SUFFICIENT CONDITIONS FOR OUTPUT REGULATION ON METRIC AND BANACH SPACES - A SET-THEORETIC APPROACH

by

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Abstract

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Output regulation problems are a class of problems that has high importance in systems control engineering. The solutions of such problems generally involve the design of an internal model based controller with error feedback structure that can also provide stability to the closed-loop system. Most of previous studies of such problems, however, are based on dynamical systems described by differential equations for continuous-time and by difference equations for discrete-time on the space of $\mathbb{R}^n$. Few results have been obtained for dynamical systems with more abstract descriptions on sets with more general topologies. In this thesis, we use a set-theoretic approach with commutative diagrams to describe a dynamical system and its properties. Output regulation problems will also be defined based on such dynamical systems. We will present sufficient conditions for output regulation problems on complete metric spaces and Banach spaces.
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Chapter 1

Introduction

In systems control engineering, output regulation is a central problem that aims to guide a plant output to match a specific reference signal, perhaps subject to external disturbances. Most previous studies of output regulation problems are based on dynamical systems described by differential equations for continuous-time and by difference equations for discrete-time on the space \( \mathbb{R}^n \). Very few results have been obtained for dynamical systems with more abstract descriptions on sets with more general topologies. The goal of this thesis is to determine sufficient conditions for output regulation using a set-theoretic description of dynamical systems on more general topological spaces than \( \mathbb{R}^n \) such as complete metric spaces and Banach spaces.

In this chapter, we will first discuss the motivation of our study of such problems, as well as output regulation applications to many real challenges in the world. We will then briefly review the literature on output regulation: from linear systems to certain classes of nonlinear systems. Finally, we will state the contribution and the organization of this thesis.
Chapter 1. Introduction

1.1 Motivation

Output regulation is a common problem in control systems, which aims to guide a plant output to match a specific reference signal, perhaps subject to external disturbances. Reference signals and external disturbance signals are typically assumed to be modeled as generated by an external autonomous dynamical system known as the “exosystem”. In the past, the term “servomechanism problem” has also been used to indicate the same class of problems in control theoretic research. Output regulation problems are considered to be more difficult than stabilization problems in the control systems literature because a controller not only needs to stabilize the plant that it regulates, but also provide asymptotic tracking of the reference signal and disturbance rejection of the regulated output of the plant. Traditionally, these problems are solved by performing stability analysis and designing feedback controllers to compensate the plant with dynamics to achieve the objective. Such feedback controllers are also referred to as output regulators. A total system with an exosystem, a plant and an output regulator in which the plant has the desired output with respect to the exosystem as the total system evolves is said to have the regulation property.

Output regulation problems or servomechanism problems have played a crucial role in many industries owing to their wide range of applications such as trajectory execution of a robotic manipulator, guidance of a tactical missile toward a moving target and spacecraft attitude control in aerospace systems. A good example of a space application is [18]. Other interesting examples of output regulation applications are [19] in neural systems, [21] in solar power systems and [20] in multi-agent consensus problems. For this reason, output regulation has attracted significant effort by control systems researchers. As, for instance, industrial automation using robots and the development of national defence systems for missile attacks have become pressing issues, output regulation will continue to play a significant role.

The concept of regulating a mechanical plant output can be dated back to several
centuries ago. An enormous amount of research has been conducted for output regulation problems of dynamical systems since the 1960’s. Rigorous mathematical formulations of such problems, however, were not available until the middle of 1970s. Among the various studies on the output regulation problems, a major break-through was made by B. A. Francis and W. M. Wonham in their 1976 paper \cite{1}, in which they introduced the concept of Internal Model Principle (IMP). The statement of the Internal Model Principle can be roughly summarized as that “an ideal feedback regulator will include in its internal structure a dynamic model of the external world behavior that the regulator tracks or regulates against” \cite{2}.

Many researchers have followed the internal-model concept of Francis and Wonham in studying the output regulation problems in various cases, including such problems as nonlinear systems with constant reference signals, time-varying plants and systems with saturation-type nonlinearities. The efforts of researchers have yielded fruitful results in this area of control systems research. In fact, most of the results focus on solving the output regulation problem for a particular class of linear or nonlinear systems. In other words, the problem is usually that given a class of plants or exosystems with certain properties, determine a control law such that the regulation property of the total system holds. To my knowledge, the solutions to such problems generally involve designing an internal model based controller with error feedback structure that can also provide stability to the “endosystem”, which is the composite of both the plant and the controller. Alternatively, most of the results indicate that by suitably incorporating an internal model and an error feedback structure, a sufficient condition for regulation in various cases can be achieved. Thus, the Francis-Wonham theory with some case-specific assumptions has been investigated to solve different output regulation problems.

Although sufficient conditions for regulation have been determined for different system settings studied, most of the previous results consider dynamical systems described by differential equations for continuous-time cases and difference equations for discrete-time
cases in the space of $\mathbb{R}^n$. There still is needed a sufficient condition that will guarantee the regulation property of a dynamical system in a broader sense; especially in dynamical systems with more abstract mathematical models in more general topologies. This motivation will lead us to investigate the output regulation problems using a set-theoretic approach.

In this thesis, a sufficient condition for output regulation will be presented based on [2], together with fixed point theorems and functional analysis tools. Using a set-theoretic description of dynamical systems in more general topological spaces than $\mathbb{R}^n$ such as metric spaces and Banach spaces, we propose to extend the output regulation analysis of the existing literature to abstract dynamical systems. Based on this type of formulation, a dynamical system can be defined as an arbitrary set together with a transition function defined on it. This scheme, as expressed in a setting of ordinary sets and functions by a series of commutative diagrams, is intended to avoid dependence on the special conditions imposed in previous research. The scheme provides both visual intuition for output regulation problems using the Internal Model Principle, and a universal agenda for research in such abstract algebraic settings.

1.2 Literature Review

Researchers started to mathematically rigorously define the output regulation problems in the middle of the 1970s. Such problems with linear time-invariant (LTI) exosystems and plants were first studied during this period of time. Among the pioneers of this field, Francis and Wonham established a theoretical basis for solving the output regulation problems for LTI systems in [1] in 1976, with a theory we refer to as the Internal Model
Principle. The Internal Model Principle was summarized in [1] as follows:

A regulator synthesis is structurally stable only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process.

In [1], Francis and Wonham defined a regulator synthesis to be a combination of a plant and a controller that has closed-loop stability and regulation properties. A synthesis is said to be structurally stable if the aforementioned two properties of a synthesis are preserved when the system is subjected to parameter perturbations [1]. The solvability of the linear output regulation problem was further discussed by Francis in [4]. In [4], Francis proved that the solvability of the linear output regulation problem was equivalent to the solvability of a pair of matrix equations along with stabilizability and detectability conditions. The pair of matrix equations is often referred to as the Regulator Equations. Based on this concept, many nonlinear extensions of the problem, some of them involving robust output regulation, were developed by other researchers. In particular, the work of Isidori in [9] has identified a condition for local solvability of the nonlinear output regulation problems, which is a significant extension of [4] in the nonlinear realm. A more general framework for solving nonlinear output regulation problems was presented by Huang in [5] in 2004. This general framework consists of two major steps: convert the output regulation problem for a target plant into a stabilization problem for the composite system which includes the plant itself and an internal model of the exosystem, then use existing nonlinear stabilization techniques to solve the stabilization problem.

In parallel with the continuous time dynamical systems, output regulation problems for discrete-time systems have also been studied intensively. Similar to the continuous-time systems, researchers such as Castillo and Di Gennaro, Huang and Lin, Castillo et.al, Lan and Huang have provided insightful works based on Francis’s work in [4], [11], [10],
and [6], respectively. Usually, a new result in the continuous setting will be extended to the discrete-time counterpart. For example, the work by Lan and Huang in [6] is a direct extension of the work of Huang in [5].

Other than studies on solving particular output regulation problems based on the Internal Model Principle, researchers have also provided alternative views on the necessity of internal model in output regulation problems. In [17], Sontag presented a result showing that under reasonable assumptions, including a signal detection property, if a dynamical system “adapts” to a class of external reference or disturbance signals, say $\mathcal{R}$, in the sense of regulation, then the system must necessarily possess a subsystem which is capable of reproducing all the signals in $\mathcal{R}$. Instead of looking at a plant as given and a controller that is designed to compensate the plant, [17] assumes a system that is not partitioned and studies its properties under regulation.

Wonham also provided an alternative view of the Internal Model Principle based on a more generic approach using state sets with state-transition functions defined on them in [3] in 1976. In [3], Wonham was able to show the necessity of the existence of the internal model in output regulation problems based on several assumptions, including the error feedback structure, detectability of the exosystem-induced subset in the overall state set and freedom of restriction on the controller state during regulation. This abstract algebraic approach enables us to describe the output regulation problem without using linear systems methods, and in a coordinate independent manner.

Wonham further explained his approach in Section 1.5 of [2]. Here, an internal model concept was paraphrased as faithful simulation of the exosystem by the controller under perfect regulation. The existence of such internal model under perfect regulation was shown using a commutative diagram and injection mapping approach. Furthermore, Wonham extended this approach to parameter-perturbed plants. With an additional assumption of “richness of parametrization”, it is shown that under the assumption of internal model and regulation condition, the controller must incorporate error feedback
1.3 Contribution and Organization

In this thesis, we use a set-theoretic approach with commutative diagrams to describe a dynamical system and its properties, based on Section 1.5 of [2]. Output regulation problems will also be defined based on such dynamical systems. In Section 1.5 of [2], Wonham proved a necessary condition for the regulation property of the total system, which is the existence of an internal model. The proof in [2] was based on structural assumptions such as closed-loop stability, detectability of the exosystem, controller-independent regulation set and feedback structure in a set-theoretic framework. It was not mentioned in [2], however, whether these structural assumptions along with a suitable internal model controller will be a sufficient condition for output regulation.

In the proof of necessity of the internal model in [2], the stability of the total system depended on the existence of a function, denoted henceforth by $i_E$, which maps an exosystem state to a state in the total system. The proof also depended on the attractiveness of the image of $i_E$ in the total system state set. It is not shown in [2], however, what conditions on the endosystem dynamics will guarantee such results. In addition, sufficiency of the internal model controller for output regulation must not only consider the dynamics on the attractor of the total system, but also the dynamics of the total system away from the attractor.

In this thesis, using fixed point theorems we present two set of results that will guarantee the existence and uniqueness (up to automorphism) of the $i_E$ function as well as attractivity of its image in the total system; one in a complete metric space setting and
the other in a Banach space setting. Later, we will present a sufficient condition for output regulation problems based on the two settings discussed. We will also present without proof a proposed iterative method for obtaining a correct internal model for regulation purposes.

The significance of this thesis is that it provides a method for formulating output regulation problems in an abstract setting. Examples of such abstract dynamical systems are discrete-event systems and discrete-time dynamical systems in non-Euclidean spaces.

The organization of this thesis will be as follows. In Chapter 2, we briefly review mathematical preliminaries for the purpose of this thesis. In Chapter 3, a summary of the proof presented in [2] on the necessity of internal model under total system regulation will be reviewed. In Chapter 4, we will present a sufficient condition on the endosystem transition function that will guarantee the existence of $i_E$, its uniqueness up to automorphism and attractiveness of its image in a complete metric space. In Chapter 5, we take the state sets to be subsets of a Banach space and a less restrictive condition on the endosystem transition function will be presented to guarantee the same outcome concluded in Chapter 4. In Chapter 6, based on the previous two chapters, sufficient conditions for total system regulation will be presented under several assumptions. Finally, in Chapter 7, we propose an iterative scheme for obtaining a suitable internal model in output regulation problems.
Chapter 2

Mathematical Preliminaries

In this chapter, mathematical preliminaries that will be useful in later chapters will be reviewed. In the first two sections, we will review definitions related to metric spaces and normed spaces. In Section 2.3, we will review definitions related to dynamical systems. In Section 2.4, we will cover contraction maps and related fixed point theorems.

2.1 Metric Spaces

The concept of metric is a generalization of distance in Euclidean space $\mathbb{R}^n$. It is a crucial concept in analysis of stability properties of dynamical systems.

**Definition 2.1.1.** Let $X$ be a set. A metric on a set $X$ is a function $d : X \times X \rightarrow [0, \infty)$ with the following properties:

- **Positive Definiteness**, $(\forall x, y \in X) d(x, y) = 0$ iff $x = y$

- **Symmetry**, $(\forall x, y \in X) d(x, y) = d(y, x)$

- **Triangle Inequality**, $(\forall x, y, z \in X) d(x, z) \leq d(x, y) + d(y, z)$

A metric space is a set $X$ with a metric $d$.

Let $X$ be a metric space with metric $d$ from now on.
Definition 2.1.2. [16] A \textit{neighbourhood} of a point \( p \in X \) is a set \( N_r(p) \) consisting of all points \( q \) such that \( d(p, q) < r \), for some \( r > 0 \). The number \( r \) is called the \textit{radius} of \( N_r(p) \).

Definition 2.1.3. [16] A point \( p \) is a \textit{limit point} of a subset \( E \subseteq X \) if every neighbourhood of \( p \) contains a point \( q \neq p \) such that \( q \in E \).

Definition 2.1.4. [16] A subset \( C \subseteq X \) is \textit{closed} if every limit point of \( C \) is a point in \( C \).

Definition 2.1.5. [16] A point \( p \) of a subset \( E \subseteq X \) is called an \textit{interior point} of \( E \) if there is a neighbourhood \( N \) of \( p \) such that \( N \subseteq E \).

Definition 2.1.6. [16] A subset \( O \subseteq X \) is \textit{open} if every point of \( O \) is an interior point of \( O \).

Definition 2.1.7. [16] An \textit{open cover} of a subset \( E \subseteq X \) is a collection \( \{G_\alpha\} \) of open subsets of \( X \) such that \( E \subseteq \bigcup \alpha G_\alpha \), where \( \bigcup \alpha G_\alpha \) is the union of all sets in \( \{G_\alpha\} \).

Definition 2.1.8. [16] A subset \( K \subseteq X \) is said to be \textit{compact} if every open cover of \( K \) contains a finite subcover.

The following theorem relates compactness to closedness of a subset of a metric space. Its proof can be found in [16].

Theorem 2.1.9. [16] Compact subsets of metric spaces are closed.

Definition 2.1.10. [16] A sequence \( \{p_n\} \) in \( X \) is said to \textit{converge} if there exists a \( p \in X \) with the following property: For every \( \epsilon > 0 \), there is \( N \in \mathbb{Z}^+ \) such that \( n \geq N \) implies \( d(p_n, p) < \epsilon \). We say that \( \{p_n\} \) converges to \( p \), or that \( p \) is the \textit{limit} of \( \{p_n\} \), and we write

\[
\lim_{n \to \infty} p_n = p
\]
Definition 2.1.11. [16] A sequence \( \{p_n\} \) in \( X \) is said to be a Cauchy sequence if for every \( \epsilon > 0 \) there exists \( N \in \mathbb{Z}^+ \) such that \( m, n \geq N \) implies \( d(p_m, p_n) < \epsilon \).

With the definition of Cauchy sequence, we are now ready to define completeness of a metric space.

Definition 2.1.12. [16] A metric space in which every Cauchy sequence converges is said to be complete.

Next, we define properties of functions on metric spaces. Let \( X, Y \) be metric spaces with metrics \( d_X, d_Y \), respectively.

Definition 2.1.13. [16] Let \( E \subseteq X \). Let \( f : E \to Y \) and let \( p \) be a limit point of \( E \). We write

\[
\lim_{x \to p} f(x) = q
\]

if there exists \( q \in Y \) with the following property: For every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x \in E \), \( d_X(x, p) < \delta \) implies \( d_Y(f(x), q) < \epsilon \). We say \( q \) is the limit of \( f \) at \( p \).

Definition 2.1.14. [16] Let \( E \subseteq X \). Let \( f : E \to Y \) and let \( p \in E \). \( f \) is said to be continuous at \( p \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x \in E \), \( d_X(x, p) < \delta \) implies \( d_Y(f(x), f(p)) < \epsilon \). If \( f \) is continuous at every point of \( E \), then \( f \) is said to be continuous on \( E \).

Theorem 2.1.15. [16] In the situation given in Definition 2.1.14, assume \( p \) is a limit point of \( E \). Then \( f \) is continuous at \( p \) if and only if \( \lim_{x \to p} f(x) = f(p) \).

The following theorem, which relates continuous functions and compact subsets of metric spaces, will be used in later chapters.
Theorem 2.1.16. [10] Let $E$ be a compact subset of $X$. If $f : E \to Y$ is a continuous map, then $f(E)$ is compact.

2.2 Normed Spaces

In this section, we will cover definitions related to normed spaces. Before we define a normed space, we need to provide the definition of a vector space. For the purpose of this thesis, we will limit the definition of vector spaces to real vector spaces.

Definition 2.2.1. [8] A (real) vector space consists of a set $V$ with elements called vectors and two operations with the following properties:

vector addition: for each pair $u, v \in V$, there is a vector $u + v \in V$, which satisfies:

- Commutativity, $u + v = v + u$ for all $u, v \in V$
- Associativity, $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$
- Zero, there exists a vector $0 \in V$ such that $0 + u = u = u + 0$ for all $u \in V$.
- Inverses, for each $u \in V$, there exists a vector $-u \in V$ such that $u + (-u) = 0$.

scalar multiplication: for each vector $v \in V$ and real number $r \in \mathbb{R}$, there is a vector $rv \in V$. This satisfies, for all $u, v \in V$ and all $r, s \in \mathbb{R}$

- $(r + s)v = rv + sv$
- $r(sv) = (rs)v$
- $r(u + v) = ru + rv$
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1. $v = v$
2. $0v = 0$
3. $(-1)v = -v$

Now, we are ready to define norms and normed spaces. The idea of norms is a generalization of the absolute value in $\mathbb{R}$ to higher dimensional spaces. It provides a means to inform us of the length of a given vector in a vector space.

Definition 2.2.2. [8] Let $V$ be a real vector space. A norm on $V$ is a function $\| \cdot \| : V \to [0, \infty)$ that has the following properties:

- **Positive Definiteness**, $(\forall v \in V)\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
- **Homogeneity**, $(\forall v \in V)(\forall \alpha \in \mathbb{R})\|\alpha v\| = |\alpha|\|v\|$
- **Triangle Inequality**, $(\forall v, w \in V)\|v + w\| \leq \|v\| + \|w\|$

A vector space $V$ with a norm $\| \cdot \|$ is called a normed vector space.

In a normed vector space $(V, \| \cdot \|)$, we can define a metric

$$d(v, w) = \|v - w\|$$  \hspace{1cm} (2.1)

for $v, w \in V$. It can be easily verified that (2.1) satisfies the definition of a metric in (2.1.1). Therefore, a normed vector space is a metric space. The topological concepts such as closedness, compactness and completeness introduced in the previous section can be extended to normed spaces by taking for the general metric defined in a metric space the specific metric defined in (2.1).

