DEVELOPMENT OF METHODS FOR RETROSPECTIVE ULTRASOUND TRANSMIT FOCUSING

by

Renée K. Warriner

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy

The Edward S. Rogers Sr. Department of Electrical and Computer Engineering

Collaborative program with the Institute of Biomaterials and Biomedical Engineering

University of Toronto

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ABSTRACT

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Renée K. Warriner
Doctor of Philosophy, 2012
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Single frame ultrasound B-mode image quality is largely governed by the ability to focus the ultrasound beam over a range in depths both in transmission and reception. By developing a comprehensive understanding of acoustic wave propagation two signal processing methods were identified for solving the transmission problem. We made use of both the impulse response using the classical point spread function (PSF) and the spatial sensitivity function (SSF) which describes the spatial distribution at a particular time.

Using the angular spectrum method, an accurate analytical model was developed for the field distribution arising from a finite geometry, apodized and focused, plane piston transducer. While there is a thorough understanding of the radiated field arising from uniformly excited plane piston transducers, the focused equivalent (i.e., one that allows a continuous change in phase over the plane piston surface) is incomplete and assumes the Fresnel approximation. Our model addresses the effects of diffraction and evanescent waves without the use of the Fresnel approximation and is applicable at all near- and far-field locations in a lossless medium. The model was analyzed to identify new insights into wave propagation and compared with the Fresnel approximation and the spherically-focused, concave transducer.

The piston transducer model was then extended to an attenuating and dispersive medium. After analysing existing models of power-law frequency dependent attenuation, a causal,
spherical wave Green’s function was derived from the Navier-Stokes equation for a classical viscous medium. Modifications to the angular spectrum method were presented and used to analyze the radiated field of a focused, planar piston transducer.

Finally, after presenting our signal processing strategy for improving imaging spatial resolution through minimization of the SSF, two signal processing methods were derived and analysed in simulation: a deconvolution technique to remove the effects of the ultrasound excitation wave and suppress additive noise from the received ultrasound signal, and a retrospective transmit focusing method that changed the response from a predefined transmit focus to an arbitrary transmit focal depth. Proof-of-concept simulations were presented using a variable number of scatterers and compared with the traditional matched filtering and envelope detection technique.
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TABLE OF CONTENTS

ABSTRACT ......................................................................................................................... ii

ACKNOWLEDGEMENTS ................................................................................................. iv

TABLE OF CONTENTS .................................................................................................... v

LIST OF TABLES ............................................................................................................. viii

LIST OF FIGURES ........................................................................................................... ix

LIST OF SYMBOLS AND ABBREVIATIONS ................................................................. xiii

CHAPTER 1 INTRODUCTION ......................................................................................... 1

1.1 Measures of image quality ......................................................................................... 2
1.2 Strategies to improve image quality ........................................................................... 4
1.3 Outline of thesis study ............................................................................................... 5
  1.3.1 Research motivation and hypothesis .................................................................. 5
  1.3.1.1 Ultrasound hardware circuitry ...................................................................... 6
  1.3.1.2 Hypothesis .................................................................................................. 7
  1.3.1.3 Linear model .............................................................................................. 7
  1.3.2 Research objectives ............................................................................................ 9
1.4 Organization of thesis .............................................................................................. 10

CHAPTER 2 ACOUSTIC WAVE PROPAGATION THEORY ......................................... 11

2.1 Navier-Stokes and Helmholtz equations ................................................................... 11
2.2 Angular spectrum method ....................................................................................... 13
  2.2.1 Velocity and velocity potential ....................................................................... 15
  2.2.2 Velocity impulse response ............................................................................. 16
  2.2.3 Wave number .................................................................................................. 17
  2.2.3.1 Case 1: Homogenous wave ....................................................................... 18
  2.2.3.2 Case 2: Evanescent wave ......................................................................... 18
  2.2.3.3 Case 3: Singularity ................................................................................... 19
  2.2.4 Green's function ............................................................................................. 19
  2.2.5 Angular spectrum in cylindrical coordinates ................................................. 20
2.3 Assumptions and simplifications ............................................................................. 22
2.4 Conclusions ............................................................................................................. 22

