Automorphic L-functions and their applications to Number Theory

by

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Graduate Department of Mathematics
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Abstract

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The main part of the thesis is applications of the Strong Artin conjecture to number theory. We have two applications. One is generating number fields with extreme class numbers. The other is generating extreme positive and negative values of Euler-Kronecker constants.

For a given number field \( K \) of degree \( n \), let \( \hat{K} \) be the normal closure of \( K \) with \( \text{Gal}(\hat{K}/\mathbb{Q}) = G \). Let \( \text{Gal}(\hat{K}/K) = H \) for some subgroup \( H \) of \( G \). Then,

\[
L(s, \rho, \hat{K}/\mathbb{Q}) = \frac{\zeta_K(s)}{\zeta(s)}
\]

where \( \text{Ind}_{H}^{G} 1_{H} = 1_{G} + \rho \).

When \( L(s, \rho) \) is an entire function and has a zero-free region \([\alpha, 1] \times [-\log N]^2, (\log N)^2\]\ where \( N \) is the conductor of \( L(s, \rho) \), we can estimate \( \log L(1, \rho) \) and \( L'(L)/(1, \rho) \) as a sum over small primes:

\[
\log L(1, \rho) = \sum_{p \leq (\log N)^k} \lambda(p)p^{-1} + O_{l,k,\alpha}(1)
\]

\[
\frac{L'}{L}(1, \rho) = -\sum_{p \leq x} \frac{\lambda(p)\log p}{p} + O_{l,x,\alpha}(1).
\]

where \( 0 < k < \frac{16}{1-\alpha} \) and \( (\log N)^{\frac{16}{1-\alpha}} \leq x \leq N^{\frac{4}{1-\alpha}} \). With these approximations, we can study extreme values of class numbers and Euler-Kronecker constants.

Let \( \mathcal{R}(n, G, r_1, r_2) \) be the set of number fields of degree \( n \) with signature \((r_1, r_2)\) whose normal closures are Galois \( G \) extension over \( \mathbb{Q} \). Let \( f(x, t) \in \mathbb{Z}[t][x] \) be a parametric
polynomial whose splitting field over $\mathbb{Q}(t)$ is a regular $G$ extension. By Cohen’s theorem, most specialization $t \in \mathbb{Z}$ corresponds to a number field $K_t$ in $\mathcal{R}(n, G, r_1, r_2)$ with signature $(r_1, r_2)$ and hence we have a family of Artin L-functions $L(s, \rho, t)$. By counting zeros of L-functions over this family, we can obtain L-functions with the zero-free region above.

In Chapter 1, we collect the known cases for the Strong Artin conjecture and prove it for the cases of $G = A_4$ and $S_4$. We explain how to obtain the approximations of $\log(1, \rho)$ and $\frac{L'}{L}(1, \rho)$ as a sum over small primes in detail. We review the theorem of Kowalski-Michel on counting zeros of automorphic L-functions in a family.

In Chapter 2, we exhibit many parametric polynomials giving rise to regular extensions. They contain the cases when $G = C_n$, $3 \leq n \leq 6$, $D_n$, $3 \leq n \leq 5$, $A_4, A_5, S_4, S_5$ and $S_n$, $n \geq 2$.

In Chapter 3, we construct number fields with extreme class numbers using the parametric polynomials in Chapter 2.

In Chapter 4, we construct number fields with extreme Euler-Kronecker constants also using the parametric polynomials in Chapter 2.

In Chapter 5, we state the refinement of Weil’s theorem on rational points of algebraic curves and prove it.

The second topic in the thesis is about simple zeros of Maass L-functions. We consider a Hecke Maass form $f$ for $SL(2, \mathbb{Z})$. In Chapter 6, we show that if the L-function $L(s, f)$ has a non-trivial simple zero, it has infinitely many simple zeros. This result is an extension of the result of Conrey and Ghosh [14].
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Chapter 1

Artin L-functions

1.1 The strong Artin Conjecture

Let $\rho$ be an $n$-dimensional irreducible representation of $G = Gal(K/Q)$ for an Galois extension $K/Q$. Artin conjectured that the Artin L-function $L(s, \rho)$ is a holomorphic function on the complex plane and it is called the Artin Conjecture. Langlands conjectured that $L(s, \rho)$ actually is automorphic, i.e. $L(s, \rho) = L(s, \pi)$ for a cuspidal automorphic representation $\pi$ of $GL(n)/Q$. Since a cuspidal automorphic L-function is entire, Langlands’s prediction implies the Artin Conjecture so it is called the strong Artin Conjecture.

For several cases, the strong Artin Conjecture is known to be true. For the one-dimensional representations, the strong Artin conjecture is true. By Langlands and Tunnell, the strong Artin Conjecture is true for 2-dimensional representations of solvable groups. When the projective image of two-dimensional odd representation $\rho$ is isomorphic to $A_5$, then $L(s, \rho)$ is automorphic by the recent work of Khare and Wintenberger [35].

Here we prove that the strong Artin Conjecture is true for the three-dimensional representations for $Gal(K/Q) = S_4$ and $A_4$.

Proposition 1.1. Let $K$ be a number field such that $G = Gal(K/Q)$ is isomorphic
to $S_4$ or $A_4$. Then, for any irreducible representation $\rho$ of $G = \text{Gal}(K/\mathbb{Q})$, its Artin L-function $L(s, K/\mathbb{Q}, \rho)$ is automorphic. Especially, the strong Artin Conjecture for $S_4$ and $A_4$ case is true.

Before proving Proposition 1.1, we introduce another proposition. We find an idea of this proof from Wang [70].

**Proposition 1.2.** Let $\phi$ be an octahedral (resp, a tetrahedral) representation of $G_\mathbb{Q}$ and let $G$ denote $\phi(G_\mathbb{Q})$. Then $G$ is isomorphic to $(GL(2, F_3) \times C_{2m})/\{\pm (I, 1)\}$ (resp, $(SL(2, F_3) \times C_{2m})/\{\pm (I, 1)\}$) where $Z(G) \cong C_{2m}$, the cyclic group of order $2m$.

Hence each irreducible representation $\Phi$ of $G$ can be expressed uniquely as $(\Phi_0, \mu)$, where $\Phi_0 = \Phi \mid_{GL(2, F_3)}$ is an irreducible representation of $GL(2, F_3)$ (resp, $SL(2, F_3)$) and $\mu = \Phi \mid_{C_{2m}}$ is a character of $C_{2m}$ and such that $\Phi_0(-I) = \mu(-1)I$. Furthermore, each such pair $(\Phi_0, \mu)$ gives an irreducible representation of $G$.

**Proof.** First $S_4$ case.

We consider the 2-dimensional representation $\Psi$ of $GL(2, F_3) = SL(2, F_3) \rtimes < a >$ defined by

\[
\Psi\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Psi\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -\sqrt{-2} & -1 + \sqrt{-2} \end{pmatrix}.
\]

This induces an embedding $\Sigma$ of $PGL(2, F_3) = PSL(2, F_3) \rtimes < \bar{a} >$ into $PGL(2, \mathbb{C})$, where $\bar{a}$ is the projective image of $a$ in $PGL(2, F_3)$.

Now let $\overline{\phi}$ be the projective representation from an octahedral representation $\phi$. Then $\overline{\phi}$ factors through a Galois group isomorphic to $PGL(2, F_3)$. Via isomorphisms, we identify it with $PGL(2, F_3)$ and $\overline{\phi}(PGL(2, F_3))$ with $\Sigma(PGL(2, F_3))$.

The image of $G' = [G, G]$, a commutator subgroup of $G$, in $PGL(2, \mathbb{C})$ is $\Sigma(PSL(2, F_3))$ because $[S_4, S_4] = A_4$. Since $\text{det}(G') = 1$ and $A_4$ has no irreducible 2-dimensional representation, we have $Z(G') = \{\pm I\}$. So $G'$ is $\Psi(PSL(2, F_3))$ which is isomorphic to the central extension $SL(2, F_3)$ of $PSL(2, F_3)$ by $\{\pm I\}$. Then $\{G' \times C_{2m}\}/\{\pm (I, 1)\}$, where $C_{2m}$ is
Since $PGL(2, F_3) = PSL(2, F_3) \rtimes < \bar{a} >$, $\Sigma(PGL(2, F_3)) = \Sigma(PSL(2, F_3)) \rtimes \Sigma(< \bar{a} >)$. Since the image in $PGL(2, \mathbb{C})$ is determined by modulo $Z(G) \cong C_{2m}$, $G$ also contains $\Psi(< a >)$. So finally we have $G \cong \{G' \times C_{2m}\}/ \pm (I, 1) \times \Psi(< a >) \cong \{(\Psi(PGL(2, F_3)) \rtimes \Psi(< a >)) \times C_{2m}\}/ \pm (I, 1) \cong (GL(2, F_3) \times C_{2m})/\{\pm (I, 1)\}$.

Second $A_4$ case.

If $det(g) = 1$ for all $g \in G$, then we have $G \cong SL(2, F_3)$. If not, we claim that $G' \cong Q_8$, where $Q_8$ is a quaternion group. To see this, its image in $PGL(2, \mathbb{C})$ is isomorphic to $V$, the Klein 4-group. Since $V$ can not be embedded into $GL(2, \mathbb{C})$ with property $det = 1$, $G'$ is isomorphic to a central extension of $V$ by $\mathbb{Z}/2\mathbb{Z}$. There are 3 non-split central extensions of $V$ by $\mathbb{Z}/2\mathbb{Z}$: $C_4 \times C_2, D_8$ and $Q_8$. Since $C_4 \times C_2$ is abelian, it should be excluded. And if $D_8 = \{r^4 = 1, s^2 = 1, srs = r^3\}$ is embedded into $GL(2, \mathbb{C})$ with property $det = 1$, it induce a contradiction. Because $det(s) = 1, s^2 = I$ implies $s = -I$, we have $r^2 = I$. So it is a contradiction.

Since $SL(2, F_3) = H \rtimes K$ where $H \cong Q_8$ and $K = < b >$ for some $b$ of order 3, by the similar argument above, we have $G \cong (SL(2, F_3) \times C_{2m})/\{\pm (I, 1)\}$

The other part of the proposition is easily followed.

Proof of Proposition 1.1:

Since $Gal(K/Q) \cong S_4$, using an embedding of $S_4$ into $PGL(2, \mathbb{C})$, we have a projective representation $\overline{\omega} : G_{Q} \rightarrow PGL(2, \mathbb{C})$. Since $H^2(G_{Q}, \mathbb{C}^*) = \{1\}$, there exists a lifting $\omega$ of $\overline{\omega}$. Then by Proposition 4, $\omega$ factor through $Gal(\overline{K}/Q)$ which is isomorphic to $(GL(2, F_3) \times C_{2m})/\{\pm (I, 1)\}$ for some $m \geq 1$.

By Proposition 1.2, the irreducible representations of $Gal(\overline{K}/Q)$ are suitable twists of irreducible representations of $GL(2, F_3)$. Since every irreducible representation of $GL(2, F_3)$ is automorphic (See Kim [37] for relevant references), so is that of $Gal(\overline{K}/Q)$. Since $(\chi_1, 1), (\chi'_1, 1), (\chi_2, 1), (\chi_3, 1), (\chi'_3, 1)$ are irreducible representations of $Gal(\overline{K}/Q)$ which factor through $S_4$, Proposition 1.1 follows. The proof of $A_4$ case is essentially the
Remark 1.3. Proposition 1.1 is well-known to the experts, at least implicitly, under the assumption of the existence of the double cover $\tilde{K}/K$ where $Gal(\tilde{K}/\mathbb{Q})$ is $GL(2, F_3)$ and it is stated in Kim ([37], p.100). What is new is the observation that $K$ always has a central extension $\tilde{K}$ such that $Gal(\tilde{K}/\mathbb{Q}) \cong (GL(2, F_3) \times C_{2m})/\pm(I, 1)$ and so every 3-dimensional representation attached to $K$ is equivalent to a twist of a symmetric square of 2 dimensional representation of $GL(2, F_3)$.

Recently, Calegari obtained the modularity of $S_5$ Galois representations for a special case.

Theorem 1.4 (Calegari [5]). Let $K/\mathbb{Q}$ be a quintic extension with $Gal(\tilde{K}/\mathbb{Q}) = S_5$. Furthermore, we assume that

1. Complex conjugation in $Gal(\tilde{K}/\mathbb{Q}) = S_5$ has conjugacy class $(12)(34)$.

2. The extension $\tilde{K}/\mathbb{Q}$ is unramified at 5 and the Frobenius element $Frob_5$ has conjugacy class $(12)(34)$.

If $\rho : Gal(\tilde{K}/\mathbb{Q}) \rightarrow GL_n(\mathbb{C})$ be an irreducible representation of dimension 4 or 6, then $\rho$ is modular.

Remark 1.5. The modularity of the 4 dimensional representation $\rho$ is considered by Y. Zhang [72] in his PhD thesis. He observed that $\rho$ is equivalent to a character twist of Asai lift of $\sigma$, where $\sigma$ is an icosahedral representation over $F$, the quadratic subextension over $\mathbb{Q}$. Calegari also proved it and used the result of S. Sasaki [57] on the modularity of $\sigma$ to prove the modularity of $\rho$.

1.2 Approximation of $\log L(1, \rho)$ and $\frac{L'}{L}(1, \rho)$

When an Artin L-function $L(s, \rho)$ has a "good" zero-free region, we can approximate $\log L(1, \rho)$ and $\frac{L'}{L}(1, \rho)$ as a sum over small primes. The following proposition by Daieda
is crucial for these two approximations.

**Proposition 1.6** (Daieda, Corollary 4 in [18]). Let $F/Q$ be a finite Galois extension and let $\rho$ be an $l$-dimensional complex representation of $\text{Gal}(F/Q)$ of conductor $N$. Let $6/7 < \alpha < 1$. If $L(s, \rho)$ is entire and is free from zeros in the rectangle $[\alpha, 1] \times [-({\log N})^2, ({\log N})^2]$ and $N$ is sufficiently large, then

$$\frac{1}{2\pi i} \int_{(2)} \frac{L'}{L} (s + u, \rho) \Gamma(s) x^s ds - \frac{L'}{L}(u, \rho) \ll_l \frac{x^2}{(1-\alpha)^2 \sqrt{N}} + \frac{({\log N})^2}{(1-\alpha)^3 x^{(1-\alpha)/8}}$$

for $1 \leq u \leq 3/2$ and $x \geq 1$.

As an application of Proposition 1.6, Daieda approximates $\log L(1, \rho)$. We copy the proof in [18] for the sake of completeness. Let

$$L(s, \rho) = \prod_p L(s, \rho)_p = \sum_{n=1}^{\infty} \lambda(n) n^{-s}, \quad L(s, \rho)_p = \prod_{i=1}^{n} (1 - \alpha_i(p)p^{-s})^{-1}.$$  

Then $\lambda(p) = \sum_{i=1}^{n} \alpha_i(p)$, and $|\lambda(p)| \leq n$.

By Mellin inversion of $\Gamma(s)$ and logarithmic derivative of $L(s, \rho)$,

$$\frac{1}{2\pi i} \int_{(2)} \frac{L'}{L} (s + u, \rho) \Gamma(s) x^s ds = -\sum_{i=1}^{l} \sum_{p} \log p \sum_{k=1}^{\infty} \alpha_i(p) k p^{-k u} e^{-p^k/x}.$$  

Substitute this to Proposition 1.6 and integrate from $u = 1$ to $u = 3/2$. Then we have

$$\sum_{p} \lambda(p) p^{-1} e^{-p^k/x} - \log L(1, \rho) + \log L(3/2, \rho) \ll_l \frac{x^2}{(1-\alpha)^2 \sqrt{N}} + \frac{({\log N})^2}{(1-\alpha)^3 x^{(1-\alpha)/8}}.$$  

Here we used the fact that the terms for $k \geq 2$ converge absolutely.

If $y < x$ then,

$$\sum_{p \leq y} p^{-1} (1 - e^{-p^k/x}) < 1, \quad \sum_{p > x^2} p^{-1} e^{-p^k/x} \ll 1 \quad \text{and} \quad \sum_{y \leq p \leq x^2} p^{-1} e^{-p^k/x} = \log \left( \frac{2 \log x}{\log y} \right) + O(1).$$  

Take $x = (\log N)^{16/(1-\alpha)}$ and $y = (\log N)^k$ with $0 < k < 16/(1-\alpha)$. Since $\log L(3/2, \rho) \ll 1$, we have

$$\log L(1, \rho) = \sum_{p \leq (\log N)^k} \lambda(p) p^{-1} + O_{l,k,\alpha}(1)$$  

and it is summarized as follows.
Proposition 1.7 (Daïleda, Proposition 2 in [18]). Let \( L(s, \rho) \) and \( N \) be as above. Let \( \frac{6}{7} < \alpha < 1 \). Suppose that \( L(s, \rho) \) is zero-free in the rectangle \([\alpha, 1] \times [-\log N, (\log N)^2] \).

If \( N \) is sufficiently large, then for any \( 0 < k < \frac{16}{1 - \alpha} \),

\[
\log L(1, \rho) = \sum_{p \leq (\log N)^k} \lambda(p)p^{-1} + O_{L,k,\alpha}(1).
\]

Now we want to approximate \( \frac{L'}{L}(1, \rho) \). Set \( u = 1 \) in Proposition 1.6. Then

\[
-\frac{L'}{L}(1, \rho) + \frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s + 1, \rho)x^s \Gamma(s) ds \ll \frac{x^2}{(1 - \alpha)^2 \sqrt{N}} + \frac{(\log N)^2}{(1 - \alpha)^3 x^{(1 - \alpha)/2}}. \tag{1.8}
\]

By taking logarithmic derivative of \( L(s, \rho) \) and the Mellin inversion of \( \Gamma(s) \), we have

\[
-\frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s + 1, \rho) \Gamma(s) x^s ds = \sum_{i=1}^{l} \sum_{p} \log p \sum_{k=1}^{\infty} \alpha_i(p) p^{-k} e^{-p/x}.
\]

Since the terms for \( k \geq 2 \) converge absolutely, we only need to estimate

\[
\sum_{p} \lambda(p) \frac{\log p}{p} e^{-p/x}.
\]

Let \( x \) be a constant with \( (\log N)^{\frac{16}{1 - \alpha}} \leq x \leq N^{\frac{\alpha}{2}} \). Then the error term in (1.8) is \( O_{l, x, \alpha}(1) \). On the other hand,

\[
\sum_{p \leq x} \frac{\log p}{p} (1 - e^{-p/x}) < 1, \quad \sum_{p > x} \frac{\log p}{p} e^{-p/x} \ll 1.
\]

Hence we obtain an approximation of \( \frac{L'}{L}(1, \rho) \) as a sum over a short interval and it is summarized as follows:

Proposition 1.9. Suppose that \( L(s, \rho) \) is entire and free from zeros in the rectangle \([\alpha, 1] \times [-\log N, (\log N)^2] \). If \( N \) is sufficiently large, then for any constant \( x \) with

\( (\log N)^{\frac{16}{1 - \alpha}} \leq x \leq N^{\frac{\alpha}{2}} \),

\[
\frac{L'}{L}(1, \rho) = -\sum_{p \leq x} \frac{\lambda(p) \log p}{p} + O_{x, \alpha}(1).
\]
Due to lack of GRH, we cannot use Proposition 1.7 and 1.9 directly. We extend the result of Kowalski-Michel [39] to isobaric automorphic representations of $GL(n)$.

Let $n = n_1 + \cdots + n_r$, and let $S(q)$ be a set of isobaric representations $\pi^i = \pi_1^i \boxplus \cdots \boxplus \pi_r^i$, where $\pi_j^i$ is a cuspidal automorphic representation of $GL(n_j)/\mathbb{Q}$ and satisfies the Ramanujan-Petersson conjecture at the finite places for each $i$. We assume that $\pi_j^i \not\cong \pi_j^k$ for $i \neq k$. Moreover, $S(q)$ holds the following conditions:

1. There exists $e > 0$ such that for $\pi^i = \pi_1^i \boxplus \cdots \boxplus \pi_r^i \in S(q)$, $\text{Cond}(\pi_1^i) \cdots \text{Cond}(\pi_r^i) \leq q^e$;

2. There exists $d > 0$ such that $|S(q)| \leq q^d$.

3. The $\Gamma$ factors of $\pi_j^i$ are of the form $\prod_{k=1}^{n_j} \Gamma(\frac{s}{2} + \alpha_k)$, where $\alpha_k \in \mathbb{R}$.

Let, for $\alpha \geq \frac{3}{2}$, $T \geq 2$,

$$N(\pi^i; \alpha, T) = |\{\rho : L(\rho, \pi^i) = 0, \text{Re}(\rho) \geq \alpha, |\text{Im}(\rho)| \leq T\}|.$$

Here zeros are counted with multiplicity. Clearly, $N(\pi^i; \alpha, T) = N(\pi_1^i; \alpha, T) + \cdots + N(\pi_r^i; \alpha, T)$.

**Theorem 1.10.** For some $B \geq 0$,

$$\sum_{\pi^i \in S(q)} N(\pi^i; \alpha, T) \ll T^B q^{c_0 \frac{1-\alpha}{1+\alpha}}.$$

One can choose any $c_0 > c'_0$, where $c'_0 = 5n' + \frac{d}{2}$ and $n' = \max\{n_i\}_{1 \leq i \leq r}$.

**Proof.** Let $S(q)_j$ be the set of the cuspidal automorphic representations consisting of the $j$-th component of $\pi^i$. Since $\pi_j^i \not\cong \pi_j^k$ for $i \neq k$, $|S(q)_j| = |S(q)|$ for all $j = 1, 2, \cdots, r$.

Then clearly, $\text{Cond}(\pi_i) \leq q^e$ and $|S(q)| \leq q^d$. So

$$\sum_{\pi^i \in S(q)} N(\pi^i; \alpha, T) = \sum_{\pi^i \in S(q)} \sum_{i=j}^r N(\pi_j^i; \alpha, T) = \sum_{j=1}^r \sum_{\pi_j^i \in S(q)} N(\pi_j^i; \alpha, T).$$
Now we apply the result of Kowalski-Michel [39] to the inner sum. They assumed that the Gamma factors of $\pi^i_j$ are the same. However, the assumption is used only to obtain the convexity bound (Lemma 10 of [39]), and our $\Gamma$-factors provide the same convexity bound. Hence our result follows.

\[ \square \]

**Remark 1.11.**

1. In the following, we apply the above result to a family of Artin $L$-functions. In this case, the $\Gamma$-factors are a product of $\Gamma\left(\frac{s}{2}\right)$ and $\Gamma\left(\frac{s+1}{2}\right)$.

2. We consider a very specific Galois representation of a Galois group $G$. If $G$ is a symmetric or alternating group, the representation of our concern is irreducible. If $G$ is a dihedral or cyclic group, it is not irreducible any longer. In that case, we will differentiate them by computing their Artin conductors.
Chapter 2

Regular extensions and Parametric polynomials

This chapter is based on [6], [7] and [8]. We follow closely [6], [7] and [8].

A finite extension $E$ of the rational function field $\mathbb{Q}(t)$ is called regular if $\mathbb{Q} \cap E = \mathbb{Q}$.

Suppose $f(x,t)$ is an irreducible polynomial of degree $n$, and gives rise to a regular Galois extension over $\mathbb{Q}(t)$ with the Galois group $G$. Let $K_t$ be a field obtained by adjoining to $\mathbb{Q}$ a root of $f(x,t)$ with a specialization $t \in \mathbb{Z}$ and let $\hat{K}_t$ be the Galois closure of $K_t$. Let $C$ be any conjugacy class of $G$. Serre observed the following important fact, regarding distribution of Frobenius elements in a regular Galois extension ([63], page 45).

**Theorem 2.1.** There is a constant $k > 0$ depending on $f(x,t)$ such that for any prime $p \geq k$, there is $t_C \in \mathbb{Z}$ so that for any $t \equiv t_C \pmod{p}$, $p$ is unramified in $\hat{K}_t/\mathbb{Q}$, and $\text{Frob}_p \in C$.

We want to construct regular extensions $E$ over $\mathbb{Q}(t)$ as a splitting field of $f(x,t) \in \mathbb{Z}[t][x]$ over $\mathbb{Q}(t)$. Let $\mathcal{K}(n,G,r_1,r_2)$ be the set of number fields of degree $n$ with signature $(r_1,r_2)$ whose normal closures are $G$ Galois extension over $\mathbb{Q}$. Assume that $f(x,t)$ of degree $n$ gives rise to an $G$ Galois extension over $\mathbb{Q}(t)$ which is also regular. By Hilbert’s irreducibility theorem, for infinitely many specialization $t \in \mathbb{Z}$, we have number fields
of degree $n$ whose normal closures are $G$ Galois extensions over $\mathbb{Q}$. Moreover, Cohen obtained a quantitative version of Hilbert’s irreducibility theorem. His theorem is valid in a very general setting. We paraphrase it into our special case. For $f(x,t) \in \mathbb{Z}[t][x]$, we define the height $|f|$ of $f$ to be the maximum of absolute values of the integral coefficients unless this quantity is less than 8, in which case put $|f| = 8$ and let $\|f\| = \log |f|$.

Theorem 2.2 (Cohen [12], Theorem 2.1). Let $f(x,t) \in \mathbb{Z}[t][x]$ be a non-zero polynomial of total degree not exceeding $n$ in the indeterminate $x$ with Galois group $G$ over $\mathbb{Q}(t)$. Then, provided $N > |f|^c$, the number of $\alpha \in \{t \in \mathbb{Z}||t| \leq N\}$ for which $G(\alpha)$, the Galois group of $f(x,\alpha)$ over $\mathbb{Q}$, differs from $G$ does not exceed

$$\|f\|^c/3 N^{1/2} \log N$$

where $c = c(n)$. Indeed, if actually $N > \exp(c\|f\|^2)$, then the number of exceptional $\alpha$ does not exceed $cN^{1/2} \log N$.

By Cohen’s theorem, once we find a polynomial $f(x,t)$ giving rise to a regular $G$ Galois extension, we also have number fields $K \in \mathcal{K}(n,G,r_1,r_2)$ for some suitable signature $(r_1,r_2)$.

Remark 2.3. To apply Theorem 2.2, the total degree of $f(x,t)$ should not exceed the degree of $x$. In section 2.1.3, the total degree of the simplest quintic polynomial is 6 which is bigger than 5. However, we don’t need Theorem 2.2 in the case because the Galois group of $f(x,\alpha)$ is $C_5$ for all $\alpha \in \mathbb{Z}$.

Also we have to compute an upper bound of regulators of number fields we constructed when we estimate the class numbers. In many cases, it is crucial to know the locations of roots of the polynomial $f(x,t)$ to determine independence of units and find an upper bound of regulators. We introduce the following lemma from [52],[60].

Lemma 2.4. Let $f$ be a polynomial of degree $m$ and $f(\alpha) \neq 0$, $f'(\alpha) \neq 0$. Then for every circle $C$ passing through $\alpha$, $\alpha - \frac{mf'(\alpha)}{f(\alpha)}$, at least one root of $f$ is inside of $C$, and one root outside of $C$. 
2.1 Cyclic groups

Let \( f(x, t) = x^n + a_1(t)x^{n-1} + \cdots + a_{n-1}(t)x \pm 1 \) be an irreducible polynomial over \( \mathbb{Q}(t) \) such that \( a_i(t) \in \mathbb{Z}[t] \). Suppose \( f(x, t) \) gives rise to a cyclic extension over \( \mathbb{Q}(t) \), and if \( t \in \mathbb{Z} \), it gives rise to a totally real extension over \( \mathbb{Q} \). For each integer \( t > 0 \), let \( K_t \) be the cyclic extension over \( \mathbb{Q} \). Let \( \text{Gal}(K_t/\mathbb{Q}) = \{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\} \). Let \( \theta \) be a root of \( f(x, t) \). Then \( \theta, \sigma(\theta), \ldots, \sigma^{n-1}(\theta) \) are roots of \( f(x, t) \).

We show that if \( n = p \) is a prime, \( \sigma(\theta), \ldots, \sigma^{p-1}(\theta) \) form independent units, and the regulator of \( K_t \) is small. By definition, the regulator of \( \mathbb{Z}[\theta] \) is

\[
R = |\det(\log |\sigma^{i+j}(\theta)|)_{1 \leq i, j \leq p-1}|.
\]

**Theorem 2.5.** (1) \( R \neq 0 \), and (2) \( R \ll (\log t)^{p-1} \).

**Proof.** By Lemma 5.26 in [71], we have

\[
R = \frac{1}{p} \prod_{\chi \neq 1} \left( \sum_{i=0}^{p-1} \chi(\sigma^i) \log |\sigma^i(\theta)| \right),
\]

where the product runs over the nontrivial characters of \( \text{Gal}(K_t/\mathbb{Q}) \).

Since \( t^{-c} \ll \sigma^i(\theta) \ll t^d \) for some \( c, d > 0 \) depending only on \( f(x, t) \), \( |\log |\sigma^i(\theta)|| \ll \log t \). Hence (2) follows.

Since \( \theta \cdot \sigma(\theta) \cdots \sigma^{p-1}(\theta) = \pm 1 \),

\[
\log |\theta| + \log |\sigma(\theta)| + \cdots + \log |\sigma^{p-1}(\theta)| = 0.
\]

Hence we write

\[
\sum_{i=0}^{p-1} \chi(\sigma^i) \log |\sigma^i(\theta)| = \sum_{i=1}^{p-1} (\chi(\sigma^i) - 1) \log |\sigma^i(\theta)|.
\]

Since \( p \) is a prime, by [19], \( \sigma(\theta), \ldots, \sigma^{p-1}(\theta) \) are multiplicatively independent. Hence \( \log |\sigma(\theta)|, \ldots, \log |\sigma^{p-1}(\theta)| \) are linearly independent over \( \mathbb{Q} \). By Baker’s theorem [2], they are linearly independent over \( \overline{\mathbb{Q}} \).
Since $\chi(\sigma^i)$ are roots of unity and $\chi \neq 1$, one of $\chi(\sigma^i) - 1$ is not zero. Hence
\[ \sum_{i=1}^{p-1} (\chi(\sigma^i) - 1) \log |\sigma^i(\theta)| \neq 0. \]

**Remark 2.6.** If $n$ is not a prime, we still have $R \ll (\log t)^{n-1}$. However, $R = 0$ for a composite number $n$. We show this for simplest quartic and sextic fields.

### 2.1.1 Simplest Cubic Fields

It was Shanks [64] who introduced Simplest cubic fields parameterized by
\[ f(x, t) = x^3 - tx^2 - (t + 3)x - 1, \]
for $t \in \mathbb{Z}^+$. Its discriminant is $g(t)^2$ with $g(t) = t^2 + 3t + 9$. If $g(t)$ is square-free, then $d_{K_t} = g(t)^2$ and $g(t)$ is the conductor of two non-trivial characters for the cubic cyclic extension.