A complete normed space is called a Banach space [8].
In $\mathbb{R}^n$, one of the norms that is often used is the infinity norm $\|\cdot\|_{\infty}$, which is defined as follows: for all $v = (v_1, ..., v_n) \in \mathbb{R}^n$,

$$\|v\|_{\infty} = \max_i |v_i|$$

(2.2)

We conclude this section by providing the definition of a convex set in a vector space.

**Definition 2.2.3.** Let $V$ be a real vector space. A subset of $C \subseteq V$ is called convex if

$$\forall v, w \in C)(\forall \alpha \in [0, 1])\alpha v + (1 - \alpha)w \in C$$

(2.3)

### 2.3 Discrete Dynamical Systems

In this section, a set theoretic definition of a dynamical system will be provided based on [2]. Several key concepts associated with this definition of a dynamical system will also be introduced.

**Definition 2.3.1.** Let $X$ be a set. A dynamical system on $X$ is a pair $(X, \alpha)$, where $\alpha$ is a function mapping $X$ into itself. The elements of $X$ are called the states, and the function $\alpha : X \rightarrow X$ is called the state transition function on $X$.

A dynamical system’s state evolution starts with a state in $x_0 \in X$, called the initial state, and the system state evolves successively through states, $x_1 = \alpha(x_0), x_2 = \alpha(x_1) =$
\(\alpha \circ \alpha(x_0), \ldots, x_n = \alpha^n(x_0), \ldots\), where \(\alpha^n\) denotes the \(n\)-fold composition of the state transition function \(\alpha : X \to X\). It should be noted that \(\alpha^0 = \text{id}_X\), the identity function on \(X\), and \(\alpha^1 = \alpha\). An initial state \(x_0\) with discrete-time evolution generates a sequence of states defined by the set \(\{\alpha^t(x_0) | t \in \mathbb{N}\}\), which is called the path of the dynamical system \((X, \alpha)\) with initial state \(x_0\) \([2]\).

For this thesis, we take the state set \(X\) to be a metric space, in which the notion of distance is well-defined. In traditional control literature, Euclidean vector spaces such as \(\mathbb{R}^n\) are used. The concept of distance from the fixed points or equilibrium states allows us to determine the stability properties of fixed points in a dynamical system. Fixed points or equilibrium states are an essential concept in the analysis of dynamical systems.

**Definition 2.3.2.** Let \(Y\) be a nonempty set and let \(f : Y \to Y\) be a function from \(Y\) to \(Y\). A **fixed point** of \(f\) is a point \(y \in Y\) such that \(f(y) = y\).

Now, we define a fixed point of a dynamical system and two stability properties of fixed points in a dynamical system.

**Definition 2.3.3.** \([10]\) A state \(x^* \in X\) is a **fixed point** or an **equilibrium state** of the dynamical system \((X, \alpha)\) if \(x^*\) is a fixed point of \(\alpha\).

**Definition 2.3.4.** \([8]\) A fixed point \(x^* \in X\) is said to be **attractive** if there is a neighbourhood \(U \subseteq X\) containing \(x^*\) such that for all \(x \in U\), the path of \((X, \alpha)\) with initial state \(x\) converges to \(x^*\).

**Definition 2.3.5.** \([8]\) A fixed point \(x^* \in X\) is said to be **repelling** if for any open neighbourhood \(U \subseteq X\) containing \(x^*\), there exists \(x \in U\) such that either the path of \((X, \alpha)\) with initial state \(x\) does not converge to a limit or it converges to a point \(x^{**} \in X\) and \(x^{**} \neq x^*\).

Finally, we define an attractor of a dynamical system \((X, \alpha)\). Given a metric space \(X\) with metric \(d\), we can define a point-to-set distance using the metric \(d\). Let \(S \subseteq X\). Then, the distance between a point \(x \in X\) to \(S\) can be defined as
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\[ d(x, S) := \inf_{s \in S} d(x, s) \]  \hfill (2.4)

We state the definition of attractor using (2.4).

**Definition 2.3.6.** Let \((X, \alpha)\) be a dynamical system defined on a metric space with metric \(d\). Let \(A \subseteq X\). \(A\) is called an attractor in \(X\) if for every initial condition \(x_0 \in X\)

\[ \lim_{t \to \infty} d(\alpha^t(x_0), A) = 0 \]  \hfill (2.5)

\[ . \]

2.4 Fixed Point Theorems

In this section, we briefly review some of the fixed point theorems that we will use in this thesis. We will cover the concept of contraction maps, the Contraction Mapping Principle of Stefan Banach and the Schauder Fixed Point Theorem. Contraction maps play a very important role in discrete dynamical systems because they allow us to determine the existence of a unique fixed point. At the same time, they also provide an algorithm to compute this unique equilibrium point up to arbitrary precision based on an iterative scheme.

**Definition 2.4.1.** *Contraction* [8]
Let $X$ be a metric space with metric $d$. Let $f$ map $X$ into $X$. If

$$
(\exists c \in [0, 1]) (\forall x, y \in X) d(f(x), f(y)) \leq cd(x, y) \tag{2.6}
$$

then $f$ is said to be a contraction of $X$ into $X$. $c$ is called the contraction constant of $f$.

The importance of contraction maps lies in the following principle, which guarantees both existence and uniqueness of a fixed point.

**Theorem 2.4.2. Contraction Mapping Principle** [8]

Let $X$ be a complete metric space with metric $d$. If $f : X \to X$ is a contraction with contraction constant $c$, then $f$ has a unique fixed point $x^*$. In addition, for all $x \in X$, the following holds:

$$
\lim_{n \to \infty} d(f^n(x), x^*) = 0 \tag{2.7}
$$

and

$$
d(f^n(x), x^*) \leq c^n d(x, x^*) \leq \frac{c^n}{1 - c} d(f(x), x) \tag{2.8}
$$

A proof of the Contraction Mapping Principle can be found in most books on real analysis such as [16] and [8] or fixed point theory such as [13].

A less powerful fixed point theorem which applies to Banach spaces and only guarantees the existence of a fixed point but can be applied to more cases is the Schauder Fixed Point Theorem.

**Theorem 2.4.3. Schauder-Tychonoff Fixed Point Theorem** [13]

If $K$ is a nonempty compact convex set in a Banach space $X$, and $f : K \to K$ is continuous, then $f(p) = p$ for some $p \in K$. 

Chapter 3

Internal Model Principle

In this chapter, a brief summary of Chapter 1 of [2] will be presented to provide readers with previous contributions based on which this thesis is developed.

3.1 Partially Ordered Sets

Let us begin with the definition of partially ordered sets, or posets. First, we need to introduce the definition of a binary relation since a partial order on a set is a binary relation on that set.

**Definition 3.1.1. Binary Relation [2]**

Let $X$ be a set. A **binary relation** on $X$ is a subset of $X \times X$, the cartesian product of $X$ with itself.

Let $\leq$ be a binary relation on $X$. For $x, y \in X$, write $x \leq y$ to mean that the ordered pair $(x, y) \in X \times X$ belongs to the relation $\leq$.

**Definition 3.1.2. Posets [2]**

A binary relation $\leq$ is a **partial order (p.o.)** on $X$ if it has the following properties:

- Reflexivity, $x \leq x$ for all $x \in X$
• Transitivity, \( x \leq y \) and \( y \leq z \) implies \( x \leq z \) for all \( x, y, z \in X \).

• Antisymmetry, \( x \leq y \) and \( y \leq x \) implies \( x = y \) for all \( x, y \in X \).

\( x, y \in X \) are said to be comparable if either \( x \leq y \) or \( y \leq x \). A partial order is a total ordering if every two elements of \( X \) are comparable. If \( x, y \in X \) are not comparable, we denote it by writing \( x <> y \). If \( \leq \) is a partial order on \( X \), the pair \( (X, \leq) \) is a partially ordered set (poset).

Let \( (X, \leq) \) be a poset. Let \( x, y \in X \). An element \( a \in X \) is a lower bound for \( x \) and \( y \) if \( a \leq x \) and \( a \leq y \).

**Definition 3.1.3. Meet** [2]

An element \( l \in X \) is called a meet for \( x, y \in X \) if

- \( l \) is a lower bound for \( x \) and \( y \)
- For all \( a \in X \) such that \( a \) is a lower bound for \( x \) and \( y \), \( a \leq l \)

In other words, if \( l \in X \) is a meet for \( x, y \in X \), \( l \) is the “greatest lower bound” for \( x \) and \( y \). It is easy to verify that if a meet exists for two elements, then it is unique. Use \( x \wedge y \) to denote the meet of elements \( x \) and \( y \).

Similarly, we can define an upper bound for \( x, y \in X \) such that an element \( b \in X \) is an upper bound for \( x \) and \( y \) if \( x \leq b \) and \( y \leq b \).

**Definition 3.1.4.** [2]

An element \( u \in X \) is called a join for \( x, y \in X \) if

- \( u \) is an upper bound for \( x \) and \( y \)
- For all \( b \in X \) such that \( b \) is an upper bound for \( x \) and \( y \), \( u \leq b \)

Dually, if \( u \in X \) is a join for \( x, y \in X \), then \( u \) is the “least upper bound” for \( x \) and \( y \). If a join for two elements exists, again, it is unique. Use \( x \vee y \) to denote the join of \( x \) and \( y \).
Finally, let \( \perp \) stand for bottom element (if it exists) of \( X \): \( \perp \in X \) and \( \perp \leq x \) for all \( x \in X \). Similarly let \( \top \) stand for top element: \( \top \in X \) and \( x \leq \top \) for all \( x \in X \).

### 3.2 Lattices

Next, we review the definition of lattices.

**Definition 3.2.1. Lattice**

A lattice is a poset \( L \) in which the meet and join of any two elements always exist.

In a lattice, the binary operators \( \wedge \) and \( \vee \) define functions

\[
\wedge : L \times L \to L, \quad \vee : L \times L \to L
\]  

(3.1)

The functions defined in (3.1) satisfy the following properties, which can be verified easily:

- Idempotent, \( x \wedge x = x \) for all \( x \in L \)
- Commutative, \( x \wedge y = y \wedge x \) for all \( x, y \in L \)
- Associative, \( (x \wedge y) \wedge z = x \wedge (y \wedge z) \) for all \( x, y, z \in L \)

The above properties also apply to the \( \vee \) operator.

Let \( (L, \wedge, \vee) \) be a lattice and let \( S \) be a nonempty subset of \( L \).

**Definition 3.2.2. Infimum**

An element \( l \in L \) is an infimum of a subset \( S \subseteq L \) if
• \( l \leq y \) for all \( y \in S \)

• \((\forall z \in L)[(\forall y \in S) z \leq y] \Rightarrow z \leq l\)

Dually, we have the definition for supremum of a set \( S \).

**Definition 3.2.3. Supremum**

An element \( u \in L \) is a **supremum** of a subset \( S \subseteq L \) if

• \( y \leq u \) for all \( y \in S \)

• \((\forall z \in L)[(\forall y \in S) y \leq z] \Rightarrow u \leq z\)

It is worthwhile to note that the infimum or the supremum of a subset \( S \) are not necessarily in \( S \). The infimum and supremum of a subset \( S \) are unique, if they exist. Finally, a lattice \( (L, \wedge, \vee) \) is said to be **complete** if for any subset \( S \subseteq L \), the infimum and supremum of \( S \) exist in \( L \). 

### 3.3 Equivalence Relations

In this section, we review an important concept from \([2]\), the equivalence relations on a set \( X \).

**Definition 3.3.1. Equivalence Relation**

Let \( X \) be a nonempty set and \( E \subseteq X \times X \) be a binary relation. \( E \) is an **equivalence relation** if

• Reflexive, \( xEx \) for all \( x \in X \)

• Symmetric, \( xEy \) iff \( yEx \) for all \( x, y \in X \)
• Transitive, $x Ey$ and $y Ez$ implies $x Ez$ for all $x, y, z \in X$.

We write $x \equiv y \pmod{E}$ to mean $x Ey$.

Let $\mathcal{P}$ be a family of subsets of $X$ indexed by $\alpha$ in some index set $A$:

$$\mathcal{P} = \{C_\alpha | \alpha \in A\}$$

**Definition 3.3.2.** The family $\mathcal{P}$ is a **partition** of $X$ if

- Each subset $C_\alpha$ is nonempty, $(\forall \alpha \in A) C_\alpha \neq \emptyset$
- $C_\alpha$’s cover $X$, $X = \bigcup\{C_\alpha | \alpha \in A\}$
- Subsets with distinct indices are pairwise disjoint, $(\forall \alpha, \beta \in A) \alpha \neq \beta \implies C_\alpha \cap C_\beta = \emptyset$

The subsets $C_\alpha$ are called the **cells** of $\mathcal{P}$.

It can be easily shown that an equivalence relation on a set $X$ is also a partition on $X$ and vice versa [2]. Thus, we will speak of equivalence relations and partitions interchangeably throughout this thesis.

Let $\mathcal{E}(X)$ be the set of all equivalence relations on $X$. We assign a partial order to $\mathcal{E}(X)$ such that $\mathcal{E}(X)$ becomes a complete lattice in the following way.

$$(\forall E_1, E_2 \in \mathcal{E}(X)) \left[ E_1 \leq E_2 \iff (\forall x, y \in X)xE_1y \Rightarrow xE_2y \right] \quad (3.2)$$

If $E_1 \leq E_2$, we say $E_1$ is **finer** than $E_2$ or $E_2$ is **coarser** than $E_1$. Next, we introduce the meet and join of equivalence relations.

**Proposition 3.3.3.** **Meet of Equivalence Relations** [2]

In the poset $(\mathcal{E}(X), \leq)$, the meet of $E_1, E_2 \in \mathcal{E}(X)$, $E_1 \land E_2$ always exists and is given by
The join of two equivalence relations is more complicated.

**Proposition 3.3.4. Join of Equivalence Relations** \[2\]

In the poset \((\mathcal{E}(X), \leq)\), the join of \(E_1, E_2 \in \mathcal{E}(X)\), \(E_1 \vee E_2\) always exists and is given by

\[
(\forall x, x' \in X) x \equiv x' \pmod{E_1 \vee E_2} \iff \exists k \in \mathbb{Z}^+ (\exists x_0, \ldots, x_k \in X) x_0 = x \text{ and } x_k = x' \text{ and } \text{and } (\forall i \in \{1, \ldots, k\}) x_i \equiv x_{i-1} \pmod{E_1} \text{ or } x_i \equiv x_{i-1} \pmod{E_2} \quad (3.4)
\]

Now, we can derive the infimum and supremum of a subset of \(\mathcal{E}(X)\) based on the meet and join of equivalence relations.

**Proposition 3.3.5. Infimum of a Subset of \(\mathcal{E}(X)\)** \[2\]

Let \(F \subseteq \mathcal{E}(X)\). Then \(\inf(F)\) exists and is given by

\[
(\forall x, x' \in X) x \equiv x' \pmod{\inf(F)} \iff (\forall F \in F) x \equiv x' \pmod{F} \quad (3.5)
\]

**Proposition 3.3.6. Supremum of a Subset of \(\mathcal{E}(X)\)** \[2\]

Let \(F \subseteq \mathcal{E}(X)\). Then \(\sup(F)\) exists and is given by
(\forall x, x' \in X) x \equiv x' \pmod{\sup(\mathcal{F})} \iff 
(\exists k \in \mathbb{Z}^+)(\exists F_1, \ldots, F_k \in \mathcal{F})(\exists x_0, \ldots, x_k \in X) 
\quad x_0 = x \text{ and } x_k = x' \text{ and } (\forall i \in \{1, \ldots, k\})x_i \equiv x_{i-1} \pmod{F_i} \quad (3.6)

Note that \( \bot = \inf(\mathcal{E}(X)) \) and \( \top = \sup(\mathcal{E}(X)) \). They are given by

\[ x \equiv x' \pmod{\bot} \iff x = x' \quad (3.7) \]

and

\[ x \equiv x' \pmod{\top} \iff \text{True} \quad (3.8) \]

Thus, we conclude that the lattice of equivalence relations on \( X \) is complete.

### 3.4 Congruences and Observers of Dynamical System

We will review the concept of congruence of dynamical systems and observers from [2] in this section.

**Definition 3.4.1. Congruence of Dynamical System [2]**

Let \((X, \alpha)\) be a dynamical system and \(\pi\) be a partition or an equivalence relation on \(X\). Let \(\bar{X}\) to be the set of all cells of \(X\) under partition \(\pi\) and \(P_\pi : X \to \bar{X}\) be the projection
of an element \( x \in X \) to its corresponding cell under \( \pi \). \( \pi \) is called a congruence for \( \alpha \) if there exists a map \( \bar{\alpha} : \bar{X} \to \bar{X} \), such that \( \bar{\alpha} \circ P_\pi = P_\pi \circ \alpha \).

A key proposition that relates to the congruence of a dynamical system is the following, which is stated in Section 1.4 of [2].

**Proposition 3.4.2.** \( \pi \) is a congruence for \( \alpha \) if and only if

\[
(\forall x, \bar{x} \in X) x \equiv \bar{x} \pmod{\pi} \implies \alpha(x) \equiv \alpha(\bar{x}) \pmod{\pi} \quad (3.9)
\]

If a partition \( \pi \) on \( X \) is a congruence for \( \alpha : X \to X \), then \( \bar{\alpha} \) is the map ‘induced’ by \( \alpha \) on \( \bar{X} \). This condition says that \( \alpha \) ‘respects’ the partition corresponding to \( \pi \) in that cells are mapped under \( \alpha \) consistently into cells. For this reason, the dynamical system \((\bar{X}, \bar{\alpha})\) can be regarded as a high-level model of \((X, \alpha)\).

Next, we introduce the definition of the equivalence kernel of a function.

**Definition 3.4.3.** **Equivalence Kernel** [2]

Let \( X \) and \( Y \) be two nonempty sets. Let \( f : X \to Y \). The **equivalence kernel** of \( f \), denoted as \( \ker f \), is an equivalence relation on \( X \) that is defined as follows

\[
(\forall x, x' \in X) x \equiv x' \pmod{\ker f} \iff f(x) = f(x') \quad (3.10)
\]

Using the equivalence kernel, we introduce the concept of an observer. The observer of a dynamical system introduced in [2] is different from most of the state-space based definitions. The underlying concept, however, is very similar.

Let \((X, \alpha)\) be a dynamical system and let \( \omega \) be a partition on \( X \) that is a congruence for \( \alpha \). From the last section, we know that if \( x \equiv x' \pmod{\omega} \), then \( \alpha(x) \equiv \alpha(x') \pmod{\omega} \). Define a new partition \( \omega \cdot \alpha \) such that \( x \equiv x' \pmod{\omega \cdot \alpha} \) if \( \alpha(x) \equiv \alpha(x') \pmod{\omega} \).

