BASES FOR INVARIANT SPACES AND GEOMETRIC REPRESENTATION THEORY

by

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Let $G$ be a simple algebraic group. Labelled trivalent graphs called webs can be used to produce invariants in tensor products of minuscule representations. For each web, a configuration space of points in the affine Grassmannian is constructed. This configuration space gives a natural way of calculating the invariant vectors coming from webs.

In the case of $G = \text{SL}_3$, non-elliptic webs yield a basis for the invariant spaces. The non-elliptic condition, which is equivalent to the condition that the dual diskoid of the web is $\text{CAT}(0)$, is explained by the fact that affine buildings are $\text{CAT}(0)$. In the case of $G = \text{SL}_n$, a sufficient condition for a set of webs to yield a basis is given. Using this condition and a generalization of a technique by Westbury, a basis is constructed for $\text{SL}_n$.

Due to the geometric Satake correspondence there exists another natural basis of invariants, the Satake basis. This basis arises from the underlying geometry of the affine Grassmannian. There is an upper unitriangular change of basis from the basis constructed above to the Satake basis. An example is constructed showing that the Satake, web and dual canonical basis of the invariant space are all different.

The natural action of rotation on tensor factors sends invariant space to invariant space. Since the rotation of web is still a web, the set of vectors coming from webs is fixed by this action. The Satake basis is also fixed, and an explicit geometric and combinatorial description of this action is developed.
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Chapter 1

Introduction

1.1 Spiders

Let $G$ be a simple, simply-connected complex algebraic group. In previous work [Kup96], Kuperberg defined a tensor category with generators and relations called a “spider”, for $G$ of rank 2. (The term “spider” was originally intended to mean any pivotal category, but in common usage only these categories are called spiders.) The Karoubi envelope of this category is equivalent to the category $\text{rep}(G)$ of finite-dimensional representations of $G$. (Actually, the spider comes with a parameter $q$ making it equivalent to the quantum deformation $\text{rep}_q(G)$.) These results in rank 2 are analogous to the influential result of Kauffman [Kau90] and Penrose [Pen71] that the Karoubi envelope of the Temperley-Lieb category (the category of planar matchings) is equivalent to $\text{rep}_q(\text{SL}_2)$. The Temperley-Lieb category can thus be called the SL$_2$ spider. Conjectural generalizations of spiders were proposed for SL$_4$ by Kim [Kim03] and for SL$_n$ by Morrison [Mor07].

In this thesis, for any $G$ as above, we define the free spider for $G$ generated by the minuscule representations of $G$. Recall that a minuscule representation is an irreducible representation such that all its weights are contained in the Weyl orbit of the highest weight. A minuscule representation is always fundamental and for SL$_n$ all fundamental representations are minuscule.
A morphism in the free spider is given by a (linear combination) of labelled trivalent graphs called webs. For each web $w$ with boundary edges labelled $\vec{\lambda}$, there is an invariant vector

$$\Psi(w) \in \text{Inv}(V(\vec{\lambda})) = \text{Inv}_G(V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots \otimes V(\lambda_n)).$$

In the case where $G$ has rank 1 or 2, the vectors $\Psi(w)$ coming from non-elliptic webs $w$ (those whose faces have non-positive combinatorial curvature) form a basis of each invariant space $\text{Inv}(V(\vec{\lambda}))$ of $G$, called a web basis. The web basis for $\text{SL}_2$ is well-known as the basis of planar matchings and it is known to be the same as Lusztig’s dual canonical basis [FK95]. On the other hand, the $\text{SL}_3$ web bases are eventually not dual canonical [KK99], even though many basis vectors are dual canonical.

### 1.2 Affine Grassmannians

The goal of this thesis is to introduce a new geometric interpretation of webs and spiders using the geometry of affine Grassmannians.

Let $\mathcal{O} = \mathbb{C}[t]$ and $\mathcal{K} = \mathbb{C}((t))$. In order to study the representation theory of $G$, we consider the affine Grassmannian of its Langlands dual group

$$\text{Gr} = \text{Gr}(G^\vee) = G^\vee(\mathcal{K})/G^\vee(\mathcal{O}).$$

Note that when $G = \text{SL}_n$ then $G^\vee = \text{PGL}_n$. The geometric Satake correspondence of Lusztig [Lus83], Ginzburg [Gin], and Mirković-Vilonen [MV07] is our main tool in this thesis.

**Theorem 1.2.1.** The representation category $\text{rep}(G)$ is equivalent as a pivotal category to the category of $G^\vee(\mathcal{O})$ equivariant perverse sheaves on the affine Grassmannian $\text{Gr}$.

Our main use of the the geometric Satake correspondence is the description it gives of any invariant space $\text{Inv}(V(\vec{\lambda}))$ for $G$ in terms of the geometry of $\text{Gr}$. Given a vector $\vec{\lambda}$ of dominant weights of $G$, there is a convolution morphism

$$m_{\vec{\lambda}} : \overline{\text{Gr}(\vec{\lambda})} = \overline{\text{Gr}(\lambda_1)} \times \overline{\text{Gr}(\lambda_2)} \times \cdots \times \overline{\text{Gr}(\lambda_n)} \longrightarrow \text{Gr},$$
where each \( \text{Gr}(\lambda) \) is a sphere of radius \( \lambda \) (in the sense of weight-valued distances [KLM08]) in \( \text{Gr} \). The fiber \( F(\vec{\lambda}) = m^{-1}(t^0) \) is a projective variety that we call the Satake fiber. In particular, we use the following corollary of the geometric Satake correspondence.

**Theorem 1.2.2.** Every invariant space in \( \text{rep}(G) \) is canonically isomorphic to the top homology of the corresponding geometric Satake fiber with complex coefficients:

\[
\text{Inv}(V(\vec{\lambda})) \cong H_{\text{top}}(F(\vec{\lambda}), \mathbb{C}).
\]

Each top-dimensional component \( Z \subseteq F(\vec{\lambda}) \) thus yields a vector \([Z] \in \text{Inv}(V(\vec{\lambda}))\). These vectors form a basis, the Satake basis.

A goal of this thesis is to understand how the invariant vectors coming from webs expand in this basis. (Throughout, we assume complex coefficients for homology and cohomology.)

In order to meet this goal it is necessary to understand the top dimensional components of \( F(\vec{\lambda}) \) when \( \vec{\lambda} \) is a minuscule sequence. Given a sequence \( \vec{\mu} \) of dominant weights, we say that it is a minuscule Littelmann path of type \( \vec{\lambda} \) if \( \mu_i - \mu_{i-1} \) is a weight in the Weyl orbit of \( \lambda_i \).

Then we have the following theorem:

**Theorem 1.2.3.** Let \( Z_{\vec{\mu}} = \{(L_0, \cdots, L_n) \in F(\vec{\lambda})|L_i \in \text{Gr}(\mu_i)\} \). Then the set of top dimensional components of \( F(\vec{\lambda}) \) is exactly \( Z_{\vec{\mu}} \) where \( \vec{\mu} \) ranges over all minuscule Littelmann paths of type \( \vec{\lambda} \) starting and ending at 0.

### 1.3 Diskoids

The orbits of \( G(\mathcal{X}) \) on the affine Grassmannian defines a notion of distance on \( \text{Gr} \) with values in the set of dominant weights for \( G \). Thus, we can interpret \( F(\vec{\lambda}) \) as the (contractive, based) configuration space in \( \text{Gr} \) of an abstract polygon \( P(\vec{\lambda}) \) whose side lengths are

\[
\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n).
\]
CHAPTER 1. INTRODUCTION

One of our ideas is to generalize this type of configuration space from polygons to diskoids. For us, a diskoid \( D \) is a contractible piecewise linear subset of the plane; in many cases it is a disk. (See Section 2.2.2.) If \( D \) is tiled by polygons and its edges are labelled by dominant weights, then its vertices are a weight-valued metric space. We define a (based) configuration space \( Q(D) \) which consists of maps from the vertices of \( D \) to \( \text{Gr} \) that preserves the lengths of edges of \( D \). We also define a special subset \( Q_g(D) \) that consists of maps that preserve all distances (globally isometric embeddings).

Assume that \( \lambda \) is a vector of minuscule highest weights. If \( w \) is a web with boundary \( \lambda \), then it has a dual diskoid \( D = D(w) \). The boundary of this diskoid is a polygon \( P(\lambda) \) and so we get a map of configuration spaces \( \pi : Q(D) \to F(\lambda) \). Our first main result is that we can recover the vector \( \Psi(w) \) using this geometry.

**Theorem 1.3.1.** There exists a homology class \( c(w) \in H_*(Q(D)) \) such that

\[
\pi_*(c(w)) \in H_{\text{top}}(F(\lambda))
\]

corresponds to \( \Psi(w) \) under the isomorphism from Theorem 1.2.2.

We prove this theorem as an application of the geometric Satake correspondence. In many cases, the class \( c(w) \) is the fundamental class of \( Q(D) \), so that the coefficients of \( \pi_*(c(w)) \) (and hence \( \Psi(w) \)) in the Satake basis are just the degrees of the map \( \pi \) over the components of \( F(\lambda) \).

### 1.4 Buildings

The affine Grassmannian \( \text{Gr} \) embeds isometrically into the affine building \( \Delta = \Delta(G^\vee) \) [Ron09]. This is a simplicial complex which has the affine Grassmannian as a subset of its vertices. We can use this perspective to gain greater insight into the variety \( Q(D) \).

If \( G = \text{SL}_2 \), then a basis web is a planar matching (or cup diagram) and its dual diskoid \( D \) is a finite tree. The affine Grassmannian \( \text{Gr} \) is the set of vertices of the affine building \( \Delta \), which is an infinite tree with infinite valence. The configuration space \( Q(D) \) is the space of colored,
based simplicial maps $f : D \to \Delta$; see Figure 1.1. It is known that

$$Q(D) = \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

is a twisted product of $\mathbb{P}^1$'s, and that these twisted products are the components of the Satake fiber $F(\vec{\lambda})$. Moreover, $Q_g(D)$ is the open dense subvariety of points in $Q(D)$ which are contained in no other component of $F(\vec{\lambda})$. Figure 1.1 is an illustration of the construction.

Figure 1.1: From a non-elliptic $A_1$ web, to a tree, to part of an affine $A_1$ building.

Figure 1.2: From a non-elliptic $A_2$ web, to a CAT(0) diskoid, to part of an affine $A_2$ building.
Our other main results are a generalization of this fact to \( G = \text{SL}_n \). In this case, \( G_r \) is again the vertex set of \( \Delta \). If \( w \) is a web with boundary \( \vec{\lambda} \), then \( Q(D(w)) \) is again the space of colored, based simplicial maps \( f : D \to \Delta \), as in Figure 1.2. Then in the case of \( \text{SL}_3 \) we have the following specific result:

**Theorem 1.4.1.** Let \( G = \text{SL}_3 = A_2 \) and let \( w \) be a non-elliptic web with minuscule boundary \( \vec{\lambda} \) and dual diskoid \( D \). Then the global isometry configuration space \( Q_g(D) \) is mapped isomorphically by \( \pi \) to a dense subset of a component of the Satake fiber \( F(\vec{\lambda}) \). This inclusion yields a bijection between non-elliptic webs and the components of \( F(\vec{\lambda}) \).

Our construction can be viewed as an explanation of why basis webs are non-elliptic. A web is non-elliptic if and only if its diskoid is \( \text{CAT}(0) \), essentially by definition. It is well-known that every affine building is a \( \text{CAT}(0) \) space [BT72]. Moreover, every convex subset of a \( \text{CAT}(0) \) space, such as a diskoid which is isometrically embedded in a building, is necessarily \( \text{CAT}(0) \). We also show that the image of each diskoid embedding \( f : D \to \Delta \) in \( Q_g(D) \) has a least area property. Likewise, the elliptic relations of the \( A_2 \) spider can be viewed as area-decreasing transformations.

### 1.5 Rotation

In [Kup96] it is shown that the rotation of webs corresponds to rotation of tensor factors. Since the set of non-elliptic webs is rotationally invariant, it follows that the Satake basis for \( \text{SL}_3 \) is rotationally invariant as well. In fact, this is a special case of the following more general result which we prove:

**Theorem 1.5.1.** Let \( G \) be a simple, simply connected algebraic group. Let \( \vec{\lambda} \) be a sequence of dominant minuscule weights and \( \vec{\lambda}^{(i)} \) its rotation by \( i \) positions. Then, up to sign, the Satake basis of \( \text{Inv}(V(\vec{\lambda})) \) is sent to the Satake basis of \( \text{Inv}(V(\vec{\lambda}^{(1)})) \) under the action of rotation of tensor factors.
On the other hand by Theorem 1.2.3, the Satake basis is naturally labelled by the set minuscule Littelmann paths. We introduce a combinatorial definition of rotation of minuscule Littelmann paths where \( \vec{\mu}^{(i)} \) is the rotation of \( \vec{\mu} \) by \( i \) positions. We then establish the following result:

**Theorem 1.5.2.** Let \( Z_{\vec{\mu}} \) be the component of \( F(\vec{\lambda}) \) labelled by a minuscule Littelmann path \( \vec{\mu} \). Under Theorem 1.5.1, the component of \( F(\vec{\lambda}^{(1)}) \) corresponding to \( Z_{\vec{\mu}} \) under rotation of tensor factors is \( Z_{\vec{\mu}^{(1)}} \).

The rotation of Littelmann paths is related to promotion on rectangular semistandard Young tableaux and the work of Brendon Rhoades [Rho10] and [PPR09]. It also can be realized via the theory of crystals.

### 1.6 Web Basis

Another goal of this thesis is to establish some general tools for producing a set of webs that form a basis for the corresponding invariant space. The essential result in the SL\(_3\) case is the idea of the non-elliptic condition. This condition is considerably less obvious in SL\(_m\) for higher \( m \). In general if \( w \) is an SL\(_m\) web, then \( Q(D) \) is sometimes the closure of \( Q_g(D) \) and hence maps to a single component of \( F(\vec{\lambda}) \). Eventually, \( Q(D) \) has other components and maps to more than one component of \( F(\vec{\lambda}) \). These other components seem related to the phenomenon that web bases are not dual canonical.

However, we can define a combinatorial property which we call coherence. A web with this property has a unique associated minuscule Littelmann path. If we can construct webs associated to each minuscule Littelmann path, then one can show the following result:

**Theorem 1.6.1.** Let \( B \) be a set of coherent webs as described above. Then both the Satake basis and \( B \) are labelled by the same set and the change of basis in \( \text{Inv}(V(\vec{\lambda})) \) from \( B \) to the Satake basis is upper unitriangular, relative to a partial ordering on the set of minuscule Littelmann paths.
We then give a construction of such a set of webs for $\text{SL}_m$. This construction generalizes the work of Westbury. For each (possibly non-dominant) minuscule weight $\lambda$ of $\text{SL}_m$, we construct what Westbury calls an irreducible triangular diagram $T_{\tilde{\lambda}}$ of length 1 and weight $\lambda$. Then, using Westbury’s product of triangular diagrams, we show that:

**Theorem 1.6.2.** Let $\tilde{\mu}$ be a minuscule Littelmann path starting and ending at 0, then $T_{\mu_1-\mu_0} \otimes \cdots \otimes T_{\mu_n-\mu_{n-1}}$ yields a coherent web associated to $\tilde{\mu}$.

This construction agrees with previous work in the cases $\text{SL}_2$, $\text{SL}_3$ and $\text{SL}_m$ when $\tilde{\lambda}$ is a sequence of the first fundamental weight and its dual.

Finally, we give an example showing that the web basis, the Satake basis, and the dual canonical basis for $\text{SL}_3$ are all eventually different.

### 1.7 Satake fibers and Springer fibers

When $G = \text{SL}_m$ and $\tilde{\lambda} = (\omega_1, \ldots, \omega_1)$ is an $n = mk$ tuple consisting of $\omega_1$ (the highest weight of the standard representation), then $F(\tilde{\lambda})$ is isomorphic to the $(k, k, \ldots, k)$ Springer fiber. In other words, $F(\tilde{\lambda})$ is the variety of flags in $\mathbb{C}^n$ invariant under a nilpotent endomorphism with $m$ Jordan blocks all of size $k \times k$. We have already mentioned the well-known description of the components of the Springer or Satake fiber in terms of planar matchings when $m = 2$. This Springer fiber formalism and this description of it have been used as a model of Khovanov homology [Kho04, Str09]. One motivation for the present work is to generalize this result to case $m = 3$ and obtain a description of the components of the Springer or Satake fiber using non-elliptic webs. Theorem 1.4.1 accomplishes this task. (See also the end of the introduction of [Tym].)
Chapter 2

Background Material

2.1 Spiders

2.1.1 Pivotal categories

A pivotal category $\mathcal{C}$ is a monoidal tensor category such that each object $A$ has a two-sided dual object $A^*$ [Mac98, FY89, BW99]. The correspondence $A \mapsto A^*$ should be a contravariant functor from $\mathcal{C}$ to itself and an involution. It should also be an order-reversing tensor functor, i.e.,

$$(A \otimes B)^* = B^* \otimes A^*.$$

Moreover, for each object $A$, there are “cup” and “cap” morphisms

$$b_A : I \longrightarrow A^* \otimes A \quad d_A : A^* \otimes A \longrightarrow I$$

where $I$ denotes the unit object, such that

$$(1_A \otimes d_A)(b_A \otimes 1_A) = (d_{A^*} \otimes 1_{A^*})(1_{A^*} \otimes b_{A^*}) = 1_A,$$

or, graphically,

$$\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node (A) {$A$};
  \node (I) [left of=A] {$I$};
  \node (A^*) [above of=A] {$A^*$};
  \draw[->] (A) -- (I);
  \draw[->] (A^*) -- (A);
  \draw[->] (A) -- (A^*)
\end{tikzpicture}
\end{array}
\end{align*}$$
A pivotal functor is a tensor functor that preserves the above structure.

Every object $A$ in a monoidal category has an invariant space

$$\text{Inv}(A) \overset{\text{def}}{=} \text{Hom}(I, A).$$

If the category is pivotal, then each invariant space has two other important properties. First, every space of morphisms is an invariant space by the relation

$$\text{Hom}(A, B) \cong \text{Inv}(A^* \otimes B).$$

Second, the relation

$$\text{Inv}(A \otimes B) \cong \text{Inv}(B \otimes A)$$

induces a cyclic action on the invariant spaces of tensor products.

Another way to describe a pivotal category (already suggested in equation (2.1)) is that it has the structure to evaluate a planar graph $w$ drawn in a disk with a marked boundary point, if the edges of $w$ are oriented and labelled by objects and the vertices are labelled by invariants in the tensor product of the associated edges. (The literature uses the words “labelled” and “colored” interchangeably here.) The value of such a graph $w$ is another invariant, taking values in the invariant space of the tensor product of the objects labelling the boundary edges of $w$ read clockwise from the marked point. The graph is considered up to isotopy rel boundary, and an edge labelled by $A$ is equivalent to the opposite edge labelled by $A^*$. It is possible to write axioms for a pivotal category using invariants and planar graphs rather than morphisms. From this viewpoint, we make the following definition:

**Definition 2.1.1.** A web is a directed planar graph in a disk with a marked point. Its edges are labelled by objects and its vertices are labelled by invariants in the tensor product of the objects labelling the incident edges. A web is only considered up to isotopy rel boundary and up to the the rule that changing the orientation of an edge dualizes the label.

A web is a special case of a ribbon graph [RT90], the difference being that a ribbon graph can also have crossings; a ribbon category is a pivotal category with distinguished morphisms for crossings.
There is a circumstance in which a type of edge in \( w \) can be unoriented. Suppose that \( A \cong A^* \) is self-dual, and suppose further that the isomorphism \( \phi \in \text{Hom}(A,A^*) \) is cyclically invariant if interpreted as an element of \( \text{Inv}(A \otimes A) \). Then an unoriented edge can be defined by a replacement:

\[
\begin{array}{c}
A \\
\text{def} \\
A \rightarrow A
\end{array}
\]

This replacement is necessary for understanding the \( A_1 \) spider as a pivotal category; but see the discussion after Theorem 2.1.2.

A fundamental example of a pivotal category is the category \( \text{vect}(k) \) of finite-dimensional vector spaces over a field \( k \). In this example, a web can be interpreted as the graph of a tensor calculus expression (or a “spin network”). For example, if \( \varepsilon_{abc} \) is a trilinear determinant form on a 3-dimensional vector space \( V \), and if \( \varepsilon^{abc} \) is the dual form on \( V^* \), then the tensor \( \varepsilon_{abc}\varepsilon^{cde} \) (with repeated indices summed) can be drawn as

\[
\begin{array}{c}
  a \\
  \rightarrow \\
  \rightarrow \\
  b \\
  \leftarrow \\
  \leftarrow
\end{array}
\]

with the convention in this case that the vertex labels can be inferred from context. If \( G \) is a group (or a Lie group, Lie algebra, or algebraic group), then \( \text{rep}(G,k) \), the category of finite-dimensional representations (or continuous or algebraic representations) over \( k \) is a pivotal category with a pivotal functor to \( \text{vect}(k) \). For the remainder of the thesis, we let \( G \) be a simple, simply connected algebraic group over \( \mathbb{C} \) (and later we specialize to \( G = \text{SL}_n \)). We study the pivotal category \( \text{rep}(G) = \text{rep}(G,\mathbb{C}) \).

There is a deformation \( \text{rep}_q(G) \) of \( \text{rep}(G) = \text{rep}_1(G) \) that consists of representations of the quantum group \( U_q(\mathfrak{g}) \), when the parameter \( q \) is not a root of unity. (The deformation also exists when \( q \) is a root of unity, but there is more than one standard choice for it.) This deformation is also a pivotal category, although it has no pivotal functor to \( \text{vect} \), because the cup and cap morphisms deform. Even though many ideas in this thesis are clearly related to quantum representations, we concentrate on \( \text{rep}_1(G) \).
We are interested in one more variation of $\text{rep}(G)$. Recall that the irreducible representations $V(\lambda)$ of $G$ are labelled by the set of dominant weights. For a dominant weight $\lambda$, we write $\lambda^*$ for the dominant weight such that $V(\lambda)^* \cong V(\lambda^*)$. From now on, we fix an isomorphism between these two representations. Recall that a dominant weight $\lambda$ is called minuscule if $\langle \alpha^\vee, \lambda \rangle \leq 1$ for every positive coroot $\alpha^\vee$. If $\lambda$ is a minuscule dominant weight, then $V(\lambda)$ is called a minuscule representation. These representations have the special property that all of their weights are in the Weyl orbit of the highest weight. We define $\text{rep}(G)_{\text{min}}$ to be the pivotal subcategory generated by minuscule representations (it is neither an additive nor an abelian category). So the objects of $\text{rep}(G)_{\text{min}}$ are tensor products of minuscule representations. In the case of $G = \text{SL}_n$, $\text{rep}(G)$ can be recovered as the Karoubi envelope of $\text{rep}(G)_{\text{min}}$, although we do not use this reconstruction in this thesis.

The other main pivotal category which we study in this paper is the category of $G^\vee(\mathcal{O})$-equivariant perverse sheaves $\text{perv}(\text{Gr})$ on $\text{Gr}$. We regard the geometric Satake correspondence as an equivalence of pivotal categories between $\text{perv}(\text{Gr})$ and $\text{rep}(G)$. Thus, we are ignoring the (more delicate) “commutativity constraint” or “braiding” on $\text{perv}(\text{Gr})$ which was defined by Ginzburg [Gin] and Mirković-Vilonen [MV07] in two different ways. We are actually more interested in the minuscule analog of $\text{perv}(\text{Gr})$, which we will explore in Section 3.3.

### 2.1.2 Free spiders and presentations

As noted earlier, it is natural to present pivotal categories by generators and relations. If the pivotal category is additive-linear over a ring or a field, then it can presented in the same sense, using linear combinations of words in the generators. In general there are generating objects (or directed edges) and generating morphisms (or invariants or vertices), while the relations are all morphisms. Relations in a pivotal category are also known as planar skein relations.

