \mathcal{O}_a(X) \).

Chapter 6 is devoted to the problem of extension within \( \mathcal{O}_a(X) \), properties of divisors of
More precisely, the problem of extension within subalgebra \( \mathcal{O}_a(X) \) (from complex submanifolds of \( X \) that are locally the zero sets of functions in \( \mathcal{O}_a(X) \)) is studied Section 6.1.1. Our approach to extension via Cartan type theorems A and B (we formulate the latter in Section 6.1.3) requires algebra \( a \) to be self-adjoint with respect to the complex conjugation. However, using a different approach via associated holomorphic Banach vector bundles and extension for Banach-valued holomorphic functions (which, unfortunately, significantly restricts the variety of sets over which one can interpolate within \( \mathcal{O}_a(X) \)), we show that the self-adjointness of \( a \) should not be essential.

In Section 6.1.2 we solve the second Cousin problem within \( \mathcal{O}_a(X) \), and describe the class of holomorphic functions on \( X \) whose (principal) divisors exhaust the class of all divisors of \( \mathcal{O}_a(X) \).

In Section 6.2 we introduce the algebra \( \mathcal{O}(c_aX) \) of holomorphic functions on \( c_aX \) ("the fibrewise \( a \)-compactification of covering \( p : X \to X_0 \)"), and equip \( c_aX \) with the ‘infrastructure’ needed to do complex function theory on \( c_aX \), such as holomorphic maps between complex manifolds and \( c_aX \), coordinate atlas, etc.

Section 6.3 is devoted to coherent sheaves on \( c_aX \).

In Section 6.4 we obtain a corona type theorem for \( \mathcal{O}_a(X) \).

In Section 6.5 we construct complex \( a \)-submanifolds with certain properties by doing ‘algebraic geometry’ on \( c_aX \).

In Chapter 7 we describe the uniqueness sets of holomorphic functions in \( \mathcal{O}_a(X) \), and obtain a Hartogs type theorem for \( \mathcal{O}_a(X) \).

Chapter 8 is devoted to approximation within \( \mathcal{O}_a(X) \). Using a Runge type approximation theorem for \( \mathcal{O}_a(X) \), we obtain a ‘holomorphic Peter-Weyl theorem’.

The proofs of results of Chapters 6–8 are contained in Chapter 9.
Chapter 4

Preliminaries

4.1 Stein manifolds and strictly pseudoconvex sets

(See, e.g., [GrR] for details.)

Definition 4.1.1. A complex manifold $X_0$ is called a Stein manifold if

1. the holomorphic functions $\mathcal{O}(X_0)$ separate points of $X_0$, i.e., for any points $x_1 \neq x_2$ in $X_0$ there exists a function $f \in \mathcal{O}(X_0)$ such that $f(x_1) \neq f(x_2)$, and

2. for any compact subset $Y \subset X_0$ the set

$$\hat{Y} = \left\{ x \in X_0 : |f(x)| \leq \sup_{y \in Y} |f(y)|, f \in \mathcal{O}(X_0) \right\}$$

is also compact.

Example 4.1.2. The following are the examples of Stein manifolds:

1. $\mathbb{C}^n$, product domains in $\mathbb{C}^n$, any non-compact Riemann surface.

2. A closed complex submanifold of a Stein manifold.

3. $X_0 \setminus Z$, where $X_0$ is a Stein manifold and $Z \subset X_0$ is a complex submanifold.

4. An open subset of a Stein manifold satisfying (2).

5. An unbranched covering (see below) of a Stein manifold.

Let $D_0$ be a subdomain of $X_0$ with a $C^2$-smooth boundary $\partial D_0$, i.e., in some open neighbourhood $V_0$ of $\partial D_0$ there exists a real-valued $C^2$-smooth function $q$ such that $D_0 \cap V_0 = \{ z \in V_0 : q(z) < 0 \}$ and $dq(z) \neq 0$ for all $z \in \partial D_0$. 
**Definition 4.1.3** (see, e.g., [GR]). A subdomain $D_0 \subset X_0$ with $C^2$-smooth boundary as above is said to be (Levi) strictly pseudoconvex if for each $z \in \partial D_0$ in some local coordinates about $z$

$$L_z(r, w) := \sum_{j,k=1}^{n} \frac{\partial^2 r(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k > 0$$

(the Levi form of $q$) for all $0 \neq w \in T_z^{1,0}(\partial D_0)$.

Here

$$T_z^{1,0}(\partial D_0) := \left\{ w \in \mathbb{T}_z^{1,0}(\mathbb{C}^n) \simeq \mathbb{C}^n : \sum_{k=1}^{n} \frac{\partial q}{\partial z_k}(z)w_k = 0 \right\}$$

is the holomorphic tangent space to $\partial D_0$ at $z$.

Every strictly pseudoconvex subset of a Stein manifold is Stein.

### 4.2 Fibre bundles and unbranched coverings

(See, e.g., [Hz, ZK] for detailed exposition.)

1. Let $E, X_0, F$ be topological spaces, $p : E \to X_0$ be a continuous surjective map such that for every point in $X_0$ there exists a neighbourhood $U_0$ for which there exists a trivialization $\varphi = \varphi_{U_0} : p^{-1}(U_0) \to U_0 \times F$, i.e., a homeomorphism that maps $p^{-1}(x)$ to $\{x\} \times F$ for each $x \in U_0$; the neighbourhood $U_0$ will be called a trivializing neighbourhood.

**Definition 4.2.1.** $(E, X_0, F, p)$ is called fibre bundle.

We also denote the fibre bundle by $p : E \to X_0$ or by $E$ if no confusion arises. The space $E$ is called the total space, $X_0$ the base, $F$ the fibre and $p$ the projection of the fibre bundle.

Let $\{U_{0,\alpha}\}$ be an open cover of $X_0$ by trivializing neighbourhoods, $\varphi_{0,\alpha} : p^{-1}(U_{0,\delta}) \to U_{0,\delta} \times F$ be corresponding trivializations. The functions

$$\varphi_\delta \varphi^{-1}_\gamma : (U_{0,\delta} \cap U_{0,\gamma}) \times F \to (U_{0,\delta} \cap U_{0,\gamma}) \times F$$

are called the transition functions.

2. Suppose that we are given a right (or left) action of a topological group $G$ on $F$,

$$F \times G \to F, \ (\omega, g) \to \omega \cdot g$$
**Definition 4.2.2.** We say that a fibre bundle \( p : E \to X_0 \) has structure group \( G \) if there exists a trivializing cover \( \{ U_{0,\delta} \} \) of \( X_0 \) and trivialization maps \( \varphi_\delta \) such that

\[
(\varphi_\delta \varphi_\gamma^{-1})(x, \omega) = (x, \omega \cdot c_{\delta,\gamma}(x)), \quad x \in U_{0,\delta} \cap U_{0,\gamma}
\]

for some continuous functions \( c_{\delta,\gamma} : U_{0,\delta} \cap U_{0,\gamma} \to G \).

Every fibre bundle may be viewed as a fibre bundle with structure group \( \text{Hom}_\mathcal{O}(F, F) \).

**Definition 4.2.3.** A fibre bundle \( p : E \to E_0 \) with structure group \( G \) is called principal if \( F = G \) and \( G \) acts on itself by right translations.

Let \( H \) be another topological space on which \( G \) acts continuously from the right.

**Definition 4.2.4.** The disjoint union \( \bigsqcup_\delta U_{0,\delta} \times H \) modulo the equivalence relation

\[
U_{0,\delta} \times H \ni (x, \eta) \sim (x, \eta \cdot c_{\delta,\gamma}(x)) \in U_{0,\gamma} \times H \quad \text{for all} \quad x \in U_{0,\delta} \cap U_{0,\gamma}
\]

forms a fibre bundle with fibre \( H \) and projection induced by canonical projections \( U_{0,\delta} \times H \to U_{0,\delta} \).

This fibre bundle is called the associated fibre bundle to \( p : E \to X_0 \).

3. Let \( X, X_0 \) be smooth manifolds.

**Definition 4.2.5.** A fibre bundle \( p : X \to X_0 \) with at most countable discrete fibre \( F \) is called an (unbranched) covering of \( X_0 \).

Since \( G \) has discrete topology, functions \( c_{\delta,\gamma} \) are locally constant.

In what follows, we assume that \( X_0 \) is connected. If \( X_0 \) is a complex manifold, we assume that projection \( p \) and trivializing maps \( \{ \varphi_\delta \} \) are holomorphic. Then \( X_0 \) is also a complex manifold.

The push-forward \( p_* \pi_1(X) \) of the fundamental group of \( X \) is a subgroup of \( \pi_1(X_0) \).

**Definition 4.2.6.** A covering \( p : X \to X_0 \) is said to be regular if \( p_* \pi_1(X) \) is a normal subgroup of \( \pi_1(X_0) \).
If $p : X \to E_0$ is a regular covering, then the group $G$ of smooth automorphism of $X$ that preserve fibres $p^{-1}(x)$ for all $x \in E_0$ (called the deck transformation group of covering $p : X \to X_0$) acts on $p^{-1}(x)$, $x \in X_0$, freely and transitively from the right. We also have

(a) group $G$ is isomorphic to the quotient group $\pi_1(X_0)/p_*\pi_1(X)$, and

(b) $p : X \to X_0$ is a principal fibre bundle with structure group $G$.

4. Next, we recall the notion of a Banach vector bundle.

**Definition 4.2.7.** A fibre bundle $p : E \to X_0$ is called a **Banach vector bundle** if $F$, $p^{-1}(x)$ $(x \in X_0)$ are (complex) Banach spaces, and

1. there exists a trivializing cover $\{U_{0,\alpha}\}$ of $X_0$ such that the corresponding trivialization functions $\varphi_{\alpha}$ determine isomorphisms of Banach spaces $p^{-1}(x)$ $(x \in E_0)$ and $F$, and

2. the maps from $U_{0,\delta} \cap U_{0,\gamma}$ to the space of bounded linear operators $\mathcal{L}(F,F)$ determined by the transition functions $\varphi_{\delta}\varphi^{-1}_{\gamma}$ are continuous in operator norm topology.

The structure group of a Banach vector bundle is the Banach space $\mathcal{L}(F,F)$.

**Definition 4.2.8.** If $X_0$ is a complex manifold, and the corresponding functions $c_{\delta\gamma}$ are $\mathcal{L}(F,F)$-valued holomorphic functions, then $p : E \to X_0$ is called a **holomorphic Banach vector bundle**.

In particular, if $X_0$ is a complex manifold, and $E \to X_0$ is a Banach vector bundle that is associated to a regular covering $p : X \to X_0$, then functions $c_{\delta\gamma}$ are locally constant and, hence, holomorphic, so $E$ is a holomorphic Banach vector bundle.

We will need the following result.

**Theorem 4.2.9** (See, e.g., [ZK]). **Let $X_0$ be a Stein manifold. Then there exist holomorphic Banach vector bundles $p_1 : E_1 \to X_0$ and $p_2 : E_2 \to X_0$ with fibres $B_1$ and $B_2$, respectively, such that $E_2 = E_1 \oplus E$ (the Whitney sum) and $E_2$ is holomorphically trivial, i.e., $E_2 \cong X_0 \times B_2$.**

Thus, any holomorphic section of $E_2$ can be naturally identified with a $B_2$-valued holomorphic function on $E_0$.

By $q : E_2 \to E$ and $\iota : E \to E_2$ we denote the corresponding quotient and embedding homomorphisms of the bundles so that $q \circ \iota = \text{Id}$. 

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Chapter 5

Subalgebras of holomorphic functions on coverings of complex manifolds

5.1 Definitions

1. Let a connected complex manifold $X$ admit a free and properly discontinuous left holomorphic action of a discrete group $G$. Then the orbit space $X_0 := X/G$ is also a complex manifold and the projection $p : X \to X_0$ is holomorphic and determines a regular covering of $X_0$ with deck transformation group $G$.

Let $a = a(G)$ be a closed unital subalgebra of the algebra $\ell_\infty(G)$ of bounded complex functions on $G$ (with pointwise multiplication and sup-norm) invariant with respect to the action of $G$ on $\ell_\infty(G)$ by right translations: $R_g(f)(h) := f(hg)$, $f \in \ell_\infty(G)$, $g, h \in G$.

**Definition 5.1.1.** A function $f \in \mathcal{O}(X)$ is called a holomorphic $a$-function if it is bounded on subsets $p^{-1}(U_0)$, $U_0 \subseteq X_0$, and for each $x \in X$ the function $G \ni g \mapsto f(g \cdot x)$ belongs to $a$.

The (sub) algebra of holomorphic $a$-functions on $X$ is denoted by $\mathcal{O}_a(X)$ and is endowed with the Frechet topology of uniform convergence on subsets $p^{-1}(U_0)$, $U_0 \subseteq X_0$.

**Remark 5.1.2.** In the case covering $p : X \to X_0$ is not regular we may assume that algebra $a$
is defined on the fibre $F_x := p^{-1}(x)$ ($x \in X_0$) and is invariant with respect to the right action of fundamental group $\pi_1(X_0)$ on $F_x$. Now, the algebra $\mathcal{O}_a(X)$ consists of functions $f \in \mathcal{O}(X)$ bounded on subsets $p^{-1}(U_0)$, $U_0 \Subset X_0$, and such that $f|_{F_x} \in a$ for all $x \in X_0$. The results of the present paper remain valid: since $a$ does not have to separate points of $F_x$, we can lift $\mathcal{O}_a(X)$ to the universal (hence, regular) covering of $X_0$, and carry out all constructions there.

In what follows, we assume that $X_0$ is a connected manifold equipped with a path metric $d_0$ determined by a (smooth) hermitian metric. Let $d$ be a semi-metric on $X$ defined by

$$d(x_1, x_2) := d_0(p(x_1), p(x_2)), \quad x_1, x_2 \in X.$$ 

**Definition 5.1.3.** A function $f \in C(X)$ is called a continuous $a$-function if it is bounded and uniformly continuous with respect to semi-metric $d$ on subsets $p^{-1}(U_0)$, $U_0 \Subset X_0$, and is such that for each $x \in X$ the function $G \ni g \rightarrow f(g \cdot x)$ belongs to $a$.

We denote by $C_a(X)$ the algebra of continuous $a$-functions on $X$. Clearly, algebra $C_a(X)$ does not depend on the choice of the hermitian metric, and we have

$$C_a(X) \cap \mathcal{O}(X) = \mathcal{O}_a(X).$$

2. We also consider continuous and holomorphic $a$-functions on closed subsets of $X$.

**Definition 5.1.4.** A subset $Z \subset X$ of the form

$$Z = \{ x \in X : \rho_a(x) = 0, a \in A \},$$

where $\{ \rho_a : a \in A \} \subset C_a(X)$, is called a closed $a$-subset of $X$.

The basic example of a closed $a$-subset is a set of the form $Z = p^{-1}(Z_0)$, where $Z_0 \subset X_0$ is a closed subset.

**Definition 5.1.5.** A function $f \in C(Z)$ on a closed $a$-subset $Z \subset X$ is called a continuous $a$-function on $Z$ if there exists a function $F \in C_a(X)$ such that $f = F|_Z$.

The algebra of continuous $a$-functions on $Z$ is denoted by $C_a(Z)$. 44
In particular, if $Z = p^{-1}(\bar{D}_0)$, where $D_0$ is a relatively compact subdomain of $X_0$, then a function $f : Z \rightarrow \mathbb{C}$ is in $C_a(X)$ if and only if it is uniformly continuous with respect to semi-metric $d$ on $Z$, and for each $x \in X$ the function $G \ni g \rightarrow f(g \cdot x)$ belongs to $a$.

We can now define holomorphic $a$-functions on subsets of $X$ of the form $\bar{D} := p^{-1}(\bar{D}_0)$, where $D_0$ is a relatively compact subdomain of $X_0$.

**Definition 5.1.6.** A function $f \in \mathcal{O}(D) \cap C_a(\bar{D})$ (cf. Section 7.1) is called a holomorphic $a$-function.

The Banach algebra of holomorphic $a$-functions on $\bar{D}$, endowed with sup-norm, is denoted by $A_a(D)$.

**3.** Now, we define continuous and holomorphic $a$-functions on open subsets of $X$.

Let $\mathfrak{T}_a$ denote the weakest topology on $X$ in which all functions in $C_a(X)$ are continuous. In Section 6.2.6 we describe the base of topology $\mathfrak{T}_a$, and show that if $a$ separates points of $G$, then $(X, \mathfrak{T}_a)$ is a Hausdorff space.

We denote by $a^* = a^*(G)$ the closed subalgebra of $\ell_\infty(G)$ generated by algebra $a(G)$ and its complex conjugate $\bar{a}(G)$.

**Definition 5.1.7.** We say that $U \in \mathfrak{T}_a$ is a $\mathfrak{T}_a$-proper open subset of $V \in \mathfrak{T}_a$ if it can be ‘separated’ from the complement of $V$, i.e., there exists a set in $\mathfrak{T}_a$ of the form

$$\{x \in X : f(x) < 0, \ f \in C_{a^*}(X)\} \subset V$$

and $U \subset \{x \in X : f(x) < -c\}$ for some $c > 0$.

**Lemma 5.1.8.** For any $V \in \mathfrak{T}_a$ there are $\mathfrak{T}_a$-proper subsets $V_\alpha \in \mathfrak{T}_a$ such that $V = \cup_\alpha V_\alpha$.

**Definition 5.1.9.** A function $f \in C(V)$, $V \in \mathfrak{T}_a$, is called a continuous $a$-function on $V$ if for some collection of $\mathfrak{T}_a$-proper open subsets $V_\alpha$ of $V$ such that $V = \cup_\alpha V_\alpha$ there exist functions $F_\alpha \in C_a(X)$ such that $F_\alpha|_{V_\alpha} = f|_{V_\alpha}$, for all $\alpha$.

The algebra of continuous $a$-functions on $V$ is denoted by $C_a(V)$.

**Definition 5.1.10.** A function $f \in \mathcal{O}(V) \cap C_a(V)$, $V \in \mathfrak{T}_a$, is called a holomorphic $a$-function.

The algebra of holomorphic $a$-functions on $V$ is denoted by $\mathcal{O}_a(V)$. 

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Definition 5.1.11. An open cover $\mathcal{V} = \{V_\alpha : V_\alpha \in \Sigma_a\}$ of $X$ is called $\Sigma_a$-fine if there exists a refinement $\mathcal{U} = \{U_\beta : U_\beta \in \Sigma_a\}$ of $\mathcal{V}$ such that each $U_\beta$ is a $\Sigma_a$-proper subset of some $V_\alpha$, and for each $x \in X_0$ there are only finitely many sets $U_\beta \in \mathcal{U}$ such that $U_\beta \cap p^{-1}(x) \neq \emptyset$.

We note that each point in $X$ has a neighbourhood $V \in \Sigma_a$ such that $V \subset p^{-1}(U_0) (\cong U_0 \times G)$ for some simply connected coordinate chart $U_0 \in X_0$. We equip such a $V$ with local coordinates pulled back from $U_0$.

5.2 Examples

Example 5.2.1 (Examples of algebras $a$). (1) The algebra $\ell_\infty(G)$ of bounded functions on $G$, and the algebra $c_0(G)$ of functions that admit extension to the one-point compactification of $G$.

(2) If group $G$ is residually finite (respectively, residually nilpotent), i.e., for any element $t \in G$, $t \neq e$, there exists a normal subgroup $G_t$ such that $G/G_t$ is finite (resp., nilpotent), we consider the closed algebra $\hat{\ell}_\infty(G) \subset \ell_\infty(G)$ generated by the pullbacks of algebras $\ell_\infty(G/G_t)$, for all $G_t$ as above.

(3) Recall that a (continuous) bounded function $f$ on a (topological) group $G$ is called almost periodic if the families of its left and right translates

$$\{t \to f(st)\}_{s \in G}, \quad \{t \to f(ts)\}_{s \in G}$$

are relatively compact in $\ell_\infty(G)$ (J. von Neumann [Ne]). (It was proved in [Ma] that the relative compactness of either the left of the right family of translates already gives the almost periodicity on $G$.) The algebra of almost periodic functions on $G$ is denoted by $AP(G)$.

The basic example of almost periodic functions on $G$ is given by the matrix elements of the finite-dimensional irreducible unitary representations of $G$.

Recall that group $G$ is called maximally almost periodic if its finite-dimensional irreducible unitary representations separate points.

Equivalently, $G$ is maximally almost periodic iff it admits a monomorphism into a compact topological group. Any residually finite group, i.e., a group such that the intersection of all its
finite index normal subgroups is trivial, belongs to this class. In particular, \( \mathbb{Z}^n \), finite groups, free groups, finitely generated nilpotent groups, pure braid groups, fundamental groups of three dimensional manifolds are maximally almost periodic.

We denote by \( AP_0(G) \subset AP(G) \) the space of functions

\[
t \to \sum_{k=1}^{m} c_k \sigma^k_{ij}(t), \quad t \in G, \quad c_k \in \mathbb{C}, \quad \sigma^k = (\sigma^k_{ij}),
\]

where \( \sigma^k (1 \leq k \leq m) \) are finite-dimensional irreducible unitary representations of \( G \). The von Neumann’s approximation theorem states that \( AP_0(G) \) is dense in \( AP(G) \) [Ne].

In particular, the algebra \( AP(\mathbb{Z}^n) \) of almost periodic functions on \( \mathbb{Z}^n \) contains as a dense subset the exponential polynomials \( t \to \sum_{k=1}^{m} c_k e^{i \langle \lambda_k, t \rangle} \), \( t \in \mathbb{Z}^n \), \( \lambda_k \in \mathbb{R}^n \). Here \( \langle \lambda_k, \cdot \rangle \) denotes the linear functional defined by \( \lambda_k \).

4 The algebra \( AP_\mathbb{Q}(\mathbb{Z}^n) \) of almost periodic functions on \( \mathbb{Z}^n \) with rational spectra. This is the subalgebra of \( AP(\mathbb{Z}^n) \) generated over \( \mathbb{C} \) by functions \( t \to e^{i \langle \lambda, t \rangle} \) with \( \lambda \in \mathbb{Q} \).

Example 5.2.2. The algebra \( O_{\ell, \infty}(X) \) consists of holomorphic functions on \( X \) that are bounded on the preimages by \( p \) of the relatively compact subsets of \( X_0 \). This algebra arises, e.g., in study of corona-type problems for bounded holomorphic functions on multiple-connected domains in \( \mathbb{C} \) (more generally, on coverings of bordered Riemann surfaces, see, e.g., [Br1] and references therein).

Example 5.2.3 (Bohr’s holomorphic almost periodic functions on tube domains).

Definition 5.2.4 (S. Bochner). A function \( f \in O(T) \) on a tube domain \( T = \mathbb{R}^n + i \Omega \subset \mathbb{C}^n \), \( \Omega \subset \mathbb{R}^n \) is open and convex, is called holomorphic almost periodic if the family of translates

\[
\{ z \to f(z + s), \; z \in T \}_{s \in \mathbb{R}^n}
\]

is relatively compact in the topology of uniform convergence on tube subdomains \( T' = \mathbb{R}^n + i \Omega', \\Omega' \subset \Omega \).

The Bohr approximation theorem states that every holomorphic almost periodic function is the uniform limit (on tube subdomains \( T' \) of \( T \)) of exponential polynomials

\[
z \to \sum_{k=1}^{m} c_k e^{i \langle \lambda_k, z \rangle}, \quad z \in T, \quad c_k \in \mathbb{C}, \quad (5.2.1)
\]
where \( \langle \lambda_k, \cdot \rangle \) is a complex linear functional defined by \( \lambda_k \in \mathbb{R}^n \).

The classical approach to the study of holomorphic almost periodic functions exploits the fact that \( T \) is the trivial bundle with base \( \Omega \) and fibre \( \mathbb{R}^n \) (see, e.g., [Lev], [JT], [Ron], [FR]). Considering \( T \) as a regular covering \( p : T \rightarrow T_0 (:= p(T) \subset \mathbb{C}^n) \) with the deck transformation group \( \mathbb{Z}^n \),

\[
p(z) := (e^{iz_1}, \ldots, e^{iz_n}), \quad z = (z_1, \ldots, z_n) \in T \tag{5.2.2}
\]

(for \( n = 1 \) this is a complex strip covering an annulus), one obtains the following

\textbf{Theorem 5.2.5.} A function \( f \in \mathcal{O}(T) \) is almost periodic if and only if \( f \in \mathcal{O}_{AP(\mathbb{Z}^n)}(T) \).

(cf. Example 5.2.1(3) for the definition of algebra \( AP(\mathbb{Z}^n) \)).

Note that \( T_0 \) is a relatively complete Reinhardt domain and is a Stein manifold. The latter follows easily from the fact that \( p \) is a local biholomorphism and \( T_0 \) admits exhaustion by strictly pseudoconvex sets, as \( \Omega \) can be exhausted by strictly convex sets.

In fact, we can consider a more general projection of the form

\[
p_\lambda(z) := (e^{i\lambda z_1}, \ldots, e^{i\lambda z_n}), \quad z = (z_1, \ldots, z_n) \in T, \quad \lambda > 0, \tag{5.2.3}
\]

so that \( p_\lambda : T \rightarrow T_{0,\lambda} (:= p_\lambda(T) \subset \mathbb{C}^n) \) forms a regular covering with deck transformation group \( \mathbb{Z}^n \). We have an analogue of Theorem 5.2.5, hence algebra \( \mathcal{O}_{AP}(T) \) defined for covering \( p_\lambda : T \rightarrow T_{0,\lambda} \) does not depend on the value of \( \lambda > 0 \).

\textbf{Example 5.2.6} (Holomorphic almost periodic functions on coverings of complex manifolds). The holomorphic \( AP \)-functions on covering \( X \) are called \textit{holomorphic almost periodic functions} on \( X \) (see Example 5.2.1(3) for the definition of algebra \( AP(\mathbb{G}) \) of almost periodic functions on deck transformation group \( \mathbb{G} \)).

Equivalently, a function \( f \in \mathcal{O}(X) \) is called holomorphic almost periodic if each \( G \)-orbit in \( X \) has a neighbourhood \( U \) that is invariant with respect to the (left) action of \( G \), such that the family of translates \( \{ z \rightarrow f(g \cdot z), z \in U \}_{g \in G} \) is relatively compact in the topology of uniform convergence on \( U \) (see [BrK1] for the proof of equivalence).

This is a variant of definition in [We], where \( G \) is taken to be the group of all biholomorphic automorphisms of the complex manifold \( X \) (an interesting result in [Ves] states that on Siegel
domains of the second kind there are no non-constant holomorphic almost periodic functions in the sense of [We], although on Siegel domains of the first kind (i.e., on tube domains in \( \mathbb{C}^n \)) the holomorphic almost periodic functions even separate points).

For instance, let \( X_0 \) be a non-compact Riemann surface, \( p : X \to X_0 \) be a regular covering with a maximally almost periodic deck transformation group \( G \) (for instance, \( X_0 \) is hyperbolic, then \( X = \mathbb{D} \) is its universal covering, and \( G = \pi_1(X_0) \) is a free (not necessarily finitely generated) group); the functions in \( \mathcal{O}_{AP}(X) \) arise, e.g., as linear combinations over \( \mathbb{C} \) of matrix entries of fundamental solutions of certain linear differential equations on \( X \) (see Example 8.0.4 below for details).
Chapter 6

Extension and divisors. Cartan type theorems A and B

6.1 Main results

6.1.1 Extension from complex submanifolds within a holomorphic algebra

We will need the following

Definition 6.1.1. A closed subset $Z \subset X$ is called a complex $\alpha$-submanifold of codimension $k \leq n$ if there exists a $\Sigma_\alpha$-fine cover $\mathcal{V}$ of $X$ such that for each $V \in \mathcal{V}$ there exist functions $h_1, \ldots, h_k \in O_\alpha(V)$ such that

(1) $Z \cap V = \{ x \in V : h_1(x) = \cdots = h_k(x) = 0 \}$;

(2) maximum of moduli of determinants of all $k \times k$ submatrices of the Jacobian matrix of the map $x \rightarrow (h_1(x), \ldots, h_k(x))$ (with respect to local coordinates on $V$ pulled back from $X_0$) is uniformly bounded away from zero on $Z \cap V$.

If $k = 1$, then $Z$ is called a complex $\alpha$-hypersurface.

If all $V$ in Definition 6.1.1 have the form $V = p^{-1}(U_0)$ with open $U_0 \subset X_0$, then $Z$ is called a cylindrical complex $\alpha$-submanifold.

(See Example 6.1.9 below for constructions of such submanifolds.)
Theorem 6.1.2 (Characterization of complex $\alpha$-submanifolds). Suppose that $\alpha$ is self-adjoint, $X_0$ is a Stein manifold. A closed subset $Z \subset X$ is a complex $\alpha$-submanifold of codimension $k \leq n$ if and only if there exists at most countable collection of functions $f_i \in \mathcal{O}_\alpha(X)$, $i \in I$, such that

1. $Z = \{x \in X : f_i(x) = 0 \text{ for all } i \in I\}$,

2. for each $x_0 \in Z$ there exists a neighbourhood $V \in \mathfrak{X}_\alpha$ and functions $f_{i_1}, \ldots, f_{i_k}$ such that $Z \cap V = \{x \in V : f_{i_1} = \cdots = f_{i_k} = 0\}$ and the maximum of moduli of determinants of all $k \times k$ submatrices of the Jacobian matrix of the map $x \to (f_{i_1}(x), \ldots, f_{i_k}(x))$ (with respect to local coordinates on $V$ pulled back from $X_0$) is non-zero at $x_0$.

Definition 6.1.3. A function $f \in \mathcal{O}(Z)$ on a complex $\alpha$-submanifold $Z \subset X$ is called a holomorphic $\alpha$-function if it admits an extension to a function in $C_\alpha(X)$.

The algebra of holomorphic $\alpha$-functions on $Z$ is denoted by $\mathcal{O}_\alpha(Z)$.

There is an equivalent ‘analytic’ definition of algebra $\mathcal{O}_\alpha(Z)$ in terms of currents.

Let $\Lambda^{t,s}_c(X)$ be the space of smooth $(t,s)$-forms with compact supports on $X$.

Proposition 6.1.4. Let $\alpha$ be self-adjoint. A function $f \in \mathcal{O}(Z)$ on a complex $\alpha$-submanifold $Z \subset X$ is a holomorphic $\alpha$-function if and only if it is bounded on subsets $Z \cap p^{-1}(U_0)$, $U_0 \in X_0$, and the corresponding current

\[(f, \varphi) := \int_Z f \varphi, \quad \varphi \in \Lambda^{k,k}_c(X), \quad k := \text{codim}_\mathbb{C}Z, \quad (6.1.1)\]

is an $\alpha$-current; the latter means that for any $\varphi$ the function $G \ni g \to (f, \varphi_g)$ belongs to algebra $\alpha$. Here $\varphi_g(x) := \varphi(g \cdot x)$ ($x \in X$).

In the setting of Example 5.2.3 (holomorphic almost periodic functions on tube domains) the almost periodic currents were studied, e.g., in [FRR2] (see further references therein).

Theorem 6.1.5. Suppose that $\alpha$ is self-adjoint, $X_0$ is a Stein manifold, $Z \subset X$ is a complex $\alpha$-submanifold, and $f \in \mathcal{O}_\alpha(Z)$. Then there is $F \in \mathcal{O}_\alpha(X)$ such that $F|_Z = f$.

The key component of our proofs of Theorems 6.1.2 and 6.1.5 is an analogue of Cartan theorem B for coherent sheaves on the fibrewise compactification of covering $X$, which we describe in Section 6.1.3 below.
For instance, in the setting of Example 5.2.3 (holomorphic almost periodic functions on a tube domain $T$), suppose that $Z_1, Z_2 \subset T$ are non-intersecting smooth complex hypersurfaces that are periodic, possibly with different periods, and $f_1 \in \mathcal{O}(Z_1), f_2 \in \mathcal{O}(Z_2)$ are holomorphic periodic with respect to these periods functions. Then there is a holomorphic almost periodic function $F \in \mathcal{O}_{AP}(T)$ such that $F|_{Z_i} = f_i, i = 1, 2$.

**Remark 6.1.6.** We have an equivalent presentation of functions in $\mathcal{O}_a(X)$ as holomorphic sections of a holomorphic Banach vector bundle $\tilde{p} : C_0X_0 \to X_0$, defined as follows. The regular covering $p : X \to X_0$ is a principal fibre bundle with structure group $G$, so there exists an open cover $\{U_0,\gamma\}$ of $X_0$ and a locally constant cocycle $\{c_{\delta\gamma} : U_{0,\gamma} \cap U_{0,\delta} \to G\}$, so that the covering $p : X \to X_0$ can be obtained from the disjoint union $\bigsqcup_{\gamma} U_{0,\gamma} \times G$ by the identification $U_{0,\delta} \times G \ni (x,g) \sim (x, g c_{\delta\gamma}(x)) \in U_{0,\gamma} \times G$ for all $x \in U_{\gamma} \cap U_{0,\delta}$, where projection $p$ is induced by the projections $U_{0,\gamma} \times G \to U_{0,\gamma}$ (see, e.g., [Hz]). Then $C_0X_0$ is a fibre bundle associated to $p : X \to X_0$ and having fibre $a$. By definition (see Section 4.2), bundle $C_0X_0$ is obtained from the disjoint union $\bigsqcup_{\gamma} U_{0,\gamma} \times a$ by the identification $U_{\delta} \times a \ni (x,f) \sim (x, R_{c_{\delta\gamma}(x)}(f)) \in U_{\gamma} \times a$ for all $x \in U_{\gamma} \cap U_{\delta}$. The projection $\tilde{p}$ is induced by projections $U_{\gamma} \times a \to U_{\gamma}$. This is a holomorphic Banach vector bundle.

Let $\mathcal{O}(C_0X_0)$ be the set of (global) holomorphic sections of $C_0X_0$. This is a Frechet algebra with respect to the usual pointwise operations and the topology of uniform convergence on compact subsets of $X_0$.

**Proposition 6.1.7.** $\mathcal{O}_a(X) \cong \mathcal{O}(C_0X_0)$.

As a consequence of Proposition 6.1.7, we obtain the following result of extension within the class of holomorphic $a$-functions.

**Proposition 6.1.8.** Let $M_0$ be a closed complex submanifold of a Stein manifold $X_0$, $M := p^{-1}(M_0), D_0 \subset X_0$ is Levi strictly pseudoconvex (see, e.g., [GR]), $D := p^{-1}(D_0)$, and $f \in \mathcal{O}_a(M \cap D)$ is bounded. Then there exists a bounded function $F \in \mathcal{O}_a(D)$ such that $F|_{M \cap D} = f|_{M \cap D}$.
Indeed, the algebra \( \mathcal{O}_a(M) \) is isomorphic to the algebra \( \mathcal{O}(C_aX)|_{M_0} \) of holomorphic sections of bundle \( CX \) over \( M_0 \) (cf. the proof of Proposition 6.1.7). By Theorem 4.2.9, since \( X_0 \) is Stein, there exist holomorphic Banach vector bundles \( p_1 : E_1 \to X_0 \) and \( p_2 : E_2 \to X_0 \) with fibres \( B_1 \) and \( B_2 \), respectively, such that \( E_2 = E_1 \oplus C_aX_0 \) (the Whitney sum) and \( E_2 \) is holomorphically trivial, i.e., \( E_2 \cong X_0 \times B_2 \). Thus, any holomorphic section of \( E_2 \) can be naturally identified with a \( B_2 \)-valued holomorphic function on \( X_0 \). By \( q : E_2 \to C_aX_0 \) and \( \iota : C_aX_0 \to E_2 \) we denote the corresponding quotient and embedding homomorphisms of the bundles so that \( q \circ \iota = \text{Id} \).

(Similar identifications hold for bundle \( CD \).) For a given function \( f \in \mathcal{O}(C_aX_0)|_{M_0} \) consider its image \( \tilde{f} := \iota(f) \), a \( B_2 \)-valued holomorphic function on \( M_0 \), and apply to it the integral representation formula from [HL] asserting the existence of a bounded function \( \tilde{F} \in \mathcal{O}(D_0, B_2) \) such that \( \tilde{F}|_{M_0 \cap D_0} = \tilde{f}|_{M_0 \cap D_0} \). Finally, we define \( F := q(\tilde{F}) \).

The algebra \( a \) in Proposition 6.1.8 does not have to be self-adjoint (cf. Theorem 6.1.5). In fact, this method allows to obtain the extension results for holomorphic functions on \( X \) whose restriction to each fibre belongs to some Banach space, and is possibly unbounded, see [Br4]. It is not yet clear to what extent the analogue of Theorem 6.1.5 is valid for such holomorphic functions.

**Example 6.1.9** (Examples of complex \( a \)-submanifolds). (1) If \( Z_0 \subset X_0 \) is a complex submanifold of codimension \( k \), then \( Z := p^{-1}(Z_0) \subset X \) is a cylindrical complex \( a \)-submanifold of codimension \( k \).

(2) The disjoint union of a finite collection of complex \( a \)-submanifolds \( Z_i \) of \( X \) separated by the functions in \( C_a(X) \) (i.e., for each \( i \) there is \( f \in C_a(X) \) such that \( f = 1 \) on \( Z_i \), and \( f = 0 \) on \( Z_j \) for \( j \neq i \)) is a complex \( a \)-submanifold. In particular, in the setting of Example 5.2.3, holomorphic almost periodic functions on a tube domain \( T \), the finite union of non-intersecting complex submanifolds periodic with respect to the action of group \( \mathbb{R}^n \) on \( T \) by translations and having different periods is an almost periodic (i.e., AP-) complex submanifold of \( T \).

(3) Let \( Z_0 = \{ x \in X_0 : f_1(x) = \cdots = f_k(x) = 0 \} \) for some \( f_i \in \mathcal{O}(X_0) \) (\( 1 \leq i \leq k \)) such that the rank of the Jacobian matrix of the map \( x \to (f_1(x), \ldots, f_k(x)) \) with respect to some local coordinates on \( X_0 \) is maximal on \( Z_0 \). Set \( Z := p^{-1}(Z_0) \).
For an open subset $X_0' \Subset X_0$ and functions $h_1, \ldots, h_k \in \mathcal{O}_a(X)$ we define

$$X' := p^{-1}(X_0'), \quad \delta := \sup_{x \in X} \max_{1 \leq i \leq k} |h_i(x)|.$$  

**Definition 6.1.10.** The set

$$Z_h := \{x \in X': p^*f_1(x) + h_1(x) = \cdots = p^*f_k(x) + h_k(x) = 0\}$$

will be called the $a$-perturbation of $Z \cap X'$ by $h = (h_i)_{i=1}^k$.

**Proposition 6.1.11.** $Z_h$ is a complex $a$-submanifold of $X'$ provided that $\delta > 0$ is sufficiently small.

(4) A complex $a$-submanifold of $X$ is called *cylindrical* if each open set $V$ in Definitions 6.1.1 has form $V = p^{-1}(V_0)$ for some open $V_0 \subset X_0$.

In Section 6.5 we construct a non-cylindrical $a$-hypersurface in $X$ in the case $a = AP(\mathbb{Z})$ (cf. Example 5.2.6) and $p : X \to X_0$ is a regular covering of a Riemann surface $X_0$ having deck transformation group $\mathbb{Z}$. We assume that $X_0$ has finite type and is a relatively compact subdomain of a larger (non-compact) Riemann surface $\tilde{X}_0$ whose fundamental group satisfies $\pi_1(\tilde{X}_0) \cong \pi_1(X_0)$ (e.g., the covering of Example 5.2.3 with $n = 1$, i.e., a complex strip covering an annulus, is a regular covering of this form).

Let us briefly describe this construction.

The covering $X$ of $X_0$ admits an injective holomorphic map into a holomorphic fibre bundle over $X_0$ having fibre $(\mathbb{C}^*)^2$, $\mathbb{C}^* := \mathbb{C}\setminus \{0\}$, defined as follows. First, note that the regular covering $p : X \to X_0$ admits presentation as a principal fibre bundle with fibre $\mathbb{Z}$, see Remark 6.1.6. We choose two characters $\chi_1, \chi_2 : \mathbb{Z} \to \mathbb{S}^1 \cong \mathbb{R}/(2\pi \mathbb{Z})$ such that the homomorphism $(\chi_1, \chi_2) : \mathbb{Z} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is an embedding with dense image. Consider the fibre bundle $b_{\mathbb{T}^2}X$ over $X_0$ with fibre $\mathbb{T}^2$ associated with the principal fibre bundle $p : X \to X_0$ via the homomorphism $(\chi_1, \chi_2)$. The bundle $b_{\mathbb{T}^2}X$ is embedded into a holomorphic fibre bundle $b_{(\mathbb{C}^*)^2}X$ with fibre $(\mathbb{C}^*)^2$ associated with the composite of the embedding homomorphism $\mathbb{T}^2 \hookrightarrow (\mathbb{C}^*)^2$ and $(\chi_1, \chi_2)$.

Now, the covering $X$ of $X_0$ admits an injective $C^\infty$ map into $b_{\mathbb{T}^2}X$ with dense image and the composite of this map with the embedding of $b_{\mathbb{T}^2}X$ into $b_{(\mathbb{C}^*)^2}X$ is an injective holomorphic map $X \to b_{(\mathbb{C}^*)^2}X$. Further, the bundle $b_{(\mathbb{C}^*)^2}X$ admits a holomorphic trivialization $\eta : b_{(\mathbb{C}^*)^2}X \to \mathbb{C}^2$.\]
We choose $\chi_1(1)$ and $\chi_2(1)$ so close to $1 \in S^1$ that the image $\eta(b_{T^2}X) \subset X_0 \times (\mathbb{C}^*)^2$ is sufficiently close to $X_0 \times T^2$. Thus identifying $X$ (by means of holomorphic injection $X \hookrightarrow b((\mathbb{C}^*)^2)X \xrightarrow{\eta} X_0 \times (\mathbb{C}^*)^2$) with a subset of $X_0 \times (\mathbb{C}^*)^2$, we obtain that $X$ is sufficiently close to $X_0 \times T^2$.

Next, we construct a smooth complex hypersurface in $X_0 \times (\mathbb{C}^*)^2$ such that in each cylindrical coordinate chart $U_0 \times (\mathbb{C}^*)^2$ on $X_0 \times (\mathbb{C}^*)^2$ for $U_0 \in X_0$ simply connected it cannot be determined as the set of zeros of a holomorphic function on $U_0 \times (\mathbb{C}^*)^2$. Intersecting this hypersurface with $X$ we obtain a non-cylindrical almost periodic hypersurface in $X$.

To construct such a hypersurface in $X_0 \times (\mathbb{C}^*)^2$, we determine a smooth divisor in $(\mathbb{C}^*)^2$ that has a non-zero Chern class (i.e., it cannot be given by a holomorphic function on $(\mathbb{C}^*)^2$), and whose support intersects the real torus $T^2 \subset (\mathbb{C}^*)^2$ transversely. Then we take the pullback of this divisor with respect to the projection $X_0 \times (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ to get the desired hypersurface.

### 6.1.2 Divisors

Recall that an effective (Cartier) divisor $E$ on $X$ is given by an open cover $\{U_\alpha\}$ of $X$ and functions $f_\alpha \in \mathcal{O}(U_\alpha)$, $f_\alpha$ not identically equal to zero on any open subset of $U_\alpha$, such that $f_\alpha = d_{\alpha\beta}f_\beta$ on $U_\alpha \cap U_\beta$ for a nowhere zero function $d_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$, for all $\alpha, \beta$.

The collection of effective divisors on $X$ is denoted by $\text{Div}(X)$.

The divisors $E = \{(U_\alpha, f_\alpha)\}, E' = \{(V_\beta, g_\beta)\}$ in $\text{Div}(X)$ are said to be equivalent (in $\text{Div}(X)$) if there exists a refinement $\{W_\gamma\}$ of both covers $\{U_\alpha\}$ and $\{V_\beta\}$ and nowhere zero holomorphic functions $c_\gamma$ on $W_\gamma$ such that $f_\alpha|_{W_\gamma} = c_\gamma g_\beta|_{W_\gamma}$ for $W_\gamma \subset U_\alpha \cap V_\beta$.

Let $T_E$ be the integration current of a divisor $E \in \text{Div}(X)$, i.e.,

$$(T_E, \varphi) := \int_E \varphi, \quad \varphi \in \Lambda^{n-1,n-1}(X).$$

Here we use the fact that locally (in the usual topology on $X$) divisor $E$ admits presentation as a collection of analytic hypersurfaces with prescribed multiplicities.

It is easy to see that divisors $E, E' \in \text{Div}(X)$ are equivalent if and only if $T_E = T_{E'}$. 

**Definition 6.1.12.** A divisor $E \in \text{Div}(X)$ is called an (effective) $a$-divisor if in the above definition of a divisor on $X$ we have

\[X_0 \times (\mathbb{C}^*)^2.\]
(1) the open cover \( \{U_\alpha\} \) is \( T_a \)-fine (cf. Definition 5.1.11),

(2) \( f_\alpha \in \mathcal{O}_a(U_\alpha) \),

(3) \( f_\alpha = d_{\alpha\beta} f_\beta \) on \( U_\alpha \cap U_\beta \neq \emptyset \) for some \( d_{\alpha\beta} \in \mathcal{O}_a(U_\alpha \cap U_\beta) \) uniformly bounded away from zero on every \( T_a \)-proper open subset of \( U_\alpha \cap U_\beta \), for all \( \alpha, \beta \).

The \( a \)-divisors \( E = \{(U_\alpha, f_\alpha), E' = \{(V_\beta, g_\beta)\} \) are said to be equivalent in \( \text{Div}_a(X) \) if they are equivalent in \( \text{Div}(X) \), with the refinement \( \{W_\gamma\} \) (cf. definition above) being \( T_a \)-fine, and functions \( c_\gamma \in \mathcal{O}(W_\gamma) \) being uniformly bounded away from zero on every \( T_a \)-proper open subset of \( W_\gamma \).

For some algebras \( a \) the \( a \)-divisors can be defined equivalently in terms of their currents of integration, cf. Remark 6.1.16(1).

The collection of \( a \)-divisors is denoted by \( \text{Div}_a(X) \).

The basic example of an \( a \)-divisor is the divisor \( E_f \) of a function \( f \in \mathcal{O}_a(X) \); such divisors are called \( a \)-principal. There are, however, \( a \)-divisors that are not \( a \)-principal, e.g., the pullbacks by \( p \) of non-principal divisors in \( \text{Div}(X_0) \) (cf. remark in Remark 6.1.16(3)).

We consider the following problem: does there exist a class of functions \( \mathcal{C}_a \subset \mathcal{O}(X) \) such that each function from \( \mathcal{C}_a \) determines a divisor that is equivalent in \( \text{Div}(X) \) to a divisor in \( \text{Div}_a(X) \), and conversely, any divisor in \( \text{Div}_a(X) \) is equivalent in \( \text{Div}(X) \) to a divisor determined by a function in \( \mathcal{C}_a \)?

If \( X = \{z \in \mathbb{C} : a < \text{Im}(z) < b\} \) and \( a = \text{AP}(\mathbb{Z}) \) (cf. Example 5.2.3), then by the result in [FRR]

\[
\mathcal{C}_{\text{AP}} = \{f \in \mathcal{O}(X) : |f| \in \mathcal{C}_{\text{AP}}(X)\};
\]

the proof in [FRR] uses the properties of almost periodic currents (see also [Fav2] for the multidimensional generalization of this result).

Using a sheaf-theoretic approach, we extend this result as follows.

**Theorem 6.1.13.** Let \( a \) be self-adjoint.

(1) Suppose that \( X_0 \) is a non-compact Riemann surface, and \( X \) is the universal covering of \( X_0 \). Then for every divisor \( E \in \text{Div}_a(X) \) there exists a function \( f \in \mathcal{O}(X) \) with \( |f| \in \mathcal{C}_a(X) \) such that \( E \) is equivalent to the principal divisor \( E_f \in \text{Div}(X) \).
Conversely, for any complex manifold $X_0$, let $f \in \mathcal{O}(X)$, $|f| \in C_\alpha(X)$, and suppose that there exists an open subset $Y \Subset X$ such that for any net $\{g_\alpha\} \subset G$ the translates $\{x \mapsto f(g_\alpha \cdot x)\}$ do not converge uniformly on $\bar{Y}$ to zero. Then $E_f$ is equivalent to a divisor $E \in \text{Div}_\alpha(X)$.

For the algebra $\mathcal{O}_{AP}(X)$ of holomorphic almost periodic functions on $X$ (cf. Example 5.2.6) Theorem 6.1.13 can be somewhat refined:

**Corollary 6.1.14.** (1) Let $X_0$ be a non-compact Riemann surface, $X$ be a universal covering of $X_0$. Then for every divisor $E \in \text{Div}_{AP}(X)$ there exists a function $f \in \mathcal{O}(X)$ with $|f| \in C_{AP}(X)$ such that $E$ is equivalent to $E_f \in \text{Div}(X)$.

(2) If $f \in \mathcal{O}(X)$, $|f| \in C_{AP}(X)$, then $E_f$ is equivalent to a divisor in $\text{Div}_{AP}(X)$.

The second Cousin problem asks whether a given divisor in $\text{Div}(X)$ is equivalent to a principal divisor. We consider a similar problem: given a divisor $E \in \text{Div}_\alpha(X)$, does there exist a function $f \in \mathcal{O}_\alpha(X)$ such that $E_f$ is equivalent to $E$?

**Theorem 6.1.15.** Let $\mathfrak{a}$ be self-adjoint, $E$ be an $\mathfrak{a}$-divisor, $X_0$ be a Stein manifold.

If $X_0$ is homotopically equivalent to an open subset $Y_0 \subset X_0$ such that the restriction of $E$ to $Y := p^{-1}(Y_0)$ is equivalent to an $\mathfrak{a}$-principal divisor, then $E$ itself is equivalent to an $\mathfrak{a}$-principal divisor.

In particular, if $\text{supp}(E) \cap Y = \emptyset$, then $E$ is equivalent to an $\mathfrak{a}$-principal divisor.

Here $\text{supp}(E)$ is the set of zeros of functions $f_\alpha$ in the definition of divisor.

In the case $X$ and $Y$ are tube domains and $\mathfrak{a} = AP(\mathbb{Z}^n)$ (cf. Example 5.2.3) this theorem is due to [FRR] ($n = 1$) and [Fav1] ($n \geq 1$). The proof in [FRR] uses Arakelyan’s theorem and gives an explicit construction of the holomorphic almost periodic function that determines the principal divisor. Our proof, similarly to the proof in [Fav1], is sheaf-theoretic.

**Remark 6.1.16.** (1) One can show that if $E \in \text{Div}(X)$ is an $AP$-divisor (cf. Example 5.2.6), then its integration current $T_E$ is an $AP$–current (cf. Proposition 6.1.4 for definition). Conversely, if the integration current $T_E$ of a divisor $E \in \text{Div}(X)$ is an $AP$-current, then $E$ is equivalent in $\text{Div}(X)$ to an $AP$-divisor. In particular, it follows that for Bohr’s holomorphic
almost periodic functions on tube domains (cf. Example 5.2.3) AP-divisors coincide with the classical almost periodic divisors, see, e.g., [Lev, FRR2].

(2) In the case $X_0$ is an arbitrary Stein manifold, Corollary 6.1.14(1) is no longer true, see [Fav2] where an explicit necessary and sufficient condition for existence of such a function $f$ is obtained in the setting of Example 5.2.3 (holomorphic almost periodic functions in tube domains).

(3) The $\mathfrak{a}$-principal divisors are contained in a broader class of cylindrical $\mathfrak{a}$-divisors, i.e., the divisors $E \in \text{Div}_{\mathfrak{a}}(X)$ such that in Definition 6.1.12 we have $U_\alpha = p^{-1}(U_{0,\alpha})$ for some open $U_{0,\alpha} \subset X_0$, and $f_\alpha \in \mathcal{O}_\mathfrak{a}(U_\alpha)$.

For some self-adjoint algebras $\mathfrak{a}$ the cylindrical $\mathfrak{a}$-divisors exhaust $\text{Div}_{\mathfrak{a}}(X)$ up to equivalence: if the covering dimension of the maximal ideal space $M_\mathfrak{a}$ of $\mathfrak{a}$ is zero, then every $\mathfrak{a}$-divisor is equivalent to a cylindrical $\mathfrak{a}$-divisor (this follows from Definitions 6.2.11, 6.2.12 and Proposition 6.2.13 below). In particular, all $\ell_\infty$, $AP_Q$-divisors are equivalent to cylindrical divisors (cf. Examples 6.2.1(3), (4) below). There are, however, non-cylindrical $AP$-divisors, see Example 6.1.9(4).

For an arbitrary self-adjoint algebra $\mathfrak{a}$, Theorem 6.1.15 implies that if $E$ is an $\mathfrak{a}$-divisor, then:

1. If $E$ is not equivalent to a cylindrical $\mathfrak{a}$-divisor, then the projection of $\text{supp}(E)$ to $X_0$ is everywhere dense.

2. If there exists a function $f \in \mathcal{O}(U)$, where $U = p^{-1}(U_0)$, $U_0 \subset X_0$ is open, such that $E|_U$ is determined by $f$, then $E$ is equivalent to a cylindrical divisor.

The converse to assertion (1) is not true, e.g., one can modify Example 6.1.9(4).

### 6.1.3 Cartan type theorems A and B

(See Sections 6.2 and 6.3 for details.)

Recall that $p : X \to X_0$ is a regular covering with a deck transformation group $G$ of a complex manifold $X_0$. Let $\mathfrak{a} \subset \ell_\infty(G)$ be a closed subalgebra invariant with respect to the action $R$ of $G$ on $\ell_\infty(G)$ (see section 1.2), and $M_\mathfrak{a}$ be its maximal ideal space, i.e., the space of non-zero continuous homomorphisms $M_\mathfrak{a} \to \mathbb{C}$ endowed with weak* topology. There exists
a continuous map $G \to M_a$ sending each element of $G$ to its point evaluation homomorphism, which is an injection if and only if $a$ separates points of $G$. Let $\hat{G}_a$ denote the closure of the image of $G$ in $M_a$ under this map. Using invariance of $a$ with respect to the action $R$, one obtains that the action of $G$ on itself by multiplications from the right is extended to a continuous action $r_a$ of $G$ on $M_a$. Clearly, $r_a$ fixes $\hat{G}_a$. It is well known that the covering $p : X \to X_0$ can be regarded as a principal bundle over $X_0$ with fibre $G$. By $c_aX$ we denote the fibre bundle over $X_0$ with fibre $\hat{G}_a$ associated with the action $r_a$ and call it the \textit{fibrewise $a$-compactification of $X$}. There exists a continuous map $\iota : X \hookrightarrow c_aX$. By definition, $G$ is dense in $\hat{G}_a$, hence $\iota(X)$ is dense in $c_aX$.

A continuous function $f$ on an open subset $U \subset c_a$ is called \textit{holomorphic} if its pullback $\iota^*f$ is holomorphic on $\iota^{-1}(U)$. The space of holomorphic on $U$ functions is denoted by $\mathcal{O}(U)$. We will show (Proposition 6.2.8 below) that there exists a linear extension operator from $\mathcal{O}_a(X)$ into $\mathcal{O}(c_aX)$ which is an algebraic isomorphism if $a$ is self-adjoint.

Let $\mathcal{O}$ be the sheaf of (rings of) germs of holomorphic functions on $c_aX$.

A sheaf of modules $\mathcal{A}$ on an open subset $U \subset c_aX$ over $\mathcal{O}|_U$ is called an analytic sheaf.

A homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between analytic sheaves $\mathcal{A}, \mathcal{B}$ is called an analytic homomorphism.

A \textit{coherent sheaf} $\mathcal{A}$ on $c_aX$ is an analytic sheaf such that every point in $c_aX$ has a neighbourhood $W$ so that for any $N \geq 1$ there exists a free resolution of $\mathcal{A}$ over $W$ of length $N$, i.e., an exact sequence of sheaves of modules of the form

$$
\mathcal{O}^{m_N}|_W \xrightarrow{\varphi^{N-1}} \cdots \xrightarrow{\varphi_2} \mathcal{O}^{m_2}|_W \xrightarrow{\varphi_1} \mathcal{O}^{m_1}|_W \xrightarrow{\varphi_0} \mathcal{A}|_W \to 0.
$$

Let $X_0$ be a Stein manifold, $\mathcal{A}$ a coherent analytic sheaf on $c_aX$. Let $\mathcal{A}$ be the stalk of sheaf $\mathcal{A}$ at $x \in c_aX$.

**Theorem 6.1.17** (Cartan A). \textit{For every $x \in c_aX$ the stalk $\mathcal{A}$ is generated by the global sections $\Gamma(c_aX, \mathcal{A})$ as an $x\mathcal{O}$-module.}

**Theorem 6.1.18** (Cartan B). \textit{The Čech cohomology groups $H^i(c_aX, \mathcal{A}) = 0$, $i \geq 1$.}
6.2 Fibrewise compactification of covering

Our proofs of Theorems 6.1.17 and 6.1.18 use the construction of fibrewise compactification of covering $p : X \to X_0$ with respect to algebra $a$, which we describe below.

6.2.1 Compactification of deck transformation group

Let $M_a$ denote the maximal ideal space of algebra $a = a(G)$, i.e., the space of non-zero continuous homomorphisms $a \to \mathbb{C}$ endowed with weak* topology (of $a^*$). The space $M_a$ is compact and Hausdorff, and every element $f$ of $a$ determines a function $\hat{f} \in C(M_a)$,

$$\hat{f}(\eta) := \eta(f), \quad \eta \in M_a.$$  

Since algebra $a$ is uniform (i.e., $\|f^2\| = \|f\|^2$) and hence semi-simple, the homomorphism $a \to C(M_a)$ (called Gelfand transform) is an isometric embedding (see, e.g., [Gam]).

We have a continuous map $j = j_a : G \to M_a$ defined by

$$j(g)(f) := f(g), \quad (f \in a),$$

i.e., we associate to each point in $G$ a point evaluation homomorphism in $M_a$. This map is an injection if and only if algebra $a$ separates points of $G$.

Let $\hat{G}_a$ denote the closure of $j(G)$ in $M_a$, also a compact Hausdorff space. If algebra $a$ is self-adjoint, then $a \cong C(M_a)$ and hence $\hat{G}_a = M_a$. The (right) action of group $G$ on itself by right multiplication extends to the right action of $G$ on $M_a$ by the formula

$$\hat{R}_g(\eta)(f) := \eta(R_g(f)), \quad \eta \in M_a, \quad f \in a, \quad g \in G.$$  

Then

$$\hat{R}_g(j(t)) = j(tg), \quad t, g \in G. \quad (6.2.2)$$

From here it follows that $\hat{G}_a$ is invariant with respect to action $\hat{R}$.

The basis $\mathcal{Q}$ of topology of $\hat{G}_a$ consists of sets of the form

$$\left\{ \eta \in \hat{G}_a : \max_{1 \leq i \leq m} |h_i(\eta) - h_i(\eta_0)| < \varepsilon \right\}, \quad (6.2.3)$$

where $\eta_0 \in \hat{G}_a$, $h_1, \ldots, h_m \in C(\hat{G}_a)$, and $\varepsilon > 0$. 
Example 6.2.1. (1) Let \( a := c_0(G) \) (cf. Example 5.2.1(1)). Then \( \hat{G}_{c_0} \) is the one-point compactification of \( G \).

(2) Let \( a = AP(G) \) (cf. Example 5.2.1(3)). Then \( \hat{G}_{AP} \) is homeomorphic to a compact group \( bG \) (called the Bohr compactification of \( G \)), determined by the following universal property: there exists a homomorphism \( \mu : G \to bG \) such that

\[
\begin{array}{ccc}
G & \xrightarrow{\mu} & bG \\
\downarrow{\nu} & & \downarrow{\nu} \\
C & \xrightarrow{j(G)} & bG
\end{array}
\]

for any compact group \( C \) and any (continuous) homomorphism \( \nu \). Applying the universal property to unitary groups \( H := U_n, n \geq 1 \), we obtain that group \( G \) is maximally almost periodic (cf. Example 5.2.1(3)) if and only if \( \mu \) is an embedding. The universal property implies that there exists a bijection between sets of finite-dimensional irreducible unitary representations of \( G \) and \( bG \). It turns, the Peter-Weyl theorem for \( C(bG) \) and von Neumann’s approximation theorem for \( AP(G) \) (cf. Example 5.2.1(3)) imply that \( AP(G) \cong C(bG) \). In particular, it follows that \( bG \) is homeomorphic to the maximal ideal space of algebra \( AP(G) \), and \( \mu(G) \) is dense in \( bG \). Under this homeomorphism \( j(G) \) is identified with the subgroup \( \mu(G) \subset bG \) (if no confusion arises, we also denote \( \mu(G) \) by \( G \subset bG \)), and under this identification the action of \( G \) on \( \hat{G}_{AP} \) coincides with the action of \( G \) on \( bG \) by right translations.

By Peter-Weil theorem group \( bG \) can be presented as the inverse limit of an inverse system of finite-dimensional compact Lie groups. In particular, the Bohr compactification \( b\mathbb{Z} \) of integers \( \mathbb{Z} \) is the inverse limit of a family of compact Abelian Lie groups \( T^k \times \oplus_{i=1}^{m} \mathbb{Z}/(n_i \mathbb{Z}) \), \( k, m, n_i \in \mathbb{N} \), where \( T^k := (S^1)^k \) is the real \( k \)-torus. It follows that \( b\mathbb{Z} \) is disconnected and has infinite covering dimension. The projections (homomorphisms) \( b\mathbb{Z} \to T^k \times \oplus_{i=1}^{m} \mathbb{Z}/(n_i \mathbb{Z}) \) are defined by finite families of characters \( \mathbb{Z} \to S^1 \). For instance, let \( \lambda_1, \lambda_2 \in \mathbb{R} \setminus \mathbb{Q} \) be linearly independent over \( \mathbb{Q} \) and \( \chi_{\lambda_i} : \mathbb{Z} \to S^1 \), \( \chi_{\lambda_i}(n) := e^{2\pi i \lambda_i n}, i = 1, 2 \), be the corresponding characters. Then the map \( (\chi_{\lambda_1}, \chi_{\lambda_2}) : \mathbb{Z} \to T^2 \) is extended by continuity to a continuous surjective homomorphism \( b\mathbb{Z} \to T^2 \). If \( \lambda_1, \lambda_2 \) are linearly dependent over \( \mathbb{Q} \), then the corresponding extended homomorphism has image in \( T^2 \) isomorphic to \( S^1 \times \mathbb{Z}/(m\mathbb{Z}) \) for some \( m \in \mathbb{N} \).
(3) Let $a = APQ(Z^n)$ (cf. Example 5.2.1(4)). Then $\hat{G}_{APQ}$ is homeomorphic to the profinite completion of group $Z^n$, it admits presentation as the inverse limit of groups $\oplus_{l=1}^n Z/(nlZ)$, $m, n_l \in \mathbb{N}$. It follows that the covering dimension of $\hat{G}_{APQ}$ is zero.

(4) Let $a = \ell_\infty(G)$ (cf. Example 5.2.1(1)). Then $\hat{G}_{\ell_\infty} \cong \beta G$, the Stone–Čech compactification of group $G$. The covering dimension of $\hat{G}_{\ell_\infty}$ is zero (see, e.g., [Hz]).

### 6.2.2 Fibrewise compactification of covering

The regular covering $p : X \rightarrow X_0$ is a principal fibre bundle with structure group $G$, so there exist an open cover $\{U_\gamma\}$ of $X_0$ and a locally constant cocycle $\{c_{\delta\gamma} : U_\gamma \cap U_\delta \rightarrow G\}$, so that $p : X \rightarrow X_0$ is obtained from the disjoint union $\bigsqcup_\gamma U_\gamma \times G$ by identification

$$ U_\delta \times G \ni (x, g) \sim (x, g \cdot c_{\delta\gamma}(x)) \in U_\gamma \times G \quad \text{for all } x \in U_\gamma \cap U_\delta, $$

where projection $p$ is induced by projections $U_\gamma \times G \rightarrow U_\gamma$ (see, e.g., [Hz]).

Let $\bar{p} : c_a X \rightarrow X_0$ be the fibre bundle associated to $p : X \rightarrow X_0$ and having fibre $\hat{G}_a$, i.e., $\bar{p} : c_a X \rightarrow X_0$ is obtained from the disjoint union $\bigsqcup_\gamma U_\gamma \times \hat{G}_a$ by identification

$$ U_\gamma \times \hat{G}_a \ni (x, \omega) \sim (x, \hat{R}_{c_{\delta\gamma}(x)}(\omega)) \in U_\delta \times \hat{G}_a, \quad \text{for all } x \in U_\gamma \cap U_\delta, $$

where $\bar{p}$ is induced by projections $U_\gamma \times \hat{G}_a \rightarrow U_\gamma$.

**Definition 6.2.2.** The bundle $c_a X$ is called the fibrewise $a$-compactification of $X$.

Note that $c_a X$ is a paracompact (Hausdorff) space, as a fibre bundle with paracompact base and compact fibre.

The map $j : G \rightarrow \hat{G}_a$ induces maps $U_\gamma \times G \rightarrow U_\gamma \times \hat{G}_a$, $(x, g) \rightarrow (x, j(g))$, which, in turn, induce a (continuous) bundle morphism

$$ \iota = \iota_a : X \rightarrow c_a X. \quad (6.2.4) $$

It is immediate that $\iota(X)$ is dense in $c_a X$, and if algebra $a$ separates points of $G$, then $\iota$ is an injection.
6.2.3 Structure of fibrewise compactification

As a set, $c_a X$ admits presentation as the disjoint union of connected complex manifolds, each being a covering of $X_0$.

Indeed, let $\Upsilon := \hat{\mathcal{G}}_a/G$ (the set of orbits elements of $\hat{\mathcal{G}}_a$ by the action of $G$); since any orbit $H \in \Upsilon$ is closed with respect to the action of $G$, we may consider the associated fibre bundle $p_H : X_H \rightarrow X_0$ obtained from the disjoint union $\sqcup \gamma U_\gamma \times H$ by identification $U_\gamma \times H \ni (x, \omega) \sim (x, \hat{R}_{c_\gamma(x)}(\omega)) \in U_\delta \times H$, for all $x \in U_\gamma \cap U_\delta$ (see notation above), where $p_H$ is induced by projections $U_\gamma \times H \rightarrow U_\gamma$. We assume that $H$ is endowed with discrete topology. Then $p_H : X_H \rightarrow X_0$ is a covering of $X_0$ (in general non-regular). Since $X$ is connected, $X$ is a covering of $X_H$, the complex manifold $X_H$ is connected as well. It follows that as a set

$$c_a X = \bigsqcup_{H \in \Upsilon} X_H.$$ 

For each $H \in \Upsilon$ we have an injective map

$$\iota_H : X_H \hookrightarrow c_a X$$

determined by inclusion $H \hookrightarrow \hat{\mathcal{G}}_a$. We denote $\hat{X}_H := \iota_H(X_H)$.

In view of (6.2.2), we have $j(G) \in \Upsilon$. Hence, if $j$ is injective (i.e., if $a$ separates points), then $X = X_{j(G)}$ and $\iota = \iota_{j(G)}$ (cf. (6.2.4)).

6.2.4 Examples

Example 6.2.3. Let $a := c_0(G)$ (cf. Example 5.2.1(1)). Then $\hat{\mathcal{G}}_{c_0}$ is the one-point compactification of $G$, and the action of $G$ on $\hat{\mathcal{G}}_{c_0}$ fixes the ‘point at infinity’, so here $\Upsilon = \{G, '\infty'\}$. It follows that as a set the fibrewise compactification $c_{c_0} X$ is the disjoint union of $X$ and $X_0$.

Example 6.2.4. Let $a = AP(G)$ (cf. Example 5.2.1(3)). In what follows, we assume that $\hat{\mathcal{G}}_{AP}$ is endowed with the group structure of Bohr compactification $bG$, cf. Example 6.2.1(2).

Recall that $G$ is identified with a subgroup of $bG$ (also denoted by $G$), so $G$ acts on $bG$ by right translations (see Example 6.2.1(2)). Therefore, every orbit $H \in \Upsilon$ (cf. Section 6.2.3) is a right coset of $G$ in $bG$. Hence, $X_H = X$ for all $H \in \Upsilon$, and each set $\hat{X}_H$ is dense in $c_{AP} X$. 

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The fibre bundle \( c_{AP}X \) can be presented as the inverse limit of smooth fibre bundles. Indeed, by Peter-Weil theorem \( bG \) can be presented as an inverse limit of an inverse system of finite-dimensional compact Lie groups \( \{ C_s : s \in S \} \) (cf. Example 6.2.1(2)), where \( \pi_s : bG \to C_s \) denote the corresponding projection homomorphisms. These Lie groups can be taken as the fibres of the required smooth fibre bundles:

\[
U_\delta \times C_s \ni (x, h) \sim (x, h \cdot \pi_s(c_\delta\gamma(x))) \in U_\gamma \times C_s \quad \text{for all } x \in U_\gamma \cap U_\delta
\]

(see Remark 6.1.6 for notation). (Note that bundle \( b_{\pi^2}X \) constructed in Example 6.1.9(4) (there \( G = \mathbb{Z} \)) is of this form.)

**Example 6.2.5.** Let \( a = AP\mathbb{Q}(\mathbb{Z}^n) \) (cf. Example 5.2.1(4)). Since the covering dimension of \( \hat{G}_{AP\mathbb{Q}} \) is zero (cf. Example 6.2.1(3)), the covering dimension of \( c_{AP\mathbb{Q}}X \) is equal to \( \dim \mathbb{R} X_0 \).

**Example 6.2.6.** Let \( a = \ell_\infty(G) \) (cf. Example 5.2.1(1)). Then \( \hat{G}_{\ell_\infty} \cong \beta G \), the Stone–Čech compactification of group \( G \). Since the covering dimension of \( \hat{G}_{\ell_\infty} \) is zero, the covering dimension of \( c_{\ell_\infty}X \) coincides with the real dimension of \( X_0 \).

It is easy to see that \( c_{\ell_\infty}X \) is the maximal fibrewise compactification of covering \( X \), in the sense that if \( a \) is any other algebra satisfying assumptions of Section 5.1, then there is a surjective bundle morphisms \( c_{\ell_\infty}X \to c_aX \). Indeed, there is a surjective map \( \kappa : \hat{G}_{\ell_\infty} \to \hat{G}_a \) adjoint to inclusion \( a \hookrightarrow \ell_\infty(G) \); since this map is equivariant with respect to the action of \( G \), the existence of surjective bundle morphism follows.

Below we will need the following fact: using the axiom of choice, we can find a right inverse \( \lambda : \hat{G}_a \to \hat{G}_{\ell_\infty} \) to \( \kappa \), i.e., \( \kappa \circ \lambda = \text{Id} \).

**6.2.5 Holomorphic functions on fibrewise compactification**

**Definition 6.2.7.** A function \( f \in C(U) \) on an open set \( U \subset c_aX \) is called holomorphic if \( \iota^* f \) is holomorphic on \( \iota^{-1}(U) \subset X \) in the usual sense.

Let \( \mathcal{O}(U) \) denote the algebra of holomorphic functions on \( U \), endowed with the topology of uniform convergence on compact subsets of \( U \). It is immediate that a function \( f \in C(c_aX) \) is in \( \mathcal{O}(c_aX) \) if and only if each point in \( c_aX \) has a neighbourhood \( U \) such that \( f|_U \in \mathcal{O}(U) \).

The next proposition gives another characterization of holomorphic \( a \)-functions.
Proposition 6.2.8. The following is true:

1. A function \( f \) in \( \mathcal{C}_a(X) \) determines a unique function \( \hat{f} \) in \( \mathcal{C}(c_aX) \) such that \( \iota^* \hat{f} = f \);
   we have \( f \in \mathcal{O}_a(X) \) if and only if \( \hat{f} \in \mathcal{O}(c_aX) \). Thus, there are continuous embeddings \( \mathcal{C}_a(X) \hookrightarrow \mathcal{C}(c_aX) \), \( \mathcal{O}_a(X) \hookrightarrow \mathcal{O}(c_aX) \).

2. If \( a \) is self-adjoint, then \( \mathcal{C}_a(X) \cong \mathcal{C}(c_aX) \) and \( \mathcal{O}_a(X) \cong \mathcal{O}(c_aX) \).

6.2.6 Holomorphic maps, coordinate charts and topology

Holomorphic maps

We now introduce the notion of a holomorphic map between complex manifolds and/or open subsets of \( c_aX \).

Let \( U_0 \subset X_0 \), \( K \subset \hat{G}_a \) be open. We say that a function \( f \in \mathcal{C}(U_0 \times K) \) is holomorphic on \( U_0 \times K \) if \( f(\cdot, j(g)) \in \mathcal{O}(U_0) \) for all \( g \in j^{-1}(K) \subset G \).

We denote the algebra of holomorphic functions on \( U_0 \times K \) by \( \mathcal{O}(U_0 \times K) \).

Let \( M_i \) (\( i = 1, 2 \)) be either a complex manifold, or open subset of \( c_aX \), or a set \( U_0 \times K \) as above. Let \( \mathcal{O}_{M_i} \) be the sheaf of germs of holomorphic functions on \( M_i \).

Definition 6.2.9. A map \( F \in \mathcal{C}(M_1, M_2) \) is called holomorphic if \( F^*\mathcal{O}_{M_2} \subset \mathcal{O}_{M_1} \).

We denote the collection of holomorphic maps \( M_1 \to M_2 \) by \( \mathcal{O}(M_1, M_2) \). If a holomorphic map in \( \mathcal{O}(M_1, M_2) \) has inverse in \( \mathcal{O}(M_2, M_1) \), then it is called a biholomorphism.

Coordinate charts

Over each simply connected open subset \( U_0 \subset X_0 \) there exists a biholomorphic trivialization \( \psi = \psi_{U_0} : \tilde{p}^{-1}(U_0) \to U_0 \times G \) of covering \( p : X \to X_0 \), which is a morphism of fibre bundles with fibres \( G \). In what follows, we fix some system of biholomorphic trivializations of \( p : X \to X_0 \).

There exists a biholomorphic trivialization \( \tilde{\psi} = \tilde{\psi}_{U_0} : \tilde{p}^{-1}(U_0) \to U_0 \times \hat{G}_a \) of bundle \( c_aX \), which is a morphism of fibre bundles with fibre \( \hat{G}_a \) such that \( \tilde{\psi} : \tilde{p}^{-1}(U_0) \to U_0 \times j(G) \).
We have a commutative diagram

\[
\begin{array}{ccc}
p^{-1}(U_0) & \overset{\iota}{\longrightarrow} & \tilde{p}^{-1}(U_0) \\
\psi \downarrow & & \tilde{\psi} \downarrow \\
U_0 \times G & \overset{\text{Id} \times j}{\longrightarrow} & U_0 \times \tilde{G}_a
\end{array}
\]

**Notation**

For a given subset \( S \subseteq G \) we denote

\[ \Pi(U_0, S) := \psi^{-1}(U_0 \times S) \quad (6.2.5) \]

and identify \( \Pi(U_0, S) \) with \( U_0 \times S \) where appropriate. Note that \( \Pi(U_0, G) = p^{-1}(U_0) \).