### 2.1.2 Simplest Quartic Fields

Consider totally real cyclic quartic fields $K_t$ generated by a root of
\[ f(x, t) = x^4 - tx^3 - 6x^2 + tx + 1, \quad t \in \mathbb{Z}^+. \]
Here $\text{Disc}(f(x, t)) = 4(t^2 + 16)^3$. This polynomial is studied by Lazarus. We refer to his PhD thesis [41] for the detail.

We can express the 4 roots of $f(x, t)$ explicitly.
\[ \theta_{1,2,3,4} = \pm \frac{\sqrt{t^2 + 16} + \sqrt{t^2 + 16} \pm t}{2\sqrt{2}} \pm \frac{\sqrt{t^2 + 16}}{4} + \frac{t}{4}, \quad t \neq 0, 3 \]
where the second and third ambiguous signs agree.

Let $\theta_1$ be the largest root by choosing $+$ for all signs. The Galois group action on the roots is given by
\[ \sigma : \theta_j \mapsto \frac{\theta_j - 1}{\theta_j + 1} = \theta_{j+1}, \quad j = 1, 2, 3, 4. \]
Now $\theta_1, \theta_2 = \sigma(\theta_1)$ and $\theta_3 = \sigma^2(\theta_1)$ are not multiplicatively independent. The regulator

$$R = \frac{1}{4} \prod_{\chi \neq 1} \left( \sum_{i=0}^{3} \chi(\sigma^i) \log |\sigma^i(\theta)| \right)$$

vanishes because the term corresponding to $\chi(\sigma) = e^{\pi i}$ is zero.

It is known that $\theta_1, \theta_2$ and $\epsilon_t$ are independent units where $\epsilon_t$ is the fundamental units of $\mathbb{Q}(\sqrt{t^2 + 16})$ (See p.10 in [41]). When $t$ is even, we can find $\epsilon_t$.

**Proposition 2.7** (Lazarus). When $t$ is even, $\epsilon_t$ is given by

$$\epsilon_t = \begin{cases} 
\frac{t/2 + \sqrt{(t/2)^2 + 4}}{2}, & t \equiv 2 \mod 4 \\
\frac{1 + \sqrt{5}}{2}, & t = 8 \\
(t/4) + \sqrt{(t/4)^2 + 1}, & \text{otherwise.}
\end{cases}$$

We replace $t$ in $f(x, t)$ by $2t$ because it is convenient to consider only even $t$. Then

$$f(x, t) = x^4 - 2tx^3 - 6x^2 + 2tx + 1$$

with $\text{disc}(f(x, t)) = 2^8(t^2 + 4)^3$. When $t^2 + 4$ is square-free, the field discriminant $d_{K_t}$ equals $2^4(t^2 + 4)^3$. $K_t$ has the unique quadratic subfield $M_t = \mathbb{Q}(\sqrt{t^2 + 4})$ and we have $Ind_{C_4}^{C_4} 1_{C_2} = 1 + \chi^2$ where $\chi$ the generator of the group of characters for $C_4$. Hence the Artin conductor $f(\chi^2)$ of $\chi^2$ equals $t^2 + 4$ when $t^2 + 4$ is square-free. Since $d_{K_t} = f(\chi)f(\chi^2)f(\chi^3)$ and $\chi^3 = \overline{\chi}$, we have $f(\chi) = f(\chi^3) = 2^2(t^2 + 4)$.

Hence by Proposition 2.7 and the value of $\theta_1$ and $\theta_2$, when $t^2 + t$ is square-free, the regulator $R_{K_t}$ is

$$R_{K_t} = O(\log^3 t) \text{ for } t^2 + 4 \text{ square-free.}$$

and since $d_{K_t} = 2^4(t^2 + 4)^3$, the regulator is $O((\log d_{K_t})^3)$ for $t^2 + 4$ square-free.

From the root formula for $f(x, t)$, it is clear that $f(x, t)$ gives rise to a $C_4$ regular extension over $\mathbb{Q}(t)$. 

{}
2.1.3 Simplest Quintic Fields

Emma Lehmer [42] introduced a family of quintic polynomials \( f(x,t) \) for \( t \in \mathbb{Z} \):

\[
f(x,t) = x^5 + t^2 x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 + (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2 + (t^3 + 4t^2 + 10t + 10)x + 1.
\]

Here \( \text{disc}(f(x,t)) = (t^3 + 5t^2 + 10t + 7)^2(t^4 + 5t^3 + 15t^2 + 25t + 25)^4 \).

It is easy to show that \( f(x,t) \) is irreducible for all \( t \in \mathbb{Z} \) when we observe it modulo 2. And it is also known that the zeros of \( f(x,t) \) generate a cyclic extension \( K_t \) of degree 5 over \( \mathbb{Q} \). Let \( G = \text{Gal}(K_t/\mathbb{Q}) \) be the Galois group and \( \sigma \) be a generator of \( G \) given by

\[
\sigma(\theta) = \frac{(t + 2) + t\theta - \theta^2}{1 + (t + 2)\theta}
\]

for a root \( \theta \) of \( f(x,t) \). For the detail, we refer to [59]. Also it is obvious that the Galois groups of \( f(x,t) \) over \( \mathbb{Q}(t) \) and over \( \overline{\mathbb{Q}}(t) \) are both \( C_5 \) generated by \( \sigma \). Hence \( f(x,t) \) gives rise to a regular \( C_5 \) extension over \( \mathbb{Q}(t) \). Also, Smith [67] showed that \( f(x,t) \) gives rise to a \( C_5 \) regular extension.

Schoof and Washington studied these simplest quintic fields when \( P_t = t^4 + 5t^3 + 15t^2 + 25t + 25 \) is a prime number. When \( P_t \) is a prime, then the zeros of \( f(x,t) \) form a fundamental system of units in \( K_t \). Gaál and Pohst extended this result for square-free \( P_t \) (see the proof of Theorem 3.5 in [59]).

In this case, by Theorem 2.5, \( R_{K_t} \ll (\log t)^4 \). It also follows from [59]: Let \( U \) denote the group of units generated by the zeros modulo \( \{\pm 1\} \). Define \( i_{\theta} = [O_{K_t}^*/\{\pm 1\} : U] \). Schoof and Washington [59] showed that \( i_{\theta} \leq 11 \) if \( |t + 1| \geq 20 \) and

\[
R = |\det(\log|\sigma^{i+j}(\theta)|)_{1 \leq i,j \leq 4}| \leq \left( 71 + \frac{36}{\log |t + 1|} \right) \log^4 |t + 1|.
\]

Jeannin [33] found the prime factorization of \( P_t \).

**Theorem 2.9** (Jeannin). The number \( P_t \) is written in a unique way: \( P_t = 5^c q^5 \prod_{i=1}^{n} p_i^{x_i}, c \in \{0,2\}, q \in \mathbb{N}, p_i \text{ distinct primes}, x_i \in [1,4] \). So the conductor of \( K_t \) is \( f_t = 5^c \prod_{i=1}^{n} p_i \).
Especially if $P_t$ is cubic-free, then $P_t = 5^c \prod_{i=1}^{n} p_i^{x_i}$ and $x_1 = 1$ or 2 and

$$t^4 \ll P_t \leq 5^c (\prod_{i=1}^{n} p_i)^2 \leq (D_{K_t})^2.$$  

Hence for cubic-free $P_t$, when we combine all these arguments, we have

$$R_{K_t} \ll \log^4(D_{K_t}) \quad (2.10)$$

### 2.1.4 Simplest Sextic Fields

It was Gras [22] who introduced the simplest sextic polynomial $f(x, t)$ first, given by

$$f(x, t) = x^6 - \frac{t - 6}{2} x^5 - \frac{5 + 6}{4} t^2 x^4 - 20 t^3 + \frac{5 - 6}{4} t^2 x + \frac{t + 6}{2} x + 1$$

and discriminant of $f(x, t)$ is $\frac{3^6}{2^7} (t^2 + 108)^5$.

Let $K_t = \mathbb{Q}(\theta)$, where $\theta$ is a root of $f(x, t)$. She showed the following properties:

1. If $t \in \mathbb{Z} - \{0, \pm 6, \pm 26\}$, then $f(x, t)$ is irreducible in $\mathbb{Q}[X]$, and $K_t$ is a real cyclic sextic field; a generator $\sigma$ of its Galois group is characterized by the relation $\sigma(\theta) = (\theta - 1)/(\theta + 2)$. We have $K_{-t} = K_t$ for all $t \in \mathbb{Z}$, and we can suppose that $t \in \mathbb{N} - \{0, 6, 26\}$.

2. The quadratic subfield of $K_t$ is $k_2 = \mathbb{Q}(\sqrt{t^2 + 108})$.

3. The cubic field of $K_t$ is $k_3 = \mathbb{Q}(\phi)$, where

$$\phi = \theta^{-1 - \sigma^3} = -\frac{2\theta + 1}{\theta(\theta + 2)}$$

and

$$\text{Irr}(\phi, \mathbb{Q}) = x^3 - \frac{t - 6}{4} x^2 - \frac{t + 6}{4} x - 1;$$

the discriminant of this polynomial is $((t^2 + 108)/16)^2$. If $t \equiv 2 \pmod{4}$, $k_3$ is the simplest cubic field.
4. The conductor $f$ of $K_t$ is given by the following procedure: Let $m$ be the product of primes, difference from 2 and 3, dividing $t^2 + 108$ with an exponent not congruent to 0 modulo 6; then $f = 4^k 3^l m$, where

$$k = 0 \text{ if } t \equiv 1 \text{ mod } 2 \quad \text{ or } t \equiv \pm 6 \text{ mod } 16, \quad k = 1 \text{ if not.}$$

$$l = 0 \text{ if } t \equiv 1 \text{ mod } 3 \quad l = 1 \text{ if } t \equiv 0 \text{ mod } 27, \quad l = 2 \text{ if not.}$$

We replace $t$ by $4t + 2$ in the definition of $f(x, t)$. Then we have that

$$f(x, t) = x^6 - 2(t - 1)x^5 - 5(t + 2)x^4 - 20x^3 + 5(t - 1)x^2 + 2(t + 2)x + 1$$

with $\text{disc}(f(x, t)) = 2^6 3^6 (t^2 + t + 7)^5$. We assume that $t^2 + t + 7$ is square-free. Then the cubic field $L_t$ of $K_t$ is the simplest cubic field with the field discriminant $(t^2 + t + 7)^2$ and the quadratic field $M_t$ is $\mathbb{Q}(\sqrt{t^2 + t + 7})$.

Since $\sigma$ be the generator of $\text{Gal}(K_t/\mathbb{Q}) \simeq C_6$, $L_t = K_t^{<\sigma^3>}$ and $M_t = K_t^{<\sigma^2>}$. Let $\chi$ be the generator of the group of characters for $\text{Gal}(K_t/\mathbb{Q})$ with $\chi(\sigma) = e^{2\pi i/6}$. Then $\text{Ind}_{<\sigma^2>} <\sigma^3> 1_{<\sigma^3>} = 1_{<\sigma>} + \chi^3$, $\text{Ind}_{<\sigma^3>} <\sigma^3> 1_{<\sigma^3>} = 1_{<\sigma>} + \chi^2 + \chi^4$ and $\text{Ind}_{<\sigma^3>} \varphi = 1_{<\sigma>} + \chi + \chi^5$ where $\varphi$ is the non-trivial representation for $< \sigma^3 >$. Hence the Artin conductor of $\chi^3$ equals the field discriminant of $M_t$ and the Artin conductors of $\chi^2$ and $\chi^4$ are both $t^2 + t + 7$.

The Artin conductors of $\chi$ and $\chi^5$ equals $(t^2 + t + 7) \sqrt{N(b)}$ where $b$ is the Artin conductor of $\varphi$. Since the product of Artin conductors of $\chi, \chi^2, \cdots, \chi^5$ is at most $2^6 3^6 (t^2 + t + 7)^5$, $N(b)$ is a bounded constant. Hence for $t^2 + t + 7$ square-free, as $t$ increases, the Artin conductors also increases. Hence we verified that the Galois representations for the simplest sextic fields satisfies the Hypotheses of Theorem 1.10.

As in the case of simplest quartic fields, $\theta, \sigma(\theta), \sigma^2(\theta), \sigma^3(\theta)$ and $\sigma^4(\theta)$ do not form independent units. The regulator

$$R = \frac{1}{6} \prod_{\chi \neq 1} \left( \sum_{i=0}^{5} \chi(\sigma^i) \log |\sigma^i(\theta)| \right)$$

vanishes because the term corresponding to $\chi(\sigma) = e^{\pi i}$ is zero.
We need to find a set of 5 independent units in $K_t$ to compute an upper bound of the regulator of $K_t$. Let $E$ be the group of units in $K_t$. A unit in $K_t$ is called a relative unit if its norm is $\pm 1$ over $k_2$ and $k_3$. The group of the relative units in $K_t$ is $<\pm 1> \oplus E^\ast$ where

$$E^\ast = \{ u \in E \mid u^{1-\sigma+\sigma^2} = 1 \}.$$ 

and there is a unit $\xi$ such that every unit $u \in E^\ast$ may be written

$$u = \xi^{\lambda+\mu\sigma}$$

for some $\lambda, \mu \in \mathbb{Z}$.

When $t$ is of some specific form, Gras [22] found a generator $\xi$ of $E^\ast$. Define $S(X)$ be a finite set of positive numbers.

$$S(X) = \{ 0 < r < X \mid 3r^2 + 3r + 1 \text{ and } 12r^2 + 12r + 7 \text{ square-free} \}.$$ 

For all $r \in S(X)$, let $t = (6r + 3)(36r^2 + 36r + 18)$ and we consider fields $L_r = K_t = \mathbb{Q}(w)$ where $w = \theta^{1-\sigma^3} = -\frac{\theta^{(2g+1)}}{\theta^{+2}}$. Then there exits a unit $v$ such that $w = v^{1+\sigma}$. Hence $v = \frac{(w+1)-\sqrt{(w+1)^2-8w}}{2}$. Then $v$ is a generator $\xi$ of $E^\ast$. Gras also shows that if $r \in S(X)$, then the conductor of $k_2$ is $f_2 = 36r^2 + 36r + 21$, and the fundamental unit of $k_2$ is

$$\epsilon_2 = \frac{(12r^2 + 12r + 5) + (2r + 1)(36r^2 + 36r + 21)}{2}.$$ 

Since $t = (6r + 3)(36r^2 + 36r + 18) \equiv 2 \mod 4$, the field $k_3$ is a simplest cubic field. Hence for $r \in S$, we have an explicit set of independent units:

$$\{ \epsilon_2, \tau, \tau^{\sigma^2}, v, v^\sigma \}$$

where $\tau$ is a root of $x^3 - \frac{t-6}{4}x^2 - \frac{t+6}{4}x - 1$. Hence, for $t = (6r + 3)(36r^2 + 36r + 18)$ with $r \in S(X)$,

$$R_{K_t} \ll (\log d_{K_t})^5$$

Now we show that $f(x, (6r + 3)(36r^2 + 36r + 18))$ gives rise to a regular $C_6$ extension over $\mathbb{Q}(r)$. If $\theta_r$ is a root of $f(x, (6r + 3)(36r^2 + 36r + 18))$, then it is clear $\mathbb{Q}(r)(\theta_r)$ is the
splitting field of \( f(x, (6r + 3)(36r^2 + 36r + 18) \) over \( \mathbb{Q}(r) \) with Galois group \( C_6 = \langle \sigma \rangle \). By the same argument, the Galois group of \( f(x, (6r + 3)(36r^2 + 36r + 18)) \) over \( \overline{\mathbb{Q}}(t) \) is also \( C_6 = \langle \sigma \rangle \). Hence the claim follows.

2.2 Dihedral groups

2.2.1 \( D_3 \) extensions

Consider a cubic polynomial

\[
f(x, t) = x^3 + tx - 1
\]

with \( \text{disc}(f(x, t)) = -(4t^3 + 27) \). This polynomial was studied by Ishida [31] first.

Theorem 2.12 (Ishida). Let \( K_t = \mathbb{Q}(\eta) \) be the cubic field of signature (1,1), where \( \eta \) is the real root of the cubic equation

\[
x^3 + tx - 1 = 0, \quad (t \in \mathbb{Z}, t \geq 2).
\]

If \( 4t^3 + 27 \) is square-free or \( t = 3m \) and \( 4m^3 + 1 \) is square-free, then \( \eta \) is the fundamental unit of \( K \).

It is easy to show that the real root \( \eta \) is located between \(-\frac{1}{1+t}\) and \(-\frac{1}{t}\) for any \( \epsilon > 0 \). Hence for \( t \) with square-free \( 4t^3 + 27 \), the regulator \( R_{K_t} \) is

\[
\log t < R_{K_t} < (1 + \epsilon) \log t.
\]

Since \( d_{K_t} = -(4t^3 + 27) \),

\[
R_{K_t} \ll \log |d_{K_t}|. \quad (2.13)
\]

It is obvious that \( f(x, t) \) is irreducible over \( \overline{\mathbb{Q}}(t) \) and its discriminant is not a square in \( \overline{\mathbb{Q}}(t) \). Hence \( f(x, t) \) gives rise to a \( D_3 \) regular extension.
2.2.2 $D_4$ extensions

Nakamula [51] constructed quartic fields with small regulators whose Galois closures have $D_4$ as the Galois group. Nakamula uses a polynomial with 3 parameters

$$f = x^4 - sx^3 + (t + 2u)x^2 - usx + 1$$

where $(s, t, u) \in \mathbb{N} \times \mathbb{Z} \times \{\pm 1\}$, $(s, t, u) \neq (1, -1, 1)$.

The discriminant $D_f$ of $f$ is given by

$$D_f = D_1^2D_2$$

with $D_1 = s^2 - 4t$, $D_2 = (t + 4u)^2 - 4us^2$.

For a zero $\epsilon$ of $f$ with $|\epsilon| \geq 1$, we define $\alpha := \epsilon + u\epsilon^{-1}$. Put

$$K = \mathbb{Q}(\epsilon), \ F = \mathbb{Q}(\sqrt{D_1}), \ L = \mathbb{Q}(\sqrt{D_2}), \ M = \mathbb{Q}(\sqrt{D_1D_2})$$

Then $F = \mathbb{Q}(\alpha) \subseteq K = F(\epsilon) = F(\sqrt{\alpha^2 - 4u})$.

With signs of $D_1$ and $D_2$ we can determine the signature of $K$. More precisely,

**Lemma 2.14.** [51] Assume $F \neq \mathbb{Q}$ and $L \neq \mathbb{Q}$. Then $K$ is a non-CM quartic field with a quadratic subfield $F$, and $|\epsilon| > 1$. If $F = L$, then $K$ is cyclic over $\mathbb{Q}$. If $F \neq L$, then $K$ is non-Galois over $\mathbb{Q}$, and the composite $MK$ is dihedral over $\mathbb{Q}$ and cyclic over $M$.

Moreover

$$\begin{cases} (r_1, r_2) = (0, 2) & \text{if } D_1 < 0 \\ (r_1, r_2) = (2, 1) & \text{if } D_2 < 0 \\ (r_1, r_2) = (4, 0) & \text{otherwise.} \end{cases}$$

Moreover, if $F \neq L, D_F = D_1$ and $D_L = D_2$, then $D_K = D_f$.

Note that if $K$ is not totally complex, the quadratic subfield $F$ is real.

$D_4$ extension with signature $(0,2)$

We specify that $s = u = 1$. Then we have, for positive integer $t$,

$$f(x, t) = x^4 - x^3 + (t + 2)x^2 - x + 1$$
with $D_1 = 1 - 4t$, $D_2 = t^2 + 8t + 12$. If $D_1, D_2$ are square-free for odd integer $t$, $D_K$ equals $(1 - 4t)^2(t + 2)(t + 6)$. For a positive integer $t$, $D_1$ is negative, by Lemma 2.14, $(r_1, r_2) = (0, 2)$ and $MK/\mathbb{Q}$ is a $D_4$ Galois extension.

Nakamula estimated the regulator $R_K$ of the field $K$.

$$R_K = \frac{1}{4} \log \frac{D_K}{16} + o(1) \text{ as } D_K \to \infty.$$ (2.15)

To show that $f(x, t)$ gives rise to a regular $D_4$ extension, we briefly recall how to determine the Galois group of a quartic polynomial over an arbitrary field $F$ in [11], page 358. We write a quartic polynomial $f$ in the form

$$f = x^4 - c_1 x^3 + c_2 x^2 - c_3 x + c_4$$

and we define the Ferrari resolvent of $f$ to be

$$\theta_f(y) = y^3 - c_2 y^2 + (c_1 c_3 - 4c_4)y - c_3^2 - c_1^2 c_4 + 4c_2 c_4.$$  

**Theorem 2.16.** Let $F$ have characteristic $\neq 2$, and $f \in F[x]$ be monic and irreducible of degree 4. Then Galois group of $f$ over $F$ is determined as follows:

(a) If $\theta_f(y)$ is irreducible over $F$, then

$$G = \begin{cases} S_4, & \text{if } \text{disc}(f) \notin F^2 \\ A_4, & \text{if } \text{disc}(f) \in F^2 \end{cases}$$

(b) If $\theta_f(y)$ splits completely over $F$, then $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(c) If $\theta_f(y)$ has a unique root $\beta$ in $F$, then

$$G \text{ is isomorphic to } \begin{cases} D_4, & \text{if } 4\beta + c_1^2 - 4c_2 \neq 0 \text{ and } \text{disc}(f)(4\beta + c_1^2 - 4c_2) \notin (F^*)^2 \\ or \ 4\beta + c_1^2 - 4c_2 = 0 \text{ and } \text{disc}(f)(\beta^2 - 4c_4) \notin (F^*)^2 \\ C_4, & \text{otherwise.} \end{cases}$$

The Ferrari resolvent of $f(x, t)$ is

$$y^3 - (t + 2)y^2 - 3y + 4t + 6 = (y - 2)(y^2 - ty - (2t + 3)).$$
Then \( \text{disc}(f(x,t))(4\beta + c_1^2 - 4c_2) = (1 - 4t)^3(t + 2)(t + 6) \notin (\mathbb{Q}(t)^*)^2 \) and \( (\mathbb{Q}(t)^*)^2 \). By Theorem 2.16, the Galois group of \( f(x,t) \) over both \( \mathbb{Q}(t) \) and \( \mathbb{Q}(t) \) is \( D_4 \). Hence \( f(x,t) \) gives rise to a \( D_4 \) regular extension over \( \mathbb{Q}(t) \).

**D.4 extension with signature (2,1)**

We specify that \( u = 1, t = 1 \). Then we have \( f(x,s) = x^4 - sx^3 + 3x^2 - sx + 1 \) and \( D_1 = s^2 - 4 \) and \( D_2 = 25 - 4s^2 = (5 + 2s)(5 - 2s) \). Assume that \( D_1 \) and \( D_2 \) are square-free for odd integers \( s \). Then \( D_F = D_1, D_L = D_2 \) and by Lemma 2.14 we have \( D_K = D_f = (s^2 - 4)^2(5 + 2s)(5 - 2s) \). For a positive integer \( s \) bigger than 3, \( D_1 \) is positive and \( D_2 \) is negative, by Lemma 2.14, \( (r_1, r_2) = (2, 1) \) and \( MF/\mathbb{Q} \) is a \( D_4 \) Galois extension.

Nakamula showed for the field generated by \( f(x,t) \),

\[
\frac{QR_K}{R_F} = \frac{1}{2} \log \frac{|D_K|}{4} + o(1)
\]

\[
R_F = \frac{1}{2} \log D_F + o(1)
\]

as \( |D_K| \) and \( D_F \rightarrow \infty \). Here \( Q \) is 1 or 2 depending on \( K \) and \( F \). Hence

\[
R_K \ll (\log |D_K|)^2.
\]   \hspace{1cm} (2.17)

The Ferrari resolvent of \( f(x,s) \) is

\[
y^3 - 3y^2 + (s^2 - 4)y - 2(s^2 - 6) = (y - 2)(y^2 - y + (s^2 - 6)).
\]

Then \( \text{disc}(f(x,s))(4\beta + c_1^2 - 4c_2) = (s^2 - 4)^3(5 + 2s)(5 - 2s) \notin (\mathbb{Q}(s)^*)^2 \) and \( (\mathbb{Q}(s)^*)^2 \). Hence \( f(x,s) \) gives rise to a \( D_4 \) regular extension over \( \mathbb{Q}(s) \).

**D.4 extension with signature (4,0)**

We specify that \( u = -1, t = 1 \) and \( s > 6 \). Then we have \( f(x,s) = x^4 - sx^3 - x^2 + sx + 1 \) and \( D_1 = s^2 - 4 \) and \( D_2 = 9 + 4s^2 \). Assume that \( D_1 \) and \( D_2 \) are square-free for odd integer \( s \). Then \( D_F = D_1, D_L = D_2 \) and by Lemma 2.14 we have \( D_K = D_f = (s^2 - 4)^2(9 + 4s^2) \).
For square-free $s^2 - 4$ and $4s^2 + 9$, $K, L$ are always distinct. Hence, by Lemma 2.14, $(r_1, r_2) = (4, 0)$ and $MK/\mathbb{Q}$ is a $D_4$ Galois extension.

Nakamula showed for the field generated by $f(x, s)$

$$\frac{QR_K}{RF} = \frac{1}{18} \log \frac{D_K}{4} \log \frac{D_F}{2^9} + o(1)$$

$$RF = \frac{1}{2} \log D_F + o(1)$$

as $D_K$ and $D_F \to \infty$. Here $Q$ is 1 or 2 depending on $K$ and $F$.

Hence

$$R_K \ll (\log |D_K|)^3. \quad (2.18)$$

The Ferrari resolvent of $f(x, s)$ is

$$y^3 + y^2 - (s^2 + 4)y - 2(s^2 + 2) = (y + 2)(y^2 - y - (s^2 + 2)).$$

Then $\text{disc}(f(x, s))(4\beta + c_1^2 - 4c_2) = (s^2 - 4)^3(9 + 4s^2) \notin \mathbb{Q}(s)^* \text{ and } (\mathbb{Q}(s)^*)^2$. Hence $f(x, s)$ gives rise to a $D_4$ regular extension over $\mathbb{Q}(s)$.

### 2.2.3 $D_5$ extensions

Consider the polynomial in [60] for the case of $D_5$ extension;

$$f(x, t) = x^5 - tx^4 + (2t - 1)x^3 - (t - 2)x^2 - 2x + 1.$$  

Its discriminant is $(4t^3 - 28t^2 + 24t - 47)^2$ and the signature is $(1, 2)$ for $t \leq 6$ and $(5, 0)$ for $n \geq 7$. We claim that $f(x, t)$ gives rise to a regular $D_5$ extension over $\mathbb{Q}(t)$, i.e., if we consider $f(x, t)$ as a polynomial over $\mathbb{Q}(t)$ and $E$ is the splitting field, then $E \cap \overline{\mathbb{Q}} = \mathbb{Q}$. This is equivalent to the fact that $Gal(E/\overline{\mathbb{Q}}(t)) \simeq Gal(E/\mathbb{Q}(t))$.

By [34], page 41, the Weber sextic resolvent of $f(x, t)$ is

$$G(z) = (z^3 + b_4z^2 + b_2z + b_0)^2 - 2^{10}(4t^3 - 28t^2 + 24t - 47)^2z,$$

where $b_4 = -4t^2 - 4t + 37, b_2 = 64t^3 - 312t^2 + 328t + 115, b_0 = 64t^3 - 2884t^2 + 4348t - 249$.

It factors as

$$G(z) = (z - 4t^2 + 12t - 9)(z^5 - (4t^2 + 20t - 83)z^4 + (128t^3 - 368t^2 - 816t + 2346)z^3$$
\[-(1024t^4 - 7040t^3 + 16536t^2 + 3448t - 29126)z^2 \]
\[+(2048t^4 + 51072t^3 - 201328t^2 + 18640t + 256933)z \]
\[-(1024t^4 - 89216t^3 + 1948548t^2 - 231404t + 6889)). \]

Therefore, the Galois group of \(f(x,t)\) over \(\mathbb{Q}(t)\) and \(\overline{\mathbb{Q}}(t)\) is either \(D_5\) or \(C_5\). In order to distinguish it, we use the criterion in [34], page 42. Namely, the Galois group is \(C_5\) if and only if the resolvent \(R(x_1 - x_2, f(x,t))(X)\) factors into irreducible polynomials of degree 5. Here

\[R(x_1 - x_2, f(x,t))(X) = X^{-5} \text{Res}(f(Y - X, t), f(Y, t))\]
\[= (X^{10} + X^8(-2t^2 + 12t - 8) + X^6(t^4 - 12t^3 + 46t^2 - 56t - 4) \]
\[+ X^4(-2t^4 + 16t^3 - 8t^2 - 112t + 127) + X^2(t^4 - 60t^2 + 128t + 6) - 4t^3 + 28t^2 - 24t + 47) \]
\[= (X^{10} + X^8(-2t^2 + 8t - 2) + X^6(t^4 - 8t^3 + 20t^2 - 4t + 7) + X^4(-2t^4 + 36t^2 - 4t + 41) \]
\[+ X^2(t^4 - 4t^3 + 66t^2 - 36t + 103) - 4t^3 + 28t^2 - 24t + 47) \]

It is clear that the above factors cannot be factored into irreducible polynomials of degree 5. Hence the Galois group of \(f(x,t)\) over both \(\mathbb{Q}(t)\) and \(\overline{\mathbb{Q}}(t)\) is \(D_5\).

Let \(\theta_t\) be a root of \(f(x,t)\). Schöpp found the fundamental units in the equation order \(\mathbb{Z}[\theta_t]\) for \(t \leq 6\). More precisely, he shows

**Theorem 2.19 (Schöpp).** The elements \(\theta_t, \theta_t - 1\) form a system of independent units in the order \(\mathbb{Z}[\theta_t]\). Moreover, they are fundamental units in \(\mathbb{Z}[\theta_t]\) for \(t \leq 6\).

However, Schöpp could not show that \(\mathbb{Z}[\theta_t]\) is the maximal order of \(\mathbb{Q}(\theta_t)\). Lavallee, Spearman, Williams and Yang [40] found a parametric family of quintics with a power integral basis. The parametric polynomial \(F_b(x)\) is given by

\[F_b(x) := x^5 - 2x^4 + (b + 2)x^3 - (2b + 1)x^2 + bx + 1, b \in \mathbb{Z}.\]

and its discriminant is \((4b^3 + 28b^2 + 24b + 47)^2\).
They showed that when $4b^3 + 28b^2 + 24b + 47$ is square-free, then the field $Q(\theta_b)$ generated by a root $\theta_b$ of $F_b(x)$ has a power integral basis $\mathbb{Z}[\theta_b]$. Since $x^5 F_{-t} \left( \frac{1}{x} \right) = f(x, t)$, this implies that $\theta_t, \theta_t - 1$ are fundamental units of $Q(\theta_t)$ when $4t^3 - 28t^2 + 24t - 47$ is square-free.

Schöpp found the locations of roots of $f(x, t)$.

