Abstract

A Local Twisted Trace Formula and Twisted Orthogonality Relations

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Around 1990, Arthur proved a local (ordinary) trace formula for real or p-adic connected reductive groups. The local trace formula is a powerful tool in the local harmonic analysis of reductive groups. One of the aims of this thesis is to establish a local twisted trace formula for certain non-connected reductive groups, which is a twisted version of Arthur’s local trace formula [10].

As an application of the local twisted trace formula, we will prove some twisted orthogonality relations, which are generalizations of Arthur’s results about orthogonality relations for tempered elliptic characters [11]. To establish these relations, we will also give a classification of twisted elliptic representations.
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Chapter 1

Introduction

In [10], Arthur established a local trace formula for real or $p$-adic connected reductive groups. The local trace formula is a powerful tool for dealing with problems (e.g., [9], [11], [12]) in the local harmonic analysis of reductive groups. For convenience, we call the local trace formula in [10] the local ordinary trace formula. At the end of [8], Arthur suggested that there should be a local twisted trace formula of which the ordinary formula would be a special case. In Chapter 3, we will derive Arthur’s suggested local twisted trace formula. As one of its applications, we will prove some twisted orthogonality relations, which are generalizations of Arthur’s other results about orthogonality relations for tempered elliptic characters [11].

To understand the motivation for the local twisted trace formula, we will first review the ordinary one. Let $G^0$ be a connected reductive algebraic group over $F$, where $F$ is a local field of characteristic 0. Consider the regular representation

$$(R(y_1,y_2)\phi)(x) = \phi(y_1^{-1}xy_2), \quad \phi \in L^2(G^0(F)), \quad x, y_1, y_2 \in G^0(F),$$

of $G^0(F) \times G^0(F)$ on the Hilbert space $L^2(G^0(F))$. Let $f$ be a smooth, compactly
supported function on $G^0(F) \times G^0(F)$ of the form

$$f(y_1, y_2) = f_1(y_1)f_2(y_2), \quad y_1, y_2 \in G^0(F).$$

We then define an operator $R(f)$ to be

$$\int_{G^0(F)} \int_{G^0(F)} f_1(y_1)f_2(y_2)R(y_1, y_2)dy_1dy_2.$$ 

The operator $R(f)$ maps any function $\phi \in L^2(G^0(F))$ to

$$(R(f)\phi)(x) = \int_{G^0(F)} \int_{G^0(F)} f_1(u)f_2(v)\phi(u^{-1}xv)dudv$$

$$= \int_{G^0(F)} K_0(x, y)\phi(y)dy,$$

where

$$K_0(x, y) = \int_{G^0(F)} f_1(xu)f_2(uy)du = \int_{G^0(F)} f_1(u)f_2(x^{-1}uy)du.$$ 

Therefore, $R(f)$ is an integral operator on $L^2(G^0(F))$ with a smooth kernel $K_0(x, y)$. Through a careful analysis of the trace of the operator $R(f)$, Arthur derived the local ordinary trace formula that is Theorem 12.2 in [10].

The local ordinary trace formula is used for the connected algebraic reductive groups over $F$, while the local twisted trace formula is used for certain non-connected algebraic reductive groups over $F$. Let $\theta$ be an (outer) automorphism of $G^0(F)$ defined over $F$, having finite order $n$. We can construct a non-connected algebraic reductive group $G^+(F)$ to be $G^0(F) \rtimes \langle \theta \rangle$ with multiplication

$$(g, \theta^i)(h, \theta^j) = (g\theta^i(h), \theta^{i+j}).$$

Let $G(F)$ be $G^0(F) \rtimes \theta$, which is a connected component of $G^+(F)$. If $\theta$ is non-trivial,
then \( G(F) \) is not the identity component \( G^0(F) \). Set the ring \( A_F \) to be

\[
F \oplus F.
\]

Then, \( F \) can be diagonally embedded in \( A_F \), and the set \( G(A_F) \) of \( A_F \)-valued points in \( G \) is just \( G(F) \times G(F) \).

The local twisted trace formula is derived from the twisted regular representation, which is described below. Let \( \phi \) be a function in \( L^2(G^0(F)) \). We then define the twisted regular representation

\[
(R(y_1, y_2)\phi)(x) = \phi(y_1^{-1}xy_2), \ x \in G^0(F), \ y_1, y_2 \in G(F),
\]

of \( G(A_F) = G(F) \times G(F) \) on the Hilbert space \( L^2(G^0(F)) \). This provides a canonical extension of the regular representation of \( G^0(A_F) = G^0(F) \times G^0(F) \) to the group generated by \( G(A_F) \). Let \( f \) be a smooth, compactly supported function on \( G(A_F) \) of the form

\[
f(y_1, y_2) = f_1(y_1)f_2(y_2), \ y_1, y_2 \in G(F).
\]

Then,

\[
R(f) = \int_{G(F)} \int_{G(F)} f_1(y_1)f_2(y_2)R(y_1, y_2)dy_1dy_2.
\]

The operator \( R(f) \) maps any function \( \phi \in L^2(G^0(F)) \) to

\[
(R(f)\phi)(x) = \int_{G(F)} \int_{G(F)} f_1(u)f_2(v)\phi(u^{-1}xv)dudv
= \int_{G^0(F)} \int_{G(F)} f_1(xu)f_2(uy)\phi(y)dudy
= \int_{G^0(F)} K(x, y)\phi(y)dy,
\]
where
\[ K(x, y) = \int_{G(F)} f_1(xu) f_2(uy) du = \int_{G(F)} f_1(u) f_2(x^{-1}uy) du. \]

Therefore, \( R(f) \) is an integral operator on \( L^2(G^0(F)) \) with a smooth kernel \( K(x, y) \). In §3.1, we can interpret \( R(f) \) in another form. Based on this interpretation, we will then define a more general operator \( R(f)R(\omega) \), where \( \omega \) is a quasicharacter on \( G^0(F) \). The local twisted trace formula in this thesis is derived by studying the trace of the operator \( R(f)R(\omega) \). To make the proofs simpler, for the most part we treat only the special case in which the twisted operator is \( R(f) \), i.e., \( \omega \) is trivial. However, all the results for \( R(f) \) can be applied to the more general case of \( R(f)R(\omega) \) with some modifications.

The local twisted trace formula contains the geometric side and the spectral side. To obtain both sides in the twisted case, we will apply Arthur’s method from [10]. The geometric side is easier than the spectral side and can be obtained by following Arthur’s method directly. In fact, to derive the geometric side in the twisted case, it is sufficient to replace the ordinary objects in [10] by the twisted analogies. However, we must make a more substantial effort to derive the spectral side, where new phenomena arise in the twisted case. In particular, we will prove Proposition 3.5.2 and Lemma 3.5.4 and define a twisted inner product in §3.5, and we will also use a lemma of Langlands. The local twisted trace formula (Theorem 3.6.1) will be established in §3.6; it is an identity between two expressions

\[ \sum_{M \in \mathcal{L}} |W_M^r| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{dt}(M)} J_M(\gamma, f) d\gamma \]  

(1.1) 

and

\[ \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Pi_{disc}(M)} a_{disc}(\pi) J_M(\pi, f) d\pi, \]  

(1.2) 

which is the first main result of this thesis.
Remark. At first glance, the twisted trace formula is the same as the trace formula in [10], but here, $G$ is a connected component of $G^+$, which may not be the identity component. If $G = G^0$, then the twisted trace formula is just the ordinary one. In addition, as in the ordinary case, this local twisted trace formula is not invariant. To make the formula more useful, in §4.3, we establish an invariant local twisted trace formula by imitating the process used in the ordinary case ([8], §8 and [11], §4).

As mentioned by Arthur, the weighted orbital integrals $J_M(\gamma, f)$ and the weighted characters $J_M(\pi, f)$ are essentially the local terms that occur in the global trace formula. Furthermore, the local trace formula has interesting applications in the local harmonic analysis. As one important application, Arthur ([11], §6) proved orthogonality relations for elliptic tempered characters by applying the local ordinary trace formula. To generalize his results, we will derive twisted orthogonality relations in Chapter 4, which will ultimately be consequences of the local twisted trace formula.

Before we introduce the twisted orthogonality relations, let us briefly review orthogonality relations for characters. There are Schur orthogonality relations for characters of finite or compact groups. For a non-compact (reductive) group $H$, it is difficult to establish similar relations. The first obstacle is how to define the characters of representations of $H$. There is no definition for the character of a tempered irreducible representation $\pi$ of $H$ in the classical sense because such a representation is infinite-dimensional. To overcome this, Harish-Chandra established the theory of characters of infinite-dimensional representations via the language of distributions. In detail, the character of $\pi$ is defined as a distribution

$$\Theta(\pi, f) = \text{tr}(\int_H f(x)\pi(x)dx), \quad f \in C^\infty_c(H),$$

which can then be identified with a function on $H$. That is, there is a locally integrable function $\Theta(\pi, x)$ on $H$ that is smooth on the open dense subset $H_{\text{reg}}$ of regular elements.
such that
\[ \Theta(\pi, f) = \int_H f(x)\Theta(\pi, x)dx, \quad f \in C^\infty_c(H). \]

For the characters of discrete series, Harish-Chandra established some orthogonality relations, which were very hard to derive at the time. For more general characters, similar orthogonality relations have been elusive. Harish-Chandra ([16]) conjectured that there should be a vanishing property for elliptic characters of a connected reductive group, and this was proved by Kazhdan ([23], Corollary to Proposition 5.4). Subsequently, Arthur ([11]) proved some more general and explicit orthogonality relations for elliptic characters of a connected reductive group.

Our aim in Chapter 4 is to establish some orthogonality relations for elliptic characters of \( G^+(F) \), which are generalizations of Arthur’s results. Let us now sketch these relations. Let \( P^0(F) \) be a parabolic subgroup of \( G^0(F) \) and \( M^0(F) \) be a Levi component of \( P^0(F) \). Assume that \( \sigma \) is an irreducible representation of \( M^0(F) \) such that \( \theta_s(\sigma) = \sigma \) for some element \( s \) in the Weyl group of \( G^0(F) \). In Theorem 4.1.4 (a), there is a 1-1 correspondence

\[ \rho^+ \leftrightarrow \pi^+_{\rho^+} \]

between the irreducible representations of a certain \( R \)-group determined by \( \sigma \) and irreducible constituents of the induced representation \( I^+_{P^0}(\sigma) \). Suppose that \( \pi^+_{\rho} \) and \( \pi^+_{\rho'} \) are two irreducible constituents of the induced representation \( I^+_{P^0}(\sigma) \). Let \( \Theta(\pi^+_{\rho}, \gamma) \) and \( \Theta(\pi^+_{\rho'}, \gamma) \) be the characters of \( \pi^+_{\rho} \) and \( \pi^+_{\rho'} \), respectively. We can then form the twisted elliptic inner product

\[ \sum_{(T)} |W(G(F), T(F))|^{-1} \int_{T(F)/A_G(F)} |D(\gamma)|\Theta(\pi^+_{\rho}, \gamma)\Theta(\pi^+_{\rho'}, \gamma)d\gamma, \quad (1.3) \]

where \( T \) runs over \( G^0(F) \)-conjugacy classes of elliptic maximal tori of \( G(F) \). In Corollary
4.5.3, we will prove that (1.3) is equal to a parallel inner product

$$|R_\sigma|^{-1} \sum_{t \in R_{\sigma,\text{reg}}} |d(t)| \theta(\rho^+, t) \overline{\theta(\rho'^+, t)}$$ (1.4)

on the $R$-group, where $\theta(\rho^+)$ and $\theta(\rho'^+)$ are the corresponding characters of $\rho^+$ and $\rho'^+$.

The thesis is organized as follows. In Chapter 2, we will first review the basic theory about twisted algebraic reductive groups, including the structure theory, $(G, M)$-families, the twisted Weyl Integration Formula, and representations of twisted groups. In Chapter 3, we will study the local twisted trace formula, investigating the geometric side and the spectral side. The formula is stated in Theorem 3.6.1. In Chapter 4, as one of the applications of the trace formula, we will prove twisted orthogonality relations (Theorem 4.5.1 and its corollaries). Also in Chapter 4, to prove these relations, we first need to classify the twisted irreducible tempered representations of $G^+(F)$ via the language of $R$-groups and to study twisted weighted characters and twisted weighted orbital integrals.
Chapter 2

Twisted Reductive Algebraic Groups

2.1 Preliminaries

Most of the results in this section can be found in [4], [5], [8], [13], and [15].

Let $F$ be a local field of characteristic 0. Assume that $G^0(F)$ is a connected algebraic group over $F$ and its Lie algebra is $G$. Let $\theta$ be an (outer) automorphism of $G^0(F)$ defined over $F$, having finite order $n$. As in Chapter 1, it is possible to construct a non-connected algebraic reductive group $G^+(F)$ to be $G^0(F) \rtimes \langle \theta \rangle$ with multiplication

$$(g, \theta^i)(h, \theta^j) = (g\theta^i(h), \theta^{i+j}).$$

Let $G(F)$ denote $G^0(F) \rtimes \theta$. If $\theta$ is non-trivial, then $G(F)$ is not the identity component $G^0(F)$. Note that if $\theta$ is an inner automorphism, the local twisted trace formula is essentially the ordinary trace formula. See the remark at the end of §3.1 for the details. Without loss of generality, we assume that $\theta$ preserves a pair $(B^0, T^0)$ in $G^0$, where $B^0$ is a Borel subgroup of $G^0$ and $T^0$ is a maximal torus of $B^0$, i.e.,

$$\theta(B^0) = B^0, \quad \theta(T^0) = T^0. \quad (2.1)$$
For convenience, let us review some basic definitions and properties stated in [4], [5], [7], and [15]. Twisted harmonic analysis applies to the non-identity component $G$. Many of the definitions and properties for connected groups extend to $G(F)$.

Let us start with the definition of regular elements in $G$. For any $x \in G$, form the polynomial of $t$ depending on $x$:

$$\det((t + 1) - Ad(x)) = \sum_k D_k(x)t^k, \quad x \in G,$$

where $Ad(x)$ is the adjoint action of $x$ on the Lie algebra $G$ of $G_0(F)$.

The smallest integer $r$ for which $D_r(x)$ does not vanish identically is called the rank of $G$.

**Definition 2.1.1 ([5], p. 227 or [15], p. 151)**: A point $x$ in $G$ is called $G$-regular if $D_r(x) \neq 0$. We denote the set of all $G$-regular elements in $G$ by $G_{\text{reg}}$, which is an open dense subset in $G(F)$.

L. Clozel proved the following result:

**Lemma 2.1.2 ([15], Lemma 1)**: If $\gamma \in G$ is $G$-regular, then $\gamma$ is semi-simple and the neutral component of the centralizer of $\gamma$ in $G_0$ is a torus.

For any $\gamma \in G$, we denote the neutral component of the centralizer of $\gamma$ in $G_0$ by $G_0^\gamma$. (Note that if $\gamma$ is regular, then the dimension of the torus $G_0^\gamma$ equals $r$, i.e., the rank of $G$.) In general, the rank $r$ of $G$ may not equal the rank $r_0$ of $G_0$, but we always have the inequality $r \leq r_0$ by the above lemma.

Assume that $\gamma \in G$ is $G$-regular. We then set $T_0$ to be $G_0^\gamma$. It has been shown in the above lemma that $T_0$ is a torus in $G_0$.

**Definition 2.1.3 ([5], p. 227)**: The variety $T = T_0^\gamma$ is called a maximal torus in $G$.

Obviously, $T$ itself is not an algebraic torus but an affine variety on which $T_0$ acts simply transitively. Set $T_{\text{reg}} = T \cap G_{\text{reg}}$. It is a fact that the map $T_{\text{reg}} \times (T_0 \setminus G_0) \to G$
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given by
\[(\gamma, x) \mapsto x^{-1} \gamma x\]

is an open immersion.

It is always possible to define the \textit{conjugacy class} for any \(\gamma \in G(F)\) to be
\[
\{\gamma\} = \{x\gamma x^{-1} \mid x \in G^0(F)\}.
\]

For the maximal torus, Arthur proved the following lemma:

\textbf{Lemma 2.1.4 ([5], pp. 227-228):} If \(T \subset G\) is a maximal torus, then \(T_0\) contains a \(G^0\)-regular element. Furthermore, let \(H^0_T\) be the centralizer of \(T_0\) in \(G^0\); then \(H^0_T\) is a maximal torus in \(G^0\).

We next turn to some definitions and properties concerning parabolic subgroups (subsets). Many definitions related to parabolic subgroups of the connected reductive group \(G^0\) are well known. For the non-connected reductive group \(G^+\), we have the following definitions:

\textbf{Definitions 2.1.5 ([5], p. 228):} A \textit{parabolic subgroup} of \(G^+\) over \(F\) is the normalizer in \(G^+\) of a parabolic subgroup of \(G^0\) that is defined over \(F\). If \(P^+\) is a parabolic subgroup of \(G^+\) over \(F\) and \(P = P^+ \cap G\) is not empty, we call \(P\) a \textit{parabolic subset} of \(G\). Put \(P^0 = P^+ \cap G^0\), and then let \(M^0\) be a Levi subgroup of \(P^0\) and \(\tilde{M}\) be the normalizer in \(G^+\) of \(M^0\). We call \(M = \tilde{M} \cap P\) a \textit{Levi subgroup} of \(G\). Let \(M^+\) be the group generated by \(M\). We call \(M^+\) a \textit{Levi subgroup} of \(G^+\), and the identity component of \(M^+\) is \(M^0\). Of course, \(M^+\) is always an algebraic reductive group over \(F\).

It is easy to see that \(P = MN_P\), where \(N_P\) is the unipotent radical of \(P^0\). If \(P^+ \cap G^0\) is a minimal parabolic subgroup of \(G^0\) over \(F\), then \(P^+\) meets every connected component of \(G^+\). We call such \(P^+\) a \textit{minimal parabolic subgroup} of \(G^+\). Assume that \(P^+_0\) is the identity component of a minimal parabolic subgroup \(P^+_0\) of \(G^+\). Then, \(P_0 = P^+_0 \cap G\) is non-empty and is called a \textit{minimal parabolic subset} of \(G\).
Let $M^+$ be a Levi subgroup of $G^+$. We shall use $M$, without further comment, to denote a Levi subset of $G$ given by $M = M^+ \cap G$, and $M^0$ to denote $M^+ \cap G^0$. We write $\mathcal{F}(M)$ for the set of all parabolic subsets of $G$ which contain $M$, and $\mathcal{L}(M)$ for the set of all Levi subsets of $G$ that contain $M$. Then, for any $P \in \mathcal{F}(M)$, there exists a unique Levi component $M_P$ of $P$ in $\mathcal{L}(M)$. We denote the set of all parabolic subsets of $G$ with $M_P = M$ by $\mathcal{P}(M)$. If $L \in \mathcal{L}(M)$, then $M$ is a Levi subset of $L$. We shall denote the above sets by $\mathcal{F}^L(M)$, $\mathcal{L}^L(M)$, and $\mathcal{P}^L(M)$, respectively, but with $G$ replaced by $L$. If $L = G$, we always omit the superscript $L$ in the above symbols. Sometimes we will emphasize the theory relative to $G^0(F)$; we then write $\mathcal{F}(M^0)$ for the set of all parabolic subgroups of $G^0$ that contain $M^0$ and $\mathcal{L}(M^0)$ for the set of all Levi subgroups of $G^0$ that contain $M^0$. We also denote the set of all parabolic groups of $G^0$ with $M_P = M^0$ by $\mathcal{P}(M^0)$.

From now on, we fix a minimal parabolic subgroup $P^0_0$ of $G^0$ and a Levi subset $M^0_0$ of $P^0_0$ of $G$, and satisfying

$$\theta(P^0_0) = P^0_0, \quad \theta(M^0_0) = M^0_0$$

(see [26], §1). This is the same as assuming that $\theta$ preserves the $(B^0, T^0)$ as in (2.1). Then,

$$P^+_0 = P^0_0 \rtimes \langle \theta \rangle, \quad P_0 = P^0_0 \rtimes \theta, \quad M^+ = M^0_0 \rtimes \langle \theta \rangle, \quad \text{and} \quad M_0 = M^0_0 \rtimes \theta.$$

We write $\mathcal{F} = \mathcal{F}(M_0)$ and $\mathcal{L} = \mathcal{L}(M_0)$. Let $K$ be a maximal compact subgroup of $G^0(F)$ that satisfies the following conditions. If $F$ is Archimedean, we assume that the Lie algebra of $K$ and $M^0_0(F)$ are orthogonal relative to the Killing form; if $F$ is $p$-adic, $K$ is special in the sense of [29] and the vertex of $K$ in the Bruhat-Tits building lies in the apartment of $M^0_0$. In this situation, we say that $K$ and $M^0_0(F)$ are in good relative
position. We then have the decomposition

\[ G^0(F) = KM_0^0(F)K. \]

If \( P^+ \) is a parabolic subgroup of \( G^+ \) that is defined over \( F \) and contains \( M_0 \) (equivalently, contains \( M_0^+ \) or \( M_0^0 \)), then

\[ G^0(F) = P^0(F)K = M^0_P(F)N_P(F)K, \]
\[ G^+(F) = P^+(F)K = M^+_P(F)N_P(F)K, \]

and

\[ G^0(F) = P^0(F)K = M^0_P(F)N_P(F)K, \]

where \( P^0 = P^+ \cap G^0, P = P^+ \cap G, \) and \( M^0_P \) is the unique Levi subgroup of \( P^0 \) that contains \( M_0^0 \).

In the remaining part of this section, we will discuss some related vector spaces and roots. Let \( A_{M^0} \) be the split component of the center of \( M^0 \). We have the real vector space

\[ \mathfrak{a}_{M^0} = Hom(X(M^0)_F, \mathbb{R}), \]

where \( X(M^0)_F \) is the module of \( F \)-rational characters of \( M^0 \). There is a canonical homomorphism

\[ H_{M^0} : M^0(F) \to \mathfrak{a}_{M^0} \]

defined by

\[ e^{<H_{M^0}(x), \chi>} = |\chi(x)|, \quad x \in M^0(F), \chi \in X(M^0)_F. \]

Set

\[ \mathfrak{a}_{M^0,F} = H_{M^0}(M^0(F)), \quad \tilde{\mathfrak{a}}_{M^0,F} = H_{M^0}(A_{M^0}(F)). \]
Let $M^0(F)^1$ denote the kernel of $H_{M^0}$.

Similarly, let $A_M$ be the split component of the centralizer of $M(F)$ in $M^0(F)$. We claim that $A_M \subset A_{M^0}$. In fact, fixing an $m \in M$, we know that $M = M^0 m$. For any $a \in A_M$, $am = ma$. Consequently, $a(xm) = (xm)a$ for any $x \in M^0$ since $xm$ is in $M$. It is easy to see that $ax = xa$, so $A_M \subset A_{M^0}$. If $M^0$ is $\theta$-stable, then

$$A_M = \{a \in A_{M^0} : \theta(a) = a\}^0,$$

where $X^0$ means the neutral component of a group $X$. If $\theta$ acts on a set $H$, we typically denote the subset of $\theta$-fixed points in $H$ by $H^\theta$. Therefore, we can write $A_M = (A_{M^0}^\theta)^0$ in this case.

We also have the real vector space

$$a_M = Hom(X(M)_F, \mathbb{R}),$$

where $X(M)_F$ is the module of $F$-rational characters of $M^+$. Then, $a_M$ is a real vector space whose dimension equals that of $A_M$. Certainly, $a_M \subset a_{M^0}$. There is also a canonical homomorphism

$$H_M : M^+(F) \to a_M$$

defined by

$$e^{<H_M(x), \chi>} = |\chi(x)|, \quad x \in M^+(F), \quad \chi \in X(M)_F.$$

We also write $M(F)^{1,+}$ for the kernel of $H_M$ and $M(F)^1 = M(F)^{1,+} \cap M(F)$. Also set

$$a_{M,F} = H_M(M^+(F)), \quad \tilde{a}_{M,F} = H_M(A_M(F)).$$

If $F$ is Archimedean, then $\tilde{a}_{M,F} = a_{M,F} = a_M$. If $F$ is $p$-adic, $\tilde{a}_{M,F}$ and $a_{M,F}$ are lattices in $a_M$ and $\tilde{a}_{M,F} \subset a_{M,F}$. There are similar results for $\tilde{a}_{M^0,F}$, $a_{M^0,F}$ and $a_{M^0}$.
For any \( x \in G(F) \), we can write \( x = m_P(x)n_P(x)k_P(x) \) according to the decomposition \( G(F) = P(F)K = M_P(F)N_P(F)K \), where \( m_P(x) \in M_P(F), n_P(x) \in N_P \), and \( k_P(x) \in K \). We can then define

\[
H_P(x) = H_M(m_P(x)).
\]

We can now discuss the basic theory of roots. The theory of roots related to \( G^0 \) is well known, so we consider the corresponding results for the twisted case. Suppose that \( P \in \mathcal{P}(M) \). We shall write \( A_P = A_{M_P} \) and \( a_P = a_{M_P} \). The roots of \( (P, A_P) \) are defined with respect to the adjoint action of \( A_P \) on the Lie algebra of \( N_P \). We write \( \Phi_P \) for the set of all roots of \( (P, A_P) \) and \( \Phi^r_P \) for the set of all reduced roots. It will be convenient for us to regard them either as characters on \( A_P \) or as elements in the dual space \( a_P^* \) of \( a_P \). As in the connected case, we can define the set of simple roots \( \Delta_P \) of \( (P, A_P) \), and the chamber \( a_P^+ \) in \( a_P \) is defined by

\[
a_P^+ = \{ H \in a_P : \alpha(H) > 0, \alpha \in \Delta_P \}.
\]

Let \( Q \) be a parabolic subgroup of \( G^0 \). The modular function \( \delta_Q \) of \( Q = M_QN_Q \) is given by

\[
\delta_Q(mn) = e^{2\rho_Q(H_{M_Q}(m))}, \quad m \in M_Q(F), \; n \in N_Q(F),
\]

where \( 2\rho_Q \) is the usual sum of the roots (with multiplicity) of \( (Q, A_Q) \).

We note that the restriction homomorphism \( X(M^+_F) \to X(A_M)_F \) is injective and has finite cokernel. Hence, we have

\[
a_P^* = X(M_P)_F \otimes \mathbb{R} \cong X(A_P)_F \otimes \mathbb{R}.
\]

Let \( P \subset Q \), where \( Q \) is another parabolic set in \( \mathcal{P}(M) \). There are canonical embed-
nings $a_Q \subset a_P$ and $a_Q^* \subset a_P^*$. Let $\Delta^0_P$ denote the set of roots in $\Delta_P$ that vanish on $a_Q$. It can be identified with the set of simple roots of the parabolic subset $P \cap M_Q$ of $M_Q$.

Let us recall the definition of the co-root in the twisted case. Let $P_0 = P_0^+ \cap G$ be a minimal parabolic subset of $G$, and $P_0^0 = P_0^+ \cap G^0$ the fixed minimal parabolic subgroup of $G^0$. If we write $a_0 = a_{P_0^0}$, $\Phi_0 = \Phi_{P_0^0}$ and $\Delta_0 = \Delta_{P_0^0}$, then the pair $(V, R) = ((a_0)^*, \Phi_0 \cup (-\Phi_0))$ is a root system as defined in [27], for which $\Phi_0$ is a system of positive roots. Here, $\Delta_0$ is a basis of the real vector space $(a_0)^*$. Let $W_0$ be the Weyl group of $(V, R)$. then $W_0$ is a finite group generated by the reflections associated to roots in $\Delta_0$.

We write $s_\alpha$ for the reflection associated to $\alpha \in \Phi_0$. Given any $\alpha \in \Delta_0$, it is possible to define the co-root $\alpha^\vee$ as the following: $\alpha^\vee$ is the unique element in $a_0$ such that

$$s_\alpha = 1 - \alpha^\vee \otimes \alpha \quad \langle \alpha^\vee, \alpha \rangle = 2.$$ 

We know that $\Delta_{P_0}$ can be regarded as a subset of $\Delta_0$ by restriction ([§5, [13]). The definition of the co-root corresponding to $\alpha \in \Delta_P$ is given by

$$\alpha^\vee = \sum \beta^\vee,$$

where $\beta$ ranges over the roots in $\Delta_{P_0}$ whose restriction to $a_P$ equals $\alpha$. Then,

$$\Delta^\vee_P = \{ \alpha^\vee : \alpha \in \Delta_P \}$$

is a basis of $a_P^G$. In fact, the definition of the co-root is somewhat arbitrary for non-minimal parabolics, but we will see that many objects related to co-roots are independent of how the co-roots are chosen. Also, we define the chamber in $a_P^*$ associated to $P$:

$$(a_P^*)^+ = \{ \Lambda \in a_P^* : \Lambda(\alpha^\vee) > 0, \alpha \in \Delta_P \}.$$
As in the ordinary case in §1 of [10], we have a natural surjective projection

\[ h_{MG} : a_M \to a_G. \]

Let \( a^*_M \) be the kernel of \( h_{MG} \). Then, the decomposition \( a_M = a^*_M \oplus a_G \) is orthogonal relative to the restriction to \( a_M \) of the \( W_0 \)-invariant inner product on \( a_0 \). Similarly, \( ia^*_M = (ia^*_M)^G \oplus ia^*_G \).

For future reference, we set

\[ \Delta^\Lambda_P = \{ \alpha \in \Delta_P : \Lambda (\alpha) < 0 \}, \]

where \( \Lambda \) is a point in \( a^*_M \) whose real part \( \Lambda_R \in a^*_M \) is in general position. Let

\[ \hat{\Delta}_P = \{ \omega_\alpha : \alpha \in \Delta_P \} \]

be the basis of \( (a^*_M)^* \) that is dual to \( \{ \alpha : \alpha \in \Delta_P \} \). There is a more detailed discussion for a connected reductive group in §5 of [13].

Before studying \((G, M)\)-families in the next section, we need to define the function

\[ \theta^G_P(\lambda) = \text{vol}(a^*_P/\mathbb{Z}(\Delta^\vee_P))^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee), \quad \lambda \in a^*_P, \]

which is independent of the choice of co-roots, where \( \mathbb{Z}(\Delta^\vee_P) \) is the lattice spanned by the basis \( \Delta^\vee_P \) of \( a^*_P \) and \( \text{vol} \) means the volume. For simplicity, we denote \( \theta^G_P(\lambda) \) by \( \theta_P(\lambda) \).

### 2.2 \((G, M)\)-families

The \((G, M)\)-family is a convenient language for studying the refined (both local and global) trace formulas. A good reference is §17 of [13]. In this section, we review some basic definitions and properties of \((G, M)\)-families. Our discussion is relevant to con-
nected and non-connected reductive groups, but we shall concentrate on the latter.

We say that two parabolic subsets \( P, P' \in \mathcal{P}(M) \) are \textit{adjacent} if their chambers share a common wall.

**Definition 2.2.1:** A \( (G, M) \)-\textit{family} is a set of smooth functions

\[
\{c_P(\lambda) : P \in \mathcal{P}(M)\}
\]

of \( \lambda \in ia_M^* \) with the property that if \( P \) and \( P' \) are adjacent, and \( \lambda \) belongs to the hyperplane spanned by the common wall of two associated chambers \( i(a_P^*)^+ \) and \( i(a_{P'}^*)^+ \) in \( ia_M^* \), then \( c_P(\lambda) = c_{P'}(\lambda) \).

We see that if \( Q \) is a parabolic subset that contains \( P \) and \( P' \), then there is a well-defined function \( c_Q(\lambda) \) for \( \lambda \in ia_Q^* \) given by the obvious restriction of \( c_P \).

Let \( \theta_P(\lambda) \) be the function defined in §2.1. Given any \( (G, M) \)-\textit{family}

\[
\{c_P(\lambda) : P \in \mathcal{P}(M)\},
\]

we can define an important function

\[
c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1}, \quad \lambda \in ia_M^*.
\]

(2.2)

It satisfies the following lemma.

**Lemma 2.2.2** ([2], Lemma 6.2): \( c_M(\lambda) \) is a smooth function of \( \lambda \in ia_M^* \).

We often write \( c_M = c_M(0) \) for the value of \( c_M(\lambda) \) at \( \lambda = 0 \). In fact, \( c_M \) is the limit value of \( c_M(\lambda) \) as \( \lambda \) approaches 0. It is in this form that the \( (G, M) \)-families usually appear in harmonic analysis.

Let us describe a basic example that is important to Arthur’s truncation (see §3.3)
in the study of the local trace formula. Assume that

\[ \mathcal{Y} = \{ Y_P : P \in \mathcal{P}(M) \} \]

is a family of points in \( a_M \) parameterized by \( \mathcal{P}(M) \). The family \( \mathcal{Y} \) is called \( (G,M) \)-orthogonal if for every pair \( P \) and \( P' \) of adjacent groups in \( \mathcal{P}(M) \),

\[ Y_P - Y_{P'} = r_\alpha \alpha^\vee, \]

where \( \alpha \) is the simple root in \( \Delta_P \) uniquely determined by the common wall of the two chambers associated to \( P \) and \( P' \), and \( r_\alpha \) is a real number. In particular, \( \mathcal{Y} \) is called positive if each \( r_\alpha \) is nonnegative. We always suppose that \( \mathcal{Y} \) is positive. Obviously,

\[ \{ c_P(\lambda, \mathcal{Y}) = e^{\lambda(Y_P)} : P \in \mathcal{P}(M) \} \]

is a \( (G,M) \)-family of smooth functions. In general, we denote by \( S_M(\mathcal{Y}) \) the convex hull in \( a_M^G \) of a positive \( (G,M) \)-orthogonal set \( \mathcal{Y} \).

Let us review the decomposition of the characteristic function of \( S_M(\mathcal{Y}) \) stated in §3 of [10], which also holds in the twisted case. Let \( \varphi_P^\Lambda \) be the characteristic function of the set

\[ \{ H \in a_M^G | \varpi_\alpha(H) > 0, \forall \alpha \in \Delta_P^\Lambda \text{ and } \varpi_\alpha(H) \leq 0, \forall \alpha \in \Delta_P - \Delta_P^\Lambda \}. \]

We then define

\[ \sigma_M(H, \mathcal{Y}) = \sum_{P \in \mathcal{P}(M)} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(H - Y_P). \tag{2.3} \]

The function \( \sigma_M(\cdot, \mathcal{Y}) \) vanishes on the complement of \( S_M(\mathcal{Y}) \). Furthermore, if \( \mathcal{Y} \) is positive, then \( \sigma_M(H, \mathcal{Y}) \) is exactly equal to the characteristic function of \( S_M(\mathcal{Y}) \). Note that \( \sigma_M(H, \mathcal{Y}) \) is independent of the point \( \Lambda \).
If $\Lambda$ is in general position and the real part of $\lambda \in a_{M,C}^*$ is $\Lambda$, then the function
\[
(-1)^{|\Delta_{\lambda}|} \varphi_P^\Lambda(H - Y_P)e^{\lambda(H)}, \quad H \in a_M^G,
\]
is rapidly decreasing. We can calculate
\[
\int_{a_M^G} (-1)^{|\Delta_{\lambda}|} \varphi_P^\Lambda(H - Y_P)e^{\lambda(H)}dH = c_P(\lambda, \mathcal{Y})\theta_P(\lambda)^{-1}.
\]

By (2.2), the function $c_M(\lambda, \mathcal{Y})$ equals
\[
\sum_{P \in \mathcal{P}(M)} c_P(\lambda, \mathcal{Y})\theta_P(\lambda)^{-1}, \quad \lambda \in ia_M^*,
\]
which can be identified with the Fourier transform of the characteristic function $\sigma_M(H, \mathcal{Y})$ of the convex hull $S_M(\mathcal{Y})$. The value $c_M(\mathcal{Y}) = c_M(\mathcal{Y}, 0)$ is exactly the volume of $S_M(\mathcal{Y})$.

There are some important product formulas for $(G, M)$-families. For the details, we refer the reader to §17 of [13].

### 2.3 Haar Measures and Twisted Weyl Integration Formula

We define the Haar measure on a non-connected reductive group in the canonical way, as was done for a connected group in [10]. For the connected reductive group $G^0$, we fix compatible Haar measures on $G^0(F)$ and $M^0(F)$ as in §1 [10]. For the details, we refer the reader to pp. 11-13 of [10]. In the twisted case, it is sufficient to translate the Haar measure on $G^0(F)$ to $G^+(F)$ by a finite cyclic group. Here are some assumptions:

a) The Haar measure on a compact group will be normalized to have total volume 1.

b) Fix a Haar measure on each of the spaces $a_M$ and $a_{M^0}$. Then, take the dual measure on $ia_M^*$ and $ia_{M^0}^*$. 

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c) If $F$ is $p$-adic, we know that $\tilde{a}_{M,F} = H_M(A_M(F))$ is a lattice in $a_M$. Normalize the measure on $a_M$ so that $vol(a_M/\tilde{a}_{M,F}) = 1$. Similarly, require $vol(a_{M^0}/\tilde{a}_{M^0,F}) = 1$. Then, the volumes of

$$ia_M^*/Hom(\tilde{a}_{M,F}, 2\pi i\mathbb{Z}) \quad \text{and} \quad ia_{M^0}^*/Hom(\tilde{a}_{M^0,F}, 2\pi i\mathbb{Z})$$

both equal 1.

d) The volume of the quotient

$$ia_{M,F}^* = ia_M^*/Hom(a_{M,F}, 2\pi i\mathbb{Z})$$

is equal to the index $|a_{M,F}/\tilde{a}_{M,F}|$. Similarly, The volume of the quotient

$$ia_{M^0,F}^* = ia_{M^0}^*/Hom(a_{M^0,F}, 2\pi i\mathbb{Z})$$

is equal to the index $|a_{M^0,F}/\tilde{a}_{M^0,F}|$.

e) The Haar measure on $A_M(F)$ is determined by those on $ker(H_M) \cap A_M(F)$ and $H_M(A_M(F))$ since $ker(H_M) \cap A_M(F)$ is compact and $H_M(A_M(F))$ is either discrete or equal to $a_M$, where $ker(H_M)$ is the kernel of $H_M$. Similarly, we can define the Haar measure on $A_{M^0}(F)$.

f) As in e), we can determine the Haar measures on $M^0(F)^1$ and $M(F)^1$.

Next, we will obtain a twisted version of the Weyl Integration Formula to deal with the geometric side of the local twisted trace formula. It is necessary to give the definition of elliptic element in $G(F)$. The definition of an elliptic element in $G^0(F)$ is well known and can be extended to $G(F)$.