CHAPTER 3 ANALYTICAL TRANSDUCER MODEL ............................................... 23

3.1 Background .............................................................................................................. 23
3.2 Angular spectrum method ....................................................................................... 27
3.3 Transducer velocity ................................................................................................. 28
  3.3.1 Finite geometry and diffraction ....................................................................... 28
  3.3.2 Gaussian apodization ..................................................................................... 29
  3.3.3 Focusing ......................................................................................................... 29
  3.3.4 Transducer velocity ....................................................................................... 30
3.4 Angular spectrum solution .................................................. 31
  3.4.1 Case A: Accurate focusing equation ................................. 31
  3.4.2 Case B: Fresnel approximation ........................................... 34
3.5 Comparison with spherically-shaped concave piston transducer .......... 35
  3.5.1 Focused plane piston .................................................. 36
  3.5.2 Spherically-shaped concave piston transducer .............. 37
3.6 Assumptions and simplifications ........................................ 38
3.7 Simulation results .......................................................... 38
  3.7.1 Transducer angular spectrum ......................................... 40
  3.7.2 Observation plane angular spectrum—Phase ...................... 43
  3.7.3 Velocity potential ..................................................... 45
  3.7.4 The point spread function and spatial sensitivity function .... 48
  3.7.5 Fresnel approximation ............................................... 50
  3.7.6 Spherically-shaped concave transducer comparison ........ 51
3.8 Discussion ........................................................................ 57
3.9 Conclusions ........................................................................ 59

CHAPTER 4 ATTENUATION AND DISPERSION ...................................... 60

  4.1 Background .................................................................... 61
    4.1.1 Complex wave number ............................................... 62
  4.2 Models of attenuation ...................................................... 68
    4.2.1 Power-law frequency dependent attenuation .................. 68
    4.2.2 Classical viscous loss ............................................... 71
      4.2.2.1 Spherical wave propagation in a classical viscous medium .. 72
  4.3 Modified angular spectrum method .................................... 77
  4.4 Assumptions and simplifications ...................................... 78
  4.5 Simulation results .......................................................... 78
    4.5.1 Green’s function ..................................................... 78
    4.5.2 Focused, plane piston transducer ................................ 80
      4.5.2.1 Angular spectrum ............................................... 80
      4.5.2.2 Velocity potential ............................................. 83
      4.5.2.3 Point spread function and spatial sensitivity function .... 85
  4.6 Discussion ....................................................................... 89
  4.7 Conclusions ..................................................................... 90

CHAPTER 5 RETROSPECTIVE TRANSMIT FOCUSING ............................. 92
  5.1 Strategy ........................................................................ 92
  5.2 Excitation voltage and noise suppression ............................ 94
    5.2.1 Wiener deconvolution ............................................... 95
    5.2.2 Linear Minimum Mean Squared Error (LMMSE) .......... 96
  5.3 Retrospective transmit focusing method .................................. 97
    5.3.1 Complex dot product ............................................... 103
  5.4 Assumptions and simplifications ....................................... 106
  5.5 Results ........................................................................... 107
    5.5.1 Ultrasound simulation system ..................................... 107
    5.5.2 Excitation voltage and noise suppression .................. 113
      5.5.2.1 SSF .............................................................. 119
<table>
<thead>
<tr>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.5.3 Validation of complex dot product mathematics</td>
</tr>
<tr>
<td>5.5.4 Retrospective transmit focusing method</td>
</tr>
<tr>
<td>5.5.4.1 Retrospective transmit focusing – Solution</td>
</tr>
<tr>
<td>5.6 Discussion</td>
</tr>
<tr>
<td>5.7 Conclusions</td>
</tr>
</tbody>
</table>