Lemma 2.20 (Schöpp). Let $\theta^{(1)}_t$ be the real root and let $\theta^{(2)}_t = \overline{\theta^{(3)}_t}, \theta^{(4)}_t = \overline{\theta^{(5)}_t}$ be the pairs of complex roots of $f(x, t)$. Then we have the following approximations:

(i) $-t + 2 + \frac{2}{t} < |\theta^{(1)}_t| < -t + 2 + \frac{1}{t}$ for $t < -4$

(ii) $-t + 3 + \frac{2}{t} < |\theta^{(1)}_t| - 1 < -t + 3 + \frac{1}{t}$ for $t < -4$

(iii) $\frac{1}{2\sqrt{-1}} < |\theta^{(2)}_t| < \frac{2}{\sqrt{-1}}$ for $t < -4$

(iv) $\sqrt{1 - \frac{3}{4t}} < |\theta^{(2)}_t| - 1 < \sqrt{1 - \frac{6}{5t}}$ for $t < -144$

(v) $\sqrt{1 + \frac{3}{7}} < |\theta^{(4)}_t| < \sqrt{1 - \frac{1}{t}}$ for $t < -174$

(vi) $\sqrt{-\frac{5}{6t}} < |\theta^{(4)}_t| - 1 < \sqrt{-\frac{14}{13t}}$ for $t < -139$.

Let $K_t$ be the quintic field by adjoining $\theta^{(1)}_t$ to $Q$. Since we know the absolute value of roots, it is easy to show that the regulator $R_{K_t}$ of a quintic field $K_t$.

Lemma 2.21. For $t$ with $4t^3 - 28t^2 + 24t - 47$ square-free, $R_{K_t} \ll (\log d_{K_t})^2$.

2.3 Alternating groups

2.3.1 $A_4$ extensions

Let

$$f(x, t) = x^4 - 8tx^3 + 18t^2x^2 + 1.$$ 

Its discriminant is $16^2(1+27t^4)^2$. We claim that the splitting field of $f(x, t)$ over $Q(t)$ is a regular extension with Galois group $A_4$. We have to show that the Galois groups of $f(x, t)$ over $Q(t)$ and $\mathbb{Q}(t)$ are both $A_4$. 

First, we have to show that \( f(x,t) \) is irreducible over \( \overline{\mathbb{Q}}(t) \). By Gauss Lemma, it is enough to check the irreducibility of \( f(x,t) \) over \( \mathbb{Q}[t] \). It is easy to check that \( f(x,t) \) has no root in \( \mathbb{Q}[t] \). If \( f(x,t) \) is decomposed into a product of two quadratic polynomials, we have

\[
x^4 - 8tx^3 + 18t^2x^2 + 1 = (x^2 + bx + c)(x^2 + dx + 1/c)
\]

for some \( b, d \in \mathbb{Q}[t] \) and \( c \in \overline{\mathbb{Q}} \) and we can induce a contradiction easily. Hence \( f(x,t) \) is irreducible over \( \mathbb{Q}[t] \).

Its Ferrari resolvent \( \theta(y) \) equals \( y^3 - 18t^2y^2 - 4y + 8t^2 \) and it is irreducible over \( \mathbb{Q}[t] \). Since the discriminant of \( f(x,t) \) is a square in \( \mathbb{Q}(t) \) and \( \overline{\mathbb{Q}}(t) \), the Galois groups over \( \mathbb{Q}(t) \) and \( \overline{\mathbb{Q}}(t) \) are both \( A_4 \).

Let \( \theta_t \) be a root of \( f(x,t) \), and \( K_t = \mathbb{Q}(\theta_t) \). Then \( \theta_t \) is a unit in \( \mathbb{Z}_{K_t} \). We prove that \( K_t \) has the smallest possible regulator. We can see easily that \( f(x,t) \) has 4 complex roots.

**Proposition 2.22.** The regulator \( R_{K_t} \) satisfies \( R_{K_t} \ll \log t \).

**Proof.** By considering \( \alpha = 6t \) in Lemma 2.4, we can see that

\[
2t - \frac{1}{54t^3} < |\theta_t| < 6t.
\]

Therefore

\[
\text{Reg}(\mathbb{Z}_t) \leq \log |\theta_t| \ll \log t.
\]

**Lemma 2.23.** If \( 27t^4 + 1 \) is cubic-free, then \( d_{K_t} \gg t^2 \). Hence \( R_{K_t} \ll \log(d_{K_t}) \).

**Proof.** First note that \( f(x,t) = (x + t)(x - 3t)^3 + 27t^4 + 1 \). Take \( t \) such that \( 27t^4 + 1 \) is cubic-free. Let \( p \mid (27t^4 + 1), p > 3 \). Then \( p \nmid t \) and \( f(x,t) \equiv (x + t)(x - 3t)^3 \mod p \). The vertices of the Newton polygon with respect to \( x - 3t \) are \( (0,0), (1,0), (4,1), (4, i) \) with \( i = 1, 2 \).

By Cohen [11], page 315, Newton polygon argument shows that

\[ p\mathbb{Z}_{K_t} = p_1p_2^3 \]
with prime ideals $p_1, p_2$. Hence $p|\text{Disc}(K_t)$.

Therefore, $d_{K_t} \geq \prod_{p|(27t^4+1), p\neq 3} p$. But $27t^4 + 1 \leq \left( \prod_{p|(27t^4+1)} p \right)^2$. Hence $d_{K_t} \gg t^2$. \qed

Consider a quartic polynomial for positive integers $t \in \mathbb{Z}^+$,

$$f_t(x) = x^4 + 18x^2 - 4tx + t^2 + 81.$$  

with $\text{disc}(f_t(x)) = 2^8t^2(t^2 + 108)^2$. This polynomial is studied by Spearman [68] first. Let $K_t$ be the quartic field obtained by adjoining a root $\theta_t$ of $f_t(x)$ to the rational numbers. It is easy to check that the signature of $K_t$ is $(0, 2)$. Spearman showed, under some square-free condition, that $K_t$ is monogenic. Hence they are distinct. More precisely;

**Theorem 2.24** (Spearman). Suppose that $t$ is a positive integer and that $t(t^2 + 108)$ is square-free. Then $K_t$ is a monogenic $A_4$ quartic extensions of $\mathbb{Q}$. Moreover the fields $K_t$ are distinct.

We claim that the Galois group of $f_t(x)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both $A_4$. i.e, the splitting field $E$ of $f(x, t)$ over $\mathbb{Q}(t)$ is a regular extension. It is enough to show that the Galois group of $f_t(x)$ over $\overline{\mathbb{Q}}(t)$ is $A_4$. First we have to show that $f_t(x)$ is irreducible over $\overline{\mathbb{Q}}[t]$. This can be shown by checking that $f(x, t)$ has no root in $\overline{\mathbb{Q}}[t]$ and it can not be a product of two quadratic polynomials.

The Ferrari resolvent of $f_t(x)$ is

$$y^3 - 18y^2 - (4t^2 + 324)y + (56t^2 + 1296).$$

Since $\text{disc}(f_t(x))$ is a square in $\overline{\mathbb{Q}}[t]$, if the Ferrai resolvent has no root in $\overline{\mathbb{Q}}[t]$, the Galois group of $f_t(x)$ is $A_4$. The only possibilities for a root are of the forms $a, b(t \pm i\sqrt{\frac{162}{7}})$ and $c(t^2 + 162)$ for some $a, b, c \in \overline{\mathbb{Q}}$. We can check that they are not a root of the Ferrai resolvent. Hence we showed that $f_t(x)$ gives rise to an $A_4$ regular extension.
2.3.2 $A_5$ extensions

Consider the polynomial $f(x,t) = x^5 + 5(5t^2 - 1)x - 4(5t^2 - 1)$, where $5t^2 - 1$ is square free. Here $\text{disc}(f(x,t)) = 2^{50}5^6t^2(5t^2 - 1)^4$.

For a non-zero integer $t$, $f(x,t)$ has one real root and four complex roots. Let $K_t$ be the quintic field obtained by adjoining the real root $\theta_t$ of $f(x,t)$.

We claim that the Galois groups of $f(x,t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both $A_5$. First we have to check that $f(x,t)$ is irreducible over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$. This is true because $f(x,t)$ is an Eisenstein polynomial with respect to $5t^2 - 1$ (resp. $\sqrt{5}t + 1$) as a polynomial over $\mathbb{Q}(t)$ (resp. $\overline{\mathbb{Q}}(t)$). Since the discriminant is a square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, the Galois group is a subgroup of $A_5$. It is enough to show that the below sextic resolvent of $f(x,t)$ has no root in $\overline{\mathbb{Q}}(t)$.

$$\theta(y) = (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\text{disc}(f(x,t))y$$

where $b_2 = -100(5t^2 - 1), b_4 = 6000(5t^2 - 1)^2$ and $b_6 = 4000(5t^2 - 1)^3$. If $\theta(y)$ has a root $\alpha$ in $\overline{\mathbb{Q}}(t)$ which is a divisor of $b_6^2$, then we have

$$(\alpha^3 + b_2\alpha^2 + b_4\alpha + b_6)^2 = 2^{10}\text{disc}(f(x,t))\alpha$$

(2.25)

Hence $\alpha$ should be a square. The possible degrees of $\alpha$ are 0, 2, 4, 6. When the degree is 0, then such $\alpha$ can not be a root of $\theta(y)$. If the degree is 4 or 6, then the degree of LHS and the degree of RHS in (2.25) does not match. Hence the only possible forms of $\alpha$ are

$$a(\sqrt{5}t + 1)^2 \text{ or } a(\sqrt{5}t - 1)^2$$

for some algebraic number $a \in \overline{\mathbb{Q}}$. By the help of computer algebra such as PARI, we can check that they can not be a root of $\theta(y)$. Hence, we have

**Lemma 2.26.** $f(x,t)$ gives rise to an $A_5$ regular extension.

Assume that $5t^2 - 1$ is square-free. For every odd prime divisor $p$ of $5t^2 - 1$, $f(x,t)$ is an Eisenstein polynomial with respect to $p$ and $p$ does not divide the index of $\theta_t$. This implies $d_{K_t}$ is at least divided by $(5t^2 - 1)^4$ or $\left(\frac{5t^2-1}{2}\right)^4$. 
Lemma 2.27. For square-free $5t^2 - 1$,

$$\log d_{K_t} \gg \log |t|.$$ 

2.4 Symmetric groups

2.4.1 $S_4$ extensions

$S_4$ extensions with signature (2,1)

Let $t > 1$ be a positive square-free integer and $f(x,t) = x^2(x-10t)(x-18t) + t$. Then the discriminant $\text{Disc}(f(x,t))$ of $f(x,t)$ is $-256t^3(12t+1)(15t-1)(144t^2 - 12t + 1)(225t^2 + 15t + 1) < 0$. Since $f(x,t)$ is an Eisenstein polynomial for each prime divisor of $t$, $\text{Disc}(f(x,t))$ is divided by $t^3$. (See [51], page 60.)

Consider $f(x,t) = x^2(x-10t)(x-18t) + t$ over $\mathbb{Q}(t)$. It is easy to see that the cubic resolvent $y^3 - 180t^2y^2 - 4ty - 64t^3$, is irreducible over $\mathbb{Q}[t]$. By Gauss Lemma, it is irreducible over $\mathbb{Q}(t)$. Hence the Galois group of $f(x,t)$ over $\mathbb{Q}(t)$ is $S_4$. Therefore, $f(x,t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$.

Note that $f'(x,t) = 4x(x-6t)(x-15t)$, and we can see easily that $f(x,t)$ has two real roots $\theta_1, \theta_2$ and two complex roots $\theta_3, \theta_4 = \overline{\theta_3}$. For sufficiently large $t$, we can see that $10t + \frac{1}{t} < \theta_1 < 10t + \frac{1}{t}, 18t - \frac{1}{t} < \theta_2 < 18t - \frac{1}{t}$. Also by taking $\alpha = \frac{1}{t}$ and applying Lemma 2.4, we can see that $\theta_3$ and its conjugate $\theta_4$ are inside of the circle of radius $1/9$ centered at the origin.

Let $K_t = \mathbb{Q}[\theta_1]$. Then we can find two independent units in $\mathbb{Z}_{K_t}$.

Lemma 2.28. $\frac{\theta_4^4}{t}$ and $\frac{(\theta_1 - 10t)^4}{t}$ are independent units in $\mathbb{Z}_{K_t}$.

Proof. Since $\theta_1^4 - 28t\theta_1^3 + 180t^2\theta_1^3 + t = 0, \frac{\theta_4^4}{t} = 28\theta_1^3 - 180t\theta_1^2 - 1$. Hence $\frac{\theta_4^4}{t}$ is an algebraic integer. Also

$$\frac{(\theta_1 - 10t)^2(\theta_1 - 18t)^2}{t} = \frac{\theta_1^2 - t(28\theta_1 - 180t))^2}{t} = \frac{\theta_1^4}{t} - 2\theta_1^2(28\theta_1 - 180t) + t(28\theta_1 - 180t)^2.$$
So \(\frac{(\theta_1-10t)^2(\theta_1-18t)^2}{t}\) is an algebraic integer. Now we have
\[
\frac{\theta_1^4}{t} \cdot \frac{(\theta_1-10t)^2(\theta_1-18t)^2}{t} = 1.
\]
Hence \(\frac{\theta_1^4}{t}\) is a unit. By considering \(y = x - 10t\) or \(y = x - 18t\), we can see that \(\frac{(\theta_1-10t)^4}{t}\) and \(\frac{(\theta_1-18t)^4}{t}\) are algebraic integers. We have
\[
\frac{\theta_1^8}{t^2} \cdot \frac{(\theta_1-10t)^4}{t} \cdot \frac{(\theta_1-18t)^4}{t} = 1.
\]
Hence \(\frac{(\theta_1-10t)^4}{t}\) is a unit.

Assume that \(\frac{\theta_1^4}{t}\) and \(\frac{(\theta_1-10t)^4}{t}\) are dependent. Then
\[
\left(\frac{\theta_1^4}{t}\right)^k = \left(\frac{(\theta_1-10t)^4}{t}\right)^m
\]
for some integers \(k\) and \(m\). Without the loss of generality, we can assume that \(k\) is positive. When we consider the size of \(\theta_1\), \(m\) should be a negative integer. But when we replace \(\theta_1\) by \(\theta_2\), \(m\) should be a positive integer and it induces a contradiction. \(\Box\)

**Lemma 2.29.** For positive square-free \(t\), \(R_{K_t} \ll (\log |d_{K_t}|)^2\).

**Proof.** By definition,
\[
R_{K_t} \leq \left| \det \begin{array}{cc}
\log |\frac{\theta_1^4}{t}| & \log |\frac{\theta_2^4}{t}| \\
\log |\frac{(\theta_1-10t)^4}{t}| & \log |\frac{(\theta_2-10t)^4}{t}| \\
\end{array} \right|
\]
By the above estimates on \(\theta_1, \theta_2\), it is clear that \(R_{K_t} \leq (\log t)^2\). Since \(t^3 \mid d_{K_t}\), we prove the claim. \(\Box\)

**S_4 extensions with signature** \((0, 2)\)

Let \(t > 1\) be a positive square-free integer and \(f(x, t) = x^4 + tx^2 + tx + t\). Then the discriminant \(Disc(f(x, t))\) of \(f(x, t)\) is \(t^3(12t^2 - 11t + 256)\). Since \(f(x, t)\) is an Eisenstein polynomial for each prime divisor of \(t\), \(Disc(f(x, t))\) is divided by \(t^3\). (See [51], page 60.)
The cubic resolvent \( y^3 - ty^2 - 4ty + 3t^2 \) of \( f(x, t) \) has three real roots. One of them is located between \( t + 1 \) and \( t + 2 \) hence it is not an integer. So if the cubic resolvent has an integer root, we can show that the integer root should be divided by \( t \). Since \( \pm t, \pm 3t, \pm t^2 \) and \( \pm 3t^2 \) are not a root of the cubic resolvent, the cubic resolvent is irreducible. Hence \( f(x, t) \) gives rise to an \( S_4 \) Galois extension for each positive square-free integer \( t \).

Consider \( f(x, t) = x^4 + tx^2 + tx + t \) over \( \mathbb{Q}(t) \). It is easy to see that the cubic resolvent \( y^3 - ty^2 - 4ty + 3t^2 \), is irreducible over \( \mathbb{Q}(t) \). By Gauss Lemma, it is irreducible over \( \mathbb{Q}(t) \). Hence the Galois group of \( f(x, t) \) over \( \mathbb{Q}(t) \) is \( S_4 \). Therefore, \( f(x, t) \) gives rise to a regular Galois extension over \( \mathbb{Q}(t) \).

Note that \( f'(x, t) = 4x^3 + 2tx + t \) has only one real root \( x_0 \), and we can easily check that \( f(x_0) > 0 \). Hence \( f(x, t) \) has four complex roots \( \theta_1, \theta_2 = \overline{\theta}_1, \theta_3 \) and \( \theta_4 = \overline{\theta}_3 \). For sufficiently large \( t \), when we apply Lemma 2.4 with \( \alpha = i\sqrt{t} \), we can see that one root lies inside of the circle of radius 1 centered at \( 1 + i\sqrt{t} \). Let \( \theta \) be the root.

Let \( K_t = \mathbb{Q}[\theta] \). Then since \( \frac{\theta^4}{t} = -(\theta^2 + \theta + 1) \), \( \frac{\theta^4}{t} \) is an algebraic integer. Here \( N_{K_t/\mathbb{Q}}(\theta) = t \). Hence \( \frac{\theta^4}{t} \) has norm 1, and it is a unit in \( \mathbb{Z}_{K_t} \).

Since \( |\theta| \ll \sqrt{t} \), \( \log |\frac{\theta^4}{t}| \ll \log t \). Since \( t^3 | d_{K_t} \), we have

**Lemma 2.30.** For positive square-free \( t \), \( R_{K_t} \ll \log d_{K_t} \).

### 2.4.2 \( S_5 \) extensions

Consider a quintic polynomial for positive square-free integers \( t \in \mathbb{Z}^+ \) with \( t \equiv 1 \pmod{5} \),

\[
f(x, t) = x^5 + tx + t
\]

with the discriminant \( disc(f(x, t)) = t^4(256t + 3125) \). We claim that the Galois groups of \( f(x, t) \) over \( \mathbb{Q}(t) \) and \( \overline{\mathbb{Q}}(t) \) are both \( S_5 \). Since \( f_t(x) \) is an Eisenstein polynomial for an irreducible element \( t \), it is irreducible over \( \mathbb{Q}(t) \) and \( \overline{\mathbb{Q}}(t) \) and it is clear that \( disc(f(x, t)) \) is not an square in \( \mathbb{Q}(t) \) and \( \overline{\mathbb{Q}}(t) \). If the sextic resolvent has no root in \( \mathbb{Q}(t) \) and \( \overline{\mathbb{Q}}(t) \),
then the Galois group is $S_5$ over both fields.

The sextic resolvent of $f(x, t)$ is given by

$$\theta_t(y) = (y^3 + b_2 y^2 + b_4 y + b_6)^2 - 2^{10}\text{disc}(f_t(x)) y$$

where $b_2 = -20t$, $b_4 = 240t^2$, and $b_6 = 320t^3$.

We have to show that $\theta_t(y)$ does not have a root in $\overline{\mathbb{Q}}(t)$. If $\alpha$, a divisor of $b_6^2$, is a root of $\theta_t(y)$, then

$$(\alpha^3 + b_2 \alpha^2 + b_4 \alpha + b_6)^2 = 2^{10}\text{disc}(f(x, t)) \alpha.$$ 

Even though $\text{disc}(f(x, t))$ and $b_6$ has a common divisor $t^3$, but the other factors are coprime, it induces a contradiction.

Let $K_t$ be a quintic field obtained by adjoining a root of $f_t(x)$ to $\mathbb{Q}$. Then the signature of $K_t$ is $(1, 2)$. Since $f_t(x)$ is an Eisenstein polynomial for a square-free integer $t$, the field discriminant $d_{K_t}$ of $K_t$ is divided by $t^4$. This polynomial is used to generate extreme values of logarithmic derivatives of Artin L-functions.

Consider, for a positive square-free integer $t$ with $t \equiv 1 \pmod{5}$,

$$f(x, t) = (x + t)(x^2 + 5t)(x^2 + 10t) + t$$

with the discriminant $\text{disc}(f(x, t)) = t^4(5000000t^{10} + 15000000t^9 + 162350000t^8 + 746700000t^7 + 1234759600t^6 + 7714500t^5 - 394744t^4 + 5143500t^3 + 162500t^2 + 3125)$.

We claim that the Galois groups of $f(x, t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both $S_5$. Since $f(x, t)$ is an Eisenstein polynomial for an irreducible element $t$, it is irreducible over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ and it is clear that $\text{disc}(f(x, t))$ is not an square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$. If the sextic resolvent has no root in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, then the Galois group is $S_5$ over both fields.

The sextic resolvent of $f(x, t)$ is given by

$$\theta_t(y) = (y^3 + b_2 y^2 + b_4 y + b_6)^2 - 2^{10}\text{disc}(f_t(x)) y$$

where $b_2 = 5t^2(24t - 335)$, $b_4 = t^3(400t^3 - 192000t^2 + 661811t - 2400)$, and $b_6 = 5^2t^3(12400t^5 + 3069000t^4 + 17775t^3 + 168480t^2 - 64t + 2400)$. As in the previous case, we
can show that $\theta_t(y)$ does not have a root in $\mathbb{Q}(t)$. Hence $f(x, t)$ gives rise to a $S_5$ regular extension over $\mathbb{Q}(t)$.

Let $K_t = \mathbb{Q}(\theta_1)$ be a quintic field obtained by adjoining the real root $\theta_1$ of $f(x, t)$ to the rational numbers $\mathbb{Q}$. Moreover, $f_t(x) \equiv x^4(x + 1) + 1 \equiv (x + 2)(x^2 + x + 1)(x^2 + 3x + 3) \pmod{5}$ and the signature of $K_t$ is $(1, 2)$. Hence the Galois extensions $\overline{K}_t/\mathbb{Q}$ satisfy the hypothesis of Theorem 1.4, and Artin L-functions $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ are cuspidal automorphic L-functions of $GL(4)/\mathbb{Q}$.

We claim that $(\theta_1 + t)^5$ and $(\theta_1^2 + 5t)^5$ are two independent units in $K_t$: Since $f_t(x) = x^5 + tx^4 + 15tx^3 + 15t^2x^2 + 50t^2x + 50t^3 + t$,

$$\frac{\theta_5^5}{t} = -(\theta_1^4 + 15\theta_1^3 + 15t\theta_1^2 + 50t\theta_1 + 50t^2 + 1).$$

Hence $\frac{\theta_5^5}{t}$ is an algebraic integer. From this, it is easy to show that $(\theta_1 + t)^5$ and $(\theta_1^2 + 5t)^5$ are algebraic integers. Now we have

$$\frac{(\theta_1 + t)^5}{t} \cdot \frac{(\theta_1^2 + 5t)^5}{t^2} \cdot \frac{(\theta_1^2 + 10t)^5}{t^2} = -1.$$

Hence $(\theta_1 + t)^5$ and $(\theta_1^2 + 5t)^5$ are units. To show that they are independent, we need to know the locations of 5 roots of $f_t(x)$. For the case of the real root $\theta_1$, we have $-t - \frac{1}{t} < \theta_1 < -t$.

For complex roots, we apply Lemma 2.4 to $f_t(x)$ with $\alpha = i\sqrt{5t}$ then we can see that another root $\theta_2$ of $f_t(x)$ converges to $i\sqrt{5t}$ as $t$ increases. More precisely,

$$|\theta_2 - i\sqrt{5t}| = O \left( \frac{1}{t^{1.5}} \right).$$

Put $\theta_3 = \overline{\theta}_2$. Apply Lemma 2.4 again to $f_t(x)$ with $\alpha = i\sqrt{10t}$. Then we can find the fourth root $\theta_4$ with

$$|\theta_4 - i\sqrt{10t}| = O \left( \frac{1}{t^{1.5}} \right)$$

and put $\theta_5 = \overline{\theta}_4$.

Assume that $(\theta_1 + t)^5$ and $(\theta_1^2 + 5t)^5$ are dependent. Then $\left(\frac{(\theta_1 + t)^5}{t}\right)^k = \left(\frac{(\theta_1^2 + 5t)^5}{t^2}\right)^l$ for some integers $k, l$. Then it holds when we replace $\theta_1$ by $\theta_4$, and when we consider the size of $\theta_1, \theta_4$, we obtain contradiction. By definition,
\[ R_{K_t} \ll \begin{vmatrix} \log \frac{(\theta_1 + t)^5}{t} & \log \frac{(\theta_4 + t)^5}{t} \\ \log \frac{(\theta_1 + 5t)^5}{t^4} & \log \frac{(\theta_4 + 5t)^5}{t^4} \end{vmatrix}. \]

By the above estimates on \( \theta_1, \theta_4 \), it is clear that \( R_{K_t} \ll (\log t)^2 \). Since \( f_t(x) \) is an Eisenstein polynomial, \( d_{K_t} \) is divisible by \( t^4 \) if \( t \) is square-free. Hence \( \log d_{K_t} \gg \log t \). We have proved

**Lemma 2.31.** For a square-free positive integer \( t \),

\[ R_{K_t} \ll (\log d_{K_t})^2. \]

### 2.4.3 \( S_n \) extensions

Duke [16] considered a polynomial of the form

\[ f(x, t) = (x - t)(x - 2^2 t)(x - 3^2 t) \cdots (x - n^2 t) - t. \]

Let \( K_t \) is a number field obtained by adjoining a root \( \theta_t \) of \( f(x, t) \). He showed that the regulator of \( K_t \) is minimal up to a constant.

**Proposition 2.32 (Duke).** Let \( t \in \mathbb{Z}^+ \) be a square-free integer. Then \( f(x, t) \) is irreducible. If \( t \) is sufficiently large, then \( K_t \) is totally real of degree \( n \) and the regulator \( R_{K_t} \) of \( K_t \) satisfies \( R_{K_t} \ll_f (\log d_{K_t})^{n-1} \).

Moreover, he showed

**Proposition 2.33 (Duke).** For \( n \geq 1 \), the splitting field of \( f(x, t) \) over \( \mathbb{Q}(t) \) is a regular \( S_n \) extension.
Chapter 3

Number fields with extreme class numbers

This Chapter is based on Cho [6], Cho and Kim [7] and Cho and Kim [8]. Except two sections 3.3.2 and 3.3.5, they are the joints works with H. Kim. We follow closely [6],[7] and [8].

3.1 Upper bound of class numbers

Let $\mathcal{K}(n,G,r_1,r_2)$ be the set of number fields of degree $n$ with signature $(r_1,r_2)$ whose normal closures have $G$ as their Galois group. Then by the class number formula, the class number $h_K$ for $K \in \mathcal{K}(n,G,r_1,r_2)$ is given by

$$h_K = \frac{w_K|d_K|^\frac{1}{2}}{2^{r_1}(2\pi)^{r_2}R_K}L(1,\rho),$$

where $w_K$ is the number of roots of unity in $K$, $d_K$ is the discriminant of $K$ and $R_K$ is its regulator and $L(s,\rho) = \frac{\zeta_K(s)}{\zeta(s)}$ is the Artin L-function.

Silverman [66] obtained a lower bound of regulator $R_K$ of number fields $K$:

$$R_K > c_n(\log \gamma_n |d_K|)^{r-r_0},$$

(3.1)
where $c_n, \gamma_n$ are positive constant depending on degree $n$ of $K$ and $r = r_1 + r_2 - 1$ and $r_0$ is the maximum of unit ranks of subfields of $K$.

It is easy to prove that under the Generalized Riemann Hypothesis (GRH) for $L(s, \rho)$, $L(1, \rho) \ll (\log \log |d_K|)^{n-1}$. Hence we obtain the upper bound for the class numbers:

$$h_K \ll |d_K|^\frac{1}{2} (\log \log |d_K|)^{n-1} \frac{(\log |d_K|)^{r-r_0}}{(\log |d_K|)^{r-r_0}}.$$

Now the question is whether the upper bound is sharp. Namely, are there number fields with the largest possible class number of the size

$$|d_K|^\frac{1}{2} (\log \log |d_K|)^{n-1} \frac{(\log |d_K|)^{r-r_0}}{(\log |d_K|)^{r-r_0}}?$$

For real quadratic fields, this is a classical result of Montgomery and Weinberger [47]. Ankeny, Brauer, and Chowla [1] constructed unconditionally, for any $n, r_1, r_2$, number fields with arbitrarily large discriminants and $h_K \gg |d_K|^\frac{1}{2} - \epsilon$. Under the GRH and Artin conjecture for $L(s, \rho)$, Duke [16] constructed totally real fields of degree $n$ whose Galois closure has the Galois group $S_n$ with the largest possible class numbers. Daileda [18] showed Duke’s result unconditionally when $n = 3$.

We show that the upper bound is sharp up to a constant for many families $\mathfrak{R}(n, G, r_1, r_2)$ of number fields.

**Remark 3.2.** If $K$ has at least one real embedding, $w_K = 2$. If $K$ has no real embedding, then the degree $\varphi(w_K)$ of $w_k$-th cyclotomic polynomial is less than or equal to $n$, the degree of $K$. Since there are only finitely many integers $m$ with $\varphi(m) \leq n$, $w_K$ is bounded for each $n$.

### 3.2 Extreme class numbers

In this section we explain how to generate number fields with extreme class numbers in a very general setting. Consider an irreducible polynomial $f(x, t)$ with parameter $t$,

$$f(x, t) = x^n + a_{n-1}(t)x^{n-1} + a_{n-2}(t)x^{n-2} + \cdots + a_1(t)x + a_0(t) \in \mathbb{Z}[t][x].$$
Assume that the splitting field $E$ of $f(x,t)$ over $\mathbb{Q}(t)$ is regular with Galois group $G$ for some finite group $G$. For a specialization $t \in \mathbb{Z}$, let $K_t$ be a number field of degree $n$ obtained by adjoining a root $\theta_t$ of $f(x,t)$ and $\hat{K}_t$ be the Galois closure of $K_t$. Then by Cohen’s theorem, $\text{Gal}(\hat{K}_t/\mathbb{Q}) = G$ in most cases.

Assume that the regulator $R_{K_t}$ is minimal up to constant, i.e., $R_{K_t} \ll (\log |d_{K_t}|)^{r-r_0}$. We will show that this is true for the cases we consider. However, we could not find number fields with minimal regulators in the cases of $\mathfrak{A}(4, D_4, 2, 1)$ and $\mathfrak{A}(4, D_4, 4, 0)$ due to the presence of subfields.

Since $f(x,t)$ gives rise to a $G$ regular extension, by Theorem 2.1, there is a constant $c_f$ depending on $f$ such that for every $p \geq c_f$, there is a residue class $s_p$ modulo $p$ so that the Frobenius element $\text{Frob}(p)$ at $p$ in $\text{Gal}(\hat{K}_t/\mathbb{Q})$ is trivial for all $t \equiv s_p \mod p$.