We now define the free spider $\text{fsp}(G)$ to be the free $\mathbb{C}$-linear pivotal category generated by a directed edge for each minuscule representation of $G$ and a trivalent vertex with inwards
pointing edges for each triple $\lambda, \mu, \nu$ of minuscule dominant weights such that

$$\text{Inv}_G(V(\lambda, \mu, \nu)) \neq 0.$$  

Note that the minuscule condition forces this vector space to be at most one-dimensional. In $\text{fsp}(G)$, we also impose that the dual of the $\lambda$ edge is $\lambda^*$ in the same direction. In [Mor07], $\text{fsp}(\text{SL}_n)$ was denoted $\text{Sym}_n$. The only relations in the free spider are those imposed by the pivotal category: two webs are equal in $\text{fsp}(G)$ if and only if they are isotopic rel boundary and the directed edge corresponding to a representation is equal to the edge corresponding to the dual representation in the opposite direction.

Let us fix $q \in \mathbb{C}$, non-zero and not a root of unity (but possibly equal to 1). There is a pivotal functor

$$\Psi : \text{fsp}(G) \to \text{rep}_q(G)_{\text{min}},$$

which is defined by choosing a non-zero element in each invariant space

$$\text{Inv}_{U_q(G)}(V(\lambda, \mu, \nu)).$$

In particular, for each web $w$ with boundary $\vec{\lambda}$, we obtain an element

$$\Psi(w) \in \text{Inv}_{U_q(G)}(V(\vec{\lambda})).$$

Actually, since webs are a notation for words in any pivotal category, we could say also say that $w$ “is” $\Psi(w)$, or that its value is $\Psi(w)$. But the distinction between $w$ and $\Psi(w)$ is useful for us. The first result is that $\Psi$ is surjective when $G = \text{SL}_n$ [Mor07, Prop. 3.5.8]. (This follows from Weyl’s fundamental theorem of invariant theory.) Thus, the vectors $\Psi(w)$ of webs $w$ span the invariant spaces.

It is an open problem to generate the kernel of $\Psi$ with planar skein relations in $\text{fsp}(G)$. This problem has been solved when $G$ has rank 1 or 2 by the Kuperberg [Kup96]. Kim [Kim03] has conjectured an answer for $\text{SL}_4$ in [Kim03] and Morrison [Mor07] has done so for $\text{SL}_n$. Once these planar skein relations (which must depend on $q$) are determined, then the resulting presented pivotal category can be called a spider and we denote it $\text{spd}_q(G)$. 

We now review the known solutions for $SL_2$ and $SL_3$. The Temperley-Lieb category or $A_1$ spider $spd_q(SL_2)$ is the quotient of $fsp(SL_2)$ by the single relation

\[ \mathbf{= -q - q^{-1}}. \]  

(2.2)

(Since $SL_2$ has a single, self-dual minuscule representation, $fsp(SL_2)$ and $spd_q(SL_2)$ have unoriented edges with a single color or label.) The $A_2$ spider $spd_q(SL_3)$ is the quotient of $fsp(SL_3)$ by the relations

\[ \mathbf{= q^2 + 1 + q^{-2}} \]

\[ \mathbf{= (-q - q^{-1})} \]

\[ \mathbf{= +} \]

(2.3)

(Since $SL_3$ has two minuscule representations which are dual to each other, $fsp(SL_3)$ and $spd_q(SL_3)$ have oriented edges with one label or color. By convention, the edge is labelled by the first fundamental representation $\omega_1$ in the direction that it is oriented.) The other two known spiders, $spd_q(B_2)$ and $spd_q(G_2)$, have similar but more complicated presentations.

**Theorem 2.1.2** (Kauffman [Kau90]). If $q$ is not a root of unity, then $spd_q(SL_2)$ is equivalent to the pivotal category $rep_q(SL_2)_{min}$ of minuscule representations.

In the statement of Theorem 2.1.2, it is necessary to modify $rep_q(SL_2)_{min}$ slightly to make $rep'_q(SL_2)_{min}$. The alteration is to make the minuscule representation $V$ an odd-graded vector space, so that it becomes symmetrically self-dual rather than anti-symmetrically self-dual. This allows its edge in $spd_q(SL_2)$ to be unoriented.

**Theorem 2.1.3.** [Kup96] If $q$ is not a root of unity, then $spd_q(SL_3)$ is equivalent to the pivotal category $rep_q(SL_3)_{min}$ of minuscule representations.

A main property of the spider relations (2.3) is that they are confluent or Gröbner type. In the free pivotal category generated by the generating edges and vertices, each web can be
graded by the number of its faces. Then each relation has exactly one leading term, an elliptic face. (In the $A_2$ spider, a face is elliptic if it has fewer than six sides. In the other two rank 2 spiders, a face is elliptic if the total angle of the corresponding dual vertex is less than $2\pi$, so that the vertex is $\text{CAT}(0)$; see Section 2.2.3.) A web that has that face can be expressed, modulo the relation, as a linear combination of lower-degree webs. The Gröbner property, proved using a diamond lemma, is that any two sequences of simplifications of the same web lead to the same final expression. This means that the webs that cannot be simplified, i.e., the webs without elliptic faces or the non-elliptic webs, form a basis of each invariant space. There is an extended version of this result, but we restrict our attention to the minuscule case, summarized in the following theorem.

**Theorem 2.1.4.** [Kup96] If $\vec{\lambda}$ is a sequence of dominant minuscule weights of $\text{SL}_3$, then the non-elliptic type $A_2$ webs with boundary $\vec{\lambda}$ are a basis of $\text{Inv}(V(\vec{\lambda}))$.

Theorem 1.6.1 implies Theorem 2.1.4 as a corollary. However, it is much more complicated than other proofs of Theorem 2.1.4 [Wes07, KK99].

### 2.2 Affine geometry

#### 2.2.1 Weight-valued metrics and linkages

In the usual definition of a metric space, distances take values in the non-negative real numbers $\mathbb{R}_{\geq 0}$. However, Kapovich, Leeb, Millson [KLM08] have a theory of metric spaces in which distances take values in the dominant Weyl chamber of $G$. Two of the axioms of such a generalized metric space are easy to state:

\[ d(x,x) = 0 \quad d(x,y) = d(y,x)'. \]

The third axiom, the triangle inequality, is different. The main results of Kapovich, Leeb, and Millson are generalized triangle inequalities that are satisfied in buildings and generalized
symmetric spaces. On the one hand, the triangle inequalities in the $A_1$ case are the usual triangle inequality. On the other hand, the inequalities in higher rank cases are decidedly non-trivial.

In this thesis, we adopt the viewpoint of weight-valued metric spaces in order to discuss isometries and distance comparisons. We do not need the generalized triangle inequalities. The definition of an isometry is straightforward. As for distance comparisons, we say that $\mu \leq \lambda$ as a distance if and only if $\mu \leq \lambda$ in the usual partial ordering on dominant weights, namely that $\lambda - \mu$ is a non-negative integer combination of simple roots. Thus, a ball of radius $\lambda$ is then a finite union of spheres of radius $\mu \leq \lambda$. For one construction we define distances that take values in the dominant Weyl chamber, instead of integral weights; and then we say that $\mu \leq \lambda$ when $\lambda - \mu$ is a non-negative real combination of simple roots.

In addition to isometries, we are interested in partial isometries in which only some distances are preserved. For this purpose, we define a linkage to be an oriented graph $\Gamma$ whose edges are labelled by dominant weights. As with webs, an edge labelled by $\lambda$ is equivalent to the opposite edge labelled by $\lambda^*$. Let $\nu(\Gamma)$ be the set of vertices of $\Gamma$. Then one may attempt to define a distance $d(p,q)$ between any two points $p, q \in \nu(\Gamma)$ by taking the shortest total distance of a connecting path. However, since weights are only partially ordered, this minimum may not be unique. We say that $\Gamma$ has coherent geodesics if the minimum distance $\min(d(p,q))$ between any two vertices $p$ and $q$ is unique, and if that minimum distance is the length of the edge $(p,q)$ when $\Gamma$ has that edge. In this case $\Gamma$ can be completed to another linkage $\Gamma_\bar{g}$ which is a complete graph, using all distances as weights.

### 2.2.2 Configuration spaces

Let $X$ be a weight-valued metric space, and let $\Gamma$ be a linkage as in Section 2.2.1. Let $\nu(\Gamma)$ be the set of vertices of $\Gamma$. Then we define the linkage configuration space $Q(\Gamma,X)$ to be the set of maps

$$f : \nu(\Gamma) \to X$$
such that \( d(f(p), f(q)) \) equals the weight of the edge from \( p \) to \( q \), when there is such an edge. If \( X \) and \( \Gamma \) have base points \( t \) and \( b \) respectively, then \( Q(\Gamma, X, b) \) is instead the configuration space of based maps, i.e. maps such that \( f(b) = t \). Another possibility is that \( \Gamma \) has a base edge of length \( \lambda \) and \( X \) has an edge \( e \) labelled \( \lambda \); then \( Q(\Gamma, X, e) \) is again the configuration space of based maps. We are interested in four types of linkages \( \Gamma \):

1. A path or polyline.
2. A cycle or polygon.
3. The 1-skeleton \( \Gamma(D) \) of a tiled diskoid \( D \) (Section 2.2.3) with edges labelled by weights.
4. The complete linkage \( \Gamma_{g}(D) \), if \( \Gamma(D) \) has coherent geodesics.

There is one final type of configuration space that is sometimes useful. If an edge \((p, q)\) has weight \( \lambda \), then we can ask that

\[
d(f(p), f(q)) \leq \lambda
\]

instead of

\[
d(f(p), f(q)) = \lambda.
\]

The result is the contractive configuration space \( Q_{c}(\Gamma, X) \).

Suppose that \( X = G/H \) for some group \( G \) with a subgroup \( H \), and that each sphere \( X(\lambda) \) around the base point is a double coset of \( H \). Let \( \Gamma \) be a linkage and let \( \Gamma_{0} \) be the same linkage with a chosen base point \( 0 \). Then there is a fibration

\[
Q(\Gamma_{0}, X, 0) \longrightarrow Q(\Gamma, X) \longrightarrow X.
\]

Similarly, if \( \Gamma_{e} \) denotes the same linkage with a base edge \( e \) of length \( \lambda \) incident to \( 0 \), then there is also a fibration

\[
Q(\Gamma_{e}, X, e) \longrightarrow Q(\Gamma_{0}, X, 0) \longrightarrow X(\lambda).
\] (2.4)
For $Q(\Gamma, X, e)$, the two base points of $X$ are the original base point and an arbitrary point of $X(\lambda)$. We will drop the base edge or vertex from the notation when the choice is clear.

If $f : \Gamma_2 \rightarrow \Gamma_1$ is a map between linkages, then there is a restriction map,

$$\pi^{\Gamma_1}_{\Gamma_2} : Q(\Gamma_1, X) \rightarrow Q(\Gamma_2, X)$$

(2.5)

between their configuration spaces. This map is of some interest, particularly when $\Gamma_1$ is a sublinkage of $\Gamma_2$ (for example its boundary).

Suppose now that $\Gamma = \Gamma_1 \cup \Gamma_2$, and that $\Gamma_1 \cap \Gamma_2$ is either an edge or a vertex. If we base $\Gamma_2$ (but not $\Gamma_1$) at this intersection, then the configuration space $Q(\Gamma, X)$ is a twisted product:

$$Q(\Gamma, X) = Q(\Gamma_1, X) \times Q(\Gamma_2, X).$$

Informally, $\Gamma_2$ is either an arm attached to $\Gamma_1$ at a point which can swing freely in any direction, or a flap attached to $\Gamma_1$ along a 1-dimensional hinge which can swing freely in the remaining directions.

### 2.2.3 Diskoids

Recall that a piecewise-linear diskoid is a contractible, compact, piecewise-linear subset of the plane. (We do not need diskoids that are not piecewise-linear. But if one were to consider them, the most natural definition could be to make it a planar, cell-like continuum.) Any diskoid $D$ has a polygonal boundary $P$ with a boundary map $P \rightarrow D$, which however is not an inclusion unless $D$ is either a point or a disk. Figure 2.1 shows an example of a diskoid $D$ with its boundary $P$.

Note that since a diskoid comes with an embedding in the plane, its boundary $P$ is implicitly oriented, so that the edges of $P$ are cyclically ordered. We assume a clockwise orientation in this thesis. Trees are diskoids, and Figure 1.1 has an example of the polygonal boundary of a tree; the polygon traverses each edge twice.

A diskoid $D$ can be tiled by polygons. Formally, a tiling of $D$ is a piecewise-linear CW complex structure on $D$ with embedded 2-cells. If $D$ is decorated in this way, then we define
the graph $\Gamma(D)$ to be its 1-skeleton. Then, as above, $\Gamma(D)$ can be made into a linkage, which means, explicitly, that the edges of $D$ are labelled by distances. In this thesis we do not need to the label the faces (or 2-cells) of a tiled diskoid to define its configuration space, but only because the corresponding representation theory is multiplicity-free. In future work, the faces could also be labelled in order to define more restrictive configuration spaces. We write $Q(D)$ for $Q(\Gamma(D))$ and $Q_g(D)$ for $Q(\Gamma_g(D))$.

Our interest in diskoids arises from the fact that they are geometrically dual to webs. As in the introduction, let $w$ be a web in $\mathfrak{fsp}(G)$ with boundary $\vec{\lambda}$. Then it has a dual diskoid $D = D(w)$, and with a natural base point. To be precise, $D$ has a vertex for every internal or external face of $w$; two vertices are connected by an edge when the faces of $w$ are adjacent; and there is a triangle glued to three edges whenever the dual edges of $w$ meet at a vertex. We label the edges of $D$ using the labels of the corresponding edges of $w$; also, if an edge of $w$ is oriented, we transfer it to an orientation of the dual edge of $D$ by rotating it counterclockwise. As a result, the boundary of the diskoid $D$ is the polygon $P(\vec{\lambda})$.

The following is an example of an $A_1 = \text{SL}_2$ web and its dual diskoid, which in the $A_1$ case is always a tree.

In the $A_2 = \text{SL}_3$ case we see the dual diskoid, which happens to be a disk because the
corresponding web is connected.

In this construction, $D$ is always triangulated because $w$ is always trivalent. The vertices of $D$ are a weight-valued metric space, and by linear extension the whole of $D$ is a Weyl-chamber-valued metric space. We can also simplify this metric to an ordinary metric space by taking the Euclidean length of the vector-valued distance. Finally, suppose that $w$ is an $A_2$ web (or a $B_2$ or $G_2$ web). Then $w$ is non-elliptic if and only if $D$, in its ordinary metric, is CAT(0) in the sense of Gromov [Gro87]. This follows from the fact that $D$ is contractible and the condition that all complete angles in $D$ are at least $2\pi$.

### 2.2.4 Affine Grassmannians and buildings

As before, let $G$ be a simple, simply-connected complex algebraic group and let $G^\vee$ be its Langlands dual group. Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series over $\mathbb{C}$ and let $\mathcal{K} = \mathbb{C}(t)$ be its fraction field. Then

$$\text{Gr} = \text{Gr}(G^\vee) = G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$$

is the **affine Grassmannian** for $G^\vee$ with residue field $\mathbb{C}$. It is an ind-variety over $\mathbb{C}$, meaning that it is a direct limit of algebraic varieties (of increasing dimension). The affine Grassmannian Gr is also a weight-valued metric space: The double cosets $G^\vee(\mathcal{O})\backslash G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ are bijective with the cone $\Lambda_+$ of dominant coweights of $G^\vee$, which is the same as the cone of dominant weights of $G$. More precisely, for each coweight $\mu$ of $G^\vee$, there is an associated point $t^\mu$ in the affine Grassmannian. If $p, q$ are two arbitrary points of the affine Grassmannian, then we can find $g \in G^\vee(\mathcal{K})$ such that $gp = t^0$ and $gq = t^\mu$ for some unique dominant coweight $\mu$. Under this circumstance, we write $d(p, q) = \mu$. So the action of $G^\vee(\mathcal{K})$ preserves distances and $d(t^0, t^\mu) = \mu$ for any dominant weight $\mu$. 
The affine Grassmannian $\text{Gr}$ is also a subset of the vertices $\text{Gr}' = v(\Delta)$ of an associated simplicial complex called an *affine building* $\Delta = \Delta(G^\vee)$ [Ron09] whose type is the extended Dynkin type of $G^\vee$. The simplices of this affine building are given by parahoric subgroups of the affine Kac-Moody group $\widehat{G}^\vee$. For a detailed description of affine buildings from this perspective, see [GL05].

An affine building $\Delta$ satisfies the following axioms:

1. The building $\Delta$ is a non-disjoint union of *apartments*, each of which is a copy of the Weyl alcove simplicial complex of $G^\vee$.

2. Any two simplices of $\Delta$ of any dimension are both contained in at least one apartment $\Sigma$.

3. Given two apartments $\Sigma$ and $\Sigma'$ and two simplices $\alpha, \alpha' \in \Sigma \cap \Sigma'$, there is an isomorphism $f : \Sigma \rightarrow \Sigma'$ that fixes $\alpha$ and $\alpha'$ pointwise.

The axioms imply that the vertices of $\Delta$, denoted $\text{Gr}'$, are canonically colored by the vertices of the extended Dynkin diagram $\hat{I} = I \cup \{0\}$ of $G^\vee$, or equivalently the vertices of the standard Weyl alcove $\delta$ of $G^\vee$. Moreover, every maximal simplex of $\Delta$ is a copy of $\delta$; it has exactly one vertex of each color. The affine Grassmannian consists of those vertices colored by 0 and by minuscule nodes of the Dynkin diagram of $G^\vee$.

The axioms also imply that $v(\Delta)$, and more generally the realization $|\Delta|$ of $\Delta$, have a metric taking values in Weyl chamber. (But not necessarily integral weights as one sees in $\text{Gr}$.) Namely, if $p, q \in |\Delta|$, then $p, q \in |\Sigma|$ for an apartment $\Sigma$, and after a suitable automorphism $p = q + \lambda$ for some vector $\lambda$ in the dominant Weyl chamber. We then define $d(p, q) = \lambda$. (The metric has coherent geodesics, and it extends the metric defined above for $\text{Gr}$.) We need the following fact.

**Lemma 2.2.1.** If $p, q \in |\Delta|$, then every geodesic path $\gamma$ from $p$ to $q$ is contained in every apartment $\Sigma$ such that $p, q \in |\Sigma|$.

A subtle feature of the above affine building $\Delta$ is that it has two very different geometries. As an ordinary simplicial complex, its vertex set $\text{Gr}'$ is discrete, and $\text{Gr}'$ has a combinatorial,
weight-valued metric. The vertex set \( \text{Gr}' \) is also naturally an algebraic ind-variety over \( \mathbb{C} \), as is the set of vertices of any given color or the set of simplices of \( \Delta \) of any given type. This second geometry endows \( \text{Gr}' \) with both a Zariski topology and an analytic topology. Among the relations between these two geometries, the following fact is important.

**Proposition 2.2.2.** The algebraic-geometric closure \( \overline{\text{Gr}'(\lambda)} \) of the sphere \( \text{Gr}'(\lambda) \) of radius \( \lambda \) is the set of all points in the metric ball of radius \( \lambda \) that have the same color as \( \lambda \).

An affine building \( \Delta \) has a third geometry which is related to the weight-valued metric but is not the same. Namely, we can give the Weyl alcove \( \delta \) its standard Euclidean structure, and consider the induced metric on the realization \( |\Delta| \) of \( \Delta \). This locally Euclidean metric can also be defined as \( ||d(p,q)||_2 \), where \( d(p,q) \) is the weight-valued metric on \( |\Delta| \).

**Theorem 2.2.3** (Bruhat-Tits [BT72]). *Every affine building is a CAT(0) space with respect to its locally Euclidean metric.*

If \( G = \text{SL}_n \) and thus \( G^\vee = \text{PGL}_n \), then \( \text{Gr} = \text{Gr}' \), and there is a simple description of \( \Delta \). Namely, a finite set of vertices in \( \text{Gr} \) subtends a simplex if and only if the distances between them are all minuscule.

Finally, to close a circle, let \( L(\vec{\lambda}) \) be a polyline whose sides are labelled by

\[
\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n),
\]

based at the first vertex. Let \( P(\vec{\lambda}) \) be the corresponding polygon, based at the vertex between \( \lambda_n \) and \( \lambda_1 \). Then the contractive polyline configuration space

\[
\text{Gr}(\vec{\lambda}) = Q_c(L(\vec{\lambda}), \text{Gr})
\]

is the domain of the convolution morphism. The restriction map coming from the projection onto the boundary \( L(\vec{\lambda}) \rightarrow \text{pt} \), or

\[
\pi^{L(\vec{\lambda})}_{\text{pt}} : Q_c(L(\vec{\lambda}), \text{Gr}) \rightarrow \text{Gr},
\]
is the convolution morphism. In keeping with the standard notation, we denote it by

\[ m_\lambda = \pi_{pt}^{L(\lambda)} \]

Meanwhile the contractive polygon configuration space

\[ Q_c(P(\lambda), \text{Gr}) = F(\lambda) = m_\lambda^{-1}(t^0) \]

is the Satake fiber. As another bit of notation, if \( \Gamma \) is a linkage, we elide the Gr and write \( Q(\Gamma) \) for \( Q(\Gamma, \text{Gr}) \), etc.
Chapter 3

Geometric Satake for tensor products of minuscule representations

3.1 Minuscule paths and components of Satake fibres

The full geometric Satake correspondence, Theorem 1.2.1, simplifies considerably when the weights are minuscule. In this special case, Haines [Hai06, Thm. 3.1] showed that all components of $F(\bar{\lambda})$ are of maximal dimension. We can use his ideas to give an explicit description of these components using minuscule paths. In addition to previous notation, let $W$ be the Weyl group of $G$.

Let $\lambda$ be a minuscule dominant weight. Then there are no dominant weights less than $\lambda$, so the sphere of radius $\lambda$ equals the ball of radius $\lambda$. Hence the sphere $\text{Gr}(\lambda)$ is closed in the algebraic geometry of $\text{Gr}$ by Proposition 2.2.2, and thus it is projective and smooth. In fact, $G^\vee$ acts transitively on $\text{Gr}(\lambda)$. The stabilizer of $t^\lambda$ is $M(\lambda)$, the opposite maximal proper parabolic subgroup corresponding to the minuscule weight $\lambda$. Thus $\text{Gr}(\lambda)$ is isomorphic to the partial flag variety $G^\vee / M(\lambda)$.

More generally, if $\Gamma$ is a minuscule linkage, meaning that all of its edges are minuscule,
then
\[ Q(\Gamma) = Q_c(\Gamma) = \overline{Q(\Gamma)}. \]

Let
\[ \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \]
be a sequence of minuscule dominant weights. A *minuscule path* (ending at 0) of type \( \vec{\lambda} \) is a sequence of dominant weights
\[ \vec{\mu} = (\mu_0, \mu_1, \mu_2, \ldots, \mu_n) \]
such that \( \mu_k - \mu_{k-1} \in W_{\lambda_k} \) for every \( k \), and such that
\[ \mu_0 = \mu_n = 0. \]

In other words, the \( k \)th step of the path \( \vec{\mu} \) is a weight of \( V(\lambda_k) \), and the path is restricted to the dominant Weyl chamber \( \Lambda_+ \). Minuscule paths are a special case of Littelmann paths [Lit95], but it was much earlier folklore knowledge that the number of minuscule paths of type \( \vec{\lambda} \) is the dimension of \( \text{Inv}(V(\vec{\lambda})) \). (See Humphreys [Hum72, Ex. 24.9], and use induction.)

![Figure 3.1: The fan diskoid \( A(\vec{\lambda}, \vec{\mu}) \).](image)

Let \( P_{\vec{\lambda}}(\nu) \) be the complete set of minuscule paths of type \( \vec{\lambda} \) ending the weight \( \nu \). Since we usually speak of paths ending at 0, we use the short hand \( P_{\vec{\lambda}} = P_{\vec{\lambda}}(0) \).
Given $\vec{\mu} \in P_\vec{\lambda}$, we define a based diskoid $A(\vec{\lambda}, \vec{\mu})$ in the shape of a fan, whose the boundary is the polygon $P(\vec{\lambda})$ and whose ribs are labelled by $\vec{\mu}$, as in Figure 3.1. Then there is a natural inclusion

$$Q(A(\vec{\lambda}, \vec{\mu})) \subseteq F(\vec{\lambda}).$$

The result of Theorem 1.2.3 is implicit in the work of Haines [Hai06], but we give the following explicit construction:

**Theorem 3.1.1.** For each $\vec{\mu} \in P_\vec{\lambda}$, the fan configuration space $Q(A(\vec{\lambda}, \vec{\mu}))$ is a dense subset of one component of $F(\vec{\lambda})$. The induced correspondence is a bijection between minuscule paths and components of $F(\vec{\lambda})$.