Let \( \gamma : X \to Y \) be the output map of the dynamical system \((X, \alpha)\) where the set \( Y \)
is the output set. Define the observer for the dynamical system with output \((X, \alpha, \gamma)\) to be the equivalence relation or partition on \(X\)

\[
\omega_o := \sup \{ \omega \in \mathcal{E}(X) | \omega \leq (\ker \gamma) \wedge (\omega \cdot \alpha) \}
\]  

(3.11)

where \(\mathcal{E}(X)\) is the set of all partitions or equivalence relations on \(X\). The above definition of the observer can be interpreted as that the observer is the coarsest congruence for \(\alpha\) that is finer than \(\ker \gamma\). It is shown in \([2]\) that

\[
\omega_o := \inf \{(\ker \gamma) \cdot \alpha^{i-1} | i = 1, 2, ... \}
\]

(3.12)

where \(\alpha^j\) is the \(j\)-fold composition of \(\alpha\) with itself.

### 3.5 Injection Maps and Subsets

Injection maps are a key in relating a subset of a general set \(X\) to \(X\) itself in Wonham’s proof of the Internal Model Principle in \([2]\). In this section, we will review the basics of injection maps.

Let \(\alpha : X \rightarrow X\) be a dynamical system, and let \(Z \subseteq X\) be a subset of \(X\). Initially, regard \(Z\) as a set that has no special relation to \(X\). Define an injection map

\[
in_Z : Z \rightarrow X : z \mapsto in_Z(z) \in X
\]

(3.13)

In other words, \(in_Z\) allows a point in \(Z\) to transform from ‘local’ coordinates (in \(Z\)) to ‘global’ coordinates (in \(X\)). In order to distinguish the set \(Z\) as both a ‘local’ entity and
a ‘global’ entity in $X$, denote $Z_{loc}$ as the local set and $Z \subseteq X$ to be the subset in the
global coordinates. Thus, $in_Z : Z_{loc} \rightarrow X$ and $in_Z(Z_{loc}) = Z \subseteq X$.

Using this injection map concept, we can define an $\alpha$-invariance condition on a subset
of $X$. We say $Z \subseteq X$ is an $\alpha$-invariant subset of $X$ if [2]

$$(\exists \alpha_Z : Z_{loc} \rightarrow Z_{loc}) \alpha \circ in_Z = in_Z \circ \alpha_Z$$  \hspace{1cm} (3.14)$$

Diagramatically, it is equivalent to say that the following diagram commutes [2].

\[
\begin{array}{ccc}
Z_{loc} & \xrightarrow{\alpha_Z} & Z_{loc} \\
\downarrow{in_Z} & & \downarrow{in_Z} \\
X & \xrightarrow{\alpha} & X \\
\end{array}
\]

In traditional literature, $\alpha$-invariance would be defined such that $Z \subseteq X$ is $\alpha$-invariant if

$$(\forall z \in Z) \alpha(z) \in Z$$  \hspace{1cm} (3.15)$$

The following simple theorem, which is a supplement to Section 1.5 of [2], shows that
these two definitions are equivalent.

**Theorem 3.5.1.** Definition 3.14 and Definition 3.15 are equivalent.

**Proof**

**3.14 $\Rightarrow$ 3.15**

Let $z \in Z$. Because $Z = in_Z(Z_{loc})$, there exists $z_{loc} \in Z_{loc}$ such that $in_Z(z_{loc}) = z$.

By assumption, $\alpha(z) = \alpha \circ in_Z(z_{loc}) = in_Z \circ \alpha_Z(z_{loc})$. Because $\alpha_Z(z_{loc}) \in Z_{loc}$ and

$in_Z(Z_{loc}) = Z$, $in_Z \circ \alpha_Z(z_{loc}) \in Z$. Thus, $\alpha(z) \in Z$.

**3.15 $\Rightarrow$ 3.14**

Let $z_{loc} \in Z_{loc}$, then $in_Z(z_{loc}) \in Z$. By assumption, $\alpha \circ in_Z(z_{loc}) \in Z$. Therefore, there
exists \( y_{\text{loc}} \in Z_{\text{loc}} \) such that \( \text{in}_Z(y_{\text{loc}}) = \alpha \circ \text{in}_Z(z_{\text{loc}}) \). Define a function \( \alpha_Z : Z_{\text{loc}} \to Z_{\text{loc}} : z_{\text{loc}} \mapsto y_{\text{loc}} \). This function is legitimate because if there exists \( y^*_{\text{loc}} \in Z_{\text{loc}} \) which also satisfies \( \text{in}_Z(y^*_{\text{loc}}) = \alpha \circ \text{in}_Z(z_{\text{loc}}) \); it must be that \( y^*_{\text{loc}} = y_{\text{loc}} \) by the injectivity of \( \text{in}_Z \). Now, \( \alpha \circ \text{in}_Z(z_{\text{loc}}) = \text{in}_Z(y_{\text{loc}}) = \text{in}_Z \circ \alpha_Z(z_{\text{loc}}) \). Thus, \( \alpha_Z : Z_{\text{loc}} \to Z_{\text{loc}} \) satisfies \( \alpha \circ \text{in}_Z = \text{in}_Z \circ \alpha_Z \).

The map \( \alpha_Z \) is called the restriction of \( \alpha \) to \( Z \). It is only defined when \( Z \) is \( \alpha \)-invariant. Similarly, if \( f : X \to Y \) and \( Z \subseteq X \), then the restriction of \( f \) to \( Z \), denoted \( f_Z \) or \( f|_Z \), is defined as \( f_Z := f \circ \text{in}_Z \). Thus, \( f_Z : Z_{\text{loc}} \to Y \).

Let \( \pi \in \mathcal{E}(X) \) be a partition on \( X \), and restrict \( \pi \) to \( Z \subseteq X \) by intersecting the cells of \( \pi \) with \( Z \). Alternatively, \( \pi|_Z \in \mathcal{E}(Z) \) is defined such that for all \( z, z' \in Z \), \( z \equiv z' \pmod{\pi|_Z} \) if \( \text{in}_Z(z) \equiv \text{in}_Z(z') \pmod{\pi} \). Since the cells of \( Z/\pi|_Z \) are subsets of the cells of \( X/\pi \), we introduce an injection function \( \text{in}_{Z/\pi|_Z} : Z/\pi|_Z \to X/\pi \) such that each cell in \( Z/\pi|_Z \) is mapped to its corresponding superset in \( X/\pi \).

In the last section, we have seen the definition of the observer of a dynamical system \((X, \alpha, \gamma)\). If \( Z \subseteq X \) and \( \alpha(Z) \subseteq Z \), two different ways to compute the local observer on \( Z \) are available:

- Compute the ‘local observer’ \( \omega_Z \in \mathcal{E}(Z) \) with \((Z, \alpha|Z, \gamma|Z)\).
- Compute the ‘global observer’ \( \omega \in \mathcal{E}(X) \) with \((X, \alpha, \gamma)\), then restrict it to \( Z \) to get \( \omega|_Z \in \mathcal{E}(Z) \).

**Theorem 3.5.2.** The two results from above are identical. Namely, \( \omega_Z = \omega|_Z \).

The subset \( Z \subseteq X \) with \( \alpha(Z) \subseteq Z \) is said to be **detectable** with respect to \((X, \alpha, \gamma)\) if \( \omega_Z = \bot \), where \( \bot \in \mathcal{E}(Z) \) is defined such that \( z \equiv z' \pmod{\bot} \) iff \( z = z' \).
3.6 Internal Model Principle

In this section, a proof of the Internal Model Principle based on [2] will be presented.

Consider the following dynamical system with three components, the exosystem (E), the plant (P) and the controller (C). Let the exosystem (E) be an autonomous dynamical system \((X_E, \alpha_E)\). The total system, which consists of \(E\), \(P\) and \(C\), is a dynamical system \((X, \alpha)\), where \(X = X_E \times X_P \times X_C\) and \(\alpha = \alpha_E \times \alpha_P \times \alpha_C\). Here, the transition functions are defined such that

\[
\begin{align*}
\alpha_E : & X_E \to X_E \quad (3.16) \\
\alpha_P : & X_E \times X_P \times X_C \to X_P \quad (3.17) \\
\alpha_C : & X_E \times X_P \times X_C \to X_C \quad (3.18)
\end{align*}
\]

Notice that \(\alpha_P\) and \(\alpha_C\) not only depend on their own states \(x_P \in X_P\) and \(x_C \in X_C\) but also depend on the states of other components. The goal of regulation is to have the total system state converge to a regulation set \(K \subseteq X := X_E \times X_P \times X_C\) eventually. When this objective is met, we say that the total system is under regulation.

We make the following basic assumptions [2]:

**Assumption 1**: The exosystem \(E\) `induces` a unique attractive \(\alpha\)-invariant subset of \(X\). This can be paraphrased as the exosystem eventually impresses its own behavior on the total system, or the exosystem “drives” the total system. Thus, the total system behavior on this induced \(\alpha\)-invariant subset is a faithful reflection of the behavior of the exosystem. Using the invariant subset definition from the previous section, this means that there exists an injection \(i_E : X_E \to X\) such that \(\alpha \circ i_E = i_E \circ \alpha_E\) and the image of \(i_E\), \(i_E(X_E) \subseteq X\) is unique and attractive. Figure 3.1 illustrates this assumption pictorially.

We assume that \(i_E\) is unique up to automorphism. For any automorphism \(\nu\) such that \(\alpha_E \circ \nu = \nu \circ \alpha_E\), the map \(\tilde{i}_E = i_E \circ \nu\) also satisfies \(\alpha \circ \tilde{i}_E = \tilde{i}_E \circ \alpha_E\) and \(\tilde{i}_E(X_E) = \)
i_E \circ \nu(X_E) = i_E(X_E)$ because $\nu$ is bijective. This is illustrated in Figure 3.2. For simplicity, let $\tilde{X}_E := i_E(X_E)$ from now on.

**Assumption 2:** Let $\gamma : X \to X_C$ be the projection onto $X_C$, so that if $x = (x_E, x_P, x_C) \in X$, then $\gamma(x) = x_C \in X_C$. Let the target regulation subset be $K \subseteq X$, where the tracking error is zero in traditional control literature. Assume that $x \in K$ imposes no restriction on $\gamma(x) \in X_C$, or the mere fact that $x \in K$ provides no information about the control state $x_C = \gamma(x) \in X_C$. This is equivalent to saying that $\gamma(K) = X_C$, or every control state $x_C \in X_C$ corresponds to some $x \in K$.

**Assumption 3:** We assume that $\tilde{X}_E$ is detectable with respect to $(X, \alpha, \gamma)$. This means that we are able to identify the state in $\tilde{X}_E$ induced by the exosystem from the information of the controller state $x_C$. For example, let $x(0) = x_0 \in X$ and $x(t+1) = \alpha(x(t))$ where $t \in \mathbb{N}$; if $x_0 \in \tilde{X}_E$ and given that $x_C(t) = \gamma(x(t))$, then Assumption 3 implies that $x_0$ can be determined uniquely from the information $\{x_C(t) | t \in \mathbb{N}\}$.

**Assumption 4:** The last assumption is a consequence of error feedback structure. We assume that the controller is externally driven only when the total system state deviates from the regulation set $K \subseteq X$. This means that as long as $x \in K$, the controller behaves
as an autonomous dynamical system. Thus, we can say the successor controller state \( x'_C \), which is equal to \( \gamma(x') = \gamma \circ \alpha(x) \), depends only on \( x_C = \gamma(x) \). Equivalently, we have \( \ker \gamma | K \leq \ker(\gamma \circ \alpha | K) \), which means that if \( x_1 \equiv x_2 \pmod{\ker(\gamma | K)} \), then \( x_1 \equiv x_2 \pmod{\ker(\gamma \circ \alpha | K)} \). Note that this assumption does not imply \( \alpha(x) \in K \) if \( x \in K \) because \( K \subseteq X \) is not \( \alpha \)-invariant in general.

Finally, we state the regulation condition based on the assumptions above; the regulation condition is equivalent to saying that \( \tilde{X}_E \subseteq K \). Since \( K \subseteq X \), \( K \) can be modeled by an injection \( \kappa : K_{loc} \to X \) and \( \kappa(K_{loc}) = K \subseteq X \); then, the regulation condition is equivalent to saying that there exists another injection \( j_E : X_E \to K_{loc} \) such that \( i_E = \kappa \circ j_E \), as in the following diagram.

\[
\begin{array}{ccc}
X_E & \xrightarrow{i_E} & X \\
\Downarrow{j_E} & & \Downarrow{\kappa} \\
K_{loc} & \rightarrow & X \\
\end{array}
\]

Let us use the following notation: \( \tilde{\alpha}_E := \alpha | \tilde{X}_E \) and \( \tilde{\gamma}_E := \gamma | \tilde{X}_E \). Now, we are ready to state the Internal Model Principle.

**Theorem 3.6.1. Internal Model Principle \^[2] \**

Under Assumptions 1, 2, 3 and 4 above, given that the regulation property holds on the total dynamical system \((X, \alpha)\), there exists \( \alpha_I : X_C \to X_C \) such that \( \alpha_I \circ \tilde{\gamma}_E = \tilde{\gamma}_E \circ \alpha_E \) and \( \tilde{\gamma}_E \) is injective. Equivalently, the controller dynamics contain a copy of the exosystem under regulation.

Diagrammatically, the following diagram commutes:

**Lemma 3.6.2.** There exists a unique mapping \( \alpha_I : X_C \to X_C \) such that \( \alpha_I \circ \gamma | K = \gamma \circ \alpha | K \).

**Proof of 3.6.2:**

Let \( x_C \in X_C \). By Assumption 2, \( \exists x \in K \subseteq X ) \gamma(x) = x_C \). Define \( \alpha_I(x) := \gamma \circ \alpha(x) \). This
definition is unambiguous because if there exists \( x' \in K \) and \( \gamma(x') = x_C \), then \( x' \equiv x \) (mod \( \ker(\gamma|K) \)). By Assumption 4, \( x' \equiv x \) (mod \( \ker(\gamma \circ \alpha|K) \)). Thus, \( \gamma \circ \alpha(x') = \gamma \circ \alpha(x) \).

This lemma is depicted in the following commutative diagram.

\[
\begin{array}{ccc}
X_E & \xrightarrow{\alpha_E} & X_E \\
\gamma_E & \downarrow & \gamma_E \\
X_C & \xrightarrow{\alpha_I} & X_C
\end{array}
\]

Lemma 3.6.3. \( \alpha_I \circ \tilde{\gamma}_E = \tilde{\gamma}_E \circ \tilde{\alpha}_E \), where \( \alpha_I \) is the mapping from [3.6.2].

Proof of [3.6.3]:

Let \( x \in \tilde{X}_E \). Because \( \tilde{X}_E \subseteq K \), we get \( \alpha_I \circ \gamma(x) = \gamma \circ \alpha(x) \) from [3.6.2]. Since \( \alpha(x) = \tilde{\alpha}_E(x) \) and \( \alpha(x) \in \tilde{X}_E \), we have \( \alpha_I \circ \tilde{\gamma}_E(x) = \tilde{\gamma}_E \circ \tilde{\alpha}_E(x) \) as needed.

Lemma 3.6.4. \( \tilde{\gamma}_E \) is injective.

Proof of [3.6.4]:

Let \( \omega \in \mathcal{E}(X) \) be the observer for \( (\gamma, \alpha) \), and \( \tilde{\omega}_E := \omega|\tilde{X}_E \), which is the restriction of \( \omega \) to \( \tilde{X}_E \). Using Assumption 3, because \( \tilde{X}_E \) is detectable, we have \( \tilde{\omega}_E = \sup\{ \omega' \in \mathcal{E}(\tilde{X}_E) | \omega' \leq \ker(\gamma \wedge \omega' \circ \tilde{\alpha}_E) \} = \bot \). Also, if we restrict the condition \( \ker(\gamma|K) \leq \ker(\gamma \circ \alpha|K) \) from Assumption 4 to \( \tilde{X}_E \), we get \( \ker(\tilde{\gamma}_E \circ \tilde{\alpha}_E) = (\ker(\tilde{\gamma}_E)) \circ \tilde{\alpha}_E \). Thus, \( \ker(\tilde{\gamma}_E \leq \tilde{\omega}_E = \bot \), which is equivalent to say \( \tilde{\gamma}_E \) is injective as claimed.
Denote $\gamma_K := \gamma \circ \kappa$; then $\tilde{\gamma}_E = \gamma \upharpoonright X_E = \gamma \circ i_E = \gamma \circ (\kappa \circ j_E) = (\gamma \circ \kappa) \circ j_E = \gamma_K \circ j_E$.

Using the three lemmas above, we can construct the following commutative diagram

\[ \begin{array}{ccc}
X_E & \xrightarrow{\alpha_E} & X_E \\
\downarrow j_E & & \downarrow i_E \\
K_{loc} & \xrightarrow{\kappa} & X \\
\downarrow \gamma_K & & \downarrow \alpha \\
X_C & \xrightarrow{\alpha_I} & X_C \\
\end{array} \]

Thus, we have $\alpha_I \circ \tilde{\gamma}_E = \tilde{\gamma}_E \circ \alpha_E$ and $\tilde{\gamma}_E$ is injective as asserted in Theorem 3.6.1.
Chapter 4

Contractive Endosystem

In Chapter 3, we have seen that the output regulation result depends on the existence of the maps $i_E$ and $j_E$. In this chapter, a sufficient condition on the existence and uniqueness (up to automorphism) of $i_E$ maps will be demonstrated under basic assumptions on the exosystem and the endosystem dynamics. At the same time, examples that verify the conditions will be shown. The examples presented in this chapter use periodic or constant exosystems. Nevertheless, they provide a general intuition for the core idea that is going to be presented in this chapter.

4.1 Discrete Integrator Exosystems

The existence of the $i_E$ map guarantees an induced invariant subset in the total system state set by the exosystem. If $i_E$ exists and its image is a global attractor in $X$, then total system state will eventually converge to this invariant set induced by the exosystem. It was not shown in [2], however, under what circumstances there will exist an $i_E$ map and its image is an attractor. In this section, we consider the case when the exosystem is a discrete integrator.