**Definition 2.3.1**: A point $x \in G(F)$ is called elliptic if the centralizer $G_x^0$ of $x$ in $G^0(F)$ is compact modulo $A_G(F)$.

Let $\Gamma_{el}(G(F))$ denote the set of conjugacy classes $\{\gamma\}$ in $G(F)$ of elliptic elements
\( \gamma \in G(F) \). It is known that \( G_{\text{reg}}(F) \) is open and dense in \( G(F) \), so it suffices to consider \( \Gamma_{\text{ell}}(G(F)) \cap G_{\text{reg}}(F) \) for integration over \( G(F) \). There is a canonical measure \( d\gamma \) on \( \Gamma_{\text{ell}}(G(F)) \) described as follows. This measure vanishes on the complement of \( G_{\text{reg}}(F) \) in \( \Gamma_{\text{ell}}(G(F)) \). Let \( T \) be a maximal torus (Definition 2.1.3) of \( G \) over \( F \) with \( T_0(F)/A_G(F) \) compact. Hence, there is a canonical Haar measure on \( T_0(F)/A_G(F) \). Recall that we have fixed a Haar measure on \( A_G(F) \) as in e) above, so we can define a measure on \( T(F) \).

The Weyl group of \( ((G^0(F), T(F)) \) is given by

\[ W((G^0(F), T(F)) = N_{G^0}(T)/Z_{G^0}(T), \]

where \( N_{G^0}(T) \) is the normalizer of \( T \) in \( G^0(F) \) and \( Z_{G^0}(T) \) the centralizer of \( T \) in \( G^0(F) \).

We have the following formula:

\[
\int_{\Gamma_{\text{ell}}(G(F))} \phi(\gamma) d\gamma = \sum_{\{T\}} |W((G^0(F), T(F)))|^{-1} \int_{T(F)} \phi(t) dt,
\]

for any function \( \phi \in C_c(\Gamma_{\text{ell}}(G(F)) \cap G_{\text{reg}}(F)). \)

Before stating the twisted Weyl Integration Formula, we need to define other Weyl groups. Let \( M_0 = M_0^0 \rtimes \theta \) be the fixed minimal Levi subset in \( G(F) \). The twisted Weyl group of \( (G^0, M_0) \) is given by

\[ W^G_0 = Norm_{G^0}(M_0)/M_0^0, \]

where \( Norm_{G^0}(M_0) \) is the normalizer of \( M_0 \) in \( G^0(F) \). Here, \( W^G_0 \) acts by conjugation on \( \mathcal{L} = \mathcal{L}(M_0) \). In particular, \( M_0^+ \) and \( M_0 \) are invariant under the action of \( W^G_0 \). Similarly, for any \( M \in \mathcal{L} \), we can define

\[ W^M_0 = Norm_{M^0}(M_0)/M_0^0. \]
We note that $W_0^G$ is the group consisting of $\theta$-fixed elements in $W_0 = W_0^{G^0}$, where

$$W_0 = W_0^{G^0} = \text{Norm}_{G^0}(M_0^0)/M_0^0$$

is the ordinary Weyl group of $(G^0, A_{M_0^0})$. Also, $W_0$ can be defined in terms of the root system as in §2.1.

Any $G$-regular conjugacy class in $G(F)$ is the image of a class $\{\gamma\} \in \Gamma_{\text{ell}}(M(F))$ for some Levi subset $M \in \mathcal{L}$. The pair $(M, \{\gamma\})$ is uniquely determined only up to the action of the Weyl group $W_0^G$, so the number of such pairs equals $|W_0^G||W_0^M|^{-1}$. Thus, we have the Twisted Weyl Integration Formula for $G(F)$.

**Proposition 2.3.2 (Twisted Weyl Integration Formula)**: If $h$ is a function in $C^\infty_c(G(F))$, then

$$\int_{G(F)} h(x) dx = \sum_{M \in \mathcal{L}} \frac{|W_0^M|}{|W_0^G|} \int_{\Gamma_{\text{ell}}(M(F))} |D(\gamma)| \left( \int_{A_M(F) \setminus G^0(F)} h(x^{-1}\gamma x) dx \right) d\gamma, \quad (2.4)$$

where

$$D(\gamma) = \det(1 - Ad(\gamma))_{G/G},$$

is the Weyl discriminant. Note that $M_0^0 = M_0^+ \cap G^0$, $M_0 = M_0^+ \cap G$, $M^0 = M^+ \cap G^0$, and $M = M^+ \cap G$.

### 2.4 The Representations of Twisted Groups

In this section, we study the relations between irreducible representations of $M^+(F)$ and $M^0(F)$. Some results have been stated in [7] and [26]. For future reference, we point out that the results stated in this section are also true if $M^0(F)$ is replaced by a finite or compact group.

Let $\Pi(M^+(F))$ and $\Pi(M^0(F))$ denote the sets of equivalence classes of irreducible
admissible representations of $M^+(F)$ and $M^0(F)$, respectively. For example, it is possible to write $M^+_0 = M^0_0 \rtimes \langle \theta \rangle$ and $M_0 = M^0_0 \rtimes \theta$, where $\theta$ is an (outer) automorphism of $M^0_0$ of finite order $n$. We note that if $\theta$ is an inner automorphism, most results about representation theory in this section will be trivial. Let $\Upsilon_M$ be the finite cyclic group

$$\langle \theta \rangle \cong M^+(F)/M^0(F).$$

Set

$$\Xi_M = \text{Hom}(\Upsilon_M, \mathbb{C}^*).$$

Then, there is a natural action of $\Xi_M$ on $\Pi(M^+(F))$ that is given by

$$\pi^+_\zeta(m) = \pi^+(m)\zeta(m), \quad \pi^+ \in \Pi(M^+(F)), \quad \zeta \in \Xi_M, \quad m \in M^+(F), \quad (2.5)$$

where $m$ is the projection of $m$ onto $\Upsilon_M$. We may assume that $\Xi_M$ is generated by $\xi$.

Next, we study the relations between irreducible representations of $M^+(F)$ and $M^0(F)$. We may assume that $M^+(F) = M^0(F) \rtimes \langle \theta \rangle$ (transformed by a conjugation if necessary). Given any (not necessarily irreducible) representation $(\pi^+, V^+)$ of $M^+(F)$ of finite length, it is easy to check that its restriction $\pi$ to $M^0(F)$ is invariant under the finite group $\Upsilon_M$. We see that

$$\pi \cong \theta(\pi),$$

where $\alpha(\pi)(g) = \pi(\alpha^{-1}(g))$ for any $g \in M^0(F)$ and $\alpha \in \text{Aut}(M^0)$ [It is possible to can check that $(\alpha\beta)(\pi) = \alpha(\beta(\pi))]$. Conversely, if a representation $\pi$ of $M^0(F)$ satisfies $\pi \cong \theta(\pi)$ and there is an intertwining operator $A$ from $\pi$ to $\theta(\pi)$ satisfying $A^n = 1$, then $\pi$ can be lifted to a representation $\pi^+$ of $M^+(F)$ such that $\pi^+|_{M^0(F)} = \pi$. In fact,

$$A \circ \pi(\theta^{-1}(g)) = \pi(g) \circ A, \quad \forall g \in M^0(F).$$
We note that the above relation is the same as

\[ A \circ \pi(g) = \pi(\theta(g)) \circ A, \quad \forall g \in M^0(F). \]

It is possible to verify that

\[ \pi(g) \circ A^n = A^n \circ \pi(\theta^{-n}(g)) = A^n \circ \pi(g), \quad \forall g \in M^0(F). \]

We have assumed that \( A^n = 1 \). We can define the lifting \( \pi^+ \) of the representation \( \pi \) given by

\[ \pi^+(x) = \begin{cases} \pi(x) & \text{if } x \in M^0(F), \\ A & \text{if } x = 1 \rtimes \theta \in M(F), \end{cases} \]

which can be extended to the whole group \( M^+(F) \). In fact, it suffices to check that

\[ \pi^+(1 \rtimes \theta) \circ \pi^+(m \rtimes 1) = \pi^+(\theta(m) \rtimes 1) \circ \pi^+((1 \rtimes \theta)) \quad (2.6) \]

which can be deduced from the relation

\[ (1 \rtimes \theta)(m \rtimes 1) = (\theta(m) \rtimes 1)(1 \rtimes \theta), \quad \forall m \in M^0(F), \]

on the group \( M^+(F) \). The relation \( (2.6) \) is exactly

\[ A \circ \pi(m) = \pi(\theta(m)) \circ A, \quad \forall m \in M^0(F). \]

Hence, we have

**Proposition 2.4.1:** (a). Given any (not necessarily irreducible) representation \( (\pi^+, V^+) \) of \( M^+(F) \) of finite length, its restriction \( \pi \) to \( M^0(F) \) satisfies

\[ \pi \cong \theta(\pi). \quad (2.7) \]
(b). Conversely, if the representation $\pi$ of $M^0(F)$ satisfies $\pi \cong \theta(\pi)$ and the intertwining operator $A$ satisfies $A^n = 1$, then $\pi$ can be lifted to a representation $\pi^+$ of $M^+(F)$ such that $\pi^+|_{M^0(F)} = \pi$ and $\pi^+(1 \rtimes \theta)$ is an intertwining operator between $\theta(\pi)$ and $\pi$.

From now on, we assume that $\pi^+$ is irreducible. Then, the restriction $\pi$ is an admissible representation of $M^0(F)$ which has a finite decomposition

$$\pi = \bigoplus_i \pi_i, \quad \text{for some } \pi_i \in \Pi(M^0(F)).$$

(2.8)

By (2.7), if $\pi_i \in \Pi(M^0(F))$ appears in the decomposition (2.8) of $\pi$, then $\theta^j(\pi_i)$ also appears in the decomposition (2.8) for any $\theta^j \in \Upsilon_M$. For $(\pi_1, V) \in \Pi(M^0(F))$, let $\{\pi_1\}_{\Upsilon_M}$ denote the orbit of $\pi_1$ under $\Upsilon_M$. We can assume that

$$\{\pi_1\}_{\Upsilon_M} = \{\theta^j(\pi_1) \mid j = 0, 1, 2, \ldots, k - 1\},$$

where $k$ is the smallest positive integer such that $\pi_1 \cong \theta^k(\pi_1)$. We know that such a $k$ always exists. In fact, since the cardinality $|\Upsilon_M|$ of $\Upsilon_M$ is $n$, then $\pi_1 \cong \theta^n(\pi_1)$ is always true. Hence, $k$ is not greater than $n$ and is actually a positive divisor of $n$. By Lemma 2.1 of [26], let $T$ be the restriction of $\pi^+(1 \rtimes \theta)^k$ to $V$; then, $T$ is an intertwining operator of order $n/k$ between $\theta^k(\pi_1)$ and $\pi_1$, i.e.,

$$\theta^k(\pi_1) = T^{-1}\pi_1T.$$

Furthermore, $(\pi = \pi^+|_{M^0}, \ V^+)$ is isomorphic to

$$\bigoplus_{i=0}^{k-1} \theta^i(\pi_1), \bigoplus_{i=0}^{k-1} \theta^i(V).$$
We shall describe \( \pi^+ \) explicitly. As a representation of \( M^+ \), we have

\[
(\pi^+, V^+) \cong \bigoplus_{i=0}^{k-1} \theta^i(\pi_1), \bigoplus_{i=0}^{k-1} \theta^i(V).
\] (2.9)

The isomorphism in (2.9) is given by the following. Let \((v_0, v_1, \ldots, v_{k-1})\) be a vector in \( \bigoplus_{i=0}^{k-1} \theta^i(V) \). For any \( m \rtimes 1 \in M^0(F) \rtimes \Upsilon_M, \)

\[
\bigoplus_{j=0}^{k-1} \theta^j(\pi_1))(m \rtimes 1)(v_0, v_1, \ldots, v_{k-1}) = (\pi_1(m)v_0, \theta(\pi_1)(m)v_1, \ldots, \theta^{k-1}(\pi_1)(m)v_{k-1}).
\]

For \( \theta = 1 \rtimes \theta \) in \( M(F) = M^0(F) \rtimes \theta \), we define

\[
\bigoplus_{j=0}^{k-1} \theta^j(\pi_1))(\theta)(v_0, v_1, \ldots, v_{k-1}) = (Tv_{k-1}, v_0, v_1, \ldots, v_{k-2}).
\] (2.10)

Let \( B \) be the isomorphism from \( \bigoplus_{i=0}^{k-1} \theta^i(V) \) to \( V^+ \) given by

\[(v_0, v_1, \ldots, v_{k-1}) \rightarrow v_0 + \pi^+(\theta)v_1 + \ldots + \pi^+(\theta^{k-1})v_{k-1}.\]

Then, \( B \) is an intertwining operator between \( \bigoplus_{i=0}^{k-1} \theta^i(\pi_1), \bigoplus_{i=0}^{k-1} \theta^i(V) \) and \( (\pi^+, V^+) \).

We know that

\[
\pi^+|_{M^0} = \bigoplus_{i=0}^{k-1} \theta^i(\pi_1) = \pi^+|_{M^0}, \quad \forall j \in \mathbb{Z},
\]

where \( \xi^j \) is the \( j \)-th power of \( \xi \) and \( \pi_{\xi^j} \) is defined by (2.5). Any two representations in

\[
\{\pi^+_i \mid i = 0, 1, \ldots, \frac{n}{k} - 1\}
\]

are not equivalent to each other, but one has

\[
\pi^+ \cong \pi^+_{\xi^{n/k}}.
\] (2.11)
Therefore, there is a 1-1 correspondence between the $\Xi_M$-orbits in $\Pi(M^+(F))$ and the $\theta$-orbits in $\Pi(M^0(F))$. For convenience, we use $\Pi(M^+(F))_{\Xi_M}$ and $\Pi(M^0(F))_{<\theta>}$ to denote these two orbits, respectively. Let $\text{Ind}^{M^+}_{M^0} \pi_1$ be the induced representation of $M^0(F)$. By Frobenius reciprocity, it is not difficult to show that

$$\text{Ind}^{M^+}_{M^0} \pi_1 \cong \bigoplus_{j=0}^{2^{k-1}} \pi^+_\xi^j.$$  

Summarizing the above discussion, we obtain

**Theorem 2.4.2:**  
(a). If $\pi^+$ belongs to $\Pi(M^+(F))$, then its restriction $\pi$ to $M^0$ satisfies $\pi = \bigoplus_{i=0}^{k-1} \theta^i(\pi_1)$. In addition, there is a 1-1 correspondence

$$\pi^+ \leftrightarrow \pi_1$$

between the orbits $\Pi(M^+(F))_{\Xi_M}$ and $\Pi(M^0(F))_{<\theta>}$, which is given by (2.9).

(b). An irreducible representation $\pi_1$ of $M^0(F)$ can be lifted to a representation $\pi^+$ of $M^+(F)$ with $\pi^+_|_{M^0} = \pi_1$ if and only if $\pi_1 \cong \theta(\pi_1)$.

(c). The induced representation $\text{Ind}^{M^+}_{M^0} \pi_1$ has the decomposition

$$\text{Ind}^{M^+}_{M^0} \pi_1 \cong \bigoplus_{j=0}^{2^{k-1}} \pi^+_\xi^j. \hspace{1cm} (2.12)$$

Let $\Pi(M(F))$ denote the subset of $\Pi(M^+(F))$ consisting of those $\pi^+$ whose restriction $\pi$ to $M^0(F)$ remains irreducible, i.e., the decomposition (2.8) has only one component. Later, we will see that only the representations in $\Pi(M(F))$ occur on the spectral side of the local twisted trace formula. Let $tr(\pi^+)$ denote the restriction of the character (as a distribution) of $\pi^+$ to $M(F)$ for any $\pi^+ \in \Pi(M^+(F))$. At least for $p$-adic groups, we know that $tr(\pi^+)$ is actually a function on $M(F)_{\text{reg}}$ by Clozel’s work [15]. The set $\Pi(M(F))$ consists of the representations $\pi^+$ in $\Pi(M^+(F))$ for which $tr(\pi^+)$ does not
vanish. In fact, using the relation (2.11) for \(\pi^+\), if the number \(k\) is greater than 1, it is not hard to show \(tr(\pi^+) \equiv 0\). If \(\pi^+ \in \Pi(M(F))\), then for the orbit \(\{\pi^+\}_{\Xi_M}\) of \(\pi\) under \(\Xi_M\), the functions \(\{tr(\pi^+)\}_{\Xi_M}\) are linearly independent. Thus, \(\Pi(M(F))\) is exactly the subset of \(\Pi(M^+(F))\) on which \(\Xi_M\) acts freely. Let \(\{\Pi(M(F))\}_{\Xi_M}\) denote the orbits of \(\Xi_M\) on \(\Pi(M(F))\). Combining these facts with Proposition 2.4.1 and Theorem 2.4.2, it is possible to the following.

**Corollary 2.4.3 ([7], p. 28):** There is a bijection

\[
\pi^+ \leftrightarrow \pi
\]

between the set \(\{\Pi(M(F))\}_{\Xi_M}\) and \(\{\pi \in \Pi(M^0(F)) \mid \pi \cong \theta(\pi)\}\) such that the restriction \(\pi\) of \(\pi^+\) to \(M^0(F)\) satisfies \(\pi \cong \theta(\pi)\) and \(\pi^+(1 \times \theta)\) is an intertwining operator between \(\theta(\pi)\) and \(\pi\).

We shall write \(\Pi_{\text{temp}}(M^0(F))\), \(\Pi_{\text{temp}}(M^+(F))\) and \(\Pi_{\text{temp}}(M(F))\) for the subsets of tempered representations in \(\Pi(M^0(F))\), \(\Pi(M^+(F))\) and \(\Pi(M(F))\), respectively.
Chapter 3

A Local Twisted Trace Formula

In this chapter, we shall establish the local twisted trace formula, which is one of the main results of this thesis. The chapter is organized as follows. In §3.1, we will describe the motivation for the problem of the local twisted trace formula. To get a feeling for the formula, in §3.2, we will begin with the easy case where $G^0(F)$ is compact. Next, we will study the case where $G^0(F)$ is a connected reductive group. Arthur’s truncation, which is introduced in §3.3, can deal with certain divergence problems that appear in the study of the trace formula. Also in §3.3, we will make the problem of the local twisted trace formula more explicit. To use the truncation process and the theory of harmonic analysis on $G^0$ from [10] directly, we need to obtain a modified expression for the kernel function $K(x, y)$. In §3.4, the geometric side of the local twisted trace formula will be discussed. To deal with the spectral side, we will introduce a twisted inner product in §3.5. Based on this inner product, we will deduce the spectral side of the trace formula. In §3.6, we will obtain the local twisted trace formula (Theorem 3.6.1).
3.1 Motivation

We reviewed the origin of the local ordinary trace formula ([10]) in the Introduction. In this section, we will state the problem of the local twisted trace formula in two explicit forms. The notation from Chapter 2 will be used freely.

The local twisted trace formula concerns certain non-connected algebraic reductive groups over the local field $F$. As in Chapter 2, we assume that $\theta$ is an (outer) automorphism of $G^0(F)$ of finite order $n$, which preserves a pair $(P_0^0, M_0^0)$ in $G^0$, where $P_0^0$ is a fixed minimal parabolic subgroup of $G^0$ and $M_0^0$ a fixed Levi subgroup of $P_0^0$.

Recall that

$$G^+(F) = G^0(F) \rtimes \langle \theta \rangle, \quad G(F) = G^0(F) \rtimes \theta.$$

We will focus on the connected component $G$ of $G^+$. The set $G(A_F)$ of $A_F$-valued points in $G$ is just $G(F) \times G(F)$, where $A_F = F \oplus F$.

The local twisted trace formula is derived from the twisted regular representation, which is described below. Let $\phi$ be a function in $L^2(G^0(F))$. We can then define the twisted regular representation

$$(R(y_1, y_2)\phi)(x) = \phi(y_1^{-1}xy_2), \ x \in G^0(F), \ y_1, y_2 \in G(F),$$

of $G(A_F) = G(F) \times G(F)$ on the Hilbert space $L^2(G^0(F))$. This provides a canonical extension of the regular representation of $G^0(A_F)$ to the group generated by $G(A_F)$. Let $f$ be a smooth, compactly supported function on $G(A_F)$ of the form

$$f(y_1, y_2) = f_1(y_1)f_2(y_2), \ y_1, y_2 \in G(F).$$

Then, the kernel of $R(f) = \int_{G(F)} \int_{G(F)} f_1(y_1)f_2(y_2)R(y_1, y_2)dy_1dy_2$ is

$$K(x, y) = \int_{G(F)} f_1(xu)f_2(uy)du = \int_{G(F)} f_1(u)f_2(x^{-1}uy)du,$$
which is a smooth function on $G^0(F) \times G^0(F)$. The local twisted trace formula is derived by studying the geometric and spectral expansions of the trace of $R(f)$.

To apply directly the theory of local harmonic analysis on $G^0$ from [10], next, we will describe a correspondence between the functions on $G$ and the functions on $G^0$ (see (3.1)). Based on this correspondence, we can develop another picture of the statements above.

As in Chapter 2, we assume that $P_0^0$ is the identity component of a minimal parabolic subgroup $P_0^+$ of $G^+$ over $F$ and $P_0 = P_0^0 \rtimes \theta$. Fix a maximal compact subgroup $K$ of $G^0(F)$. Assume that $K$ and $A_{M_0^0}(F)$ are in good relative position in the sense of §2.1. Write $\mathcal{P}(M^0)$ for the set of parabolic subgroups $Q$ of $G^0$ over $F$ with Levi component $M_Q = M^0$. Let $\mathcal{P}(M^0)$ and $\mathcal{L}(M^0)$ denote the finite sets of parabolic subgroups and Levi subgroups of $G^0$ that contain $M^0$. In the special case where $M^0 = M_0^0$, we will generally write $\mathcal{P}^0 = \mathcal{P}(M_0^0)$ and $\mathcal{L}^0 = \mathcal{L}(M_0^0)$. As in §2.1, we write $\mathcal{P} = \mathcal{P}(M_0)$ and $\mathcal{L} = \mathcal{L}(M_0)$, which are the set of all Levi subsets of $G$ which contain $M_0$ and the set of all parabolic subsets $P$ of $G$ with $M_P = M_0$, respectively.

To state the problem of the local twisted trace formula in another form, we now give the correspondence between the functions on $G$ and the functions on $G^0$. Any function $f(x \rtimes \theta)$ on $G(F)$ can be regarded as a function $f_0$ on $G^0(F)$ defined by

$$f_0(x) := f(x \rtimes \theta), \quad \forall \ x \in G^0.$$  \hspace{1cm} (3.1)

For convenience, we still use $f$ to denote the function $f_0$ on $G^0(F)$. This correspondence will be used freely.
The kernel function $K(x, y)$ can be written as

$$
K(x, y) = \int_{G^0(F)} f_1(u)f_2(x^{-1}uy)du \quad (u = u_0 \times \theta)
$$

$$
= \int_{G^0(F)} f_1(u_0 \times \theta)f_2(x^{-1}(u_0 \times \theta)y)du_0
$$

$$
= \int_{G^0(F)} f_1(u_0 \times \theta)f_2(x^{-1}u_0\theta(y) \times \theta))du_0 \quad (u_0 \to u)
$$

$$
= \int_{G^0(F)} f_1(u)f_2(x^{-1}u\theta(y))du.
$$

It is also equal to

$$
\int_{G^0(F)} f_1(xu)f_2(u\theta(y)))du.
$$

There is another way to obtain the kernel function $K(x, y)$. Let

$$(R_\theta(y_1, y_2)\phi)(x) = \phi(\theta^{-1}(y_1^{-1}xy_2)),$$ 

$\phi \in L^2(G^0(F))$, $x, y_1, y_2 \in G^0(F)$,

be the twisted representation of $G^0(F) \times G^0(F)$ on $L^2(G^0(F))$.

Let $f$ be a smooth, compactly supported function on $G^0(A_F)$ of the form

$$
f(y_1, y_2) = f_1(y_1)f_2(y_2), \quad y_1, y_2 \in G^0(F).
$$

Define an operator $R_\theta(f)$ on $L^2(G^0(F))$ by

$$(R_\theta(f)\phi)(x) = \int_{G^0(F)} \int_{G^0(F)} f_1(u)f_2(y)\phi(\theta^{-1}(u^{-1}xy))dudy.$$

It is possible to check that

$$
K(x, y) = \int_{G^0} f_1(xu)f_2(u\theta(y)))du
$$
is also the kernel function of $R_\theta(f)$.

Based on the above interpretation, we can consider some more general operators that are introduced in [17] for the global trace formula. They are also studied in [25] for the theory of twisted endoscopy. Let $\omega$ be a quasicharacter of $G^0(F)$ and $R_\omega$ the operator of pointwise multiplication by $\omega$. We define a more general operator $R_{\theta, \omega}(f)$ on $L^2(G^0(F))$ given by

$$R_\theta(f)R_\omega.$$

More precisely, we have

$$\begin{align*}
(R_{\theta, \omega}(f)\phi)(x) &= \int_{G^0(F)} \int_{G^0(F)} f_1(u)f_2(y)\omega(\theta^{-1}(u^{-1}xy))\phi(\theta^{-1}(u^{-1}xy))dudy \\
&= \int_{G^0(F)} K(x,y)\phi(y)dy,
\end{align*}$$

where

$$K(x, y) = \omega(y) \int_{G^0(F)} f_1(xu)f_2(u\theta(y))du = \omega(y) \int_{G^0(F)} f_1(u)f_2(x^{-1}u\theta(y))du.$$ 

The twisted trace formula presented in this thesis was derived by studying the trace of the operator $R_{\theta, \omega}(f)$, i.e., the integral of $K(x, x)$ on $G^0(F)$. To simplify the proofs, we assumed that $\omega = 1$ for the most part. However, all the proofs for $\omega = 1$ can be applied to the case of non-trivial $\omega$ with some modifications.

Remark. Let Inn$(G^0(F))$ denote the group of inner automorphisms of $G^0(F)$. If $\theta \in$ Inn$(G^0(F))$, the local twisted trace formula is essentially the ordinary one. In fact, if $\theta(x)$ equals $hxh^{-1}$ for some $h \in G^0(F)$, then it is sufficient to replace $f(x, y)$ by another function $f_\theta(x, y) = f(xh^{-1}, yh^{-1})$ in the ordinary local trace formula. Consider
the twisted kernel function $K(x, y)$. It is possible to verify that

$$K(x, y) = \int_{G_0(F)} f_1(xu) f_2(u\theta(y))du$$

$$= \int_{G_0(F)} f_1(xuh^{-1}) f_2(uyh^{-1})du$$

$$= \int_{G_0(F)} f_{\theta,1}(xu) f_{\theta,2}(uy)du.$$ 

That is, if $\theta$ is inner, the twisted trace formula becomes the ordinary formula, so the local twisted trace formula is a generalization of the local ordinary trace formula \[10\].

We will develop the theory of the local twisted trace formula as follows. For the geometric side, we will use the first picture, i.e., the language of $G^+$; for the spectral side, the second picture, i.e., the language of $G^0$ and $\theta$, will be applied. At the end of this chapter, we will merge the two pictures.

### 3.2 The Compact Case

In this section, we study the local twisted trace formula when the group is compact. Let $G^0$ be a connected compact Lie group and $\theta$ be an (outer) automorphism of $G^0$ with finite order $n$. Recall that we can construct $G^+ = G^0 \rtimes \langle \theta \rangle$ and $G = G^0 \rtimes \theta$.

Consider the twisted representation of $G \times G$

$$\phi(x, y_1, y_2) = \phi(y_1^{-1}xy_2), \quad \phi \in L^2(G^0), \quad x \in G_0, y_1, y_2 \in G,$$

on the Hilbert space $L^2(G^0)$. Assume that $f$ is a smooth function on $G \times G$ of the form

$$f(y_1, y_2) = f_1(y_1)f_2(y_2), \quad y_1, y_2 \in G.$$
Then, the operator $R(f)$ has a kernel $K(x, y)$ that is equal to

$$
\int_G f_1(u)f_2(x^{-1}uy)du. \quad (3.2)
$$

In the compact case, the local twisted trace formula is directly derived from the calculation of the trace of $R(f)$. Let $tr(R(f))$ denote the trace of $R(f)$. It is well known that

$$
tr(R(f)) = \int_{G^0} K(x, x)dx.
$$

First, we calculate the geometric side of $tr(R(f))$. From §4.5 of [14], we know that the set of all conjugacy classes of $G$ can be chosen in a maximal torus $T = T_0\gamma_0$ in the sense of Definition 2.1.3, where $\gamma_0$ is a fixed regular element in $G$ and $T_0$ the neutral component of the centralizer of $\gamma_0$ in $G^0$. Recall the twisted Weyl Integration Formula for $G$ :

$$
\int_G h(x)dx = \frac{1}{|W(G^0, T)|} \int_T |D(\gamma)| (\int_{T_0\setminus G^0} h(v^{-1}\gamma v)dv) d\gamma, \quad (3.3)
$$

where $W(G^0, T) = N_{G^0}(T)/Z_{G^0}(T)$ is the Weyl group of $(G^0, T)$ and

$$
D(\gamma) = det(1 - Ad(\gamma))_{G/\gamma},
$$
is the Weyl discriminant.

Using (3.2) and (3.3), we obtain

$$
Tr(R(f)) = \int_{G^0} K(x, x)dx
$$

$$
= \int_{G^0} \left( \int_G f_1(u)f_2(x^{-1}ux)du \right) dx
$$

$$
= \int_{G^0} \left( \frac{1}{|W(G^0, T)|} \int_T |D(\gamma)| \int_{T_0\setminus G^0} f_1(x_1^{-1}\gamma x_1)f_2(x_1^{-1}x_1^{-1}\gamma x_1)dx_1 \right) d\gamma \, dx
$$

$$
= \frac{1}{|W(G^0, T)|} \int_T (|D(\gamma)| \int_{T_0\setminus G^0} \int_{T_0\setminus G^0} f_1(x_1^{-1}\gamma x_1)f_2(x_2^{-1}\gamma x_2)dx_2 dx_1) \, d\gamma.
$$
Hence, the geometric side of $tr(R(f))$ is

$$J_{geom}(f) = \frac{1}{|W(G^0, T)|} \int_T J(\gamma, f)d\gamma, \quad (3.4)$$

where

$$J(\gamma, f) = |D(\gamma)| \int_{T^0 \backslash G^0} \int_{T^0 \backslash G^0} f_1(x_1^{-1} \gamma x_1) f_2(x_2^{-1} \gamma x_2) dx_2 dx_1.$$

Next, we compute the spectral side of $tr(R(f))$. To clarify the calculation, let us state another expansion of $K(x, y)$, which is equal to

$$\int_{G^0} f_1(u) f_2(x^{-1} u \theta(y)) du. \quad (3.5)$$

Let $\Pi(G^0)$ be the set of all equivalence classes of irreducible representations of $G^0$. If $\pi$ belongs to $\Pi(G^0)$, we write $n_\pi$ for the dimension of $\pi$. The Plancherel formula for $G^0$ is

$$\phi(x) = \sum_{\pi \in \Pi(G^0)} n_\pi \cdot tr(\pi(x^{-1}) \pi(\phi)), \quad (3.6)$$

where $\phi(x)$ is a smooth function on $G^0$. Set

$$h(v) = \int_{G^0} f_1(u) f_2(x^{-1} v u \theta(y)) du.$$

Then, $h(v)$ is a function on $G^0$ depending on $x$ and $y$. We see that $h(1) = K(x, y)$. Applying the Plancherel formula (3.6) to $h(v)$, we have

$$K(x, y) = h(1) = \sum_{\pi \in \Pi(G^0)} n_\pi \cdot tr(\pi(x) \pi(f_2) \pi(\theta(y^{-1})) \pi(f_1^\gamma)),$$

where $f_1^\gamma(x) = f_1(x^{-1})$. 
Therefore, the spectral side of $tr(R(f))$ is

$$
\sum_{\pi \in \Pi(G^0)} n_\pi \cdot tr((\int_{G^0} (\pi(x)\pi(f_2)\pi(\theta(x^{-1}))dx) \circ \pi(f_1^\vee))).
$$

(3.7)

For fixed $\pi$ in $\Pi(G^0)$, set $A_\pi = \int_{G^0} \pi(x)\pi(f_2)\pi(\theta(x^{-1}))dx$. It is easy to check that $A_\pi \circ \pi(g) = \pi(\theta^{-1}(g)) \circ A_\pi$ for any $g \in G^0$, i.e., $A_\pi$ is an intertwining operator between $\pi$ and $\theta(\pi)$. By Schur’s Lemma, we know that

$$
A_\pi = \begin{cases} 
\text{an element in } Hom(\pi, \theta(\pi)) \cong \mathbb{C} & \text{if } \pi \cong \theta(\pi), \\
0 & \text{otherwise}.
\end{cases}
$$

Consequently, if $\pi \not\cong \theta(\pi)$, then the corresponding term on the spectral side will disappear. Therefore, we can assume that $\pi \cong \theta(\pi)$ on the spectral side of the local twisted trace formula.

Choose a unitary operator $B_\pi \in Hom(\pi, \theta(\pi))$ with $B_\pi^n = 1$; then,

$$
\pi(\theta(x^{-1})) = B_\pi^{-1} \circ \pi(x^{-1}) \circ B_\pi, \quad \forall g \in G^0.
$$

(3.8)

Substituting (3.8) into (3.7) and using Schur’s orthogonality relations, we get

$$
tr(R(f)) = \sum_{\pi \cong \theta(\pi) \atop \pi \in \Pi(G^0)} tr(\pi(f_2) \circ B_\pi^{-1}) \cdot tr(B_\pi \circ \pi(f_1^\vee)).
$$

Here, $tr(A)$ means the trace of $A$ if $A$ is an operator of trace class on a Hilbert space. This equation is independent of the choice of $B_\pi$.

By Corollary 2.4.3, an irreducible representation $\pi$ of $G^0$ satisfying $\pi \cong \theta(\pi)$ corresponds to a unique irreducible representation $\pi^+$ of $G^+$ in $\{\Pi(G)\}_{\Xi_G}$ such that the restriction of $\pi^+$ to $G^0$ is $\pi$. We know that $\pi^+(1 \times \theta)$ is an intertwining operator between $\theta(\pi)$ and $\pi$. Hence, $B_\pi$ can be chosen as $\pi^+(1 \times \theta)^{-1}$. By (3.1), we can regard $f_i \ (i = 1, 2)$
as functions on $G$:

$$f_i(x \rtimes \theta) = f_i(x), \quad \forall x \in G^0.$$  

Then, $f_1$ and $f_2$ are the same functions on $G$ as on the geometric side. Let $(\pi^+)^\vee$ be the contragredient of $\pi^+$; then, the spectral side can then be written as

$$J_{\text{spec}}(f) = \sum_{\pi^+ \in \mathcal{P}(G)_{\Xi_G}} J(\pi^+, f),$$

where $J(\pi^+, f) = \text{tr}((\pi^+)^\vee(f_1)) \cdot \text{tr}(\pi^+(f_2)).$

Finally, combining (3.4) and (3.9), we obtain the following theorem.

**Theorem 3.2.1:** The local twisted trace formula for the compact Lie group $G^+$ is

$$\frac{1}{|W(G^0, T)|} \int_T J(\gamma, f) \, d\gamma = \sum_{\pi^+ \in \mathcal{P}(G)_{\Xi_G}} J(\pi^+, f).$$

### 3.3 Arthur’s Truncation

We now turn to the main object of study, the connected component $G = G^0(F) \rtimes \theta$ of $G^+ = G^0(F) \rtimes \langle \theta \rangle$. We know that $G^+$ is the group generated by $G$.

The twisted trace formula will be derived by studying the trace of the operator $R(f)$ i.e., the integral of $K(x, x)$ on $G^0(F)$. However, $K(x, x)$ is not integrable on $G^0(F)$ in general. We must use Arthur’s truncation ([10], §3) to deal with this divergence problem. For convenience, we recall the fundamentals of Arthur’s truncation.

The truncation is based on the decomposition $G^0(F) = KM_0^0(F)K$. For any point $T \in a_0 = \text{Hom}(X(M_0^0)F, \mathbb{R})$ and $P_0^0 \in \mathcal{D}(M_0^0)$, let $T_{P_0^0}$ be the unique $W_0^0$-translate of $T$ that lies in the closure of the chamber $a_{P_0^0}^+$, where $W_0^0$ is the Weyl group of $(G^0, A_{M_0^0})$. 
We know that

\[ \{ T_{P_0^0} : P_0^0 \in \mathcal{P}(M_0^0) \} \]

is a positive \((G^0, M_0^0)\)-orthogonal set, which will be denoted by \(T\). For any \((G^0, M^0)\)-orthogonal set \(Y\), a determined convex hull \(S_M(Y)\) in \(\mathfrak{a}_{M^0}/\mathfrak{a}_{G^0}\) was determined in §2.2.

Let \(S_{M_0^0}(T)\) be the convex hull in \(\mathfrak{a}_0^0/\mathfrak{a}_{G^0}\) that is associated to \(T\).

We shall require that \(T\) be highly regular in the sense that the number

\[ d(T) = \inf \{ \alpha(T_{Q_0^0}) : \alpha \in \Delta_{Q_0^0}, Q_0^0 \in \mathcal{P}(M_0^0) \} \]

is suitably large. Note that

\[ \{ T_{P_0^0} : P_0^0 \in \mathcal{P}(M^0) \}, \text{ for any } M^0 \in \mathcal{L}(M_0^0), \]

is a \((G^0, M^0)\)-orthogonal set, where \(T_{P_0^0}\) is the projection onto \(\mathfrak{a}_{M^0}\) of any point \(T_{Q_0^0}\) and \(Q_0^0\) is any group in \(\mathcal{P}(M_0^0)\) that is contained in \(P^0\). For any Levi subset \(M \in \mathcal{L}(M_0)\), we can also associate a \((G, M)\)-orthogonal set

\[ \{ T_P : P \in \mathcal{P}(M) \}, \]

where \(T_P\) is the projection onto \(\mathfrak{a}_M\) of any point \(T_{Q_0^0}\), and \(Q_0^0\) is any group in \(\mathcal{P}(M_0^0)\) that is contained in \(P^+\).