For a subset \( K \subseteq \tilde{G}_a \) we denote

\[ \tilde{\Pi}(U_0, K) := \tilde{\psi}^{-1}(U_0 \times K). \quad (6.2.6) \]

A subset of the form \( \tilde{\Pi}(U_0, K) \) will be called a *coordinate chart* for \( c_aX \). Similarly, we identify \( \tilde{\Pi}(U_0, K) \) with \( U_0 \times K \). Clearly, if \( K \subseteq \tilde{G}_a \) is open, then \( \mathcal{O}(\tilde{\Pi}(U_0, K)) \cong \mathcal{O}(U_0 \times K) \).

**Basis of topology**

Let us define

\[ \mathcal{B} := \{ \tilde{\Pi}(V_0, L) \subseteq c_aX : V_0 \text{ is open simply connected in } X_0 \text{ and } L \in \Omega \}. \quad (6.2.7) \]

It is easy to see that \( \mathcal{B} \) is a basis of topology of \( c_aX \) (cf. (6.2.3)). In Section 6.1.1 we defined \( \Sigma_a \) to be the weakest topology on \( X \) in which all functions in \( C_a(X) \) are continuous. In Proposition 6.2.10 below we relate the notions of continuous and holomorphic \( a \)-functions on open subsets in \( \Sigma_a \) (see Section 6.1.1) with the notions of continuous and holomorphic functions on open subsets of \( c_aX \) introduced above.

Topology \( \Sigma_a \) on \( X \) coincides with the pullback by \( \iota \) of the topology on \( c_aX \). Indeed, since the pullback by \( j \) of the topology on \( \tilde{G}_a \) generated by basis \( \Omega \) (cf. 6.2.3) coincides with the weakest topology on \( G \) in which all functions in \( a \) are continuous, we obtain that \( \iota^*\mathcal{B} \) is the basis of topology \( \Sigma_a \), which implies the required. It follows that topology \( \Sigma_a \) is Hausdorff.
Let \( U \in \mathfrak{T}_a \). By definition, there exists an open subset \( U' \subset c_a(X) \) such that \( \iota^{-1}(U') = U \) and, since \( \iota(X) \) is dense in \( c_a(X) \), \( \iota(U) \) is dense in \( U' \). Among such open set there is the maximal one with respect to inclusion, namely, the union of all such \( U' \). We denote this union by \( \hat{U} \). Since \( c_aX \) is a normal space (as a paracompact Hausdorff space), it follows that \( U \) is a \( \mathfrak{T}_a \)-proper subset of \( V \) if and only if \( \hat{U} \subset \hat{V} \). Taking into account the results of the present section, the following sharpening of Proposition 6.2.8 is immediate.

**Proposition 6.2.10.** Let \( V \in \mathfrak{T}_a \).

1. A function \( f \) in \( C_a(V) \) determines a unique function \( \hat{f} \) in \( C(\hat{V}) \) such that \( \iota^* \hat{f} = f \); we have \( f \in O_a(V) \) if and only if \( \hat{f} \in C(\hat{V}) \). Thus, we have \( C_a(V) \hookrightarrow C(\hat{V}) \) and \( O_a(V) \hookrightarrow O(\hat{V}) \).

2. If \( a \) is self-adjoint, then \( C_a(V) = \iota^* C(\hat{V}) \) and \( O_a(V) = \iota^* O(\hat{V}) \).

### 6.2.7 Divisors on fibrewise compactification

For the proofs of Theorems 6.1.13 and 6.1.15 we will need the following definitions and results.

**Definition 6.2.11.** An (effective) divisor \( D \) on \( c_aX \) is given by an open cover \( \{U_\alpha\} \) of \( c_aX \) and a collection of functions \( f_\alpha \in O_a(U_\alpha) \), \( f_\alpha \) is not identically equal to zero on any open subset of \( U_\alpha \), such that \( f_\alpha = d_{\alpha\beta}f_\beta \) on \( U_\alpha \cap U_\beta \neq \emptyset \) for a nowhere zero function \( d_{\alpha\beta} \in O(U_\alpha \cap U_\beta) \), for all \( \alpha, \beta \).

We say that divisors \( D = \{(U_\alpha, f_\alpha)\} \) and \( D' = \{(U'_\beta, f'_\beta)\} \) in \( \text{Div}(c_aX) \) are equivalent if there exists a refinement \( \{V_\gamma\} \) of both covers \( \{U_\alpha\} \) and \( \{U'_\beta\} \) and nowhere zero holomorphic functions \( c_\gamma \) on \( V_\gamma \) such that \( f_\alpha|_{V_\gamma} = c_\gamma f'_\beta|_{V_\gamma} \) for \( V_\gamma \subset U_\alpha \cap U'_\beta \).

The divisors on \( c_aX \) form a multiplicative semigroup with identity, which we denote by \( \text{Div}(c_aX) \).

Similarly, we introduce the notion of principal divisors as those divisors \( \text{Div}(c_aX) \) that are determined by functions in \( O(c_aX) \).

Clearly, the pullback \( E = \iota^* D \) of \( D \in \text{Div}(c_aX) \) is a divisor in \( \text{Div}(X) \) (cf. Section 6.2 for the definition of map \( \iota : X \to c_aX \)). Using Hurwitz theorem one can show that divisor \( D \) is determined by its pullback \( E \) uniquely.
**Definition 6.2.12.** A divisor $E \in \text{Div}(X)$ of the form $E = \iota^* D$, $D \in \text{Div}(c_\alpha X)$, is called an $\alpha$-divisor on $X$.

The $\alpha$-divisors $E := \iota^* D$ and $H := \iota^* G$ are said to be equivalent if divisors $D$ and $G$ are equivalent in $\text{Div}(c_\alpha X)$.

The relation between Definitions 6.1.12 and 6.2.12 is as follows.

**Proposition 6.2.13.** Suppose that $\alpha$ is self-adjoint.

(1) A divisor $E \in \text{Div}(X)$ is an $\alpha$-divisor in the sense of Definition 6.1.12 if and only if $E$ is an $\alpha$-divisor in the sense of Definition 6.2.12.

(2) The $\alpha$-divisors $D, E \in \text{Div}_\alpha(X)$ are equivalent in the sense of Definition 6.2.12 if and only if they are equivalent in the sense of Definition 6.1.12.

The proof is immediate from Proposition 6.2.10.

Proposition 6.2.8 implies that if $\alpha$ is self-adjoint, then a divisor $D \in \text{Div}(c_\alpha X)$ is principal if and only if $E = \iota^* D$ is a principal $\alpha$-divisor.

### 6.3 Coherent sheaves

Recall (cf. Section 6.1.3) that by $\mathcal{O} := \mathcal{O}_{c_\alpha X}$ we denote the sheaf of (rings of) germs of holomorphic functions on $c_\alpha X$. A sheaf of modules $\mathcal{A}$ on an open subset $U \subset c_\alpha X$ over the sheaf of rings $\mathcal{O}|_U$ is called an **analytic sheaf**. A homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between analytic sheaves $\mathcal{A}, \mathcal{B}$ is called an **analytic homomorphism**.

**Definition 6.3.1.** A coherent sheaf $\mathcal{A}$ on $c_\alpha X$ is an analytic sheaf such that every point in $c_\alpha X$ has a neighbourhood $W$ so that for any $N \geq 1$ there exists a free resolution of $\mathcal{A}$ over $W$ of length $N$, i.e., an exact sequence of sheaves of modules of the form

$$
\mathcal{O}^{m_N}|_W \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_2} \mathcal{O}^{m_2}|_W \xrightarrow{\varphi_1} \mathcal{O}^{m_1}|_W \xrightarrow{\varphi_0} \mathcal{A}|_W \xrightarrow{0}.
$$

A free sheaf $c_\alpha X \mathcal{O}^m$, the sheaf of ideals of a complex $\alpha$-submanifold of $c_\alpha X$ (defined in the proof of Theorem 6.1.5 below, cf. Lemma 9.5.9) are the examples of coherent sheaves.
An analytic sheaf $\mathcal{A}$ on $\mathcal{O}_X$ will be called a Frechet sheaf if for each open set $U \in \mathcal{B}$ (cf. (6.2.7)) the module of sections $\Gamma(U, \mathcal{A})$ of $\mathcal{A}$ over $U$ is endowed with a topology of Frechet space.

**Proposition 6.3.2.** Every coherent sheaf can be turned in a unique way into a Frechet sheaf so that the following conditions are satisfied:

1. If $\mathcal{A}$ is a coherent subsheaf of $\mathcal{O}$ then for any open subset $U \in \mathcal{B}$ the module of sections $\Gamma(U, \mathcal{A})$ of $\mathcal{A}$ over $U$ is endowed with the topology of uniform convergence on compact subsets of $U$, for all $k$.

2. If $\mathcal{A}, \mathcal{B}$ are coherent sheaves on $\mathcal{O}_X$, then for any $U \in \mathcal{B}$ the spaces $\Gamma(U, \mathcal{A}), \Gamma(U, \mathcal{B})$ are Frechet spaces, and any analytic homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is continuous in the sense that the homomorphisms of sections of $\mathcal{A}$ and $\mathcal{B}$ over sets $U \in \mathcal{B}$ induced by $\varphi$ are continuous.

The topology on $\Gamma(U, \mathcal{A})$ can be defined by a family of semi-norms

$$
\|f\|_{V_k} := \inf \left\{ \sup_{x \in V_k} |h(x)| : h \in \Gamma(V_k, \mathcal{O}^{m_1}), f = (\varphi_0)_*(h) \right\},
$$

where $(\varphi_0)_*$ is the homomorphism of sections induced by $\varphi_0$ in (6.3.8), and open sets $V_k \in \mathcal{B}$ are such that $V_k \subset V_{k+1} \subset U$ for all $k$, and $U = \bigcup V_k$ (see Lemma 9.11.4(2) below).

We denote by $\mathcal{A}_x$ the stalk of an analytic sheaf $\mathcal{A}$ at $x \in \mathcal{O}_X$.

Our main result concerning coherent sheaves on $\mathcal{O}_X$ is the following

**Theorem 6.3.3.** Let $X_0$ be a Stein manifold, $\mathcal{A}$ a coherent analytic sheaf on $\mathcal{O}_X$. Then the following is true:

1. For every $x \in \mathcal{O}_X$ the stalk $\mathcal{A}_x$ is generated by the global sections in $\Gamma(\mathcal{O}_X, \mathcal{A})$ as an $\mathcal{A}_x\mathcal{O}$-module.

2. For all $i \geq 1$ the Čech cohomology groups $H^i(\mathcal{O}_X, \mathcal{A}) = 0$.

3. (Runge-type) Suppose that $Y_0 \subset X_0$, $\hat{Y} \subset \mathcal{O}_X$ are open and such that either (1) $Y_0$ is holomorphically convex in $X_0$ and $\hat{Y} = \bar{p}^{-1}(Y_0)$, or (2) $Y_0$ is holomorphically convex in $X_0$ and is contained in a simply connected open subset of $X_0$, and $\hat{Y} = \bar{p}(Y_0, K)$ for some $K \in \mathcal{Q}$ (cf. 6.2.3).

Then the restriction map $\Gamma(\mathcal{O}_X, \mathcal{A}) \to \Gamma(\hat{Y}, \mathcal{A})$ has dense image.

Statements (A) and (B) of Theorem 6.3.3 coincide with Theorems 6.1.17 and 6.1.18, respectively.
6.4 Maximal ideal space

We now relate the notions of the fibrewise $\mathfrak{a}$-compactification $\mathfrak{c}_a X$ of covering $X$, and of the maximal ideal space $M_X$ of algebra $O_a(X)$, i.e., the space of non-zero continuous homomorphisms $O_a(X) \to \mathbb{C}$ endowed with weak* topology (of $O_a(X)^*$).

**Theorem 6.4.1.** Suppose that algebra $\mathfrak{a}$ is self-adjoint, and $X_0$ is a Stein manifold. Then $M_X$ is homeomorphic to $\mathfrak{c}_a X$.

Since $\iota(X)$ is dense in $\mathfrak{c}_a X$, and the natural mapping of $X$ into $M_X$, sending each point of $X$ to its point evaluation homomorphism, coincides with $\iota$ under the homeomorphism of Theorem 6.4.1, we obtain the following Corona-type theorem.

**Corollary 6.4.2.** If $\mathfrak{a}$ is self-adjoint, and $X_0$ is Stein, then $X$ is dense in $M_X$.

6.5 Construction of a non-cylindrical almost periodic hypersurface

Recall that a complex $\mathfrak{a}$-submanifold of $X$ is called *cylindrical* if open sets $V$ in Definitions 6.1.1 have form $V = p^{-1}(V_0)$ for some open $V_0 \subset X_0$; a complex $\mathfrak{a}$-submanifold of codimension 1 is called an $\mathfrak{a}$-hypersurface.

In Example 6.1.9(4) we described a construction of a non-cylindrical $\mathfrak{a}$-hypersurface in the case $p : X \to X_0$ is a regular covering of a Riemann surface $X_0$ having deck transformation group $\mathbb{Z}$, and $\mathfrak{a} = AP(\mathbb{Z}) (=: AP)$ (cf. Example 5.2.6). Below we present the details of this construction.

We assume that $X_0$ has finite type and is a relatively compact subdomain of a larger (non-compact) Riemann surface $\tilde{X}_0$ whose fundamental group satisfies $\pi_1(\tilde{X}_0) \cong \pi_1(X_0)$ (e.g., the covering of Example 5.2.3 with $n = 1$, i.e., a complex strip covering an annulus, is a regular covering of this form).

In general, since a regular covering $X' \to X_0$ with a maximally almost periodic deck transformation group whose commutator subgroup has infinite index (e.g., the universal covering of $X_0$) can be factorized via a regular covering $p : X \to X_0$ with the deck transformation group
$Z$, the pullback of $Z$ by the factorizing covering map is a smooth non-cylindrical hypersurface in $X'$.

Note that the covering of Example 5.2.3 with $n = 1$, i.e., a complex strip covering an annulus, is the regular covering of the above form.

I. First, we relate the notions of a complex AP-hypersurface and an AP-divisor in $X$. We will use notation and results of Section 6.2.7 in the case $a = AP(Z)$.

**Definition 6.5.1.** A divisor $D \in \text{Div}(cAPX)$ is called **cylindrical** if the open sets $U_\alpha$ in Definition 6.2.11 have form $U_\alpha = \hat{\pi}^{-1}(U_{0,\alpha})$ for some open $U_{0,\alpha} \subset X_0$.

**Definition 6.5.2.** A divisor $D \in \text{Div}(cAPX)$ is called **smooth** if in Definition 6.2.11 sets $U_\alpha = \hat{\Pi}(U_{0,\alpha}, K_\alpha) \cong U_{0,\alpha} \times K_\alpha$ for some open $U_{0,\alpha} \subset X_0$, $K_\alpha \subset \hat{G}_a$ (cf. Section 6.2.6), and the functions $f_\alpha \in \mathcal{O}(U_\alpha)$ satisfy: $\nabla_z f_\alpha(z, \eta) \neq 0$ for all $(z, \eta) \in U_{0,\alpha} \times K_\alpha$.

**Definition 6.5.3.** The set of zeros of functions $f_\alpha$ in Definition 6.2.11 is called the support of divisor $D$, and is denoted by $\text{supp}(D) \subset cAPX$.

We give analogous definitions for AP-divisors on $X$ (these are pullbacks by $\iota$ of divisors on $cAPX$, cf. Section 6.2.7).

We will need results formulated in the beginning of Section 9.5 below. There we introduce the notion of a complex hypersurface in $cAPX$, and show that a closed subset $Z \subset X$ is a complex AP-hypersurface if and only if $Z = \iota^{-1}(Y)$ for some complex hypersurface $Y \subset cAPX$.

We say that a complex hypersurfaces $Z \subset cAPX$ is **cylindrical** if the open sets $U$ in Definition 9.5.1 have form $U = p^{-1}(U_0)$ for some open $U_0 \subset X_0$. Clearly, $Z \subset cAPX$ is a cylindrical complex hypersurface if and only if $Y = \iota^{-1}(Z)$ is a cylindrical AP-hypersurface, see above.

**Proposition 6.5.4.** Let $Y \subset cAPX$ be a complex hypersurface, $U_\alpha \subset cAPX$ and $h_\alpha \in \mathcal{O}(U_\alpha)$ be open sets and holomorphic functions determining $Y$ in Definition 9.5.1.

1. The collection $\{(U_\alpha, h_\alpha)\}$ determines a smooth divisor $D_Y \in \text{Div}(cAPX)$ such that $Y = \text{supp}(D_Y)$.

2. If $D_1, D_2 \in \text{Div}(cAPX)$ are smooth divisors such that $\text{supp}(D_1) = \text{supp}(D_2)$, then $D_1, D_2$ are equivalent. In particular, $D_Y$ is determined by the hypersurface $Y$ uniquely up to equivalence.
(3) The support $Y := \text{supp}(D)$ of a smooth divisor $D \in \text{Div}(c_{AP}X)$ is a complex hypersurface in $c_{AP}X$, and any representing divisor $D_Y$ is equivalent to $D$.

(4) The hypersurface $Y$ is cylindrical if and only if $D_Y$ is equivalent to a smooth cylindrical divisor.

Similar to (1)–(4) correspondences hold between complex AP-hypersurfaces and the supports of AP-divisors in $X$.

It follows from Proposition 6.5.4 that we need to construct a smooth divisor $D \in \text{Div}(c_{AP}X)$ that is not equivalent to a cylindrical divisor; then the support of the pullback $\iota^*D \in \text{Div}_{AP}(X)$ will be the required non-cylindrical $AP$-hypersurface in $X$.

II. We will carry out our construction for a smooth fibre bundle (instead of bundle $c_{AP}X$), defined as follows.

We will use notation and results of Examples 6.1.9(4), 6.2.1(2), 6.2.4.

Characters $\chi_1, \chi_2 : \mathbb{Z} \to S^1 \subset \mathbb{C}^*$ in the definition of the fibre bundle $b_{T^2}X$ can be chosen in the form $\chi_i := e^{i\lambda_i \varphi}$, $\lambda_i \in \mathbb{R}$, $i = 1, 2$, with $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ linearly independent over $\mathbb{Q}$. This guarantees that the homomorphism $\chi := (\chi_1, \chi_2) : \mathbb{Z} \to T^2 = S^1 \times S^1$ is an embedding with dense image. Recall that $p_{b_{T^2}X} : b_{T^2}X \to X_0$ is the bundle with fibre $T^2$ associated with $\chi$. The homomorphism $\chi$ induces a $C^\infty$ injection $\iota^\chi$ of the covering $p : X \to X_0$ into $b_{T^2}X$ such that $\iota^\chi(X)$ is dense in $b_{T^2}X$. Extending $\chi$ to a continuous surjective homomorphism $b\mathbb{Z} \to T^2$ we extend by continuity $\iota^\chi$ to a surjective homomorphism of fibre bundles $Q : c_{AP}X \to b_{T^2}X$. In particular,

$$\iota^\chi = Q \circ \iota,$$

(6.5.9)

cf. Section 6.2.2.

Next, we introduce notions of a holomorphic function and a divisor on $b_{T^2}X$.

A function $f \in C(U)$ on an open subset $U \subset b_{T^2}X$ is called \textit{holomorphic} if its pullback $(\iota^\chi)^*f$ is holomorphic on $X \cap (\iota^\chi)^{-1}(U)$ (cf. Section 6.2.5).

The definition of a divisor on $b_{T^2}X$ (or on its open subset) is analogous to that of a divisor on $c_{AP}X$ (cf. Definition 6.2.11).

Let $\text{Div}(b_{T^2}X)$ denote the collection of (effective) divisors on $b_{T^2}X$. Clearly, the pullback
by $Q$ of a divisor from $\text{Div}(b_{T^2}X)$ belongs to $\text{Div}(c_{AP}X)$ (cf. Definition 6.2.11). In particular, it follows from (6.5.9) that if $H \in \text{Div}(b_{T^2}X)$, then $(\iota^*)^*H \in \text{Div}_{AP}(X)$.

Similarly, we define equivalence relation in $\text{Div}(b_{T^2}X)$, cylindrical and smooth divisors (cf. Definitions 6.5.1 and 6.5.2).

The following proposition is immediate.

**Proposition 6.5.5.** Let $H \in \text{Div}(b_{T^2}X)$.

1. If divisor $H$ is smooth, then the AP-divisor $(\iota^*)^*H$ is smooth.
2. If divisor $H$ is cylindrical, then the AP-divisor $(\iota^*)^*H$ is cylindrical.

The next statement justifies the choice of the bundle $b_{T^2}X$ (instead of $c_{AP}X$) in our construction.

**Proposition 6.5.6.** Suppose that divisor $H \in \text{Div}(b_{T^2}X)$ is not equivalent to a cylindrical divisor on $b_{T^2}X$. Then the AP-divisor $(\iota_0^*)^*H \in \text{Div}_{AP}(X)$ is not equivalent to a cylindrical AP-divisor on $X$.

It follows from Propositions 6.5.5(1) and 6.5.6 that it suffices to construct a smooth divisor $H \in \text{Div}(b_{T^2}X)$ not equivalent to a cylindrical divisor on $b_{T^2}X$. Then its pullback $D := (\iota^*)^*H$ is a smooth non-cylindrical AP-divisor on $X$. By Proposition 6.5.4(3) (cf. last statement), the support $Z := \text{supp}(D)$ of divisor $D$ is a complex AP-hypersurface in $X$. Moreover, $Z$ is non-cylindrical, for otherwise, by Proposition 6.5.4(4) its representing divisor $D_Z \in \text{Div}_{AP}(X)$ is equivalent to a (smooth) cylindrical divisor. However, by Proposition 6.5.4(3) divisor $D$ is equivalent to divisor $D_Z$ giving a contradiction.

**III.** Now we construct divisor $H$ on $b_{T^2}X$ with the required properties.

(A) First, note that the bundle $b_{T^2}X$ is embedded into a holomorphic fibre bundle $p_{b,(C^*)^2} : b_{(C^*)^2} \rightarrow X_0$ with fibre $(C^*)^2$ associated with the composite of the embedding homomorphism $T^2 \hookrightarrow (C^*)^2$ and $\chi$. Moreover, the composite of this embedding with $\iota^X$ is a holomorphic injective map $X \rightarrow b_{(C^*)^2}X$. 

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Proposition 6.5.7. For any neighbourhood $U \Subset (\mathbb{C}^*)^2$ of $T^2$ there exist sufficiently small $\lambda_1, \lambda_2$ in the definition of $\chi$ and a holomorphic trivialization $\eta : b(\mathbb{C}^*)^2 X \to X_0 \times (\mathbb{C}^*)^2$ of the corresponding to $\chi$ bundle $b(\mathbb{C}^*)^2 X$ such that $\eta(b_{T^2} X) \subset X_0 \times U$ (i.e., under a suitable definition, $\eta(b_{T^2} X)$ is sufficiently close to $X_0 \times T^2$).

In what follows, we assume that $\max\{\lambda_1, \lambda_2\} > 0$ and is sufficiently small.

(B) We define the equivalence relation on the set of divisors on $(\mathbb{C}^*)^2$ analogously to that for divisors on $c_{AP} X$ (cf. Section 6.2.7).

Let $G$ be a divisor on $(\mathbb{C}^*)^2$ whose support intersects the real torus $T^2 \subset (\mathbb{C}^*)^2$ transversely (cf. Proposition 6.5.9 below). Let $\pi : X_0 \times (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ be the natural projection. We define a smooth divisor $E$ on the complex manifold $b(\mathbb{C}^*)^2 X$ as pullback with respect to the holomorphic map $\pi \circ \eta$ of the divisor $G$ on $(\mathbb{C}^*)^2$. By Proposition 6.5.7, since $\text{supp}(G)$ intersects $T^2 \subset (\mathbb{C}^*)^2$ transversely, for all sufficiently small $\lambda_1, \lambda_2 > 0$ the support of the divisor $E$ has a non-empty intersection with the subbundle $b_{T^2} X$ of $b(\mathbb{C}^*)^2 X$.

We define divisor $H$ as the restriction of divisor $E$ to $b_{T^2} X$. Specifically, if divisor $G$ on $(\mathbb{C}^*)^2$ is determined by an open cover $(U_\alpha)$ and functions $\{f_\alpha \in O(U_\alpha)\}$, then divisor $H$ is determined by the open cover $(b_{T^2} X \cap (\pi \circ \eta)^{-1}(U_\alpha))$ and the functions $((\pi \circ \eta)^* f_\alpha)|_{b_{T^2} X \cap (\pi \circ \eta)^{-1}(U_\alpha)}$.

Clearly, each set $b_{T^2} X \cap (\pi \circ \eta)^{-1}(U_\alpha)$ is open in $b_{T^2} X$, and the ratios of functions $(\pi \circ \eta)^* f_\alpha$ on intersections of their domains are nowhere zeros. However, to claim that $H$ is a well-defined divisor on $b_{T^2} X$ we must prove that these functions are holomorphic.

Proposition 6.5.8. (1) Functions $(\pi \circ \eta)^* f_\alpha$ are holomorphic on open subsets $b_{T^2} X \cap (\pi \circ \eta)^{-1}(U_\alpha)$ of $b_{T^2} X$. Thus, divisor $H$ is well defined.

(2) If divisor $G \in \text{Div}(\mathbb{C}^*)^2$ is smooth, then divisor $H \in \text{Div}(b_{T^2} X)$ is also smooth.

(3) If divisor $G$ is not equivalent to a principal divisor, then divisor $H$ is not equivalent to a cylindrical divisor.

Proposition 6.5.9. There exists a divisor $G$ on $(\mathbb{C}^*)^2$ which is smooth, not equivalent to a principal divisor, and whose support intersects the real torus $T^2 \subset (\mathbb{C}^*)^2$ transversely in finitely many points.
It follows from Propositions 6.5.8 and 6.5.9 that there exists a smooth divisor $H$ on $b_{\mathbb{T}}X$ which is not equivalent to a cylindrical divisor. This concludes our construction.

**Remark 6.5.10.** In the case $X := T$ is a tube domain and $p : T \to T_0$ is the regular covering of Example 5.2.3, the construction of a non-cylindrical AP-hypersurface is much simpler. In fact, one can show that there exists $\lambda > 0$ such that any AP-hypersurface $Y$ of the form $Y := p^{-1}(Y_0) \subset T$, where $Y_0 \subset X_0$ is a hypersurface that cannot be determined by a single function in $\mathcal{O}(T_0)$, is a non-cylindrical AP-hypersurface with respect to projection $p_{\lambda}$ (cf. (5.2.3)).
Chapter 7

Uniqueness sets and Hartogs-type theorem

7.1 Uniqueness sets

A classical result by H. Bohr states that if a holomorphic function \( f \) on a complex strip \( T := \{ z \in \mathbb{C} : \text{Im}(z) \in (a,b) \} \), bounded on closed substrips, is continuous almost periodic on a horizontal line \( \mathbb{R} + ic, c \in (a,b) \) (see Example 5.2.1(3) for definition of continuous almost periodic functions on \( \mathbb{R} \)), then \( f \) is holomorphic almost periodic on \( T \) (cf. Example 5.2.3).

We extend this result to a general algebra \( \mathcal{O}_a(X) \) as follows.

Let \( X_0 \) be a Stein manifold, \( U_0 \subset X_0 \) be an open simply connected set, \( Z_0 \subset U_0 \) a uniqueness set for holomorphic functions in \( \mathcal{O}(U_0) \), and \( K \subset G \) be such that \( \cup_i \mathcal{K}g_i = G \) for some \( g_1, \ldots, g_m \in G \).

Theorem 7.1.1. Suppose that \( Z \) is a closed \( a \)-subset of \( X \) with the property that \( p^{-1}(Z_0) \cap \Pi(U_0, K) \subset Z \) for some sets \( U_0 \) and \( Z_0 \) as above. If \( f \in \mathcal{O}_{\ell, \infty(G)}(X) \) and \( f|_Z \in C_a(Z) \), then \( f \in \mathcal{O}_a(X) \).

It follows that in Bohr’s result the line \( \mathbb{R} + ic \) can be replaced, e.g., with a periodic curve \( \{ z \in T : \text{Im}(z) = a + (b-a) \sin(\text{Re}(z)) \} \).

As an example of set \( Z_0 \) we can take any real hypersurface in \( X_0 \) or, more generally, a
set of the form \( Z_0 := \{ x \in X_0 : \rho_1(x) = \cdots = \rho_d(x) = 0 \} \), where \( \rho_1, \ldots, \rho_d \) are real-valued differentiable functions on \( X_0 \), \( d \leq n \), and \( \partial \rho_1(x_0) \wedge \cdots \wedge \partial \rho_d(x_0) \neq 0 \) for some \( x_0 \in Z_0 \) (see, e.g., [Bog]).

### 7.2 Hartogs-type theorem

**Theorem 7.2.1.** Let \( n \geq 2 \), \( D_0 \subseteq X_0 \) be a subdomain with a connected piecewise smooth boundary \( \partial D_0 \) contained in a Stein open submanifold of \( X_0 \), and \( D := p^{-1}(D_0) \). Suppose that \( f \in C_a(\partial D) \) satisfies tangential CR equations on \( \partial D \), i.e., for any \( \omega \in \Lambda_c^{n,n-2}(X) \)

\[
\int_{\partial Y} f \, \bar{\omega} = 0.
\]

Then there exists a function \( F \in O_a(D) \cap C_a(\bar{D}) \) such that \( F|_{\partial D} = f \).

In particular, Theorem 7.2.1 implies that if \( n \geq 2 \), then each continuous almost periodic function on the boundary \( \partial T = \mathbb{R}^n + i\partial \Omega \) of a tube domain \( T := \mathbb{R}^n + i\Omega \subset \mathbb{C}^n \), \( \Omega \subset \mathbb{R}^n \) is a domain with piecewise-smooth boundary \( \partial \Omega \), satisfying tangential CR equations on \( \partial T \), admits a continuous extension to a holomorphic almost periodic function in \( O_{AP}(T) \cap C(\bar{T}) \).
Chapter 8

Approximation

1. To formulate the results, let $a_\iota (\iota \in I)$ be a collection of closed subspaces of $a$ such that

   (1) $a_\iota$ are invariant with respect to the action of $G$ on $a$ by right translates (i.e., if $f \in a_\iota$, then $R_g(f) \in a_\iota$, $g \in G$, cf. Section 5.1),
   
   (2) the family $\{a_\iota : \iota \in I\}$ forms a direct system ordered by inclusion, and
   
   (3) the linear space $a_0 := \bigcup_{\iota \in I} a_\iota$ is dense in $a$.

Example 8.0.1. Examples of spaces $a_\iota$ are as follows.

   (1) Let $a = \ell_\infty(G)$, $I$ be the collection of all subsets of $G$ ordered by inclusion. Given $\iota \in I$, we define $a_\iota$ to be the closed linear subspace spanned by translates $\{R_g(\chi_\iota) : g \in G\}$ of the characteristic function $\chi_\iota$ of subset $\iota$.

   (2) Let $a = AP(Z^n)$ (cf. Example 5.2.1(3)). We can take $I$ to be the collection of all finite subsets of $\mathbb{R}^n/2\pi\mathbb{Z}^n$ ordered by inclusion, and $a_\iota(Z^n) := \text{span}_{\mathbb{C}}\{e^{i(\lambda,t)}, \lambda \in \iota, \iota \in I, t \in Z^n\}$ (finite-dimensional spaces). (See also Example 5.2.6 below.)

   We can also consider $a = AP_Q(Z^n)$, the algebra of almost periodic functions on $Z^n$ having rational spectra (cf. Example 5.2.1(4)). Here we take $I$ to be the collection of all finite subsets of $Q^n$ ordered by inclusion, and similar spaces $a_\iota(Z^n)$.

   (3) Let $a = AP(G)$ (cf. Example 5.2.1(4)), $I$ consists of finite collections of finite-dimensional irreducible unitary representations of group $G$. We define $a_\iota(G)$, where $\iota = \{\sigma_1, \ldots, \sigma_m\} \in I$, to be the linear $\mathbb{C}$-hull of matrix elements $\sigma_{ij}^k \in AP(G)$ of representations $\sigma_k = (\sigma_{ij}^k), 1 \leq k \leq m$ (finite-dimensional spaces).
Let $O_\iota(X)$ be the space of holomorphic functions $f \in O_a(X)$ such that for every $x_0 \in X_0$

$$g \mapsto f(g \cdot x) \in a_\iota, \quad g \in G, \quad x \in F_{x_0},$$

and $O_0(X)$ be the $\mathbb{C}$-linear hull of spaces $O_\iota(X)$ with $\iota$ varying over $I$.

**Theorem 8.0.2.** If $X_0$ is a Stein manifold, then $O_0(X)$ is dense in $O_a(X)$.

**Example 8.0.3.** Let $O_0(T)$ be determined by the choice of spaces $a_\iota = a_\iota(\mathbb{Z}^n) (\iota \in I)$ as in Example 8.0.1(2).

Let us show that exponential polynomials (5.2.1) are dense in $O_0(T)$. We denote $e_\lambda(t) := e^{i\langle \lambda, t \rangle}$ ($\lambda \in \mathbb{R}^n/2\pi \mathbb{Z}^n, \ t \in \mathbb{Z}^n$). Clearly, $e_\lambda \in O_{\{\lambda\}}(T)$.

Now, let $\iota = \{\lambda_1, \ldots, \lambda_m\}$. Since functions $e_{\lambda_k}$ ($1 \leq k \leq m$) are linearly independent in $a_\iota$, there exist linear projections $p_{\iota,\lambda_k}: a_\iota \to a_{\{\lambda_k\}}$. Since projections $p_{\iota,\lambda_k}, 1 \leq k \leq m$, are invariant with respect to the action of $G$ on itself by right translates, they induce projections $P_{\iota,\lambda_k}: O_{\iota}(T) \to O_{\{\lambda_k\}}(T)$.

(The latter can be easily seen, e.g., from the presentation of functions in $O_{AP}(T)$ as sections of holomorphic Banach vector bundle $C_{AP}X_0$, obtained in Remark 6.1.6, where projections $P_{\iota,\lambda_k}$ become bundle homomorphisms $C_{a_\iota}X_0 \to C_{a_{\{\lambda_k\}}}X_0$.)

Therefore, there exist functions $f_{\lambda_k} \in O_{\{\lambda_k\}}(T)$, $f_{\lambda_k} := P_{\iota,\lambda_k}(f), 1 \leq k \leq m$, such that $f(z) = \sum_{k=1}^m f_{\lambda_k}(z), \ z \in T$. It is now easy to see that for each $f_{\lambda_k}$ there exists a function $h_{\lambda_k} \in O(T_0)$ such that $f_{\lambda_k}/e_{\lambda_k} = p^* h_{\lambda_k}$, hence

$$f(z) = \sum_{k=1}^m (p^* h_{\lambda_k})(z)e^{i\langle \lambda_k, z \rangle}, \quad z \in T. \quad (8.0.1)$$

Since the base $T_0$ of the covering is a relatively complete Reinhardt domain, functions $h_{\lambda_k}$ admit expansions into Laurent series (see, e.g., [S])

$$h_k(z) = \sum_{|\alpha|=-\infty}^{\infty} b_\alpha z^\alpha, \quad z \in T_0, \quad b_\alpha \in \mathbb{C},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Since $p(z) = (e^{iz_1}, \ldots, e^{iz_n}), \ z = (z_1, \ldots, z_n) \in T$ (cf. Example 5.2.3), each $p^* h_{\lambda_k}$ admits approximations by finite sums

$$\sum_{|\alpha|=-M}^{M} b_\alpha e^{i\langle \alpha, z \rangle}, \quad z \in T, \quad (8.0.2)$$
converging uniformly on subsets $p^{-1}(W_0) \subset T$, $W_0 \Subset T_0$. Together with (8.0.1) this implies that exponential polynomials (5.2.1) are dense in $\mathcal{O}_0(T)$.

Together with Theorem 8.0.2, this example gives us another proof of Theorem 5.2.5. A similar argument shows that the algebra of holomorphic almost periodic functions with rational spectra (cf. Definition 5.2.4) coincides with algebra $\mathcal{O}_{AP_0}(T)$ (cf. Example 5.2.1(4)).

**Example 8.0.4.** Let $X_0$ be a non-compact Riemann surface, $p : X \to X_0$ be a regular covering with a maximally almost periodic deck transformation group $G$ (for instance, $X_0$ is hyperbolic, then $X = \mathbb{D}$ is its universal covering, and $G = \pi_1(X_0)$ is a free (not necessarily finitely generated) group). The functions in $\mathcal{O}_{AP}(X)$ (cf. Example 5.2.6) arise, e.g., as linear combinations over $\mathbb{C}$ of matrix entries of fundamental solutions of certain linear differential equations on $X$.

Indeed, let $\mathcal{U}_G$ be the set of finite dimensional irreducible unitary representations $\sigma : G \to U_m$ ($m \geq 1$), let $I$ be the collection of finite subsets of $\mathcal{U}_G$ directed by inclusion, and for each $\iota \in I$ let $AP_{\iota}(G)$ be the (finite-dimensional) subspace generated by matrix elements of the unitary representations $\sigma \in \iota$. Then by Theorem 8.0.2 the $\mathbb{C}$-linear hull $\mathcal{O}_0(X)$ of spaces $\mathcal{O}_{\iota}(X)$, cf. Chapter 8, is dense in $\mathcal{O}_a(X)$ (note that for each $\sigma \in \mathcal{U}_G$ the space $\mathcal{O}_{\{\sigma\}}(X)$ is the $\mathbb{C}$-linear hull of coordinates of vector-valued functions $f$ in $\mathcal{O}(X, \mathbb{C}^m)$ having the property that $f(g \cdot x) = \sigma(g)f(x)$ for all $g \in G$, $x \in X$). Now, a unitary representation $\sigma : G \to U_m$, $m \geq 1$, can be obtained as the monodromy of the system $dF = \omega F$ on $X_0$, where $\omega$ is a holomorphic 1-form on $X_0$ with values in the space of $m \times m$ complex matrices $M_m(\mathbb{C})$ (see, e.g., [Fo]). In particular, the system $dF = (p^*\omega)F$ on $X$ admits a global solution $F \in \mathcal{O}(X, GL_m(\mathbb{C}))$ such that $F \circ g^{-1} = F\sigma(g)$ ($g \in G$). By definition, a linear combination of matrix entries of $F$ is an element of $\mathcal{O}_{AP}(X)$.

2. We also consider a similar problem for holomorphic $a$-functions on subsets of $X$ of the form $\bar{D} := p^{-1}(\bar{D}_0)$, where $D_0$ is a relatively compact subdomain of $X_0$ (cf. Section 5.1).

Analogously, we denote by $\mathcal{A}_{\iota}(D)$ the space of holomorphic functions $f \in \mathcal{A}_a(D)$ (cf. Definition 5.1.6) such that for every $x_0 \in \bar{D}_0$ the function $g \mapsto f(g \cdot x)$ ($g \in G$, $x \in F_{x_0}$) is in $\mathfrak{a}_{\iota}$, and by $\mathcal{A}_0(D)$ the $\mathbb{C}$-linear hull of spaces $\mathcal{A}_{\iota}(D)$, $\iota \in I$. 
Theorem 8.0.5. If $X_0$ is a Stein manifold, and $D_0 \subset X_0$ is a strictly pseudoconvex domain (cf. Section 4.1), then $A_0(D_0)$ is dense $A_a(D_0)$.

Example 8.0.6. Similarly to Definition 5.2.4, one can define holomorphic almost periodic functions on a closed tube domain $T^s$, where $T^s := \mathbb{R}^n + i\Omega_c$ and $\Omega_c \subset \mathbb{R}^n$ is open and strictly convex, as the uniform limits of exponential polynomials (5.2.1).

Now, we proceed as in Example 5.2.3. We consider a regular covering $p : T^s \to T^s_0$ with deck transformation group $\mathbb{Z}_n$, where $p$ is defined by (5.2.2), $T^s_0 := p(T^s)$. Then $T^s_0$ is a strictly pseudoconvex domain in $\mathbb{C}^n$. Thus, we obtain algebra $A_{AP}(T^s)$ (cf. Example 5.2.1(3)).

The result analogous to the one obtained in Example 8.0.3 is also valid for algebra $A_{AP}(T^s)$. Hence, by Theorem 8.0.5 the algebra of holomorphic almost periodic functions on $T^s$ coincides with algebra $A_{AP}(T^s)$.

3. Let $A, B$ be (complex) Banach spaces. By $\mathcal{L}(A, B)$ we denote the space of bounded linear operators $A \to B$. Recall that a Banach space $B$ is said to have the approximation property if for every compact set $K \subset B$ and every $\varepsilon > 0$ there is an operator $T = T_{\varepsilon,K} \in \mathcal{L}(B, B)$ of finite rank so that

$$\|Tx - x\|_B < \varepsilon \quad \text{for every} \quad x \in K.$$ 

Proposition 8.0.7. The space $AP(G)$ of almost periodic functions on a group $G$ (cf. Example 5.2.1(3) below) has the approximation property with (approximation) operators $T$ in $\mathcal{L}(AP(G), AP_0(G))$.

Many problems of Banach space theory admit especially simple solutions if one of the spaces under consideration has the approximation property. One of such problems is the problem of determination whether, given two Banach algebras $A \subset C(\mathcal{X}), B \subset C(\mathcal{Y})$ ($\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff spaces), their slice algebra

$$S(A, B) := \{f \in C(\mathcal{X} \times \mathcal{Y}) : f(\cdot, y) \in A \text{ for all } y \in \mathcal{Y}, f(x, \cdot) \in B \text{ for all } x \in \mathcal{X}\}$$

coincides with the closure in $C(\mathcal{X} \times \mathcal{Y})$ of the symmetric tensor product of $A$ and $B$. For instance, this is true if either $A$ or $B$ have the approximation property. The latter is an immediate consequence of the following result of Grothendieck.
Let $A \subset C(\mathcal{X})$ be a closed subspace, $B$ be a Banach space and $A_B \subset C_B(\mathcal{X}) := C(\mathcal{X}, B)$ be the Banach space of all continuous $B$-valued functions $f$ such that $\varphi(f) \in A$ for any $\varphi \in B^*$. By $A \otimes B$ we denote completion of symmetric tensor product of $A$ and $B$ with respect to norm

$$
\left\| \sum_{k=1}^{m} a_k \otimes b_k \right\| := \sup_{x \in \mathcal{X}} \left\| \sum_{k=1}^{m} a_k(x)b_k \right\|_B \quad \text{with} \quad a_k \in A, \ b_k \in B.
$$

(8.0.3)

**Theorem 8.0.8 ([G]).** The following statements are equivalent:

1) $A$ has the approximation property;

2) $A \otimes B = A_B$ for every Banach space $B$.

We prove a refined version of Theorem 8.0.5 as follows:

**Theorem 8.0.9.** Suppose that the following conditions are satisfied:

1) For every $\iota \in I$ space $a_\iota$ is finite-dimensional.

2) Space $a$ has the approximation property with approximation operators $S \in \mathcal{L}(a, a_0)$ invariant with respect to the action $G$ on $a$ by right translates, i.e., $S(f) = S(R_g(f))$ for all $f \in a, \ g \in G$.

If $X_0$ is a Stein manifold, $D_0 \subset X_0$ is a strictly pseudoconvex domain, then $A_a(D)$ has the approximation property with approximation operators in $\mathcal{L}(A_a(D), A_0(D))$. 

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Chapter 9

Proofs

9.1 Preliminaries

9.1.1 Notation

Given a topological space $B$ and its subspace $A$, the notation “$A \subseteq B$” stands for “the closure $\bar{A}$ of $A$ in $B$ is a compact subspace of $B$”.

For a topological space $A$ and a sheaf of Abelian groups $S$ on $A$, we denote by $\Gamma(A, S)$ the Abelian group of continuous sections of $S$ over $A$.

Further, let $U$ be an open cover of $A$. We denote by $C^i(U, S)$ the space of Čech $i$-cochains with values in $S$, by $\delta : C^i(U, S) \to C^{i+1}(U, R)$ the Čech coboundary operator (for detailed definition see, e.g., [GR]), by $Z^i(U, S) := \{\sigma \in C^i(U, S) : \delta \sigma = 0\}$ the space of $i$-cocycles, and by $B^i(U, S) := \{\sigma \in Z^i(U, S) : \sigma = \delta(\eta), \eta \in C^{i-1}(U, S)\}$ the space of $i$-coboundaries.

The Čech cohomology groups $H^i(U, S)$, $i \geq 0$, are defined by

$$H^i(U, S) := Z^i(U, S)/B^i(U, S), \quad i \geq 1,$$

and $H^0(U, S) := \Gamma(U, S)$.

9.1.2 $\bar{\partial}$-equation

Let $B$ be a complex Banach space, $D_0 \subset X_0$ be a strictly pseudoconvex domain. In what follows, we assume that we have fixed a system of local coordinates on $D_0$. Let $\{W_{0,i}\}_{i \geq 1}$ be
the cover of $D_0$ by coordinate patches. By $\Lambda_b^{(0,q)}(D_0, B)$, $q \geq 0$, we denote the space of bounded continuous $B$-valued $(0,q)$-forms on $D_0$ endowed with norm

$$\|\omega\|_{D_0} = \|\omega\|_{D_0}^{(0,q)} := \sup_{x \in U_{i,\alpha} \geq 1, \alpha} \|\omega_{\alpha,i}(x)\|_B,$$

where $\omega_{\alpha,i}$ ($\alpha$ is a multiindex) are the coefficients of form $\omega|_{W_0,i} \in \Lambda_b^{(0,q)}(W_0,i, B)$ in local coordinates on $W_{0,i}$.

The next lemma follows easily from the results in [HL] (proved for $B = \mathbb{C}$), as all integral presentations and estimates are preserved when passing to the case of Banach-valued forms.

**Lemma 9.1.1.** There exists a bounded linear operator

$$R_{D_0,B} \in \mathcal{L}\left(\Lambda_b^{(0,q)}(D_0, B), \Lambda_b^{(0,q-1)}(D_0, B)\right), \quad q \geq 1,$$

such that if form $\omega \in \Lambda_b^{(0,q)}(D_0, B)$ is $C^\infty$ and satisfies $\bar{\partial}\omega = 0$ on $D_0$, then $\bar{\partial}R_{D_0,B}\omega = \omega$ on $D_0$.

### 9.2 Proofs of Propositions 6.1.7 and 6.2.8

The proofs of our results are based on the equivalences established in Propositions 6.1.7 and 6.2.8, so we prove these propositions first.

**Proof of Proposition 6.1.7.** We prove the assertion for holomorphic functions, the proof for continuous functions is analogous.

Let us establish the first isomorphism. It is easy to see that any function $f \in \mathcal{O}_a(X)$ is locally Lipschitz with respect to the semi-metric $d$ (see Section 7.1), i.e.,

$$|f(x_1,g) - f(x_2,g)| \leq C d((x_1,g),(x_2,g)) := C d_0(x_1,x_2)$$

for all $(x_1,g), (x_2,g) \in W_0 \times G \cong p^{-1}(W_0)$, where $W_0 \subset X_0$ is a simply connected coordinate chart. (Here $C$ depends on $d_0$ and $W_0$ only.) We denote $f_{x_0} := f|_{p^{-1}(x_0)} \in a$, $x_0 \in X_0$, and define

$$\tilde{f}(x_0) := f_{x_0}, \quad x_0 \in X_0.$$
Then \( \tilde{f} \) is a section of bundle \( C_aX_0 \). From (9.2.2) for any linear functional \( \varphi \in a^* \) we have \( \varphi(\tilde{f}(g)) := \varphi(f(x,g)) \in \mathcal{O}(W_0) \), \( g \in G \), \( x \in W_0 \Subset X_0 \), a simply connected coordinate chart, cf. [Lin] for similar arguments. Thus \( \tilde{f} \) is a holomorphic section of \( C_aX_0 \). Reversing these arguments we obtain that any holomorphic section of \( C_aX_0 \) determines an almost periodic holomorphic function on \( X \). The proof of the second isomorphism is similar.

Proof of Proposition 6.2.8. Given \( f \in \mathcal{O}_a(X) \), denote \( f_{x_0} := f|_{p^{-1}(x_0)} \) and then define \( \hat{f}_{x_0} \in C(\hat{G}_a) \) to be the extension of \( f_{x_0} \) from \( p^{-1}(x_0) \cong G \) to \( \bar{p}^{-1}(x_0) \cong \hat{G}_a \) so that \( j^*\hat{f}_{x_0} = f_{x_0} \). The family of the extended functions over points of \( X_0 \) determines a function \( \hat{f} \) on \( c_aX \) such that \( \hat{f}(x) = \hat{f}_{x_0}(x) \) for \( x_0 := \bar{p}(x) \). Using a normal family argument one shows that \( \hat{f} \in \mathcal{O}(c_aX) \), see, e.g., [Lin] or [BrK3, Lemma 2.3] for similar results. Clearly, \( f \) is such that \( \iota^*\hat{f} = f \).

Since the algebra homomorphism \( a \to C(\hat{G}_a) \) is an injection, the constructed homomorphism \( i : \mathcal{O}_a(X) \to \mathcal{O}(c_aX) \) is an injection too. This completes the proof of the first assertion.

For the proof of the second assertion, suppose that \( a \) be self-adjoint. Then \( a \cong C(\hat{G}_a) \), and we can define the inverse homomorphism \( i^{-1} : \mathcal{O}(c_aX) \to \mathcal{O}(X) \) by the formula

\[
i(\hat{f}) := \iota^*\hat{f}, \quad \hat{f} \in \mathcal{O}(c_aX).
\]

Since \( i^{-1}(\hat{f})|_{p^{-1}(x_0)} = j^*(\hat{f}|_{p^{-1}(x_0)}) \in a \), \( x_0 \in X_0 \), we have \( i^{-1}(\hat{f}) \in \mathcal{O}_a(X) \), i.e., \( i^{-1} \) maps \( \mathcal{O}(c_aX) \) into \( \mathcal{O}_a(X) \).

\[\square\]

9.3 Proof of Theorem 5.2.5

Let \( T^s = \mathbb{R}^n + i\Omega^s, \bar{\Omega}^s \subset \mathbb{R}^n \) is compact. We will need the following definition.

Definition 9.3.1. A function \( f \in C(T^s) \) is called continuous almost periodic if the family of its translates \( \{ T^s \ni z \mapsto f(z + t) \}, t \in \mathbb{R}^n, \) is relatively compact in \( C_b(T^s) \).

It suffices to prove that \( APC(T^s) = C_{AP}(T^s) \).

Let \( f \in APC(T^s) \), i.e., for any sequence \( \{ t_k \} \subset \mathbb{R}^n \) there exists a subsequence of \( \{ T^s \ni z \mapsto f(z + t_k) \} \) that converges uniformly on \( T^s \). Then \( f \) is uniformly continuous on \( T^s \) and for every \( z_0 \in T^s \) and \( \{ d_k \} \subset \mathbb{Z}^n \) the family of translates \( \{ \mathbb{Z}^n \ni g \mapsto f(z_0 + g + d_k) \} \) is relatively compact, hence \( f \in C_{AP}(T^s) \).
Now, let \( f \in C_{AP}(T^s) \). Let us show that \( f \in APC(T^s) \). We fix some sequence \( \{t_k\} \subset \mathbb{R}^n \).

We have a continuous group homomorphism \( \mu : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \). Since \( \mathbb{R}^n/\mathbb{Z}^n \) is compact, \( \{\mu(s_k)\} \) has a convergent subsequence; without loss of generality we may assume that \( \{\mu(s_k)\} \) converges to 0. Hence, there exists a sequence \( \{d_k\} \subset \mathbb{Z}^n \) such that \( |t_k - d_k| \to 0 \) as \( k \to \infty \). Since \( f \) is uniformly continuous on \( T^s \),

\[
|f(z + t_k) - f(z + d_k)| \to 0 \quad \text{uniformly on } T^s \text{ as } k \to \infty.
\]

Thus, it suffices to show that \( \{T^s \ni z \to f(z + d_k)\} \) has a convergent subsequence.

Let \( C := \{z = (z_1, \ldots, z_n) \in T^s : 0 \leq \text{Re}(z_i) \leq 1, 1 \leq i \leq n\} \). Since \( f \in C_{AP}(T^s) \), for every \( z_0 \in C \) the family of translates \( \{\mathbb{Z}^n \ni g \to f(z_0 + g + d_k)\} \) is relatively compact in the topology of uniform convergence on \( \mathbb{Z}^n \), i.e., there exists a subsequence \( \{d_{k_l}\} \) (depending on \( z_0 \)) such that for every \( \varepsilon > 0 \) there exists \( N \) such that for all \( l, m > N \)

\[
|f(z_0 + g + d_{k_l}) - f(z_0 + g + d_{k_m})| < \frac{\varepsilon}{3} \quad \text{for all } g \in \mathbb{Z}^n.
\]

Now, since \( f \) is uniformly continuous on \( T^s \), there exists \( \delta > 0 \) such that

\[
|f(z + h) - f(z_0 + h)| < \frac{\varepsilon}{3} \quad \text{for all } z \in C, |z - z_0| < \delta \text{ and all } h \in \mathbb{Z}^n.
\]

It follows that for all \( l, m > N \)

\[
|f(z + g + d_{k_l}) - f(z + g + d_{k_m})| < \varepsilon \quad \text{for all } z \in C, |z - z_0| < \delta, \quad g \in \mathbb{Z}^n.
\]

Since \( C \) is compact, we need to consider only finitely many \( \delta \)-neighbourhoods of points in \( C \) that cover \( C \), i.e., passing to a subsequence of \( \{d_{k_l}\} \) finitely many times (without loss of generality this is \( \{d_{k_l}\} \) itself) we obtain that for all \( l, m > N \)

\[
|f(z + g + d_{k_l}) - f(z + g + d_{k_m})| < \varepsilon \quad \text{for all } z \in C, \quad g \in \mathbb{Z}^n,
\]

i.e., since \( \{z + g : z \in C, g \in \mathbb{Z}^n\} = T^s \), for all \( l, m > N \)

\[
|f(z + d_{k_l}) - f(z + d_{k_m})| < \varepsilon \quad \text{for all } z \in T^s.
\]

The constructed subsequence \( \{d_{k_l}\} \) depends on \( \varepsilon > 0 \). Now, let \( \varepsilon_r \to 0^+ \) as \( r \to \infty \), \( \varepsilon := \varepsilon_1 \).

We leave \( 2N + 1 \) first terms of \( \{d_{k_l}\} \) unchanged, and for \( \{d_{k_l}\}_{l \geq 2N+1} \) set \( \varepsilon := \varepsilon_2 \), and apply the
previous procedure, passing to a subsequence of \( \{d_{k_l}\}_{l \geq 2N+1} \), etc. We obtain a subsequence of \( \{d_{k_l}\} \) (without loss of generality, this is \( \{d_{k_l}\} \) itself) such that for every \( \varepsilon_r > 0 \) there exists \( N_r \) such that for all \( l, m > N_r \)

\[ |f(z + d_{k_l}) - f(z + d_{k_m})| < \varepsilon_r \quad \text{for all } z \in T^s, \]

i.e., the sequence \( \{T^s \ni z \mapsto f(z + d_{k_l})\} \) converges uniformly on \( T^s \), as needed.

9.4 Proof of Lemma 5.1.8

If \( U \in \mathcal{T}_a \), then there exists an open subset \( U' \subset c_aX \) such that \( \iota^{-1}(U') = U \) and, since \( \iota(X) \) is dense in \( c_aX \), \( \iota(U) \) is dense in \( U' \). The maximal among such sets \( U' \) with respect to inclusion (the union of all such \( U' \)) will be denoted by \( \hat{U} \).

Now, since \( C_a^v(X) \cong C(c_aX) \), if \( U \) is a \( \mathcal{T}_a \)-proper subset of some \( V \in \mathcal{T}_a \) (cf. Section 6.1.1), then clearly \( \hat{U} \subset \hat{V} \). Conversely, since space \( c_aX \) is paracompact and, therefore, is normal, continuous functions on \( c_aX \) separate closed subsets of \( c_aX \), hence if \( \hat{U} \subset \hat{V} \), then \( U \) is a \( \mathcal{T}_a \)-proper subset of \( V \). Now the assertion of lemma follows from Lemma 9.11.4(2) and the fact that any open subset of \( c_aX \) is a union of subsets in basis \( \mathfrak{B} \).

9.5 Proof of Theorem 6.1.5

We derive Theorem 6.1.5 from an analogous result (Theorem 9.5.4 below) for complex submanifolds of \( c_aX \), defined as follows.

**Definition 9.5.1.** A closed subset \( Y \subset c_aX \) is called a complex submanifold of \( c_aX \) of codimension \( k \) if for every point \( x \in c_aX \) there exist a neighbourhood \( U := \hat{\Pi}(U_0, K) \cong U_0 \times K \) of \( x \), where \( U_0 \subset X_0 \) is open and simply connected, \( K \subset \hat{G}_a \) is open (cf. (6.2.6)), and functions \( h_1, \ldots, h_k \in \mathcal{O}(U) \) such that

1. \( Y \cap U = \{x \in U : h_1(x) = \cdots = h_k(x) = 0\} \);

2. The rank of map \( z \mapsto (h_1(z, \omega), \ldots, h_k(z, \omega)) \) is \( k \) at each point \( (z, \omega) \in Y \cap U \).
Recall that by $\iota$ we denote the canonical map $X \to c_\mathfrak{a}X$ (cf. Section 6.2.2). Any complex submanifold $Y$ of $c_\mathfrak{a}X$ has the following properties:

(i) $\iota^{-1}(Y) \subset X$ is a complex submanifold of $X$ of codimension $k$.

(ii) $Y \cap \iota(X)$ is dense in $Y$ (see the proof of Lemma 9.5.6 below).

It follows from Proposition 6.2.10 that if $Z \subset X$ is a complex $\mathfrak{a}$-submanifold in the sense of Definition 6.1.1, then the closure of $\iota(Z)$ in $c_\mathfrak{a}X$ is a complex submanifold of $c_\mathfrak{a}X$.

Conversely, if algebra $\mathfrak{a}$ is self-adjoint and $Y$ is a complex submanifold of $c_\mathfrak{a}X$, then $\iota^{-1}(Y) \subset X$ is a complex $\mathfrak{a}$-submanifold of $X$ in the sense of Definition 6.1.1.

**Definition 9.5.2.** A function $f \in C(Y)$ is called holomorphic if $\iota^* f \in \mathcal{O}(\iota^{-1}(Y))$.

The algebra of holomorphic functions on a complex submanifold $Y \subset c_\mathfrak{a}X$ is denoted by $\mathcal{O}(Y)$.

**Proposition 9.5.3.** Let $Y$ be a complex submanifold of $c_\mathfrak{a}X$, set $Z := \iota^{-1}(Y)$.

Every function in $f \in \mathcal{O}_\mathfrak{a}(Z)$ (cf. Definition 6.1.3) admits a unique extension to a function $\hat{f} \in \mathcal{O}(Y)$ such that $f = \iota^* \hat{f}$, i.e., $\mathcal{O}_\mathfrak{a}(Z)$ embeds into $\mathcal{O}(Y)$.

If also $\mathfrak{a}$ is self-adjoint, then $\mathcal{O}_\mathfrak{a}(Z) \cong \iota^* \mathcal{O}(Y)$.

**Theorem 9.5.4.** Suppose that $X_0$ is a Stein manifold. Let $Y \subset c_\mathfrak{a}X$ be a complex submanifold, $f \in \mathcal{O}(Y)$. Then there exists a function $F \in \mathcal{O}(c_\mathfrak{a}X)$ such that $F|_Y = f$.

In view of Propositions 6.2.8, 9.5.3 and the remark before Definition 9.5.2, Theorem 6.1.5 is a special case of Theorem 9.5.4.

**9.5.1 Proof of Theorem 9.5.4**

Our proof relies on Theorem 6.3.3(B) asserting the vanishing of cohomology of coherent sheaves on $c_\mathfrak{a}X$, and the fact that the sheaf $I_Y$ of germs of holomorphic functions vanishing on a complex submanifold $Y \subset c_\mathfrak{a}X$ is coherent (cf. Corollary 9.5.9 below).

We will need the following results.
**Lemma 9.5.5** (Inverse function theorem). Let $U'_0, U''_0 \subset X_0$ be simply connected coordinate charts, $K', K'' \subset \hat{G}_a$ be open, we set $U' := U'_0 \times K', U'' := U''_0 \times K''$. Suppose that $H \in \mathcal{O}(U', U'')$ is a holomorphic map (cf. Definition 6.2.9) of the form

$$H(z, \omega) = (\Theta_\omega, \theta), \quad \Theta_\omega \in \mathcal{O}(U'_0, U''_0), \quad \omega \in K', \quad \theta \in C(K', K'')$$

such that for a certain point $y_0 = (z_0, \omega_0) \in U'$ we have

1. the rank of the Jacobian matrix of map $z \to \Theta_{\omega_0}$ (with respect to the local coordinates on $U'_0$) is maximal at $z_0$.
2. The restriction of $\theta$ to a neighbourhood $L' \subset K'$ of $\omega_0$ is a homeomorphism between $L'$ and $L'' := \theta(L')$.

Then there exists an open neighbourhood $V' \subset U'$ of $y_0$ such that $V' := H(V')$ is open, and $H|_{V'} \in \mathcal{O}(V', V'')$ is a biholomorphism.

**Proof.** We may assume, without loss of generality, that $U'_0, U''_0$ are open subsets of $\mathbb{C}^n$, $L' = L''$ and $\theta|_L' = \text{Id}$.

Since $H$ is continuous, we may shrink $L'$, if necessary, so that the Jacobian matrix of the map $z \to \Theta_{\omega}(z)$ at $z_0$, as a function of $\omega \in L'$, is bounded, and the maximum of moduli of the determinants of all its $k \times k$ submatrices is bounded away from zero on $L'$.

Now, it follows from the proof of the usual inverse function theorem (see, e.g., [Bre]) that (possibly after further shrinking of $L'$) for each $\omega \in L'$ there exists a holomorphic inverse map $\Theta_{\omega}^{-1}$ whose domain contains an open ball $B_r(\zeta_0) \subset U''_0$ centered at $\zeta_0 := \Theta_{\omega_0}(z_0)$ having radius $r > 0$ independent of $\omega$, and that $\Theta_{\omega}^{-1}$ depends on $\omega$ continuously in the topology of uniform convergence on compact subsets.

We shrink $L''$, so that $L'' = L'$, and set $V'' := B_r(\zeta_0) \times L'' \subset U''$, define

$$H^{-1}(\zeta, \omega) := (\Theta_{\omega}^{-1}(\zeta), \omega), \quad (\zeta, \omega) \in V'',$$

and $V' := H^{-1}(V'') \subset U'$. Clearly, $V'$ is an open neighbourhood of $y_0$, the map $H^{-1}$ is in $\mathcal{O}(V'', V')$, and $H|_{V'}, H^{-1}$ are inverses of each other, as needed.
Lemma 9.5.6 (Local structure of complex submanifolds of $c_aX$). Let $Y \subset c_aX$ be a complex submanifold of codimension $k$, $y_0 \in Y$.

There exist an open neighbourhood $V \subset c_aX$ of $y_0$, open subsets $V_0 \subset X_0$ and $L \subset \hat{G}_{a}$, a closed complex submanifold $Z_0$ of $V_0$ of codimension $k$, and a biholomorphic map $\Phi \in \mathcal{O}(V_0 \times L, V)$ such that $\Phi^{-1}(V \cap Y) = Z_0 \times L$.

Proof. Let us fix a neighbourhood $U := \Pi(U_0, K) \subset c_aX$ of point $y_0$, where $U_0 \subset X_0$ is a simply connected coordinate chart, $K \subset \hat{G}_{a}$ is open. Using the corresponding trivialization of bundle $c_aX$ (cf. Section 6.2.6) we identify $U$ with $U_0 \times K$.

Without loss of generality, we may assume that $U_0$ is an open subset of $\mathbb{C}^n$.

We will use the notation of Definition 9.5.1: we have

$$Y \cap U = \{y \in U : h_1(y) = \cdots = h_k(y) = 0\},$$

where $h_i \in \mathcal{O}(U)$, $(1 \leq i \leq k)$ satisfy the conditions of the definition.

Let us define

$$H(z, \omega) := (h_1(z, \omega), \ldots, h_k(z, \omega), z_{k+1}, \ldots, z_n, \omega), \quad z := (z_1, \ldots, z_n), \quad (z, \omega) \in U.$$

We can present map $H$ in the form $H(\cdot, \omega) = (\Theta_\omega(\cdot), \theta(\omega))$, $\omega \in K$, where

$$\Theta_\omega(z_1, \ldots, z_n) := (h_1(z, \omega), \ldots, h_k(z, \omega), z_{k+1}, \ldots, z_n), \quad \theta := \text{Id}.$$

Let $y_0 = (z_0, \omega_0) \in U$. Since functions $h_i$ ($1 \leq i \leq k$) determine a complex submanifold of $c_aX$, the rank of the map $z \to \Theta_{\omega_0}(z)$ is maximal at $z_0$ (cf. Definition 9.5.1). Hence, by Lemma 9.5.5 there exist open subsets $V \subset U$, $y_0 \in V$, and $V_0 \times L$, where $V_0 \subset \mathbb{C}^n$, $L \subset K$ are open, and a biholomorphic map $\Phi \in \mathcal{O}(V_0 \times L, V)$ such that $H|_V \circ \Phi$ is the identity map. Therefore,

$$(h_l \circ \Phi)(\zeta_1, \ldots, \zeta_n, \omega) = \zeta_l, \quad (\zeta, \omega) \in V_0 \times L, \quad 1 \leq l \leq n - k.$$

It follows that $\Phi^{-1}(V \cap Y)$, i.e., the zero set of map $(h_1, \ldots, h_k) \circ \Phi$, is

$$\{(0, \ldots, 0, \zeta_{k+1}, \ldots, \zeta_n) : (\zeta_1, \ldots, \zeta_n) \in V_0 \times L\}.$$  

Hence, we can set $Z_0 := \{(0, \ldots, 0, \zeta_{k+1}, \ldots, \zeta_n) : (\zeta_1, \ldots, \zeta_n) \in V_0\}$. 

\qed
Lemma 9.5.7. Let $Y$ be a complex submanifold of $c_\alpha X$, and $f \in \mathcal{O}(Y)$ (cf. Definition 9.5.2). Then for every $y_0 \in Y$ there exist a neighbourhood $U \subset c_\alpha X$ of $y_0$ and a function $F_U \in \mathcal{O}(U)$ such that $F_U|_{U \cap Y} = f|_{U \cap Y}$.

Proof. It follows from the proof of Lemma 9.5.6 that there exist an open neighbourhood $V \subset c_\alpha X$ of point $y_0$, open subsets $V_0 \subset \mathbb{C}^n$, $L \subset \hat{G}_\alpha$, and a biholomorphic map $\Phi \in \mathcal{O}(V_0 \times L, V)$ such that $\Phi^{-1}(V \cap Y) = Z_0 \times L$, where $Z_0 = \{(0, \ldots, 0, z_{k+1}, \ldots, z_n) : (z_1, \ldots, z_n) \in V_0\}$. Let $\tilde{f} := \Phi^* f \in \mathcal{O}(Z_0 \times L)$; define

$$\tilde{F}_U(z_1, \ldots, z_n, \omega) := \tilde{f}(z_{k+1}, \ldots, z_n, \omega), \quad (z_1, \ldots, z_n, \omega) \in V_0 \times L,$$

Now, we set $F_U := (\Phi^{-1})^* \tilde{F}_U$. \hfill \Box

Let $V_0 \subset X_0$, $L \subset \hat{G}_\alpha$ be open subsets. The sheaf $I_{Z_0 \times L}$ of germs of holomorphic functions on $V_0 \times L$ vanishing on $Z_0 \times L$ will be called the sheaf of ideals of $Z_0 \times L$.

Lemma 9.5.8. Let $Z_0 \subset V_0$ be a complex analytic subset. Then sheaf $I_{Z_0 \times L}$ is coherent (cf. Definition 6.3.1).

Proof. We denote by $\mathcal{O}$ the sheaf of germs of holomorphic functions on $V_0 \times L$, and by $\hat{\mathcal{O}}$ the sheaf of germs of holomorphic functions on $V_0$. Let $Z := Z_0 \times L$. By $\hat{I}_{Z_0} \subset \hat{\mathcal{O}}$ we denote the sheaf of ideals of $Z_0$.

By Cartan theorem (see, e.g., [Gun3]) sheaf $\hat{I}_{Z_0}$ is coherent: every point in $V_0$ has an open neighbourhood (without loss of generality, we assume that it coincides with $V_0$) such that there exists a free resolution

$$0 \to \hat{\mathcal{O}}^m_{N} \xrightarrow{\hat{\varphi}^N_{N-1}} \ldots \xrightarrow{\hat{\varphi}^1_{1}} \hat{\mathcal{O}}^m_{1} \xrightarrow{\hat{\varphi}^0_{0}} \hat{I}_{Z_0} \to 0,$$  

(9.5.3)

where sheaf homomorphisms $\hat{\varphi}_k$ are given by functions in $\mathcal{O}(V_0, M_{m_k \times m_{k-1}}(\mathbb{C}))$, $0 \leq k \leq N - 1$ (here $m_0 := 1$). The exact sequence (9.5.3) can be rewritten as a collection of short exact sequences

$$0 \to \hat{R}_k \xrightarrow{\iota} \hat{\mathcal{O}}^m_k \xrightarrow{\hat{\varphi}^k_{k-1}} \hat{R}_{k-1} \to 0, \quad 1 \leq k \leq N - 1,$$  

(9.5.4)

where $\hat{R}_k := \text{Im} \hat{\varphi}_k$ ($1 \leq k \leq N - 1$), $\hat{R}_0 := \hat{I}_{Z_0}$, and $\iota$ stands for inclusion.
Let $y_0 = (x_0, \omega_0) \in V_0 \times L$, $f_{y_0} \in \mathcal{O}(U_0 \times K)$ be a section of sheaf $\mathcal{O}^{mk}$ over an open neighbourhood $U_0 \times K \subset V_0 \times L$ of $y_0$. Let $(\varphi_k)_*(f_{y_0})$, $0 \leq k \leq N - 1$, be the map

$$U_0 \times K \ni (x, \omega) \mapsto (\tilde{\varphi}_k)_*(f_{y_0}(x, \omega)), $$

where $(\tilde{\varphi}_k)_*$ is the homomorphism of sections over $U_0$ induced by $\tilde{\varphi}_k$. This is a section over $U_0 \times K$ of sheaf $\mathcal{O}^{mk-1}$ if $k > 1$, or sheaf $I_Z$ if $k = 1$.

The homomorphisms of sections $(\varphi_k)_*$ induce sheaf homomorphisms $\varphi_k$, $0 \leq k \leq N - 1$. Thus, we get a chain complex

$$0 \longrightarrow \mathcal{O}^N \xrightarrow{\varphi_N} \ldots \xrightarrow{\varphi_1} \mathcal{O}^1 \longrightarrow I_Z \longrightarrow 0, \quad (9.5.5)$$

or, equivalently, a collection of chain complexes

$$0 \longrightarrow \mathcal{R}_k \xrightarrow{\iota} \mathcal{O}^k \xrightarrow{\varphi_{k-1}} \mathcal{R}_{k-1} \longrightarrow 0, \quad 1 \leq k \leq N - 1, \quad (9.5.6)$$

where $\mathcal{R}_k := \text{Im} \varphi_k$ ($1 \leq k \leq N - 1$), $\mathcal{R}_0 := I_Z$, and $\iota$ stands for inclusion. We have to show that sequence (9.5.5) is exact: this would imply that $I_Z$ is coherent. The latter is equivalent to exactness of sequences (9.5.6). Since $\iota$ is an inclusion, it is injective and its image coincides with the kernel of $\varphi_{k-1}$. It remains to show that homomorphisms $\varphi_{k-1}$, $1 \leq k \leq N - 1$, in (9.5.6) are surjective.

We fix $1 \leq k \leq N - 1$. Let $y_0 = (x_0, \omega_0) \in V_0 \times L$, let $g_{y_0} \in \Gamma(U_0 \times K, \mathcal{R}_{k-1})$ be a section of sheaf $\mathcal{R}_{k-1}$ over an open neighbourhood $U_0 \times K \subset V_0 \times L$ of $y_0$. By shrinking $K$ we may assume that $g_{y_0}$ is defined also over $U_0 \times \bar{K}$, and that $\bar{K} \subset L$ ($\bar{K}$ is compact as $\hat{G}_a$ is compact).

To show that $\varphi_{k-1}$ is surjective, it suffices to prove that there exists a section $f_{y_0} \in \Gamma(U_0 \times K, \mathcal{O}^{mk})$ such that $(\varphi_{k-1})_*(f_{y_0}) = g_{y_0}$, where $(\varphi_{k-1})_*$ is the homomorphism of sections induced by $\varphi_{k-1}$.

We may assume, without loss of generality, that $U_0 \subset V_0$ are polydisks in $\mathbb{C}^n$.

Let $\Gamma(U_0, \mathcal{R}_{k-1})$ and $\Gamma(U_0, \mathcal{O}^{mk})$ denote the Frechet spaces of sections over $U_0$ of sheaves $\mathcal{R}_{k-1}$ and $\mathcal{O}^{mk}$, respectively, endowed with the topology of uniform convergence on compact subsets of $U_0$. Since (9.5.4) is exact, the sequence of sections

$$0 \longrightarrow \Gamma(U_0, \mathcal{R}_k) \longrightarrow \Gamma(U_0, \mathcal{O}^k) \longrightarrow \Gamma(U_0, \mathcal{R}_{k-1}) \longrightarrow 0,$$
is also exact (see, e.g., [Gun3]), therefore by the open mapping theorem homomorphism \( \tilde{\varphi}_{k-1} \) induces an isomorphism of Frechet spaces

\[
\Gamma(U_0, \tilde{O}_m^k) / \Gamma(U_0, \tilde{R}_k) \cong \Gamma(U_0, \tilde{R}_{k-1}).
\]

(9.5.7)

Now, section \( g_{y_0} \) determines a continuous map \( \hat{g} \in C(\bar{K}, \Gamma(U_0, \tilde{O}_m^k)) \),

\[
\hat{g}(\omega) := g_{y_0}(\cdot, \omega), \quad \omega \in \bar{K}.
\]

By Michael selection theorem, in view of (9.5.7), there exists a continuous selection \( \hat{f} \) for \( \hat{g} \) in \( \Gamma(U_0, \tilde{O}_m^k) \), i.e., a map \( \hat{f} \in C(\bar{K}, \Gamma(U_0, \tilde{O}_m^k)) \) such that

\[
(\varphi_{k-1})_* \circ \hat{f} = \hat{g}.
\]

We set \( f_{y_0}(\cdot, \omega) := \hat{f}(\omega), \ \omega \in K \). Clearly, \( f_{y_0} \in \Gamma(U_0 \times K, \mathcal{O}_m^k) \). This implies that \( \varphi_{k-1} \) is a surjective sheaf homomorphism, as needed.

**Corollary 9.5.9.** The sheaf \( I_Y \) of ideals of a complex submanifold \( Y \subset c_a X \) is coherent.

**Proof.** In the notation of Lemma 9.5.6, we have \( \Phi^*(I_Y|_V) = I_{Z_0 \times L} \). Thus, it suffices to prove the coherence of sheaf \( I_{Z_0 \times L} \). But the latter follows from Lemma 9.5.8.