**CHAPTER 6 CONCLUSIONS**

<table>
<thead>
<tr>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1 Summary and conclusions</td>
</tr>
<tr>
<td>6.2 Primary research contributions</td>
</tr>
<tr>
<td>6.3 Suggestions for future work</td>
</tr>
<tr>
<td>6.3.1 Transducer model</td>
</tr>
<tr>
<td>6.3.2 Attenuation and dispersion model</td>
</tr>
<tr>
<td>6.3.3 Signal processing solutions</td>
</tr>
<tr>
<td>6.4 Final remarks</td>
</tr>
</tbody>
</table>

**REFERENCES**

**APPENDIX A ASSUMPTIONS AND SIMPLIFICATIONS**

<table>
<thead>
<tr>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1 Angular spectrum method</td>
</tr>
<tr>
<td>A.2 Piston transducer model in lossless medium</td>
</tr>
<tr>
<td>A.3 Attenuation and dispersion model</td>
</tr>
<tr>
<td>A.4 Signal processing algorithms</td>
</tr>
</tbody>
</table>

**APPENDIX B HANKEL CONVOLUTION**

**APPENDIX C HANKEL TRANSFORM OF FOCUSING TERM**
LIST OF TABLES

Table 3.1: Finite geometry piston transducer simulation parameters ........................................... 39
Table 3.2: Simulation test cases for a finite geometry piston transducer. .................................... 39
Table 3.3: Rayleigh distance and FWHM results. ........................................................................... 48
Table 3.4: Focused plane piston and concave piston simulation parameters. .............................. 52
Table 4.1: Attenuating medium simulation parameters................................................................. 64
Table 5.1: Apodized and focused plane piston transducer simulation parameters ...................... 108
Table 5.2: LMMSE filtering simulation parameters........................................................................ 114
Table 5.3: Retrospective focusing simulation parameters .............................................................. 124
LIST OF FIGURES

Figure 1.1: Sketch describing the B-mode ultrasound imaging system.................................2
Figure 1.2: Sketch showing the relationship between a point source object and its image, the
point spread function..................................................................................................................3
Figure 1.3: Cross-sectional view of the point spread function to identify the spatial resolution and
contrast.........................................................................................................................................3
Figure 1.4: Ultrasound hardware circuitry flowchart.................................................................6
Figure 1.5: Sketch illustrating the point spread function (PSF) and the spatial sensitivity function
(SSF). The PSF identifies the impulse response over time for a single spatial location. The
SSF describes the spatial distribution of the wave at a single point in time..............................8
Figure 3.1: Sketch of a focused and apodized plane piston with a focal point F, showing the
cylindrical geometry used in the analysis. It is assumed that the amplitude \( \alpha \) and phase \( \phi \) of
the velocity vary over the piston surface to minimize diffraction effects and to produce
focusing.........................................................................................................................................26
Figure 3.2: Sketch of a spherically-shaped concave piston and a focused plane piston transducer
that have the same aperture and focal point..............................................................................35
Figure 3.3: Transducer velocity magnitude (apodization, \( \alpha(r) \)) and phase (focusing, \( \phi(r) \))
distributions for the assumed values of \( F \) and \( \sigma \) given in Table 3.2.....................................40
Figure 3.4: Angular spectra results on the transducer plane \((z=0)\). The normalized magnitudes
are shown for (a) Cases 1 and 2 (unfocused) and for (b) Cases 3 and 4 (focused) together
with the response for the focusing term in equation (3.13). (c) Phase characteristics on the
transducer plane..........................................................................................................................42
Figure 3.5: Phase characteristics for Case 4, in addition to the focusing term from (3.13) on
several observation planes. Note that on the focal plane \((z = 4.5 \text{ cm})\) the phase is almost
constant for \( k_r < 120 \text{ rad/cm} \)...............................................................................................44
Figure 3.6: Normalized velocity potential \([\text{dB}]\) around the focus. (a) Case 1, (b) Case 2, (c) Case
3, (d) Case 4................................................................................................................................46
Figure 3.7: Velocity potential phase for Case 4 on three observation planes around the focus.
The radial range results were shown corresponding to normalized velocity potential
magnitudes to \(-40\text{dB}\). ....................................................................................................................46
Figure 3.8: Magnitude of velocity potential \([\text{dB}]\) using an expanded field of view. (a) Case 1
and (b) Case 2...............................................................................................................................47
Figure 3.9: SSF results \([\text{dB}]\) for (a) Case 1, (b) Case 2, (c) Case 3, and (d) Case 4, displaying an
overlay of three time responses around the focus........................................................................49
Figure 3.10: Velocity potential of the apodized, focused piston transducer using the Fresnel
approximation. (a) On-axis velocity potential. (b) Velocity potential magnitude \([\text{dB}]\).
Compare with the accurate focusing solution from Figure 3.6d....................................................51
Figure 3.11: Showing the impulse responses at various axial locations for the spherically-shaped concave piston and focused plane piston transducer at: (a) 35 mm and (b) 55 mm, (c) closer to the focus: at 43 mm and 47 mm. Note that the vertical scale is in km/s and that the focus for both transducers is at 45 mm. .................................................................53