Consider the following problem. Let $X_E, X_S$ be subsets of a complete metric space with metric $d$. Let $x_E \in X_E$, $x_S \in X_S$, $\alpha : X_E \times X_S \to X_E \times X_S : (x_E, x_S) \mapsto$
Lemma 4.1.1. Given \( \alpha_E = \text{id}_E \), where \( \text{id}_E \) is the identity map on \( X_E \), let \( \alpha_S : X_E \times X_S \to X_S \) be a contraction with respect to \( X_S \) uniformly over \( X_E \), meaning that

\[
(\exists c \in [0, 1])(\forall x_E \in X_E)(\forall x_{S1}, x_{S2} \in X_S)d\left(\alpha_S(x_E, x_{S1}), \alpha_S(x_E, x_{S2})\right) \leq cd\left(x_{S1}, x_{S2}\right)
\] (4.1)

Then there exists a unique map \( i_{ES} : X_E \to X_S \) such that

\[
\alpha_S \circ (\text{id}_E \times i_{ES}) = i_{ES} \circ \alpha_E
\] (4.2)

Furthermore, for all \( x_E \in X_E \), for all initial conditions \( x_S(0) \in X_S \), \( d\left(x_S(t) - i_{ES}(x_E)\right) \to 0 \) as \( t \to \infty \). In other words, \( i_{ES}(X_E) \subseteq X_S \) is an attractor in \( X_S \).

Proof:

If \( \alpha_S(x_E, x_S) \) is a contraction with respect to \( x_S \) for every \( x_E \in X_E \), by the Banach Contraction Mapping Principle, we know that for every \( x_E \in X_E \), there exists a unique \( \tilde{x}_S \in X_S \) such that \( \alpha_S(x_E, \tilde{x}_S) = \tilde{x}_S \). Define a function \( i_{ES}(x_E) : X_E \to X_S \) such that \( \tilde{x}_S = i_{ES}(x_E) \). This definition is unambiguous because for each \( x_E \in X_E \), there is a unique \( \tilde{x}_S \in X_S \) satisfying \( \alpha_S(x_E, \tilde{x}_S) = \tilde{x}_S \). Since \( \alpha_E = \text{id}_E \), \( i_{ES} \circ \alpha_E(x_E) = i_{ES}(x_E) \). Thus, (4.2) becomes \( \alpha_S(x_E, i_{ES}(x_E)) = i_{ES}(x_E) \), which is true by the definition of \( i_{ES} \). Suppose there exists another map \( i_{ES}^\prime : X_E \to X_S \) such that \( \alpha_S \circ (\text{id}_E \times i_{ES}^\prime) = i_{ES}^\prime \circ \alpha_E \), then for any \( x_E \in X_E \), \( i_{ES}^\prime(x_E) \) satisfies \( \alpha_S(x_E, i_{ES}^\prime(x_E)) = i_{ES}^\prime(x_E) \). Thus, \( i_{ES}^\prime(x_E) \) is a fixed point of \( \alpha_S(x_E, \cdot) \). But the fixed point of \( \alpha_S(x_E, \cdot) \) is unique due to the nature of contraction. Therefore, \( i_{ES}^\prime(x_E) = i_{ES}(x_E) \). Because the choice of \( x_E \in X_E \) was arbitrary, we have
$i_{ES} \equiv i_{ES}$. Thus, $i_{ES}$ is unique. Given $x_E \in X_E$, the convergence of $x_S(t)$ to $i_{ES}(X_E)$ under $\alpha_S(x_E, \cdot)$ follows directly from the contraction property. 

Let $i_E = id_E \times i_{ES}$, where $id_E$ is the identity map on $X_E$. Such $i_E$ function satisfies the commutative diagram \[4.1\]}

\[
\begin{array}{ccc}
X_E & \xrightarrow{\alpha_E = id_E} & X_E \\
\downarrow{iid E \times i_{ES}} & & \downarrow{iid E \times i_{ES}} \\
X_E \times X_S & \xrightarrow{\alpha = (\alpha_E, \alpha_S)} & X_E \times X_S 
\end{array}
\]

\[\text{Figure 4.1: Discrete Integrator Exosystem}\]

Obviously, $i_E(X_E) \subset X$ is an $\alpha$-invariant subset of $X = X_E \times X_S$ and an attractor in $X$ since $i_{ES}(X_E)$ is an attractor in $X_S$ and satisfies (4.2).

### 4.2 Harmonic Oscillator Exosystems

We have shown that when $\alpha_S : X_E \times X_S \to X_S$ is a contraction with respect to $X_S$ uniformly over $X_E$, there exists a unique $i_{ES}$ function such that $\alpha_S \circ (id_E \times i_{ES}) = i_{ES} \circ \alpha_E$ for the discrete-integrator exosystem case. Now, let us consider a harmonic oscillator exosystem with period $N > 1$, where $N \in \mathbb{Z}^+$; i.e.

\[(\alpha_E)^N = id_E \quad (4.3)\]
Equivalently, for any \( x \in X \), \( \alpha^N_E(x) = x \).

**Theorem 4.2.1.** Given a harmonic oscillator exosystem \((X_E, \alpha_E)\) with period \( N > 1 \), or \( \alpha^N_E = \text{id}_E \), let \( \alpha_S : X_E \times X_S \to X_S : (x_E, x_S) \mapsto \alpha_S(x_E, x_S) \) be a contraction with respect to \( X_S \) uniformly over \( X_E \); then there exists a unique map \( i_{ES} : X_E \to X_S \) such that \( \alpha_S \circ (\text{id}_E \times i_{ES}) = i_{ES} \circ \alpha_E \). Furthermore, for all initial conditions \( x_E(0) \in X_E \) and \( x_S(0) \in X_S \), \( d(x_S(t), i_{ES}(x_E(t))) = 0 \) as \( t \to \infty \). In other words, \( i_{ES}(X_E) \subset X_S \) is an attractor in \( X_S \).

**Proof:**

Let \( x_E \in X_E \) be arbitrary. Given that for all \( x_E \in X_E \), \( \alpha_S \) is a contraction with respect to \( x_S \), we know that \( \alpha_S(x_E, \cdot) : X_S \to X_S \), \( \alpha_S(\alpha_E(x_E), \cdot) : X_S \to X_S \), \( \alpha_S(\alpha_E^2(x_E), \cdot) : X_S \to X_S \), ... , \( \alpha_S(\alpha_E^{N-1}(x_E), \cdot) : X_S \to X_S \) are all contraction maps on \( X_S \). For convenience in this proof later on, we introduce the following notation; let

\[
\alpha_S[x_E]_i^j(x_S) = \alpha_S\left(\alpha_E^{j-1}(x_E), \alpha_S\left(\alpha_E^{j-2}(x_E), \ldots \alpha_S(\alpha_E^i(x_E), \alpha_S(\alpha_E^{i-1}(x_E), x_S))\right)\ldots\right) \tag{4.4}
\]

for any \( i, j \in \mathbb{N} \) such that \( j \geq i \). For \( j = i \),

\[
\alpha_S[x_E]_i^i(x_S) = \alpha_S(\alpha_E^{i-1}(x_E), x_S) \tag{4.5}
\]
For example,

\[
\alpha_S[x_E]_1^1(x_S) = \alpha_S(\alpha_E^{1-1}(x_E), x_S) = \alpha_S(x_E, x_S)
\]  

(4.6)

and

\[
\alpha_S[x_E]_2^4(x_S) = \alpha_S\left(\alpha_E^3(x_E), \alpha_S(\alpha_E^2(x_E), \alpha_S(\alpha_E(x_E), x_S))\right)
\]  

(4.7)

Let \(x_S \in X_S\) and \(x_E \in X_E\). We define

\[
\alpha_S[x_E]_0^0(x_S) := x_S
\]  

(4.8)

Note that for all \(i, j, k \in \mathbb{Z}^+\) such that \(i \geq j \geq k\)

\[
\alpha_S[x_E]_j^{i+1} \circ \alpha_S[x_E]_k^{j} = \alpha_S[x_E]_k^{j}
\]  

(4.9)

Therefore, by (4.8) and (4.9),

\[
\alpha_S[x_E]_0^1(x_S) = \alpha_S[x_E]_1^1 \circ \alpha_S[x_E]_0^0(x_S)
\]  

(4.10)

\[
= \alpha_S[x_E]_1^1(x_S)
\]  

(4.11)

Another fact that should be noted here is the following: for all \(i, j \in \mathbb{Z}^+\),

\[
\alpha_S[x_E]_j^{i+j}(x_S) = \alpha_S[\alpha_E^j(x_E)]_1^i(x_S)
\]  

(4.12)

for all \(x_E \in X_E\) and \(x_S \in X_S\), which can be easily verified algebraically. A consequence of (4.12) is the following.
where $N$ is the period of $\alpha_E$. To show (4.13), let $i \in \mathbb{N}$. (4.13) is obviously satisfied for $i = 1$. Suppose it satisfies
\[
\alpha_S[x_E]|_{iN}^1 = \left( \alpha_S[x_E]|_1^N \right)^i \tag{4.14}
\]
then
\[
\alpha_S[x_E]|_{i+1}|_{1}^{i+1} = \alpha_S[x_E]|_{iN+1}^1 \circ \alpha_S[x_E]|_1^N \tag{4.15}
\]
\[
= \alpha_S[x_E]|_{iN}^1 \circ \alpha_S[x_E]|_1^N \tag{4.16}
\]
\[
= \alpha_S[x_E]|_{iN}^1 \circ \alpha_S[x_E]|_1^N \tag{4.17}
\]
\[
= \left( \alpha_S[x_E]|_1^N \right)^i \circ \alpha_S[x_E]|_1^N \tag{4.18}
\]
\[
= \left( \alpha_S[x_E]|_1^N \right)^{i+1} \tag{4.19}
\]
where (4.15) and (4.16) are consequences of (4.9) and (4.12), respectively, and (4.18) is from the inductive hypothesis. Thus, (4.13) is proved.

Since a composition of contraction maps is still a contraction, which can be easily proved, a composition of any combination from the set of functions
\[
\mathcal{A}_S[x_E]|_1^N := \left\{ \alpha_S(\alpha_E^{N-1}(x_E), \ldots), \alpha_S(\alpha_E^{N-2}(x_E), \ldots), \ldots, \alpha_S(\alpha_E(x_E), \ldots), \alpha_S(x_E, \ldots) \right\} \tag{4.20}
\]
or for any $i, j \in \mathbb{N}$, $\alpha_S[x_E]|_{ij}^i : X_S \to X_S$ is a contraction. Therefore, there exists a unique
attractive fixed point in $X_S$ for all possible compositions of functions from $A_S[x_E]|_1^N$. In particular, let $y_{x_E} \in X_S$ denote the fixed point of $\alpha_S[x_E]|_1^N$. Define a function $i_{ES} : X_E \to X_S$ such that $i_{ES}(x_E) := y_{x_E} \in X_S$. This function is well defined and the domain of this function is the entire set $X_E$ since for all $x_E \in X_E$, $\alpha_S[x_E]|_1^N : X_S \to X_S$ is a contraction and hence $i_{ES}(x_E) = y_{x_E}$ exists uniquely for all $x_E$ by the contraction property. This function, by definition, satisfies

$$(\forall x_E \in X_E) \alpha_S[x_E]|_1^N \circ i_{ES}(x_E) = i_{ES}(x_E) \quad (4.21)$$

Now, we need to show that this $i_{ES}$ satisfies $\alpha_S \circ (id_E \times i_{ES}) = i_{ES} \circ \alpha_E$, or $\alpha_S(x_E, i_{ES}(x_E)) = i_{ES}(\alpha_E(x_E))$ for all $x_E \in X_E$. Again, let $x_E \in X_E$ be arbitrary and let $z_{x_E} := \alpha_S(x_E, i_{ES}(x_E)) \in X_S$. Our objective is to show $z_{x_E} = i_{ES} \circ \alpha_E(x_E)$. Applying the mappings $\alpha_S[x_E]|_2^N$ to $z_{x_E}$, we get

$$\alpha_S[x_E]|_2^N(z_{x_E}) = \alpha_S[x_E]|_2^N \circ \alpha_S[x_E]|_1^1 \circ i_{ES}(x_E) \quad (4.22)$$

$$= \alpha_S[x_E]|_1^N \circ i_{ES}(x_E) \quad (4.23)$$

By the definition of $i_{ES}$, we have

$$\alpha_S[x_E]|_1^N \circ i_{ES}(x_E) = i_{ES}(x_E) \quad (4.24)$$

Applying $\alpha_S(x_E,.) = \alpha_S(\alpha_E(x_E),.)$ to both sides of (4.24), we get

$$\alpha_S[x_E]|_1^{N+1} \circ \alpha_S[x_E]|_1^N \circ i_{ES}(x_E) = \alpha_S[x_E]|_1^{N+1} \circ i_{ES}(x_E) \quad (4.25)$$

$$= \alpha_S(x_E, i_{ES}(x_E)) \quad (4.26)$$

$$=: z_{x_E} \quad (4.27)$$
But the left-hand side of (4.25) is simply $\alpha_S[\alpha_E(x_E)]_1^N \circ \alpha_S(x_E, i_{ES}(x_E))$, so (4.25) is equivalent to

$$\alpha_S[\alpha_E(x_E)]_1^N(z_{x_E}) = z_{x_E} \quad (4.28)$$

Therefore, $z_{x_E} \in X_S$ is a fixed point for $\alpha_S[\alpha_E(x_E)]_1^N$. Recall that $\alpha_S[\alpha_E(x_E)]_1^N$ is a contraction and therefore, it has a unique fixed point. By definition of $i_{ES}$, the unique fixed point is $i_{ES} \circ \alpha_E(x_E)$. Therefore, it must be true that $z_{x_E} = i_{ES} \circ \alpha_E(x_E)$. Thus, $\alpha_S(x_E, i_{ES}(x_E)) = i_{ES}(\alpha_E(x_E))$ as we needed to show.

The uniqueness of $i_{ES}$ can be shown similarly to the discrete integrator case. Assume that there exists another function $\tilde{i}_{ES} : X_E \to X_S$ which also satisfies

$$\alpha_S \circ (id_E \times \tilde{i}_{ES}) = \tilde{i}_{ES} \circ \alpha_E \quad (4.29)$$

Let $x_E \in X_E$. We must show that $\tilde{i}_{ES}(x_E) = i_{ES}(x_E)$. We make the following Claim to aid our proof of the uniqueness of $i_{ES}$:

$$\forall x_E \in X_E)(\forall n \in \mathbb{Z}^+)\alpha_S[x_E]|_1^n \circ i_{ES}(x_E) = i_{ES} \circ \alpha_E^n(x_E) \quad (4.30)$$

We prove the claim by induction. Let $x \in X_E$. By definition, $\alpha_S[x]|_1^1 \circ i_{ES}(x) = \alpha_S(x, i_{ES}(x))$ and $\alpha_S(x, i_{ES}(x)) = i_{ES} \circ \alpha_E(x)$ by (4.29). Thus, $\alpha_S[x]|_1^1 \circ i_{ES}(x) = i_{ES} \circ \alpha_E(x)$ and (4.30) is true for the base case $i = 1$. Now, assuming that (4.30) is true for $n = i - 1$, we must show that it is also true for $n = i$. For $n = i$, $\alpha_S[x]|_1^i \circ i_{ES}(x) = \alpha_S(\alpha_E^{i-1}(x), \alpha_S[x]|_0^{i-1} \circ i_{ES}(x))$. By the inductive hypothesis,

$$\alpha_S[x]|_0^{i-1} \circ i_{ES}(x) = i_{ES} \circ \alpha_E^{i-1} \quad (4.31)$$

Therefore,
\[ \alpha_S\left(\alpha_E^{i-1}(x), \alpha_S[x]_0^{i-1} \circ \tilde{i}_{ES}(x)\right) = \alpha_S\left(\alpha_E^{i-1}(x), \tilde{i}_{ES} \circ \alpha_E^{i-1}(x)\right) \quad (4.32) \]

But
\[ \alpha_S\left(\alpha_E^{i-1}(x), \tilde{i}_{ES} \circ \alpha_E^{i-1}(x)\right) = \tilde{i}_{ES} \circ \alpha_E^{i-1+1}(x) \quad (4.33) \]
\[ = \tilde{i}_{ES} \circ \alpha_E^{i}(x) \quad (4.34) \]
by (4.29). Thus, \( \alpha_S[x]_1^{i} \circ \tilde{i}_{ES}(x) = \tilde{i}_{ES} \circ \alpha_E^{i}(x) \) and this completes the proof of Claim (4.30).

Using Claim (4.30), we know that
\[ \alpha_S[x]_1^{N} \circ \tilde{i}_{ES}(x_E) = \tilde{i}_{ES} \circ \alpha_E^{N}(x_E) \quad (4.35) \]
\[ = \tilde{i}_{ES}(x_E) \quad (4.36) \]
Therefore, \( \tilde{i}_{ES}(x_E) \) is a fixed point of \( \alpha_S[x]_1^{N} \). But \( \alpha_S[x]_1^{N} \) has a unique fixed point, and it is \( \tilde{i}_{ES}(x_E) \) by the definition of \( \tilde{i}_{ES} \). Thus, it must be that \( \tilde{i}_{ES}(x_E) = \tilde{i}_{ES}(x_E) \). Since \( x_E \in X_E \) was arbitrary, we have \( \tilde{i}_{ES} \equiv i_{ES} \).