For the truncation process in [10], it is necessary to fix a highly regular point \(T \in \mathfrak{a}_0^0 = \mathfrak{a}_{M_0^0}\). The convex hull \(S_{M_0^0}(T)\) is a large compact subset of \(\mathfrak{a}_0^0/\mathfrak{a}_{G^0}\). Define \(u(x, T)\) to be the characteristic function of the set of points

\[ x = k_1 m k_2, \quad m \in A_{G^0}(F) \setminus M_0^0(F), \quad k_1, k_2 \in K, \]

in \(A_{G^0}(F) \setminus G^0(F)\) such that \(H_{M_0^0}(m)\) lies in the convex hull \(S_{M_0^0}(T)\). We know that the
support of $u(x, T)$ is compact in $A_{G^0}(F) \setminus G^0(F)$.

The ordinary kernel function is

$$K_0(x, y) = \int_{G^0(F)} f_1(u) f_2(x^{-1}uy) du,$$

which is a function on $A_{G^0}(F) \setminus G^0(F)$, i.e., $K_0(x, y)$ is invariant under the change of variables $(x, y) \rightarrow (ax, ay)$ for any $a \in A_{G^0}$. It is obvious that the integral

$$K_T^0(f) = \int_{A_{G^0(F)} \setminus G^0(F)} K_0(x, x) u(x, T) dx$$

converges.

The twisted kernel function

$$K(x, y) = \int_{G(F)} f_1(u) f_2(x^{-1}uy) du$$

is invariant under the change of variables $(x, y) \rightarrow (ax, ay)$ for any $a \in A_G(F) \subset A_{G^0}(F)$. We can define a similar truncated integration

$$K^T(f) = \int_{A_G(F) \setminus G^0(F)} K(x, x) u(x, T) dx. \quad (3.11)$$

In Chapter 2, $A_G$ is defined to be the split component of the centralizer of $G$ in $G^0$ and $A_{G^0}$ the split component of the center of $G^0$. We have the relation $A_G \subset A_{G^0}$. Explicitly,

$$A_G = \{ a \in A_{G^0} : \theta(a) = a \}^0,$$

where $X^0$ means the neutral component of a group $X$.

It is not obvious that the truncated integration $K^T(f)$ converges since $u(x, T)$ is only a characteristic function of a compact set of $A_{G^0}(F) \setminus G^0(F)$. However, the integration domain for $K^T(f)$ is $A_G(F) \setminus G^0(F)$. The next step is to prove the convergence of $K^T(f)$. 
For convenience, we shall define a modified kernel function \( \tilde{K}(x, y) \) that is invariant under the change of variables \((x, y) \to (ax, ay)\) for any \( a \in \mathcal{G}_0(F) \). Set
\[
\tilde{K}(x, y) = \int_{\mathcal{G}_0(F)/\mathcal{G}(F)} K(ax, ay) da.
\]

We show that \( \tilde{K}(x, y) \) is well defined in the following lemma.

**Lemma 3.3.1 :** \( \tilde{K}(x, y) \) is well defined, i.e., the integration
\[
\int_{\mathcal{G}_0(F)/\mathcal{G}(F)} K(ax, ay) da
\]
is convergent for any \((x, y) \in \mathcal{G}_0(F) \times \mathcal{G}_0(F)\). Furthermore, \( \tilde{K}(x, x) \) is a function on \( \mathcal{G}_0(F) \setminus \mathcal{G}(F) \).

**Proof.** Using the facts that \( f_i \in C_c^\infty(\mathcal{G}_0(F)) \) \((i = 1, 2)\) and \( \varphi(h) = (I - \theta)(h) \) for \( h \in a_{\mathcal{G}}/a_G \) is an automorphism of \( a_{\mathcal{G}}/a_G \), we find that
\[
K(ax, ay) = \int_{\mathcal{G}_0(F)} f_1(u)f_2(a\theta^{-1}(a)x^{-1}u\theta(y)) du
\]
belongs to \( C_c^\infty(\mathcal{G}_0(F)/\mathcal{G}(F)) \) as a function of \( a \) for fixed \( x, y \in \mathcal{G}_0(F) \).

Hence,
\[
\tilde{K}(x, y) = \int_{\mathcal{G}_0(F)/\mathcal{G}(F)} K(ax, ay) da
\]
converges. Therefore, \( \tilde{K}(x, y) \) is invariant under the change of variables
\[(x, y) \to (ax, ay) \quad \forall a \in \mathcal{G}_0(F)\].

Consequently, \( \tilde{K}(x, x) \) is a function on \( \mathcal{G}_0(F) \setminus \mathcal{G}(F) \).
Obviously, the truncated integration \( K^T(f) = \int_{A G_0(F)} K(x, x) u(x, T) dx \) equals
\[
\int_{A G_0(F)} \tilde{K}(x, x) u(x, T) dx. \tag{3.13}
\]
Combining the above facts, we can now state the important result of this section.

**Proposition 3.3.2:** The truncated integral \( K^T(f) \) converges.

Our next goal is to study the distribution \( K^T(f) \) to obtain the local twisted trace formula.

### 3.4 The Geometric Side

In this section, we study the geometric side of the local twisted trace formula. To obtain a geometric expansion of \( K(x, y) \), it is necessary to apply the twisted Weyl Integration Formula. We shall then use Arthur’s method to derive a suitable expression \( J^T(f) \) which is an asymptotic formula for \( K^T(f) \). The main result of this section is Proposition 3.4.3.

#### 3.4.1 The Geometric Expansions for \( K(x, y) \) and \( K^T(f) \)

Let \( M_0 = M_0^+ \cap G(F) \) be the minimal Levi subset of \( G(F) \) and \( \mathcal{L} = \mathcal{L}(M_0) \) be the finite set of Levi subsets of \( G(F) \) which contain \( M_0 \). Recall that the twisted Weyl Integration Formula is Proposition 2.3.2:
\[
\int_{G(F)} h(x) dx = \sum_{M \in \mathcal{L}} \frac{|W_0^M|}{|W_0^G|} \int_{\Gamma_M(F)} |D(\gamma)| \int_{A_M(F) \setminus G_0(F)} h(x^{-1} \gamma x) dx d\gamma.
\]
for any \( h \) in \( C_c^\infty(G(F)) \).
Applying this formula to $K(x, y)$, we see that $K(x, y)$ equals

$$
\sum_{M \in \mathcal{L}} \frac{|W_0^M|}{|W_0^G|} \int_{\Gamma_{\text{ell}}(M(F))} |D(\gamma)| \left( \int_{A_M(F) \setminus G_0(F)} f_1(z^{-1} \gamma z) f_2(x^{-1} z^{-1} \gamma y) dz \right) d\gamma.
$$

(3.14)

We can now calculate the geometric side of

$$
K^T(f) = \int_{A_G(F) \setminus G_0(F)} K(x, x) u(x, T) dx.
$$

Substituting (3.14) into (3.11), we calculate

$$
K^T(f) = \sum_{M \in \mathcal{L}} \frac{|W_0^M| |W_0^G|^{-1}}{|W_0^G|} \int_{\Gamma_{\text{ell}}(M(F))} K^T(\gamma, f) d\gamma,
$$

(3.15)

where $K^T(\gamma, f)$ equals

$$
|D(\gamma)| \int_{A_M(F) \setminus G_0(F)} \int_{A_M(F) \setminus G_0(F)} f_1(x_1^{-1} \gamma x_1) f_2(x_2^{-1} \gamma x_2) u_M(x_1, x_2, T) dx_2 dx_1
$$

and

$$
u_M(x_1, x_2, T) = \int_{A_G(F) \setminus A_M(F)} u(x_1^{-1} ax_2, T) da.
$$

At this point, we have the geometric side of $K^T(f)$. However, this expansion is not suitable for our final goal. The next step is to find an asymptotic formula for $K^T(f)$, which will be achieved in the following subsection.

3.4.2 Asymptotic Formulas: $J^T(\gamma, f)$ and $J^T(f)$

To find a suitable asymptotic formula for $K^T(f)$, we need to define a new weight function $v_M(x_1, x_2, T)$ which is an asymptotic expansion for $u_M(x_1, x_2, T)$.

For any $P \in \mathcal{P}(M)$, it can be seen that the opposite parabolic subset $\overline{P}$ of $P$ is also
in $\mathcal{P}(M)$. Set

$$Y_P(x_1, x_2, T) = T_P + H_P(x_1) - H_P(x_2), \quad P \in \mathcal{P}(M);$$

then,

$$\mathcal{Y}_M(x_1, x_2, T) = \{Y_P(x_1, x_2, T) : P \in \mathcal{P}(M)\}$$

form a $(G, M)$-orthogonal set. If $d(T)$ is large relative to $x_1$ and $x_2$, then $\mathcal{Y}_M(x_1, x_2, T)$ is positive.

If $\mathcal{Y}$ is positive, recall that $\sigma(\cdot, \mathcal{Y})$ is the characteristic function of $S_M(\mathcal{Y})$. Set the weight function $v_M(x_1, x_2, T)$ equal to

$$\int_{A_G(F) \setminus A_M(F)} \sigma_M(H_M(a), \mathcal{Y}_M(x_1, x_2, T))da,$$

which is an asymptotic formula for $u_M(x_1, x_2, T)$. This will be proved in Lemma 3.4.1. If $\mathcal{Y}_M(x_1, x_2, T)$ is positive, then $\sigma_M(H_M(a), \mathcal{Y}_M(x_1, x_2, T))$ is the characteristic function of the convex hull $S_M(\mathcal{Y}_M(x_1, x_2, T))$ in $a_M / a_G$.

Let $J^T(\gamma, f)$ be

$$|D(\gamma)| \int_{A_M(F) \setminus G_0(F)} \int_{A_M(F) \setminus G_0(F)} f_1(x_1^{-1} \gamma x_1) f_2(x_2^{-1} \gamma x_2) v_M(x_1, x_2, T) dx_2 dx_1;$$

then, $J^T(\gamma, f)$ is an asymptotic formula for $K^T(\gamma, f)$.

We can now define the desired geometric expansion $J^T(f)$ by

$$J^T(f) = \sum_{M \in \mathcal{L}} |W_0^M| W_0^G |^{-1} \int_{\Gamma_{cell}(M(F))} J^T(\gamma, f) d\gamma.$$

(3.16)

In Proposition 3.4.2, we will see that $J^T(f)$ is an asymptotic expression for $K^T(f)$.

As in the ordinary case in [10], we must make use of a height function $\| \cdot \|$ on $G^+(F)$ that is determined by a finite dimensional representation $\Lambda_0 : G^+ \to GL(V_0)$ of $G^+$ on
This was originally defined in [1], and some related properties are summarized in §4 of [10]. These properties also hold for the twisted case.

The essential step is to estimate the difference between the two weight functions. Lemma 4.4 of [10] is one of the main technical results in the ordinary case. This result also holds for the twisted case and the proof is quite similar. We simply state the lemma.

**Lemma 3.4.1**: Given any $\delta > 0$, there exist positive constants $C$, $\varepsilon_1$, and $\varepsilon_2$ with the following property: for any $T$ with $\delta \|T\| \leq d(T)$ and all $x_1, x_2$ in the set

$$\{ x \in G^0(F) : \|x\| \leq e^{\varepsilon_2\|T\|} \},$$

we have the inequality

$$|u_M(x_1, x_2, T) - v_M(x_1, x_2, T)| \leq Ce^{-\varepsilon_1\|T\|}.$$

It is possible to show a relation for $K^T(f)$ and $J^T(f)$ in the twisted case similar to Proposition 4.5 of [10] in the ordinary case. We state the relation here without proof.

**Proposition 3.4.2**: For fixed $\delta > 0$, there exist positive constants $C$, $\varepsilon$ with the following property: for any $T$ with $\delta \|T\| \leq d(T)$, we have the inequality

$$|K^T(f) - J^T(f)| \leq Ce^{-\varepsilon\|T\|}.$$

In fact, for the $p$-adic case, the weight factors $u_M(x_1, x_2, T)$ and $v_M(x_1, x_2, T)$ are actually equal ([10], p. 39). Consequently, $K^T(f)$ is equal to $J^T(f)$. Therefore, we have a strong version of Proposition 3.4.2 in the $p$-adic case.

In applications, the most important case is the one where the support of $f_1$ or $f_2$ is contained in $\Gamma_{ell}(G(F))$. Under this condition, we see that the geometric side of the trace
formula is

\[
K(f) = \int_{\Gamma_{\ell}(G(F))} K(\gamma, f) d\gamma, \tag{3.17}
\]

where \(K(\gamma, f)\) equals

\[
|D(\gamma)| \int_{A_G(F) \backslash G_0(F)} \int_{A_G(F) \backslash G_0(F)} f_1(x_1^{-1} \gamma x_1) f_2(x_2^{-1} \gamma x_2) dx_2 dx_1.
\]

### 3.4.3 Explicit Geometric Expansions for \(J^T(f)\) and \(\tilde{J}(f)\)

By Proposition 3.4.2, we know that \(J^T(f)\) is an asymptotic geometric expansion for \(K^T(f)\). In §3.5, we will obtain a parallel asymptotic spectral expansion for \(K^T(f)\). One of our aims is to identify the asymptotic geometric expansion \(J^T(f)\) with the spectral expansion. For this purpose, we need to obtain a more explicit expression for \(J^T(f)\) as a function of \(T\).

If \(F\) is Archimedean, it is possible to check that \(v(x_1, x_2, T)\) is a polynomial in \(T\) as in the ordinary case (p. 43 of [10]). However, in the case where \(F\) being \(p\)-adic, the situation is more complicated. As in the ordinary case, we can use Arthur’s method to deal with the twisted case.

Here, we give only some definitions and list the main results for the geometric side of the local twisted trace formula. The details are analogous to the process for the ordinary case described on pp. 43-46 of [10].

If \(F\) is \(p\)-adic, as in §2.1, we have the surjective map

\[
H_M : A_M(F)/A_G(F) \rightarrow \tilde{a}_{M,F}/\tilde{a}_{M,F}.
\]

The kernel of this map is a compact group with volume 1. Let \(T\) be a point in the lattice \(a_{M_0,F}\), which is a sub-lattice of \(a_{M_0}\). For any \(P \in P(M)\), the point \(T_P\) belongs to \(a_{M,F}\).
We summarize the formulas and definitions as follows (" := " means definition):

(1) \( \tilde{\mathcal{L}}_M := (\tilde{a}_{M,F} + a_G)/a_G \),

(2) \( \mathcal{L}_M := (a_{M,F} + a_G)/a_G \),

(3) \( \mathcal{L}^\vee := \text{Hom}(\mathcal{L}, 2\pi i \mathbb{Z}) \), \( \mathcal{L} = \tilde{\mathcal{L}}_M \) or \( \mathcal{L}_M \),

(4) \( Y_P = Y_P(x_1, x_2, T) = T_P + H_P(x_1) - H_P(x_2) \).

It is possible to check that \( v_M(x_1, x_2, T) \) is equal to

\[
\sum_{v \in \mathcal{L}^\vee_M / \mathcal{L}_M} \lim_{\Lambda \to 0} \left( \sum_{P \in \mathfrak{P}(M)} \frac{1}{|\mathcal{L}_M / \mathcal{L}_M|} \sum_{X \in \mathcal{L}_M} (-1)^{|\Delta_P^M|} \varphi_P^\Lambda(X - Y_P) e^{(\Lambda + v)(X)} \right). \tag{3.18}
\]

For any positive integer \( k \), we shall write

\[ \mu_{\alpha,k} := k \log(q_F)\alpha^\vee, \]

where \( q_F \) is the order of the residue class field of \( F \) and \( \alpha \in \Delta_P \). The set

\[ \mathcal{L}_{M,k} := k \log(q_F)\mathbb{Z}(\Delta_P^\vee) = \{ \sum_{\alpha \in \Delta_P} n_\alpha \mu_{\alpha,k} : n_\alpha \in \mathbb{Z} \} \]

is a lattice in \( a_M^G \), which is independent of \( P \). If \( Y \) is any point in \( \mathcal{L}_M \) and \( X \) belongs to \( \mathcal{L}_M / \mathcal{L}_{M,k} \), let \( X_P(Y) \) be the representative of \( X \) in \( \mathcal{L}_M \) such that

\[ X_P(Y) - Y = \sum_{\alpha \in \Delta_P} r_{\alpha \mu_{\alpha,k}}, \tag{3.19} \]
with real numbers \( r_\alpha \) with \(-1 < r_\alpha \leq 0 \). Set

\[
\theta_{P,k}(\lambda) = \text{vol}(a_{M,F}/\mathcal{L}_{M,k})^{-1} \prod_{\alpha \in \Delta_P} (1 - e^{X_{\lambda,\alpha,k}}), \quad \lambda \in a_{M,F}^*.
\] (3.20)

If \( F \) is Archimedean, we have defined

\[
\theta_P(\lambda) = \text{vol}(a_P/F(\mathcal{\Delta}_P^\vee))^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee), \quad \lambda \in a_{M,F}^*
\]
in §2.1.

It is possible to establish that \( v_M(x_1, x_2, T) \) equals

\[
\sum_{\nu \in \mathcal{L}_M/\mathcal{L}_M} \frac{1}{|\mathcal{L}_M/\mathcal{L}_M|} \sum_{X \in \mathcal{L}_M/\mathcal{L}_{M,k}} e^{X+\nu}(X_P+Y_P)\theta_{P,k}(\Lambda + \nu)^{-1},
\] (3.21)

where \( X_P = X_P(0) \).

We can assume that \( T \) lies in the intersection of the lattice \( \mathcal{L}_0 \) and a fixed chamber \( a_0^+ \) of \( a_{M,F}/a_G \), where \( \mathcal{L}_0 = \mathcal{L}_{M_0} = (a_{M,F} + a_G)/a_G \).

It is possible to show that

\[
v_M(x_1, x_2, T) = \sum_{\xi \in (\frac{1}{2}\mathcal{L}_0^\vee)/\mathcal{L}_0^\vee} q_{\xi}(T)e^{\xi(T)}, \quad T \in \mathcal{L}_0 \cap a_0^+.
\] (3.22)

The constant term \( q_0(0) \) is a well-defined function of \( v_M(x_1, x_2, T) \). By (3.21), the explicit expansion of \( q_0(0) \) is

\[
\lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}(M)} \frac{1}{|\mathcal{L}_M/\mathcal{L}_M|} \sum_{X \in \mathcal{L}_M/\mathcal{L}_{M,k}} e^{X+P(x_1)-P(x_2)}\theta_{P,k}(\Lambda)^{-1},
\] (3.23)

which is denoted by \( \tilde{v}_M(x_1, x_2) \).

The following proposition holds for \( F \) \( p \)-adic or Archimedean.
Proposition 3.4.3: We have a decomposition for $J^T(f)$:

$$J^T(f) = \sum_{\xi \in (\mathbb{L}_0')^{N}/\mathbb{L}_0'} p_{\xi}(T, f)e^{\xi(T)}, \quad T \in \mathcal{L}_0 \cap a_0^\perp,$$

where $N$ is a fixed positive integer, and $p_{\xi}(T, f)$ is a polynomial in $T$. Setting $\xi = 0$ and $T = 0$, we obtain the constant term of $J^T(f)$:

$$\tilde{J}(f) = p_0(0, f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\text{Fell}(M(F))} \tilde{J}_M(\gamma, f)d\gamma,$$

where $\tilde{J}_M(\gamma, f)$ equals

$$|D(\gamma)| \int_{A_M(F)\backslash G^0(F)} \int_{A_M(F)\backslash G^0(F)} f_1(x_1^{-1}\gamma x_1)f_2(x_1^{-1}\gamma x_2)\tilde{v}_M(x_1, x_2)dx_2dx_1.$$

In §3.6, We will see that the geometric side of the local trace formula will be derived from $\tilde{J}(f)$, the constant term of $J^T(f)$.

3.5 The Spectral Side

In this section, we will study the spectral side of the local twisted trace formula. Recall that $G^+(F)$ is isomorphic to $G^0(F) \rtimes \langle \theta \rangle$ and $G$ is $G^0(F) \rtimes \theta$. It is easy to check that the split component of the centralizer of $G(F)$ in $G^0(F)$ is the neutral component of the subgroup consisting of $\theta$-invariant points in $A_{G^0}$. In §3.6, we will obtain the parallel spectral expansions.
3.5.1 The Spectral Expansions for $K(x, y)$ and $K^T(f)$

In this subsection, we will obtain the spectral expansion of the twisted kernel function $K(x, y)$. The notation in Chapter 2 will be used in this subsection freely.

Let $M^0 \in \mathcal{L}^0 = \mathcal{L}(M_0^0)$ be a Levi subgroup of $G^0$ over $F$. We write $\mathcal{P}(M^0)$ for the set of parabolic subgroups $P$ of $G^0$ over $F$ with the Levi component $M_P = M^0$ (in other sections, $P$ usually denotes a parabolic subset of $G$ over $F$). Recall that $\mathcal{F}(M^0)$ and $\mathcal{L}(M^0)$ denote the finite sets of parabolic subgroups and Levi subgroups of $G^0$ that contain $M^0$, respectively. Given $\sigma \in \Pi_{\text{temp}}(M^0)$ and $\lambda \in \mathfrak{a}_{M^0}^*$, we write

$$I_P(\sigma_\lambda, x), \quad P \in \mathcal{P}(M^0), \ x \in G^0(F)$$

for the corresponding parabolically induced representation of $G^0(F)$ as usual. There are two realizations of the induced representation: the induced and the compact picture. They are equivalent but have different uses. The details of the theory can be found in [24] for Lie groups and [28] for $p$-adic groups.

For convenience, we now recall these two pictures. Let $(\sigma, V_\sigma)$ be an irreducible tempered representation of $M^0$ and $\lambda \in \mathfrak{a}_{M^0}^*$.

(1). Induced picture:

A dense subspace of the induced representation space is

$$\mathcal{H}_P(\sigma_\lambda) = \{ f : G^0 \rightarrow V_\sigma \text{ continuous} \mid f(nmx) = e^{(\lambda + \rho)(H_P(m))} \sigma(m)f(x) \}$$

with the norm $||f||^2 = \int_K |f(k)|^2 dk$, where $n \in N_P$, $m \in M^0$, and $x \in G^0$. $G^0$ acts on $\mathcal{H}_P(\sigma_\lambda)$ by

$$I_P(\sigma_\lambda, g)f(x) = f(xg).$$

(2). Compact picture:

Define $K_{M^0} = K \cap M^0$. In this picture, a dense subspace of the induced representation
The space is

$$\mathcal{H}_P(\sigma) = \{ f : K \to V_\sigma \text{ continuous} \mid f(nmk) = \sigma(m)f(k) \}$$

with the norm $$||f||^2 = \int_K |f(k)|^2 \, dk$$, where $$n \in K \cap N_p$$, $$m \in K_{M^0}$$, and $$k \in K$$. If $$g$$ decomposes as

$$g = N_P(g)M^0(g)K_P(g), \quad N_P(g) \in N_P, \quad M^0(g) \in M^0, \quad K_P(g) \in K$$

through the decomposition $$G^0 = N_P M^0 K$$, then $$G^0$$ acts on $$\mathcal{H}_P(\sigma)$$ by

$$I_P(\sigma, g)f(k) = e^{(\lambda + \rho)(H_P(kg))}\sigma(M_P(kg))f(K_P(kg)).$$

This Hilbert space $$\mathcal{H}_P(\sigma)$$ of vector valued functions on $$K$$ is independent of $$\lambda$$.

For the non-connected group $$G^+(F)$$, there is an analogous twisted theory of induced representations. We refer the reader to [7].

Next, we recall some points concerning irreducible representations. There is a locally free action of $$i\mathfrak{a}_{M^0}^*$$ on $$\Pi_2(M^0(F))$$ defined by

$$\sigma_\lambda(m) = \sigma(m)e^{\lambda(H_{M^0}(m))}, \quad m \in M^0(F).$$

We shall write $$\{\Pi_2(M^0(F))\} = \Pi_2(M^0(F))/i\mathfrak{a}_{M^0}^*$$ for the set of $$i\mathfrak{a}_{M^0}^*$$-orbits in $$\Pi_2(M^0(F))$$ and $$\{\Pi_{\text{temp}}(M^0(F))\} = \Pi_{\text{temp}}(M^0(F))/i\mathfrak{a}_{M^0}^*$$ for the set of $$i\mathfrak{a}_{M^0}^*$$-orbits in $$\Pi_{\text{temp}}(M^0(F))$$.

To obtain the spectral expansion of $$K(x, y)$$, we need to apply Harish-Chandra’s Plancherel formula [21], [22]. It can be written as

$$h(1) = \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) tr(\pi(h)) \, d\pi;$$
more precisely,

\[ h(1) = \sum_{M_0 \in \mathcal{L}^0} \frac{|W_0^{M_0}|}{|W_0^{G^0}|} \int_{\Pi_2(M_0(F))} m(\sigma) \text{tr} (\mathcal{I}_P(\sigma, h)) d\sigma \tag{3.24} \]

for any \( h \in C_\infty^c(G^0(F)) \), where \( m(\sigma) \) is the Plancherel density and \( P \) is any parabolic subgroup in \( \mathcal{P}(M^0) \).

Recall that \( K(x, y) \) is equal to

\[ \int_{G^0(F)} f_1(u) f_2(x^{-1} u \theta(y)) du. \]

We can use the Plancherel formula to determine that \( K(x, y) \) equals

\[ \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) \text{tr} (\pi(f_1) \pi(x) \pi(f_2) \pi(\theta(y))^\ast) d\pi; \tag{3.25} \]

more precisely,

\[ \sum_{M_0 \in \mathcal{L}^0} \frac{|W_0^{M_0}|}{|W_0^{G^0}|} \int_{\Pi_2(M_0(F))} m(\sigma) \sum_{S \in \mathcal{B}_P(\sigma)} \text{tr} (\mathcal{I}_P(\sigma, x) S(f)) \text{tr} (\mathcal{I}_P(\sigma, \theta(y)) S \ast) d\sigma, \tag{3.26} \]

where \( \mathcal{B}_P(\sigma) \) is a fixed \( K \)-finite basis of the Hilbert space of the Hilbert-Schmidt operators on \( \mathcal{H}_P(\sigma) \) and \( S(f) = \mathcal{I}_P(\sigma, f_2) S \mathcal{I}_P(\sigma, f_1^\ast) \).

The next goal is to compute the spectral expansion of the modified kernel function \( \tilde{K}(x, y) \). We have shown that \( K^T(f) = \int_{A_G \setminus G^0(F)} K(x, x) u(x, T) dx \) equals

\[ \int_{A_G \setminus G^0(F) \setminus G^0(F)} \tilde{K}(x, x) u(x, T) dx, \]

where

\[ \tilde{K}(x, y) = \int_{A_G^0 / A_G} K(ax, ay) da \]
is defined in \[3.12\].

To obtain a suitable spectral expansion of $K^T(f)$, we need to study the spectral expansion of $\tilde{K}(x,y)$. First, we present some basic statements.

Let $\chi_\sigma$ be the central character of a representation $\sigma$ in $\Pi_{\text{temp}}(M^0)$. We define unramified characters and ramified characters.

**Definitions 3.5.1**: A (unitary) character $\chi$ of $A_{M^0}(F)$ is called *unramified* if there exists $\mu \in i(a_{M^0})^*$ such that $\chi(a) = e^{\mu(H_{M^0}(a))}$ for any $a \in A_{M^0}(F)$; otherwise, $\chi$ is called *ramified*.

Let $A^1_{M^0}$ denote $\ker(H_{M^0}) \cap A_{M^0}(F)$, which is a compact abelian subgroup of $A_{M^0}(F)$. It follows that a character $\chi$ of $A_{M^0}(F)$ is *unramified* if and only if the restriction of $\chi$ to $A^1_{M^0}$ is trivial.

Write

$$\tilde{\Pi}_{\text{temp}}(M^0(F)) = \{\sigma \in \Pi_{\text{temp}}(M^0(F)) \mid \sigma(z) = \sigma(\theta(z)) \text{ for any } z \in A_{G^0}\}$$

and

$$\tilde{\Pi}_2(M^0(F)) = \{\sigma \in \Pi_2(M^0(F)) \mid \sigma(z) = \sigma(\theta(z)) \text{ for any } z \in A_{G^0}\}.$$

If we set

$$i(a_{M^0})_{\theta}^* = \{\lambda \in i(a_{M^0})^* \mid e^{\lambda(a)} = e^{\lambda(\theta(a))}, \text{ for any } a \in a_{G^0,F}\},$$

then $\{\tilde{\Pi}_2(M^0(F))\} = \tilde{\Pi}_2(M^0(F))/i(a_{M^0})_{\theta}^*$ can be regarded as the set of $i(a_{M^0})_{\theta}^*$-orbits in $\tilde{\Pi}_2(M^0(F))$ and $\{\tilde{\Pi}_2(M^0(F))\}$ as a subset of $\{\Pi_2(M^0(F))\}$.

Recall that

$$\tilde{K}(x,y) = \int_{A_{G^0}(F)/A_G(F)} K(ax,ay)da.$$

Using the measure $d\sigma$ on $\Pi_2(M^0(F))$ defined by (2.4) in [10], we have the Plancherel formula for $\tilde{K}(x,y)$ in the following proposition.
Proposition 3.5.2: \( \tilde{K}(x,y) \) equals

\[
\frac{1}{|\det(1-\theta)_{G_0}|} \int_{\tilde{\Pi}_{\text{temp}}(G^0(F))} m(\pi) \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*)d\pi; \tag{3.27}
\]

more precisely, it is the product of

\[
\sum_{M^0 \in \mathcal{L}^0} \frac{|W_0^{M^0}|}{|W_0^{G^0}|} \int_{\tilde{\Pi}_2(M^0(F))} m(\sigma) \sum_{S \in \mathcal{B}_P(\sigma)} \text{tr}(\mathcal{I}_P(\sigma,x)S(f))\overline{\text{tr}(\mathcal{I}_P(\sigma,\theta(y))S)}d\sigma, \tag{3.28}
\]

where \( \tilde{\Pi}_{\text{temp}}(G^0(F)) \) and \( \tilde{\Pi}_2(M^0(F)) \) are defined above.

**Proof.** It is sufficient to prove (3.27).

Using the Plancherel formula (3.25) for \( K(x,y) \), we see that \( \tilde{K}(x,y) \) equals

\[
\int_{\mathcal{A}_{G^0}(F)/\mathcal{A}_G(F)} \left( \int_{\tilde{\Pi}_{\text{temp}}(G^0(F))} m(\pi)\chi_{\pi}(a\theta(a^{-1}))\text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*)d\pi \right)da.
\]

Next, we write \( \Pi_{\text{temp}}(G^0(F)) \) as a disjoint union of two suitable subsets.

Let \( i(\tilde{\mathfrak{a}}_{G^0})^* \) be the (unitary) dual group of \( \tilde{\mathfrak{a}}_{G^0,F}/\tilde{\mathfrak{a}}_{G,F} \). Set

\[
\Pi'_{\text{temp}}(G^0(F)) = \tilde{\Pi}_{\text{temp}}(G^0(F)) \times i(\tilde{\mathfrak{a}}_{G^0})^*,
\]

and

\[
\Pi^c_{\text{temp}}(G^0(F)) = \Pi_{\text{temp}}(G^0(F)) - \Pi'_{\text{temp}}(G^0(F)).
\]

Therefore, \( \Pi_{\text{temp}}(G^0(F)) \) is the disjoint union

\[
\Pi'_{\text{temp}}(G^0(F)) \sqcup \Pi^c_{\text{temp}}(G^0(F)).
\]

We always assume that the Plancherel measures are compatible.
It is possible to write

$$\tilde{K}(x, y) = \int_{A_{G_0}(F)/A_G(F)} K_1(a) \, da + \int_{A_{G_0}(F)/A_G(F)} K_2(a) \, da,$$

where

$$K_1(a) = \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) \chi_\pi(a \theta(a^{-1})) \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*) \, d\pi),$$

and

$$K_2(a) = \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) \chi_\pi(a \theta(a^{-1})) \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*) \, d\pi).$$

We shall deal with $\int_{A_{G_0}(F)/A_G(F)} K_1(a) \, da$ and $\int_{A_{G_0}(F)/A_G(F)} K_2(a) \, da$ separately.

1. We claim that $\int_{A_{G_0}(F)/A_G(F)} K_2(a) \, da = 0$.

In fact,

$$\int_{A_{G_0}(F)/A_G(F)} K_2(a) \, da = \int_{A_{G_0}(F)/A_G(F)} \left( \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) \chi_\pi(a \theta(a^{-1})) \times \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*) \, d\pi) \, da \right)$$

$$= \int_{A_{G_0}(F)/A_{G_0}(F)} \left( \int_{A_{G_0}(F)/A_G(F)} \left( \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) \chi_\pi(a \theta(a^{-1})) \chi_\pi(b \theta(b^{-1})) \right. \right.$$

$$\times \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*) \, d\pi) \, db) \, da$$

$$= \int_{A_{G_0}(F)/A_{G_0}(F)} \left( \int_{\Pi_{\text{temp}}(G^0(F))} \left( \int_{A_{G_0}(F)/A_G(F)} \chi_\pi(b \theta(b^{-1})) \, db \right) \times \right.$$

$$\left. m(\pi) \chi_\pi(a \theta(a^{-1})) \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(y))^*) \, d\pi) \right) \, da.$$

For $\pi \in \Pi_{\text{temp}}(G^0(F))$, since the restriction of $\chi_\pi(b \theta(b^{-1}))$ to $A_{G_0}^1/A_G^1$ is non-trivial, the component $\int_{A_{G_0}(F)/A_G(F)} \chi_\pi(b \theta(b^{-1})) \, db$ in the last expression is 0 by the orthogonality relation for characters of compact abelian groups. Consequently, $\int_{A_{G_0}(F)/A_G(F)} K_2(a) \, da = 0$.

2. To prove this proposition, it is sufficient to show that

$$\int_{A_{G_0}(F)/A_G(F)} K_1(a) \, da = \frac{1}{|\det(1-\theta)_{G_0^C}|} \int_{\Pi_{\text{temp}}(G^0(F))} m(\pi) \text{tr}(\pi(f_1^\vee)\pi(x)\pi(f_2)\pi(\theta(x))^*) \, d\pi).$$
For $\pi \in \Pi_{\text{temp}}^G(\mathcal{G}^0(F))$, there exists $\mu_\pi \in i(\tilde{\mathcal{G}}_{\mathcal{G}^0})^*$ such that $\chi_\pi(a\theta(a^{-1})) = e^{\mu_\pi(H_{\mathcal{G}^0}(a\theta(a^{-1})))}$ for any $a \in A_{\mathcal{G}^0}(F)$. Then,

$$
\int_{A_{\mathcal{G}^0}(F)/A_G(F)} K_1(a)da = \int_{A_{\mathcal{G}^0}(F)/A_G(F)} \left( \int_{\Pi_{\text{temp}}^G(\mathcal{G}^0(F))} m(\pi)\chi_\pi(a\theta(a^{-1})) \times tr(\pi(f_1^{\pi(y)})\pi(x)\pi(f_2)\pi(\theta(y))^*)d\pi \right)da
$$

$$
= \int_{A_{\mathcal{G}^0}(F)/A_G(F)} \left( \int_{i(\tilde{\mathcal{G}}_{\mathcal{G}^0})^*} \left( \int_{\Pi_{\text{temp}}^G(\mathcal{G}^0(F))} m(\pi_\lambda)\chi_\pi_\lambda(a\theta(a^{-1})) \times tr(\pi_\lambda(f_1^{\pi(y)})\pi_\lambda(x)\pi_\lambda(f_2)\pi_\lambda(\theta(y))^*)d\pi \right)d\lambda \right)da
$$

$$
= \int_{A_{\mathcal{G}^0}(F)/A_G(F)} \left( \int_{i(\tilde{\mathcal{G}}_{\mathcal{G}^0})^*} e^{\lambda(H_{\mathcal{G}^0}(a\theta(a^{-1})))} \left( \int_{\Pi_{\text{temp}}^G(\mathcal{G}^0(F))} m(\pi_\lambda) \times tr(\pi_\lambda(f_1^{\pi(y)})\pi_\lambda(x)\pi_\lambda(f_2)\pi_\lambda(\theta(y))^*)d\pi \right)d\lambda \right)da.
$$

After a change of variable in the last expression

$$
a \rightarrow z = H_{\mathcal{G}^0}(a\theta(a^{-1})),$$

since $\tilde{\mathcal{G}}_{\mathcal{G}^0,F}/\tilde{\mathcal{G}}_{G,F} = H_{\mathcal{G}^0}(A_{\mathcal{G}^0}(F)/A_G(F))$, we see that

$$
\int_{A_{\mathcal{G}^0}(F)/A_G(F)} K_1(a)da
$$

$$
= \frac{1}{|\text{det}(1-\theta)_{\tilde{\mathcal{G}}_{\mathcal{G}^0}}|} \int_{\tilde{\mathcal{G}}_{\mathcal{G}^0,F}/\tilde{\mathcal{G}}_{G,F}} \left( \int_{i(\tilde{\mathcal{G}}_{\mathcal{G}^0})^*} e^{\lambda(z)} \left( \int_{\Pi_{\text{temp}}^G(\mathcal{G}^0(F))} m(\pi_\lambda) \times tr(\pi_\lambda(f_1^{\pi(y)})\pi_\lambda(x)\pi_\lambda(f_2)\pi_\lambda(\theta(y))^*)d\pi \right)d\lambda \right)dz
$$

$$
= \frac{1}{|\text{det}(1-\theta)_{\tilde{\mathcal{G}}_{\mathcal{G}^0}}|} \int_{\Pi_{\text{temp}}^G(\mathcal{G}^0(F))} m(\pi)tr(\pi(f_1^{\pi(y)})\pi(x)\pi(f_2)\pi(\theta(y))^*)d\pi.
$$

In the last equality, we used

$$
\int_{\tilde{\mathcal{G}}_{\mathcal{G}^0,F}/\tilde{\mathcal{G}}_{G,F}} \int_{i(\tilde{\mathcal{G}}_{\mathcal{G}^0})^*} e^{-\lambda(z)}g(\lambda)d\lambda dz = g(0)
$$

via the Plancherel formula for the abelian group $\tilde{\mathcal{G}}_{\mathcal{G}^0,F}/\tilde{\mathcal{G}}_{G,F}$. 
In summary, we have shown that \( \tilde{K}(x, y) \) equals

\[
\frac{1}{|\det(1 - \theta)|_{G_0}} \int_{\Pi_{temp}(G^0(F))} m(\pi)\text{tr}(\pi(f_1\lambda)\pi(x)\pi(f_2)\pi(\theta(y))^*)d\pi.
\]

The Plancherel formula for \( \tilde{K}(x, y) \) is important for defining a twisted inner product. For later use, we shall get a more explicit expansion for \( \tilde{K}(x, y) \).