The key to the proof of this theorem is the following lemma.

**Lemma 3.1.2.** Let

$$T_e(\mu, \lambda, \nu) = \mu \rightarrow \lambda \rightarrow \nu$$

be a triangle with a minuscule edge $\lambda$, based at the edge $e$ of length $\mu$. Then $Q(T_e(\mu, \lambda, \nu))$ is non-empty if there exists $w \in W$ such that $\mu + w\lambda = \nu$. If it is non-empty, then it is smooth and has complex dimension $\langle \nu - \mu + \lambda, \rho \rangle$.

**Proof.** Let $W_\mu$ denote the stabilizer of $\mu$ in the Weyl group. It is a parabolic subgroup of $W$.

Let us choose the base edge in Gr to be the edge connecting $t^{-\mu}$ and $t^0$. Then the edge based configuration space $Q(T_e(\mu, \lambda, \nu))$ is a subvariety of $\text{Gr}(\lambda)$ since there is only one free vertex. In fact

$$Q(T_e(\mu, \lambda, \nu)) = \{ p \in \text{Gr}(\lambda) | d(t^{-\mu}, p) = \nu \}. $$

Let $A$ denote the set $W/W_\lambda$, which we regard as a poset using the opposite Bruhat ordering. With this ordering, $A$ becomes the poset of $B$-orbits on $\text{Gr}(\lambda) = G^\vee / M(\lambda)$, where $B$ is the Borel subgroup of $G$. We are interested in the action of $W_\mu$ on $A$ by left multiplication. The quotient $W_\mu \backslash A$ is the set of $M_+(\mu)$ orbits on $\text{Gr}(\lambda)$, where $M_+(\mu) = \text{Stab}_{G^\vee}(t^{-\mu})$ is the parabolic subgroup corresponding to the minuscule weight $\mu$. 
Hence we can write any point \( p \) of \( \text{Gr}(\lambda) \) as 

\[ p = g r^{a \lambda} \]

where \( g \in M_+(\mu) \) and \( a \in A \) is chosen to be a maximal representative under the opposite Bruhat order for the orbit of \( W_\mu \). The action of \( M_+(\mu) \) on \( \text{Gr} \) stabilizes \( t^{-\mu} \) so

\[ d(t^{-\mu}, g r^{a \lambda}) = d(t^{-\mu}, t^{a \lambda}) = d(t^0, t^{\mu + a \lambda}). \]

Now, we claim that \( \mu + a \lambda \) is always dominant. Let us write \( a = [w] \) for \( w \in W \). We must check that

\[ \langle \mu + w \lambda, \alpha_i^\vee \rangle = \langle \mu, \alpha_i^\vee \rangle + \langle \lambda, w^{-1} \alpha_i^\vee \rangle \geq 0 \]

for all simple coroots \( \alpha_i^\vee \). We break this calculation into two cases.

First, suppose that \( s_i \mu = \mu \). Then \( \langle \mu, \alpha_i^\vee \rangle = 0 \). On the other hand \( s_i w > w \), since \( a \) is minimal in the usual Bruhat order in the \( W_\mu \)-orbit. This implies that \( w^{-1} \alpha_i^\vee \) is a positive coroot, which implies that \( \langle \lambda, w^{-1} \alpha_i^\vee \rangle \) is non-negative (since \( \lambda \) is dominant). Hence

\[ \langle \mu, \alpha_i^\vee \rangle + \langle \lambda, w^{-1} \alpha_i^\vee \rangle \geq 0. \]

Next, suppose that \( s_i \mu \neq \mu \). Then since \( \mu \) is dominant, \( \langle \mu, \alpha_i^\vee \rangle \geq 1 \). On the other hand,

\[ |\langle \lambda, w^{-1} \alpha_i^\vee \rangle| \leq 1 \] since \( w \alpha_i^\vee \) is a coroot and \( \lambda \) is minuscule. Hence

\[ \langle \mu, \alpha_i^\vee \rangle + \langle \lambda, w \alpha_i^\vee \rangle \geq 0 \]

in this case as well.

Since \( \mu + a \lambda \) is always dominant, we conclude that

\[ d(t^{-\mu}, g r^{a \lambda}) = \mu + a \lambda. \]

Hence, \( Q(T_e(\mu, \lambda, \nu)) \) is non-empty iff there exists \( w \in W \) such that \( \mu + w \lambda = \nu \). (The above argument shows that \( [w] \) is necessarily a maximal length representative for the \( W_\mu \) action on \( A \).)

If such \( w \) exists, then the configuration space \( Q(T_e(\mu, \lambda, \nu)) \) is simply the \( M(\mu) \)-orbit through \( t^{w \lambda} \). Hence it is smooth and its dimension is given by the length of \( [w] \) in \( A \) because it is of the same dimension as the \( B \)-orbit through \( t^{w \lambda} \). Since \( \lambda \) is minuscule, this equals \( \langle w \lambda + \lambda, \rho \rangle \) as desired. \( \square \)
Proof of Theorem 3.1.1. It is easy to show by induction that the fan configuration space

\[ Q(A(\vec{\lambda}, \vec{\mu})) = Q(P_e(\mu_0, \lambda, \mu_1)) \times \cdots \times Q(P_e(\mu_{n-1}, \lambda_n, \mu_n)) \]

is an iterated twisted product of triangle configuration spaces. Since each factor has a minuscule edge, Lemma 3.1.2 tells us that \( Q(A(\vec{\lambda}, \vec{\mu})) \) is also a smooth variety. Moreover, the dimensions add to tell us that

\[ \dim \mathbb{C} Q(A(\vec{\lambda}, \vec{\mu})) = \langle \lambda_1 + \cdots + \lambda_n, \rho \rangle = \dim \mathbb{C} F(\vec{\lambda}). \]

On the other hand, \( F(\vec{\lambda}) = Q(P(\vec{\lambda})) \) is partitioned as a set by the subvarieties \( Q(A(\vec{\lambda}, \vec{\mu})) \), simply by taking the distances between the vertices of \( P(\vec{\lambda}) \) and the origin. If \( X \) is any algebraic variety with an equidimensional partition into smooth varieties \( X_1, \ldots, X_N \), then \( X \) has pure dimension and its components are the closures of the parts \( X_k \). In our case, \( X = F(\vec{\lambda}) \). \( \square \)

It will be convenient later to abbreviate the dimension of \( F(\vec{\lambda}) \) as:

\[ d(\vec{\lambda}) \overset{\text{def}}{=} \langle \lambda_1 + \cdots + \lambda_n, \rho \rangle = \dim \mathbb{C} F(\vec{\lambda}). \]

The same integers also arise in a different dimension formula:

\[ \dim \mathbb{C} \text{Gr}(\vec{\lambda}) = 2d(\vec{\lambda}). \]

(Indeed, \( \text{Gr}(\vec{\lambda}) \) is a top-dimensional component of \( F(\vec{\lambda} \sqcup \vec{\lambda}^*) \), given by collapsing the polygon \( P(\vec{\lambda} \sqcup \vec{\lambda}^*) \) onto the polyline \( L(\vec{\lambda}) \).)

Another important corollary of Lemma 3.1.2 is the following:

**Theorem 3.1.3.** Suppose that \( D \) is a diskoid with boundary \( \vec{\lambda} \) with no internal vertices, and suppose that all edges of \( D \) (including the terms of \( \vec{\lambda} \)) are minuscule. Then \( Q(D) \) is smooth and projective, and therefore a single component of \( F(\vec{\lambda}) \).

**Proof.** Let \( T_e(\mu, \lambda, \nu) \) be a triangle of \( D \) with three minuscule edges, and let the base edge \( e \) be any of the edges. Then by Lemma 3.1.2, \( Q(T_e(\mu, \lambda, \nu)) \) is smooth. Likewise \( T_p(\mu, \lambda, \nu) \), based
at a point \( p \) instead, is smooth. By construction, \( Q(D) \) is a twisted product of configuration spaces of this form, so it is also smooth. It is also projective since \( D \) is a minuscule linkage.

There is one delicate point in the inference that \( Q(D) \) is a component of \( F(\tilde{\lambda}) \): Is the restriction map \( Q(D) \rightarrow F(\tilde{\lambda}) \) injective? As in the proof of Lemma 3.1.2, the restriction map
\[
\pi : Q(T_e(\mu, \lambda, \nu)) \rightarrow \text{Gr}(\lambda)
\]
is injective, and so is the restriction map
\[
\pi : Q(T(\mu, \lambda, \nu)) \rightarrow \text{Gr}(\mu, \lambda).
\]
The diskoid \( D \) must have a triangle with at least two edges on the boundary, so by induction its restriction map to \( F(\tilde{\lambda}) \) is also injective.

### 3.2 A homological state model

The pivotal category isomorphism of Theorem 1.2.1 is in spirit a type of state model or counting model to evaluate webs in \( \text{rep}(G) \). If \( w \) is a web with dual diskoid \( D \), then there is a map of linkages
\[
P(\tilde{\lambda}) = \partial D \rightarrow \Gamma(D)
\]
given by the inclusion of the boundary. This gives rise to a restriction map
\[
\pi = \pi_{P(\tilde{\lambda})}^{\Gamma(D)} : Q(D) \rightarrow F(\tilde{\lambda}).
\]
A point in \( Q(D) \) is a “state” of \( D \) in the sense of mathematical physics, in which each vertex of \( D \) (or each face of \( w \)) is assigned an element of \( \text{Gr} \). We would like to count the number of states of \( D \) with some fixed boundary, or in other words the cardinality of a diskoid fiber \( \pi^{-1}(f) \) for \( f \in F(\tilde{\lambda}) \). If \( f \) is chosen generically in a top-dimensional component of \( F(\tilde{\lambda}) \), then optimistically this cardinality will be the coefficient of \( \Psi(w) \) in the Satake basis.

However, this sketch is naive. The diskoid fiber \( \pi^{-1}(f) \) often has a complicated geometry for which it is hard to define “counting”. The first and main solution for us is to replace
counting by a homological intersection. In particular, for each web $w$, we give a description of a homology class $c(w) \in H_{\text{top}}(Q(D))$ such that $\pi_*(c(w))$ equals $\Psi(w)$.

### 3.3 The homology convolution category

If $M$ is an algebraic variety over $\mathbb{C}$, we consider its intersection cohomology sheaf $IC_M$ as a simple object in the category of perverse sheaves on $M$. If $M$ is smooth, then $IC_M$ is isomorphic to $\mathbb{C}[\dim M]$, the constant sheaf shifted by the complex dimension of $M$. For brevity, we write this perverse sheaf as $\mathbb{C}[M]$.

The geometric Satake correspondence is a tensor functor that takes the usual product on $\text{rep}(G)$ to the convolution tensor product on $\text{perv}(\text{Gr})$. In particular, the tensor product $V(\tilde{\lambda})$ of irreducible minuscule representations corresponds to the convolution tensor product of the simple perverse sheaves $\mathbb{C}[\text{Gr}(\lambda_i)]$ on minuscule spheres, which are closed in the algebraic geometry. By definition this convolution tensor product is given by the pushforward $(m_{\tilde{\lambda}})_*(\mathbb{C}[\text{Gr}(\tilde{\lambda})])$ along the convolution morphism.

Let $\text{perv}(\text{Gr})_{\text{min}}$ denote the subpivotal category of $\text{perv}(\text{Gr})$ consisting of such pushforwards. By construction, $\text{perv}(\text{Gr})_{\text{min}}$ is equivalent to $\text{rep}(G)_{\text{min}}$. Following ideas of Ginzburg, our goal is to study $\text{perv}(\text{Gr})_{\text{min}}$ using convolutions in homology. We begin by reviewing some generalities, following [CG97, Sec. 2.7].

Let $\{M_i\}$ be a set of connected, smooth complex varieties and let $M_0$ be a possibly singular, stratified variety with strata $\{U_\alpha\}$. For each $i$, let $\pi_i : M_i \to M_0$ be a proper semismall map. In this context, the statement that $\pi_i$ is semismall means that $\pi_i$ restricts to a fiber bundle over each stratum $U_\alpha$ and that the dimensions of these fibers is given by

$$\dim_{\mathbb{C}} \pi_i^{-1}(u) = \frac{\dim_{\mathbb{C}} M_i - \dim_{\mathbb{C}} U_\alpha}{2}$$

for $u \in U_\alpha$. Let $d_i = \dim_{\mathbb{C}} M_i$.

With this setup, let $Z_{ij} = M_i \times_{M_0} M_j$. The semismallness condition implies that $\dim_{\mathbb{C}} Z_{ij} =$
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Let \( d_i + d_j \). Let

\[
H_{\text{top}}(Z_{ij}) = H_{2d_i+2d_j}(Z_{ij})
\]

be the top homology of \( Z_{ij} \). If the \( M_i \) are proper, which they are in our situation, then we will obtain a valid definition of the convolution product using the ordinary singular homology of \( Z_{ij} \). (Otherwise the correct type of homology would be Borel-Moore homology.)

Define a homological convolution product

\[
* : H_{\text{top}}(Z_{ij}) \otimes H_{\text{top}}(Z_{jk}) \to H_{\text{top}}(Z_{ik})
\]

by the formula

\[
c_1 * c_2 = (\pi_{ik})_* (\pi_{ij}^*(c_1) \cap \pi_{jk}^*(c_2)),
\]

where “\( \cap \)” denotes the intersection product (with support), relative to the ambient smooth manifold \( M_i \times M_j \times M_k \). This may be defined using the cup product in cohomology via Poincaré duality. For more details about this construction, see [CG97, Sec. 2.6.15] or [Ful98, Sec. 19.2].

Note that because

\[
\dim \mathbb{C} Z_{ij} = d_i + d_j = \frac{\dim \mathbb{C} M_i + \dim \mathbb{C} M_j}{2},
\]

the correct homological degree is preserved by the convolution product.

This construction is relevant for us because of a theorem of Ginzburg that relates \( H_{\text{top}}(Z_{ij}) \) to morphisms in the category \( \text{perv}(M_0) \) of perverse sheaves on \( M_0 \).

**Theorem 3.3.1.** [CG97, Thm. 8.6.7] With the above setup, there is an isomorphism

\[
H_{\text{top}}(Z_{ij}) \cong \text{Hom}_{\text{perv}(M_0)} \left( (\pi_i)_* \mathbb{C}[M_i], (\pi_j)_* \mathbb{C}[M_j] \right).
\]

This isomorphism identifies convolution products on the left side with compositions of morphisms on the right side.

We apply this setup by letting \( M_0 = \text{Gr} \) and by letting each \( M_i \) be \( \text{Gr}(\vec{\lambda}) \) for a sequence \( \vec{\lambda} \) of dominant minuscule weights. The convolution morphism \( m_{\vec{\lambda}} : \text{Gr}(\vec{\lambda}) \to \text{Gr} \) is semismall. (See [MV07, Lem. 4.4]; it also follows from the proof of Theorem 3.1.1.) Then \( Z_{ij} \) becomes

\[
Z(\vec{\lambda}, \vec{\mu}) = \text{Gr}(\vec{\lambda}) \times_{\text{Gr}} \text{Gr}(\vec{\mu}) = Q(P(\vec{\lambda}^* \sqcup \vec{\mu})),
\]
where $P(\vec{\lambda}^* \sqcup \vec{\mu})$ is this polygon:

$$P(\vec{\lambda}^* \sqcup \vec{\mu}) = \begin{array}{c}
\vec{\lambda} \\
\vec{\mu}
\end{array}$$

Theorem 3.3.1 motivates the following construction of a category $\text{hconv}(\text{Gr})$. The objects in $\text{hconv}(\text{Gr})$ are the polyline varieties $\text{Gr}(\vec{\lambda})$, where $\vec{\lambda}$ is a sequence of minuscule weights. The tensor product on objects is, by definition, given by convolution on objects, so

$$\text{Gr}(\vec{\lambda}) \otimes \text{Gr}(\vec{\mu}) \overset{\text{def}}{=} \text{Gr}(\vec{\lambda} \sqcup \vec{\mu}),$$

where $\sqcup$ denotes concatenation of sequences. So the identity object is the point $\text{Gr}(\emptyset)$. Finally the dual object $\text{Gr}(\vec{\lambda})^* = \text{Gr}(\vec{\lambda}^*)$ of $\text{Gr}(\vec{\lambda})$ is given by reversing $\vec{\lambda}$ and taking the dual of each of its terms.

We define the morphism spaces of $\text{hconv}(\text{Gr})$ as

$$\text{Hom}_{\text{hconv}(\text{Gr})}(\text{Gr}(\vec{\lambda}), \text{Gr}(\vec{\mu})) \overset{\text{def}}{=} H_{\text{top}}(Z(\vec{\lambda}, \vec{\mu})).$$

The composition of morphisms is given by the convolution product. Note that the identity morphism $1_{\vec{\lambda}} \in H_{\text{top}}(Z(\vec{\lambda}, \vec{\lambda}))$ is given by the class $[\text{Gr}(\vec{\lambda})_\Delta]$ of the diagonal

$$\text{Gr}(\vec{\lambda})_\Delta \subseteq Z(\vec{\lambda}, \vec{\lambda}) \subseteq \text{Gr}(\vec{\lambda}) \times \text{Gr}(\vec{\lambda}),$$

i.e., it is the configurations in which the polygon $P(\vec{\lambda}^* \sqcup \vec{\lambda})$ has collapsed onto the polyline $L(\vec{\lambda})$.

To describe the tensor structure on morphisms, it is enough to describe how to tensor with the identity morphism. So let $\vec{\lambda}, \vec{\mu}, \vec{\nu}$ be three sequences of dominant minuscule weights and let $c \in H_{\text{top}}(Z(\vec{\mu}, \vec{\nu}))$. Our goal is to construct a class

$$1_{\vec{\lambda}} \otimes c \in H_{\text{top}}(Z(\vec{\lambda} \sqcup \vec{\mu}, \vec{\lambda} \sqcup \vec{\nu}))$$
For the moment, let $\Gamma$ be a $\rho$-shaped graph with a tail of type $\vec{\lambda}$ and a loop of type $\vec{\mu}^* \sqcup \vec{v}$, based at the end of the tail:

Let $X = Q(\Gamma)$ be its based configuration space. We describe two fibration constructions related to $X$. First, there is a restriction map

$$\pi_{L(\vec{\lambda} \sqcup \vec{\mu})}^{L(\vec{\lambda})} : \text{Gr}(\vec{\lambda} \sqcup \vec{\mu}) \to \text{Gr}(\vec{\lambda}) \times \text{Gr}$$

given by restricting to the polyline $L(\vec{\lambda})$ and the free endpoint of $L(\vec{\lambda} \sqcup \vec{\mu})$. Then $X$ is the fibered product

$$X = \text{Gr}(\vec{\lambda} \sqcup \vec{\mu}) \times_{\text{Gr}(\vec{\lambda}) \times \text{Gr}} \text{Gr}(\vec{\lambda} \sqcup \vec{v}).$$

Second, there is a projection

$$\pi^\Gamma_{L(\vec{\lambda})} : X \to \text{Gr}(\vec{\lambda})$$

given by restricting from $\Gamma$ to $L(\vec{\lambda})$. The fibers of this projection are $Z(\vec{\mu}, \vec{v})$.

Since $\text{Gr}(\vec{\lambda})$ is simply connected, we get an isomorphism

$$H_{\text{top}}(X) \cong H_{\text{top}}(\text{Gr}(\vec{\lambda})) \otimes H_{\text{top}}(Z(\vec{\mu}, \vec{v}))$$

and thus we obtain an isomorphism

$$H_{\text{top}}(Z(\vec{\mu}, \vec{v})) \xrightarrow{\cong} H_{\text{top}}(X)$$

given by $c \mapsto [\text{Gr}(\vec{\lambda})] \otimes c$.

There is also an inclusion

$$i = \pi^\Gamma_{P(\vec{\lambda} \sqcup \vec{\mu} \sqcup \vec{v}^*)} : X \to Z(\vec{\lambda} \sqcup \vec{\mu}, \vec{\lambda} \sqcup \vec{v}),$$
using the polygon which travels twice along the tail of $\Gamma$ and around the loop of $\Gamma$. Combining all this structure, we define

$$1_\lambda \otimes c \overset{\text{def}}{=} \iota_* (c \otimes [\text{Gr}(\bar{\lambda})]).$$

Tensoring by the identity morphism on the other side is similar and we leave the construction to the reader.

Finally, to define the cap and cup morphisms for any $\bar{\lambda}$, we define them for a single minuscule weight $\lambda$. Note that

$$Z(\lambda \sqcup \lambda^*, \emptyset) = Z(\emptyset, \lambda \sqcup \lambda^*) = F(\lambda, \lambda^*) \cong \text{Gr}(\lambda).$$

We define the cup $b_\lambda$ and the cap $d_\lambda$ to each be the class $[\text{Gr}(\bar{\lambda})]$ in their respective hom spaces.

**Theorem 3.3.2.** There is an equivalence of pivotal categories

$$\text{hconv}(\text{Gr}) \cong \text{perv}(\text{Gr})_{\text{min}}.$$

Combining this with the geometric Satake equivalence, we obtain an equivalence of pivotal category $\text{rep}(G)_{\text{min}} \cong \text{hconv}(\text{Gr})$. Applying this to invariant spaces, we obtain an isomorphism

$$\text{Inv}(V(\bar{\lambda})) \cong \text{Hom}_{\text{hconv}(\text{Gr})}([\text{Gr}(\emptyset), \text{Gr}(\bar{\lambda})])$$

$$= H_{\text{top}}(Z(\emptyset, \bar{\lambda})) = H_{\text{top}}(F(\bar{\lambda})), $$

which is Theorem 1.2.2.

**Proof.** By the definition, the objects in both categories are parameterized by sequences $\bar{\lambda}$, so the functor on objects is very simple. On morphisms, the functor is given by the isomorphisms from Theorem 3.3.1. By this theorem, the functor is fully faithful and is compatible with composition on both sides. *(I.e., it is a functor.*) To complete the proof this theorem, we need only to show that the functor is compatible with the tensor product and with pivotal duality.
To see that it is compatible with the tensor product, we use the same notation as above. If
\[ c \in \text{Hom}((m_\vec{\mu})_* \mathbb{C}[\text{Gr}(\vec{\mu})], (m_\vec{\nu})_* \mathbb{C}[\text{Gr}(\vec{\nu})]), \]
then with respect to the tensor structure in \( \text{perv}(\text{Gr}) \), \( I((m_\vec{\mu})_* \mathbb{C}[\text{Gr}(\vec{\mu})] \otimes c \) is given by the image of \( c \) under the map
\[
\text{Hom}_{\text{perv}(\text{Gr})} \left( (m_\vec{\mu})_* \mathbb{C}[\text{Gr}(\vec{\mu})], (m_\vec{\nu})_* \mathbb{C}[\text{Gr}(\vec{\nu})] \right) \xrightarrow{\cong} \text{Hom}_{\text{perv}(\text{Gr} \times \text{Gr})} \left( (\pi^{L(\lambda \sqcup \mu)})_* \mathbb{C}[\text{Gr}(\lambda \sqcup \mu)], (\pi^{L(\lambda \sqcup \nu)})_* \mathbb{C}[\text{Gr}(\lambda \sqcup \nu)] \right) \xrightarrow{p_*} \text{Hom}_{\text{perv}(\text{Gr})} \left( (\pi_{\lambda \sqcup \mu})_* \mathbb{C}[\text{Gr}(\lambda \sqcup \mu)], (m_{\lambda \sqcup \nu})_* \mathbb{C}[\text{Gr}(\lambda \sqcup \nu)] \right).
\]
Here \( p : \text{Gr}(\lambda \times \text{Gr}) \to \text{Gr} \) is the projection onto the second factor. This is easily seen to match our above definition.