For given $X > 0$, define $M = \prod_{c_f \leq p \leq y} p$ for $y = \frac{\log X}{\log \log X}$. Let $s_M$ be the residue class modulo $M$ for which $s_M \equiv s_p \mod p$ for $c_f \leq p \leq y$. We assume the modularity of the $n-1$ dimensional Galois representation $\rho$ for which $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$.

Then we define a set

$$L(X) = \{ \frac{X}{2} < t < X \mid t \equiv s_M \mod M \}$$

Each $t \in L(X)$ gives rise to an cuspidal automorphic $L$-functions $L(s, \rho, t)$ of $GL(n-1)/\mathbb{Q}$. There is an important question which we may fail to notice. It is possible that two different $t_1$ and $t_2$ correspond to the same $L$-function. This is equivalent to the fact that $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$. When $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$, we say that $K_{t_1}$ and $K_{t_2}$ are arithmetically equivalent. For number fields of small degree, they should be conjugate. More precisely;

**Theorem 3.3** (Klingen [38]). Let $K|k$ be an extension of number fields of degree $n \leq 11$ and assume there exists some non-conjugate field $K'$ being arithmetically equivalent to $K$.
over \( k \). Then up to conjugacy only the following four cases are possible for \( G = G(\overline{K}|k) \):

- \( n = 7 : \quad G = GL_3(2) \)
- \( n = 8 : \quad G = \mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/8\mathbb{Z})^\times, G = GL_2(3) \)
- \( n = 11 : \quad G = PSL_2(11) \)

On the other hand, since the conductor of \( L(s, \rho, t) \) is \( |d_{K_t}| \), different discriminant \( d_{K_t} \) distinguishes the L-functions \( L(s, \rho, t) \). To determine the discriminant or its factor, we impose some extra conditions on \( t \) which depends on \( f(x, t) \). Assume that the discriminant of \( f(x, t) \) is a polynomial in \( t \) of degree \( D \). Then there is a constant \( C \) such that

\[
|d_{K_t}| \leq Ct^D.
\]

For \( A = C^{1/D}X \), we define a set \( L(A) \) of positive integers

\[
L(A) = \{ \frac{X}{2} < t < X \mid t \equiv s_M \mod M, \text{ some condition on } t \}.
\]

In the cases in consideration, we will show that \( |L(A)| \gg X^{1-\epsilon} \) for any \( \epsilon > 0 \).

Let \( c_0 = \frac{5(n-1)D}{2} + 1 \). Or we may replace \( (n-1) \) in \( c_0 \) by a smaller constant if \( \rho \) is not irreducible. Choose \( \alpha \) with \( c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with \( e = D, d = 1 \) and \( T = (\log A^D)^2 \), every automorphic L-function excluding exceptional \( O(A^{98/100}) \) L-functions has a zero-free region \([\alpha, 1] \times [- (\log |d_{K_t}|)^2, (\log |d_{K_t}|)^2]\). Let us denote by \( \widehat{L}(A) \), the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.6 to this L-function \( L(s, \rho, t) \) in \( \widehat{L}(A) \), we have

\[
\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1)
\]

\[
= (n - 1) \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n, \alpha}(1) = (n - 1) \log \log \log |d_{K_t}| + O_{n, \alpha}(1).
\]

where we use the fact that \( \sqrt{\log |d_{K_t}|} < y = \frac{\log X}{\log \log X} \) for large \( X \).

By the class number formula and the size of regulator \( R_{K_t} \), we have the required result

\[
h_{K_t} \gg |d_{K_t}|^{1/2} (\log \log |d_{K_t}|)^{n-1} (\log |d_{K_t}|)^{r-\alpha}.
\]
3.3 Symmetric Groups

3.3.1 $S_5$ Quintic Extensions with signature $(1, 2)$

In section 2.4.2, we showed that

$$f(x, t) = (x + 5)(x^2 + 5t)(x^2 + 10t) + t$$

gives rise to an $S_5$ regular extension. Since $disc(f(x, t))$ is a polynomial of degree 14, there is a constant $C$ with $disc(f(x, t)) \leq Ct^{14}$. For given $X > 0$, let $A = C^{1/14}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \left\{ \frac{X}{2} < t < X \mid t: \text{square-free } t \equiv 1 \mod 5, t \equiv s_M \mod M, \text{Gal}(\mathcal{K}_t/\mathbb{Q}) = S_5 \right\}$$

where $s_M$ and $M$ are defined as in section 3.2. Then we have

$$|L(A)| = \frac{6}{\pi^2} \frac{25}{24} \prod_{p|N} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{2M} + O(X^{1/2} \log X).$$

Every $t$ in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_{\mathcal{K}_t}(s)}{\zeta(s)}$ which is actually an cuspidal automorphic L-function of $GL(4)/\mathbb{Q}$ since $\rho$ is modular by Theorem 1.4.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that $\mathcal{K}_t$ are not conjugate each other.

**Lemma 3.5.** For a square-free $t$ with $t \equiv 1 \mod 5$,

$$p \text{ totally ramifies in } \mathcal{K}_t \text{ if and only if } p \text{ divides } t.$$ 

**Proof.** Since $f(x, t)$ is an Eisenstein polynomial, if $p \mid t$, $p$ ramifies totally. See Corollary 6.2.4 in [11]. If $p = 5$, then $f(x, t) \equiv (x + 1)^5 \mod 5$. However, by Newton polygon method, $5O_{\mathcal{K}_t} = p_1^5p_2$.

Now assume that $p$ ramifies totally and does not divide 5 and $t$. Then

$$f(x, t) \equiv (x + a)^5 \mod p.$$
By comparing coefficients of $f(x, t)$ and $(x + a)^5 \mod p$, we can induce a contradiction and we finish the proof.

\[ \square \]

Let $c_0 = 141$. Choose $\alpha$ with $c_0 \frac{1-\alpha}{2-\alpha} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 14$, $d = 1$ and $T = (\log A^{14})^2$, every automorphic $L$-function excluding exceptional $O(A^{98/100}) L$-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_K|)^2, (\log |d_K|)^2]$. Let us denote by $\hat{L}(A)$, the set of the automorphic $L$-functions with the zero-free region.

Applying Proposition 1.7 to this $L$-function $L(s, \rho, t)$ in $\hat{L}(A)$, we have

$$
\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log |d_K|}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) \quad (3.6)
$$

$$
= 4 \sum_{c_f \leq p \leq \sqrt{\log |d_K|}} \frac{1}{p} + O_{n, \alpha}(1) = 4 \log \log |d_K| + O_{n, \alpha}(1).
$$

By the class number formula and Lemma 2.31,

$$
h_K \gg d_K^{1/2} \frac{(\log \log d_K)^4}{(\log d_K)^2}.
$$

We summarize as follows:

**Theorem 3.7.** There is a constant $c > 0$ such that there exist $K \in \mathcal{R}(5, S_5, 1, 2)$ with arbitrary large discriminant $d_K$ for which

$$
h_K > cd_K^{1/2} \frac{(\log \log d_K)^4}{(\log d_K)^2}.
$$

**3.3.2 S$_4$ Quartic Extensions with signature (4, 0)**

In section 2.4.3, we stated the fact that

$$
f(x, t) = (x - t)(x - 2^2 t)(x - 3^2 t)(x - 4^2 t) - t
$$
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gives rise to an $S_4$ regular extension. Since $\text{disc}(f(x,t))$ is a polynomial of degree 12, there is a constant $C$ with $\text{disc}(f(x,t)) \leq Ct^{12}$. For given $X > 0$, let $A = C^{1/12}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \left\{ \frac{X}{2} < t < X \mid t: \text{square-free}, t \equiv s_M \mod M, \text{Gal}(\widehat{K_t}/\mathbb{Q}) = S_4 \right\}$$

where $s_M$ and $M$ are defined as in section 3.2. Then we have

$$|L(A)| = \frac{6}{\pi^2} \prod_{p|M} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{2M} + O\left(X^{1/2} \log X\right).$$

Every $t$ in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_K(s)}{\zeta(s)}$ which is actually a cuspidal automorphic L-function of $GL(3)/\mathbb{Q}$ since $\rho$ is modular by Proposition 1.1.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that $K_t$ are not conjugate each other.

Lemma 3.8. For a square-free $t$,

$p$ totally ramifies in $K_t$ if and only if $p$ divides $t$.

Proof. Since $f(x,t)$ is an Eisenstein polynomial, if $p | t$, $p$ ramifies totally. See Corollary 6.2.4 in [11]. Now assume that $p$ ramifies totally and does not divide $t$. If $t$ is even, then $p$ is not equal to $2$. If $t$ is odd, then $p$ can be even. However, if $p = 2$, then $f_t(x) \equiv (x+a)^4 \pmod{2}$ and we have $6a^2 \equiv 273t^3 \pmod{2}$ hence it induces a contradiction. So we can assume that $p \neq 2$. Also we can see easily that $a \not\equiv 0 \pmod{p}$ for totally ramified $p$.

If $p = 3$, we have $a \equiv 0 \pmod{p}$, hence $3$ should be excluded. Since $p \neq 2, 3$, we have the following system of equations mod $p$.

$$a \equiv -\frac{15}{2} t, \quad a^2 \equiv \frac{91}{2} t^2, \quad a^3 \equiv -205t^3 \pmod{p}$$

and we can check that this system is inconsistent and we finish the proof.
Let $c_0 = 91$. Choose $\alpha$ with $c_0^{1-\alpha} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 12$, $d = 1$ and $T = (\log A^{12})^2$, every automorphic $L$-function excluding exceptional $O(A^{98/100})$ $L$-functions has a zero-free region $[\alpha, 1] \times [-\log d_{K_t}, \log d_{K_t}]$. Let us denote by $\hat{L}(A)$, the set of the automorphic $L$-functions with the zero-free region.

Applying Proposition 1.7 to this $L$-function $L(s, \rho, t)$ in $\hat{L}(A)$, we have

$$ \log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) $$ (3.9)

$$ = 3 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log d_{K_t} + O_{n, \alpha}(1). $$

By the class number formula and Lemma 2.32,

$$ h_{K_t} \gg d_{K_t}^{1/2} \left( \frac{\log \log d_{K_t}}{\log d_{K_t}} \right)^3. $$

We summarize as follows:

**Theorem 3.10.** There is a constant $c > 0$ such that there exist $K \in \mathcal{K}(4, S_4, 4, 0)$ with arbitrary large discriminant $d_K$ for which

$$ h_K > cd_K^{1/2} \left( \frac{\log \log d_K}{\log d_K} \right)^3. $$

### 3.3.3 $S_4$ Quartic Extensions with signature $(2,1)$

In section 2.4.1, we showed that

$$ f(x, t) = x^2(x - 10t)(x - 18t) + t $$

gives rise to an $S_4$ regular extension. Since $\text{disc}(f(x, t))$ is a polynomial of degree 9, there is a constant $C$ with $|\text{disc}(f(x, t))| \leq Ct^9$. For given $X > 0$, let $A = C^{1/14}X$. We define a set $L(A)$ of square-free integers

$$ L(A) = \left\{ \frac{X}{2} < t < X \mid t: \text{square-free}, t \equiv 1 \bmod 2, t \equiv s_M \bmod M, \text{Gal}(\overline{K_t}/\mathbb{Q}) = S_4 \right\} $$
where $s_M$ and $M$ are defined as in section 3.2. Then we have

$$|L(A)| = \frac{6}{\pi^2} \frac{4}{3} \prod_{p|\mathcal{M}} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{2M} + O(X^{1/2} \log X).$$

Every $t$ in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ which is actually an cuspidal automorphic L-function of $GL(3)/\mathbb{Q}$ since $\rho$ is modular by Proposition 1.1.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that $K_t$ are not conjugate to each other.

**Lemma 3.11.** For a square-free $t$,

$$p \text{ totally ramifies in } K_t \text{ if and only if } p \mid t \text{ or } p = 2.$$

**Proof.** Since $f(x, t)$ is an Eisenstein polynomial, if $p \mid t$, $p$ ramifies totally. See Corollary 6.2.4 in [11]. When $p = 2$, we can not determine the prime decomposition of 2 in $K_t$.

Now assume that $p$ ramifies totally and does not divide 2 and $t$.

Then

$$f(x, t) \equiv (x + a)^4 \mod p.$$

By comparing coefficients of $f(x, t)$ and $(x + a)^4 \mod p$, we can induce a contradiction and we finish the proof.

Let $c_0 = 68$. Choose $\alpha$ with $c_0 \frac{1-\alpha}{2n-1} < \frac{95}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 9$, $d = 1$ and $T = (\log A^9)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, (\log |d_{K_t}|)^2]$. Let us denote by $\hat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\hat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\chi(p, t)}{p} + O_{n, \alpha}(1)$$

$$= 3 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log \log |d_{K_t}| + O_{n, \alpha}(1).$$
By the class number formula and Lemma 2.29,

\[ h_{K_t} \gg |d_{K_t}|^{1/2} \left( \frac{\log \log |d_{K_t}|}{\log |d_{K_t}|} \right)^3. \]

We summarize as follows:

**Theorem 3.13.** There is a constant \( c > 0 \) such that there exist \( K \in \mathfrak{R}(4, S_4, 2, 1) \) with arbitrary large discriminant \( d_K \) for which

\[ h_K > c|d_K|^{1/2} \left( \frac{\log \log |d_K|}{\log |d_K|} \right)^3. \]

### 3.3.4 \( S_4 \) Quartic Extensions with signature \((0, 2)\)

In section 2.4.1, we showed that

\[ f(x, t) = x^4 + tx^2 + tx + t \]

gives rise to an \( S_4 \) regular extension. Since \( \text{disc}(f(x, t)) \) is a polynomial of degree 5, there is a constant \( C \) with \( \text{disc}(f(x, t)) \leq Ct^5 \). For given \( X > 0 \), let \( A = C^{1/5}X \). We define a set \( L(A) \) of square-free integers

\[ L(A) = \{ \frac{X}{2} < t < X \mid t : \text{square-free}, t \equiv s_M \mod M, \text{Gal}(\overline{K_t}/\mathbb{Q}) = S_4 \} \]

where \( s_M \) and \( M \) are defined as in section 3.2. Then we have

\[ |L(A)| = \frac{6}{\pi^2} \prod_{p|M} \left( 1 - \frac{1}{p^2} \right)^{-1} \frac{X}{M} + O(X^{1/2} \log X). \]

Every \( t \) in \( L(A) \) corresponds an Artin L-function \( L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)} \) which is actually an cuspidal automorphic L-function of \( GL(3)/\mathbb{Q} \) since \( \rho \) is modular by Proposition 1.1.

We claim that every \( L(s, \rho, t) \) for \( t \in L(A) \) is distinct. It follows from the fact that \( K_t \) are not conjugate to each other.

**Lemma 3.14.** For a square-free \( t \),

\[ p \text{ totally ramifies in } K_t \text{ if and only if } p \text{ divides } t. \]
Proof. Since \( f(x, t) \) is an Eisenstein polynomial, if \( p \mid t \), \( p \) ramifies totally. See Corollary 6.2.4 in [11].

Now assume that \( p \) ramifies totally and does not divide \( t \).

Then

\[
 f(x, t) \equiv (x + a)^4 \mod p.
\]

By comparing coefficients of \( f(x, t) \) and \((x+a)^4 \mod p\), we can see that the only possibility is \( p = 3 \) and \( t \equiv 1 \mod 3 \). However, \( x^4 + x^2 + 1 \) is irreducible mod 3, this case is also excluded and we finish the proof. \( \square \)

Let \( c_0 = 39 \). Choose \( \alpha \) with \( c_0^{\frac{1}{25} - 1} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with \( e = 5, \ d = 1 \) and \( T = (\log A)^2 \), every automorphic \( L \)-function excluding exceptional \( O(A^{98/100}) \) \( L \)-functions has a zero-free region \([\alpha, 1] \times [-\log(\log|d_K|)]^2, (\log|d_K|)^2]\). Let us denote by \( \hat{L}(A) \), the set of the automorphic \( L \)-functions with the zero-free region.

Applying Proposition 1.7 to this \( L \)-function \( L(s, \rho, t) \) in \( \hat{L}(A) \), we have

\[
 \log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_K}} \frac{\lambda (p, t)}{p} + O_{n, \alpha}(1) \quad (3.15)
\]

\[
 = 3 \sum_{c_f \leq p \leq \sqrt{\log d_K}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log \log d_K + O_{n, \alpha}(1).
\]

By the class number formula and Lemma 2.30,

\[
h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^3}{(\log d_{K_t})}.
\]

We summarize as follows:

**Theorem 3.16.** There is a constant \( c > 0 \) such that there exist \( K \in \mathcal{K}(4, S_4, 0, 2) \) with arbitrary large discriminant \( d_K \) for which

\[
h_K > cd_K^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)}.
\]
3.3.5  $S_n$ number fields of degree $n$ with $(n,0)$

In section 2.4.3, we stated the fact that

$$f_t(x) = (x - t)(x - 2^2 t)(x - 3^2 t) \cdots (x - n^2 t) - t$$

gives rise to an $S_n$ regular extension. Since $\text{disc}(f_t(x)) = t^{n-1}g(t)$ is a polynomial of degree $n^2 - n$, there is a constant $C$ with $\text{disc}(f_t(x)) \leq Ct^{n^2 - n}$. For given $X > 0$, let $A = C^{\frac{1}{n^2 - n}}X$.

We define a set $L(A)$ of square-free integers

$$L(A) = \{ \frac{X}{2} < t < X \mid t : \text{square-free}, \nu(t) \leq 2, t \equiv s_M \mod M, \text{Gal}(\hat{K}_t/Q) = S_n \}$$

where $\nu(t)$ is the number of distinct prime divisors of $t$.

Here the reason why we impose the extra condition $\nu(t) \leq 2$ in $L(A)$ is to distinguish $L(s, \rho, t)$ for $t \in L(A)$. It would be possible to prove $p$ ramifies totally in $K_t$ if and only if $p \mid t$. However, we don’t know how to prove it. We explain below how to distinguish $L(s, \rho, t)$ with the condition $\nu(t) \leq 2$ in detail.

The following Proposition makes us able to estimate $|L(A)|$.

**Proposition 3.17.** Let $x$ be sufficiently large positive integer and $l$ be a positive integer of the size of $x^u$ for some real number $0 < u \leq 0.3$. For any integer $b$ with $(b, l) = 1$,

$$| \{ 0 < n < x \mid n : \text{square-free} n \equiv b \mod l, \nu(n) \leq 2 \} | \gg \frac{x}{\log x \phi(l)}$$

where $\phi$ is the Euler phi-function.

The proof of Proposition 3.17 can be found in the appendix.

By Theorem 2.2, we have

$$| \{ \frac{X}{2} < t < X \mid \text{Gal}(\hat{K}_t/Q) \cong S_4 \} | = \frac{X}{2} + O(X^{1/2} \log X).$$

By Proposition 3.17, we have

$$| \{ \frac{X}{2} < t < X \mid t : \text{square-free}, \nu(t) \leq 2, t \equiv t_M \mod M \} | \gg X^{1-\epsilon}.$$
hence we have that $|L(A)| \gg X^{1-\epsilon}$.

We must assume that the $n-1$ dimensional representation $\rho$ is modular because we don’t have the modularity theorem for $S_n$ with $n \geq 5$ except some special cases in $S_5$.

Now there is possibility that different square-free integers in $L(A)$ may correspond to an same automorphic L-function. Since the conductor of $L(s, \rho, t)$ is the discriminant of a number field $K_t$, distinct discriminants distinguish L-functions in $L(A)$. Recall that $\text{disc}(f_t(s)) = t^{n-1}g(t)$. Assume that $K_{t_1}$ and $K_{t_2}$ have the same discriminant for $t_1, t_2 \in L(A)$. If they are co-prime, $g(t_1)$ is divided by $t_2^3$. If they have a common prime divisor, then $g(t_1)$ is divided by $\left(\frac{t_2}{(t_1, t_2)}\right)^3$. The number of all possible repetition of the same L-function in $L(A)$ is at most $\ll \nu(g(t))^2$. This means that $K_t$ has at most $\nu(g(t))^2$ number fields which have the same discriminant. It is well-known that $\nu(n) < \log n$. (See p. 167 in [56]). Hence $\nu(g(t))^2 = O(\log^2 X)$ for all $t$ in $L(A)$. Hence $L(A)$ has at least $\gg X^{1-\epsilon}$ distinct L-functions. Let $\tilde{L}(A)$ be the set of distinct L-functions coming from $L(A)$.

Let $c_0 = \frac{5(n-1)(n^2-n)}{2} + 1$. Choose $\alpha$ with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $\tilde{L}(A)$ with $e = n^2 - n$, $d = 1$ and $T = (\log A^{n^2-n})^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\hat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\hat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1)$$

$$= (n-1) \sum_{c_i \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = (n-1) \log \log \log d_{K_t} + O_{n, \alpha}(1).$$

By the class number formula and Lemma 2.32,

$$h_{K_t} \gg d_{K_t}^{1/2} \left(\frac{\log \log d_{K_t}}{\log d_{K_t}}\right)^{n-1}.$$
We summarize as follows:

**Theorem 3.19.** Assume the Strong Artin conjecture for $S_n$, $n \geq 5$ is true. Fix $n$. There is a constant $c > 0$ such that there exist $K \in \mathcal{R}(n, S_n, n, 0)$ with arbitrary large discriminant $d_K$ for which

$$h_K > cd_K^{1/2} \left( \frac{\log \log d_K}{\log d_K} \right)^{n-1}.$$

### 3.4 Alternating Groups

#### 3.4.1 $A_4$ quartic extension with $(0, 2)$

In section 2.3.1, we showed that

$$f(x, t) = x^4 - 8tx^3 + 18t^2x^2 + 1$$

gives rise to an $A_4$ regular extension. Since $\text{disc}(f(x, t))$ is a polynomial of degree 8, there is a constant $C$ with $\text{disc}(f(x, t)) \leq Ct^8$. For given $X > 0$, let $A = C^{1/8}X$. We define a set $L(A)$ of cubic-free integers:

$$L(A) = \{0 < t < X : 27t^4 + 1 \text{ cubic-free and } t \equiv s_M \mod M \}.$$

We prove the following lemma, which is a direct consequence of [28], page 69.

**Lemma 3.20.** Let $f(x)$ be an irreducible polynomial of degree $d \geq 3$ in $\mathbb{Z}[x]$. Suppose that if $p|M$, then $f(a) \not\equiv 0 \pmod{p}$. Let $N(X, f, M)$ be the number of integers $1 \leq n < X$ and $n \equiv a \pmod{M}$, with the property that $f(n)$ is $(d-1)$-free. Then

$$N(X, f, M) = C(M) \frac{X}{M} + O \left( \frac{X}{M} \log \frac{X}{M} \frac{1}{\log p^k} \right),$$

where $C(M) = \prod_{p|M} (1 - \frac{\rho(p^k)}{p^k})$, and $\rho(p^k)$ is the number of solutions for $f(x) \equiv 0 \pmod{p^k}$. 


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Proof. Let \( n = Mm + a \), and \( g(m) = f(Mm + a) \). Then \( 1 \leq m < \frac{X}{M} \), and \( g(x) \) is an irreducible polynomial of degree \( d \). Hooley [28], page 69 showed that the number of \( 1 \leq m < \frac{X}{M} \) with the property that \( g(m) \) is \((d - 1)\)-free, is

\[
C(M) \frac{X}{M} + O \left( \frac{X}{M} (\log \frac{X}{M})^{\frac{2}{d+1}} \right),
\]

where \( C(M) = \prod_p (1 - \frac{\rho(p^d)}{p^d}) \), and \( \rho(p^d) \) is the number of solutions for \( g(m) \equiv 0 \) (mod \( p^d \)). If \( p|M, g(m) \equiv f(a) \neq 0 \) (mod \( p \)). Hence \( \rho(p^d) = 0 \). If \( p \nmid M \), then since \( Mm + a \equiv 0 \) (mod \( p^d \)) has a unique solution mod \( p^d \), \( \rho(p^d) \) = \( \rho(p^d) \). Our result follows.

Note that by the definition of \( M, t_M \), if \( p|M, \) then \( c \leq p \leq y, \) \( t_M \equiv t_p \) (mod \( p \)), and \( p \) splits completely in \( \hat{K}_t \). Then \( 27(Mm + t_M)^4 + 1 \equiv 27t_p^4 + 1 \neq 0 \) (mod \( p \)). Hence \( \rho(p) = 0 \). This implies \( \rho(p^3) = 0 \). If \( p \nmid M \), by Nagell [49], page 87, \( \rho(p^3) = \rho(p) \leq 4 \). So

\[
\prod_{p|M} (1 - \frac{\rho(p^3)}{p^3}) \gg \prod_{p|M} (1 - 4p^{-3}) \geq \prod_{p|M} (1 - p^{-3})^5 \geq \zeta(3)^{-5}.
\]

Hence by the above lemma, \( |L(A)| = C(M) \frac{X}{M} + O \left( \frac{X}{M} (\log \frac{X}{M})^{\frac{2}{d+1}} \right) \), and \( |L(A)| \gg \frac{X}{M} \). In the remark below, we use the recent result of Heath-Brown [27] to obtain a better error term in \( |L(A)| \).

Here different \( t_1, t_2 \in L(A) \) may give rise to a same L-function. We claim that at most 32 different \( t \)'s correspond to the same L-function. We owe Ankeny, Brauer and Chowla [1] for this idea.

First, we need to know the locations of the roots more precisely. By applying Lemma 2.4 with \( \alpha = 4t + 1.4ti \), for sufficiently large \( t \), we find a complex root inside of the circle of radius \( 0.03\sqrt{2}t \) centered at \( 4.015t + 1.385ti \). Again by applying Lemma 2.4 with \( \alpha = \frac{0.23i}{t} \), for sufficiently large \( t \), we find a complex root inside of the circle of radius \( \frac{0.0115}{t} \) centered at \( \frac{0.2415i}{t} \). Here we use PARI to choose a suitable \( \alpha \).

We need to order the roots of \( f(x, t) \) in the following way. Let \( \theta_1^t \) be the root near the origin whose imaginary part is positive and \( \theta_2^t = \overline{\theta_1^t} \). Let \( \theta_3^t \) be the other root whose
imaginary part is positive and \( \theta_1^t = \overline{\theta_3^t} \). Let \( \rho \) be the complex embedding of \( K_t \) which maps \( \theta_1^t \) to \( \theta_3^t \).

If \( t_1, t_2 \in L(A) \) correspond to the same L-function, \( \mathbb{Q}(\theta_1^{t_1}) \) and \( \mathbb{Q}(\theta_1^{t_2}) \) are isomorphic, since they are quartic fields ([38], page 93). Hence \( \mathbb{Q}(\theta_1^{t_1}) = \mathbb{Q}(\theta_1^{t_2}) \) for some \( 1 \leq j \leq 4 \). Assume that 33 different \( t_1, t_2, \ldots, t_{33} \) give rise to the same L-function. Then we can see that there are at least nine \( t_{i1}, t_{i2}, \ldots, t_{i9} \) with \( \mathbb{Q}(\theta_1^{t_{i1}}) = \mathbb{Q}(\theta_1^{t_{i2}}) = \cdots = \mathbb{Q}(\theta_1^{t_{i9}}) \) for some \( 1 \leq k \leq 4 \). Without the loss of the generality, we assume \( k = 1 \). Then there are at least two \( t_{i1}, t_{i2} \) such that \( \rho : \theta_1^{t_{i1}} \to \theta_3^{t_{i1}} ; \rho : \theta_1^{t_{i2}} \to \theta_3^{t_{i2}} \). Now we further assume that \( 0 < \theta_1^{t_{i1}} - \theta_1^{t_{i2}} \neq 0 \), it induces a contraction. So there are at most 32 \( t \)'s corresponding to the same L-function.

Let \( \tilde{L}(A) \) be the set of distinct L-functions coming from \( L(A) \). Then, we have

\[
A^{1-\epsilon} \ll |L(A)| \ll A.
\]

Let \( c_0 = 61 \). Choose \( \alpha \) with \( c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100} \). By applying Theorem 1.10 to \( \tilde{L}(A) \) with \( e = 8, d = 1 \) and \( T = (\log A^8)^2 \), every automorphic L-function excluding exceptional \( O(A^{98/100}) \) L-functions has a zero-free region \([\alpha, 1] \times [-\log d_{K_t}^2, (\log d_{K_t})^2]\). Let us denote by \( \hat{L}(A) \), the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function \( L(s, \rho, t) \) in \( \hat{L}(A) \), we have

\[
\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) \quad (3.21)
\]

\[
= 3 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log \log |d_{K_t}| + O_{n, \alpha}(1).
\]

By the class number formula and Lemma 2.23,

\[
h_{K_t} \gg d_{K_t}^{1/2} \left( \frac{\log \log d_{K_t}}{\log d_{K_t}} \right)^3.
\]
We summarize as follows:

**Theorem 3.22.** There is a constant $c > 0$ such that there exist $K \in \mathcal{K}(4, A_4, 0, 2)$ with arbitrary large discriminant $d_K$ for which

$$h_K > cd_K^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)}.$$ 

**Remark 3.23.** Let $M \sim X^{\delta'}$ with $0 < \delta' < \delta$, where $\delta$ is the constant in [27]. Then we have

$$|L(A)| = \prod_{p \mid M} \left(1 - \frac{\rho(p^3)}{p^3}\right) \frac{X}{M} + O(X^{1-\delta}).$$

where $\rho(p^3)$ is the number of solutions to $27t^4 + 1 \equiv 0 \pmod{p^3}$.

**Proof.** We have

$$|L(A)| = N(X, t_M, M) = \sum_{b, (b, M) = 1} \mu(b)N(b; X, t_M, M)$$

with $N(b; X, t_M, M) = \#\{0 < t < X : b^3 \mid 27t^4 + 1 \text{ and } t \equiv t_M \pmod{M}\}$. Note that if $p|M$, by the definition of $t_M$, $27t^4 + 1 \equiv 27t_M^4 + 1 \not\equiv 0 \pmod{p}$.