The following comments concerning the spaces \( a_{\mathfrak{M}_0} \) will be useful. If \( F \) is \( p \)-adic, we know that \( a_{\mathfrak{M}_0,F} \) is a lattice. Set

\[
a_{\mathfrak{M}_0,F}^\vee = \text{Hom}(a_{\mathfrak{M}_0,F}, 2\pi i\mathbb{Z}).
\]

We also define \( a_{\mathfrak{M}_0,\sigma}^\vee \) for the stabilizer of \( \sigma \) in \( i a_{\mathfrak{M}_0}^\vee \). Write

\[
i a_{\mathfrak{M}_0,F}^\vee = i a_{\mathfrak{M}_0,F}^\vee / a_{\mathfrak{M}_0,F}^\vee, \quad i a_{\mathfrak{M}_0,F}^\vee / a_{\mathfrak{M}_0,\sigma}^\vee = i a_{\mathfrak{M}_0,\sigma}^\vee.
\]

Then, \( K(x, y) \) can be written as

\[
\sum_{M_0 \in S_0} \frac{|W_0^M|}{|W_0^G|} \sum_{\sigma \in \Pi_2(M_0(F))} |a_{\mathfrak{M}_0,\sigma}^\vee / a_{\mathfrak{M}_0,F}^\vee|^{-1} \int_{i a_{\mathfrak{M}_0,F}^\vee} K_\theta(\sigma_\lambda, x, y) d\lambda,
\]

where

\[
K_\theta(\sigma_\lambda, x, y) = m(\sigma_\lambda) \sum_{S \in \mathcal{B}_P(\sigma)} \text{tr}(\mathcal{I}_P(\sigma_\lambda, x)S_\lambda(f))\overline{\text{tr}(\mathcal{I}_P(\sigma_\lambda, \theta(y))S)},
\]

\( \mathcal{B}_P(\sigma) \) is a fixed \( K \)-finite basis of the Hilbert space of the Hilbert-Schmidt operators on \( \mathcal{H}_P(\sigma) \), and \( S_\lambda(f) = \mathcal{I}_P(\sigma_\lambda, f_2)S\mathcal{I}_P(\sigma_\lambda, f_1) \).

Similarly, let \( (a_{\mathfrak{M}_0,\sigma})_\theta \) denote the stabilizer of \( \sigma \) in \( i(a_{\mathfrak{M}_0})_\theta \), and set

\[
i(a_{\mathfrak{M}_0,F})_\theta = i(a_{\mathfrak{M}_0})_\theta / a_{\mathfrak{M}_0,F}^\vee, \quad i(a_{\mathfrak{M}_0,\sigma})_\theta = i(a_{\mathfrak{M}_0})_\theta / (a_{\mathfrak{M}_0,\sigma})_\theta.
\]
Thus, \( \widetilde{K}(x, y) \) can also be written as the product of \( |\det(1 - \theta)_{G_0^G}|^{-1} \) with

\[
\sum_{M^0 \in Z^0} \frac{|W_0^{M^0}|}{|W_0^{G_0^G}|} \sum_{\sigma \in \{ \Pi_2(M^0(F)) \}} |(a_{M^0,\sigma}^\vee)_{\theta}/a_{M^0,F}^\vee|^{-1} \int_{i(a_{M^0,F})^\vee} K_\theta(\sigma, x, y) d\lambda, \tag{3.30}
\]

where

\[
K_\theta(\sigma, x, y) = m(\sigma) \sum_{S \in \mathcal{B}_P(\sigma)} tr(I_P(\sigma, x)S_\lambda(f))\overline{tr(I_P(\sigma, \theta(y))S)}. \tag{3.31}
\]

Finally, we obtain an explicit expression for

\[
K^T(f) = \int_{A_{G_0^G}(F)\backslash G_0^G(F)} \widetilde{K}(x, x) u(x, T) dx.
\]

It equals the product of \( |\det(1 - \theta)_{G_0^G}|^{-1} \) with

\[
\sum_{M^0 \in Z^0} \frac{|W_0^{M^0}|}{|W_0^{G_0^G}|} \sum_{\sigma \in \{ \Pi_2(M^0(F)) \}} |(a_{M^0,\sigma}^\vee)_{\theta}/a_{M^0,F}^\vee|^{-1} \int_{i(a_{M^0,F})^\vee} K_\theta^T(\sigma, f) d\lambda, \tag{3.32}
\]

where \( K_\theta^T(\sigma, f) \) is equal to

\[
\mu(\sigma) \sum_{S \in \mathcal{B}_P(\sigma)} \int_{A_{G_0^G}(F)\backslash G_0^G(F)} tr(I_P(\sigma, x)S_\lambda(f))\overline{tr(I_P(\sigma, \theta(x))S)} u(x, T) dx,
\]

and \( \mu(\sigma) \) is defined in §3.5.2.

We can imagine that the expansion

\[
\int_{A_{G_0^G}(F)\backslash G_0^G(F)} tr(I_P(\sigma, x)S_\lambda(f))\overline{tr(I_P(\sigma, \theta(x))S)} u(x, T) dx \tag{3.32}
\]

is an important object to study.
3.5.2 A Twisted Inner Product

To deal with the spectral side of the local twisted trace formula, it is necessary to obtain a twisted inner product for (3.32). We shall also obtain an asymptotic formula for the inner product in this section. Essentially, the theory of the twisted inner product is derived from the theory of the inner product in §8 of [10].

We first recall some preliminaries about the local harmonic analysis stated in §7 of [10]. If \( \pi \) is an admissible tempered representation of \( G^0(F) \), let \( \mathcal{A}_\pi(G^0) \) stand for the space of functions on \( G^0(F) \) spanned by \( K \)-finite matrix coefficients of \( \pi \). Let \( \mathcal{A}_{\text{temp}}(G^0) \) and \( \mathcal{A}_2(G^0) \) denote the sum of \( \mathcal{A}_\pi(G^0) \) over all \( \pi \in \Pi_{\text{temp}}(G^0(F)) \) and \( \pi \in \Pi_2(G^0(F)) \), respectively. Suppose that \( \tau \) is a unitary, two-sided representation of \( K \) on a finite dimensional Hilbert space \( V \), which can be regarded as a subspace of \( C^\infty(K \times K) \) (see [24], §14.2). Let \( \mathcal{A}_\pi(G^0, \tau) \) denote the space of functions \( f \in \mathcal{A}_\pi(G^0) \otimes V \) such that

\[
 f(k_1 x k_2) = \tau(k_1) f(x) \tau(k_2), \quad x \in G^0(F), \quad k_1, k_2 \in K.
\]

If \( \pi \in \Pi_2(G^0(F)) \), there is an inner product defined on \( \mathcal{A}_\pi(G^0, \tau) \) given by

\[
 (\psi', \psi) = \int_{A_{G^0(F) \setminus G^0(F)}} (\psi'(x), \psi(x)) dx, \quad \psi', \psi \in \mathcal{A}_\pi(G^0, \tau).
\]

Similarly, we can define the spaces \( \mathcal{A}_{\text{temp}}(G^0, \tau) \) and \( \mathcal{A}_2(G^0, \tau) \).

Fix a Levi subgroup \( M^0 \in \mathcal{L}^0 \) and write \( K_{M^0} = K \cap M^0(F) \). Let \( \tau_{M^0} \) be the restriction of \( \tau \) to \( K_{M^0} \). Given any \( f \) in \( \mathcal{A}_{\text{temp}}(G^0, \tau) \) and \( P \in \mathcal{P}(M^0) \), Harish-Chandra ([20], §21 and [22], §3) showed that there is a uniquely determined function \( C^P f \) in \( \mathcal{A}_{\text{temp}}(M^0, \tau_{M^0}) \) such that

\[
 \delta_P(ma)^{1/2} f(ma) - (C^P f)(ma), \quad m \in M^0(F), \quad a \in A_{M^0}(F),
\]

is asymptotic to 0 as \( a \) approaches infinity along the chamber of \( P \). The function \( C^P f \)
is called the weak constant term of \( f \) related to \( M^0 \) and \( P \).

If \( P' \) and \( P \) are parabolic groups in \( \mathcal{P}(M^0) \), we can define \( \tau_{P' \mid P} \) to be the subrepresentation of \( \tau_{M^0} \) on the invariant subspace

\[
V_{P' \mid P} = \{ v \in V \mid \tau(n')v\tau(n) = v, \ n' \in N_{P'}(F) \cap K, \ n \in N_P(F) \cap K \}
\]

of \( V \). Let \( \Gamma \) be a finite set of classes of irreducible representations of \( K \) and \( \mathcal{H}_P(\sigma)_\Gamma \) be the subspace of vectors in \( \mathcal{H}_P(\sigma) \) that transform under \( K \) according to representations in \( \Gamma \). We can also define a finite dimensional subspace \( V_\Gamma \) of \( L^2(K \times K) \) and a two-sided representation \( \tau_\Gamma \) on \( V_\Gamma \) associated to \( \Gamma \) ([10], §7). Harish-Chandra defined an isomorphism \( S \to \psi_S \) from \( \text{End}(\mathcal{H}_P(\sigma)_\Gamma) \) onto \( \mathcal{A}_\sigma(M, (\tau_\Gamma)_{P \mid P}) \) ([10], §7 and [21], §7).

For any \( \psi \in \mathcal{A}_\sigma(M, \tau_{P \mid P}) \), we can define the Eisenstein integral that depends on a parameter \( \lambda \in i\mathfrak{a}^*_M \) by the following:

\[
E_P(x, \psi, \lambda) = \int_K \tau(k)^{-1}\psi_P(kx)e^{(\lambda+\rho_P)(H_P(kx))}dk, \quad (3.33)
\]

where

\[
\psi_P(nmk) = \psi(m)\tau(k), \ n \in N_P(F), \ m \in M^0(F), \ k \in K.
\]

Harish-Chandra established the following proposition.

**Proposition 3.5.3 (Harish-Chandra [21])**: For the objects defined above, we have

\[
(\psi_S, \psi_T) = d^{-1}_\sigma \text{tr}(ST^*), \quad S, T \in \text{End}(\mathcal{H}_P(\sigma)_\Gamma), \quad (3.34)
\]

and

\[
E_P(x, \psi_T, \lambda)_{k_1, k_2} = \text{tr}(I_P(\sigma_{\lambda}, k_1 x k_2)T), \quad (3.35)
\]

where \( d_\sigma \) is the formal degree of \( \sigma \).

For convenience, let us recall the notation and some conclusions from §7 of [10].
(1) Let $\psi \in \mathcal{A}_2(M, \tau|_P)$. For another $M_0^0 \in \mathcal{L}^0$, $P_1 \in \mathcal{P}(M_0^0)$, and $m_1 \in M_0^0$, the weak constant term $(C_{P_1} E_P)(m_1, \psi, \lambda)$ is equal to

$$
\sum_{s \in W(a_{M_0^0}, a_{M_1^0})} \left( c_{P_1|P}(s, \lambda) \psi \right)(m_1) e^{(s\lambda)(H_{M_1^0}(m_1))},
$$

where $W(a_{M_0^0}, a_{M_1^0})$ is the set of all possible isomorphisms of $a_{M_0^0}$ onto $a_{M_1^0}$ obtained by restricting elements in $W_0^G$ to $a_{M_0^0}$, and $c_{P_1|P}(s, \lambda)$ are the Harish-Chandra $c$-functions. There are also Harish-Chandra auxiliary $c$-functions $c^0_{P_1|P}(s, \lambda)$ and $0^c_{P_1|P}(s, \lambda)$ ([3], §I.2).

(2) For $P, P' \in \mathcal{P}(M^0)$,

$$(J_{P'|P}(\sigma, \lambda))(k) = \int_{N_{P'}(F) \cap N_P(F) \setminus N_{P'}(F)} \sigma(M_P(n)) \phi(K_{P'}(n)k)e^{(\lambda + \rho_P)(H_P(n))} dn$$

is an unnormalized intertwining operator between $\mathcal{I}_P(\sigma, \lambda)$ and $\mathcal{I}_{P'}(\sigma, \lambda)$.

(3)

$$\gamma(P) := \int_{N_P(F)} e^{(2\rho_M)(H_P(n))} dn.$$

(4)

$$R_{P'|P}(\sigma, \lambda) = r_{P'|P}(\sigma, \lambda)^{-1} J_{P'|P}(\sigma, \lambda)$$

is the normalized intertwining operator between $\mathcal{I}_P(\sigma, \lambda)$ and $\mathcal{I}_{P'}(\sigma, \lambda)$ ([7], §2), where

$$r_{P'|P}(\sigma, \lambda) = \prod_{\beta \in \Sigma'_{P'} \cap \Sigma'_{P}} r_\beta(\sigma, \lambda).$$

(5) Using the notation in (1), for $s \in W(a_{M_0^0}, a_{M_1^0})$, we define the operators

$$R_{P_1|P}(s, \sigma, \lambda) = w_s R_{s^{-1}P_1|P}(\sigma, \lambda),$$

where $w_s$ is a representative of $s$ in $K$. 

(6) \[
\mu_{P'|P}(\sigma_\lambda) := (J_{P'|P}(\sigma_\lambda)J_{P'|P}(\sigma_\lambda))^{-1}
\]
is a scalar valued meromorphic function of \(\lambda\) that is analytic on \(i\mathfrak{a}_M^*\).

(7) \[
\mu(\sigma_\lambda) := \gamma(P)\mu_{P'|P}(\sigma_\lambda).
\]

(8) The Plancherel density \(m(\sigma_\lambda)\) is equal to \(d_\sigma \mu(\sigma_\lambda)\).

(9) Define
\[
N(\lambda) = \begin{cases} 
  1 + ||\lambda||, & \text{if } F \text{ is Archimedean}, \\
  0 & \text{otherwise}.
\end{cases}
\]

For simplicity, we use \(M\) to replace \(M^0\) through the end of this section. In §3.6, we will change the notation back.

Motivated by (3.32), we define a twisted inner product
\[
\Omega_{P_0^0,\theta}(\lambda', \lambda, \psi_S, \psi_S') = \int_{A_{G_0(F)} \backslash G_0(F)} tr(I_{P'}(\sigma_\lambda', x)S')tr(I_P(\sigma_\lambda, \theta(x))S)u(x, T)dx, \quad (3.36)
\]
where \(P_0^0\) is the fixed minimal parabolic subgroup of \(G^0\) over \(F\), \(M, M' \in \mathcal{L}^0\), \(\sigma \in \Pi_2(M(F))_\theta\), \(\sigma' \in \Pi_2(M'(F))_\theta\), \(\lambda' \in i(\mathfrak{a}_{M'})^*_\theta\), \(\lambda \in i(\mathfrak{a}_M)^*_\theta\), \(S \in \text{End}(\mathcal{H}_P(\sigma)_{T})\), \(S' \in \text{End}(\mathcal{H}_{P'}(\sigma')_{T})\), and
\[
\psi_S \in \mathcal{A}_{\sigma}(M, (\tau_T)_{P|P}), \quad \psi'_S \in \mathcal{A}_{\sigma'}(M', (\tau_T)_{P'|P'})
\]
corresponding to \(S\) and \(S'\), respectively.

By (3.31) and the definition of \(\Omega_{P_0^0,\theta}(\lambda, \lambda, \psi_{S_\lambda(f)}, \psi_S)\), we see that the spectral side of the truncated expansion
\[
K^T(f) = \int_{A_{G_0^0} \backslash G_0^0(F)} \tilde{K}(x, x)u(x, T)dx
\]
equals the product of $|\det(1 - \theta)_{G G_0}|^{-1}$ with

$$\sum_M \frac{|W^M_0|}{|W^G_0|} \sum_\sigma \sum_S |(a^\gamma_{M,\sigma})_\theta / a^\gamma_{M,F}|^{-1} \int_{(a_{M,F})^*_\theta} \Omega^{T}_{P_0,\theta}(\lambda, \lambda, \psi_{S_\lambda(f)}, \psi_S) \mu(\sigma, \lambda) d\lambda,$$  \hspace{1cm} (3.37)

where the sums are over $M \in \mathcal{L}^0$, $\sigma \in \{\tilde{\Pi}_2(M(F))\}$, and $S \in \mathcal{B}_P(\sigma)$.

**Remark.** Although the definition of the twisted inner product $\Omega^{T}_{P_0,\theta}(\lambda', \lambda, \psi'_{S'}, \psi_S)$ is not defined directly by some kind of inner product of Eisenstein integrals, later, we will see that it actually has the usual form of the inner product.

To apply the inner product in §8 of [10] to the twisted case, we need the following lemma.

**Lemma 3.5.4 :** For any $\sigma \in \Pi_2(M)$, we can define a representation $\theta^{-1}(\sigma)$ in $\Pi_2(\theta^{-1}(M))$ by $\theta^{-1}(\sigma)(m) = \sigma(\theta(m))$ for $m \in \theta^{-1}(M)$. Then,

$$\mathcal{I}_{P}(\sigma, \theta(x)) \cong \mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma), x), \hspace{1cm} x \in G^0(F).$$  \hspace{1cm} (3.38)

**Proof.** We use the induced picture (see the beginning of §3.5.1). Define a map

$$B_{\sigma} : \mathcal{H}_{\theta^{-1}(P)}(\theta^{-1}(\sigma)) \rightarrow \mathcal{H}_P(\sigma)$$

by $(B_{\sigma} f)(x) = f(\theta^{-1}(x))$.

It is possible to check that $B_{\sigma}$ is an isomorphism and an intertwining operator between $\mathcal{I}_{P}(\sigma, \theta(x))$ and $\mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma), x)$, i.e.,

$$\mathcal{I}_{P}(\sigma, \theta(x)) \circ B_{\sigma} = B_{\sigma} \circ \mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma), x).$$  \hspace{1cm} (3.39)

We also need to show that the operator $B_{\sigma}$ is an isometry between the norm $|| \cdot ||_{\theta^{-1}(\sigma)}$ on $\mathcal{H}_{\theta^{-1}(P)}(\theta^{-1}(\sigma))$ and $|| \cdot ||_{\sigma}$ on $\mathcal{H}_P(\sigma)$. In fact, one can use the standard technique.
(see [24], §7.2) to prove this claim. Choose any $\phi \geq 0$ on $G^0(F)$ satisfying
\[
\int_{N_PM_P} \phi(nmx)d_r(nm) = 1, \quad \text{for all } x \in G^0(F),
\]
where $d_r(nm)$ is the compatible right Haar measure on $P$. We use $d_l$ to denote the left Haar measure on $P$ and $d_r$ the right Haar measure.

Note that $P_0$ is $\theta$-invariant, $P_0 \subset P$, and $G^0(F) = PK$. It is easy to check that
\[
G^0(F) = P\theta^i(K), \quad \forall \ i \in \mathbb{Z}.
\]

For any $f$ in $\mathcal{H}_{\theta^{-1}(P)(\theta(\sigma))}$, we have
\[
||B_{\sigma}f||_{\sigma}^2 = \int_K |f(\theta^{-1}(k))|^2dk
= \int_K \left( \int_{N_PM_P} \phi(nm\theta^{-1}(k))d_r(nm) |f(\theta^{-1}(k))|^2dk \right)
= \int_{N_PM_PK} \phi(nm\theta^{-1}(k)) |f(\theta^{-1}(k))|^2e^{2\rho_P(H_P(m))}d_l(nm)dk
= \int_{N_PM_PK} \phi(nm\theta^{-1}(k)) |f(nm\theta^{-1}(k))|^2d_l(nm)dk.
\]

Using the fact that $G^0(F) = P\theta(K)$, we calculate
\[
||B_{\sigma}f||_{\sigma}^2 = \int_{N_PM_P\theta^{-1}(K)} \phi(nmk)|f(nmk)|^2d_l(nm)dk
= \int_{G^0(F)} \phi(x)|f(x)|^2dx
= \int_{N_PM_PK} \phi(nmk)|f(k)|^2e^{2\rho_P(H_P(m))}d_l(nm)dk
= \int_K \left( \int_{N_PM_P} \phi(nmk)d_r(nm) |f(k)|^2dk \right)
= ||f||_{\theta^{-1}(\sigma)}^2.
\]
$B_{\sigma}$, defined in Lemma 3.5.4, can induce an intertwining operator between $\mathcal{I}_P(\sigma, \theta(x))$ and $\mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma), x)$ in the compact picture. We still use $B_{\sigma}$ to denote this induced intertwining operator. We require that the central characters of $\sigma'$ and $\theta^{-1}(\sigma)$ coincide on $A_{G^0}(F)$ and $(\lambda' - \theta^{-1}(\lambda))$ belongs to the space $(i a_{M}^*)_G$. Note that $\theta(\lambda)$ satisfies $\theta(\sigma_\lambda) = \theta(\sigma)_{\theta(\lambda)}$.

By Lemma 3.5.4, we can rewrite the twisted inner product $\Omega^T_{P_0, \theta}(\lambda', \lambda, \psi_{S'}, \psi_S)$ as

$$\int_{A_{G^0}(F)\backslash G^0(F)} tr(\mathcal{I}_{P'}(\sigma'_\lambda', x)S') tr(\mathcal{I}_P(\sigma_{\lambda}, \theta(x))S) u(x, T) dx$$

$$= \int_{A_{G^0}(F)\backslash G^0(F)} tr(\mathcal{I}_{P'}(\sigma'_\lambda', x)S') tr(B_{\sigma_{\lambda}}\mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma_{\lambda}), x)B_{\sigma_{\lambda}}^{-1}S) u(x, T) dx$$

$$= \int_{A_{G^0}(F)\backslash G^0(F)} tr(\mathcal{I}_{P'}(\sigma'_\lambda', x)S') tr(\mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma_{\lambda}), x)B_{\sigma_{\lambda}}^{-1}SB_{\sigma_{\lambda}}) u(x, T) dx$$

$$= \int_{A_{G^0}(F)\backslash G^0(F)} tr(\mathcal{I}_{P'}(\sigma'_\lambda', x)S') tr(\mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma)_{\theta^{-1}(\lambda)}, x)\theta(S)) u(x, T) dx,$$

where $\theta(S) = B_{\sigma_{\lambda}}^{-1}SB_{\sigma_{\lambda}}$ and $B_{\sigma_{\lambda}}$ is the intertwining operator defined in Lemma 3.5.4.

In [10], Arthur defined an inner product $\Omega^T_{P_0}(\lambda', \lambda, \psi_{S'}, \psi_S)$ given by

$$\int_{A_{G^0}(F)\backslash G^0(F)} tr(\mathcal{I}_{P'}(\sigma'_\lambda', x)S') tr(\mathcal{I}_P(\sigma_{\lambda}, x)S) u(x, T) dx$$

$$= \int_{A_{G^0}(F)\backslash G^0(F)} (E_{P'}(x, \psi_{S'}, \lambda'), E_P(x, \psi_S, \lambda)) u(x, T) dx$$

By the calculation above, we obtain the following theorem.

**Theorem 3.5.5**: The twisted inner product $\Omega^T_{P_0, \theta}(\lambda', \lambda, \psi_{S'}, \psi_S)$ satisfies

$$\Omega^T_{P_0, \theta}(\lambda', \lambda, \psi_{S'}, \psi_S) = \Omega^T_{P_0}(\lambda', \theta(\lambda), \psi_{S'}, \psi_{\theta(S)}).$$
By Corollary 8.2, [10], we know that \( \Omega_{P_0}^{T}(\lambda', \lambda, \psi_{S'}, \psi_S) \) has an asymptotic expression

\[
\omega_{P_0}^{T}(\lambda', \lambda, \psi_{S'}, \psi_S) = \sum_{P_1 \supset P_0} r_{P_1}^{T}(C_{P_1} E)(\psi'_{S'}, \lambda'), (C_{P_1} E)(\psi_S, \lambda)),
\]

where \( P_1 \) is summed over \( \mathcal{P}(M_1) \) and \( r_{P_1}^{T}(\cdot, \cdot) \) is a bilinear form over \( \mathcal{A}_0(M_1, \tau_{M_1}) \) defined in §8 of [10].

Therefore, we obtain

**Corollary 3.5.6:** Given \( \delta > 0 \), we can choose the constants \( C, k \) and \( \epsilon \) in Theorem 8.1 of [10] so that

\[
|\Omega_{P_0}^{T}(\lambda', \theta^{-1}(\lambda), \psi'_{S'}, \psi_{\theta(S)}) - \omega_{P_0}^{T}(\lambda', \theta^{-1}(\lambda), \psi'_{S'}, \psi_S)| \leq C N_k(\lambda', \theta^{-1}(\lambda), \psi'_{S'}, \psi_{\theta(S)}) e^{-\epsilon||T||}
\]

for all \( (\lambda', \theta^{-1}(\lambda)), \psi'_{S'}, \psi_{\theta(S)} \), and \( T \) satisfying the conditions in Theorem 8.1 of [10].

\[\square\]

**Remark.** In the theory about \( \Omega_{P_0}^{T}(\lambda', \lambda, \psi_{S'}, \psi_S) \) and \( \omega_{P_0}^{T}(\lambda', \lambda, \psi_{S'}, \psi_S) \), we need assume that the central characters of \( \sigma' \) and \( \sigma \) coincide on \( A_{G^0}(F) \) and \( (\lambda' - \lambda) \) belongs to \( (i\mathfrak{a}_M^*)^{G^0} \) (see [10], p. 59). Fortunately, the representation \( \sigma_{\lambda} \) in (3.32) at the end of the previous subsection satisfies these conditions.

We can now deduce an explicit expression for \( \omega_{P_0}^{T}(\lambda', \lambda, \psi_{S'}, \psi_S) \).

For fixed \( \pi \) and \( \pi' \) in \( \tilde{\Pi}_2(M(F)) \), let \( \mathcal{E}^{G_0}(\pi', \pi) \) denote the set of points

\[
\nu \in (i\mathfrak{a}_M^*)^{G_0} \cong i(\mathfrak{a}_M/\mathfrak{a}_{G_0})^*\]

such that \( \pi' \) is equivalent to \( \pi_\nu \). Recall that \( W(\mathfrak{a}_M, \mathfrak{a}_{M_1}) \) is the set of all possible isomorphisms of \( \mathfrak{a}_M \) onto \( \mathfrak{a}_{M_1} \) obtained by restricting elements in \( W_0^{G_0} \) to \( \mathfrak{a}_M \). We always require that \( W(\mathfrak{a}_M, \mathfrak{a}_{M_1}) \) is nonempty.

The explicit expression for \( \omega_{P_0}^{T}(\lambda', \lambda, \psi_{S'}, \psi_S) \) is (8.10) in [10]. We state it as a theorem:

**Theorem 3.5.7 (J. Arthur [10]):** The asymptotic expression of the inner product...
Chapter 3. A Local Twisted Trace Formula

\[ \Omega^T_{P_0}(\lambda', \lambda, \psi'_{S'}, \psi_S) \text{ equals} \]

\[
\omega^T_{P_0}(\lambda', \lambda, \psi'_{S'}, \psi_S) = \sum_{P_1 \supset P_0} \sum_{s',s} \sum_{\Lambda_1} (\Psi'_{s'}, \Psi_s)_{\Lambda_1} e^{\Lambda_1(T_{P_1})} \theta_{P_1,l}(\Lambda_1)^{-1}, \quad (3.40)
\]

where the sums are taken over

\[ s' \in W(a_{M'}, a_{M_1}), \quad s \in W(a_M, a_{M_1}), \quad \text{and} \quad \Lambda_1 \in \mathcal{E}^{G_0}(s' \sigma', s\sigma)/L^{\gamma}_{M_1,l}. \]

The functions \( \Psi'_{s'} \) and \( \Psi_s \) are defined on p. 59 of [10] by

\[
\Psi'_{s'}(m_1) = (c(s', \lambda')\psi')(m_1)e^{(s'\lambda')(H_{M_1}(m_1))}
\]

and

\[
\Psi_s(m_1) = (c(s, \lambda)\psi)(m_1)e^{(s\lambda)(H_{M_1}(m_1))},
\]

which are in \( \mathcal{A}_{s'\sigma'}(M_1, \tau_{P_1|P_1}) \) and \( \mathcal{A}_{s\sigma}(M_1, \tau_{P_1|P_1}) \), respectively. The inner product \( (\Psi'_{s'}, \Psi_s)_{\Lambda_1} \) is defined on p. 56 of [10] and \( \theta_{P_1,l}(\Lambda_1) \) is defined in §3.4.3.

Observe that

\[
\mathcal{E}^{G_0}(s' \sigma', s\sigma) = s' \lambda' - s\lambda + \mathcal{E}^{G_0}(s' \sigma', s\sigma)
\]

and if \( \Lambda_1 = s' \lambda' - s\lambda + \nu_1 \) for \( \nu_1 \in \mathcal{E}^{G_0}(s' \sigma', s\sigma) \), then

\[
(\Psi'_{s'}, \Psi_s)_{\Lambda_1} = (c(s', \lambda')\psi', c(s, \lambda)\psi)_{\nu_1}.
\]

Combining Corollary 3.5.6 and Theorem 3.5.7, we obtain the main result in this subsection:

**Corollary 3.5.8**: The asymptotic expression of the twisted inner product
\[ \Omega_{P_0, \theta}^T(\lambda', \lambda, \psi_{S'}, \psi_S) \text{ is equal to} \]
\[ \omega_{P_0}^T(\lambda', \theta^{-1}(\lambda), \psi_{S'}, \psi_{\theta(S)}) = \sum_{P_1 \supset P_0} \sum_{s'} \sum_{\Lambda_1} (\Psi_{s'}, \Psi_s) \Lambda_1 e^{\Lambda_1(T_{P_1})} \theta_{P_1,l}(\Lambda_1)^{-1}, \] (3.41)

where the sums are taken over
\[ s' \in W(a_{M'}, a_{M_1}), \quad s \in W(a_{\theta^{-1}(M)}, a_{M_1}) \text{ and} \quad \Lambda_1 \in \mathcal{E}_{G_0}^0(s', \sigma_{M'}, s\theta^{-1}(\sigma_M))/\mathcal{L}_{M_1,l}. \]

Note that
\[ \Psi_{s'}(m_1) = (c(s', \lambda')\psi_{S'})(m_1)e^{(s'\lambda')(H_{M_1}(m_1))} \]
and
\[ \Psi_s(m_1) = (c(s, \theta^{-1}(\lambda))\psi_{\theta(S)})(m_1)e^{(s\theta^{-1}(\lambda))(H_{M_1}(m_1))}. \]

### 3.5.3 An Asymptotic Formula: \( k^T(f) \)

We will follow Arthur’s method from §10 and §11 of [10].

In the last subsection, we have seen that \( K^T(f) \) is equal to the product of \( |\text{det}(1 - \theta)_{G_0}^0|^{-1} \) with

\[ \sum_M \frac{|W_M^0|}{|W_0^0|} \sum_{\sigma} \sum_{S |(a_{M,\sigma})_\theta/a_{M,F}^\gamma|^{-1} \int \Omega_{P_0, \theta}^T(\lambda, \lambda, \psi_{S_{\lambda}(f)}, \psi_S) \mu(\sigma_\lambda) d\lambda, \] (3.42)

where the sums are over \( M \in \mathcal{L}^0, \sigma \in \{\tilde{\Pi}_2(M(F))\} \), and \( S \in \mathfrak{B}_P(\sigma) \).

Motivated by Corollary 3.5.6, we set \( k^T(f) \) equal to the product of \( |\text{det}(1 - \theta)_{G_0}^0|^{-1} \) with

\[ \sum_M \frac{|W_0^0|}{|W_0^0|} \sum_{\sigma} \sum_{S |(a_{M,\sigma})_\theta/a_{M,F}^\gamma|^{-1} \int \omega_{P_0}^T(\lambda, \theta^{-1}(\lambda), \psi_{S_{\lambda}(f)}, \psi_{\theta(S)}) \mu(\sigma_\lambda) d\lambda, \] (3.43)

where the sums are over \( M \in \mathcal{L}^0, \sigma \in \{\tilde{\Pi}_2(M(F))\} \) and \( S \in \mathfrak{B}_P(\sigma) \).

By Corollary 3.5.6, we have
Lemma 3.5.9: There are positive constants $C$ and $\varepsilon$ such that

$$|K^T(f) - k^T(f)| \leq Ce^{-\varepsilon||T||}$$

(3.44)

for all $T$ with $d(T) \geq \delta||T||$.

For simplicity, we write

$$\sigma_\theta = \theta^{-1}(\sigma), \quad \lambda_\theta = \theta^{-1}(\lambda), \quad M_\theta = \theta^{-1}(M), \quad \text{and} \quad P_\theta = \theta^{-1}(P).$$

The integrand in (3.43) is a limiting value of

$$\omega_T^{\pi}(\lambda', \lambda_\theta, \psi_{S_\lambda(f)}, \psi_{s(S)})\mu(\sigma_\lambda),$$

for points $\lambda, \lambda' \in i(a_{M,F})_\theta^*$ in general position. It is equal to

$$\sum_{P_1 \supset P_0} \sum_{s'} \sum_s \sum_{\nu_1} (c(s', \lambda')\psi_{S_\lambda(f)}, c(s, \lambda_\theta)\psi_{s(S)}),\nu_1 e^{A_1(T_{P_1})}\theta_{P_1,t}(\Lambda_1)^{-1}\mu(\sigma_\lambda),$$

(3.45)

where the sums are taken over $P_1 \in \mathcal{P}(M_1)$, $s' \in W(a_M, a_{M_1})$, $s \in W(a_{M_\theta}, a_{M_1})$, and $\nu_1 \in \mathcal{E}^{\sigma}(s', \sigma, \sigma_\theta) / \mathcal{L}_{M_1,t}^{\sigma}$, and $\Lambda_1$ equals $s'\lambda' - s\lambda_\theta + \nu_1$.

We now make a change of variables

$$s' = st, \quad t \in W(a_M, a_{M_\theta}).$$

Setting $Q = s^{-1}(P_1)$, we can replace the sum over $P_1$ and $s$ by a sum over the groups $Q \in \mathcal{P}(M_\theta)$. It follows from the definitions that

$$e^{A_1(T_{P_1})}\theta_{P_1,t}(\Lambda_1)^{-1} = e^{A(T_Q)}\theta_{Q,t}(\Lambda)^{-1},$$

where $\Lambda = s^{-1}(\Lambda_1) = t(\lambda') - \lambda_\theta + \nu$ and $\nu = s^{-1}(\nu_1)$. 
We see that $\nu = s^{-1}(\nu_1) \in \mathcal{E}^G(t\sigma, \sigma_\theta)$. Consequently, $t\sigma \cong (\sigma_\theta)\nu$. Because $\mu(\sigma_\lambda)$ is independent of any choice of measures and any choice of $P \in \mathcal{P}(M)$ ([10], (7.4)), it is possible to check that

$$\mu(\sigma_\lambda) = \mu((\sigma_\theta)\lambda_\theta). \quad (3.46)$$

By (7.8) in [10], we can write

$$\mu(\sigma_\lambda) = \mu((t\sigma)\lambda_\theta - \nu) = c_{P|P}(1, \lambda_\theta - \nu)^{-1} c_{P|P}(1, \lambda_\theta - \nu)^*_{t\sigma}.$$  \quad (3.47)

Using the functional equations of $c$-functions (§I.2, [3]), we have

$$c(st, \lambda') = c_{P|P}(st, \lambda') = s c_{Q|P}(t, \lambda'), \quad c(s, \lambda_\theta) = c_{P|P}(s, \lambda_\theta) = sc_{Q|P}(1, \lambda_\theta),$$

and

$$c_{Q|P}(t, \lambda') = c_{Q|P}(1, t\lambda') = 0 c_{P|P}(t, \lambda').$$

Also, it is possible to check that $s : \mathcal{A}_2(M_\theta, \tau_{M_\theta}) \rightarrow \mathcal{A}_2(M_1, \tau_{M_1})$ is an isometry between the inner products $(\cdot, \cdot)_\nu$ and $(\cdot, \cdot)_{\nu_1}$ and there is an equality

$$(\psi', c_{Q|P}(1, \lambda_\theta)\sigma_\theta \psi)_v = \left(c_{Q|P}(1, \lambda_\theta - \nu)^*_{t\sigma} \psi', \psi\right)_v.$$ 

For simplicity, we write

$$c_{Q|P}(t\lambda') = c_{Q|P}(1, t\lambda').$$

Combining these relations, we find that (3.45) equals the sum over $t \in W(a_M, a_{M_\theta})$ of

$$\sum_Q \sum_{\nu} (c_{Q|P}(\lambda_\theta - \nu)^{-1} c_{Q|P}(t\lambda')^0 c_{P|P}(t, \lambda') \psi_{S_\lambda(f)}, \psi_{\theta(S)}_v e^{\Lambda(Q_0)} \theta_{Q,t}(\Lambda)^{-1}) = (3.48)$$

where $Q$ is summed over $\mathcal{P}(M_\theta)$, $\nu$ is summed over $\mathcal{E}^G(t\sigma, \sigma_\theta)/\mathcal{L}_{M_\theta,t}^\nu$, and

$$\Lambda = s^{-1}(\Lambda_1) = t(\lambda') - \lambda_\theta + \nu.$$
If $F$ is $p$-adic, we need to use an idea of Waldspurger to deal with the associated combinatorial problems.

**Lemma 3.5.10 (Lemma 10.2, [10]:)** If $F$ is a $p$-adic field we can choose a family of functions

$$\{ u_P \in C_c^\infty(i\mathbf{a}_P^*/i\mathbf{a}_{G_0}^*): P \in \mathcal{P}(M_0^0) \}$$

with the following properties:

(i) \[ \sum_{\nu \in \mathcal{L}_{M_0,p,l}^\vee} u_P(\Lambda + \nu) = 1, \quad \Lambda \in i\mathbf{a}_P^*/i\mathbf{a}_{G_0}^*. \]

(ii) For each $P$, the function

$$u_{P,l} = u_P(\Lambda) \theta_P(\Lambda) \theta_{P,l}(\Lambda)^{-1}, \quad \Lambda \in i\mathbf{a}_P^*/i\mathbf{a}_{G_0}^* \quad (3.49)$$

is smooth on $\Lambda \in i\mathbf{a}_P^*/i\mathbf{a}_{G_0}^*$.