To check compatibility with the pivotal duality, we must show that the cap and cup morphisms are preserved by the functor. It suffices to check this for the simple objects of our category. Hence we need to check that the cap morphism
\[ \mathbb{C}[\text{Gr}(\emptyset)] \to \mathbb{C}[\text{Gr}(\lambda, \lambda^*)] \]
is given by \([F(\lambda, \lambda^*)] \in H_{\text{lop}}(F(\lambda, \lambda^*))\) under the isomorphism from Theorem 3.3.1 (and similarly for the cup morphism). To see this, note that the cap morphism is actually defined over \( \mathbb{Z} \) and hence corresponds to a generator of \( H_{\text{lop}}(F(\lambda, \lambda^*), \mathbb{Z}) \). Hence it must correspond to \([F(\lambda, \lambda^*)]\) or its negative. Let us assume that it actually corresponds to \([F(\lambda, \lambda^*)]\). If it actually corresponds to the negative, then the rest of the paper is unaffected with the exception of the introduction of some signs. \( \square \)

### 3.4 From the free spider to the convolution category

Section 2.1 describes a pivotal functor
\[ \Psi : \text{fsp}(G) \to \text{rep}(G)_{\text{min}}. \]
On the other hand, the geometric Satake correspondence and Theorem 3.3.2 yield equivalences

$$\text{rep}(G)_{\text{min}} \cong \text{perv}(\text{Gr})_{\text{min}} \cong \text{hconv}(\text{Gr}).$$

The composition is a functor $\text{fsp}(G) \to \text{hconv}(\text{Gr})$ which we also denote by $\Psi$. Our goal now is to describe this functor and in particular its action on invariant vectors.

Let $\lambda, \mu, \nu$ be a triple of dominant minuscule weights such that

$$\text{Inv}_G(V(\lambda, \mu, \nu)) \neq 0.$$

There is a simple web $w \in \text{Inv}_{\text{fsp}(G)}(\lambda, \mu, \nu)$ which contains a single vertex. On the other hand,

$$\text{Inv}_{\text{hconv}(\text{Gr})}(\lambda, \mu, \nu) \cong H_{\text{top}}(F(\lambda, \mu, \nu))$$

is one-dimensional with canonical generator $[F(\lambda, \mu, \nu)]$. Recall, from Section 2.1, that in the construction of the functor $\text{fsp}(G) \to \text{rep}(G)_{\text{min}}$, there was some freedom to choose the image of the simple web $w$ (it was only defined up to a non-zero scalar). Now, we fix this choice by setting

$$\Psi(w) \overset{\text{def}}{=} [F(\lambda, \mu, \nu)].$$

The functor $\Psi$ is now determined by what it does on vertices and the fact that it preserves the pivotal structure on both sides.

We are now in a position to prove Theorem 1.3.1, which we restate as follows. Recall that

$$d(\bar{\lambda}) = \dim_{\mathbb{C}} F(\bar{\lambda}).$$

**Theorem 3.4.1.** Let $w$ be a web with boundary $\bar{\lambda}$ and dual diskoid $D = D(w)$. Let

$$\pi : Q(D) \to F(\bar{\lambda})$$

be the boundary restriction map. There exists a homology class $c(w) \in H_{2d(\bar{\lambda})}(Q(D))$ such that $\pi_*(c(w)) = \Psi(w)$. Moreover, when $Q(D)$ has dimension $d(\bar{\lambda})$ and is reduced as a scheme, then $c(w)$ is the fundamental class $[Q(D)]$. 
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**Proof.** We begin by picking a isotopy representative for $w$ such that the height function is a Morse function and so that the boundary of $w$ is at the top level. We assume a sequence of horizontal lines $\ell_0, \ldots, \ell_m$ such that in between each pair, $w$ has only a single cap, cup, or a vertex. We assume further that each vertex is either an ascending Y (it is in the shape of a Y) or a descending Y (an upside-down Y).

![Figure 3.2: A web for SL$_9$ in Morse position.](image)

Let $\vec{\lambda}^{(k)}$ be the vector of labels of the edges cut by the horizontal line $\ell_k$. Then $\vec{\lambda}^{(0)} = \emptyset$ and $\vec{\lambda}^{(m)} = \vec{\lambda}$. For example, in Figure 3.2 shows an SL$_9$ web in Morse position, with edges labelled by its minuscule weights $\omega_k$ with $1 \leq k \leq 8$. In this example,

$\vec{\lambda}^{(1)} = \{\omega_4, \omega_5\}$

$\vec{\lambda}^{(3)} = \{\omega_7, \omega_6, \omega_1, \omega_4\}$.

(Note that in SL$_n$ in general, $\omega^*_k = \omega_{n-k}$; if an edge points down as it crosses a line, then we must take the dual weight.)

Let

$$w_k \in \text{Hom}_{\text{sp}(G)}(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)})$$

denote the web in the horizontal strip between the lines $\ell_{k-1}$ and $\ell_k$. By examining the above definition, we see that for each $1 \leq k \leq m$, there exists a component $X_k \subset Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)})$ such...
that $\Psi(w_k) = [X_k]$. We would like to describe this component explicitly. For convenience, if

$$\vec{p} = (p_0, p_1, \ldots, p_m) \in \text{Gr}^{m+1}$$

(with $p_0 = t^0$ for us), define $\sigma_i(\vec{p})$ by omitting the term $p_i$.

(i) If $w_k$ is an ascending Y vertex that connects the $i$th point on $\ell_{k-1}$ to the $i$th and $i+1$st points on $\ell_k$, then

$$X_k = \{(\vec{p}, \vec{p}') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) | \vec{p} = \sigma_i(\vec{p}') \}.$$  

(ii) If $w_k$ is a descending Y vertex that connects the $i$th and $i+1$st points on $\ell_{k-1}$ to the $i$th point on $\ell_k$, then

$$X_k = \{(\vec{p}, \vec{p}') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) | \vec{p}' = \sigma_i(\vec{p}) \}.$$  

(iii) If $w_k$ is a cup that connects the $i$th and $i+1$st points on $\ell_k$, then

$$X_k = \{(\vec{p}, \vec{p}') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) | \vec{p} = \sigma_i(\sigma_i(\vec{p}')) \}.$$  

(iv) If $w_k$ is a cap that connects the $i$th and $i+1$st points on $\ell_{k-1}$, then

$$X_k = \{(\vec{p}, \vec{p}') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) | \vec{p}' = \sigma_i(\sigma_i(\vec{p})) \}.$$  

Then $w = w_m \circ \cdots \circ w_1$. Since $\Psi$ is a functor,

$$\Psi(w) = \Psi(w_m) \ast \cdots \ast \Psi(w_1) = [X_m] \ast \cdots \ast [X_1].$$

Now, compositions of convolutions can be computed as a single convolution as

$$[X_m] \ast \cdots \ast [X_1] = (\pi_{0,m})_*(\pi_{0,1}^*[X_1] \cdots \pi_{m-1,m}^*[X_m]),$$

where the intersection products take place in the ambient smooth manifold

$$X = \text{Gr}(\vec{\lambda}^{(0)}) \times \cdots \times \text{Gr}(\vec{\lambda}^{(m)}).$$
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Here $\pi_{k-1,k}$ denotes the projection from $X$ to $\text{Gr}(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)})$.

From the definitions, we see that the diskoid configuration spaces $Q(D)$ can be obtained as

$$Q(D) = \pi^{-1}_{0,1}(X_1) \cap \cdots \cap \pi^{-1}_{m-1,m}(X_m).$$

Let

$$c(w) = \pi^*_0[X_1] \cap \cdots \cap \pi^*_m[X_m]$$

$$= [\pi^{-1}_{0,1}(X_1)] \cap \cdots \cap [\pi^{-1}_{m-1,m}(X_m)].$$

Because we are using the intersection product with support, $c(w)$ lives in $H_d(\vec{\lambda})(Q(D))$, the homology of the intersection. When $Q(D)$ is reduced of the expected dimension, then the intersection product of the homology classes corresponds to the fundamental class of the intersection (see [Ful98, Sec. 8.2]), so $c(w) = [Q(D)]$.

Finally, $\pi : Q(D) \to F(\vec{\lambda})$ is the restriction of $\pi_{0,m}$ to $Q(D)$. Hence we conclude that

$$\Psi(w) = \pi_*(c(w)).$$

Because $\pi_*(c(w))$ is supported on $\pi(Q(D))$, we immediately obtain the following.

**Corollary 3.4.2.** $\Psi(w)$ is a linear combination of the fundamental classes of the components of $F(\vec{\lambda})$ which are in the image of $\pi$.

It may not seem clear that $c(w)$ depends only on the web $w$, and not on the Morse position of $w$ used to construct it. However, a posteriori, this must be verified by checking that it is invariant under basic isotopy moves (for example, straightening out a cup/cap pair).
Chapter 4

Webs

4.1 Coherent Webs

If the results of Theorem 1.4.1 were to hold for any web, we would have an easy test to see if a set of webs was in fact a basis. This theorem requires the existence of global isometries from the dual diskoid, which in turn would need coherent geodesics between any vertices. In general there are not enough webs with coherent geodesics to form a basis for $\text{Inv}(V(\tilde{\lambda}))$ for arbitrary $\tilde{\lambda}$. In order to obtain a basis we choose more relaxed conditions on webs.

Consider the dual diskoid $D(w)$ of a web $w$. We say it has coherent geodesics at a vertex $p$ if for each vertex $q$ there is a unique minimal distance $d(p,q)$ between $p$ and $q$. Note that when we consider a web as an element of $\text{Inv}(V(\tilde{\lambda}))$ there is a choice of distinguished external face in the web and distinguished vertex in the dual diskoid. A web is then said to be coherent if:

(i) The dual has coherent geodesics at the distinguished vertex, $o$.

(ii) Every internal vertex is contained along some geodesic between the distinguished vertex and the boundary.

(iii) For every edge $a \xrightarrow{\gamma} b$ the condition $d(o,b) − d(o,a) ∈ W\gamma$.

As before, let $P_{\tilde{\lambda}}(v)$ be the set of minuscule paths of type $\tilde{\lambda}$ starting at 0 and ending at
ν. If ν = 0, we just denote this set \( P_\lambda \). Then, to any coherent web with boundary \( \bar{\lambda} \), we can associate \( \bar{\mu} \in P_\lambda \) by taking \( \mu_i = d(o, v_i) \) where \( v_i \) are the external vertices of the dual diskoid read clockwise from \( o \). These are also known as the set of cut-weights for the web. In [Kup96], Kuperberg defines the set of non-elliptic webs for \( \text{SL}_3 \), those webs with no internal faces of less than degree 6. It follows from [FKK11, Thm. 5.1] that non-elliptic webs are coherent, but this is also a consequence of the work which follows.

The following lemma on geodesics is useful:

**Lemma 4.1.1.** Any subpath of a geodesic is still a geodesic. If the pair \((a, b)\) has coherent geodesics, than any pair \((c, d)\) of vertices lying on a geodesic of \((a, b)\) also has coherent geodesics.

**Proof.** Let \( c \) and \( d \) be two vertices lying along a geodesic \( \gamma \). If there exists a path \( \delta \) between \( c \) and \( d \) with lower total weight than the section of \( \gamma \) between \( c \) and \( d \), replacing that section of \( \gamma \) with \( \delta \) would result in a smaller path. But \( \gamma \) is already a geodesic.

For the other statement, suppose that \( \gamma \) is a geodesic between a pair of vertices \((a, b)\) with coherent geodesics and \( \delta \) any path between \( c \) and \( d \). Replacing the segment of \( \gamma \) between \( c \) and \( d \) with \( \delta \) results in a path with a total weight greater than or equal to that of \( \gamma \) since the pair \((a, b)\) has coherent geodesics.

At this point, given a dominant minuscule sequence \( \bar{\lambda} \) and a web with associated path \( \bar{\mu} \in P_\lambda \), there is the Satake basis element \( \overline{[Q(A(\bar{\lambda}, \bar{\mu}))]} \). The goal now is to determine the relationship between \( Q(D(w)), \Psi(w) \) and \( \overline{[Q(A(\bar{\lambda}, \bar{\mu}))]} \). There is a natural partial ordering on \( P_\lambda \): given \( \bar{\mu}, \bar{\mu}' \in P_\lambda \), we say \( \bar{\mu}' \leq \bar{\mu} \) iff \( \mu'_i \leq \mu_i \) for all \( i \). We wish to understand the expansion of a coherent web in terms of the Satake basis. The next result shows that a coherent web expands as the corresponding Satake component (via the identification with elements of \( P_\lambda \)) plus lower order terms.

**Lemma 4.1.2.** Let \( w(\bar{\lambda}, \bar{\mu}) \) be a coherent web with boundary \( \bar{\lambda} \) and associated path \( \bar{\mu} \in P_\lambda \). Suppose that \( \bar{v} \in P_\lambda \) and \( \bar{v} \not\leq \bar{\mu} \), then the coefficient of \( \overline{[Q(A(\bar{\lambda}, \bar{v}))]} \) in \( \Psi(\bar{\lambda}, \bar{\mu}) \) is 0.
Proof. By Corollary 3.4.2, it suffices to show that if \( Q(A(\vec{\lambda}, \vec{\nu})) \) is contained in \( \pi(Q(D(\vec{\lambda}, \vec{\mu}))) \), then \( \vec{\nu} \leq \vec{\mu} \).

Let \( f \in Q(D(\vec{\lambda}, \vec{\mu})) \). If \( v_i \) is the \( i \)th boundary vertex of the diskoid \( D(\vec{\lambda}, \vec{\mu}) \), then \( f(v_i) \in \overline{\text{Gr}(\mu_i)} \). On the other hand, if \( \pi(f) \in Q(A(\vec{\lambda}, \vec{\nu})) \), then \( f(v_i) \in \text{Gr}(v_i) \). Thus \( v_i \leq \mu_i \) for all \( i \) as desired.

Thus we see that although we relaxed conditions on webs and cannot hope to prove Theorem 1.4.1 in general, we at least have some evidence this has only introduced some error terms. Of course the above theorem says nothing about the coefficient of \( [Q(A(\vec{\lambda}, \vec{\mu}))] \), it may even be zero. For this to prove the result we need, Theorem 1.6.1, we can show, using the geometry of affine buildings, that this coefficient is 1. The first step is to show that \( \pi \) generically one to one over \( Q(A(\vec{\lambda}, \vec{\mu})) \).

**Theorem 4.1.3.** For \( G = \text{SL}_n \), let \( w(\vec{\lambda}, \vec{\mu}) \) be a coherent web with boundary \( \vec{\lambda} \) and associated path \( \vec{\mu} \in P_{\vec{\lambda}} \), then any configuration \( f \in Q(A(\vec{\lambda}, \vec{\mu})) \) extends uniquely to a configuration \( f \in Q(D(w(\vec{\lambda}, \vec{\mu}))) \).

**Proof.** Let \( f \in Q(A(\lambda, \mu)) \). Since the boundary cycles of \( A(\vec{\lambda}, \vec{\mu}) \) and \( D(w) \) are the same, we only need extend \( f \) to the interior vertices. Let \( v_0, \ldots, v_n \) be the exterior vertices of \( D(w) \), then \( d(f(\bullet), f(v_i)) = \mu_i \) by definition and \( \mu_i \) is the length of any geodesic \( \gamma \) between \( \bullet \) and \( v_i \).

Considering \( \text{Gr} \) as the vertices of associated affine building, let \( \Sigma \) be an apartment containing \( f(\bullet) = t^0 \) and \( f(v_i) \). We can choose coordinates on \( \Sigma \) such that \( f(v_i) \) has position \( \mu_i \) and \( t^0 \) has position 0. It follows that there is a unique geodesic \( \tilde{\gamma} \in \Sigma \) between \( t^0 \) and \( f(v_i) \) with the same sequence of edge weights as \( \gamma \). Thus we can extend \( f \) by setting \( f(\gamma_i) = \tilde{\gamma}_i \). Since \( w \) is coherent, it follows from condition 2 in the definition of coherence that \( f \) is now defined for any vertex of \( D(w) \).

Next we must show that this extension is consistent on vertices \( v \) in the interior of \( D(w) \). Suppose that we have two geodesics \( \gamma \) and \( \gamma' \) from \( \bullet \) to \( v_i \) and \( v_j \) for some \( i < j \) such that \( v \in \gamma \cap \gamma' \). By induction on \( j - i \), we show that \( f(v) \) is well defined. In the case that \( j - i \leq 1 \), this
is obvious since there exists an apartment $\Sigma$ containing $t^0$, $f(v_i)$ and $f(v_j)$ since $d(f(v_i), d(v_j))$ is either zero or minuscule, so $\{f(v_i), f(v_j)\}$ is a 0 or 1 simplex in $\Delta$. Since $\Sigma$ contains the convex hull of $t^0$, $f(v_i)$ and $f(v_j)$, both $\gamma$ and $\gamma'$ lie in $\Sigma$ so they both define $f(v)$ in the same way. In the case that $j - i > 1$, consider any geodesic $\gamma''$ from $t^0$ to $f(v_{i+1})$. Since the paths $\gamma$ from $v_i$ to $v$ and $\gamma'$ from $v_j$ to $v$ form a closed region with the boundary of the web, it follows that $\gamma''$ intersects either $\gamma$ or $\gamma'$ at some point between $v$ and $v_i$ or $v$ and $v_j$. Without loss of generality, assume that $\gamma$ and $\gamma''$ intersect at some point $v'$ after $v$. Then the path along $\gamma$ from $\bullet$ to $v'$ and then from $v'$ to $v_{i+1}$ is a geodesic passes through $v$, and so by induction we are done.

Since $f$ is now well defined, we need to check that $f$ is indeed a configuration. That is, we must show that

$$d(f(a), f(b)) = d(a, b)$$

for $a, b$ adjacent vertices. Without loss of generality, there exists geodesics $\gamma$ and $\gamma'$ from $\bullet$ to $v_i$ and $v_j$ for some $i \leq j$ such that $a \in \gamma$ and $b \in \gamma'$. If $j - i \leq 1$, then there exists an apartment $\Sigma$ containing $f(\bullet)$, $f(v_i)$ and $f(v_j)$ as above and $\Sigma$ contains both geodesics $\gamma$ and $\gamma'$. There are coordinates on this apartment such that $t^0$ has position 0, $f(v_i)$ has position $d(\bullet, v_i)$ and $f(v_j)$ has position $d(\bullet, v_j)$. By construction $f(a)$ has position $d(\bullet, a)$ and $f(b)$ has position $d(\bullet, b)$, but then since $w$ is coherent $d(\bullet, b) - d(\bullet, a) \in Wd(a, b)$, $f(a)$ and $f(b)$ have adjacent positions. Thus it follows that $d(f(a), f(b)) = d(a, b)$ as needed. If $j - i > 1$ consider any geodesic $\gamma''$ from $\bullet$ to $v_{i+1}$. As before it intersects either $\gamma$ in between $v_i$ and $a$ or $\gamma'$ in between $v_j$ and $b$. In either case, we can replace one of $v_i$ or $v_j$ with $v_{i+1}$ and a suitable geodesic formed from $\gamma''$ and $\gamma$ or $\gamma'$. By induction we see that $f \in Q(D(w))$.

Finally, this extension is unique since there was a unique choice of $f(v)$ for any internal vertex.

Note that this proof should follow in general for other simple simply-connected groups, but some attention needs to be paid to the existence of non-special vertices in the building. Theorem 1.6.1 now follows after some scheme theoretic considerations.
**Lemma 4.1.4.** The coefficient of \( [Q(A(\lambda, \mu))] \) in \( \Psi(w(\lambda, \mu)) \) is 1.

**Proof.** Let \( Z = \pi^{-1}(Q(A(\lambda, \mu))) \). Then \( Z \) is a component of \( Q(D(\lambda, \mu)) \), and it has dimension \( d(\lambda) \) by Theorem 4.1.3. Recall that from Theorem 3.4.1, that we have a homology class \( c(w) \in H_{d(\lambda)}(D(\lambda)) \) such that \( \pi_*(c(w)) = \Psi(w) \). Using the notation of the proof of Theorem 3.4.1,

\[
Q(D(\lambda, \mu)) = \pi_{0,1}^{-1}(X_1) \cap \cdots \cap \pi_{n-1,n}^{-1}(X_n)
\]

and

\[
c(w) = [\pi_{0,1}^{-1}(X_1)] \cap \cdots \cap [\pi_{n-1,n}^{-1}(X_n)]
\]

Since \( Z \) is a component of the expected dimension, we see that the coefficient of \( [Z] \) in \( c(w) \) is the length of the local ring of \( Q(D(\lambda, \mu)) \) along \( Z \) (by [Ful98], Proposition 8.2). This length equals 1 since following lemma shows that the scheme \( \pi^{-1}(Q(A(\lambda, \mu))) \) is isomorphic to the reduced scheme \( Q(A(\lambda, \mu)) \).

The degree \( \pi|_Z \) is 1, so \( \pi_*(Z) = [Q(A(\lambda, \mu))] \). Moreover, \( Z \) is the only component of \( Q(D(\lambda, \mu)) \) which maps onto \( Q(A(\lambda, \mu)) \), so we conclude that the coefficient of \( [Q(A(\lambda, \mu))] \) in \( \pi_*c(w) \) is also 1, as desired. \( \square \)

**Lemma 4.1.5.** The restriction of the map \( \pi : Q(D(\lambda, \mu)) \to F(\lambda) \) to \( \pi^{-1}(Q(A(\lambda, \mu))) \) is an isomorphism of schemes onto the reduced scheme \( Q(A(\lambda, \mu)) \).

**Proof.** First note that \( Q(A(\lambda, \mu)) \) is reduced since it is isomorphic to an iterated fibered product of varieties by the proof of Theorem 3.1.1.

Let \( X = \pi^{-1}(Q(A(\lambda, \mu))) \), \( Y = Q(A(\lambda, \mu)) \). We have already shown in Theorem 4.1.3 that the map \( \pi : X \to Y \) gives a bijection at the \( \mathbb{C} \)-points. Now, let \( S \) be any scheme of finite-type over \( \mathbb{C} \). The proof of Theorem 4.1.3 uses some building-theoretic arguments which don’t obviously work for \( S \)-points. However, the argument in the first paragraph of the proof does work for any \( S \), as follows. Following the notation in that paragraph, let \( \gamma \) be a geodesic in \( \Gamma \) from the base point \( p \) of \( D(\lambda, \mu) \) to the \( k \)-th boundary vertex \( q \) and let \( \bar{v} \) be the lengths along this geodesic (by definition \( \sum v_i = \mu_k \)). Let \( f \in X(S) \). Then the restriction of the map \( m : \text{Gr}(\bar{v}) \to \text{Gr} \)
to $m^{-1}(\text{Gr}(\mu_k))$ is an isomorphism of schemes, and in particular is an injection on $S$-points. Hence we see that $f(r)$ is determined by $f(q)$ for all $r$ along the geodesic. Since every internal vertex of the diskoid lies on some geodesic, $f \in X(S)$ is determined by its restriction to the boundary. Thus, the map $X(S) \to Y(S)$ is injective.

So we have a map from a scheme to a smooth variety which is a bijection on $\mathbb{C}$-points and is an injection on $S$-points. By the following lemma, the map is an isomorphism.

Lemma 4.1.6. Let $X, Y$ be finite-type schemes over $\mathbb{C}$. Assume that $Y$ is reduced and normal. Let $\phi : X \to Y$ be a morphism which induces a bijection on $\mathbb{C}$-points and an injection on $S$-points for all finite-type $\mathbb{C}$-schemes $S$. Then $\phi$ is an isomorphism.

Proof. Consider the maps

$$X_{\text{red}} \to X \to Y.$$ 

The composition $X_{\text{red}} \to Y$ is a bijection on $\mathbb{C}$-points and hence it is an isomorphism [Kum02, Thm. A.11]. This allows us to construct a map $\psi : Y \to X$ such that $\phi \psi = \text{id}_Y$.