Denote the solutions to $27t^4 + 1 \equiv 0 \pmod{b^3}$ by $n_1, n_2, \cdots, n_r$. Then

$$N(b; X, t_M, M) = \sum_{i \leq r} \#\{0 < t < X : t \equiv n_i \pmod{b^3} \text{ and } t \equiv t_M \pmod{M}\}$$

$$= \sum_{i \leq r} \left(\frac{X}{b^3 M} + O(1)\right) = \frac{X}{b^3 M} \rho(b^3) + O(\rho(b^3)).$$

It is easy to show

$$\sum_{\substack{b \leq X^{1/2} \\ (b, M) = 1}} \frac{\mu(b)\rho(b^3)}{b^3} = \prod_{p \mid M} \left(1 - \frac{\rho(p^3)}{p^3}\right) + O(X^{-1+\epsilon})$$

and

$$\sum_{b \leq X^{1/2}} \rho(b^3) = O(X^{1/2+\epsilon}).$$
Hence
\[
\sum_{b \leq X^{1/2}} \mu(b)N(b; X, t_M, M) = \prod_{p|M} \left(1 - \frac{\rho(p^3)}{p^3}\right) \frac{X}{M} + O(X^{1/2+\epsilon}).
\]

Note that \(27t^4 + 1 = (27)^{-3}((27t)^4 + 27^3)\). Hence by [27], we obtain
\[
\sum_{b \geq X^{1/2}} \mu(b)N(b; X, t_M, M) = O(X^{1-\delta}),
\]
where \(\delta\) is the constant in [27]. This proves our assertion.

\[\square\]

### 3.5 Dihedral Groups

#### 3.5.1 Representations of Dihedral Groups

Let’s review irreducible representations of \(D_n\): If \(n\) is odd, \(D_n = \langle r, s : r^n = s^2 = e, srs = r^{-1} \rangle\). Let \(H = \{1, s\}\). Then irreducible representations of \(D_n\) are: 2 one-dimensional representations \(1, \chi\), and \(\frac{n-1}{2}\) two-dimensional representations \(\rho_1, \ldots, \rho_{\frac{n-1}{2}}\), where \(\chi\) is the character of \(H\). We have \(\text{Ind}^G_H 1 = 1 + \rho_1 + \cdots + \rho_{\frac{n-1}{2}}\).

If \(n\) is even, \(D_n = \langle r, s : r^n = s^2 = e, srs = r^{-1} \rangle\). Let \(H_1 = \{1, s\}, H_2 = \{1, r^{\frac{n}{2}}\}, H_3 = \{1, r^{\frac{n}{2}}s\}\) be three order 2 subgroups. Then irreducible representations of \(D_n\) are: 4 one-dimensional representations \(1, \chi_1, \chi_2, \chi_3\), and \(\frac{n-2}{2}\) two-dimensional representations \(\rho_1, \ldots, \rho_{\frac{n-2}{2}}\), where \(\chi_i\) is the character of \(H_i\). We have, for each \(i\), \(\text{Ind}^G_{H_1} 1 = 1 + \chi_i + \rho_1 + \cdots + \rho_{\frac{n-2}{2}}\).

Let \(K/\mathbb{Q}\) be a degree \(n\) extension and \(\hat{K}/\mathbb{Q}\) be the Galois closure such that \(\text{Gal}(\hat{K}/\mathbb{Q}) \simeq D_n\). Then if \(n\) is odd,
\[
\frac{\zeta_K(s)}{\zeta(s)} = L(s, \rho_1) \cdots L(s, \rho_{\frac{n-1}{2}}).
\]
Especially when \(n = 5\), \(\frac{\zeta_K(s)}{\zeta(s)} = L(s, \rho_1)L(s, \rho_2)\), the Artin conductors of \(\rho_1\) and \(\rho_2\) are \(|d_K|^\frac{1}{2}\).

If \(n\) is even,
\[
\frac{\zeta_K(s)}{\zeta(s)} = L(s, \chi)L(s, \rho_1) \cdots L(s, \rho_{\frac{n-2}{2}}),
\]
where $H = \text{Gal}(\bar{K}/K)$ is one of the order 2 subgroup of $D_n$, and $\chi$ is the non-trivial character of $H$.

Especially, when $n = 4$, \( \zeta_{\hat{K}}(s) = L(s, \chi)L(s, \rho) \) where $\rho$ is the unique two-dimensional representation of $D_4$. Then we can see that the Artin conductor of $\chi$ is the absolute value of discriminant $d_k$ of quadratic subfield of $K$ and the Artin conductor of $\rho$ equals to $\left| \frac{d_k}{d_k^*} \right|$.

### 3.5.2 $D_5$ quintic extensions with signature $(1, 2)$

In section 2.2.3, we showed that

\[
f(x, t) = x^5 - tx^4 + (2t - 1)x^3 - (t - 2)x^2 - 2x + 1
\]

gives rise to an $D_5$ regular extension. When $t \leq 6$, the signature of $K_t$ is $(1, 2)$. We prefer that $t$ varies in positive integers, we use $f(x, t) = x^5 + tx^4 - (2t + 1)x^3 + (t + 2)x^2 - 2x + 1$ from now on. Then its discriminant is $(4t^3 + 28t^2 + 24t + 47)^2$. Since $\text{disc}(f(x,t))$ is a polynomial of degree 6, there is a constant $C$ with $\text{disc}(f_t(x)) \leq Ct^6$. For given $X > 0$, let $A = C^{1/6}X$. We define a set $L(A)$ of square-free integers for $A = C^{1/6}X$,

\[
L(A) = \{ \frac{X}{2} < t < X \mid 4t^3 + 28t^2 + 24t + 47 \text{ square-free, } t \equiv s_M \text{ mod } M \}.
\]

By Lemma 3.20, $|L(A)| = \beta \frac{X}{2M} + O\left( \frac{X}{M(\log \frac{X}{M})^{1/2}} \right)$ for some constant $\beta$. Hence $|L(A)| \gg A^{1-\epsilon}$. Note that every $t \in L(A)$ corresponds to a distinct automorphic $L$-function $L(s, \rho_t) = \sum_{n=1}^{\infty} \lambda_t(n)n^{-s}$ of $GL(4)/\mathbb{Q}$ with $\lambda_t(q) = 4$ for all $k \leq q \leq y$.

Here $\rho_t$ is not irreducible any longer but a sum of two 2-dimensional representations $\sigma_{1,t}$ and $\sigma_{2,t}$ of $D_5$. $D_5$ has the cyclic subgroup $C_5$ of order 5 and let $M_t$ be the fixed field by $C_5$. Then $\sigma_{1,t}, \sigma_{2,t}$ are induced by non-trivial characters for $\hat{K}_t/M_t$. Hence Artin conductors of $\sigma_1$ and $\sigma_2$ equal $|d_M|N_{M/Q}(b_t)$ where $b_t$ is the Artin conductor of nontrivial characters of $C_5$. Since $d_{K_t} = (4t^3 + 28t^2 + 24t + 47)^2 = (|d_{M_t}|N_{M_t/Q}(b_t))^2$ for $4t^3 + 28t^2 + 24t + 47$ square-free, the Artin conductors of $\sigma_{1,t}, \sigma_{2,t}$ are both $4t^3 + 28t^2 + 24t + 47$. 
Hence, the Hypotheses of Theorem 1.10 are satisfied. Let \( c_0 = 31 \). Choose \( \alpha \) with \( c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with \( e = 6 \), \( d = 1 \) and \( T = (\log A^6)^2 \), every automorphic \( L \)-function excluding exceptional \( O(A^{98/100}) \) \( L \)-functions has a zero-free region \([\alpha, 1] \times [-\log d_{K_t}]^2, (\log d_{K_t})^2 \]. Let us denote by \( \hat{L}(A) \), the set of the automorphic \( L \)-functions with the zero-free region.

Applying Proposition 1.7 to this \( L \)-function \( L(s, \rho, t) \) in \( \hat{L}(A) \), we have

\[
\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1)
\]

(3.24)

\[
= 4 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 4 \log \log d_{K_t} + O_{n, \alpha}(1).
\]

By the class number formula and Lemma 2.21,

\[
h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^4}{(\log d_{K_t})^2}.
\]

We summarize as follows:

**Theorem 3.25.** There is a constant \( c > 0 \) such that there exist \( K \in \mathcal{R}(5, D_5, 1, 2) \) with arbitrary large discriminant \( d_K \) for which

\[
h_K > cd_K^{1/2} \frac{(\log \log d_K)^4}{(\log d_K)^2}.
\]

### 3.5.3 \( D_4 \) quartic extensions with signature \((0, 2)\)

In section 2.2.2, we showed that

\[
f(x, t) = x^4 - x^3 + (t + 2)x^2 - x + 1
\]

gives rise to a \( D_4 \) regular extension. Since \( \text{disc}(f(x, t)) = (t + 2)(t + 6)(1 - 4t)^2 \) is a polynomial of degree 4, there is a constant \( C \) with \( \text{disc}(f_t(x)) \leq Ct^4 \). For given \( X > 0 \), let \( A = C^{1/4}X \). We define a set \( L(A) \) of square-free integers for \( A = C^{1/4}X \),

\[
L(A) = \left\{ \frac{X}{2} < t < X \mid (t + 2)(t + 6)(1 - 4t) \text{ square-free}, t \equiv s_M \mod M \right\}.
\]
For \( t \in L(A) \), \(|1 - 4t|\) is the Artin conductor of the one-dimensional representation and 
\(|(t+2)(t+6)(1-4t)|\) is the Artin conductor of the two-dimensional representation. Hence 
we can apply Theorem 1.10 to \( L(A) \).

To estimate \(|L(A)|\), we introduce Nair’s work.[50]. For an polynomial \( f(x) \in \mathbb{Z}[x] \) of 
degree \( d \), we define,

\[
N_k(f, x, h) = N_k(x, h) = |\{n : x < n \leq x + h | f(n) : \text{k-free}\}|
\]

He showed

**Theorem 3.26 (Nair).** If

\[
f(x) = \prod_{i=1}^{m} (a_ix - b_i)^{\alpha_i}, \text{ and } \alpha = \max_i \alpha_i,
\]

then

\[
N_k(x, h) = \prod_p \left(1 - \frac{\rho(p^2)}{p^2}\right) h + O\left(\frac{h}{(\log h)^{k-1}}\right)
\]

for \( h = x^{(\alpha/2k)+\epsilon} \) if \( k > \alpha \) and \( \epsilon > 0 \).

Theorem 3.26 implies

\[
|L(A)| = \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^2}\right) \frac{X}{2M} + O\left(\frac{X}{M(\log \frac{X}{M})}\right) \gg A^{1-\epsilon}.
\]

Let \( c_0 = 21 \). Choose \( \alpha \) with \( c_0^{\frac{1-\alpha}{2\alpha-1}} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with 
\( e = 4 \), \( d = 1 \) and \( T = (\log A^4)^2 \), every automorphic \( L \)-function excluding exceptional 
\( O(A^{98/100}) \) \( L \)-functions has a zero-free region \([\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]\). Let us 
denote by \( \hat{L}(A) \), the set of the automorphic \( L \)-functions with the zero-free region.

Applying Proposition 1.7 to this \( L \)-function \( L(s, \rho, t) \) in \( \hat{L}(A) \), we have

\[
\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) \tag{3.27}
\]

\[
= 3 \sum_{e_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log d_{K_t} + O_{n, \alpha}(1).
\]
By the class number formula and (2.15),
\[ h_{K_t} \gg d_{K_t}^{-1/2} \frac{(\log \log d_{K_t})^3}{(\log d_{K_t})}. \]
We summarize as follows:

**Theorem 3.28.** There is a constant \( c > 0 \) such that there exist \( K \in \mathcal{K}(4, D_4, 0, 2) \) with arbitrary large discriminant \( d_K \) for which
\[ h_K > cd_{K}^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)}. \]

### 3.5.4 \( D_4 \) quartic extensions with signature \((2, 1)\)

In section 2.2.2, we showed that
\[ f(x, s) = x^4 - sx^3 + 3x^2 - sx + 1 \]
gives rise to a \( D_4 \) regular extension. Since \( \text{disc}(f(x, s)) = (s^2 - 4)^2(25 - 4s^2) \) is a polynomial of degree 6, there is a constant \( C \) with \( \text{disc}(f(x, s)) \leq Cs^6 \). For given \( X > 0 \), let \( A = C^{1/6}X \). We define a set \( L(A) \) of square-free integers
\[ L(A) = \{ \frac{X}{2} < s < X \mid (s^2 - 4)(25 - 4s^2) \text{ square-free}, s \equiv s_M \mod M \}. \]
For \( s \in L(A) \), \( s^2 - 4 \) is the Artin conductor of the one-dimensional representation and \( |(s^2 - 4)(25 - 4s^2)| \) is the Artin conductor of the two-dimensional representation. Hence we can apply Theorem 1.10 to \( L(A) \).

Theorem 3.26 implies
\[ |L(A)| = \prod_{p \mid M} \left( 1 - \frac{\rho(p^2)}{p^2} \right) \frac{X}{2M} + O \left( \frac{X}{M(\log \frac{X}{M})} \right) \gg A^{1-\epsilon}. \]

Let \( c_0 = 31 \). Choose \( \alpha \) with \( c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with \( e = 6, d = 1 \) and \( T = (\log A^6)^2 \), every automorphic \( L \)-function excluding exceptional \( O(A^{98/100}) \) \( L \)-functions has a zero-free region \([\alpha, 1] \times [-(\log |d_{K_1}|)^2, (\log |d_{K_1}|)^2] \) \( L \)-function \( L(s, \rho, s) \) in \( \hat{L}(A) \), we have
\[ \log L(1, \rho, s) = \sum_{p \leq \sqrt{\log |d_{K_s}|}} \frac{\lambda(p, s)}{p} + O_{n, \alpha}(1) \]  
\[ = 3 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_s}|}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log \log |d_{K_s}| + O_{n, \alpha}(1). \]

By the class number formula and (2.17),

\[
 h_{K_s} \gg |d_{K_s}|^{1/2} \left( \frac{\log \log |d_{K_s}|}{(\log |d_{K_s}|)^2} \right)^3.
\]

We summarize as follows:

**Theorem 3.30.** There is a constant \( c > 0 \) such that there exist \( K \in \mathfrak{R}(4, D_4, 2, 1) \) with arbitrary large discriminant \( d_K \) for which

\[
h_K > c|d_K|^{1/2} \left( \frac{\log \log |d_K|}{(\log |d_K|)^2} \right)^3.
\]

### 3.5.5 \( D_4 \) quartic extensions with signature \((4, 0)\)

In section 2.2.2, we showed that

\[
f(x, s) = x^4 - sx^3 - x^2 + sx + 1
\]
gives rise to a \( D_4 \) regular extension. Since \( \text{disc}(f(x, s)) = (s^2-4)(4s^2+9) \) is a polynomial of degree 6, there is a constant \( C \) with \( \text{disc}(f(x, s)) \leq Cs^6 \). For given \( X > 0 \), let \( A = C^{1/6}X \). We define a set \( L(A) \) of square-free integers

\[
L(A) = \{ \frac{X}{2} < s < X \mid (s^2-4)(4s^2+9) \text{ square-free, } s \equiv s_M \mod M \}\].

Note that we need Theorem 4.15 to estimate \( |L(A)| \) since \( 4s^2+9 \) is an irreducible quadratic polynomial. For \( s \in L(A) \), \( s^2 - 4 \) is the Artin conductor of the one-dimensional representation and \( (s^2-4)(4s^2+9) \) is the Artin conductor of the two-dimensional representation. Hence we can apply Theorem 1.10 to \( L(A) \).

As in the previous section, we can show that \( \mathfrak{R}(4, D_4, 4, 0) \) has infinitely many number fields with extreme class numbers and it is summarized as follows:
Theorem 3.31. There is a constant \( c > 0 \) such that there exist \( K \in \mathfrak{R}(4, D_4, 4, 0) \) with arbitrary large discriminant \( d_K \) for which

\[
h_K > cd_K^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)^3}.
\]

3.5.6 \( D_3 \) cubic extensions with signature \((1, 1)\)

In section 2.2.1, we showed that \( f(x, t) = x^3 + tx - 1 \) gives rise to a \( D_3 \) regular extension. Since \( \text{disc}(f(x, t)) = -(4t^3 + 27) \) is a polynomial of degree 3, there is a constant \( C \) with \( |\text{disc}(f_t(x))| \leq Ct^3 \). For given \( X > 0 \), let \( A = \frac{1}{3} X \).

We define a set \( L(A) \) of square-free integers

\[
L(A) = \left\{ \frac{X}{2} < t < X \mid 4t^3 + 27 \text{ square-free}, t \equiv s_M \mod M \right\}.
\]

Note that each \( t \) in \( L(A) \) corresponds to a distinct \( L \)-function \( L(s, \rho, t) \).

By Lemma 3.20, \( |L(A)| = \beta \frac{X}{2M} + O \left( \frac{X}{M \log X} \right)^{3/2} \) for some \( \beta, |L(A)| \gg A^{1-\epsilon} \).

Let \( c_0 = 16 \). Choose \( \alpha \) with \( c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with \( e = 3, d = 1 \) and \( T = (\log A^3)^2 \), every automorphic \( L \)-function excluding exceptional \( O(A^{98/100}) \) \( L \)-functions has a zero-free region \([\alpha, 1] \times [- \log |d_{K_i}|^2, (\log |d_{K_i}|)^2] \). Let us denote by \( \hat{L}(A) \), the set of the automorphic \( L \)-functions with the zero-free region.

Applying Proposition 1.7 to this \( L \)-function \( L(s, \rho, s) \) in \( \hat{L}(A) \), we have

\[
\log L(1, \rho, s) = \sum_{p \leq \sqrt{\log |d_{K_i}|}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) \quad (3.32)
\]

\[
= 2 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_i}|}} \frac{1}{p} + O_{n, \alpha}(1) = 2 \log \log \log |d_{K_i}| + O_{n, \alpha}(1).
\]

By the class number formula and (2.13),

\[
h_{K_i} \gg |d_{K_i}|^{1/2} \frac{(\log \log |d_{K_i}|)^2}{(\log |d_{K_i}|)}.
\]
We summarize as follows:

**Theorem 3.33.** There is a constant $c > 0$ such that there exist $K \in \mathcal{K}(3, D_3, 1, 1)$ with arbitrary large discriminant $d_K$ for which

$$h_K > c|d_K|^{1/2}\frac{(\log \log |d_K|)^2}{(\log |d_K|)}.$$ 

### 3.6 Cyclic Groups

#### 3.6.1 Simplest Sextic Fields

For $f(x, t) = f(x, t) = x^6 - \frac{t-6}{2}x^5 - 5\frac{t+6}{4}x^4 - 20x^3 + 5\frac{t-6}{4}x^2 + \frac{t+6}{2}x + 1$, we checked that $f(x, (6r + 3)(36r^2 + 36r + 18))$ gives rise to a $C_6$ regular extension.

Since $\text{disc}(f(x, (6r + 3)(36r^2 + 36r + 18))) = 2^63^{21}(3r^2 + 3r + 1)^{10}(12r^2 + 12r + 7)^5$, there is a constant $C$ with $\text{disc}(f(x, (6r + 3)(36r^2 + 36r + 18))) \leq Cr^{30}$. Now define a set $L(A)$ for $A = C^{1/30}X$.

$$L(A) = \left\{ \frac{X}{2} < r < X \mid 3r^2 + 3r + 1 \text{ and } 12r^2 + 12r + 7 \text{ square-free, } r \equiv s_M \mod M \right\}.$$ 

we can show that $A^{1-\epsilon} \ll |L(A)| \ll A$. Note that each $r$ in $L(A)$ corresponds to a distinct $L$-function.

Let $c_0 = 76$. Choose $\alpha$ with $c_0\frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 30$, $d = 1$ and $T = (\log A^{30})^2$, every automorphic $L$-function excluding exceptional $O(A^{98/100})$ $L$-functions has a zero-free region $[\alpha, 1] \times [-d_{K_t}^2, (d_{K_t}^2)]$. Let us denote by $\hat{L}(A)$, the set of the automorphic $L$-functions with the zero-free region.

Applying Proposition 1.7 to this $L$-function $L(s, \rho, t)$ in $\hat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1)$$

$$= 5 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 5 \log \log d_{K_t} + O_{n, \alpha}(1).$$
By the class number formula and (2.11),
\[ h_{K_i} \gg d_{K_i}^{1/2} \frac{(\log \log d_{K_i})^5}{(\log d_{K_i})^5}. \]

We summarize as follows

**Theorem 3.35.** There is a constant \( c > 0 \) such that there exist \( K \in \mathcal{K}(6, C_6, 6, 0) \) with arbitrary large discriminant \( d_K \) for which
\[ h_K > cd_K^{1/2} \frac{(\log \log d_K)^5}{(\log d_K)^5}. \]

### 3.6.2 Simplest Quintic fields

Smith [67] showed that Lehmer’s polynomial \( f(x, t) = x^5 + t^2 x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 + (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2 + (t^3 + 4t^2 + 10t + 10)x + 1 \) generates a \( C_5 \) regular extension. Since the discriminant is a polynomial of degree 22, there is a constant \( C \) with \( \text{disc}(f(x, t)) \leq Ct^{22} \).

For \( A = C^{1/22}X \), let \( L(A) \) be a finite set given by
\[ L(A) = \left\{ \frac{X}{2} < t < X \mid t^4 + 5t^3 + 15t^2 + 25t + 25 \text{ cubic-free and } t \equiv t_M \mod M \right\}. \]

Then by Lemma 3.20, we have \( A^{1-\epsilon} \ll L(A) \ll A \).

For cyclic extensions of prime degree, the conductor of a cyclic extension equals the Artin conductors of characters for the extension. In section 2.1.3, we state that the product of prime divisors of \( P_t = t^4 + 5t^3 + 15t^2 + 25t + 25 \) is the conductor. But it would be possible that the sets of distinct prime divisors of \( P_t \) coincide for different \( t \)s. Let \( \nu(n) \) be the number of distinct prime divisors of \( n \). For each \( t \), the number of possible repetition is bounded by \( 2^{\nu(P_t)} \) because we assume that \( P_t \) is cubic-free. It is known that
\[ \nu(n) \ll \frac{\log n}{\log \log n}. \] (See page 167 in [56]) Hence, for all \( t < X \), \( 2^{\nu(P_t)} \ll 2^{\frac{\log X}{\log \log X}} \ll X^{\epsilon}. \) After removing the possible repetition, we can say that the Artin conductors are distinct.

Let \( c_0 = 56 \). Choose \( \alpha \) with \( c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100} \). By applying Theorem 1.10 to \( L(A) \) with \( e = 22, d = 1 \) and \( T = (\log A^{22})^2 \), every automorphic \( L \)-function excluding exceptional
$O(A^{98/100})$ $L$-functions has a zero-free region $[\alpha, 1] \times [-\log d_{K_t}^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic $L$-functions with the zero-free region.

Applying Proposition 1.7 to this $L$-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) \quad (3.36)$$

$$= 4 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 4 \log \log d_{K_t} + O_{n, \alpha}(1).$$

By the class number formula and (2.10),

$$h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^4}{(\log d_{K_t})^4}.$$ We summarize as follows

**Theorem 3.37.** There is a constant $c > 0$ such that there exist $K \in \mathcal{R}(5, C_5, 5, 0)$ with arbitrary large discriminant $d_K$ for which

$$h_K > cd_K^{1/2} \frac{(\log \log d_K)^4}{(\log d_K)^4}.$$

### 3.6.3 Simplest Quartic fields

In section 2.1.1, we showed that $f(x, t) = x^4 - 2tx^3 - 6x^2 + 2tx + 1$ generates a $C_4$ regular extension. Since the discriminant $\text{disc}(f(x, t)) = 2^6(t^2 + 4)^3$ is a polynomial of degree 6, there is a constant $C$ with $\text{disc}(f(x, t)) \leq Ct^6$.

For $A = C^{1/6}X$, let $L(A)$ be a finite set given by

$$L(A) = \{ \frac{X}{2} < t < X \mid t^2 + 4 \text{ square-free, } t \equiv t_M \text{ mod } M \}.$$ Then we have $A^{1-c} \ll L(A) \ll A$. Since $d_{K_t} = 2^4(t^2 + 4)^3$, each $t \in L(A)$ corresponds to a distinct $L$-function.

Since the Artin conductors of the characters for the simplest Quartic fields are increasing functions in $t$, they satisfy the Hypothesis in Theorem 1.10. Let $c_0 = 16$. 
Choose $\alpha$ with $c_0 \frac{1 - \alpha}{2\alpha - 1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 6$, $d = 1$ and $T = (\log A^6)^2$, every automorphic $L$-function excluding exceptional $O(A^{98/100})$ $L$-functions has a zero-free region $[\alpha, 1] \times [-\log d_{K_t}, (\log d_{K_t})^2]$. Let us denote by $\hat{L}(A)$, the set of the automorphic $L$-functions with the zero-free region.

Applying Proposition 1.7 to this $L$-function $L(s, \rho, t)$ in $\hat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1)$$

(3.38)

$$= 3 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log d_{K_t} + O_{n, \alpha}(1).$$

By the class number formula and (2.8),

$$h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^3}{(\log d_{K_t})^3}.$$ 

We summarize as follows

**Theorem 3.39.** There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, C_4, 4, 0)$ with arbitrary large discriminant $d_K$ for which

$$h_K \geq cd_K^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)^3}.$$
Chapter 4

Logarithmic derivatives of Artin L-functions

This chapter is written based on the joint work [9] with H. Kim. We follow [9] closely.

4.1 Euler-Kronecker Constants

Let $K$ be a number field of degree $n$ with discriminant $d_K$ and $\zeta_K(s)$ be the Dedekind zeta function of $K$, with the Laurent expansion at $s = 1$:

$$
\zeta_K(s) = c_{-1}(s - 1)^{-1} + c_0 + c_1(s - 1) + c_2(s - 1)^2 + \cdots
$$

Then $\gamma_K = \frac{c_0}{c_{-1}}$ is called the Euler-Kronecker constant of $K$. If $K = \mathbb{Q}$, then $\gamma_{\mathbb{Q}}$ is just the Euler constant $\gamma = 0.57721566\cdots$. When $K$ is an imaginary quadratic field, the Kronecker limit formula express $\gamma_K$ in terms of special values of the Dedekind $\eta$-function.

It was Ihara who began a study of Euler-Kronecker constant systematically. We refer to [29] for the detail.

We can see that $\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + (s - 1)h(s)$, for some holomorphic function $h(s)$ at $s = 1$. Let $\tilde{K}$ be the Galois closure of $K$. Then we have $\zeta_{\tilde{K}}(s) = \zeta(s)L(s, \rho)$ for some
\( n - 1 \) dimensional complex representation \( \rho \) of the Galois group \( \text{Gal}(\widehat{K}/\mathbb{Q}) \). So we have

\[
\gamma_K = \gamma + \frac{L'}{L}(1, \rho). \tag{4.1}
\]

This is one of the motivations to study the logarithmic derivatives at \( s = 1 \). Ihara [29] found a upper bound and a lower bound for \( \gamma_K \) under GRH. The main term of his upper and lower bound under GRH are

\[
2 \log \log \sqrt{|d_K|}, \quad -2(n - 1) \log \left( \frac{\log \sqrt{|d_K|}}{n - 1} \right).
\]

In [30], page 260, the authors remarked that in the case of Dirichlet characters, the coefficient 2 can be replaced by \( 1 + o(1) \). In section 4.7, we prove that under the Artin conjecture, GRH and certain zero density hypothesis (Conjecture 4.19), the upper and lower bound are

\[
\log \log |d_K| + O(\log \log \log |d_K|), \quad -(n - 1) \log \log |d_K| + O(\log \log \log |d_K|),
\]

resp. The lower bound comes from number fields where almost all small primes split completely. This agrees with Ihara’s observation ([29], page 409) that number fields with many primes having small norms have the lower bound.

When \( K \) is a quadratic field \( \mathbb{Q}(\sqrt{d}) \), the value \( \frac{L'}{L}(1, \chi_d) \) determines \( \gamma_{\mathbb{Q}(\sqrt{d})} \) where \( \chi_d \) is the Dirichlet character attached to the quadratic field \( \mathbb{Q}(\sqrt{d}) \). Recently, Mourtada and Murty [48] showed unconditionally that there are infinitely many Dirichlet \( L \)-functions of quadratic characters whose logarithmic derivative at \( s = 1 \) have large values. Namely, there are infinitely many fundamental discriminants \( d \) such that \( \left| -\frac{L'}{L}(1, \chi_d) \right| \gg \log \log |d| \).

This implies that \( \left| \gamma_{\mathbb{Q}(\sqrt{d})} \right| \gg \log \log |d| \) for infinitely many quadratic fields \( \mathbb{Q}(\sqrt{d}) \). We realized that the techniques we used to obtain extreme values of \( L(1, \rho) \) in [6], [7], [8], can be applied to generalize their result to arbitrary Artin \( L \)-functions.

For the several families of number fields, in sections 4.4 to 4.6, we show unconditionally that there are infinitely many number fields such that

\[
\frac{L'}{L}(1, \rho) \geq \log \log |d_K| + O(\log \log \log |d_K|), \tag{4.2}
\]
and infinitely many number fields such that

$$\frac{L'(1, \rho)}{L(1, \rho)} \leq -(n - 1) \log \log |d_K| + O(\log \log \log |d_K|).$$  \hspace{1cm} (4.3)

### 4.2 Extreme values of $L'(1, \rho)$

In this section, we describe how to obtain extreme positive or negative values of $L'(1, \rho)$ in a general setting. Suppose $K_t \in \mathcal{K}(n, G, r_1, r_2)$. Let $\rho_t$ be the $n-1$ dimensional complex representation of the Galois group $\text{Gal}\left(\hat{K} / \mathbb{Q}\right)$ given by $\zeta_{K_t}(s) = \zeta(s)L(s, \rho_t)$. Then the conductor of $\rho_t$ is $|d_{K_t}|$. Now we assume that $\rho_t$ is modular, i.e., an automorphic representation of $GL_{n-1}$. The discriminant of $f(x, t)$ is a polynomial in $t$. We expect that regular Galois extension property implies that the field discriminant $|d_{K_t}|$ will increase with respect to $t$.

**Assumption 4.4.** $\log |d_{K_t}| \gg f \log |t|$.

#### 4.2.1 Extreme positive values of $L'(1, \rho)$

We say that a finite group $G$ has property $\text{Gal}_T$ if $G$ is realized as a Galois group of a regular extension over $\mathbb{Q}(t)$. Let $G$ be a finite group having property $\text{Gal}_T$ and let $f(x, t) \in \mathbb{Z}[t][x]$ be an irreducible polynomial of degree $n$ whose splitting field over $\mathbb{Q}(t)$ is a regular extension with Galois group $G$. Let $K_t$ be the number field of signature $(r_1, r_2)$ obtained by adjoining a root of $f(x, t)$ to the rational number field $\mathbb{Q}$ for a specialization $t \in \mathbb{Z}$ and let $\hat{K}_t$ be its Galois closure. Let $L(s, \rho, t) = \sum_{l=1}^{\infty} \lambda(l, t)l^{-s}$ be the Artin L-function $\frac{\zeta_{K_t}(s)}{\zeta(s)}$.

Note that the conductor of $L(s, \rho, t)$ is $|d_{K_t}|$, and for unramified prime $p$, $\lambda(p, t) = N(p, t) - 1$, where $N(p, t)$ is the number of solutions of $f(x, t) \equiv 0 \pmod{p}$. Hence $-1 \leq \lambda(p, t) \leq n - 1$.