(iii) If $P$ contains $P_1$, $u_P$ and $u_{P,l}$ are the restrictions of $u_{P_1}$ and $u_{P_1,l}$ to $i\mathbf{a}_P^*$.

We now return to the expression (3.48). It depends on an element $t \in W(\mathbf{a}_M, \mathbf{a}_{M_0})$, which will be fixed until further notice. Write (3.48) as

$$\sum_{Q \in \mathcal{P}(M_0)} \sum_{\nu \in \mathcal{L}_{M_0,p,l}^\vee} \Delta_{\theta,Q}(\nu) e^{\Lambda(T_Q)} \theta_{Q,l}(\Lambda)^{-1}, \quad (3.50)$$

where

$$\Delta_{\theta,Q}(\nu) = (c_{Q|p_0}(\lambda_\theta - \nu)^{-1} c_{Q|p_0}(t \lambda') c_{P_{\nu|p}(t, \lambda') \psi_{S_\lambda}(f), \psi_{\theta(S)})_{\nu}$$

and

$$\Lambda = \Lambda(\lambda', \lambda, \nu) = t(\lambda') - \lambda_\theta + \nu.$$ 

Applying Lemma 3.5.10, we can rewrite (3.50) as

$$\sum_{Q \in \mathcal{P}(M_0)} \sum_{\nu \in \mathcal{L}_{M_0,p,l}^\vee} \Delta_{\theta,Q}(\nu) e^{\Lambda(T_Q)} \theta_{Q,l}(\Lambda)^{-1} \sum_{\xi \in \mathcal{L}_{M_0,p,l}^\vee} u_Q(\Lambda + \xi),$$
which is also equal to
\[
\sum_{Q \in \mathcal{P}(\mathfrak{p})} \sum_{\nu} \sum_{\xi} \Delta_{\theta,Q}(\nu + \xi)e^{(\Lambda + \xi)(T_Q)}\theta_{Q,I}(\Lambda + \xi)^{-1}u_Q(\Lambda + \xi).
\]

Noting that \(T_Q\) is a point in \(\mathfrak{a}_{M_0,F}\) and
\[
\Lambda + \xi = t(\lambda') - \lambda_\theta + (\nu + \xi),
\]
we can combine the double sum over \((\nu, \xi)\) into a single sum over \(\mathcal{E}_\mathfrak{c}(t\sigma, \sigma_\theta)\). Then, (3.50) equals
\[
\sum_{Q \in \mathcal{P}(\mathfrak{p})} \sum_{\nu \in \mathcal{E}_\mathfrak{c}(t\sigma, \sigma_\theta)} \Delta_{\theta,Q}(\nu)e^{\Lambda(T_Q)}u_{Q,I}(\Lambda)\theta_{Q,I}(\Lambda)^{-1}. \tag{3.51}
\]

One of our goals is to reduce the general \(\lambda \in i\mathfrak{a}_M^*\) to some special \(\lambda\) such that \(\theta t(\lambda) = \lambda\). If \(\theta t(\lambda) = \lambda\), then \(\theta t(\sigma_\lambda) = \sigma_\lambda\). Set
\[
\mathfrak{a} = \{H \in \mathfrak{a}_M : \theta t(H) = H\}. \tag{3.52}
\]

We shall use a lemma of Langlands. Assume that \(L\) is a Levi subgroup in \(\mathcal{L}(M)\) and \(\mathfrak{a} \subset \mathfrak{a}_L\). We say that the pair \((L, \mathfrak{a})\) is \(\theta\)-special if it is conjugate to \((L', \mathfrak{a}_{L'}^\theta)\) and \(L'\) is the Levi factor of a \(\theta\)-stable parabolic subgroup. Applying the lemma of Langlands ([17], Lemma 3), for
\[
\mathfrak{a} = \{H \in \mathfrak{a}_M : \theta t(H) = H\},
\]
there exists a unique Levi subgroup \(L\) in \(\mathcal{L}(M)\) such that \((L, \mathfrak{a})\) is \(\theta\)-special. That is, we can find a Levi factor \(L'\) of a \(\theta\)-stable parabolic subgroup and an element \(g_0 \in G^0(F)\) such that
\[
(L, \mathfrak{a}) = g_0(L', \mathfrak{a}_{L'}^\theta)g_0^{-1}. \tag{3.53}
\]

We see that \(i\mathfrak{a}^* = \{\mu \in i(\mathfrak{a}_M)^{\mathfrak{c}} : \theta t(\mu) = \mu\} \).
From now on, we will take $\lambda' = \lambda + \zeta$, where $\zeta$ is restricted to lie in the subspace $i\mathfrak{a}^*$. Then,

$$t(\zeta) = \theta^{-1}(\zeta) = \zeta_\theta$$

and

$$\Lambda = t(\lambda) - \lambda_\theta + \zeta_\theta + \nu.$$

We shall write $\lambda_L$ for the projection of $\lambda$ onto $i\mathfrak{a}^*$, relative to the canonical decomposition $i(\mathfrak{a}_M)_\theta^* = i\mathfrak{a}_M^{L*} \oplus i\mathfrak{a}^*$. Note that $i\mathfrak{a}_M^{L*}$ is a different symbol from $(i\mathfrak{a}_M)^L$ and they have different meanings.

The map

$$(\lambda, \zeta, \nu) \rightarrow (\Lambda, \lambda_L, \nu), \quad \text{where} \quad \Lambda = t(\lambda') - \lambda_\theta + \nu, \quad \lambda' = \lambda + \zeta, \quad (3.54)$$

is a bijection from $i(\mathfrak{a}_M)_\theta^* \times i\mathfrak{a}^* \times \mathcal{E}^{G_\theta}(t\sigma, \sigma_\theta)$ onto $i(\mathfrak{a}_M)_\theta^* \times i\mathfrak{a}^* \times \mathcal{E}^{G_\theta}(t\sigma, \sigma_\theta)$.

Define

$$C_Q(\Lambda, T) = e^{A(TQ)} \quad (3.55)$$

and

$$D_{\theta, Q}(\Lambda, \lambda_L, \nu) = \Delta_{\theta, Q}(\Lambda, \lambda_L, \nu) u_{Q, l}(\Lambda), \quad (3.56)$$

where

$$\Delta_{\theta, Q}(\Lambda, \lambda_L, \nu) = \Delta_{\theta, Q}(\nu) = (c_Q|P_\theta(\lambda_\theta - \nu)^{-1}c_Q|P_\theta(t\lambda')^0c_{P_\theta}|P(t, \lambda')\psi_{S_\lambda(f)}, \psi_{\theta(S)})_\nu.$$  

By the definitions of $C_Q(\Lambda, T)$ and $D_{\theta, Q}(\Lambda, \lambda_L, \nu)$, the formula (3.48) becomes

$$\sum_{Q \in \mathcal{P}(M_\theta)} \sum_{\nu \in \mathcal{E}^{G_\theta}(t\sigma, \sigma_\theta)} C_Q(\Lambda, T) D_{\theta, Q}(\Lambda, \lambda_L, \nu) \theta_{Q}(\Lambda)^{-1}. \quad (3.57)$$

We have seen that

$$\{C_Q(\Lambda, T) : Q \in \mathcal{P}(M_\theta)\}$$
is a \((G^0, M_\theta)\)-family of functions of \(\Lambda \in i(a_{M_\theta})^*_\theta\) \([2, \S 6]\). Note that we need to use a slightly more general definition of \((G^0, M_\theta)\)-family since the domain of \(\Lambda\) is \(i(a_{M_\theta})^*_\theta\), which is a subspace of \(ia^*_M\). But the extended definition is clear.

Next, we shall prove that \(D_{\theta, Q}(\Lambda, \lambda_L, \nu)\) is also a \((G^0, M_\theta)\)-family of functions of \(\Lambda \in i(a_{M_\theta})^*_\theta\). We need the following lemma.

**Lemma 3.5.11 :**

a) For any \(\Lambda \in i(a_{M_\theta})^*_\theta\), the set of points \(\lambda_L \in ia^*\) such that \(D_{\theta, Q}(\cdot, \cdot, \nu)\) is regular at \((\Lambda, \lambda_L)\) is an open dense subset of \(ia^*\).

b) \(D_{\theta, Q}(\Lambda, \lambda_L, \nu)\) can be regarded as a smooth function of \(\Lambda \in i(a_{M_\theta})^*_\theta\) with values in a topological vector space of meromorphic functions of \(\lambda_L\).

c) The set

\[
\{D_{\theta, Q}(\Lambda, \lambda_L, \nu) : Q \in P(M_\theta)\}
\]

is a \((G^0, M_\theta)\)-family of functions of \(\Lambda \in i(a_{M_\theta})^*_\theta\).

**Proof.** Set

\[
r_{\theta, Q}(\Lambda, \lambda_L, \nu) = r_{\overline{Q}Q}(\nu) r_{\overline{Q}Q}(\sigma_{\lambda \theta - \nu})^{-1} r_{\overline{Q}Q}(\sigma_{\lambda \theta})^{-1} \]

and

\[
d_{\theta, Q}(\Lambda, \lambda_L, \nu) = r_{\theta, Q}(\Lambda, \lambda_L, \nu)^{-1} D_{\theta, Q}(\Lambda, \lambda_L, \nu).
\]

Then, \(d_{\theta, Q}(\Lambda, \lambda_L, \nu)\) is regular for all \(\Lambda \in ia_M^*\) and \(\lambda_L \in ia^*\). It suffices to prove the assertion a) with \(D_{\theta, Q}(\Lambda, \lambda_L, \nu)\) replaced by \(r_{\theta, Q}(\Lambda, \lambda_L, \nu)\).

It is easy to check that \((t\sigma)_{\lambda \theta - \nu} \cong (\sigma_{\lambda \theta})\).

Consequently,

\[
r_{\overline{Q}Q}(\nu) r_{\overline{Q}Q}(\sigma_{\lambda \theta - \nu}) = r_{\overline{Q}Q}(\sigma_{\lambda \theta}).
\]

Let \(\pi = \sigma_\theta = \theta^{-1}(\sigma)\). We see that

\[
r_{\overline{Q}Q}(\sigma_{\lambda \theta - \nu}) = r_{\overline{Q}Q}(\pi \lambda_{\theta}) \quad \text{and} \quad r_{\overline{Q}Q}(\sigma_{\lambda \theta}) = r_{\overline{Q}Q}(\nu) r_{\overline{Q}Q}(\sigma_{\lambda \theta}) = r_{\overline{Q}Q}(\nu) r_{\overline{Q}Q}(\pi \lambda_{\theta}).
\]

Now \(\theta\) acts on the Weyl group \(W_{0G^0}\) by the assumption that \(\theta \in Aut(G), \theta(M^0_0) = M^0_0\)
and $\theta(P_0) = P_0$. For $t \in W(a_M, a_{M_0})$, it is possible to choose the rank one normalizing factors satisfying

$$r_\beta((t\theta)(\pi)) = r_{(t\theta)^{-1}(\beta)}(\pi_{\lambda_0}).$$

By the above relations, we get

$$r_{\theta,Q}(\Lambda, \lambda, \nu) = \prod_{\beta \in \Sigma_Q} r_{\beta}(\pi_{\lambda_0})^{-1} r_{(t\theta)^{-1}(\beta)}(\pi_{\lambda_0}).$$

Next, it is sufficient to replace $t$ by $\theta t^{-1}$ in the proof of Lemma 10.3 of [10]. The assertion a) of this lemma follows.

b) is essentially a restatement of a).

Finally, we show that $\{D_{\theta,Q}(\Lambda, \lambda, \nu) : Q \in \mathcal{P}(M_\theta)\}$ is a $(G_0, M_\theta)$-family of functions of $\Lambda \in i(a_{M_\theta})^\ast$. The essential step is checking the compatibility of functions attached to adjacent groups $Q$ and $Q'$ in $\mathcal{P}(M_\theta)$. Let $\alpha$ be the unique simple root of $(Q, A_{M_\theta})$ that is not a root of $(Q', A_{M_\theta})$. We must show that

$$D_{\theta,Q'}(\Lambda) = D_{\theta,Q}(\Lambda)$$

for any $\Lambda \in i(a_{M_\theta})^\ast$, where $i(a_{M_\theta})^\ast$ is the hyperplane

$$\{ \Lambda \in i(a_{M_\theta})^\ast : \Lambda(\alpha^\vee) = 0 \}$$

in $i(a_{M_\theta})^\ast$.

It is possible to verify that the following equalities hold:

1. $u_{Q',l}(\Lambda) = u_{Q,l}(\Lambda)$ for any $\Lambda \in i(a_{M_\theta})^\ast$.
2. $t(\lambda') = \lambda_\theta - \nu + \Lambda$.
3. $c_{Q',Q}^\circ(t\lambda') = c_{Q',Q}^\circ(\lambda_\theta - \nu)$.

Hence, $D_{\theta,Q'}(\Lambda) = D_{\theta,Q}(\Lambda)$. In other words, $\{D_{\theta,Q}(\Lambda, \lambda, \nu) : Q \in \mathcal{P}(M_\theta)\}$ is a
We have a corollary to the previous lemma, which is a twisted analogue of Corollary 10.4 of [10].

**Corollary 3.5.12** : Suppose that $R$ is a group in $\mathcal{F}(M_\theta)$. Then, the limit

$$D_{\theta,M_\theta}^R(t\lambda - \lambda_\theta + \nu, \lambda_L, \nu) = \lim_{\Lambda \to(t\lambda - \lambda_\theta + \nu)} \sum_{Q \in \mathcal{B}(M_\theta)} D_{\theta,Q}(\Lambda, \lambda_L, \nu)\theta_Q(\Lambda)^{-1}$$

extends to a smooth function of $\lambda \in i(a_{M_\theta})^*$. If $F$ is Archimedean, the function belongs to the Schwartz space on $i(a_{M_\theta})^*$.

We now return to the study of $k^T(f)$. We have written (3.48) as the sum over $\nu \in \mathcal{E}^0(t\sigma, \sigma_\theta)$ of the expression

$$\sum_{Q \in \mathcal{B}(M_\theta)} C_Q(\Lambda, T)D_{\theta,Q}(\Lambda, \lambda_L, \nu)\theta_Q(\Lambda)^{-1}. \quad (3.60)$$

Similar to the discussion on p. 78 of [10], the limit of (3.59) becomes

$$\sum_{L_1, L_2 \in \mathcal{L}(M_\theta)} d_{M_\theta}^G(L_1, L_2)C_{M_\theta}^Q(t\lambda - \lambda_\theta + \nu, T)D_{\theta,M_\theta}^Q(t\lambda - \lambda_\theta + \nu, \lambda_L, \nu). \quad (3.61)$$

This is an application of the splitting formula for a $(G^0, M)$-family stated in Corollary 7.4 of [6]. Here, we use the same notation as §7 of [6].

Finally, we can obtain a result about $k^T(f)$, which is a twisted analogical result of Lemma 10.5 of [10].

**Lemma 3.5.13** : The distribution $k^T(f)$ equals the sum over $M \in \mathcal{L}^0$, $\sigma \in \overline{\Pi}_2(M(F))$, $S \in \mathfrak{B}_P(\sigma)$, and $t \in W(a_M, a_{M_\theta})$, of the product of

$$|\det(1 - \theta)_{C^0}|^{-1}|W_0^M||W_0^{G^0}|^{-1}|(a_{M,\sigma})_\theta/a_{M,F}^\gamma|^{-1}$$
with

\[
\sum_{L_1, L_2 \in \mathcal{L}(M_\theta)} d_{M_\theta}^G(L_1, L_2) \int_{i(a_{M,F})^\oplus} d\lambda \sum_\nu C_{M_\theta}^{Q_1}(t \lambda - \lambda_\theta + \nu, T) D_{M_\theta}^{Q_2}(t \lambda - \lambda_\theta + \lambda_L, \nu) d\lambda,
\] (3.62)

where \((L_1, L_2)\) and \(\nu\) are summed over \(\mathcal{L}(M_\theta) \times \mathcal{L}(M_\theta)\) and \(\mathcal{E}^G(t\sigma, \sigma_\theta)\), respectively.

### 3.5.4 The Spectral Expansion for \(J^T(f)\)

In this subsection, we proceed with a reduction theory for the local twisted trace formula. Let \(\bar{a}_M^L\) be the dual space of \(i a_M^L\), where \(i a_M^L\) is defined by the canonical decomposition \(i(a_M)^\oplus = i a_M^{L^*} \oplus i a^*\). Let \(a_{F,F}\) denote \(a \cap a_{L,F}\) and set

\[
a_{M,\theta t}^\vee = (\theta t - 1) a_{M,F}^\vee = \{\theta t(\nu) - \nu : \nu \in a_{M,F}^\vee\}.
\]

We can write (3.62) as the product of \(|det(\theta t - 1)\bar{a}_M^L|^{-1}\) with

\[
\sum_{L_1, L_2 \in \mathcal{L}(M_\theta)} d_{M_\theta}^G(L_1, L_2) \int_{i a_{M,F}^\oplus} d\mu \sum_\nu C_{M_\theta}^{Q_1}(\mu_\theta + \nu, T) D_{M_\theta}^{Q_2}(\mu_\theta + \nu, \lambda_L, \nu),
\] (3.63)

where \(\mu\) is integrated over \(i a_{M,F}^{L^*}/a_{M,\theta t}^\vee\) and \(\nu\) is summed over \(\mathcal{E}^G(t\sigma, \sigma_\theta)\).

Modifying the discussion on pp. 79-81 of [10], we obtain that there exists a constant \(c_n\) for each \(n\), such that the difference between (3.63) and the expression

\[
\sum_{\nu} \sum_{L_1 \in \mathcal{L}(L_\theta)} \sum_{L_2 \in \mathcal{L}(M_\theta)} d_{M_\theta}^G(L_1, L_2) C_{M_\theta}^{Q_1}(\nu_{L_\theta}, T) \int_{i a_{M,F}^\oplus} D_{M_\theta}^{Q_2}(\nu_{L_\theta}, \lambda_{L, \nu}) d\lambda_L
\] (3.64)

is bounded in absolute value by \(c_n ||T||^{-n}\), where \(\nu\) is summed over \(\mathcal{E}^G(t\sigma, \sigma_\theta)/\theta^{-1}(a_{M,\theta t}^\vee)\) and \(\theta^{-1}(a_{M,\theta t}^\vee) = a_{M_\theta,t\theta}^\vee\).

To account for the dependence of the group \(L\) on the element \(t \in W(a_M, a_{M_\theta})\), we
shall write $W(a_M, a_{M_0})$ as the disjoint union over $L \in \mathcal{L}(M)$ of the sets

$$W(a_M^L, a_{M_0}^L)_{\text{reg}} = \{ t \in W(a_M^L, a_{M_0}^L) : \det(\theta t - 1)_{a_{M_0}^L} \neq 0 \}. $$

We can now define a distribution $J^T_{\theta, \text{spec}}(f)$ from which we will get the final expression for the spectral side. This is analogous to Lemma 11.1 of [10].

**Lemma 3.5.14:** Define $J^T_{\theta, \text{spec}}(f)$ to be the sum over $M \in \mathcal{L}^0, \sigma \in \{\tilde{\Pi}_2(M(F))\}$, $S \in \mathcal{B}_P(\sigma)$, $L \in \mathcal{L}(M)$ and $t \in W(a_M^L, a_{M_0}^L)_{\text{reg}}$ of the product of

$$|\det(1 - \theta)_{a_{G_0}^L}|^{-1}|W_0^M||W_0^{G_0}|^{-1}|(a_{M,\sigma}^\vee)_{\theta}/a_{M,F}^\vee|^{-1}|\det(\theta t - 1)_{a_{M_0}^L}|^{-1}$$

with the expression (3.64). It is equal to

$$\sum_{\xi \in \frac{1}{N} \mathcal{Z}_0^\vee \cap \mathcal{L}_0^\vee} p_{\theta, \xi}(T,f)e^{\xi(T)}$$

where $N$ is a fixed positive integer, and $p_{\theta, \xi}(T,f)$ is a polynomial in $T$.

The spectral side of the local twisted trace formula is derived from the constant term

$$\tilde{J}_\theta(f) = p_{\theta, 0}(0,f)$$

of $J^T_{\theta, \text{spec}}(f)$. Write $\mathcal{L}_L^\vee$ for $\mathcal{L}_L^\vee \cap ia^*$. We have the following lemma, which is is an analogous result of Corollary 11.2 of [10].

**Corollary 3.5.15:** The distribution $\tilde{J}_\theta(f)$ equals the sum over $M \in \mathcal{L}^0, \sigma \in \{\tilde{\Pi}_2(M(F))\}$, $S \in \mathcal{B}_P(\sigma)$, $L \in \mathcal{L}(M)$ and $t \in W(a_M^L, a_{M_0}^L)_{\text{reg}}$ of the product of

$$|\det(1 - \theta)_{a_{G_0}^L}|^{-1}|W_0^M||W_0^{G_0}|^{-1}|(a_{M,\sigma}^\vee)_{\theta}/a_{M,F}^\vee|^{-1}|\det(\theta t - 1)_{a_{M_0}^L}|^{-1}$$
with the expression

\[
\sum_{\nu^L} \sum_{\nu_L \in \mathcal{L}_{L,\theta}^\vee} \int_{i\mathfrak{a}_F^\vee} D_{\theta,L_\theta}(\nu_{L_\theta}, \lambda_L, \nu^L + \nu_L) d\lambda_L, \tag{3.68}
\]

where \(\nu^L\) is summed over \(\mathcal{E}^{G_\theta}(t\sigma, \sigma_\theta)/\mathfrak{a}^\vee_{M_\theta,t\theta}\).

We want to interpret Corollary 3.5.15 as an elementary identity involving induced representations and intertwining operators. Write \(\mathcal{L}_{L,k,\theta}^\vee\) for \(\mathcal{L}_{L,k}^\vee \cap i\mathfrak{a}^*\). Let \(\mathcal{L}_{L,\theta}\) and \(\mathcal{L}_{L,k,\theta}\) be the dual lattices of \(\mathcal{L}_{L,\theta}^\vee\) and \(\mathcal{L}_{L,k,\theta}^\vee\), respectively. Imitating the discussion on pp. 83-85 of [10], we can show that \(\tilde{J}_\theta(f)\) equals the sum over \(M \in \mathcal{L}_0\), \(\sigma \in \{\tilde{\Pi}_2(M(F))\}, \ L \in \mathcal{L}(M), \ t \in W(\mathfrak{a}_M^\vee, \mathfrak{a}_{M_\theta}^\vee)_{reg}, \) and \(\nu^L \in \mathcal{E}^{G_\theta}(t\sigma, \sigma_\theta)/\mathfrak{a}^\vee_{M_\theta,t\theta}\) and the integral over \(\lambda_L \in i\mathfrak{a}_L^*\) of the product of (3.65) with the expression

\[
\lim_{\zeta \to 0} \sum_{R \in \mathcal{P}(L_\theta)} \left( \sum_{S \in \mathfrak{M}(\sigma)} \Delta_{\theta,Q_R}(\zeta, \lambda_L, \nu^L) \left( \frac{1}{|\mathcal{L}_{L,\theta}|} \sum_{X \in \mathcal{L}_{L,\theta}/\mathcal{L}_{L,k,\theta}} e^{\zeta(X_R)} \theta_{R,k}(\zeta^{-1}) \right) \right). \tag{3.69}
\]

The representation theoretic objects are wrapped up in \(\Delta_{\theta,Q_R}(\zeta, \lambda_L, \nu^L)\). Fix \(t\) and \(\nu^L\). Let \(\mu\) be the uniquely determined point in \(i\mathfrak{a}_M^\vee\) such that \(\nu^L = \mu_\theta + t\mu\). Set \(\lambda = \xi + \mu + \lambda_L\) and \(\lambda' = \lambda + \zeta\), where \(\xi \in i\mathfrak{a}_M^\vee\) is in general position and approaches 0. Then, the point

\[
\Lambda = t\lambda' - \lambda_\theta + \nu^L = t\xi - \xi_\theta + t\zeta
\]

approaches \(t\zeta\) as \(\xi\) approaches 0. It follows from the definition that \(\Delta_{\theta,Q}(t\zeta, \lambda_L, \nu)\) is the limit as \(\xi\) approaches 0 of

\[
(c_{Q|P_\theta}(\lambda_\theta - \nu^L)^{-1}c_{Q|P_\theta}(t\lambda')^0c_{P_\theta|P}(t, \lambda')^0S_{\lambda}(f), \psi_{\theta(S)})_{\nu}.
\]

Set \(\bar{\sigma} = \sigma_\mu\). Since \(\nu^L \in \mathcal{E}^{G_\theta}(t\sigma, \sigma_\theta)\), we see that

\[
\theta t(\bar{\sigma}) = \sigma_\theta,
\]
i.e., \( \bar{\sigma} \) is invariant under \( \theta t \).

Let \( A_\mu \) be the map that sends any \( \psi \in \mathcal{A}_\sigma(M, \tau|_P) \) to the function

\[
\psi_\mu(m) = \psi(m) e^{\mu(H_M(m))}, \quad m \in M(F)
\]

in \( \mathcal{A}_\sigma(M, \tau|_P) \). If \( S \in \text{End}(\mathcal{H}_P(\sigma)_\Gamma) \), we define \( \bar{S} \) to be the corresponding operator in \( \text{End}(\mathcal{H}_P(\bar{\sigma})_\Gamma) \) such that \( A_\mu(\psi_S) = \psi_{\bar{S}} \).

Therefore,

\[
A_{\mu_\theta}(\psi_{\theta(S)}) = \psi_{\theta(\bar{S})}.
\] (3.70)

In this equation, we use the notation \( \mu_\theta = \theta^{-1}(\mu) \), \( \theta(S) = B_{\sigma_\lambda}^{-1}SB_{\sigma_\lambda} \) and \( \theta(\bar{S}) = B_{\sigma_\lambda}^{-1}\bar{S}B_{\sigma_\lambda} \), where \( B_{\sigma_\lambda} \) is defined in Lemma 3.5.4. The above equation is just \( A_{\mu_\theta}(\psi_{B_{\sigma_\lambda}^{-1}SB_{\sigma_\lambda}}) = \psi_{B_{\sigma_\lambda}^{-1}\bar{S}B_{\sigma_\lambda}} \).

Using the definition of \( c \)-functions and the invariance of \( 0c(t, \cdot, \cdot) \) under translation by \( ia^* \) (note that \( t \mu = \mu_\theta - \nu^L \)), we calculate

\[
(c_Q|P_b(\lambda_\theta - \nu^L)^{-1}c_Q|P_b(t\lambda') 0c_{P_b|P}(t, \lambda') \psi_{S_{\lambda}(f)}, \chi_\theta(S))_0
\]

\[
= (A_{t\mu}c_Q|P_b(\lambda_\theta - \nu^L)^{-1}c_Q|P_b(t\lambda') 0c_{P_b|P}(t, \lambda') \psi_{S_{\lambda}(f)}, \chi_\theta(S))_0
\]

\[
= (c_Q|P_b(\lambda_\theta - \nu^L - t\mu)^{-1}c_Q|P_b(t\lambda' - t\mu) 0c_{P_b|P}(t, \lambda' - \mu) A_\mu(\psi_{S_{\lambda}(f)}), \chi_\theta(S))_0
\]

\[
= (c_Q|P_b(\xi_\theta + (\lambda L)_\theta)^{-1}c_Q|P_b(t\xi + (\lambda L + \zeta)_\theta) 0c_{P_b|P}(t, \xi) \psi_{\bar{S}_{\lambda}(f)}, \chi_\theta(S))_0.
\]

We know that

\[
\bar{S}_{\lambda}(f) = d_\theta I_P(\sigma_{\lambda - \mu}, f_2) \bar{S} I_P(\sigma_{\lambda - \mu}, f'_2).
\]

Hence (note that \( \psi_{\theta(S)} = \psi_{B_{\sigma_\lambda}^{-1}\bar{S}B_{\sigma_\lambda}} \)),

\[
\Delta_{\theta,q}(\zeta, \lambda_L, \nu) = \lim_{\xi \to 0}(d_\theta(\psi_{T_2(\xi)\bar{S}T_1(\xi)}, \psi_{B_{\sigma_\lambda}^{-1}\bar{S}B_{\sigma_\lambda}})_0)
\]

\[
= \lim_{\xi \to 0}(tr(T_2(\xi)\bar{S}T_1(\xi) B_{\sigma_\lambda}^{-1}\bar{S}B_{\sigma_\lambda})).
\]
where \( T_1(\xi) = \mathcal{I}_P(\bar{\sigma}_{\xi + \lambda_L}, f_1) R_{P|P}(t, \bar{\sigma})^{-1} J_{P|Q}((t\bar{\sigma})_{(\lambda_L + \zeta)_{\theta}}) J_{P|Q}((t\bar{\sigma})_{(\lambda_L)_{\theta}})^{-1} \)
and \( T_2(\xi) = J_{Q|P}(((t\bar{\sigma})_{(\xi + \lambda_L)_{\theta}})^{-1} J_{Q|P}((t\bar{\sigma})_{(\lambda_L + \zeta)_{\theta}}) R_{P|P}(t, \bar{\sigma}) \mathcal{I}_P(\bar{\sigma}_{\xi + \lambda_L}, f_2). \)

Accordingly, \( \{ S : S \in \mathfrak{B}_P(\sigma) \} \) is an orthonormal basis of the space of Hilbert-Schmidt operators on \( \mathcal{H}_P(\bar{\sigma}). \)

It is possible to obtain
\[
\sum_{S \in \mathfrak{B}_P(\sigma)} \Delta_{\theta,Q}(\zeta, \lambda_L, \nu) = \lim_{\xi \to 0} (tr(T_1(\xi) B_{\theta,1}) tr(B_{\theta,1} T_2(\xi))). \tag{3.71}
\]

This equation (3.71) is a basic form of the spectral side of the local twisted trace formula.

Let
\[
\Pi = P \cap L
\]
be a fixed parabolic subgroup of \( L. \) Let \( R(\Pi) \) be the unique such parabolic group in \( \mathcal{P}(M) \) whose intersection with \( L \) equals \( \Pi. \) We take \( Q \) to be \( R(\Pi)_{\theta}. \) There is an important constant \( \varepsilon_{\theta}(t) \) that will appear on the spectral side of the trace formula; it is defined as follows. Let \( \Sigma_{r_{\Pi,\theta}} \) denote the set of roots \( \beta \in \Sigma_{r_{\Pi,\theta}} \) such that the function \( r_{\beta}(t\bar{\sigma}_{\xi}) \) has a simple pole at \( \xi = 0. \) We define \( \varepsilon_{\theta}(t) \) to be
\[
(-1)^{t \theta(\Sigma_{r_{\Pi,\theta}}) \cap \Sigma_{\Pi,\theta}}
\]
where \( |t \theta(\Sigma_{r_{\Pi,\theta}}) \cap \Sigma_{\Pi,\theta}| \) means the cardinality of the set \( t \theta(\Sigma_{r_{\Pi,\theta}}) \cap \Sigma_{\Pi,\theta}. \) Through a discussion similar to that on pp. 86-87 of [10], it is possible to show that
\[
\sum_{S \in \mathfrak{B}_P(\sigma)} \Delta_{\theta,Q}(\zeta, \lambda_L, \nu)
\]
equals the product of \( \varepsilon_{\theta}(t) \) with the traces of the operators
\[
\mathcal{I}_P(\bar{\sigma}_{\lambda_L}, f_1) R_{P|P}(t, \bar{\sigma})^{-1} J_{P|R(\Pi)_{\theta}}((t\bar{\sigma})_{(\lambda_L + \zeta)_{\theta}}) J_{P|R(\Pi)_{\theta}}((t\bar{\sigma})_{(\lambda_L)_{\theta}})^{-1} B_{\theta,1}
\]
and
\[
B_{\theta,1} J_{R(\Pi)_{\theta}|P}(((t\bar{\sigma})_{(\lambda_L)_{\theta}})^{-1} J_{R(\Pi)_{\theta}|P}((t\bar{\sigma})_{(\lambda_L + \zeta)_{\theta}}) R_{P|P}(t, \bar{\sigma}) \mathcal{I}_P(\bar{\sigma}_{\lambda_L}, f_2). \]
In general, for any $t \in W(\tilde{a}_M^L, \tilde{a}_M^{L_\theta})$, let $\Pi_2(M(F))^{\theta t}$ be the set of representations in $\tilde{\Pi}_2(M(F))$ that are fixed by $\theta t$. We also write $\Pi_2(M(F))^{\theta t}/ia^*$ for the set of orbits in $\Pi_2(M(F))^{\theta t}$ under the action of $ia^*$. Fix $t \in W(\tilde{a}_M^L, \tilde{a}_M^{L_\theta})_{reg}$ as above. Given any $\sigma \in \tilde{\Pi}_2(M(F))$, we have associated a representation $\tilde{\sigma} = \sigma_\mu$ in $\Pi_2(M(F))^{\theta t}$ to each point $\nu^L = \mu_\theta - t\mu$ in $E_{G^0}(t\sigma, \sigma_\theta)$. Now, the original sum over $\nu^L \in E_{G^0}(t\sigma, \sigma_\theta)$ was taken only modulo the action of $a_{G^0,\theta}$. The isotropy group $a_{G^0,\theta}$ is isomorphic under the map $\mu \rightarrow \nu^L = \mu_\theta - t\mu$ to the group

$$(a_{M,F}^\vee + ia^*)/ia^* \cong a_{M,F}^\vee/a_{M,F}^\vee \cap ia^* = a_{M,F}^\vee/a_F^\vee.$$ \hfill (3.65)$$

On the other hand, two representations $\sigma_\mu$ and $\sigma_{\mu'}$ define the same object in $\Pi_2(M(F))^{\theta t}/ia^*$ if and only if $\mu - \mu'$ belongs to the group

$$((a_{M,\sigma}^\vee)_\theta + ia^*_\theta)/(ia^*)_\theta \cong (a_{M,\sigma}^\vee)_\theta/(a_{M,\sigma}^\vee)_\theta \cap ia_F^* = (a_{M,\sigma})_\theta/a_{\sigma}^\vee,$$

where $a_{\sigma}^\vee$ is the stabilizer of $\sigma$ in $a^\vee$. We need to sum the product of (3.65) and (3.69) over $\nu^L \in E_{G^0}(t\sigma, \sigma_\theta)/a_{M,\theta}^\vee$. We can then convert this into a sum over $\tilde{\sigma} \in \Pi_2(M(F))^{\theta t}/ia^*$ provided that we multiply the summand by

$$|((a_{M,\sigma})_\theta/a_{\sigma}^\vee)/(a_{M,F}^\vee/a_F^\vee)| = |(a_{M,\sigma})_\theta/a_{M,F}^\vee|a_{\sigma}^\vee/a_F^\vee|^{-1}.$$ \hfill (3.69)$$

Hence, we see that the product of (3.65) with this number equals

$$|det(1 - \theta)_{G^0}||W^*_{M}||W^*_{G^0}|^{-1}|a_{\sigma}^\vee/a_F^\vee|^{-1}|det(\theta t - 1)_{G^0}|^{-1}.$$
Assume that $\sigma \in \Pi_2(M(F))^{\theta t}$ and $\lambda \in i\mathfrak{a}^*$. Define $\tilde{J}_{\theta,L}(\sigma, t, f)$ to be

$$\lim_{\xi \to 0} \sum_{R \in \mathcal{P}(L)} \tau_{\theta,1,R}(\xi) \frac{1}{|[L,\theta]/L,k,\theta|} \sum_{X \in [L,\theta]/L,k,\theta} e^{\xi(X_R)} \theta_{R,k}(\xi)^{-1}, \quad (3.72)$$

where

$$\tau_{\theta,1,R}(\xi) = \text{tr}(I_P(\sigma, f_1^\vee) R_{P_\theta}|_{P(t, \sigma)}^{-1} J_{P_\theta}|_{R(\Pi)^\theta}((t\sigma)(\lambda+\zeta)_\theta) J_{P_\theta}|_{R(\Pi)^\theta}((t\sigma)\lambda_\theta)^{-1} B_{\sigma_\lambda}^{-1})$$

and

$$\tau_{\theta,2,R}(\xi) = \text{tr}(B_{\sigma_\lambda} J_{R(\Pi)^\theta}|_{P_\theta} ((t\sigma)\lambda_\theta)^{-1} J_{R(\Pi)^\theta}|_{P_\theta} ((t\sigma)(\lambda+\zeta)_\theta) R_{P_\theta}|_{P(t, \sigma)} I_P(\sigma, f_2))$$

for any points $\zeta \in i\mathfrak{a}^*$.

Finally, we have a twisted version of Proposition 11.3 of [10].

**Proposition 3.5.16**: The distribution $\tilde{J}_\theta(f)$ is equal to

$$\frac{1}{|\text{det}(1-\theta)\varepsilon|_{G^0}} \sum_{M,L,t,\sigma} |W_M^0| \frac{|\varepsilon_\sigma(t)|}{|W_0^0| |\mathfrak{a}_L^\vee/\mathfrak{a}_F^\vee||\text{det}(\theta t - 1)\bar{a}_\theta^L|} \int_{i\mathfrak{a}_F^*} \tilde{J}_{\theta,L}(\sigma, t, f) d\lambda \quad (3.73)$$

with the sums being taken over

$M \in \mathcal{L}^0, \ L \in \mathcal{L}(M), \ t \in W(\bar{a}_M^L, \bar{a}_M^{L_0})_{\text{reg}}, \ and \ \sigma \in \Pi_2(M(F))^{\theta t}/i\mathfrak{a}^*$.

In (3.73), we see that $t(\sigma, \lambda) = \theta^{-1}(\sigma, \lambda)$ for any $\sigma \in \Pi_2(M(F))^{\theta t}/i\mathfrak{a}^*$ and $\lambda \in i\mathfrak{a}_F^*$. By Langlands’ classification theory for tempered representations of reductive groups (see [11], Proposition 1.1), we have

$$I_P(\sigma, x) \cong I_{\theta^{-1}(P)}(\theta^{-1}(\sigma), x).$$
In Lemma 3.5.4, we have also seen that $\mathcal{I}_P(\sigma, \theta(x)) \cong \mathcal{I}_{\theta^{-1}(P)}(\theta^{-1}(\sigma), x)$, Therefore, we obtain

$$\mathcal{I}_P(\sigma, x) \cong \mathcal{I}_P(\sigma, \theta(x)). \quad (3.74)$$

Later, we shall see that only certain irreducible representations $\pi$ satisfying

$$\pi \cong \theta(\pi)$$

appears on the spectral side of the local twisted trace formula, which we will discuss in detail in the next subsection. This result is similar to that for the compact case (see (3.9)). We know that an irreducible representation $\pi$ of $G^0$ satisfying $\pi \cong \theta(\pi)$ can be lifted to a representation of $G^+$ (Corollary 2.4.3).