The fact that $\phi$ induces an injection on $S$-points means that the map

$$\text{Hom}_{\text{Sch}}(X, X) \xrightarrow{\phi^\circ} \text{Hom}_{\text{Sch}}(X, Y)$$

is injective. Consider what happens to $\text{id}_X$ and $\psi \phi$ under this map. They are sent to $\phi$ and $\phi \psi \phi$ respectively. But since $\phi \psi = \text{id}_Y$, these two elements of $\text{Hom}_{\text{Sch}}(X, Y)$ are equal. Hence by the injectivity, $\text{id}_X = \psi \phi$ and hence $\phi$ is an isomorphism.

The following version of Theorem 1.6.1 is a direct consequence of the previous results:

Theorem 4.1.7. Let $G = \text{SL}_n$ and $\vec{\lambda}$ be a sequence of dominant minuscule weights. Let $B$ be a set of coherent webs, each with a unique corresponding minuscule path in $P_{\vec{\lambda}}$. Then $\Psi(B)$ forms a basis for $\text{Hom}_{\text{SL}_n}(\mathbb{C}, V_{\vec{\lambda}})$ and the change of basis from $\Psi(B)$ to the Satake basis is upper unitriangular with respect to the partial ordering of $P_{\vec{\lambda}}$. 
4.2 Triangular Diagrams

Our goal now is to produce a set of coherent webs and to do this we adapt Westbury’s [Wes07] method of triangular diagrams.

**Definition 4.2.1.** A triangular diagram is a coherent web for $\text{SL}_n$ with 2 extra marked points. The original marked point distinguished from the first 2 and is labelled $A$. The other two marked points are labelled $X$ and $Y$ clockwise from $A$. As with webs they are considered up to isotopy rel boundary and the orientation switching relation.

The set of triangular diagrams can be constructed from the set coherent webs by placing the two extra marked points in every possible way on the boundary. In this way a web with $k$ boundary edges results in $\binom{k+1}{2}$ distinct triangular diagrams since the boundary edges and original marked point cut the boundary into $k + 1$ regions. We embed the set of webs into the set of triangular diagrams by placing one marked point on the boundary to the left of $A$ and the other to the right. We will typically straighten the boundary of the disk between the marked points and draw a triangular diagram as a triangle with the vertex $A$ at the bottom and the edge $XY$ at the top. We call the face containing $A$ the exterior face and the faces along the boundary $XY$ are the topmost faces.

We can recover a web from a triangular diagram by dropping the marked points $X$ and $Y$. In this way a triangular diagram has a dual diskoid and the marked boundary points $X$ and $Y$ label the vertices of the dual diskoid corresponding to the face of the web they each was contained in. Note that a vertex may have multiple labels, for example the labels $A$, $X$ and $Y$ coincide for a triangular diagram in the image of the above embedding of the set of webs. We will restrict our attention to the set of triangular diagrams such that the path in the dual diskoid corresponding to the edges $AX$ and $AY$ are geodesics.

For such a triangular diagram, we can then define the *length* of the diagram to be the number of boundary edges incident to the edge $XY$ and the *weight* of the diagram to be the length (as a weight) of the geodesic $AY$ minus that of the geodesic $AX$. We can then translate Westbury’s
notion of irreducible diagrams into the language of geodesics in the dual diskoid:

**Definition 4.2.2.** An irreducible triangular diagram is one such that there exists no vertex $B$ in the dual diskoid with the property that there exists geodesics connecting $A$ with $X$ and $Y$ both passing through $B$.

The key insight of Westbury is that for triangular diagrams $A$ and $B$ there is a natural product $A \otimes B$. This is described in subsection 4.2.2. For each minuscule weight of $SL_n$, i.e. $\omega$ in the Weyl orbit of some fundamental weight, we construct an irreducible triangular diagram $T_\omega$ of length 1 and weight $\omega$. We will then show that for $\vec{\mu} \in P_1$, the diagram

$$T_{\vec{\mu}} = \bigotimes T_{\mu_i - \mu_{i-1}}$$

is in the image of the above embedding of webs into triangular diagrams and has corresponding path $\vec{\mu}$. The resulting set of webs is then a basis by Theorem 4.1.7. This gives a construction rule for a basis of $SL_n$ webs similar to that of Kuperberg for the non-elliptic $SL_3$ webs.

### 4.2.1 Irreducible diagrams of length one

For each weight in the Weyl orbit of a dominant minuscule weight we must produce an irreducible triangular diagram of length one. These are our building blocks to construct diagrams with larger lengths. Recall that if we consider the weights of $SL_n$ to be the set $\mathbb{Z}^n / (1, 1, \ldots, 1)$, a weight that lies in the Weyl orbit of dominant minuscule weight $\omega_k$ is a sequence of $k$ ones and $n-k$ zeroes. Let $\omega$ be such a weight, then $\omega$ can be written uniquely as a sum of fundamental weights with coefficients in $\{-1, 0, 1\}$ in the following way: let $\omega = (c_1, c_2, \ldots, c_n)$, with $c_i \in \{0, 1\}$ then $\omega = \sum_j (c_j - c_{j+1}) \omega_j$. Let $l_1 < l_2 < \cdots$ be the sequence of $j$ such that $c_j - c_{j+1} = -1$ and $r_1 < r_2 < \cdots$ be the sequence of $j$ such that $c_j - c_{j+1} = 1$. Then we have either $r_1 < l_1 < r_2 < l_2 < \cdots$ or $l_1 < r_1 < l_2 < r_2 < \cdots$. We produce the triangular diagram for the first case, the second is similar. For convenience, edges of a web will be labelled with integers $0 < i < n$ corresponding to a label $\omega_i$ or sometimes 0, in which case the edge can be
considered an edge of weight 0 which does not exist in the corresponding web. The condition on the edge labels at each vertex in a web becomes the condition that the sum of the labels of the incoming edges must be equal to the sum of the outgoing edges modulo \(n\).

**Theorem 4.2.3.** The following is an irreducible triangular diagram which we denote \(T_\omega\):

\[
\begin{array}{c}
X \\
| \\
| \\
| \\
| \\
| \\
| \\
| \\
| \\
| \\
\end{array}
\]

It should be noted that the diagram above is by no means the only irreducible length one diagram corresponding to the given weight. In general there is at least one other diagram: the proof below follows if all inequalities and order of the edge labels are reversed. Note that there may be no irreducible diagram corresponding to an arbitrary ordering of the labels \(l_i\) and \(r_i\).

For \(n \leq 3\) this ambiguity does not appear since the lengths of the sequences \(l_i\) and \(r_i\) are at most 1. In section 4.5, it is possible to minimize the number of vertices in the triangular diagram by carefully choosing which diagram should be used for \(T_\omega\) in the SL\textsubscript{4} case.

**Proof.** The edges from the left to the middle are labelled \(l_i\), those from the middle to the right \(r_i\) and the interior edges are labelled \(f(r_i)\) or \(f(l_i)\) with \(f(r_1) = r_1\). At each vertex the sum of the incoming edges equals the sum of the outgoing edges mod \(n\). This uniquely specifies the \(f(r_i)\) and \(f(l_i)\) and we show that \(0 < f(r_i) \leq r_i\) and \(0 < f(l_i) \leq l_i\) for all \(i\). Clearly this is true for \(i = 1\), since \(0 < f(r_1) = r_1\) and then since \(f(l_1) = l_1 - f(r_1)\) and \(l_1 > f(r_1) = r_1 > 0\), we have \(0 < f(l_1) < l_1\). By induction, it is true for all \(i\): if \(0 < f(r_{i-1}) \leq r_{i-1}\) and \(0 < f(l_{i-1}) \leq l_{i-1}\), then we have \(f(r_i) = r_i - f(l_{i-1})\) and since \(r_i > l_{i-1} \geq f(l_{i-1}) > 0\), we have \(0 < f(r_i) < r_i\) and
similarly $0 < f(l_i) < l_i$. Note that since none of the edges are labelled 0, they all appear in the web.

To establish that this is a triangular diagram we must show that $A$ has coherent geodesics and that $AX$ and $AY$ are geodesics. Consider figure 4.1, the dual diskoid for the above diagram. By Lemma 4.1.1, since the diagram has length 1 and there are no internal vertices, it suffices to show that $AX$ and $AY$ are geodesics and that the pairs $(A,X)$ and $(A,Y)$ have coherent geodesics.

Suppose we have a path $\gamma$ between $A$ and $X$ in the dual diskoid. Then we want to show that the total weight of $\gamma$ is greater than the total weight of $AX$. We apply a series of operations to $\gamma$ that reduce its weight and result in the path $AX$. If $\gamma$ has any loops, remove them, this lowers the total weight of $\gamma$. Suppose that the path $\gamma$ travels along two edges of a triangular face:

![Diagram](image)

Then using the edge labelled $a+b$ rather than the two edges labelled $a$ and $b$ results in lowering the weight of that segment of the path from $\omega_a + \omega_b$ to $\omega_{a+b}$ where $a+b$ is reduced modulo $n$. These are called triangle moves in [FKK11]. Thus we can assume that $\gamma$ has no loops or available triangle moves. Thus if $\gamma$ is not already $AX$, it travels some number of edges up the left side (possibly 0) then crosses to the right. It must cross on an edge labelled $f(r_i)$ otherwise the path would contain a loop or triangle move. Then the path cannot travel along $f(l_i)$ or $r_i$ otherwise there would also be a triangle move. Thus the path travels along $r_{i+1}$. So the path
travels along the left side of the following diamond:

\[
\begin{array}{c}
\vdots \\
\sum_i \\
\end{array}
\]

By doing a diamond move, and swapping the path to the other side, \( \gamma \) would travel further along the left side. For this move to lower the weight of the path, we must have \( \omega_{f(r_i)} + \omega_{r_{i+1}} \geq \omega_{l_i} + \omega_{f(r_{i+1})} \). Since \( l_i - f(r_i) = f(l_1) = r_{i+1} - f(r_{i+1}) \) we have \( f(r_i) + r_{i+1} = l_i + f(r_{i+1}) \) so the weights differ by a number of roots. We have \( r_{i+1} > l_i > r_i \geq f(r_i) \) so \( r_{i+1} > l_i > f(r_i) \) and thus \( r_{i+1} > f(r_{i+1}) > f(r_i) \). But this means that we have \( \omega_{f(r_i)} + \omega_{r_{i+1}} \geq \omega_{l_i} + \omega_{f(r_{i+1})} \) as needed. Thus by a sequence of removing loops, triangle and diamond moves we can reduce any path from \( A \) to \( X \) so that it only travels along \( AX \). Thus \( AX \) is a geodesic and the pair \( \langle A, X \rangle \) has coherent geodesics.

The proof for \( AY \) is similar.

\[4.2.2 \text{ Product of Triangular Diagrams}\]

Given two triangular diagrams \( A \) and \( B \) we can define their product \( A \otimes B \). Placing the diagrams for \( A \) and \( B \) side by side as in Figure 4.2, we can fill the diamond \( C \) so that the resultant diagram is a triangular diagram.

\[
\begin{array}{c}
A \\
\downarrow \\
\text{C} \\
\uparrow \\
B \\
\vdots
\end{array}
\]

Figure 4.2: Product of triangular diagrams \( A \) and \( B \) via the diamond \( C \)

Let \( i_1, \ldots, i_l \) be the labels of the (outgoing) edges on the right side of triangle \( A \) from bottom to top and \( i'_1, \ldots, i'_{l'} \) the labels on the (incoming) edges on the left side of \( B \). Then the diamond
$C$ is defined to be the result of taking the diamond:

![Diamond Diagram]

and replacing each vertex of degree 4 via the following rule (evaluated from the top most vertex to the bottom most):

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
=\begin{cases}
\begin{pmatrix} a \\ a \end{pmatrix}, & a = b, \\
\begin{pmatrix} a \\ b \end{pmatrix}, & a \neq b.
\end{cases}
\]

Note that, as previously mentioned, an edge with 0 weight does not appear in the web. In order to simplify arguments we leave it in. Taking the dual diskoid still makes sense with edges labelled 0, the corresponding web variety is unchanged since then we can contract edges of weight 0 and delete the resulting 2 faces to obtain the dual diskoid of the web. We can now
define the left and right diamond moves. Consider a path in the dual diskoid of a diamond:

The small diamonds inside are given by the dual diskoid equivalent to the rule above:

\[
\begin{align*}
    & a \quad b \\
    & b-a \quad \text{if } a \neq b \text{ or } \\
    & b \quad a \quad 0 \quad 0 \quad \text{otherwise}
\end{align*}
\]

Then, if a path travels along the right two edges of one of the above small diamonds, replacing that portion of the path with the left edges is a left diamond move and replacing the left edges with the right edges is a right diamond move. In either of the cases, doing a left or right \(H\) move does not change the weight of the path.

**Proposition 4.2.4.** The product \(A \otimes B\) is a triangular diagram.

**Proof.** We abuse notation and denote the vertex of \(A\) by \(A\), the vertex of \(B\) by \(B\) and the vertex of \(A \otimes B\) as \(C\):

We show that any path \(\gamma\) in the dual diskoid starting at \(C\) and ending at a vertex corresponding to a face in \(A\) has weight greater than or equal to that of the path starting with \(CA\) and then
following a geodesic in the dual diskoid of $A$ to the endpoint. First, assume that the given path never intersects itself at any point during the proof, otherwise we would trim the loop in the path to obtain a path with lower weight.

Since the path $\gamma$ must end at a vertex in $A$, it follows that the path must cross $AY$. $AY$ is a geodesic and has the same weight in both the triangle $A$ and the diamond $C$, thus $AY$ is a geodesic in $A \otimes B$. If $\gamma$ crosses $AY$ more than once, then by Lemma 4.1.1 we can replace the portion of $\gamma$ starting from the first intersection with $AY$ to the last with the corresponding portion of the path $AY$, reducing the weight of $\gamma$. Thus we may assume that $\gamma$ starts at the vertex of the product, eventually enters triangle $A$ and never enters the diamond $C$ again.

Now we consider the segment of the path in the diamond and apply all possible left diamond and triangle moves to this path. Note that this is a finite process and never enters a cycle since triangle moves reduce number of edges in the path and left diamond moves result in a path strictly to the left of the previous path. Since left diamond moves maintain the weight and triangle moves drop the weight, it follows that the resulting path has weight less than or equal to that of the original path. Thus any path in $C$ has weight greater than or equal to a path between the same vertices with no left diamond moves or triangle moves. We show that such a path must follow $CA$ and then travel some distance along $AY$.

Consider the diamond $C$ oriented as above in the plane. Suppose the first coordinate function $\pi_1$ has local maxima, this can only happen at a vertex and thus $\gamma$ has one of the following three forms at that vertex:

In each of these situations, there is a triangle or left diamond move available. Thus $\pi_1$ has no local maxima and thus since $\gamma$ ends along $AY$, $\gamma$ travels along $CA$ and then it enters $A$ where it continues until it ends. The above argument holds for paths starting at $C$ and ending at a vertex in $B$ after replacing $A$ with $B$, left with right and maxima with minima.

Now, to see that the geodesics of $A \otimes B$ at $C$ are coherent, consider any geodesic $\gamma$ from
C to a vertex $D$ in $A$. Since $\gamma$ is a geodesic, by above argument $\gamma$ has the same weight as a path $\gamma'$ that follows $CA$ and then stays in $A$ until it reaches $D$. But then $\gamma'$ is a geodesic and by Lemma 4.1.1 the portion of $\gamma'$ from $A$ to $D$ is also a geodesic. But the weight of this geodesic is independent of the path because $A$ is a triangular diagram and itself has coherent geodesics. Thus the weight of $\gamma$ is the weight of $CA$ plus the weight of any geodesic from $A$ to $D$ and it does not depend on $\gamma$. The same holds for paths from $C$ to a vertex on $D$ in $B$ after replacing $A$ by $B$. It then follows that $CX$ and $CY$ are geodesics.

\textbf{Proposition 4.2.5.} The operator $\otimes$ on triangular diagrams is associative.

\textit{Proof.} We must show that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for any triangular diagrams $A, B, C$, but this is obvious. \qed

\subsection*{4.2.3 Diagrams from Minuscule Paths}

In general, given a minuscule sequence $\overrightarrow{\lambda}$, a minuscule path of type $\overrightarrow{\lambda}$ is a sequence of weights $\overrightarrow{\mu}$ of $G$ such that $\mu_i - \mu_{i-1} \in W \lambda_i$ and $\mu_0 = 0$. Thus the set $P_{\overrightarrow{\lambda}}$ is the set of dominant minuscule path ending at 0. To each minuscule path we can assign the triangular diagram $T_{\overrightarrow{\mu}} = \bigotimes T_{\mu_{i+1}-\mu_i}$.

\textbf{Lemma 4.2.6.} Given a minuscule path $\overrightarrow{\mu}$ of length $m$, let $l_{\overrightarrow{\mu}}$ and $r_{\overrightarrow{\mu}}$ be the weights of the geodesics along the left and right sides of $T_{\overrightarrow{\mu}}$, we have

$$\mu_m = r_{\overrightarrow{\mu}} - l_{\overrightarrow{\mu}}.$$

\textit{Proof.} This is true by construction if $m = 1$, i.e. when $T_{\overrightarrow{\mu}}$ is an irreducible triangular diagram of length 1. Otherwise consider the subsequence $\overrightarrow{\mu}'$ of the first $m - 1$ elements of $\overrightarrow{\mu}$, we have $T_{\overrightarrow{\mu}'} \otimes T_{\mu_{m-\mu_{m-1}}} = T_{\overrightarrow{\mu}}$ and $T_{\overrightarrow{\mu}'}$ satisfies the lemma by induction on the length of the sequence. We have the following diagram:
If $l_{\vec{\mu}}$ and $r_{\vec{\mu}}$ are the weights of the geodesics along the left and right sides of $T_{\vec{\mu}}$ we then have $\mu_{m-1} = r_{\vec{\mu}} - l_{\vec{\mu}}$. If $C$ is the diamond that appears in $T_{\vec{\mu}} \otimes T_{\mu_m - \mu_{m-1}}$ then let $l_C$ and $r_C$ be the weights of the geodesics along the bottom left and bottom right sides of $C$ respectively. We have $l_{\vec{\mu}} = l_{\vec{\mu}} + l_C, r_{\vec{\mu}} = r_{\mu_m} + r_C$ and also $\mu_m - \mu_{m-1} = r_{\mu_m} - l_{\mu_m}$.

Thus

$$r_{\vec{\mu}} - l_{\vec{\mu}} = r_{\mu_m} + r_C - l_{\vec{\mu}} - l_C$$

$$= \mu_m - \mu_{m-1} + l_{\mu_m} + r_C - l_{\vec{\mu}} - l_C$$

$$= \mu_m - (r_{\vec{\mu}} - l_{\vec{\mu}}) + l_{\mu_m} + r_C - l_{\vec{\mu}} - l_C$$

$$= \mu_m - r_{\vec{\mu}} + l_{\mu_m} + r_C - l_C$$

$$= \mu_m.$$

since $l_{\mu_m} + r_C = r_{\vec{\mu}} + l_C$ as they are weights of the geodesics along the left and right side of $C$. 

The next theorems show that for $\vec{\mu} \in P_\lambda$, the triangular diagram $T_{\vec{\mu}}$ is a coherent web with associated path $\vec{\mu}$. To check that the associated path in $P_\lambda$ is $\vec{\mu}$ it is sufficient to check that the weight of the geodesics along the left and right sides of the diagram are both 0.

**Theorem 4.2.7.** Let $\vec{\mu}$ be a dominant minuscule path, then the geodesic along the left side of $T_{\vec{\mu}}$ is of weight 0.

**Proof.** As usual, we proceed by induction on the length of the path. For a path of length 1, the first weight of the path is a dominant minuscule weight, i.e. it is $\omega_j$ for some $j$. The irreducible
diagram for this weight has only one edge that exits on the right side of the triangle, proving the base case.

Suppose that we have some dominant minuscule path \( \mu \) of length \( m \), we proceed using the conventions of Lemma 4.2.6. By induction \( l_{\vec{\mu'}} = 0 \) so by Lemma 4.2.6 we have \( r_{\vec{\mu'}} = \mu_{m-1} \). Since \( l_{\vec{\mu'}} = 0 \), what we must prove is \( l_C = 0 \). Now, \( r_{\vec{\mu'}} - l_{\mu_m} \) can be written as a difference of dominant weights which share no common fundamental weights, by construction this decomposition is simply \( r_C - l_C \). Thus if \( r_{\vec{\mu'}} - l_{\mu_m} \) is dominant, \( l_C = 0 \). Suppose that \( r_{\vec{\mu'}} - l_{\mu_m} \) was not dominant, then since \( m \mu_m = \mu_{m-1} + (\mu_{m-1} - \mu_{m-1}) = r_{\vec{\mu'}} + r_{\mu_m} - l_{\mu_m} \) is dominant, the set of fundamental weights that sum to \( r_{\mu_m} \) would have to intersect those that sum to \( l_{\mu_m} \). This is impossible by the construction of \( T_{\mu_m-\mu_{m-1}} \).

**Corollary 4.2.8.** Any triangular diagram \( T_{\vec{\mu}} \) with \( \vec{\mu} \in P_{\vec{\lambda}} \) has no edges along the left or right side of the triangle.

**Proof.** By the previous theorem, the left side has weight 0 and \( \vec{\mu} \) ends at 0, so by Lemma 4.2.6, the right side has weight 0.

**Theorem 4.2.9.** For \( \vec{\mu} \in P_{\vec{\lambda}} \), \( T_{\vec{\mu}} \) is a coherent web.

**Proof.** In fact, this is true for any triangular diagram \( T \) that is a product of the diagrams described in Theorem 4.2.3 or those with the opposite ordering of labels.

Since \( T \) has coherent geodesics at its vertex, we must show that the other two criteria for a coherent web are satisfied. First we must show that any internal vertex of \( T \) occurs along some geodesic between the vertex of \( T \) and a vertex corresponding to a top most face. We proceed by induction. In the case of a single diagram of length one this is true by design, since all vertices of the dual lie on left or right sides of the diagram which are both geodesics. Using the notation of Proposition 4.2.4, suppose that both \( A \) and \( B \) as above. For any vertex \( D \) in \( A \), there exists a geodesic \( \gamma' \) between \( A \) and a vertex along the boundary \( XY \) that passes through \( D \). Then the path \( \gamma = CA \sqcup \gamma' \) is a geodesic from \( C \) to the boundary of the dual diskoid passing through \( D \). Similarly we can construct a geodesic from \( C \) to the boundary of the dual diskoid
passing through any vertex of $B$. This leaves the internal vertices of the diamond. Consider the geodesic $CAY$, by right diamond moves it passes to the geodesic $CBY$ and any vertex in $C$ is on one of the intermediate geodesics. Thus any vertex in the dual diskoid of $A \otimes B$ lies on a geodesic between $C$ and a vertex on $XZ$.

Let $A$ be the vertex of $T$. The second condition that we need to check is that

$$d(A, b) - d(A, a) \in Wd(a, b)$$

for adjacent vertices $a$ and $b$ where $A$ is the marked vertex of the dual diskoid. If the edge joining $a$ and $b$ is a part of a geodesic from $A$, then the statement is true since either $d(A, b) + d(b, a) = d(A, a)$ or $d(A, b) = d(A, a) + d(a, b)$. Otherwise we can break this into two cases: $a$ and $b$ are either both contained in a triangle $T'$ of length 1 that appears in the construction of $T$ or not. In the first case, we can pick geodesics $\gamma_a$ and $\gamma_b$ from $A$ to $a$ and $A$ to $b$ that pass through the vertex $B$ of $T'$, then it follows that $d(A, b) - d(A, a) = d(B, b) - d(B, a)$. Since $a$ and $b$ are not joined by an edge contained in a geodesic from $A$ (or $B$), following the construction of $T$ in figure 4.1, we have $a$ appearing as the $i$-th vertex on the left and $b$ as the $j$-th vertex on the right with $|i - j| \leq 1$. Thus $d(B, a) = \sum_{k=1}^{i} \omega_{k}$ and $d(B, b) = \sum_{k=1}^{j} \omega_{k}$. Since the labels on the left and right are all unique,

$$d(B, b) - d(B, a) = \sum_{k=1}^{j} \omega_{k} - \sum_{k=1}^{i} \omega_{k} \in W \sum_{k=1}^{i} r_{k} - \sum_{k=1}^{j} l_{k}.$$ 

But $\sum_{k=1}^{i} r_{k} - \sum_{k=1}^{j} l_{k}$ is the label of the edge between $a$ and $b$ by construction.