**Proof of Theorem 9.5.4.** By Lemma 9.5.7 there exist an open cover \( U = \{U_j\} \) of \( c_a X \) and functions \( f_j \in \mathcal{O}(U_j) \) such that

\[
f_j|_{Y \cap U_j} = f|_{Y \cap U_j} \text{ if } Y \cap U_j \neq \emptyset,
\]

If \( Y \cap U_j = \emptyset \), we define \( f_j := 0 \). Then \( \{g_{ij} := f_i - f_j \text{ on } U_i \cap U_j \neq \emptyset\} \) is a 1-cocycle with values in the sheaf \( I_Y \) of ideals of complex submanifold \( Y \).

By Corollary 9.5.9 the ideal sheaf \( I_Y \) is coherent. Hence, by Theorem 6.3.3(B) the cohomology group \( H^1(c_a X, I_Y) = 0 \). We may deduce from here, possibly after passing to a refinement of cover \( U \), that there are holomorphic functions \( h_j \in \Gamma(U_j, I_Y) \) such that \( g_{ij} = h_i - h_j \) on \( U_i \cap U_j \neq \emptyset \). Now, we can define the extension \( F \) of \( f \) by the formula \( F := f_j - h_j \) on \( U_j \), for all \( j \).
9.6 Proof of Theorem 6.1.2

Our proof is based on Theorem 6.3.3(A), and the equivalence of notions of a complex a-
submanifold of $X$ and a complex submanifold of $c_a X$ established in Section 9.5.

It suffices to prove that, given a complex submanifold $Y \subset c_a X$ of codimension $k$, there
exists a countable collection of functions $f_i \in \mathcal{O}(c_a X)$, $i \in I$, such that

(i) $Y = \{y \in c_a X : f_i(y) = 0 \text{ for all } i \in I\}$, and

(ii) for each $y_0 \in Y$ there exists a neighbourhood $U = \check{\Pi}(U_0, K)$ (cf. (6.2.6) for notation)
and functions $f_{i_1}, \ldots, f_{i_k}$ such that $Y \cap U = \{y \in U : f_{i_1}(y) = \cdots = f_{i_k} = 0\}$, and the rank of
map $z \to (f_1(z, \omega), \ldots, f_k(z, \omega))$, $(z, \omega) \in U$, is maximal at each point of $Y \cap U$.

The sheaf of ideals $I_Y$ of $Y$ is coherent (cf. the proof of Corollary 9.5.9), hence by Theorem
6.3.3(A) there exists a countable collection of sections $f_i \in \Gamma(c_a X, I_Y)$ ($\subset \mathcal{O}_{c_a X}$), $i \in I$, that
generate $I_Y$ at every point. By definition, condition (i) is satisfied, and for every point $y_0 \in Y$
there is a neighbourhood $U = \check{\Pi}(U_0, K)$, sections $f_{i_1}, \ldots, f_{i_m}$, and functions $u_{jl} \in \mathcal{O}(c_a X)$ such that

$$h_j = u_{j1} \tilde{f}_1 + \cdots + u_{jm} \tilde{f}_m, \quad 1 \leq j \leq k,$$

(9.6.8)

where $\tilde{f}_l := f_{i_l}$, $1 \leq l \leq m$, and $h_j$ are the generators of $I_Y|_U$ (such that in appropriate local
coordinates $h_j(z, \omega) = z_j$, the $j$-th component of $z \in U_0 \subset \mathbb{C}^n$, $\omega \in K$, cf. Lemma 9.5.6).

It is immediate that $Y \cap U = \{y \in U : \tilde{f}_1(y) = \cdots = \tilde{f}_m = 0\}$. It remains to show that
functions $\tilde{f}_l$ can be chosen in such a way that $m = k$, and condition (ii) is satisfied.

Indeed, let $\nabla h_j$, $\nabla \tilde{f}_l$ denote the vector-valued functions $\nabla_z h_j(z, \omega)$, $\nabla_z \tilde{f}_l(z, \omega)$, $(z, \omega) \in U$.
Then

$$\nabla h_j = u_{j1} \nabla \tilde{f}_1 + \cdots + u_{jm} \nabla \tilde{f}_m \quad \text{on } Y \cap U, \quad 1 \leq j \leq k.$$  

Since $(\nabla h_j)_{j=1}^k$ has rank $k$ on $U$, we obtain that $k \leq m$, and $(u_{jl})_{1 \leq j \leq k, 1 \leq l \leq m}$, $(\nabla \tilde{f}_l)_{l=1}^m$ have rank $k$. Assuming that vectors $(u_{jl})_{j=1}^k, \ldots, (u_{jl})_{j=1}^k$ and $\nabla \tilde{f}_1, \ldots, \tilde{f}_k$ are linearly independent, we can apply the holomorphic Inverse function theorem to the matrix identity (9.6.8) (possibly,
after shrinking $U$ and, as a result, extending collection $\{f_i\}_{i \in I}$ to represent functions $\tilde{f}_l$, $l \neq l_i$,
$1 \leq i \leq k$ via $\tilde{f}_1, \ldots, \tilde{f}_k$, so that we can choose as $f_{i_1}, \ldots, f_{i_k}$ the functions $\tilde{f}_1, \ldots, \tilde{f}_k$. This
completes the proof.
9.7 Proof of Proposition 6.1.4

(1) First, let us show that, given a holomorphic $a$-function in the sense of Definition 6.1.3, its current in an $a$-current. We refer to [Dem] for the basic results of the theory of currents (distributions).

We will need the following notation. Let $Y \subset c\alpha X$ be a complex submanifold, $f \in O(Y)$. By Tietze-Urysohn theorem there exists a function $F \in C(c\alpha X)$ such that $F|_Y = f$. Further, let $U_0 \subset X_0$ be open simply connected, let $i_\omega : U_0 \hookrightarrow U_0 \times \hat{G}_a$, $i_\omega (z) := (z, \omega)$. Define $F_\omega := i_\omega^* F \in C(U_0)$, and set

$$Y_\omega := i_\omega^{-1} (Y \cap \bar{p}^{-1}(U_0)) \subset U_0,$$

where we identify $\bar{p}^{-1}(U_0)$ with $U_0 \times \hat{G}_a$.

Using the equivalence of notions of a complex $\alpha$-submanifold of $X$ and a complex submanifold of $c\alpha X$ established in Section 9.5, Proposition 9.5.3, and a partition of unity argument, it is not difficult to see that the assertion (1) will follow once we prove that the function

$$\omega \mapsto \int_{Y_\omega} F_\omega \varphi, \quad \omega \in \hat{G}_a$$

is continuous for any $\varphi \in \Lambda_{c,\alpha}^{k,k}(U_0)$ (we also use the fact that $\alpha \cong C(M_\alpha)$ since $\alpha$ is self-adjoint, cf. Section 6.2.1).

Indeed, denote

$$(T_{Y_\omega}, \varphi) := \int_{Y_\omega} \varphi, \quad \varphi \in \Lambda_{c,\alpha}^{k,k}(U_0)$$

(the current of integration of $Y_\omega$). By Poincaré-Lelong formula

$$T_{Y_\omega} = \frac{i}{\pi} \partial_z \bar{\partial}_z \log \sum_{l=1}^k |h_l(z, \omega)|^2,$$

where $h_1(\cdot, \omega), \ldots, h_k(\cdot, \omega)$ are the functions that determine $Y_\omega \subset U_0$ for all $\omega \in K$ (we use notation of Definition 9.5.1). In particular, since map $\omega \mapsto (h_1(\cdot, \omega), \ldots, h_k(\cdot, \omega)) \in O(U_0, \mathbb{C}^n)$ is continuous on $K$, the map $\omega \mapsto T_{Y_\omega}$ (taking values in the space of $(k,k)$-currents) is continuous on $K$; the sets $K$ cover $\hat{G}_a$, hence $\omega \mapsto T_{Y_\omega}$ is continuous on $\hat{G}_a$.

Now, it is immediate that $T_{Y_\omega}$ is a current of order 0, therefore the product $F_\omega T_{Y_\omega}$ is well defined. Further, the map $\omega \to F_\omega$ is a continuous $C(U_0)$-valued map on $\hat{G}_a$, where space $C(U_0)$
is endowed with the topology of uniform convergence on $U_0$. Therefore, the map $\omega \mapsto F_\omega T_{Y_\omega}$ is continuous as well, and hence the function

$$\omega \mapsto (F_\omega T_{Y_\omega}, \varphi) = \int_{Y_\omega} F_\omega \varphi, \quad \varphi \in \Lambda_{c}^{k,k}(U_0)$$

is continuous on $\hat{G}_a$, as required.

(2) Now, we have to show that a holomorphic function as in Proposition 6.1.4 whose current is an $a$-current is a holomorphic $a$-function. We will use the equivalence of notions of a complex $a$-submanifold of $X$ and a complex submanifold of $c_a X$ established in Section 9.5, Proposition 9.5.3, as well as the notation introduced in part (1). So, we have to show that, given a function $f$ on $Y$ that is bounded on subsets $Y \cap \bar{p}^{-1}(V_0)$, $V_0 \Subset X_0$, such that $i^* f \in \mathcal{O}(i^{-1}(Y))$ and for every open simply connected $U_0 \subset X_0$ the map $\omega \mapsto \int_{Y_\omega} f(\cdot, \omega) \varphi$ is continuous for every $\varphi \in \Lambda_{c}^{k,k}(U_0)$, the function $f$ itself is continuous.

Since this is a local statement, it suffices to show that for every $y_0 \in Y$ there exists an open subset $K \subset \hat{G}_a$ such that $y_0 \in \hat{\Pi}(U_0, K)$ (cf. (6.2.6) for notation), and $f$ is continuous on $Y \cap \hat{\Pi}(U_0, K)$. (Recall that $\hat{\Pi}(U_0, K)$ can be identified with $U_0 \times K$ (cf. (6.2.6))).

Suppose that we have fixed some $y_0 \in Y$. Using Lemma 9.5.6, possibly after shrinking $U_0$, we may assume (i.e. after applying a biholomorphic transformation) that for some $K \in \hat{G}_a$ as above, we have

$$Y \cap \hat{\Pi}(U_0, K) = \hat{\Pi}(Y_0, K),$$

where $Y_0 \subset U_0$ is a complex submanifold of codimension $k$. In particular, we may assume that $Y_\omega = Y_0$ for all $\omega \in K$. By our assumption, we have

(a) the map $\omega \mapsto \int_{Y_0} f(\cdot, \omega) \varphi$ is continuous on $K$,

(b) $\{f(\cdot, \omega)\}_{\omega \in K}$ is uniformly bounded with respect to sup-norm on $Y_0 \cap V_0$ for every open $V_0 \Subset U_0$, and

(c) $f(\cdot, j(g)) \in \mathcal{O}(Y_0)$, $g \in G$ (cf. Section 6.2.1 for notation).

Therefore, we can apply Montel theorem to $\{f(\cdot, j(g))\}_{g \in G}$: the compactness argument implies that $f(\cdot, \omega) \in \mathcal{O}(Y_0)$ for all $\omega \in K$, and $f$ is continuous on $Y_0 \times K$, as needed.
9.8 Proof of Proposition 6.1.11

We will use the notation and results of Section 6.2. Let $F_i := p^* f_i$, $G_i := p^* f_i + h_i \in \mathcal{O}_a(X)$, $1 \leq i \leq k$. We denote by $\tilde{F}_i, \tilde{G}_i$ the extensions of these functions from $X$ to $c_aX$ (cf. Proposition 6.2.8(1)). We set
\[
\hat{Z} := \{y \in c_aX : \tilde{F}_1(y) = \cdots = \tilde{F}_k(y) = 0\},
\]
\[
\hat{Z}_h := \{y \in c_aX : \tilde{G}_1(y) = \cdots = \tilde{G}_k(y) = 0\}.
\]
Clearly, $Z = \iota^{-1}(\hat{Z}), Z_h = \iota^{-1}(\hat{Z}_h)$.

It suffices to show that every point $y \in \hat{Z}_h$ has a neighbourhood $\tilde{\Pi}(U_0, K)$ (cf. Section 6.2.6 for notation), which we may identify with $U_0 \times K$, such that the rank of the Jacobian of map $x \to (\tilde{G}_1(x, \omega), \ldots, \tilde{G}_k(x, \omega))$ is maximal at every point $(x, \omega) \in (U_0 \times K) \cap \hat{Z}_h$. Note that an analogous condition for $\hat{Z}$ is obviously satisfied.

We may assume without loss of generality that $U_0 \subset \mathbb{C}^n$ is an open ball centered at the origin. Furthermore, using Lemma 9.5.6 we may assume that $\tilde{F}_i|_{U_0 \times K}(x, \omega) = x_i, 1 \leq i \leq k$, where $x = (x_1, \ldots, x_n)$. Thus, for every $\omega \in K$ the set $(U_0 \times \{\omega\}) \cap \hat{Z}_h$, viewed as a subset of $U_0$, is contained in $\delta$-neighbourhood (with respect to Euclidean metric on $U_0 \subset \mathbb{C}^n$) of hyperplane $\{x \in U_0 : x_1 = \cdots = x_k = 0\}$. Therefore, by continuity of the Jacobian of map $x \to (\tilde{F}_1(x, \omega), \ldots, \tilde{F}_k(x, \omega))$ (in fact, it is identically equal to identity matrix by our assumption), using Cauchy estimates, we can choose $\delta > 0$ so small that the Jacobian of map $x \to (\tilde{G}_1(x, \omega), \ldots, \tilde{G}_k(x, \omega))$ will be maximal at every point $(x, \omega) \in (U_0 \times K) \cap \hat{Z}_h$, as required.

9.9 Proof of Proposition 9.5.3

First, let $f$ be a holomorphic $a$-function on $Z$ in the sense of Definition 6.1.3, i.e., there is a function $F \in C_a(X)$ such that $F|_Z = f$. By Proposition 6.2.8(1) there exists a function $\hat{F} \in C(c_aX)$, such that $\iota^* \hat{F} = F$. We set $\hat{f} := \hat{F}|_Y$. Since $\iota^* \hat{f} = f$, we obtain $\hat{f} \in \mathcal{O}(Y)$ (cf. Definition 9.5.2), as required.

Now, suppose that $a$ is self-adjoint. Let $\hat{f} \in \mathcal{O}(Y)$. Since $c_aX$ is a normal space, by Tietze-Urysohn extension theorem there exists a function $\hat{F} \in C(c_aX)$ such that $\hat{F}|_Y = \hat{f}$. 

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By Proposition 6.2.8(2) $F := i^*\hat{F}$ belongs to $C_a(X)$. Since $F|_Z = f$, function $f (= i^*\hat{f})$ is a holomorphic $a$-function on $Z$ in the sense of Definition 6.1.3.

9.10 Proofs of Theorems 6.1.13 and 6.1.15

We will need notation and results of Section 6.2.7. We will also need the following

**Lemma 9.10.1.** Every open cover of $c_aX$ has an at most countable refinement by open sets $\hat{\Pi}(U_{0,\alpha}, K_\alpha)$ (cf. (6.2.6)), where $U_{0,\alpha} \subset X_0$ is open and simply connected, and $K_\alpha \in \mathcal{Q}$ (cf. (6.2.3)).

In particular, we may assume in Definition 6.2.11 that $\{U_\alpha\}$ is at most countable, and $U_\alpha = \hat{\Pi}(U_{0,\alpha}, K_\alpha)$, for all $\alpha$.

9.10.1 Proof of Theorem 6.1.15

In view of Proposition 6.2.13, we need to prove the following reformulation of Theorem 6.1.15:

Let $X_0$ be a Stein manifold, $D \in \text{Div}(c_aX)$. If $X_0$ is homotopically equivalent to an open subset $Y_0 \subset X_0$ such that the restriction of divisor $D$ to $Y := \hat{p}^{-1}(Y_0)$ is equivalent to a principal divisor in $\text{Div}(c_aX)$, then $D$ itself is equivalent to a principal divisor.

Let $O_*(U)$, where $U \subset c_aX$ is open, denote the multiplicative group of nowhere zero holomorphic functions on $U$. We denote by $O_*$ the multiplicative sheaf of germs of nowhere zero holomorphic functions on $c_aX$.

Let $D \in \text{Div}(c_aX)$. We will use the notation of Definition 6.2.11.

The functions $\{d_{\alpha\beta} \in O_*(U_\alpha \cap U_\beta)\}$ are determined uniquely, and define a 1-cocycle with values in sheaf $O_*$.

The divisor $D$ is equivalent to a principal divisor if and only if there exist functions $d_\alpha \in O_*(U_\alpha)$ such that

$$d_{\alpha\beta} = d_\beta d_\alpha^{-1} \quad \text{on } U_\alpha \cap U_\beta \neq \emptyset,$$

for all such $\alpha$, $\beta$. Indeed, since $f_\alpha = d_{\alpha\beta}f_\beta$, we have $d_\alpha f_\alpha = d_\beta f_\beta$. We set $f := d_\alpha f_\alpha$ on $U_\alpha$, for all $\alpha$. Then $f \in O(c_aX)$ and the principal divisor $D_f$ is equivalent to $D$. 

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Let us show that there exists a presentation (9.10.9) (after passing to a refinement of \( \{ U_\alpha \} \)). We follow the standard argument (see, e.g., [GH]). We have an exact sequence of sheaves

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_e^{\oplus i} \longrightarrow \mathcal{O}_s \longrightarrow 0,
\]

which induces an exact sequences of cohomology groups

\[
\ldots \longrightarrow H^1(c_\alpha X, \mathbb{Z}) \longrightarrow H^1(c_\alpha X, \mathcal{O}) \longrightarrow H^1(c_\alpha X, \mathcal{O}_s) \overset{\sigma}{\longrightarrow} H^2(c_\alpha X, \mathbb{Z}) \longrightarrow \ldots \tag{9.10.10}
\]

Clearly, if \( c \in H^1(c_\alpha X, \mathcal{O}_s) \) is trivial, then \( \sigma(c) = 0 \).

We can repeat these arguments over a subset \( Y := \bar{p}^{-1}(Y_0) \), where \( Y_0 \subset X_0 \) is open and is homotopically equivalent to \( X_0 \). In particular, over such \( Y \) we get an exact sequence of the form (9.10.10). Let \( \sigma_Y : H^1(Y, \mathcal{O}_s) \longrightarrow H^2(Y, \mathbb{Z}) \) denote the corresponding group homomorphism.

Since \( X_0 \) is a Stein manifold, by Theorem 6.3.3(B) we have \( H^1(c_\alpha X, \mathcal{O}) = 0 \). Hence, if \( c \in H^1(c_\alpha X, \mathcal{O}_s) \) is such that \( \sigma(c) = 0 \), then \( c \) is trivial.

Now, let \( D \in \text{Div}(X) \). We denote by \( c_D \) the image in \( H^1(c_\alpha X, \mathcal{O}_s) \) of the 1-cocycle \( C_D := \{ d_{\alpha \beta} \in \mathcal{O}_s(U_\alpha \cap U_\beta) \} \), and by \( c_{D,Y} \in H^1(Y, \mathcal{O}_s) \) the image of the restriction \( C_{D,Y} := \{ d_{\alpha \beta}|_Y \in \mathcal{O}_s(U_\alpha \cap U_\beta \cap Y) \} \). If the restriction \( D|_Y \) is equivalent to a principal divisor, then by the homotopy axiom for locally constant sheaves (see, e.g., [Bre, Ch. II.11]) we have \( \sigma_Y(c_{D,Y}) = \sigma(c_D) \), hence \( c_D \) is trivial. Therefore, there exists a refinement \( \{ V_\gamma \} \) of cover \( \{ U_\alpha \} \) such that for the restrictions of functions \( d_{\alpha \beta} \) to the corresponding \( V_\gamma \) there exists a presentation (9.10.9).

**9.10.2 Proof of Theorem 6.1.13**

1. For the proof of assertion (1) we will need the following notation and results. Let \( U \subset c_\alpha X \) be open.

**Definition 9.10.2.** We denote by \( \mathcal{O}_\sigma(V) \) the multiplicative semigroup of functions \( h : U \to \mathbb{C} \) such that for every subset \( V := \hat{\Pi}(V_0, K) \subset U \), where \( V_0 \subset X_0 \) is open and simply connected, \( K \subset \hat{G}_a \) is open, we have

(i) \( h(\cdot, \omega) \) is in \( \mathcal{O}(V_0) \) for every \( \omega \in K \), and

(ii) \( |h| \in C(V) \).
The next lemma is immediate.

**Lemma 9.10.3.** If \( h \in \mathcal{O}_\sigma(U) \), then \( \iota^* h \in \mathcal{O}(\iota^{-1}(U)) \), \( |\iota^* h| \in C^*_\sigma(\iota^{-1}(U)) \).

Let \( (\mathcal{O}_\sigma)_*(V) \) denote the holomorphic functions in \( \mathcal{O}_\sigma(V) \) that are nowhere zero; this is a multiplicative Abelian group.

**Definition 9.10.4.** A \( \sigma \)-divisor \( H \) is determined by an open cover \( \{U_\alpha\} \) of \( c_aX \) and functions \( f_\alpha \in \mathcal{O}_\sigma(U_\alpha) \) such that \( f_\alpha \not\equiv 0 \) on any open subset of \( U_\alpha \), and \( f_\alpha = d_{\alpha\beta} f_\beta \) on \( U_\alpha \cap U_\beta \neq \emptyset \) for some \( d_{\alpha\beta} \in (\mathcal{O}_\sigma)_*(U_\alpha \cap U_\beta) \), for all \( \alpha, \beta \).

The divisors \( D = \{(U_\alpha, f_\alpha)\}, D' = \{(U'_\beta, f'_\beta)\} \in \text{Div}_\sigma(c_aX) \) are said to be equivalent if there exists a refinement \( \{V_\gamma\} \) of both covers \( \{U_\alpha\} \) and \( \{U'_\beta\} \) and functions \( c_\gamma \in (\mathcal{O}_\sigma)_*(V_\gamma) \) such that \( f_\alpha|_{V_\gamma} = c_\gamma f'_\beta|_{V_\gamma} \) for \( V_\gamma \subset U_\alpha \cap U'_\beta \neq \emptyset \), for all \( \alpha, \beta \).

The multiplicative semigroup of \( \sigma \)-divisor divisors is denoted by \( \text{Div}_\sigma(c_aX) \).

The divisor \( D_h = \{(c_aX, h)\} \in \text{Div}_\sigma(c_aX) \), where \( h \in \mathcal{O}_\sigma(c_aX) \), is called principal.

We have a monomorphism of semigroups

\[
\text{Div}(c_aX) \hookrightarrow \text{Div}_\sigma(c_aX).
\] (9.10.11)

Assertion (1) would follow once we prove

**Theorem 9.10.5.** Under the assumption of Theorem 6.1.13(1), every divisor \( D \in \text{Div}_\sigma(c_aX) \) is equivalent to a divisor \( D' = \{(U_\alpha', f'_\alpha)\} \in \text{Div}_\sigma(c_aX) \) such that

\[
f'_\alpha / f'_\beta|_{\iota(X) \cap U_\alpha \cap U_\beta} \equiv 1
\]

for all \( \alpha, \beta \).

Indeed, given a divisor \( E \in \text{Div}_a(X) \), by Proposition 6.2.13(1) there exists a divisor \( D \in \text{Div}(c_aX) \) such that \( E = \iota^* D \). In view of monomorphism (9.10.11), by Theorem 9.10.5 \( D \) is equivalent to a divisor \( D' \) in \( \text{Div}_\sigma(c_aX) \) whose pullback \( E' = \iota^* D' \) is determined by a function in \( \mathcal{O}_\sigma(X) \) (cf. Lemma 9.10.3), and is equivalent in \( \text{Div}(X) \) to \( E \), so assertion (1) follows.
9.10.3 Proof of Theorem 9.10.5

We will need the following definitions and results.

**Definition 9.10.6.** A divisor \( D = \{ (U_\alpha, f_\alpha) \} \in \text{Div}_\sigma(c_aX) \) is called *cylindrical* if for every \( \alpha \) we have \( U_\alpha = \bar{p}^{-1}(U_{0,\alpha}) \) for some open \( U_{0,\alpha} \subset X_0 \).

**Proposition 9.10.7.** Every divisor \( D = \{ (U_\alpha, f_\alpha) \} \in \text{Div}_\sigma(c_aX) \) is equivalent to a cylindrical divisor in \( \text{Div}_\sigma(c_aX) \).

Let \( Z_d \) be the additive sheaf associated to the presheaf of functions defined similarly to \( O_\sigma(U), U \subset c_aX \) is open, but with condition (ii) omitted, and (i) replaced with the assumption that \( f(\cdot, \omega) \) is identically equal to an integer, for every \( \omega \in K \).

Let \( \mathbb{R}_d := \mathbb{R} \otimes_{\mathbb{Z}} Z_d \). (This is the additive sheaf defined similarly to \( Z_d \), but now instead of the functions with values in \( \mathbb{Z} \) we consider functions with values in \( \mathbb{R} \). )

**Proposition 9.10.8.** Let \( U_0 \subset X_0 \) be open, \( U := \bar{p}^{-1}(U_0) \). Let \( D = \{ (U_\alpha, f_\alpha) \} \in \text{Div}_\sigma(U) \) be a cylindrical divisor such that

(i) the sets \( U_{0,\alpha} \subset U_0 \) are open and simply connected, and

(ii) \( U := \{ U_{0,\alpha} \} \) is an open cover of of \( U_0 \) such that the connected components of the intersection of any two sets in \( U \) are simply connected, and the intersection of any three sets in \( U \) is empty.

Then \( D \) is equivalent to a cylindrical divisor \( D' = \{ (U_\alpha, f'_\alpha) \} \in \text{Div}_\sigma(c_aX) \) such that

\[
\frac{f'_{\alpha}}{f'_{\beta}}|_{(X) \cap U_\alpha \cap U_\beta} \equiv 1
\]

for all \( \alpha, \beta \).

We will also need the following version of Proposition 9.10.8.

**Proposition 9.10.9.** Let \( U_0 \subset X_0 \) be open and simply connected, \( U := \bar{p}^{-1}(U_0) \). Let \( D = \{ (U_\alpha, f_\alpha) \} \in \text{Div}_\sigma(U) \) be a cylindrical divisor such that

(i) the sets \( U_{0,\alpha} \subset U_0 \) are open and simply connected, and
(ii) $U := \{U_{0,\alpha}\}$ is an open cover of $U_0$ such that the connected components of the intersection of any two sets in $U$ are simply connected, and the intersection of any three sets in $U$ is empty.

Then there exists a function $h \in \mathcal{O}_\sigma(U)$ such that $D$ is equivalent to $D_h \in \text{Div}_\sigma(U)$.

**Corollary 9.10.10.** Let $U_0 \subset X_0$ be open and simply connected, $U := \bar{p}^{-1}(U_0)$, let $D = \{(U_\alpha, f_\alpha)\} \in \text{Div}_\sigma(U)$ be a cylindrical divisor. Then $D$ is equivalent to a cylindrical divisor $D' = \{(U_\alpha, f'_\alpha)\} \in \text{Div}_\sigma(c_aX)$ such that $f'_\alpha/f'_\beta|_{(X)_{\cap U_\alpha \cap U_\beta}} \equiv 1$ for all $\alpha, \beta$.

We prove Propositions 9.10.7, 9.10.8 and Corollary 9.10.10 in the next section.

Let us complete the proof of Theorem 9.10.5. Since $X_0$ is a Riemann surface, it admits a strong deformation retract $S_t : X_0 \rightarrow X_0$ ($t \in [0, 1]$) to a 1-dimensional CW-complex $\Gamma \subset X_0$. By definition, $S_0 = \text{Id}$, $S_1(X_0) = \Gamma$. Since $\Gamma$ is locally contractible, there exists an open cover $V = \{V_\alpha\}$ of $\Gamma$ by contractible open sets, such that the intersection of any two sets in $V$ is contractible, and the intersection of any three sets in $V$ is empty. We define $U_{0,\alpha} := S^{-1}_0(V_{0,\alpha})$. Then $U := \{U_{0,\alpha}\}$ forms an open cover of $X_0$ that satisfies conditions of Proposition 9.10.8. The proof now follows by consecutive application of Propostion 9.10.7, Corollary 9.10.10 (to each $U_{0,\alpha}$) and Proposition 9.10.8.

### 9.10.4 Proofs of Propositions 9.10.7–9.10.9 and Corollary 9.10.10

We will need the following notation and results.

Let $(\mathcal{O}_\sigma)_*$ denote the multiplicative sheaf associated to the presheaf of functions $(\mathcal{O}_\sigma)_*(U)$, $U \subset c_aX$ open.

We denote by $\mathcal{O}_{\text{Re}}$ the additive sheaf on $c_aX$ associated to the presheaf of functions that are defined similarly to the functions in $\mathcal{O}_\sigma(U)$, $U \subset c_aX$ open, but now instead of condition $(ii)$ we require that $\text{Re} \ f \in \mathcal{O}(V)$.

The next three statements are obvious.

(a) We have an exact sequence

$$0 \rightarrow \mathbb{Z}_d \xrightarrow{i} \mathcal{O}_{\text{Re}} \xrightarrow{\text{Re}} (\mathcal{O}_\sigma)_* \rightarrow 0,$$

(9.10.12)
where $i$ is the composition of the multiplication by $\sqrt{-1}$ and inclusion.

(b) Using Lemma 9.10.1, below we may assume that $\{U_\alpha\}$ is at most countable, and each $U_\alpha = \hat{\Pi}(U_{0,\alpha},K_\alpha)$ (cf. (6.2.6)), where $U_{0,\alpha} \subset X_0$ is open and simply connected, $K_\alpha \in \mathcal{Q}$ (cf. (6.2.3)). Clearly, $\{U_{0,\alpha}\}$ is an open cover of $X_0$, and $\{K_\alpha\}$ is an open cover of $\hat{G}_a$.

(c) We will need Lemma 9.10.11.

Let $U := \hat{\Pi}(U_0 \times K)$, where $U_0 \subset X_0$ is open and simply connected, and $K \in \mathcal{Q}$. Then $H^k(U,\mathbb{R}_d) = 0$, $k \geq 1$.

Proof of Lemma 9.10.11. We may identify $U$ with $U_0 \times K$ (cf. Section 6.2.6). By Lemma 9.11.4(2) space $U_0 \times K$ is paracompact. Therefore, it suffices to show that, given an at most countable open cover $\{W_l\}$ of $U_0 \times K$, every $k$-cocycle $\sigma = \{\sigma_I \in \Gamma(W_{l_1} \cap \cdots \cap W_{l_k},\mathbb{R}_d) : I = (l_1,\ldots,l_k)\} \in Z^k(\{W_l\},\mathbb{R}_d)$ (cf. Section 9.1.1) can be presented in the form $\sigma = \delta \eta$ for some $k-1$-cochain $\eta \in C^{k-1}(\{W_l\},\mathbb{R}_d)$.

Indeed, let $i_\omega : U_0 \to U_0 \times K$, $i_\omega(x) := (x,\omega)$ ($x \in U_0$, $\omega \in K$) be the natural inclusion. For each $\omega \in K$ we denote $W_{l,\omega} := i_\omega^{-1}(W_l)$ for all $l$ (some of these sets will be empty); this is an open cover of $U_0$. Next, define

$$\sigma_{I,\omega} := i_{\omega}^* \sigma_I$$

for all $I$ (by definition, if $W_{l_1,\omega} \cap \cdots \cap W_{l_k,\omega} = \emptyset$, then $\sigma_{I,\omega} := 0$). Then

$$\sigma_{\omega} := \{\sigma_{I,\omega}\} \in Z^k(\{W_{l,\omega}\},\mathbb{R}).$$

Now, since $U_0$ is contractible, we have $H^k(U_0,\mathbb{R}) = 0$, $k \geq 1$. We may assume without loss of generality that the connected components of any intersection $W_{l_{i,\omega}} \cap \cdots \cap W_{l_{k-1,\omega}}$ are simply connected. Hence, using Leray theorem (see, e.g., [Gun3]) we obtain that for every $\omega \in K$ there exists a $k-1$-cochain $\eta_\omega = \{\eta_{J,\omega} \in \Gamma(W_{l_{i,\omega}} \cap \cdots \cap W_{l_{k-1,\omega}},\mathbb{R}) : J = (l_1,\ldots,l_{k-1})\} \in C^{k-1}(\{W_{l,\omega}\},\mathbb{R})$ such that

$$\sigma_\omega = \delta \eta_\omega,$$

(9.10.13)

where operator $\delta = \delta_{\{W_{l,\omega}\}}$ has the same formal definition as $\delta_{\{W_l\}}$ since there is a bijective correspondence between the elements of covers $\{W_l\}$ and $\{W_{l,\omega}\}$. We define

$$\eta_J(z,\omega) := \eta_{J,\omega}(z)$$
for all $J = (l_1, \ldots, l_{k-1})$, $\omega \in K$ and $z \in W_{l_1, \omega} \cap \cdots \cap W_{l_{k-1}, \omega}$. It is immediate from the definition of sheaf $\mathbb{R}_d$ that $\eta := \{\eta_J\}$ is a $k - 1$-cochain in $C^{k-1}(\{W_i\}, \mathbb{R}_d)$. Further, since (9.10.13) holds for all $\omega \in K$, we obtain that $\sigma = \delta \eta$, as required.

\textbf{Proof of Proposition 9.10.7.} We have to show that $D$ is equivalent to a divisor $H = \{(V_{\beta, h_{\beta}}) \in \text{Div}_\sigma(c_\sigma X), \text{ where } V_{\beta} = \bar{p}^{-1}(V_{0, \beta})$ for some open simply connected sets $V_{0, \beta} \subset X_0$.

Let $x_0 \in X_0$. Since $\bar{p}^{-1}(x_0) \cong \hat{G}_a$ is compact, there exist finitely many sets $U_{\alpha_i} = \hat{\Pi}(U_{0, \alpha_i}, K_{\alpha_i})$ (cf. (b)), $1 \leq i \leq m$, such that $\bar{p}^{-1}(x_0) \subset \cup_{i=1}^{m} U_{\alpha_i}$. Hence, there exists an open simply connected neighbourhood $V_0 \subset \cap_{i=1}^{m} U_{0, \alpha_i}$ of $x_0$ such that $V := \bar{p}^{-1}(V_0) \subset \cup_{i=1}^{m} U_{\alpha_i}$.

We set $V_i := U_{\alpha_i} \cap V$, $K_i := K_{\alpha_i}$, and $f_i := f_{\alpha_i}|_{V_i} \in \mathcal{O}_\sigma(V_i)$ ($1 \leq i \leq m$).

Since $x_0 \in X_0$ was chosen arbitrarily, it suffices to prove that for the restriction $D|_V = \{(V_i, f_i)\}$ there exists a function $h \in \mathcal{O}_\sigma(V)$ such that $D|_V$ is equivalent in $\text{Div}_\sigma(V)$ to the principal divisor $D_h \in \text{Div}_\sigma(V)$. (The definition of $\text{Div}_\sigma(V)$ is completely analogous to the definition of $\text{Div}_\sigma(c_\sigma X)$.) We will need

\textbf{Lemma 9.10.12.} Let $W := \hat{\Pi}(W_0, K)$, where $W_0 \subset X_0$ is open and simply connected, and $K \in \Omega$. Then $H^k(W, \mathbb{Z}_d) = 0$, $k \geq 1$.

The proof is similar to the proof of Lemma 9.10.11.

It follows from Lemma 9.10.12 (with $W := V_i$, $1 \leq i \leq m$) that the exact sequence of cohomology groups induced by the short exact sequence (9.10.12) has form

$$0 \longrightarrow \Gamma(V_i, \mathbb{Z}_d) \longrightarrow \Gamma(V_i, \mathcal{O}_{Re}) \longrightarrow \Gamma(V_i, (\mathcal{O}_\sigma)_*) \longrightarrow 0 \longrightarrow \ldots \quad (9.10.14)$$

($1 \leq i \leq m$). An argument similar to the one in the proof of Lemma 9.10.11 shows that we may replace set $V_i$ in the formulation of the lemma and in (9.10.14) with an intersection of sets $V_i$, $1 \leq i \leq m$. Hence, we have an exact sequence of cochain complexes

$$0 \longrightarrow C^k(\{V_i\}, \mathbb{Z}_d) \longrightarrow C^k(\{V_i\}, \mathcal{O}_{Re}) \longrightarrow C^k(\{V_i\}, (\mathcal{O}_\sigma)_*) \longrightarrow 0, \quad k \geq 0. \quad (9.10.15)$$

The exact sequence (9.10.15) induces an exact sequence of Čech cohomology groups corresponding to cover $\{V_i\}$

$$\ldots \longrightarrow H^1(\{V_i\}, \mathbb{Z}_d) \longrightarrow H^1(\{V_i\}, \mathcal{O}_{Re}) \longrightarrow H^1(\{V_i\}, (\mathcal{O}_\sigma)_*) \longrightarrow H^2(\{V_i\}, \mathbb{Z}_d) \longrightarrow \ldots \quad (9.10.16)$$
An argument similar to the one in the proof of Lemma 9.10.11 yields $H^2(\{V_i\}, \mathbb{Z}_d) = 0$. By definition, functions $f_i$ satisfy $f_i = d_{ij} f_j$ on $V_i \cap V_j \neq \emptyset$ for some $d_{ij} \in (\mathcal{O}_\sigma)_*(V_i \cap V_j)$, and $d_D := \{d_{ij}\}$ is a 1-cocycle. We have to show that there exist $d_i \in (\mathcal{O}_\sigma)_*(V_i)$ such that $d_{ij} = d_j d_i^{-1}$, then we will replace $f_i$ with $f_i d_i$, thus obtaining a globally defined function $h := f_i d_i$ in $V$, as needed.

Since (9.10.16) is exact and $H^2(\{V_i\}, \mathbb{Z}_d) = 0$, there exists a 1-cocycle $\{g_{ij} \in \mathcal{O}_{\text{Re}}(V_i \cap V_j)\}$ such that $d_{ij} = e^{2\pi g_{ij}}$, for all $1 \leq i, j \leq m$.

Let us show that there exists a 0-cochain $\{g_i \in \mathcal{O}_{\text{Re}}(V_i)\}$ such that $g_{ij} = g_j - g_i$ on $V_i \cap V_j \neq \emptyset$, for all $1 \leq i, j \leq m$. Once we find such $g_i$, we will set $d_i = e^{2\pi g_i} \in (\mathcal{O}_\sigma)_*(V_i)$, $1 \leq i \leq m$, thus completing the proof.

We construct functions $g_i$, $1 \leq i \leq m$, as follows. Let $\{\rho_i\}_{i=1}^m \subset C(\hat{\Gamma}_\alpha)$ be a partition of unity subordinate to the open cover $\{K_i\}_{i=1}^m$. Now, we define

$$g_i := \sum_{i,j=1}^m \rho_j g_{ij} \quad \text{on} \quad V_i, \quad 1 \leq i \leq m.$$

Clearly, the 0-cochain $\{g_i\}$ gives a resolution of the 1-cocycle $\{g_{ij}\}$. Under the identification of $V = \hat{p}^{-1}(V_0)$ and $V_0 \times \hat{\Gamma}_\alpha$ (cf. Section 6.2.3), for every $1 \leq i \leq m$ and every $\omega \in \hat{\Gamma}_\alpha$ the function $\rho_i(\cdot, \omega)$ is constant and hence is holomorphic on $V_0$; also, functions $\rho_i$ viewed as functions on $V$, are continuous, hence $\rho_i \in \mathcal{O}(V)$ ($1 \leq i \leq m$). It follows that for every $\omega \in K_i$ we have $g_i(\cdot, \omega) \in \mathcal{O}(V_0)$, and the real part of $g_i$ is continuous on $V$, as functions $\rho_i$ are real-valued, and the real parts of functions $g_{ij}$ are continuous on $V$. Therefore, $g_i \in \mathcal{O}_{\text{Re}}(V_i)$, for all $1 \leq i \leq m$, as required.

**Proof of Proposition 9.10.8.** We will use the notation introduced in the proof of Proposition 9.10.7. By Lemma 9.10.12 (there we take $W := U_\alpha$) the short exact sequence (9.10.12) induces an exact sequence of the form

$$0 \longrightarrow \Gamma(U_\alpha, \mathbb{Z}_d) \longrightarrow \Gamma(U_\alpha, \mathcal{O}_{\text{Re}}) \longrightarrow \Gamma(U_\alpha, (\mathcal{O}_\sigma)_*) \longrightarrow 0 \longrightarrow \ldots, \quad \text{for all} \quad \alpha.$$

Thus, we obtain an exact sequence of chain complexes

$$0 \longrightarrow C^l(U_\alpha, \mathbb{Z}_d) \longrightarrow C^l(U_\alpha, \mathcal{O}_{\text{Re}}) \longrightarrow C^l(U_\alpha, (\mathcal{O}_\sigma)_*) \longrightarrow 0, \quad l \geq 0. \quad (9.10.17)$$

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In turn, sequence (9.10.17) induces an exact sequence of Čech cohomology groups

$$
\ldots \longrightarrow H^1(\{U_\alpha\}, \mathbb{Z}_d) \longrightarrow H^1(\{U_\alpha\}, \mathcal{O}_{\mathcal{R}_0}) \xrightarrow{\partial^2\alpha} H^1(\{U_\alpha\}, (\mathcal{O}_\sigma)_*) \longrightarrow H^2(\{U_\alpha\}, \mathbb{Z}_d) \longrightarrow \ldots
$$

(9.10.18)

We may assume without loss of generality that the connected components of any intersection of sets $U_{0,\alpha}$ are simply connected. Hence, by Leray theorem (see, e.g., [Gun3]) and Lemma 9.10.12 $H^2(\{U_\alpha\}, \mathbb{Z}_d) = 0$. Now, it follows from (9.10.18) that

$$
H^1(\{U_\alpha\}, \mathcal{O}_{\mathcal{R}_0}) \xrightarrow{\partial^2\alpha} H^1(\{U_\alpha\}, (\mathcal{O}_\sigma)_*) \text{ is surjective.}
$$

(9.10.19)

Let $d = \{d_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, (\mathcal{O}_\sigma)_*)$ be the 1-cocycle determined by divisor $D$. Using (9.10.19), we obtain that (possibly after replacing divisor $D$ with an equivalent one) there exists a 1-cocycle $c = \{c_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, \mathcal{O}_{\mathcal{R}_0})$ such that $d_{\alpha\beta} = e^{2\pi c_{\alpha\beta}}$, for all $\alpha$, $\beta$.

Let us show that there exists a 1-cocycle $c' = \{c'_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, \mathcal{O}_{\mathcal{R}_0})$ that is equivalent to $c$, i.e., $c_{\alpha\beta} - c'_{\alpha\beta} = c_{\beta} - c_{\alpha}$ for some 0-cochain $\{c_{\alpha}\} \in C^0(\{U_\alpha\}, \mathcal{O}_{\mathcal{R}_0})$, and is such that

$$
c'_{\alpha\beta} = 0 \quad \text{on } \iota(X) \cap U_\alpha \cap U_\beta
$$

for all $\alpha$, $\beta$. Once we find such a cocycle, we can replace functions $f_\alpha$ determining divisor $D$ by $f'_\alpha := f_\alpha e^{2\pi c_{\alpha}}$, so that $f'_\alpha = d'_{\alpha\beta} f'_\beta$ for all $\alpha$, $\beta$, where $d'_{\alpha\beta} := e^{2\pi c'_{\alpha\beta}}$. Since $d'_{\alpha\beta} = f'_\alpha / f'_\beta \equiv 1$ on $\iota(Y) \cap U_\alpha \cap U_\beta$, the divisor $D' := \{(U_\alpha, f'_\alpha)\}$ is the one required in assertion (1).

To find $\{c'_{\alpha\beta}\}$ we will need the following result.

**Lemma 9.10.13.** There exists a presentation

$$
c = \tilde{c} + ir,
$$

where $\tilde{c} \in Z^1(\{U_\alpha\}, \mathcal{O})$, $r \in Z^1(\{U_\alpha\}, \mathbb{R}_d)$.

**Proof of Lemma 9.10.13.** Let $C_{\alpha\beta}^j$, $1 \leq j \leq t = t(\alpha, \beta)$, denote the connected components of $U_\alpha \cap U_\beta$. These are simply connected open subsets of $U_0$. We can identify $U_\alpha$, $U_\beta$ with $U_{0,\alpha} \times \hat{G}_\alpha$, $U_{0,\alpha} \times \hat{G}_\alpha$, respectively, and $U_\alpha \cap U_\beta$ with $(U_{0,\alpha} \cap U_{0,\beta}) \times \hat{G}_\alpha$ (cf. Section 6.2.6). Let us fix some points $x_{\alpha\beta}^j \in C_{\alpha\beta}^j$, for all $1 \leq j \leq t$. We define

$$
r_{\alpha\beta}|_{C_{\alpha\beta}^j} := \text{Im } c_{\alpha\beta}(x_j, \omega), \quad \omega \in \hat{G}_\alpha.
$$
It follows from (ii) that \( r = \{r_{\alpha\beta}\} \) is a 1-cocycle in \( Z^1(\{U_\alpha\}, \mathbb{R}_d) \).

Next, we define
\[
\tilde{c}_{\alpha\beta} := c_{\alpha\beta} - iv_{k-1,k}.
\]

Let us show that \( \tilde{c}_{\alpha\beta} \in O(U_\alpha \cap U_\beta) \), i.e., that for each \( 1 \leq j \leq t \) we have \( \tilde{c}_{k-1,k} \vert_{C^j_{\alpha\beta}} \in O(C^j_{\alpha\beta} \times \hat{G}_a) \). Indeed, by definition \( \tilde{c}_{\alpha\beta} \vert_{C^j_{\alpha\beta}} \cdot \omega \in O(C^j_{\alpha\beta}) \) for all \( \omega \in \hat{G}_a \). It remains to show that the map
\[
\hat{G}_a \ni \omega \mapsto \tilde{c}_{\alpha\beta} \vert_{C^j_{\alpha\beta}}(\cdot, \omega) \in O(C^j_{\alpha\beta})
\]
is continuous, where space \( C^j_{\alpha\beta} \) is endowed with the topology of uniform convergence on compact subsets of \( C^j_{\alpha\beta} \). By definition \( \omega \mapsto \text{Re} \tilde{c}_{\alpha\beta} \vert_{C^j_{\alpha\beta}}(\cdot, \omega) \) is continuous. Since \( \text{Im} \tilde{c}_{\alpha\beta} \vert_{C^j_{\alpha\beta}}(x, \omega) = 0 \) for all \( \omega \in \hat{G}_a \), the continuity of \( \omega \mapsto \text{Im} \tilde{c}_{\alpha\beta} \vert_{C^j_{\alpha\beta}}(\cdot, \omega) \) follows from the integral formula presentation for conjugate harmonic function and the condition that \( \text{Im} \tilde{c}_{\alpha\beta}(x, \omega) = 0 \) for all \( \omega \in \hat{G}_a \).

Next, we have \( H^1(\{U_\alpha\}, \mathcal{O}) = 0 \) (see, e.g., [ZK]). Therefore, there exists a 0-cochain \( \{\tilde{c}_a\} \in C^0(\{U_\alpha\}, \mathcal{O}) \) such that \( \tilde{c}_{\alpha\beta} = \tilde{c}_{\beta} - \tilde{c}_{\alpha} \).

Further, since \( H^1(X, \mathbb{R}) = 0 \), we obtain (using Leray theorem) that there exist a 0-cochain \( \{r_\alpha\} \in C^0(\{U_\alpha\}, \mathbb{R}_d) \) such that \( r_\alpha = 0 \) on \( U_\alpha \setminus \iota(X) \), and \( r_\beta - r_\alpha = r_{\alpha\beta} \) on \( \iota(X) \cap U_\alpha \cap U_\beta \).

We define \( c_\alpha := \tilde{c}_\alpha + ir_\alpha \), for all \( \alpha \), and set \( c'_{\alpha\beta} := c_{\alpha\beta} - c_\beta + c_\alpha \).

This completes the proof of Proposition 9.10.8.

Proof of Proposition 9.10.9. The proof follows closely the proof of Proposition 9.10.8, except for the last step, where, instead of replacing the 1-cocycle \( c_{\alpha\beta} \) with an equivalent 1-cocycle we resolve it (applying Lemma 9.10.11 to \( \{r_{\alpha\beta}\} \)).

Proof of Corollary 9.10.10. We conduct our proof is two steps.

1. First, assume that

   (i) cover \( \{U_\alpha\} \) of \( U_0 \) is finite, i.e. \( \{U_\alpha\} = \{U_{\alpha_i}\}_{i=1}^m \); we denote \( U_{0,i} := U_{0,\alpha_i}, U_i := U_{\alpha_i} (= \tilde{p}^{-1}(U_{0,\alpha_i})), f_i := f_{\alpha_i}, 1 \leq i \leq m. \)

   (ii) We have ordered sets \( U_i \) in such a way that the sets \( V^k_0 := \bigcup_{i=1}^k U_{0,i}, 1 \leq k \leq m, \) are simply connected, and

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(iii) the connected components of the intersection of \( U_{0,k} \) and \( V^{k-1} \) are simply connected, for all \( 2 \leq k \leq m \).

We define \( V^k := \bar{p}^{-1}(V_{0,k}) \). Clearly, \( V^m = U_0 \).

Next, we proceed by induction over \( k \), and find functions \( h_k \in \mathcal{O}_\sigma(V^k) \), \( 1 \leq k \leq m \), such that \( D|_{V^k} = \{(U_i, f_\beta) : 1 \leq i \leq k\} \) is equivalent to \( D_{h_k} \). Once such functions are found, we can set \( h := h_m \) to complete the proof.

For \( k = 1 \) we take \( h_1 := f_1 \).

Now, suppose that we have constructed \( h_{k-1} \), let us find \( h_k \).

Since \( D|_{V^{k-1}} \) is equivalent to \( D_{h_{k-1}} \), we obtain that \( \tilde{D}_k := \{(V^{k-1}, h_{k-1}), (U_k, f_k)\} \) is a \( \sigma \)-divisor on \( V^k \) equivalent to \( D|_{V^k} \). Since the open cover \( \{V^{k-1}, U_k\} \) of \( V^k \) satisfies the conditions of Proposition 9.10.9, there exists a function \( h_k \in \mathcal{O}_\sigma(V^k) \) such that \( \tilde{D}_k \) is equivalent to \( D_{h_k} \). This completes the proof of case 1.

2. We now consider the case of a general cover \( \{U_\alpha\} \). Let us fix some \( x_0 \in U_0 \). Let \( V_0 \subset U_0 \) be a neighbourhood of \( x_0 \) such that \( \bar{V}_0 \) is contained in some coordinate chart \( \bar{V}_0 \) of \( U_0 \), and let \( \varphi : \bar{V}_0 \rightarrow \mathbb{C} \) be the corresponding biholomorphic coordinate map such that \( \varphi(x_0) = 0 \). We may assume without loss of generality that \( \varphi(\bar{V}_0) = \bar{B}_r = \{z \in \mathbb{C} : |z| \leq r\} \). Furthermore, since \( U_0 \) is simply connected, we can choose \( V_0 \) in such a way that there exists a strong deformation retract \( S_t : U_0 \rightarrow U_0 \) of \( U_0 \) to \( \bar{V}_0 \).

By definition, \( S_0 = \text{Id}, S_1(U_0) = \bar{V}_0 \).

Now, let \( \{A_j\}_{j=0}^m \) be an open cover of \( \bar{B}_r \setminus \{0\} \) by open cones \( A_j \) with vertices at 0. We set

\[
\Omega_{0,j} := (\varphi \circ S_0)^{-1}(A_j), \quad 0 \leq j \leq m.
\]

Then \( \{\Omega_{0,j}\}_{j=0}^m \) is an open cover of \( U_0 \setminus \{x_0\} \).

We define \( W_0' := \bigcup_{j=1}^m \Omega_{0,j} \) (i.e. we exclude \( \Omega_{0,0} \)). This is an open simply connected subset of \( U_0 \). Next, we can choose a simply connected open neighbourhood \( W_0'' \) of \( \Omega_{0,0} \) so that \( W_0' \cap W_0'' \) is simply connected. Then \( \{W_0', W_0''\} \) forms an open cover of \( U_0 \) that satisfies conditions of Proposition 9.10.9 (note that since \( U_0 \) is simply connected, we have \( H^1(\bar{p}^{-1}(U_0), \mathbb{R}_d) = 0 \) by Lemma 9.10.11). Thus, if we can show that \( D \) is equivalent to a cylindrical \( \sigma \)-divisor.
\{(W', h'), (W'', h'')\}, where \(W' := \tilde{p}^{-1}(W_0')\), \(W'' := \tilde{p}^{-1}(W_0'')\), and \(h' \in \mathcal{O}_\sigma(W')\), \(h'' \in \mathcal{O}_\sigma(W'')\), then applying Proposition 9.10.9 we can obtain the required function \(h\).

Indeed, note that the intersection \(\Omega_{0,j} (0 \leq j \leq m)\) with any relatively compact subset of \(U_0\) is relatively compact in \(U_0\). Hence, for every fixed \(0 \leq j_0 \leq m\) there exists an open cover \(\{\Omega_{0,j_0}\}\) that satisfies conditions of Proposition 9.10.9, and is such that each set \(\Omega_{0,j_0}\) is relatively compact in \(U_0\). Hence, using the result of case 1 (clearly, every relatively compact subset of \(U_0\) can be covered by open sets satisfying conditions (i)-(iii)), we may assume without loss of generality that sets \(U_{0,\alpha}\) in the definition of divisor \(D\) are sufficiently large, so that \(\{\Omega_{0,j}\}\) is a refinement of the open cover \(\{U_{0,\alpha}\}\).

Now, for each fixed \(1 \leq j_0 \leq m\) we apply Proposition 9.10.9 to the open cover \(\{\Omega_{0,j_0}\}\) of \(\Omega_{0,j_0}\), thus obtaining a function \(f_{j_0} \in \mathcal{O}_\sigma(\Omega_{j_0})\), where \(\Omega_{j_0} := \tilde{p}^{-1}(\Omega_{0,j_0})\), such that \(D|_{\Omega_{j_0}}\) is equivalent to \(D_{f_{j_0}} \in \text{Div}_\sigma(\Omega_{j_0})\).

The same argument applied to the open cover of \(W_0''\), obtained from the open cover \(\{\Omega_{0,0}\}\) of \(\Omega_0\) by enlarging sets \(\Omega_{0,0}\) slightly, implies that there exists a function \(h'' \in \mathcal{O}_\sigma(W'')\) such that \(D|_{W''}\) is equivalent to \(D_{h_0} \in \text{Div}_\sigma(W'')\).

To obtain function \(h'\), note that the open cover \(\{\Omega_{0,j}\}_{j=1}^m\) of \(W_0'\) itself satisfies conditions of Proposition 9.10.9. Therefore, there exists a function \(h_1 \in \mathcal{O}_\sigma(W')\) such that divisor \(\{(\Omega_j, f_j)\}_{j=1}^m\) is equivalent to \(D_{h_1} \in \text{Div}_\sigma(W')\). The proof is complete.

2. Next, we prove assertion (2). We will use the notation of part 1.

It suffices to show that, for a given open simply connected subset \(U_0 \subset X_0\), the divisor in \(\text{Div}(p^{-1}(U_0))\) determined by the restriction \(f|_{p^{-1}(U_0)}\) is equivalent to a divisor in \(\text{Div}_a(p^{-1}(U_0))\).

Let us recall some results from Section 6.2.1. There exist equivariant maps

\[ j_a : G \to \hat{G}_a, \quad j_{\ell_\infty} : G \to \hat{G}_{\ell_\infty} \]

and a surjection \(\kappa : \hat{G}_{\ell_\infty} \to \hat{G}_a\) (adjoint to the inclusion \(a \hookrightarrow \ell_\infty(G)\)) such that

\[ \kappa \circ j_{\ell_\infty} = j_a. \]

Using the axiom of choice, we can find a right inverse \(\lambda : \hat{G}_a \to \hat{G}_{\ell_\infty}\) to \(\kappa\), i.e., \(\kappa \circ \lambda = \text{Id}\).
In what follows, we identify \( p^{-1}(U_0) \) with \( U := U_0 \times G \), and \( \overline{p}^{-1}(U_0) \) with \( \hat{U}_a := U_0 \times \hat{G}_a \) (cf. Section 6.2.6). We denote \( \hat{U}_{t_\infty} := U_0 \times \hat{G}_{t_\infty} \).

We conduct the rest of the proof in two steps.

1. First, let us show that there exists a function \( F : \hat{U}_a \to \mathbb{C} \) such that \( |F| \in C(\hat{U}_a) \), \( 0 \neq F(\cdot, \omega) \in \mathcal{O}(U_0) \) for all \( \omega \in \hat{G}_a \), and \( f|_U = j^*_a F \).

   Since \( |f|_U \in \mathcal{C}_a(U) \), there is a function \( M_f \in C(\hat{U}_a) \) such that \( |f|_U = j^*_a M_f \). Further, we have \( f|_U \in \mathcal{O}_{t_\infty}(U) \), hence there exists a function \( S_f \in \mathcal{O}(\hat{U}_{t_\infty}) \) such that \( f|_U = j^*_{t_\infty} S_f \).

   Note that \( S_f(\cdot, \eta) \neq 0 \) for all \( \eta \in \hat{G}_{t_\infty} \), for otherwise, since \( S_f \) is continuous, there would exist a set \( \{g_\alpha\} \subset G \), \( j_{t_\infty}(g) \to \eta_0 \in \hat{G}_{t_\infty} \), such that the sequence \( \{S_f(\cdot, j_{t_\infty}(g_\alpha))\} \) would converge to zero uniformly on compact subsets of \( U_0 \), which is not possible by our assumption.

   We define a function \( F : \hat{U}_a \to \mathbb{C} \) by the formula

   \[
   F(x, \omega) := S_f(x, \lambda(\omega)), \quad (x, \omega) \in \hat{U}_a.
   \]

   We have \( |F| = M_f \). Indeed, let \( \tilde{M}_f := (\text{Id} \times \kappa)^* M_f \in C(U_0 \times \hat{G}_{t_\infty}) \). By definition, we have

   \[
   |S_f(x, j_{t_\infty}(g))| = \tilde{M}_f(x, j_{t_\infty}(g)) = M_f(x, j_a(g)), \quad (x, g) \in U.
   \]

   Since \( j_{t_\infty}(G) \) is dense in \( \hat{G}_{t_\infty} \) (cf. Section 6.2.1) and both functions \( |S_f| \) and \( \tilde{M}_f \) are continuous, we obtain that \( |S_f| = \tilde{M}_f \). By definition, we have \( \tilde{M}(x, \lambda(\omega)) = M_f(x, \omega), \quad (x, \omega) \in \hat{U}_a \), hence \( |F| = M_f \). It follows that \( |F| \in C(\hat{U}_a) \), as required.

2. Now, let \( D_F \in \text{Div}_\sigma(\hat{U}_a) \) be the \( \sigma \)-divisor determined by function \( F \), and \( E_f \in \text{Div}(U) \) be the divisor determined by \( f|_U \). By our construction,

   \[
   (\text{Id} \times j_a)^* D_H = E_f.
   \]

   It remains to show that \( D_H \) is equivalent in \( \text{Div}_\sigma(\hat{U}_a) \) to a divisor \( D \in \text{Div}(\hat{U}_a) \); then \( E_f \) would be equivalent in \( \text{Div}(X) \) to \( E := (\text{Id} \times j_a)^* D \in \text{Div}_a(U) \), as needed.

   For a given \( x \in U_0 \) we denote

   \[
   I_x := \{ \omega \in \hat{G}_a : F(x, \omega) \neq 0 \}.
   \]

   Since \( |F| \) is continuous, each set \( I_x \) is open. We have \( \hat{G}_a = \cup_{x \in U_0} I_x \). Indeed, for otherwise there exists \( \omega_0 \in \hat{G}_a \) such that \( F(\cdot, \omega_0) \equiv 0 \), which is a contradiction.
Since \( \hat{G}_a \) is compact, there exist points \( x_i, 1 \leq i \leq m \), such that \( \hat{G}_a = \bigcup_{i=1}^{m} I_{x_i} \).

We fix some \( 1 \leq i_0 \leq m \), and define

\[
F_{i_0}(x, \omega) := F(x, \omega)e^{-\text{Arg}F(x_{i_0}, \omega)}, \quad x \in U_0, \quad \omega \in I_{x_{i_0}}. \tag{9.10.20}
\]

Since \( F(x_{i_0}, \omega) \neq 0 \) for all \( \omega \in I_{x_{i_0}} \), this function is well defined.

By definition, \( F_{i_0}(\cdot, \omega) \in \mathcal{O}(U_0) \) for every \( \omega \in I_{x_{i_0}} \).

**Lemma 9.10.14.** \( F_{i_0} \in C(U_0 \times I_{x_{i_0}}) \).

**Proof of Lemma.** We fix \( \omega_0 \in I_{x_{i_0}} \). Let \( \{\omega_\alpha\} \subset I_{x_{i_0}} \) be a net such that \( \omega_\alpha \to \omega_0 \). Using Montel theorem, it suffices to prove that all partial limits \( \{c_\beta\} \subset \mathcal{O}(U_0) \) of \( \{F_{i_0}(\cdot, \omega_\alpha)\} \subset \mathcal{O}(U_0) \) coincide with \( F_{i_0}(\cdot, \omega_0) \). Indeed, since \( |F_{i_0}| = |F| \) is continuous on \( U_0 \times I_{x_{i_0}} \), we obtain that a partial limit \( c_\beta \) differs from \( F_{i_0}(\cdot, \omega_0) \) by a constant multiple of modulus 1. By (9.10.20) \( F_{i_0}(x_{i_0}, \omega) \in \mathbb{R} \), hence this multiple must be equal to 1, i.e., \( c_\beta = F_{i_0}(\cdot, \omega_0) \) for all \( \beta \).

It follows that \( D := \{ (U_0 \times I_{x_i}, F_i) \}, 1 \leq i \leq m \), is a divisor in \( \text{Div}(\hat{U}_a) \). By our construction, \( D \) is equivalent to \( D_H \) in \( \text{Div}\sigma(\hat{U}_a) \), which completes the proof.

### 9.11 Proof of Theorem 6.3.3

In what follows, all polydisks have finite polyradii.

We prove part (B). In the proof we will use the following results.

**Proposition 9.11.1.** Let \( U := \hat{\Pi}(U_0, K) \), where \( U_0 \subset X_0 \) is open and biholomorphic to an open polydisk in \( \mathbb{C}^n \), and \( K \in \mathcal{O} \) (cf. (6.2.3)).

The following is true:

1. Let \( \mathcal{R} \) be an analytic sheaf over \( U \) having a free resolution of length \( 4N \)

\[
\mathcal{O}^{k_{4N}}|_U \xrightarrow{\varphi_{4N}^{-1}} \ldots \xrightarrow{\varphi_2} \mathcal{O}^{k_2}|_U \xrightarrow{\varphi_1} \mathcal{O}^{k_1}|_U \xrightarrow{\varphi_0} \mathcal{R}|_W \rightarrow 0. \tag{9.11.21}
\]

If \( N \geq n := \dim_{\mathbb{C}} U_0 \), then the induced sequence of sections truncated to \( N \)-th term

\[
\Gamma(U, \mathcal{O}^{k_N}) \xrightarrow{\varphi_N^{-1}} \ldots \xrightarrow{\varphi_2} \Gamma(U, \mathcal{O}^{k_2}) \xrightarrow{\varphi_1} \Gamma(U, \mathcal{O}^{k_1}) \xrightarrow{\varphi_0} \Gamma(U, \mathcal{R}) \rightarrow 0
\]

is exact.
Suppose that free resolution (9.11.21) exists for every \( N \). Then \( H^i(U, \mathcal{R}) = 0, \ i \geq 1 \).

Let \( \mathcal{A} \) be a coherent sheaf on \( c_aX \).

**Proposition 9.11.2.** Every point \( x_0 \in X_0 \) has a neighbourhood \( U_0 \) such that for each \( N \geq 1 \) there exists a free resolution of sheaf \( \mathcal{A} \) over \( \bar{p}^{-1}(U_0) \) having length \( N \) (cf. Definition 6.3.1).

(In other words, we may assume that the open sets \( W \) in Definition 6.3.1 have form \( U = \bar{p}^{-1}(U_0), U_0 \subset X_0 \) is open.)

We prove Propositions 9.11.1 and 9.11.2 in Sections 9.11.2 and 9.11.3, respectively.

Now, let \( \hat{\mathcal{A}} := \bar{p}_* \mathcal{A} \) be the direct image of sheaf \( \mathcal{A} \) under projection \( \bar{p} : c_aX \to X_0 \). By definition, \( \hat{\mathcal{A}} \) is a sheaf of modules over the sheaf of rings \( \mathcal{O}^{C(\hat{G}_a)} \) of germs of holomorphic functions on \( X_0 \) taking values in Banach space \( C(\hat{G}_a) \). By Propositions 9.11.2 and 9.11.1(2) every \( x_0 \in X_0 \) has a basis of neighbourhoods \( U_0 \) such that \( H^i(U, \mathcal{A}) = 0, \ i \geq 1, \ U := \bar{p}^{-1}(U_0) \).

Therefore,

\[
H^i(c_aX, \mathcal{A}) \cong H^i(X_0, \hat{\mathcal{A}}), \quad i \geq 0
\]

(9.11.22)

(see, e.g., [Gun3, Ch. F, Cor. 6]). We have

\[
\Gamma(U, \mathcal{A}) \cong \Gamma(U_0, \hat{\mathcal{A}}), \quad \Gamma(U, \mathcal{O}) \cong \Gamma(U_0, \mathcal{O}^{C(\hat{G}_a)}).
\]

It follows from Proposition 9.11.2 and Proposition 9.11.1(1) that for every \( x_0 \in X_0 \) and each \( N \geq 1 \) there exist a neighbourhood \( U_0 \) of \( x_0 \) and an exact sequence of sections

\[
\Gamma(U_0, (\mathcal{O}^{C(\hat{G}_a)})^{k_N}) \to \cdots \to \Gamma(U_0, (\mathcal{O}^{C(\hat{G}_a)})^{k_1}) \to \Gamma(U_0, \hat{\mathcal{A}}) \to 0.
\]

Then it follows that we have an exact sequence of sheaves

\[
(\mathcal{O}^{C(\hat{G}_a)})^{k_N}|_{U_0} \to \cdots \to (\mathcal{O}^{C(\hat{G}_a)})^{k_1}|_{U_0} \to \hat{\mathcal{A}}|_{U_0} \to 0.
\]

(9.11.23)

For every open set \( U_0 \subset X_0 \) the spaces of sections \( \Gamma(U_0, \hat{\mathcal{A}}), \Gamma(U_0, \mathcal{O}^{C(\hat{G}_a)}) \) can be endowed with Frechet topology, so that the homomorphisms of sections induced by sheaf homomorphisms in (9.11.23) are continuous; indeed, since \( \Gamma(U_0, \hat{\mathcal{A}}) \cong \Gamma(U, \mathcal{A}), \Gamma(U_0, \mathcal{O}^{C(\hat{G}_a)}) \cong \Gamma(U, \mathcal{O}) \), this follows from Proposition 6.3.2 with \( U = \bar{p}^{-1}(U_0) \). Hence, in the terminology of [Lt1] \( \hat{\mathcal{A}} \) is
a Banach coherent analytic Frechet sheaf. Therefore, according to Theorem 2.3(iii) in [Lt1] $H^i(X_0, \hat{A}) = 0$, $i \geq 1$. Isomorphism (9.11.22) now implies the required statement.

(C) Case (1). Due to the argument in the proof of (B), we have isomorphisms of Frechet spaces $\Gamma(c_a X, \mathcal{A}) \cong \Gamma(X_0, \hat{A})$, $\Gamma(\hat{Y}, \mathcal{A}) \cong \Gamma(Y_0, \hat{A})$. Now the result follows from Theorem 2.3(iv) in [Lt1] applied to $\hat{A}$.

Case (2). It suffices to show that the restriction map $\Gamma(p^{-1}(Y_0), \mathcal{A}) \rightarrow \Gamma(\hat{Y}, \mathcal{A})$ has dense image, and then apply the result of case (1).

We have $\hat{Y} = \hat{\Pi}(Y_0, K)$ for some $Y_0 \subseteq X_0$ open simply connected, and $K \in \mathcal{Q}$. Since $\hat{Y} \in \mathcal{B}$, we may use the last assertion of Proposition 6.3.2: it suffices to show that given a section $f \in \Gamma(\hat{Y}, \mathcal{A})$ for every $\varepsilon > 0$ and every $k$ there exists a section $\tilde{f}_k \in \Gamma(p^{-1}(Y_0), \mathcal{A})$ such that $\|f - \tilde{f}_k\|_{V_k} < \varepsilon$.

Without loss of generality we may identify $\hat{Y}$ with $Y_0 \times K$, and $p^{-1}(Y_0)$ with $Y_0 \times \hat{G}_a \hat{G}_a$ (see Section 6.2.6). Then sets $V_k$ have form $V_k = V_{0,k} \times N_k$, where each set $V_{0,k}$ is open and simply connected, and sets $N_k \in \mathcal{Q}$ are such that $N_k \subseteq N_{k+1} \subseteq K$ for all $k$, and $K = \bigcup_k N_k$ (see Lemma 9.11.4(1) below). Since space $\hat{G}_a$ is compact and, therefore, is normal, for each $k$ there exists a function $\rho_k \in C(\hat{G}_a)$ such that $0 \leq \rho_k \leq 1$ on $\hat{G}_a$, $\rho_k \equiv 1$ on $N_k$, and $\rho_k \equiv 0$ on $\hat{G}_a \setminus \hat{G}_a$. By definition, $\Gamma(Y_0 \times K, \mathcal{A})$ is a module over $\Gamma(Y_0 \times K, \mathcal{O})$, hence we can define $\tilde{f}_k := \rho_k f \in \Gamma(Y_0 \times \hat{G}_a, \mathcal{A})$. Then $f - \tilde{f}_k = 0$ on $Y_0 \times N_k$, so $\|f - \tilde{f}_k\|_{V_k} = 0$. Thus, $\tilde{f}_k$ is the required approximation.

(A) Let $N \geq n$. Since sheaf $\mathcal{A}$ is coherent, there exists a neighbourhood $U$ of $x$ over which there is a free resolution

$$
\mathcal{O}^{m_{1}N}|_U \xrightarrow{\varphi_4 N^{-1}} \cdots \xrightarrow{\varphi_2} \mathcal{O}^{m_2}|_U \xrightarrow{\varphi_1} \mathcal{O}^{m_1}|_U \xrightarrow{\varphi_0} \mathcal{A}|_U \rightarrow 0 \quad (9.11.24)
$$

of length $4N$. It follows from the exactness of sequence (9.11.24) that there exist sections $h_1, \ldots, h_{m_1} \in \Gamma(U, \mathcal{A})$ that generate $x\mathcal{A}$ as an $x\mathcal{O}$-module. Now, it suffices to show that there exist a neighbourhood $V \subset U$ of $x$, global sections $f_1, \ldots, f_{m_1} \in \Gamma(c_a X, \mathcal{A})$ and functions $r_{ij} \in \mathcal{O}(V)$, $1 \leq i,j \leq m_1$, such that

$$
h_i|_V = \sum_{j=1}^{m_1} r_{ij} f_j|_V, \quad 1 \leq i \leq m_1. \quad (9.11.25)
$$
Without loss of generality, we may assume that \( U = \hat{\Pi}(U_0, K) \in \mathfrak{B} \), where \( U_0 \subset X_0 \) is biholomorphic to an open polydisk in \( \mathbb{C}^n \) and is holomorphically convex in \( X_0 \), and \( K \in \mathfrak{Q} \). By Proposition 6.3.2 the topology on \( \Gamma(W, A) \) is determined by semi-norms

\[
\|h\|_V := \inf \left\{ \sup_{x \in V_k} |g(x)| : g \in \Gamma(V_k, \mathcal{O}^{m_1}), \ h = \varphi_0^* (g) \right\},
\]

where \( \varphi_0^* \) is the homomorphism of sections induced by \( \varphi_0 \) in (9.11.24), and open sets \( V_k \in \mathfrak{B} \) are such that \( V_k \subset V_{k+1} \subset W \) for all \( k \), and \( W = \bigcup_k V_k \), cf. Lemma 9.11.4(2) below; by definition, \( V_k = V_{0,k} \times N_k \), where \( V_{0,k} \subset U_0, N_k \subset K \) are open. Without loss of generality, we may assume that each set \( V_{0,k} \) is biholomorphic to an open polydisk in \( \mathbb{C}^n \) and is holomorphically convex in \( X_0 \).

Let \( V := V_{k_0} \), where \( k_0 \) is chosen so that \( x \in V_{k_0} \). It follows from (C) (case (2), for \( \hat{Y} := U \)) that for every \( \varepsilon > 0 \) there exist sections \( f_1, \ldots, f_{m_1} \in \Gamma(c_n X, A) \) such that \( \|h_i - f_i\|_V < \varepsilon \). Now, by Proposition 9.11.1(1) the sequence of sections corresponding to (9.11.24)

\[
\ldots \longrightarrow \Gamma(V, \mathcal{O}^{m_1}) \xrightarrow{\varphi_0^*} \Gamma(V, A) \longrightarrow 0
\]

is exact. Note that \( \Gamma(V, \mathcal{O}^{m_1}) \) consists of \( m_1 \)-tuples of holomorphic functions on \( V \). Let \( \tilde{h}_i := (0, \ldots, 1, \ldots, 0) \) (1 is in the \( i \)-th position), \( 1 \leq i \leq m_1 \). Without loss of generality we may assume that \( h_i|_V = \varphi_0^*(\tilde{h}_i) \). Since \( \varphi_0^* \) is surjective, there exist functions \( \tilde{f}_i \in \Gamma(V, \mathcal{O}^{m_1}) \) such that \( \varphi_0^*(\tilde{f}_i) = f_i|_V \). It follows from the definition of semi-norm \( \|\cdot\|_V \), cf. (9.11.26), that functions \( \tilde{f}_i \) can be chosen in such a way that

\[
\sup_{x \in V} |\tilde{h}_i(x) - \tilde{f}_i(x)| < 2\varepsilon.
\]

Since \( \varphi_0^* \) is a \( \mathcal{O}(V) \)-module homomorphism, the required identity (9.11.25) would follow once we find functions \( r_{ij} \in \Gamma(V, \mathcal{O}), 1 \leq i, j \leq m_1, \) such that

\[
\tilde{h}_i = \sum_{j=1}^{m_1} r_{ij} \tilde{f}_j, \quad 1 \leq i \leq m_1.
\]

The latter system of linear equations (with respect to \( r_{ij} \)) can be rewritten as a matrix equation \( H = FR \) with respect to \( R = (r_{ij})_{i,j=1}^{m_1} \in \mathcal{O}(V, M_n(\mathbb{C})) \), where \( M_n(\mathbb{C}) \) denotes the set of \( n \times n \) complex matrices, \( H = (\tilde{h}_i)_{i=1}^{m_1} \in \mathcal{O}(V, GL_n(\mathbb{C})) \) (\( \tilde{h}_i \) are the columns of \( H \)) is the
identity matrix, where $GL_n(\mathbb{C}) \subset M_n(\mathbb{C})$ is the group of invertible matrices, and $F = (\tilde{f}_i)_{i=1}^m \in \mathcal{O}(V, M_n(\mathbb{C}))$ ($\tilde{f}_i$ are the columns of $F$). Since \( \varepsilon > 0 \) can be chosen arbitrarily small, in view of (9.11.28) we may assume that $F \in \mathcal{O}(V, GL_n(\mathbb{C}))$. Hence, we can define $R := F^{-1}H$. This completes the proof of (A).