Figure 3.12: Comparing the transient response of spherically-shaped concave piston and focused, plane piston transducer. The excitation was a 5 MHz Gaussian modulated pulse with a 100% bandwidth. The pulse shape is shown at the top right. (a) Response at 43 mm (focus is at 45 mm), (b) Response at 47 mm. .................................................................54

Figure 3.13: On-axis velocity potential for 5MHz excitation illustrating the differences between the two transducers in the frequency domain. (a) Normalized magnitude, (b) Phase difference between the two transducers. .................................................................56

Figure 3.14: Sketches showing the incremental sections of (a) spherically-shaped concave piston and (b) focused, plane piston transducer that contribute to the impulse response over a small increment of time. The dashed lines in (b) represent the duration of the delay times. .................................57

Figure 4.1: (a) Attenuation coefficient $\alpha(f)$ for power-law frequency dependent attenuation and classical viscous media from (4.2) and (4.5). (b,c) Corresponding fractional change in wave speed from (4.4) and (4.5). The red and magenta dashed curves are coincident. .......................66

Figure 4.2: Frequency dependent attenuation coefficient and fractional change in wave speed over $\omega \gamma$ in a viscous medium for $\gamma = 10^{-7}$ s (Buckingham 2005). Since the curves refer to different metrics, as specified in the descriptions adjacent to each curve, the vertical axis remains dimensionless. The horizontal axis is in terms of $\omega \gamma$ since these two parameters are paired in the attenuation and wave speed equations. .................................................................67

Figure 4.3: Normalized time Green's function results at $R$ for both classical viscous media and frequency dependent attenuation.................................................................79

Figure 4.4: Angular spectrum magnitude on the imaginary axis $s_r = \alpha_r + jk_r =jk_r$ for several observation planes at 5MHz. While the shape of the curve is consistent regardless of observation plane depth, the entire observation plane curve decays with $z$. The results are normalized to that on the transducer plane. Results are shown for frequency dependent attenuation using $n = 1$, $\alpha = 0.5$ dB/cm at 1MHz. .................................................................82

Figure 4.5: Angular spectrum for a classical viscous medium $(\gamma = 10^{-7}$ s) (a) Normalized magnitude on focal plane at 5MHz. (b) Phase on various observation planes. On the focal plane, the phase is not constant due to dispersion. However, the 5 MHz signal will focus near $z = 3$ cm due to the nearly constant phase over $k_r < 40$ rad/cm. .................................................................83

Figure 4.6: Velocity potential results [dB] at 5 MHz for the three viscosity and power-law frequency dependent attenuation cases assuming an unapodized, focused piston transducer. ..84

Figure 4.7: PSF results at several positions around the focus for the low and medium viscosity, and power-law frequency dependent attenuation cases. Note that while the magnitude of the response at the focus can be significantly attenuated due to the loss of high spatial frequency content, its effect further from the focus can be minimal. .................................................................86
Figure 4.8: PSF results for high viscosity and power-law frequency dependent attenuation. Note that the focal response is not distinguishable in the responses, unlike that observed in Figure 4.7.