The Galois group $\text{Gal}(\hat{K}_t / \mathbb{Q}) \simeq G$ acts on the set $X = \{x_1, x_2, \ldots, x_n\}$ of roots of $f(x, t)$ transitively. Let $G_0$ be the set of $g \in G$ with no fixed point. Then $G_0$ is not empty.
and \( \frac{|G_0|}{|G|} \geq \frac{1}{n} \) (see [62], page 430). Choose any \( g_0 \in G_0 \) and let \([g_0]\) be the conjugacy class of \( g_0 \) in \( G \). If the Frobenius element of \( p \) belongs to \([g_0]\), then \( f(x, t) \equiv 0 \) (mod \( p \)) has no root and hence \( \lambda(p, t) = -1 \).

Since \( f(x, t) \) gives rise to a regular extension, by Theorem 2.1, there is a constant \( c_f \) depending on \( f \) such that for any prime \( p \geq c_f \), there is an integer \( i_p \) so that for any \( t \equiv i_p \) (mod \( p \)), the Frobenius element of \( p \) belongs to \([g_0]\). For \( X > 0 \), let \( y = \frac{\log X}{\log \log X} \) and \( M = \prod_{c_f \leq p \leq y} p \). Note that \( M \ll e^y = e^{\frac{\log X}{\log \log X}} \ll \epsilon X^{\epsilon} \) for any \( \epsilon > 0 \).

Let \( i_M \) be an integer such that \( i_M \equiv i_p \) (mod \( p \)) for all \( c_f \leq p \leq y \). So if \( t \equiv i_M \) (mod \( M \)), for all \( c_f \leq p \leq y \), \( p \) belongs to \([g_0]\) and \( \lambda(p, t) = -1 \).

Assume that the discriminant of \( f(x, t) \) is a polynomial in \( t \) of degree \( D \). Then there is a constant \( C \) such that \( |d_{K_t}| \leq Ct^D \). For \( A = C^{1/D}X \), we define a set \( L(A) \) of positive numbers given by

\[
L(A) = \{ \frac{X}{2} < t < X \mid t \equiv i_M \pmod{M}, \text{Gal}(\hat{K_t}/\mathbb{Q}) \simeq G \}.
\]

Under the strong Artin Conjecture, every \( t \) in \( L(A) \) corresponds to an automorphic L-function of \( GL(n-1) \) over \( \mathbb{Q} \). But it is possible that different \( t \in L(A) \) correspond to the same automorphic L-function. See Theorem 3.3.

**Assumption 4.5.** There exists a finite set \( T \subset \mathbb{Z} \), depending only on \( f \), such that \( L(s, \rho, t) \)'s are distinct for all \( t \in L(A) \setminus T \).

Let \( \tilde{L}(A) \) be the set of automorphic L-functions coming from \( L(A) \) after removing the possible repetition of the same L-functions among them. In sections 4.4 through 4.6, we consider explicit examples of families of number fields. In those cases, we may have to put more conditions in \( L(A) \) in order to satisfy Assumption 4.5, or replace it by some other set. In any case, we show that \( A^{1-\epsilon} \ll |\tilde{L}(A)| \ll A \) for any fixed \( \epsilon > 0 \).

Let \( c_0 = \frac{5(n-1)D}{2} + 1 \). Or we may replace \( (n-1) \) in \( c_0 \) by a smaller constant if \( \rho \) is not irreducible. Choose \( \alpha \) with \( c_0^{1-\alpha} < \frac{98}{100} \). By applying Theorem 1.10 to \( \tilde{L}(A) \) with \( e = D, d = 1 \) and \( T = (\log A^D)^2 \), every automorphic L-function excluding exceptional
Chapter 4. Logarithmic derivatives of Artin L-functions

$O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-\log |d_{K_i}|^2, \log |d_{K_i}|^2]$. Let us denote by $\hat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.9 to this L-function $L(s, \rho, t)$ in $\hat{L}(A)$ with $x = (\log CX^D)^{16}$, we have

$$\frac{L'}{L}(1, \rho, t) = -\sum_{p \leq x} \frac{\lambda(p, t) \log p}{p} + O_{n, \alpha}(1)$$

(4.6)

$$= \sum_{c_f \leq p \leq y} \frac{\log p}{p} - \sum_{y < p \leq x} \frac{\lambda(p, t) \log p}{p} + O_{n, \alpha}(1)$$

$$= \log \log X - \sum_{y < p \leq x} \frac{\lambda(p, t) \log p}{p} + O(\log \log \log X).$$

(Here we use the fact that $\sum_{p \leq y} \log p = \log y + O(1)$, and $y = \frac{\log X}{\log \log X}$.)

Now we sum the logarithmic derivative $\frac{L'}{L}(1, \rho, t)$ over $\hat{L}(A)$, namely, consider

$$\sum_{L(s, \rho, t) \in \hat{L}(A)} \frac{L'}{L}(1, \rho, t).$$

We need to deal with the sum

$$\sum_{L(s, \rho, t) \in \hat{L}(A)} \sum_{y < p \leq x} \frac{\lambda(p, t) \log p}{p} = \sum_{y < p \leq x} \frac{\log p}{p} \sum_{L(s, \rho, t) \in \hat{L}(A)} \lambda(p, t).$$

In the next section, we prove the following proposition:

**Proposition 4.7.** For all $y < p \leq x$,

$$\sum_{L(s, \rho, t) \in \hat{L}(A)} \lambda(p, t) \ll \frac{\hat{L}(A)}{\sqrt{p}} + \frac{\hat{L}(A)}{(\log X)^{1/2}}.$$

where the implied constant is independent of $p$ for $y < p \leq x$.

Proposition 4.7 implies

$$\sum_{L(s, \rho, t) \in \hat{L}(A)} \sum_{y < p \leq x} \frac{\lambda(p, t) \log p}{p} \ll \frac{\hat{L}(A)}{y^{1/2}} + \frac{\hat{L}(A) \log \log X}{(\log X)^{1/2}}.$$
Hence we have
\[ \sum_{L(s, \rho, t) \in \mathcal{L}(A)} \frac{L'}{L}(1, \rho, t) = |\hat{L}(A)| \log \log X + O(|\hat{L}(A)| \log \log \log X). \]

Now note that \(|d_{K_i}| \leq C t^D\) and \(t < X\). So if there are only finitely many \(L\)-functions with \(\frac{L'}{L}(1, \rho, t) \geq \log \log |d_{K_i}| + O(\log \log \log |d_{K_i}|)\), they cannot reach the average value \(\log \log X\) as \(X\) increases and it is summarized as follows:

**Theorem 4.8.** Assume that \(G\) has property \(\text{Gal}_T\), it is realized as the Galois group of a polynomial \(f(x, t)\) of degree \(n\) and the signature \((r_1, r_2)\) of \(K_t\) is fixed for sufficiently large \(t\). Furthermore, assume Proposition 4.7 holds for \(f(x, t)\). Then Under the Strong Artin conjecture and Assumptions 4.4 and 4.5, there are infinitely many number fields \(K \in \mathcal{R}(n, G, r_1, r_2)\) with
\[ \frac{L'}{L}(1, \rho) \geq \log \log |d_K| + O(\log \log \log |d_K|). \]

### 4.2.2 Extreme negative values of \(\frac{L'}{L}(1, \rho)\)

To generate a negative \(\frac{L'}{L}(1, \rho, t)\), but whose absolute value is large, we need to manipulate \(\lambda(p, t)\) so that \(\lambda(p, t) = n - 1\) for all primes \(p\) between \(c_f\) and \(y = \frac{\log X}{\log \log X}\) in (4.6).

Since \(f(x, t)\) gives rise to a regular Galois extension, by Theorem 2.1, for any prime \(p \geq c_f\), there is an integer \(s_p\) so that for any \(t \equiv s_p \pmod p\), the Frobenius element of \(p\) is the identity in \(G\). For \(X > 0\), let \(M = \prod_{c_f \leq p \leq y} p\). Let \(s_M\) be an integer such that \(s_M \equiv s_p \pmod p\) for all \(c_f \leq p \leq y\). So if \(t \equiv s_M \pmod M\), for all \(c_f \leq p \leq y\), \(p\) splits completely in \(\hat{K}_t\), and \(\lambda(p, t) = n - 1\).

For \(A = C^{1/D} X\), we define a set \(L(A), \tilde{L}(A)\) and \(\hat{L}(A)\) as in the previous section. Then as in (4.6),
\[ \frac{L'}{L}(1, \rho, t) = -(n - 1) \log \log X - \sum_{y < p \leq x} \frac{\lambda(p, t) \log p}{p} + O(\log \log \log X). \]
Then by Proposition 4.7,

$$\sum_{L(s, \rho, t) \in \hat{L}(A)} \frac{L'}{L}(1, \rho, t) = -(n - 1)|\hat{L}(A)| \log \log X + O(|\hat{L}(A)| \log \log \log X).$$

Hence we proved the following under the Strong Artin Conjecture and Assumptions 4.4 and 4.5,

**Theorem 4.9.** Assume that $G$ has property $\text{Gal}_T$, it is realized as the Galois group of a polynomial $f(x, t)$ of degree $n$ and the signature $(r_1, r_2)$ of $K_t$ is fixed for sufficiently large $t$. Furthermore, assume Proposition 4.7 holds for $f(x, t)$. Then Under the Strong Artin Conjecture and Assumptions 4.4 and 4.5, there are infinitely many number fields $K \in \mathcal{H}(n, G, r_1, r_2)$ with

$$\frac{L'}{L}(1, \rho) \leq -(n - 1) \log \log |d_K| + O(\log \log \log |d_K|).$$

### 4.3 Proof of Proposition 4.7

For a fixed prime $p$, consider the equation $f(x, t) \equiv 0 \pmod{p}$. Now we consider $f(x, t)$ as an algebraic curve over $\mathbb{Z}/p\mathbb{Z}$. Let $A_i$ be the number of $t \pmod{p}$ such that $\lambda(p, t) = i$, i.e., $f(x, t) \equiv 0 \pmod{p}$ has $i + 1$ roots. Then we have

$$\sum_{i=-1}^{n-1} A_i = p + O(1),$$

where $O(1)$ is bounded by $D$, the degree of discriminant of $f(x, t)$.

Recall Weil’s celebrated theorem on rational points of a curve over a finite field. ([58], page 75):

**Theorem 4.10.** Let $f(x, y) \in \mathbb{F}_p[x, y]$ be absolutely irreducible and of total degree $d > 0$. Let $N$ be the number of zeros of $f$ in $\mathbb{F}_p \times \mathbb{F}_p$. Then

$$|N - p| \leq (d - 1)(d - 2)\sqrt{p} + c(d),$$

for a constant $c(d)$. 
Weil’s theorem implies
\[ \sum_{i=-1}^{n-1} (i+1)A_i = p + O(\sqrt{p}). \]

Hence we obtain
\[ \sum_{i=-1}^{n-1} iA_i = O(\sqrt{p}). \] (4.11)

Now we define \( Q_i = \{ \frac{1}{2} X < t < X \mid t \in L(A) \text{ and } t \equiv i \pmod{p} \} \) and write
\[ L(A) = Q_0 \cup Q_1 \cup \cdots \cup Q_{p-1}. \]

Let \( R \) be a finite subset of \( \{0, 1, 2, \ldots, p-1\} \) for which \( k \in R \) if and only if \( p \) is ramified for \( t \in Q_k \). Then we prove the following in the examples in sections 4.4 through 4.6:
\[ |Q_i| = c_p \frac{|L(A)|}{p} + O \left( \frac{|L(A)|}{p (\log X)^{\frac{1}{2}}} \right) \quad \text{for } \notin R \] (4.12)

where \( c_p \) is a constant close to 1, independent of \( i \). (We can show that \( \frac{1}{2} < c_p < 2 \).)

Since
\[ \sum_{L(s,\rho,t) \in \hat{L}(A)} \lambda(p, t) = \sum_{L(s,\rho,t) \in L(A)} \lambda(p, t) + O(X^{98/100}), \]

in order to prove Proposition 4.7, it is enough to show that
\[ \sum_{L(s,\rho,t) \in L(A)} \lambda(p, t) \ll \frac{|L(A)|}{\sqrt{p}} + \frac{|L(A)|}{(\log X)^{\frac{1}{2}}}. \]

When \( k \in R \),
\[ \left| \sum_{L(s,\rho,t) \in Q_k} \lambda(p, t) \right| \leq (n-1) \frac{|L(A)|}{p} + O(1). \]

If \( k \notin R \), \( p \) is unramified for all \( t \in Q_k \), and \( \lambda(p, t) = j(k) \) for a unique \( j(k) \). In that case,
\[ \sum_{L(s,\rho,t) \in Q_k} \lambda(p, t) = j(k)c_p \frac{|L(A)|}{p} + O \left( \frac{|L(A)|}{p (\log X)^{\frac{1}{2}}} \right). \]

Hence
\[ \sum_{L(s,\rho,t) \in L(A)} \lambda(p, t) = \sum_{k \in R} \sum_{L(s,\rho,t) \in Q_k} \lambda(p, t) + \sum_{k \notin R} \sum_{L(s,\rho,t) \in Q_k} \lambda(p, t). \]
Here
\[ \sum_{k \in R} \sum_{L(s, \rho, t) \in Q_k} \lambda(p, t) \ll \frac{|L(A)|}{p}, \]
where the implied constant is independent of \( p \). On the other hand,
\[ \sum_{k \notin R} \sum_{L(s, \rho, t) \in Q_k} \lambda(p, t) = \sum_{k \notin R} j(k) |Q_k| = c_p \frac{|L(A)|}{p} \sum_{k \notin R} j(k) + O \left( \frac{|L(A)|}{(\log X)^\frac{1}{2}} \right) \]
\[ = c_p \frac{|L(A)|}{p} \sum_{j = -1}^{n-1} j A_j + O \left( \frac{|L(A)|}{(\log X)^\frac{1}{2}} \right). \]

4.4 Cyclic and Dihedral extensions

Cyclic and dihedral extensions satisfy \( \text{Gal}_T \) property. Hence given \( G \), a cyclic or dihedral group, there exists a polynomial \( f(x, t) \in \mathbb{Z}[t][x] \) whose splitting field over \( \mathbb{Q}(t) \) is a regular Galois extension and whose Galois group is \( G \). We give some details for quadratic and cyclic cubic extensions.

4.4.1 Quadratic extensions

Consider \( K_t = \mathbb{Q}[\sqrt{t}] \) for \( t \) square free and \( t \equiv 1 \pmod{4} \). Consider, for \( M = 4 \prod_{3 \leq p \leq y} p \),
\[ L(X)_1 = \{ \frac{X}{2} < t < X \mid t \text{ square-free and } t \equiv s_M \pmod{M} \} \]
\[ L(X)_2 = \{ \frac{X}{2} < t < X \mid t \text{ square-free and } t \equiv i_M \pmod{M} \}. \]

Then Assumptions 4.4 and 4.5 are clear. We verify (4.12) in the case of \( L(X)_2 \):
\[ Q_i = \{ \frac{X}{2} < t < X \mid t \text{ square free, } t \equiv i_M \pmod{M}, \ t \equiv i \pmod{p} \}. \]

Since \( p > y \), \( (p, M) = 1 \) and if \( i \neq 0 \), by [18], page 248,
\[ |Q_i| = \frac{3}{\pi^2} \prod_{q | M} (1 - q^{-2})^{-1} \frac{X}{M} (1 - p^{-2})^{-1} \frac{1}{p} + O(X^{\frac{1}{2}}) = c_p \frac{|L(X)_2|}{p} + O(X^{\frac{1}{2}}), \]
where \( c_p = (1 - p^{-2})^{-1} \) and \( 1 < c_p < 2 \). Since \( p \ll (\log X)^{\frac{32}{15\alpha}} \), \( X^{\frac{1}{2}} \ll \frac{|L(X)|}{p(\log X)^{\frac{15}{2}}} \). Here we considered real quadratic fields. However, the same argument is applicable to imaginary quadratic fields. So Theorem 4.8 and Theorem 4.9 are now stated as follows:

**Theorem 4.13.** (1) There are infinitely many real quadratic fields \( \mathbb{Q}(\sqrt{t}) \) (resp, imaginary quadratic fields \( \mathbb{Q}(\sqrt{t}) \)) with

\[
\frac{L'(s, \chi_t)}{L(s, \chi_t)} \leq - \log \log |t| + O(\log \log \log |t|).
\]

(2) There are infinitely many real quadratic fields \( \mathbb{Q}(\sqrt{t}) \) (resp, imaginary quadratic fields \( \mathbb{Q}(\sqrt{t}) \)) with

\[
\frac{L'(s, \chi_t)}{L(s, \chi_t)} \geq \log \log |t| + O(\log \log \log |t|).
\]

### 4.4.2 Cyclic Cubic extensions

Consider

\[ f(x, t) = x^3 - tx^2 - (t + 3)x - 1, \]

for \( t \in \mathbb{Z}^+ \). Its discriminant is \( g(t)^2 \) with \( g(t) = t^2 + 3t + 9 \). Then \( K_t/\mathbb{Q} \) is a \( C_3 \) Galois extension, and \( L(s, \rho, t) = L(s, \chi_t)L(s, \overline{\chi_t}) \), where \( \chi_t, \overline{\chi_t} \) are two non-principal characters of \( C_3 \). The conductor \( f_{\chi_t} \) of \( \chi_t \) is \( g(t) \) when \( g(t) \) is square-free. Note also that

\[
\frac{L'(s, \rho, t)}{L(s, \rho, t)} = 2Re \left( \frac{L'(s, \chi_t)}{L(s, \chi_t)} \right).
\]

Consider, for \( M = 6 \prod_{5 \leq p \leq y} p \),

\[
\begin{align*}
L(A)_1 &= \{ \frac{X}{2} < t < X \mid g(t) \text{ square-free}, t \equiv s_M \pmod{M} \} \\
L(A)_2 &= \{ \frac{X}{2} < t < X \mid g(t) \text{ square-free}, t \equiv i_M \pmod{M} \}.
\end{align*}
\]

Then Assumptions 4.4 and 4.5 are clear. We verify (4.12) in the case of \( L(A)_2 \):

\[ Q_i = \{ \frac{X}{2} < t < X \mid g(t) \text{ square free}, t \equiv i_M \pmod{M}, t \equiv i \pmod{p} \}. \]

Define \( R' \) be the set of solutions \( t \pmod{p} \) for \( g(t) \equiv 0 \pmod{p} \). Then \( R' \) has at most 2 elements. So it is enough to consider \( i \notin R' \). Since \( p > y \), \( (p, M) = 1 \) and for \( i \notin R' \),
by [17],

\[
|Q_1| = \prod_{q \mid M} \left(1 - \left(1 + \left(-\frac{3}{q}\right)\right)q^{-2}\right) \frac{X}{2M} \left(1 - \left(1 + \left(-\frac{3}{p}\right)\right)p^{-2}\right)^{-1} \frac{1}{p} + O(X^{\frac{2}{3}} \log X)
\]

\[
= c_p \frac{|L(A)_{2}|}{p} + O(X^{\frac{2}{3}} \log X),
\]

where \(c_p = (1 - (1 + \left(-\frac{3}{p}\right))p^{-2})^{-1}\) and \(\frac{1}{2} < c_p < 2\). Since \(p \ll (\log X)^{\frac{32}{15}}\), \(X^{\frac{2}{3}} \log X \ll \frac{|L(A)_{2}|}{p(\log X)^{\frac{7}{8}}}\). So Theorem 4.8 and Theorem 4.9 are now stated as follows:

**Theorem 4.14.** (1) There are infinitely many \(L(s, \rho, t)\) with

\[
\frac{L'}{L}(1, \rho, t) \leq -2 \log \log |d_{K_1}| + O(\log \log \log |d_{K_1}|), \quad Re \left(\frac{L'}{L}(1, \chi_t)\right) \leq - \log f_{\chi_t} + O(\log \log f_{\chi_t}).
\]

(2) There are infinitely many \(L(s, \rho, t)\) with

\[
\frac{L'}{L}(1, \rho, t) \geq \log \log |d_{K_1}| + O(\log \log \log |d_{K_1}|), \quad Re \left(\frac{L'}{L}(1, \chi_t)\right) \geq \frac{1}{2} \log f_{\chi_t} + O(\log \log f_{\chi_t}).
\]

### 4.4.3 Dihedral and cyclic extensions

For higher degree extensions, we recall the explicit examples from Chapter 2.

\(\mathfrak{A}(6, C_6, 6, 0): f(x, t) = x^6 - 2(t - 1)x^5 - 5(t + 2)x^4 + 20x^3 + 5(t - 1)x^2 + 2(t + 2)x + 1\)

\(\mathfrak{A}(5, D_5, 5, 0): f(x, t) = x^5 - tx^4 + (2t - 1)x^3 - (t - 2)x^2 - 2x + 1, t > 7\)

\(\mathfrak{A}(5, D_5, 1, 2): f(x, t) = x^5 + tx^4 - (2t + 1)x^3 + (t + 2)x^2 - 2x + 1, t \geq 0\)

\(\mathfrak{A}(4, D_4, 4, 0): f(x, t) = x^4 - tx^3 - x^2 + tx + 1\)

\(\mathfrak{A}(4, D_4, 2, 1): f(x, t) = x^4 - tx^3 + 3x^2 - tx + 1\)

\(\mathfrak{A}(4, D_4, 0, 2): f(x, t) = x^4 - x^3 + (t + 2)x^2 - x + 1\)

\(\mathfrak{A}(4, C_4, 4, 0): f(x, t) = x^4 - 2tx^3 - 6x^2 + 2tx + 1\)

\(\mathfrak{A}(3, D_3, 3, 0): f(x, t) = (x - t)(x - 4t)(x - 9t) - t\)

\(\mathfrak{A}(3, D_3, 1, 1): f(x, t) = x^3 + tx - 1\)

We do not include the simplest quintic fields because we cannot verify Assumption 4.5 for these fields. In the case of \(C_6\), we do not need to specialize \(t\) as in section 3.6.1 since we do not need to find units.
The strong Artin conjecture is valid in all of the above cases. We recall the definition of the sets \( L(A) \) in each cases. We only write for the extreme positive value case.

\[
\mathfrak{R}(6, C_6, 6, 0) : L(A) = \left\{ \frac{X}{2} < t < X \mid t^2 + t + 7 \text{ square-free, and } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(5, D_5, 5, 0) : L(A) = \left\{ \frac{X}{2} < t < X \mid 4t^3 - 28t^2 + 24t - 47 \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(5, D_5, 1, 2) : L(A) = \left\{ \frac{X}{2} < t < X \mid 4t^3 + 28t^2 + 24t + 47 \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(4, D_4, 4, 0) : L(A) = \left\{ \frac{X}{2} < t < X \mid (t^2 - 4)(4t^2 + 9) \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(4, D_4, 2, 1) : L(A) = \left\{ \frac{X}{2} < t < X \mid (t^2 - 4)(25 - 4t^2) \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(4, D_4, 0, 2) : L(A) = \left\{ \frac{X}{2} < t < X \mid (t + 2)(t + 6)(1 - 4t) \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(4, C_4, 4, 0) : L(A) = \left\{ \frac{X}{2} < t < X \mid t^2 + 4 \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(3, D_3, 3, 0) : L(A) = \left\{ \frac{X}{2} < t < X \mid t \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]
\[
\mathfrak{R}(3, D_3, 1, 1) : L(A) = \left\{ \frac{X}{2} < t < X \mid 4t^3 + 27 \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]

Except the case of \( \mathfrak{R}(3, D_3, 3, 0) \), Assumptions 4.4 and 4.5 are shown in Chapter 3. Since \( f(x, t) = (x - t)(x - 4t)(x - 9t) - t \) is an Eisenstein polynomial for square-free \( t \), \( \log d_{K_i} \gg \log t \). We prove Assumption 4.5 as follows: We claim that \( p \mid t \) if and only if \( p \) is totally ramified in \( K_i \): If \( p \mid t \), by [11], page 315, \( p \) is totally ramified. Conversely, if \( p \) is totally ramified and \( p \nmid t \), \( f(x, t) \equiv (x + a)^3 \pmod{p} \). If we compare the coefficients of \( f(x, t) \pmod{p} \), we obtain contradiction. Therefore, \( K_i \)'s are distinct for all \( t \in L(A) \) and showed Assumption 4.5.

Now we show that (4.12) holds for these cases. For the cases of \( C_6 \) and \( C_4 \), it can be verified as in the cyclic cubic cases. For the case of \( D_3, (3, 0) \), it is checked as in the quadratic fields.

Consider the case of \( D_5, (1, 2) \):

\[
Q_i = \left\{ \frac{X}{2} < t < X \mid 4t^3 + 28t^2 + 24t + 47 \text{ square-free and } t \equiv i_M \pmod{M}, \text{ and } t \equiv i \pmod{p} \right\}
\]

Let \( R' \) be the set of solutions \( t \pmod{p} \) for \( 4t^3 + 28t^2 + 24t + 47 \equiv 0 \pmod{p} \). Then \( R' \) has at most 3 elements. Hence it is enough to consider \( i \notin R' \). For \( i \notin R' \), by Lemma 3.20,
\[
|Q_i| = \prod_{q \n M} \left( 1 - \frac{\rho(q^2)}{q^2} \right) \left( 1 - \frac{\rho(p^2)}{p^2} \right)^{-1} \frac{X}{pM} + O \left( \frac{X}{pM} \left( \log \frac{X}{pM} \right)^{-\frac{1}{2}} \right) \\
= c_p \frac{|L(A)|}{p} + O \left( \frac{X}{pM} \left( \log \frac{X}{pM} \right)^{-\frac{1}{2}} \right),
\]

where \(\rho(p^2)\) is the number of solutions for \(4t^3 + 28t^2 + 24t + 47 \equiv 0 \pmod{p^2}\), and \(c_p = \left(1 - \frac{\rho(p^2)}{p^2} \right)^{-1}\). Here \(1 < c_p < 2\). Since \((1 - \epsilon) \log X < \log \frac{X}{pM} < \log X\) for any \(\epsilon > 0\), we have
\[
\frac{X}{pM} \left( \log \frac{X}{pM} \right)^{-\frac{1}{2}} \ll \frac{|L(A)|}{p(\log X)^{\frac{1}{2}}}. 
\]

For the cases of \(D_3, (1, 1)\) and \(D_5, (5, 0)\), (4.12) is verified similarly as in the case of \(D_5, (1, 2)\).

Consider the case of \(D_4, (4, 0)\):

\[
Q_i = \{ \frac{X}{2} < t < X | (t^2 - 4)(4t^2 + 9) \text{ square-free and } t \equiv i \pmod{M}, \text{ and } t \equiv i \pmod{p} \}.
\]

Let \(R'\) be the set of solutions \(t \pmod{p}\) for \((t^2 - 4)(4t^2 + 9) \equiv 0 \pmod{p}\). Then \(R'\) has at most 4 elements. Hence it is enough to consider \(i \notin R'\). For \(i \notin R'\), by Theorem 3.26,
\[
|Q_i| = \prod_{q \n M} \left( 1 - \frac{\rho(q^2)}{q^2} \right) \left( 1 - \frac{\rho(p^2)}{p^2} \right)^{-1} \frac{X}{pM} + O \left( \frac{X}{pM} \left( \log \frac{X}{pM} \right)^{-1} \right) \\
= c_p \frac{|L(A)|}{p} + O \left( \frac{X}{pM} \left( \log \frac{X}{pM} \right)^{-1} \right),
\]

where \(\rho(q^2)\) is the number of solutions for \((t^2 - 4)(4t^2 + 9) \equiv 0 \pmod{q^2}\), and \(c_p = \left(1 - \frac{\rho(q^2)}{p^2} \right)^{-1}\). Here \(1 < c_p < 2\). Clearly, we have
\[
\frac{X}{pM} \left( \log \frac{X}{pM} \right)^{-1} \ll \frac{|L(A)|}{p(\log X)^{\frac{1}{2}}}.
\]

For the cases of \(D_4, (2, 1)\) and \(D_4, (0, 2)\), (4.12) is verified similarly as in the \(D_4, (4, 0)\) case. Hence Theorem 4.8 and Theorem 4.9 are valid for the above examples.
4.5 Alternating Groups

4.5.1 $A_4$ Galois extensions

Consider from section 2.3.1 and [67]:

$K(4, A_4, 0, 2) : f(x, t) = x^4 + 18x^2 - 4tx + t^2 + 81$

$K(4, A_4, 4, 0) : f(x, t) = x^4 + 18tx^3 + (81t^2 + 2)x^2 + 2t(54t^2 + 1)x + 1$

All these polynomials generate regular Galois extensions and the strong Artin conjecture is true. In the first case, Assumption 4.4 was verified in 2.3.1. The second case is similar: Note that $\text{disc}(f(x, t)) = 16^2t^2(27t^2 - 4)^2(27t^2 + 4)^2$. So if $t$ is square free, Newton polygon argument shows that $t|d_{K_t}$. (If $p|t$, then $p\mathcal{O}_{K_t} = p^2$ for a prime ideal $p$.)

Hence $\log d_{K_t} \gg \log |t|$.

$K(4, A_4, 0, 2) : L(A) = \{ \frac{X}{2} < t < X : t(t^2 + 108) \text{ square-free and } t \equiv i_M \pmod{M} \}$

$K(4, A_4, 4, 0) : L(A) = \{ \frac{X}{2} < t < X : t \text{ is square free and } t \equiv i_M \pmod{M} \}$

In the case of $A_4, (0, 2), t(t^2 + 108)$ is no longer irreducible so we can’t apply Theorem 3.20. We need Nair’s work [50] Here. He showed

**Theorem 4.15** (Nair). If

$$f(x) = \prod_{i=1}^{m} (f_i(x))^{\alpha_i} \in \mathbb{Z}[x],$$

where each $f_i$ is irreducible, $\alpha = \max \alpha_i$ and $\deg f_i(x) = g_i$, then

$$N_k(x, h) = \prod_p \left(1 - \frac{\rho(p^k)}{p^k}\right) h + O \left(\frac{h}{(\log h)^{k-1}}\right)$$

for $h = x^\theta$ where $0 < \theta < 1$ and $k \geq \max \{\lambda g_i \alpha_i\}$, $(\lambda = \sqrt{2} - 1/2)$ provided that at least one $g_i \geq 2$.

Theorem 4.15 implies that

$$|L(A)| = \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^2}\right) \frac{X}{2M} + O \left(\frac{X}{M \log \frac{X}{M}}\right).$$
For both cases, (4.12) is verified easily. For the first case, Assumption 4.5 is true. However, for the second polynomial Assumption 4.5 remains to be proved. Hence Theorem 4.8 and Theorem 4.9 are valid modulo Assumption 4.5.

4.5.2 $A_5$ Galois extension

Consider the polynomial $f(x,t) = x^5 + 5(5t^2 - 1)x - 4(5t^2 - 1)$ from section 2.3.2.

If $K_t = \mathbb{Q}[\alpha_t]$ for $t \in \mathbb{Z}$, $K_t$ has signature $(1,2)$. Let $\widehat{K}_t$ be the Galois closure. Then $G$ has a subgroup $H$ isomorphic to $A_4$ such that $\widehat{K}_t^H = K_t$. Let $\text{Ind}^G_H 1_H = 1 + \rho$ be the induced representation of $G$ by the trivial representation of $H$ where $\rho$ is the 4-dimensional representation of $A_5$, so that $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$.