### 3.6 The Local Twisted Trace Formula

We will use a process similar to that in §12 of [10] to obtain the final non-invariant local twisted trace formula.

Because we want to express some results in §3.5 in the language related to $G^+$, we need to change the notation for Levi subgroups to the original form. Use $M^0$ and $L^0$ in place of $M$ and $L$ in §3.5, respectively. Also, $P$ is changed back to $P^0$ for a parabolic subgroup of $G^0(F)$ in $\mathcal{P}(M^0)$. Except in §3.5, we still use $M$ and $L$ to denote Levi subsets of $G(F)$, and $P$ to denote a parabolic subset of $G(F)$. We can restate Proposition 3.5.16 as the following: $\tilde{J}_\theta(f)$ equals the product of $|det(1 - \theta)_{\mathcal{C}}|^{-1}$ with

$$\sum_{M^0, L^0, t, \sigma, \overline{a}_{\mathcal{G}}|\mathcal{C}|} |W_0^{M^0} / W_0^{L^0}| |a_\sigma / a_\mathcal{F}| |det(\theta t - 1)|_{\mathcal{A}_{\mathcal{L}}^0} | \times \int_{ia_\mathcal{F}} \tilde{J}_{\theta, L^0}(\sigma, t, f) d\lambda, \quad (3.75)$$

where the sums are taken over $M^0 \in \mathcal{L}^0$, $L^0 \in \mathcal{L}(M^0)$, $t \in W(\overline{a}_{\mathcal{M}^0}^{L^0}, \overline{a}_{\mathcal{M}^0}^{L^0})_{\text{reg}}$, and $\sigma \in \Pi_2(M^0(F))^{\theta t / ia^*}$. 
Assume that there exists an element $t$ in $W(a_{M^0}, a_{M^0})$ for $M^0 \in \mathcal{L}^0$ in (3.75). If $\sigma$ is in $\Pi_2(M^0(F))^\theta t$, then we can lift the representation $\mathcal{I}_{p^0}(\sigma)$ of $G^0(F)$ to a representation $\mathcal{I}_{p^0}^+(\theta t, \sigma)$ of $G^+(F)$. Keep in mind that $\sigma_\theta = \theta^{-1}(\sigma)$ and $\sigma_\theta = t\sigma$. We have the following commutative diagram (for simplicity, we replace $P^0$ by $P$ in this diagram):

$$
\begin{array}{cccc}
\mathcal{H}_P(\sigma) & \xrightarrow{R_{P\theta}(t,\sigma)} & \mathcal{H}_P(\sigma_\theta) & \xrightarrow{B_\sigma} & \mathcal{H}_P(\sigma) \\
\downarrow \mathcal{I}_P(\sigma, x) & & \downarrow \mathcal{I}_P(\sigma_\theta, x) & & \downarrow \mathcal{I}_P(\sigma, \theta(x)) \\
\mathcal{H}_P(\sigma) & \xrightarrow{R_{P\theta}(t,\sigma)} & \mathcal{H}_P(\sigma_\theta) & \xrightarrow{B_\sigma} & \mathcal{H}_P(\sigma).
\end{array}
$$

Hence, we see that $B_\sigma R_{p^0\theta}^0(t, \sigma)$ is an intertwining operator between $\mathcal{I}_{p^0}(\sigma)$ and $\mathcal{I}_{p^0}(\sigma)_\theta$, where $\mathcal{I}_{p^0}(\sigma)_\theta$ is the same as $\theta^{-1}(\mathcal{I}_{p^0}(\sigma))$. For simplicity, set

$$T(\theta t, \sigma) = B_\sigma R_{p^0\theta}^0(t, \sigma).$$

Next, we will show that $T(\theta t, \sigma)^n = c(0) \cdot 1$, where $c(0)$ is a non-zero constant defined below and $1$ is the identity map. In fact, let $\lambda$ be a regular point in $i\mathfrak{a}_{M^0}^*$ (the condition $\theta t(\lambda) = \lambda$ is not necessary). Obviously,

$$t(\sigma_\lambda) = (\theta t(\sigma_\lambda))_\theta.$$

Then, $T(\theta t, \sigma_\lambda)$ is an intertwining operator between $\mathcal{I}_{p^0}(\sigma_\lambda)$ and $\mathcal{I}_{p^0}(\theta t(\sigma_\lambda))_\theta$. We observe that

$$T(\sigma_\lambda) = T(\theta t, \sigma_\lambda) \circ T(\theta t, \theta t(\sigma_\lambda)) \circ ... \circ T(\theta t, (\theta t)^{n-1}(\sigma_\lambda))$$

is an intertwining operator between $\mathcal{I}_{p^0}(\sigma_\lambda)$ and $\mathcal{I}_{p^0}((\theta t)^n(\sigma_\lambda))_{\theta^n} = \mathcal{I}_{p^0}(\sigma_\lambda)$. By Schur’s Lemma, we know that $T(\sigma_\lambda) = c(\lambda) \cdot 1$, where $c(\lambda)$ is a non-zero smooth function of $\lambda$. Using $\theta t(\sigma_\lambda) = \sigma_\lambda$ for $\lambda = 0$, i.e., $\theta t(\sigma) = \sigma$, it is easy to see that

$$T(\theta t, (\theta t)^k(\sigma)) = T(\theta t, \sigma), \quad \forall \ k = 0, 1, ..., n - 1.$$
Consequently, $T(\theta t, \sigma)^n = T(\sigma_0) = c(0) \cdot 1$. It would be interesting to study the function $c(\lambda)$. However, in this thesis, we do not need to know explicit information about $c(\lambda)$. Choose an $n$-th root $b$ of $c(0)$ (the absolute value of $c(0)$ is 1). Define a new intertwining operator $R(\theta t, \sigma)$ between $I_{P0}(\sigma)$ and $I_{P0}(\sigma)_{\theta}$ by

$$R(\theta t, \sigma) = \frac{T(\theta t, \sigma)}{b}.$$ 

Then,

$$R(\theta t, \sigma)^n = 1.$$ 

We can now define the lifting $I_{P0}(\theta t, \sigma)$ of the representation $I_{P0}(\sigma)$. The representation $I_{P0}(\theta t, \sigma)$ is given by

$$I_{P0}(\theta t, \sigma, x) = \begin{cases} 
I_{P0}(\sigma, x) & \text{if } x \in G^0(F), \\
R(\theta t, \sigma) & \text{if } x = 1 \rtimes \theta,
\end{cases}$$

which can be extended to $G^+(F)$ (see Theorem 2.4.1).

To obtain a suitable spectral side of the trace formula, we shall define a distribution that is called the "discrete part" of the spectral side. Let

$$W_\sigma = \{ t \in W(a_{M^0}, a_{M^0}) : \theta t(\sigma) \cong \sigma \}$$

be the twisted stabilizer of $\sigma$ in the torsor $W(a_{M^0}, a_{M^0})$ of the Weyl group of $a_{M^0}$ (note that Arthur uses $W_\sigma$ for the ordinary stabilizer of $\sigma$ in the Weyl group $W(a_{M^0})$ in [I]). If $t \in W_\sigma$, then we have the normalized intertwining operator

$$R(\theta t, \sigma^\vee \otimes \sigma) = R(\theta t, \sigma^\vee) \otimes R(\theta t, \sigma)$$
from
\[ I_{P^0}(\sigma^\vee \otimes \sigma) = I_{P^0}(\sigma^\vee) \otimes I_{P^0}(\sigma) \]
to
\[ I_{P^0}(\sigma^\vee \otimes \sigma)_\theta = I_{P^0}(\sigma^\vee)_\theta \otimes I_{P^0}(\sigma)_\theta. \]

This operator depends only on the orbit of \( \sigma \) in \( \Pi_2(M(F))^{\theta t}/i(a_{G^0})_\theta \). It can be seen that
\[ J_{G^0, \theta}(\sigma_\lambda, t, f) = \text{tr}(R(\theta t, \sigma^\vee \otimes \sigma)I_{P^0}(\sigma^\vee_\lambda \otimes \sigma_\lambda, f)). \tag{3.76} \]

The functions \( f_i \) (\( i = 1 \) or \( 2 \)) can also be regarded as functions on \( G(F) \) in the expression \( J_{\theta, G^0}(\sigma_\lambda, t, f) \) by the correspondence between functions on \( G^0(F) \) and functions on \( G(F) \) in §3.1 (see Equation (3.1)). For convenience, we recall that a function \( h \) on \( G^0(F) \) can be regarded as a function \( h|_G \) on \( G(F) \) by the definition
\[ h|_G(g \times \theta) = h(g), \quad g \in G^0(F). \]

For the function \( f(x_1, x_2) = f_1(x_1) \times f(x_2) \) on \( G^0(F) \times G^0(F) \), the corresponding function \( f|_G \) on \( G(F) \times G(F) \) is defined to be \( f_1|_G \times f_2|_G \). For simplicity, we still use \( f \) to denote \( f|_G \). Under the above correspondence, we can regard \( J_{\theta, G^0}(\sigma_\lambda, t, f) \) as a distribution on \( G(F) \times G(F) \). When we consider \( J_{\theta, G^0}(\sigma_\lambda, t, f) \) as a distribution on \( G(F) \times G(F) \), we use a different symbol \( J_{\theta}(\sigma_\lambda, t, f) \) to replace \( J_{\theta, G^0}(\sigma_\lambda, t, f) \). We can write \( J_{\theta}(\sigma_\lambda, t, f) \) as
\[ \text{tr}(I_{P^0}(\theta t, \sigma^\vee_\lambda \otimes \sigma_\lambda, f)), \]
where \( I_{P^0}(\theta t, \sigma^\vee_\lambda \otimes \sigma_\lambda) = I_{P^0}(\theta t, \sigma^\vee_\lambda) \otimes I_{P^0}(\theta t, \sigma_\lambda) \). It is easy to check that
\[ i(a_{G^0})_\theta^* = i a_G^*. \]
If \( t \) belongs to \( W(\mathfrak{a}_{M^0}^{G_0}, \mathfrak{a}_{M^0}^{G_0})_{\text{reg}} \), then \( \mathfrak{a} = \mathfrak{a}_G \) and \( \mathfrak{a}_{M^0}^{G_0} \cong \mathfrak{a}_{M^0}^{G_0} \).

We now define the discrete part \( I_{\text{disc}}(f) = I_{\text{disc}}^G(f) \) of the spectral side to be

\[
\frac{1}{|\text{det}(1 - \theta)|_{\mathfrak{a}_G^{G_0}}} \sum_{M^0, t, \sigma} \|W_0^{M^0}\|_{W_0^{G_0}} \|a_{G, \sigma}^G/a_{G, F}^\vee\| |\text{det}(\theta t - 1)|_{\mathfrak{a}_{M^0}^{G_0}} \int_{ia_G^*} \bar{J}_G(\sigma, t, f) d\lambda,
\]

where the sums are taken over

\[
M^0 \in \mathcal{L}^0, \quad t \in W(\mathfrak{a}_{M^0}^{G_0}, \mathfrak{a}_{M^0}^{G_0})_{\text{reg}}, \quad \text{and} \quad \sigma \in \Pi_2(M(F))^\theta t/ia_G^*.
\]

Here, \( I_{\text{disc}}(f) = I_{\text{disc}}^G(f) \) is a distribution, which depends only on the restriction \( f^1 \) of \( f \) to the subset

\[
G(A_F)^1 = \{ (y_1, y_2) \in G(A_F) : H_G(y_1) = H_G(y_2) \}.
\]

In fact, \( I_{\text{disc}}(f) \) is a finite linear combination of characters of irreducible representations. More precisely, these irreducible representations are in \( \Pi_{\text{temp}}(G(A_F)) \), which is a subset of

\[
\Pi_{\text{temp}}(G^+(A_F))
\]

(see §2.4). When we use the notation \( \bar{J}_{\theta, G^0}(\sigma, t, f) \) in the definition of \( I_{\text{disc}}^G(f) \), the distribution \( I_{\text{disc}}^G(f) \) can be regarded as a finite linear combination of characters of \( \theta \)-fixed irreducible representations of \( G^0(F) \) by Corollary 2.4.3 and the discussion on pp. 27-28 of [7]. Observe that

\[
|\text{det}(1 - \theta)|_{\mathfrak{a}_G^{G_0}} |\text{det}(\theta t - 1)|_{\mathfrak{a}_{M^0}^{G_0}} = |\text{det}(\theta t - 1)|_{\mathfrak{a}_{M^0}^{G_0}} |
\]

We conclude that the discrete part \( I_{\text{disc}}(f) = I_{\text{disc}}^G(f) \) of the spectral side equals

\[
\sum_{M^0, t, \sigma} \|W_0^{M^0}\|_{W_0^{G_0}} \|a_{G, \sigma}^G/a_{G, F}^\vee\| |\text{det}(\theta t - 1)|_{\mathfrak{a}_{M^0}^{G_0}} \int_{ia_G^*} \bar{J}_G(\sigma, t, f) d\lambda,
\]  

(3.77)
where the sums are taken over

\[ M^0 \in \mathcal{L}^0, \ t \in W(\mathfrak{a}_{M^0}^G, \mathfrak{a}_{M^0}^G)_{\text{reg}}, \ \text{and} \ \sigma \in \Pi_2(M(F))^{et}/i\mathfrak{a}_G^*. \]

Next, we deal with the general terms on the spectral side. To state the results in a simpler form, we define \( \Pi_{\text{disc}}(G) \) to be the set of equivalence classes of irreducible representations \( \pi = \pi_1^\vee \otimes \pi_2 \) of \( G^+(A_F) \) that are constituents of induced representations

\[ I_{P^0}(\theta t, \sigma^\vee \otimes \sigma) = I_{P^0}(\theta t, \sigma^\vee) \otimes I_{P^0}(\theta t, \sigma), \quad P^0 \in \mathcal{P}(M^0), \]

in which \( \sigma \) is a representation in \( \Pi_2(M(F))^{et} \) for some element \( t \in W(\mathfrak{a}_{M^0}^G, \mathfrak{a}_{M^0}^G)_{\text{reg}} \) and both \( \pi_1 \) and \( \pi_2 \) are in \( \Pi_{\text{temp}}(G(F)) \).

There is an action of \( i\mathfrak{a}_G^* \) on \( \Pi_{\text{disc}}(G) \) given by

\[ \pi \to \pi_\lambda = \pi_1^{\sigma^\vee} \otimes \pi_2, \quad \lambda \in i\mathfrak{a}_G^*. \]

We write \( \Pi_{\text{disc}}(G)/i\mathfrak{a}_G^* \) for the set of orbits of this action. Then, we can define a measure \( d\pi \) on \( \Pi_{\text{disc}}(G) \) by setting

\[
\int_{\Pi_{\text{disc}}(G)} \phi(\pi) d\pi = \sum_{\Pi_{\text{disc}}(G)/i\mathfrak{a}_G^*} \int_{i\mathfrak{a}_G^*} \phi(\pi_\lambda) d\lambda, \tag{3.78}
\]

where \( \phi \) is a function in \( C_c(\Pi_{\text{disc}}(G)) \).

It follows from the definitions that we can write

\[ I_{\text{disc}}(f) = \int_{\Pi_{\text{disc}}(G)} a^G_{\text{disc}}(\pi) tr(\pi(f)) d\pi, \tag{3.79} \]

where each \( a^G_{\text{disc}}(\pi) \) is a uniquely determined complex number that depends only the \( i\mathfrak{a}_G^* \)-orbit of \( \pi \). Generally, for any Levi subset \( M \in \mathcal{L} \) and \( \pi \in \Pi_{\text{disc}}(M)/i\mathfrak{a}_M^* \), we can define the number \( a^M_{\text{disc}}(\pi) \) in the same way as in the definition of \( a^G_{\text{disc}}(\pi) \).
Suppose that $M \in \mathcal{L}$ is a general Levi subset and that $\sigma$ is a representation in $\Pi_{\text{temp}}(M)$. For any parabolic subset $P, Q \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$, we can define the induced representation $I_P(\sigma_\lambda)$ of $G^+(F)$ and the standard unnormalized intertwining operators $J_{Q|P}(\sigma_\lambda)$ between $I_P(\sigma_\lambda)$ and $I_Q(\sigma_\lambda)$ from the vector space $\mathcal{H}_P(\sigma)$ to $\mathcal{H}_Q(\sigma)$ (see [7], §1, where the space $\mathcal{H}_P(\sigma)$ is called $\mathcal{V}_P(\sigma)$). If $\pi_1 \otimes \pi_2$ is a representation in $\Pi_{\text{disc}}(M)$, we can form the induced representation

$$I_P(\pi_\lambda, f) = I_P(\pi_1^\vee, f_1) \otimes I_P(\pi_2, f_2)$$

of the Hecke algebra $\mathcal{H}(G^+(A_F))$ and the standard unnormalized intertwining operators

$$J_{Q|P}(\pi_\lambda) = J_{Q|P}(\pi_1^\vee) \otimes J_{Q|P}(\pi_2, \lambda).$$

Observe that the operators

$$\mathcal{J}_Q(\Lambda, \pi_\lambda, P) = J_{Q|P}(\pi_\lambda)^{-1}J_{Q|P}(\pi_{\lambda+\Lambda})$$

map $\mathcal{H}_P(\pi) = \mathcal{H}_P(\pi_1^\vee) \otimes \mathcal{H}_P(\pi_2)$ to itself. In fact, the set

$$\{ \mathcal{J}_Q(\Lambda, \pi_\lambda, P) : Q \in \mathcal{P}(M) \}$$

is a $(G, M)$-family of functions of $\lambda \in i\mathfrak{a}_M^*$ with values in the space of operator-valued meromorphic functions of $\lambda$ (see [7], §6). In particular, the limit

$$\mathcal{J}_M(\pi_\lambda, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} \mathcal{J}_Q(\Lambda, \pi_\lambda, P)\theta_Q(\Lambda)^{-1}$$

exists and is a meromorphic function of $\lambda$. It is possible to show that the matrix coefficients of the operator $\mathcal{J}_M(\pi_\lambda, P)$ are analytic functions of $\lambda \in i\mathfrak{a}_M^*$ whose derivatives are slowly increasing. This is analogous to Lemma 12.1 of [10].
Let $J_M(\pi, P)$ denote the value of $J_M(\pi_\lambda, P)$ at $\lambda = 0$. We then define the twisted weighted character

$$J_M(\pi, f) = \text{tr}(J_M(\pi, P)I_P(\pi_\lambda, f)).$$

The distributions $J_M(\pi, f)$, which depend only on the equivalence classes of representations $\pi \in \Pi_{disc}(M)$, will be the main ingredients of the spectral side of the local twisted trace formula.

To describe the terms on the geometric side, we shall first define weighted functions $v_M(x)$ for Levi subsets $M \in \mathcal{L}$. Assume that $x = (x_1, x_2)$ belongs to $G(A_F)$. Then, the functions

$$v_P(\Lambda, x) = e^{-\Lambda(H_P(x_2) - H_T(x_1))}, \quad \Lambda \in i\mathfrak{a}_M^*, \; P \in \mathcal{P}(M)$$

form a $(G, M)$-family. The limit

$$v_M(x) = \lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\Lambda, x)\theta_Q(\Lambda)^{-1}$$

exists ([2], Lemma 6.2) and equals the volume in $\mathfrak{a}_M/\mathfrak{a}_G$ of the convex hull of the points

$$\{(-H_P(x_2) + H_T(x_1)) : P \in \mathcal{P}(M)\}.$$

The function $v_M(x)$ is invariant under left translation by $M(A_F) = M(F) \times M(F)$. If $\gamma$ is a $G$-regular element in $M(F)$ embedded diagonally in $M(A_F)$, we can define the twisted weighted orbital integral

$$J_M(\gamma, f) = |D(\gamma)| \int_{A_M(A_F) \backslash G_0(A_F)} f(x^{-1} \gamma x) v_M(x) dx,$$

which also equals

$$|D(\gamma)| \int_{A_M(F) \backslash G_0(F)} \int_{A_M(F) \backslash G_0(F)} f_1(x_1^{-1} \gamma x_1)f_2(x_2^{-1} \gamma x_2) v_M(x_1, x_2) dx_2 dx_1.$$
We write $\Gamma_{\text{ell}}(M)$ for the set of conjugacy classes in $M(A_F)$ of the form $(\gamma, \gamma)$, where $\gamma$ is an $F$-elliptic conjugacy class in $M(F)$. The distributions $J_M(\gamma, f)$, evaluated at the $G$-regular elements in $\Gamma_{\text{ell}}(M)$, will be the main ingredients on the geometric side of the formula.

We can now state the local twisted trace formula.

**Theorem 3.6.1**: Assume that $f \in \mathcal{H}(G(A_F))$; then, the expression

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\text{ell}}(M)} J_M(\gamma, f) d\gamma$$

(3.84)

equals

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Pi_{\text{disc}}(M)} a_{\text{disc}}(\pi) J_M(\pi, f) d\pi.$$  

(3.85)

Before giving the proof of Theorem 3.6.1, let us restate Langlands’ Lemma about $\theta$-special pairs $(L, a)$ (see §3.5.3).

**Lemma 3.6.2 (Langlands [17], §15, Lemma 3)**: Assume that $M^0 \in \mathcal{L}^0$ and $t$ is in $W(a_{M^0}, a_{M^0})$. Set $a$ to be the subspace $\{H \in a_{M^0} : \theta t(H) = H\}$ of $a_{M^0}$. Then, there exists a unique Levi subgroup $L^0$ in $\mathcal{L}(M)$ such that $a \subset a_{L^0}$ and $(L^0, a)$ is $\theta$-special. That is, we can find $L^0$ that is the Levi factor of an $\theta$-stable parabolic subgroup $Q^0$ and an element $g_0 \in G^0(F)$ such that

$$(L^0, a) = g_0(L^0, a_{L^0})g_0^{-1}.$$  

(3.86)

Set $Q = Q^0 \rtimes \theta$ for the parabolic subset. Let $L' = \text{Norm}_{G^+}(L^0) \cap Q$ be the Levi subset of $Q$ corresponding to $L^0$. We see that $a_{L'} = a_{L^0}^\theta = Ad(g_0)^{-1}(a)$. Then, there is a correspondence

$$L^0 \longrightarrow L'.$$
as described above. Obviously, \( M' = g_0^{-1}M^0g_0 \subset L^0 \). Consequently, through the conjugation \( \text{Ad}(g_0) \), we can replace the sum over the Levi subgroups \( M^0 \in \mathcal{L}^0 \) and \( L^0 \in \mathcal{L}(M^0) \) by a sum over the Levi subsets \( L' \in \mathcal{L} \) and \( M^0 \subset L^0 \). For simplicity, we use the notation \( L, L^0 \), and \( M^0 \) in place of \( L', L_0^0 \) and \( M_0' \), respectively. Let \( L^+ \) be the group generated by \( L \). We can define \( \tilde{J}_L(\sigma, s, f) \) in the same way we define \( \tilde{J}_G(\sigma, t, f) \), where \( t \in W(\bar{a}^{L_0}_{M^0}, \bar{a}^{L_0}_{M^0})_{\text{reg}} \) and \( s \in W(\bar{a}^{G_0}_{M^0}, \bar{a}^{G_0}_{M^0})_{\text{reg}} \). Using (3.75), we observe that \( \tilde{J}_\theta(f) \) can be written as the product of \( |\text{det}(1-\theta)_{a^G_{M^0}}|^{-1} \) with

\[
\sum_{L, M^0, t, s, \sigma} \frac{|W_L|}{|W_0^G|} \left| a_L^\vee_{\sigma} / a_{L,F}^\gamma \right| |\text{det}(\theta t - 1)_{a^L_{M^0}}| \times \int_{i\sigma_{L,F}} \tilde{J}_L(\sigma, t, f) d\lambda \tag{3.87}
\]

with the sums being taken over

\[ L \in \mathcal{L}, M^0 \in \mathcal{L}^0, t \in W(\bar{a}^{L_0}_{M^0}, \bar{a}^{L_0}_{M^0})_{\text{reg}}, \text{and} \, \sigma \in \prod_2(M^0(F))^\theta / i\bar{a}^*_L. \]

We note that \( W(\bar{a}^{L_0}_{M^0}, \bar{a}^{L_0}_{M^0})_{\text{reg}} = W(\bar{a}^{L_0}_{M^0}, \bar{a}^{L_0}_{M^0})_{\text{reg}} \). Observe that

\[ |\text{det}(1-\theta)_{a^G_{M^0}}| |\text{det}(\theta t - 1)_{a^L_{M^0}}| = |\text{det}(\theta t - 1)_{a^L_{M^0}}|. \]

Then, \( \tilde{J}(f) \) can be simplified to

\[
\sum_{L, M^0, t, s, \sigma} \frac{|W_L|}{|W_0^G|} \left| a_L^\vee_{\sigma} / a_{L,F}^\gamma \right| |\text{det}(\theta t - 1)_{a^L_{M^0}}| \times \int_{i\sigma_{L,F}} \tilde{J}_L(\sigma, t, f) d\lambda \tag{3.88}
\]

with the sums being taken over

\[ L \in \mathcal{L}, M^0 \in \mathcal{L}^0, t \in W(\bar{a}^{L_0}_{M^0}, \bar{a}^{L_0}_{M^0})_{\text{reg}} \text{ and } \sigma \in \prod_2(M^0(F))^\theta / i\bar{a}^*_L. \]

We now return to the proof of Theorem 3.6.1.

First, the integrals in (3.84) and (3.85) are absolutely convergent. In fact, the integrand on the geometric side is a locally integrable function of compact support (cf. Lemma 3.4.3); on the spectral side, the \( K \)-finiteness of \( f \) implies that the expression is a
finite linear combination of integrals

$$\int_{i\mathfrak{a}_{M,F}^*} J_M(\pi, f) d\lambda, \quad \pi \in \Pi_{\text{disc}}(M).$$

The integral in (3.85) is absolutely convergent since $J_M(\pi, f)$ is a Schwartz function of $\lambda \in i\mathfrak{a}_{M,F}^*$.

Next, it suffices to take

$$f(x) = f_1(x_1) f_2(x_2), \quad f_i \in \mathcal{H}(G(F)),$$

as before. For general $f \in \mathcal{H}(G(A_F))$, it suffices to use a standard approximation argument.

We want to prove this theorem by induction. If $G^0(F)$ equals $M_0^0(F)$, it is easy to check that (3.84) and (3.85) are equal. Hence, the theorem is true in this case.

We now make the induction assumption that the theorem holds if $G$ is replaced by any proper Levi subset. We write $J_{\text{geom}}(f) = J_G^{\text{geom}}(f)$ and $J_{\text{spec}}(f) = J_G^{\text{spec}}(f)$ for the respective expressions (3.84) and (3.85). Our aim now is to convert the geometric and spectral expression of $\tilde{J}(f)$ into two parallel linear combinations of distributions $\{J_{\text{geom}}(f_Q)\}$ and $\{J_{\text{spec}}(f_Q)\}$, in which $Q$ ranges over the parabolic subsets in $\mathcal{F}$ and

$$f_Q(m) = \delta_Q(m)^{1/2} \int_{K \times K} \int_{N_Q(A_F)} f(k^{-1}mnk) dndk, \quad m \in M_Q(A_F),$$

where $\delta_Q(m)$ is the modular function of $Q$ ([6], §1). We will then be able to exploit the induction hypothesis in the form

$$J_{\text{geom}}^{M_Q}(f_Q) = J_{\text{spec}}^{M_Q}(f_Q), \quad Q \neq G.$$ (3.89)
Motivated by the definition (3.23) of $\tilde{v}_M(x)$, we set

$$c_P(\Lambda) = \frac{1}{|L_M/L_{M,k}|} \sum_{X \in L_M/L_{M,k}} e^{\Lambda(X_P)} \theta_{P,k}(\Lambda)^{-1} \theta_P(\Lambda),$$

(3.90)

for $M \in \mathcal{L}$, $P \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$. 

On the geometric side, it is possible to check that

$$\tilde{J}(f) = \sum_{Q \in \mathcal{F}} |W_0^{MQ}||W_0^G|^{-1}(-1)^{\dim(A_Q/A_G)} J_{geom}^M(f_Q)c_Q'$$

(3.91)

using the product formula for $(G,M)$-families, where $c_Q'$ is defined by the twisted version of (6.3) in [2].

We now consider the spectral side.

We can apply the definition (3.77) of $I_{disc}^L$, (modifying the process for the ordinary case described on pp. 93-94 of [10]). Then, we find that (3.88) equals

$$\int_{\Pi_{disc}(L)} a_{disc}^L(\pi) \tilde{J}_L(\pi, f) d\pi,$$

where

$$\tilde{J}_L(\pi, f) = (-1)^{\dim(A_L/A_G)} \lim_{\zeta \to 0} \sum_{R \in \mathcal{P}(L)} tr(\mathcal{I}_R(\zeta, \pi, S) \mathcal{I}_S(\pi, f))c_R(\zeta)\theta_R(\zeta)^{-1}.$$

As in the ordinary case, we have

$$\tilde{J}(f) = \sum_{Q \in \mathcal{F}} |W_0^{MQ}||W_0^G|^{-1}(-1)^{\dim(A_Q/A_G)} J_{spec}^M(f_Q)c_Q'.$$

(3.92)

By the induction hypothesis (3.89), the terms corresponding to $Q \neq G$ are pairwise equal. This leaves the two terms with $Q = G$, which are just $J_{geom}(f)$ and $J_{spec}(f)$. 
Consequently, we obtain

\[ J_{\text{geom}}(f) = J_{\text{spec}}(f), \]

which is just what we wanted to prove. \qed
Chapter 4

The Twisted Orthogonality Relations

As an application of the local ordinary trace formula, Arthur proved some orthogonality relations for elliptic tempered characters [11]. In this chapter, we shall generalize Arthur’s orthogonality relations to the twisted elliptic tempered characters. In §4.1, we will classify the twisted irreducible tempered representations of $G^+(F)$ via the language of $R$-groups. In §4.3, we will derive an invariant local twisted trace formula. After studying twisted weighted characters and twisted weighted orbital integrals, we will prove the orthogonality relations at the end of this chapter. We will apply a process similar to that used in §1-§6 of [11].

4.1 Elliptic Tempered Representations

Recall that $G = G^0 \rtimes \theta$ is a connected component of the non-connected reductive group $G^+ = G^0 \rtimes \langle \theta \rangle$ over $F$.

It is known that, in general, a representation $\pi \in \Pi_{\text{temp}}(G^+(F))$ is infinite-dimensional and does not have a character in the classical sense. Fortunately, Harish-Chandra estab-
lished the theory of characters of infinite-dimensional representations in the language of distributions. It was proved by L. Clozel (15) that the character of $\pi$ is defined as a distribution

$$\Theta(\pi, f) = \text{tr} \left( \int_{G^+(F)} f(x) \pi(x) dx \right), \quad f \in C_\infty^c(G^+(F)),$$

which can then be identified with a function on $G^+(F)$. In other words, there exists a locally integrable function $\Theta(\pi, x)$ on $G^+(F)$ that is smooth on the open dense subset $G^+_{\text{reg}}(F)$ of regular elements such that

$$\Theta(\pi, f) = \int_{G^+(F)} f(x) \Theta(\pi, x) dx, \quad f \in C_\infty^c(G^+(F)).$$

Let $\Theta(\pi)$ denote the character of a general representation $\pi \in \Pi_{\text{temp}}(G^+(F))$. We define the normalized character to be

$$\Phi(\pi, \gamma) = |D(\gamma)|^{1/2} \Theta(\pi, \gamma), \quad \pi \in \Pi_{\text{temp}}(G^+(F)), \quad \gamma \in G^+_{\text{reg}}(F),$$

where $D(\gamma) = \text{det}(1 - \text{Ad}(\gamma))$ is the Weyl discriminant.

We can now give the definition of a $G(F)$-elliptic representation of $G^+(F)$.

**Definition 4.1.1**: A tempered irreducible representation $\pi$ of $G^+(F)$ is said to be $G(F)$-elliptic if its character does not vanish identically on the regular elliptic set of the connected component $G(F)$.

Let

$$\Pi_{\text{ell}}(G(F))$$

denote the set of $G(F)$-elliptic representations of $G^+(F)$. It follows that $\pi$ is $G(F)$-elliptic if and only if $\pi$ belongs to $\Pi_{\text{temp}}(G(F))$ and $\Phi(\pi)$ does not vanish identically on the regular elliptic set of $G(F)$.

In the rest of this section, we want to use $R$-groups to describe the set $\Pi_{\text{ell}}(G(F))$. First, let us review some results from the theory of $R$-groups stated in §2 of [11]. Assume
that $M^0 \in \mathcal{L}^0$ is a Levi subgroup of $G^0(F)$ and $\sigma$ is a representation in $\Pi_2(M^0(F))$. For any $P^0 \in \mathcal{P}(M^0)$, we can construct the induced representation $\mathcal{I}_{P^0}(\sigma)$. Recall that the stabilizer of $\sigma$ in the Weyl group of $a_{M^0}$ is given by

$$W_{\sigma}^{G^0} = \{ r \in W(a_{M^0}) : r\sigma \cong \sigma \}.$$ 

For every $\omega$ in $W_{\sigma}^{G^0}$, we can define the normalized intertwining operator $R(w, \sigma)$ from $\mathcal{I}_{P^0}(\sigma)$ to itself as follows. In fact, $\sigma$ can be extended to a representation of the group $M^+_\omega(F)$ generated by $M^0(F)$ and $\tilde{\omega}$, where $\tilde{\omega}$ is a representative of $\omega$ in $K$. We still use $\sigma$ to denote the extended representation. Let $A(\sigma)$, which is given by

$$A(\sigma)\phi(x) = \sigma(\omega)\phi(\omega^{-1}x), \quad \phi \in \mathcal{H}_{P^0}(\sigma),$$

be an intertwining operator between $\mathcal{I}_{P^0}(\sigma)$ and $\mathcal{I}_{P^0}(\sigma)$. Then, the composition

$$R(w, \sigma) = A(\sigma)R_{P^0}(\sigma)$$

is the desired intertwining operator for $\mathcal{I}_{P^0}(\sigma)$.

We can now define the $R$-group of $\sigma$. Let $W'_{\sigma}$ be

$$\{ \omega \in W_{\sigma}^{G^0} : R(\omega, \sigma) is a scalar \},$$

which is a normal subgroup of $W_{\sigma}^{G^0}$. Then, the quotient

$$R^0_{\sigma} = R_{\sigma}^{G^0} = W_{\sigma}^{G^0}/W'_{\sigma}$$

is the $R$-group of $\sigma$. It is known that $W'_{\sigma}$ is the Weyl group of a root system, composed of scalar multiples of those reduced roots of $(G^0, A_{M^0})$ for which the reflection $w$ belongs to $W'_{\sigma}$. These roots divide the vector space $a_{M^0}$ into chambers. Fixing such a chamber
\(a^+_\sigma\), we identify \(R_\sigma^0\) with the subgroup of elements in \(W_\sigma\) which preserve \(a^+_\sigma\). We can then write \(W_{\sigma^0}\) as a semi-direct product

\[
W_{\sigma^0} = W' \rtimes R_\sigma^0.
\]

In general, the product satisfies

\[
R(r_1r_2, \sigma) = \eta_\sigma(r_1, r_2)R(r_1, \sigma)R(r_2, \sigma), \quad r_1, r_2, \in R_\sigma,
\]

where \(\eta_\sigma(r_1, r_2)\) is a 2-cocycle for \(R_\sigma^0\) with values in \(\mathbb{C}^*\).

Arthur defined a finite central extension of \(R_\sigma^0\)

\[
1 \to Z_\sigma \to \tilde{R}_\sigma^0 \to R_\sigma^0 \to 1
\]

over which \(\eta_\sigma\) splits. That is, there is a function \(\xi_\sigma : \tilde{R}_\sigma^0 \to \mathbb{C}^*\) such that

\[
\eta_\sigma(r_1, r_2) = \xi_\sigma(r_1r_2)\xi_\sigma(r_1)^{-1}\xi_\sigma(r_2)^{-1}, \quad r_1, r_2, \in \tilde{R}_\sigma.
\]

Consequently, there is a linear character \(\chi_\sigma\) on the central subgroup \(Z_\sigma\) which is induced from \(\eta_\sigma\). In fact, the extension group \(\tilde{R}_\sigma^0\) is not unique. To make the results in the ordinary case and in the twisted case coincide, we shall require that \(\tilde{R}_\sigma^0\) satisfy Lemma 4.1.3.

We can then define a homomorphism

\[
\tilde{R}(r, \sigma) = \xi_\sigma(r)^{-1}R(r, \sigma), \quad r, \in \tilde{R}_\sigma^0,
\]

of \(\tilde{R}_\sigma^0\) into the group of unitary intertwining operators for \(I_{P^0}(\sigma)\). We find that

\[
R^0(r, x) = \tilde{R}(r, \sigma)I_{P^0}(\sigma, x), \quad r, \in \tilde{R}_\sigma^0, \quad x \in G^0(F),
\]
Chapter 4. The Twisted Orthogonality Relations

is a representation of $\tilde{R}_0^0 \times G^0(F)$ on $\mathcal{H}_{p_0}(\sigma)$. Arthur showed that there is a 1-1 correspondence

$$\rho \leftrightarrow \pi_\rho,$$

between $\Pi(\tilde{R}_0^0, \chi_\sigma)$ and $\Pi_\sigma(G^0(F))$, where $\Pi(\tilde{R}_0^0, \chi_\sigma)$ is the set of irreducible representations of $\tilde{R}_0^0$ with $Z_\sigma$-central character $\chi_\sigma$, and $\Pi_\sigma(G^0(F))$ is the set of irreducible constituents of the induced representation $I_{p_0}(\sigma)$. Furthermore, $\mathcal{R}^0$ has the decomposition

$$\mathcal{R}^0 = \bigoplus_{\rho \in \Pi(\tilde{R}_0^0, \chi_\sigma)} (\rho^\vee \otimes \pi_\rho). \quad (4.1)$$

Expressed in terms of characters, the bijection is

$$tr(\tilde{R}(r, \sigma)I_{p_0}(\sigma, h)) = \sum_{\rho \in \Pi(\tilde{R}_0^0, \chi_\sigma)} tr(\rho^\vee(r))tr(\pi_\rho(h)), \quad (4.2)$$

for any $r$ in $\tilde{R}_0^0$ and $h$ in the Hecke algebra $\mathcal{H}(G^0(F))$.