Figure 4.9: SSF results [dB] for three time points as the wave passes through the 4.5 cm focal point. Results were normalized to the peak magnitude during the time range shown. Since attenuation was observed to significantly affect the focal point PSF response in Figure 4.7, the relative SSF magnitudes at positions near the focus become more significant, thereby increasing the lateral and axial span in subplots (b,d,e).

Figure 5.1: Spatial sensitivity functions (SSF) a) Transducer impulse response on transmission into the medium. b) Received ultrasound signal after temporal convolving several terms in (1.1).

Figure 5.2: Graphical representation of equation (5.11).

Figure 5.3: Graphical representation of $G(r,z_1 : \omega) # [B(r_1,z_1 : \omega)\delta (\sqrt{(r_1-x)^2+y^2} , z-z_1 , \omega)]$ from (5.13).

Figure 5.4: Final graphical representation of (5.13). The summation of all scatterer planes followed by element wise multiplication and summation of the transducer surface velocity generates the scalar received signal, $D(\omega)$.

Figure 5.5: Graphical representation of the relationship between vectors $F_{vec}$, $M_{vec}$, and $V_{vec}$, and angles $\zeta_{FM}$, $\alpha_{VM}$, and $\gamma_{VF}$ in two 2D views. a) Sideview. Note that the three vectors are not necessarily coplanar. b) Frontview. The sideview plot was rotated to show vector $V_{vec}$ pointing out of the page. Vector $V_{vec}$ can lie anywhere on the dotted circle around $F_{vec}$.

Figure 5.6: 5-cycle, tapered cosine excitation wave at 5 MHz used in ultrasound simulation system.

Figure 5.7: SSF [dB] of transmit transducer impulse response for three times: 32.7 $\mu$s, 36.0 $\mu$s, and 39.3 $\mu$s, surrounding the 4.5 cm focal point. Results are shown in cylindrical coordinates over the axial $z$ and radial $r$ axes.

Figure 5.8: SSF [dB] of total transducer impulse response (transmit and receive) for three times: 65.5 $\mu$s, 72.1 $\mu$s, and 78.7 $\mu$s, surrounding wave propagation through the 4.5 cm focal point.

Figure 5.9: SSF of total transducer transmit/receive impulse response with excitation wave for three times: 65.5 $\mu$s, 72.1 $\mu$s, and 78.7 $\mu$s, corresponding to wave propagation through the focal point. Results are normalized.

Figure 5.10: Normalized received ultrasound signal (a) with excitation wave; (b) without excitation wave.

Figure 5.11: Received ultrasound signal including the effects of additive white Gaussian noise (AWGN) at an SNR of 20 dB. (a) Response with the excitation wave, (b) Response without the excitation wave. The legend applies to both plots.

Figure 5.12: Simulation results for the LMMSE and matched filtering techniques. While the signal recovery capabilities of the LMMSE method are excellent as the results are nearly
coincident with the ideal ultrasound response without the excitation wave and noise, the matched filter results cannot recover such high resolution.

Figure 5.13: SSF plots for three time points around the 4.5 cm focal point. The oscillating black and white bars correspond to the local minima and maxima of the excitation wave, while the speckle pattern represents the variation in backscattering cross-section over the insonated medium. Results are normalized.

Figure 5.14: SSF results [dB] for three time points around the 4.5 cm focal point after filtering the excitation wave and additive noise using the LMMSE method. (a) SSF at 65.5 µs, (b) SSF at 72.1 µs, (c) SSF at 78.7 µs.