Now by [36], page 498, $\rho$ is equivalent to a twist of $\sigma \otimes \sigma^\tau$ by a character, where $\sigma, \sigma^\tau$ are the icosahedral 2-dimensional representations of $\tilde{A}_5 \simeq SL_2(\mathbb{F}_5)$. Since $K_t$ is not totally real, $\sigma$ and $\sigma^\tau$ are odd. Hence by [35], Corollary 10.2, $\sigma, \sigma^\tau$ are modular, i.e., they are attached to cuspidal representations $\pi, \pi^\tau$ of $GL_2/\mathbb{Q}$. By [53], the functorial product $\pi \boxtimes \pi^\tau$ is a cuspidal representation of $GL_4/\mathbb{Q}$. Hence $L(s, \rho, t)$ is a cuspidal automorphic $L$-function of $GL_4/\mathbb{Q}$.

Let

$$L(A) = \{ \frac{X}{2} < t < X \mid 5t^2 - 1 \text{ square-free, } t \text{ even, } t \equiv i_M \text{ (mod } M) \}.$$ 

Now we show Assumption 4.5.

To verify Assumption 4.5, it is enough that $K_t$ are not isomorphic for odd square-free $5t^2 - 1$. Suppose $5t^2 - 1$ is square free for even $t$. Now we prove that the primes $p$ who ramifies totally in $K_t$ are exactly prime divisors of $5t^2 - 1$. Since $f(x,t)$ is an Eisenstein polynomial with respect to each prime divisor of $5t^2 - 1$, $p$ ramifies totally in $K_t$. Conversely, assume that a prime $p$ ramifies totally and is not a prime divisor of $5t^2 - 1$. If $p = 2$, then $f(x,t) \equiv x(x^4 + 1) \text{ mod } 2$. Hence $p$ does not ramifies totally. Now assume that $p$ is not 2 and not a prime divisor of $5t^2 - 1$. Then we should have that
Chapter 4. Logarithmic derivatives of Artin L-functions

\( f(x, t) \equiv (x + a)^5 \mod p \) with \( a \not\equiv 0 \mod p \) and it forces that \( p = 5 \) and \( a = 4 \). However, by Newton polygon method, \( 5\mathcal{O}_{K_t} = p_1 p_2^4 \).

In this case, (4.12) is verified as in \( D_4, (4, 0) \) case. So Theorem 4.8 and Theorem 4.9 are valid.

4.6 Symmetric Groups

4.6.1 \( S_4 \) Galois extensions

Consider from section 2.4.1:

\[
\mathcal{R}(4, S_4, 4, 0) : f(x, t) = (x - t)(x - 4t)(x - 9t)(x - 16t) - t \\
\mathcal{R}(4, S_4, 2, 1) : f(x, t) = x^2(x - 10t)(x - 18t) + t \\
\mathcal{R}(4, S_4, 0, 2) : f(x, t) = x^4 + tx^2 + tx + t
\]

All these polynomials generate regular Galois extensions and the strong Artin conjecture is true. We define \( L(A) \) as

\[
L(A) = \left\{ \frac{X}{2} < t < X \mid t \text{ square-free, } t \equiv i_M \pmod{M} \right\}
\]

Assumptions 4.4 and 4.5 are shown in Chapters 2 and 3.

In these cases, (4.12) is verified as in the quadratic case.

Hence Theorem 4.8 and Theorem 4.9 are valid unconditionally.

4.6.2 \( S_5 \) Galois extension

Consider from section 2.4.2:

\[ f(x, t) = x^5 + tx + t \]

Since 5 does not divide \( \text{disc}(f(x, t)) = t^4(256t + 3125) \), 5 is unramified in \( \widehat{K}_t/Q \). In addition, \( f(x, t) \equiv x^5 + x + 1 \equiv (x + 3)(x^2 + x + 1)(x^2 + x + 2) \pmod{5} \), the Galois extensions \( \widehat{K}_t/Q \) satisfy the hypothesis of Theorem 1.4. Hence Artin L-functions \( L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)} \) are cuspidal automorphic L-functions of \( GL(4)/\mathbb{Q} \).
On the other hand, it is easy to check, for square-free \( t \),

\[
p \text{ is totally ramified in } K_t \iff p | t.
\]

Hence \( K_t \) are not isomorphic for all square-free \( t \) and this fact implies that \( L(s, \rho, t) \) are distinct. Hence we verified Assumption 4.5 for this case.

Let

\[
L(A) = \left\{ \frac{X}{2} < t < X \mid t : \text{square-free and } t \equiv i_M \pmod{M} \right\}
\]

Then we can see that (4.12) holds.

So Theorem 4.8 and Theorem 4.9 are valid.

### 4.7 Conditional result under zero density hypothesis

Until now, we obtained the average value of logarithmic derivatives of Artin \( L \)-functions in a family. In this section, we assume a zero density hypothesis and evaluate the logarithmic derivative of a single Artin \( L \)-function.

We use the same notation as in section 4.2: Let \( f(x, t) \in \mathbb{Z}[t][x] \) be an irreducible polynomial of degree \( n \) whose splitting field over \( \mathbb{Q}(t) \) is a regular extension with Galois group \( G \). Let \( K_t, \hat{K}_t \) be as in section 4.2. Let \( L(s, \rho, t) = \sum_{l=1}^{\infty} \lambda(l, t) l^{-s} \) be the Artin \( L \)-function \( \frac{\zeta_{K_t}(s)}{\zeta(s)} \). For simplicity of notation, let \( L(s, \rho) = L(s, \rho, t) \), \( \lambda(p) = \lambda(p, t) \), and \( N = |d_{K_t}| \).

If we assume the Artin conjecture and GRH for \( L(s, \rho) \), then by [16],

\[
\frac{L'(1, \rho)}{L(1, \rho)} = -\sum_{p \leq (\log N)^{2+\varepsilon}} \frac{\lambda(p) \log p}{p} + O_{n, x, \alpha}(1).
\]

We show under certain zero hypothesis (Conjecture 4.19) that if \( w = (\log N)(\log \log N)^2 \), \( x = (\log N)^{2+\varepsilon} \),

\[
\sum_{w < p < x} \frac{\lambda(p) \log p}{p} = O(1).
\] (4.16)
Proof of (4.16): By partial summation,
\[
\sum_{w < p < x} \frac{\lambda(p) \log p}{p} = -\frac{1}{w} \sum_{p < w} \lambda(p) \log p + \frac{1}{x} \sum_{p < x} \lambda(p) \log p + \int_w^x \frac{\sum_{p < u} \lambda(p) \log p}{u^2} du. \tag{4.17}
\]
Then \(\frac{1}{w} \sum_{p < w} \lambda(p) \log p = O(1)\), and \(\frac{1}{x} \sum_{p < x} \lambda(p) \log p = O(1)\) since \(\sum_{p < x} \log p = O(x)\).

Let \(\rho = \beta + i\gamma\) run over the zeros of \(L(s, \rho)\) in the critical strip of height up to \(T\), with \(1 \leq T \leq u\). Then by [32], page 112,
\[
\psi(\rho, u) = \sum_{n \leq u} \lambda(n) \Lambda(n) = -\sum_{|\gamma| \leq T} \frac{u^\rho - 1}{\rho} + O\left(\frac{u \log u}{T} \log(u^{n-1}N)\right). \tag{4.18}
\]
Here \(\psi(\rho, u) = \sum_{p \leq u} \lambda(p) \log p + \sum_{p^k \leq u, k \geq 2} \lambda(p^k) \log p\). Since \(\lambda(l) \leq d_{n-1}(l)\), where \(\zeta(s)^{n-1} = \sum_{l=1}^\infty d_{n-1}(l) l^{-s}\) and \(d_{n-1}(l) \leq d(l)^{n-1}\), \(\lambda(p^k) \leq (k + 1)^{n-1}\). Hence
\[
\sum_{p^k \leq u, k \geq 2} \lambda(p^k) \log p \ll \sum_{p \leq \sqrt{u}} \log p \sum_{k < \log u / \log p} (k + 1)^{n-1} \ll \sqrt{u} (\log u)^n.
\]
So this error term contributes to \(O(1)\) in the integral in (4.17). Therefore, we can use \(\psi(\rho, u)\) in the integral in (4.17). We apply (4.18) with \(T = (\log N)(\log \log N)^2\). The error term \(O\left(\frac{u \log u}{T} \log(u^{n-1}N)\right)\) gives rise to
\[
\int_w^x \left( (n - 1) \frac{u(\log u)^2}{(\log N)(\log \log N)^2} + \frac{u \log u}{(\log N)^2} \right) \frac{du}{u^2},
\]
which is \(O(1)\).

The sum \(\sum_{|\gamma| \leq T} \frac{1}{\rho}\) is bounded by \((\log N) \sum_{k=1}^T \frac{1}{k} \ll \log N \log T\) and it gives rise to
\[
(\log N)(\log T) \int_w^x \frac{du}{u^2} \ll \frac{\log N \log T}{w} = O(1).
\]

Now we assume the following zero density hypothesis for \(L(s, \rho)\). (cf. [55], page 6)

**Conjecture 4.19.** For \(u \geq (\log(n-1) \log N)^\kappa\), and \(\kappa \geq 1\)
\[
\sum_{|\gamma| \leq T} \frac{u^\rho}{\rho} \leq u^{1-\frac{c}{\log(n-1) \log N}} \frac{T}{\log(n-1) \log N}^\kappa
\]
for some positive constants \(c, d\) which are independent of \(L(s, \rho)\).
**Remark 4.20.** Conjecture 4.19 follows from GRH if \( u \) is large. However, if \( u \) is small, of size \((\log N)^{a}\) which is under consideration, it does not follow from GRH.

If \( T = (\log N)(\log \log N)^2 \),

\[
T^{\frac{d}{\log(n-1)\log N}} = O(1).
\]

Let \( b = \frac{c}{(\log((n-1)\log N))} \), then under Conjecture 4.19,

\[
\int_{w}^{x} \left( \sum_{|\gamma| \leq T} \frac{u^\gamma}{\vartheta^{2}} \right) \frac{du}{u^2} \ll \int_{w}^{x} u^{-1-b} \; du \ll w^{-b} = O(1).
\]

Hence under the zero density hypothesis, we proved (4.16). \(\square\)

Then since

\[
\sum_{\frac{\log N}{\log \log N} \leq p \leq w} \frac{\log p}{p} \ll \log \log N,
\]

we have

\[
\frac{L'(1, \rho)}{L(1, \rho)} = - \sum_{c_{f} \leq p \leq \frac{\log N}{\log \log N}} \frac{\lambda(p) \log p}{p} + O(\log \log \log N). \quad (4.21)
\]

Since \(-1 \leq \lambda(p) \leq n - 1\), we have

**Theorem 4.22.** Under the Artin conjecture and GRH and Conjecture 4.19 for \( L(s, \rho) \), the upper and lower bound for \( \frac{L'(1, \rho, t)}{\frac{L}{2}} \) are

\[
\log \log |d_{K_{t}}| + O(\log \log \log |d_{K_{t}}|), \quad -(n - 1) \log \log |d_{K_{t}}| + O(\log \log \log |d_{K_{t}}|),
\]

resp.

For \( X > 0 \), let \( y = \frac{1}{100} \log X \) and define \( M, i_{M}, s_{M} \) as in section 4.2. So for all \( c_{f} \leq p \leq y \), if \( t \equiv s_{M} \pmod{M} \), \( p \) splits completely in \( \hat{K}_{t} \), and \( \lambda(p, t) = n - 1 \); if \( t \equiv i_{M} \pmod{M} \), \( \lambda(p, t) = -1 \). Assume that the discriminant of \( f(x, t) \) is a polynomial in \( t \) of degree \( D \). Then there is a constant \( C \) such that \( |d_{K_{t}}| \leq C t^{D} \). So \( \log |d_{K_{t}}| \ll \log t \). For \( A = C^{1/D} X \), we define a set \( L(A)_{i} \) by

\[
L(A)_{i} = \{ \frac{X}{2} < t < X \mid t \equiv i_{M} \pmod{M}, \text{Gal}(\hat{K}_{t}/\mathbb{Q}) \simeq G \}.
\]
Similarly we define \( L(A)_s \). Note that for \( \frac{1}{2}X < t < X \), \( \frac{\log |d_{K_t}|}{\log \log |d_{K_t}|} \leq y = \frac{1}{100} \log X \) for sufficiently large \( X \). Hence we can control \( \lambda(p) \) for \( c_f \leq p < \frac{\log N}{\log \log N} \), namely, for \( t \in L(A)_s, \lambda(p) = n - 1; \) for \( t \in L(A)_i, \lambda(p) = -1 \). Hence we have proved

**Theorem 4.23.** Under the Artin conjecture and GRH and Conjecture 4.19 for \( L(s, \rho, t) \), for all \( t \in L(A)_i, \frac{L'}{L}(1, \rho, t) = \log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|) \), and for all \( t \in L(A)_s, \frac{L'}{L}(1, \rho, t) = -(n - 1) \log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|) \).

Let’s assume the strong Artin conjecture for \( L(s, \rho, t) \) instead of the Artin conjecture and GRH. In addition we assume that Assumptions 4.4 and 4.5 are true. They by applying Theorem 1.10 to \( L(A)_s \) and \( L(A)_i \), we obtain sets \( \hat{L}(A)_s \) and \( \hat{L}(A)_i \), where every automorphic \( L \)-function excluding possible exceptional \( O(A^{98/100}) \) \( L \)-functions, has a zero-free region \( [\alpha, 1] \times [-(\log |d_{K_t}|)^2, \log |d_{K_t}|)^2] \). Then we can prove

**Theorem 4.24.** Under the strong Artin conjecture, Assumptions 4.4, 4.5 and Conjecture 4.19 for \( t \in \hat{L}(A)_s, \frac{L'}{L}(1, \rho, t) = -(n - 1) \log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|) \). For \( t \in \hat{L}(A)_i, \frac{L'}{L}(1, \rho, t) = \log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|) \).
A refinement of Weil’s Theorem

This chapter is written based on the joint work [10] with H. Kim. We follow [10] closely.

5.1 A refinement of Weil’s Theorem

First, let us recall Weil’s celebrated theorem on rational points of algebraic curves over finite fields. ([58], page 75):

**Theorem 5.1** (Weil). Let \( f(x, y) \in \mathbb{F}_p[x, y] \) be absolutely irreducible and of total degree \( d > 0 \). Let \( N \) be the number of zeros of \( f \) in \( \mathbb{F}_p \times \mathbb{F}_p \). Then

\[
|N - p| \leq (d - 1)(d - 2)\sqrt{p} + c(d),
\]

for a constant \( c(d) \).

Let \( f_t(x) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t) \), where \( a_i(t) \in \mathbb{Z}[t] \). Suppose \( f_t \) is irreducible over \( \mathbb{Q}(t) \). Furthermore, we assume that the splitting field \( E \) of \( f_t(x) \) over \( \mathbb{Q}(t) \) is regular. Let \( p \) be a prime and \( N_t(p) \) be the number of solutions \( f_t(x) \equiv 0 \pmod{p} \), and let \( \lambda_t(p) = N_t(p) - 1 \). Let \( G = \text{Gal}(\widehat{K_t}/\mathbb{Q}) \). Recall that

\[
L(s, \rho_t) = \frac{\zeta_{K_t}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda_t(n)n^{-s},
\]

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and $\rho_t$ is the $(n - 1)$-dimensional representation of $G$, and $\lambda_t(p)$ is the value of the character of $\rho_t$ at the conjugacy class of $p$. Note that $-1 \leq \lambda_t(p) \leq n - 1$ for unramified primes $p$.

Let $A_i$ be the number of $t \pmod{p}$ such that $\lambda_t(p) = i$, i.e., $f_t(x) \equiv 0 \pmod{p}$ has $i + 1$ roots. Then clearly,

$$N = \sum_{i=-1}^{n-1} (i + 1)A_i + O(1).$$

When $G$ is abelian, then $A_i = 0$ for $i \neq -1, n - 1$. So $N = nA_{n-1} + O(1)$.

**Theorem 5.2.** Fix a prime $p$. Let $C_i$ be the union of conjugacy classes such that $\lambda_t(p) = i$. Then

$$A_i = \frac{|C_i|}{|G|}p + O(\sqrt{p}).$$

**Proof.** This is essentially Chebotarev density theorem for function field, and is proved by Ree [54]. In [54], Theorem 2, it is stated only for when $f(x, t) \mod{p}$ is irreducible. (He needs to assume that $f(x, t)$ gives rise to a regular Galois extension $E$ over $\mathbb{Q}(t)$.) But it is straightforward to generalize it. We follow his exposition in [54]. Let $k = \mathbb{F}_p$, and $K$ be the splitting field of $f(x, t)$ over $k(t)$. Since $f(x, t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$, $\text{Gal}(K/k(t)) \simeq \text{Gal}(E/\mathbb{Q}(t))$ for sufficiently large $p$. Let $D$ be the set of elements $a \in k$ such that the place $p_a$ of $k(t)$ corresponding to $t - a$ does not ramify in $K$. Then the conjugacy class $C_a$ of the Frobenius at $p_a$ in $K$ is the same as factorization of $f(x, a) \pmod{p}$. For any conjugacy class $C$ in $G$, let $N_n(C)$ be the number of elements $a \in D$ such that $C_a = C$. Then the density theorem of Weil says that there exists a constant $\alpha_n$, depending only on $n$, such that

$$\left| N_n(C) - \frac{|C|}{|G|}p \right| < \alpha_n \sqrt{p}. \tag{5.3}$$

Hence (5.3) implies our result.

**Remark 5.4.** Theorem 5.2 can be thought of a refinement of Theorem 5.1. Indeed, Theorem 5.2 implies Theorem 5.1: By class equation, $\sum_{i=-1}^{n-1} |C_i| = |G|$, and so Theorem
5.2 implies
\[ \sum_{i=1}^{n-1} (i+1)A_i = \sum_{i=1}^{n-1} (i+1) \left( \left\lfloor \frac{|C_i|}{|G|} \right\rfloor p + O(\sqrt{p}) \right) = \frac{p}{|G|} \sum_{i=1}^{n-1} i|C_i| + \frac{p}{|G|} \sum_{i=1}^{n-1} |C_i| + O(\sqrt{p}). \]

Here \( \sum_{i=1}^{n-1} i|C_i| = 0 \). We can prove this as follows: Note that \( \chi_{\rho_i} \) is the sum of irreducible characters \( \chi_1, ..., \chi_k \), and
\[ \sum_{i=1}^{n-1} i|C_i| = \sum_{g \in G} \chi_{\rho_i}(g) = \sum_{j=1}^{k} \sum_{g \in G} \chi_j(g). \]

By orthogonality of characters, \( \sum_{g \in G} \chi_j(g) = 0 \) for each \( j = 1, ..., k \). Therefore, \( \sum_{i=1}^{n-1} (i+1)A_i = p + O(\sqrt{p}) \).

This implies that \( \sum_{i=1}^{n-1} iA_i = O(\sqrt{p}) \). This played a crucial role in the previous chapter to find the error term of logarithmic derivatives of Artin L-functions in a family.

In the special case \( f_t(x) = x^2 - g(t) \), where \( g \in \mathbb{Z}[t] \) and \( g \) is square free, we recover the result of Davenport and Burgess [2] that \( \sum_{t \pmod{p}} (\frac{g(t)}{p}) = O(\sqrt{p}) \).

For the several cyclic polynomials, we determine \( A_i \) up to a constant in section 5.2.

### 5.2 Cyclic groups

In the case of cyclic extension, we need to determine only \( A_{-1} \) and \( A_{n-1} \). In the case of simplest cubic fields, Duke [17] already obtained the result. The simplest cubic fields are the cubic fields parameterized by the polynomial
\[ f_t(x) = x^3 - tx^2 - (t+3)x - 1, \text{ for } t \in \mathbb{Z}^+. \]

For a prime \( p \geq 5 \), Duke computed the number of residue classes \( t \) modulo \( p \) for which \( f_t(x) \) splits completely, remains inert or ramifies respectively.

<table>
<thead>
<tr>
<th></th>
<th>split</th>
<th>inert</th>
<th>ramified</th>
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<tbody>
<tr>
<td>( p \equiv 1 \pmod{3} )</td>
<td>( \frac{p-4}{3} )</td>
<td>( \frac{2p-2}{3} )</td>
<td>2</td>
</tr>
<tr>
<td>( p \equiv 2 \pmod{3} )</td>
<td>( \frac{p-2}{3} )</td>
<td>( \frac{2p+2}{3} )</td>
<td>0</td>
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This can be paraphrased as

\[ A_{-1} = \frac{2p - 2}{3}, \text{ and } A_2 = \frac{p - 4}{3} \text{ if } p \equiv 1 \mod 3 \]

and

\[ A_{-1} = \frac{2p + 2}{3}, \text{ and } A_2 = \frac{p - 2}{3} \text{ if } p \equiv 2 \mod 3. \]

With Duke’s idea, we can extend the analogous result to simplest quartic and sextic fields. The simplest quartic fields are the fields parameterized by

\[ f_t(x) = x^4 - tx^3 - 6x^2 + tx + 1 \]

with the discriminant \( \text{disc}(f_t(x)) = 4(t^2 + 16)^3 \). Note that for odd primes \( p \), \( \text{disc}(f_t(x)) \equiv 0 \mod p \) has a solution if and only if \( p \equiv 1 \mod 4 \).

We consider the polynomial \( f_t(x) \) over a finite field \( \mathbb{F}_p \) of \( p \) elements. Then

\[ t = \frac{x^4 - 6x^2 + 1}{x(x^2 - 1)} \]

hence \( x \) should belong to \( \mathbb{F}_p \setminus \{0, \pm 1\} \).

If \( p \equiv 3 \mod 4 \), \( \text{disc}(f_t(x)) \) has no root mod \( p \). Hence \( 4 \times A_3 = p - 3 \). Hence we obtain that

\[ A_{-1} = \frac{3p + 3}{4} \text{ and } A_3 = \frac{p - 3}{4} \]

for primes \( p \) with \( p \equiv 3 \mod 4 \) and \( p \geq 3 \).

If \( p \equiv 1 \mod 4 \), \( \text{disc}(f_t(x)) \) has two roots \( \pm 4\sqrt{-1} \mod p \). For each \( t \) corresponding a root of \( \text{disc}(f_t(x)) \), \( f_t(x) \) has at most 3 roots mod \( p \), so there is a constant \( 0 \leq c_p \leq 6 \) depending on \( p \) with \( 4 \times A_3 = p - 3 - c_p \). Since \( A_3 \) is of integer value, \( c_p \) is 2 or 6. For example, we can check easily that \( c_5 = 2 \). Hence we obtain that

\[ A_{-1} = \frac{3p - 5 + c_p}{4} \text{ and } A_3 = \frac{p - 3 - c_p}{4} \]

for primes \( p \) with \( p \equiv 1 \mod 4 \) and \( p \geq 5 \).

Next, we consider the simplest sextic fields. They are parametrized by

\[ f_t(x) = x^6 - 2tx^5 - 5(t + 3)x^4 - 20x^3 + 5tx^2 + 2(t + 3)x + 1 \]
with the discriminant \( \text{disc}(f_t(x)) = 2^63^6(t^2 + 3t + 9)^5 \). Note that for odd primes \( p \geq 5 \), \( \text{disc}(f_t(x)) \equiv 0 \pmod{p} \) has a solution if and only if \( p \equiv 1 \pmod{3} \).

We consider the polynomial \( f_t(x) \) over a finite field \( \mathbb{F}_p \) of \( p \) elements. Then

\[
t = \frac{x^6 - 15x^4 - 20x^3 + 6x + 1}{x(x^2 - 1)(2x + 1)(x + 2)}
\]

hence \( x \) should belong to \( \mathbb{F}_p \setminus \{0, \pm1, -2, -1/2\} \).

If \( p \equiv 2 \pmod{3} \), \( \text{disc}(f_t(x)) \) has no root \( \pmod{p} \). Hence \( 6 \times A_5 = p - 5 \). Hence we obtain that

\[
A_{-1} = \frac{5p + 5}{6} \quad \text{and} \quad A_5 = \frac{p - 5}{6}
\]

for primes \( p \) with \( p \equiv 2 \pmod{3} \) and \( p \geq 5 \).

If \( p \equiv 1 \pmod{3} \), \( \text{disc}(f_t(x)) \) has two roots \( \frac{-3 \pm \sqrt{-3}}{2} \pmod{p} \). For each \( t \) corresponding a root of \( \text{disc}(f_t(x)) \), \( f_t(x) \) has at most 5 roots \( \pmod{p} \), so there is a constant \( 0 \leq c_p \leq 10 \) with \( 6 \times A_5 = p - 5 - c_p \). Since \( A_5 \) is of integer value, \( c_p \) must be 2 or 8. Especially we can check that \( c_7 = 2 \). Hence we obtain that

\[
A_{-1} = \frac{5p - 7 + c_p}{6} \quad \text{and} \quad A_3 = \frac{p - 5 - c_p}{6}
\]

for primes \( p \) with \( p \equiv 1 \pmod{3} \) and \( p \geq 7 \).
Chapter 6

Simple zeros of Maass L-functions

6.1 Introduction

Many people have had an interest in zeros of various kinds of L-functions. One of the most famous open question is the Riemann Hypothesis. In 1942 Selberg [61] proved that a positive proportion of zeros lie on the critical line \( \text{re}(s) = 1/2 \). Levinson [44] showed that more than one third of the zeros are on the critical line. Heath-Brown [26] was the first man who showed that the Riemann zeta function has infinitely simple zeros by observing the work of Levinson. Montgomery [46], assuming the Riemann Hypothesis, showed that more than two thirds of the zeros are simple and lie on the critical line. Unconditionally, Conrey [15] showed that more than two fifths of zeros are simple and lie on the critical line.

In 1983 Hafner [23] showed that L-functions attached to Hecke cusp form for \( SL(2, \mathbb{Z}) \) has a positive proportion of zeros with odd multiplicity on the critical line. Later he [24] obtained the same conclusion for the L-functions attached to even Maass forms.

Except the case of the Riemann zeta function, people did not know about simple zeros of L-functions. In 1986, Conrey, Ghosh and Gonek [13] showed that the zeta function of a quadratic field has infinitely many simple zeros. In 1988, Conrey and Ghosh [14]
got a breakthrough about simple zeros of modular L-functions. They showed that the L-function attached to Ramanujan tau-function has infinitely many simple zeros. In fact, they showed that one simple zero of a Heck cusp L-function implies infinitely many simple zeros if it has no non-trivial real zero.

The purpose of this article is to extend Conrey and Ghosh [14] to Maass L-functions. First, we recall a Maass form and its L-function briefly.

Let $\Gamma = PSL(2, \mathbb{Z})$ act on the upper half-plane $\mathfrak{H} = \{ z : Imz > 0 \}$ by linear fractional transformations. The Maass cusp forms which are also eigenfunctions of the Hecke operators is functions $f$ in $L^2(\Gamma \backslash \mathfrak{H})$ satisfying

1. $\Delta f = (\frac{1}{4} + r^2)f, \Delta = -y^2(\partial^2_x + \partial^2_y),$
2. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma,$
3. $\int_0^1 f(z)dx = 0,$
4. $T_n f = \lambda_n f$ for $n \geq 1$

where $T_n$ is the nth Hecke operator for $\Gamma$.

Then we have the Fourier-Whittaker expansion of $f(z),$

$$f(z) = \sum_{n \neq 0} a_n y^{1/2}K_{ir}(2\pi ny)e^{2\pi inx}.$$  

Since $f$ is an eigenfunction of the $T_n$’s, it follows that if we normalize $a_1 = 1,$ then $a_n = \lambda_n,$ which is real, and

$$a_na_m = \sum_{d|\langle n,m \rangle} a_{nm/d^2}.$$  

The L-function attached to $f$ is defined by

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$  

and it satisfies the functional equation

$$\Lambda_f(s) := \pi^{-s}\Gamma \left( \frac{s+\epsilon + ir}{2} \right) \Gamma \left( \frac{s+\epsilon - ir}{2} \right) L_f(s) = (-1)^\epsilon \Lambda_f(1 - s)$$  

where $\epsilon = 0$ if $f$ is even and $\epsilon = -1$ if $f$ is odd.
Strömbergsson [69] found simple zeros of three different L-functions attached to even Maass forms on $SL(2, \mathbb{Z})$. He considered three even Hecke eigenforms corresponding to $r = 13.779751351891, 17.738563381058$ and $125.313840177018$. He showed that there are 156, 157 and 170 nontrivial zeros under height $t = 200$ in the three respective cases. The zeros are all found to be simple and to lie on the critical line $\text{Re}(s) = 1/2$. We will show that at least these three Maass L-functions have infinitely many simple zeros.

Throughout this article, we assume that our Maass form is even except in section 6.5. This is because a minor modification of the proof is required when the form is odd. Another reason is that Strömbergsson [69] found simple zeros of even Maass L-function. In section 5, we explain about the difference when the form is odd in detail.

We imitate Conrey and Ghosh [14] and Booker [3]. In case of Conrey and Ghosh [14], the gamma factor of a Hecke modular L-function consists of one gamma function. On the other hand, the gamma factor of a Maass L-function is the product of 2 gamma functions, that is $H(s) = \pi^{-s} \Gamma \left( \frac{s + ir}{2} \right) \Gamma \left( \frac{s + ir}{2} \right)$.

Via Stirling’s formula, we can see

$$\pi^{-s} \Gamma \left( \frac{s + ir}{2} \right) \Gamma \left( \frac{s - ir}{2} \right) (s - 1/2)$$

$$= \sqrt{8\pi(2\pi)^{-s}} \Gamma (s + 1/2) + b(2\pi)^{-s} \Gamma (s - 1/2) + (2\pi)^{-s} \Gamma (s - 1/2) E_{(1,r)}(s)$$

where $b$ is some constant constants and $E_{(1,r)}(s)$ is holomorphic and $O(1/s)$ in $\text{Re}(s) > 1$.

First we consider the difference of two integrals,

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} L'(1-s) \frac{L'}{L}(s) H(s)L(s)(s - 1/2)e^{i(\pi/2 - \delta)s} ds$$

$$- \frac{1}{2\pi i} \int_{(-\epsilon)} L'(1-s) \frac{L'}{L}(s) H(s)L(s)(s - 1/2)e^{i(\pi/2 - \delta)s} ds$$

And this equals, by the functional equation ,

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} L'(1-s) \frac{L'}{L}(s) H(s)L(s)(s - 1/2)f(s, \delta) ds \quad (6.1)$$
where \( f(s, \delta) = e^{i(\pi/2-\delta)s} + e^{i(\pi/2-\delta)(1-s)} \). The integral is, via Cauchy’s theorem, expressed as the following sum over zeros together with \( O(1) \) term.

\[
- \sum_{0 < \text{re}(\rho) < 1} L'(\rho)H(\rho)(\rho - 1/2)e^{i(\pi/2-\delta)\rho} + O(1)
\]

where \( O(1) \) is contributed by trivial zeros \( 1 \pm ir \) of \( L(1-s) \) and simple poles \( \pm ir \) of \( H(s) \).