9.11.1 Auxiliary topological results

For the proofs of Propositions 9.11.1 and 9.11.2 we will need the following results.

Let $L = \{L_i\}$ be an open cover of $\hat{G}_a$. We define a refinement of $L$ to be an open cover $L' = \{L'_j\}$ of $\hat{G}_a$ such that each $L'_j \subset L_i$ for some $i = i(j)$.

Note that since $\hat{G}_a$ is compact, each open cover of $\hat{G}_a$ has a finite subcover.

**Lemma 9.11.3.** Let $L$ be a finite open cover of $\hat{G}_a$. There exist finite refinements $L^k = \{L_j^k : L_j^k \in \Omega\}$ of $L$ having the same cardinality and such that $L_j^{k+1} \subset L_j^k$ for all $j, k$.

**Proof of Lemma 9.11.3.** Since $\hat{G}_a$ is compact, there exists a finite refinement $L' = \{L'_j\}$ of $L = \{L_i\}$ such that every $L'_j \subset L_i$ for some $i = i(j)$, and functions $\{\rho_j\} \subset C(\hat{G}_a)$ such that $\rho_j \equiv 1$ on $L'_j$, $\rho_j \equiv 0$ on $\hat{G}_a \setminus L_i$. We set $L_j^k := \{\eta \in \hat{G}_a : \rho_j(\eta) > 1 - \frac{1}{2k}\}$, $k \geq 1$. By definition, $L_j^k \in \Omega$ for all $j, k$ (cf. (6.2.3)). It follows that $L^k := \{L_j^k\}$ are the required refinements of $L$. \hfill \Box

**Lemma 9.11.4.** Let $K \in \Omega$, $U_0 \subset X_0$ be open, set $U := U_0 \times K$. The following is true:

1. There exist open subsets $N_k \in \Omega$, $1 \leq k < \infty$, such that $N_k \subset N_{k+1} \subset K$ for all $k$, and $K = \cup_k N_k$.

2. There are open subsets $V_k = V_{0,k} \times N_k$, $1 \leq k < \infty$ such that $V_k \subset V_{k+1} \subset U$ for all $k$, and $U = \cup_k V_k$. Here $V_{0,k} \subset U_0$ is open, and $N_k \in \Omega$, for all $k$.

3. Let $L \in \Omega$ be such that $L \subset K$. Then there exists a collection of sets $L^m \in \Omega$, $m \geq 1$, such that $L \subset \cdots \subset L^{m+1} \subset L^m \subset \cdots \subset L^1 \subset K$ for all $m$. 115
(4) Let \( N \subseteq K \), and \( \{L_i\} \) be a finite collection of open subsets of \( K \) such that \( N \subseteq \cup_i L_i \).

Then there exists a finite number of open subsets \( L'_j \subseteq K \), \( L'_j \in \Omega \), such that \( N \subseteq \cup_j L'_j \), and for each \( j \) we have \( L'_j \subseteq L_i \) for some \( i = i(j) \).

Proof. (1) Recall that the basis \( \Omega \) of topology of \( \hat{G}_a \) consists of sublevel sets of functions in \( C(\hat{G}_a) \), cf. (6.2.3), so \( K = \{ \eta \in \hat{G}_a : \max_{1 \leq i \leq m} |h_i(\eta)) - h_i(\eta_0)| < \varepsilon \} \) for some \( \eta_0 \in \hat{G}_a \), \( h_1, \ldots, h_m \in C(\hat{G}_a) \) and \( \varepsilon > 0 \). Let \( \mathfrak{a}' \) be the subalgebra of \( C(\hat{G}_a) \) generated by functions \( h_1, \ldots, h_m, \bar{h}_1, \ldots, \bar{h}_m \). Since algebra \( \mathfrak{a}' \) is finitely generated, the maximal ideal space \( M_{\mathfrak{a}'} \) of \( \mathfrak{a}' \) is a compact subset of some \( \mathbb{C}^p \), and we have \( \mathfrak{a}' \cong C(M_{\mathfrak{a}'}). \) The map \( \pi : \hat{G}_a \to M_{\mathfrak{a}'} \) adjoint to inclusion \( \mathfrak{a}' \subseteq C(\hat{G}_a) \) is proper and surjective. By definition, there exists an open subset \( K' \subseteq M_{\mathfrak{a}'} \) such that \( K = \pi^{-1}(K') \). Since \( M_{\mathfrak{a}'} \) is a compact metric space (as a compact subset of \( \mathbb{C}^p \)), there exist open subsets \( N'_k \subseteq M_{\mathfrak{a}'} \) such that \( N'_{k-1} \subseteq N'_k \subseteq K' \) for all \( k \), and \( K' = \cup_k N'_k \).

We define \( N_k := \pi^{-1}(N'_{k}) \subseteq \Omega \). Clearly, each sets \( N'_k \) can be chosen to be a set of the form \( N'_k = \{ y \in M_{\mathfrak{a}'} : \max_{1 \leq i \leq r_k} |f_{ik}(y) - f_{ik}(y_0)| < \varepsilon \} \) for some \( y_0 \in M_{\mathfrak{a}'} \), \( f_{ik} \in C(M_{\mathfrak{a}'} \) and \( \varepsilon > 0 \). Since \( \pi^*C(M_{\mathfrak{a}'}) \subseteq C(\hat{G}_a) \), we have \( N_k \in \Omega \) (cf. (6.2.3)).

A similar argument yields (3).

(2) It is clear that there exists a sequence of open sets \( V_{0,k} \) such that \( V_{0,k} \subseteq V_{0,k+1} \subseteq U_0 \) for all \( k \), and \( U_0 = \cup_k V_{0,k} \). We set \( V_k := V_{0,k} \times N_k \).

(4) We apply Lemma 9.11.3 to the finite open cover of \( \hat{G}_a \) consisting of the sets \( L_i \) and set \( \hat{G}_a \setminus \bar{N} \), to obtain a finite refinement \( \{L'_j\} \subseteq \Omega \) of this cover. We exclude subsets \( L'_j \) such that \( L'_j \subseteq \hat{G}_a \setminus \bar{N} \). Then \( \bar{N} \subseteq \cup_j L'_j \) and by definition of the refinement for each \( j \) we have \( L'_j \subseteq L_i \) for some \( i \), as required.

\[ \square \]

9.11.2 Proof of Proposition 9.11.1

Let \( U_0 \subseteq \mathbb{C}^n \) be an open polydisk, \( K \in \Omega \) (cf. (6.2.3)).

The sets \( U_0 \times K \) and \( \hat{\Pi}(U_0, K) \subseteq c_a \mathbb{X} \) are biholomorphic (cf. Section 6.2.6). The definitions of analytic homomorphism and free resolution (of an analytic sheaf over an open subset of \( c_a \mathbb{X} \), cf. Section 6.3) are transferred naturally to analytic sheaves over \( U_0 \times K \). Thus, it suffices to prove Proposition 9.11.1 in the assumption that analytic sheaf \( \mathcal{R} \) and free resolution (9.11.21) are given over \( U_0 \times K \).
We set $U := U_0 \times K$.

A function $f \in C(U)$ is said to be $C^\infty$-smooth if all its derivatives with respect to variable $x \in U_0$ (in some local coordinates on $U_0$) are in $C(U)$. The algebra of $C^\infty$-smooth function on $U$ will be denoted by $C^\infty(U)$.

Let $\Lambda^{p,q}(U_0)$ be the collection of all $C^\infty$-smooth $(p,q)$-forms on $U_0$. We define the space $\Lambda^{p,q}(U)$ of $C^\infty$-smooth $(p,q)$-forms on $U$ by the formula $\Lambda^{p,q}(U) := C^\infty(U_0) \otimes \Lambda^{p,q}(U_0)$. We have operator $\bar{\partial} : \Lambda^{p,q}(U) \to \Lambda^{p,q+1}(U)$, defined as follows: suppose that $\omega \in \Lambda^{p,q}(U)$ is given (in local coordinates on $U_0$) by the formula

$$\omega = \sum_{|I|=p} \sum_{|J|=q} f_{IJ} dz_I \wedge d\bar{z}_J, \quad f_{IJ} \in C^\infty(U),$$

where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$; then

$$\bar{\partial}f := \sum_{|I|=p} \sum_{|J|=q} \bar{\partial}f_{IJ} \wedge dz_I \wedge d\bar{z}_J,$$  \hspace{1cm} (9.11.29)

where

$$\bar{\partial}f_{IJ}(z, \eta) := \sum_{J=1}^n \frac{\partial f_{IJ}(z, \xi)}{\partial \bar{z}_j} d\bar{z}_j, \quad z = (z_1, \ldots, z_n), \quad (z, \xi) \in U = U_0 \times K.$$

A form $\omega \in \Lambda^{p,q}(U)$ is called $\bar{\partial}$-closed if $\bar{\partial}\omega = 0$.

Let $\Lambda^{p,q}$ be the sheaf of germs of $C^\infty$-smooth $(p,q)$-forms on $U$, and $Z^{p,q} \subset \Lambda^{p,q}$ be the subsheaf of germs of $\bar{\partial}$-closed $(p,q)$-forms. Note that $Z^{0,0} = \mathcal{O}$.

Notation. We fix an open polydisk $V_0 \Subset U_0$.

Let $W_0 \subset \tilde{V}_0$ be open in $\tilde{V}_0$ and such that $W_0 = \tilde{V}_0 \cap \tilde{W}_0$ for some product domain $\tilde{W}_0 = \tilde{W}_0^1 \times \cdots \times \tilde{W}_0^n \subset U_0$, where each $\tilde{W}_0^i \Subset \mathbb{C}$ ($1 \leq i \leq n$) is simply connected and has smooth boundary (clearly, given any open neighbourhood of $\tilde{W}_0$ in $U_0$, we can find such a set $\tilde{W}_0$ contained in this neighbourhood).

Fix a subset $W'_0 \subset W_0$ open in $\tilde{V}_0$ and satisfying the same intersection condition as $W_0$.

Let $S \subset K$ be a closed subset, and let $L' \Subset L \subset S$ be open in $S$.

**Lemma 9.11.5.** For every $\omega \in \Gamma(W_0 \times L, Z^{0,q})$ there exists $\eta \in \Gamma(W'_0 \times L', \Lambda^{0,q-1})$ such that $\bar{\partial}\eta = \omega$. 

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Proof. By definition, a section of sheaf $Z^{\alpha,q}$ over $W_0 \times L$ is the restriction of a section of $Z^{\alpha,q}$ over some open neighbourhood of $W_0 \times L$. Therefore, we may assume that $L$ is open in $K$, and $\omega \in \Gamma(\tilde{W}_0 \times L, Z^{\alpha,q})$ for some product domain $\tilde{W}_0$ as above.

Clearly, there exists a product domain $W_0 \in \tilde{W}_0$ open in $U_0$, where $\tilde{W}_0 = \tilde{W}_0^1 \times \cdots \times \tilde{W}_0^n$ and each domain $\tilde{W}_0^i \in \tilde{W}_0^i$ has smooth boundary, such that $W_0' \in W_0$. Further, since $G_a$ is a normal space, there exists an open set $L'' \in L$ such that $L' \subseteq L''$.

Let $C(\tilde{L}'')$ be the Banach space of continuous functions on $\tilde{L}''$ endowed with sup-norm, $\Lambda^{0,q}(\tilde{W}_0, C(\tilde{L}''))$ be the space of $C^\infty$-smooth $C(\tilde{L}'')$–valued $(0, q)$-forms on $\tilde{W}_0$, and $Z^{0,q}(\tilde{W}_0, C(\tilde{L}'')) \subseteq \Lambda^{0,q}(\tilde{W}_0, C(\tilde{L}''))$ be the subspace of $\bar{\partial}C(\tilde{L}'')$-closed forms on $\tilde{W}_0$. Here

$$\bar{\partial}C(\tilde{L}'') : \Lambda^{0,q}(\tilde{W}_0, C(\tilde{L}'')) \to Z^{0,q+1}(\tilde{W}_0, C(\tilde{L}''))$$

is the usual operator of differentiation of $C(\tilde{L}'')$-valued forms.

It is easy to see that every form in $\Gamma(\tilde{W}_0 \times L, \Lambda^{0,q})$ defines a unique form in $\Lambda^{0,q}(\tilde{W}_0, C(\tilde{L}''))$ and, since $\tilde{W}_0 \times \tilde{L}''$ is a neighbourhood of $\tilde{W}_0' \times \tilde{L}'$, every form in $\Lambda^{0,q}(\tilde{W}_0, C(\tilde{L}''))$ determines a unique form in $\Gamma(\tilde{W}_0' \times \tilde{L}', \Lambda^{0,q})$; these maps commute with the actions of operators $\bar{\partial}$ and $\bar{\partial}C(\tilde{L}'')$. In particular, form $\omega$ determines a form $\hat{\omega} \in Z^{0,q}(\tilde{W}_0, C(\tilde{L}''))$. Note that since $\tilde{W}_0 \subseteq \mathbb{C}^n$ is a product domain, it is pseudoconvex. Hence $W_0$ admits an exhaustion by strictly pseudoconvex subdomains (see, e.g., [Kra]). Therefore, there exists a strictly pseudoconvex domain $D_0 \subseteq \tilde{W}_0$ such that $\tilde{W}_0 \subseteq D_0$. We restrict form $\hat{\omega}$ to $D_0$ (clearly, $\hat{\omega}|_{D_0}$ is bounded), and apply Lemma 9.1.1, where we take $B := C(\tilde{L}'')$. We obtain that there exists a form $\hat{\eta} \in \Lambda^{0,q-1}(\tilde{W}_0, C(\tilde{L}''))$ such that $\bar{\partial}C(\tilde{L}'')\hat{\eta} = \hat{\omega}$ over $\tilde{W}_0$. It follows that form $\eta \in \Gamma(\tilde{W}_0' \times \tilde{L}', \Lambda^{0,q-1})$ determined by $\hat{\eta}$ is the required one. \hfill $\Box$

**Definition 9.11.6.** We say that a finite open cover $\mathcal{U} = \{U_\alpha\}$ of $\tilde{V}_0 \times S$ is of class $(P)$ if the following conditions are satisfied:

1. $U_\alpha = U_{0,l} \times L_j$, $\alpha = (l, j)$, where $\{U_{0,l}\}$ and $\{L_j\}$ are finite open covers of, respectively, $\tilde{V}_0$ and $S$, for all $\alpha$;

2. $L_j = S \cap \tilde{L}_j$ for some $\tilde{L}_j \in \Omega$ such that $\tilde{L}_j \subseteq K$, for all $j$;
(3) $U_{0,l} = \bar{V}_0 \cap \tilde{U}_{0,l}$ for some product domain $\tilde{U}_{0,l} = \tilde{U}_{0,l}^1 \times \cdots \times \tilde{U}_{0,l}^n \subseteq U_0$, where each domain $\tilde{U}_{0,l}^i \subseteq \mathbb{C}$ ($1 \leq i \leq n$) is simply connected and has smooth boundary, for all $l$.

**Lemma 9.11.7.**  (1) Each open cover of $\bar{V}_0 \times S$ has a refinement of class $(P)$.

(2) Each open cover of $\bar{V}_0 \times S$ of class $(P)$ has a refinement of class $(P)$ having the same cardinality.

**Proof.** (1) There exists a refinement of a given open cover of $\bar{V}_0 \times S$ by open sets of the form $U_{0,l} \times M_j$, $\alpha = (l,j)$, where $\{U_{0,l}\}$ and $\{M_j\}$ are finite open covers of, respectively, $\bar{V}_0$ and $S$. By the definition of induced topology on $S$, there exist open sets $\tilde{M}_i \subset K$ such that $M_i = S \cap \tilde{M}_i$. Now, we apply Lemma 9.11.4(4) to $\{\tilde{M}_i\}$ (there we take $\tilde{N} := S$) to obtain open sets $\{\tilde{L}_j\}$ such that $L_j \subseteq L_i$ for some $i = i(j)$ and $\tilde{L}_j \in \Omega$, for all $j$. Finally, we set $L_j := S \cap \tilde{L}_j$. The sets $U_{0,l} \times L_j$ form the required refinement of class $(P)$.

(2) Follows from assertions (3) and (4) of Lemma 9.11.4. \qed

Let $\mathcal{U} = \{U_\alpha := U_{0,l} \times L_j\}$ be a finite open cover of $\bar{V}_0 \times S$ of class $(P)$, and $\mathcal{U}' = \{U'_\alpha := U'_{0,l} \times L'_j\}$ be a refinement of $\mathcal{U}$ of class $(P)$ having the same cardinality (cf. Lemma 9.11.7(2)). By definition, $\{U'_\alpha\}$, $\{L'_j\}$ are refinements of open covers $\{U_{0,l}\}$ and $\{L_j\}$, respectively.

We have a refinement map $\iota_{\mathcal{U},\mathcal{U}'} : \mathcal{Z}(\mathcal{U}, \mathcal{R}) \rightarrow \mathcal{Z}(\mathcal{U}', \mathcal{R})$ (see Section 9.1.1 for notation). If no confusion arises, for a given $\sigma \in \mathcal{Z}(\mathcal{U}, \mathcal{R})$ we denote the image $\iota_{\mathcal{U},\mathcal{U}'}(\sigma)$ again by $\sigma$.

**Lemma 9.11.8.** The following is true:

(1) Let $\sigma \in \mathcal{Z}(\mathcal{U}, \mathcal{O})$, $i \geq 1$. Then $\sigma \in \mathcal{B}(\mathcal{U}', \mathcal{O})$.

(2) $H^i(\bar{V}_0 \times S, \mathcal{O}) = 0$, $i \geq 1$.

**Proof.** (1) We will prove a more general result: if $\sigma \in \mathcal{Z}(\mathcal{U}, \mathcal{Z}^{0,q})$, $i \geq 1$, $q \geq 0$, then $\sigma \in \mathcal{B}(\mathcal{U}', \mathcal{Z}^{0,q})$. In particular, taking $q = 0$ we obtain assertion (1).

Let $i = 1$, $\sigma_1 \in \mathcal{Z}(\mathcal{U}, \mathcal{Z}^{0,q})$. Since $\bar{V}_0 \times S$ is a paracompact space, there exist partitions of unity $\{\lambda_l\}$ and $\{\rho_j\}$ subordinate to covers $\{U'_{0,l}\}$ and $\{L'_j\}$ ($C^\infty$-smooth and continuous, respectively). We define a 0-cocycle $\sigma_0^\infty \in \mathcal{C}^0(\mathcal{U}', \Lambda^{0,q})$ by the formula

$$
(\sigma_0^\infty)_\alpha(x, \xi) := \sum_{\beta = (l,j)} \rho_j(\xi) \lambda_l(x)(\sigma_1)_{\beta,\alpha}(x, \xi), \quad (x, \xi) \in U'_\alpha, \quad \text{for all } \alpha.
$$

(9.11.30)
Since \((\sigma_1)_{\alpha,\beta} = (\delta\sigma_0^\infty)_{\alpha,\beta} = (\sigma_0^\infty)_\alpha - (\sigma_0^\infty)_\beta\), and \(\partial(\sigma_1)_{\alpha,\beta} = 0\), it follows that \(\omega := \partial(\sigma_0^\infty)_\alpha\) on \(U'_\alpha\), for all \(\alpha\), determines a section in \(\Gamma(\bar{V}_0 \times S, Z^{0,q+1})\) such that \(\partial\omega = 0\). By Lemma 9.11.5 (there we take \(W'_0 = W_0 = \bar{V}_0\), and \(L' = L = S\)) there exists \(\eta \in \Gamma(\bar{V}_0 \times S, \Lambda^{0,q})\) such that \(\partial\eta = \omega\). We define a 0-cochain \(\sigma_0 \in C^0(\mathcal{U}, Z^{0,q})\) by the formula \((\sigma_0)_\alpha = (\sigma_0^\infty)_\alpha - \eta\). It follows that \(\sigma_1 = \delta\sigma_0\), therefore \(\sigma_1 \in \mathcal{B}^1(\mathcal{U}, Z^{0,q})\).

Using Lemma 9.11.7(2) we may assume that there exists a refinement \(\mathcal{U}'' = \{U''_\alpha := U''_{0,l} \times L''_j\}\) of cover \(\mathcal{U}\) of class \((P)\), having the same cardinality as \(\mathcal{U}\), and such that \(\mathcal{U}'\) is a refinement of \(\mathcal{U}''\).

Now, let \(i > 1\), assume that we have shown for all \(1 \leq l < i\), \(q \geq 0\), that each \(\sigma \in \mathcal{Z}^1(\mathcal{U}, Z^{0,q})\) belongs to \(\mathcal{B}^1(\mathcal{U}'', Z^{0,q})\). For a given \(\sigma_i \in \mathcal{Z}^i(\mathcal{U}, Z^{0,q})\) we define an \(i - 1\)-cocycle \(\sigma_i^{\infty} \in C^{i-1}(\mathcal{U}'', \Lambda^{0,q})\) by the formula

\[
(\sigma_i^{\infty})_{\alpha_1,\ldots,\alpha_i}(x, \xi) := \sum_{\beta=(\ell,j)} \rho_j(\xi)\lambda_i(x)(\sigma_i)_{\beta,\alpha_1,\ldots,\alpha_i}(x, \xi), \quad (x, \xi) \in U''_{\alpha_1,\ldots,\alpha_i}
\]

for all \(\alpha_1,\ldots,\alpha_i\), where \(U''_{\alpha_1,\ldots,\alpha_i} := \cap_{r=1}^i U''_{\alpha_r} \neq \emptyset\). We have \(\delta(\sigma_i^{\infty}) = \sigma_i\), so \(\partial\delta(\sigma_i^{\infty}) = \delta(\partial\sigma_i^{\infty}) = 0\). Define \(\mu_{i-1} := \partial\sigma_i^{\infty} \in C^{i-1}(\mathcal{U}'', Z^{0,q+1})\). Since \(\delta(\mu_{i-1}) = \partial\mu_{i-1} = 0\), by the induction assumption there exists an \(i - 2\)-cochain \(\mu_{i-2} \in C^{i-2}(\mathcal{U}'', Z^{0,q})\) such that \(\delta(\mu_{i-2}) = \mu_{i-2}\) and \(\partial\mu_{i-2} = 0\). Now, by Lemma 9.11.5(1) there exists an \(i - 2\)-cochain \(\eta_{i-2} \in C^{i-2}(\mathcal{U}', \Lambda^{0,q})\) such that \(\partial\eta_{i-2} = \mu_{i-2}\). We define \(\sigma_{i-1} := \sigma_i^{\infty} - \delta(\eta_{i-2})\). Then \(\delta(\sigma_{i-1}) = \sigma_i\), so \(\sigma_i \in \mathcal{B}^i(\mathcal{U}', Z^{0,q})\).

(2) By Lemma 9.11.7(1) any open cover of \(\bar{V}_0 \times S\) has a finite refinement of class \((P)\), hence the required result follows from (1). 

Let \(\{V_k\}_{k=1}^{\infty}\) be the exhaustion of \(\mathcal{U}\) by open sets obtained in Lemma 9.11.4(2). By definition, each set \(V_k\) has form \(V_k = V_{0,k} \times N_k\), where \(V_{0,k} \subset U_0\), \(N_k \subset K\) are open, and \(N_k \in \Omega\), for all \(k\). Since \(U_0\) is an open polydisk in \(\mathbb{C}^n\), we may choose \(V_{0,k}\) also to be an open polydisk, for all \(k\).

**Definition 9.11.9** (cf. [GrR]). We say that an analytic sheaf \(\mathcal{R}\) on \(\mathcal{U}\) satisfies the **Runge condition** if the following holds for every \(k \geq 1\):

(a) The space of sections \(\Gamma(\bar{V}_k, \mathcal{R})\) is endowed with a semi-norm \(|\cdot|_k\) such that \(\Gamma(\mathcal{U}, \mathcal{R})|_{\bar{V}_k}\) is dense in \(\Gamma(\bar{V}_k, \mathcal{R})\).
(b) There exist constants $M_k > 0$ such that for every $f \in \Gamma(\tilde{V}_{k+1}, \mathcal{R})$ we have $|f|_{\tilde{V}_k} \leq M_k |f|_{k+1}$.

(c) If $\{f_j\}$ is a Cauchy sequence in $\Gamma(\tilde{V}_{k+1}, \mathcal{R})$, then $\{f_j|_{\tilde{V}_k}\}$ has a limit in $\Gamma(\tilde{V}_k, \mathcal{R})$.

(d) If $f \in \Gamma(\tilde{V}_{k+1}, \mathcal{R})$ and $|f|_{k+1} = 0$, then $f|_{\tilde{V}_k} = 0$.

Lemma 9.11.10 ([GrR]). Let $\mathcal{R}$ be an analytic sheaf on $U$. The following is true:

1. Suppose that $H^i(\tilde{V}_k, \mathcal{R}) = 0$ for all $i \geq 1$, $k \geq 1$. Then $H^i(U, \mathcal{R}) = 0$ for all $i \geq 2$.

2. If $\mathcal{R}$ satisfies the Runge condition and $H^1(\tilde{V}_k, \mathcal{R}) = 0$ for all $k \geq 1$, then $H^1(U, \mathcal{R}) = 0$.

Lemma 9.11.11. The sheaf $\mathcal{O}|_U$ satisfies the Runge condition.

Proof. For a given section $f \in \Gamma(\tilde{V}_k, \mathcal{O})$ let us denote by $\hat{f}(\omega) \in \mathbb{C}$ the value of germ $f(\omega)$ at point $\omega \in \tilde{V}_k$.

We endow each space $\Gamma(\tilde{V}_k, \mathcal{O})$ with semi-norm $|f|_k := \sup_{\omega \in \tilde{V}_k} |\hat{f}(\omega)|$. Conditions (b)–(d) are trivially satisfied. For the proof of (a), let us fix a section $f \in \Gamma(\tilde{V}_k, \mathcal{O})$. By the definition, a section of sheaf $\mathcal{O}$ over $\tilde{V}_k := \tilde{V}_{0,k} \times \tilde{N}_k$ is the restriction of a section of $\mathcal{O}$ over an open neighbourhood of $\tilde{V}_k$. In particular, there exists an open neighbourhood $L \subset K$ of $\tilde{N}_k$ such that section $f|_{\tilde{V}_k}$ admits a bounded extension to $\tilde{V}_{0,k} \times L$. Since $\tilde{G}_a (\supset K)$ is a normal space, there exists a function $\rho_k \in C(K)$ such that $\rho_k \equiv 1$ on $\tilde{N}_k$, and $\rho_k \equiv 0$ on $K \setminus L$. We set $\tilde{f} := f \rho_k \in \Gamma(V_{0,k} \times K, \mathcal{O})$. Then function $\tilde{f}$ determines a holomorphic function $\hat{f}$ defined in a neighbourhood of $\tilde{V}_{0,k}$ and with values in the Banach space $C_b(K)$ of bounded continuous functions on $K$ endowed with sup-norm $\| \cdot \|$. We now apply the Runge-type approximation theorem for Banach-valued holomorphic functions, see [Bu2], to obtain that for every $\varepsilon > 0$ there is a function $\tilde{F} \in \mathcal{O}(U_0, C_b(K))$ such that $\sup_{x \in \tilde{V}_{0,k}} \| \hat{f}(x) - \tilde{F}(x) \| < \varepsilon$. Then $\tilde{F}$ determines a function $F \in \mathcal{O}(U)$ such that $\sup_{\omega \in \tilde{V}_k} |f(\omega) - F(\omega)| < \varepsilon$, which implies (a).

Corollary 9.11.12. $H^i(U, \mathcal{O}) = 0$, $i \geq 1$.

Proof. Follows from Lemmas 9.11.8(2), 9.11.10 and 9.11.11.

Lemma 9.11.13. Let $\mathcal{B}$, $\mathcal{R}$ be analytic sheaves on $U$, let $V_0 \subseteq U_0$ be an open polydisk, $S \subset K$ a closed subset. Suppose that sequence

$$
\mathcal{B} \xrightarrow{\psi} \mathcal{R} \longrightarrow 0
$$

(9.11.31)
Proof. We denote $\tilde{\mathcal{B}} := q_* (\mathcal{B}|_{\tilde{V}_0 \times S})$, $\tilde{\mathcal{R}} := q_* (\mathcal{R}|_{\tilde{V}_0 \times S})$, $\hat{\psi} := q_\ast \psi$. We have to show that $\hat{\psi}$ is surjective. Given open subsets $W_0 \subset \tilde{V}_0$, $L \subset S$, we denote by $\Psi_{W_0 \times L}$ the homomorphism of sheaves of sections $\Gamma(W_0 \times L, \mathcal{B}) \to \Gamma(W_0 \times L, \mathcal{R})$ induced by $\psi$, and by $\hat{\Psi}_{W_0}$ the homomorphism of sheaves of sections $\Gamma(W_0, \tilde{\mathcal{B}}) \to \Gamma(W_0, \tilde{\mathcal{R}})$ induced by $\hat{\psi}$. By the definition of direct image sheaf (see, e.g., [Gun3, Ch. F])

$$\Gamma(W_0 \times S, \mathcal{B}) \cong \Gamma(W_0, \tilde{\mathcal{B}}), \quad \Gamma(W_0 \times S, \mathcal{R}) \cong \Gamma(W_0, \tilde{\mathcal{R}}).$$

(9.11.33)

To prove exactness of (9.11.32) it suffices to show that for every point $x_0 \in \tilde{V}_0$, a neighbourhood $W_0 \subset \tilde{V}_0$ of $x_0$, and a section $\hat{f}_{x_0} \in \Gamma(W_0, \tilde{\mathcal{B}})$, there exists a section $\hat{g}_{x_0} \in \Gamma(\tilde{W}_0, \tilde{\mathcal{B}})$ over a neighbourhood $\tilde{W}_0 \subset W_0$ of $x_0$ such that $\hat{\Psi}_{\tilde{W}_0}(\hat{g}_{x_0}) = \hat{f}_{x_0}|_{\tilde{W}_0}$.

Let $f_{x_0} \in \Gamma(W_0 \times S, \mathcal{R})$ be the section corresponding to $\hat{f}_{x_0}$ under the second isomorphism in (9.11.33). By definition, a section of sheaf $\mathcal{R}$ over $W_0 \times S$ is the restriction of a section of $\mathcal{R}$ over an open neighbourhood of $W_0 \times S$. Therefore, shrinking $W_0$ if necessary, we obtain that $f_{x_0}$ can be extended to a section of $\mathcal{R}$ over $W_0 \times M_1$, where $M_1 \subset K$ is an open neighbourhood of $S$. Since $\psi$ is a surjective sheaf homomorphism, for each point $y \in \{x_0\} \times M_1$ there exist open sets $W_{0,y} \subset W_0$, $L_y \subset M_1$ and a section $s_y \in \Gamma(W_{0,y} \times L_y, \mathcal{B})$ such that $y \in W_{0,y} \times L_y$ and $\Psi_{W_{0,y} \times L_y}(s_y) = f_{x_0}|_{W_{0,y} \times L_y}$. Since space $\tilde{G}_a (\supset M_1)$ is compact Hausdorff and, hence, is normal, there exists an open subset $M_2 \subset M_1$ such that $S \subset M_2$, and $\tilde{M}_2 \subset M_1$. Since $\tilde{M}_2$ is compact, there exist finitely many points $\{y_j\}_{j=1}^m \subset S$ such that $\tilde{M}_2 \subset \cup_j L_{y_j}$. We set $\tilde{L}_{y_j} := \tilde{M}_2 \cap L_{y_j}$, for all $j$. There exists a partition of unity $\{\rho_j\} \subset C(\tilde{M}_2)$ subordinate to $\{\tilde{L}_{y_j}\}$. We define $\tilde{W}_0 := \cap_j W_{0,y_j}$, and set

$$g_{x_0}(z, \eta) := \sum_j \rho_j(\eta)s_{y_j}(z, \eta), \quad (z, \eta) \in \tilde{W}_0 \times S.$$
Then \( g_{x_0} \in \Gamma(\tilde{W}_0 \times M_2, \mathcal{B}) \). We have

\[
\Psi_{\tilde{W}_0 \times S}(g_{x_0}) = \sum_j \rho_j \Psi_{\tilde{W}_0 \times L_{y_j}}(s_{y_j}) = \sum_j \rho_j f_{x_0} |_{\tilde{W}_0 \times L_{y_j}} = f_{x_0} |_{\tilde{W}_0 \times S}.
\]

Let \( \hat{g}_{x_0} \) denote the section in \( \Gamma(\tilde{W}_0, \hat{\mathcal{B}}) \) corresponding to \( g_{x_0} \) under the first isomorphism in (9.11.33). Then \( \hat{\Psi}_{\tilde{W}_0}(\hat{g}_{x_0}) = \hat{f}_{x_0} |_{\tilde{W}_0} \), as needed.

**Definition 9.11.14.** We say that an analytic sheaf \( R \) (on \( U \)) admits a free resolution of length \( N \geq 1 \) over \( U \) if there exists an exact sequence

\[
F_N |_U \xrightarrow{\varphi_{N-1}} \ldots \xrightarrow{\varphi_2} F_2 |_U \xrightarrow{\varphi_1} F_1 |_U \xrightarrow{\varphi_0} R \rightarrow 0, \tag{9.11.34}
\]

where \( F_i \) are free sheaves, i.e., sheaves of the form \( \mathcal{O}^k \) for some \( k \geq 0 \) (by definition, \( \mathcal{O}^0 = \{0\} \)).

**Lemma 9.11.15.** Let \( R \) be an analytic sheaf on \( U \) having a free resolution of length \( 3N \)

\[
F_{3N} |_U \xrightarrow{\varphi_{3N-1}} \ldots \xrightarrow{\varphi_2} F_2 |_U \xrightarrow{\varphi_1} F_1 |_U \xrightarrow{\varphi_0} R \rightarrow 0. \tag{9.11.35}
\]

If \( N \geq n = \dim \mathbb{C} U_0 \), then for each \( k \) the induced sequence of sections

\[
\Gamma(\bar{V}_k, F_N) \xrightarrow{\varphi_{N-1}} \ldots \xrightarrow{\varphi_2} \Gamma(\bar{V}_k, F_2) \xrightarrow{\varphi_1} \Gamma(\bar{V}_k, F_1) \xrightarrow{\varphi_0} \Gamma(\bar{V}_k, R) \rightarrow 0 \tag{9.11.36}
\]

is exact.

**Proof.** Let us fix \( k \geq 1 \). Let \( q : \bar{V}_k \rightarrow \bar{V}_{0,k} \) be the projection, \( q(x, \eta) = x \), \( (x, \eta) \in V_k := V_{0,k} \times N_k \) (cf. notation before Definition 9.11.9). Let \( q_* \) denote the direct image functor, set \( \hat{F}_i := q_*(F_i |_{\bar{V}_k}) \), \( \hat{R} := q_*(R |_{\bar{V}_k}) \), \( \hat{\varphi}_i := q_* \varphi_i \). Applying \( q_* \) to (9.11.35) we obtain a complex of sheaf homomorphisms

\[
\hat{F}_{3N} \xrightarrow{\hat{\varphi}_{3N-1}} \ldots \xrightarrow{\hat{\varphi}_2} \hat{F}_2 \xrightarrow{\hat{\varphi}_1} \hat{F}_1 \xrightarrow{\hat{\varphi}_0} \hat{R} \rightarrow 0 \tag{9.11.37}
\]

(a priori this sequence is not exact). By the definition of a direct image sheaf, the sequence of sections of (9.11.37) over \( \bar{V}_{0,k} \), truncated to \( N \)-th term

\[
\Gamma(\bar{V}_{0,k}, \hat{F}_N) \rightarrow \cdots \rightarrow \Gamma(\bar{V}_{0,k}, \hat{F}_1) \rightarrow \Gamma(\bar{V}_{0,k}, \hat{R}) \rightarrow 0 \tag{9.11.38}
\]

coincides with sequence (9.11.36). Hence, the assertion would follow once we prove that sequence (9.11.38) is exact.
Now, exact sequence (9.11.35) yields a collection of short exact sequences

$$0 \to \mathcal{R}_i|_{\tilde{V}_k} \to \mathcal{F}_i|_{\tilde{V}_k} \xrightarrow{\varphi_{i-1}} \mathcal{R}_{i-1}|_{\tilde{V}_k} \to 0, \quad 1 \leq i \leq 3N - 1,$$

(9.11.39)

where $\mathcal{R}_i := \text{Im} \varphi_i$ ($0 \leq i \leq 3N - 1$), $\mathcal{R}_0 := \mathcal{R}$, and $\iota$ stands for inclusion. We apply to (9.11.39) the direct image functor $q_*$ (recall that $q_*$ is left exact, see, e.g., [Gun3, Ch. F]) and Lemma 9.11.13 to obtain a collection of short exact sequences

$$0 \to \mathcal{T}_i \xrightarrow{\iota} \hat{\mathcal{F}}_i \xrightarrow{\varphi_{i-1}} \mathcal{T}_{i-1} \to 0, \quad 1 \leq i \leq 3N - 1.$$

(9.11.40)

An argument similar to the one in the proof of Lemma 9.11.8 implies $H^l(\tilde{V}_{0,k}, \hat{\mathcal{F}}_i) = 0$, $l \geq 1$, $k \geq 1$, $1 \leq i \leq 3N$. Hence, each short exact sequence (9.11.40) yields a long exact sequence of the form

$$0 \to \Gamma(\tilde{V}_{0,k}, \mathcal{T}_i) \to \Gamma(\tilde{V}_{0,k}, \hat{\mathcal{F}}_i) \to \Gamma(\tilde{V}_{0,k}, \mathcal{T}_{i-1}) \to H^1(\tilde{V}_{0,k}, \mathcal{T}_i) \to H^1(\tilde{V}_{0,k}, \mathcal{T}_{i-1}) \to H^2(\tilde{V}_{0,k}, \mathcal{T}_i) \to \ldots$$

Thus, $H^m(\tilde{V}_{0,k}, \mathcal{T}_i) \cong H^m+1(\tilde{V}_{0,k}, \mathcal{T}_{i+1})$, $m \geq 1$, $1 \leq i \leq 3N - 2$, and so

$$H^m(\tilde{V}_{0,k}, \mathcal{T}_i) \cong H^{m+l+1}(\tilde{V}_{0,k}, \mathcal{T}_{i+l+1}), \quad l \geq -1.$$

Let us take $m = 1$, $1 \leq i \leq N$, $l := 2n - 2$. Then

$$H^1(\tilde{V}_{0,k}, \mathcal{T}_i) \cong H^{2n+1}(\tilde{V}_{0,k}, \mathcal{T}_{i+2n-1}), \quad 1 \leq i \leq N.$$

Since $N \geq n$, we have $i + 2n - 1 \leq 3N - 1$ for all $1 \leq i \leq N$, hence $\mathcal{T}_{i+2n-1}$ is well defined for all $1 \leq i \leq N$. Since the topological dimension of $\tilde{V}_{0,k}$ is equal to $2n$, we have $H^{2n+1}(\tilde{V}_{0,k}, \mathcal{T}_{i+2n-1}) = 0$, therefore $H^1(\tilde{V}_{0,k}, \mathcal{T}_i) = 0$, $1 \leq i \leq N$. Therefore, we obtain collection of short exact sequences

$$0 \to \Gamma(\tilde{V}_{0,k}, \mathcal{T}_i) \to \Gamma(\tilde{V}_{0,k}, \hat{\mathcal{F}}_i) \to \Gamma(\tilde{V}_{0,k}, \mathcal{T}_{i-1}) \to 0, \quad 1 \leq i \leq N,$$

which yields the exactness of sequence (9.11.38). The proof is complete. \[\square\]

**Lemma 9.11.16.** Let $\mathcal{R}$ be an analytic sheaf on $U$ having a free resolution of length $3N$

$$\mathcal{F}_{3N}|_U \xrightarrow{\varphi_{3N-1}} \ldots \xrightarrow{\varphi_2} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{R} \to 0.$$  

(9.11.41)

If $N \geq n$, then $\mathcal{R}$ satisfies the Runge condition.
Proof. For a given section $h$ of sheaf $\mathcal{O}^m|_U$ we denote by $\hat{h}(\omega) \in \mathbb{C}^m$ the value of germ $h(\omega)$ at $\omega \in U$. We have a short exact sequence

$$0 \longrightarrow \text{Ker } \varphi_0 \longrightarrow \mathcal{F}_1|_U \longrightarrow \mathcal{R} \longrightarrow 0,$$

where $\iota$ stands for inclusion. In the proof of Lemma 9.11.15 we have shown that, under the present assumptions, for each $k \geq 1$ the sequence of sections

$$0 \longrightarrow \Gamma(\tilde{V}_k, \text{Ker } \varphi_0) \xrightarrow{\iota^*} \Gamma(\tilde{V}_k, \mathcal{F}_1) \xrightarrow{\varphi_0^*} \Gamma(\tilde{V}_k, \mathcal{R}) \longrightarrow 0$$

is exact. Given a section $h \in \Gamma(\tilde{V}_k, \mathcal{F}_1)$, we define semi-norm $|h|_k = \sup_{x \in \tilde{V}_k} ||\hat{h}(x)||$, where $|| \cdot ||$ is a Euclidean norm in $\mathbb{C}^m$, where $\mathcal{F}_1 = \mathcal{O}^m$. Now, for a section $h \in \Gamma(\tilde{V}_k, \mathcal{R})$ we set

$$|f|_k := \inf\{|h|_k : h \in \Gamma(\tilde{V}_k, \mathcal{F}_1), \varphi_0^*(h) = f\}. \quad (9.11.42)$$

We obtain a family of semi-norms $\{| \cdot |_k : k \geq 1\}$ on $\Gamma(U, \mathcal{R})$. Let us show that conditions (a)-(d) are satisfied.

(a) Let $f \in \Gamma(\tilde{V}_k, \mathcal{R})$. There exists a section $h \in \Gamma(\tilde{V}_k, \mathcal{F}_1)$ such that $f = \varphi_0^*(h)$. Using the same argument as in the proof of Lemma 9.11.11, we obtain that for any $\varepsilon > 0$ there exists a section $\tilde{h} \in \Gamma(U, \mathcal{F}_1)$ such that $|\tilde{h} - h|_k < \varepsilon$. We set $\tilde{f} := \varphi_0^*(\tilde{h}) \in \Gamma(U, \mathcal{R})$. By definition, $|\tilde{f} - f|_k < \varepsilon$, as required.

(b) Let $f \in \Gamma(\tilde{V}_{k+1}, \mathcal{R})$. Since

$$\{h \in \Gamma(\tilde{V}_{k+1}, \mathcal{F}_1), f = \varphi_0^*(h)\}|_{\tilde{V}_k} \subset \{g \in \Gamma(\tilde{V}_k, \mathcal{F}_1), f|_{\tilde{V}_k} = \varphi_0^*(g)\},$$

and for every $h \in \Gamma(\tilde{V}_{k+1}, \mathcal{F}_1)$ we have $|h|_k \leq |h|_{k+1}$, condition (b) is satisfied with $M_k = 1$ ($k \geq 1$) (cf. (9.11.42)).

(c) Let $\{f_j\}$ be a Cauchy sequence in $\Gamma(\tilde{V}_{k+1}, \mathcal{R})$. Then there exists a Cauchy sequence $\{h_j\} \subset \Gamma(\tilde{V}_{k+1}, \mathcal{O}^m)$ such that $f_j = \varphi_0^*h_j$ for all $j$. Clearly, there exists a function $h \in \mathcal{O}(\tilde{V}_{k+1}, \mathbb{C}^m) \cap C(\tilde{V}_{k+1}, \mathbb{C}^m)$ such that

$$\sup_{\omega \in \tilde{V}_{k+1}} |h(\omega) - \hat{h}_j(\omega)| \to 0 \quad \text{as } j \to \infty. \quad (9.11.43)$$

Then $h \in \Gamma(\tilde{V}_k, \mathcal{O}^m)$, and by (9.11.43) $|h - h_j|_k \to 0$ ($j \to \infty$). Now, we set $f := \varphi_0^*h \in \Gamma(\tilde{V}_k, \mathcal{R})$, so by continuity $|f - f_j|_k \to 0$ ($j \to \infty$).
(d) Let \( f \in \Gamma(\bar{V}_{k+1}, \mathcal{R}) \), \( |f|_{k+1} = 0 \). By definition, there exists a sequence of sections \( h_l \in \Gamma(\bar{V}_{k+1}, \mathcal{F}_1) \) such that \( f = \varphi_0^*(h_l) \) for all \( l \), and \( \sup_{\omega \in \bar{V}_{k+1}} \| \hat{h}_l(\omega) \| \to 0 \) as \( l \to \infty \). Let \( g_l := h_1 - h_l, l \geq 1 \). Then \( g_l \in \Gamma(\bar{V}_{k+1}, \text{Ker} \varphi_0) \), and

\[
\hat{g}_l(x) \to \hat{h}_1(\omega) \quad \omega \in \bar{V}_{k+1} \text{ uniformly as } l \to \infty.
\] (9.11.44)

Now, suppose that \( f|_{\bar{V}_k} \neq 0 \). Then \( h_1|_{\bar{V}_k} \notin \Gamma(\bar{V}_k, \text{Ker} \varphi_0) \).

Consider the second fragment of the free resolution of \( \mathcal{R} \),

\[
0 \longrightarrow \text{Ker} \varphi_1 \xrightarrow{\iota} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \text{Ker} \varphi_0 \longrightarrow 0,
\]

and the corresponding sequence of sections (cf. Lemma 9.11.15)

\[
0 \longrightarrow \Gamma(\bar{V}_{k+1}, \text{Ker} \varphi_1) \xrightarrow{\iota^*} \Gamma(\bar{V}_{k+1}, \mathcal{F}_2) \xrightarrow{\varphi_1^*} \Gamma(\bar{V}_{k+1}, \text{Ker} \varphi_0) \longrightarrow 0,
\] (9.11.45)

where \( \varphi_1^* \) is given by a matrix with entries in \( \Gamma(\bar{V}_{k+1}, \mathcal{O}) \). Recall that \( \Gamma(\bar{V}_{k+1}, \text{Ker} \varphi_1) \) is endowed with semi-norm

\[
|g|_{k+1} = \sup_{\omega \in \bar{V}_{k+1}} \| \hat{g}(\omega) \|, \quad g \in \Gamma(\bar{V}_{k+1}, \text{Ker} \varphi_1).
\] (9.11.46)

Each section in space \( \Gamma(\bar{V}_{k+1}, \text{Ker} \varphi_1) \) determines a continuous function on \( \bar{V}_{k+1} \) holomorphic in \( V_{k+1} \). Let \( \mathcal{A}(\bar{V}_{k+1}, \text{Ker} \varphi_1) \) be the completion of the space of these functions with respect to norm defined by (9.11.46). We introduce similar notation \( \mathcal{A}(\bar{V}_{k+1}, \mathcal{F}_2), \mathcal{A}(\bar{V}_{k+1}, \text{Ker} \varphi_0) \), for Banach spaces of holomorphic functions corresponding to two other terms in (9.11.45), so (9.11.45) yields an exact sequence of Banach spaces

\[
0 \longrightarrow \mathcal{A}(\bar{V}_{k+1}, \text{Ker} \varphi_1) \xrightarrow{\iota^*} \mathcal{A}(\bar{V}_{k+1}, \mathcal{F}_2) \xrightarrow{(\varphi_1^*)'} \mathcal{A}(\bar{V}_{k+1}, \text{Ker} \varphi_0) \longrightarrow 0,
\]

where \( (\varphi_0^*)' \) is given by a matrix with entries in \( \mathcal{A}(\bar{V}_{k+1}, \mathcal{O}) \), holomorphic functions on \( V_{k+1} \) continuous on \( \bar{V}_{k+1} \). It follows from (9.11.44) that sequence \( \{g_l\} \) whose elements are viewed as functions in \( \mathcal{A}(\bar{V}_{k+1}, \text{Ker} \varphi_0) \) is a Cauchy sequence and hence has a limit \( g \in \mathcal{A}(\bar{V}_{k+1}, \text{Ker} \varphi_0) \). Then there exists \( r \in \mathcal{A}(\bar{V}_{k+1}, \mathcal{F}_2) \) such that \( g = (\varphi_1^*)'(r) \). Note that both \( g|_{\bar{V}_k}, (\varphi_1^*)'|_{\bar{V}_{k+1}} \) are the sections of analytic sheaves, and in particular \( g|_{\bar{V}_k} \in \Gamma(\bar{V}_k, \text{Ker} \varphi_0) \), \( (\varphi_1^*)'|_{\bar{V}_k} = \varphi_1^*|_{\bar{V}_k} \). It follows that \( h_1|_{\bar{V}_k} = g|_{\bar{V}_k} \), so \( h_1|_{\bar{V}_k} \in \Gamma(\bar{V}_k, \text{Ker} \varphi_0) \), which is a contradiction.

\[\square\]
Lemma 9.11.17. Let $\mathcal{R}$ be an analytic sheaf over $U$ admitting a free resolution of length $4N$

$$
\mathcal{F}_4|_U \xrightarrow{\phi^4_{N-1}} \ldots \xrightarrow{\phi^2_2} \mathcal{F}_2|_U \xrightarrow{\phi^1_1} \mathcal{F}_1|_U \xrightarrow{\phi^0_0} \mathcal{R} \rightarrow 0. \tag{9.11.47}
$$

If $N \geq n$, and for each $k$ the sequence of sections

$$
\Gamma(V_k, \mathcal{F}_N) \xrightarrow{\phi^N_k} \ldots \xrightarrow{\phi^2_2} \Gamma(V_k, \mathcal{F}_2) \xrightarrow{\phi^1_1} \Gamma(V_k, \mathcal{F}_1) \xrightarrow{\phi^0_0} \Gamma(V_k, \mathcal{R}) \rightarrow 0 \tag{9.11.48}
$$

is exact, then the sequence of sections

$$
\Gamma(U, \mathcal{F}_N) \xrightarrow{\phi^N_k} \ldots \xrightarrow{\phi^2_2} \Gamma(U, \mathcal{F}_2) \xrightarrow{\phi^1_1} \Gamma(U, \mathcal{F}_1) \xrightarrow{\phi^0_0} \Gamma(U, \mathcal{R}) \rightarrow 0 \tag{9.11.49}
$$

is also exact.

Proof. The exact sequence (9.11.47) yields a collection of short exact sequences

$$
0 \rightarrow \mathcal{R}_i \xrightarrow{\iota} \mathcal{F}_i|_U \xrightarrow{\phi^i_{i-1}} \mathcal{R}_{i-1} \rightarrow 0, \quad 1 \leq i \leq N - 1, \tag{9.11.50}
$$

where $\mathcal{R}_i := \text{Im} \varphi_i$ ($0 \leq i \leq N-1$), $\mathcal{R}_0 := \mathcal{R}$, and $\iota$ stands for inclusion. Recall that the section functor $\Gamma$ is left exact (see, e.g., [Gun3, Ch. 3]), hence we have a collection of exact sequences

$$
0 \rightarrow \Gamma(U, \mathcal{R}_i) \xrightarrow{\iota^*} \Gamma(U, \mathcal{F}_i) \xrightarrow{\phi^i_{i-1}} \Gamma(U, \mathcal{R}_{i-1}), \quad 1 \leq i \leq N - 1.
$$

It suffices to show that $\varphi^i_{i-1}$ is surjective; this would imply that (9.11.49) is exact.

It follows from the exactness of sequence (9.11.48) that for each $k$ the sequences

$$
0 \rightarrow \Gamma(V_k, \mathcal{R}_i) \xrightarrow{\iota^*} \Gamma(V_k, \mathcal{F}_i) \xrightarrow{\phi^i_{i-1}} \Gamma(V_k, \mathcal{R}_{i-1}) \rightarrow 0, \quad 1 \leq i \leq N - 1, \tag{9.11.51}
$$

are exact. By Lemma 9.11.8 $H^1(V_k, \mathcal{F}_i) = 0$, $1 \leq i \leq N$, for all $k \geq 1$, therefore the long exact sequence for (9.11.50) over $\bar{V}_k$ has the form

$$
0 \rightarrow \Gamma(\bar{V}_k, \mathcal{R}_i) \rightarrow \Gamma(\bar{V}_k, \mathcal{F}_i) \rightarrow \Gamma(\bar{V}_k, \mathcal{R}_{i-1}) \rightarrow H^1(\bar{V}_k, \mathcal{R}_i) \rightarrow H^1(\bar{V}_k, \mathcal{R}_{i-1}) \rightarrow H^2(\bar{V}_k, \mathcal{R}_i) \rightarrow \ldots, \quad 1 \leq i \leq N - 1.
$$

Now it follows from (9.11.51) that $H^1(\bar{V}_k, \mathcal{R}_i) = 0$ for all $k \geq 1$, $1 \leq i \leq N - 1$.

The long exact sequence for (9.11.50) over $U$ has form

$$
0 \rightarrow \Gamma(U, \mathcal{R}_i) \rightarrow \Gamma(U, \mathcal{F}_i) \rightarrow \Gamma(U, \mathcal{R}_{i-1}) \rightarrow H^1(U, \mathcal{R}_i) \rightarrow H^1(U, \mathcal{R}_{i-1}) \rightarrow H^2(U, \mathcal{R}_i) \rightarrow \ldots, \quad 1 \leq i \leq N - 1. \tag{9.11.52}
$$

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Each sheaf $\mathcal{R}_i$, $1 \leq i \leq N - 1$, has free resolution of length $3N$, hence by Lemma 9.11.16 it satisfies the Runge condition. It follows from Lemma 9.11.10(2) that $H^i(U, \mathcal{R}_i) = 0$ for all $1 \leq i \leq N - 1$. We obtain from (9.11.52) that sequences

$$0 \longrightarrow \Gamma(U, \mathcal{R}_i) \overset{\iota^*_i}{\longrightarrow} \Gamma(U, \mathcal{F}_i) \overset{\varphi^*_i \iota^*_i}{\longrightarrow} \Gamma(U, \mathcal{R}_{i-1}) \longrightarrow 0, \quad 1 \leq i \leq N - 1,$$

are exact, which implies the exactness of sequence (9.11.49).

**Proof of Proposition 9.11.1.** (1) Follows from Lemmas 9.11.15 and 9.11.17.

(2) According to Lemma 9.11.16 sheaf $\mathcal{R}$ satisfies the Runge condition. Hence, by Lemma 9.11.10 we only have to show that $H^i(V_k, \mathcal{R}) = 0$ for all $i \geq 1$ and $k \geq 1$.

Let $V$ be an open cover of $\bar{V}_k := \bar{V}_{0,k} \times K$. It suffices to show that, given an $i$-cocycle $\sigma \in Z^i(V, \mathcal{R})$ (cf. notation before Lemma 9.11.8), there exists a refinement $V'$ of $V$ such that the image of $\sigma$ by the refinement map $Z^i(V, \mathcal{R}) \to Z^i(V', \mathcal{R})$ belongs to $B^i(V', \mathcal{R})$.

By Lemma 9.11.7(1) there exists a finite refinement $\mathcal{U} = \{U_\alpha\}$, $U_\alpha := U_{0,l} \times L_j$, $\alpha = (l, j)$, of cover $V$ of class $(P)$ (cf. Definition 9.11.6). Let $s = s_\mathcal{U}$ be the number of elements of $\mathcal{U}$, and let $N \geq \max\{n, s\}$ be the length of the free resolution of $\mathcal{R}$ over $U$. By the definition of open cover of class $(P)$, a section of sheaf $\mathcal{R}$ over an element $U_\alpha$ of $\mathcal{U}$ admits extension to $\tilde{U}_\alpha = \tilde{U}_{0,l} \times L_j$, where $\tilde{U}_{0,l} = \tilde{U}_{0,1} \times \cdots \times \tilde{U}_{0,n} \subseteq U_0$ is a product domain such that each $\tilde{U}_{0,i} \subset \mathbb{C}$ $(1 \leq i \leq n)$ is simply connected and has smooth boundary, and $U_{0,l} = \bar{V}_{0,k} \cap \tilde{U}_{0,l}$. By part (1) of the proposition over each $U_\alpha$ the sequence of sections $U_\alpha$ corresponding to (9.11.21) is exact (there we can take product domain $\tilde{U}_{0,l}$ instead of polydisk $U_0$) Hence, we have a sequence of cochain complexes

$$C^\cdot(\mathcal{U}, \mathcal{F}_N) \longrightarrow \cdots \longrightarrow C^\cdot(\mathcal{U}, \mathcal{F}_1) \longrightarrow C^\cdot(\mathcal{U}, \mathcal{R}) \longrightarrow 0,$$

By Lemma 9.11.7(2) there exists a refinement $\mathcal{U}'$ of cover $\mathcal{U}$ of class $(P)$ having the same cardinality. We have a commutative diagram with exact rows

$$\begin{array}{cccccccc}
C^\cdot(\mathcal{U}, \mathcal{F}_N) & \longrightarrow & \cdots & \longrightarrow & C^\cdot(\mathcal{U}, \mathcal{F}_1) & \longrightarrow & C^\cdot(\mathcal{U}, \mathcal{R}) & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
C^\cdot(\mathcal{U}', \mathcal{F}_N) & \longrightarrow & \cdots & \longrightarrow & C^\cdot(\mathcal{U}', \mathcal{F}_1) & \longrightarrow & C^\cdot(\mathcal{U}', \mathcal{R}) & \longrightarrow & 0
\end{array}$$

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or, equivalently, the collection of commutative diagrams with exact rows

\[
\begin{array}{c}
0 \rightarrow C(U, \mathcal{R}_i) \rightarrow C(U, \mathcal{F}_i) \rightarrow C(U, \mathcal{R}_{i-1}) \rightarrow 0 \\
0 \rightarrow C(U', \mathcal{R}_i) \rightarrow C(U', \mathcal{F}_i) \rightarrow C(U', \mathcal{R}_{i-1}) \rightarrow 0
\end{array}
\]

where \( \mathcal{R}_i := \text{Im } \varphi_i \) (0 ≤ \( i \) ≤ \( N - 1 \)), \( \mathcal{R}_0 := \mathcal{R} \). Each row yields a long exact sequence

\[
0 \rightarrow \Gamma(\bar{V}_k, \mathcal{R}_i) \rightarrow \Gamma(\bar{V}_k, \mathcal{F}_i) \rightarrow \Gamma(\bar{V}_k, \mathcal{R}_{i-1}) \rightarrow \\
H^1(U, \mathcal{R}_i) \rightarrow H^1(U, \mathcal{F}_i) \xrightarrow{\varphi_{i-1}^1} H^1(U, \mathcal{R}_{i-1}) \xrightarrow{\psi_i^2} H^2(U, \mathcal{R}_i) \rightarrow \ldots, \quad 1 \leq i \leq N - 1
\]

(and a similar one for \( U' \)), where \( H^1(U, \mathcal{R}_i) := Z^1(U, \mathcal{R}_i)/B^1(U, \mathcal{R}_i) \) are the Čech cohomology groups corresponding to cover \( U \). These sequences form a commutative diagram

\[
\begin{array}{c}
\ldots \rightarrow H^1(U, \mathcal{R}_i) \rightarrow H^1(U, \mathcal{F}_i) \xrightarrow{\varphi_{i-1}^1} H^1(U, \mathcal{R}_{i-1}) \xrightarrow{\psi_{i}^2} H^2(U, \mathcal{R}_i) \rightarrow \ldots \\
\quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\ldots \rightarrow H^1(U', \mathcal{R}_i) \rightarrow H^1(U', \mathcal{F}_i) \xrightarrow{(\varphi_{i-1}^1)'} H^1(U', \mathcal{R}_{i-1}) \xrightarrow{(\psi_{i}^2)'} H^2(U', \mathcal{R}_i) \rightarrow \ldots
\end{array}
\]

where \( t_i', \gamma_{i-1}^l, \gamma_{i+1}^l \) are the corresponding refinement maps.

We have to show that, given \( \sigma \in H^l(U, \mathcal{R}) \), \( l \geq 1 \), there exists a refinement \( \mathcal{W} \) of cover \( U \) such that the image of \( \sigma \) in \( H^l(\mathcal{W}, \mathcal{R}) \) is zero. We construct this refinement using the following algorithm.

Suppose that there exists a non-zero \( \sigma \in H^l(U, \mathcal{R}_i) \). Let \( U'' \) be a finite refinement of cover \( U' \) of class \( (P) \) having the same cardinality \( s \) as \( U \) and \( U' \) (cf. Lemma 9.11.7(2)).

We consider two cases:

(a) \( \psi_{l-1}^{l+1}(\sigma) = 0 \). Then there exists \( \eta \in H^l(U, \mathcal{F}_i) \) such that \( \sigma = \varphi_{i-1}^l(\eta) \). We have \( \gamma_{i-1}^l(\sigma) = (\varphi_{i-1}^l)'(t_i^l(\eta)) \). By Lemma 9.11.8 \( t_i^l(H^l(U, \mathcal{F}_i)) = 0 \), hence the image of \( \sigma \) by the refinement map \( \gamma_{i-1}^l(\sigma) = 0 \).

(b) \( \sigma' := \psi_{l+1}^{l+1}(\sigma) \neq 0 \). If \( \psi_{i+2}^{l+2}(\sigma') = 0 \), then \( \gamma_{i+1}^{l+1}(\sigma') = 0 \), so by case (a) the image of \( \sigma \) by the refinement map \( H^l(U, \mathcal{R}_{i-1}) \rightarrow H^l(U'', \mathcal{R}_{i-1}) \) is zero. If \( \psi_{i+2}^{l+2}(\sigma') \neq 0 \), then we apply case (b) to \( \psi_{i+2}^{l+2}(\sigma') \), etc.

We apply this algorithm to \( \mathcal{R}_0 = \mathcal{R} \) assuming that there exists a non-zero \( \sigma \in H^l(U, \mathcal{R}) \), \( l \geq 1 \). Note that case (b) can not occur after \( s \) steps: assuming the opposite, we obtain a
finite refinement $W$ of $U$ of class $(P)$ having the same cardinality as $U$ and a non-zero element of $H^s(W, R_i);$ however, since the cardinality of $U$ is $s$, we have $H^s(W, R) = 0$, which is a contradiction. Thus, after at most $s$ steps we arrive to case (a), and hence the image of $\sigma$ under the corresponding refinement map is zero.

9.11.3 Proof of Proposition 9.11.2

The proof of Proposition 9.11.2 is based on the following lemma.

**Lemma 9.11.18.** Let $U_0 \Subset C^n$ be an open polydisk, and $K_1, K_2 \in \mathfrak{Q}$. Let $\mathcal{R}$ be an analytic sheaf over $U_0 \times (K_1 \cup K_2)$. Let $x_0 \in U_0$.

Suppose that for every $N \geq 1$ sheaf $\mathcal{R}$ admits free resolutions of length $N$ over $U_0 \times K_1$ and $U_0 \times K_2$. Then for any open subsets $L_1 \Subset K_1$, $L_2 \Subset K_2$ such that $L_i \in \mathfrak{Q}$ ($i = 1, 2$) there exists a neighbourhood $V_0 \subset U_0$ of $x_0$ such that for every $N \geq 1$ sheaf $\mathcal{R}$ admits a free resolution of length $N$ over $V_0 \times (L_1 \cup L_2)$.

We prove Lemma 9.11.18 below. Let us now complete the proof of the proposition.

Let $U_0 \Subset C^n$ be an open polydisk, $x_0 \in U_0$. Since sets $\bar{p}^{-1}(U_0)$ and $U_0 \times \hat{G}_a$ are biholomorphic (cf. Section 6.2.6), it suffices to prove Proposition 9.11.2 for a coherent sheaf $\mathcal{A}$ over $U_0 \times \hat{G}_a$.

By definition of a coherent sheaf (cf. Definition 6.3.1), there exist a finite open cover of $\{x_0\} \times \hat{G}_a$ by sets $W_{0,i} \times L_i$, where $W_{0,i} \subset U_0$ is a neighbourhood of $x_0$, $\cup_i L_i = \hat{G}_a$, and for every $N \geq 1$ sheaf $\mathcal{A}$ admits free resolutions of length $N$ over each $W_{0,i} \times L_i$.

By Lemma 9.11.3 there exist a collection of finite refinements

$$L^k(m) = \{L^k_j : L^k_j \in \mathfrak{Q}, 1 \leq j \leq m\}, \quad k \geq 1$$

of open cover $\{L_i\}$, such that $L^k_{j+1} \Subset L^k_j$ for all $1 \leq j \leq m$, $k \geq 1$.

Let $k = 1$. We apply Lemma 9.11.18 to sheaf $\mathcal{A}$ with $K_1 := L^1_{m-1}$, $K_2 := L^1_m$, $L_1 := L^2_{m-1}$, $L_2 := L^2_m$, $V_0 := V_{0,m} \subset \cap_i W_{0,i}$, obtaining that for each $N \geq 1$ sheaf $\mathcal{A}$ has a free resolution of length $N$ over $V_{0,m} \times (L^2_{m-1} \cup L^2_m)$.

We replace $L^k(m)$, $k \geq 2$, with $L^k(m-1) := \{L^k_1, \ldots, L^k_{m-2}, L^k_{m-1} \cup L^k_m\}$, $k \geq 2$. 

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Now, taking \( k = 2 \), we apply the above procedure to cover \( L^2(m - 1) \), obtaining that for each \( N \geq 1 \) sheaf \( A \) has a free resolution of length \( N \) over \( V_{0,m-1} \times (L_{m-2}^3 \cup L_{m-1}^3 \cup L_{m-1}^3) \) for some open \( V_{0,m-1} \subset V_{0,m} \), etc.

After \( m \) steps we obtain that there exists an open subset \( V_{0,1} \subset \cap_j W_{0,j} \) such that for each \( N \geq 1 \) sheaf \( A \) has a free resolution over \( V_{0,1} \times \hat{G}_0 \), as required.

### 9.11.4 Proof of Lemma 9.11.18

We will use the following notation.

Let \( M_{l \times k}(\mathbb{C}) \) be the space of \( l \times k \) matrices \( C = (c_{ij}) \) with entries \( c_{ij} \in \mathbb{C} \), endowed with norm \( |C| := \max\{|c_{ij}|\}_{i,j=1}^{l,k} \). We set \( M_k(\mathbb{C}) := M_{k \times k}(\mathbb{C}) \).

Let \( GL_k(\mathbb{C}) \subset M_k(\mathbb{C}) \) be the group of invertible matrices. We denote by \( I = I_k \in GL_k(\mathbb{C}) \) the identity matrix.

Let \( U_0 \subset \mathbb{C}^n \) be an open polydisk, \( K \in \Omega \), set \( U := U_0 \times K \). The space \( \mathcal{O}(U, M_k(\mathbb{C})) \) of holomorphic \( M_k(\mathbb{C}) \)-valued functions is endowed with norm

\[
\|F\|_U := \sup_{x \in U} |F(x)|, \quad F \in \mathcal{O}(U, M_k(\mathbb{C})).
\]

The subset \( \mathcal{O}(U, GL_k(\mathbb{C})) \subset \mathcal{O}(U, M_k(\mathbb{C})) \) of holomorphic \( GL_k(\mathbb{C}) \)-valued maps on \( U \) has the induced topology of uniform convergence on compact subsets of \( U \) (cf. Lemma 9.11.4(2)).

The identity map \( (z, \omega) \to I, (z, \omega) \in U \), will be denoted also by \( I \).

**Lemma 9.11.19.** Let \( U' := U_0 \times K', U'' := U_0 \times K'' \), where \( K', K'' \in \Omega \). Suppose that \( H \in \mathcal{O}(U' \cap U'', GL_k(\mathbb{C})) \) belongs to the connected component of the identity map \( I \) in \( \mathcal{O}(U' \cap U'', GL_k(\mathbb{C})) \).

Then for any open polydisk \( V_0 \in U_0 \) and open subsets \( L' \subset K', L'' \subset K'' \) there exists a function \( H' \in \mathcal{O}(V' \cup V'', GL_k(\mathbb{C})) \), where \( V' := V_0 \times L', V'' := V_0 \times L'' \), such that \( H'|_{V' \cap V''} = H|_{V' \cap V''} \).

**Proof.** We may assume without loss of generality that polydisks \( V_0, U_0 \) are centered at the origin \( 0 \in \mathbb{C}^n \).

First, suppose that \( \|I - H\|_{V' \cap V''} < \frac{1}{2} \), so we can define \( F := \ln H \in \mathcal{O}(V' \cap V'', M_k(\mathbb{C})) \cap C(V' \cap V'', M_k(\mathbb{C})) \). Let us show that, after shrinking \( V_0 \), there exists a function \( F' \in \mathcal{O}(V' \cup V'') \) and
Now, we set compact space \( \hat{b} F \) which gives us identity (9.11.53). Provided that \( \max_1 H \) where each map belonging to the connected component of the identity map \( I \) path \( H \), Let us show that map \( \in O \cup_{L} \) belongs to the connected component of the identity map \( I \bar{1} \), \( GL_{L} \) \( \in O \), \( \omega' \in C \) \( \in O \), \( |b_m(\omega)| = \sup_{\omega \in L \cap L'} |\tilde{b}_m(\omega)| \), and define (possibly after shrinking \( V_0 \))

\[
F'(z, \omega) := \sum_{m=0}^{\infty} \tilde{b}_m(\omega) z^m, \quad z \in V_0, \quad \omega \in L' \cup L''.
\]

Now, we set \( H' := \exp(F') \in O(V' \cup V'', GL_k(C)) \).

Further, let \( H \in O(U' \cap U'', GL_k(C)) \) be an arbitrary \( GL_k(C) \)-valued bounded holomorphic map belonging to the connected component of the identity map \( I \) of \( O(U' \cap U'', GL_k(C)) \).

Let us show that map \( H|_{V' \cap V''} \) can be presented in the form

\[
H|_{V' \cap V''} = H^1 \cdots H^l, \tag{9.11.53}
\]

where each \( H^i \in O(V' \cap V'', GL_k(C)) \), \( 1 \leq i \leq l \), satisfies

\[
\|I - H^i\|_{V' \cap V''} < \frac{1}{2}. \tag{9.11.54}
\]

Since \( H \) belongs to the connected component of the identity map \( I \), there exists a continuous path \( H_t \in O(U' \cap U'', GL_k(C)) \) \( t \in [0,1] \) such that \( H_0 = I, H_1 = H \). Consider a partition \( 0 = t_0 < t_1 < \cdots < t_l = 1 \) of the unit interval \([0,1]\), and define

\[
H^i(z, \omega) = H_{t_{i-1}}^{-1}(z, \omega) H_{t_i}(z, \omega), \quad (z, \omega) \in V' \cap V'', \quad 1 \leq i \leq l,
\]

which gives us identity (9.11.53). Provided that \( \max_{1 \leq i \leq l-1} |t_{i+1} - t_i| \) is sufficiently small, inequality (9.11.54) holds for all \( 1 \leq i \leq l \).

Now, shrinking \( V_0 \) if necessary, we obtain according to the first case that there exist maps \( H^\nu \in O(V' \cup V'', GL_k(C)) \) such that \( H^\nu|_{V' \cap V''} = H^i|_{V' \cap V''} \). We define \( H' := H^1 \cdots H^l \in O(V' \cup V'', GL_k(C)) \).
**Corollary 9.11.20.** In the notation of Lemma 9.11.19, for any open polydisk $V_0 \subset U_0$ and open subsets $L' \subset K'$, $L'' \subset K''$ there exist functions $h' \in \mathcal{O}(V', GL_k(C))$, $h'' \in \mathcal{O}(V'', GL_k(C))$ such that

$$H = h'h'', \quad \text{on } V' \cap V''.$$ 

**Proof.** Let $H' \in \mathcal{O}(V' \cup V'', GL_k(C))$ be as in Lemma 9.11.19. Since $H'|_{V' \cap V''} = H|_{V' \cap V''}$, we can choose $h' := H'|_{V'}$, $h'' := I$. □

**Lemma 9.11.21.** Any analytic homomorphism $\varphi : \mathcal{O}^k|_U \to \mathcal{O}^l|_U$ is determined by a holomorphic function $\Phi \in \mathcal{O}(U, M_{l \times k}(C))$.

The proof is immediate.

**Definition 9.11.22** (cf. [Lem]). Let $\mathcal{R}, \mathcal{B}_i$, $1 \leq i \leq N$, be analytic sheaves over $U$. We say that sequence

$$\mathcal{B}_N \to \cdots \to \mathcal{B}_2 \to \mathcal{B}_1 \to \mathcal{R} \to 0 \quad (9.11.55)$$

is completely exact if for any $m \geq 1$ the sequence of sections

$$\Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{B}_N)) \to \cdots \to \Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{B}_1)) \to \Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{R})) \to 0$$

or, equivalently,

$$\Gamma(U, \mathcal{B}_N^m) \to \cdots \to \Gamma(U, \mathcal{B}_1^m) \to \Gamma(U, \mathcal{R}^m) \to 0, \quad (9.11.56)$$

is exact.

Here $\mathcal{B}_i^m$, $\mathcal{R}^m$ stand for direct product of $m$ copies of, respectively, $\mathcal{B}_i$, $\mathcal{R}$, with itself, and $\text{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{B}_i)$, $\text{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{R})$ are the sheaves of germs of analytic homomorphisms $\mathcal{O}^m \to \mathcal{B}_i$, $\mathcal{O}^m \to \mathcal{R}$, respectively.

Note that if sequence (9.11.56) is exact for $m = 1$, then it is exact for all $m \geq 1$.

The next two lemmas are due to [Lem].

**Lemma 9.11.23.** Let $\mathcal{B}, \mathcal{C}$ be analytic sheaves on $U$. If sequence $\mathcal{B} \to \mathcal{C} \to 0$ is completely exact, and $\varphi : \mathcal{O}^k|_U \to \mathcal{C}$ is an analytic homomorphism, then there is an analytic homomorphism $\psi : \mathcal{O}^k|_U \to \mathcal{B}$ such that $\varphi = \gamma \psi$. 

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Proof. We can take $\psi$ in the preimage of $\varphi$ under the surjective homomorphism
\[
\gamma_* : \Gamma(U, \text{Hom}_O(O^k, B)) \to \Gamma(U, \text{Hom}_O(O^k, C))
\]
(cf. Definition 9.11.22).

**Lemma 9.11.24** (Three lemma). Let $A$, $B$ and $C$ be analytic sheaves on $U$. Suppose that sequence
\[
0 \to A \xrightarrow{\beta} B \xrightarrow{\gamma} C \to 0
\]
is completely exact. If two among $A$, $B$ and $C$ have free resolutions of length $N + n$, where $n := \dim\mathbb{C} U_0$, $N \geq n + 2$, then the third has a free resolution of length $N - n - 1$.

The proof of Lemma 9.11.24 repeats the proof of an analogous result in [Lem]. For the sake of completeness, we provide the proof below.

**Proof of Lemma 9.11.18.** We denote $U_1 := U_0 \times K_1$, $U_2 := U_0 \times K_2$. Let $N \geq n + 1$, consider free resolutions of $R$ of length $M \geq 4N$,
\[
O^{k_{M,i}} \mid_{U_i} \longrightarrow \ldots \longrightarrow O^{k_{1,i}} \mid_{U_i} \xrightarrow{\alpha_i} R \mid_{U_i} \longrightarrow 0, \quad i = 1, 2. \tag{9.11.57}
\]
Consider the end portions of (9.11.57):
\[
O^{k_{i}} \mid_{U_i} \xrightarrow{\alpha_i} R \mid_{U_i} \longrightarrow 0, \quad i = 1, 2. \tag{9.11.58}
\]
Let $U := U_0 \times (K_1 \cup K_2)$. We denote by $\pi_i : O^{k_1} \mid_U \oplus O^{k_2} \mid_U \to O^{k_i} \mid_U$, $i = 1, 2$, the natural projection homomorphisms.