If $\Pi$ is a set of equivalence classes of irreducible representations of a group, let $C(\Pi)$ denote the complex vector space of virtual characters generated by $\Pi$. According to the bijection $\rho \leftrightarrow \pi_\rho$, there is an isomorphism

$$\vartheta \rightarrow \Theta$$

from $C(\Pi(\tilde{R}_0^0, \chi_\sigma))$ to $C(\Pi_\sigma(G^0(F)))$. The inverse of the formula (4.2) is

$$\Theta(h) = \frac{1}{|\tilde{R}_0^0|} \sum_{r \in \tilde{R}_0^0} \vartheta(r)tr(\tilde{R}(r, \sigma)I_{p_0}(\sigma, h)). \quad (4.3)$$

In fact, the correspondence $\vartheta \rightarrow \Theta$ behaves well under induction (Theorem 4.1.2 (b)). Let $L^0$ be a Levi subgroup in $\mathcal{L}(M^0)$. We write $R_\sigma^L$ for the $R$-group of $\sigma$ relative to $L^0$. 
Similarly, we construct the central extension of $R^L_\sigma$

$$1 \to Z_\sigma \to \tilde{R}^L_\sigma \to R^L_\sigma \to 1$$

such that $\tilde{R}^L_\sigma$ is the inverse image of $R^L_\sigma$ in $\tilde{R}^0_\sigma$.

To classify the elliptic representations of $G^0(F)$, we need to define the regular subset of $R^0_\sigma$ to be

$$R^0_{\sigma, \text{reg}} = \{ r \in R^0_\sigma : a^r_{M^0} = a_{G^0} \},$$

where

$$a^\omega_{M^0} = \{ H \in a_{M^0} : \omega(H) = H \}$$

denotes the space of fixed vectors of an element $\omega \in W(a_{M^0})$. Let $\tilde{R}^0_{\sigma, \text{reg}}$ be the inverse image in $\tilde{R}^0_\sigma$ of the set $R^0_{\sigma, \text{reg}}$.

We can now state Arthur’s results ([11], Proposition 2.1) about the classification of the representations in $\Pi_\sigma(G^0(F))$.

**Theorem 4.1.2 (Arthur, [11]):**

(a) There is a unique bijection $\rho \to \pi_\rho$ from $\Pi(\tilde{R}^0_\sigma, \chi_\sigma)$ onto $\Pi_\sigma(G^0(F))$ which satisfies the character identity (4.2).

(b) A sum of characters in $\Pi(\tilde{R}^0_\sigma, \chi_\sigma)$ is induced from a proper subgroup $\tilde{R}^L_\sigma$ of $\tilde{R}^0_\sigma$ if and only if the corresponding sum of characters in $\Pi_\sigma(G^0(F))$ is induced from a parabolic subgroup with Levi component $L^0(F)$.

(c) A representation $\pi_\rho$ in $\Pi_\sigma(G^0(F))$ is elliptic if and only if the character of $\rho$ does not vanish identically on $\tilde{R}^0_{\sigma, \text{reg}}$.

Next, we study the twisted case. Let $M^0 \in \mathcal{L}^0$ be a Levi subgroup of $G^0(F)$ and $\sigma$ be a representation in $\Pi_2(M^0(F))$. Assume that there exists an element $s \in W(a_{M^0}, a_{M^0})$ such that

$$\theta_s(\sigma) \cong \sigma.$$
We defined
\[ W_\sigma = \{ \omega \in W(a_{M^0}, a_{M^0}) : \theta \omega(\sigma) \cong \sigma \} \]
in §3.6, which also can be regarded as the torsor \( W_\sigma^{G^0} \cdot s \) of \( W_\sigma^{G^0} \) for some \( s \in W_\sigma \). Since \( \theta s \) can act on \( W_\sigma^{G^0} \) by conjugation, we define
\[ W_\sigma^+ = W_\sigma^{G^0} \rtimes \langle \theta s \rangle. \]

Given \( \sigma \) as above, we shall fix an element \( s \in W_\sigma \) satisfying \( \theta s(a^+_\sigma) = a^+_\sigma \) (note that we can always find such an \( s \) since we can replace \( s \) by \( sr \) for some suitable \( r \) in \( W_\sigma^{G^0} \)). Thus, for any \( r \in W_\sigma^{G^0}, \sigma \) belongs to \( \Pi_2(M^0(F))^{\theta sr} \). For convenience, we define
\[ \Pi_{2,\theta}(M^0(F)) = \{ \sigma \in \Pi_2(M^0(F)) : \theta r(\sigma) = \sigma \ for \ some \ r \in W(a_{M^0}, a_{M^0}) \}. \]

Given any \( \sigma \in \Pi_{2,\theta}(M^0(F)) \), there exists an element \( s \in W(a_{M^0}, a_{M^0}) \) such that \( \theta s(\sigma) = \sigma \) and \( \theta s(a^+_\sigma) = a^+_\sigma \). In §3.6, we defined the representation \( \mathcal{I}_{p_0}(\theta s, \sigma) \), which is a lifting of the induced representation \( \mathcal{I}_{p_0}(\sigma) \). Recall that \( R(\theta s, \sigma) \) is an intertwining operator between \( \mathcal{I}_{p_0}(\sigma) \) and \( \mathcal{I}_{p_0}(\sigma)_\theta \) with \( R(\theta s, \sigma)^n = 1 \). Then, \( \mathcal{I}_{p_0}(\theta s, \sigma) \) is given by
\[ \mathcal{I}_{p_0}(\theta s, \sigma, x) = \begin{cases} 
\mathcal{I}_{p_0}(\sigma, x) & \text{if } x \in G^0(F), \\
R(\theta s, \sigma) & \text{if } x = 1 \rtimes \theta,
\end{cases} \]
which can be extended to the group \( G^+(F) \). In general, the induced representation \( \mathcal{I}_{p_0}(\sigma_\lambda) \) of \( G^0(F) \) is not irreducible, so its lifting to \( G^+(F) \) is not unique, even up to a constant multiple. The lifting representation \( \mathcal{I}_{p_0}(\theta s, \sigma) \) is dependent on the element \( s \). Later, we will find a suitable lifting \( \mathcal{I}_{p_0}^+(\sigma) \) of \( \mathcal{I}_{p_0}(\sigma) \) that is independent of \( s \). We will then decompose \( \mathcal{I}_{p_0}^+(\sigma) \) into a direct sum of irreducible representations of \( G^+(F) \) and also determine which irreducible component of \( \mathcal{I}_{p_0}^+(\sigma) \) is in \( \Pi_{\text{temp}}(G(F)) \).
By the decomposition (4.1) of $R^0_\sigma$, $\mathcal{I}_{P^0}(\sigma)$ can be decomposed as

$$\mathcal{I}_{P^0}(\sigma) = \bigoplus_{\rho \in \Pi(R^0_\sigma, \chi^0_\sigma)} \dim(\rho)\pi_\rho.$$ 

Observe that $\theta$s can act on $R^0_\sigma$ by conjugation, so we can define a group

$$R^+_\sigma = R^0_\sigma \rtimes \langle \theta \rangle,$$

which is independent of the choice of $s$.

To study the twisted virtual character $\text{tr}(R(\theta t, \sigma)\mathcal{I}_{P^0}(\sigma, f))$, we need to associate an intertwining operator $R(q, \sigma)$ to any element $q$ in $R^+_\sigma$. If $q$ is in $R^0_\sigma$ or $q$ equals $\theta$, we have defined the operator $R(q, \sigma)$. For any $q = (r, (\theta)^k)$ in $R^+_\sigma$, we define

$$R((r, (\theta)^k), \sigma) = R(r, \sigma)R(\theta, \sigma)^k.$$ 

We must deal with the possibility that the map

$$q \rightarrow R(q, \sigma), \quad q \in R^+_\sigma$$

is not a homomorphism. In general, we have

$$R(r_1r_2, \sigma) = \eta_\sigma(r_1, r_2)R(r_1, \sigma)R(r_2, \sigma), \quad r_1, r_2 \in R^0_\sigma, \quad (4.4)$$

where $\eta_\sigma$ is a 2-cocycle for $R^0_\sigma$ with values in $\mathbb{C}^*$. Furthermore,

$$R(\theta, \sigma)^{-1}R(r, \sigma)R(\theta, \sigma) = \chi_0(r)R((\theta)^{-1}(r), \sigma) \quad r \in R^0_\sigma, \quad (4.5)$$

where $\chi_0$ is a character of $R^0_\sigma$, and can be non-trivial.\footnote{The author thanks Paul Mezo for pointing out the possibility that $\chi_0$ can be non-trivial.} The relation (4.5) is equivalent
to Lemma 5 (iv) of [19] if \( F \) is Archimedean; if \( F \) is \( p \)-adic, it is possible to show the existence of \( \chi_0 \) via a discussion similar to that in the proof of Lemma 5 (iv) of [19]. Note that \( \chi_0 \) can be defined by minimal \( K \)-types in [19]; for the \( p \)-adic case, it is sufficient to know the existence of \( \chi_0 \). The relations (4.4) and (4.5) can be combined as

\[
R(q_1 q_2, \sigma) = \eta_\sigma^+(q_1, q_2) R(q_1, \sigma) R(q_2, \sigma), \quad q_1, q_2 \in R_\sigma^+,
\]

where \( \eta_\sigma^+ \) is a 2-cocycle for \( R_\sigma^+ \) with values in \( \mathbb{C}^* \) and can be regarded as an element of \( H^2(R_\sigma^+, \mathbb{C}^*) \). To deal with the possibility that the map \( q \to R(q, \sigma) \) is not a homomorphism, we need the following Lemma.

**Lemma 4.1.3**: There is a finite central extension of \( R_\sigma^0 \)

\[
1 \to Z_\sigma \to \tilde{R}_\sigma^0 \to R_\sigma^0 \to 1
\]

satisfying

(i) \( \langle \theta_s \rangle \) acts on \( \tilde{R}_\sigma^0 \). Set \( \tilde{R}_\sigma^+ = \tilde{R}_\sigma^0 \rtimes \langle \theta_s \rangle \);

(ii) \( \eta_\sigma^+ \) splits over \( \tilde{R}_\sigma^+ \).

**Proof.** We will prove this Lemma using the method from §53 of [18].

We define \( \tilde{R}_\sigma^+ \) directly. Let \( Z_\sigma = H^2(R_\sigma^+, \mathbb{C}^*) \) (note that the classical language for the 2nd cohomology in [18] is Schur’s multiplier). Define

\[
\tilde{R}_\sigma^+ = \{(r, z, (\theta s)^k) \mid r \in R_\sigma^0, \ z \in Z_\sigma, \ (\theta s)^k \in \langle \theta s \rangle \}
\]

and the multiplication on \( \tilde{R}_\sigma^+ \) by

\[
(r, z, (\theta s)^{k_1})(r', z', (\theta s)^{k_2}) = (r(\theta s)^{k_1}(r'), \eta_\sigma^+(r \rtimes (\theta s)^{k_1}, r' \rtimes (\theta s)^{k_2})zz', (\theta s)^{k_1+k_2}).
\]

Using (53.11) in [18], we know that

\[
\eta_\sigma^+(q_1, q_2 q_3) \eta_\sigma^+(q_2, q_3) = \eta_\sigma^+(q_1, q_2) \eta_\sigma^+(q_1 q_2, q_3), \quad q_i \in R_\sigma^+.
\]
Then, $\tilde{R}_\sigma^+$ is a group under this multiplication. In addition, we can set

$$\tilde{R}_\sigma^0 = \{(r, z) \mid r \in R_\sigma^0, \, z \in Z_\sigma\},$$

which can be regarded as a subgroup of $\tilde{R}_\sigma^+$. Also, $\tilde{R}_\sigma^0$ is given by the central extension

$$1 \rightarrow Z_\sigma \rightarrow \tilde{R}_\sigma^0 \rightarrow R_\sigma^0 \rightarrow 1.$$

The action of $\theta s$ on $\tilde{R}_\sigma^0$ is given by

$$\theta s(r, z) = (\theta s(r), \eta_\sigma^+(1 \rtimes \theta s, r)\eta_\sigma^+(\theta s(r) \rtimes \theta s, 1 \rtimes (\theta s)^{-1})z).$$

Consequently,

$$\tilde{R}_\sigma^+ = \tilde{R}_\sigma^0 \rtimes \langle \theta s \rangle.$$

Hence, we have proved (i).

Then, (ii) follows from (53.7) in [18].

We can choose a function $\xi_\sigma : \tilde{R}_\sigma^+ \rightarrow \mathbb{C}^*$ that splits $\eta_\sigma^+$. This means that the function $\xi_\sigma$ satisfies

$$\eta_\sigma(q_1, q_2) = \xi_\sigma(q_1 q_2)\xi_\sigma(q_1)^{-1}\xi_\sigma(q_2)^{-1}, \quad q_1, q_2 \in \tilde{R}_\sigma^+.$$

We see that

$$\xi_\sigma(zr) = \chi_\sigma(z)\xi_\sigma(r), \quad z \in Z_\sigma, \, r \in \tilde{R}_\sigma^0,$$

where $\chi_\sigma$ is a linear character on the central subgroup $Z_\sigma$ of $\tilde{R}_\sigma^0$.

Use $\xi_\sigma$ to twist the intertwining operators as follows:

$$\tilde{R}(q, \sigma) = \xi_\sigma(q)^{-1} R(q, \sigma), \quad q \in \tilde{R}_\sigma^+.$$
Then, \( q \rightarrow \tilde{R}(q, \sigma) \) is a homomorphism of \( \tilde{R}_\sigma^+ \) into the group of unitary operators on the space \( \mathcal{H}_{p0}(\sigma) \). It can easily be seen that

\[
\tilde{R}(zr, \sigma) = \chi_\sigma(z)^{-1}\tilde{R}(r, \sigma), \quad z \in Z_\sigma, \ r \in \tilde{R}_\sigma^0.
\]

As \( G(F) = G^0(F) \times \theta \) is one component of the non-connected group \( G^+ \), for the groups \( R_\sigma^+ \) and \( \tilde{R}_\sigma^+ \), we write

\[
R_\sigma = R_\sigma^G = R_\sigma^0 \times \theta s \quad \text{and} \quad \tilde{R}_\sigma = \tilde{R}_\sigma^G = \tilde{R}_\sigma^0 \times \theta s,
\]

which are both independent of the choice of \( s \). We can also write \( \tilde{R}_\sigma \) as \( \theta s \cdot \tilde{R}_\sigma^0 \). For any \( t \) in \( \tilde{R}_\sigma \), \( t \) equals \( \theta sr \) for some \( r \) in \( \tilde{R}_\sigma^0 \).

**Remark.** There are some differences between the notation in this thesis and Arthur’s in [11]. For example, we use \( R_\sigma^0 \) to denote the \( R \)-group of the representation \( \sigma \), for which Arthur used \( R_\sigma \). In this thesis, \( R_\sigma \) means the torsor \( R^0 \times \theta s \) of the \( R \)-group of \( \sigma \).

We know that

\[
\mathcal{I}_{p0}(\sigma) \cong \mathcal{I}_{p0}(\sigma)_\theta
\]

by Langlands’ classification of tempered representations ([11], Proposition 1.1). Consider the representation

\[
\mathcal{R}_\theta^0(r, x) = \tilde{R}(r, \sigma)\mathcal{I}_{p0}(\sigma, \theta(x)), \quad r \in \tilde{R}_\sigma^0, \ x \in G^0(F),
\]

of \( \tilde{R}_\sigma^0 \times G^0(F) \) on \( \mathcal{H}_{p0}(\sigma) \). By (4.1), we have the decomposition

\[
\mathcal{R}_\theta^0 = \bigoplus_{\rho \in \Pi(\tilde{R}_\sigma^0, \chi_\sigma)} (\rho^\vee \otimes \theta(\pi_\rho)). \quad (4.7)
\]

\( \mathcal{R}_\theta^0(r, x) \) also equals

\[
\tilde{R}(r, \sigma)\tilde{R}(\theta s, \sigma)\mathcal{I}_{p0}(\sigma, x)\tilde{R}(\theta s, \sigma)^{-1},
\]
which is equivalent to
\[ \tilde{R}(\theta s, \sigma)^{-1} \tilde{R}(r, \sigma) \tilde{R}(\theta s, \sigma) \mathcal{I}_{po}(\sigma, x). \]

We find that \( \tilde{R}(\theta s, \sigma)^{-1} \tilde{R}(r, \sigma) \tilde{R}(\theta s, \sigma) = \tilde{R}((\theta s)^{-1}(r), \sigma) \). Consequently, we obtain that \( \mathcal{R}^0_\theta(r, x) \) equals
\[ \mathcal{R}^0((\theta s)^{-1}(r), x). \]
Also by (4.1), we see that
\[ \mathcal{R}^0_\theta = \bigoplus_{\rho \in \Pi(\tilde{R}^0_\sigma, \chi_\sigma)} (\theta s(\rho^\vee) \otimes \pi_\rho). \] (4.8)

Combining (4.7), (4.8), and Arthur’s correspondence (Theorem 4.1.2 (a)), we have more explicit relations for the twisted case: If \( \rho \leftrightarrow \pi_\rho \) holds in Arthur’s correspondence, so does
\[ \theta s(\rho) \leftrightarrow \theta(\pi_\rho). \]

Note that \( \theta s(\rho) \) is independent of the choice of \( s \). Then,
\[ \pi_\rho \cong \theta(\pi_\rho) \iff \rho \cong \theta s(\rho). \] (4.9)

Set
\[ \Pi_\sigma(G^0(F))^\theta = \{ \pi \in \Pi_\sigma(G^0(F)) : \pi \cong \theta(\pi) \} \]
and
\[ \Pi(\tilde{R}^0_\sigma, \chi_\sigma)^\theta = \{ \rho \in \Pi(\tilde{R}^0_\sigma, \chi_\sigma) : \rho \cong \theta s(\rho) \}, \]
which is independent of the choice of \( s \). From (4.9), we see that there is a bijection from \( \Pi(\tilde{R}^0_\sigma, \chi_\sigma)^\theta \) onto \( \Pi_\sigma(G^0(F))^\theta \), which is just the restriction of Arthur’s correspondence to \( \Pi(\tilde{R}^0_\sigma, \chi_\sigma)^\theta \).

Motivated by the twisted weighted character (3.76), which occurs in the local twisted
trace formula, we need to study

\[ tr(R(\theta t, \sigma)I_{P^0}(\sigma, f)). \]  \hfill (4.10)

Define a group

\[ (\widetilde{R}_\sigma G(F))^+ = \langle \widetilde{R}_\sigma^0 \times G^0(F) \rangle \rtimes (\theta s \times \theta), \]

which is generated by \( \widetilde{R}_\sigma \times G(F) \). The component \( \widetilde{R}_\sigma^0 \times G^0(F) \) of \( (\widetilde{R}_\sigma G(F))^+ \) contains the identity. We can lift the representation \( \mathcal{R}^0 \) of \( \widetilde{R}_\sigma^0 \times G^0(F) \) to a representation \( \mathcal{R}^+ \) of \( (\widetilde{R}_\sigma G(F))^+ \). In fact, we calculate

\[
\begin{align*}
\widetilde{R}(\theta s, \sigma)\mathcal{R}^0(r, x)\widetilde{R}(\theta s, \sigma)^{-1} \\
= \widetilde{R}(\theta s, \sigma)\widetilde{R}(r, \sigma)I_{P^0}(\sigma, x)\widetilde{R}(\theta s, \sigma)^{-1} \\
= (\widetilde{R}(\theta s, \sigma)\widetilde{R}(r, \sigma)\widetilde{R}(\theta s, \sigma)^{-1})(\widetilde{R}(\theta, s, \sigma)I_{P^0}(\sigma, x)\widetilde{R}(\theta s, \sigma)^{-1}) \\
= \widetilde{R}(\theta s(r), \sigma)I_{P^0}(\sigma, \theta(x)) \\
= \mathcal{R}^0(\theta s(r), \theta(x)) \\
= (\theta s \times \theta)\mathcal{R}^0(r, x)).
\end{align*}
\]

That is, \( \mathcal{R}^0 \) is equivalent to \( (\theta s \times \theta)\mathcal{R}^0 \). Obviously, \( \widetilde{R}(\theta s, \sigma)^n = 1 \), hence we can lift \( \mathcal{R}^0 \) to be a representation \( \mathcal{R}^+ \) of \( (\widetilde{R}_\sigma G(F))^+ \) by Theorem 2.4.1. Specifically, we define

\[
\mathcal{R}^+(y) = \begin{cases} 
\mathcal{R}^0(r, x) & \text{if } y = (r, x) \in \widetilde{R}_\sigma^0 \times G^0(F), \\
\widetilde{R}(\theta s, \sigma) & \text{if } x = (\theta s, \theta) \in \widetilde{R}_\sigma \times G(F).
\end{cases}
\]

This can be extended to the whole group \( (\widetilde{R}_\sigma G(F))^+ \). Let \( \Pi(\widetilde{R}_\sigma^+, \chi_\sigma)_{\Xi_{\widetilde{R}_\sigma}} \) be the set of orbits of \( \Pi(\widetilde{R}_\sigma^+, \chi_\sigma) \) under the action \( \Xi_{\widetilde{R}_\sigma} \), where \( \Xi_{\widetilde{R}_\sigma} \) is defined in §2.4. Then, there is a 1-1 correspondence

\[ \rho \rightarrow \rho^+ \]  \hfill (4.11)
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between the set $\Pi(\tilde{R}_\sigma^0, \chi_\sigma)/<\theta s>$ and $\Pi(\tilde{R}_\sigma^+, \chi_\sigma)\Xi_{\tilde{R}_\sigma^+}$ such that $\rho^+|_{\tilde{R}_\sigma^+} = \bigoplus_i (\theta s)^i(\rho)$. Combining (4.11), (4.1), and Theorem 2.4.2, we see that $\mathcal{R}^+$ has a decomposition:

$$\mathcal{R}^+ = \bigoplus_{\rho^+ \in \Pi(\tilde{R}_\sigma^+, \chi_\sigma)\Xi_{\tilde{R}_\sigma^+}} (\rho^+)^\vee \otimes \pi^+_{\rho}.$$  

(4.12)

As we claimed before, we can now define a suitable lifting $\mathcal{I}^+_p(\sigma)$ of $\mathcal{I}_p(\sigma)$ that is independent of $s$. Let $\mathcal{I}^+_p(\sigma)$ be the restriction of $\mathcal{R}^+$ to the group $G^+(F)$.

Write

$$\Pi_\sigma(G^+(F)) = \{\pi^+_{\rho} : \rho^+ \in \Pi(\tilde{R}_\sigma^+, \chi_\sigma)\Xi_{\tilde{R}_\sigma^+}\},$$

where $\pi^+_{\rho}$ is determined by $\rho^+$ in the decomposition (4.12). Then, there is a bijection

$$\rho^+ \rightarrow \pi^+_{\rho}$$

(4.13)

between the sets $\Pi(\tilde{R}_\sigma^+, \chi_\sigma)\Xi_{\tilde{R}_\sigma^+}$ and $\Pi_\sigma(G^+(F))$. The set $\Pi_\sigma(G^+(F))$ consists of all irreducible constituents of $\mathcal{I}^+_p(\sigma)$. Write

$$\Pi_\sigma(G(F)) = \Pi_\sigma(G^+(F)) \cap \Pi_{temp}(G(F)).$$

That is, if $\pi^+ \in \Pi_\sigma(G(F))$, then $\pi^+$ is in $\Pi_\sigma(G^+(F))$, and its restriction to $G^0(F)$ is also irreducible. Let $\Pi_\sigma(G^0(F))/<\theta>$ be the set of $<\theta>$-orbits in $\Pi_\sigma(G^0(F))$. Then, there is a 1-1 correspondence

$$\pi_{\rho} \longrightarrow \pi^+_{\rho}$$

between the sets $\Pi_\sigma(G^0(F))/<\theta>$ and $\Pi_\sigma(G^+(F))$ such that $\pi^+_{\rho}|_{G^0(F)} = \bigoplus_i \theta^i(\pi_{\rho})$. In particular, there is a 1-1 correspondence

$$\pi_{\rho} \longrightarrow \pi^+_{\rho}.$$
between the sets \( \Pi_\sigma(G^0(F))^\theta \) and \( \Pi_\sigma(G(F)) \). In other words,

\[
\Pi_\sigma(G(F)) = \{ \pi^+ : \pi_\rho \in \Pi_\sigma(G^0(F))^\theta \}.
\]

For any \( t \) in \( \widetilde{R}_\sigma \), we define \( I^+_P_0(t, \sigma, x) \) by

\[
I^+_P_0(t, \sigma, x) = \begin{cases} 
I^+_P_0(\sigma, x) & \text{if } x \in G^0(F), \\
\widetilde{R}(t, \sigma) & \text{if } x = 1 \rtimes \theta,
\end{cases}
\]

which can be extended to \( G^+(F) \).

Observe that

\[
\mathcal{R}^+(\theta s, x \rtimes \theta) = \mathcal{I}^+_P_0(\theta s, \sigma, x \rtimes \theta) = \mathcal{I}^+_P_0(\sigma, x)\widetilde{R}(\theta s, \sigma), \quad x \in G^0(F);
\]

hence,

\[
tr(\widetilde{R}(\theta s, \sigma)I^+_P_0(\sigma, f)) = tr(I^+_P_0(\sigma, f)) = tr(\mathcal{R}^+(\theta s, f)).
\]

Consequently, the bijection (4.13) expressed in terms of characters is

\[
tr(I^+_P_0(\sigma, f)) = tr(\mathcal{R}^+(\theta s, f)) = \sum tr((\rho^+)^\vee(\theta s))tr(\pi^+_\rho(f)), \quad (4.14)
\]

where the sum is over \( \rho^+ \) in \( \Pi(\widetilde{R}^+_\sigma, \chi_\sigma)_{\widetilde{R}_0} \). We know that if \( \pi^+_\rho \) is not in \( \Pi_\sigma(G(F)) \), then \( tr(\pi^+_\rho(f)) = 0 \). Hence, we can make (4.14) into a simpler form. First, write

\[
\Pi(\widetilde{R}_\sigma, \chi_\sigma) = \{ \rho^+ \in \Pi(\widetilde{R}^+_\sigma, \chi_\sigma)_{\widetilde{R}_0} : \rho^+|_{\widetilde{R}_0} \text{ irreducible} \}.
\]

We can then define a sub-representation of \( \mathcal{R}^+ \) by

\[
\mathcal{R}^+_0 = \bigoplus_{\rho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)} (\rho^+)^\vee \otimes \pi^+_\rho.
\]
By (4.9) and (4.13), we get a 1-1 correspondence
\[ \rho^+ \rightarrow \pi^+_\rho \] (4.15)
between the sets \( \Pi(\tilde{R}_\sigma, \chi_\sigma) \) and \( \Pi_\sigma(G(F)) \). We can now write the relation (4.14) as a simpler expression:
\[ \text{tr}(\tilde{R}(\theta s, \sigma)I_{P^0}(\sigma, f)) = \text{tr}(I_{P^0}(\theta s, \sigma, f)) = \sum_{\rho^+ \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \text{tr}((\rho^+)\vee(\theta s))\text{tr}(\pi^+_{\rho}(f)), \] (4.16)
where \( \pi^+_{\rho} \) runs through the set \( \Pi_\sigma(G(F)) \).

As in the ordinary case, there is an isomorphism
\[ \vartheta \rightarrow \Theta \]
from \( C(\Pi(\tilde{R}_\sigma, \chi_\sigma)) \) to \( C(\Pi_\sigma(G(F))) \). The inverse of the formula (4.16) is
\[ \Theta(f) = \frac{1}{|\tilde{R}_\sigma|} \sum_{t \in \tilde{R}_\sigma} \vartheta(t)\text{tr}(\tilde{R}(t, \sigma)I_{P^0}(\sigma, f)), \] (4.17)
which can be deduced by applying the twisted orthogonality relation for the finite group \( \tilde{R}_\sigma^+ \). The formula (4.17) can also be written as
\[ \Theta(f) = \frac{1}{|\tilde{R}_\sigma|} \sum_{t \in \tilde{R}_\sigma} \vartheta(t)\text{tr}(I_{P^0}^+(t, \sigma, f)). \] (4.18)

The correspondence \( \vartheta \rightarrow \Theta \) also behaves well under induction in the twisted case (Theorem 4.1.4 (b)), which will be explained below.

Assume that \( L^+ \) is a Levi subgroup of \( G^+(F) \) such that \( L^0 = L^+ \cap G^0(F) \) is in \( \mathcal{L}(M^0) \), and \( L = L^+ \cap G(F) \) is a non-empty Levi subset. By Lemma 3.6.2 and the explanation after the lemma, we can assume that \( M^0 \in \mathcal{L}^{L^0} \). Let \( W^{L^0}(a_{M^0}) \) denote the
the Weyl group of $a_{M^0}$ relative to $L^0$ instead of $G^0$. We also assume that the closure $\overline{a_\sigma^+}$ of the chamber $a_\sigma^+$ contains an open subset of $a_{L^0}$ and that there exists an element $s_L \in W^{L^0}(a_{M^0}, a_{M^0})$ such that $\theta s_L(\sigma) \cong \sigma$. As before, we write $R^L_\sigma$ for the $R$-group of $\sigma$ relative to $L^0$ and make the central extension of $R^L_\sigma$

$$1 \to Z_\sigma \to \tilde{R}^L_\sigma \to R^L_\sigma \to 1$$

such that $\tilde{R}^L_\sigma$ is the inverse image of $R^L_\sigma$ in $\tilde{R}_\sigma$.

Given $L^+$ above, let $\rho^+_L$ be a representation in $\Pi(\tilde{R}^L_\sigma, \chi_\sigma)$. We can induce this representation from $\tilde{R}^L_\sigma$ to $\tilde{R}^+_\sigma$, thereby obtaining a character $\tilde{\vartheta}$ in $C(\Pi(\tilde{R}^+_\sigma, \chi_\sigma))$. Let $\vartheta$ be the projection of $\tilde{\vartheta}$ in $C(\Pi(\tilde{R}_\sigma, \chi_\sigma))$. We still call $\vartheta$ the character induced from the representation $\rho^+_L$. On the other hand $\rho^+_L$ determines a representation $\pi^+_L = \pi^+_{\rho_L}$ in $\Pi(\rho_L(L(F)))$.

For any $Q \in \mathcal{P}(L)$, we can obtain an induced representation $I_Q(\pi^+_L)$ of $G^+(F)$ ([7], §1). This induced representation gives a character $\tilde{\Theta}$ in $C(\Pi(\tilde{R}_\sigma(G(F)))$. Also, let $\Theta$ be the projection of $\tilde{\Theta}$ in $C(\Pi(\rho_{\rho_L}(G(F))))$. We claim that $\vartheta$ and $\Theta$ correspond to the bijection described above and, in particular, are related by (4.17). To see this, we first apply (4.17) to the characters of $\pi_L$ and $\rho_L$. Take $P \in \mathcal{P}(M^0)$ to be any group contained in $Q^0$, where $Q^0$ in $\mathcal{P}(L^0)$ corresponds to $Q$. Since $tr(I_Q(\pi^+_L(f))) = tr(\pi^+_L(f_Q))$, we can obtain

$$\Theta(f) = tr(I_Q(\pi^+_L, f)) = \frac{1}{|\tilde{R}^L_\sigma|} \sum_{t \in \tilde{R}^L_\sigma} tr(\rho^+_L(t)) tr(I^+_Q(t, \sigma, f)), \quad f \in \mathcal{H}(G(F)),$$

from the transitivity properties of induction. Since

$$t \to tr(I^+_Q(t, \sigma, f)), \quad t \in \tilde{R}_\sigma,$$

is a (twisted) class function on $\tilde{R}_\sigma$, the last expression becomes the right-hand side of (4.17) when we apply the standard formula for the induced character $\vartheta$. 
Let us write $C_{\text{ind}}(\Pi(\tilde{R}_{\sigma}, \chi_{\sigma}))$ for the submodule of $C(\Pi(\tilde{R}_{\sigma}, \chi_{\sigma}))$ generated by all characters $\vartheta$ of $\tilde{R}^{+}_{\sigma}$ induced from representations $\rho_{L}^{+} \in \Pi(\tilde{R}_{\sigma}, \chi_{\sigma})$, where $L^{+} = L^{0} \rtimes (\theta)$ ranges over proper Levi subgroups of $G^{+}(F)$ satisfying the condition that the closure $\overline{a^{+}_{\sigma}}$ of the chamber $a^{+}_{\sigma}$ contains an open subset of $a_{L^{0}}$ and there exists an element $s_{L} \in W^{L^{0}}(a_{M^{0}}, a_{M^{0}})$ such that $\theta s_{L}(\sigma) \cong \sigma$. Let us also write $C_{\text{ind}}(\Pi_{\sigma}(G(F)))$ for the submodule of $C(\Pi_{\sigma}(G(F)))$ generated by all characters

$$\Theta(f) = \text{tr}(I_{Q}(\pi^{+}_{L}, f)), \quad Q \in \mathcal{P}(L), \pi^{+}_{L} \in \Pi_{\sigma}(L(F)),$$

where $L$ ranges over all proper Levi subsets of $G(F)$ with $L^{0} \in \mathcal{L}(M^{0})$. For any such non-zero $\Theta$, we can replace $\pi^{+}_{L}$ by a representation $\omega \pi^{+}_{L}$ in $\Pi_{\sigma}((\omega L)(F))$ for an element $w \in W^{0}$. This means that we can replace the space $a_{L^{0}}$ by $\omega a_{L^{0}}$. We can therefore assume that $L^{+}$ satisfies the condition that the closure $\overline{a^{+}_{\sigma}}$ of the chamber $a^{+}_{\sigma}$ contains an open subset of $a_{L^{0}}$, and there exists an element $s_{L} \in W^{L^{0}}(a_{M^{0}}, a_{M^{0}})$ such that $\theta s_{L}(\sigma) \cong \sigma$. It follows that the bijection $\vartheta \rightarrow \Theta$ maps a set of generators of $C(\Pi(\tilde{R}_{\sigma}, \chi_{\sigma}))$ to a set of generators of $C(\Pi_{\sigma}(G(F)))$. Consequently, the image of $C(\Pi(\tilde{R}_{\sigma}, \chi_{\sigma}))$ is $C(\Pi_{\sigma}(G(F)))$.

As in the ordinary case, an element in $C(\Pi_{\sigma}(G(F)))$, regarded as a locally integrable class function on $G_{\text{reg}}(F)$, vanishes on the elliptic set $\Gamma_{\text{ell}}(G(F))$ when the usual formula of an induced character is applied. Therefore, there is a map $p$ from

$$C(\Pi_{\sigma}(G(F)))/C_{\text{ind}}(\Pi_{\sigma}(G(F)))$$

into a space of functions on $\Gamma_{\text{ell}}(G(F)) \cap G_{\text{reg}}(F)$. The map $p$ is injective, as can be seen from the twisted orthogonality relations (see the explanation at the end of §4.5). It follows that the elliptic representations $\Pi_{\sigma,\text{ell}}(G(F))$ are precisely the representations in $\Pi_{\sigma}(G(F))$ whose characters do not lie in $C_{\text{ind}}(\Pi_{\sigma}(G(F)))$.

To classify the $G(F)$-elliptic representations of $G^{+}(F)$, we define the twisted regular
subset of $R_\sigma$ to be
\[ R_{\sigma, \text{reg}} = \{ \theta t \in R_\sigma : a_{M^0}^{\theta t} = a_G \} , \]
where
\[ a_{M^0}^{\theta \omega} = \{ H \in a_{M^0} : \theta \omega(H) = H \} \]
denotes the space of fixed vectors of an element $\theta \omega$. Let $\tilde{R}_{\sigma, \text{reg}}$ be the inverse image in $\tilde{R}_\sigma$ of the set $R_{\sigma, \text{reg}}$. If $\theta t$ is an arbitrary element in $R_\sigma$, the space $a_{M^0}^{\theta t}$ is of the form $a_L$ (under conjugation) for some Levi subgroup $L^+$ whose identity component $L^0$ is in $\mathcal{L}(M^0)$. By Lemma 3.6.2, we know that $L^+ = L^0 \times \langle \theta \rangle$. Observe that $\tilde{R}_\sigma$ is the disjoint union of $\tilde{R}_{\sigma, \text{reg}}$ with the set
\[ \tilde{R}_\sigma^e = \bigcup_{L^+ \neq G^+} \tilde{R}_\sigma^L. \]
Now, $C_{\text{ind}}(\Pi(\tilde{R}_\sigma, \chi_\sigma))$ is the space of $\chi_\sigma$-equivariant class functions on $\tilde{R}_\sigma$ which are supported on $\tilde{R}_{\sigma, \text{reg}}$. This follows from the usual formula for an induced character. There is consequently an isomorphism from the quotient
\[ C(\Pi(\tilde{R}_\sigma, \chi_\sigma))/C_{\text{ind}}(\Pi(\tilde{R}_\sigma, \chi_\sigma)) \]
on to the space of $\chi_\sigma$-equivariant class functions on $\tilde{R}_{\sigma, \text{reg}}$. Since $C_{\text{ind}}(\Pi(\tilde{R}_\sigma, \chi_\sigma))$ corresponds to $C_{\text{ind}}(\Pi_\sigma(G(F)))$, we conclude that the elliptic representations in $\Pi_\sigma(G(F))$ are given by the irreducible characters in $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ that do not vanish identically on $\tilde{R}_{\sigma, \text{reg}}$.

Summarizing the discussion above, we can now state the theorem about the classification of the representations in $\Pi_\sigma(G(F))$.

**Theorem 4.1.4**:

(a) There is a unique bijection $\rho^+ \rightarrow \pi^+_\rho$ from the set $\Pi(\tilde{R}_{\sigma, \text{reg}}^+, \chi_\sigma)_{z_{\tilde{R}_\sigma}}$ onto $\Pi_\sigma(G^+(F))$ which satisfies the character identity [4.14]. Also, we have a unique bijection $\rho^+ \rightarrow \pi^+_\rho$ from $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ onto $\Pi_\sigma(G(F))$ that satisfies the simpler character identity [4.16].

(b) A sum of characters in $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ is induced from a proper subgroup $\tilde{R}_\sigma^{L_0^+}$ of $\tilde{R}_\sigma^+$ if
and only if the corresponding sum of characters in $\Pi_{\sigma}(G(F))$ is induced from a parabolic subgroup with Levi component $L(F)$.