Otherwise, the edge appears in the dual diskoid of some diamond used in the construction of $T$. In any diamond, the only edges that are not part of a geodesic from $A$ are the horizontal edges. These appear in between the small diamonds outlined by the left and right diamond moves:
There is a geodesic $\gamma$ from $A$ to $a$ passing through $e$ and $\gamma'$ from $A$ to $b$ passing through $e$. Thus

\[
d(A, b) - d(A, a) = (d(A, e) + \omega_c) - (d(A, e) + \omega_d)
= \omega_c - \omega_d \in W \omega_{c-d} = Wd(a, b).
\]

Until this result the internal structure of the diagrams of length one was not important. The need for the resulting web to be coherent forces us to pick irreducible triangular diagrams of length one that satisfy the coherence conditions for webs. Both the diagram introduced in Theorem 4.2.3 and the similar diagram with the ordering of the labels switched satisfy these properties by construction. By Theorem 4.1.7 the set of webs coming from the triangular diagrams \(\{T_{\bar{\mu}}\}_{\bar{\mu} \in P_{\bar{\lambda}}}\) form a basis of Inv\((V(\bar{\lambda}))\).

### 4.3 Webs as a basis for \(\text{Hom}_{\text{SL}_n}(V_V, V_{\bar{\lambda}})\)

In the previous section, given $\bar{\mu} \in P_{\bar{\lambda}}$ with length $k$, we constructed a corresponding web:

\[
\bigotimes_{i=0}^{k} T_{\mu_i - \mu_{i-1}}.
\]

Now we consider the partial products

\[
\bigotimes_{i=0}^{j} T_{\mu_i - \mu_{i-1}}
\]

for $j \leq k$. The underlying web is still coherent and thus results in a class in top homology. Since the irreducible representation $V_{\mu_j}$ of highest weight $\mu_j$ occurs in $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_j}$ with multiplicity vector space

\[
\text{Hom}_{\text{SL}_n}(V_{\mu_j}, V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_j}) \cong H_{\text{top}}(\text{Gr}(\mu_j) \times \text{Gr}(\lambda_1, \cdots, \lambda_j))
\]

it is reasonable to ask if the classes coming from these partial products form a basis for this space.
Denote the set of dominant minuscule paths of type $\vec{\lambda}$ ending at $\nu$ by $P_{\vec{\lambda}}(\nu)$. Using this notation $P_{\vec{\lambda}} = P_{\vec{\lambda}}(0)$. For $\vec{\mu} \in P_{\vec{\lambda}}(\nu)$, by theorem 4.2.7, the triangular diagram $T_{\vec{\mu}}$ has no incident edges on the left. Thus the geodesic on the right has weight $\nu$ and we let $\tilde{\nu}$ be the sequence of weights corresponding the labels of the edges ending on the right side of the triangle from bottom to top. By the previous section, the web underlying $T_{\vec{\mu}}$ is coherent. Let $\tilde{\vec{\mu}} \in P_{\vec{\lambda} \sqcup \vec{\nu}}(0)$ be the associated path. Let $\phi : V_\nu \hookrightarrow V_{\tilde{\nu}}$ be the canonical inclusion, then $\Psi_\nu = (id_{V_\lambda} \otimes \phi^*) \circ \Psi$ is a map from webs of type $\vec{\lambda} \sqcup \vec{\nu}^*$ to $\text{Hom}_{\text{SL}_n}(\mathbb{C}, V_\lambda \otimes V_{\nu}^*)$. The goal of this section is to prove the following result:

**Theorem 4.3.1.** Let $\vec{\lambda}$ be a sequence of dominant minuscule weights. Relative to the usual ordering on $P_{\vec{\lambda}}(\tilde{\nu})$, there is an upper unitriangular change of basis between $\Psi_\nu(T_{\vec{\mu}})$ for $\vec{\mu} \in P_{\vec{\lambda}}(\tilde{\nu})$ and the Satake basis of $\text{Hom}_{\text{SL}_n}(\mathbb{C}, V_\lambda \otimes V_{\nu}^*)$. In particular, the set of webs arising from the triangular diagrams $T_{\vec{\mu}}$ form a basis.

This result follows directly from the next lemma:

**Lemma 4.3.2.**

$$
\Psi_\nu(T_{\vec{\mu}}) = [Q(A(\vec{\lambda} \sqcup \vec{\nu}^*, \vec{\mu}))] + \sum_{\vec{\gamma} \in P_{\vec{\lambda}}(\nu)} c_{\vec{\mu}}^{\vec{\gamma}} [Q(A(\vec{\lambda} \sqcup \vec{\nu}^*, \vec{\gamma}))].
$$

To prove this result requires the use of the geometric Satake correspondence of Lusztig [Lus83], Ginzburg [Gin], and Mirković-Vilonen [MV07]:

**Theorem 4.3.3.** The representation category $\text{rep}(G)$ is equivalent as a pivotal category to the category $\text{perv}(\text{Gr})$ of perverse sheaves on the affine Grassmannian $\text{Gr}$ constructible with respect to the stratification by the orbits $\text{Gr}(\vec{\lambda})$.

Using this theorem we can transport the problem to the category $\text{perv}(\text{Gr})$. Recall that the orbits $\text{Gr}(\vec{\lambda})$ are smooth and simply connected of complex dimension $2\langle \vec{\lambda}, \rho \rangle$ where $\rho$ is half the sum of the positive roots of $\text{SL}_n$. For any stratified algebraic variety $M$ there is a
pervasive sheaf $\mathbf{IC}_M$, the intersection cohomology sheaf which restricts to the shifted constant sheaf $\mathbb{C}[\dim_{\mathbb{C}} M]$ on the open stratum. In the case that $M$ is smooth, $\mathbf{IC}_M = \mathbb{C}_M[\dim_{\mathbb{C}} M] = \mathbb{D}_M[-\dim_{\mathbb{C}} M]$ where $\mathbb{C}_M$ is the locally constant sheaf on $M$ and $\mathbb{D}_M$ is the dualizing sheaf on $M$. Then for each orbit $\text{Gr}(\lambda)$, there is one simple object in $\text{perv}(\text{Gr})$: $\mathbf{IC}_{\text{Gr}(\lambda)}$ which we extend by zero to all of $\text{Gr}$. Under the geometric Satake correspondence, $\mathbf{IC}_{\text{Gr}(\lambda)}$ corresponds to a sequence of dominant weights, $V_{\lambda}$ is sent to $\mathbf{IC}_{\text{Gr}(\lambda)}$ and if $\tilde{\lambda}$ is a sequence of dominant weights, $V_{\tilde{\lambda}}$ corresponds to $(m_{\tilde{\lambda}})_* \mathbf{IC}_{\text{Gr}(\lambda)}$ where $m_{\tilde{\lambda}} : \text{Gr}(\tilde{\lambda}) \to \text{Gr}$ is the multiplication morphism. By general principles we then have

\[
\text{Hom}_{\text{SL}_n}(C, V_{\tilde{\lambda}}) \cong \text{Hom}_{\text{perv}(\text{Gr})}(\mathbb{C}_{\text{Gr}(0)}, (m_{\tilde{\lambda}})_* \mathbf{IC}_{\text{Gr}(\lambda)}) \cong \\
\text{Hom}_{\text{perv}(\text{Gr})}(\mathbb{C}_{\text{Gr}(0)}, (m_{\tilde{\lambda}})_* \mathbb{D}_{\text{Gr}(\lambda)}[-\dim_{\mathbb{C}} \text{Gr}(\lambda)]) \cong \\
H_{\text{top}}(\text{Gr}(\tilde{\lambda})) = H_{\text{top}}(F(\tilde{\lambda})).
\]

Here $t$ is canonical map between $\mathbf{IC}_{\text{Gr}(\lambda)}$ and $* \mathbb{D}_{\text{Gr}(\lambda)}[-\dim_{\mathbb{C}} \text{Gr}(\lambda)]$.

**Proof of Lemma 4.3.2.** Under the geometric Satake correspondence, $\tilde{\phi}$ corresponds to a map

\[
\tilde{\phi} : \text{Hom}(\mathbb{C}_{\text{Gr}(0)}, (m_{\lambda}^\mu)_* \mathbb{D}_{\text{Gr}(\lambda)}[-2d]) \to \text{Hom}(\mathbb{C}_{\text{Gr}(0)}, (m_{\lambda}^\mu_\nu)_* \mathbf{IC}_{\text{Gr}(\lambda)}).
\]

Here

\[
d = \langle |\lambda| \sqcup \nu^* |, \rho \rangle = \langle \lambda_1 + \cdots + \lambda_k + \nu_1^* + \cdots + \nu_l^*, \rho \rangle = \frac{1}{2} \dim_{\mathbb{C}} \text{Gr}(\lambda \sqcup \nu^*).
\]

On the other hand, we consider the map $\pi : \text{Gr}(\lambda \sqcup \nu^*) \to \text{Gr}(\lambda \sqcup \nu^*)$, the projection onto the first $k$ and the last factors. Then $\pi$ restricts to the map

\[
t : F(\lambda \sqcup \nu^*) \to F(\lambda \sqcup \nu^*),
\]

which projects onto the first $k$ factors.

This in turn defines a push forward in homology

\[
t_* : H_{2d}(F(\lambda \sqcup \nu^*)) \to H_{2d}(F(\lambda \sqcup \nu^*)).
\]

$F(\lambda \sqcup \nu^*)$ has pure dimension $d$ and its components are $Q(A(\lambda \sqcup \nu^*, \bar{\mu}))$ for $\bar{\mu} \in P_{\lambda \sqcup \nu^*}(0)$. The variety $F(\lambda \sqcup \nu^*)$ is no longer pure dimensional, but its irreducible components of dimension $d$
are $Q(A(\bar{\lambda} \sqcup v^*, \bar{\mu}))$ where $\bar{\mu} \in P_\lambda(v)$. If $\bar{\mu} \in P_{\bar{\lambda} \sqcup \bar{v}^*}(0)$ then let $\bar{\mu}_t$ be its truncation at position $k$. Then

$$t(Q(A(\bar{\lambda} \sqcup \bar{v}^*, \bar{\mu}))) = Q(A(\bar{\lambda} \sqcup \mu_k^*, \bar{\mu}_t)).$$

When $\mu_k < v$, the dimension of $Q(A(\bar{\lambda} \sqcup \mu_k^*, \bar{\mu}_t))$ is $\langle |\bar{\lambda} \sqcup \mu_k^*|, \rho \rangle < d$, thus it follows that $t_*$ sends $[Q(A(\bar{\lambda} \sqcup \bar{v}^*, \bar{\mu}))]$ to 0. Otherwise $t$ is an isomorphism from $Q(A(\bar{\lambda} \sqcup \bar{v}^*, \bar{\mu}))$ to its image $Q(A(\bar{\lambda} \sqcup v^*, \bar{\mu}_t))$, so $t$ has degree 1 here and it sends $[Q(A(\bar{\lambda} \sqcup \bar{v}^*, \bar{\mu}))]$ to $[Q(A(\bar{\lambda} \sqcup v^*, \bar{\mu}_t))]$.

Thus if $\bar{\phi} = t_*$ on homology we will be done: if we take $\bar{\mu} \in P_\lambda(v)$ and let $\bar{\mu} \in P_{\bar{\lambda} \sqcup \bar{v}^*}(0)$ be its unique extension then we have:

$$\Psi_v(T_{\bar{\mu}}) = t_*(\Psi(T_{\bar{\mu}}))$$

$$= t_*([Q(A(\bar{\lambda} \sqcup \bar{v}^*, \bar{\mu}))]) + \sum_{\bar{\gamma} \in P_{\bar{\lambda} \sqcup \bar{v}^*}(0)} \sum_{\bar{\gamma} \prec \bar{\mu}} c_{\bar{\mu}}^{\bar{\gamma}} t_*([Q(A(\bar{\lambda} \sqcup \bar{v}^*, \bar{\gamma}^v))])$$

$$= [Q(A(\bar{\lambda} \sqcup v^*, \bar{\mu}))] + \sum_{\bar{\gamma} \in P_\lambda(v)} \sum_{\bar{\gamma} \prec \bar{\mu}} c_{\bar{\mu}}^{\bar{\gamma}} [Q(A(\bar{\lambda} \sqcup v^*, \bar{\gamma}))]$$

Note that $\bar{\gamma} \in P_\lambda(v)$ uniquely extends to $\bar{\gamma} \in P_{\bar{\lambda} \sqcup \bar{v}^*}(0)$ so the coefficient $c_{\bar{\mu}}^{\bar{\gamma}}$ in the last line is the same as in the second.

Now recall that $\bar{\phi}$ is composition with the map

$$id_{(m_{\bar{\lambda}})_* \mathbb{D}_{Gr(\bar{\lambda})}[-2(\bar{\lambda}, \rho)]} \otimes \bar{\phi} \in Hom((m_{\bar{\lambda} \sqcup \bar{v}^*})_* \mathbb{D}_{Gr(\bar{\lambda} \sqcup \bar{v}^*)}[-2d], (m_{\bar{\lambda} \sqcup \bar{v}^*})_* \mathbb{IC}_{Gr(\bar{\lambda} \sqcup \bar{v}^*)}),$$

where $\phi$ is the canonical map in

$$Hom((m_{\bar{\lambda} \sqcup \bar{v}^*})_* \mathbb{D}_{Gr(\bar{\lambda} \sqcup \bar{v}^*)}[-2(\bar{\lambda}, \rho)], \mathbb{IC}_{Gr(\bar{\lambda} \sqcup \bar{v}^*)}).$$

Since $m_{\bar{\lambda} \sqcup \bar{v}^*} = m_{\bar{\lambda} \sqcup \bar{v}^*} \pi$ we have $(m_{\bar{\lambda} \sqcup \bar{v}^*})_* = (m_{\bar{\lambda} \sqcup \bar{v}^*})_* \pi$, which gives the following claim:

**Lemma 4.3.4.** There exists a map

$$\psi \in Hom(\pi_* \mathbb{D}_{Gr(\bar{\lambda} \sqcup \bar{v}^*)}[-2d], \mathbb{IC}_{Gr(\bar{\lambda} \sqcup \bar{v}^*)})$$

such that

$$(m_{\bar{\lambda} \sqcup \bar{v}^*})_* \psi = id_{(m_{\bar{\lambda}})_* \mathbb{D}_{Gr(\bar{\lambda})}[-2(\bar{\lambda}, \rho)]} \otimes \phi.$$
This map exists because by the definition of the convolution tensor product we have
\[
\text{Hom}(m^\ast v^\ast, D_{\text{Gr}(v^\ast)}[-2(|v^\ast|, \rho)]) \cong \text{Hom}(\pi^! D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d], IC_{\text{Gr}(\lambda \sqcup v^\ast)})
\]
and the image of $\phi$ under this map is $\psi$. In fact, this map is characterized by the fact that it is
the identity when restricted to the open dense stratum of $\lambda \sqcup v^\ast$. Thus we have the following
diagram of complexes of constructible sheaves on $\text{Gr}(\lambda \sqcup v^\ast)$:

\[
\begin{array}{ccc}
\pi^! D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d] = \pi^! D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d] & \xrightarrow{\psi} & IC_{\text{Gr}(\lambda \sqcup v^\ast)} \\
\psi & \downarrow{\tilde{\pi}} & \downarrow{t} \\
D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d]
\end{array}
\]

Once again $t$ is the canonical map between the $\text{IC}$ sheaf of a variety and the shifted dualizing sheaf and $\tilde{\pi}$ is the map obtained from applying the adjunction $\pi^! \vdash \pi^1$ to the identity morphism in $\text{Hom}(D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d], D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d])$ after realizing that $D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d] = \pi^! D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d]$.

**Lemma 4.3.5.** The above diagram commutes.

**Proof.** First we notice that
\[
\text{Hom}(D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d], D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d]) \cong \text{Hom}(\pi^! D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d], D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d])
\]
via the adjunction. Since $\text{Hom}(D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d], D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d])$ is one dimensional, so is $\text{Hom}(\pi^! D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d], D_{\text{Gr}(\lambda \sqcup v^\ast)}[-2d])$. Thus the above diagram must commute up to some constant $c$. By restricting to the open dense stratum $\text{Gr}(\lambda \sqcup v^\ast) \subset \text{Gr}(\tilde{\lambda} \sqcup v^\ast)$, we can discover what this constant is. After restricting, each complex becomes the constant sheaf $\mathbb{C}_{\text{Gr}(\tilde{\lambda} \sqcup v^\ast)}[2d]$ and the maps $\tilde{\pi}$ and $t$ both become the identity map. As stated before, the map $\Psi$ is also the identity map. Thus the constant is one and the diagram commutes. \qed
After applying the functors \((m_{\lambda \cup \nu^*})_*\) and \(\text{Hom}(C_{Gr(0)}, \cdot)\) we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}(C_{Gr(0)}, (m_{\lambda \cup \nu^*})_* \mathbb{D}_{Gr(\lambda \cup \nu^*)}[-2d]) & \xrightarrow{\bar{\phi}} & \text{Hom}(C_{Gr(0)}, (m_{\lambda \cup \nu^*})_* \mathbb{IC}_{Gr(\lambda \cup \nu^*)}) \\
\bar{\pi} & \downarrow \bar{t} & \\
\text{Hom}(C_{Gr(0)}, (m_{\lambda \cup \nu^*})_* \mathbb{D}_{Gr(\lambda \cup \nu^*)}[-2d])
\end{array}
\]

The map \(t\) becomes an isomorphism and we have \(\bar{\phi} = \bar{\pi}\). We have

\[
\text{Hom}(C_{Gr(0)}, (m_{\lambda \cup \nu^*})_* \mathbb{D}_{Gr(\lambda \cup \nu^*)}[-2d]) \cong H_{\text{top}}(F(\lambda \cup \nu^*))
\]

and

\[
\text{Hom}(C_{Gr(0)}, (m_{\lambda \cup \nu^*})_* \mathbb{D}_{Gr(\lambda \cup \nu^*)}[-2d]) \cong H_{\text{top}}(F(\lambda \cup \nu^*)),
\]

and under these isomorphisms \(\bar{\pi}\) becomes \(t_*\). Thus \(\bar{\phi}\) is \(t_*\) on homology as needed. 

\[\square\]

### 4.4 Agreement with Earlier Work

#### 4.4.1 SL\(_2\)

In the case of SL\(_2\), due to Frenkel and Khovanov [FK95], the set of cup diagrams of length \(2n\) is in bijection with Lusztig’s dual canonical basis for \((V_1 \otimes 2n)^{SL_2}\). The set of cup diagram or crossless planar matchings is the ways of connecting an even number of points on a line via edges with no crossings.

Such a planar matching is associated with a 2 by \(n\) Young Tableaux and hence with a minuscule Littelmann path. If position \(i\) is the start of an edge, \(i\) is in the top row of the Young Tableaux and the \(i\)-th weight in the path is \((l_i, 0)\) where \(l_i\) is the number of arcs that start at before position \(i\) and end after.

We must check that from the given minuscule Littelmann path we recreate the same cup diagram. Since SL\(_2\) has only one minuscule weight \(\omega_1 = (1, 0)\), it follows \(W \omega_1 = \{(1, 0), (0, 1)\}\)
and thus we have two possible irreducible triangles of length 1:

```
(1,0)  (0,1)
\[ \omega_1 \]
```

Thus the three possible diamonds are

```
\[ \omega_1 \]
```

Here we have not shown the edges that are labelled zero by the construction. Since the dual weight to $\omega_1$ is $\omega_1$ the orientations that appear in the construction do not matter, thus we can drop the orientation and labels all together. Since the diagram corresponding to minuscule Littelmann path had no edges incident to the left or right sides of the triangle and there are no crossings in the diamonds, we get a crossless planar matching. This edges in this matching start at the triangles labelled $(1,0)$ and end at those labelled $(0,1)$, so each minuscule Littelmann path gives a unique crossless planar matching. Since the number of minuscule Littelmann paths is $C_n$, which is also the number of crossless planar matchings, the two constructions agree.

### 4.4.2 $\text{SL}_3$

For $\text{SL}_3$, the work of Greg Kuperberg gives a set of webs, i.e. the graphs generated by two vertices

```
```

The set of basis webs are those graphs with no faces of degree 2 or 4. Since $\omega_1 = \omega_2^*$, we have oriented all edges so that they are labelled with the weight $\omega_1$. Each web has coherent
geodesics and a path of minimal weight in the dual diskoid is also one of minimal length. Each graph corresponds to a unique minuscule Littelmann path and given a minuscule Littelmann path, there is an algorithm in [KK99] which generates the basis web from the given path.

**Lemma 4.4.1.** Given a minuscule Littelmann path $\mu$, $T_\mu$ is a non-elliptic web, i.e. it has no faces of degree 4 or lower.

*Proof.* Consider an internal face $f$ and the triangle of length one, or diamond closest to the top of $T_\mu$ that also contains part of the selected face. Since the face is internal there are two possibilities:

Either of these possibilities would add one vertex to the face. Considering a diamond that contains a left or rightmost point, it follows that the diamonds would be as above and thus each add 2 more vertices to the face. Thus an internal face must have at least 5 vertices. Then $T_\mu$ is an $SL_3$ web with no faces of degree 4 or 2 and thus it is non-elliptic. $\square$

Thus given a minuscule sequence $\tilde{\lambda}$, the set of webs $\{T_\vec{\mu} | \vec{\mu} \in P_\tilde{\lambda} \}$ is contained in the set of basis webs and is the same size since they are both index by $P_\tilde{\lambda}$, thus every basis web is generated. For each minuscule Littelmann path $\vec{\mu}$ the growth algorithm of Khovanov-Kuperberg gives the same web as $T_\vec{\mu}$.
4.4.3 \( \text{SL}_n \) for weights \( \omega_1 \) and \( \omega_{n-1} \)

In [Wes] Westbury produces a set of basis webs for \( \text{Hom}_{\text{SL}_n}( \mathbb{C}, V_{\vec{\lambda}} ) \) when \( \vec{\lambda} \) is a sequence of weights \( \omega_1 \), and \( \omega_{n-1} \). In this case, the minimal diagrams of length 1 are simply:

\[
\begin{align*}
\omega_1 & \quad \omega_{n-1} \\
\omega_{j-1} & \quad \omega_j & \quad \omega_{j-1} \\
& \quad \omega_j
\end{align*}
\]

Since there is only one diagram of length one for each element in the orbits \( W \omega_1 \) and \( W \omega_{n-1} \), there is a unique triangular diagram for each minuscule path. He proves using combinatorial methods that the set of webs resulting from these two diagrams form a web basis. As in the \( \text{SL}_3 \) and \( \text{SL}_2 \) cases, the image of this basis in \( \text{Hom}_{\text{SL}_n}( \mathbb{C}, V_{\vec{\lambda}} ) \) is invariant under rotation.

4.5 Specialization to \( \text{SL}_4 \)

To simplify the notation in this section, an unoriented double edge in a triangular diagram is considered to have label \( \omega_2 \) and any directed edge is considered to have label \( \omega_1 \). This is equivalent to the previous description of webs since \( \omega_2 \) is self dual, so the edges labelled \( \omega_2 \) have no inherent orientation, and \( \omega_1 \) is dual to \( \omega_3 \), so all edges with label \( \omega_3 \) can be reversed to have label \( \omega_1 \). As mentioned earlier, for \( \text{SL}_2 \), \( \text{SL}_3 \) there was a single irreducible triangular diagram of length 1 for each weight in the Weyl orbit. Under the construction of webs via triangular diagrams this means that there was only one possibility for a set of basis webs. On the other hand when we include \( \omega_2 \) in \( \text{SL}_4 \), we have the following choices:

\[
\begin{align*}
(1,0,1,0) & \quad (1,0,1,0) & \quad (0,1,0,1) & \quad (0,1,0,1)
\end{align*}
\]

When forming a triangular diagram using \( T_{(1,0,1,0)} \) or \( T_{(0,1,0,1)} \) it is reasonable to pick either of these diagrams. In most cases there is a unique choice for the diagrams \( T_{(1,0,1,0)} \) and \( T_{(0,1,0,1)} \).
that produces a triangular diagram with the least number of vertices. When the choice is not unique, we show that all choices produce the same web vector. This happens only when we have a minuscule Littelmann path $\vec{\lambda}$ such that there exists indices $i$ and $j$ with the property that $\lambda_{i+1} - \lambda_i = (1, 0, 1, 0), \lambda_{j+1} - \lambda_j = (0, 1, 0, 1)$ and the triangular diagram corresponding to the sub path $\lambda_{i+1}, \ldots, \lambda_{j-1}$ has only edges of weight $\omega_2$ appearing along its sides. For each such pair $(i, j)$ we show that there are two choices that result in a minimal number of vertices.