The attractiveness of \( i_{ES}(X_E) \) is shown by the following argument. Given an initial state \( x_E \in X_E \), the N-periodic exosystem generates a periodic orbit \( \{x_E, \alpha_E(x_E), \ldots, \alpha_E^{N-1}(x_E)\} \subset X_E \) for the exosystem state to follow. We claim that for any initial condition \( x_S \in X_S \), the endosystem eventually converges to the orbit \( \{i_{ES}(x_E), i_{ES} \circ \alpha_E(x_E), \ldots, i_{ES} \circ \alpha_E^{N-1}(x_E)\} \) and
\[ (\forall x_S(0) \in X_S) \lim_{t \to \infty} d\left(x_S(t), i_{ES}(x_E(t))\right) = 0 \quad (4.37) \]
To prove (4.37), let \( x_{S0} := x_S(0) \in X_S \) and \( x_{E0} := x_E(0) \in X_E \) be arbitrary. Let the discrete-time variable \( t \) be such that \( t = iN + j \), where \( i \in \mathbb{Z}^+ \), \( j \in \{0, ..., N - 1\} \) and \( j = n \mod N \). Denote \( x_{E0} = x_E(0 \mod N) \), \( x_{E1} = x_E(1 \mod N) \), \( ..., x_{Ej} = x_E(j \mod N) \), \( ..., x_{E(N-1)} = x_E(N - 1 \mod N) \). Note that \( x_{S}(t) = \alpha_S[x_E(0)]^i(x_{S0}) \). Since \( t = iN + j \), using (4.9) and (4.12), we get

\[
x_S(t) = \alpha_S[x_E0]^i(x_{S0})
\]

(4.38)

\[
= \alpha_S[x_E0]^{iN+j}(x_{S0})
\]

(4.39)

\[
= \alpha_S[x_E0]^{iN+j} \circ \alpha_S[x_{E0}]^j(x_{S0})
\]

(4.40)

\[
= \alpha_S[x_E^j(x_{E1})]^{iN} \circ \alpha_S[x_{E0}]^j(x_{S0})
\]

(4.41)

\[
= \alpha_S[x_E]^i \circ \alpha_S[x_{E0}]^j(x_{S0})
\]

(4.42)

On the other hand, \( i_{ES}(x_{E}(t)) = i_{ES}(x_{Ej}) \) satisfies

\[
i_{ES}(x_{Ej}) = \alpha_S[x_{Ej}]^N \circ i_{ES}(x_{Ej})
\]

(4.43)

by the definition of \( i_{ES} \). It can be easily verified using (4.9) and (4.12) that

\[
\alpha_S[x_{Ej}]^N \circ i_{ES}(x_{Ej}) = \alpha_S[x_{Ej}]^{iN} \circ i_{ES}(x_{Ej})
\]

(4.44)

for any \( k \in \mathbb{Z}^+ \). In particular, we know

\[
i_{ES}(x_{Ej}) = \alpha_S[x_{Ej}]^N \circ i_{ES}(x_{Ej}) = \alpha_S[x_{Ej}]^{iN} \circ i_{ES}(x_{Ej})
\]

(4.45)

Thus,

\[
d\left(x_S(t), i_{ES}(x_{E}(t))\right) = d\left(\alpha_S[x_{Ej}]^i \circ \alpha_S[x_{E0}]^j(x_{S0}), \alpha_S[x_{Ej}]^{iN} \circ i_{ES}(x_{Ej})\right)
\]

(4.46)
Since $\alpha_S[x_{Ej}]_1^N$ is a contraction and its contraction constant is $c^N \in [0, 1)$, where $c \in [0, 1)$ is the uniform contraction constant of $\alpha_S$ with respect to $X_S$, it follows that for all $x_S, \tilde{x}_S \in X_S$, we have $d\left(\alpha_S[x_{Ej}]_1^N(x_S), \alpha_S[x_{Ej}]_1^N(\tilde{x}_S)\right) \leq c^N d\left(x_S, \tilde{x}_S\right)$. Therefore,

$$d\left(\alpha_S[x_{Ej}]_1^N \circ \alpha_S[x_{E0}]_1^j(x_{S0}), \alpha_S[x_{Ej}]_1^N \circ i_{ES}(x_{Ej})\right) \leq (c^N)^i d\left(\alpha_S[x_{E0}]_1^j(x_{S0}), i_{ES}(x_{Ej})\right)$$

(4.47)

In other words, $d\left(x_S(t), i_{ES}(x_E(t))\right)$ is bounded by $(c^N)^i d\left(\alpha_S[x_{E0}]_1^j(x_{S0}), i_{ES}(x_{Ej})\right)$. As $t \to \infty$, $i \to \infty$, and $(c^N)^i \to 0$. Therefore,

$$\lim_{t \to \infty} d\left(x_S(t), i_{ES}(x_E(t))\right) \leq \lim_{i \to \infty} (c^N)^i d\left(\alpha_S[x_{E0}]_1^j(x_{S0}), i_{ES}(x_{Ej})\right) = 0$$

(4.48)

This completes the proof of \ref{4.37} and shows the convergence of $x_S(t)$ with any initial condition $x_S(0) \in X_S$ to the image of $i_{ES}$.

\[\boxempty\]

### 4.3 Continuity of $i_{ES}$

In the previous sections, we found that a contraction property on $\alpha_S : X_E \times X_S \to X_S$ with respect to $X_S$ uniformly over $X_E$ guarantees the existence and uniqueness of a function $i_{ES} : X_E \to X_S$ that satisfies $\alpha_S \circ (id_E \times i_{ES}) = i_{ES} \circ \alpha_E$ and the image of $i_{ES}$ is an attractor in $X_S$ for periodic ecosystems. We can not, however, conclude functional properties such as continuity of $i_{ES}$ based merely on the contraction condition. In this
section, we present a condition on \( \alpha_S : X_E \times X_S \to X_S \) that results in continuity of \( i_{ES} \).

**Lemma 4.3.1. Continuity of \( i_{ES} \)**

Let the assumptions on \( \alpha_E \) and \( \alpha_S \) in Theorem 4.2.1 hold. Suppose further that \( \alpha_E : X_E \to X_E \) is continuous and \( \alpha_S : X_E \times X_S \to X_S : (x_E, x_S) \mapsto \alpha_S(x_E, x_S) \) is continuous with respect to \( x_E \); then, the resulting \( i_{ES} : X_E \to X_S \) from Theorem 4.2.1 is continuous.

**Proof:**

We need to show that for all \( \bar{x} \in X_E \),

\[
\lim_{x \to \bar{x}} d\left(i_{ES}(x), i_{ES}(\bar{x})\right) = 0 \quad (4.49)
\]

Suppose the exosystem is a discrete-integrator, or \( \alpha_E = id_E \); then, \( i_{ES} \) satisfies

\[
\alpha_S(x, i_{ES}(x)) = i_{ES} \circ \alpha_E(x) \quad (4.50)
\]

for all \( x \in X_E \). Now, to prove (4.49),

\[
d\left(i_{ES}(x), i_{ES}(\bar{x})\right) = d\left(\alpha_S(x, i_{ES}(x)), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right) \leq d\left(\alpha_S(x, i_{ES}(x)), \alpha_S(x, i_{ES}(\bar{x}))\right) + d\left(\alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right) \quad (4.51)
\]

by the definition of \( i_{ES} \) and the triangle inequality. Let \( c \in [0, 1) \) be the uniform contraction constant of \( \alpha_S \) with respect to \( X_S \). Then,

\[
d\left(\alpha_S(x, i_{ES}(x)), \alpha_S(x, i_{ES}(\bar{x}))\right) \leq cd\left(i_{ES}(x), i_{ES}(\bar{x})\right) \quad (4.53)
\]
Thus,

\[ d(i_{ES}(x), i_{ES}(\bar{x})) \leq cd(i_{ES}(x), i_{ES}(\bar{x})) + d(\alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x}))) \] (4.54)

If we move \( cd(i_{ES}(x), i_{ES}(\bar{x})) \) to the left-hand side, we get

\[ (1 - c)d(i_{ES}(x), i_{ES}(\bar{x})) \leq d(\alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x}))) \] (4.55)

Now, let us take the limit as \( x \to \bar{x} \) on both sides of the inequality above. Since \( \alpha_S \) is continuous with respect to \( X_E \), the right-hand side satisfies

\[ \lim_{x \to \bar{x}} d(\alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x}))) = 0 \] (4.56)

Therefore, the left-hand side limit

\[ \lim_{x \to \bar{x}} (1 - c)d(i_{ES}(x), i_{ES}(\bar{x})) \] (4.57)

exists, and as \( (1 - c) > 0 \), it must be that

\[ \lim_{x \to \bar{x}} d(i_{ES}(x), i_{ES}(\bar{x})) = 0 \] (4.58)

For exosystem with period \( N > 1 \), we will only show the case with \( N = 2 \) due to the complexity of algebraic manipulations. For cases with \( N > 2 \), readers can easily use the same arguments used to prove the case with \( N = 2 \).

Suppose \( \alpha^2_E = id_E \). Then \( i_{ES} : X_E \to X_S \) satisfies

\[ \alpha_S(\alpha_E(x), \alpha_S(x, i_{ES}(x))) = i_{ES}(x) \] (4.59)
for all \( x \in X_E \) by the definition of \( i_{ES} \). To show (4.49), we use the triangle inequality and the contraction property again as follows

\[
d(i_{ES}(x), i_{ES}(\bar{x})) = d\left(\alpha_S\left(\alpha_E(x), \alpha_S(x, i_{ES}(x))\right), \alpha_S\left(\alpha_E(\bar{x}), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right)\right) \tag{4.60}
\]

\[
\leq d\left(\alpha_S\left(\alpha_E(x), \alpha_S(x, i_{ES}(x))\right), \alpha_S\left(\alpha_E(x), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right)\right) + d\left(\alpha_S\left(\alpha_E(x), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right), \alpha_S\left(\alpha_E(\bar{x}), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right)\right) \tag{4.61}
\]

\[
\leq c d\left(\alpha_S(x, i_{ES}(x)), \alpha_S(x, i_{ES}(\bar{x}))\right) + d\left(\alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right) \tag{4.62}
\]

\[
\leq c^2 d\left(i_{ES}(x), i_{ES}(\bar{x})\right) + c d\left(\alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right) + d\left(\alpha_S(\alpha_E(x), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right), \alpha_S(\alpha_E(\bar{x}), \alpha_S(\bar{x}, i_{ES}(\bar{x}))\right) \tag{4.63}
\]

\[
\leq c^2 d\left(i_{ES}(x), i_{ES}(\bar{x})\right) \tag{4.64}
\]

Moving \( c^2 d\left(i_{ES}(x), i_{ES}(\bar{x})\right) \) to the left-hand side, we get
(1 - c^2)d\left( i_{ES}(x), i_{ES}(\bar{x}) \right) \leq cd\left( \alpha_S(x, i_{ES}(\bar{x})), \alpha_S(\bar{x}, i_{ES}(\bar{x})) \right) \\
\quad + d\left( \alpha_S(\alpha_E(x), \alpha_S(\bar{x}, i_{ES}(\bar{x}))), \alpha_S(\alpha_E(\bar{x}), \alpha_S(\bar{x}, i_{ES}(\bar{x}))) \right) \\
(4.65)

We take the limit as \( x \to \bar{x} \) on both sides of the inequality above. Because \( \alpha_S \) is continuous with respect to \( X_E \) and \( \alpha_E \) is continuous, both terms on the right-hand side of the inequality approach zero as \( x \to \bar{x} \). Therefore,

\[
\lim_{x \to \bar{x}} (1 - c^2)d\left( i_{ES}(x), i_{ES}(\bar{x}) \right) = 0 \\
(4.66)
\]

which implies \((4.49)\) as in the case with \( N = 1 \).

Readers can repeat the steps from \((4.60)\) to \((4.64)\) to obtain the same conclusion for the cases with \( N > 2 \).

\[\Box\]

### 4.4 An Example for the Harmonic Oscillator Case

Let \( x \in X_E \) and \( y \in X_S \), and \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \). Choose \( \alpha_S(x, y) \) to be

\[
\alpha_S(x, y) = x + \left( \frac{1}{2} \right)y \\
(4.67)
\]

which is one of the simplest forms that is a contraction with respect to \( y \) for all \( x \in X_E \).

Let the exosystem be a four-state oscillator such that \( \alpha_E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). The periodic
orbit induced by this exosystem on $X_E$ is $\{x, \alpha_E(x), \alpha_E^2(x), \alpha_E^3(x)\}$, which is equal to $\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}\}$ in this example. Since our $\alpha_S(x, y)$ is relatively simple, the injection function $i_{ES}: X_E \rightarrow X_S$ can be computed manually. By definition, $i_{ES}(x)$ is the unique fixed point of $\alpha_S\left(\alpha_E^3(x), \alpha_S(\alpha_E^2(x), \alpha_S(\alpha_E(x), \alpha_S(x, .)))\right): X_S \rightarrow X_S$, so $i_{ES}(x)$ can be determined by the following equation

$$i_{ES}(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + \frac{1}{2} \left( \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} + \frac{1}{2} \left( \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} + \frac{1}{2} \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} i_{ES}(x) \right) \right) \right) \quad (4.68)$$

After manipulation of the terms, we obtain

$$i_{ES}(x) = \begin{bmatrix} -\frac{3}{8}x_1 - \frac{3}{4}x_2 \\ \frac{3}{4}x_1 - \frac{3}{8}x_2 \end{bmatrix} + \frac{1}{16} i_{ES}(x) \quad (4.69)$$

so that

$$i_{ES}(x) = \begin{bmatrix} -\frac{2}{5}x_1 - \frac{4}{5}x_2 \\ \frac{4}{5}x_1 - \frac{2}{5}x_2 \end{bmatrix} \quad (4.70)$$

Choose our exosystem dynamics to be on the unit circle and the initial state to be the point $x_{E0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in X_E$. The evolution of $x_E(t)$ can then be completely described using modulo 4 operation on $k \in \mathbb{N}$ as

$$x_{E0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_{E1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, x_{E2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, x_{E3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4.71)$$

where $x_{Ej} = \alpha_E^j(x_{E0})$. Now, we can compute the subset induced in $X_S$ by $i_{ES}$. Based on
(4.70) and (4.71), we can calculate the following:

\[ i_{ES}(x_{E0}) = \begin{bmatrix} -0.4 \\ 0.8 \end{bmatrix} , i_{ES}(x_{E1}) = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix} , i_{ES}(x_{E2}) = \begin{bmatrix} 0.4 \\ -0.8 \end{bmatrix} , i_{ES}(x_{E3}) = \begin{bmatrix} -0.8 \\ -0.4 \end{bmatrix} \]

(4.72)

Based on our proofs from previous sections, we would expect the endosystem state \( x_S(t) \) to behave similarly to the dynamics of \( x_E(t) \) but to oscillate through the points listed in (4.72) as \( t \to \infty \). Using MATLAB, we simulated the following results for different initial conditions of \( x_S(t) \). Here, the simulation step number is set to 40. Figure 4.3 shows the evolution of \( x_E(t) \) with initial condition \( x_E(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Three different initial conditions of \( x_S(t) \) were considered; they are \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} 2 \\ 4 \end{bmatrix} \) and \( \begin{bmatrix} -3.789 \\ -6.748 \end{bmatrix} \). Figure 4.4, 4.5 and 4.6 show the behaviour of \( x_S(t) \) for each initial condition, respectively. Observe that in all cases, \( x_S(t) \) will oscillate through the four points we indicated in (4.72) after a sufficiently long time. Note that the endosystem converges to the image of the \( x_E \)-orbit under \( i_{ES} \). It is clear that the simulations verify our claims.
Figure 4.3: $x_E(t)$ Evolution

Figure 4.4: $x_S(t)$ with $x_S(0) = (1, 0)^T$
Figure 4.5: $x_S(t)$ with $x_S(0) = (2,4)^T$

Figure 4.6: $x_S(t)$ with $x_S(0) = (-3.789, -6.748)^T$
Chapter 5

Eventually Contractive Endosystem

In the previous chapter, we have shown that when the endosystem transition function is
a contraction with respect to the endosystem state uniformly for any exosystem state,
then a unique function \( i_{ES} : X_E \rightarrow X_S \) always exists to ensure that the following equation
holds

\[
\alpha_S \circ (id_E \times i_{ES}) = i_{ES} \circ \alpha_E
\]

(5.1)

and the image of this \( i_{ES} \) function is an attractor in \( X_S \). The contraction condition on
\( \alpha_S : X_E \times X_S \rightarrow X_S \), however, is a very strong and restrictive condition. In reality,
not many dynamical systems satisfy this requirement. For example, in discrete-time
linear systems, it is well-known that an asymptotically stable (or Schur stable) system
transition matrix is not a contraction operator on the system state space in general, with
the exception of diagonal system transition matrices.

In this chapter, we will be working on dynamical systems on a Banach space instead
of just a complete metric space. By taking the \( X_E, X_S \) to be subsets of a Banach space,
a less restrictive condition on the endosystem transition function will be introduced to
guarantee the existence and uniqueness of the map \( i_{ES} : X_E \rightarrow X_S \). With this new con-
dition, our regulation analysis can be extended to a much broader class of endosystems.
5.1 Eventual Contraction

Before we introduce the main theorem of this chapter, we need to present the following definition and lemma.

**Definition 5.1.1. Eventual Contraction** [14]

Let $X$ be a subset of a Banach space with norm $\|\cdot\|$. Let $f$ be a map from $X$ to $X$. $f$ is said to be an eventual contraction on $X$ if

\[(\exists N \in \mathbb{Z}^+)(\exists c \in [0,1])(\forall n \geq N)(\forall x_1, x_2 \in X)\|f^n(x_1) - f^n(x_2)\| \leq c\|x_1 - x_2\|\]  

where $f^n$ is the $n$-fold composition of $f$. The number $N$ is called the contractive composition number of $f$.

In other words, $f$ is an eventual contraction if there exists a positive integer $N$ such that $f^n$ is a contraction for any $n \geq N$. The definition of eventual contraction has been slightly modified from the definition in [14] to suit the context in this thesis. Now, we are ready to state our first lemma.

**Lemma 5.1.2.** Let $X \subseteq \mathbb{R}^n$ and $f : X \rightarrow X$. If $f$ is an eventual contraction on $X$, then $f$ is uniformly continuous on $X$.

**Proof:**

First, we claim that if $f$ is an eventual contraction on $X$, then $f$ is Lipschitz on $X$; that is,

\[(\exists L \in \mathbb{R}^+)(\forall x_1, x_2 \in X)\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|\]  

To prove the claim, we prove its contraposition; that is, we want to show that if $f$ is not Lipschitz on $X$, then it is not an eventual contraction on $X$. Equivalently, in logic notation
$(\forall L \in \mathbb{R}^+)(\exists x_1, x_2 \in X)\|f(x_1) - f(x_2)\| > L\|x_1 - x_2\|
\Rightarrow (\forall N \in \mathbb{Z}^+)(\forall c \in [0, 1])(\exists n \geq N)(\exists x_1, x_2 \in X)\|f^n(x_1) - f^n(x_2)\| > c\|x_1 - x_2\|

Let $N \in \mathbb{Z}^+$ and $c \in [0, 1)$. We must find $n \geq N$ and $x_1, x_2 \in X$ such that $\|f^n(x_1) - f^n(x_2)\| > c\|x_1 - x_2\|$. By assumption, we know that there exist $u_1, u_2 \in X$ such that $\|f(u_1) - f(u_2)\| > c^{\frac{1}{N+1}}\|u_1 - u_2\|$. Thus, $\|f^2(u_1) - f^2(u_2)\| > c^{\frac{2}{N+1}}\|u_1 - u_2\|$. By induction, we get $\|f^{N+1}(u_1) - f^{N+1}(u_2)\| > c\|u_1 - u_2\|$. Let $n = N + 1$ and $(x_1, x_2) = (u_1, u_2)$. Therefore, $f$ is not an eventual contraction on $X$ and our claim is proved.

Now, it is a well-known fact that if a function is Lipschitz on $X$, then it is uniformly continuous on $X$, which proves the lemma.

The next lemma shows how the eventual contraction property guarantees the existence and uniqueness of a fixed point.

**Lemma 5.1.3.** Let $X$ be a convex and compact subset of a Banach space. If $f : X \to X$ is an eventual contraction on $X$, then there exists a unique attractive fixed point of $f$ in $X$ with $X$ as the basin of attraction.