First, let us show that there is an injective analytic homomorphism $H : O^{k_1} \mid_U \oplus O^{k_2} \mid_U \to O^{k_1} \mid_U \oplus O^{k_2} \mid_U$ such that $\alpha_1 \pi_1 H = \alpha_2 \pi_2$. By Proposition 9.11.1(1) sequence (9.11.57) and, hence, sequence (9.11.58), truncated to $N$-th term, are completely exact. By Lemma 9.11.23 we can factor $\alpha_1 = \alpha_2 \psi$, $\alpha_2 = \alpha_1 \varphi$ on $U_1 \cap U_2$ for some analytic homomorphisms $\psi : O^{k_1} \mid_{U_1 \cap U_2} \to O^{k_2} \mid_{U_1 \cap U_2}$, $\varphi : O^{k_2} \mid_{U_1 \cap U_2} \to O^{k_1} \mid_{U_1 \cap U_2}$. Now, identifying sheaf homomorphisms $\psi$, $\varphi$ with the holomorphic matrix functions that determine them (cf. Lemma 9.11.21), we define
\[
H = \begin{pmatrix} I_{k_1} & \varphi \\ 0 & I_{k_2} \end{pmatrix} \begin{pmatrix} I_{k_1} & 0 \\ \psi & I_{k_2} \end{pmatrix}^{-1} \in O(U_1 \cap U_2, GL_k(\mathbb{C})),
\]
where \( k := k_1 + k_2 \). It is immediate that \( \alpha_1 \pi_1 H = \alpha_2 \pi_2 \). The map \( H \) belongs to the connected component of the identity map in \( \mathcal{O}(U_1 \cap U_2, GL_k(\mathbb{C})) \). Indeed, consider a path \( H_t \in \mathcal{O}(U_1 \cap U_2, GL_k(\mathbb{C})) \) \((t \in [0,1])\),

\[
H_t := \begin{pmatrix} I_{k_1} & t \varphi \\ 0 & I_{k_2} \end{pmatrix} \begin{pmatrix} I_{k_1} & 0 \\ t \psi & I_{k_2} \end{pmatrix}^{-1},
\]

so that \( H_0 = I_k, \ H_1 = H \).

Next, let \( L_i \in K_i, \ L_i \in \Omega \ (i = 1,2) \) and \( V_0 \subseteq U_0 \) be an open polydisk, \( x_0 \in V_0 \). Let \( L_i^m \in \Omega \ (i = 1,2), m \geq 1 \), be the collection of open subsets of \( K \) such that \( L_i \subseteq L_i^{m+1} \subseteq L_i^m \subseteq K_i \) for all \( m \geq 1 \ (i = 1,2) \), obtained in Lemma 9.11.4(4).

Let \( \{V_0^m\} \) be a collection of open polydisks such that \( V_0 \subseteq V_0^{m+1} \subseteq V_0^m \subseteq U_0 \) for all \( m \geq 1 \ (i = 1,2) \).

We set \( V_i^m := V_0^m \times L_i^m \ (i = 1,2), m \geq 1 \).

We now amalgamate the free resolutions of \( R \).

Let \( m = 1 \). By Corollary 9.11.20 there exist functions \( h_i \in \mathcal{O}(V_i^1, GL_k(\mathbb{C})) \ (i = 1,2) \), such that \( H = h_1 h_2 \) on \( V_1^1 \cap V_2^1 \). Since \( \alpha_1 \pi_1 H = \alpha_2 \pi_2 \), the sheaf homomorphisms

\[
\alpha_1 \pi_1 h_1 : \mathcal{O}^{k_1}|_{V_1^1} \oplus \mathcal{O}^{k_2}|_{V_1^1} \to R|_{V_1^1} \to 0,
\]

\[
\alpha_2 \pi_2 h_2^{-1} : \mathcal{O}^{k_1}|_{V_2^1} \oplus \mathcal{O}^{k_2}|_{V_2^1} \to R|_{V_2^1} \to 0
\]

coincide over \( V_1^1 \cap V_2^1 \); they induce an analytic homomorphism

\[
\alpha : \mathcal{O}^{k_1}|_{V_1^1 \cup V_2^1} \oplus \mathcal{O}^{k_2}|_{V_1^1 \cup V_2^1} \to R|_{V_1^1 \cup V_2^1}.
\]

Let \( R_1 := \text{Ker} \ \alpha \). The sequence

\[
0 \to R_1|_{V_1^1 \cup V_2^1} \to \mathcal{O}^{k_1}|_{V_1^1 \cup V_2^1} \oplus \mathcal{O}^{k_2}|_{V_1^1 \cup V_2^1} \xrightarrow{\alpha} R|_{V_1^1 \cup V_2^1} \to 0
\]

is completely exact over sets \( V_1^1 \) and \( V_2^1 \) since sequences (9.11.58) are. By Lemma 9.11.24 the analytic sheaf \( R_1 \) has free resolutions over \( V_1^1 \) and \( V_2^1 \) (of length \( 4N - 2n - 1 \)) because two other sheaves do.

Provided that \( M \) was chosen sufficiently large, we can repeat this construction \( N - 1 \) times more over subsets \( V_1^m, V_2^m, 1 \leq m \leq N - 1 \), obtaining in the end a free resolution of \( R \) over \( V_1 \cup V_2 \) having length \( N \). Since \( V_0, L_1, L_2 \) and \( N \) were arbitrary, the required result follows. \( \square \)
9.11.5 Proof of Lemma 9.11.24

We will need the following lemmas.

**Lemma 9.11.25.** Let $A$ be an analytic sheaf on $U$ that admits a free resolution of length $N$

$$F_N|_U \xrightarrow{\varphi_{N-1}} \cdots \xrightarrow{\varphi_1} F_2|_U \xrightarrow{\varphi_0} F_1|_U \xrightarrow{\varphi_0} A \rightarrow 0,$$  \hfill (9.11.59)

Given a completely exact sequence of analytic sheaves $B_i$ on $U$, $0 \leq i \leq N$,

$$B_N \xrightarrow{\beta_{N-1}} \cdots \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_0} B_0 \rightarrow 0, \quad (9.11.60)$$

a sheaf homomorphism $\Phi_0 : A \rightarrow B_0$ can be extended to a homomorphism $\Phi_j : F_j|_U \rightarrow B_j$ ($0 \leq j \leq N$) of sequences (9.11.34), (9.11.60).

**Proof.** The proof is by induction. We put $\varphi_{-1} := 0$ (cf. (9.11.59)), $\beta_{-1} := 0$. Suppose that for $0 \leq j \leq r$, $r \leq N - 1$, the homomorphisms $\Phi_j : F_j|_U \rightarrow B_j$ have been constructed, so that $\Phi_j \varphi_j = \beta_j \Phi_j$. If $r = N - 1$, then we are done. For $r < N - 1$ we have $\beta_{r-1}(\Phi_r \varphi_r) = \Phi_{r-1} \varphi_{r-1} \varphi_r = 0$. The sequence

$$\Gamma(U, \text{Hom}_O(F_{r+1}, B_{r+1})) \rightarrow \cdots \rightarrow \Gamma(U, \text{Hom}_O(F_1, B_0)) \rightarrow 0$$

is exact since (9.11.60) is completely exact (cf. Definition 9.11.22), hence there is a homomorphism $\Phi_{r+1} \in \Gamma(U, \text{Hom}_O(F_{r+1}, B_{r+1}))$ such that $\Phi_r \varphi_r = \beta_r \Phi_{r+1}$ over $U$, as required. \hfill $\square$

**Lemma 9.11.26.** Given a free resolution (9.11.34) of an analytic sheaf $A$ on $U$ of length $N$, the sheaf $\text{Ker} \varphi_{n-1} = \text{Im} \varphi_n$ on $U$, $1 \leq n \leq N - 1$, has a free resolution of length $N - n$.

**Proof.** Immediate from the Definition 9.11.14 of free resolution of an analytic sheaf. \hfill $\square$

**Lemma 9.11.27.** Let $A_0$ be an analytic sheaf over $U$. Suppose that for a given $N \geq 1$ there exists a completely exact sequence of analytic sheaves $A_i$ on $U$, $1 \leq i \leq 2N + 2$,

$$A_{2N+2} \xrightarrow{\alpha_{2N+1}} \cdots \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_0} A_0 \rightarrow 0,$$  \hfill (9.11.61)

such that sheaves $A_i$, $1 \leq i \leq 2N + 2$, have free resolutions of length $n + N$, where $n := \dim \mathbb{C} U_0$. Then $A_0$ has a free resolution of length $N$.  

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(1) First, we construct a completely exact sequence of length $M - 2$ of the form

$$B_{M-2} \xrightarrow{\beta_{M-3}} \ldots \xrightarrow{\beta_2} B_2 \xrightarrow{\beta_1} B_1 \xrightarrow{e_0} A_0 \to 0,$$

where $B_1 = \mathcal{O}^k|_U$ for some $k \geq 0$, a free sheaf, and $B_k$, $2 \leq k \leq M - 2$, are analytic sheaves on $U$ having free resolutions of length $N + n - 1$. Let

$$\mathcal{F}_{n+N,k} \to \cdots \to \mathcal{F}_{1,k} \xrightarrow{\omega_k} A_k \to 0$$

be a free resolution of $A_k$, $1 \leq k \leq M$. By Lemma 9.11.25 there exist analytic homomorphisms $\psi_k$ such that the diagram

$$\begin{array}{ccc}
A_M & \xrightarrow{\alpha_{M-1}} & \ldots & \xrightarrow{\alpha_2} & A_2 & \xrightarrow{\alpha_1} & A_1 & \xrightarrow{\alpha_0} & A_0 & \to 0 \\
\downarrow{\omega_M} & & & & \downarrow{\omega_2} & & \downarrow{\omega_1} & & \\
F_{1,M} & \xrightarrow{\psi_{M-1}} & \ldots & \xrightarrow{\psi_2} & F_{1,2} & \xrightarrow{\psi_1} & F_{1,1}
\end{array}$$

is commutative. Let us show that sequence

$$\mathcal{F}_{1,M} \oplus \ker \omega_{M-1} \xrightarrow{\beta_{M-1}} \ldots \xrightarrow{\beta_2} F_{1,2} \oplus \ker \omega_1 \xrightarrow{\beta_1} F_{1,1} \xrightarrow{\beta_0} A_0 \to 0,$$

truncated to term $\mathcal{F}_{1,M-2} \oplus \ker \omega_{M-3}$, is completely exact. Here $\beta_0 := \alpha_0 \omega_1$, $\beta_1 := \psi_1 - \iota_1$, where $\iota_k : \ker \omega_k \hookrightarrow F_k$ is an inclusion, and $\beta_k = (\iota_k \oplus \psi_{k-1})(\psi_k - \iota_k)$, $k \geq 2$. We apply to (9.11.63) and (9.11.64) a left exact functor $\Gamma(U, \text{Hom}_\mathcal{O}(\mathcal{E}, \cdot))$, where $\mathcal{E}$ is a free sheaf: let

$$A_k := \Gamma(U, \text{Hom}_\mathcal{O}(\mathcal{E}, A_k)), \quad (0 \leq k \leq M),$$

$$F_k := \Gamma(U, \text{Hom}_\mathcal{O}(\mathcal{E}, F_k)), \quad (1 \leq k \leq M).$$

We obtain commutative diagrams of Abelian groups

$$\begin{array}{ccc}
A_M & \xrightarrow{a_{M-1}} & \ldots & \xrightarrow{a_2} & A_2 & \xrightarrow{a_1} & A_1 & \xrightarrow{a_0} & A_0 & \to 0 \\
\downarrow{w_M} & & & & \downarrow{w_2} & & \downarrow{w_1} & & \\
F_M & \xrightarrow{p_{M-1}} & \ldots & \xrightarrow{p_2} & F_2 & \xrightarrow{p_1} & F_1
\end{array}$$
and
\[ F_M \oplus \text{Ker } w_{M-1} \xrightarrow{b_{M-1}} \cdots \xrightarrow{b_2} F_2 \oplus \text{Ker } w_1 \xrightarrow{b_1} F_1 \xrightarrow{b_0} A_0 \rightarrow 0. \] (9.11.66)

By Definition 9.11.22 the middle row of (9.11.65) is exact. Also, by Proposition 9.11.1(1) each \( w_k, 1 \leq k \leq M \), is surjective, so the columns of (9.11.65) are exact. Hence, we have analogous identities
\[ b_0 = a_0 w_1, \quad b_1 = p_1 - i_1, \quad b_k = (i_k \oplus p_{k-1})(p_k - i_k), \] (9.11.67)
where \( i_k : \text{Ker } w_k \hookrightarrow F_k \) is an inclusion. Let us show that (9.11.66) is exact up to term \( F_{M-2} \oplus \text{Ker } w_{M-3} \). First, note that \( b_0 \) is surjective because both \( a_0 \) and \( w_1 \) are. Second, if \( \xi \in \text{Ker } b_0 \), then \( w_1(\xi) \in \text{Ker } a_0 = \text{Im } a_1 = \text{Im } a_2 = \text{Im } w_1 p_1 \). Here we have used the fact that \( w_2 \) is surjective. Let \( w_1(\xi) = p_1(\zeta) \) and \( \tau := p_1(\zeta) - \xi \in \text{Ker } w_1 \), so that \( \xi = b_1(\zeta, \tau) \in \text{Im } b_1 \). Third, if \( 1 \leq k \leq M-3 \), and \( (\xi, \eta) \in \text{Ker } b_k = \text{Ker } (p_k - i_k) \), then \( \eta = p_k(\xi) \) and \( 0 = w_k(p_k(\xi)) = a_k(w_{k+1}(\xi)) \), hence \( w_{k+1}(\xi) \in \text{Im } a_{k+1} = \text{Im } a_{k+1} w_{k+2} = \text{Im } w_{k+1} p_{k+1} \). Choose \( \zeta \) so that \( w_{k+1}(\xi) = w_{k+1}(p_{k+1}(\xi)) \). Then \( \tau := p_{k+1}(\zeta) - \xi \in \text{Ker } w_{k+1} \). We conclude that \( (\xi, \eta) = b_{k+1}(\zeta, \tau) \in \text{Im } b_{k+1} \), as required.

By Lemma 9.11.26 each \( F_k \oplus \text{Ker } w_{k-1} \) has a free resolution of length \( N + n - 1 \). Hence, if we take
\[ B_1 := F_1, \quad \varepsilon_0 := \beta_0, \quad B_k := F_k \oplus \text{Ker } w_{k-1}, \quad 2 \leq k \leq M - 2, \]
we obtain the required completely exact sequence (9.11.62).

(2) Now, consider completely exact sequence (cf. (9.11.62))
\[ B_M \xrightarrow{\beta_{M-1}} \cdots \xrightarrow{\beta_2} B_2 \xrightarrow{\beta_1} \text{Ker } \varepsilon_0 \rightarrow 0. \]

We have proved that there is a completely exact sequence
\[ \mathcal{D}_{M-4} \rightarrow \cdots \rightarrow \mathcal{D}_3 \rightarrow \mathcal{D}_2 \xrightarrow{\varepsilon_1} \text{Ker } \varepsilon_0 \rightarrow 0, \]
where \( \mathcal{D}_2 \) is a free sheaf, and each sheaf \( \mathcal{D}_k, 3 \leq k \leq M - 4 \), has free resolution of length \( N - n - 2 \). Therefore, we have a completely exact sequence
\[ \mathcal{D}_{M-4} \rightarrow \cdots \rightarrow \mathcal{D}_3 \rightarrow \mathcal{D}_2 \xrightarrow{\varepsilon_1} B_1 \rightarrow A_0 \rightarrow 0. \]
Continuing in this way we obtain a free resolution of \( A_0 \) of length \( N \). □
Proof of Lemma 9.11.24. We can assume that $A \subset B$ and that $\beta$ is the inclusion map.

(a) If $A$ and $B$ have free resolutions of length $N + n$, then Lemma 9.11.27 implies that $C$ has a free resolution of length $N$ (and, in particular, of length $N - n - 1$).

Consider two remaining cases. Sheaf $C$ has a free resolution of length $N + n$,

$$F_{N+n} \rightarrow \cdots \rightarrow F_1 \xrightarrow{\varphi} C \rightarrow 0$$  \hspace{1cm} (9.11.68)

for some open $V_0 \subset U_0$. By Proposition 9.11.1(1) sequence (9.11.68) is completely exact. By Lemma 9.11.23 there is a commutative diagram

Let $\iota : \ker \varphi \rightarrow F_1$ denote the inclusion. Let us show that the sequence

$$0 \rightarrow \ker \varphi \xrightarrow{\psi} A \oplus F_1 \xrightarrow{\beta - \psi} B \rightarrow 0$$  \hspace{1cm} (9.11.70)

is completely exact.

We apply functor $\Gamma(U, \text{Hom}_\mathcal{O}(\mathcal{E}, \cdot))$ to (9.11.69) and (9.11.70), where $\mathcal{E}$ is a free sheaf. We obtain diagrams of Abelian groups

and

$$0 \rightarrow \ker f \xrightarrow{\varphi} A \oplus F_1 \xrightarrow{b - \varphi} B \rightarrow 0.$$  \hspace{1cm} (9.11.71)

The first diagram is commutative, its top row is exact (cf. Definition 9.11.22). By Proposition 9.11.1(1) we may assume that $f$ is surjective. The latter sequence is a complex and is exact at
Ker $f$. We have to check that it is exact at the next two terms. If $(\xi, \eta) \in \text{Ker } (b - p)$ then $p(\eta) = b(\xi) = \xi, 0 = c(p(\eta)) = f(\eta)$. Thus, $\eta \in \text{Ker } f$ and $(\xi, \eta) = (p \oplus i)(\eta) \in \text{Im } (p \oplus i)$, hence (9.11.71) is exact in the middle term. On the other hand, if $\zeta \in B$, then with some $\eta \in F$

$$-c(\zeta) = f(\eta) = c(p(\eta)),$$

i.e.,

$$\zeta + p(\eta) = \xi \in \text{Ker } c = A.$$

Thus, $\zeta = \xi - p(\eta) \in \text{Im } (b - p)$. We obtain that sequence (9.11.71) is exact, hence sequence (9.11.70) is completely exact.

Note that by Lemma 9.11.26 Ker $\varphi$ has a free resolution of length $N + n - 1$.

(b) The sheaf $A \oplus F_1$ has a free resolution of length $N + n - 1$ over $U_0 \times K$. By Lemma 9.11.27 sheaf $B$ has a free resolution of length $N - 1$ over $U_0 \times K$ (in particular, of length $N - n - 1$).

(c) We may assume that $B$ has a free resolution of length $N + n - 1$ over $V_0 \times K$. Since Ker $\varphi$ has a free resolution of length $N + n - 1$, by (b) $A \oplus F$ has a free resolution of length $N - 1$. Since sequence $0 \to F \to A \oplus F \to A \to 0$ is completely exact as $F$ is a free sheaf (cf. Corollary 9.11.12), we obtain by part (a) that $A$ has a free resolution of length $N - n - 1$. \hfill \Box

9.12 Proof of Proposition 6.3.2

The proof essentially repeats the proof of an analogous result for coherent analytic sheaves on complex manifolds, see, e.g., [GR].

First, let $\mathcal{A}$ be a coherent subsheaf of $\mathcal{O}^k$, let $U \in \mathcal{B}$ (cf. (6.2.7)). By Lemma 9.11.4(2) there exist open sets $V_k \in \mathcal{B}$ such that $V_k \subseteq V_{k+1} \subseteq U$ for all $k$, and $U = \cup_k V_k$. We endow space $\Gamma(U, \mathcal{A})$ of sections of sheaf $\mathcal{A}$ over $U$ with the topology of uniform convergence on $\bar{V}_k$, for all $k$. Then $\Gamma(U, \mathcal{A})$ becomes a metrizable vector space. We have to show that space $\Gamma(U, \mathcal{A})$ is complete, i.e., it is a Frechet space.

It is easy to see that space $\Gamma(U, \mathcal{O}^k)$ endowed with such topology is complete. Since $\mathcal{A}$ is coherent, we may assume that there exists a free resolution (6.3.8) of $\mathcal{A}$ over $U$ of length $4N$,
\( N > n := \dim_{\mathbb{C}} X_0. \) Therefore, we have a short exact sequence

\[
0 \to \text{Ker } \varphi \overset{i}{\to} \mathcal{O}^m|_U \overset{\varphi}{\to} \mathcal{A}|_U \to 0,
\]

where \( i \) denotes the inclusion. In the proof of Proposition 9.11.1(1) we have shown that the sequence of sections

\[
0 \to \Gamma(U, \text{Ker } \varphi) \overset{i}{\to} \Gamma(U, \mathcal{O}^m) \overset{\varphi}{\to} \Gamma(U, \mathcal{A}) \to 0
\]

(9.12.72) is exact (cf. Lemmas 9.11.15 and 9.11.17). By our assumption \( \Gamma(U, \mathcal{A}) \subset \Gamma(U, \mathcal{O}^k) \). By Lemma 9.11.21 the \( \Gamma(U, \mathcal{O}) \)-module homomorphism \( \varphi_* : \Gamma(U, \mathcal{O}^m) \to \Gamma(U, \mathcal{O}^k) \) is determined by a \( k \times m \) matrix with entries in \( \mathcal{O}(U) \), hence it is continuous; further, \( \iota_* \) is continuous. Since sequence (9.12.72) is exact, \( \Gamma(U, \text{Ker } \varphi) \cong \text{Ker } \varphi_* \), hence \( \Gamma(U, \text{Ker } \varphi) \) is closed. Therefore, \( \Gamma(U, \mathcal{A}) \), being a quotient of a complete space by its closed subspace, is a complete space.

We note that by the open mapping theorem the topology in \( \Gamma(U, \mathcal{A}) \) coincides with the quotient topology determined by (9.12.72).

Now, let \( \mathcal{A} \) be an arbitrary coherent sheaf on \( c_a X \). Similarly, we have a free resolution (6.3.8) of \( \mathcal{A} \) over a neighbourhood \( U \) of length \( 4N, N > n \), which yields a short exact sequence of sheaves

\[
0 \to \text{Ker } \varphi \overset{i}{\to} \mathcal{O}^m|_U \overset{\varphi}{\to} \mathcal{A}|_U \to 0
\]

(9.12.73) and an exact sequence of sections

\[
0 \to \Gamma(U, \text{Ker } \varphi) \overset{i}{\to} \Gamma(U, \mathcal{O}^m) \overset{\varphi}{\to} \Gamma(U, \mathcal{A}) \to 0.
\]

(9.12.74)

Using Lemma 9.11.24, we obtain that \( \text{Ker } \varphi \) is a coherent subsheaf of \( \mathcal{O}^m|_U \), so by the previous part the subspace \( \Gamma(U, \text{Ker } \varphi) \subset \Gamma(U, \mathcal{O}^m) \) is closed. We introduce in \( \Gamma(U, \mathcal{A}) \) the quotient topology defined by (9.12.74), which makes it a complete (i.e., Frechet) space, and also implies the last assertion of the proposition concerning the semi-norms determining the topology in \( \Gamma(U, \mathcal{A}) \).

Let us show that thus defined topology on \( \Gamma(U, \mathcal{A}) \) does not depend on the choice of resolution (9.12.73). Suppose that there is another resolution

\[
0 \to \text{Ker } \varphi' \overset{i}{\to} \mathcal{O}^m'|_U \overset{\varphi'}{\to} \mathcal{A}|_U \to 0.
\]

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By Lemma 9.11.25 there is a homomorphism \( \psi : \mathcal{O}^m|_U \to \mathcal{O}^{m'}|_U \) such that the diagram of exact sequences of sheaves

\[
\begin{array}{ccc}
\mathcal{O}^m|_U & \xrightarrow{\varphi} & \mathcal{A}|_U \\
\downarrow \psi & & \downarrow \lambda \\
\mathcal{O}^{m'}|_U & \xrightarrow{\varphi'} & \mathcal{A}|_U
\end{array}
\]

is commutative. Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
\Gamma(U, \mathcal{O}^m) & \xrightarrow{\varphi^*} & \Gamma(U, \mathcal{A}) \\
\downarrow \psi_* & & \downarrow \lambda_* \\
\Gamma(U, \mathcal{O}^{m'}) & \xrightarrow{\varphi'_*} & \Gamma(U, \mathcal{A})
\end{array}
\]

of exact sequences of sections. By our construction \( \varphi_* \), \( \varphi'_* \) are continuous and surjective, \( \psi_* \) is continuous as a homomorphism of sections of free sheaves. By the open mapping theorem the preimage of an open set by \( \lambda_*^{-1} = \varphi_* (\psi_*)^{-1} (\varphi'_*)^{-1} \) is open, so \( \lambda_* \) is continuous, hence it is a homeomorphism.

Finally, let \( \gamma : \mathcal{A} \to \mathcal{B} \) be an analytic homomorphism. Let us show that \( \gamma \) is continuous. Analogously to the previous part, applying Lemma 9.11.25, we obtain a commutative diagram of exact sequences of sheaves, which yields a commutative diagram of exact sequences

\[
\begin{array}{ccc}
\Gamma(U, \mathcal{O}^m) & \xrightarrow{\varphi^*} & \Gamma(U, \mathcal{A}) \\
\downarrow \psi_* & & \downarrow \gamma_* \\
\Gamma(U, \mathcal{O}^{m'}) & \xrightarrow{\varphi'_*} & \Gamma(U, \mathcal{B})
\end{array}
\]

As before, the continuity of \( \gamma_* \) can be deduced from the continuity of the other homomorphisms, as before. This completes the proof of the proposition.

### 9.13 Proof of Theorem 6.4.1

By Proposition 6.2.8(2) \( M_X \) is homeomorphic to the maximal ideal space of \( \mathcal{O}(c_a X) \). Further, it follows, e.g. from Theorem 6.1.5, that algebra \( \mathcal{O}(c_a X) \) separates points of \( c_a X \), therefore we have a continuous injection \( c_a X \hookrightarrow M_X \) via point evaluation homomorphisms. Let us show that it is surjective.
Let \( \varphi \in M_X \). We identify \( \mathcal{O}(X_0) \) with \( \bar{p}^*\mathcal{O}(X_0) \subset \mathcal{O}(c_a X) \). The restriction \( \varphi|_{\mathcal{O}(X_0)} \) belongs to the maximal ideal space of \( \mathcal{O}(X_0) \). Since \( X_0 \) is a Stein manifold, there exists a point \( x_0 \in X_0 \) such that \( \varphi|_{\mathcal{O}(X_0)} = \psi_{x_0} \), where \( \psi_{x_0}(u) := u(x_0), u \in \mathcal{O}(X_0), \) is the evaluation homomorphism at point \( x_0 \) (see, e.g., [GR]).

There exists a function \( h \in \mathcal{O}(X_0) \) such that \( X_0^{n-1} := \{ x \in X_0 : h(x) = 0 \} \) is a non-singular hypersurface, and \( x_0 \in X_0^{n-1} \) [For]. We set \( X_0^{n-1} := p^{-1}(X_0^{n-1}) \) and \( c_a X^{n-1} := \bar{p}^{-1}(X_0^{n-1}) \).

If \( f \in \mathcal{O}(X_0) \) is identically zero on \( c_a X^{n-1} \), then \( \varphi(f) = 0. \) Indeed, by Lemma 9.14.2 the function \( \tilde{f} := f/\bar{p}^*h \in \mathcal{O}(c_a X) \), hence \( \varphi(f) = \varphi(\tilde{f}) \varphi(\bar{p}^*h) = \varphi(\tilde{f}) \psi_{x_0}(h) = 0. \)

Thus, homomorphism \( \varphi \) is well defined on the quotient algebra \( \mathcal{O}(c_a X)/I_{c_a(X^{n-1})} \), where \( I_{c_a(X^{n-1})} \) is the ideal of holomorphic functions vanishing on \( c_a X^{n-1} \). We have

\[
\mathcal{O}(c_a X)/I_{c_a X^{n-1}} \cong \mathcal{O}(c_a X^{n-1}),
\]

hence the homomorphism \( \varphi \) can be identified with an element of the maximal ideal space of algebra \( \mathcal{O}(c_a X^{n-1}). \)

Proceeding in this way, we define sets \( X_k^0, X^k, c_a X^k, 0 \leq k \leq n-1 \), and obtain that homomorphism \( \varphi \) may be viewed as an element of the maximal ideal space of algebra \( \mathcal{O}(c_a X^0) \), where \( X_0^0 = \{ x_0, x_1, \ldots \} \) is a discrete set. By definition, \( \mathcal{O}(c_a X^0) \cong \sqcup_{i \geq 0} C(\bar{p}^{-1}(x_i)), \) so \( \varphi \) coincides with the evaluation homomorphism at a point of \( \bar{p}^{-1}(x_0) \subset c_a X \), as needed.

Finally, it is easy to see that the topology in \( c_a X \) is the weakest topology in which all point evaluation homomorphisms of \( \mathcal{O}(c_a X) \) are continuous, i.e., the continuous bijection between \( c_a X \) and \( M_X \) is a homeomorphism.

### 9.14 Proofs of Propositions 6.5.4, 6.5.6, 6.5.7, 6.5.8 and 6.5.9

#### 9.14.1 Proof of Proposition 6.5.4

(1) Immediate from the definition of a complex hypersurface in \( c_{AP} X \).

(2) The proof follows from the next two lemmas.

**Lemma 9.14.1.** Let \( U_0 \subset \mathbb{C}^n, K \subset \hat{G}_a \) be open, \( f \in \mathcal{O}(U_0 \times K) \) (cf. Section 6.2.6). Then

\[
\frac{\partial f}{\partial z_k} \in \mathcal{O}(U_0 \times K), \quad 1 \leq k \leq n, \quad z = (z_1, \ldots, z_n) \in U_0.
\]
Proof of Lemma 9.14.1. By definition, \( f(\cdot, \xi) \in \mathcal{O}(U_0) \) for every \( \xi \in K \). Let \( \Delta_1 \subseteq \Delta_2 \subseteq U_0 \) be open polydisks. By Cauchy formula
\[
\frac{\partial f}{\partial z_k}(z, \xi) = -\frac{1}{(2\pi i)^n} \int_{\partial \Delta_1} \frac{f(\zeta, \xi)}{(z_1 - \zeta_1) \cdots (z_k - \zeta_k)^2 \cdots (z_n - \zeta_n)^2} d\zeta, \quad z \in \Delta_1, \tag{9.14.75}
\]
where \( \partial \Delta_1 \) stands for the boundary torus of \( \Delta_1 \). Since \( f \) is continuous and \( \Delta_1 \) is relatively compact in \( U_0 \), by Montel theorem \( f(\cdot, \xi_k) \to f(\cdot, \xi) \) uniformly on \( \Delta_1 \) if \( \xi_k \to \xi \) in \( K \). Therefore, by (9.14.75)
\[
\frac{\partial f}{\partial z_k}(\cdot, \xi_k) \to \frac{\partial f}{\partial z_k}(\cdot, \xi) \quad \text{uniformly on } \Delta_2.
\]
It follows that \( \frac{\partial f}{\partial z_k} \in \mathcal{C}(U_0 \times K) \) and \( \frac{\partial f}{\partial z_k}(\cdot, \xi) \in \mathcal{O}(U_0) \) (\( \xi \in K \)), hence \( \frac{\partial f}{\partial z_k} \in \mathcal{O}(U_0 \times K) \).

Lemma 9.14.2. Let \( U_0 \subset X_0, K \subset \hat{G}_a \) be open, \( f \in \mathcal{O}(U_0 \times K) \) be such that \( \nabla_z f(z, \eta) \neq 0 \) for all \( (z, \eta) \in Z_f := \{(z, \eta) \in U_0 \times K: f(z, \eta) = 0\} \). Suppose that \( g \in \mathcal{O}(U_0 \times K) \) vanishes on \( Z_f \). Then \( h := g/f \in \mathcal{O}(U_0 \times K) \).

Proof of Lemma 9.14.2. Since this is a local statement, by Proposition 9.5.6 we may assume that \( U_0 \) is an open polydisk in \( \mathbb{C}^n \), and \( f(z_1, \ldots, z_n, \xi) = z_1 \) on \( U_0 \times K \). Now, for each \( \eta \in K \) we have
\[
g(z_1, \ldots, z_n, \eta) = z_1 \int_0^1 \frac{\partial g(tz_1, \ldots, z_n, \xi)}{\partial z_1} dt, \quad (z_1, \ldots, z_n) \in U_0. \tag{9.14.76}
\]
By Lemma 9.14.1 function
\[
(z_1, \ldots, z_n, \eta) \to \frac{\partial g(tz_1, \ldots, z_n, \xi)}{\partial z_1}
\]
belongs to \( \mathcal{O}(U_0 \times K) \) for all \( t \in [0, 1] \). By Montel’s theorem, if \( \eta_k \to \eta \) in \( K \), then
\[
\frac{\partial g(z_1, \ldots, z_n, \eta_k)/\partial z_1}{\partial g(z_1, \ldots, z_n, \eta)/\partial z_1}
\]
uniformly over a smaller polydisk \( \Delta \subseteq U_0 \). Hence,
\[
\frac{\partial g(tz_1, \ldots, z_n, \eta_k)/\partial z_1}{\partial g(tz_1, \ldots, z_n, \eta)/\partial z_1}
\]
as \( k \to \infty \) uniformly over \( [0, 1] \times \Delta \). Therefore, by shrinking \( U_0 \) if necessary, we obtain that function \( h(z, \eta) \) (\( z \in U_0, \eta \in K \)), defined to be the second multiple in (9.14.76), is continuous in \( \eta \) and, therefore belongs to \( \mathcal{O}(U_0 \times K) \), as required.
Now, if $D = \{(U_\alpha, f_\alpha)\}$, $G = \{(W_\gamma, g_\gamma)\}$ are smooth divisors in $\text{Div}(c_{AP}X)$ having the same supports, then by Lemma 9.14.2 we have $f_\alpha/g_\gamma, g_\gamma/f_\alpha \in \mathcal{O}(U_\alpha \cap W_\gamma)$ (here we use the fact that every point in $U_\alpha \cap W_\gamma$ has a neighbourhood of the form $\tilde{\Pi}(U_0, K) \cong U_0 \times K$ for some open $U_0 \subset X_0$, $K \subset \hat{G}_a$, cf. Section 6.2.6) for all $\alpha, \gamma$, i.e., divisors $D, G$ are equivalent.

(3) The first assertion is immediate from the definition of a smooth divisor on $c_{AP}X$. The second assertion follows from (2).

(4) If the complex hypersurface $Y$ is cylindrical, then by definition $D_Y$ is a cylindrical divisor. Conversely, suppose that $D_Y$ is equivalent to a cylindrical divisor $D \in \text{Div}(c_{AP}X)$; then the open cover and holomorphic functions determining $D$ (cf. Definition 6.2.11) determine a cylindrical hypersurface that coincides with $Y$, as needed.

9.14.2 Proof of Proposition 6.5.6

Our proof consists of three parts.

1. By definition, divisor $H$ is not equivalent to a cylindrical divisor if there exists a cylindrical neighbourhood $U := p_{b,T^2}^{-1}(U_0) \subset b_{T^2}X$, where $U_0 \subset X_0$ is open simply connected, such that the restriction $H|_U$ of $H$ to $U$ is not equivalent to any principal divisor on $U$ (i.e., a divisor defined by a single function in $\mathcal{O}(U)$).

We can reformulate the latter statement in terms of the Chern classes of divisors on $U$. By definition the Chern class $c_{U,T^2}(D) \in H^2(U, \mathbb{Z})$ of a divisor $D$ on $U$ is the Chern class of the line bundle associated with $D$ (i.e., constructed by transition functions which are the ratios of holomorphic functions on an open cover of $U$ determining $D$). If $D$ is principal, then the associated line bundle is topologically trivial; hence, $c_{U,T^2}(D) = 0$.

Therefore, if $H|_U$ is equivalent to a principal divisor, then $c_{U,T^2}(H|_U) = 0$.

2. We have a surjective map $Q : c_{AP}X \to b_{T^2}X$, cf. Section 6.5(II), such that if $D \in \text{Div}(b_{T^2}X)$, then $Q^*D \in \text{Div}(c_{AP}X)$. By definition, divisor $(i^X)^*H \in \text{Div}_{AP}(X)$ is cylindrical if and only if $Q^*H \in \text{Div}(c_{AP}X)$ is cylindrical (cf. Definition 6.5.1).

As in part 1, we define Chern classes $c_{Q^{-1}(U)}(D) \in H^2(Q^{-1}(U), \mathbb{Z})$ of $D \in \text{Div}(c_{AP}X)$ (note that $Q^{-1}(U) = p_b^{-1}(U_0)$). Therefore, if divisor $Q^*D|_{Q^{-1}(U)}$ is equivalent to a principal divisor,
then $c_{Q^{-1}(U)}(Q^*D|_{Q^{-1}(U)}) = 0$. By the functoriality of Chern classes we have

$$c_{Q^{-1}(U)}(Q^*D|_{Q^{-1}(U)}) = Q^*(c_{U,T^2}(H|_U)),$$

where $Q^* : H^2(U, \mathbb{Z}) \to H^2(Q^{-1}(U), \mathbb{Z})$ is the map induced by $Q|_U$.

3. Thus, we must prove that if $c_{U,T^2}(H|_U) \neq 0$, then $Q^*(c_{U,T^2}(H|_U)) \neq 0$. Without loss of generality we may assume that $U_0$ is contractible. Hence, $U \cong U_0 \times T^2$ is homotopic to $T^2$ and $Q^{-1}(U) \cong U_0 \times b\mathbb{Z}$ is homotopic to $b\mathbb{Z}$.

Further, for a fixed point $x \in U_0$, the embeddings $T^2 \cong p_{b,T^2}^{-1}(x) \hookrightarrow U$ and $b\mathbb{Z} \cong p_b^{-1}(x) \hookrightarrow Q^{-1}(U)$ induce isomorphisms of the corresponding cohomology groups. Identifying under these isomorphisms $c_{U,T^2}(H|_U)$ with a non-zero element $\eta \in H^2(T^2, \mathbb{Z})$ and $Q^*(c_{U,T^2}(H|_U))$ with $q^*(\eta) \in H^2(b\mathbb{Z}, \mathbb{Z})$, where $q := Q|_{Q^{-1}(x)} : b\mathbb{Z} \to T^2$ is the restriction (surjective) homomorphism, we see that it suffices to show that $q^*$ is injective.

To do this, we present $b\mathbb{Z}$ as inverse limit of an inverse family of compact Abelian Lie groups $\mathcal{Z}_\alpha$, $\alpha \in \Lambda$ (a partially ordered set with the infimal element 0), equipped with surjective homomorphisms $q_\alpha^\beta : \mathcal{Z}_\beta \to \mathcal{Z}_\alpha$ for $\alpha \leq \beta$ and such that $\mathcal{Z}_0 := T^2$ and the limit homomorphism $q_0 : b\mathbb{Z} \to \mathcal{Z}_0$ coincides with $q$. Then if $q^*(\omega) = 0$ for some non-zero $\omega \in H^2(T^2, \mathbb{Z}) \cong \mathbb{Z}$, there exists an index $\beta > 0$ such that $(q_0^\beta)^*(\omega) = 0 \in H^2(\mathcal{Z}_\beta, \mathbb{Z})$. By definition,

$$\mathcal{Z}_\beta = T^m \times \bigoplus_{l=1}^k \mathbb{Z}/n_l \quad \text{for certain } m \geq 2, k, n_l \in \mathbb{Z}_+, \quad (9.14.77)$$

where $T^m := (S^1)^m$ is the real $m$-torus.

Since $q_0^\beta$ is surjective, its restriction $\hat{q}$ to $T^m$ is surjective as well, and the restriction of $(q_0^\beta)^*(\omega)$ to $T^m$ is 0 in $H^2(T^m, \mathbb{Z})$. Thus it suffices to prove that the surjective homomorphism $\hat{q} : T^m \to T^2$ induces the injective map of 2-cohomology groups.

Indeed, the kernel of $\hat{q}$ is isomorphic to $T^{m-2} \oplus \Gamma$, where $\Gamma$ is a finite Abelian group, and the regular covering $r : M \to T^2$ of $T^2$ with the transformation group $\Gamma$ is also isomorphic to $T^2$. Moreover, there exists a surjective homomorphism $s : T^m \to M$ with connected fibres isomorphic to $T^{m-2}$ such that $\hat{q} = r \circ s$. One easily shows that the exact sequence of groups

$$0 \to T^{m-2} \cong \text{Ker}(s) \to T^m \xrightarrow{s} M \to 0$$

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splits, i.e., there exists a monomorphism \( \hat{s} : M \to \mathbb{T}^m \) such that \( \mathbb{T}^m = \text{Ker}(s) \oplus \hat{s}(M) \). Thus if \( \hat{q}^*(\omega) = 0 \) for some non-zero \( \omega \in H^2(\mathbb{T}^2, \mathbb{Z}) \), then the restriction of \( q^*(\omega) \) to \( \hat{s}(M) \cong M \) is 0 in \( H^2(\hat{s}(M), \mathbb{Z}) \). This implies that it suffices to prove that \( r : M \to \mathbb{T}^2 \) induces injection of 2-cohomology groups. But the map \( r^* : \mathbb{Z} \cong H^2(\mathbb{T}^2, \mathbb{Z}) \to H^2(M, \mathbb{Z}^2) \cong \mathbb{Z} \) is multiplication by \( \#\Gamma \) (= the degree of \( r \)); that is, \( r^* \) is injective, as required.

### 9.14.3 Proof of Proposition 6.5.7

Recall that \( X_0 \) is relatively compact in a Riemann surface \( \tilde{X}_0 \) such that \( \pi_1(X_0) \cong \pi_1(\tilde{X}_0) \). Consider the composite \( \varphi \) of the identity homomorphism \( \mathbb{Z} \to \mathbb{Z} \) and the embedding \( \mathbb{Z} \hookrightarrow \mathbb{C} \), where \( \mathbb{Z} \) is the quotient group of \( \pi_1(\tilde{X}_0) \) corresponding to the deck transformation group of the covering \( p : X \to X_0 \). The holomorphic bundle over \( \tilde{X}_0 \) associated with homomorphism \( \varphi \) has fibre \( \mathbb{C} \) and is given on an acyclic open cover \((U_i)_{i \in I}\) of \( \tilde{X}_0 \) by an integer-valued additive cocycle \( \{c_{ij}\}_{i,j \in I} \). Since \( H^1(\tilde{X}, \mathcal{O}) = 0 \) and the cover is acyclic, by the Leray theorem this cocycle can be resolved, i.e., there are \( c_i \in \mathcal{O}(U_i) \) such that \( c_i - c_j = c_{ij} \) on \( U_i \cap U_j \). Further, let us consider a refinement \((V_j)_{j \in J}\) of \((U_i)_{i \in I}\) such that each \( V_j \) is a relatively compact coordinate chart in some \( U_{i(j)} \). Taking a finite cover \((W_k)\) of \( X_0 \) by elements \( V_j \) and restricting the above cocycle and its resolution to this cover (retaining the same symbols for the restrictions) we get

\[
c_k - c_{\ell} = c_{k\ell} \quad \text{on} \quad W_k \cap W_\ell \tag{9.14.78}
\]

and, moreover, each \( c_k \in \mathcal{O}(W_k) \) is bounded. According to the result of [GN] there exists a function \( f \in \mathcal{O}(\tilde{X}_0) \) without critical points. Adding to functions \( c_k \) the function \( cf \) with a sufficiently large \( c \in \mathbb{C} \), we may and will assume that each holomorphic 1-form \( dc_k \) is nowhere zero on \( W_k \).

By definition the bundle \( b_{(\mathbb{C}^*)^2}X \) is defined on the cover \((W_k)\) of \( X_0 \) by the 1-cocycle

\[
\varphi_{k\ell}(z) := \begin{pmatrix} e^{i\lambda_1 c_{k\ell}} & 0 \\ 0 & e^{i\lambda_2 c_{k\ell}} \end{pmatrix}, \quad z \in W_k \cap W_\ell.
\]

According to (9.14.78) we have

\[
\varphi_{k\ell} = \varphi_k \cdot \varphi_\ell^{-1} \quad \text{on} \quad W_k \cap W_\ell,
\]
Thus $b_{(\mathbb{C}^*)^2}X$ is isomorphic to the trivial bundle $X_0 \times (\mathbb{C}^*)^2$ and this isomorphism is given by the formulas

$$
\eta(z, z_1, z_2) = \left(z, z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}\right), \quad z \in W_k, \quad (z_1, z_2) \in (\mathbb{C}^*)^2.
$$

This formula shows that if $\lambda_1, \lambda_2$ are sufficiently small, then $\eta(b_{T^2}X)$ belongs to $X_0 \times U$ for a given neighbourhood $U \subseteq (\mathbb{C}^*)^2$ of $T^2$.

### 9.14.4 Proof of Proposition 6.5.8

(1) By results of part II (A) of Section 6.5, the complex manifold $X$ admits a holomorphic injective map into $b_{(\mathbb{C}^*)^2}(X)$. Without loss of generality we identify $X$ with its image in $b_{(\mathbb{C}^*)^2}(X)$. The functions $(\pi \circ \eta)^* f_\alpha$ are holomorphic on subsets $(\pi \circ \eta)^{-1}(U_\alpha) \subset b_{(\mathbb{C}^*)^2}(X)$, hence their restrictions to $X$ are holomorphic on open subsets $X \cap (\pi \circ \eta)^{-1}(U_\alpha) \subset X$. This means that the restrictions of functions $(\pi \circ \eta)^* f_\alpha$ to $b_{T^2}X$ are holomorphic on subsets $b_{T^2}X \cap (\pi \circ \eta)^{-1}(U_\alpha) \subset b_{T^2}X$.

(2) Fix a neighbourhood $U \subseteq (\mathbb{C}^*)^2$ such that the projection $q : (\mathbb{C}^*)^2 \to \mathbb{C}^2/\Lambda$ maps $U$ biholomorphically onto its image. Then there exist a finite open cover $(B_\alpha)$ of $\bar{U}$ by open balls and holomorphic functions $f_\alpha \in \mathcal{O}(2B_\alpha)$, where $2B_\alpha$ is the ball with the same center as $B_\alpha$ and of twice the radius of $B_\alpha$, with norms of gradients bounded away from zero on $2B_\alpha$ such that divisor $G$ on $2B_\alpha$ is defined as the set of zeros of $f_\alpha$. By definition, $E$ on $(\pi \circ \eta)^{-1}(U)$ is determined by the family of pullbacks $f_\alpha \circ \eta$. In local coordinates $(z, z_1, z_2)$ on $W_k \times B_\alpha$ (considered as a subset of $b_{(\mathbb{C}^*)^2}X$ with $W_k$ as in (9.14.79) the divisor $E$ is given (for sufficiently small $\lambda_j$) by the equation

$$
g_\alpha(z, z_1, z_2) := f_\alpha(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}) = 0. \quad (9.14.80)
$$

Next, for such $\lambda_j$ the preimage $\eta^{-1}(X_0 \times U) \subset b_{(\mathbb{C}^*)^2}X$ contains the bundle $b_{T^2}X$ (see (9.14.79)). Therefore the intersection of $\text{supp}(E)$ with $b_{T^2}X$ is defined by the above equations with $(z_1, z_2) \in T^2$. Also for such $\lambda_j$, since $\text{supp}(G)$ intersects $T^2$ transversely in finitely many points, for a fixed
$z \in X_0$ the manifold $\text{supp}(E)$ intersects the fibre (torus) over $z$ in $b_{T^2}X$ transversely in finitely many points as well. Moreover, all points $(z_1, z_2) \in T^2$ satisfying (9.14.80) are sufficiently close to the points of the intersection of $\text{supp}(G)$ with $U_\alpha$ and tend to these points uniformly in $z$ as $\lambda_1, \lambda_2$ tend to 0 (this follows from the implicit function theorem).

By $O_\alpha \subset \mathbb{C}^2$ we denote an open ball with center at 0 in the space of parameters $\lambda_1, \lambda_2$ such that for $\lambda = (\lambda_1, \lambda_2) \in O_\alpha$ the expression on the left hand-side of (9.14.80) is well defined (i.e., $(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}) \in 2B_\alpha$ for such $\lambda$ and all $z \in W_k$, $(z_1, z_2) \in B_\alpha$). Then we have for $\lambda \in O_\alpha$, $(z, z_1, z_2) \in W_k \times B_\alpha$,

$$
\frac{\partial g_\alpha}{\partial z}(z, z_1, z_2)
$$

$$
= \frac{\partial f_\alpha}{\partial \xi_1}(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}) \cdot z_1 e^{-i\lambda_1 c_k(z)} \cdot (-i\lambda_1) \cdot \frac{dc_k}{dz}(z)
$$

$$
+ \frac{\partial f_\alpha}{\partial \xi_2}(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}) \cdot z_2 e^{-i\lambda_2 c_k(z)} \cdot (-i\lambda_2) \cdot \frac{dc_k}{dz}(z)
$$

(9.14.81)

$$
= - \frac{dc_k}{dz}(z) \left( \lambda_1 \frac{\partial f_\alpha}{\partial \xi_1}(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)})
$$

$$
+ \lambda_2 \frac{\partial f_\alpha}{\partial \xi_2}(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}) \right).
$$

Suppose that a point $\theta = (\theta_1, \theta_2) \in T^2$ of the intersection of $\text{supp}(G)$ with $T^2$ belongs to $B_\alpha$. The equation

$$
\lambda_1 \frac{\partial f_\alpha}{\partial \xi_1}(\theta_1, \theta_2) + \lambda_2 \frac{\partial f_\alpha}{\partial \xi_2}(\theta_1, \theta_2) = 0
$$

determines a one-dimensional complex subspace $\ell$ in the space $\mathbb{C}^2$ of parameters $\lambda = (\lambda_1, \lambda_2)$. Fix a compact subset $K_\theta \subset \partial O_\alpha \setminus \ell$ on the boundary $\partial O_\alpha$ of $O_\alpha$. Since the numbers of indices $\alpha$ and $k$ in the covers are finite, without loss of generality (decreasing each $O_\alpha$, if necessary) we may assume that all $O_\alpha$ coincide (denote this set by $O$). Also, since the number of points of intersection of $\text{supp}(G)$ with $T^2$ is finite, the above argument shows, that we may choose the sets $K_\theta$ so that $K := \cap K_\theta \neq \emptyset$ and, moreover, $K$ contains points from $\partial O \cap (\mathbb{R}_+)^2$. We set $K_+ := K \cap (\mathbb{R}_+)^2$. Then by the continuity of derivatives of $f_\alpha$ there exist a number $0 < t \leq 1$ and neighbourhoods $N_\beta$ in $U$ of points of intersection of $\text{supp}(G)$ with $T^2$ such that each $N_\beta$
is a subset of some $B_\alpha$ and for $z \in W_k$ and $(z_1, z_2) \in N_\beta$ and for any $(\lambda_1, \lambda_2) \in t_0 K_+$ with $0 < t_0 \leq t$,
$$\left| \frac{\partial g_\alpha}{\partial z}(z, z_1, z_2) \right| \geq c t_0 > 0,$$
where $c$ is a constant independent of the choice of $(z, z_1, z_2)$ and indices $k, \alpha$. Here we have used that $\left| \frac{d g_\alpha}{d z} \right|$ is bounded away from zero by a numerical constant by our choice of $c_k$, see (9.14.78).

As we have noticed before for sufficiently small $t_0$ the sets of solutions of equations (9.14.80) belong to unions of open sets $W_k \times N_\beta$.

Thus we have proved that $\text{supp}(H) := \text{supp}(E) \cap b_{T^2} X$ can be covered by finitely many sets in $b_{(\mathbb{C}^*)^2}(X)$ of the form $W_k \times N_\beta$ (in suitable local coordinates on $b_{(\mathbb{C}^*)^2}(X)$) and there exist functions $g_{k\beta} \in \mathcal{O}(W_k \times N_\beta)$ such that $\text{supp}(H) \cap (W_k \times N_\beta)$ is the set of zeros of $g_{k\beta}$ and the modulus of the derivative of $g_{k\beta}$ with respect to $z \in W_k$ is bounded away from zero on $W_k \times N_\beta$. This shows that $H \in \text{Div}(b_{T^2} X)$ is smooth.

(3) We use the results of part 1 of the proof of Proposition 6.5.6. It suffices to show that there exists a contractible coordinate chart $U_0 \in X_0$ such that the Chern class $c_{U, T^2}(H|_U) \neq 0$ with $U := p_{b_{(\mathbb{C}^*)^2}}^{-1}(U_0)$.

Since divisor $G$ on $(\mathbb{C}^*)^2$ is not equivalent to a principal divisor, its Chern class $c(G) \neq 0$ in $H^2((\mathbb{C}^*)^2, \mathbb{Z})$ (see, e.g., [GH]). For any contractible coordinate chart $U_0 \in X_0$ the divisor $G_{U_0}$ on $U_0 \times (\mathbb{C}^*)^2$ is defined as the pullback of $G$ with respect to the projection $\pi : U_0 \times (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$. In particular, its Chern class $c(G_{U_0})$ equals $\pi^*(c(G))$ and is non-zero because $\pi$ induces isomorphisms of cohomology groups. Since for such $U_0$ the open set $U' := p_{b_{(\mathbb{C}^*)^2}}^{-1}(U_0) \subset b_{(\mathbb{C}^*)^2} X$ is biholomorphic by means of $\eta$ to $U_0 \times (\mathbb{C}^*)^2$, from our construction we obtain that the restriction $E|_{U'}$ of divisor $E := (\pi \circ \eta)^* G$ to $U'$ is the same as $\eta^* G_{U_0}$. Thus the Chern class $c(E|_{U'}) \neq 0$ in $H^2(U', \mathbb{Z})$. Finally, $U := p_{b_{T^2}}^{-1}(U_0) \subset b_{T^2} X$ is a deformation retract of $U'$ and therefore because $H|_{U'}$ coincides with $(E|_{U'})|_{U}$ the Chern class $c_{U, T^2}(H|_{U}) \neq 0$ (in fact, it coincides with $c(E|_{U'})$ under the identification of $H^2(U', \mathbb{Z})$ with $H^2(U, \mathbb{Z})$).

### 9.14.5 Proof of Proposition 6.5.9

We construct a smooth divisor $G$ on $(\mathbb{C}^*)^2$ that has a non-zero Chern class and whose support intersects the real torus $T^2$ transversely.
Let \( \Lambda := (2\pi \mathbb{Z} + 2\pi i\mathbb{Z})^2 \), \( \Gamma := (2\pi i\mathbb{Z})^2 \subset \mathbb{C}^2 \). Then \( \mathbb{C}^2/\Lambda \) (with respect to the action of \( \Lambda \) on \( \mathbb{C}^2 \) by translations) is a complex two-dimensional torus and \( \mathbb{C}^2/\Gamma \) is the product of two infinite cylinders. Let \( c : \mathbb{C}^2 \to \mathbb{C}^2/\Gamma \) be the (holomorphic) quotient map. Then there exists a biholomorphic map \( q_1 : (\mathbb{C}^*)^2 \to \mathbb{C}^2/\Gamma \) defined by the formula

\[
q_1(\zeta_1, \zeta_2) := c((\log \zeta_1, \log \zeta_2)), \quad (\zeta_1, \zeta_2) \in (\mathbb{C}^*)^2;
\]

here \( \log : \mathbb{C}^* \to \mathbb{C} \) is the multi-valued logarithmic function.

Further, denoting by \( q_2 : \mathbb{C}^2/\Gamma \to (\mathbb{C}^2/\Gamma)/(2\pi \mathbb{Z})^2 = \mathbb{C}^2/\Lambda \) the corresponding holomorphic quotient map, we obtain that the regular covering \( q : (\mathbb{C}^*)^2 \to \mathbb{C}^2/\Lambda \) with the deck transformation group \( \mathbb{Z}^2 \) can be obtained as the composite \( q_2 \circ q_1 \).

We start by constructing a smooth divisor \( V \) on \( \mathbb{C}^2/\Lambda \) with a non-zero Chern class. Let \( z_1 = x_1 + iy_1, \ z_2 = x_2 + iy_2 \) be standard complex coordinates on \( \mathbb{C}^2 \). They produce local coordinates on \( \mathbb{C}^2/\Lambda \) denoted analogously. It follows that

\[
\omega_0 = dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2 = 2(dx_1 \wedge dx_2 + dy_1 \wedge dy_2)
\]

is a \( d \)-closed \((1,1)\)-form on \( \mathbb{C}^2/\Lambda \) with a non-zero de Rham cohomology class \([\omega_0] \in H^2_{DR}(\mathbb{C}^2/\Lambda)\). We also consider a positive \( d \)-closed \((1,1)\)-form

\[
\eta := ki(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = 2k(dx_1 \wedge dy_1 + dx_2 \wedge dy_2), \quad k \in \mathbb{N}.
\]

Since \( \omega_0 \) and \( \eta \) have integer coefficients, they represent integral cohomology classes, i.e., \([\omega_0]\) and \([\eta]\) belong to the image of \( H^2(\mathbb{C}^2/\Lambda, \mathbb{Z}) \) in \( H^2_{DR}(\mathbb{C}^2/\Lambda) \) under the de Rham isomorphism (see, e.g., [GH, Ch. 2.6]). By taking \( k \) sufficiently large one obtains that \( \omega := \omega_0 + \eta \) is a positive \( d \)-closed \((1,1)\)-form representing an integral cohomology class. By the Lefschetz \((1,1)\)-theorem (observe that \( \mathbb{C}^2/\Lambda \) is projective) the cohomology class \([\omega]\) is the Chern class of a positive line bundle \( L_\omega \) on \( \mathbb{C}^2/\Lambda \). Increasing \( k \), if necessary, we can embed (using the Kodaira theorem) \( \mathbb{C}^2/\Lambda \) into a projective space \( \mathbb{CP}^N \) by means of holomorphic sections of the bundle \( L_\omega \) so that \( L_\omega \) is the pullback of the hyperplane bundle on \( \mathbb{CP}^N \). By the Bertini theorem the preimage in \( \mathbb{C}^2/\Lambda \) of a generic hyperplane \( H \subset \mathbb{CP}^N \) determines a smooth divisor \( V \) in \( \mathbb{C}^2/\Lambda \) with Chern class \([\omega]\). We claim that
Lemma 9.14.3. Under a suitable choice of \( H \) the support of the constructed divisor \( V \) intersects the image \( q(\mathbb{T}^2) \subset \mathbb{C}^2/\Lambda \) transversely in finitely many points.

Proof. Clearly, \( q \) embeds \( \mathbb{T}^2 \) into \( \mathbb{C}^2/\Lambda \) so that \( q(\mathbb{T}^2) \) is a (real) analytic submanifold of \( \mathbb{C}^2/\Lambda \). It is also totally real meaning that \( T_x \cap iT_x = 0 \) and \( T_x + iT_x = T_x^C \) at each \( x \in q(\mathbb{T}^2) \), where \( T_x \) is the tangent space to \( q(\mathbb{T}^2) \) at \( x \) and \( T_x^C \) is the minimal complex subspace of the tangent space to \( \mathbb{C}^2/\Lambda \) at \( x \) containing \( T_x \) (in our case it coincides with this tangent space). This follows from the fact that \( \mathbb{T}^2 \subset (\mathbb{C}^*)^2 \) is totally real and \( q \) is biholomorphic in a neighbourhood of \( \mathbb{T}^2 \).

Without loss of generality we will identify \( \mathbb{C}^2/\Lambda \) with its image in \( \mathbb{C}P^N \).

Let us choose a generic hyperplane \( H \subset \mathbb{C}P^N \) transversely intersecting \( \mathbb{C}^2/\Lambda \) and intersecting \( q(\mathbb{T}^2) \) transversely at least at one point. To do that we pick \( x \in q(\mathbb{T}^2) \) and decompose the complex tangent space \( T_x(\mathbb{C}P^N) \) of \( \mathbb{C}P^N \) at \( x \) as \( T_x^C \oplus L \), where \( L \) is a complex subspace of codimension 2 of \( T_x(\mathbb{C}P^N) \). Further, in \( T_x^C \) choose a one-dimensional complex subspace \( L' \) which intersects the real part \( T_x \) of \( T_x^C \) by 0. Then \( L + L' \) is a complex hyperplane in \( T_x(\mathbb{C}P^N) \) transversely intersecting \( T_x \). Let \( H' \subset \mathbb{C}P^N \) be the hyperplane whose tangent space at \( x \) coincides with \( L + L' \). Then \( H' \) intersects \( q(\mathbb{T}^2) \) transversely at \( x \). Further, by the Bertini theorem we can perturb \( H' \) to get a hyperplane \( H \) that also transversely intersects \( q(\mathbb{T}^2) \) at least at one point and transversely intersects \( \mathbb{C}^2/\Lambda \).

Next, consider a projection \( \pi \) in \( \mathbb{C}P^N \) along \( H \) onto a one-dimensional projective subspace \( \ell \subset \mathbb{C}P^N \) transversely intersecting \( H \) at a single point. In fact, \( \pi \) is a meromorphic map of \( \mathbb{C}P^N \) onto \( \ell \cong \mathbb{C}P^1 \) and so it is defined outside a projective subspace of \( \mathbb{C}P^N \) of (complex) codimension two. Perturbing \( H \), if necessary, we may assume that the latter subspace does not intersect \( q(\mathbb{T}^2) \) so that \( \pi \) is well defined on \( q(\mathbb{T}^2) \). By our construction of \( H \) the image \( \pi(q(\mathbb{T}^2)) \) contains interior points (because in a neighbourhood of a point of transversal intersection of \( H \) and \( q(\mathbb{T}^2) \), the map \( \pi|_{q(\mathbb{T}^2)} \) is diffeomorphic). Therefore by Sard’s theorem for almost each interior point \( z \in \pi(q(\mathbb{T}^2)) \) (with respect to the measure on \( \mathbb{C}P^1 \) determined by the Fubini-Study volume form) the preimage \( \pi^{-1}(z)|_{q(\mathbb{T}^2)} \) consists of finitely many non-critical points of \( \pi|_{q(\mathbb{T}^2)} \). But this means that the complex hyperplane \( \pi^{-1}(z) \) in \( \mathbb{C}P^N \) intersects \( q(\mathbb{T}^2) \) transversely. Finally, we can perturb \( \pi^{-1}(z) \) (using the Bertini theorem) so that the perturbed hyperplane intersects \( q(\mathbb{T}^2) \) transversely in finitely many points and also intersects transversely \( \mathbb{C}^2/\Lambda \) (determining...
the required divisor $V$).

Now, we define a smooth divisor $G$ on $(\mathbb{C}^*)^2$ as the pullback by map $q$ of the divisor $V$. Let us show that the Chern class $[q^*\omega] = [q^*\omega] \in H^2_{DR}(\mathbb{C}^2/\Lambda)$ of $G$ is non-zero. First, $[q^*\eta] = 0$ by the definition of $\eta$ (because each term of $q^*\eta$ is a $d$-closed 2-form on $\mathbb{C}^*$ which homotopic to $S^1$). Thus we must check that $[q^*_0\omega] \neq 0$. Since $q_1$ is a biholomorphism, it suffices to check that $[q^*_0\omega] \neq 0$. Using coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ on $\mathbb{C}^2/\Gamma$ induced from the standard coordinates on $\mathbb{C}^2$ we easily obtain

$$[q^*_0\omega] = 2[dx_1 \wedge dx_2 + dy_1 \wedge dy_2] = 2[dy_1 \wedge dy_2].$$

The latter cohomology class is non-zero; to see this one integrates $dy_1 \wedge dy_2$, e.g., over an embedded real torus $\{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2/\Gamma : x_1, x_2 = \text{const}\}$ getting a non-zero value which, as follows from the Stokes theorem, contradicts to the assumption that $dy_1 \wedge dy_2$ is exact.

Lemma 9.14.3 implies that the support of $G$ intersects $T^2 \subset (\mathbb{C}^*)^2$ transversely in finitely many points.

### 9.15 Proofs of Theorems 7.1.1 and 7.2.1

Let us recall notation and some results of Section 6.2.1 and Example 6.2.6. There exist equivariant maps

$$j_a : G \to \hat{G}_a, \quad j_{\ell,\infty} : G \to \hat{G}_{\ell,\infty}.$$  

We denote by $\hat{K}_a \subset \hat{G}_a$ and $\hat{K}_{\ell,\infty} \subset \hat{G}_{\ell,\infty}$ the closures of sets $j_a(K)$ and $j_{\ell,\infty}(K)$ in $\hat{G}_a$ and $\hat{G}_{\ell,\infty}$, respectively. There exist a commutative diagram

$$\begin{array}{ccc}
\Pi(U_0, K) & \xrightarrow{\text{Id} \times j_a} & \hat{\Pi}_a(U_0, \hat{K}_a) \\
\downarrow_{\equiv} & & \downarrow_{\equiv} \\
\Pi(U_0, K) & \xrightarrow{\text{Id} \times j_{\ell,\infty}} & \hat{\Pi}_{\ell,\infty}(U_0, \hat{K}_{\ell,\infty})
\end{array} \quad \lambda \quad = \quad \kappa \quad = \quad (9.15.82)$$

All maps, except possibly for $\lambda$, are continuous.
9.15.1 Proof of Theorem 7.1.1

For the proof, we will need the following results.

(a) There exists a unique function \( \hat{f} \in \mathcal{O}(\hat{\Pi}_a(U_0, \hat{K}_a)) \) such that

\[
\hat{f}|_{\Pi(U_0, K)} = (\text{Id} \times j_a)^* \hat{f}.
\] (9.15.83)

Indeed, since \( f \in \mathcal{O}_{\ell_\infty}(X) \), there exists a function \( \hat{f} \in \mathcal{O}(\hat{\Pi}_{\ell_\infty}(U_0, \hat{K}_a)) \) such that \( f|_{\Pi(U_0, K)} = (\text{Id} \times j_{\ell_\infty})^* \hat{f} \). We set

\[
\hat{f} := (\text{Id} \times \lambda)^* \hat{f} : \hat{\Pi}_a(U_0, \hat{K}_a) \to \mathbb{C}.
\]

Clearly, (9.15.83) is satisfied. Identifying \( \hat{\Pi}_a(U_0, \hat{K}_a) \) with \( U_0 \times \hat{K}_a \) (cf. (6.2.6)), we obtain that \( \hat{f}(\cdot, \omega) \in \mathcal{O}(U_0) \) for all \( \omega \in \hat{K}_a \). It remains to show that \( \hat{f} \) is continuous. By our assumption, there exists a function \( F \in C(\hat{\Pi}_a(Z_0, \hat{K}_a)) \) such that \( f|_{\Pi(Z_0, K)} = (\text{Id} \times j_a)^* F \). Since \( (\text{Id} \times j_a)(\Pi(Z_0, K)) \) is dense in \( \hat{\Pi}_a(Z_0, \hat{K}_a) \), and diagram (9.15.82) is commutative, we have

\[
\hat{f}|_{\Pi_a(Z_0, \hat{K}_a)} = F.
\] (9.15.84)

We identify \( \hat{\Pi}_a(U_0, \hat{K}_a) \) with \( U_0 \times \hat{K}_a \), and \( \hat{\Pi}_a(Z_0, \hat{K}_a) \) with \( Z_0 \times \hat{K}_a \). Suppose that \( \hat{f} \) is discontinuous, i.e., there exists a net \( \{\omega_\alpha\} \subset \hat{K}_a \), \( \omega_\alpha \to \omega \in \hat{K}_a \), such that \( \hat{f}(\cdot, \omega_\alpha) \not\to \hat{f}(\cdot, \omega) \) in \( \mathcal{O}(U_0) \).

Using Montel theorem, we obtain that there exists a partial limit of \( \{\hat{f}(\cdot, \omega_\alpha)\} \) in \( \mathcal{O}(U_0) \) that does not coincide with \( \hat{f}(\cdot, \omega) \). However, since \( \hat{f}|_{Z_0 \times \hat{K}_a} \) is continuous, cf. (9.15.84), and \( Z_0 \) is a uniqueness set for holomorphic functions in \( \mathcal{O}(U_0) \), this partial limit must coincide with \( \hat{f}(\cdot, \omega) \) in \( \mathcal{O}(U_0) \), which is a contradiction.

Therefore, \( \hat{f} \in C(\hat{\Pi}_a(Z_0, \hat{K}_a)) \), and hence \( \hat{f} \in \mathcal{O}(\hat{\Pi}_a(Z_0, \hat{K}_a)) \).

(b) We have \( \cup_{i=1}^m \hat{K}_a \cdot g_i = \hat{G}_a \). Indeed, assuming the opposite, we obtain that there exists \( \omega_0 \in \hat{G}_a \setminus \cup_{i=1}^m \hat{K}_a \cdot g_i \). Since \( j_\alpha(G) \) is dense in \( \hat{G}_a \) (cf. Section 6.2.1), there exists a sequence \( \{h_i\} \subset G \) such that \( j_\alpha(h_i) \to \omega_0 \) as \( l \to \infty \). Then there exists \( 1 \leq i_0 \leq m \) such that \( \{j(h_i)\} \cap \hat{K} \cdot g_{i_0} \) has infinitely many elements. Since \( \hat{K} \cdot g_{i_0} \) is closed and \( \hat{G}_a \) is compact, \( \hat{K} \cdot g_{i_0} \) is compact as well. Therefore, \( \omega_0 \in \hat{K} \cdot g_{i_0} \), which contradicts to the assumption that \( \cup_{i=1}^m \hat{K}_a \cdot g_i \subset \hat{G}_a \).

(c) Since covering \( p : X \to X_0 \) is regular, for each \( 1 \leq i \leq m \) there exists \( h_i \in \pi_1(X_0) \) such that \( h_i \cdot \Pi(U_0, K) = \Pi(U_0, K \cdot g_i) \) (i.e., if \( x_0 \in U_0 \) is the base point for \( \pi_1(X) \), and we have fixed
By the result in [Br3] there exists a sequence of open sets \( \Pi(U_0, l, K_l) \subset X \), \( 1 \leq l \leq s \), such that

1. \( \Pi(U_0, 1, K_1) = \Pi(U_0, K) \), \( \Pi(U_0, s, K_s) = \Pi(U_0, K \cdot g_i) \),
2. each \( U_0, l \subset X_0 \) is connected, \( U_0, l \cap U_0, l+1 \neq \emptyset \), and
3. \( \Pi(U_0, l \cap U_0, l+1, K_l) = \Pi(U_0, l \cap U_0, l+1, K_{l+1}) \), \( 1 \leq l \leq s - 1 \).

Now, let \( \hat{K}_{a,l} \) denote the closure of \( K_l \) in \( \hat{C}_a \). Clearly, the sequence of sets \( \hat{K}_{a}(U_0, \hat{K}_{a,l}) \), \( 1 \leq l \leq s \), has properties analogous to (1) and (3).

We now complete the proof of Theorem 7.1.1 using an analytic continuation-type argument. Starting with set \( \Pi(U_0, K) \) as in the formulation of the theorem, by (a) we can extend \( f|_{\Pi(U_0, K)} \) to a (unique) function \( \hat{f}_1 \in \mathcal{O}(\hat{\Pi}_a(U_0, \hat{K}_a)) \).

Now, let us fix some \( 1 \leq i_0 \leq m \). Let sets \( \{U_0, l\} \) be as in (c). Since \( U_0 \cap U_0, 2 \neq \emptyset \) is a uniqueness set for holomorphic functions in \( \mathcal{O}(U_0, 2) \), we can extend \( f|_{\Pi(U_0, 2, K)} \) to a function \( \hat{f}_2 \in \mathcal{O}(\hat{\Pi}_a(U_0, 2, \hat{K}_{a,2})) \) such that \( \hat{f}_1 = \hat{f}_2 \) on \( \hat{\Pi}_a(U_0 \cap U_0, 2, \hat{K}_{a,2}) \).

Repeating this argument for the remaining sets \( \{U_0, l\} \), \( 3 \leq l \leq s \), we obtain a (unique) extension \( \hat{f}_s \in \mathcal{O}(\hat{\Pi}_a(U_0, 2, \hat{K}_a \cdot g_{i_0})) \) of \( f|_{\Pi(U_0, K \cdot g_{i_0})} \),

The functions \( \hat{f} \) and \( \hat{f}_s \) coincide on the intersection of their domains \( \hat{\Pi}_a(U_0, \hat{K}_a) \cap \hat{\Pi}_a(U_0, \hat{K}_a \cdot g_{i_0}) \), since they are continuous and coincide on a dense subset \( \iota(\Pi(U_0, K) \cap \Pi(U_0, K \cdot g_{i_0})) \).

Moreover, \( \hat{f}_s \) does not depend on the choice of sequence \( \{U_0, l\} \), as any two paths \( p^*h_i, p^*h_i' \) joining \( y_0 \) and \( y_0 \cdot g_i \), cf. (c), are homotopic, hence, if \( \{U_0, l\} \), \( \{U'_{0, l}\} \) denote the corresponding sequences of open subsets of \( X_0 \), we may assume that \( U_0, l \cap U'_{0, l} \neq \emptyset \) for all \( 1 \leq l \leq s \); in turn, each set \( U_0, l \cap U'_{0, l} \) is a uniqueness sets for functions in \( \mathcal{O}(U_0, l), \mathcal{O}(U'_{0, l}) \); from here the uniqueness of continuation follows.

By (b) we obtain in this way a (unique) extension of \( f|_{p^{-1}(U_0)} \) to a function in \( \mathcal{O}(p^{-1}(U_0)) \).

Arguing similarly, we extend function \( f \) to a function in \( \mathcal{O}(c_aX) \). Now, Proposition 6.2.8(2) implies that \( f \in \mathcal{O}_a(X) \), as required.

**9.15.2 Proof of Theorem 7.2.1**

By the result in [Br3] there exists a (unique) function \( F \in \mathcal{O}_{\ell \infty}(D) \cap C_{\ell \infty}(\bar{D}) \) such that \( F|_{\partial D} = f \).

It remains to show that \( F \in \mathcal{O}_a(D) \).
An argument similar to the one in the proof of Proposition 6.2.8(2) implies that there exists an extension \( \tilde{F} \in \mathcal{O}(c_{\ell}D) \cap C(c_{\ell}D) \) of function \( F \) from \( \bar{D} \) to the fibrewise compactification \( c_{\ell}D \subset c_{\ell}X \) of covering \( p|_{\bar{D}}: \bar{D} \to \bar{D}_0. \) Let \( U_0 \subset D_0 \) be an open simply connected subset such that \( \bar{U}_0 \cap \partial D_0 \neq \emptyset \) is open in \( \partial D_0. \) Below we identify \( \Pi(U_0,G) \) with \( U_0 \times G, \bar{\Pi}_a(U_0,\hat{G}_a) \) with \( U_0 \times \hat{G}_a, \) and \( \bar{\Pi}_{\ell}(U_0,\hat{G}_{\ell}) \) with \( U_0 \times \hat{G}_{\ell} \) (cf. Section 6.2.6). We define

\[
\hat{F}_{U_0 \times \hat{G}_a} := (\text{Id} \times \lambda)^* (\hat{F}|_{U_0 \times \hat{G}_{\ell}}).
\]

We have

\[
F|_{U_0 \times G} = (\text{Id} \times j_a)^* \hat{F}_{U_0 \times \hat{G}_a}
\]

(as similar relation holds for \( \tilde{F} \)). Hence, if we can show that \( \hat{F}_{U_0 \times \hat{G}_a} \in \mathcal{O}(U_0 \times \hat{G}_a) \), then \( F|_{U_0 \times G} \in \mathcal{O}_a(U_0 \times G) \) by Proposition 6.2.10(2); since \( U_0 \subset D_0 \) was chosen arbitrarily, this would imply that \( F \in \mathcal{O}_a(D) \), as needed.