Call a path $\vec{\mu}$ $\gamma$-dominant if $\mu_i + \gamma$ is dominant for all $i$. A sequence of minuscule weights $\vec{\nu}$ is $\gamma$-dominant if its sequence of partial sums is $\gamma$ dominant.

**Lemma 4.5.1.** If $\vec{\nu}$ is a sequence of minuscule weights, the multiset of edges weights on the left and right edges of $\bigotimes_i T_{\nu_i}$ is independent of the choice of $T_{\nu_i}$ when $\nu_i = (1, 0, 1, 0)$ or $(0, 1, 0, 1)$.

**Proof.** Since the multiset of weights on the left and right sides of the irreducible triangles of length 1 does not depend on this choice and the diagram, let $L_i$ and $R_i$ be the multiset of weights on the left and right of $T_{\nu_i}$. Then the rule for the product of diagrams tells us that the left and right side of $\bigotimes_i T_{\nu_i}$ have multisets of weights $\cup L_i \cup R_i$ and $\cup R_i \cup L_i$ respectively.

**Corollary 4.5.2.** Let $\vec{\nu}$ be a $\gamma$-dominant minuscule path. The multiset of weights on the left of $T_{\vec{\nu}}$ is contained in the multiset of fundamental weights that sum to $\gamma$.

**Theorem 4.5.3.** Let $\vec{\lambda}$ be a sequence of dominant minuscule weights and $\vec{\mu} \in P_{\vec{\lambda}}$. Then there exists a sequence $T_i$ of minuscule diagrams of length 1 such that $\bigotimes T_i$ is associated to $\vec{\mu}$ and has the least possible number of vertices. This sequence is unique up to the following technical condition: Let $\vec{\nu}$ be the sequence of successive differences of $\vec{\mu}$. For each pair $(i, j)$ such that $\nu_i = (1, 0, 1, 0), \nu_{j+1} = (0, 1, 0, 1), \nu_{i+1}, \ldots, \nu_j$ is $k\omega_2$ dominant and $\sum_{l=i+1}^{j} \nu_l = k'\omega_2$ for some $k, k'$ there are two choices for the diagrams $T_i$ and $T_{j+1}$.

**Proof.** For any $\nu_i \not\in \{(1, 0, 1, 0), (0, 1, 0, 1)\}$, we only have one choice of irreducible diagram of length one, set $T_i = T_{\nu_i}$. For $\nu_i = (1, 0, 1, 0)$, let $j$ be maximal such that $\nu_{i+1}, \ldots, \nu_j$ is $k\omega_2$ dominant for some $k$. Thus by the above corollary $\bigotimes_{k=i+1}^{j} T_k$ has only unoriented double edges.
on the left. Since $j$ is maximal, this implies that there is at least one oriented edge on the left of $\otimes_{k=i+1}^{j+1} T_k$. There can be at most 2, since no irreducible diagram of length 1 has more than 2 edges on the left. Suppose there is 1 oriented edge on the left: if it is oriented from left to right we choose $T_i$ to be the first of the two diagram for $\nu_i$, otherwise we chose the second. The previous lemma implies that this edge is independent of any choices for $T_k$ for $i < k \leq j$. In the case that two oriented edges are present we must have $\nu_{j+1} = (0,1,0,1)$ since any other weight would add 0 or 1 edges to the left. In this case there are two choices which minimize crossings: both the first or both the second choices for $T_i$ and $T_{j+1}$. The argument is similar for $\nu_i = (0,1,0,1)$.

In order to see that these choices result in the least number of vertices, making the other choice at any stage results in a diagram with a strictly greater number of vertices. By the previous lemma, these choices do not interfere with each other, so each time we switch the choice in the above construction, we increase the number of vertices.

\begin{lemma}
The invariant vector assigned to the preceding web is independent of the choices made during the construction of the triangular diagram.
\end{lemma}

Figure 4.3 shows one possible choice for the strip of diamonds that join the two triangles. By supposition the triangular diagram taking the place of $A$ in Figure 4.3 has only edges of weight $\omega_2$ incident on the left and right sides. Thus, in general, the only possible diamonds.
that appear in the strip are those pictured in Figure 4.3. In [Kim03] Kim proposes a set of
generators of the kernel of the map $\Psi$ from linear combinations of webs to invariant vectors.
Since these relations lie in the kernel, we can use them to see that two diagrams result in the
same invariant vector. Of the relations that Kim defines, the following 3 are needed to prove
the lemma:

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
\end{tikzpicture}
\end{array}
\quad & = \quad \\
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
\end{tikzpicture}
\end{array}
\end{align}

(4.1)

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}
\quad & = \quad \\
\quad & \quad \quad \quad \text{and} \quad \\
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}
\quad & = \quad \\
\end{align}

(4.2)

Proof. The following sequence of diagrams all share the same invariant vector:

The first and last are the two choices we could make in the construction of the triangular
diagrams. Since the internal edges of weight $\omega_2$ have all other edges adjacent incoming or
outgoing, we can apply equation 4.2. This shows that the second and third diagrams result
in the same invariant vector as the first and last respectively. The resulting two diagrams are a
sequence of squares linking by edges of weight $\omega_2$. We can apply equation 4.1 to move between
these two diagrams. Thus the second and third diagrams have the same invariant vector. \qed
Figure 4.4: The triangular diagram associated to the minuscule Littelmann path $(0, \omega_1, \omega_3 + \omega_1, \omega_2 + \omega_3, \omega_2, 0)$.

Figure 4.5: A diagram associated to the minuscule Littelmann path $(0, \omega_1, \omega_3 + \omega_1, \omega_2 + \omega_3, \omega_2, 0)$ with a minimal number of faces.

Since the resulting set of webs is a basis, the hope would be to use this basis to prove the conjecture of Kim [Kim03] on the generators of the kernel of the map $\Psi$. If it can be shown that any web can be reduced to a sum of the given basis webs by Kim’s relations, then the set of kernel relations would be complete. Unfortunately there does not seem to be any global criterion which picks out the basis webs from the set of coherent webs. For instance the webs in Figures 4.4 and 4.5 both have associated sequence $(0, \omega_1, \omega_3 + \omega_1, \omega_2 + \omega_3, \omega_2, 0)$. Figure 4.4 is build via triangular diagrams is neither minimal with respect to the number of vertices or faces.
4.6 Consequences of the cyclic action

The goal of this section is to prove Theorem 1.4.1 and Theorem 1.5.1 and then derive some corollaries. The proof is based on Theorem 4.1.3. However, we first need to understand the cyclic action on webs and Satake fibers, i.e., the action that results from changing the base point of a polygon or a diskoid.

4.6.1 Rotation of minuscule Littelmann paths

Let $G$ be a simple, simply connected algebraic group. Fix a minuscule sequence $\vec{\lambda}$ of length $n$ and regard the indices of the sequence $\vec{\lambda}$ as lying in $\mathbb{Z}/n$. For each $i \in \mathbb{Z}/n$, we define $\vec{\lambda}(i) = (\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_n = \lambda_0, \lambda_1, \ldots, \lambda_i)$ to be the $i$th cyclic permutation of $\vec{\lambda}$ (so that $\vec{\lambda}(0) = \vec{\lambda}$). We begin by constructing a combinatorial bijection between $P_{\vec{\lambda}}$ and $P_{\vec{\lambda}(1)}$.

Let $W$ be the Weyl group of $G$ and for a weight $\lambda$ of $G$, let $W_\lambda$ be the stabilizer of $\lambda$ in $W$ and $W\lambda$ be the $W$ orbit of $\lambda$. Define a minuscule path $\vec{\nu}_0$ of type $\vec{\lambda}(1)$ by $\nu_0 = \mu_{i+1} - \mu_1$. The resulting path may not be dominant, so we inductively define a path $\vec{\nu}_j$ by setting $a_j \in W$ to be the minimal length word such that $a_j \nu_j - 1$ is dominant. Note that $a_j$ is often the identity. Then let

$$v_i^j = \begin{cases} v_i^{j-1} & \text{if } i < j \\ v_i^{j-1} + (a_j v_j^{j-1} - v_j^{j-1}) & \text{if } i \geq j \end{cases}$$

We set $\mu_i^{(1)} = v_i^n$ for $0 \leq i < n$ and $\mu_n^{(1)} = 0$.

**Lemma 4.6.1.** Let $\gamma, \beta$ be weights of $G$ with $\beta$ dominant and $\gamma - \beta \in W\lambda$ for some minuscule weight $\lambda$. Then the word $w$ of minimal length such that $w\gamma$ is dominant is contained in $W\beta$.

**Proof.** In general, the length of the minimal word $w$ such that $w\gamma$ is dominant is the number of positive roots $\alpha$ such that $\langle \gamma, \alpha^\vee \rangle < 0$. If $\gamma$ is dominant, then we are done. Otherwise there exists a simple root $\kappa$ with $\langle \gamma, \kappa^\vee \rangle < 0$. Since $\beta$ is dominant we have $\langle \beta, \kappa^\vee \rangle \geq 0$ and since
\( \lambda \) is minuscule \( \langle \gamma - \beta, \kappa^\vee \rangle \in \{-1, 0, 1\}. Thus \( \langle \gamma, \kappa^\vee \rangle = -1 \) and \( \langle \beta, \kappa^\vee \rangle = 0 \). This means that \( s_k \beta = \beta \) and \( s_k \in W_\beta \), so we have \( s_k \gamma - \beta \in W \lambda \). Note that \( \langle s_k \gamma, \alpha^\vee \rangle = \langle \gamma, s_k \alpha^\vee \rangle \) and recall that \( s_k \) permutes the positive roots that are not \( \kappa \) and sends \( \kappa \) to \( -\kappa \). Thus the minimal word \( \tilde{w} \) such that \( \tilde{w}s_k \gamma \) is dominant satisfies \( l(\tilde{w}) + 1 = l(w) \). By induction on \( l(w) \), \( \tilde{w} \in W_b \) and we can take \( w = \tilde{w}s_k \).

**Lemma 4.6.2.** We have \( \tilde{\mu}^{(1)} \in P_\lambda^{(1)} \).

**Proof.** Suppose that \( \tilde{\nu}^j \) are defined as above, then we show by induction that the first \( j \) terms of \( \tilde{\nu}^j \) are dominant. The case \( j = 0 \) is clear. If \( \nu_j^{j-1} \) is dominant for \( i < j \), then since \( \nu_j^{j} = \nu_i^{j-1} \) for \( i < j \) we only need to check the dominance of \( \nu_j^{j} \), but \( \nu_j^{j} = a_j \nu_j^{j-1} \) where \( a_j \in W \) is the minimal word such that \( a_j \nu_j^{j-1} \) is dominant. Thus the first \( j \) terms of \( \tilde{\nu}^j \) are dominant as required. Thus \( \tilde{\nu}^1 = \tilde{\nu}^n \) is a dominant path.

We see that \( \tilde{\nu}^0 \) is a path of type \( \tilde{\lambda}^{(1)} \) by construction. If \( \tilde{\nu}^{j-1} \) is a path of type \( \tilde{\lambda}^{(1)} \) then from the definition of \( \tilde{\nu}^j \), we only need check that \( \nu_j^{j} - \nu_j^{j-1} \in W \lambda_j^{(1)} \). Since \( \nu_j^{j-1} - \nu_j^{j-1} \in W \lambda_j^{(1)} \) and \( a_j \) is minimal length, by Lemma 4.6.1 we have

\[
v_j^{j} - \nu_j^{j-1} = a_j \nu_j^{j-1} - \nu_j^{j-1} = a_j(\nu_j^{j-1} - \nu_j^{j-1}) \in W \lambda_j^{(1)}. \tag{4.3}
\]

Thus \( \tilde{\mu}^{(1)} \) is almost a dominant minuscule path of type \( \tilde{\lambda} \), since the only difference between \( \tilde{\mu}^{(1)} \) and \( \tilde{\nu}^n \) is the last term.

Since \( \tilde{\mu}^{(1)} \) and \( \tilde{\nu}^n \) are same except at the last term, we must check that \( 0 - \mu^{(1)}_{n-1} = -\nu_{n-1}^n \in W(\lambda_{n-1}^{(1)}) \). We show by induction that \( \nu_{n-1}^j \in W(-\lambda_{n-1}^{(1)}) = W(-\lambda_1) \) for all \( j \). For \( j = 0 \) \( \nu_{n-1}^0 = -\lambda_1 \). The difference

\[
\nu_{n-1}^{j-1} - \nu_{n-1}^{j-1} = \nu_0^0 - \nu_{n-1}^0 = 0
\]

is a dominant weight so \( \langle \nu_{n-1}^{j-1}, \nu_{n-1}^{j-1}, \alpha^\vee \rangle \geq 0 \) for all simple roots \( \alpha \). By induction we have \( \nu_{n-1}^{j-1} \in W(-\lambda_1) \), so \( \langle \nu_{n-1}^{j-1}, \alpha^\vee \rangle \in \{-1, 0, 1\} \). Since \( \nu_{n-1}^{j-1} \) is a minuscule distance away from a dominant weight, if \( \langle \nu_{n-1}^{j-1}, \alpha^\vee \rangle \leq 0 \) then it must be \( -1 \). But then \( \langle \nu_{n-1}^{j-1}, \alpha^\vee \rangle = -1 \), otherwise \( \nu_{n-1}^{j-1} - \nu_{n-1}^{j-1} \) would not be dominant. This shows that if \( \langle \nu_{n-1}^{j-1}, \alpha^\vee \rangle = -1 \) then \( \langle \nu_{n-1}^{j-1}, \alpha^\vee \rangle = -1 \).
thus $a_j v_j^{j-1} - v_j^j = a_j v_n^{j-1} - v_n^j$. We then have

$$v_n^j = v_{n-1}^j + (a_j v_j^{j-1} - v_j^j) = v_{n-1}^j + (a_j v_{n-1}^{j-1} - v_{n-1}^j) = a_j v_{n-1}^{j-1} \in W(-\lambda_1).$$

It follows that $\bar{\mu}^{(1)}$ is a dominant minuscule path of type $\vec{\lambda}$ and since $\mu_n^{(1)} = 0$, $\bar{\mu}^{(1)} \in P_{\alpha}^{(1)}$. \qed

The generalization to $\bar{\mu}^{(i)}$ is obvious, just apply the above operation $i$ times. Note that at this point it is not clear that $\bar{\mu}^{(n)} = \bar{\mu}$, but this is forced by the connection with the underlying geometry shown in the next section.

In the case that $G = \text{SL}_m$, this algorithm has a more classical interpretation. There is a one to one map from $P_{\alpha}$ to a certain set of tableaux and the map is equivariant with respect to promotion on tableaux and rotation of minuscule paths.

**Lemma 4.6.3.** Let $G = \text{SL}_m$ and $\vec{\lambda} = (\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_n})$, there is a one to one correspondence between rectangular row strict tableaux with $m$ rows and content $(i_1, i_2, \cdots, i_n)$ and the elements of $P_{\vec{\lambda}}$.

**Proof.** Given a tableaux with $m$ rows and content $i$, define the $\mu_j$ as the shape of the tableaux supported on entries 1 to $j$. $\mu_j - \mu_{j-1}$ records the rows in which the entry $j$ appears, and thus $\mu_j - \mu_{j-1} \in W \omega_j$. This is a bijection between the two sets. \qed

Recall that the operation of promotion on row strict tableaux works as follows: delete all boxes labelled 1, then perform jeu-de-taquin. In the $j$-th step, slide the boxes labelled $j + 1$ to the left to fill any gaps and then up fill gaps. Relabel these boxes $j$. At the end, the last column of the tableaux has $i_1$ empty boxes, which get filled with the number $k$. The relationship between promotion on tableaux and rotation on $P_{\vec{\lambda}}$ is as follows: the choice of reflection at each step in rotation on the minuscule path $\bar{\mu}$ are exactly implements the the upwards slides of jeu-de-taquin.

**Theorem 4.6.4.** Under this correspondence, rotation on the elements of $P_{\vec{\lambda}}$ corresponds to promotion of the associated tableaux.
Proof. Let $\vec{\mu} \in P_{\vec{\lambda}}$, $T$ be its corresponding tableaux and $T^p$ its promotion. We show by induction on $j$ that $\vec{\mu}^{(1)}_j$ is the sequence corresponding to $T^p$. At $j = 1$, promotion on the tableaux moves the $i_2$ boxes labelled 2 into the first column of and relabels them 1. The $\mu_1^{(1)} - \mu_0^{(1)}$ is the dominant weight $\omega_{i_2}$, so the correspondence agrees up to the entry 1. Suppose that it agrees up to the entry $j$, then the subtableaux with entries 1 to $j$ corresponds to the subsequence $\mu_0^{(1)}$ to $\mu_j^{(1)}$. The boxes currently labelled $j + 2$ are in rows specified by $\mu_j^{(1)} - \mu_{j+1}^{(1)}$. After relabeling with $j + 1$ and shifting left, the 'tableaux' on entries 1 to $j + 1$ has corresponding path

$\mu_0^{(1)}, \ldots, \mu_j^{(1)}, \mu_j^{(1)} + (\mu_{j+2} - \mu_{j+1})$

the operation of shifting the blocks labelled $j + 1$ up to fill any empty spaces corresponds to taking the dominant weight in the Weyl orbit of $\mu_j^{(1)} + (\mu_{j+2} - \mu_{j+1})$ which is $\mu_{j+1}^{(1)}$. 

In [Rho10] and [PPR09], Rhoades explores the correspondence between promotion and rotation of tensor factors. It should also be noted that the above operation of rotation can naturally be defined on the level of crystals. Let $B(\lambda)$ be the highest weight crystal of highest weight $\lambda$. Let $B(\vec{\lambda}) = B(\lambda_1) \otimes \cdots \otimes B(\lambda_n)$, then the set $\text{Hom}(B(0), B(\vec{\lambda}))$ picks out the disconnected elements of $B(\vec{\lambda})$. These elements $b_1 \otimes \cdots \otimes b_n \in B(\vec{\lambda})$ have the property that $b_1 \otimes \cdots \otimes b_j$ is a highest weight element in $B(\lambda_1) \otimes \cdots \otimes B(\lambda_j)$ and $b_{j+1} \otimes \cdots \otimes b_n$ is a lowest weight element in $B(\lambda_{j+1}) \otimes \cdots \otimes B(\lambda_n)$. Thus we can naturally define a bijection from $\text{Hom}(B(0), B(\vec{\lambda}))$ to $\text{Hom}(B(0), B(\vec{\lambda}^{(1)}))$ by sending $b_1 \otimes \cdots \otimes b_n$ to $c_1 \otimes \cdots \otimes c_n$ where $c_1 \otimes \cdots \otimes c_{n-1}$ is $b_2 \otimes \cdots \otimes b_n$ raised to the highest weight path in its crystal and $c_n$ is $b_1$ lowered to the lowest weight element of its crystal.

In the case of the Littelmann path model of $B(\vec{\lambda})$, by definition we have $\text{Hom}(B(0), B(\vec{\lambda})) = P_{\vec{\lambda}}$ and the operation defined above is the same as the one defined on the level of crystals.

### 4.6.2 Rotation of the Satake basis

As seen above, there is a natural map between $P_{\vec{\lambda}^{(i-1)}}$ and $P_{\vec{\lambda}^{(i)}}$. By Theorem 3.1.1 these sets give rise to the components of $F(\vec{\lambda}^{(i-1)})$ and $F(\vec{\lambda}^{(i)})$, and hence we have a natural correspondence
between the components of these two Satake fibres. On the other hand, we now see that, up to sign, the action of rotation of tensor factors sends the Satake basis of \( \text{Inv}(V(\tilde{\lambda}^{(i-1)})) \) to the Satake basis of \( \text{Inv}(V(\tilde{\lambda}^{(i)})) \).

We now want to make a correspondence between \( F(\tilde{\lambda}^{(i-1)}) \) and \( F(\tilde{\lambda}^{(i)}) \). To do so, we write \( P_i(\tilde{\lambda}) \) for the polygon with edges lengths \( \tilde{\lambda} \) based at the edge labelled \( \lambda_i \). Let us introduce two edge based configuration spaces \( Q(P_i(\tilde{\lambda})) \) and \( Q'(P_i(\tilde{\lambda})) \). In \( Q(P_i(\tilde{\lambda})) \), we require that the endpoints of the based edge in \( P_i(\tilde{\lambda}) \) are sent to \( t^0, t^{\lambda_i} \), whereas in \( Q(P_i(\tilde{\lambda})) \), we require that these endpoints be sent to \( t^{-\lambda_i}, t^0 \). Clearly, multiplication by \( t^{\lambda_i} \) defines an isomorphism between these two configuration spaces.

We have two fibrations coming from the general setup (2.4):

\[
\begin{align*}
Q(P_i(\tilde{\lambda})) & \rightarrow F(\tilde{\lambda}^{(i-1)}) \rightarrow \text{Gr}(\lambda_i) \\
Q(P_i(\tilde{\lambda})) & \cong Q'(P_i(\tilde{\lambda})) \rightarrow F(\tilde{\lambda}^{(i)}) \rightarrow \text{Gr}(\lambda_i^*).
\end{align*}
\]

Since \( \text{Gr}(\lambda_i) \) and \( \text{Gr}(\lambda_i^*) \) are simply connected and irreducible, this gives us bijections between the irreducible components

\[
\text{Irr}(F(\tilde{\lambda}^{(i-1)})) \cong \text{Irr}(Q(P_i(\tilde{\lambda}))), \quad \text{Irr}(F(\tilde{\lambda}^{(i)})) \cong \text{Irr}(Q(P_i(\tilde{\lambda})))
\]

(4.4)

and compatible isomorphisms of vector spaces

\[
H_{\text{top}}(F(\tilde{\lambda}^{(i-1)})) \cong H_{\text{top}}(Q(P_i(\tilde{\lambda}))), \quad H_{\text{top}}(F(\tilde{\lambda}^{(i)})) \cong H_{\text{top}}(Q(P_i(\tilde{\lambda}))).
\]

From the definition of duality in the category \( \text{perv}(\text{Gr}) \), we deduce that the diagram

\[
\begin{align*}
H_{\text{top}}(F(\tilde{\lambda}^{(i-1)})) & \rightarrow \text{Inv}(V(\tilde{\lambda}^{(i-1)})) \\
H_{\text{top}}(Q(P_i(\tilde{\lambda}))) & \rightarrow \text{Hom}(V(\lambda_i^*), V(\lambda_{i+1}, \ldots, \lambda_{i-1})) \\
H_{\text{top}}(F(\tilde{\lambda}^{(i)})) & \rightarrow \text{Inv}(V(\tilde{\lambda}^{(i)}))
\end{align*}
\]

(4.5)
commutes, where the horizontal edges come from the composition of the geometric Satake correspondence with the equivalence between $\text{perv}(\text{Gr})_{\text{min}}$ and $\text{hconv}(\text{Gr})$.

Now let $Z \subseteq F(\hat{\lambda})$ be a component. Using (4.4), we can produce irreducible components $Z_i \subseteq F(\hat{\lambda}(i))$ for all $i \in \mathbb{Z}/n$.