**Proof:**

Since $f$ is an eventual contraction on $X$, we know that it is uniformly continuous on $X$ by Lemma 5.1.2. Because $X$ is convex and compact, and $f$ is continuous on $X$, there exists $x^* \in X$ such that $f(x^*) = x^*$ by Theorem 2.4.3. Note that if $x^*$ is a fixed point for $f$, then it is also a fixed point for $f^n$ for any $n \in \mathbb{N}$. To show the uniqueness of $x^*$, suppose there exists another fixed point $x^{**}$ of $f$; then, $f(x^{**}) = x^{**}$, $f^2(x^{**}) = x^{**}$, ..., $f^N(x^{**}) = x^{**}$, where $N$ is the contractive composition number of $f$. Thus, $x^{**}$ is also a fixed point of $f^N$. But $f^N$ is a contraction and it has a unique fixed point by Theorem 2.4.2, it must be that $x^* = x^{**}$. Therefore, $x^*$ is unique.
To show that \( x^* \in X \) is attractive, we need to show

\[
(\forall x_0 \in X) \lim_{n \to \infty} \|f^n(x_0) - x^*\| = 0 \tag{5.4}
\]

Let \( x_0 \in X \). Note that \( f^n(x^*) = x^* \) for any \( n \in \mathbb{Z}^+ \). Let \( n = iN + j \), where \( j = n \mod N \), and \( i \to \infty \) as \( n \to \infty \). Let \( c \) denote the contraction constant for \( f^N \). Then, we have

\[
\lim_{n \to \infty} \|f^n(x_0) - x^*\| = \lim_{n \to \infty} \|f^n(x_0) - f^n(x^*)\| = \lim_{i \to \infty} \|f^{iN+j}(x_0) - f^{iN+j}(x^*)\| \tag{5.5}
\]

\[
= \lim_{i \to \infty} \|f^{iN+j}(x_0) - f^{iN+j}(x^*)\| \tag{5.6}
\]

\[
= \lim_{i \to \infty} \|(f^N)^i \circ f^j(x_0) - (f^N)^i \circ f^j(x^*)\| \tag{5.7}
\]

\[
\leq \lim_{i \to \infty} (c)^i \|f^j(x_0) - f^j(x^*)\| \tag{5.8}
\]

\[
= 0 \tag{5.9}
\]

because \( c \in [0, 1) \). Thus,

\[
\lim_{n \to \infty} \|f^n(x_0) - x^*\| = 0 \tag{5.10}
\]

This proves the attractiveness of \( x^* \) on \( X \). \( \square \)
5.2 Eventual Contraction for Multivariable Maps

In this section, we extend the eventual contraction definition to functions of more than one variable.

Definition 5.2.1. *Eventual Contraction (Multivariable)* \[14\]

Let $X, Y$ be subsets of a Banach space and $g : X \times Y \to Y$ be a function. The function $g$ is said to be an eventual contraction with respect to $Y$ uniformly over $X$ if there exist $M \in \mathbb{Z}^+$ and $c \in [0, 1)$ such that

$$
\left( \forall m \geq M \right) \left( \forall x_1, \ldots, x_m \in X \right) \left( \forall y_1, y_2 \in Y \right) \left\| \left( \prod_{i=1}^{m} g_{x_i} \right)(y_1) - \left( \prod_{i=1}^{m} g_{x_i} \right)(y_2) \right\| \leq c \left\| y_1 - y_2 \right\| 
$$

(5.11)

where $\prod_{i=1}^{m} g_{x_i} := g_{x_m} \circ \ldots \circ g_{x_1}$ and $g_{x_i} := g(x_i,.) : Y \to Y$ for all $i \in \{1, \ldots, m\}$. The number $M$ is called the contractive composition number of $g$ with respect to $Y$.

This definition can be paraphrased as follows: if $g : X \times Y \to Y$ is an eventual contraction with respect to $Y$ uniformly over $X$, there exists a positive integer $M$ such that if we pick $m$ arbitrary points $\{x_1, \ldots, x_m\}$ from $X$, where $m \geq M$, then the composition of the functions $g(x_1,.)$, $g(x_2,.)$, ... , $g(x_m,.)$ is a contraction. Again, the definition of eventual contraction has been modified from the definition in \[14\] to suit the context in this thesis.

5.2.1 Example

To illustrate the idea of eventually contractive functions with respect to one argument uniformly over another argument, consider the following example:

Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^n$ and $g : X \times Y \to Y$ be as follows:

$$
g(x, y) = Ay + x
$$

(5.12)
where $x \in X$, $y \in Y$ and $A$ is a $n \times n$ matrix such that all of its eigenvalues lie in the open unit circle. We want to show that $g$ is an eventual contraction with respect to $Y$ uniformly over $X$; i.e. we need to find a positive integer $M$ such that if we pick $m$ points $x_1, \ldots, x_m$ from $X_E$, where $m \geq M$, then the composition of the functions $g(x_1, \cdot) : Y \rightarrow Y, \ldots, g(x_m, \cdot) : Y \rightarrow Y$ is a contraction with respect to $Y$.

From matrix theory, we know that if a square matrix has all of its eigenvalues in the open unit disk, then there exists a positive integer $N$ such that the $k$-fold composition of that matrix is a contraction with respect to its argument, where $k \geq N$. Thus, we can conclude that

\[
(\exists N \in \mathbb{Z}^+)(\exists c \in [0, 1))\left(\forall k \geq N\right)\left(\forall y_1, y_2 \in \mathbb{R}^n\right)\|A^k y_1 - A^k y_2\| \leq c\|y_1 - y_2\| \quad (5.13)
\]

or $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an eventual contraction on $\mathbb{R}^n$ and $N$ is the contractive composition number of $A$. We claim that $g$ defined in (5.12) is an eventual contraction with respect to $Y$ uniformly over $X$ with $M = N$. Let $k \geq N$ and let us pick $k$ arbitrary points $x_1, x_2, \ldots, x_k$ from $X$. These points determine functions $g(x_1, \cdot) : Y \rightarrow Y, \ldots, g(x_k, \cdot) : Y \rightarrow Y$, where

\[
g(x_i, y) = Ay + x_i \quad (5.14)
\]

for all $i \in \{1, 2, \ldots, k\}$ and $y \in Y$. The composition of all these functions is equal to

\[
G_k(y) = g(x_k, \cdot) \circ \ldots \circ g(x_1, y)
\]

\[
= A\left(\ldots A\left(A y + x_1\right) + x_2\ldots\right) + x_{k-1} + x_k
\]

\[
= A^k y + \sum_{i=1}^{k} A^{k-i} x_i \quad (5.17)
\]
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Let \( y_1, y_2 \in Y \) be arbitrary. Then,

\[
\|G_k(y_1) - G_k(y_2)\| = \|A^k y_1 + \sum_{i=1}^{k} A^{k-1} x_i - A^k y_2 - \sum_{i=1}^{k} A^{k-1} x_i\| = \|A^k y_1 - A^k y_2\| \quad (5.18)
\]

Since we know that \( A \) is eventually contractive with contractive composition number \( N \) and \( k \geq N \), \( A^k \) is a contraction and \( \|A^k y_1 - A^k y_2\| \leq c\|y_1 - y_2\| \), where \( c \) is the contraction constant for \( A^k \). Thus, we have shown that the map \( g \) in (5.12) is an eventual contraction with respect to \( Y \) uniformly over \( X \).

5.3 Main Theorem

In this section, we state the main theorem of this chapter. Throughout this section, we take \( X_E \) and \( X_S \) to be subsets of a finite-dimensional Banach space with norm \( \| \cdot \| \), where \( X_E \) and \( X_S \) are the exosystem state set and the endosystem state set, respectively.

**Theorem 5.3.1.** Let \( (X_E, \alpha_E) \) be an exosystem such that \( \alpha_E \) is periodic with period \( N \in \mathbb{Z}^+ \). Let \( X_S \) be a convex and compact, and \( \alpha_S : X_E \times X_P \to X_P \) be an eventual contraction with respect to \( X_S \) uniformly over \( X_E \); then, there exists a unique function \( i_{ES} : X_E \to X_S \) such that

\[
\alpha_S \circ (id_E \times i_{ES}) = i_{ES} \circ \alpha_E \quad (5.19)
\]

and the image of \( i_{ES} \) is an attractor on \( X_S \).

**Proof:**
First, we prove the case when \( N = 1 \); i.e. the exosystem is a discrete integrator.

Let \( x \in X_E \). Since \( \alpha_S : X_E \times X_S \to X_S \) is an eventual contraction with respect to \( X_S \) uniformly over \( X_E \), there exists a positive integer \( M \) such that if we pick \( n \) elements \( x_1, \ldots, x_n \) from \( X_E \), where \( n \geq M \), then the composition of \( \alpha_S(x_1,.) : X_S \to X_S, \ldots, \alpha_S(x_n,.) : X_S \to X_S \) is a contraction. Let \( n = M \) and choose \( x_1 = x, \ldots, x_n = x \). Thus, \( \alpha_S(x,.) : X_S \to X_S \) is an eventual contraction with respect to \( X_S \) and pick \( \tilde{y} \in X_S \) with basin of attraction \( X_S \). Define \( i_{ES} : X_E \to X_S \) such that \( i_{ES}(x) = \tilde{y} \). This definition is unambiguous because for any \( x \in X_E \), \( \alpha_S(x,.) : X_S \to X_S \) is an eventual contraction and thus there is a unique fixed point of \( \alpha_S(x,.) \) in \( X_S \). Now, \( \alpha_S(x,i_{ES}(x)) = \alpha_S(x,\tilde{y}) = \tilde{y} = i_{ES}(x) = i_{ES} \circ \alpha_E(x) \), where \( \alpha_E = id_E \). Thus, \( i_{ES} \) satisfies (5.19).

Suppose there exists another function \( \tilde{i}_{ES} : X_E \to X_S \) that satisfies (5.19); then for \( x \in X_E \), \( \alpha_S(x,\tilde{i}_{ES}(x)) = \tilde{i}_{ES} \circ \alpha_E(x) = \tilde{i}_{ES}(x) \). Thus, \( \tilde{i}_{ES}(x) \) is a fixed point of \( \alpha_S(x,.) \). But the fixed point of \( \alpha_S(x,.) \) is unique by Lemma 5.1.3 and is given by \( i_{ES}(x) \). Therefore, \( \tilde{i}_{ES}(x) = i_{ES}(x) \) and \( i_{ES} \) is unique.

The attractiveness of \( i_{ES}(X_E) \subseteq X_S \) follows directly from the attractiveness of each fixed point of the \( \alpha_S(x,.) \) maps, where \( x \in X_E \).

Next, we prove the case when \( N > 1 \). Given that \( \alpha_S : X_E \times X_S \to X_S \) is an eventual contraction with respect to \( X_S \) uniformly over \( X_E \), we let \( M \in \mathbb{Z}^+ \) be its contractive composition number.

**Case 1: \( N \geq M \).** Let \( x \in X_E \). Let \( n = N \) and pick \( x_1 = x, x_2 = \alpha_E(x), \ldots, x_n = \alpha_E^{N-1}(x) \); then, the functional composition \( \alpha_S(x_n,.) \circ \ldots \circ \alpha_S(x_2,.) \circ \alpha_S(x_1,.) = \alpha_S(\alpha_E^{N-1}(x),.) \circ \ldots \circ \alpha_S(\alpha_E(\ldots,.) \circ \alpha_S(x,.) = \alpha_S[x]_1^N : X_S \to X_S \) is a contraction and it has a unique fixed point \( \bar{y} \in X_S \) that is attractive on \( X_S \). Let \( i_{ES}(x) = \bar{y} \); the validity of this function can be shown similarly to the discrete integrator case. Now, we need to
show that this $i_{ES} : X_E \rightarrow X_S$ satisfies (5.19).

$$\alpha_S(x, i_{ES}(x)) = \alpha_S(x_1, i_{ES}(x)) = \alpha_S(x_1, \cdot) \circ \alpha_S(x_n, \cdot) \circ \ldots \circ \alpha_S(x_2, \cdot) \circ \alpha_S(x_1, i_{ES}(x)) \quad (5.20)$$

Note that if we pick $\alpha_E(x) \in X_E$ as our initial exosystem state, then the $n = N$ points we pick from $X_E$ according to the definition of $i_{ES}$ are $\bar{x}_1 = \alpha_E(x)$, $\bar{x}_2 = \alpha_E^2(x)$, ..., $\bar{x}_{N-1} = \alpha_E^{N-1}(x)$, $\bar{x}_N = \alpha_E^N(x) = x$. Thus, $i_{ES}(\alpha_E(x))$ is the unique fixed point of the functional composition $\alpha_S[\alpha_E(x)]_1^N : X_S \rightarrow X_S$, which is equivalent to the function $\alpha_S(x, \cdot) \circ \ldots \circ \alpha_S(\alpha_E(x), \cdot) : X_S \rightarrow X_S$ and $\alpha_S(x_1, \cdot) \circ \alpha_S(x_n, \cdot) \circ \ldots \circ \alpha_S(x_2, \cdot) : X_S \rightarrow X_S$. Notice that from (5.20), $\alpha_S(x_1, i_{ES}(x))$ is a fixed point of $\alpha_S(x_1, \cdot) \circ \alpha_S(x_n, \cdot) \circ \ldots \circ \alpha_S(x_2, \cdot) : X_S \rightarrow X_S$ is unique and equals $i_{ES}(\alpha_E(x))$. Thus, we have $\alpha_S(x, i_{ES}(x)) = i_{ES} \circ \alpha_E(x)$.

The uniqueness of $i_{ES} : X_E \rightarrow X_S$ and the attractiveness of $i_{ES}(X_E)$ follows by the same proof as for Theorem 4.2.1.

**Case 2:** $N < M$. There exists $l \in \mathbb{Z}^+$ such that $lN \geq M$. Let $n = lN$ and $x \in X_E$. The sequence of $n$ points from $X_E$ will be $x_1 = x$, ..., $x_N = \alpha_E^{N-1}(x)$, $x_{N+1} = x$, ..., $x_{2N} = \alpha_E^{N-1}(x)$, $x_{2N+1} = x$, ..., $x_{lN} = \alpha_E^{N-1}(x)$. The composition of the functions by this list of $n$ points in $X_E$ is

$$\alpha_S(x_n, \cdot) \circ \ldots \circ \alpha_S(x_1, \cdot) = \alpha_S[x]_1^{lN} = (\alpha_S[x]_1^N)^l \quad (5.21)$$

which is a contraction on $X_S$ since $lN \geq M$. Thus, the above function has a unique attractive fixed point in $X_S$, say $\bar{y}$. Define $i_{ES}(x) = \bar{y}$. Again, this definition is unambiguous for a similar reason to the discrete integrator case. This function also satisfies (5.19) with a proof similar to the case when $N \geq M$ with $\alpha_S[x]_1^N$ and $\alpha_S[\alpha_E(x)]_1^N$ replaced by $(\alpha_S[x]_1^N)^l$ and $(\alpha_S[\alpha_E(x)]_1^N)^l$.

The uniqueness and convergence to $i_{ES}(X_E)$ also follow the same argument as for the
Therefore, there exists a unique $i_{ES} : X_E \rightarrow X_S$ such that (5.19) holds and its image is an attractor on $X_S$ for all cases. 

\[ N \geq M. \]

\[ \therefore \exists \text{unique } i_{ES} : X_E \rightarrow X_S \text{ such that } (5.19) \text{ holds and its image is an attractor on } X_S \text{ for all cases.} \]

\[ \square \]

### 5.4 Continuity of $i_{ES}$

The continuity of the $i_{ES}$ function defined in this chapter can also be guaranteed by the continuity of $\alpha_S : X_E \times X_S \rightarrow X_S : (x, y) \mapsto \alpha_S(x, y)$ with respect to $x \in X_E$ and the continuity of $\alpha_E : X_E \rightarrow X_E$. This can be proved similarly to the proof of Lemma 4.3.1.

\[ \text{Lemma 5.4.1. Let the assumptions on } \alpha_E \text{ and } \alpha_S \text{ in Theorem 5.3.1 hold. Suppose further that } \alpha_E : X_E \rightarrow X_E \text{ is continuous and } \alpha_S : X_E \times X_S \rightarrow X_S : (x_E, x_S) \mapsto \alpha_S(x_E, x_S) \text{ is continuous with respect to } x_E; \text{ then, the resulting } i_{ES} : X_E \rightarrow X_S \text{ from Theorem 5.3.1 is continuous.} \]

\[ \text{Proof:} \]

First, we choose a large enough $l \in \mathbb{Z}^+$ such that $lN \geq M$, where $N$ is the periodicity of the exosystem and $M$ is the contractive composition number of $\alpha_S$. Then, for any $\bar{x} \in X_E$,
\[ \| i_{ES}(x) - i_{ES}(\bar{x}) \| = \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(x) - \left( \alpha_S[\bar{x}]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| \] (5.22)

\[ = \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(x) - \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| 
+ \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(\bar{x}) - \left( \alpha_S[\bar{x}]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| \] (5.23)

\[ \leq \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(x) - \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| 
+ \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(\bar{x}) - \left( \alpha_S[\bar{x}]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| \] (5.24)

\[ \leq c \left\| i_{ES}(x) - i_{ES}(\bar{x}) \right\| + \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(\bar{x}) - \left( \alpha_S[\bar{x}]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| \] (5.25)

where \( c \in [0, 1) \) is the contraction constant. Moving \( c \left\| i_{ES}(x) - i_{ES}(\bar{x}) \right\| \) to the left-hand side, we get

\[ (1 - c) \left\| i_{ES}(x) - i_{ES}(\bar{x}) \right\| \leq \left\| \left( \alpha_S[x]_1^N \right)^I \circ i_{ES}(\bar{x}) - \left( \alpha_S[\bar{x}]_1^N \right)^I \circ i_{ES}(\bar{x}) \right\| \] (5.26)

Taking the limit as \( x \to \bar{x} \) on both sides, we conclude similarly to Lemma 4.3.1 that

\[ \lim_{x \to \bar{x}} \left\| i_{ES}(x) - i_{ES}(\bar{x}) \right\| = 0 \] (5.27)

from the assumptions of the lemma.
5.5 An Example of Nonlinear Eventual Contraction

Consider the following weakly nonlinear example, which illustrates that Theorem 5.3.1 can be applied to a non-empty class of nonlinear dynamical systems.