Indeed, by definition \( \hat{F}_{U_0 \times \hat{G}_a}(\cdot,\omega) \in \mathcal{O}(U_0) \cap C(\bar{U}_0) \) for all \( \omega \in \hat{G}_a \), hence we only need to show that \( \hat{F}_{U_0 \times \hat{G}_a} \) is continuous (cf. Section 6.2.6). Suppose that \( \hat{F}_{U_0 \times \hat{G}_a} \) is discontinuous, i.e., there exists a net \( \{\omega_\alpha\} \subset \hat{G}_a, \omega_\alpha \to \omega \in \hat{G}_a \), such that \( \hat{F}_{U_0 \times \hat{G}_a}(\cdot,\omega_\alpha) \neq \hat{F}_{U_0 \times \hat{G}_a}(\cdot,\omega) \) in \( \mathcal{O}(U_0) \).

Using Montel theorem, we obtain that there exists a partial limit of \( \{\hat{F}_{U_0 \times \hat{G}_a}(\cdot,\omega_\alpha)\} \) in \( \mathcal{O}(U_0) \) that does not coincide with \( \hat{F}_{U_0 \times \hat{G}_a}(\cdot,\omega) \). Since \( \hat{F}_{U_0 \times \hat{G}_a}|_{U_0 \cap \partial D_0 \times \hat{G}_a} \) is continuous, and \( \bar{U}_0 \cap \partial D_0 \) is a uniqueness set for functions in \( \mathcal{O}(U_0) \cap C(\bar{U}_0) \) (see, e.g., [Bog]), this partial limit must coincide with \( \hat{F}_{U_0 \times \hat{G}_a} \) in \( \mathcal{O}(U_0) \), which is a contradiction.

**9.16 Proofs of Theorems 8.0.2, 8.0.5 and 8.0.9 and Proposition 8.0.7**

**9.16.1 Proof of Theorem 8.0.2**

We deduce Theorem 8.0.2 from Theorem 8.0.5. We fix the following

*Notation.* We denote by \( C_a,X_0 (i \in I) \) the holomorphic Banach vector bundle associated to the principal fibre bundle \( p : X \to X_0 \) and having fibre \( a_i \) (cf. Remark 6.1.6).

For a given open subset \( D_0 \subset X_0 \) we denote by \( \mathcal{O}(D_0,C_a,X_0) \) the space of holomorphic sections of bundle \( C_a,X_0 \) over \( D_0 \), endowed with the topology of uniform convergence on compact
subsets of $D_0$, which makes it a Frechet space. We have an isomorphism of Frechet spaces

$$\mathcal{O}_a(D) \cong \mathcal{O}(D_0, C_a X_0), \tag{9.16.85}$$

where $D := p^{-1}(D_0)$ (the proof repeats literally the proof of Proposition 6.1.7).

We will need the following result.

**Proposition 9.16.1.** Let $Y_0 \subseteq X_0$ be open and such that $\bar{Y}_0$ is a holomorphically convex compact subset of $X_0$, let $D_0 \subset X_0$ be an open neighbourhood of $\bar{Y}_0$. We set $Y := p^{-1}(Y_0)$ and $D := p^{-1}(D_0)$.

Let $X_0$ be a Stein manifold, $f \in \mathcal{O}_a(D)$. For any $\varepsilon > 0$ there exists a function $h \in \mathcal{O}_a(X)$ such that $\sup_{z \in Y} |f(z) - h(z)| < C\varepsilon$ for some $C > 0$ independent of $\bar{f}$ and $\varepsilon > 0$.

**Proof.** We will need the following approximation result due to [Bu2, Theorem C].

Let $B$ be a complex Banach space, let $\mathcal{O}(X_0, B)$ be the space of $B$-valued holomorphic functions on $X_0$.

**Lemma 9.16.2.** Suppose that $\bar{f} \in \mathcal{O}(D_0, B)$. Then for any $\varepsilon > 0$ there exists a function $\bar{h} \in \mathcal{O}(X_0, B)$ such that $\sup_{z \in Y_0} \|\bar{f}(z) - \bar{h}(z)\|_B < \varepsilon$ for some $C > 0$ independent of $\bar{f}$ and $\varepsilon > 0$.

Next, by Theorem 4.2.9, since $X_0$ is a Stein manifold, there exist holomorphic Banach vector bundles $p_1 : E_1 \to X_0$ and $p_2 : E_2 \to X_0$ with fibres $B_1$ and $B_2$, respectively, such that $E_2 = E_1 \oplus C_a X_0$ (the Whitney sum) and $E_2$ is holomorphically trivial, i.e., $E_2 \cong X_0 \times B_2$. Thus, any holomorphic section of $E_2$ can be naturally identified with a $B_2$-valued holomorphic function on $X_0$. By $q : E_2 \to C_a X_0$ and $i : C_a X_0 \to E_2$ we denote the corresponding quotient and embedding homomorphisms of these bundles, so that $q \circ i = \text{Id}$.

Now, given a function $f \in \mathcal{O}_a(D)$, by $\bar{f} \in \mathcal{O}(D_0, C_a X_0)$ we denote its image under isomorphism (9.16.85). Define $\tilde{f} := i(\bar{f}) \in \mathcal{O}(D_0, B_2)$. By Lemma 9.16.2 for every $\varepsilon > 0$ there exists a function $\tilde{h} \in \mathcal{O}(X_0, B_2)$ such that $\sup_{z \in Y_0} \|\tilde{f}(z) - \tilde{g}(z)\|_{B_2} < \varepsilon$. We define $\bar{h} := q(\tilde{h}) \in \mathcal{O}(X_0, C_a X_0)$, and denote by $h \in \mathcal{O}_a(X)$ the image of $\bar{h}$ under the inverse isomorphism (9.16.85). The continuity of $i$ and $q$, and the compactness of $Y_0$ now imply that $\sup_{z \in Y} |f(z) - h(z)| < C\varepsilon$ for some $C > 0$ independent of $\bar{f}$ and $\varepsilon > 0$. \qed
Using this proposition, we complete the proof of Theorem 8.0.2. Let \( f \in O_a(X) \). We have to show that for a sequence of open subsets \( Y_{0,k} \Subset Y_{0,k+1} \Subset X_0, \ k \geq 1 \), for any \( \varepsilon > 0 \) there exist functions \( h_k \in O_a(X) \) such that \( \sup_{x \in Y_k} |f(x) - h_k(x)| < \frac{\varepsilon}{k} \), where \( Y_k := p^{-1}(Y_{0,k}) \).

Since \( X_0 \) is a Stein manifold, we may assume without loss of generality that each \( \bar{Y}_{0,k}, k \geq 1 \), is holomorphically convex in \( X_0 \). There is a strictly pseudoconvex open neighbourhood \( D_{0,k} \Subset X_0 \) of \( \bar{Y}_{0,k}, k \geq 1 \) (see, e.g., [HL]). Since the restriction \( f|_{D_k} \in A_a(D_k) \), where \( D_k := p^{-1}(D_{0,k}) \), by Theorem 8.0.5 there exist functions \( h_{k}^{'} \in A_{a_0}(D_k), k \geq 1 \), such that \( \sup_{x \in D_k} |f(x) - h_{k}^{'}(x)| < \frac{\varepsilon}{2k} \).

By the definition of space \( A_{a_0}(D_k) \), there exists \( \iota_k \in I \) such that \( h_{k}^{'} \in A_{a_k}(D_k), k \geq 1 \).

Now, by Proposition 9.16.1 there exists a function \( h_k \in O_{a_k}(X) \) such that \( \sup_{x \in Y_k} |h_{k}^{'}(x) - h_k(x)| < \frac{\varepsilon}{2k} \).

Therefore, \( \sup_{x \in Y_k} |f(x) - h_k(x)| < \frac{\varepsilon}{2k} \). Since \( O_{a_k}(X) \subset O_{a_0}(X) \), this implies the required, modulo Theorem 8.0.5.

### 9.16.2 Proof of Theorem 8.0.5

Notation. We denote by \( A(D_0, C_a X_0) \) and \( A(D_0, C_a X_0) \) the spaces of sections of bundles \( C_a X_0 \) and \( C_a, X_0 \), respectively, continuous over \( D_0 \) and holomorphic on \( D_0 \).

Space \( A(D_0, C_a X_0) \) is endowed with norm \( \|f\| := \sup_{x \in D_0} \|f(x)\|_a \), which makes it a Banach space. Then \( A(D_0, C_a X_0) \) is a closed subspace of \( A(D_0, C_a X_0) \).

Also, we define the linear space \( A_0(D_0, C_a X_0) := \bigcup_{\iota \in I} A(D_0, C_a, X_0) \).

We have isomorphisms of Banach spaces

\[
A_{a_0}(D) \xrightarrow{\cong} A(D_0, C_a, X_0), \quad A_0(D) \xrightarrow{\cong} A_0(D_0, C_a X_0)
\]  

(9.16.86)

(the proof is similar to the proof of Proposition 6.1.7).

In view of (9.16.86) Theorem 8.0.5 can be restated as follows:

(*) Suppose that \( X_0 \) is a Stein manifold, and the subdomain \( D_0 \Subset X_0 \) is strictly pseudoconvex.

Then \( A_0(D_0, C_a X_0) \) is dense \( A(D_0, C_a X_0) \).

For the proof of this claim, we will need the following notation and results.
Let $E$ be a holomorphic Banach vector bundle on $D_0$. Similarly, we denote by $\Lambda_b^{(0,q)}(D_0, E)$, $q \geq 0$, the Banach space of bounded continuous $(0,q)$-forms on $D_0$ with values in the bundle $E$, endowed with sup-norm defined analogously to (9.1.1). Using Theorem 4.2.9 (cf. Remark 6.1.6) and Lemma 9.1.1 we obtain the following

**Corollary 9.16.3.** There exists a bounded linear operator

$$R_{D_0,E} \in \mathcal{L}\left(\Lambda_b^{(0,q)}(D_0, E), \Lambda_b^{(0,q-1)}(D_0, E)\right), \quad q \geq 1,$$

such that if form $\omega \in \Lambda_b^{(0,q)}(D_0, E)$ is $C^\infty$ and satisfies $\bar{\partial}\omega = 0$ on $D_0$, then $\bar{\partial}R_{D_0,E}\omega = \omega$ on $D_0$.

We define $\mathcal{A}(D_0, B) := \mathcal{O}(D_0, B) \cap C(\bar{D}_0, B)$, and endow space $\mathcal{A}(D_0, B)$ with norm

$$\|f\|_{D_0} := \sup_{x \in D_0} \|f(x)\|_B.$$

The next result follows easily from the results in [HL] (proved for the scalar case $B = \mathbb{C}$), since all integral presentations and estimates are preserved when passing to Banach-valued forms.

**Lemma 9.16.4.** Let $K \subset \mathcal{A}(D_0, B)$ be compact. Then for every $\varepsilon > 0$ there exists an open neighbourhood $D'_0 \subset X_0$ of $\bar{D}_0$ and an operator $A_{K,\varepsilon} = A_{D_0,K,\varepsilon} \in \mathcal{L}(\mathcal{A}(D_0, B), \mathcal{A}(D'_0, B))$ such that for each $f \in K$ we have $\|f - Af\|_{D_0} < \varepsilon$.

We prove claim (\*) in three steps (i)-(iii).

(i) Let $f \in \mathcal{A}(D_0, C_aX_0)$. Combining the result in [ZK] (cf. Remark 6.1.6) and Lemma 9.16.4, we may assume that $f \in \mathcal{O}(D'_0, C_aX_0)$, where $D'_0 \subset X_0$ is an open neighbourhood of $\bar{D}_0$.

We have to show that for every $\varepsilon > 0$ there exists a section $F \in \mathcal{A}_0(D_0, C_aX_0)$ such that $\sup_{x \in D_0} \|f(x) - F(x)\|_a < \varepsilon$.

(ii) Let $\mathcal{U} = \{U_k\}_{k=1}^M$, where each $U_k \subset D'_0$ is open and biholomorphic to an open polydisk in $\mathbb{C}^n$, and $D_0 \subset \bigcup_{k=1}^M U_k$.  

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Lemma 9.16.5. For every $\varepsilon > 0$ there exist a subspace $a_{\varepsilon} \subset a$ ($\iota_{\varepsilon} \in I$) and sections $F_{\varepsilon,k} \in A(U_k, C_{a_{\varepsilon}} X_0)$ such that

$$\|f(x) - F_{\varepsilon,k}(x)\|_a < \varepsilon \quad \text{for all } x \in U_k, \quad 1 \leq k \leq M.$$  (9.16.87)

Proof. Since each $U_k$, $1 \leq k \leq M$, is simply connected, bundles $C_a X_0$, $C_{a_{\varepsilon}} X_0$ ($\iota \in I$) admit holomorphic trivializations over $U_k$. Throughout the proof, we identify sections of these bundles over $U_k$ with corresponding $a$-valued (respectively, $a_{\varepsilon}$-valued) functions on $U_0$.

By our assumption, for every $1 \leq k \leq M$ there exists a biholomorphism $\psi_k$ between $U_k$ and open polydisk $\Delta \subset \mathbb{C}^n$ centered at 0. Without loss of generality, we may assume that $f|_{U_k}$ is defined over an open neighbourhood of $\bar{\Delta}$. We denote $f|_{U_k}$ by $f_k$, so that $f_k$ can be identified by means of the corresponding holomorphic trivialization of bundle $C_a X_0$ with a holomorphic $a$-valued function defined on an open neighbourhood of $\bar{\Delta}$.

For a given function $h \in O(\Delta, a)$ we denote by $T_0^N h$ its Taylor polynomial at $x = 0$ of order $N$.

Without loss of generality we may assume that $N$ is chosen in such a way that

$$\|f_k(x) - T_0^N f_k(z)\|_a < \varepsilon \quad \text{for all } x \in \Delta, \quad 1 \leq k \leq M,$$

where $T_0^N f_k(x) := \sum_{|\alpha| \leq N} a_{k,\alpha} x^\alpha$, $a_{k,\alpha} \in a$, and $\alpha$ is a multi-index.

Since $a_0$ is dense in $a$, for every $\delta > 0$ and all $1 \leq k \leq M$, $|\alpha| \leq N$, there exist $a_{\varepsilon}^{k,\alpha} \in a_0$ such that $\|a_{k,\alpha} - a_{\varepsilon}^{k,\alpha}\|_a < \delta$. We choose $\delta > 0$ to be sufficiently small, so that

$$\left\| \sum_{|\alpha| \leq N} a_{k,\alpha} x^\alpha - F_{\varepsilon,k}(x) \right\|_a < \frac{\varepsilon}{2},$$

where $F_{\varepsilon,k}(x) := \sum_{|\alpha| \leq N} a_{\varepsilon}^{k,\alpha} x^\alpha$. Therefore,

$$\|f_k(x) - F_{\varepsilon,k}(x)\|_a < \varepsilon, \quad \text{for all } x \in \Delta, \quad 1 \leq k \leq M.$$  

By definition, there exists $\iota_{\varepsilon} \in I$ such that $a_{\iota_{\varepsilon}}$ contains all $a_{\varepsilon}^{k,\alpha}$ ($1 \leq k \leq M$, $|\alpha| \leq N$); hence $F_{\varepsilon,k}(x) \in A(\Delta, a_{\iota_{\varepsilon}})$. □

(iii). We will also need the following result.
Lemma 9.16.6. In the notation of Lemma 9.16.5, for every \( \varepsilon > 0 \) there exists a section \( F \in \mathcal{A}(D_0, C_{a\varepsilon} X_0) \subset \mathcal{A}_0(D_0, C_0 X_0) \) such that
\[
\| F(x) - F_{x,k}(x) \|_a < C \varepsilon, \quad \text{for all } x \in U_k \cap \tilde{D}_0, \quad 1 \leq k \leq M
\]
for a certain \( C > 0 \) independent of the section \( f \in \mathcal{A}_0(D_0', C_0 X_0) \) and \( \varepsilon > 0 \).

Proof. There exists an open neighbourhood \( D_0'' \subset D_0' \) of \( \tilde{D}_0 \) such that \( D_0'' \in \cup_{k=1}^M U_k \). We may assume without loss of generality that \( D_0'' \) is strictly pseudoconvex.

Let \( \{ \rho_k \} \subset C^\infty(X_0) \) be a collection of functions such that \( \text{supp}(\rho_k) \subset U_k, 1 \leq k \leq M \), and \( \sum_{k=1}^m \rho_k \equiv 1 \) on \( D_0'' \). Let \( c := \max_{x \in \tilde{D}_0''} \sum_{k=1}^M |\partial \rho_k(x)| < \infty \) (in some local coordinates on \( D_0' \)).

We define holomorphic 1-cocycle: if \( U_k \cap U_l \neq \emptyset \) then
\[
g_{k,l} := F_{x,k}|_{U_k \cap U_l D_0''} - F_{x,l}|_{U_k \cap U_l D_0''} / \mathcal{A}(U_k \cap U_l \cap D_0'', C_{a\varepsilon} X_0),
\]
and \( g_{k,l} := 0 \) if \( U_k \cap U_l \cap D_0'' = \emptyset \). According to Lemma 9.16.5
\[
\sup_{x \in U_k \cap U_l \cap D_0''} \| g_{k,l} \|_a < 2 \varepsilon.
\]
Here we assume without loss of generality that \( \{ U_k \cap D_0'' \}_{k=1}^M \) are the coordinate charts in the definition of norm in \( \Lambda^{0,1}_b(D_0'', C_{a\varepsilon} X_0) \), cf. (9.1.1). We resolve cocycle \( \{ g_{x,k,l} \} \) by the formula \( \tilde{g}_l := \sum_{k=1}^M \rho_k g_{k,l} \in C^\infty(U_l \cap D_0'') \), so that \( g_{k,l} = \tilde{g}_k - \tilde{g}_l \) on \( U_k \cap U_l \cap D_0'' \). It follows that the (0,1)-form \( \omega \) defined by the formula \( \omega := \bar{\partial} \tilde{g}_l \) on \( U_l \cap D_0'' \) is a \( \bar{\partial} \)-closed form on \( D_0'' \) taking values in bundle \( C_{a\varepsilon} X_0 \).

We have \( \| \omega \|_{D_0''} \leq 2c \varepsilon \). By Corollary 9.16.3 there exists a function \( \eta \in \Lambda^{0,0}_b(D_0'', E) \) such that \( \bar{\partial} \eta = \omega \) and \( \| \eta \|_{D_0''} \leq C_1 \| \omega \|_{D_0''} \leq 2C_1 c \varepsilon \) for some constant \( C_1 > 0 \) independent of \( \omega \).

Since \( D_0 \subset D_0'' \), the restriction \( \eta|_{D_0} \) is continuous on \( \tilde{D}_0 \).

We define
\[
F := F_{x,k}|_{U_k \cap \tilde{D}_0} - \tilde{g}_k|_{U_k \cap \tilde{D}_0} + \eta|_{U_k \cap \tilde{D}_0}, \quad 1 \leq k \leq M.\]
It follows that \( F \in \mathcal{A}(D_0, C_{a\varepsilon} X_0) \) and
\[
\sup_{x \in D_0} \| F - F_{x,k} \|_a \leq 2M \varepsilon + 2C_1 c \varepsilon,
\]
so we may set \( C := 2M + 2C_1 c \).

The proof of claim (\( \ast \)) now follows from Lemmas 9.16.5 and 9.16.6.
9.16.3 Proof of Proposition 8.0.7

The proof follows closely the argument in [Sh].

Recall that by $bG$ we denote the Bohr compactification of group $G$ (cf. Example 6.2.4). This is a compact group. We have

$$AP(G) \cong C(bG). \tag{9.16.88}$$

Let $C_0(G)$ denote the image of $AP_0(G)$ in $C(bG)$ under isomorphism (9.16.88). It suffices to show that space $C(bG)$ endowed with sup-norm $\| \cdot \|_{C(bG)}$ has the approximation property with approximation operators in $\mathcal{L}(C(bG), C_0(bG))$.

To define the required approximation operators, we now construct a $\delta$-shape net of functions in $C_0(bG)$.

Let $\mu$ denote the Haar measure on $bG$. Since $\mu$ is a Radon measure, we may consider $\mu$ as a linear continuous functional in $C^\ast(bG)$. It follows from Tietze-Urysohn extension theorem and the properties of Haar measure $\mu$ that for every open neighbourhood $U$ of $1 \in bG$ there exists a function $\varphi_U \in C(bG)$ such that

$$\varphi_U \geq 0, \quad \varphi_U|_{bG \setminus U} = 0, \quad \mu(\varphi_U) = 1.$$

By von Neumann approximation theorem (cf. Example 5.2.1(3)), for every $\varepsilon > 0$ there exists a function $\psi_{U,\varepsilon} \in C_0(bG)$ such that $\|\varphi_U - \psi_{U,\varepsilon}\|_{C(bG)} < \varepsilon$. We define

$$0 < \eta_{U,\varepsilon} := \psi_{U,\varepsilon} + \varepsilon \in C_0(bG). \tag{9.16.89}$$

Let $\{\gamma := (U, \varepsilon)\}$ be a directed set ordered by relation $\gamma = (U, \varepsilon) < \gamma' = (U', \varepsilon')$ if $U' \subset U$, $\varepsilon' \leq \varepsilon$. We obtain a net $\{\eta_{\gamma} = (U, \varepsilon)\}$ such that

1. $\eta_{\gamma} \geq 0$,
2. $\mu(\eta_{\gamma}) = 1 + \varepsilon_{\gamma}$ for a certain $\varepsilon_{\gamma}$, where $\varepsilon_{\gamma} \xrightarrow{\gamma} 0$, and
3. for every neighbourhood $V$ of $1$ we have $\sup_{\omega \in bG \setminus V} \eta_{\gamma}(\omega) \xrightarrow{\gamma} 0$.

For every $\gamma$ we define approximation operator $T_\gamma$ by the formula

$$(T_\gamma f)(s) := \mu_t(f(t) \eta_{\gamma}(st^{-1})), \quad s \in bG, \quad f \in C(bG)$$

(here $\mu_t$ denotes the functional $\mu$ acting in variable $t$). It is immediate that $T$ is linear.
By definition, $\eta$ is a linear combination of matrix entries $\sigma_{kl}^j$ of some irreducible finite dimensional unitary representations $\sigma$ of group $bG$. Since $\sigma(ts^{-1}) = \sigma(t)\sigma(s^{-1})$, and hence $\sigma_{kl}^j(ts^{-1})$ is a finite sum of matrix entries of $\sigma(t)$ and $\sigma(s^{-1})$, the operator $T_\gamma$ takes values in $C_0(bG)$ and has finite rank. The boundedness of $T_\gamma$ follows from (1), (2) and the invariance of Haar measure $\mu$ with respect to translations:

$$\|T_\gamma f\|_{C(bG)} \leq \sup_s \mu_t(\eta_\gamma(st^{-1})) \|f\|_{C(bG)} \leq \mu_t(\eta_\gamma) \|f\|_{C(bG)} \leq (1 + \varepsilon_\gamma) \|f\|_{C(bG)}, \quad f \in C(bG). \quad (9.16.90)$$

It remains to show that for every compact subset $K \subset C(bG)$ and every $\varepsilon > 0$ there exists $\gamma$ such that for any $f \in K$ the value $T_\gamma f$ is an $\varepsilon$-approximation to $f$ in $C(bG)$.

Indeed, let us fix some $f \in C(bG)$; we have

$$\|T_\gamma f - f\|_{C(bG)} = \sup_s |\mu_t(f(t)\eta_\gamma(st^{-1})) - f(s)| \leq \sup_s \mu_t(|f(t) - f(s)|\eta_\gamma(st^{-1})).$$

Since $f$ is uniformly continuous on group $bG$, there exists a neighbourhood $U$ of $1 \in bG$ such that $|f(t) - f(s)| < \varepsilon$ whenever $st^{-1} \in U$. Let $\gamma$ be such that $\text{supp}(\eta_\gamma) \subset U$, see (9.16.89). Then by the previous estimate

$$\|T_\gamma f - f\|_{C(bG)} \leq \varepsilon \sup_s \mu_t(\eta_\gamma(st^{-1})) \leq \varepsilon(1 + \varepsilon_\gamma).$$

Clearly, if $\gamma' > \gamma$ then $\|T_{\gamma'} f - f\|_{C(bG)} < \|T_\gamma f - f\|_{C(bG)}$.

Now, let $\{f_k\}_{k=1}^l$ be an $\varepsilon$-net for $K$, i.e., for every $f \in K$ there exists $1 \leq k \leq l$ such that $\|f - f_k\|_{C(bG)} < \varepsilon$. We obtain that there exist $\gamma$ such that

$$\|T_\gamma f_k - f_k\|_{C(bG)} < \varepsilon(1 + \varepsilon_\gamma), \quad 1 \leq k \leq l. \quad (9.16.91)$$

Increasing $\gamma$, if necessary, we may assume that $\varepsilon_\gamma < \varepsilon$. Therefore, by (9.16.90), (9.16.91) and the definition of $\varepsilon$-net

$$\|T_\gamma f - f\|_{C(bG)} \leq \|T_\gamma(f - f_k)\|_{C(bG)} + \|T_\gamma f_k - f_k\|_{C(bG)} + \|f_k - f\|_{C(bG)} < (1 + \varepsilon)\varepsilon + 2\varepsilon + \varepsilon < 5\varepsilon, \quad f \in K,$$

from which the required result follows.
9.16.4 Proof of Theorem 8.0.9

In view of isomorphism (9.16.86), it suffices to prove that under the assumptions of Theorem 8.0.9 \(A(D_0, C_aX_0)\) has the approximation property with approximation operators in \(L\left(A(D_0, C_aX_0), A_0(D_0, C_aX_0)\right)\).

Let \(\|\cdot\|_{\bar D_0}\) denote the norm in the Banach space \(A(D_0, C_aX_0)\).

Let \(K \subset A(D_0, C_aX_0)\) be a compact subset. We fix \(\varepsilon > 0\).

Now, we follow the proof of Theorem 8.0.5, making the following changes:

1. We replace step (i) with a construction of a bounded linear operator

\[ T_{1, K, \varepsilon} : L(A(D_0, C_aX_0), A(D_0', C_aX_0)) \]

where \(D_0' \subset X_0\) is a strictly pseudoconvex neighbourhood of \(\bar D_0\), such that for every \(f \in K\) we have \(\|f - (T_{1, K, \varepsilon} f)\|_{D_0} < \varepsilon\).

The operator \(T_{1, K, \varepsilon}\) is defined as follows.

Recall that by the result in \([Lt2]\) (see Remark 6.1.6), since \(X_0\) is a Stein manifold, there exist holomorphic Banach vector bundles \(p_1 : E_1 \to X_0\) and \(p_2 : E_2 \to X_0\) with fibres \(B_1\) and \(B_2\), respectively, such that \(E_2 = E_1 \oplus C_aX_0\) (the Whitney sum) and \(E_2\) is holomorphically trivial, i.e., \(E_2 \cong X_0 \times B_2\). Thus, any holomorphic section of \(E_2\) can be identified with a \(B_2\)-valued holomorphic function on \(X_0\). By \(q : E_2 \to C_aX_0\) and \(i : C_aX_0 \to E_2\) we denote the corresponding quotient and embedding homomorphisms of these bundles, so that \(q \circ i = \text{Id}\).

Further, let \(A_{K, \varepsilon} \in L(A(D_0, B_2), A(D_0', B_2))\) be the operator constructed in Lemma 9.16.4, where \(D_0' \subset X_0\) is an open neighbourhood of \(D_0\). Without loss of generality, we may assume that \(D_0'\) is strictly pseudoconvex (see, e.g., \([HL]\)).

We set

\[ T_{1, K, \varepsilon} := q \circ A_{K, \varepsilon} \circ i. \]

Let us denote \(K^1 := T_{1, K, \varepsilon}(K) \subset A(D_0', C_aX_0)\). Since operator \(T_{1, K, \varepsilon}\) is bounded, \(K^1\) is a compact subset.

2. In step (ii) we replace Lemma 9.16.5 with the following proposition:
There exist a subspace \( a \subseteq \mathfrak{a} \) (\( \iota \in I \)) and finite rank bounded linear operators \( \tilde{T}_{K^1, k, \varepsilon}^n \in \mathcal{L}(\mathcal{A}(U_k, C_a X_0), \mathcal{A}(U_k, C_{a, \varepsilon} X_0)) \) \( (1 \leq k \leq M) \) such that sections \( F_{K^1, k, \varepsilon} := \tilde{T}_{K^1, k, \varepsilon}^n(f|_{U_k}) \), \( f \in K^1 \), satisfy

\[
\|f(x) - F_{K^1, k, \varepsilon}(x)\|_a < (1 + C_1)\varepsilon \quad \text{for all } x \in U_k, \ 1 \leq k \leq M,
\]

where \( C_1 > 0 \) is independent of \( f, k \) and \( \varepsilon > 0 \).

**Proof of claim (\( \star \)).** Let \( f \in \mathcal{A}(D'_0, C_a X_0) \).

Similarly to the proof of Lemma 9.16.5, we may assume without loss of generality that \( f|_{U_k} \) is defined over an open neighbourhood of a closed polydisk \( \bar{\Delta} \subset \mathbb{C}^n \). We denote \( f_k := f|_{U_k} \), and identify \( f_k \) via the corresponding holomorphic trivialization of bundle \( C_a X_0 \) with a holomorphic \( a \)-valued function defined in an open neighbourhood of \( \Delta \). In particular, \( f_k|_{\Delta} \in \mathcal{A}(\Delta, a) \).

We now apply the argument of the proof of Lemma 9.16.5 with the following modifications.

A. There is Taylor series expansion

\[
T_0^N f_k(x) := \sum_{|\alpha| \leq N} a_{k, \alpha, f} x^\alpha, \quad x \in \Delta, \ 1 \leq k \leq M,
\]

where \( a_{k, \alpha, f} \in a \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index, \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

Since \( K^1 \) is a compact subset of a metric space, there exists a finite \( \varepsilon \)-net \( \{f^1, \ldots, f^m\} \) for \( K^1, m = m(\varepsilon) \), i.e., a subset of \( K^1 \) such that for every \( f \in K^1 \) there is \( f^s \) so that

\[
\|f - f^s\|_{D_0} < \varepsilon \quad \text{(9.16.92)}
\]

(see, e.g., [Yos]). We can choose \( N \) such that

\[
\|f_k^s(x) - T_0^N f_k^s(z)\|_a < \varepsilon, \quad \text{for all } x \in \Delta, \ 1 \leq k \leq M, \ 1 \leq s \leq m. \quad \text{(9.16.93)}
\]

Then it follows from (9.16.92), (9.16.93) and the Cauchy integral formula that

\[
a_{k, \alpha, f} = \frac{|\alpha|!}{2\pi i} \int_{b\Delta} \frac{f_k(\zeta)}{\zeta^{\alpha_1+1} \cdots \zeta^{\alpha_n+1}} d\zeta, \quad \text{(9.16.94)}
\]

where \( b\Delta \) is the boundary torus of \( \Delta \), that

\[
\|f_k(x) - T_0^N f_k(z)\|_a < C_1\varepsilon, \quad \text{for all } x \in \Delta, \ 1 \leq k \leq M, \ f \in K^1, \quad \text{(9.16.95)}
\]

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where $C_1 > 0$ is independent of $f$ and $k$.

We denote $\hat{K} := \{ a_{k,\alpha,f} : 1 \leq k \leq M, |\alpha| \leq N, f \in K^1 \} \subset a$. Let us show that $\hat{K}$ is compact. Indeed, since map $I_{k,\alpha} : f_k \rightarrow a_{k,\alpha,f}$ defined by (9.16.94) is continuous, the image $I_{k,\alpha}(K^1)$ is a compact subset of $a$. By definition, $\hat{K} = \bigcup_{1 \leq k \leq M, |\alpha| \leq N} I_{k,\alpha}(K^1)$, therefore $\hat{K}$ is a compact subset as well.

B. For every $f \in \mathcal{A}(D_0, C_a X_0)$ we set

$$a_{k,\alpha,f}^\varepsilon := S_{\hat{K},\varepsilon}(a_{k,\alpha,f}), \quad \delta > 0,$$

where $S_{\hat{K},\varepsilon} \in L(a, a_0)$ are the (finite rank) approximation operators as in the formulation of the theorem, and $\delta > 0$ is chosen to be sufficiently small, so that we have for all $f \in K^1$

$$\left\| \sum_{|\alpha| \leq N} a_{k,\alpha,f} x^\alpha - \sum_{|\alpha| \leq N} a_{k,\alpha,f}^\varepsilon x^\alpha \right\|_{a} < \varepsilon, \quad 1 \leq k \leq M. \quad (9.16.96)$$

It follows from the definition of space $a_0$ (cf. Section 8) that there is $\iota \varepsilon \in I$ such that $S_{\hat{K},\varepsilon}(a) \subset a_{\iota \varepsilon}$.

Now, we set

$$\tilde{T}_{K^1,k,\varepsilon}^2(f_k) := \sum_{|\alpha| \leq N} a_{k,\alpha,f}^\varepsilon x^\alpha, \quad 1 \leq k \leq M, \quad f \in \mathcal{A}(D_0, C_a X_0).$$

Clearly, $\tilde{T}_{K^1,k,\varepsilon}^2 \in L(\mathcal{A}(U_k, C_a X_0), \mathcal{A}(U_k, C_{a_{\iota \varepsilon}} X_0)), 1 \leq k \leq M$.

Since space $a_{\iota \varepsilon}$ is finite dimensional and $N$ is independent of $f$, operators $\tilde{T}_{K^1,k,\varepsilon}^2$ have finite rank.

Further, it follows from (9.16.95) and (9.16.96) that

$$\| f(x) - \tilde{T}_{K^1,k,\varepsilon}^2(f_k)(x) \|_{a} < (1 + C_1)\varepsilon, \quad x \in U_k, \quad 1 \leq k \leq M, \quad f \in K^1, \quad (9.16.97)$$

as required.
Since operators $\tilde{T}_{K^1,k,\varepsilon}^2 (1 \leq k \leq M)$ have finite rank, operator $T_{K^1,\varepsilon}^2$ has finite rank as well.

We set $K^2 := T_{K^1,\varepsilon}^2 (K^1) \subset \bigoplus_{k=1}^M A(U_k, C_{a,\varepsilon} X_0)$. Since operator $T_{K^1,\varepsilon}^2$ is bounded, $K^2$ is compact.

C. In step (iii) we proceed as in the proof of Lemma 9.16.6. Since all transformations used in the proof are given by bounded linear operators, we have, in fact, the following result:

There exists a bounded linear operator

$$T_{K^2,\varepsilon}^3 \in \mathcal{L} \left( \bigoplus_{k=1}^M A(U_k, C_{a,\varepsilon} X_0), A(D_0, C_{a,\varepsilon} X_0) \right)$$

such that if $(F_{\varepsilon,k})_{k=1}^M \in K^2$, and $F := T_{K^2,\varepsilon}^3 \left( (F_{\varepsilon,k})_{k=1}^M \right)$, then

$$\|F(x) - F_{\varepsilon,k}(x)\|_a < C_2 \epsilon, \quad \text{for all } x \in U_k \cap \bar{D}_0, \quad 1 \leq k \leq M,$$

for a certain $C_2 > 0$ independent of $(F_{\varepsilon,k})_{k=1}^M$ and $\varepsilon > 0$.

Note that by definition $A(D_0, C_{a,\varepsilon} X_0) \subset A_0(D_0, C_{a} X_0)$. We define the required approximation operator $T_{K,\varepsilon} \in \mathcal{L}(A(D_0, C_{a} X_0), A_0(D_0, C_{a} X_0))$ as the composition

$$T_{K,\varepsilon} := T_{K^2,\varepsilon}^3 T_{K^1,\varepsilon}^2 T_{K^1,\varepsilon}^1.$$

It follows from the above estimates that

$$\|f - Tf|_{D_0} \|_{D_0} < (1 + C_1 + C_2)\varepsilon, \quad f \in K.$$ 

Since operator $T_{K^1,\varepsilon}^2$ has finite rank, operator $T_{K,\varepsilon}$ has finite rank as well. The proof is complete.
Part IV

Bounded holomorphic functions on unit disk whose moduli can have only first-kind boundary discontinuities (2007-2009)
Chapter 10

Introduction

We study the Banach algebras of holomorphic semi-almost periodic functions, i.e., bounded holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ whose boundary values belong to the algebra $SAP(\partial \mathbb{D}) \subset L^\infty(\partial \mathbb{D})$ of semi-almost periodic functions on the unit circle $\partial \mathbb{D}$.

A function $f \in L^\infty(\partial \mathbb{D})$ is called semi-almost periodic if for any $z_0 \in \partial \mathbb{D}$ and any $\varepsilon > 0$ there exist functions $f_k : \partial \mathbb{D} \to \mathbb{C}$ ($k \in \{-1, 1\}$) and arcs $\gamma_k$ with $z_0$ being their right (if $k = -1$) or left (if $k = 1$) endpoint with respect to the counterclockwise orientation of $\partial \mathbb{D}$ such that the functions $x \mapsto f_k(z_0 e^{ikx})$, $-\infty < x < 0$, $k \in \{-1, 1\}$, are restrictions of Bohr’s almost periodic functions on $\mathbb{R}$ (cf. Definition 11.0.1 below) and

$$\sup_{z \in \gamma_k} |f(z) - f_k(z)| < \varepsilon, \quad k \in \{-1, 1\}.$$

Algebra $SAP(\partial \mathbb{D})$ contains as a subalgebra the algebra introduced by D. Sarason [Sar] in connection with some problems in the theory of Toeplitz operators.

The graph of a real-valued semi-almost periodic function discontinuous at a single point has a form
Our interest to holomorphic semi-almost periodic functions is motivated by the problem of description of the weakest possible boundary discontinuities of functions in $H^\infty(D)$, the Hardy algebra of bounded holomorphic functions on $D$. (Recall that a function $f \in H^\infty(D)$ has radial limits almost everywhere on $\partial D$, the limit function $f|_{\partial D} \in L^\infty(\partial D)$, and $f$ can be recovered from $f|_{\partial D}$ by means of the Cauchy integral formula.)

In the general form this problem is as follows:

*Given a continuous function $\Phi : \mathbb{C} \to \mathbb{C}$ to describe the minimal Banach subalgebra $A_\Phi \subset H^\infty(D)$ containing all elements $f \in H^\infty(D)^*$ such that $\Phi(f)|_{\partial D}$ is piecewise Lipschitz having finitely many first-kind discontinuities.*

Here $H^\infty(D)^*$ is the group of invertible elements of $H^\infty(D)$. Clearly, each $A_\Phi$ contains the disk-algebra $A(D)$ (i.e., the algebra of holomorphic functions continuous up to the boundary). Moreover, if $\Phi(z) = z$, $\text{Re}(z)$ or $\text{Im}(z)$, then the Lindelöf theorem, see, e.g., [Gar], implies that $A_\Phi = A(D)$. In contrast, if $\Phi$ is constant on a closed simple curve which does not encompass $0 \in \mathbb{C}$, then $A_\Phi = H^\infty(D)$. (This result is obtained by consequent applications of the Carathéodory conformal mapping theorem, the Mergelyan theorem and the Marshall theorem, see, e.g., [Gar].)

In the thesis, we study the case $\Phi(z) = |z|$ and show that

$$A_\Phi = SAP(\partial D) \cap H^\infty(D)$$

(10.0.1)

(cf. Theorem 12.1.6 below). Although our results are valid for the particular choice of $\Phi(z) = |z|$, the methods developed here can be applied further to a more general class of functions $\Phi$.

In what follows, we use presentation (10.0.1) to describe the analytic and topological structure of the maximal ideal space of algebra $A := A_\Phi$, including corona theorem, Silov boundary
and Čech cohomology groups (cf. Section 12.3 below), and prove that space $A$ has the approximation property.

**Definition 10.0.1.** A Banach space $B$ is said to have the approximation property if for every compact set $K \subset B$ and every $\varepsilon > 0$ there is an operator $T : B \to B$ of finite rank so that

$$\|Tx - x\|_B < \varepsilon \quad \text{for every} \quad x \in K.$$

(Throughout the thesis all Banach spaces are assumed to be complex.)

Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces which appear naturally in analysis, it is not known yet even for the space $H^\infty(\mathbb{D})$ (see, e.g., the paper of Bourgain and Reinov [BR] for some results in this direction). The first example of a space which fails to have the approximation property was constructed by Enflo [E]. Since Enflo’s work several other examples of such spaces were constructed, for the references see, e.g., [L].

Many problems of Banach space theory admit especially simple solutions if one of the spaces under consideration has the approximation property. One of such problems is the problem of determination whether given two Banach algebras $D \subset C(X)$, $B \subset C(Y)$ ($X$ and $Y$ are compact Hausdorff spaces) their slice algebra

$$S(D, B) := \{f \in C(X \times Y) : f(\cdot, y) \in D \text{ for all } y \in Y, f(x, \cdot) \in B \text{ for all } x \in X\}$$

coincides with $D \otimes B$, the closure in $C(X \times Y)$ of the symmetric tensor product of $D$ and $B$. For instance, this is true if either $D$ or $B$ have the approximation property. The latter is an immediate consequence of the following result of Grothendieck.

Let $D \subset C(X)$ be a closed subspace, $B$ be a Banach space and $D_B \subset C_B(X) := C(X, B)$ be the Banach space of all continuous $B$-valued functions $f$ such that $\varphi(f) \in D$ for any $\varphi \in B^*$. By $D \otimes B$ we denote completion of symmetric tensor product of $D$ and $B$ with respect to norm

$$\left\| \sum_{k=1}^m a_k \otimes b_k \right\| := \sup_{x \in X} \left\| \sum_{k=1}^m a_k(x)b_k \right\|_B \quad \text{with} \quad a_k \in D, \ b_k \in B.$$

**Theorem 10.0.2 ([G]).** The following statements are equivalent:

1) $D$ has the approximation property;

2) $D \otimes B = D_B$ for every Banach space $B$. 

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Chapter 11

Preliminaries on Bohr’s almost periodic functions

1. We first recall the definition of Bohr almost periodic functions on $\mathbb{R}$. In what follows, by $C_b(\mathbb{R})$ we denote the algebra of bounded continuous functions on $\mathbb{R}$ endowed with sup-norm.

**Definition 11.0.1.** A function $f \in C_b(\mathbb{R})$ is said to be almost periodic if the family of its translates $\{S_\tau f\}_{\tau \in \mathbb{R}}$, $S_\tau f(x) := f(x + \tau)$, $x \in \mathbb{R}$, is relatively compact in $C_b(\mathbb{R})$.

The basic example of an almost periodic function is given by the formula

$$x \mapsto \sum_{l=1}^{m} c_l e^{i\lambda_l x}, \quad c_l \in \mathbb{C}, \quad \lambda_l \in \mathbb{R}.$$  

Let $AP(\mathbb{R})$ denote the Banach algebra of almost periodic functions endowed with sup-norm.

An almost periodic function $f \in AP(\mathbb{R})$ is uniquely determined by its Bohr-Fourier coefficients $a_\lambda(f)$ and spectrum $\text{spec}(f)$, defined in terms of the mean value

$$M(f) := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x)dx.$$  

(11.0.1)

Namely,

$$a_\lambda(f) := M(f e^{-i\lambda x}), \quad \lambda \in \mathbb{R}.$$  

(11.0.2)

Then $a_\lambda(f) \neq 0$ for at most countably many values of $\lambda$, see, e.g., [Bes]. These values constitute the spectrum $\text{spec}(f)$ of $f$. In particular, if $f = \sum_{l=1}^{\infty} c_l e^{i\lambda_l x}$ ($c_l \neq 0$ and $\sum_{l=1}^{\infty} |c_l| < \infty$), then $\text{spec}(f) = \{\lambda_1, \lambda_2, \ldots\}$.  

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One of the main results of the theory of almost periodic functions states that each function \( f \in \text{AP}(\mathbb{R}) \) can be uniformly approximated by functions of the form \( \sum_{l=1}^{m} c_l e^{i\lambda_l x} \) with \( \lambda_l \in \text{spec}(f) \).

Let \( \Gamma \subset \mathbb{R} \) be a unital additive semi-group (i.e., \( 0 \in \Gamma \)). It follows easily from the cited approximation result that the space \( \text{AP}_\Gamma(\mathbb{R}) \) of almost periodic functions with spectra in \( \Gamma \) forms a unital Banach subalgebra of \( \text{AP}(\mathbb{R}) \).

We will need the following result.

**Proposition 11.0.2.** \( \text{AP}_\Gamma(\mathbb{R}) \) has the approximation property.

For a unital semi-group \( \Gamma \subset \mathbb{R} \) by \( b_\Gamma(\mathbb{R}) \) we denote the maximal ideal space of algebra \( \text{AP}_\Gamma(\mathbb{R}) \). (E.g., for \( \Gamma = \mathbb{R} \) the space \( b\mathbb{R} := b_\mathbb{R}(\mathbb{R}) \), called the *Bohr compactification* of \( \mathbb{R} \), is a compact abelian topological group that admits presentation as the inverse limit of compact finite-dimensional tori. The group \( \mathbb{R} \) admits a canonical embedding into \( b\mathbb{R} \) as a dense subgroup.)

**Notation.** Here and below, in case \( \Gamma = \mathbb{R} \) we omit index \( \Gamma \).

2. Let \( T := \{ z \in \mathbb{C} : \text{Im} \;(z) \in [0, \pi]\} \), we set \( T_0 := \{ z \in \mathbb{C} : \text{Im} \;(z) \in (0, \pi)\} \) – the interior of \( T \).

Let \( C_b(T) \) denote the Banach algebra of bounded continuous functions on strip \( T \) endowed with sup-norm. We recall that following

**Definition 11.0.3.** A function \( f \in C_b(T) \) is called *holomorphic almost periodic* if it is holomorphic in \( T_0 \) and the family of its translates \( \{S_x f\}_{x \in \mathbb{R}} \), \( S_x f(z) := f(z + x), \; z \in T \), is relatively compact in \( C_b(T) \).

Equivalently, a function \( f \in C_b(T) \) is called *holomorphic almost periodic* if it is uniformly continuous, holomorphic in \( T_0 \), and almost periodic on each horizontal line in \( T \).

We denote by \( \text{APH}(T) \) the Banach algebra of holomorphic almost periodic functions endowed with sup-norm. Any function in \( \text{APH}(T) \) is uniformly continuous on \( T \).

The mean value of a function \( f \in \text{APH}(T) \) is defined by the formula

\[
M(f) := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x + iy)dx \in \mathbb{C}
\] (11.0.3)
$(M(f)$ does not depend on $y, \text{ see, e.g., [Bes]}$). Further, the Bohr-Fourier coefficients of $f$ are defined by

$$a_\lambda(f) := M(f e^{-i\lambda z}), \quad \lambda \in \mathbb{R}.$$  \hfill (11.0.4)

Then $a_\lambda(f) \neq 0$ for at most countably many values of $\lambda$, these values form the spectrum $\text{spec}(f)$ of $f$. For instance, if $f = \sum_{l=1}^{\infty} c_l e^{i\lambda_l z}$ ($c_l \neq 0, \sum_{l=1}^{\infty} |c_l| < \infty$), then $\text{spec}(f) = \{\lambda_1, \lambda_2, \ldots\}$.

Similarly to the functions in $AP(\mathbb{R})$, each function $f \in APH(T)$ can be uniformly approximated by functions of the form $\sum_{l=1}^{m} c_l e^{i\lambda_l z}$ with $\lambda_l \in \text{spec}(f)$.

Let $\Gamma \subset \mathbb{R}$ be a unital additive semi-group. The space $APH_{\Gamma}(T)$ of holomorphic almost periodic functions with spectra in $\Gamma$ is a unital Banach algebra.

**Proposition 11.0.4.** $APH_{\Gamma}(T)$ has the approximation property.

By $b_{\Gamma}(T)$ we denote the maximal ideal space of algebra $APH_{\Gamma}(T)$ and by $\iota_{\Gamma} : T \rightarrow b_{\Gamma}(T)$ the continuous map associating to each point of $T$ the corresponding point evaluation homomorphism. Note that $\iota = \iota_{\mathbb{R}}$ is an injection, although $\iota_{\Gamma}$ is not an injection in general.

3. We will also need the following definitions and results.

**Definition 11.0.5.** A function $f \in C_b(T)$ is called continuous almost periodic if the family of its translates $\{S_x f\}_{x \in \mathbb{R}}, S_x f(z) := f(z + x), z \in T$, is relatively compact in $C_b(T)$.

Equivalently, a function $f \in C_b(T)$ is called continuous almost periodic if it is uniformly continuous and almost periodic on each horizontal line in $T$.

We denote by $APC(T)$ the Banach algebra of continuous almost periodic functions on $T$ endowed with sup-norm.

**Lemma 11.0.6.** The maximal ideal space of algebra $APC(T)$ is homeomorphic to $bT$.

4. Using the results of Section 6.2 (in particular, Example 6.2.4), we obtain that there is a family of continuous injections $\iota_{\xi} : T \rightarrow bT$ parametrized by the elements $\xi \in b\mathbb{Z}$ whose images in $b\mathbb{Z}/\mathbb{Z}$ are mutually distinct. As a set, $bT = \bigcup_{\xi} \iota_{\xi}(T)$.

**Definition 11.0.7.** A function $f \in C(bT)$ is called holomorphic if

$$(\iota_{\xi}^* f)|_{T_0} \in \mathcal{O}(T_0) \quad \text{for all } \xi.$$
We denote by $O(bT)$ the Banach algebra of holomorphic functions on $bT$ endowed with sup-norm.

**Lemma 11.0.8.** There is an isometric isomorphism $APH(T) \cong O(bT)$.

5. Since every function $f \in APH(T)$ can be approximated uniformly on $T$ by the polynomials in variables $e^{i\lambda z}$, $\lambda \in \mathbb{R}$, using the inverse limit construction for maximal ideal spaces of uniform algebras (see [R]) we obtain that the base of topology of $bT$ is generated by functions $e^{i\lambda z}$.

Precisely, the base of topology on $bT$ consists of open sets of the form

$$U(\lambda_1, \ldots, \lambda_l, \xi, \epsilon) := \{\eta \in bT : \max_{1 \leq k \leq l} |e_{\lambda_k}(\eta) - e_{\lambda_k}(\xi)| < \epsilon\}$$

where $e_\lambda$ denotes the extension of $z \mapsto e^{i\lambda z}$ to $bT$ by means of the Gelfand transform. In what follows, we identify $\iota(T)$ (see above) with $T$.

**Theorem 11.0.9.** $T$ is dense in $bT$. 
Chapter 12

Definitions

12.1 Semi-almost periodic functions

In what follows, we consider the boundary $\partial \mathbb{D}$ of $\mathbb{D}$ with counterclockwise orientation.

**Definition 12.1.1.** We denote by $A^S$ the closed subalgebra of $H^\infty(\mathbb{D})$ generated by the disk-algebra $A(\mathbb{D})$ and the holomorphic functions of the form $(z + z_0)e^{\lambda h}$ ($z \in \mathbb{D}$), where $\text{Re}(h)|_{\partial \mathbb{D}}$ is the characteristic function of the closed arc going from the initial point at $z_0$ to the endpoint at $-z_0$ such that $z_0 \in S$, and $\lambda \in \mathbb{R}$ (in particular, $(z + z_0)e^{\lambda h}$ has discontinuity at $z_0$ only).

The moduli of functions in $A^S_{\Sigma}$ can have only first-kind boundary discontinuities.

The main result of this part is the characterization of functions in $A^S_{\Sigma}$ as those functions in $H^\infty(\mathbb{D})$ that have semi-almost periodic boundary values (cf. Definition 12.1.2 and Theorem 12.1.6 below). This presentation allows us to use the results of Bohr’s theory of almost periodic functions to study approximation in space $A^S_{\Sigma}$, and describe the topological structure of the maximal ideal space of algebra $A^S_{\Sigma}$, including corona theorem, Šilov boundary and Čech cohomology groups (cf. Section 12.3 below).

We will need the following notation. For $z_0 := e^{it_0}$, $t_0 \in [0, 2\pi)$, let

$$\gamma^k_{z_0}(\delta) := \{z_0e^{ikx} : 0 \leq x < \delta < 2\pi\}, \quad k \in \{-1, 1\},$$

(12.1.1)

be two open arcs having $z_0$ as the right and the left endpoints (with respect to the counterclockwise orientation of $\partial \mathbb{D}$), respectively.
Definition 12.1.2. A function $f \in L^\infty(\partial \mathbb{D})$ is called semi-almost periodic if for any $z_0 \in \partial \mathbb{D}$, and any $\varepsilon > 0$ there exist a number $\delta = \delta(z_0, \varepsilon) \in (0, \pi)$ and functions $f_k : \gamma_{z_0}(\delta) \to \mathbb{C}$, $k \in \{-1, 1\}$, such that functions

$$\tilde{f}_k(x) := f_k(z_0 e^{ik\delta e^x}), \quad -\infty < x < 0, \quad k \in \{-1, 1\},$$

are restrictions of some almost periodic functions from $AP(\mathbb{R})$, and

$$\sup_{z \in \gamma_{z_0}(\delta)} |f(z) - f_k(z)| < \varepsilon, \quad k \in \{-1, 1\}.$$  

We denote by $SAP(\partial \mathbb{D})$ the Banach algebra of semi-almost periodic functions on $\partial \mathbb{D}$ endowed with sup-norm. It is easy to see that the set of points of discontinuity of a function in $SAP(\partial \mathbb{D})$ is at most countable.

Let $SAP(S) \subset SAP(\partial \mathbb{D})$ be the closed subalgebra consisting of semi-almost periodic functions on $\partial \mathbb{D}$ that are continuous on $\partial \mathbb{D} \setminus S$. (Note that the Sarason algebra introduced in [Sar] is isomorphic to $SAP(\{z_0\}, z_0 \in \partial \mathbb{D}$.)

Example 12.1.3. A function $g$ defined on $\mathbb{R} \sqcup (\mathbb{R} + i\pi)$ is said to belong to the space $AP(\mathbb{R} \sqcup (\mathbb{R} + i\pi))$ if the functions $g(x)$ and $g(x + i\pi)$, $x \in \mathbb{R}$, belong to $AP(\mathbb{R})$. The space $AP(\mathbb{R} \sqcup (\mathbb{R} + i\pi))$ is a function algebra (with respect to sup-norm).

Given $z_0 \in \partial \mathbb{D}$ consider the map

$$\varphi_{z_0} : \partial \mathbb{D} \setminus \{-z_0\} \to \mathbb{R}, \quad \varphi_{z_0}(z) := \frac{2i(z_0 - z)}{z_0 + z},$$

and define a linear isometric embedding $L_{z_0} : AP(\mathbb{R} \sqcup (\mathbb{R} + i\pi)) \to L^\infty(\partial \mathbb{D})$ by the formula

$$(L_{z_0} g)(z) := (g \circ \text{Log} \circ \varphi_{z_0})(z),$$  \hspace{1cm} (12.1.2)

where $\text{Log}(z) := \ln |z| + i \text{Arg}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}_{-}$, and $\text{Arg} : \mathbb{C} \setminus \mathbb{R}_{-} \to (-\pi, \pi)$ stands for the principal branch of the multi-function arg. Then the range of $L_{z_0}$ is a subspace of $SAP(\{-z_0, z_0\})$.

Definition 12.1.4. The functions in $SAP(S) \cap H^\infty(\mathbb{D})$ are called holomorphic semi-almost periodic.

Example 12.1.5. For $z_0 \in \partial \mathbb{D}$ consider the map

$$\varphi_{z_0} : \tilde{D} \setminus \{-z_0\} \to \tilde{H}_{+}, \quad \varphi_{z_0}(z) := \frac{2i(z_0 - z)}{z_0 + z}.$$
Here $\mathbb{H}^+$ is the upper half-plane. Then $\varphi_{z_0}$ maps $\mathbb{D}$ conformally onto $\mathbb{H}_+$ and $\partial\mathbb{D} \setminus \{-z_0\}$ diffeomorphically onto $\mathbb{R}$ (the boundary of $\mathbb{H}_+$) so that $\varphi_{z_0}(z_0) = 0$.

Let $T_0$ be the interior of the strip $T$. Consider the conformal map $\text{Log} : \mathbb{H}_+ \to T_0$, $z \mapsto \text{Log}(z) := \ln|z| + i\text{Arg}(z)$, where $\text{Arg} : \mathbb{C} \setminus \mathbb{R}_- \to (-\pi, \pi)$ is the principal branch of the multifunction $\text{arg}$. The function $\text{Log}$ is extended to a homeomorphism of $\mathbb{H}_+ \setminus \{0\}$ onto $T$. Let $g \in A\mathcal{P}(T)$. Then the function

$$(L_{z_0}g)(z) := (g \circ \text{Log} \circ \varphi_{z_0})(z), \quad z \in \mathbb{D},$$

belongs to $\text{SAP}(-z_0, z_0) \cap H^\infty(\mathbb{D})$.

**Theorem 12.1.6.** $\text{SAP}(S) \cap H^\infty(\mathbb{D}) = A^S$.

(cf. Theorem 13.1.3 below.)

### 12.2 Spectra of semi-almost periodic functions

We will need the following

**Proposition 12.2.1.** Let $z_0 \in \partial \mathbb{D}$. There exists a homomorphism of Banach algebras

$$E_{z_0} : \text{SAP}(\partial \mathbb{D}) \to \text{AP}(\mathbb{R} \sqcup (\mathbb{R} + i\pi))$$

(cf. Example 12.1.3 for notation) of norm 1 such that for each $f \in \text{SAP}(\partial \mathbb{D})$ the function $f - L_{z_0}(E_{z_0}f) \in \text{SAP}(\partial \mathbb{D})$ is continuous and equal to 0 at $z_0$.

**Definition 12.2.2.** We define the left $(k = -1)$ and the right $(k = 1)$ mean values $M^k_{z_0}(f)$ of a function $f \in \text{SAP}(\partial \mathbb{D})$ over $z_0 \in \partial \mathbb{D}$ by the formula

$$M^k_{z_0}(f) := M(f_{k,z_0}).$$

It is not difficult to see that the mean value $M^k_{z_0}$ is a complex continuous linear functional on $\text{SAP}(\partial \mathbb{D})$ of norm 1.

**Definition 12.2.3.** We define the left $(k = -1)$ and the right $(k = 1)$ Bohr-Fourier coefficients and the spectrum of function $f$ over $z_0$ by the formulas

$$a^k_{\lambda}(f, z_0) := a_{\lambda}(f_{k,z_0}).$$

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and

\[ \text{spec}^k_{z_0}(f) := \{ \lambda \in \mathbb{R} : a^k_\lambda(f, z_0) \neq 0 \}, \]

respectively.

It follows from the properties of spectrum of an almost periodic function on \( \mathbb{R} \) that \( \text{spec}^k_{z_0}(f) \) is at most countable.

Let \( S \) be a closed subset of \( \partial \mathbb{D} \), and

\[ \Sigma : S \times \{-1, 1\} \rightarrow 2^\mathbb{R} \]

be a set-valued map that associates to each \( z_0 \in S, k \in \{-1, 1\} \) a unital semi-group \( \Sigma(z_0, k) \subset \mathbb{R} \). We denote by \( SAP_{\Sigma}(S) \subset SAP(S) \) the closed subalgebra of semi-almost periodic functions \( f \) with \( \text{spec}^k_{z_0}(f) \subset \Sigma(z_0, k), z_0 \in S, k \in \{-1, 1\} \).

**Notation.** If \( \Sigma(s, \pm 1) = \mathbb{R} \) for all \( z_0 \in S \), then we omit writing index \( \Sigma \).

**Definition 12.2.4.** We call the maximal ideal space \( b^S_{\Sigma}(\partial \mathbb{D}) \) of algebra \( SAP_{\Sigma}(S) \) the **Bohr compactification** of circle \( \partial \mathbb{D} \) (relative to \( S \) and \( \Sigma \)).

Let

\[ r^S_{\Sigma} : b^S_{\Sigma}(\partial \mathbb{D}) \rightarrow \partial \mathbb{D} \]

be the map transpose to the embedding \( C(\partial \mathbb{D}) \hookrightarrow SAP_{\Sigma}(S) \).

**Theorem 12.2.5.** The following is true:

1. The map transpose to the restriction of homomorphism \( E_{z_0} \) to \( SAP_{\Sigma}(S) \) determines an embedding

   \[ h^0_{\Sigma} : b_{\Sigma(z_0, -1)}(\mathbb{R}) \sqcup b_{\Sigma(z_0, 1)}(\mathbb{R}) \hookrightarrow b^S_{\Sigma}(\partial \mathbb{D}) \]

   (cf. Section 11 for notation) whose image coincides with \( (r^S_{\Sigma})^{-1}(z_0) \).

2. The restriction

   \[ r^S_{\Sigma} : b^S_{\Sigma}(\partial \mathbb{D}) \setminus (r^S_{\Sigma})^{-1}(S) \rightarrow \partial \mathbb{D} \setminus S \]

   is a homeomorphism.
For an \( m \) point set \( S \) and each \( \Sigma(z_0, k) \), \( z_0 \in S, k \in \{-1, 1\} \), being a group, the maximal ideal space \( b^S_{\Sigma}(\partial \mathbb{D}) \) is the union of \( \partial \mathbb{D} \setminus S \) and \( 2m \) Bohr compactifications \( b_{\Sigma(z_0, k)}(\mathbb{R}) \) that can be viewed as (finite or infinite dimensional) tori.

### 12.3 Holomorphic semi-almost periodic functions

**Definition 12.3.1.** The functions in \( SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) will be called holomorphic semi-almost periodic (cf. Definition 12.1.4).

**Proposition 12.3.2.** Suppose that \( f \in SAP(S) \cap H^\infty(\mathbb{D}) \). Then

\[
\text{spec}^{-1}_{z_0}(f) = \text{spec}^1_{z_0}(f) =: \text{spec}_{z_0}(f),
\]

and

\[
a_{\lambda}^{-1}(f, z_0) = e^{\lambda \pi} a_{\lambda}^1(f, z_0) \quad \text{for each} \quad \lambda \in \text{spec}_{z_0}(f).
\]

(Recall that the choice of the upper indices \( \pm 1 \) is determined by the orientation of \( \partial \mathbb{D} \).

**Remark 12.3.3.** Proposition 12.3.2 and Lindelöf’s theorem (see, e.g., [Gar]) imply that

\[
SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) = SAP_{\Sigma'}(S') \cap H^\infty(\mathbb{D}),
\]

where

\[
S' := \{ z_0 \in S : \Sigma(z_0, -1) \cap \Sigma(z_0, 1) \neq \{0\} \}.
\]
and

\[ \Sigma'(z_0, k) := \Sigma(z_0, -1) \cap \Sigma(z_0, 1) \quad \text{for} \quad k = -1, 1, \quad z_0 \in S'. \]

Thus, we may assume without loss of generality that

\[ \Sigma(z_0, -1) = \Sigma(z_0, 1) =: \Sigma(s) \]

and each \( \Sigma(z_0), z_0 \in S, \) is non-trivial.

**Example 12.3.4.** If \( g(z) := e^{ix}, z \in T, \) then \( L_{z_0}g = e^{\lambda h} \) (cf. Example 6.2.6 for notation), where \( h \) is a holomorphic functions whose real part \( \text{Re}(h) \) is the characteristic function of the closed arc going in the counterclockwise direction from the initial point at \( z_0 \) to the endpoint at \( -z_0, \) and such that \( h(0) = \frac{1}{2} + \frac{i\ln 2}{\pi}. \) Thus, \( \text{spec}_{z_0}(e^{\lambda h}) = \{ \lambda/\pi \}. \) Indeed, in this case

\[ (g \circ \text{Log})|_\mathbb{R} = \left( x \mapsto \exp\left( \lambda (\chi_{\mathbb{R}^+}(x) + \frac{i\ln |x|}{\pi}) \right) \right), \]

where \( \chi_{\mathbb{R}^+} \) is the characteristic function of \( \mathbb{R}^+. \) In turn, the restriction of the pullback \( e^{\lambda h} \circ \varphi_{z_0}^{-1} \) to \( \mathbb{R} \) coincides with \( e^{\lambda (\chi_{\mathbb{R}^+} + \frac{i\ln |x|}{\pi})} \) as well. This implies the required result.
Chapter 13

Main results

13.1 Approximation

Theorem 13.1.1. SAP\(\Sigma\)(S) \(\cap\) \(H^\infty(D)\) has the approximation property.

Our proof of Theorem 13.1.1 is based on the equivalence established in Theorem 10.0.2 and on an approximation result for a Banach-valued analogue of algebra SAP\(\Sigma\)(S) \(\cap\) \(H^\infty(D)\) formulated below. Namely, for a (complex) Banach space \(B\) we define

\[ SAP^B\Sigma(S) := SAP\Sigma(S) \otimes B, \]

where SAP\(\Sigma\)(S) \(\otimes\) B denotes the completion in \(L^\infty_B(\partial D)\) of the symmetric tensor product of SAP\(\Sigma\)(S) and B. Using the Poisson integral formula we can extend each function from SAP\(\Sigma\)(S) to a bounded \(B\)-valued harmonic function on \(D\) having the same sup-norm. We identify SAP\(\Sigma\)(S) with its harmonic extension. Let \(H^\infty_B(D)\) be the Banach space of bounded \(B\)-valued holomorphic functions on \(D\) equipped with sup-norm. By \((SAP\Sigma(S) \cap H^\infty(D))_B\) we denote the Banach space of all continuous \(B\)-valued functions \(f\) on the maximal ideal space \(b^S(D)\) of algebra SAP\(\Sigma\)(S) \(\cap\) \(H^\infty(D)\) such that \(\varphi(f) \in SAP\Sigma(S) \cap H^\infty(D)\) for any \(\varphi \in B^*\). In what follows we naturally identify \(D\) with a subset of \(b^S(D)\).

Proposition 13.1.2. Let \(f \in (SAP\Sigma(S) \cap H^\infty(D))_B\). Then \(f|_D \in SAP^B\Sigma(S) \cap H^\infty_B(D)\).

Let \(A^S\Sigma\) be the closed subalgebra of \(H^\infty(D)\) generated by the disk-algebra \(A(D)\) and the functions of the form \((z + z_0)e^{\lambda h}\) \((z \in D)\), where Re \((h)|_{\partial D}\) is the characteristic function of the
closed arc going in the counterclockwise direction from the initial point at \( z_0 \) to the endpoint at \( -z_0 \) such that \( z_0 \in S \), and \( \frac{\lambda}{\pi} \in \Sigma(z_0) \).

The next result combined with Proposition 13.1.2 and Theorem 10.0.2 implies Theorem 13.1.1.

**Theorem 13.1.3.** \( SAP_{\Sigma}(S) \cap H_B^\infty(\mathbb{D}) = A_S^S \otimes B \).

In particular, we obtain Theorem 12.1.6. We also obtain the following

**Corollary 13.1.4.** \( SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) is generated by algebras \( SAP_{\Sigma|F}(F) \cap H^\infty(\mathbb{D}) \) for all possible finite subsets \( F \) of \( S \).

### 13.2 Maximal ideal space

**Definition 13.2.1.** The maximal ideal space \( b_S^S(\mathbb{D}) \) of algebra \( A_S^S (= SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})) \) (cf. Theorem 13.1.3) will be called the *Bohr compactification* of disk \( \mathbb{D} \) (relative to \( S \) and \( \Sigma \)).

We denote by

\[
a_S^S: b_S^S(\mathbb{D}) \to \mathbb{D}
\]

the continuous surjective map transpose to the embedding \( A(\mathbb{D}) \hookrightarrow SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \). (Recall that the maximal ideal space of the disk-algebra \( A(\mathbb{D}) \) is homeomorphic to \( \mathbb{D} \).) The next theorem describes the topological structure of \( b_S^S(\mathbb{D}) \).

**Theorem 13.2.2.** The following is true:

1. For every \( z_0 \in S \) there exists an embedding

\[
i_S^{z_0} : b_{\Sigma(z_0)}(T) \hookrightarrow b_S^S(\mathbb{D})
\]

(cf. Section 11 for notation) whose image is \( (a_S^S)^{-1}(z_0) \), so that the pullback \( (i_S^{z_0})^* \) maps \( SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) surjectively onto \( A PH_{\Sigma(z_0)}(T) \).

Furthermore, the composition of the restriction map to \( \mathbb{R} \sqcup (\mathbb{R} + i\pi) \) and \( (i_S^{z_0} \circ \iota_{\Sigma(z_0)})^* \) coincides with the restriction of homomorphism \( E_{z_0} \) to \( SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) (see Proposition 12.2.1).
(2) The restriction \(a^S_\Sigma : b^S_\Sigma(\mathbb{D}) \setminus (a^S_\Sigma)^{-1}(S) \to \bar{\mathbb{D}} \setminus S\) is a homeomorphism.

Since \(SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})\) separates the points on \(\mathbb{D}\), the evaluation at points of \(\mathbb{D}\) determines a natural embedding \(\iota : \mathbb{D} \to b^S_\Sigma(\mathbb{D})\). In what follows, we identify \(\mathbb{D}\) and \(\iota(\mathbb{D})\).

One has the following commutative diagram:

Here the ‘dashed’ arrows stand for embeddings in the case \(\Sigma(z_0, -1) = \Sigma(z_0, 1)\) are (non-trivial) groups for all \(z_0 \in S\), and for continuous maps otherwise.

We also obtain the following Corona-type theorem.

**Theorem 13.2.3.** \(\mathbb{D}\) is dense in \(b^S_\Sigma(\mathbb{D})\) if and only if each \(\Sigma(z_0), z_0 \in S\), is a group.

Recall that the corona theorem is equivalent to the following statement: for any collection of functions \(f_1, \ldots, f_m \in SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})\) such that

\[
\max_{1 \leq k \leq m} |f_k(z)| \geq \delta > 0 \quad \text{for all} \quad z \in \mathbb{D}
\]

there exist functions \(g_1, \ldots, g_m \in SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})\) such that

\[
f_1g_1 + \cdots + f_mg_m = 1 \quad \text{on} \quad \mathbb{D}.
\]

Our next result shows that \(b^S_\Sigma(\mathbb{D}), S \neq \emptyset\), is not arcwise connected.

**Theorem 13.2.4.** Let \(F : [0, 1] \to b^S_\Sigma(\mathbb{D})\) be a continuous path. Then either \(F([0, 1]) \subset \mathbb{D} \setminus S\) or there exists \(s \in S\) such that \(F([0, 1]) \subset (a^S_\Sigma)^{-1}(s)\).

Next, we describe the Čech cohomology groups of \(b^S_\Sigma(\mathbb{D})\).
Theorem 13.2.5. The following is true:

(1) The Čech cohomology groups

\[ H^k(b^\mathcal{S}_\Sigma(D), \mathbb{Z}) \cong \bigoplus_{z_0 \in S} H^k(b_{\Sigma(z_0)}(T), \mathbb{Z}), \quad k \geq 1. \]

(2) Suppose that each \( \Sigma(z_0) \) is a subset of \( \mathbb{R}_+ \) or \( \mathbb{R}_- \). Then \( H^k(b^\mathcal{S}(\mathbb{D}), \mathbb{Z}) = 0, \ k \geq 1, \) and \( SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) is projective free.

Recall that a commutative ring \( R \) with identity is called projective free if every finitely generated projective \( R \)-module is free. Equivalently, \( R \) is projective free iff every square idempotent matrix \( F \) with entries in \( R \) (i.e., such that \( F^2 = F \)) is conjugate over \( R \) to a matrix of the form

\[ \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \]

where \( I_k \) stands for the \( k \times k \) identity matrix. Every field \( \mathbb{F} \) is trivially projective free. Quillen and Suslin proved that if \( R \) is projective free, then the rings of polynomials \( R[x] \) and formal power series \( R[[x]] \) over \( R \) are projective free as well (see, e.g., [Lam]). Grauert proved that the ring \( \mathcal{O}(\mathbb{D}^n) \) of holomorphic functions on the unit polydisk \( \mathbb{D}^n \) is projective free [Gr]. In turn, it was shown in [BrS] that the triviality of any complex vector bundle of finite rank over the connected maximal ideal space of a unital semi-simple commutative complex Banach algebra is sufficient for its projective freeness. We use this result to prove the last assertion of Theorem 13.2.5.

Note that if a unital semi-simple commutative complex Banach algebra \( A \) is projective free, then it is Hermite, i.e., every finitely generated stably free \( A \)-module is free. Equivalently, \( A \) is Hermite iff any \( k \times n \) matrix, \( k < n \), with entries in \( A \) having rank \( k \) at each point of the maximal ideal space of \( A \) can be extended to an invertible \( n \times n \) matrix with entries in \( A \), see [C]. (Here the values of elements of \( A \) at points of the maximal ideal space are defined by means of the Gelfand transform.)

If \( G \) is a compact connected abelian topological group and \( \hat{G} \) is its dual, then \( H^k(G, \mathbb{Z}) \cong \wedge^k \hat{G}, \ k \geq 1 \) (see, e.g., [HM]). Using Pontryagin duality, we obtain
Corollary 13.2.6. Assume that each $\Sigma(s)$ is a group. Then

$$H^k(b^S_\Sigma(\mathbb{D}), \mathbb{Z}) \cong \bigoplus_{z_0 \in S} \wedge^k_{\Sigma}(z_0), \quad k \geq 1.$$  

In particular, if for a fixed $n \in \mathbb{N}$ each $\Sigma(s)$ is isomorphic to a subgroup of $\mathbb{Q}^n$, then

$$H^k(b^S_\Sigma(\mathbb{D}), \mathbb{Z}) = 0$$

for all $k \geq n + 1$.

Let $K^S_\Sigma$ be the Šilov boundary of algebra $SAP_\Sigma(S) \cap H^\infty(\mathbb{D})$, that is, the minimal closed subset of $b^S_\Sigma(\mathbb{D})$ such that for every $f \in SAP_\Sigma(S) \cap H^\infty(\mathbb{D})$

$$\sup_{z \in \mathbb{D}} |f(z)| = \max_{\varphi \in K^S_\Sigma} |f(\varphi)|,$$

where $f$ is assumed to be extended to $b^S_\Sigma(\mathbb{D})$ by means of the Gelfand transform. For a non-trivial semigroup $\Gamma \subset \mathbb{R}$ by $\mathrm{cl}_\Gamma(\mathbb{R} + i\pi)$ and $\mathrm{cl}_\Gamma(\mathbb{R})$ we denote closures of $\iota_\Gamma(\mathbb{R} + i\pi)$ and $\iota_\Gamma(\mathbb{R})$ in $b_\Gamma(T)$ (the maximal ideal space of $APH_\Gamma(T)$). One can easily show that these closures are homeomorphic to $b_{\hat{\Gamma}}(\mathbb{R})$, where $\hat{\Gamma}$ is the minimal subgroup of $\mathbb{R}$ containing $\Gamma$.

We retain the notation of Theorem 13.2.2.