We show that, in Section 6.2, this integral is very large if there is a simple zero of \( L(s) \). This implies the infinity of the number of simple zeros of a Maass L-function.

### 6.2 Estimates of integrals

Define \( X(s) = H(1-s)/H(s) \). Then we have an asymmetric functional equation \( L(s) = X(s)L(1-s) \). If we replace \( H(s) \) by the above linear combination of Gamma functions, the integral (6.1) equals

\[
- \int_{(1+\epsilon)} L'(s)L'(s)(2\pi)^{-s}G_1(s)f(s, \delta)ds + \int_{(1+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}G_1(s)f(s, \delta)ds \tag{6.2}
\]

where \( G_1(s) = (\sqrt{8\pi}\Gamma(s+1/2) + b\Gamma(s-1/2) + \Gamma(s-1/2)E_{(1,\nu)}(s)) \).

We divide (6.2) into 4 integrals. We estimate the value of each integral. For this purpose, we need the following Lemma.

**Lemma 6.3.** Suppose that

\[
F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}
\]

where the sum is absolutely convergent for \( \sigma > \sigma_0 > 0 \) and \( 0 < \delta < \pi/2 \).

Then for \( l > \sigma_0 \) and \( l + c > 0 \),

\[
\frac{1}{2\pi i} \int_{(l)} F(s)(2\pi)^{-s}\Gamma(s+c)e^{i(\pi/2-\delta)s}ds = \frac{1}{2\pi i} \int_{(l)} F(s)(2\pi)^{-s}\Gamma(s+c)(2\sin(\delta/2))^{-s-c} \left(e^{\pm i\delta/2}\right)^{-s+c}(\mp i)^c ds.
\]

**Proof.** The proof is almost identical with the proof of Lemma 1 in Conrey and Ghosh [14] or Lemma 2 in Booker [3]. For the sake of completeness, we re-write the proof. First recall that \( z^e e^{-z} = \oint \Gamma(z+c)z^{-s} \) for \( re(z) > 0 \) and \( l + c > 0 \).
The integral equals
\[= \sum_{n=1}^{\infty} f(n) \frac{1}{2\pi i} \int_{(t)} \Gamma(s + c) \left( \mp 2\pi i e^{\pm i\delta} \right)^{-s} ds\]
\[= \sum_{n=1}^{\infty} f(n)(\mp 2\pi i e^{\pm i\delta}) e^{\pm 2\pi ni(e^{\pm i\delta} - 1)}\]
\[= \sum_{n=1}^{\infty} f(n)(\mp 2\pi i e^{\pm i\delta}) e^{-2\pi n(2\sin(\delta/2)) e^{\pm i\delta/2}}\]
\[= (\mp i)^c (2\sin(\delta/2))^{-c} (e^{\pm i\delta/2})^c \sum_{n=1}^{\infty} f(n)(2\pi n(2\sin(\delta/2)) e^{\pm i\delta/2}) e^{-2\pi n(2\sin(\delta/2)) e^{\pm i\delta/2}}\]
\[= (\mp i)^c (2\sin(\delta/2))^{-c} (e^{\pm i\delta/2})^c \times \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{(t)} f(n)n^{-\sigma} \Gamma(s + c)(2\pi)^{-s}(e^{\pm i\delta/2})^{-s}(2\sin(\delta/2))^{-s} ds\]
which equals the right hand side. \qed

Let's consider the following integral.

\[\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}\Gamma(s + 1/2) f(s, \delta) ds.\]

By Lemma 6.3, this integral is equal to

\[\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-(s+1/2)} g(s, \delta) ds\]
where \(g(s, \delta) = \left[ (e^{i\delta/2})^{-s+1/2} (-i)^{1/2} + e^{i(\pi/2-\delta)} (e^{-i\delta/2})^{-s+1/2} (i)^{1/2} \right].\)

Let's assume that \(L(s)\) has a simple zero \(\rho = \beta + i\delta\) for \(0 < \beta < 1\). From the functional equation \(H(s) L(s) = H(1-s) L(1-s)\), we may assume that \(\beta \geq 1/2\). Since zeros of \(g(s, \delta)\) are \(\frac{3}{2} - \frac{2\pi(n+1)}{\delta}\) for \(n \in \mathbb{Z}\), the integrand has a simple pole at \(s = \beta + i\delta\).

**Lemma 6.4** (Booker [3]). Let \(\psi(s)\) be meromorphic in the complex plane, and holomorphic for \(\sigma > \sigma_0\) and of rapid decay in vertical strips in a right hand plane. If \(\psi(s)\) has a pole at \(s = \beta + i\), then for \(l > \sigma_0\)

\[\frac{1}{2\pi i} \int_{(l)} \psi(s)x^{-s} ds = \Omega_\epsilon \left( x^{-(\beta-\epsilon)} \right) \text{ as } x \to 0\]
for all \(\epsilon > 0\).
Hence by Lemma 6.4,
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'(s)}{L(s)} L'(2\pi)^{-s} \Gamma (s + 1/2) f(s, \delta) ds = \Omega (\delta^{-(\beta+1/2-\epsilon)})
\]
as \(\delta \to 0\).

Let's consider the integral
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'(s)}{L(s)} L'(2\pi)^{-s} (b \Gamma (s - 1/2) + \Gamma (s - 1/2) E_{1,1}(s)) f(s, \delta) ds
\]
The value of this integral is dominated by the value of the integral
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'(s)}{L(s)} L'(2\pi)^{-s} \Gamma (s - 1/2) e^{i(\pi/2-\delta)(-s)} ds
\]
and this integral is equal to, by Lemma 6.3,
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X} (s) L'(2\pi)^{-s} (b \Gamma (s - 1/2) + \Gamma (s - 1/2) E_{1,1}(s)) f(s, \delta) ds
\]
which is dominated by
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X} (s) L'(2\pi)^{-s} \Gamma (s - 1/2) e^{i(\pi/2-\delta)(-s)} ds
\]

If shift the contour of the integral to \(1/2 + \epsilon\), then
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X} (s) L'(2\pi)^{-s} \Gamma (s - 1/2) e^{i(\pi/2-\delta)(-s)} ds
\]
\[= \frac{1}{2\pi i} \int_{(1/2+\epsilon)} \frac{X'}{X} (s) L'(2\pi)^{-s} \Gamma (s - 1/2) e^{i(\pi/2-\delta)(-s)} ds + O(1).
\]

Meurman [45] showed that \(L(1/2+it) \ll t^{1/3+\epsilon}\). Hence the integrand is \(O \left( |t|^{5/6+\epsilon-1} e^{-\delta|t|} \right)\)

and this integral is \(O \left( (1/\delta)^{5/6+\epsilon} \right)\).

Lastly, the remaining integral is
\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} X'(s)L'(s)(2\pi)^{-s}\Gamma(s+1/2)f(s,\delta)ds \tag{\diamond}
\]

The estimate of the integral (\diamond) is the most difficult part. In section 6.3, we show

**Lemma 6.5.**

\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma(s+1/2)f(s,\delta)ds \ll \left(\frac{1}{\delta}\right)^{5/6+\epsilon}
\]

for any \(\epsilon > 0\).

Our estimates of the 4 integrals show that

\[
\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(1-s)\frac{L'}{L}(s)H(s)L(s)(s-1/2)f(s,\delta)ds = \Omega_{\epsilon}\left(\frac{1}{\delta}\right)^{\beta+1/2-\epsilon}
\]

as \(\delta \to 0\).

### 6.3 An estimate of the integral (\diamond)

Recall that \(H(s) = \pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)\). By Stirling’s formula, we have

\[H(s) = (2\pi)^{-s}\Gamma\left(s - 1/2\right)E_{(2,r)}(s)\]

where \(E_{(2,r)}(s)\) is holomorphic in the complex plane except when \(s = \pm ir - 2n\) for \(n = 0, 1, 2, \cdots\) and \(E_{(2,r)}(s) = \sqrt{8\pi} + O(1/s)\).

Then

\[X(s) = \frac{H(1-s)}{H(s)} = (2\pi)^{2s-1}\frac{\Gamma(1/2-s)E_{(2,r)}(1-s)}{\Gamma(s-1/2)E_{(2,r)}(s)} \]

\[= (2\pi)^{2s-1}\frac{\Gamma(3/2-s)(s-1/2)E_{(2,r)}(1-s)}{\Gamma(s+1/2)(1/2-s)E_{(2,r)}(s)} = -(2\pi)^{2s-1}\frac{\Gamma(3/2-s)E_{(2,r)}(1-s)}{\Gamma(s+1/2)E_{(2,r)}(s)}.
\]

Hence we have,

\[\frac{X'(s)}{X(s)} = 2\log 2\pi - \frac{\Gamma'}{\Gamma}(3/2-s) - \frac{\Gamma'}{\Gamma}(s+1/2) - \frac{E'_{(2,r)}}{E_{(2,r)}}(1-s) - \frac{E'_{(2,r)}}{E_{(2,r)}}(s).
\]
Since $E'(z) = \log z - \frac{1}{2z} + O(1/z^2)$, it is easy to see that $E'(z) \frac{E'(z)}{E(z)}(s) = O(1/s)$. Then, the contribution of $E'(z) \frac{E'(z)}{E(z)}(1 - s) + E'(z) \frac{E'(z)}{E(z)}(s)$ to the integral $\langle \rangle$ is $O\left((1/\delta)^{5/6+\epsilon}\right)$ when we move the contour of integral from $c = 1 + \epsilon$ to $c = 1/2$.

We plug in the remaining part of $X'(s) X(s)$ into the integral $\langle \rangle$, we have

$$\int_{(1+\epsilon)} L'(s)G_2(s)f(s,\delta)ds$$

where $G_2(s) = 2 \log 2\pi \Gamma(s + 1/2) - \frac{\Gamma'}{\Gamma}(3/2 - s)\Gamma(s + 1/2) - \Gamma'(s + 1/2)$.

By Lemma 6.3,

$$\int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma(s + 1/2)e^{(\pi/2-1/2)s}ds$$

$$= \int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-s-1/2}(e^{i\delta/2})^{-s+1/2}(-i)^{1/2}ds$$

$$= \int_{(\epsilon)} L'(s)(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-s-1/2}(e^{i\delta/2})^{-s+1/2}(-i)^{1/2}ds$$

$$= O\left((1/\delta)^{1/2+\epsilon}\right)$$

Secondly,

$$\int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma'(s + 1/2)e^{(\pi/2-1/2)s}ds$$

$$= \int_{(1+\epsilon)} \sum_{n=1}^{\infty} \frac{b(n)}{n^s}(2\pi)^{-s}\log\left(-2\pi ine^{i\delta}\right)(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-s-1/2}(e^{i\delta/2})^{-s}ds$$

$$= \int_{(1+\epsilon)} \sum_{n=1}^{\infty} \frac{b(n)\log(n) + \log(-2\pi ine^{i\delta})}{n^s}$$

$$\times(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-s-1/2}(e^{i\delta/2})^{-s}ds = O\left((1/\delta)^{1/2+\epsilon}\right)$$

where $L'(s) = -\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b(n)n^{-s}$.

The remaining integral to estimate is

$$\int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma'(3/2-s)\Gamma(s + 1/2)(s + 1/2)e^{((\pi/2)-\delta)s}ds.$$

For this purpose, we need analogous lemmas of Lemma 2 and Lemma 3 in [14].

**Lemma 6.6.** Assume that $|\arg(z)| < \pi/2$, then $\frac{1}{2\pi i}\int_{(1+\epsilon)} \Gamma'(3/2-s)\Gamma(s + 1/2)z^{-s} = z^{1/2}e^{-z} \left(\frac{\Gamma'}{\Gamma}(2) - \int_0^1 \frac{e^{t-1}}{t}(1-t)dt\right).$
Proof. $\Gamma(s+1/2)$ has a simple pole at $s = -1/2 - n$ for $n = 0, 1, 2, \cdots$ with residue $\frac{(-1)^n}{n!}$.

Hence the integral equals
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n z^{1/2+n}}{n!} \frac{\Gamma'(2+n)}{\Gamma(2+n)}
\]

Since $s\Gamma(s) = \Gamma(s+1)$, we have
\[
\frac{\Gamma'(2+n)}{\Gamma(2+n)} = \frac{\Gamma'(2)}{\Gamma(2)} + \sum_{l=0}^{n-1} \frac{1}{l+2}
= \frac{\Gamma'(2)}{\Gamma(2)} + \int_{0}^{1} t \frac{(1-t^n)}{(1-t)} dt.
\]

Then the integral is equal to
\[
= \frac{\Gamma'(2)}{\Gamma(2)} z^{1/2}e^{-z} + z^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \int_{0}^{1} t \frac{(1-t^n)}{(1-t)} dt
\]
\[
= \frac{\Gamma'(2)}{\Gamma(2)} z^{1/2}e^{-z} + z^{1/2} \int_{0}^{1} \frac{t}{1-t} (e^{-z} - e^{-zt}) dt
\]
and when we change $t$ by $1-t$, we obtain the right hand side.

\[\square\]

Lemma 6.7. Suppose $v \geq 1/2$. Define
\[
\omega(v, \delta) = \frac{1}{2\pi i} \int_{(1+e)} \frac{\Gamma'(3/2-s)\Gamma(s+1/2)(2\pi)^{-s}v^{-s}e^{i(\pi/2-\delta)s}}{\Gamma(3/2-s)} ds.
\]

Then
\[
\omega(v, \delta) \ll v^{1/2} \left( e^{-x} \log \frac{1}{\delta} + \min\{1, \frac{1}{x^2}\} \right)
\]
where $x = 2\pi v \sin \delta$. Moreover, we have
\[
\omega(v+1/2, \delta) + \omega(v, \delta) \ll e^{-x} \left( \delta \log \frac{1}{\delta} + \frac{1}{v^{1/2}} \log \frac{1}{\delta} \right) + \delta \log \min\{1, \frac{1}{x^2}\} + \frac{1}{v^{1/2}} \min\{1, \frac{1}{x}\}.
\]

Proof. Since the process of the proof is identical with Lemma 3 in [14], we skip some detail and refer to [14].

By Lemma 6.6, $\omega(v, \delta) = z^{1/2}e^{-z} \left( \frac{\Gamma'(2)}{\Gamma(2)} - \int_{0}^{1} \frac{e^{zt}-1}{t}(1-t) dt \right) = e^{-z}G(z)$ where $z = -2\pi v e^{i\delta} = 2\pi v \sin \delta - 2\pi i \cos \delta = x - iy$.

Since $\int_{0}^{1} \frac{e^{zt}-1}{t}(1-t) dt \ll \log \frac{|z|}{x} + e^{x} \min\{1, x^{-2}\}$, the first statement follows.
Next, \( \omega(v + 1/2, \delta) + \omega(v, \delta) = e^{-(z+\eta)}(G(z + \eta) - G(z)) + G(z)(e^{-z} + e^{-(z+\eta)}) \) where \( \eta = -\pi i \varepsilon \delta \). We have

\[
e^{-z} + e^{-z} \ll e^{-x}|e^{-\varepsilon \eta} + 1| = e^{-x}|1 - e^{\pi i (e^{2i\delta - 1})}| \ll \delta e^{-x}.
\]

Also we have

\[
G(z + \eta) - G(z) = \sqrt{z + \eta} \left( -\int_0^1 e^{t(z+\eta)} - e^{tz} \frac{1}{t}(1 - t) dt \right) + (\sqrt{z + \eta} - \sqrt{z}) \left( \frac{\Gamma'}{\Gamma}(2) - \int_0^1 (e^{tz} - 1)(1 - t) \frac{1}{t} dt \right)
\]

\[
\ll |z|^{1/2} \frac{1}{|z|^2} e^{x \min\{1, x^{-1}\}} + \frac{1}{|z|^{1/2}} \left( \log \frac{1}{\delta} + e^{x \min\{1, x^{-2}\}} \right).
\]

Then the second statement follows.

Now we are ready to estimate the integral

\[
F(\delta) = \int_{(1+\varepsilon)} L'(s)(2\pi)^{-s} \frac{\Gamma'}{\Gamma}(3/2 - s)\Gamma(s + 1/2)e^{i(\pi/2 - \delta)s} ds = -\sum_{n=1}^{\infty} a_n \log n \omega(n, \delta).
\]

By Proposition 1.1 in [21] and partial summation, we have

**Lemma 6.8.** Let \( S_m = \sum_{n=1}^{m} a_n \log n \). Then \( S_m \ll m^{1/3+\varepsilon} \).

We notice that by Lemma 6.7,

\[
\omega(n + 1, \delta) - \omega(n, \delta) = (\omega(n + 1, \delta) + \omega(n + 1/2, \delta)) - (\omega(n + 1/2, \delta) + \omega(n, \delta))
\]

\[
\ll \left( e^{-(n\delta)} \left( \delta \log \frac{1}{\delta} + \frac{1}{n^{1/2}} \log \frac{1}{\delta} \right) + \delta \min\left\{1, \frac{1}{(n\delta)^2}\right\} + \frac{1}{n^{1/2}} \min\left\{1, \frac{1}{n\delta}\right\} \right)
\]

Since by Lemma 6.7 and Lemma 6.8, \( S_n \omega(n + 1, \delta) \to 0 \) as \( n \to \infty \) and the series

\[
\sum_{n=1}^{\infty} S_n(\omega(n + 1, \delta) - \omega(n, \delta))
\]

converges absolutely. Hence we rewrite \( F(\delta) \) as

\[
F(\delta) = -\sum_{n=1}^{\infty} S_n(\omega(n + 1, \delta) - \omega(n, \delta)).
\]
Finally, we have, by Lemma 6.8,

\[ F(\delta) \ll \sum_{n=1}^{\infty} n^{1/3+\epsilon} |\omega(n+1,\delta) - \omega(n,\delta)| \]

\[ \ll \sum_{n=1}^{\infty} n^{1/3+\epsilon} \left( e^{-n\delta} \left( \delta \log \frac{1}{\delta} + \frac{1}{n^{1/2}} \log \frac{1}{\delta} \right) + \delta \min\{1, \frac{1}{(n\delta)^2}\} + \frac{1}{n^{1/2}} \min\{1, \frac{1}{n\delta}\} \right) \]

\[ \ll \delta \left( \log \frac{1}{\delta} \right) \left( \frac{1}{\delta} \right)^{4/3+\epsilon} + \left( \log \frac{1}{\delta} \right) \left( \frac{1}{\delta} \right)^{5/6+\epsilon} + \left( \frac{1}{\delta} \right)^{1/3+\epsilon} + \left( \frac{1}{\delta} \right)^{5/6+\epsilon} \ll \left( \frac{1}{\delta} \right)^{5/6+2\epsilon} . \]

Hence Lemma 6.5 follows.

### 6.4 Main Theorem and its Corollary

Now we state the main Theorem and its Corollary.

**Theorem 6.9.** Let \( L(s) \) is the L-function attached to a Maass Hecke eigenform on \( SL(2,\mathbb{Z}) \) and we assume that \( L(s) \) has a simple zero \( \rho = \beta + it \) for \( 0 < \beta < 1 \). Then, there is an arbitrary large \( T \) so that \( L(s) \) has \( \gg \epsilon T^{1/6-\epsilon} \) simple zeros in the region \( 0 < t < T \).

**Corollary 6.10.** There are at least three L-functions attached to even Maass Hecke eigenforms for \( SL(2,\mathbb{Z}) \) which have infinitely many simple zeros.

Meurman [45] showed that \( L(1/2 + it) \ll_{\epsilon} t^{1/3+\epsilon} \). By Phragmen-Lindelöf argument, we have

\[ |L(\sigma + it)| \ll_{\epsilon} |t|^{-2/3\sigma+2/3+\epsilon} \]

for \( \frac{1}{2} \leq \sigma \leq 1 \).

Now let \( \beta_0 = \sup\{\beta|\rho = \beta + it \text{ is a simple zero of } L(s)\} \).

Let’s return to our original sum

\[ \sum_{0 < \rho \in \rho(\rho) < 1} L'(\rho)H(\rho)(\rho - 1/2)e^{i(\pi/2-\delta)\rho} + O(1) \]
For any $\epsilon > 0$, we have $H(s)L'(s) \ll e^{-\frac{T}{2}|t|}|t|^{|s|\geq \frac{1}{2}+\epsilon}$. Let $\frac{1}{8} = T$ and for sufficiently large $T$, the sum is

$$\ll_{\epsilon} \sum_{|t|<T, \sigma = \beta + it \text{ simple}} T^\frac{1}{2} \delta_0 + \frac{3}{2} + \epsilon$$

By the calculation in Section 6.2, there is an arbitrary large $T$ for which this sum is \(\gg_{\epsilon} T^\delta_0 + \frac{1}{2} - \epsilon\). Hence we finish the proof.

### 6.5 In case of an odd Maass form

When a Maass form is odd, we need some modifications in our arguments.

We consider the integral

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} L'(1-s)L'(s)H(s)(s-1/2)^2(e^{i(\pi/2-\delta)s} - e^{i(\pi/2-\delta)(1-s)})ds$$

where $H(s) = \pi^{-s}\Gamma\left(\frac{s}{2} + \frac{1}{2} - s\right)\Gamma\left(\frac{s}{2} - \frac{1}{2} - s\right)$. Via Stirling’s formula, we also see

$$\pi^{-s}\Gamma\left(\frac{s}{2} + \frac{1}{2} - s\right)\Gamma\left(\frac{s}{2} - \frac{1}{2} - s\right)(s-1/2)^2$$

$$= \sqrt{8\pi}(2\pi)^{-s}\Gamma(s+1/2) + b_1(2\pi)^{-s}\Gamma(s-1/2) + b_2(2\pi)^{-s}\Gamma(s-3/2) + (2\pi)^{-s}\Gamma(s-3/2)E_{(1,r)}(s)$$

where $b_1, b_2$ are some constants and $E_{(1,r)}(s)$ is holomorphic and $O(1/s)$ in $Re(s) > 1$. Define $X(s) = \frac{H(1-s)}{H(s)}$. Then $X(s)$ equals

$$(2\pi)^{2s-1}\frac{\Gamma(-1/2-s)E_{(2,r)}(1-s)}{\Gamma(s-3/2)E_{(2,r)}(s)} = (2\pi)^{2s-1}\frac{\Gamma(3/2-s)(s-3/2)E_{(2,r)}(1-s)}{\Gamma(s+1/2)(s+1/2)E_{(2,r)}(s)}.$$ 

Then, we have

$$\frac{X'}{X}(s) = 2\log 2\pi - \frac{\Gamma'}{\Gamma}(3/2-s) - \frac{\Gamma'}{\Gamma}(1/2+s)$$

$$+ \frac{1}{s-3/2} - \frac{1}{s-1/2} - \frac{E'_{(2,r)}}{E_{(2,r)}}(1-s) - \frac{E'_{(2,r)}}{E_{(2,r)}}(s).$$

Except these differences, all calculations work similarly as in the case of an even Maass form.
Chapter 7

Appendix

7.1 Proof of Proposition 3.17

This proposition is a linear sieve problem and it is well-known, at least, to those familiar
with sieve method. However, for the sake of completeness, we provide the proof of
Proposition 3.17. We refer to Halberstam and Richert [25] for unexplained notations.

Let $\mathcal{A}$ be a finite sequence of integers. The for a square-free integer $d$, we define

$$\mathcal{A}_d := \{a \mid a \in \mathcal{A}, a \equiv 0 \mod d\}.$$ 

Let $X$ be a convenient approximation to $|\mathcal{A}|$. For example, we put $X = \text{Li}(x) = \int_2^x \frac{dt}{\ln t}$ for $|\mathcal{A}| = \pi(x)$. For each prime $p$, we choose $\omega_0(p)$ so that \frac{\omega_0(p)}{p}X approximates
to $|\mathcal{A}_p|$.

and write the remainders as

$$r_p := |\mathcal{A}_p| - \frac{\omega_0(p)}{p}X \text{ for all } p.$$ 

With these choices of $X$ and $\omega_0(p)$, we define, for each square-free integer $d$,

$$\omega_0(1) := 1, \quad \omega_0(d) = \prod_{p \mid d} \omega_0(p).$$ 

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so that \( \omega_0(d) \) is a multiplicative function. Then we introduce for each square-free integer \( d \),

\[
r_d := |A_d| - \frac{\omega_0(d)}{d} X.
\]

Let \( \mathfrak{P} = \{p\} \) be a set of primes and \( \mathfrak{P}^c \) denotes the complement of \( \mathfrak{P} \) with respect to the set of all primes. After our choice of \( \mathfrak{P} \), we define

\[
\omega(p) = \begin{cases} 
\omega_0(p), & p \in \mathfrak{P} \\
0, & p \in \mathfrak{P}^c.
\end{cases}
\]

and it is extended to the set of all square-free integers by

\[
\omega(1) := 1, \quad \omega(d) = \prod_{p|d} \omega(p).
\]

Correspondingly we introduce for a square-free integer \( d \),

\[
R_d := |A_d| - \frac{\omega(d)}{d} X.
\]

With the new function \( \omega \) we form the product

\[
W(z) := \prod_{p<z} \left( 1 - \frac{\omega(p)}{p} \right).
\]

Theorems in sieve method are formulated subject to certain basic conditions and they are expressed as symbols. We need to introduce some conditions and their symbols necessary for our purpose. Here \( A_0, A_1, A_2 \) and \( L \) are some constants bigger than or equal to 1 and \( k \) is a positive constant. Especially, if \( k = 1 \), then the corresponding sieve problem is called a linear sieve.

\[\begin{align*}
(\Omega_0) & \quad \omega(p) \leq A_0 \text{ for all } p. \\
(\Omega_1) & \quad 0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1} \text{ for all } p. \\
(R) & \quad |R_d| \leq \omega(d) \text{ if } \mu(d) \neq 0, \ (d, \mathfrak{P}) = 1. \\
(\Omega_2(k)) & \quad \sum_{w \leq p < z} \frac{\omega(p) \log p}{p} \leq k \log \frac{z}{w} + A_2 \text{ if } 2 \leq w \leq z.
\end{align*}\]
Chapter 7. Appendix

\((\Omega_2(k, L))\) \quad -L \leq \sum_{w \leq p < z} \frac{\omega(p) \log p}{p} - k \log \frac{z}{w} \leq A_2 \text{ if } 2 \leq w \leq z.

Then, our main interest is evaluating the sifting function

\[ S(\mathcal{A}; \mathcal{P}, z) := | \{ a \mid a \in \mathcal{A}, (a, P(z)) = 1 \} |. \]

We quote a lower bound theorem of a linear sieve necessary to prove Proposition 3.17.

**Theorem 7.1** (Halberstam and Richert). \((\Omega_1), (\Omega_2(1, L))\) : For \(\xi \geq z\), we have

\[ S(\mathcal{A}; \mathcal{P}, z) \geq XW(z) \left\{ f \left( \frac{\log \xi^2}{\log z} \right) - B \frac{L}{(\log \xi)^{1/4}} \right\} - \sum_{d \leq \xi^2, d \mid P(z)} 3^{\nu(d)} | R_d | \]

where \(f(u)\) is a positive function for \(u > 2\) and monotonically increasing toward 1, \(\nu(d)\) is the number of distinct prime divisors of \(d\) and \(B\) is a positive constant.

For given positive integer \(l\) of the size of \(x^u\) with \(0 < u \leq 0.3\), let define a finite set \(\mathcal{A}\) to be

\[ \mathcal{A} = \{ 0 < n < x \mid n \equiv b \mod l \} \]

for any co-prime integers \(b\) to \(l\). Put \(\mathcal{P} = \{ p \mid p : \text{prime and } p \nmid l \}\) and \(z = x^{1/3+\epsilon}\) for some \(\epsilon > 0\). If \(n \in \mathcal{A}\) and \((n, P(z)) = 1\), then the number of prime divisors of \(n\) is at most 2. Hence we have

\[ S(\mathcal{A}; \mathcal{P}, z) + O(\sqrt{x}) \leq | \{ 0 < n < x \mid n : \text{square-free, } n \equiv b \mod l, \nu(n) \leq 2 \} |. \]

It is easy to check that \(\omega(p) = 1\) if \(p \in \mathcal{P}\). Hence our sieve problem satisfies \(\Omega_1\) condition with \(A_1 = 2\).

\((\Omega_1)\) \quad \quad 0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1} \text{ for all } p.

Since, for \(2 \leq w \leq z\),

\[ \sum_{w \leq p < z} \frac{\omega(p) \log p}{p} = \sum_{w \leq p < z} \frac{\log p}{p} - \sum_{w \leq p < z, p \mid l} \frac{\log p}{p} = \log \frac{z}{w} + O(1), \]

our sieve problem satisfies \((\Omega_2(1, L))\). So we can apply Theorem 7.1 to our problem. We choose \(X\) to be \(x^u = x^{1-u}\) and we have \(W(z) = \frac{\prod_{p \leq z} (1 - \frac{1}{p})^{\nu_p}}{\phi(l)^{\nu(l)}}\). Now we put \(\xi^2 = \frac{x^1-u-\delta}{\log^4 z}\).
Then, for large \( x \) and very small \( \epsilon \) and \( \delta \),

\[
\frac{\log \xi^2}{\log z} = \frac{(1 - u - \delta) \log x - 4(\log \log x + \log(1/3 + \epsilon))}{(1/3 + \epsilon) \log x} > 2.
\]

Since, for \( x \geq 2 \) we have

\[
\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = e^{-\gamma} \left(1 + O\left(\frac{1}{\log x}\right)\right)
\]

with Euler constant \( \gamma \), we obtain the main term of Proposition 3.17.

Now we should show that the error term is smaller than the main term. We have that \( |R_d| \leq \omega(d) \leq 1 \) for all square-free \( d \) and especially \( R_d = 0 \) if \( d \) is not co-prime to \( l \). Then,

\[
\sum_{d < \xi^2, d \mid P(z)} 3^{v(d)} |R_d| \leq \sum_{d < \xi^2, d \mid P(z)} 3^{v(d)} \omega(d) \leq \xi^2 \sum_{d \mid P(z)} 3^{v(d)} \frac{\omega(d)}{d}
\]

\[
\leq \xi^2 \prod_{p \mid P(z)} \left(1 + \frac{\omega(p)}{p}\right)^3 \leq \frac{\xi^2}{W^3(z)} \leq \xi^2 W(z) \log^4 z
\]

where in the last inequality we used the fact that \( \frac{1}{W(z)} = O(\log z) \) under \((\Omega_1)\) and \((\Omega_2(1))\) conditions. By choice of \( \xi^2 \), the error term is ignorable compared with the main term.

Since there are at most \( x^{1/2} \) squares in \( \mathcal{A} \), we finish the proof.
Bibliography