Let $X_E \subseteq \mathbb{R}$ and $\alpha_E : X_E \rightarrow X_E$. Let the exosystem $(X_E, \alpha_E)$ be periodic with period $N \in \mathbb{Z}^+$ and $\alpha^N_E = id_E$. Let $X_S$ be a convex and compact subset of $\mathbb{R}^2$. Let $\alpha_S : X_E \times X_S \rightarrow X_S : (x, y) \mapsto \alpha_S(x, y)$, where $x \in X_E$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in X_S$ be as follows:

$$\alpha_S(x, y) = \begin{bmatrix} \frac{1}{2}y_1 + f(y_2) + x \\ \frac{1}{2}y_2 \end{bmatrix} =: F(x, y)$$

or equivalently,

$$y(t + 1) = F(x(t), y(t)) = \alpha_S(x(t), y(t)) = \begin{bmatrix} \frac{1}{2}y_1(t) + f(y_2(t)) + x(t) \\ \frac{1}{2}y_2(t) \end{bmatrix}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function that satisfies the following properties:

- $f$ is differentiable on $\mathbb{R}$.
- $(\exists c_1, c_2 > 0)(\forall x \in \mathbb{R})c_1 \leq \frac{df}{dx}(x) \leq c_2$

There are infinitely many nonlinear functions that satisfy the above properties. For example, $f(x) = x + \frac{1}{4}\sin(x)$ is differentiable on $\mathbb{R}$ and its derivative is bounded by $\frac{3}{4}$ and $\frac{5}{4}$. It is worthwhile to note that the above conditions on $f : \mathbb{R} \rightarrow \mathbb{R}$ do not require $f(0) = 0$.

Since we are interested in eventual contractions that are not strict contractions with respect to $X_S$ for any $x \in X_E$, we must show this is not a trivial example; i.e. we must show that there exists $x \in X_E$ such that for all $c \in [0, 1)$, there exist $y, z \in X_S$
such that \( \| \alpha_S(x, y) - \alpha_S(x, z) \| > c \| y - z \| \). Let \( y = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \), \( z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), so \( \| y - z \| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

But \( \alpha_S(x, y) - \alpha_S(x, z) = \begin{bmatrix} f(2) - f(1) \\ 1 \end{bmatrix} \) for any \( x \in X_E \). Since the derivative of \( f \) is lower bounded by a positive number, \( f \) is strictly increasing and \( f(2) - f(1) > 0 \). Therefore \( \| \begin{bmatrix} f(2) - f(1) \\ 1 \end{bmatrix} \| \geq \| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \| \) using any norm, which leads to \( \| \begin{bmatrix} f(2) - f(1) \\ 1 \end{bmatrix} \| > c \| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \| \) for any \( c \in [0, 1) \). This proves that (5.29) is not a contraction with respect to \( X_S \).

Knowing that (5.29) is not a contraction with respect to \( X_S \), our goal is to show that it is an eventual contraction with respect to \( X_S \) over \( X_E \). To show this, we need the following lemmas:

**Lemma 5.5.1.** If \( c_1 \leq \frac{df}{dx}(x) \leq c_2 \) for all \( x \in \mathbb{R} \), then \( c_1|x_1 - x_2| \leq |f(x_1) - f(x_2)| \leq c_2|x_1 - x_2| \) for all \( x_1, x_2 \in \mathbb{R} \).

**Proof:**
Without loss of generality, we assume \( x_1 > x_2 \) and integrate the inequality \( c_1 \leq \frac{df}{dx}(x) \leq c_2 \) from \( x_2 \) to \( x_1 \). Using the Fundamental Theorem of Calculus, the result is obvious. \( \square \)

We will use the infinity norm \( \| . \|_\infty \) on \( \mathbb{R}^2 \) to prove that (5.29) is an eventual contraction with respect to \( X_S \) over \( X_E \); i.e. \( \| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \|_\infty = \max\{ |x_1|, |x_2| \} \). For simplicity, we will still use the \( \| . \| \) to denote \( \| . \|_\infty \).

Note that we can find a closed form solution for (5.29) given the initial condition \( y(0) \).
\[ y_0 = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} \] as follows:

\[
y(t) = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} + \sum_{i=1}^{t} \frac{1}{2^{i-1}} f \left( \frac{y_{20}}{2^{i-1}} \right) + \sum_{j=1}^{t} \frac{1}{2^{j-1}} x(j - 1) \]

Using the closed form above, we can prove the next lemma.

**Lemma 5.5.2.** \((\exists T \in \mathbb{N})(\forall t \geq T)(\forall y_0, z_0 \in X_S)||y(t) - z(t)|| = |y_1(t) - z_1(t)|\)

**Proof:**

Since \(\|x(t) - y(t)\| = \max\{|x_1(t) - y_1(t)|, |x_2(t) - y_2(t)|\}\), it suffices to show that

\[
\lim_{t \to \infty} \frac{|y_2(t) - z_2(t)|}{|y_1(t) - z_1(t)|} = 0
\]

To show the equation above, we use the closed form solution of (5.29):

\[
\frac{|y_2(t) - z_2(t)|}{|y_1(t) - z_1(t)|} = \frac{|(y_{20} - z_{20})|}{|y_{10} + \sum_{i=1}^{t} \frac{1}{2^{i-1}} f \left( \frac{y_{20}}{2^{i-1}} \right) - \sum_{i=1}^{t} \frac{1}{2^{i-1}} f \left( \frac{z_{20}}{2^{i-1}} \right)|}
\]

Notice that the summation term involving \(x(j - 1)\) in (5.30) is common to both \(y_1(t)\) and \(z_1(t)\) so it is cancelled by taking their difference. After taking out the common factor \(2^{-t}\) from both the numerator and denominator, we obtain

\[
\frac{|y_2(t) - z_2(t)|}{|y_1(t) - z_1(t)|} = \frac{|y_{20} - z_{20}|}{|(y_{10} - z_{10}) + \sum_{i=1}^{t} 2^i \left( f \left( \frac{y_{20}}{2^{i-1}} \right) - f \left( \frac{z_{20}}{2^{i-1}} \right) \right)|}
\]

Since the numerator in the equation above is a constant that is independent of the time variable \(t\), the limit as \(t \to \infty\) of the quotient above is zero if the sum \(\sum_{i=1}^{t} 2^i \left( f \left( \frac{y_{20}}{2^{i-1}} \right) - f \left( \frac{z_{20}}{2^{i-1}} \right) \right)\) diverges as \(t \to \infty\).

Recall that \(f : \mathbb{R} \to \mathbb{R}\) is a strictly increasing function because of the positive lower bound on its derivative. Thus, if \(y_{20} > z_{20}\), \(\frac{y_{20}}{2^{i-1}} > \frac{z_{20}}{2^{i-1}}\) for all \(i\), and \(f \left( \frac{y_{20}}{2^{i-1}} \right) > f \left( \frac{z_{20}}{2^{i-1}} \right)\)
for all $i$. Similarly, for the case when $y_{20} < z_{20}$, we get $f\left(\frac{y_{20}}{2^{i-1}}\right) < f\left(\frac{z_{20}}{2^{i-1}}\right)$ for all $i$. Thus, the terms $f\left(\frac{y_{20}}{2^{i-1}}\right) - f\left(\frac{z_{20}}{2^{i-1}}\right)$ all have the same sign for any $i$. Therefore,

$$|\sum_{i=1}^{t} 2^i \left(f\left(\frac{y_{20}}{2^{i-1}}\right) - f\left(\frac{z_{20}}{2^{i-1}}\right)\right)| = \sum_{i=1}^{t} 2^i |f\left(\frac{y_{20}}{2^{i-1}}\right) - f\left(\frac{z_{20}}{2^{i-1}}\right)|$$

(5.34)

Using Lemma 5.5.1, we know that

$$\sum_{i=1}^{t} 2^i |f\left(\frac{y_{20}}{2^{i-1}}\right) - f\left(\frac{z_{20}}{2^{i-1}}\right)| \geq \sum_{i=1}^{t} 2^i c_1 |\frac{y_{20}}{2^{i-1}} - \frac{z_{20}}{2^{i-1}}| = \sum_{i=1}^{t} 2c_1 |y_{20} - z_{20}|$$

(5.35)

where $c_1 > 0$ is the lower bound of the derivative of $f$. Since the terms in the summation are independent of $i$, the expression above equals $2tc_1|y_{20} - z_{20}|$, which goes to infinity as $t \to \infty$. Thus, Lemma 5.5.2 is proved.

The next lemma shows that the infinity norm of the difference $y(t) - z(t)$ will converge to zero as $t \to \infty$ for any $y_0, z_0 \in X_S$.

**Lemma 5.5.3.** For all $y_0, z_0 \in X_S$,

$$\lim_{t \to \infty} |y_1(t) - z_1(t)| = 0$$

(5.36)

**Proof:**

In this proof, we will use the upper bound $c_2 > 0$ on the derivative of $f$.

$$|y_1(t) - z_1(t)| = \frac{1}{2^t} \left|(y_{10} - z_{10}) + \sum_{i=1}^{t} 2^i \left(f\left(\frac{y_{20}}{2^{i-1}}\right) - f\left(\frac{z_{20}}{2^{i-1}}\right)\right)\right|$$

(5.37)

Using the triangle inequality, we obtain

$$|y_1(t) - z_1(t)| \leq \frac{1}{2^t} \left(|y_{10} - z_{10}| + \sum_{i=1}^{t} 2^i |f\left(\frac{y_{20}}{2^{i-1}}\right) - f\left(\frac{z_{20}}{2^{i-1}}\right)|\right)$$

(5.38)
Now, using the derivative upper bound part of Lemma (5.5.1), we get

\[ |y_1(t) - z_1(t)| \leq \frac{1}{2^t} \left( |y_{10} - z_{10}| + \sum_{i=1}^{t} 2^i c_2 \left| \frac{y_{20}}{2^{i-1}} - \frac{z_{20}}{2^{i-1}} \right| \right) \]  (5.39)

The right hand side of the above inequality can be simplified to

\[ \frac{1}{2^t} \left( |y_{10} - z_{10}| + \sum_{i=1}^{t} 2^i c_2 \left| \frac{y_{20}}{2^{i-1}} - \frac{z_{20}}{2^{i-1}} \right| \right) = \frac{|y_{10} - z_{10}| + 2c_2 |y_{20} - z_{20}| t}{2^t} \]  (5.40)

which goes to zero as \( t \to \infty \). Therefore, \( |y_1(t) - z_1(t)| \to 0 \) as \( t \to \infty \).

From Lemma (5.5.2) we know \( (\exists N_1 \in \mathbb{N}) (\forall t \geq N_1) (\forall y_0, z_0 \in X_S) \| y(t) - z(t) \| = |y_1(t) - z_1(t)| \). Let \( c \in [0, 1) \) be arbitrary; then we know \( (\exists N_2 \in \mathbb{N}) (\forall t \geq N_2) (\forall y_0, z_0 \in X_S) |y_1(t) - z_1(t)| \leq c \| y_0 - z_0 \| \) from Lemma (5.5.3) and the definition of limit. Pick \( N = \max\{N_1, N_2\} \), then

\[ (\forall t \geq N)(\forall y_0, z_0 \in X_S) \| y(t) - z(t) \| = |y_1(t) - z_1(t)| \leq c \| y_0 - z_0 \| \]  (5.41)

Thus, \( \{5.29\} \) is an eventual contraction with respect to \( X_S \) uniformly over \( X_E \) and we have shown that this example is a valid example for which to apply Theorem (5.3.1).
Chapter 6

Output Regulation

In the last two chapters, we discussed conditions that led to the existence and uniqueness of an attractor within the endosystem state set $X_S$. We did not, however, consider whether the total system satisfies the asymptotic regulation property. The asymptotic regulation property of the total system is equivalent to saying that the attractor on $X = X_E \times X_S$ is a subset of a given regulation set $K$. In this chapter, we will provide conditions under which the regulation property of the total system holds.

6.1 Superset of Attractor

Let $X_E, X_S$ be subsets of a Banach space. Recall from the last chapter that if the exosystem $(X_E, \alpha_E)$ is periodic and the endosystem transition function $\alpha_S : X_E \times X_S \to X_S$ is an eventual contraction with respect to $X_S$ over $X_E$, then there exists a unique function $i_{ES} : X_E \to X_S$ such that $\alpha_S \circ (id_E \times i_{ES}) = i_{ES} \circ \alpha_E$ and $i_{ES}(X_E)$ is an attractor in $X_S$. Define $i_E := id_E \times i_{ES}$. Then $i_E(X_E) \subseteq X$ is an attractor in $X = X_E \times X_S$. We state a key lemma in this section to characterize any superset of $i_E(X_E)$.

**Lemma 6.1.1.** Let $V \subseteq X$. If $V$ satisfies

- $P_E(V) = X_E$, where $P_E : X \to X_E$ is the projection of $X$ onto $X_E$. 


• \( \alpha(V) \subseteq V \), or \( V \) is \( \alpha \)-invariant, where \( \alpha = \alpha_E \times \alpha_S \).

• \( V \) is closed

then, \( i_E(X_E) \subseteq V \).

**Proof:**

Suppose \( i_E(X_E) \) is not a subset of \( V \). Then, there exists \( w \in i_E(X_E) \) such that \( w \notin V \).

Because \( w \in i_E(X_E) \), there exists \( x_0 \in X_E \) such that \( w = i_E(x_0) = (x_0, i_{ES}(x_0)) \).

Since \( P_E(V) = X_E \) by assumption, there exists \( y_0 \in X_S \) such that \( (x_0, y_0) \in V \). Let \( v_0 := (x_0, y_0) \). Because \( \alpha(V) \subseteq V \) by assumption, we know \( \alpha^k(V) \subseteq V \) for all \( k \in \mathbb{N} \). In particular, \( \alpha^N(V) \subseteq V \), where \( N \) is the period of \( \alpha_E \). Let \( v(t) := \alpha^t(v_0) \). Then,

\[
v(t) = (x(t), y(t)) = (\alpha_E^t(x_0), \alpha_S^t[x_0]_1(y_0))
\]

where,

\[
\alpha_S^t[x_0]_1(y_0) = \alpha_S^t(\alpha_E^{t-1}(x_0), \alpha_S(\alpha_E^{t-2}(x_0), \alpha_S(..., \alpha_S(\alpha_E(x_0), \alpha_S(x_0, y_0))...))
\]

as in previous chapters. Now, for \( i \in \mathbb{N} \),

\[
v(iN) = (x(iN), y(iN)) = (\alpha_E^{iN}(x_0), \alpha_S^{iN}[x_0]_1(y_0)) = (x_0, \alpha_S^{[N]}_1[x_0]^{i}(y_0))
\]

where the last step is a consequence of (4.13). Define a sequence \( \{v_i\}_{i=0}^{\infty} \), where \( v_i := \)
Let \( v(iN) \); then all \( v_i \in V \) by the \( \alpha \)-invariance of \( V \). Since \( \alpha_S[x_0]|_1^N \) is a contraction or an eventual contraction, \( \left( \alpha_S[x_0]|_1^N \right)^i(y_0) \) has a limit which equals \( i_{ES}(x_0) \) by the definition of \( i_{ES} \) in the last two chapters. Thus, the limit of the sequence \( \{v_i\}_{i=0}^\infty \) exists and equals \( (x_0, i_{ES}(x_0)) \), which is equal to \( w \). By the assumption that \( V \) is closed, the sequence \( \{v_i\}_{i=0}^\infty \) will have its limit in \( V \). Therefore, \( w \in V \), which is a contradiction to \( w \notin V \).

We conclude that \( i_E(X_E) \subseteq V \).

The importance of this lemma is as follows. Given any regulation set \( K \subseteq X \), if we can find a subset of \( V \subseteq K \) such that \( V \) satisfies all the assumptions listed in the lemma above, then we conclude that \( i_E(X_E) \subseteq K \). Therefore, the total system state converges to the regulation set and the regulation problem is solved.

### 6.2 Sufficient Condition for Output Regulation

Let \( X_E, X_P \) and \( X_C \) be subsets of a Banach space. Consider the system

\[
\begin{align*}
\alpha_E &: X_E \to X_E \\
\alpha_P &: X_E \times X_P \times X_C \to X_P \\
\alpha_I &: X_C \to X_C
\end{align*}
\]

(6.7)

where \( \alpha_E \) is a periodic exosystem with period \( N \in \mathbb{Z}^+ \). \( \alpha_I : X_C \to X_C \) is a “copy” of \( \alpha_E \) such that there exists a continuous and bijective function \( \delta_c \) such that \( \alpha_I \circ \delta_c = \delta_c \circ \alpha_E \).

Thus, \( \alpha_I : X_C \to X_C \) is a periodic function with period \( N \) as well.

Suppose we make an additional assumption that the plant is pre-stabilized with re-
spect to its own state, meaning that \( \alpha_P \) is a contraction with respect to \( X_P \) over \( X_E \times X_C \).

Then, by Theorem 4.2.1 there exists a unique function \( h : X_E \times X_C \to X_P \) such that

\[
\left( \forall (x, z) \in X_E \times X_C \right) \alpha_P(x, h(x, z), z) = h(\alpha_E(x), \alpha_I(z)) \tag{6.8}
\]

and \( h(X_E \times X_C) \) is an attractor in \( X_P \). Suppose further that \( \alpha_P : X_E \times X_P \times X_C \to X_C \) is continuous with respect to \( (x, y, z) \in X_E \times X_P \times X_C \). Then, by Lemma 4.3.1 we know that \( h : X_E \times X_C \) is continuous with respect to \( (x, z) \in X_E \times X_C \). Let \( \delta_p : X_E \to X_P \) be defined as \( \delta_p(x) := h(x, \delta_c(x)) \). Since \( \delta_c : X_E \to X_C \) is continuous, \( \delta_p : X_E \to X_P \) is continuous as well. Now, the endosystem state evolution can be described by the following equations.

\[
\alpha_P\left(x, \delta_p(x), \delta_c(x)\right) = \delta_p \circ \alpha_E(x) \\
\alpha_I \circ \delta_c(x) = \delta_c \circ \alpha_E(x) \tag{6.9}
\]

Define \( \delta := id_E \times \delta_p \times \delta_c : X_E \to X_E \times X_P \times X_C \). Since a Cartesian product of continuous functions is still continuous, \( \delta : X_E \to X_E \times X_P \times X_C \) is continuous. The set \( \delta(X_E) := \{(x, \delta_p(x), \delta_c(x)) | x \in X_E\} \) is an \( \tilde{\alpha} \)-invariant set on \( X = X_E \times X_P \times X_C \), where \( \tilde{\alpha} = \alpha_E \times \alpha_P \times \alpha_I \). Thus, the internal model and the exosystem together induce an invariant subset on \( X = X_E \times X_P \times X_C \). Notice that \( P_E \circ \delta(X_E) = X_E \) from the definition of \( \delta(X_E) \). Suppose further that \( X_E \) is a compact set. Then, \( \delta(X_E) \) is compact by Theorem 2.1.16 Since compact sets are closed, we conclude that \( \delta(X_E) \) is closed. Therefore, \( \delta(X_E) \) satisfies all three requirements in Lemma 6.1.1. Thus, if we can make the endosystem \( \alpha_P \times \alpha_C \) a contraction with respect to \( X_P \times X_C \) uniformly over \( X_E \), then the attractor \( i_E(X_E) \subseteq \delta(X_E) \).