Theorem 13.2.7.

$$K^S_\Sigma = \left( \bigcup_{z_0 \in S} i^S_{\Sigma}(\mathrm{cl}_{\Sigma(z_0)}(\mathbb{R}) \cup \mathrm{cl}_{\Sigma(z_0)}(\mathbb{R} + i\pi)) \right) \cup \partial \mathbb{D} \setminus S.$$

If each $\Sigma(z_0)$, $z_0 \in S$, is a group, then the Šilov boundary $K^S_\Sigma$ is naturally homeomorphic to the maximal ideal space $b^S_\Sigma(\partial \mathbb{D})$ of algebra $SAP_\Sigma(S)$, cf. Theorem 12.2.5.

13.3 Connected components

Let $GL_n(A)$ denote the group of invertible $n \times n$ matrices with entries in a unital Banach algebra $A$. By $[GL_n(A)]$ we denote the group of connected components of $GL_n(A)$, i.e., the quotient of $GL_n(A)$ by the connected component containing the unit $I_n \in GL_n(A)$ (this is a normal subgroup of $GL_n(A)$).
We set
\[ G^n_{\Sigma(s)}(T) := GL_n(APH_{\Sigma(s)}(T)) \quad \text{and} \quad G^n_{\Sigma}(S) := GL_n(SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})). \]

Let
\[ bT(S) := \bigsqcup_{s \in S} b_{\Sigma(s)}(T). \]

According to Theorem 13.2.2 there exists a natural embedding \( i_{\Sigma}(z_0) : b_{\Sigma(z_0)}(T) \hookrightarrow b_{\Sigma}(\mathbb{D}) \) whose image is \( (a_{\Sigma}^S)^{-1}(z_0) \). Then the map
\[ I : bT(S) \to (a_{\Sigma}^S)^{-1}(S), \quad I(\xi) := i_{\Sigma}(\xi) \quad \text{for} \quad \xi \in b_{\Sigma(z_0)}(T), \]

is a bijection.

**Theorem 13.3.1.** The map transpose to the composition \( bT(S) \xrightarrow{I} (a_{\Sigma}^S)^{-1}(S) \hookrightarrow b_{\Sigma}(\mathbb{D}) \)

induces an isomorphism
\[ [G^n_{\Sigma}(S)] \cong \bigoplus_{z_0 \in S} [G^n_{\Sigma(s)}(T)]. \]

In particular, if each \( \Sigma(z_0), z_0 \in S, \) is a subset of \( \mathbb{R}_+ \) or \( \mathbb{R}_- \), then \( G^n_{\Sigma}(S) \) is connected.
Chapter 14

Proofs

14.1 Proofs of Propositions 11.0.2 and 11.0.4

We prove Proposition 11.0.4 only (the proof of Proposition 11.0.2 is similar). We refer to [Bes] for the corresponding definitions and facts from the theory of almost periodic functions.

Proof of Proposition 11.0.4. Let $K \subset \text{APH}_\Gamma(T)$ be compact. Given $\varepsilon > 0$ consider an $\varepsilon$-net $\{f_1, \ldots, f_l\} \subset K$. Let

$$K(t) := \sum_{|\nu_1| \leq n_1, \ldots, |\nu_r| \leq n_r} \left(1 - \frac{\nu_1}{n_1}\right) \cdots \left(1 - \frac{\nu_r}{n_r}\right) e^{-i(\frac{\nu_1}{n_1}\beta_1 + \cdots + \frac{\nu_r}{n_r}\beta_r)t}$$

be a Bochner-Fejer kernel such that for all $1 \leq k \leq l$

$$\sup_{z \in \Sigma} |f_k(z) - M_t\{f_k(z + t)K(t)\}| \leq \frac{\varepsilon}{3}. \quad (14.1.1)$$

Here $\beta_1, \ldots, \beta_r$ are linearly independent over $\mathbb{Q}$ and belong to the union of spectra of functions $f_1, \ldots, f_l$, $\nu_1, \ldots, \nu_r \in \mathbb{Z}$, $n_1, \ldots, n_r \in \mathbb{N}$, and

$$M_t\{f_k(z + t)K(t)\} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_k(z + t)K(t) \, dt$$

are the corresponding holomorphic Bochner-Fejer polynomials belonging to $\text{APH}_\Gamma(T)$ as well (clearly, the spectrum of the function $z \mapsto M_t\{f_k(z + t)K(t)\}$ is contained in $\text{spec}(f_k)$).

We define a linear operator $T : \text{APH}_\Gamma(T) \to \text{APH}_\Gamma(T)$ from the definition of the approximation property by the formula

$$(Tf)(z) := M_t\{f(z + t)K(t)\}, \quad f \in \text{APH}_\Gamma(T). \quad (14.1.2)$$
Then \(T\) is a bounded linear projection onto a finite-dimensional subspace of \(A\) generated by functions 
\[e^{i\left(\frac{\nu_1}{n_1}z_1 + \cdots + \frac{\nu_r}{n_r}z_r\right)}, \ |\nu_1| \leq n_1, \ldots, |\nu_r| \leq n_r.\]
Moreover, since \(K(t) \geq 0\) for all \(t \in \mathbb{R}\) and \(M_t \{K(t)\} = 1\), the norm of \(T\) is 1. Finally, given \(f \in K\) choose \(k\) such that 
\[\|f - f_k\|_{A\} \leq \frac{\varepsilon}{3}.\]

Then we have by (14.1.1)
\[\|Tf - f\|_{A\} \leq \|T(f - f_k)\|_{A\} + \|Tf_k - f\|_{A\} + \|f_k - f\|_{A\} < \varepsilon.\]

This completes the proof of the theorem. \(\square\)

### 14.2 Proof of Theorem 11.0.9

Assume that the corona theorem is not true, that is, \(T\) is not dense in \(b\). Then there exist \(\xi \in b\) and its neighbourhood \(U(\lambda_1, \ldots, \lambda_l, \xi, \epsilon)\) such that 
\[U(\lambda_1, \ldots, \lambda_l, \xi, \epsilon) \cap cl(T) = \emptyset;\]
here \(cl(T)\) denotes the closure of \(T\) in \(b\). Set \(c_k := e_{\lambda_k}(\xi), 1 \leq k \leq l.\) Then
\[\max_{1 \leq k \leq l} |e^{i\lambda_k z} - c_k| \geq \epsilon > 0 \quad \text{for all } z \in T.\] (14.2.3)

Clearly, every function \(e^{i\lambda_k z} - c_k, 1 \leq k \leq l,\) has at least one zero in \(T\). (For otherwise, if \(e^{i\lambda_k z} - c_k\) has no zeros on \(T\), then the function \(g_k(z) := \frac{1}{e^{i\lambda_k z} - c_k}, z \in T,\) belongs to \(A\) and \(g_k(z)(e^{i\lambda_k z} - c_k) = 1\) for all \(z \in T,\) which contradicts our assumption.) In particular, since the solutions of the equation 
\[e^{i\lambda_k z} = c_k, \quad \lambda_k \neq 0,\]
are given by 
\[z = -\frac{i \ln |c_k|}{\lambda_k} + \frac{\Arg c_k + 2\pi s}{\lambda_k}, \quad s \in \mathbb{Z},\]
they all belong to \(T\). Further, without loss of generality we may assume that all \(\lambda_k > 0.\) Indeed, if some \(\lambda_k < 0,\) we can replace the function \(e^{i\lambda_k z} - c_k\) by \(e^{-i\lambda_k z} - \frac{1}{c_k}\) (observe that \(c_k \neq 0\) by the above argument) so that the new family of functions also satisfies (14.2.3) (possibly with a
different $\epsilon$) and extensions of these functions to $APH(T)$ vanish at $\xi$. Since all these functions have zeros in $T$ and satisfy (14.2.3) there, we have
\[
\max_{1 \leq k \leq l} |e^{i\lambda_k z} - c_k| \geq \tilde{\epsilon} > 0, \quad \text{for all } z \in \mathbb{H}_+ \]
where $\mathbb{H}_+ \subset \mathbb{C}$ is the open upper half-plane, and all $e^{i\lambda_k z} - c_k$, $\lambda_k \in \mathbb{R}_+$, are almost periodic on $\tilde{\mathbb{H}}_+$. From the last inequality by the Böcher corona theorem [Bö] we obtain that there exist holomorphic almost periodic functions $g_1, \ldots, g_l$ on $\tilde{\mathbb{H}}_+$ such that
\[
\sum_{k=1}^l g_k(z)(e^{i\lambda_k z} - c_k) = 1 \quad \text{for all } z \in \tilde{\mathbb{H}}_+.
\]
Thus, taking the restrictions of these functions to $T$, we obtain a contradiction with our assumption. The proof is complete.

14.3 Proof of Proposition 12.2.1 and Theorem 12.2.5

We will need the following definition and results.

We define almost periodic functions on open arcs $\gamma_{z_0}^k(\delta) \subset \partial \mathbb{D}$, $z_0 = e^{it_0} \in \partial \mathbb{D}$, $k \in \{-1, 1\}$ (cf. Section 12.1 for notation).

**Definition 14.3.1.** A continuous function $f_k : \gamma_{z_0}^k(\delta) \to \mathbb{C}$, $k \in \{-1, 1\}$, is said to be almost periodic if the functions
\[
\tilde{f}_k : (-\infty, 0) \to \mathbb{C}, \quad \tilde{f}_k(x) := f_k(z_0 e^{ikx})
\]
are the restriction of almost periodic functions in $AP(\mathbb{R})$ (cf. Section 11 for notation).

In particular, an almost periodic function on $\gamma_1(z_0) \cup \gamma_{-1}(z_0)$ belongs to $SAP(\partial \mathbb{D})$.

Further, it is easy to see that the function $e^{i\lambda \log_{z_0}}$, $\lambda \in \mathbb{R}$, where
\[
\log_{z_0}(z_0 e^{ikt}) := \ln t, \quad 0 < t < 2\pi, \quad k \in \{-1, 1\},
\]
is in $SAP(\partial \mathbb{D})$. We fix a real continuous function $g$ on $\partial \mathbb{D}$ such that
\[
\lim_{t \to 0^+} g(e^{it}) = 1, \quad \lim_{t \to 2\pi^-} g(e^{it}) = 0
\]
and $g(e^{it})$ is decreasing for $0 < t < 2\pi$. We set $g_{z_0}(e^{it}) := g(z_0 e^{it})$. Clearly, $g_{z_0} \in SAP(\partial \mathbb{D})$. 190
Proposition 14.3.2. The algebra $SAP(S)$ is the uniform closure in $L^\infty(\partial\mathbb{D})$ of the algebra of complex polynomials in variables $g_{z_l}$ and $e^{i\lambda\log_{z_l}}$, $\lambda \in \mathbb{R}$, $z_0 \in S$, $k \in \{-1, 1\}$.

Proof of Proposition. Let $f \in SAP(S)$. Since $\partial\mathbb{D}$ is a compact set, for a given $\epsilon > 0$ there are finitely many points $z_l \in \partial\mathbb{D}$, numbers $\delta_l \in (0, \pi)$ and almost periodic functions $f^l_k : \gamma_{z_l}^k(\delta_l) \to \mathbb{C}$ (as in Definition 12.1.2), $k \in \{-1, 1\}$, $1 \leq l \leq n$, such that

$$\bigcup_{l=1}^{n}(\gamma_{z_l}^{-1}(\delta_l) \cup \gamma_{z_l}^1(\delta_l)) = \partial\mathbb{D} \setminus \{z_1, \ldots, z_n\} \quad \text{and for all} \quad 1 \leq l \leq n$$

(14.3.4)

$$\operatorname{ess sup}_{z \in \gamma_{z_l}^{-1}(\delta_l)} |f(z) - f^l_1(z)| < \frac{\epsilon}{2}, \quad \operatorname{ess sup}_{z \in \gamma_{z_l}^1(\delta_l)} |f(z) - f^l_{-1}(z)| < \frac{\epsilon}{2}.$$  

We set $U_l := \gamma_{z_l}^{-1}(\delta_l) \cup \gamma_{z_l}^1(\delta_l) \cup \{z_l\}$. Then $U = (U_l)_{l=1}^{n}$ is a finite open cover of $\partial\mathbb{D}$. Let $\{\rho_l\}_{l=1}^{n}$ be a continuous partition of unity subordinate to $U$ such that $\operatorname{supp} \rho_l \subset \subset U_l$ and $\rho_l(z_l) = 1$, $1 \leq l \leq n$. Consider the functions $f_l$ on $\partial\mathbb{D} \setminus \{z_l\}$ defined by the formulas

$$f_l(z) := \begin{cases} 
\rho_l(z)f^l_{-1}(z), & \text{if } z \in \gamma_{z_l}^{-1}(\delta_l), \\
\rho_l(z)f^l_1(z), & \text{if } z \in \gamma_{z_l}^1(\delta_l). 
\end{cases}$$

Since $f_l$ is continuous outside $z_l$ and coincides with $f^l_{-1}$ and $f^l_1$ in a neighbourhood of $z_l$, it belongs to $SAP(S)$. Moreover,

$$\|f - \sum_{l=1}^{n} f_l\|_{L^\infty(\partial\mathbb{D})} < \frac{\epsilon}{2}. \quad (14.3.5)$$

Thus in order to prove the theorem it suffices to approximate every $f_l$ by polynomials in $g_{z_l}$ and $e^{i\lambda\log_{z_l}}$, $k \in \{-1, 1\}$, $\lambda \in \mathbb{R}$.

Suppose first that $z_l \notin S$. Since $f$ is continuous outside the compact set $S$, we can choose the above cover $U$ and the family of functions $\{f^m_k\}_{1 \leq m \leq n}$, $k \in \{-1, 1\}$, such that in the $U_l$ the functions $f^l_k$, $k \in \{-1, 1\}$, have the same limit at $z_l$. This implies the continuity of $f_l$ on $\partial\mathbb{D}$.

Next, consider the uniform algebra $\mathbb{C}(g_{z_l})$ over $\mathbb{C}$ generated by the function $g_{z_l}$. Since by our definition $g_{z_l}$ separates points on $\partial\mathbb{D} \setminus \{z_l\}$, the maximal ideal space of $\mathbb{C}(g_{z_l})$ is homeomorphic to the closed interval $\partial\mathbb{D} \setminus \{z_l\} \cup \{z_m\}$, $k \in \{-1, 1\}$, with endpoints $z_l$ and $z_{l-1}$ identified with the counterclockwise and clockwise orientations at $z_l$. Clearly every continuous function on $\partial\mathbb{D}$
is extended to the maximal ideal space of $C(g_{z_l})$ as a continuous function having the same values at $z_l$ and $z_{l-1}$. Thus by the Stone-Weierstrass theorem $f_l$ can be uniformly approximated on $\partial \mathbb{D} \setminus \{z_l\}$ by complex polynomials in $g_{z_l}$.

Now, suppose that $z_l \in S$. Let us choose some $\delta \in (0, \delta_l)$.

**Definition 14.3.3.** By $\text{SAP}_{\{z_l\}}(\delta)$ we denote the uniform algebra of complex continuous functions $f$ on $\partial \mathbb{D} \setminus \{z_l\}$ almost periodic on the open arcs $\gamma_{z_l}^k(\delta)$, $k \in \{-1, 1\}$ (cf. Definition 14.3.1).

(Since $\delta \in (0, \pi)$, the closures of these arcs are disjoint.) By $M_{\{z_l\}}(\delta)$ we denote the maximal ideal space of algebra $\text{SAP}_{\{z_l\}}(\delta)$. Then $\partial \mathbb{D} \setminus \{z_l\}$ is dense in $M_{\{z_l\}}(\delta)$ (in the Gelfand topology).

The space $M_{\{z_l\}}(\delta)$ is constructed as follows. Consider the Bohr compactification $b\mathbb{R}$ of $\mathbb{R}$. We identify the negative ray $\mathbb{R}_-$ in $\mathbb{R} \subset b\mathbb{R}$ with $\gamma_{z_l}^1(\delta) \subset \partial \mathbb{D}$ by means of the map $t \mapsto z_l e^{-i\delta t}$, $t \in \mathbb{R}_-$. Similarly, consider another copy of $b\mathbb{R}$ and identify $\mathbb{R}_-(\subset \mathbb{R})$ in this copy with $\gamma_{z_l}^{-1}(s) \subset \partial \mathbb{D}$ by means of the map $t \mapsto z_l e^{i\delta t}$, $t \in \mathbb{R}_-$. On the identified sets we introduce the topology induced from $b\mathbb{R}$, and on $\partial \mathbb{D} \setminus (\gamma_{z_l}^{-1}(\delta) \cup \gamma_{z_l}^1(\delta))$ introduce the topology is induced from $\partial \mathbb{D}$. Then the quotient space of $b\mathbb{R} \sqcup b\mathbb{R} \sqcup \partial \mathbb{D}$ under these identifications coincides with $M_{\{z_l\}}(\delta)$.

Next, recall that by definition the algebra $\text{SAP}(\{z_l\})$ is the uniform closure in $C(\partial \mathbb{D} \setminus \{z_l\})$ of the algebra generated by the algebras $\text{SAP}_{\{z_l\}}(\delta)$, $\delta \in (0, \delta_l)$. Recall that by $b^{\{z_l\}}(\partial \mathbb{D})$ we denote the maximal ideal space of $\text{SAP}(\{z_l\})$. Since for any $\delta'' < \delta'$ we have inclusions $i_{\delta''} : \text{SAP}(\{z_l\})(\delta'') \hookrightarrow \text{SAP}(\{z_l\})(\delta')$, the space $b^{\{z_l\}}(\partial \mathbb{D})$ is the inverse limit of the spaces $M_{\{z_l\}}(\delta)$ (here the maps $p_{\delta''} : M_{\{z_l\}}(\delta'') \rightarrow M_{\{z_l\}}(\delta')$ in the definition of this limit are defined as the dual maps to $i_{\delta''}$). Also, $\partial \mathbb{D} \setminus \{z_l\}$ is dense in $b^{\{z_l\}}(\partial \mathbb{D})$ in the Gelfand topology. Since by the definition the functions $f_l$, $e^{i\lambda \log z_l^k}$ and $g_{z_l}$ admit the continuous extensions (denoted by the same symbols) to $b^{\{z_l\}}(\partial \mathbb{D})$, it suffices to show that the extended functions $e^{i\lambda \log z_l^k}$, $g_{z_l}$ separate points on $b^{\{z_l\}}(\partial \mathbb{D})$. Then we will apply the Stone-Weierstrass theorem to get a complex polynomial $p_l$ in variables $e^{i\lambda \log z_l^k}$ and $g_{z_l}$ which uniformly approximates $f_l$ on $b^{\{z_l\}}(\partial \mathbb{D})$ with an error $< \varepsilon/2n$. Therefore, $\sum_{l=1}^{n} p_l$ gives an $\varepsilon$-approximation to $f$ in $L^\infty(\partial \mathbb{D})$.

So let us show that the functions $e^{i\lambda \log z_l^k}$, $g_{z_l}$ separate points on $b^{\{z_l\}}(\partial \mathbb{D})$. By $p_{\delta} : b^{\{z_l\}}(\partial \mathbb{D}) \rightarrow M_{\{z_l\}}(\delta)$ we denote the continuous surjection determined by the inverse limit
construction. First, we consider distinct points \(x, y \in b^{(z_1)}(\partial \mathbb{D})\) for which there is \(\delta \in (0, \delta_t)\) such that \(p_\delta(x)\) and \(p_\delta(y)\) are distinct and belong to one of the Bohr compactifications of \(\mathbb{R}\) in \(M_{(z_1)}(s)\), say, e.g., to the compactification obtained by gluing \(\mathbb{R}_-\) with \(\gamma_{z_1}^-(\delta)\). Since in this case the functions \(e^{i\lambda \log z_1}\) extended to \(b\mathbb{R}\) are identified with the extensions to \(b\mathbb{R}\) of functions \(c_{\lambda\delta} e^{i\lambda t}\) on \(\mathbb{R}\), \(c_{\lambda\delta} := e^{i\lambda \ln \delta}\), by the classical Bohr theorem there is \(\lambda_0 \in \mathbb{R}\) such that the extension of \(e^{i\lambda_0 \log z_1}\) to \(b\mathbb{R}\) separates \(p_\delta(x)\) and \(p_\delta(y)\). So, the extension of \(e^{i\lambda_0 \log z_1}\) to \(b^{(z_1)}(\partial \mathbb{D})\) separates \(x\) and \(y\).

Suppose now that \(x\) and \(y\) are such that \(p_\delta(x)\) and \(p_\delta(y)\) belong to different Bohr compactifications of \(\mathbb{R}\) for all \(\delta\). This implies that \(x\) and \(y\) are limit points of the sets \(\gamma_{z_1}^k(\overline{(x)})\) and \(\gamma_{z_1}^k(\overline{(y)})\) for some \(\delta \in (0, \delta_t)\) with \(k(x) \neq k(y)\) and \(k(x), k(y) \in \{-1, 1\}\). Then the function \(g_{z_1}\) by definition equals 1 at one of the points \(x, y\) and 0 at the other one.

Finally, assume that \(x \in b^{(z_1)}(\partial \mathbb{D}) \setminus \partial \mathbb{D}\) and \(y \in \partial \mathbb{D} \setminus \{z_1\}\). Then \(g_{z_1}(x)\) equals either 0 or 1 and \(g_{z_1}(y)\) differs from these numbers because \(g_{z_1}\) is decreasing on \(\partial \mathbb{D} \setminus \{z_1\}\).

Thus we have proved that the family of functions \(e^{i\lambda \log z_1}: g_{z_1}\) separate the points of \(b^{(z_1)}(\partial \mathbb{D})\). This completes the proof of the proposition. \(\square\)

Let \(r^S : b^S(\partial \mathbb{D}) \to \partial \mathbb{D}\) be the map transpose to the embedding \(C(\partial \mathbb{D}) \hookrightarrow SAP(S)\).

**Proposition 14.3.4.** The following is true:

1. Let \(z_0 \in S\). Then the preimage \((r^S)^{-1}(z_0)\) is homeomorphic to \(b\mathbb{R} \sqcup b\mathbb{R}\).

2. The restriction

\[
r^S : b^S(\partial \mathbb{D}) \setminus (r^S)^{-1}(S) \to \partial \mathbb{D} \setminus S
\]

is a homeomorphism.

Since \((r^S)^{-1}(z_0) \cap (r^S)^{-1}(z_0') = \emptyset\) if \(z \neq z_0\), we obtain from (1) that there is an embedding

\[
h^{z_0} : b\mathbb{R} \sqcup b\mathbb{R} \hookrightarrow b^S(\partial \mathbb{D}). \tag{14.3.6}
\]

**Proof of Proposition.** (1) According to Proposition 14.3.2 the Bohr compactification \(b^S(\partial \mathbb{D})\) is homeomorphic to the inverse limit of compact spaces \(b^F(\partial \mathbb{D})\) with \(F \subset S\) finite. Let \(\tilde{p}_{F_1,F_2} : b^{F_2}(\partial \mathbb{D}) \to b^{F_1}(\partial \mathbb{D}), F_1 \subset F_2, \) be continuous maps determining this limit, and \(\tilde{p}_F : b^S(\partial \mathbb{D}) \to \)
be the corresponding limit maps. Since each $SAP(F)$ is a self-adjoint algebra, by the
Stone-Weierstrass theorem $\partial \mathbb{D} \setminus F$ is dense in $b^F(\partial \mathbb{D})$. Hence, $\tilde{p}_{F_1F_2}$ and $\tilde{p}_F$ are surjective maps.

We begin with the description of $b^F(\partial \mathbb{D})$. Suppose that $F := \{z_1, \ldots, z_n\}$ and $F_i := F \setminus \{z_i\}$, $1 \leq i \leq n$. Consider the disjoint union

$$X = \bigcup_{1 \leq i \leq n} (b^{\{z_i\}}(\partial \mathbb{D}) \setminus F_i).$$

Note that each component of $X$ contains $\partial \mathbb{D} \setminus F$ as an open subset. By $h_i : \partial \mathbb{D} \setminus F \hookrightarrow b^{\{z_i\}}(\partial \mathbb{D}) \setminus F_i$ we denote the corresponding embeddings. Then for each $z \in \partial \mathbb{D} \setminus F$ we sew together points $h_i(z)$, $1 \leq i \leq n$, and identify the obtained point with $z$. As a result we obtain the quotient space $\tilde{X}$ of $X$ and the “sewing” map $\pi : X \to \tilde{X}$. We equip $\tilde{X}$ with the quotient topology: $U \subset \tilde{X}$ is open iff $\pi^{-1}(U) \subset X$ is open.

We will need the following

**Lemma 14.3.5.** $\tilde{X}$ is homeomorphic to $b^F(\partial \mathbb{D})$.

**Proof of Lemma.** By definition each $V_i := \pi(b^{\{z_i\}}(\partial \mathbb{D}) \setminus F_i)$ is an open subset of $\tilde{X}$ homeomorphic to $b^{\{z_i\}}(\partial \mathbb{D}) \setminus F_i$. Since the latter spaces are Hausdorff, $\tilde{X}$ is Hausdorff as well. Let us cover $\partial \mathbb{D}$ by closed arcs $\gamma_1, \ldots, \gamma_n$ such that $\gamma_i \cap F = \{z_i\}$, $1 \leq i \leq n$. By $\tilde{\gamma}_i$ we denote the closure of $\gamma_i$ in $b^{\{z_i\}}(\partial \mathbb{D})$. Then $\tilde{\gamma}_i$ is a compact subset of $b^{\{z_i\}}(\partial \mathbb{D}) \setminus F_i$ and $U_i := \pi(\tilde{\gamma}_i)$ is a compact subset of $V_i$. It is easy to see that $\tilde{X} = \bigcup_{1 \leq i \leq n} U_i$. Thus $\tilde{X}$ is a compact space. Further, according to (14.3.5) each function from $SAP(F)$ is extended continuously to $\tilde{X}$ and the algebra of the extended functions separates points on $\tilde{X}$. Hence by the Stone-Weierstrass theorem, $\tilde{X}$ is homeomorphic to $b^F(\partial \mathbb{D})$. \hfill \square

As a corollary of the above construction we immediately obtain the following:

Let $F_1 \subset F_2$ be finite subsets of $S$. Consider the commutative diagram

$$
\begin{array}{ccc}
b^F(\partial \mathbb{D}) & \overset{r_F}{\longrightarrow} & \partial \mathbb{D} \\
\tilde{p}_{F_2} \downarrow & & \downarrow \\
b^{F_2}(\partial \mathbb{D}) & \overset{r_{F_2}}{\longrightarrow} & \partial \mathbb{D} \\
\tilde{p}_{F_1F_2} \downarrow & & \downarrow \\
b^{F_1}(\partial \mathbb{D}) & \overset{r_{F_1}}{\longrightarrow} & \partial \mathbb{D}.
\end{array}
$$

(14.3.7)
Here $\tilde{p}_{F_1} := \tilde{p}_{F_1F_2} \circ \tilde{p}_{F_2}$. We set $\tilde{F}_1 := r^{-1}_{F_1}(F_1)$ and $\tilde{S}_i := r^{-1}_{F_i}(S)$, $i = 1, 2$. Then

\[(A)\]

$\tilde{p}_{F_1F_2} : \tilde{p}_{F_1F_2}^{-1}(\tilde{F}_1) \to \tilde{F}_1$

is a homeomorphism;

\[(B)\]

$b^{F_2}(\partial \mathbb{D}) \setminus \tilde{S}_2 \xrightarrow{\tilde{p}_{F_1F_2}} b^{F_1}(\partial \mathbb{D}) \setminus \tilde{S}_1 \xrightarrow{r_{F_1}} \partial \mathbb{D} \setminus S$

are the identity maps.

From here by the definition of the inverse limit we obtain

\[(A1)\]

$\tilde{p}_{F_1} : \tilde{p}_{F_1}^{-1}(\tilde{F}_1) \to \tilde{F}_1$

is a homeomorphism;

\[(B1)\]

$b^{S}(\partial \mathbb{D}) \setminus r_{S}^{-1}(S) \xrightarrow{\tilde{p}_{F_1}} b^{F_1}(\partial \mathbb{D}) \setminus \tilde{S}_1 \xrightarrow{r_{F_1}} \partial \mathbb{D} \setminus S$

are the identity maps.

Assume now that $F_1 = \{z\} \subset S$. To prove (1) we have to show (according to (14.3.7) and (A), (A1) that each set $r_{\{z\}}^{-1}(z)$ is homeomorphic to $b\mathbb{R} \sqcup b\mathbb{R}$.

In the proof of Proposition 14.3.2 we have established that $b^{\{z\}}(\partial \mathbb{D})$ is the inverse limit of the maximal ideal spaces $M_{\{z\}}(\delta)$ of algebras $SAP_{\{z\}}(\delta)$ of continuous functions on $\partial \mathbb{D} \setminus \{z\}$ almost periodic on the open arcs $\gamma_{z}^{\delta}(\delta)$ where $\delta \in (0, \pi)$. Also, in that proof we have described the structure of spaces $M_{\{z\}}(\delta)$.

To every pair $0 < \delta'' < \delta' < \pi$ we associate continuous surjective map

$p_{\delta''\delta'} : M_{\{z\}}(\delta'') \to M_{\{z\}}(\delta')$

dual to the embedding $i_{\delta''\delta'} : SAP_{\{z\}}(\delta') \hookrightarrow SAP_{\{z\}}(\delta'')$. From the proof of Proposition 14.3.2 we know that every $M_{\{z\}}(\delta)$ is obtained by gluing $\partial \mathbb{D} \setminus \{z\}$ with two copies of $b\mathbb{R}$ where one copy (denoted $b\mathbb{R}_1$) is obtained by gluing with $\gamma_{z}^{-1}(\delta)$ and another one (denoted by $b\mathbb{R}_{-1}$) is obtained
by gluing with $\gamma^{-1}(\delta)$. Suppose that $\xi \in b\mathbb{R}_1 \subset M_{\{z\}}(\delta'')$. Let us compute $p_{\delta''\delta'}(\xi) \in b\mathbb{R}_1 \subset M_{\{z\}}(\delta')$. Let $\{z_\alpha\} \subset \gamma_{\{z\}}(\delta'')$ be a net converging to $\xi$. This means that the net $\{\phi_{\delta''}(z_\alpha)\} \subset \mathbb{R}_-$ converges to $\xi$ in the topology of the Bohr compactification on $b\mathbb{R}$; here $\phi_{\delta''}$ is the map inverse to the map $\psi_{\delta''} : \mathbb{R}_- \to \gamma_{\{z\}}(\delta'')$, $x \mapsto xe^{i\delta''e^x}$. Next, by definition the net $\{\phi_{\delta'}(z_\alpha)\}$ converges to $p_{\delta''\delta'}(\xi)$. A straightforward computation shows that $\phi_{\delta'}(z_\alpha) = \phi_{\delta''}(z_\alpha) + \ln(\frac{\delta'}{\delta''})$ for all $z_\alpha$.

Thus we have

$$p_{\delta''\delta'}(\xi) = \xi + \ln\left(\frac{\delta'}{\delta''}\right), \quad \xi \in b\mathbb{R}_1.$$  \hspace{1cm} (14.3.8)

Here the sum denotes the group operation in $b\mathbb{R}_1$. Similarly,

$$p_{\delta''\delta'}(\xi) = \xi + \ln\left(\frac{\delta'}{\delta''}\right), \quad \xi \in b\mathbb{R}_{-1}.$$  \hspace{1cm} (14.3.9)

Using these formulas we now prove that each $r_{\{z\}}^{-1}(z)$ is homeomorphic to $b\mathbb{R} \sqcup b\mathbb{R}$.

For a fixed $\delta_0 \in (0, \pi)$ let us consider the limit map $p_{\delta_0} : b\{z\}(\partial\mathbb{D}) \to M_{\{z\}}(\delta_0)$. Then $p_{\delta_0}$ maps $r_{\{z\}}^{-1}(z)$ into $X_{\delta_0} := b\mathbb{R}_1 \sqcup b\mathbb{R}_2 \subset M_{\{z\}}(\delta_0)$. Moreover, by definition $r_{\{z\}}^{-1}(z)$ is the inverse limit of the system $\{(X_{\delta''}, X_{\delta'}, p_{\delta''\delta'})\}$ where we write $X_{\delta}$ for $b\mathbb{R}_1 \sqcup b\mathbb{R}_2 \subset M_{\{z\}}(\delta)$. Since according to (14.3.8) every $p_{\delta''\delta'} : X_{\delta''} \to X_{\delta'}$ is a homeomorphism (even an automorphism of $b\mathbb{R}_1 \sqcup b\mathbb{R}_2$), by the definition of the inverse limit $p_{\delta_0} : r_{\{z\}}^{-1}(z) \to X_{\delta_0}$ is a homeomorphism.

This completes the proof of (1).

(2) Follows from (B1).

### 14.3.1 Proof of Proposition 12.2.1

The homomorphism $E_{z_0} : SAP(\partial\mathbb{D}) \to AP(\mathbb{R} \sqcup (\mathbb{R} + i\pi))$ is dual to the embedding $h^{z_0} : b\mathbb{R} \sqcup b\mathbb{R} \hookrightarrow b^S(\partial\mathbb{D})$, cf. (14.3.6).

### 14.3.2 Proof of Theorem 12.2.5

Immediate from Proposition 14.3.4 and the definition of Bohr-Fourier coefficients on functions in $SAP(\partial\mathbb{D})$ (cf. Section 12.2).
14.4 Proof of Proposition 12.3.2

Let \( g \in APH(T) \), put \( g_1(x) := g(x) \), \( g_2(x) := g(x + i\pi) \), \( x \in \mathbb{R} \). It follows easily from the approximation result for algebra \( APH(T) \) cited in Chapter 11 that \( \text{spec}(g_1) = \text{spec}(g_2) \) and for each \( \lambda \in \text{spec}(g_1) \)

\[
a_\lambda(g_1) = e^{\lambda \pi} a_\lambda(g_2).
\]

Suppose that \( f \in SAP(S) \cap H^\infty(\mathbb{D}) \). Then Theorem 13.2.2(1) implies that for each \( z_0 \in S \) and \( k \in \{-1, 1\} \) the functions \( f_k, z_0 \) are the boundary values of the function \((i^z_\Sigma \circ i_{\Sigma(z_0)})^*(f) \in APH(T)\). This gives us the required result.

14.5 Proof of Proposition 13.1.2

We have to show that if \( f \in (SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}))_B \) on the maximal ideal space \( b^S(\mathbb{D}) \) of algebra \( SAP(S) \cap H^\infty(\mathbb{D}) \), then \( f|_{\mathbb{D}} \in SAP^B_S(S) \cap H^\infty_B(\mathbb{D}) \).

Indeed, since \( f \in C^B(b^S(\mathbb{D})) \), and space \( C(b^S(\mathbb{D})) \) has the approximation property, \( f \in C(b^S(\mathbb{D})) \otimes B \) by Theorem 10.0.2. Next, \( C(b^S(\mathbb{D})) \) is generated by algebra \( SAP(S) \cap H^\infty(\mathbb{D}) \) and its conjugate. Therefore \( f \) can be uniformly approximated on \( b^S(\mathbb{D}) \) by a sequence of \( B \)-valued polynomials in variables from algebras \( SAP(S) \cap H^\infty(\mathbb{D}) \) and its conjugate. This easily implies that \( f|_{\partial \mathbb{D}} \) is well defined and belongs to \( SAP^B_S(S) \). In fact, \( f|_{\partial \mathbb{D}} \in SAP^B_S(S) \) because \( \phi(f) \in SAP_{\Sigma}(S) \) and the Bohr-Fourier coefficients of \( f \) satisfy \( a^k_\lambda(\varphi(f), z_0) = \varphi(a^k_\lambda(f, z_0)) \) for any \( z_0 \in S \), \( k \in \{-1, 1\} \) and \( \phi \in B^* \). Further, by the definition \( f|_{\mathbb{D}} \) is such that \( \varphi(f) \in H^\infty(\mathbb{D}) \) for any \( \varphi \in B^* \). This shows that \( f \in H^\infty_B(\mathbb{D}) \).

14.6 Auxiliary results

In what follows, we will need the following results.

14.6.1 Action of a unit circle diffeomorphism

Let \( \phi : \partial \mathbb{D} \to \partial \mathbb{D} \) be a \( C^1 \)-diffeomorphism. By \( \phi^* : C(\partial \mathbb{D}) \to C(\partial \mathbb{D}) \), \( \phi^*(f) := f \circ \phi \), we denote the pullback by \( \phi \). Set \( \bar{S} := \phi(S) \). As a consequence of Proposition 14.3.2 we obtain
Proposition 14.6.1. $\phi^*$ maps $SAP(\tilde{S})$ isomorphically onto $SAP(S)$.

14.6.2 Scalar approximaton theorem

Theorem 14.6.2. $SAP(S) \cap H^\infty(\mathbb{D}) = A^S$.

For the proof, we will need the following results and notation.

Let $z_0 \in \partial \mathbb{D}$. For $\delta \in (0, \pi)$ we set

$$\gamma_1(z_0, \delta) := \text{Log}(\phi_{z_0}(\gamma_{-1}(\delta))) \subset \mathbb{R}$$

and

$$\gamma_{-1}(z_0, \delta) := \text{Log}(\phi_{z_0}(\gamma_{-1}(\delta))) \subset \mathbb{R} + i\pi.$$  

(cf. Example 12.1.3 for notation).

Lemma 14.6.3. Let $f \in SAP(\{-z_0, z_0\})$. We set $f_k := f|_{\gamma_{z_0}(\delta)}$ and consider functions

$$h_k = f_k \circ \varphi_{z_0}^{-1} \circ \text{Log}^{-1}, \quad k \in \{-1, 1\}$$

on $\gamma_k(z_0, \delta)$. Then for any $\epsilon > 0$ there are $\delta_\epsilon \in (0, \pi)$ and almost periodic functions $h'_1$ on $\mathbb{R}$, and $h'_{-1}$ on $\mathbb{R} + i\pi$ such that

$$\sup_{z \in \gamma_k(z_0, \delta_\epsilon)} |h_k(z) - h'_k(z)| < \epsilon \quad \text{for each } k \in \{-1, 1\}.$$ 

Proof of Lemma. We prove the result for $f_1$. According to Proposition 14.3.2 it suffices to prove the lemma for $f_1 = g_{z_0}$ or $f_1 = e^{i\lambda \log z_0}$, $\lambda \in \mathbb{R}$. In the first case we can choose a sufficiently small $\delta_\epsilon$ such that on $\gamma_{z_0}^1(\delta_\epsilon)$ the function $g_{z_0}$ is uniformly approximated with an error $< \epsilon$ by a constant function. Then as $h'_1$ we can choose the corresponding constant function on $\gamma_1(z_0, \delta_\epsilon)$.

In the second case, by definition,

$$h_1(x) = e^{i\lambda \ln \left( \text{Arg} \left( \frac{2i - e^x}{2i + e^x} \right) \right)} = e^{i\lambda \ln \left( \frac{4(e^x)^2 + o(e^{3x})}{4(e^x)^2 + o(e^{3x})} \right)} = e^{i\lambda (x + o(e^x))} \quad \text{as } x \to -\infty.$$ 

From here for a sufficiently small $\delta_\epsilon$ we have $|h_1(x) - e^{i\lambda x}| < \epsilon$ for all $x \in \gamma_1(z_0, \delta_\epsilon)$. 

We will also need the following well known result.
Lemma 14.6.4. Suppose that \( f_1 \) and \( f_2 \) are continuous almost periodic functions on \( \mathbb{R} \) and \( \mathbb{R} + i\pi \), respectively. Then there exists a function \( F \in APC(T) \) harmonic in \( T_0 \) whose boundary values are \( f_1 \) and \( f_2 \).

Proof. Let \( F \) be a function harmonic in \( T_0 \) with boundary values \( f_1 \) and \( f_2 \). Since \( f_1 \) and \( f_2 \) are almost periodic, for any \( \varepsilon > 0 \) there exists \( l(\varepsilon) > 0 \) such that every interval \([t_0, t_0 + l(\varepsilon)]\) contains a common \( \varepsilon \)-period of \( f_1 \) and \( f_2 \), say, \( \tau_\varepsilon \), see, e.g., [LZ]. Thus

\[
\sup_{x \in \mathbb{R}} |f_1(x + \tau_\varepsilon) - f_1(x)| < \varepsilon \quad \text{and} \quad \sup_{x \in \mathbb{R}} |f_2(x + i\pi + \tau_\varepsilon) - f_2(x + i\pi)| < \varepsilon.
\]

Now, by the maximum principle for harmonic functions

\[
\sup_{x \in \mathbb{R}} |F(x + iy + \tau_\varepsilon) - F(x + iy)| < \varepsilon \quad \text{for each} \quad y \in [0, \pi],
\]

that is, \( F \) is almost periodic on every line \( \mathbb{R} + iy \), \( y \in [0, \pi] \). \( \square \)

Definition 14.6.5. Let \( z_0 \in \partial \mathbb{D} \) and \( U_{z_0} \) be the intersection of an open disk of some radius \( \leq 1 \) centered at \( z_0 \) with \( \mathbb{D} \setminus \{z_0\} \). We call such \( U_{z_0} \) a circular neighbourhood of \( z_0 \).

Definition 14.6.6. We say that a bounded continuous function \( f : \mathbb{D} \to \mathbb{C} \) is almost-periodic near \( z_0 \) if there exist a circular neighbourhood \( U_{z_0} \), and a function \( \tilde{f} \in APC(T) \) such that

\[
f(z) = (L^B_{z_0}\tilde{f})(z) := (\tilde{f} \circ \text{Log} \circ \varphi_{z_0})(z), \quad z \in U_{z_0}.
\]

If \( \tilde{f} \in APH(T) \), then function \( f \) is called holomorphic almost periodic near \( z_0 \).

Let \( \mathcal{A}(U_{z_0}) \) be the algebra of continuous functions on \( \mathbb{D} \) almost periodic on the circular neighbourhood \( U_{z_0} \) of \( z_0 \). By \( \mathcal{A}_{z_0} \) we denote the uniform closure of the algebra generated by all \( \mathcal{A}(U_{z_0}) \) and by \( \mathcal{M}_{z_0} \) the closure of \( \mathbb{D} \) in the maximal ideal space of \( \mathcal{A}_{z_0} \). Since the algebra \( \mathcal{A}_{z_0} \) is self-adjoint, by the Stone-Weierstrass theorem \( \mathcal{M}_{z_0} \) coincides with the maximal ideal space of \( \mathcal{A}_{z_0} \). Next, let \( p_{z_0} : \mathcal{M}_{z_0} \to \mathbb{D} \) be the continuous surjective map dual to the natural embedding \( C(\mathbb{D}) \hookrightarrow \mathcal{A}_{z_0} \).

Lemma 14.6.7. The following is true:

(a) For every neighbourhood \( U \) of the compact set \( F_{z_0} := p_{z_0}^{-1}(z_0) \) there is a circular neighbourhood \( U_{z_0} \) of \( z_0 \) such that \( U_{z_0} \cap \mathbb{D} \subset U \cap \mathbb{D} \).
(b) $F_{z_0}$ is homeomorphic to $bT$.

(c) Each function $f \in SAP(S) \cap H^\infty$ belongs to algebra $\cap_{z \in \partial D} A_z$.

Proof of Lemma. First, we prove assertions (a) and (b). Since algebra $A(U_{z_0})$ is self-adjoint, $\mathbb{D}$ is dense in its maximal ideal space $M(U_{z_0})$. Then $M_{z_0}$ is the inverse limit of the compact spaces $M(U_{z_0})$ (because $A_{z_0}$ is the uniform closure of the algebra generated by algebras $A(U_{z_0})$), see, e.g., [R]. For $U_{z_0} \subset V_{z_0}$ by $p_{U_{z_0}}V_{z_0} : M(U_{z_0}) \rightarrow M(V_{z_0})$ we denote the maps in this limit system and by $p_{U_{z_0}} : M_{z_0} \rightarrow M(U_{z_0})$ the corresponding (continuous and surjective) limit maps. Then, by the definition of the inverse limit, the base of topology of $M_{z_0}$ consists of the sets $p_{U_{z_0}}^{-1}(U)$ where $U \subset M(U_{z_0})$ is open and $U_{z_0}$ is a circular neighbourhood of $z_0$. In particular, since $F_{z_0}$ is a compact set, for a neighbourhood $U$ of $F_{z_0}$ there is a circular neighbourhood $\tilde{U}_{z_0}$ of $z_0$ and a neighbourhood $\tilde{U} \subset M(\tilde{U}_{z_0})$ of $F(\tilde{U}_{z_0}) := p_{\tilde{U}_{z_0}}(F_{z_0})$ such that $p_{\tilde{U}_{z_0}}^{-1}(\tilde{U}) \subset U$.

Recall that

1. $\mathbb{D}$ is a dense subset of $M(\tilde{U}_{z_0})$ and $M_{z_0}$,

2. $p_{\tilde{U}_{z_0}}^{-1}(\mathbb{D})$ contains $\mathbb{D} \subset M_{z_0}$.

Thus in order to prove (a) it suffices to show that there is $U_{z_0} \subset \tilde{U}_{z_0}$ such that $U_{z_0} \cap \mathbb{D} \subset \tilde{U} \cap \mathbb{D}$.

Let us describe the structure of $M(\tilde{U}_{z_0})$. Let $A^*(\tilde{U}_{z_0})$ denote the pullback to $T$ by means of the map $(\log \circ \phi_{z_0})^{-1}$ of the algebra $A(\tilde{U}_{z_0})$. Then $A^*(\tilde{U}_{z_0})$ consists of continuous functions on $T_0$ such that on $(\log \circ \phi_{z_0})(\tilde{U}_{z_0})$ these are the restrictions of the almost periodic functions on $T$. Since $A^*(\tilde{U}_{z_0})$ is isomorphic to $A(\tilde{U}_{z_0})$ we can naturally identify the maximal ideal spaces of these algebras.

Further, observe that there is $w < 0$ such that $(\log \circ \phi_{z_0})(\tilde{U}_{z_0})$ contains the subset $T_w := \{z \in \Sigma : \Re z \leq w\}$ of the strip $T$. By the definition of the topology on $bT$ (cf. Section 11) $T_w$ is dense in $bT$. Hence, space $M(\tilde{U}_{z_0})$ contains the maximal ideal space of algebra $APC(T)$ ($= bT$), cf. Chapter 11.

Next, let $K$ be the intersection of the closures of $\tilde{U}_{z_0}$ and $\mathbb{D} \setminus \tilde{U}_{z_0}$ in $\mathbb{C}$. Then $K' := \text{Log} \circ \phi_{z_0}(K)$ is a compact subset of $T$. In particular, $APC(T)|_{K'} = C(K')$. This implies (by
the Tietze extension theorem) that every bounded continuous function on \((\mathbb{D} \setminus \tilde{U}_{z_0}) \cup K\) can be extended to a function in \(\mathcal{A}(\tilde{U}_{z_0})\) having the same sup-norm.

Now, we can describe explicitly \(\mathcal{M}(\tilde{U}_{z_0})\) as follows. Let \(M\) be the maximal ideal space of the algebra of bounded continuous functions on \((\mathbb{D} \setminus \tilde{U}_{z_0}) \cup K\). We identify \(K \subset M\) with \(K' \subset bT\) by means of \(\log \circ \phi_{z_0}\). Then the quotient space of \(M \sqcup \mathcal{M}(\text{AP}_\mathcal{O}(\Sigma))\) under this identification is homeomorphic to \(\mathcal{M}(\tilde{U}_{z_0})\).

Note that fibre \(F_{z_0}\) consists of the limit points in \(\mathcal{M}_{z_0}\) of all nets converging to \(\{z_0\}\) inside \(\mathbb{D}\). This and the above construction of \(\mathcal{M}(\tilde{U}_{z_0})\) show that \(F(\tilde{U}_{z_0})\) is homeomorphic to \(bT\). Moreover, \(F_{z_0}\) is the inverse limit of compact sets \(F(\tilde{U}_{z_0})\) where the limit system is determined by the maps \(p_{U_{z_0}V_{z_0}}|_{F(U_{z_0})}\) for \(U_{z_0} \subset V_{z_0}\). Let us show that the maps \(p_{U_{z_0}V_{z_0}}|_{F(U_{z_0})} : F(U_{z_0}) \to F(V_{z_0})\) are homeomorphisms. Indeed, let \(\{z_0\} \subset U_{z_0}\) be a net converging to a point \(\xi \in F(U_{z_0})\). Since \(U_{z_0} \hookrightarrow V_{z_0}\), in our definitions of \(\mathcal{M}(U_{z_0})\) and \(\mathcal{M}(V_{z_0})\) the net \(\{z_0\}\) converges to the same point \(\xi \in F(V_{z_0})\) which gives the required statement. Since all maps \(p_{U_{z_0}V_{z_0}}|_{F(U_{z_0})}\) are homeomorphisms, by the definition of the inverse limit the map \(p_{\tilde{U}_{z_0}}|_{F_{z_0}} : F_{z_0} \to F(\tilde{U}_{z_0})\), is also a homeomorphism. This completes the proof of (b).

Now, observe that in our model of \(\mathcal{M}(\tilde{U}_{z_0})\) the intersection of \(\tilde{U}\) with \(\mathbb{D}\) contains \(\tilde{U}_{z_0} \cap \mathbb{D}\); this gives us (a).

(c) Fix a point \(z_* \in \partial \mathbb{D}\). We have to show that every function \(f \in \text{SAP}(S) \cap H^\infty(\mathbb{D})\) belongs to \(\mathcal{A}_{z_*}\). According to (14.3.5) and Lemma 14.6.3 each \(f \in \text{SAP}(\partial \mathbb{D}) \cap H^\infty(\mathbb{D})\) can be approximated locally on open arcs of the form \(\gamma_{tk}(\delta)\), \(k \in \{-1, 1\}\), \(\delta \in (0, \pi)\), \(z := e^{it} \in \partial \mathbb{D}\), by pullbacks of almost periodic functions on \(\partial T\). Using the compactness of \(\partial \mathbb{D}\), for a given \(\epsilon > 0\) we can find finitely many points \(z_1, \ldots, z_n \in \partial \mathbb{D}\), arcs \(\gamma_{\delta_k}^t(\delta_l)\), \(k \in \{-1, 1\}\), \(\delta_l \in (0, \pi)\), and functions \(f^t : \partial \mathbb{D} \setminus \{-z_l, z_l\} \to \mathbb{C}\) which are the pullbacks of the almost periodic functions on \(\partial T\) by means of map \(\log \circ \phi_{z_l}\), \(1 \leq l \leq n\), such that

\[\partial \mathbb{D} \setminus \{z_1, \ldots, z_n\} = \bigcup_{1 \leq l \leq n} (\gamma_{\delta_l}^{-1} \cup \gamma_{\delta_l}^1)\] and for all \(1 \leq l \leq n\)

\[\text{ess sup}_{z \in \gamma_{\delta_l}^{-1} \cup \gamma_{\delta_l}^1} |f(z) - f^t(z)| < \epsilon.\]

Without loss of generality we may assume that \(z_* \in \{z_1, \ldots, z_n\}\). Next, set \(V_l := \gamma_{\delta_l}^{-1} \cup \gamma_{\delta_l}^1 \cup \{z_l\}\).
Then \((V_l)_{l=1}^n\) is a finite open cover of \(\partial \mathbb{D}\). Let \(\{\rho_l\}_{l=1}^n\) be a smooth partition of unity subordinate to this cover such that \(\rho_l(z_l) = 1\). Consider the functions \(f_l\) on \(\partial \mathbb{D} \setminus \{z_l\}\) defined by the formulas
\[
f_l := \rho_l f^l, \quad 1 \leq l \leq n.
\]
Let \(F\) be the harmonic function on \(\mathbb{D}\) such that \(F|_{\partial \mathbb{D}} = \sum_{1 \leq l \leq n} f_l\). Then
\[
\|f - F\|_{L^\infty(\mathbb{D})} < \epsilon.
\]
We prove that \(F \in A_{z^*}\). Since \(\epsilon > 0\) is arbitrary, this will complete the proof of (c).

We denote by \(F_{l,1}\) and \(F_{l,2}\) the harmonic functions on \(\mathbb{D}\) with the boundary values \(f_l\) and \(f^l - f_l\), respectively. Thus \(F_l := F_{l,1} + F_{l,2}\) is the harmonic function with the boundary values \(f^l\). According to Lemma 14.6.4, every \(F_l\) is almost periodic on \(\mathbb{D} \setminus \{\pm z_l\}\). Thus if \(z_l = z_\ast\), then \(F_l \in A_{z_\ast}\). If \(z_l \neq z_\ast\), then \(F_l\) is continuous at \(z_\ast\) and so by the definition of \(A_{z_\ast}\) the function \(F_l \in A_{z_\ast}\), as well. Further, for a point \(z_l\) distinct from \(z_\ast\) the function \(F_{l,1}\) can be extended continuously in an open disk centered at \(z_\ast\) (because the support of \(f_l\) does not contain \(z_\ast\)). Hence, such \(F_{l,1} \in A_{z_\ast}\). Assume now that \(z_l = z_\ast\) for some \(l\). Then the function \(F_{l,2}\) can be extended continuously in an open disk centered at \(z_\ast\) (because the support of \(f^l - f_l\) does not contain \(z_\ast\)). Thus \(F_{l,2} \in A_{z_\ast}\) and in this case \(F_{l,1} := F_l - F_{l,2} \in A_{z_\ast}\), as well. Since \(F := \sum_{1 \leq l \leq n} F_{l,1}\), combining the above considered cases we obtain that \(F \in A_{z_\ast}\).

**Theorem 14.6.8.** Let \(f \in SAP(S) \cap H^\infty(\mathbb{D})\). Then for each \(z_0 \in \partial \mathbb{D}\) and any \(\epsilon > 0\) there is a circular neighbourhood \(U_{z_0} := U_{z_0}(f, \epsilon)\) of \(z_0\) and a holomorphic almost periodic function \(f_{z_0}\) on \(U_{z_0}\) such that
\[
\sup_{z \in U_{z_0} \cap \mathbb{D}} |f(z) - f_{z_0}(z)| < \epsilon.
\]

**Proof of Theorem.** Fix a point \(z_0 \in \partial \mathbb{D}\). According to Lemma 14.6.7(c) the function \(f \in SAP(S) \cap H^\infty(\mathbb{D})\) belongs to \(A_{z_0}\) and so it can be extended by means of the Gelfand transform to a continuous function \(\widehat{f}\) on \(M_{z_0}\). We will use the description of \(M_{z_0}\) obtained in the proof of Lemma 14.6.7. Recall that in that construction the fibre \(F_{z_0} \subset M_{z_0}\) over \(z_0\) is naturally identified with \(bT\). Then we have (cf. Definition 11.0.7)

**Lemma 14.6.9.** The function \(\widehat{f}|_{F_{z_0}}\) is holomorphic.
Proof of Lemma. In the proof we use the results of Section 11. Let us consider the map \( i_\xi : T_0 \to bT, \xi \in b\mathbb{Z} \). We must show that \( \hat{f} \circ i_\xi \) is holomorphic. We denote by \( \tilde{f} \) the pullback of \( f \) to \( T_0 \) via map \( (\log \circ \phi_{z_0})^{-1} \). Let us fix a point \( \eta \in i_\xi(T_0) \), say, \( \eta := i_\xi(w), w \in T_0 \). Then there is a horizontal line \( \mathbb{R} + iy \subset T_0, y \in (0, \pi) \), and a net \( \{z_\alpha\} \subset \mathbb{R} + iy \) that is an infinite discrete subset of \( \mathbb{R} + iy \) such that \( \{z_\alpha\} \) converges to \( \eta \) in the topology of \( bT \). Let \( B \subset \mathbb{C} \) be an open disk such that \( \{\mathbb{R} + iy\} + B \subset T_0 \). By the definition of \( i_\xi \), for each \( z \in B \) the net \( \{z_\alpha + z\} \) converges in \( bT \) to \( i_\xi(w + z) \). Also, by the definition of \( \mathcal{M}_{z_0} \) we have

\[
\lim_{\alpha} \tilde{f}(z_\alpha + z) = (\hat{f} \circ i_\xi)(w + z), \quad z \in B.
\]

But the holomorphic functions \( \tilde{f}_\alpha(z) := \tilde{f}(z_\alpha + z) \) form a normal family on \( B \). Therefore using an argument similar to that of the proof of Lemma 11.0.8 (1) we obtain that \( \hat{f} \circ i_\xi|_B \) is holomorphic. Since \( \xi \) and \( \eta \) are arbitrary, the latter implies that \( \hat{f}|_{F_{z_0}} \) is holomorphic. \( \square \)

Next, using Lemma 11.0.8 we obtain that there is a function \( \tilde{f}_{z_0} \in APH(T) \) whose extension to \( bT \) coincides with \( \hat{f}|_{F_{z_0}} \). Let us consider the function \( f_{z_0} \in A_{z_0} \) whose pullback to \( T \) via \( (\log \circ \phi_{z_0})^{-1} \) coincides with \( \tilde{f}_{z_0} \). Then by the definition of the topology of \( \mathcal{M}_{z_0} \) the extension \( \hat{f}_{z_0} \) of \( f_{z_0} \) to \( \mathcal{M}_{z_0} \) satisfies \( \hat{f}_{z_0}|_{F_{z_0}} = \hat{f}|_{F_{z_0}} \). Since \( F_{z_0} \) is a compact set, the latter implies that there is a neighbourhood \( U \) of \( F_{z_0} \) in \( \mathcal{M}_{z_0} \) such that

\[
|\hat{f}_{z_0}(x) - \hat{f}(x)| < \epsilon \quad \text{for all } x \in U.
\]

Finally, by Lemma 14.6.7(a) there is a circular neighbourhood \( U_{z_0} \) such that \( U_{z_0} \cap \mathbb{D} \subset U \cap \mathbb{D} \). Thus \( |f_{z_0}(z) - f(z)| < \epsilon \) for all \( z \in U_{z_0} \cap \mathbb{D} \). \( \square \)

We are now ready to prove Theorem 14.6.2.

Proof of Theorem 14.6.2. We have to show that \( A^S = SAP(S) \cap H^\infty(\mathbb{D}) \). We conduct our proof in several steps.

First, we prove

**Lemma 14.6.10.** \( SAP(S) \cap H^\infty \) is the uniform closure of the algebra generated by all possible subalgebras \( SAP(F) \cap H^\infty \) with finite \( F \subset S \).
Then we will prove that \( A_F = SAP(F) \cap H^\infty \) for every finite subset \( F \subset \partial \mathbb{D} \). Together with the above lemma and the fact that \( A^S \) is the uniform closure of the algebra generated by all possible subalgebras \( A^F \) with finite \( F \subset S \), this will complete the proof of the theorem.

**Proof of Lemma.** According to Theorem 14.6.8, for given \( f \in SAP(S) \cap H^\infty \) and \( \varepsilon > 0 \) we can find finitely many points \( z_1, \ldots, z_n \), circular neighbourhoods \( U_{z_1}, \ldots, U_{z_n} \) and holomorphic almost periodic functions \( f_1, \ldots, f_n \) defined on \( U_{z_1}, \ldots, U_{z_n} \), respectively, such that \( (U_{z_i})_{1 \leq i \leq n} \) forms an open cover of \( \partial \mathbb{D} \setminus \{z_1, \ldots, z_n\} \) and

\[
\max_i ||f|_{U_{z_i}} - f_i||_{L^\infty(U_{z_i})} < \varepsilon.
\]

Since the discontinuities of \( f|_{\partial \mathbb{D}} \) belong to the closed set \( S \), each function \( f_i \) with \( z_i \not\in S \) can be chosen also to be continuous on the closure \( \overline{U}_{z_i} \).

Further, we define a cocycle \( \{c_{ij}\} \) on the intersections of sets from the above cover by the formula

\[
c_{ij}(z) := f_i(z) - f_j(z), \quad z \in U_{z_i} \cap U_{z_j}.
\]

We may assume without loss of generality that all \( U_{z_i} \cap U_{z_j}, \ i \neq j \), do not contain points \( z_1, \ldots, z_n \). Then each \( U_{z_i} \cap U_{z_j}, \ i \neq j \), is a compact subset of \( \mathbb{D} \) and the corresponding \( c_{ij} \) are continuous and holomorphic in interior points of \( U_{z_i} \cap U_{z_j} \).

Let \( \{\rho_i\} \) be a smooth partition of unity subordinate to the cover \( (U_{z_i})_{1 \leq i \leq n} \). We can choose every \( \rho_i \) so that it is the restriction to \( \overline{U}_{z_i} \) of a \( C^\infty \)-function on \( \mathbb{C} \) and \( \rho_i(z_i) = 1 \). We resolve the cocycle \( \{c_{ij}\} \) by the formulas

\[
\tilde{f}_j(z) := \sum_{k=1}^n \rho_k(z)c_{jk}(z), \quad z \in \overline{U}_{z_j},
\]

so that

\[
c_{ij}(z) := \tilde{f}_i(z) - \tilde{f}_j(z), \quad z \in U_{z_i} \cap U_{z_j}.
\]

In particular, since \( c_{ij} \) are holomorphic in \( \mathbb{D} \cap U_{z_i} \cap U_{z_j} \), the formula

\[
h(z) := \frac{\partial \tilde{f}_i(z)}{\partial \overline{z}}, \quad z \in U_{z_i} \cap \mathbb{D},
\]

determines a smooth bounded function in an open annulus \( Z \subset \cup_{i=1}^n \overline{U}_{z_i} \) with the outer boundary \( \partial \mathbb{D} \). Also, by our choice of the partition of unity, \( h \) is extended continuously to the closure \( \overline{Z} \) of \( Z \).
Let us consider function $H$ defined by the formula

$$H(z) = \frac{1}{2\pi i} \int_{\zeta \in Z} \frac{h(\zeta)}{\zeta - z} \, d\zeta \wedge d\overline{\zeta}, \quad z \in \mathbb{Z}. \quad (14.6.12)$$

Passing in (14.6.12) to polar coordinates with the origin at $z$, we obtain

$$\sup_{z \in Z} |H(z)| \leq C w(Z) \sup_{z \in Z} |h(z)| \quad (14.6.13)$$

where $w(Z)$ is the width of annulus $Z$ and $C > 0$ is an absolute constant. Moreover, $H \in C(\overline{Z})$ and $\partial H/\partial \overline{z} = h$ in $Z$, see, e.g., [Gar, Ch. VIII]. Let us replace $Z$ by annulus of a smaller width (which we also denote by $Z$) such that

$$\sup_{z \in Z} |H(z)| < \epsilon.$$ 

Now we set

$$c_i(z) := \tilde{f}_i(z) - H(z), \quad z \in U_{z_i} \cap \overline{Z}.$$ 

Then each $c_i$ is continuous on $U_{z_i} \cap \overline{Z}$ holomorphic in interior points of this set and

$$c_i(z) - c_j(z) = c_{ij}(z), \quad z \in U_{z_i} \cap U_{z_j}.$$ 

Since every $|c_{ij}(z)| < 2\epsilon$ for all $z \in U_{z_i} \cap U_{z_j},$

$$|c_i(z)| < 3\epsilon, \quad z \in U_{z_i} \cap U_{z_j}.$$ 

Now, let us define a global function $f_\epsilon$ on $\overline{Z} \setminus \{z_1, \ldots, z_n\}$ by the formulas

$$f_\epsilon(z) := f_i(z) - c_i(z), \quad z \in U_i \cap \overline{Z}.$$ 

Since for $z_i \not\in S$, the function $f_i$ is continuous on $U_{z_i}$, from the above construction we obtain that $f_\epsilon \in H^\infty(Z) \cap SAP(F)$ where $F := \{z_1, \ldots, z_n\} \cap S$. Also,

$$||f - f_\epsilon||_{L^\infty(Z)} < 4\epsilon.$$ 

Let $B$ be an open disk centered at 0 whose intersection with $Z$ is an annulus of width $< \epsilon$. Consider the cocycle $c$ on $B \cap Z$ defined by

$$c(z) = f(z) - f_\epsilon(z), \quad z \in B \cap A.$$ 

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By definition, \(|c(z)| \leq 4\epsilon\) for all \(z \in B \cap Z\). Let \(Z'\) be the open annulus with the interior boundary coinciding with the interior boundary of \(Z\) and with the outer boundary \(\{z \in \mathbb{C} : |z| = 2\}\). Then \(Z' \cap B = Z \cap B\). Consider a smooth partition of unity subordinate to the cover \(\{Z', B\}\) of \(\mathbb{D}\) which consists of smooth radial functions \(\rho_1\) and \(\rho_2\) such that

\[
\max_i ||\nabla \rho_i||_{L^\infty(\mathbb{C})} \leq \frac{\tilde{C}}{w(B \cap Z)} < \tilde{C}\epsilon
\]

for some absolute constant \(\tilde{C} > 0\). Then using arguments similar to the above based on versions of (14.6.11), (14.6.12) and (14.6.13) for the cocycle \(c\) and the partition of unity \(\{\rho_1, \rho_2\}\), we can find holomorphic functions \(\varphi_1\) on \(B\) and \(\varphi_2\) on \(Z\) continuous on the corresponding boundaries such that

\[
\varphi_1(z) - \varphi_2(z) = c(z), \quad z \in B \cap Z, \quad \text{and}
\]

\[
\max\{||\varphi_1||_{H^\infty(B)}, ||\varphi_2||_{H^\infty(Z)}\} \leq \overline{C}||c||_{H^\infty(Z \cap B)}
\]

where \(\overline{C} > 0\) is an absolute constant. Finally, define

\[
F_\epsilon(z) := \begin{cases} 
  f(z) - \varphi_1(z), & \text{if } z \in B, \\
  f_\epsilon(z) - \varphi_2(z), & \text{if } z \in \mathbb{D} \setminus B.
\end{cases}
\]

Clearly we have

\[
||f - F_\epsilon||_{L^\infty(\mathbb{D})} < c\epsilon.
\]

for some absolute constant \(c > 0\), and \(F_\epsilon \in SAP(F) \cap H^\infty\), where \(F := \{z_1, \ldots, z_n\} \cap S\). Since \(\epsilon\) is arbitrary, this completes the proof of the lemma. \(\square\)

Now, we prove Theorem 14.6.2 for \(SAP(F) \cap H^\infty(\mathbb{D})\) for a finite set \(F \subset \partial S\), \(F := \{z_1, \ldots, z_n\}\).

**Lemma 14.6.11.**

\[ A^F = SAP(F) \cap H^\infty(\mathbb{D}). \]

**Proof of Lemma.** Let us show that \(SAP(F) \cap H^\infty(\mathbb{D}) \subset A^F\).

First, suppose that \(F\) contains at least two points. Let \(\psi_1 : \partial \mathbb{D} \to \partial \mathbb{D}\) be the restriction to the boundary of Möbius transformation of \(\mathbb{D}\) that maps \(-z_1\) to a point of \(F\) distinct from \(z_1\)
and preserves $z_1$. Then by the definition of Möbius transformations $\psi_1$ is a $C^1$-diffeomorphism of $\partial\mathbb{D}$. In particular, by Proposition 14.6.1 for a given $f \in SAP(F) \cap H^\infty(\mathbb{D})$ the function $f \circ \psi_1 \in SAP(F_1) \cap H^\infty(\mathbb{D})$ where $F_1 := 1/f \circ \psi_1^{-1}(F)$. Since $z_1 \in F_1$, as in the proof of Theorem 14.6.8 we can find a holomorphic almost periodic function $g_1$ on $\mathbb{D} \setminus \{\pm z_1\}$ such that $f \circ \psi_1 - g_1$ is continuous and equals 0 at $z_1$. We set

$$
\tilde{g}_1(z) := \frac{g(z)(z + z_1)}{2z_1}, \quad z \in \mathbb{D} \setminus \{z_1\}.
$$

Then $\tilde{g}_1$ has discontinuity at $z_1$ only. Let us show that $\tilde{g}_1 \in A\{z_1, z_1\}$. Indeed, by the definition of $g_1$ the function $g_1 \circ \phi_{z_1}^{-1} \circ \Log^{-1}$ belongs to $APH(T)$. Therefore, it can be uniformly approximated on $T$ by polynomials in variables $e^{i\lambda z}$, $\lambda \in \mathbb{R}$, cf. Section 11. In turn, $g_1$ can be uniformly approximated on $\mathbb{D} \setminus \{\pm z_1\}$ by complex polynomials in variables $e^{i\lambda \Log z}$. Now for $z \in \partial\mathbb{D}$ we have

$$
\Im((\Log \circ \phi_{z_1})(z)) := \begin{cases}
0, & 0 \leq \Arg(z/z_1) < \pi, \\
\pi, & 0 \leq \Arg(z/z_1) < \pi.
\end{cases}
$$

This implies that every function $e^{i\lambda \Log z}$, $\lambda \in \mathbb{R}$, belongs to $A\{z_1, z_1\}$ and so $g_1 \in A\{z_1, z_1\}$, as well. Since $(z + z_1)/2z_1 \in A(\mathbb{D})$, the function $\tilde{g}_1 \in A\{z_1, z_1\}$ Thus the function $h_1 := \tilde{g}_1 \circ \psi_1^{-1}$ belongs to $A^F$, is continuous outside $z_1$ and $f_1 := f - h_1 \in SAP(F_1) \cap H^\infty(\mathbb{D})$, where $F_1 := F \setminus \{z_1\}$. Further, using similar arguments we can find $h_2 \in A_F$, continuous outside $z_2$ such that $f_2 := f_1 - h_2 \in SAP(F_2) \cap H^\infty(\mathbb{D})$ where $F_2 := F_1 \setminus \{z_2\}$ etc. After $n$ steps we obtain functions $h_1, \ldots, h_n \in A^F$ such that $h_k$ is continuous outside $z_{k-1}$, $1 \leq k \leq n$, and the function

$$
h_{n+1} := f - \sum_{k=1}^{n} h_k \quad (14.6.14)
$$

has no discontinuities on $\partial\mathbb{D}$, that is, $h_{n+1} \in A(\mathbb{D})$. Therefore $f \in A^F$.

Next, if $F$ consists of a single point, say $z_0$, then for a given $f \in SAP(F) \cap H^\infty(\mathbb{D})$ using the above argument we can find a function $h \in A\{z_0, z_1\}$ with a fixed $z_1 \in \partial\mathbb{D}$ such that $f - h$ is continuous on $F$. Let $g \in A(\mathbb{D})$ be a function equal to 1 at $F$ and 0 at $z_1$. Then $f - gh \in A(\mathbb{D})$. This completes the first part of the proof.

Finally, let us show that $A^F \subseteq SAP(F) \cap H^\infty(\mathbb{D})$. First, suppose that $F$ contains at least two points. Let $e^\lambda \in A^F$, $\lambda \in \mathbb{R}$, where $\Re f$ is the characteristic function of an arc, say $[x, y]$,
with \(x, y \in F\). Let \(\psi : \partial \mathbb{D} \to \partial \mathbb{D}\) be the restriction to \(\partial \mathbb{D}\) of a Möbius transformation sending 1 to \(x\) and \(-1\) to \(y\). Then by the definition
\[
(f \circ \psi \circ \phi^{-1}_1)(z) = -\frac{i}{\pi} \log z + C, \quad z \in \mathbb{H}_+,
\]
for some constant \(C\). Thus we have
\[
e^{(\lambda f \circ \psi \circ (\log \circ \psi_1)^{-1})}(z) = e^{\lambda C} e^{-i\lambda z/\pi}, \quad z \in \Sigma.
\]
This means that \(e^{\lambda f \circ \psi} \in SAP(\{-1, 1\}) \cap H^\infty(\mathbb{D})\). Then by Proposition 14.6.1 we get \(e^{\lambda f} \in SAP(F) \cap H^\infty(\mathbb{D})\). Since \(A_F\) is generated by \(A(\mathbb{D})\) and such functions \(e^{\lambda f}\), we obtain the required implication.

If \(F\) is a single point, then we must show that \(ge^{\lambda f} \in SAP(F) \cap H^\infty(\mathbb{D})\), \(\lambda \in \mathbb{R}\), where \(Re f\) is the characteristic function of an arc with an endpoint at \(F\) and \(g \in A(\mathbb{D})\) is such that \(ge^f\) has discontinuity at \(F\) only. The result follows easily from the previous part of the proof, because \(e^{\lambda f}\) is almost periodic on \(\partial \mathbb{D} \setminus \{F, y\}\) for some \(y\) and is continuous at \(y\).

This completes the proof of the lemma. \(\square\)

Theorem 14.6.2 now follows from Lemmas 14.6.10 and 14.6.11. \(\square\)

### 14.6.3 Maximal ideal space

Let \(b^S(\mathbb{D})\) denote the maximal ideal space of algebra \(A^S = SAP(S) \cap H^\infty(\mathbb{D})\) (cf. Theorem 14.6.2).

Since point evaluation homomorphisms \(z(f) := f(z), z \in \mathbb{D}, f \in A^S\), belong to \(b^S(\mathbb{D})\) and \(A^S\) separates points on \(\mathbb{D}\), there is a continuous embedding \(\iota : \mathbb{D} \hookrightarrow b^S(\mathbb{D})\). In what follows we identify \(\mathbb{D}\) with \(\iota(\mathbb{D})\).

We have \(A(\mathbb{D}) \hookrightarrow A^S\), hence there is a continuous surjection of the maximal ideal spaces
\[
a^S : b^S(\mathbb{D}) \to \mathbb{D}.
\]

We have the following Corona-type theorem.

**Theorem 14.6.12.** \(\mathbb{D}\) is dense in \(b^S(\mathbb{D})\).
Remark 14.6.13. Recall that the corona theorem is equivalent to the following statement (see, e.g., [Gar, Ch. V]). For any collection of functions $f_1, \ldots, f_n \in A^S$ satisfying

$$\max_{1 \leq j \leq n} |f_j(z)| \geq \delta > 0, \quad z \in \mathbb{D},$$

there are functions $g_1, \ldots, g_n \in A^S$ such that

$$f_1g_1 + \cdots + f_ng_n = 1.$$  \hfill (14.6.16)

Proof of Theorem. Recall that $A^S$ is the uniform closure of the algebra generated by all possible $A^F$ with finite $F \subset S$. Therefore, the maximal ideal space $b^S(\mathbb{D})$ of $A^S$ is the inverse limit of the maximal ideal spaces $b^F(\mathbb{D})$ of $A^F$. Hence, if we prove that $\mathbb{D}$ is dense in each $b^F(\mathbb{D})$, then by the definition of the inverse limit this would imply that $\mathbb{D}$ is dense in $b^S(\mathbb{D})$, as needed.

Proposition 14.6.14. $\mathbb{D}$ is dense in $b^F(\mathbb{D})$.

Proof of Proposition. Let $F = \{z_1, \ldots, z_n\} \subset \partial \mathbb{D}$. We denote by $\mathcal{I}_k \subset A^F$ the closed ideal of functions that are continuous and equal to 0 at $z_k$. By $A_k$ we denote the quotient Banach algebra $A^F/\mathcal{I}_k$ equipped with the quotient norm. Recall that by $\mathcal{M}_{z_k}$ we denote the maximal ideal space of algebra $A_{z_k}$ which is the uniform closure of the algebras of continuous functions on $\mathbb{D}$ almost periodic in circular neighbourhoods of $z_k$ (cf. the proof of Theorem 14.6.2). Also, according to Lemma 14.6.7 (b) there is a natural continuous projection $p_{z_k} : \mathcal{M}_{z_k} \to \tilde{\mathbb{D}}$ and $p_{z_k}^{-1}(z_k)$ is homeomorphic to $APH(T)$. Moreover, by Lemma 14.6.7 (c) each $f \in A^F$ is extended to a continuous function on $\mathcal{M}_{z_k}$ holomorphic on $p_{z_k}^{-1}(z_k)$. Hence, there is a continuous map $H_k : \mathcal{M}_{z_k} \to b^S(\mathbb{D})$ whose image coincides with the closure of $\mathbb{D}$. Moreover, according to the decomposition obtained in the proof of Lemma 14.6.11, see (14.6.14), $H_k$ maps $p_{z_k}^{-1}(z_k)$ homeomorphically onto its image.