Given $\vec{\mu} \in P_{\hat{\lambda}}$, the component $Z = Q(A(\vec{\lambda}, \vec{\mu}))$ maps to corresponding component $Z_1 \subseteq F(\vec{\lambda}(1))$. By Theorem 3.1.1, there is some path in $P_{\vec{\lambda}(1)}$ giving rise to this component. We claim that in fact, $Z_i = Q(A(\vec{\lambda}(1), \vec{\mu}(1)))$. That is, we wish to prove Theorem 1.5.2:

**Theorem 4.6.5.** Let $Z_{\vec{\mu}}$ be the component of $F(\vec{\lambda})$ corresponding to $\vec{\mu} \in P_{\hat{\lambda}}$ so that

$$[Z_{\vec{\mu}}] \in H_{\text{top}}(F(\vec{\lambda})) = (V(\vec{\lambda}))^G.$$  

Then under rotation on $(V(\vec{\lambda}))^G$, $[Z_{\vec{\mu}}]$ gets sent to

$$[Z_{\vec{\mu}(1)}] \in H_{\text{top}}(F(\vec{\lambda}(1))).$$

Recall that the component $Z_{\vec{\mu}}$ is the closure of $\{L \in \text{Gr}(\vec{\lambda})| L_1 \in \text{Gr}(\mu_i)\}$ and $Z_{\vec{\mu}(1)}$ is the closure of $\{L \in \text{Gr}(\vec{\lambda}(1))| d(L_0, L_1) \in \text{Gr}(\mu_i^{(1)})\}$. Then it is sufficient to show that the subset of $Z_{\vec{\mu}}$ given by $\{L \in Q(A(\vec{\lambda}, \vec{\mu}))| d(L_1, L_i) = \mu_i^{(1)}\}$ is dense. The proof is a variation on Lemma 3.1.2 and Theorem 3.1.1. We need the following lemmas before we begin the proof.

**Lemma 4.6.6.** For $\mu, \nu$ dominant weights, $\lambda$ a dominant minuscule weight:

$$T(\mu, \lambda, \nu) = c \quad \begin{array}{ccc} 
\mu & \rightarrow & \lambda \\
\downarrow & & \downarrow \\
\nu & & a \\
b & & \end{array}$$

If there exists $w \in W/W_2$ with $\mu + w\lambda = \nu$, any element in $Q(T(\mu, \lambda, \nu))$ can be brought to the element $(a, b, c) = (t^0, t^{w\lambda}, t^\nu)$ by multiplication by an element in $G^\lambda(\mathcal{X})$.

**Proof.** Let $f \in Q(T(\mu, \lambda, \nu))$, then let $g_1$ be a representative of $f(a)$, $g_1^{-1}f(a) = t^0$, then since $d(f(a), f(c)) = \nu$, $d(t^0, g_1^{-1}f(c)) = \nu$ and there exists $g_2 \in G^\lambda(\mathcal{O})$ such that $g_2g_1^{-1}f(c) = t^\nu$. Note that $g_2g_1^{-1}f(a) = t^0$. Now, since $\lambda$ is minuscule $\text{Gr}(\lambda) = G^\lambda/M(\lambda)$ where $M(\lambda)$ is the stabilizer of $t^\lambda$, the opposite parabolic to the standard parabolic for $\lambda$. Let $A = W/W_\lambda$. 

\[ \text{Hence, } H_{\text{top}}(F(\vec{\lambda})) = (V(\vec{\lambda}))^G. \]
Consider the stabilizer $M(\nu)$ of $t^\nu$, also the opposite parabolic for $\nu$. Its orbits on $G^\vee/M(\lambda)$ are parametrized by $W_\nu \setminus A$. Due to Lemma 3.1.2 we know that $w \in A$ is the longest element (in the opposite order) of its coset $[w] \in W_\nu \setminus A$ and that $g_2 g_1^{-1} f(b) \in M(\nu) t^w \lambda$, the orbit of $G^\vee/M(\lambda)$ corresponding to $[w] \in W_\nu \setminus A$. Let $g_3 \in M(\nu)$ be such that $g_2 g_1^{-1} f(b) = g_3 t^w \lambda$. Then $g_3^{-1}$ fixes $t^\nu$ and $t^0$, thus $g_3^{-1} g_2 g_1^{-1} \in G^\vee(\mathfrak{X})$ sends $(f(a), f(b), f(c))$ to $(t^0, t^w \lambda, t^\nu)$ as needed.

\[\Box\]

**Lemma 4.6.7.** Let $\mu, \nu, \mu', \nu'$ be dominant weights, $\lambda, \lambda'$ dominant minuscule weights such that all triangles in the following diagram are nonempty.

![Diagram]

Then if there exists a $w \in W_\mu$ with $w(\nu' - \nu) + \mu = \mu'$, any configuration for the triangle $T(\mu, \lambda, \nu)$ can be extended to a configuration of the above diagram. In fact, the set of extensions restrict to a dense subset of the triangle $T(\nu, \lambda', \nu')$.

**Proof.** By the previous result, we only need to consider the following configuration for the base triangle:
This configuration can be obtained from the given one by multiplication by \( t^{-\nu} \). To extend the given configuration we must have \( f(p) \in Gr_{\lambda'} = G^\vee / M(\lambda) \). Consider the stabilizers of \( t^{-\mu} \) and \( t^{-\nu} \), the standard parabolics \( M_+(\mu) \) and \( M_+(\nu) \). There exists \( a \in W / W_{\lambda'} \) such that \( \nu + a\lambda' = \nu' \), so that \( f(p) \in M_+(\nu)t^{a\lambda'} \) and note that \( a \) is the longest element (under the reverse Bruhat order) of its \( W_{\nu} \setminus W / W_{\lambda'} \) orbit. On the other hand, \( \mu + wa\lambda' = \mu' \) since \( w(\nu' - \nu) + \mu = \mu' \). Thus \( f(p) \in M_+(\mu)t^{wa\lambda'} = M_+(\mu)t^{a\lambda'} \) since \( w \in W_{\mu} \). Considering the decompositions into standard Borel orbits: \( M_+(\nu)t^{a\lambda'} = \bigsqcup_{a' \in W_{\nu} a} B_+ t^{a'\lambda'} \) and \( M_+(\mu)t^{a\lambda'} = \bigsqcup_{a' \in W_{\mu} a} B_+ t^{a'\lambda'} \). Thus \( B_+ t^{a\lambda'} \subset M_+(\mu)t^{a\lambda'} \) and it is dense in \( M_+(\nu)t^{a\lambda'} \). Thus the configurations of the above diagram restrict to a dense subset of the configurations of the triangle \( T(\nu, \lambda', \nu') \).

We can now give a proof of Theorem 4.6.5:

**Proof of Theorem 4.6.5.** Given a minuscule sequence \( \bar{\mu} \) we see that \( a_j(\mu_{j+1} - \mu_j) + \mu_j^{(1)} = \mu_j^{(1)} \) where \( a_j \in W_{\mu_{j-1}} \), this follows from equation (4.3). Thus the above theorem holds for

![Diagram](image)

Then the set \( M = \{ L \in Q(A(\lambda, \bar{\mu})) | d(L_1, L_i) = \mu_{i-1}^{(1)} \} \) can be constructed as the fibre product

\[
Q(T_1) \times_{Q(T(\mu_1^{(1)}, \lambda_1, \mu_2))} Q(T_2) \times_{Q(T(\mu_2^{(1)}, \lambda_1, \mu_3))} \cdots \times_{Q(T(\mu_{n-1}^{(1)}, \lambda_1, \mu_n))} Q(T_{n-1}).
\]

The theorem is proved if we can show that \( M \) is dense in \( Z_{\bar{\mu}} \). Recall that \( Z_{\bar{\mu}} \) itself is the closure of \( Q(A(\lambda', \bar{\mu})) \) which can be written as a fibre product of triangles \( T(\mu_j, \lambda_{j+1}, \mu_{j+1}) \) by Theorem 3.1.1. The result follows since by construction, when restricted to the triangle \( T(\mu_j, \lambda_{j+1}, \mu_{j+1}), Q(T_j) \) gives a dense subset of \( Q(T(\mu_j, \lambda_{j+1}, \mu_{j+1})) \). Thus the fibre product
$M$ is a dense subset of the fibre product for $Q(A(\bar{\lambda}, \bar{\mu}))$ and thus is a dense subset of $Z_{\bar{\mu}}$ as needed.

4.6.3 $\text{SL}_3$

In the case $G = \text{SL}_3$, as previously mentioned the non-elliptic, or CAT(0) webs form a basis and from subsection 4.4.2, from $(\bar{\lambda}, \bar{\mu})$, we obtain a diskoid $D = D(\bar{\lambda}, \bar{\mu})$. In $D$, the distances from the base point to the other boundary vertices are given by $\bar{\mu}$. Now for each $i \in \mathbb{Z}/n$, let $\bar{\mu}^{(i)}$ denote the sequence of distances from the $i$th boundary vertex to the rest of the boundary. Since a rotated CAT(0) diskoid is still a CAT(0) diskoid, we see that $D = D(\bar{\lambda}^{(i)}, \bar{\mu}^{(i)})$ as well but with a different base point. Note that this $\bar{\mu}^{(i)}$ and the one previously defined are not obviously equal, but the next lemma shows that they must be. Note along with the previous subsections, this gives an alternate proof of the results from [PPR09] which shows that the rotation of webs corresponds to promotion of the tableaux associated to them via their corresponding minuscule Littelmann paths.

**Lemma 4.6.8.** For each $i$, $Z_i = Q(A(\bar{\lambda}^{(i)}, \bar{\mu}^{(i)})).$

Although this lemma may look purely formal, it is (as far as we know) a non-trivial identification of two different cyclic actions. The cyclic action used to define $Z_i$ can be defined directly from the geometric Satake correspondence; it comes from the fact that the unbased configuration space of $P(\bar{\lambda})$ fibers over Gr in more than one way. The cyclic action on the right, in particular the definition of $\bar{\mu}^{(i)}$, comes instead from rotating webs. The two cyclic actions “should be” the same because the diagram analogous to (4.5) for webs immediately commutes. However, the lemma is non-trivial because it is not true that the invariant vector $\Psi(w(\bar{\lambda}, \bar{\mu}))$ coming from the web equals the fundamental class of the corresponding component.

**Proof.** Our proof uses Theorem 1.6.1, the unitriangularity theorem. Let $M$ be the unitriangular change of basis matrix; the rows of $M$ are labelled by the web basis, while the columns are indexed by the geometric Satake basis. Since both bases are cyclically invariant as in the
diagram (4.5), there is a combinatorial cyclic action on the rows and columns of $M$ that takes $M$ to itself.

Suppose for the moment that $M$ is an abstractly unitriangular matrix whose rows and columns are labelled by two sets $A$ and $B$. In other words, there exists an unspecified bijection $A \cong B$, and a linear or partial ordering of $A$ that makes $M$ unitriangular. Then the partial ordering may not be unique, but the bijection is. If we choose any compatible linear ordering, then it is easy to see that the expansion of $\det M$ has only one non-zero term. This term selects the unique compatible bijection. Since it is unique, it intertwines the two cyclic actions in our case.

Lemma 4.6.8 allows a sharper version of Theorem 1.6.1 than the one proved in Section 4.1. Say that $\vec{v} \leq_S \vec{\mu}$ when $\vec{v}^{(i)} \leq \vec{\mu}^{(i)}$ for all $i \in \mathbb{Z}/n$. Then Theorem 1.6.1 also holds for the weaker partial ordering $\leq_S$. If $D$ and $E$ are the diskoids of $w(\vec{v})$ and $w(\vec{\mu})$, then this condition says that $d_D(p, q) \leq d_E(p, q)$ for every two vertices on their common boundary.

**Question 4.6.9.** Suppose that $D$ and $E$ are two CAT(0) diskoids of type $A_2$ with the same boundary, and suppose that $d_D(p, q) \leq d_E(p, q)$ for every two vertices $p, q$ on the common boundary, as in Theorem 1.6.1. Does it follow that either $D = E$ or $D$ has fewer triangles than $E$?

We define a subset $U \subseteq \mathbb{Z}$ as follows:

$$U = \{(L_i)_{i \in \mathbb{Z}/n} \in F(\vec{\lambda}) | d(L_i, L_j) = \mu_j^{(i)} \}.$$ 

Lemma 4.6.8 shows that $U$ is dense in $\mathbb{Z}$. The following proposition then completes the proof of Theorem 1.4.1.

**Proposition 4.6.10.** Restricting the configuration to the boundary gives an isomorphism

$$\pi : Q_8(D) \xrightarrow{\cong} U.$$ 

**Proof.** By definition, $U$ consists of those configurations of $D$ that preserve all distances between boundary vertices. By [FKK11][Lemma 5.2], any geodesic between two vertices in
$D$ can be extended to a geodesic between two boundary vertices. Thus these are exactly the configurations that preserve all distances in $D$.

If $f \in Q_g(D)$ is a global isometry, then in particular it is an embedding of $D$ into the affine building $\Delta$. This has an interesting area consequence.

**Lemma 4.6.11.** Let $K$ be a 2-dimensional simplicial complex with trivial homology, that is we have $H_*(K, \mathbb{Z}) = H_*(\text{pt})$. Then every simplicial 1-cycle $\alpha$ in $K$ is the homology boundary of a unique 2-chain $\beta$.

**Proof.** If $\beta_1$ and $\beta_2$ are two such 2-chains, then $\beta_1 - \beta_2$ is closed and therefore null homologous. Since $K$ has no 3-simplices, the only way for $\beta_1$ and $\beta_2$ to be homologous is if they are equal.

**Theorem 4.6.12.** If a CAT(0), type $A_2$ diskoid $D$ is embedded in an affine building $\Delta$, then it is the unique least area diskoid that extends the embedding of its boundary $P$.

**Proof.** Let $f$ be the embedding. Then $f_*([D])$ is a 2-chain whose 1-norm is the area of $D$. If $f' : D' \to \Delta$ is another extension of $P$, then $f'_*([D']) = f_*([D])$ and the area of $D'$ cannot be smaller than the area of $D$. Moreover, if they have equal area, then $f^{-1} \circ f$ is a bijection between the faces of $D'$ and the faces of $D$. The faces of $D'$ must be connected in the same way as those of $D$, and attached to $P$ in the same way, because each edge in $\Delta$ has at most two faces of $f(D)$.

By contrast, the $A_2$ spider relations (2.3) reduce the area of a diskoid. The following proposition is easy to check, as well as inevitable given Proposition 4.6.11 and Theorem 1.4.1:

**Proposition 4.6.13.** If $w$ is a web with a face with 2 or 4 sides, so that the dual diskoid $D$ has a vertex with 2 or 4 triangles, then in any configuration $f : D \to \text{Gr}$ these triangles land on top of each other in pairs.

Proposition 4.6.13 thus motivates the relations (2.3) as moves that locally remove area from a configuration $f$. 

4.7 Web bases are not Satake

In Section 4.1, we showed that the transformation between the web basis and the Satake basis is unitriangular with respect to the given ordering. Thus it is reasonable to ask if this transformation is the identity. As with Lustzig’s dual canonical basis, there is an early agreement between the two. For any web with no internal faces, that is, whose dual diskoid has no internal vertices, the image of the map $\pi$ is $Q(A(\vec{\lambda}, \vec{\mu}))$ by Theorem 3.1.3, and $\pi$ is injective. It follows from Corollary 3.4.2 and Lemma 4.1.4 that $[Q(A(\vec{\lambda}, \vec{\mu}))]$ is the web vector.

Now consider the following web $w(\vec{\mu})$, with the indicated base point:

In [KK99], it was shown that this is the first web whose invariant vector is not dual canonical. This is the web associated with the minuscule path

$$\vec{\mu} = (0, \omega_1, \omega_1 + \omega_2, \omega_1 + 2 \omega_2, 3 \omega_2, \omega_1 + 3 \omega_2, 2 \omega_1 + 2 \omega_2, 3 \omega_1 + \omega_2, 3 \omega_1, 2 \omega_1 + \omega_2, \omega_1 + \omega_2, \omega_2, 0)$$

of type

$$\vec{\lambda} = (\omega_1, \omega_2, \omega_1, \omega_1, \omega_2, \omega_2, \omega_1, \omega_1, \omega_2, \omega_1, \omega_2, \omega_1).$$

Let

$$\vec{v} = (0, \omega_1, 0, \omega_2, 0, \omega_1, 0, \omega_2, 0, \omega_1, 0, \omega_2, 0).$$

This is another minuscule path also of type $\vec{\lambda}$; the corresponding web $w(\vec{v})$ is much simpler and is both a Satake vector and a dual canonical vector:
In [KK99], it was shown that

$$\Psi(w(\vec{\mu})) = b(\vec{\mu}) + b(\vec{\nu}),$$

where $b(\vec{\mu})$ denotes the dual canonical basis vector indexed by $\vec{\mu}$.

**Theorem 4.7.1.** Let $w(\vec{\mu}), \vec{\lambda}, \vec{\mu},$ and $\vec{\nu}$ be as above. Then the invariant vector $\Psi(w(\vec{\mu}))$ is not in the Satake basis. More precisely, it has a coefficient of 2 for the basis vector $[Q(A(\vec{\lambda}, \vec{\nu}))].$

**Proof.** We show that the general fiber of $\pi$ over $Q(A(\vec{\lambda}, \vec{\nu}))$ is of size 2. We give the faces of the web the following labels:

![Diagram of a web with labeled points and lines]

If $f \in Q(D(\vec{\lambda}, \vec{\mu}))$ then $\pi(f) \in Q(A(\vec{\lambda}, \vec{\nu}))$ if and only if $f$ assigns $p_i \in \text{Gr}(\omega_1)$ and $\ell_i \in \text{Gr}(\omega_2)$ on those faces and assigns $r^0 \in \text{Gr}(0)$ to all empty faces. In order to determine the fiber of $\pi$ over a point in $Q(A(\vec{\lambda}, \vec{\nu}))$ we must calculate the possible choices for $p'_i, \ell'_i$ and $c$ satisfying the appropriate conditions. Since $p_i \in \text{Gr}(\omega_1)$ and $\ell_i \in \text{Gr}(\omega_2)$, this forces $p'_i \in \text{Gr}(\omega_1)$ and $\ell'_i \in \text{Gr}(\omega_2)$ and $c \in \text{Gr}(\omega_1 + \omega_2)$. We can think of the points of $\text{Gr}(\omega_1)$ and $\text{Gr}(\omega_2)$ as, respectively, the points and lines in $\mathbb{C}\mathbb{P}^2$. Then the conditions given by the edges of the web are as following: $p'_i$ is a point on the line $\ell_i$ and $\ell'_i$ is a line containing the points $p_i, p'_{i-1}$ and $p'_i$.

Suppose that either the $p_i$ are not collinear and or the $\ell_i$ are not concurrent. Then by the duality of points and lines, we may assume that the $\ell_i$ are not concurrent. Let $e_i$ be the intersection of $\ell_i$ and $\ell_{i+1}$. Then we can express the points $p'_i$ in barycentric coordinates given
by $e_i$:

\[
p_1' = (t_1, 0, 1 - t_1)
\]
\[
p_2' = (1 - t_2, t_2, 0)
\]
\[
p_3' = (0, 1 - t_3, t_3).
\]

Note that by doing this we restrict ourselves to an affine subspace of $\mathbb{P}^2$, so we may lose, but we don’t gain solutions. The collinearity condition results in the equations

\[
p_i = (1 - s_i)p_1' + s_ip_{i-1}.
\]

Solving this problem amounts to solving

\[
(1 - s_1)t_1 = p_{11} \\
(1 - s_2)t_2 = p_{22} \\
(1 - s_3)t_3 = p_{33}
\]
\[
s_1(1 - t_3) = p_{12} \\
s_2(1 - t_1) = p_{23} \\
s_3(1 - t_2) = p_{31},
\]

where $p_{ij}$ are the barycentric coordinates of the $p_i$. If none of these coordinates are 0, then we

Figure 4.6: The two solutions to the problem for the given $\ell_i$ and $p_i$. 
can eliminate all but one variable to get the relation
\[
t_1 = \frac{p_{11}}{1 - \frac{p_{12}}{1 - \frac{p_{33}}{1 - \frac{p_{31}}{1 - \frac{p_{22}}{1 - \frac{p_{23}}{1 - t_1}}}}}}.
\]
The right side of this equation is a composition of fractional linear transformations that condenses to a single fractional linear transformation
\[
t_1 = \frac{\alpha_{11}t_1 + \alpha_{12}}{\alpha_{21}t_1 + \alpha_{22}}
\]
with generic coefficients. Thus, generically, we obtain a quadratic equation for \(t_1\) with 2 solutions.

It remains to determine the face \(c\), which lies in \(\text{Gr}(\omega_1 + \omega_2)\). If \(c \not\in \text{Gr}(0)\), then the conditions given by the edges of the web would be \(p_i' = p_j'\) and \(\ell_i' = \ell_j'\) for all \(i, j\) which cannot happen since either \(p_i\) are not collinear or \(\ell_i\) are not concurrent. Thus for any solution of the above equations, we get exactly one element in \(Q(D(\tilde{\lambda}, \tilde{\mu}))\). And for any generic point \(p \in Q(A(\tilde{\lambda}, \tilde{\nu}))\), the fiber \(\pi^{-1}(p)\) has 2 points.

Let \(X\) denote the closure in \(Q(D(\tilde{\lambda}, \tilde{\mu}))\) of the union of all fibers \(\pi^{-1}(p)\) with 2 points. Then \(X\) is either a component of \(Q(D(\tilde{\lambda}, \tilde{\mu}))\) or a union of two components. Moreover, \(X\) contains all components of \(Q(D(\tilde{\lambda}, \tilde{\mu}))\) which map onto \(Q(A(\tilde{\lambda}, \tilde{\nu}))\). Since the above argument shows that the scheme-theoretic fiber of \(\pi\) over a general point of \(Q(A(\tilde{\lambda}, \tilde{\nu}))\) is two reduced points, we also know that \(X\) is generically reduced. Hence the coefficient of \([X]\) in the homology class \(c(w)\) from Theorem 3.4.1 is 1. Since the map \(\pi : X \to Q(A(\tilde{\lambda}, \tilde{\nu}))\) is of degree 2 and since \(X\) contains all components mapping to \(Q(A(\tilde{\lambda}, \tilde{\nu}))\), the coefficient of \([Q(A(\tilde{\lambda}, \tilde{\nu}))]\) in \(\pi_*(c(w))\) is 2. In particular, \(\pi_*(c(w))\) differs from \([Q(A(\tilde{\lambda}, \tilde{\mu}))]\), as desired.

In fact, we suspect that \(Q(D(\tilde{\lambda}, \tilde{\mu}))\) only has two components, which would imply that
\[
\Psi(w(\tilde{\mu})) = [Q(A(\tilde{\lambda}, \tilde{\mu}))] + 2[Q(A(\tilde{\lambda}, \tilde{\nu}))].
\]
Otherwise, \(\Psi(w(\tilde{\mu}))\) has these two terms and perhaps others. Either way, the coefficient of 2
is different from what arises in the dual canonical basis [KK99]:

\[ \Psi(w(\bar{\mu})) = b(\bar{\mu}) + b(\bar{\nu}). \]

Thus,

**Theorem 4.7.2.** The geometric Satake bases for invariants of \( G = \text{SL}_3 \) are eventually not dual canonical.

This is not such a surprising statement in light of the well-known fact that the canonical and semicanonical basis do not coincide (as a consequence of the work of Kashiwara-Saito [KS97]). In both Theorem 4.7.2 and in the canonical/semicanonical situation, a homology basis does not coincide with a basis defined using a bar-involution. The analogy between these two results could perhaps be made precise using skew Howe duality (\( \text{SL}_3, \text{SL}_n \)-duality).

It is known that \( \Psi(w(\bar{\mu})) \) is the first basis web that is not dual canonical, i.e., the only basis web up to rotation with 12 or fewer minuscule tensor factors. We conjecture that it is also the first basis web for \( \text{SL}_3 \) that is not geometric Satake. Equivalently, we conjecture that all three bases first diverge at the same position.

**Question 4.7.3.** For arbitrary \( G \), is the dual canonical basis of an invariant space \( \text{Inv}_G(V(\lambda)) \) positive unitriangular in the geometric Satake basis?
Bibliography