(c) A representation $\pi^+_{\rho}$ in $\Pi_{\sigma}(G(F))$ is $G(F)$-elliptic if and only if the character of $\rho^+$ does not vanish identically on $\tilde{R}_{\sigma,\text{reg}}$.

For future reference, we state Langlands’ Classification of tempered irreducible representations $\Pi_{\text{temp}}(G(F))$, which is a subset of $\Pi_{\text{temp}}^+(G(F))$. Combining Proposition 1.1 in [11], some facts in [26], Lemma 2 in [19], and Theorem 4.1.4, we obtain Langlands’ Classification in the twisted case.

**Proposition 4.1.5**:  
(a) As $M^0$ and $\sigma$ range over $L^0$ and $\Pi_{2,\theta}(M^0(F))$, the set $\Pi_{\sigma}(G(F))$ exhausts $\Pi_{\text{temp}}(G(F))_{\Xi_G}$.

(b) Let $(M^0, \sigma)$ and $(M^0', \sigma')$ be any two pairs in (a). If $(M^0, \sigma')$ equals $(\omega M^0, \omega \sigma)$ for an element $\omega \in W_{G^0}$, then $\Pi_{\sigma}(G(F)) = \Pi_{\sigma'}(G(F))$; conversely, if the sets $\Pi_{\sigma}(G(F))$ and $\Pi_{\sigma'}(G(F))$ have a representation in common, then there exists an element $\omega \in W_{G^0}$ such that $(M^0, \sigma') = (\omega M^0, \omega \sigma)$.

### 4.2 The Distribution $I_{\text{disc}}$

In §3.6, we found that the local twisted trace formula is an expansion of a certain distribution $I_{\text{disc}}(f', f)$ on $\mathcal{H}(G(F)) \times \mathcal{H}(G(F))$ in terms of weighted orbital integrals and weighted characters. To express the local twisted trace formula in an explicit form, we need to give a simple description of $I_{\text{disc}}(f', f)$.

Many objects attached to representations $\sigma$ in $\Pi_{2,\theta}(M^0(F))$ in §4.1 are not uniquely determined. To make further objects completely independent of the choices of certain objects (p. 92, [11]) related to $\sigma$ in $\Pi_{2,\theta}(M^0(F))$, from now on, we assume that any obvious compatibility condition stated in §3 of [11] holds. For example, we will want a symmetry condition with respect to the action of $W_G^0$. We require that conjugation of $R_{\sigma}^+$ by an element $\omega$ in $W_G^0$ extends to an isomorphism $r \rightarrow \omega r$ form $\tilde{R}_{\sigma}^+$ onto $\tilde{R}_{\omega \sigma}^+$. Of course,
the action of $\omega$ on the torsor $\tilde{R}_\sigma$ is by twisted conjugation. We also take $\tilde{R}^*_\sigma = \tilde{R}^+_\sigma$ and $\chi_{\sigma^\vee} = \chi_{\sigma}^{-1}$. For the contragredient, the ordinary correspondence \((4.2)\) becomes

$$\operatorname{tr}(\tilde{R}(r, \sigma^\vee)\mathcal{P}_0(\sigma^\vee, h)) = \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_{\sigma})} \operatorname{tr}(\rho(r))\operatorname{tr}(\pi_{\rho}(h)),$$

and the twisted correspondence \((4.16)\) becomes

$$\operatorname{tr}(\tilde{R}(\theta s, \sigma^\vee)\mathcal{P}_0(\sigma, f)) = \sum_{\rho^+ \in \Pi(\tilde{R}_\sigma, \chi_{\sigma})} \operatorname{tr}((\rho^+)(\theta s))\operatorname{tr}((\pi_{\rho}^+)^\vee(f)).$$

In what follows, we shall generally not make a notational distinction between objects defined on $R^+_\sigma$ and the corresponding $Z_\sigma$-invariant objects on $\tilde{R}^+_\sigma$. For example, $X \to r X$ could stand for the action of $\tilde{R}^+_\sigma$ on $a_{M^0}$ with isotropy subgroup $Z_\sigma$, as well as for the underlying $R^+_\sigma$-action from which it is obtained.

To formulate and prove the twisted analogues of the results in §3 of [11], for any $\sigma$ in $\Pi_{2,\theta}(M^0(F))$, we consider triplets

\[ \tau = (M^0, \sigma, t), \quad M^0 \in \mathcal{L}^0, \quad \sigma \in \Pi_{2,\theta}(M^0(F)), \quad t \in \tilde{R}_\sigma. \]

Motivated by (3.76) in $I_{\text{disc}}(f)$, for any such $\tau$, we define a distribution

$$\Theta(\tau, f) = \operatorname{tr}((\mathcal{I}_{p_0}(t, \sigma, f)), \quad f \in \mathcal{H}(G(F)), \quad (4.20)$$

which also equals $\operatorname{tr}(\tilde{R}(t, \sigma)\mathcal{I}_{p_0}(\sigma, f))$ by the definition of $\mathcal{I}_{p_0}^+(t, \sigma)$. This is the virtual character whose decomposition is given by \((4.16)\). Set $Z_\tau = Z_\sigma$ and $\chi_\tau = \chi_\sigma$. If $z$ belongs to $Z_\tau$, we obtain another triplet

\[ z\tau = (M^0, \sigma, zt) \]
which satisfies
\[ \Theta(z\tau, f) = \chi_{\tau}(z)^{-1}\Theta(\tau, f). \]

We can define an action of $W_0^G$ on the set of all triplets below. The action is given by
\[ \tau \rightarrow \omega\tau = (\omega M^0, \omega\sigma, \omega(t)), \quad \omega \in W_0^G, \]
and has the property that
\[ \Theta(\omega\tau, f) = \Theta(\tau, f). \]

Note that under this action, the outer automorphism $\theta$ may become $\omega\theta$. These two conditions force some of the distributions $\Theta(\tau)$ to vanish. We shall say that $\tau = (M^0, \sigma, t)$ is essential if the subgroup of elements in $Z_{\sigma}$ that stabilize the $\tilde{R}_{\sigma}$-conjugacy class of $t$ is contained in the kernel of $\chi_{\sigma}$. The inessential distributions $\Theta(\tau)$ are then zero and can be discarded. We shall write $\tilde{T}(G)$ for the remaining set of essential triplets, and we define $T(G)$ to be the set of $W_0^G$-orbits in $\tilde{T}(G)$. Our basic objects are then the distributions
\[ \{\Theta(\tau) : \tau \in T(G)\}. \]

Regarded modulo the equivalence relation defined by the action of the group $Z_{\tau}$, these distributions form a basis of the vector space of all tempered characters. This follows from Proposition 4.1.5 and the formula (4.16) for the irreducible representations in $\Pi_{\sigma}(G(F))$.

We are especially concerned about the subset
\[ T_{\text{ell}}(G) = \{\tau = (M^0, \sigma, t) \in T(G) : t \in \tilde{R}_{\sigma,\text{reg}}\} \]

of orbits in $T(G)$ which are elliptic.

As in the ordinary case ([Π], p. 93), $\tilde{T}(G)$ has a natural structure of an analytic manifold. More precisely, if $\tau = (M^0, \sigma, t)$ is an element in $\tilde{T}(G)$, the isotropy subspace
\( a^{\varpi}_{M^0} \) of \( a_{M^0} \) equals \( a_L \) for some Levi subgroup \( L^+ \) whose identity component \( L^0 \) includes \( M^0 \). There is a locally free action

\[
\tau \to \tau_\lambda = (M^0, \sigma_\lambda, t), \quad \lambda \in i\mathfrak{a}^*_L,
\]

of \( i\mathfrak{a}^*_L \) on the subset of elements \( \tau \in \tilde{T}(G) \) of this form. Hence, \( \tilde{T}(G) \) becomes an analytic manifold that is homeomorphic to either a disjoint union of Euclidean spaces (the Archimedean case), or a disjoint union of compact tori (the \( p \)-adic case). Then, \( T(G) \) acquires the quotient topology from the action of \( W_0^G \).

A simpler view of the set \( T(G) \) is the formulation of the twisted trace Paley-Wiener Theorem. Define \( \mathcal{I}(G(F)) \) to be the space of functions

\[
\phi : \tilde{T}(G) \to \mathbb{C}
\]

that satisfy the following four conditions:

(a) \( \phi \) is supported on finitely many components of \( \tilde{T}(G) \).

(b) \( \phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau), \tau \in \tilde{T}(G), z \in Z_\tau \).

(c) \( \phi \) is symmetric under \( W_0^G \).

(d) Let \( \phi_0 \) be the restriction of \( \phi \) to any connected component of \( \tilde{T}(G) \), then \( \phi_0 \) is in the Paley-Wiener space. Precisely, \( \phi_0 \) is a finite Fourier series in the \( p \)-adic case, or the Fourier transform of a smooth function of compact support if \( F \) is Archimedean.

There is a natural topology which makes \( \mathcal{I}(G(F)) \) into a complete topological vector space. By means of the inversion formula (4.18), we can in fact identify \( \mathcal{I}(G(F)) \) with the topological vector space of functions on \( \Pi_{\text{temp}}(G(F)) \) introduced in §11 of [7], and also denoted by \( \mathcal{I}(G(F)) \). We now want to describe the space \( \mathcal{I}(G(F)) \) by the twisted trace Paley-Wiener Theorem. The twisted trace Paley-Wiener Theorem is proved in [26] (\( p \)-adic case) and [19] (real case). The theorem is equivalent to the assertion that the
map which sends \( f \in \mathcal{H}(G(F)) \) to the function

\[ f_G(\tau) = \Theta(\tau, f), \quad \tau \in T(G), f \in \mathcal{H}(G(F)), \]

is a continuous surjective map from \( \mathcal{H}(G(F)) \) onto \( I(G(F)) \). We then have the following proposition.

**Proposition 4.2.1**: If \( \tau \) is an element in \( T_{\text{ell}}(G) \), then there exists a function \( f \) in \( \mathcal{H}(G(F)) \) with \( f_G(\tau) = 1 \), and such that \( f_G \) is supported on the \( (Z_\tau \times i a^*_G) \)-orbit of \( \tau \) in \( T(G) \). The function \( f \) is called a pseudocoefficient for \( \tau \).

For the local twisted trace formula, it is useful to take a set which lies between \( T_{\text{ell}}(G) \) and \( T(G) \). Let \( \sigma \) be an element in \( \Pi_{2,\theta}(M^0(F)) \). Write

\[ W_{\sigma,\text{reg}} = \{ \omega \in W_\sigma : a^{\theta_\omega}_{M^0} = a_G \} \]

for the set of regular elements in \( W_\sigma \). If \( t \) belongs to the \( R \)-torsor \( R_\sigma \), we set \( W_\sigma(t)_{\text{reg}} \) to be the intersection of the \( W_0^\sigma \)-coset \( W_\sigma(t) = W_\sigma t \) in \( W_\sigma^+ \) with the set \( W_\sigma,\text{reg} \). This also serves to define \( W_\sigma(t) \) and \( W_\sigma(t)_{\text{reg}} \) for elements \( t \in \tilde{R}_\sigma \), as we have agreed earlier. We define \( T_{\text{disc}}(G) \) to be the set of orbits \( (M^0, \sigma, t) \) in \( T(G) \) such that \( W_\sigma(t)_{\text{reg}} \) is not empty. It is clear that

\[ T_{\text{ell}}(G) \subset T_{\text{disc}}(G) \subset T(G). \]

To each element \( \tau = (M^0, \sigma, t) \) in \( T_{\text{disc}}(G) \), we attach a number

\[ i(\tau) = i^G(\tau) = |W'_{\sigma}|^{-1} \sum_{\omega \in W_\sigma(t)} \varepsilon_\sigma(\omega)|\det(1 - \omega) a_{G_{M^0}}^G|^{-1}. \quad (4.21) \]

We can now describe the distribution \( I_{\text{disc}}(f', f) \), where \( f' \) and \( f \) are two fixed functions in \( \mathcal{H}(G(F)) \). Through a discussion similar to that given in the ordinary case on p. 95
of \([\Pi]\), we find that \(I_{\text{disc}}(f', f)\) equals the sum over all triplets

\[
\tau = (M^0, \sigma, t), \quad M^0 \in \mathcal{L}^0, \quad \sigma \in \Pi_{2,\theta}(M(F))/i^*_{a^*_G}, \quad t \in \tilde{R}_\sigma,
\]

of the expression

\[
i(\tau)|W'_\sigma||W'^{M^0}_0||W'^{G^0}_0|^{-1}|Z_\sigma|^{-1}|a^*_G|^{-1}\int_{ia^*_G} \Theta(\tau_{\lambda'}, f')\Theta(\tau_\lambda, f)d\lambda. \tag{4.22}
\]

Let \(\tilde{R}_{\sigma, t}\) be the centralizer of \(t\) in \(\tilde{R}^0_\sigma\). Imitating the discussion on p. 96 of \([\Pi]\), we find that \(I_{\text{disc}}(f', f)\) also equals

\[
\sum i(\tau)|\tilde{R}_{\sigma, t}|^{-1}|a^*_G|^{-1}\int_{ia^*_G} \Theta(\tau_{\lambda'}, f')\Theta(\tau_\lambda, f)d\lambda,
\]

where the sum is over elements \(\tau = (M^0, \sigma, t)\) in \(\Pi_{\text{disc}}(G)/i^*_{a^*_G}\). Define a measure \(d\tau\) on \(T_{\text{disc}}(G)\) by setting

\[
\int_{T_{\text{disc}}(G)} \vartheta(\tau)d\tau = \sum_{\tau \in \Pi_{\text{disc}}(G)/i^*_{a^*_G}} |\tilde{R}_{\sigma, t}|^{-1}|a^*_G|^{-1}\int_{ia^*_G} \vartheta(\tau_\lambda)d\lambda \tag{4.23}
\]

for any function \(\vartheta \in C_c(\Pi_{\text{disc}}(G))\). Then, we have the following proposition.

**Proposition 4.2.2**: If \(f'\) and \(f\) are functions in \(\mathcal{H}(G(F))\), then

\[
I_{\text{disc}}(f', f) = \int_{T_{\text{disc}}(G)} i(\tau)\Theta(\tau_{\lambda'}, f')\Theta(\tau_\lambda, f)d\tau.
\]

We shall use the special case of Proposition 4.2.2 in which one of the functions is cuspidal. We give the definition of cuspidal functions.

**Definition 4.2.3**: A function in \(\mathcal{H}(G(F))\) is called **cuspidal** if for every proper Levi
subset $L$ of $G$, the function

$$f_L(\pi_L) = tr(\pi_L(f_Q)) = tr(I_Q(\pi_L, f)), \quad \pi_L \in \Pi_{\text{temp}}(L(F)),$$

vanishes identically, where $f_Q$ is defined in the proof of Theorem 3.6.1. If $f$ is cuspidal, it is possible to show that $I_{\text{disc}}(f', f)$ is supported on the subset $T_{\text{ell}}(G)$ of $T_{\text{disc}}(G)$ via a discussion similar to that on p. 97 of [11]. (An invariant distribution $I$ on $\mathcal{H}(G(F))$ is said to be supported on characters if $I(F) = 0$ for every function $f \in \mathcal{H}(G(F))$ such that $f_G = 0$.)

For a given $\sigma \in \Pi_{2,0}(M^0(F))$, the set $\tilde{R}_{\sigma, \text{reg}}$ could of course be empty. However, if it is not empty, then the subgroup $W_{\sigma}'$ of $W_{\sigma}^{G_0}$ is trivial. Hence, if $\tau = (M^0, \sigma, t)$ belongs to $T_{\text{ell}}(G)$, then $W_{\sigma}' = \{1\}$. Consequently, the coset $W_{\sigma}(\theta t)$ equals $t$ itself. We see that if $\varepsilon_{\sigma}(t) = 1$, then the formula (4.21) for the number $i(\tau)$ reduces simply to the inverse of the absolute value of the number

$$d(\tau) = d(t) = det(1 - t)_{M^0, G_0}. \quad (4.24)$$

We have established

Corollary 4.2.4: Suppose that $f'$ and $f$ are functions in $\mathcal{H}(G(F))$ and that $f$ is cuspidal. Then,

$$I_{\text{disc}}(f', f) = \int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau', f') \Theta(\tau, f) d\tau.$$
4.3 The Invariant Local Twisted Trace Formula

In Chapter 3, we obtained a non-invariant local twisted trace formula. Just as for the ordinary local trace formula, we need to change the formula into an invariant form to make it more useful in the local harmonic analysis. The method for obtaining an invariant local twisted trace formula is the same as that used to derive an ordinary local twisted trace formula. For the details, see §8 of [8] and §4 of [11]. We limit ourselves to defining some related objects in the twisted case and list the results without proof.

To state a new form of the non-invariant local twisted trace formula, we need to define \( J_M(\tau, f' \times f) \) to be

\[
\sum_{\rho', \rho \in \Pi(R^M_\sigma \chi_\sigma)} \text{tr}(\rho'(t)) \text{tr}((\rho^+)^\vee(t)) J_M((\pi_\rho^+)^\vee \otimes \pi_\rho^+, f' \times f),
\]

for

\[
\tau = (M^0_1, \sigma, t), \quad M^0_1 \subset M^0, \quad \sigma \in \Pi_{2,0}(M^0_1(F)), \quad t \in \tilde{R}^M_\sigma.
\]

The number \( a_{\text{disc}}^M(\pi) \) occurs on the spectral side of the trace formula in Theorem 3.6.1. It is possible to verify that the coefficient \( a_{\text{disc}}^G(\pi) \) satisfies

\[
a_{\text{disc}}^G((\pi_\rho^+)^\vee \otimes \rho_\rho^+) = \sum_t |\tilde{R}_{\sigma,t}|^{-1} i(M^0_1, \sigma, t) \text{tr}(\rho'(t)) \text{tr}((\rho^+)^\vee(t)),
\]

where the sum is taken over conjugacy classes in \( \tilde{R}_\sigma \). We can now state the non-invariant local twisted trace formula (Theorem 3.6.1) in another form.

**Proposition 4.3.1**: Suppose that \( f' \) and \( f \) are functions in \( \mathcal{H}(G(F)) \). Then, we have the following non-invariant local twisted trace formula. The geometric expansion

\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{\text{ell}}(M)} J_M(\gamma, f' \times f) d\gamma
\]
equals
\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{T_{disc}(M)} i^M(\tau)J_M(\tau, f' \times f) d\tau.
\]

To obtain an invariant trace formula, we define an invariant distribution

\[
I_M(\gamma, f) = I_M^G(\gamma, f)
\]

inductively by setting

\[
I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L \in \mathcal{L}(M)} \hat{I}_M(\gamma, \phi_L(f)) \tag{4.26}
\]

as in §8.1 of [8]. This is the invariant expression for the geometric side. For the spectral side, we will define some invariant distributions attached to the weighted characters \(J_M(\tau, f' \times f)\). Specifically,

\[
r_M(\tau, f' \times f) = r_M(\tau, P)\Theta(\tau^\vee, f'_P)\Theta(\tau, f_P), \quad f', f \in \mathcal{H}(G(F)),
\]

where \(r_M(\tau, P)\) can be defined analogously to the ordinary case on p. 102 of [11]. We can now state the invariant local twisted trace formula.

**Theorem 4.3.2:** For any \(f'\) and \(f\) in \(\mathcal{H}(G(F))\), we have the following invariant local twisted trace formula. The geometric expansion

\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{ell}(M)} I_M(\gamma, f' \times f) d\gamma
\]
equals the spectral expansion
\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{T_{disc}(M)} i^M(\tau)r_M(\tau, f' \times f) d\tau.
\]

Also we have some special cases of Theorem 4.3.2 that are important in our proof of the orthogonality relations. We state the results in the following corollary.

**Corollary 4.3.3**: Suppose that \(f'\) and \(f\) are functions in \(\mathcal{H}(G(F))\), and that \(f\) is cuspidal. The invariant local trace formula then reduces to the identity of an expression
\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{ell}(M(F))} I_G(\gamma, f') I_M(\gamma, f) d\gamma,
\]
with
\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{T_{ell}(G)} |d(\tau)|^{-1}\Theta(\tau^\vee, f')\Theta(\tau, f) d\tau.
\]

If both \(f'\) and \(f\) are cuspidal, the formula simplifies further to
\[
\int_{\Gamma_{ell}(G(F))} I_G(\gamma, f') I_G(\gamma, f) d\gamma = \int_{T_{ell}(G)} |d(\tau)|^{-1}\Theta(\tau^\vee, f')\Theta(\tau, f) d\tau. \tag{4.27}
\]

### 4.4 Twisted Elliptic Tempered Characters and Twisted Weighted Orbital Integrals

In this section, we will state the twisted analogues of the results in §5 of [11]. Since the proofs of the twisted results are very similar to the ordinary ones, we will omit them.

Define
\[
\Phi_M(\tau^\vee, \gamma) = \begin{cases} 
|D(\gamma)|^{1/2}\Theta(\tau^\vee, \gamma), & \text{if } \gamma \in M(F)_{ell}, \\
0, & \text{otherwise}.
\end{cases}
\]
We then have the following twisted result.

**Theorem 4.4.1**: Suppose that $f$ is a function in $\mathcal{H}(G(F))$. Then,

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} \int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Phi_M(\tau^\vee, \gamma) \Theta(\tau, f) d\tau,$$  \hspace{1cm} (4.28)

for any group $M \in \mathcal{L}$ and any $G$-regular point $\gamma$ in $M(F)$.

We also have two corollaries.

**Corollary 4.4.2**: The distributions $I_M(f)$ on $\mathcal{H}(G(F))$ are supported on characters.

**Corollary 4.4.3**: Suppose that $f$ is a function in $\mathcal{H}(G(F))$ such that for any triplet $(M_1^0, \sigma_1, t_1)$, $M_1^0 \in \mathcal{L}^0$, $\sigma_1 \in \Pi_{2,0}(M_1^0(F))$, $t_1 \in \tilde{R}_{\sigma_1}$, the expression

$$\text{tr}(I_{P_1^0}^+(t_1, \sigma_1, f)), \quad P_1^0 \in \mathcal{P}(M_1^0),$$

vanishes unless $(M_1^0, \sigma_1, t_1)$ belongs to the $W_0^G$-orbit of $(M^0, \sigma, t)$, in which case it equals

1. $I_M(\gamma, f)$ then equals

$$(-1)^{\dim(A_M)} |R_{\sigma,t}|^{-1} |det(1-t)|^{-1} \sum_{\rho^+ \in \Pi(R_{\sigma})} \text{tr}(\rho^+(t)) |D(\gamma)|^{1/2} \Theta((\pi_{\rho}^+)^\vee, \gamma),$$

for any $G$-regular point $\gamma$ in $M(F)_{\text{ell}}$.

### 4.5 Twisted Orthogonality Relations for Elliptic Characters

Let $\mathcal{H}'(G(F))$ be the full dual space of $\mathcal{I}(G(F))$. A test function $\vartheta$ in $\mathcal{I}'(G(F))$ is called *cuspidal* if the transpose

$$\tau \rightarrow \vartheta(\tau^\vee), \quad \tau \in T(G),$$
belongs to \( \mathcal{I}(G(F)) \) and is supported on \( T_{ell}(G) \). For such a \( \vartheta \), we define a distribution

\[
\Theta(f) = \int_{T_{ell}(G)} \vartheta(\tau)\Theta(\tau, f)d\tau, \quad f \in \mathcal{H}(G(F)),
\]

(4.29)
on \( \mathcal{H}(G(F)) \). Given the above class function \( \Theta \) on \( G(F) \), the normalization of \( \Theta(\gamma) \) is usually denoted by

\[
\Phi(\gamma) = |D(\gamma)|^{1/2}\Theta(\gamma).
\]

(4.30)

The restriction of \( \Phi(\gamma) \) to \( \Gamma_{ell}(G(F)) \) equals

\[
\Phi_{G}(\gamma) = \int_{T_{ell}(G)} \vartheta(\tau)\Phi_{G}(\tau, \gamma)d\tau.
\]

We also define the normalized function related to \( \vartheta \) as

\[
\phi(\tau) = |d(\tau)|^{1/2}\vartheta(\tau).
\]

(4.31)

We then have the following theorem.

**Theorem 4.5.1** : Suppose that \( \vartheta \) and \( \vartheta' \) are two cuspidal test functions in \( \mathcal{I}'(G(F)) \). Then, the associated pairs \( \Phi, \Phi' \) and \( \phi, \phi' \) of functions defined by (4.30) and (4.31), respectively, satisfy the inner product formula

\[
\int_{\Gamma_{ell}(G(F))} \Phi(\gamma)\overline{\Phi'(\gamma)}d\gamma = \int_{T_{ell}(G)} \phi(\tau)\overline{\phi'(\tau)}d\tau.
\]

(4.32)

**Proof.** We can modify the proof of Theorem 6.1 in [11].

The function \( \tau \to |d(\tau)|\vartheta(\tau^\vee) \) is also a function in \( \mathcal{I}(G(F)) \) which is supported on \( T_{ell}(G) \). We can then find a cuspidal function \( f \in \mathcal{H}(G(F)) \) such that

\[
f_{G}(\tau) = \Theta(\tau, f) = |d(\tau)|\vartheta(\tau^\vee), \quad \tau \in T(G),
\]
by the trace Paley-Wiener Theorem.

Applying (4.28) with $M = G$, we can calculate that, for any $G$-regular point $\gamma$,

$$I_G(\gamma, f) = \Phi_G(\gamma).$$

Given $\vartheta'$, it is possible to define a second cuspidal function $f' \in \mathcal{H}(G(F))$ in the same way. It has the property

$$\Phi_{G}^{\prime}(\gamma) = I_G(\gamma, f') = I_G(\gamma, \overline{f'}).$$

The simple version (4.27) of the local twisted trace formula tells us that the inner product

$$\int_{\Gamma_{\text{ell}}(G(F))} \Phi_G(\gamma) \overline{\Phi_G(\gamma)} d\gamma = \int_{T_{\text{ell}}(G)} I_G(\gamma, f) I_G(\gamma, \overline{f'}) d\gamma$$

equals

$$\int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau^\vee, f) \Theta(\tau, \overline{f'}) d\tau.$$

In general, it is possible to check that

$$\Theta(\tau^\vee, f') = \text{tr}(\mathcal{I}_{\text{po}}^+(t, \sigma^\vee, \overline{f'})) = \text{tr}(\mathcal{I}_{\text{po}}^+(t, \sigma, \overline{f'})) = \Theta(\tau, \overline{f'}).$$

We then see that the left-hand side of (4.32) equals

$$\int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau^\vee, f) \overline{\Theta(\tau^\vee, f')} d\tau.$$

Since $d(\tau) = d(\tau^\vee)$, this can be written as

$$\int_{T_{\text{ell}}(G)} |d(\tau)| \vartheta(\tau) \overline{\vartheta'(\tau)} d\tau = \int_{T_{\text{ell}}(G)} \phi(\tau) \overline{\phi'(\tau)} d\tau,$$

which is exactly the right-hand side of (4.32).

Any element $\tau = (M, \sigma, \theta t)$ in $T(G)$ has a central character $\zeta_\tau$ on $A_G(F)$. Suppose
that $\tau'$ is another element in $T(G)$ with the same central character. The function

$$
\Phi(\tau, \gamma) \Phi(\tau', \gamma)
$$

is invariant under $A_G(F)$ and can be integrated over the elliptic conjugacy classes in $G(F)/A_G(F)$. Define the twisted elliptic inner product

$$
\sum_{\{T\}} |W(G, T)|^{-1} \int_{T(F)/A_G(F)} \Phi(\tau, \gamma) \Phi(\tau', \gamma) d\gamma,
$$

(4.33)

where $\{T\}$ is summed over the $G(F)$-conjugacy classes of elliptic maximal tori in $G$. We have the following corollary.

**Corollary 4.5.2**: Suppose that $\tau$ and $\tau'$ are two elements in $T_{\text{ell}}(G)$ with the same central character. Then, the inner product (4.32) vanishes unless $\tau'$ is in $Z_\tau \times i\mathfrak{a}_G^*$-orbit in $T_{\text{ell}}(G)$. However, if $\tau' = \tau = (M_0, \sigma, t)$, the inner product equals

$$
|R_{\sigma, t}||d(\mathcal{T})|,
$$

where $\mathcal{T}$ is the image of $t$ in $R_\sigma$.

**Proof.** We shall apply Theorem 4.5.1 with $\vartheta$ supported on the $Z_\tau \times i\mathfrak{a}_G^*$-orbit of $\tau$ and $\vartheta'$ supported on the $Z_{\tau'} \times i\mathfrak{a}_G^*$-orbit of $\tau'$. If $\tau$ and $\tau'$ lie in different orbits, the right hand side of (4.32) vanishes since the supports of $\vartheta$ and $\vartheta'$ are disjoint. It is possible to choose suitable $\vartheta$ and $\vartheta'$ such that the left-hand side of (4.32) becomes the inner product (4.33). The vanishing of the inner product follows.

If $\tau' = \tau = (M_0, \sigma, t)$, we take $\vartheta = \vartheta'$. We first substitute the formula (4.23) for the measure on $T_{\text{ell}}(G)$ into the right-hand side of (4.32). It is possible to check that

$$
\int_{T_{\text{ell}}(G)} |\phi(\tau)|^2 d\tau = |d(\tau)||\widetilde{R}_{\sigma, t}|^{-1}|(a_{G, n, \sigma})_{\theta}^{\sigma} (a_{G, n, F})_{\theta}^{-1} \sum_{z \in Z_\tau / Z_{\tau'}} \int_{i\mathfrak{a}_G^*, F} |\vartheta(z\tau_\lambda)|^2 d\lambda,
$$

where $\widetilde{R}_{\sigma, t}$ is the image of $t$ in $R_\sigma$. 


where $Z^0_\tau$ is the stabilizer of $\tau$ in $Z_\tau$. We also have

$$|\tilde{R}_{\sigma,t}|^{-1}|Z_\tau/Z^0_\tau| = |R_{\sigma,t}|^{-1},$$

and the above integrand is independent of $z$.

The right-hand side of (4.32) becomes

$$|d(\bar{t})||R_{\sigma,t}|^{-1}|a^{\vee}_{G,\sigma}/\tilde{a}^{\vee}_{G,F}|^{-1} \int_{ia\tilde{G},F/\tilde{a}^{\vee}_{G,G,\sigma}} |\vartheta(\tau_\lambda)|^2 d\lambda$$

$$= |d(\bar{t})||R_{\sigma,t}|^{-1} \int_{ia\tilde{G},F/\tilde{a}^{\vee}_{G,G,\sigma}} |\vartheta(\tau_\lambda)|^2 d\lambda$$

$$= |d(\bar{t})||R_{\sigma,t}|^{-1}|\tilde{a}^{\vee}_{G,F}/a^{\vee}_{G,G,\sigma}| \int_{\tilde{a}^{\vee}_{G,G,\sigma}} |\hat{\vartheta}(X)|^2 dX,$$

where

$$\hat{\vartheta}(X) = |\tilde{a}^{\vee}_{G,F}/a^{\vee}_{G,G,\sigma}|^{-1} \int_{ia\tilde{G},F/\tilde{a}^{\vee}_{G,G,\sigma}} \vartheta(\tau_\lambda)e^{\lambda(X)} d\lambda$$

is the Fourier transform of $\vartheta$ relative to the normalized Haar measure on $ia\tilde{G},F/\tilde{a}^{\vee}_{G,G,\sigma}$.

On the other hand, we know that

$$\Phi(\tau_\lambda,\gamma) = \Phi(\tau,\gamma)e^{\lambda(H_G(\gamma))}, \quad \lambda \in ia\tilde{G}/\tilde{a}^{\vee}_{G,G,\sigma}.$$ 

In particular, $\Phi(\tau,\gamma)$ vanishes unless $H_G(\gamma)$ belongs to $a_{G,G,\sigma}$. We can calculate

$$\Phi(\gamma) = \int_{T_{\text{ell}}(G)} \vartheta(\tau)\Phi(\tau,\gamma)d\tau$$

$$= |d(\bar{t})||R_{\sigma,t}|^{-1}|\tilde{a}^{\vee}_{G,F}/a^{\vee}_{G,G,\sigma}|\Phi(\tau,\gamma)\hat{\vartheta}(H_G(\gamma)).$$

The left-hand side of (4.32) therefore equals

$$|R_{\sigma,t}|^{-2}|\tilde{a}^{\vee}_{G,F}/a^{\vee}_{G,G,\sigma}|^2 \int_{\Gamma_{\text{ell}}(G(F))} |\Phi(\tau,\gamma)|^2 |\hat{\vartheta}(H_G(\gamma))|^2 d\gamma.$$
which is also equal to

$$\sum_{\{T\}} |W(G, T)|^{-1} \int_{T(F)/A_G(F)} |\Phi(\tau, \gamma)|^2 \eta(H_G(\gamma)) d\gamma, \quad (4.34)$$

where

$$\eta(X) = \int_{\tilde{a}_{G,F}} |\hat{\vartheta}(H_G(\gamma) + Y)|^2 dY, \quad X \in a_{G,\sigma}.$$  

From the two expressions of (4.32) we have obtained above, we see that (4.34) equals

$$|d(\bar{t})||R_{\sigma,\bar{t}}||a_{G,\sigma} / a_{G,F}|^{-1} \int_{a_{G,\sigma}} |\hat{\vartheta}(X)|^2 dX.$$

We can choose \(\vartheta\) so that \(\eta(X) = 1\) for any \(X\) in \(a_{G,\sigma}\). Hence,

$$\int_{a_{G,\sigma}} |\hat{\vartheta}(X)|^2 dX = \sum_{X \in a_{G,\sigma} / \tilde{a}_{G,F}} \eta(X) = |\tilde{a}_{G,\sigma} / a_{G,F}|.$$

Thus, the inner product (4.33) is just (4.34), which is equal to \(|R_{\sigma,\bar{t}}||d(\bar{t})|\).

\[\square\]

**Corollary 4.5.3**: Fix \(\sigma \in \Pi_{\vartheta}(M^0(F))\), and suppose that

$$\{\pi^+_\rho, \pi^+_{\rho'} : \rho^+, \rho'^+ \in \Pi(\tilde{R}_{\sigma}, \chi_{\sigma})\}$$

are two \(G(F)\)-elliptic representations in \(\Pi_{\vartheta}(G(F))\). Then, the twisted elliptic inner product

$$\sum_{\{T\}} |W(G^0(F), T(F))|^{-1} \int_{T(F)/A_G(F)} |D(\gamma)|\Theta(\pi^+_\rho, \gamma)\Theta(\pi^+_{\rho'}, \gamma) d\gamma \quad (4.35)$$

equals

$$\frac{1}{|R_{\sigma}|} \sum_{\theta t \in R_{\sigma,\text{reg}}} |d(t)| tr(\rho^+(t)) tr(\rho'^+(t)).$$
Proof. Let $r$ denote $\theta t$. The character $\Theta(\pi^+_\rho)$ is the image of the function

$$\partial(\rho^+, r) = tr(\rho^+(r)), \quad r \in \tilde{R}_\sigma,$$

given in Theorem 4.1.4 (a). Using the formula (4.18), we obtain

$$\Theta(\pi^+_\rho, \gamma) = \frac{1}{|\tilde{R}_\sigma|} \sum_{r \in R_\sigma} tr(\rho^+(r))tr(I_{p0}^+(r, \sigma, f))$$

$$= \frac{1}{|R_\sigma|} \sum_{r \in R_\sigma} tr(\rho^+(r))tr(I_{p0}^+(r, \sigma, f))$$

$$= \sum_{r \in \Gamma(R_\sigma)} \frac{1}{|R_\sigma,r|} tr(\rho^+(r))tr(I_{p0}^+(r, \sigma, f)),$$

for any $G$-regular element in $G(F)_{ell}$, where $R_{\sigma,r}$ is the centralizer of $r$ in $R_\sigma^0$ and $\Gamma(R_\sigma)$ is the $R_\sigma^0$-conjugacy classes in $R_\sigma$. If $r$ belongs to the complement of $\tilde{R}_{\sigma,reg}$ in $\tilde{R}_\sigma$, then the virtual character $tr(I_{p0}^+(\theta t, \sigma, f))$ is a linear combination of induced characters, and it vanishes on the regular elliptic set. We may therefore take the last sum over the set $\Gamma(R_{\sigma,reg})$ of $R_\sigma^0$-conjugacy classes in $R_{\sigma,reg}$. The formula becomes

$$\Theta(\pi^+_\rho, \gamma) = \sum_{r \in \Gamma(R_{\sigma,reg})} \frac{1}{|R_{\sigma,r}|} tr(\rho^+(r))\Theta(\tau_r, f).$$

We see that the inner product (4.33) is a double sum over $r, r' \in \Gamma(R_{\sigma,reg})$ of the expression obtained by multiplying

$$\frac{1}{|R_{\sigma,r}|} \frac{1}{|R_{\sigma,r}'|} tr(\rho^+(r))tr(\rho^+(r'))$$

by the inner product

$$\sum_{\{T\}} |W(G^0(F), T(F))|^{-1} \int_{T(F)/A_G(F)} \Phi(\pi^+_\rho, \gamma)\bar{\Phi}(\pi^+_\rho', \gamma)d\gamma.$$
By Corollary 4.4.2, the last inner product vanishes unless \( r = r' \), in which case it equals \( |R_{\sigma,r}| d(r) \).

Consequently, the original inner product (4.35) equals

\[
\sum_{t \in \Gamma(R_{\sigma,\text{reg}})} \frac{1}{|R_{\sigma,t}|} |d(t)| tr(\rho^+(t)) tr(\rho'^+(t)).
\]

This is exactly the required expression

\[
\frac{1}{|R_{\sigma}|} \sum_{t \in R_{\sigma,\text{reg}}} |d(t)| tr(\rho^+(t)) tr(\rho'^+(t)).
\]

By Corollary 4.5.3, we see that if \( \rho^+ \) does not vanish on \( \tilde{R}_{\sigma,\text{reg}} \), then the corresponding representation \( \pi^+_{\rho} \) does not vanish on the elliptic set of \( G(F) \). Hence, we show that the map \( p \) defined in (4.19) is injective.

\textit{Remark.} For the twisted case, we still have analogous relations between \( G^+ \) and \( R^+ \) as in the ordinary case described in Remarks (2) at the end of §6 of [11].
Bibliography