We will need

Lemma 14.6.15. Let $\phi_k : A^F \to APh(T)$ be the composition of the extension homomorphism of functions from $A^F$ to $\mathcal{M}_{z_k}$ and the restriction homomorphism of functions on $\mathcal{M}_{z_k}$ to $p_{z_k}^{-1}(z_k)$. Then $\text{Ker } \phi_k = \mathcal{I}_k$ and $A_k$ is isomorphic to $APh(T)$. 209
Proof of Lemma. Clearly, \( \mathcal{I}_k \subset \ker \phi_k \). Let us prove the converse inclusion. Let \( f \in \ker \phi_k \). As follows from the proof of Lemma 14.6.11, see (14.6.14), there are linear continuous operators \( T_k : APH(T) \to A^{\{z_k\}} \subset A^F \) such that \( \phi_k \circ T_k = \text{Id} \). Moreover, \( T_0 := I - \sum_{k=1}^n T_k \circ \phi_k \), where \( I \) is the identity map, maps \( A^F \) onto \( A(\mathbb{D}) \). In particular, we have \( T_k(\phi_k(f)) = 0 \). Thus \( f = -T_0(f) + \sum_{s \neq k} T_s(\phi_s(f)) \). Since \( \phi_k(f) = 0 \), this implies that \( f \) is continuous and equal to 0 at \( z_k \). Now from the formula \( \phi_k \circ T_k = \text{id} \) we obtain that \( A_k \) is isomorphic to \( APH(T) \). \( \square \)

Let \( i : A(\mathbb{D}) \hookrightarrow A^F \) be the natural inclusion. Its dual determines a continuous surjective map \( a^F : b^F(\mathbb{D}) \to \bar{\mathbb{D}} \). Next, taking the dual map to \( \phi_k \) we obtain that each \( bT \) is embedded into \( b^F(\mathbb{D}) \), its image coincides with \( H_k(p_{\xi_k}^{-1}(z_k)) \) and \( a^F \) maps \( H_k(p_{\xi_k}^{-1}(z_k)) \) to \( z_k \).

Let \( \xi \in b^F(\mathbb{D}) \) and \( m := \ker \xi \subset A^F \) be the corresponding maximal ideal. First, suppose that there is \( k \) such that \( \mathcal{I}_k \subset m \). Then \( m_k = \phi_k(m) \) is a maximal ideal of \( A_k \). By \( \xi_k \in bT \) we denote the character corresponding to \( m_k \). Then \( \xi = \phi^*(\xi_k) \in H_k(p_{\xi_k}^{-1}(z_k)) \). Now, by the definition of \( H_k \) the point \( \xi \) belongs to the closure of \( \mathbb{D} \) in \( b^F(\mathbb{D}) \).

We continue with the following

**Lemma 14.6.16.** Suppose that a maximal ideal \( m \) of \( A^F \) does not contain any of \( \mathcal{I}_k \). Then \( m \) does not contain \( \cap_{1 \leq k \leq n} \mathcal{I}_k \), as well.

**Proof of Lemma.** Suppose, to the contrary, that \( \cap_{1 \leq k \leq n} \mathcal{I}_k \subset m \). Let \( x_k \in \mathcal{I}_k \), \( 1 \leq k \leq n \), be such that \( x_k \notin m \). Since \( \mathcal{I}_k \) are ideals, \( x_1 \cdots x_n \in \cap_{1 \leq k \leq n} \mathcal{I}_k \). Thus \( x_1 \cdots x_n \in m \). Since \( m \) is a prime ideal, there is some \( k \) so that \( x_k \in m \), a contradiction. \( \square \)

Thus, in order to prove the theorem it suffices to consider the case \( m \notin \cap_{1 \leq k \leq n} \mathcal{I}_k \). Observe that \( \cap_{1 \leq k \leq n} \mathcal{I}_k \) consists of all functions from \( A(\mathbb{D}) \) that vanish on \( F \). Thus there is \( f \in \cap_{1 \leq k \leq n} \mathcal{I}_k \) such that \( f(\xi) \neq 0 \). This implies that \( a^F(\xi) \notin F \). For every \( g \in A^F \), let us consider the function \( gf \). By the definition \( gf \in A(\mathbb{D}) \). Thus we have

\[
g(a^F(\xi))f(a^F(\xi)) = (gf)(a^F(\xi)) = (gf)(\xi) = g(\xi)f(\xi) = g(\xi)f(a^F(\xi)).
\]

Equivalently,

\[
g(\xi) = g(a^F(\xi)) \quad \text{for all} \quad g \in A^F.
\]
This implies that \((a^F)^{-1}(a^F(\xi)) = \{\xi\}\). Therefore \(a^F : b^F(\mathbb{D}) \setminus (a^F)^{-1}(F) \to \mathbb{D} \setminus F\) is a homeomorphism. In particular, \(\xi\) belongs to the closure of \(\mathbb{D}\).

The proof of Theorem 14.6.12 now follows from the inverse limit-type argument, see above.

Finally, we describe the structure of \(b^S(\mathbb{D})\). Recall that the Šilov boundary of algebra \(A^S\) is the smallest compact subset \(K^S \subset b^S(\mathbb{D})\) such that for each \(f \in A^S\)

\[
\sup_{z \in b^S(D)} |f(z)| = \sup_{\xi \in K} |f(\xi)|;
\]

here we assume that every function \(f \in A^S\) is also defined on the maximal ideal space \(b^S(\mathbb{D})\) via Gelfand transform.

**Theorem 14.6.17.** The following is true:

1. \(a^S : b^S(D) \setminus (a^S)^{-1}(S) \to \overline{D} \setminus S\) is a homeomorphism.

2. The Šilov boundary \(K^S\) of \(A^S\) is naturally homeomorphic to \(b^S(\partial \mathbb{D})\). Under the identification of \(K^S\) and \(b^S(\partial \mathbb{D})\) one has \(a^S|_{K^S} = r^S\) (cf. Theorem 12.2.5(2)).

3. For each \(z \in S\), the preimage \(a^{-1}_{\mathbb{S}}(z)\) is homeomorphic to the maximal ideal space of the algebra \(\text{APH}(T)\).

**Proof of Theorem.** Assertions (1) and (3) follow easily from similar results for \(a^F\) with a finite subset \(F \subset S\) obtained in the proof of Theorem 14.6.12, and the properties of the inverse limit. Let us prove (2).

We first prove the statement for a finite subset \(F \subset S\). Since \(A(\mathbb{D}) \subset A^F\) and for each function from \(A(\mathbb{D})\) it modulus attains its maximum on \(\partial \mathbb{D}\), we have \(K^F \subset (a^F)^{-1}(\partial \mathbb{D})\). As it follows from Theorem 14.6.2 \(A^F \hookrightarrow \text{SAP}(F)\). Also, the extensions of functions from \(A^F\) to \(b^F(\partial \mathbb{D})\) separate points. Therefore \(b^F(\partial \mathbb{D})\) is embedded into \(b^F(D)\). Identifying \(b^F(\partial \mathbb{D})\) with its image under this embedding we have \(b^F(\partial \mathbb{D}) \subset (a^F)^{-1}(\partial \mathbb{D})\). Since \((a^F)^{-1}(\partial \mathbb{D}) \setminus (a^F)^{-1}(F) \to \partial \mathbb{D} \setminus F\) is a homeomorphism and each \(z \in \partial \mathbb{D}\) is a peak point for \(A(\mathbb{D})\), the set \(K_F\) contains the closure of \(\mathbb{D} \setminus F\) which coincides with \(b^F(\partial \mathbb{D})\). Assume that there is \(\xi \in K_F \setminus b^F(\partial \mathbb{D})\). Then
(a_F)(\xi) := z^* \in F$. Further, identifying $(a_S)^{-1}(z^*)$ with $bT$ we obtain that $\xi \in i_\eta(T_0)$ for some $\eta \in b\mathbb{Z}$, cf. Section 11; here $T_0$ is the interior of strip $T$. Then, since $i_\eta(T_0)$ is dense in $bT$, by the maximum modulus principle each function $f \in A^F$ for which $\max_D |f| = |f(\xi)|$ is constant on $a_S^{-1}(z^*)$. Thus, $f$ also attains the maximum of its modulus on $bF(\partial \mathbb{D})$. This contradicts to the minimality of $K^F$. Therefore $K^F = bF(\partial \mathbb{D})$.

Further, $bS(\partial \mathbb{D})$ is the inverse limit of compact spaces $bF(\partial \mathbb{D})$ for all finite $F \subset S$. As before we naturally identify $bS(\partial \mathbb{D})$ with a subset of $bS(\mathbb{D})$. Then since $A^S$ is the uniform closure of the algebra generated by all possible $A^F$ with finite $F \subset S$, by the definition of the inverse limit $K^S \subset b^S(\partial \mathbb{D})$. But in fact $K^S$ coincides with $b^S(\partial \mathbb{D})$ since otherwise its projection to some $bF(\partial \mathbb{D})$ is the boundary of $A^F$ and a proper subset of $bF(\partial \mathbb{D})$, which is a contradiction. 

\section{14.7 Proof of Theorem 13.1.3}

Our proof of Theorem 13.1.3 is based on Theorem 14.6.2.

\subsection*{14.7.1 Banach-valued almost periodic and semi-almost periodic functions}

Let $C^B_b(\mathbb{R})$ and $C^B_b(T)$ denote the Banach spaces of $B$-valued bounded continuous functions on $\mathbb{R}$ and $T$, respectively, with norms $\|f\| := \sup_x \|f(x)\|_B$.

\textbf{Definition 14.7.1.} 1) A function $f \in C^B_b(\mathbb{R})$ is said to be \textit{almost periodic} if the family of its translates $\{S_\tau f\}_{\tau \in \mathbb{R}}$, $S_\tau f(x) := f(x + \tau)$, $x \in \mathbb{R}$, is relatively compact in $C^B_b(\mathbb{R})$.

2) A function $f \in C^B_b(T)$ is called \textit{holomorphic almost periodic} if it is holomorphic in the interior of $T$ and the family of its translates $\{S_x f\}_{x \in \mathbb{R}}$ is relatively compact in $C^B_b(T)$.

Let $AP^B(\mathbb{R})$ and $APH^B(T)$ denote the Banach spaces of almost periodic and holomorphic almost periodic functions on $\mathbb{R}$ and $T$, respectively, endowed with sup-norms.

Since $AP(\mathbb{R})$ and $APH(T)$ have the approximation property, it follows from Theorem 10.0.2 that the functions of the form $\sum_{l=1}^m b_l e^{i\lambda_l x}$ ($x \in \mathbb{R}, b_l \in B, \lambda_l \in \mathbb{R}$) and $\sum_{l=1}^m b_l e^{i\lambda_l z}$ ($z \in T, b_l \in B, \lambda_l \in \mathbb{R}$) are dense in $AP^B(\mathbb{R})$ and $APH^B(T)$, respectively.

As in the case of scalar almost periodic functions, a Banach-valued almost periodic function $f \in AP^B_b(\mathbb{R})$ is characterized by its Bohr-Fourier coefficients $a_\lambda(f)$ and the spectrum $\text{spec}(f)$,
defined in terms of the mean value

\[ M(f) := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x) \, dx \in B. \]  

(14.7.17)

Namely, we define

\[ a_{\lambda}(f) := M(f e^{-i\lambda x}), \quad \lambda \in \mathbb{R}. \]

It follows from the above remark and the properties of scalar almost periodic functions that

\[ a_{\lambda}(f) \neq 0 \text{ in } B \text{ for at most countably many values of } \lambda. \]

These values constitute the spectrum \( \text{spec}(f) \) (e.g. if \( f = \sum_{l=1}^{\infty} b_{l} e^{i\lambda_{l}x} \), where \( \sum_{l=1}^{\infty} \|b_{l}\| < \infty \) and all \( b_{l} \neq 0 \), then \( \text{spec}(f) = \{\lambda_{1}, \lambda_{2}, \ldots\} \)). Similarly one defines the mean-values and the spectra for functions from \( \text{APH}^{B}(T) \).

Let \( L_{B}^{\infty}(\partial \mathbb{D}) \) be the Banach space of \( B \)-valued bounded measurable functions on \( \partial \mathbb{D} \) equipped with sup-norm.

**Definition 14.7.2.** A function \( f \in L_{B}^{\infty}(\partial \mathbb{D}) \) is called semi-almost periodic if for any \( z_{0} \in \partial \mathbb{D} \) and any \( \varepsilon > 0 \) there exist a number \( \delta = \delta(z_{0}, \varepsilon) \in (0, \pi) \) and functions \( f_{k} : \gamma_{z_{0}}^{k}(\delta) \to B \),

\( \gamma_{z_{0}}^{k}(\delta) := \{z_{0} e^{ikx} : 0 \leq x < 2\pi\}, \quad k \in \{-1, 1\}, \)

such that functions

\[ x \mapsto f_{k}(z_{0} e^{ik\delta x}), \quad -\infty < x < 0, \quad k \in \{-1, 1\}, \]

are restrictions of \( B \)-valued almost periodic functions from \( \text{AP}^{B}(\mathbb{R}) \) and

\[ \sup_{z \in \gamma_{z_{0}}^{k}(\delta)} \|f(z) - f_{k}(z)\|_{B} < \varepsilon, \quad k \in \{-1, 1\}. \]

For a closed subset \( S \subset \partial \mathbb{D} \) we denote the closed subspace of semi-almost periodic functions that are continuous on \( \partial \mathbb{D} \setminus S \) by \( \text{SAP}^{B}(S) \subset L_{B}^{\infty}(\partial \mathbb{D}) \).

**Proposition 14.7.3.** \( \text{SAP}^{B}(S) = \text{SAP}(S) \otimes B \).

The statement is a consequence of Theorem 10.0.2 and the following two facts: each function \( f \in \text{SAP}^{B}(S) \) admits a norm preserving extension to the maximal ideal space \( b^{S}(\partial \mathbb{D}) \) of the algebra \( \text{SAP}(S) \) as a continuous \( B \)-valued function, and \( C(b^{S}(\partial \mathbb{D})) \) has the approximation property. The first fact follows easily from the definitions of \( \text{SAP}^{B}(S) \) and \( b^{S}(\partial \mathbb{D}) \) and the existence of analogous extensions of functions in \( \text{AP}^{B}(\mathbb{R}) \) to \( b\mathbb{R} \), while the second fact is valid.
for any algebra $C(X)$ on a compact Hausdorff topological space $X$ (it can be proved using finite partitions of unity of $X$).

Next, we introduce the Banach space $AP^B(\mathbb{R} \cup (\mathbb{R} + i\pi)) := AP(\mathbb{R} \cup (\mathbb{R} + i\pi)) \otimes B$ of $B$-valued almost periodic functions on $\mathbb{R} \cup (\mathbb{R} + i\pi)$, cf. Example 12.1.3. Also, for each $z_0 \in S$ we define a linear isometry $L_{z_0}^B : AP^B(\mathbb{R} \cup (\mathbb{R} + i\pi)) \to L_B^\infty(\partial \mathbb{D})$ by the formula (cf. (12.1.2))

$$(L_{z_0}^B g)(z) := (g \circ \log \circ \varphi_{z_0})(z), \quad g \in AP^B(\mathbb{R} \cup (\mathbb{R} + i\pi)).$$

Now, using Proposition 14.7.3 we prove a $B$-valued analogue of Proposition 12.2.1.

**Proposition 14.7.4.** For every point $s \in \partial \mathbb{D}$ there exists a bounded linear operator $E_s^B : SAP^B(\partial \mathbb{D}) \to AP^B(\mathbb{R} \cup (\mathbb{R} + i\pi))$ of norm 1 such that for each $f \in SAP^B(\partial \mathbb{D})$ the function $f - L_{z_0}^B(E_s^B f) \in SAP^B(\partial \mathbb{D})$ is continuous and equal to 0 at $z_0$.

**Proof of Proposition.** According to Proposition 14.7.3, it suffices to define the required operator $E_s^B$ on the space of functions of the form $f = \sum_{l=1}^m b_l f_l$, where $b_l \in B$, $f_l \in SAP(\partial \mathbb{D})$. In this case we set

$$E_{z_0}^B(f) := \sum_{l=1}^m b_l E_{z_0}(f_l),$$

where $E_{z_0}$ is the operator from Proposition 12.2.1. Let $B_1$ denote the unit ball in $B^*$. Then according to Proposition 12.2.1 we have

$$\|E_{z_0}^B(f)\| = \sup_{z \in \partial \mathbb{D}} \|E_{z_0}^B(f)(z)\|_B = \sup_{z \in \partial \mathbb{D}, \varphi \in B_1} |\varphi(E_{z_0}^B(f)(z))| = \sup_{z \in \partial \mathbb{D}, \varphi \in B_1} \left| \sum_{l=1}^m \varphi(b_l) E_{z_0}(f_l)(z) \right| \leq \sup_{z \in \partial \mathbb{D}, \varphi \in B_1} \left| \sum_{l=1}^m \varphi(b_l) f_l(z) \right| = \|f\|.$$ 

This implies that $E_{s}^B$ is continuous and of norm 1 on a dense subspace of $SAP^B(\partial \mathbb{D})$. Moreover, for any function $f$ from this subspace we have (by Proposition 12.2.1), $f - L_{z_0}^B(E_{s}^B(f)) \in SAP^B(\partial \mathbb{D})$ is continuous and equal to 0 at $z_0$. Extending $E_{z_0}^B$ by continuity to $SAP^B(\partial \mathbb{D})$ we obtain the operator satisfying the required properties.

We make use of the functions $f_{-1,z_0}^B(x) := (E_{z_0}^B f)(x)$ and $f_{1,z_0}^B(x) := (E_{z_0}^B f)(x + i\pi)$, $x \in \mathbb{R}$, belonging to $AP^B(\mathbb{R})$ to define the left ($k = -1$) and the right ($k = 1$) mean values of $f \in SAP^B(\partial \mathbb{D})$ over $z_0$:

$$M^k_{z_0}(f) := M(f_{k,z_0}) \in B.$$
Using formulas similar to those of the scalar case we define the Bohr-Fourier coefficients $a_k^\lambda(f, z_0) \in B$ and the spectrum $\text{spec}^k_{z_0}(f)$ of $f$ over $z_0$. It follows straightforwardly from the properties of the spectrum of a $B$-valued almost periodic function on $\mathbb{R}$ that $\text{spec}^k_{z_0}(f)$ is at most countable.

By $SAP^B_\Sigma(S) \subset SAP^B(S)$ we denote the Banach algebra of semi-almost periodic functions $f$ with $\text{spec}^k_{z_0}(f) \subset \Sigma(s, k)$ for all $z_0 \in S$, $k \in \{-1, 1\}$. Note that

$$SAP^B_\Sigma(S) = SAP_\Sigma(S) \otimes B$$

(the proof follows easily from Definition 14.7.2, using an appropriate partition of unity on $\partial \mathbb{D}$, Theorem 10.0.2 and Proposition 11.0.4).

Also, a statement analogous to Theorem 12.3.2 holds for $SAP^B_\Sigma(S) \cap H^\infty_B(\mathbb{D})$. Namely, if $f \in SAP^B_\Sigma(S) \cap H^\infty_B(\mathbb{D})$, then

$$\text{spec}^1_{z_0}(f) = \text{spec}^{-1}_{z_0}(f) =: \text{spec}_{z_0}(f).$$

### 14.7.2 Proof of the theorem

We will need the following results.

Let $APC(T)$ denote the Banach algebra of functions $f : T \to \mathbb{C}$ uniformly continuous on $T$ and almost periodic on each horizontal line. We define $APC^B(T) := APC(T) \otimes B$.

The proof of the next statement is analogous to the proof of Lemma 14.6.4.

**Lemma 14.7.5.** Suppose that $f_1 \in AP^B(\mathbb{R})$, $f_2 \in AP^B(\mathbb{R} + i\pi)$. Then there exists a function $F \in APC^B(T)$ harmonic in the interior of $\Sigma$ whose boundary values are $f_1$ and $f_2$. Moreover, $F$ admits a continuous extension to the maximal ideal space $bT$ of $APH(T)$.

The proof of the next statement uses Lemma 14.7.5 and is very similar to the proof of Lemma 14.6.3.

**Lemma 14.7.6.** Let $z_0 \in S$. Suppose that $f \in SAP^B(-z_0, z_0)$. We put $f_k = f|_{\gamma_k(z_0, \pi)}$ and define on $\gamma_k(z_0, \pi)$

$$h_k := f_k \circ \varphi_{z_0}^{-1} \circ \text{Log}^{-1}, \quad k \in \{-1, 1\}.$$
Then for any $\epsilon > 0$ there exist $\delta_\epsilon \in (0, \pi)$ and a function $H \in APC^B(T)$ harmonic in the interior $T_0$ of $T$ such that

$$\sup_{z \in \gamma_k(z_0, \delta_\epsilon)} \| h_k(z) - H(z) \|_B < \epsilon, \quad k \in \{-1, 1\}.$$ 

**Definition 14.7.7.** We say that a bounded continuous function $f : \mathbb{D} \to B$ is almost-periodic near $z_0$ if there exist a circular neighbourhood $U_{z_0}$ (cf. Definition 14.6.5), and a function $\bar{f} \in APC^B(T)$ such that

$$f(z) = (L^B_{z_0} \bar{f})(z) := (\bar{f} \circ \Log \circ \varphi_{z_0})(z), \quad z \in U_{z_0}.$$ 

(14.7.19)

In the proof of Theorem 14.6.2 (see Lemmas 14.6.7, 14.6.9) we have established, cf. Theorem 13.2.2,

(1) Any scalar harmonic function $f$ on $\mathbb{D}$ almost periodic near $s$ admits a continuous extension $f_{z_0}$ to $(a^S)^{-1}(\bar{U}_{z_0}) \subset b^S(\mathbb{D})$ for some circular neighbourhood $U_{z_0}$.

(2) For any $z_0 \in S$ and any $g \in APH(T)$ the holomorphic function $\tilde{g} := L_s g$ on $\mathbb{D}$ almost periodic near $z_0$ is such that $\tilde{g}_{z_0} \circ i^{z_0}$ coincides with the extension of $g$ to $bT$.

More generally, Lemma 14.7.5, statements (1) and (2) and the fact that $AP^B(\mathbb{R}) = AP(\mathbb{R}) \otimes B$ (see Theorem 10.0.2 and Proposition 11.0.2) imply

(3) Any $B$-valued harmonic function $f$ on $\mathbb{D}$ almost periodic near $z_0$ admits a continuous extension $f^B_{z_0}$ to $(a^S)^{-1}(\bar{U}_{z_0}) \subset b^S(\mathbb{D})$ for some circular neighbourhood $U_{z_0}$.

(4) For any $z_0 \in S$ and any $g \in APH^B(T)$ the $B$-valued holomorphic function $\tilde{g} := L^B_s g$ on $\mathbb{D}$ almost periodic near $z_0$ is such that $\tilde{g}^{B}_{z_0} \circ i^{z_0}$ coincides with the extension of $g$ to $bT$.

**Lemma 14.7.8.** Let $f \in SAP^B_\Sigma(S) \cap H^\infty_B(\mathbb{D})$ and $z_0 \in \partial \mathbb{D}$. There is a bounded $B$-valued holomorphic function $\hat{f}$ on $\mathbb{D}$ almost periodic near $z_0$ such that for any $\epsilon > 0$ there is a circular neighbourhood $U_{k,\epsilon}$ of $z_0$ so that

$$\sup_{z \in U_{k,\epsilon}} \| f(z) - \hat{f}(z) \|_B < \epsilon.$$ 

Moreover, $\hat{f} = L^B_{z_0} \hat{f}$ for some $\hat{f} \in APH^B_\Sigma(z_0)(T)$. 

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**Proof of Lemma.** Assume, first, that \( z_0 \in S \). By Lemma 14.7.6, for any \( n \in \mathbb{N} \) there exist a number \( \delta_n \in (0, \pi) \) and a function \( H_n \in APC^B(T) \) harmonic on \( T_0 \) such that

\[
\sup_{z \in \gamma_k(z_0, \delta_n)} \| f_k(z) - H_n(z) \|_B < \frac{1}{n}, \quad k \in \{-1, 1\}. \quad (14.7.20)
\]

Using the Poisson integral formula for the bounded \( B \)-valued harmonic function \( f - h_n, h_n := L_{z_0} H_n := H_n \circ \text{Log} \circ \varphi_s \), on \( D \) we easily obtain from (14.7.20) that there is a circular neighbourhood \( V_{z_0; n} \) of \( z_0 \) such that

\[
\sup_{z \in V_{z_0; n}} \| f(z) - h_n(z) \|_B < \frac{2}{n}. \quad (14.7.21)
\]

According to (3) each \( h_n \) admits a continuous extension \( \hat{h}_n \) to \((aS)^{-1}(z_0) \cong bT\). Moreover, (14.7.21) implies that the restriction of the sequence \( \{\hat{h}_n\}_{n \in \mathbb{N}} \) to \((aS)^{-1}(z_0)\) forms a Cauchy sequence in \( C^B((aS)^{-1}(z_0)) \). Let \( \hat{h} \in C^B((aS)^{-1}(z_0)) \) be the limit of this sequence.

Further, for any functional \( \phi \in B^* \) the function \( \phi \circ f \in SAP(S) \cap H^\infty(D) \) and therefore admits a continuous extension \( f_\phi \) to \((aS)^{-1}(s)\) such that on \((i^{z_0})^{-1}((aS)^{-1}(z_0))\) the function \( f_\phi \circ i^{z_0} \) belongs to \( APH(T) \). Now, (14.7.21) implies that \( f_\phi = \phi \circ \hat{h} \) for any \( \phi \in B^* \). Then it follows from Theorem 10.0.2 and Proposition 11.0.4 that \( \hat{h} \circ i^{z_0} \in APH^B(T) \). Therefore by (4) we find a bounded \( B \)-valued holomorphic function \( \hat{f} \) on \( D \) of the same sup-norm as \( \hat{h} \) almost periodic near \( z_0 \) such that its extension to \((aS)^{-1}(z_0)\) coincides with \( \hat{h} \). Next, by the definition of the topology of \( bS(D) \), cf. Lemma 14.6.7(a), we obtain that for any \( \varepsilon > 0 \) there is a number \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
\sup_{z \in V_{z_0; n}} \| \hat{f}(z) - h_n(z) \|_B < \frac{\varepsilon}{2}.
\]

Now, choose \( n \geq N \) in (14.7.21) such that the right-hand side there is \(< \frac{\varepsilon}{2} \). For this \( n \) we set \( U_{z_0; \varepsilon} := V_{s; n} \). Then the previous inequality and (14.7.21) imply the required inequality

\[
\sup_{z \in U_{z_0; \varepsilon}} \| f(z) - \hat{f}(z) \|_B < \varepsilon.
\]

Further, if \( z_0 \notin S \), then, by definition, \( f|_{\partial D} \) is continuous at \( z_0 \). In this case as the function \( \hat{f} \) we can choose the constant \( B \)-valued function equal to \( f(z_0) \) on \( D \). Then the required result follows from the Poisson integral formula for \( f - \hat{f} \).
By definition, \( \hat{f} \) is determined by formula (14.7.19) with an \( \tilde{f} \in A\overline{PH}^B(T) \). Let us show that \( \tilde{f} \in A\overline{PH}^B_{\Sigma(z_0)}(T) \). To this end it suffices to prove that \( \varphi(\tilde{f}) \in A\overline{PH}_{\Sigma(z_0)}(T) \) for any \( \varphi \in B^* \).

Indeed, it follows from the last inequality that the extension of \( \varphi(f) \in SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) to \( (a^S_{\Sigma})^{-1}(z_0) \) coincides with \( \varphi(\tilde{f}) \). By the definition of spectrum of a semi-almost periodic function, this implies that \( \text{spec}_{z_0}(\varphi(\tilde{f})) \subset \Sigma(z_0) \).

Now, we are ready to prove Theorem 13.1.3.

The inclusion \( A^S_{\Sigma} \subset SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D}) \) follows from Example 12.3.4. Indeed, for \( s \in S \) assume that the holomorphic function \( e^{\lambda h} \in H^\infty(\mathbb{D}) \) is such that \( \Re al(h)|_{\partial \mathbb{D}} \) is the characteristic function of the closed arc going in the counterclockwise direction from the initial point at \( s \) to the endpoint at \( -s \) and such that \( \frac{\lambda}{\pi} \in \Sigma(s) \). Then Example 12.3.4 implies that \( e^{\lambda h} \in SAP(\{s,-s\}) \cap H^\infty(\mathbb{D}) \) and \( \text{spec}_s(e^{\lambda h}) = \{\frac{\lambda}{\pi}\} \). In particular, \( (z+s)e^{\frac{\lambda}{\pi} h} \in SAP_{\Sigma(\{s\})}(\{s\}) \cap H^\infty(\mathbb{D}) \), as required.

Let us prove the opposite inclusion.

(A) Consider first the case \( S = F \), where \( F = \{z_i\}_{i=1}^m \) is a finite subset of \( \partial \mathbb{D} \). Let \( f \in SAP^B(\Sigma_{\{s\}}) \cap H^\infty(\mathbb{D}) \). Then according to Lemma 14.7.8 there exists a function \( f_{z_1} \in A\overline{PH}^B_{\Sigma(z_1)}(T) \) such that the bounded \( B \)-valued holomorphic function \( g_{z_1} - f \), where \( g_{z_1} := f_{z_1} \circ \log \circ \varphi_{z_1} \), on \( \mathbb{D} \) is continuous and equals 0 at \( z_1 \).

Let us show that \( g_{z_1} \in A^{\{z_1,-z_1\}} \otimes B \). Since \( f_{z_1} \in A\overline{PH}^B_{\Sigma(z_1)}(T) \), by Theorem 10.0.2 and Proposition 11.0.4 it can be approximated in \( A\overline{PH}^B_{\Sigma(z_1)}(T) \) by finite sums of functions of the form \( be^{i\lambda z} \), \( b \in B \), \( \lambda \in \Sigma(z_1) \), \( z \in T \). In turn, \( g_{z_1} \) can be approximated by finite sums of functions of the form \( be^{i\lambda \log \circ \varphi_{z_1}} \), \( b \in B \). It is not difficult to see that \( e^{i\lambda \log \circ \varphi_{z_1}} \in A^{\{z_1,-z_1\}} \).
Hence, \( g_{z_1} \in A^{\{z_1,-z_1\}} \otimes B \). We define

\[
\hat{g}_{z_1} = \frac{g_{z_1}(z)(z + z_1)}{2z_1}.
\]

Then, since the function \( z \mapsto (z+z_1)/(2z_1) \in A(\mathbb{D}) \) and equals 0 at \(-z_1\), and \( g_{z_1} \in A^{\{z_1,-z_1\}} \otimes B \), the function \( \hat{g}_{z_1} \in A^{\{z_1\}} \otimes B \). Moreover, by the construction of \( \hat{g}_1 \) and the definition of the spectrum \( \hat{g}_{z_1} \in A^{\{s_1\}} \otimes B \). By definition, the difference \( \hat{g}_{z_1} - f \) is continuous and equal to zero.
at \( z_1 \). Thus,
\[
\hat{g}_{z_1} - f \in SAP^B_{\zeta(z_1)} (F \setminus \{ z_1 \}) \cap H^\infty_B (\mathbb{D}).
\]
We proceed in this way to get functions \( \hat{g}_{z_k} \in A_{\Sigma(z_k)} \otimes B \), \( 1 \leq k \leq m \), such that
\[
f - \sum_{k=1}^m \hat{g}_{s_k} \in A^B (\mathbb{D}),
\]
where \( A^B (\mathbb{D}) \) is the Banach space of \( B \)-valued bounded holomorphic functions on \( \mathbb{D} \) continuous up to the boundary. As in the scalar case using the Taylor expansion at 0 of functions from \( A^B (\mathbb{D}) \) one can easily show that
\[
A^B (\mathbb{D}) = A (\mathbb{D}) \otimes B.
\]
Therefore, \( f \in A^F \otimes B \).

(B) Let us consider the general case of \( S \subset \partial \mathbb{D} \) being an arbitrary closed set. Let \( f \in SAP^B_{\Sigma} (S) \cap H^\infty_B (\mathbb{D}) \). As follows from Lemma 14.7.8 and the arguments presented in part (A), given an \( \varepsilon > 0 \) there exist points \( z_k \in \partial \mathbb{D} \), functions \( f_k \in A_{\Sigma(z_k)} \otimes B \) and circular neighbourhoods \( U_{z_k} \) \( (1 \leq k \leq m) \) such that \( \{ U_{z_k} \}^m_{k=1} \) forms an open cover of \( \partial \mathbb{D} \setminus \{ z_k \}^m_{k=1} \)
\[
\| f(z) - f_k(z) \|_B < \varepsilon \quad \text{on} \quad U_{z_k}, \quad 1 \leq k \leq m. \tag{14.7.22}
\]
Since \( S \) is closed, for \( z_k \notin S \) we may assume that \( f_k \) is continuous in \( \overline{U}_{z_k} \). Let us define a \( B \)-valued 1-cocycle \( \{ c_{kj} \}^m_{k,j=1} \) on intersections of the sets in \( \{ U_{z_k} \}^m_{k=1} \) by the formula
\[
c_{kj}(z) := f_k(z) - f_j(z), \quad z \in U_{z_k} \cap U_{z_j}. \tag{14.7.23}
\]
Then (14.7.22) implies \( \sup_{k,j,z} \| c_{kj}(z) \|_B < 2\varepsilon \). Let \( A \in \cup_{k=1}^m U_{z_k} \) be an open annulus with outer boundary \( \partial \mathbb{D} \). Using the argument from the proof of Lemma 14.6.10 one obtains that if the width of the annulus is sufficiently small, then there exist \( B \)-valued functions \( c_i \) holomorphic on \( U_{z_i} \cap A \) and continuous on \( \overline{U}_{z_i} \cap \overline{A} \) satisfying
\[
\sup_{z \in U_{z_i} \cap A} \| c_i(z) \|_B \leq 3\varepsilon \tag{14.7.24}
\]
and such that
\[
c_i(z) - c_j(z) = c_{ij}(z), \quad z \in U_{z_i} \cap U_{z_j} \cap A. \tag{14.7.25}
\]
For such \( A \) let us define a function \( f_\varepsilon \) on \( \overline{A} \setminus \{ z_i \}^m_{i=1} \) by formulas
\[
f_\varepsilon(z) := f_i(z) - c_i(z), \quad z \in U_{z_i} \cap \overline{A}.
\]
According to (14.7.23) and (14.7.25), \( f_\varepsilon \) is a bounded continuous \( B \)-valued function on \( \bar{A} \setminus \{z_i\}_{i=1}^m \) holomorphic in \( A \). Furthermore, since \( c_i \) is continuous on \( \bar{U}_{z_i} \cap \bar{A} \), and \( f_i \in A^{\{z_i\} \otimes B} \) for \( s_i \in S \), and \( f_i \in A^B(\mathbb{D}) \) otherwise, \( f_\varepsilon|_{\partial D} \in SAP^B_{\{z_i\} \otimes B}(F) \), where \( F = \{z_i\}_{i=1}^m \cap S \). Also, from inequalities (14.7.22) and (14.7.24) we obtain

\[
\sup_{z \in A} \|f(z) - f_\varepsilon(z)\|_B < 4\varepsilon.
\] (14.7.26)

Next, we consider a 1-cocycle subordinate to a cover of the unit disk \( \mathbb{D} \) consisting of an open annulus having the same interior boundary as \( A \) and the outer boundary \( \{z \in \mathbb{C} : |z| = 2\} \), and of an open disk centered at 0 not containing \( A \) but intersecting it by a nonempty set. Resolving this cocycle one obtains a \( B \)-valued holomorphic function \( F_\varepsilon \) on \( D \) such that for an absolute constant \( \hat{C} > 0 \)

\[
\sup_{z \in \mathbb{D}} \|f(z) - F_\varepsilon(z)\|_B < \hat{C}\varepsilon
\]

and by definition \( F_\varepsilon \in SAP^B_{\{z_i\} \otimes H^\infty_B} \), where \( F = \{z_1, \ldots, z_m\} \cap S \). The latter inequality and part (A) of the proof show that the complex vector space generated by spaces \( A^F_{\{z_i\} \otimes B} \) for all possible finite subsets \( F \subset S \) is dense in \( SAP^B_{\{z_i\} \otimes H^\infty_B} \). Since by definition the closure of all such \( A^F_{\{z_i\} \otimes B} \) is \( A^S_B \), we obtain the required: \( SAP^B_{\{z_i\} \otimes H^\infty_B} = A^S_B \).

### 14.8 Proof of Theorem 13.2.2

Follows from Theorem 14.6.17 and the definition of spectra of a (holomorphic) semi-almost periodic function (cf. Sections 12.2 and 12.3).

### 14.9 Proofs of Theorems 13.2.3 and 13.2.4

#### 14.9.1 Proof of Theorem 13.2.3

Below we identify \( \iota(\mathbb{D}) \subset b^S_\Sigma(\mathbb{D}) \) with \( \mathbb{D} \), cf. Section 12.3.

By Theorem 13.2.2, the maximal ideal space \( b^S_\Sigma(\mathbb{D}) \) is \( (\bar{\mathbb{D}} \setminus S) \cup (\bigcup_{z_0 \in S} (a_\Sigma^{-1}(z_0) \circ i^z_{\Sigma}(T))) \) (here \( i^z_{\Sigma} : b_\Sigma(z_0)(T) \to (a_\Sigma^{-1}(z_0) \circ i^z_{\Sigma}(T)) \) is a homeomorphism). For each \( z_0 \in S \) one has the natural map \( \iota_{\Sigma(z_0)} : T \hookrightarrow b_{\Sigma(z_0)} \) (determined by evaluations at points of \( T \)). Also, the argument of the proof of Theorem 14.6.12 implies that the closure of \( \mathbb{D} \) in \( b^S_\Sigma(\mathbb{D}) \) contains (as a dense subset)
Thus in order to prove the theorem, it suffices to show that \( \iota_{\Sigma(z_0)}(T) \) is dense in \( b_{\Sigma(z_0)}(T) \) if and only if \( \Sigma(z_0) \) is a group.

We will use the following result.

**Theorem 14.9.1 ([RS]).** Suppose that \( \Gamma \) is the intersection of an additive subgroup of \( \mathbb{R} \) and \( \mathbb{R}_+ \). Then the image of the upper half-plane \( \mathbb{H}^+ \) in the maximal ideal space \( b_{\Gamma}(T) \) is dense.

Observe that in this case each element of \( A\mathcal{P}H_{\Gamma}(T) \) is extended to a holomorphic almost periodic function on \( \mathbb{H}^+ \) by means of the Poisson integral. Therefore the evaluations at points of \( \mathbb{H}^+ \) of the extended algebra determine the map \( \mathbb{H}^+ \to b_{\Gamma}(T) \) of the theorem.

First, assume that \( \Sigma(s) \) is a group. We have to show that \( \iota_{\Sigma(s)}(T) \) is dense in \( b_{\Sigma(s)}(T) \).

Suppose that this is wrong. Then there exists \( \xi \in b_{\Sigma(s)}(T) \) and a neighbourhood of \( \xi \)

\[
U(\lambda_1, \ldots, \lambda_m, \xi, \varepsilon) := \{ \eta \in b_{\Sigma(s)}(T) : |\eta(e^{i\lambda_k z}) - c_k| < \varepsilon, \ 1 \leq k \leq m \},
\]

where \( \lambda_1, \ldots, \lambda_m \in \Sigma(s), \ c_k := \xi(e^{i\lambda_k z}), \) such that \( U(\lambda_1, \ldots, \lambda_m, \xi, \varepsilon) \cap \text{cl} \left( \iota_{\Sigma(s)}(T) \right) = \emptyset \), cf. the proof of Theorem 11.0.9. Therefore,

\[
\max_{1 \leq k \leq m} |e^{i\lambda_k z} - c_k| \geq \varepsilon > 0 \quad \text{for all } z \in T. \tag{14.9.27}
\]

Without loss of generality we may assume that \( c_k \neq 0 \) and \( \lambda_k > 0 \), i.e.,

\[ e^{i\lambda_k z} - c_k \in A\mathcal{P}H_{\Sigma(s) \cap \mathbb{R}_+}(T). \]

(For otherwise we replace \( e^{i\lambda_k z} - c_k \) with \( e^{-i\lambda_k z} - c_k^{-1} \). Here \( e^{-i\lambda_k z} - c_k^{-1} \in A\mathcal{P}H_{\Sigma(s) \cap \mathbb{R}_+}(T) \) since \( \Sigma(s) \) is a group. Also, \( (14.9.27) \) will be satisfied, possibly with a different \( \varepsilon > 0 \).) Note that \( e^{i\lambda_k z} - c_k \) is not invertible in \( A\mathcal{P}H_{\Sigma(s)}(T) \), since \( \xi(e^{i\lambda_k z} - c_k) = 0 \). Therefore, since each function \( e^{i\lambda_k z} - c_k \) is periodic (with period \( 2\pi / \lambda_k \)), it has a zero in \( T \). Since solutions of the equation \( e^{i\lambda_k z} = c_k \) are of the form

\[
z_k = -\frac{i \ln |c_k|}{\lambda_k} + \frac{\text{Arg} c_k + 2\pi l}{\lambda_k}, \quad l \in \mathbb{Z},
\]

all zeros of \( e^{i\lambda_k z} - c_k \) belong to \( T \). Hence, in virtue of inequality \( (14.9.27) \), one has

\[
\max_{1 \leq k \leq m} |e^{i\lambda_k z} - c_k| \geq \tilde{\varepsilon} > 0 \quad \text{for all } z \in \mathbb{H}^+. \]
This implies, by Theorem 14.9.1, that there exist \( g_1, \ldots, g_m \in APH_{\Sigma(s) \cap \mathbb{R}_+}(\mathbb{H}_+) \) such that
\[
\sum_{k=1}^{m} g_k(z)(e^{i\lambda_k z} - c_k) = 1 \quad \text{for all} \quad z \in \mathbb{H}_+.
\]
In particular, the above identity holds on \( T \). This gives a contradiction with the assumption \( \xi(e^{i\lambda_k z} - c_k) = 0, \ 1 \leq k \leq m. \)

Now, assume that \( \Sigma(z_0) \) is not a group, i.e., it contains a non-invertible element \( \lambda_0 \). Suppose that \( \iota_{\Sigma(z_0)}(T) \) is dense in \( b_{\Sigma(z_0)}T \). Then, since the modulus of \( f_1 := e^{i\lambda_0 z} \) is bounded from below on \( T \) by a positive number, there exists \( g_1 \in APH_{\Sigma(z_0)}(T) \) such that \( f_1g_1 \equiv 1 \). Therefore, \( g_1 = e^{-i\lambda_0 z} \in APH_{\Sigma(z_0)}(T) \), i.e., \( -\lambda_0 \in \Sigma(z_0) \), a contradiction.

### 14.9.2 Proof of Theorem 13.2.4

For the proof we will need the following auxiliary result.

Let \( \Gamma \subset \mathbb{R} \) be a nontrivial additive semi-group. For a subset \( X \subset T \) by \( X_\infty \) we denote the set of limit points of \( \iota_{\Gamma}(X) \) in \( b_{\Gamma}(T) \setminus \iota_{\Gamma}(T) \).

**Lemma 14.9.2.** Let \( G \in C([0,1), T) \) be such that the closure of \( G([0,1)) \) in \( T \) is non-compact. Then the set \( G_\infty \) contains more than one element.

**Proof of Lemma.** If there exists a horizontal line \( \mathbb{R} + ic, \ 0 \leq c \leq \pi, \) such that
\[
\text{dist}_T(G(t), \mathbb{R} + ic) \to 0 \quad \text{as} \quad t \to 1-,
\]
then clearly \( (\mathbb{R} + ic)_\infty = G_\infty \). Moreover, \( (\mathbb{R} + ic)_\infty \) is infinite (e.g., it contains a subset homeomorphic to interval \([0,1])\). In the case that such a line does not exist, one can find two closed substrips \( T_1, T_2 \subset T, T_1 \cap T_2 = \emptyset \), such that the closures in \( T \) of both \( G([0,1)) \cap T_1 \) and \( G([0,1)) \cap T_2 \) are non-compact. Then \( (G([0,1)) \cap T_1)_\infty \) and \( (G([0,1)) \cap T_2)_\infty \) are nonempty, while \( (T_1)_\infty \cap (T_2)_\infty = \emptyset \). This implies the required statement.

Now, we are ready to prove the theorem. Suppose that for a continuous map \( F : [0,1] \to b_S^\Sigma(\mathbb{D}) \) the assertion of the theorem is not true. First, assume that there exists a point \( c \in [0,1) \) such that \( F(c) \in \mathbb{D} \setminus S \) but \( F([0,1]) \not\subset \mathbb{D} \setminus S \). Then, since \( b_S^\Sigma(\mathbb{D}) \setminus (\mathbb{D} \setminus S) \) is a compact set (here we naturally identify \( \mathbb{D} \setminus S \) we a subset of \( b_S^\Sigma(\mathbb{D}) \)), passing to a subinterval, if necessary, we may
assume without loss of generality that \( F[0, 1) \subset \overline{D} \setminus S \) and \( F(1) \in (a^{\Sigma})^{-1}(z_0) \) for some \( z_0 \in S \). Define

\[
G(t) := (\text{Log} \circ \varphi_s)(F(t)) \subset T, \quad t \in [0, 1)
\]

(cf. Example 12.1.3). Then \( G \) satisfies conditions of Lemma 14.9.2 for \( \Gamma = \Sigma(z_0) \). Next, consider an \( f \in \text{SAP}_{\Sigma}(S) \). According to Lemma 14.7.8 there exists a (unique) function \( f_{z_0} \in \text{APH}_{\Sigma(s)}(T) \) such that the difference \( f - F_s \), where \( F_{z_0} := f_{z_0} \circ \text{Log} \circ \varphi_{z_0} \), is continuous and equal to 0 at \( z_0 \). This yields

\[
\lim_{t \to 0} (f_{z_0}(G(t)) - f(F(t))) = 0.
\]

The latter implies that the set of limit points of \( F([0, 1]) \) in \( b^{\Sigma}_{b}(D) \setminus (\overline{D} \setminus S) \) is in one-to-one correspondence with the set of limit points \( G_{\infty} \) of \( \mu_{\Sigma(s)}(G([0, 1])) \) in \( b_{\Sigma(z_0)}(T) \setminus \mu_{\Sigma(z_0)}(T) \). By our assumption the set of limit points of \( F([0, 1]) \) in \( b^{\Sigma}_{b}(D) \setminus (\overline{D} \setminus S) \) consists of the point \( F(1) \). This contradicts the assertion of Lemma 14.9.2. Hence, in this case \( F([0, 1]) \subset \overline{D} \setminus S \).

In the second case, \( F([0, 1]) \subset b^{\Sigma}_{b}(D) \setminus (\overline{D} \setminus S) \). Let \( z_0 \in S \) be such that \( F([0, 1]) \cap (a^{\Sigma})^{-1}(s) \neq \emptyset \). Consider the continuous map \( \omega_{z_0} : b^{\Sigma}_{b}(D) \to b^{\Sigma}_{\Sigma(z_0)}(D) \) transpose to the embedding \( \text{SAP}_{\Sigma(z_0)}(\{z_0\}) \cap H^\infty(D) \to \text{SAP}_{\Sigma}(S) \cap H^\infty(D) \). According to the above argument, if \( \omega_{z_0} \circ F : [0, 1] \to b^{\Sigma}_{\Sigma(z_0)}(D) \) is such that \( (\omega_{z_0} \circ F)(c) \in \overline{D} \setminus \{z_0\} \) for some \( c \in [0, 1) \), then \( (\omega_{z_0} \circ F)([0, 1]) \subset \overline{D} \setminus \{z_0\} \) which contradicts the assumption \( F([0, 1]) \cap (a^{\Sigma})^{-1}(z_0) \neq \emptyset \). Thus

\[
(\omega_{z_0} \circ F)([0, 1]) \subset b^{\Sigma}_{\Sigma(z_0)}(D) \setminus (\overline{D} \setminus \{z_0\}) = (a^{\Sigma})^{-1}(z_0).
\]

This implies that \( F([0, 1]) \subset (a^{\Sigma})^{-1}(z_0) \).

**14.10 Proof of Theorem 13.2.7**

First, we prove that for a finite \( F \subset S \)

\[
K^F_{\Sigma|F} = \left( \bigcup_{z_0 \in F} i^{\Sigma}_{\Sigma|F}(c_{\Sigma(z_0)}(R) \cup c_{\Sigma(z_0)}(R + i\pi)) \right) \cup \partial D \setminus F. \tag{14.10.28}
\]

Indeed, since each point of \( \partial D \setminus F \) is a peak point for \( A(D) (\subset \text{SAP}_{\Sigma|F}(F) \cap H^\infty(D)) \), we have \( \partial D \setminus F \subset K^F_{\Sigma|F} \). Next, the closure of \( \partial D \setminus F \) in \( b^F_{\Sigma|F}(D) \) (the maximal ideal space of
SAP\varepsilon|_{F}(F) \cap H^\infty(\mathbb{D})) coincides with the right-hand side of (14.10.28), cf. the proof of Theorem 11.0.9. Thus the right-hand side of (14.10.28) is a subset of $K^F_{\Sigma|_{F}}$. Finally, Theorem 11.0.9 implies that for each function $f \in SAP\varepsilon|_{F}(F) \cap H^\infty(\mathbb{D})$ its modulus $|f|$ attains its maximum on the set in the right-hand side of (14.10.28). This gives us the required identity.

Since $SAP\varepsilon(S) \cap H^\infty(\mathbb{D})$ is generated by algebras $SAP\varepsilon|_{F}(F) \cap H^\infty(\mathbb{D})$, where $F \subset S$ is finite, the inverse limit of $\{K^F_{\Sigma|_{F}}; \omega \} \subset S; \# F < \infty$ of the corresponding Šilov boundaries coincides with $K^S_{\Sigma}$ (see Section 3.4 for notation). Now, one can easily show that the inverse limit of the family of sets in the right-hand sides of equations (14.10.28) coincides with

$$
\left( \bigcup_{z_0 \in S} \overline{i_S^\varepsilon(\mathbb{R})} \cup \overline{i_S^\varepsilon(z_0)}(\mathbb{R} + i\pi) \right) \cup \mathbb{D} \setminus S.
$$

This completes the proof of the theorem.

### 14.11 Proofs of Theorems 13.2.5, 13.3.1 and Corollary 13.2.6

#### 14.11.1 Proof of Theorem 13.2.5

(1) Consider first the case of $S$ being a finite subset of $\partial \mathbb{D}$. For $z_0 \in S$ we define

$$
U_1 := b^S(\mathbb{D}) \setminus (a^S)^{-1}(z_0).
$$

Let $U_2$ be the union of $(a^S)^{-1}(z_0)$ and a circular neighbourhood of $z_0$ whose closure is a proper subset of $\mathbb{D}$. Both $U_1$, $U_2$ are open in $b^S(\mathbb{D})$ and $U_1 \cap U_2 = U_2 \setminus (a^S)^{-1}(z_0)$ is the circular neighbourhood of $z_0$. Since $U_1 \cap U_2$ is contractible, one has $H^k(U_1 \cap U_2, \mathbb{Z}) = 0$, $k \geq 1$. Let us show that for any $k \in \mathbb{Z}$,

$$
H^k(U_2, \mathbb{Z}) \cong H^k(b^S(T), \mathbb{Z}). \quad (14.11.29)
$$

To this end consider a sequence $V_1 \supset V_2 \supset \ldots$ of circular neighbourhoods of $z_0$ such that $\cap_{k=1}^{\infty} V_k = \{z_0\}$ and $V_1 = U_1 \cap U_2$. We set

$$
\bar{U}_k := V_k \cup (a^S)^{-1}(z_0).
$$

Let $\iota^m : \bar{U}_m \rightarrow \bar{U}_l$, $m \geq l$, be the corresponding embedding. Then $(a^S)^{-1}(z_0)$ is the inverse limit of the family $\{\bar{U}_j; \iota\}_{j \in \mathbb{N}}$. It is well known (see, e.g., [Bre], Chapter II, Corollary 14.6)
that the direct limit of Čech cohomology groups $H^k(U, \mathbb{Z})$ with respect to this family gives $H^k((a_{\Sigma})^{-1}(z_0), \mathbb{Z})$. Note also that each $U_i$ is a deformation retract of $U_1 := U_2$. Thus the maps $i'_{1}$ induce isomorphisms $H^k(U_2, \mathbb{Z}) \cong H^k(U_1, \mathbb{Z})$, $l \in \mathbb{N}$. Since $(a_{\Sigma})^{-1}(z_0) \cong b_{\Sigma(z_0)}(T)$, these facts imply (14.11.29).

Further, consider the Mayer-Vietoris sequence corresponding to cover \{U_1, U_2\} of $b_{\Sigma}(\mathbb{D})$:

$$\cdots \to H^{k-1}(b_{\Sigma}(\mathbb{D}), \mathbb{Z}) \to H^k(U_1 \cap U_2, \mathbb{Z}) \to H^k(U_1, \mathbb{Z}) \oplus H^k(U_2, \mathbb{Z}) \to H^k(b_{\Sigma}(\mathbb{D}), \mathbb{Z}) \to \cdots$$

By the above results $H^k(U_1 \cap U_2, \mathbb{Z}) = 0$ and $H^k(U_2, \mathbb{Z}) \cong H^k(b_{\Sigma(z_0)} T, \mathbb{Z})$, $k \geq 1$. Therefore,

$$H^k(b_{\Sigma}(\mathbb{D}), \mathbb{Z}) = H^k(U_1, \mathbb{Z}) \oplus H^k(b_{\Sigma(z_0)}(T), \mathbb{Z}), \quad k \geq 1.$$  

Proceeding further inductively (i.e., applying similar arguments to $U_1$ etc.) and using the fact that $H^k(b_{\Sigma}(\mathbb{D}) \setminus S, \mathbb{Z}) = 0$, $k \geq 1$, we obtain that

$$H^k(b_{\Sigma}(\mathbb{D}), \mathbb{Z}) = \bigoplus_{z_0 \in S} H^k(b_{\Sigma(z_0)}(T), \mathbb{Z}).$$

Now, if $S \subset \partial\mathbb{D}$ is an arbitrary closed subset, then since $b_{\Sigma}(\mathbb{D})$ is the inverse limit of $b_{\Sigma_{F'}}(\mathbb{D})$ for all possible finite subsets $F \subset S$, by the cited result in [Bre] $H^k(b_{\Sigma}(\mathbb{D}), \mathbb{Z})$ is the direct limit of $H^k(b_{\Sigma_{F'}}(\mathbb{D}), \mathbb{Z})$. Based on the case considered above we obtain that this limit is isomorphic to $\bigoplus_{z_0 \in S} H^k(b_{\Sigma(z_0)}(T), \mathbb{Z})$.

This proves the first part of the theorem.

(2) As is shown in [Br1], if $\Gamma \subset \mathbb{R}_+$ or $\Gamma \subset \mathbb{R}_-$, then $b_T(T)$ is contractible. Therefore under hypotheses of the theorem $H^k(b_{\Sigma(z_0)}(T), \mathbb{Z}) = 0$ for all $z_0 \in S$. The required result now follows from (1), i.e., $H^k(b_{\Sigma}(\mathbb{D}), \mathbb{Z}) = 0$ for all $k \geq 1$.

Further, according to [BrS] the connectedness of $b_{\Sigma}(\mathbb{D})$ and the topological triviality of any complex vector bundle of a finite rank over $b_{\Sigma}(\mathbb{D})$ are sufficient for projective freeness of $SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})$.

Clearly $b_{\Sigma}(\mathbb{D})$ is connected. For otherwise, according to the Shilov theorem on idempotents, see [Sh], $SAP_{\Sigma}(S) \cap H^\infty(\mathbb{D})$ contains a function $f$ not equal identically to 0 or 1 on $\mathbb{D}$ such that $f^2 = f$, a contradiction.
Next, we show that any finite rank complex vector bundle $\xi$ over $b^S_D(\mathbb{D})$ is topologically trivial.

Since $b^S_D(\mathbb{D})$ is the inverse limit of the system $\{b^F_{\\Sigma \cap F}(\mathbb{D}) ; \omega\}_{F \subset S ; \# F < \infty}$, see Section 3.4, $\xi$ is isomorphic (as a topological bundle) to pullback to $b^S_D(\mathbb{D})$ of a bundle on some $b^F_{\\Sigma \cap F}(\mathbb{D})$ with $F \subset S$ finite, see, e.g., [ES] and [Hu]. Therefore it suffices to prove the statement for $S \subset \partial \mathbb{D}$ being a finite subset. In this case, for each $z_0 \in S$ by the contractibility of $(a^S_{\Sigma})^{-1}(z_0) \cong b_{\Sigma(z_0)}(T)$ (see [Br1]) we have that the restriction of $\xi$ to $(a^S_{\Sigma})^{-1}(z_0)$ is topologically trivial. Using a finite open cover $\{U_i\}_{1 \leq i \leq m}$ of $(a^S_{\Sigma})^{-1}(s)$ such that $\xi|_{U_i} \cong U_i \times \mathbb{C}^n$, $n = \text{rank}_\mathbb{C} \xi$, for each $i$, we extend (by the Urysohn lemma) global continuous sections $t_j : (a^S_{\Sigma})^{-1}(s) \to \xi$, $1 \leq j \leq n$, determining the trivialization of $\xi$ over $(a^S_{\Sigma})^{-1}(z_0)$ to each $U_i$. Then using a continuous partition of unity subordinate to a finite refinement of $\{U_i\}_{1 \leq i \leq m}$ we glue together these extensions to get global continuous sections $\tilde{t}_j$, $1 \leq j \leq n$, of $\xi$ on a neighbourhood $U_s$ of $(a^S_{\Sigma})^{-1}(z_0)$ in $b^S_D(\mathbb{D})$ such that $\tilde{t}_j|_{(a^S_{\Sigma})^{-1}(z_0)} = t_j$ for each $j$. Since sections $t_j$, $1 \leq j \leq n$, are linearly independent at each point of $(a^S_{\Sigma})^{-1}(z_0)$, diminishing, if necessary, $U_s$ we obtain that sections $\tilde{t}_j$, $1 \leq j \leq n$, are linearly independent at each point of $U_s$. Thus $\xi$ is topologically trivial on $U_s$. Also, by the definition of the topology on $b^S_D(\mathbb{D})$ without loss of generality we may assume that $U_s \setminus (a^S_{\Sigma})^{-1}(z_0)$ is a circular neighbourhood of $z_0$.

Suppose that $S = \{z_1, \ldots, z_k\}$. Let us cover $b^S_D(\mathbb{D})$ by sets $U_j := U_{z_j}$, $1 \leq j \leq k$, described above and by $U_0 := \mathbb{D} \setminus V$, where $V \subset \bigcup_{j=1}^k U_{z_j}$ and $V \cap U_{z_j}$ is a circular neighbourhood of $z_j$ distinct from $U_{z_j} \setminus (a^S_{\Sigma})^{-1}(z_j)$, $1 \leq j \leq k$. Since $U_0$ is contractible, $\xi|_{U_0}$ is topologically trivial. Using trivializations of $\xi$ on $U_j$, $0 \leq j \leq k$, we obtain that $\xi$ is defined by a 1-cocycle $\{c_{ij}\}$ with values in $GL_n(\mathbb{C})$ defined on intersections $U_i \cap U_j$, $0 \leq i < j \leq k$. In turn, by the definition of sets $U_j$, there is an acyclic cover $\{\tilde{U}_j\}_{j=0}^k$ of $\mathbb{D}$ such that $(a^S_{\Sigma})^{-1}(\tilde{U}_j) = U_j$, $0 \leq j \leq k$. Thus there exists a cocycle $\{\tilde{c}_{ij}\}$ on $\{\tilde{U}_j\}_{j=0}^k$ such that $\tilde{c}_{ij} \circ a^S_{\Sigma} = c_{ij}$ for all $i, j$. This cocycle determines a continuous vector bundle $\tilde{\xi}$ on $\mathbb{D}$ trivial on each $\tilde{U}_i$, $0 \leq i \leq k$, such that $(a^S_{\Sigma})^*\tilde{\xi} = \xi$. Since $\mathbb{D}$ is contractible, $\tilde{\xi}$ is topologically trivial. Hence $\xi$ is topologically trivial as well.

The proof of the theorem is complete.
14.11.2 Proof of Corollary 13.2.6

Let $G \subset \mathbb{R}$ be an additive subgroup. We denote by $APC_G(T) \subset APC(T)$ the algebra of uniformly continuous almost periodic functions on $T$ having their spectrum in $G$. Here the spectrum of a function in $APC(T)$ is the union of the spectra of its restrictions to each horizontal line in $T$ (see [Bes]). The vector space of functions $\sum_{j=1}^k c_j(y)e^{i\lambda_j x}$, $x + iy \in T$, $c_j \in C([0, \pi])$, $\lambda_j \in G$, $k \in \mathbb{N}$, is dense in $APC_G(T)$ and, hence, the maximal ideal space $M(APC_G(T))$ of $APC_G(T)$ is homeomorphic to $b_G(\mathbb{R}) \times [0, \pi]$. On the other hand, $APH_G(T) \subset APC_G(T)$ and the extension of $APH_G(T)$ to $M(APC_G(T))$ separates the points of $M(APC_G(T))$. Since the image of $T$ in $b_G(T)$ is dense (see the proof of Theorem 13.2.3), the latter implies that $b_G(T) \cong M(APC_G(T))$. Hence, taking $G := \Sigma(z_0)$, $z_0 \in S$, we obtain

$$H^k(b_{\Sigma(z_0)}(T), \mathbb{Z}) \cong H^k(b_{\Sigma(z_0)}(\mathbb{R}), \mathbb{Z}).$$

Since $b_{\Sigma(z_0)}(\mathbb{R})$ is a compact connected abelian group, the required statements follow from the remark before the formulation of the corollary, and Theorem 13.2.5 (1).

14.11.3 Proof of Theorem 13.3.1

In what follows we assume that the elements of a uniform algebra are defined on its maximal ideal space via Gelfand transform.

We will need the following result.

**Lemma 14.11.1.** Assume that a set-valued map $\Sigma$ as in Section 12.2 is defined on $\{-z_0, z_0\}$, and $f \in SAP_{\Sigma}(\{-z_0, z_0\}) \cap H^\infty(\mathbb{D})$. Consider the function

$$H_{z_0}f(z) := f\left(\frac{z + z_0}{2}\right), \quad z \in \mathbb{D}.$$

Then $H_{z_0}f \in SAP_{\Sigma_{\{z_0\}}}(\{z_0\}) \cap H^\infty(\mathbb{D})$ and

$$[(i_{\Sigma_{\{z_0\}}} \circ \iota_{\Sigma(z_0)})^* H_{z_0}f](z) = [(i_{\Sigma}^z \circ \iota_{\Sigma(z_0)})^* f](z - \ln 2), \quad z \in T, \quad (14.11.30)$$

see Theorem 13.2.2.

This result states that $H_{z_0} : SAP_\Sigma(\{-z_0, z_0\}) \cap H^\infty(\mathbb{D}) \to SAP_{\Sigma_{\{z_0\}}}(\{z_0\}) \cap H^\infty(\mathbb{D})$ is a bounded linear operator which induces under the identification of the fibre $(a^z_{\Sigma_{\{z_0\}}})^{-1}(z_0)$
with \( b_{\Sigma(z_0)}(T) \) by \( i_{\Sigma(z_0)}^0 \) the map \( APH_{\Sigma(z_0)}(T) \rightarrow APH_{\Sigma(\emptyset)}(T) \) defined by \( h(z) \mapsto h(z - \ln 2) \), \( z \in T, h \in APH_{\Sigma(z_0)}(T) \).

**Proof of Lemma.** Clearly \( H_{z_0}f \) is holomorphic on \( \mathbb{D} \) and continuous on \( \partial \mathbb{D} \setminus \{z_0\} \). Let us consider the function \( g(z) := [(H_{z_0}f) \circ (\Log \circ \varphi_{z_0})^{-1}](z) - [f \circ (\Log \circ \varphi_{z_0})^{-1}](z - \ln 2), z \in T \).

Next, we have

\[
\frac{(\Log \circ \varphi_{z_0})^{-1}(z) + z_0}{2} - (\Log \circ \varphi_{z_0})^{-1}(z - \ln 2) = \frac{z_0 e^{2z}}{(2i + e^z)(4i + e^z)} \rightarrow 0
\]

as \( \Re(z) \rightarrow -\infty \). Since by definition of algebra \( SAP_{\Sigma}(\{-z_0, z_0\}) \cap H^\infty(\mathbb{D}) \) the function \( f \circ (\Log \circ \varphi_{z_0})^{-1} \) is uniformly continuous on \( T \), from the last expression we obtain that \( g(z) \rightarrow 0 \) as \( \Re(z) \rightarrow -\infty \). But \( \Re(z) \rightarrow -\infty \) if and only if \( (\Log \circ \varphi_{z_0})^{-1}(z) \rightarrow z_0 \). Therefore the function \( g \circ \Log \circ \varphi \) is continuous in a circular neighbourhood of \( z_0 \) and equals 0 at \( z_0 \). Since the pullback of the function \( [f \circ (\Log \circ \varphi_{z_0})^{-1}](z - \ln 2), z \in T \), by \( (\Log \circ \varphi_{z_0})^{-1} \) belongs to \( SAP_{\Sigma}(\{-z_0, z_0\}) \cap H^\infty(\mathbb{D}) \) (it is obtained as the composition of \( f \) with a Möbius transformation preserving points \( -z_0 \) and \( s \)), the function \( H_{z_0}f \in SAP_{\Sigma(\{z_0\})}(\{z_0\}) \cap H^\infty(\mathbb{D}) \). Now, the identity (14.11.30) follows from the fact that \( (g \circ \Log \circ \varphi)(z_0) = 0 \) by the definition of \( i_{\Sigma(z_0)}^0 \).

**Corollary 14.11.2.** Let \( f \in G^n_{\Sigma(z_0)}(T) \). Consider function

\[
F := H_{z_0}[Kf(\Log \circ \varphi_{z_0})], \quad \text{where} \quad Kf(z) := f(z - \ln 2), \quad z \in T.
\]

Then \( F \in G^n_{\Sigma(\{z_0\})}(\{z_0\}) \) and

\[
(i_{\Sigma(\{z_0\})}^0 \circ \iota_{\Sigma(z_0)})^* F = f.
\]

**Proof of Corollary.** The fact that \( F \in G^n_{\Sigma(\{z_0\})}(\{z_0\}) \) follows from the proof of Lemma 14.11.1 because the pullback by \( \Log \circ \varphi_s \) maps \( APH_{\Sigma(z_0)}(T) \) isometrically into \( SAP(\{-z_0, z_0\}) \cap H^\infty(\mathbb{D}) \) so that \( \spec_{z_0} \) of each of the pulled back function is a subset of \( \Sigma(z_0) \). The second statement of the corollary follows directly from (14.11.30) because \( (i_{\Sigma(\{z_0\})}^0 \circ \iota_{\Sigma(z_0)})^* (h \circ \Log \circ \varphi_{z_0}) = h \) for any \( h \in APH_{\Sigma(z_0)}(T) \) by Theorem 13.2.2 (1).

We are ready to prove the theorem. First we will consider the case \( S = \{z_1, \ldots, z_m\} \) a finite subset of \( \partial \mathbb{D} \).
By the definition of connected components of $GL_n(A)$, where $A$ is a Banach algebra, the map $f \mapsto ((i_{\Sigma(1)}^z \circ \iota_{\Sigma(z_1)})^* f, \ldots, (i_{\Sigma(z_m)}^z \circ \iota_{\Sigma(z_m)})^* f)$, $f \in G^m_\Sigma(S)$, induces a homomorphism

$$
\Psi_S : [G^m_\Sigma(S)] \to \bigoplus_{z_i \in S} [G^m_\Sigma(z_i)(T)].
$$

We will show that $\Psi_S$ is an isomorphism.

Suppose that $(g_1, \ldots, g_m) \in \bigoplus_{z_i \in S} G^m_\Sigma(z_i)(T)$ represents an element $[g] \in \bigoplus_{z_i \in S} [G^m_\Sigma(z_i)(T)]$. Then according to Corollary 14.11.2 for an element

$$
\tilde{g} := H_{z_i} [K(Log \circ \varphi_{z_1})^* g_1] \cdots H_{z_m} [K(Log \circ \varphi_{z_m})^* g_m] \in G^m_\Sigma(S)
$$

and each $l \in \{1, \ldots, m\}$ we have

$$
(i_{\Sigma(z_l)}^z \circ \iota_{\Sigma(z_l)})^* \tilde{g} = c_{1l} \cdots c_{l-1l} \cdot c_{ll} \cdot c_{l+1l} \cdots c_{ml},
$$

where every $c_{jl}$ is an invertible matrix. Since the matrix-function on the right-hand side is homotopic to $g_l$, for the element $[\tilde{g}] \in [G^m_\Sigma(S)]$ representing $\tilde{g}$, we obtain $\Psi_S([\tilde{g}]) = [g]$. Hence $\Psi_S$ is a surjection.

To prove that $\Psi_S$ is an injection, we require a modification of the construction of Corollary 14.11.2. So suppose that $F_{z_l} = H_{s_j}[K(Log \circ \varphi_{z_l})^* f]$, where $f \in G^m_\Sigma(z_l)(T)$. By the definition, $F_{z_l}(s_j)$, $j \neq l$, are well-defined invertible matrices. Let $M$ be a matrix-function with entries from $A(\mathbb{D})$ such that $M(z_j) = \Log(F_{z_l}(z_j))$, $j \neq l$, and $M(z_l) = 0$. (Here the logarithm of an invertible matrix $c$ is a matrix $\tilde{c}$ such that $\exp(\tilde{c}) = c$.) Then we have

1. $\tilde{F}_{z_l} := F_{z_l} \cdot \exp(-M) \in G^m_\Sigma(z_l)(\{z_l\})$ and satisfies

$$
(i_{\Sigma(z_l)}^z \circ \iota_{\Sigma(z_l)})^* \tilde{F}_{z_l} = f \quad \text{and} \quad \tilde{F}_{z_l}(z_j) = I_n, \quad j \neq l
$$

(here $I_n$ is the unit $n \times n$ matrix);

2. $\tilde{F}_{z_l}$ is homotopic to $F_{z_l}$.

Statement (2) follows from the fact that $\exp(-M)$ clearly belongs to the connected component containing $I_n$. 229
Now, suppose that $f \in G_n^\Sigma(S)$ is such that every matrix-function $g_l := (i_{\Sigma(z_l)}^\Sigma \circ \iota_{\Sigma(z_l)})^* f$, $l \in \{1, \ldots, m\}$, belongs to the connected component of $G_n^{\Sigma(z_l)}(T)$ containing the unit matrix $I_n$, (i.e., $[f] \in \ker(\Psi_S)$). We set

$$G_{zl} := H_{zl}[K(\Log \circ \varphi_{zl})^* g_l], \quad G := \prod_{1 \leq l \leq m} G_{zl}, \quad \tilde{G} := \prod_{1 \leq l \leq m} \tilde{G}_{zl},$$

where each $\tilde{G}_{zl}$ is constructed from $G_{zl}$ as $\tilde{F}_{zl}$ from $F_{zl}$.

According to property (1),

$$(i_{\Sigma(z_l)}^\Sigma \circ \iota_{\Sigma(z_l)})^* \tilde{G} = g_l, \quad \text{for} \quad l \in \{1, \ldots, m\}.$$ 

Moreover, property (2) implies that $\tilde{G}$ is homotopic to $G$. Observe also that each $G_{zl}$ is homotopic to $I_n$ (because $g_l$ satisfies this property and so the required homotopy is defined as the image of the homotopy between $g_l$ and $I_n$ under the continuous map $H_{zl} \circ K \circ (\Log \circ \varphi_{zl})^*$) and therefore $G$ and $\tilde{G}$ are homotopic to $I_n$. Finally, according to our construction $f \cdot \tilde{G}^{-1}$ is an invertible matrix with entries from $A(D)$. Since $\hat{D}$ is contractible, each such a matrix is homotopic to $I_n$. These facts imply that $f$ is homotopic to $I_n$, that is $[f] = 1 \in [G_n^\Sigma(S)]$, where $[f]$ stands for the connected component containing $f \in G_n^\Sigma(S)$.

So $\Psi_S$ is an injection which completes the proof of the theorem in the case of a finite $S$.

To prove the result in the general case we require the following lemma.

**Lemma 14.11.3.** For every $f \in G_n^\Sigma(S)$ there exists $\tilde{f} \in G_n^{\Sigma|[F]}(F)$, where $F \subset S$ is finite, such that $\tilde{f} \in [f]$.

**Proof of Lemma.** Let $M_n^\Sigma(S)$ be the Banach algebra of $n \times n$ matrix-functions with entries in $SAP_\Sigma(S) \cap H^\infty(\hat{D})$ equipped with the norm $\|h\| := \sup_{z \in \hat{D}} \|h(z)\|_2$, $h \in M_n^\Sigma(S)$, where $\| \cdot \|_2$ is the $\ell_2$ operator norm on the complex vector space $M_n(\C)$ of $n \times n$ matrices. According to Theorem 13.1.3 with $B = \C$, $f$ can be approximated in $M_n^\Sigma(S)$ by functions from $M_n^{\Sigma|[F]}(F)$ for some finite subsets $F \subset S$. Since the connected component $[f]$ is open (because $G_n^\Sigma(S) \subset M_n^\Sigma(S)$ is open), the latter implies the required statement: there exists $\tilde{f} \in G_n^{\Sigma|[F]}(F)$, where $F \subset S$ is finite, such that $\tilde{f} \in [f]$. □
This lemma implies that \([G_{\Sigma}^n(S)]\) is the direct limit of the family \(\{[G_{\Sigma}^n(F)]; F \subset S, \#F < \infty\}\). Therefore we can define a homomorphism

\[
\Psi_S : \left[ G_{\Sigma}^n(S) \right] \rightarrow \bigoplus_{z_0 \in S} \left[ G_{\Sigma(z_0)}^n(T) \right]
\]

as the direct limit of homomorphisms \(\Psi_F\) described above. Then \(\Psi_S\) is an isomorphism because each \(\Psi_F\) is an isomorphism on each image.

This proves the first statement of the theorem.

The second statement follows from the fact that if \(\Sigma(z_0) \subset \mathbb{R}^+\) or \(\mathbb{R}^\circ\), then the maximal ideal space \(b_{\Sigma(z_0)}(T)\) of algebra \(APH_{\Sigma(z_0)}(T)\) is contractible, see [Br7]. Then the result of Arens [Ar] implies that \(G_{\Sigma(z_0)}^n(T)\) is connected and therefore \([G_{\Sigma(z_0)}^n(T)]\) is trivial. From here and the first statement of the theorem we obtain that \([G_{\Sigma}^n(S)]\) is trivial, or equivalently, that \(G_{\Sigma}^n(S)\) is connected.
Bibliography


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