Suppose we let the controller transition function \( \alpha_C : X_E \times X_P \times X_C \to X_C \) be designed such that it has the following properties
Chapter 6. Output Regulation

• Autonomy on the Regulation Set

\[
\left(\forall (x, y, z) \in X\right) \alpha_C(x, y, z) = \alpha_I(z) \iff (x, y, z) \in K
\]  \tag{6.10}

• Contractive Endosystem: \(\alpha_P \times \alpha_C\) is a contraction with respect to \(X_P \times X_C\) uniformly over \(X_E\).

In other words, \((6.10)\) means the controller is autonomous if and only if the total system is in the regulation set. By the second property above, there exist unique functions \(i_{EP} : X_E \to X_P\), \(i_{EC} : X_E \to X_C\) such that

\[
\alpha_P\left(x, i_{EP}(x), i_{EC}(x)\right) = i_{EP} \circ \alpha_E(x)
\]

\[
\alpha_C\left(x, i_{EP}(x), i_{EC}(x)\right) = i_{EC} \circ \alpha_E(x)
\]  \tag{6.11}

and \(i_E(X_E) = \{(x, i_{EP}(x), i_{EC}(x))| x \in X_E\}\) is an attractor on \(X_P \times X_C\). Now, because the controller will not be autonomous and follow \(\alpha_I : X_C \to X_C\) unless it is in the regulation set; i.e. the “autonomy on the regulation set” property \((6.10)\) and the “contractive endosystem” property “force” the set \(\delta(X_E)\) to be a subset of the regulation set. Now, since \(i_E(X_E) \subseteq \delta(X_E)\), we conclude \(i_{ES}(X_E) \subseteq K\). Since the attractor is in \(K\), regulation will be achieved as \(t \to \infty\).

The above results can be similarly derived from the eventual contractive case on Banach spaces as well.
Chapter 7

Iterative Scheme for Internal Model

From previous chapters, we saw the necessity of internal model for an output regulation problem. In many real examples, however, a suitable internal model and a feedback structure are not always explicitly available to the designers due to nonlinearities. In this chapter, we will present an iterative method to determine a satisfactory controller structure for output regulation problems which contains a “correct” internal model. What we mean by a correct internal model is that it is an isomorphic copy of the exosystem; i.e. for an exosystem $(X_E, \alpha_E)$ and an internal model $(X_C, \alpha_I)$, there exists a bijection $t : X_E \rightarrow X_C$ such that $\alpha_I \circ t = t \circ \alpha_E$. This algorithm starts with an initial guess of the internal model that is “reasonably” close to a correct internal model. In Section 1, we will present this iterative scheme. In Section 2, an example to illustrate the proposed scheme will be presented.

7.1 Iterative Method for Internal Model

In this section, an iterative method to determine a suitable internal model will be proposed for output regulation problems.

Iterative Method for Internal Model:
Let \( X_E, X_P \) be subsets of a Banach space and \( X_P \) be convex and compact. Let \((X_E, \alpha_E)\) be an exosystem and \((X_P, \alpha_P)\), where \( \alpha_P : X_E \times X_P \times X_C \rightarrow X_P \), be a plant. Suppose we are given a regulation subset \( K \subseteq X := X_E \times X_P \times X_C \). Then, we conjecture that the following iterative procedure will converge to a satisfactory internal model.

**Step 1** Make an initial guess of internal model and denote it as \( \alpha_I^0 : X_C \rightarrow X_C \).

**Step 2** Construct a controller state transition function \( \alpha_C^0 : X \rightarrow X_C \) such that it has the following properties

- Autonomy on Regulation Set: \( \forall (x, y, z) \in X \), \( \alpha_C^0(x, y, z) = \alpha_I^0(z) \iff (x, y, z) \in K \)

- Endosystem Stability: \( \alpha_P \times \alpha_C : X \rightarrow X_P \times X_C \) is an eventual contraction with respect to \( (y, z) \in X_P \times X_C \) uniformly over \( x \in X_E \)

**Step 3** Solve for \( i_E^0 : X_E \rightarrow X \) such that

\[
\alpha^0 \circ i_E^0 = i_E^0 \circ \alpha_E
\]  

where \( \alpha^0 = \alpha_E \times \alpha_P \times \alpha_C^0 \).

3.1 If \( i_E^0(X_E) \subseteq K \), then the internal model candidate \( \alpha_I^0 \) is valid, and exit.

3.2 Else, compute \( \tilde{\gamma}_E^0 = \gamma \circ i_E^0 \).

**Step 4** Solve (if possible) the following equation

\[
\alpha_I^1 \circ \tilde{\gamma}_E^0 = \tilde{\gamma}_E^0 \circ \alpha_E
\]  

for \( \alpha_I^1 : X_C \rightarrow X_C \).

**Step 5** Let \( \alpha_I^1 : X_C \rightarrow X_C \) be the new internal model guess and repeat the procedure from Step 2. Repeat this process until \( i_E^i(X_E) \subseteq K \) for some \( i \in \mathbb{N} \).
7.2 Example

In this section, an example illustrating the aforementioned iterative scheme will be shown.

Let \( X_E = \mathbb{R}^2 \) and \( X_P = \mathbb{R} \). Choose \( X_C = X_E \) and let \( X = X_E \times X_P \times X_C \). Let \( x_E = [x_{E1}, x_{E2}]^T \in X_E \), \( x_P \in X_P \) and \( x_C = [x_{C1}, x_{C2}]^T \in X_C \), where \( v^T \) denotes the transpose of a given vector \( v \). Let the transition functions of the exosystem and the plant be as follows.

\[
\alpha_E(x_E) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_E \quad (7.3)
\]

\[
\alpha_P(x_E, x_P, x_C) = \frac{1}{4} x_P + x_C \quad (7.4)
\]

Let the regulation set be

\[
K := \left\{ [x_{E1}, x_{E2}, x_P, x_{C1}, x_{C2}]^T \in X \mid x_{E1} = x_P \right\} \quad (7.5)
\]

Thus, the regulation objective is to make \( x_P \) “follow” \( x_{E1} \). Now, we make an initial guess of the internal model to be

\[
\alpha_0^I(x_C) = \begin{bmatrix} 0 & \frac{9}{10} \\ -\frac{9}{10} & 0 \end{bmatrix} x_C \quad (7.6)
\]

To construct a controller that is autonomous on the regulation set and provides stability to the endosystem, the controller transition function must have the following form
Chapter 7. Iterative Scheme for Internal Model

\[
\alpha_C^0(x_E, x_P, x_C) = \begin{bmatrix} -k_1 & 0 & k_1 & 0 & \frac{9}{10} \\ -k_2 & 0 & k_2 & -\frac{9}{10} & 0 \end{bmatrix} x_C
\]  

(7.7)

for some appropriate parameters \(k_1, k_2 \in \mathbb{R}\). Let \(x_S := [x_P, x_{C1}, x_{C2}]^T\). This means that \(k_1, k_2\) must be chosen such that

\[
\alpha_S(x_E, x_S) := (\alpha_P \times \alpha_C)(x_E, x_P, x_C)
\]

(7.8)

\[
= \begin{bmatrix} \frac{1}{4} x_P + x_{C1} \\ k_1(x_P - x_{E1}) + \frac{9}{10} x_{C2} \\ k_2(x_P - x_{E1}) - \frac{9}{10} x_{C1} \end{bmatrix}
\]

(7.9)

\[
= \begin{bmatrix} \frac{1}{4} & 1 & 0 \\ k_1 & 0 & \frac{9}{10} \\ k_2 & -\frac{9}{10} & 0 \end{bmatrix} x_S + \begin{bmatrix} 0 & 0 \\ -k_1 & 0 \\ -k_2 & 0 \end{bmatrix} x_E
\]

(7.10)

is an eventual contraction with respect to \(x_S\) uniformly over \(x_E \in X_E\). As shown in Section 5.2.1, a sufficient condition for (7.10) to be an eventual contraction with respect to \(x_S\) uniformly over \(x_E\) is that the matrix

\[
A_S := \begin{bmatrix} \frac{1}{4} & 1 & 0 \\ k_1 & 0 & \frac{9}{10} \\ k_2 & -\frac{9}{10} & 0 \end{bmatrix}
\]

(7.11)

has all its eigenvalues within the open unit disk, or is Schur stable. Let us call \(A_S\) the endosystem transition matrix. Let \(k_1 = 1\) and \(k_2 = -\frac{1}{4}\). Now,
Chapter 7. Iterative Scheme for Internal Model

$A_S = \begin{bmatrix} \frac{1}{4} & 1 & 0 \\ 1 & 0 & \frac{9}{16} \\ -\frac{1}{4} & -\frac{9}{16} & 0 \end{bmatrix}$ \hfill (7.12)

$A_S$ has eigenvalues 0.5288, −0.3884 and 0.1095, which are all within the open unit disk. Thus, $A_S$ is Schur stable. We solve (7.1) for $i_E^0$ in this example. Here, $i_E^0 : X_E \rightarrow X_S$ is in the form

$$i_E^0(x_E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ i_1 & i_2 \\ i_3 & i_4 \\ i_5 & i_6 \end{bmatrix} x_E$$ \hfill (7.13)

with $i_1, \ldots, i_6$ to be determined. Now, (7.1) becomes

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 1 & 0 \\ -1 & 0 & 1 & 0 & \frac{9}{16} \\ \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{9}{16} & 0 \end{bmatrix} \begin{bmatrix} i_1 & i_2 \\ i_3 & i_4 \\ i_5 & c_6 \end{bmatrix} x_E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ i_1 & i_2 \\ i_3 & i_4 \\ i_5 & i_6 \end{bmatrix} x_E$$ \hfill (7.14)
Using MATLAB, we get

\[
\begin{bmatrix}
  x_{E1} \\
  x_{E2} \\
  i_{EP}^0(x_E) \\
  i_{EC1}^0(x_E) \\
  i_{EC2}^0(x_E)
\end{bmatrix} = \begin{bmatrix}
  x_{E1} \\
  x_{E2} \\
  \frac{200210}{238457}x_{E1} - \frac{760}{238457}x_{E2} \\
  -\frac{98585}{476914}x_{E1} + \frac{200400}{238457}x_{E2} \\
  -\frac{180170}{238457}x_{E1} - \frac{53925}{238457}x_{E2}
\end{bmatrix} \quad (7.15)
\]

Since \( i_{EP}^0(x_E) \neq x_{E1} \) unless \( x_E = 0 \in \mathbb{R}^2 \), \( i_E(X_E) \notin K \). Therefore, we continue to Step 3.2 of Section 7.1. Because \( \gamma : X \rightarrow X_C \) is the projection of the total system state onto the controller state,

\[
\gamma = \begin{bmatrix}
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (7.16)
\]

and \( \tilde{\gamma}_E(x_E) = \gamma \circ i_E^0(x_E) \) is

\[
\tilde{\gamma}_E(x_E) = \begin{bmatrix}
  -\frac{98585}{476914}x_{E1} + \frac{200400}{238457}x_{E2} \\
  -\frac{180170}{238457}x_{E1} - \frac{53925}{238457}x_{E2}
\end{bmatrix} = \begin{bmatrix}
  -\frac{98585}{476914} & \frac{200400}{238457} \\
  -\frac{180170}{238457} & -\frac{53925}{238457}
\end{bmatrix} x_E \quad (7.17)
\]

Now, to solve Equation 7.2 for \( a_i^1 \), let \( a_i^1 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \). Equation 7.2 then becomes

\[
\begin{bmatrix}
  a_1 & a_2 \\
  a_3 & a_4
\end{bmatrix} \begin{bmatrix}
  -\frac{98585}{476914} & \frac{200400}{238457} \\
  -\frac{180170}{238457} & -\frac{53925}{238457}
\end{bmatrix} = \begin{bmatrix}
  -\frac{98585}{476914} & \frac{200400}{238457} \\
  -\frac{180170}{238457} & -\frac{53925}{238457}
\end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (7.18)
\]

Using MATLAB, we obtain
Now, using the new internal model \( \alpha^I_1 : X_C \to X_C \), we repeat Step 2 of Section 7.1. First, we try with the same \( k_1, k_2 \) parameters as in the first iteration and let

\[
\alpha^C_1 = \begin{bmatrix}
-1 & 0 & 1 & 0.0497 & 1.0987 \\
\frac{1}{4} & 0 & -\frac{1}{4} & -0.9124 & -0.0497
\end{bmatrix}
\] (7.20)

The endosystem transition matrix \( A_S \) is

\[
A_S = \begin{bmatrix}
\frac{1}{4} & 1 & 0 \\
1 & 0.0497 & 1.0987 \\
-\frac{1}{4} & -0.9124 & -0.0497
\end{bmatrix}
\] (7.21)

\( A_S \) has eigenvalues 0.403529, \(-0.076764 + 0.236771i\), \(-0.076764 - 0.236771i\), which are within the open unit disk. Thus, the endosystem is an eventual contraction with respect to \( x_S \) uniformly over \( x_E \in X_E \). Now, we solve for \( i^E_1 : X_E \to X \) in Equation 7.1 or

\[
\alpha^1 \circ i^E_1 = i^E_1 \circ \alpha_E
\] (7.22)

where \( \alpha^1 := \alpha_E \times \alpha_P \times \alpha^C_1 \). Similarly to the first iteration, we let
Chapter 7. Iterative Scheme for Internal Model

\[ i_E^1(x_E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ i_1 & i_2 \\ i_3 & i_4 \\ i_5 & i_6 \end{bmatrix} x_E \]  

(7.23)

and solve for \( i_1, \ldots, i_6 \in \mathbb{R} \). We obtain

\[ i_E^1(x_E) = \begin{bmatrix} x_{E1} \\ x_{E2} \\ x_{E1} - \frac{1}{4}x_{E1} + x_{E2} \\ -\frac{1511}{1681}x_{E1} - \frac{917}{3362}x_{E2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} x_{E1} \\ x_{E2} \end{bmatrix} \]  

(7.24)

Since \( i_{EP}^1(x_E) = x_{E1} \) for all \( x_E \in X_E \), \( i_E(X_E) \subseteq K \). Thus, the iteration stops and \( \alpha_I^1 \) is a satisfactory internal model for this output regulation problem.

Notice that the internal model matrix

\[ \alpha_I^1 = \begin{bmatrix} 0.0497 & 1.0987 \\ -0.9124 & -0.0497 \end{bmatrix} \]  

(7.25)

has the same eigenvalues \( \{i, -i\} \) as the exosystem transition matrix

\[ \alpha_E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]  

(7.26)
From linear systems theory, we know that two state transition matrices are isomorphic to each other if the two matrices have the same set of eigenvalues. This confirms the correctness of $\alpha_1^I$ as an internal model.
Chapter 8

Conclusions and Future Research

To conclude, we present a summary of results of this thesis, and propose topics that can be further investigated.

The contribution of this thesis is mainly that it provides an alternative way of viewing dynamical systems and corresponding output regulation problems using a set-theoretic approach. One advantage of this approach is that the analysis of dynamical system properties is coordinate independent. In addition, the theorems developed in this thesis are not limited to dynamical systems defined on a Euclidean space. Also, it is worthwhile to mention that the descriptions of dynamical systems in this thesis are not limited to dynamical systems evolving with respect to time in the traditional literature. Using this abstract algebraic approach, we are able to apply this method of dynamical systems modelling to any deterministic dynamical systems. Therefore, the analysis presented in this thesis can be applied to a broader class of dynamical systems compared to the existing literature.

Throughout this thesis, we answered the following questions. Under what condition, will the exosystem induce an invariant set on the total system dynamics? Under what condition, is the invariant set induced by the exosystem unique? Under what condition, is the invariant set induced by the exosystem an attractor on the the total system state
set? Under what condition, is the invariant set induced by the exosystem a subset of the regulation set?

In Chapter 4, we took the exosystem state set $X_E$ and endosystem state set $X_S$ to be subsets of a complete metric space. In this topological setting, we concluded that if the endosystem transition function possesses a contraction property that is uniform over the exosystem states, then there exists a unique invariant subset in the total system state set $X = X_E \times X_S$ that is also an attractor in $X$.

In Chapter 5, we took the exosystem state set $X_E$ and endosystem state set $X_S$ to be subsets of a Banach space, where $X_S$ is convex and compact. In this topological setting, instead of using the contraction property on the endosystem, we use an eventual contraction property on the endosystem transition function with respect to the endosystem state uniformly over the exosystem state. We were able to derive the same results as in Chapter 4 using the new assumptions.

In Chapter 6, we assumed a pre-stabilized plant transition function. By stabilization, we mean that it is a contraction or an eventual contraction, depending on the topology of the state sets, with respect to its own state uniformly over the exosystem state and the controller state. We also assumed that the controller transition function is autonomous and behaves like the exosystem if and only if the total system state is in the regulation set. Under these assumptions, if the endosystem transition function, which is the Cartesian product of transition functions of the plant and the controller, is stabilized, then the regulation condition must be achieved.

In Chapter 7, we proposed without proof an iterative scheme for obtaining the right internal model for output regulation problems.

Based on this thesis, several future research topics may be proposed.

First, a natural extension of this thesis is to consider the case with plant parameter variations. In Section 1.5 of [2], Wonham introduced a “richness of parametrization” assumption that led to the conclusion of the necessity of feedback and internal model
under plant parameter variations. A simple linear system example was also shown in [2] for demonstration purposes. It was not clear from [2] that the necessity of internal model and feedback under plant parameter variations can be shown in nonlinear settings. In addition, it was not mentioned in [2] that the same “richness of parametrization” assumption, together with an internal model and feedback, would be a sufficient condition for total system regulation under plant parameter variations. Thus, a sufficient condition for regulation property of the total system under plant parameter variations must be studied rigorously.

Secondly, output regulation problems for various types of dynamical systems can be defined. For instance, to my knowledge, there has not been a well-defined output regulation problem in the realm of discrete-event dynamical systems. A discrete-event dynamical system is a dynamical system that is event-driven, whereas most of the classical control systems are clock-driven. The nondeterministic characteristic of discrete-event dynamical systems can be problematic in the analysis of output regulation problems.

Lastly, the iterative scheme for obtaining the correct internal model for output regulation needs to be justified mathematically. In Chapter 7, we have provided a linear example that illustrates our main idea about this algorithm. We do not know at this stage, however, if the algorithm proposed would work in a nonlinear setting. The main difficulty in validating this proposed algorithm on nonlinear examples is that there is no systematic procedure of designing the controller dynamics in a way that it provides both “endosystem stability” property and “autonomy on the regulation set” property. Both mathematical justification of the algorithm and nonlinear examples must be studied in the future. We do conjecture that the algorithm we have proposed can be justified mathematically and that it would work on more general nonlinear systems.
Bibliography