Figure 5.15: SSF results [dB] for three time points around the 4.5 cm focal point after matched filtering and envelope detection.

Figure 5.16: Received transducer response without excitation wave or additive noise. (a) Identical transmit and receive transducer configurations focused at 3.5 cm, (b) Results after retrospective transmit focusing \( F_{tx} \) to 5.5 cm. All relevant vector information was assumed known \textit{a priori}.

Figure 5.17: Temporal frequency response of 5 on-axis scatterers using retrospective transmit focusing. (a) Magnitude of received signal over frequency assuming \( F_{tx} = F_{rx} \) for 3.5 cm to 5.5 cm. (b) Magnitude of \( P(\omega) \) required to retrospectively modify the received signal (c) Phase of \( P(\omega) \) required to retrospectively modify the received signal.

Figure 5.18: Unwrapped phase of \( P(\omega) \) over focal depth in 250 kHz increments up to 3.25 MHz. Since the phase increases fairly monotonically with angular frequency, frequencies in between marked curves should be assumed at 250 kHz between adjacent labels.

Figure 5.19: Retrospectively modified transducer using the exact magnitude and phase of \( P(\omega) \). The vertical axis is dimensionless since each individual curve is normalized and multiple curves are superimposed in the figure. For each curve, the constant baseline response is zero, while the maximum peak is normalized to one.

Figure 5.20: Retrospectively modified transducer using only the exact phase of \( P(\omega) \). The vertical axis is dimensionless as described in the caption of Figure 5.19.

Figure 5.21: Retrospectively modified transducer using all 40 simulated transmit focal depths in the second-order polynomial regression for the phase of \( P(\omega) \). The vertical axis is dimensionless as described in the caption of Figure 5.19.

Figure 5.22: Retrospectively modified transducer using only 3 points in the second-order polynomial regression for the phase of \( P(\omega) \). The vertical axis is dimensionless as described in the caption of Figure 5.19.

Figure 5.23: PSF for a single scatterer at 5 cm before and after retrospectively transmit focusing at 5 cm. Results are shown in dB to identify mainlobe and sidelobe details. The different transmit focal depths will cause the two responses to occur at different times, however they have been shown superimposed on the red plot here. Correct timing is shown in Figure 5.22.
### LIST OF SYMBOLS AND ABBREVIATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D, 2D, 3D</td>
<td>One, two, three dimensional</td>
</tr>
<tr>
<td>$a$</td>
<td>Piston transducer radius [m]</td>
</tr>
<tr>
<td>$a$</td>
<td>Complex-valued vector</td>
</tr>
<tr>
<td>$A = \sqrt{a^2 + F^2}$</td>
<td>Substitution variable</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Amplitude of transducer surface velocity</td>
</tr>
<tr>
<td>A/D</td>
<td>Analog to digital converter</td>
</tr>
<tr>
<td>$b = \frac{-1}{2\sigma^2} + \frac{jk}{2F}$</td>
<td>Substitution variable</td>
</tr>
<tr>
<td>$b$</td>
<td>Complex-valued vector</td>
</tr>
<tr>
<td>$bsc(r)$</td>
<td>Backscattering cross-section of scatterer</td>
</tr>
<tr>
<td>$B = h_{u_s}(r : t) \cdot bsc(r)$</td>
<td>Substitution variable</td>
</tr>
<tr>
<td>$c(\omega)$</td>
<td>Frequency dependent wave speed</td>
</tr>
<tr>
<td>$c_0$</td>
<td>Equilibrium wave speed [m/s]</td>
</tr>
<tr>
<td>$C_1, C_2$</td>
<td>Constants to be solved from Helmholtz formula</td>
</tr>
<tr>
<td>$d$</td>
<td>Derivative operator</td>
</tr>
<tr>
<td>$d(t)$</td>
<td>Desired signal</td>
</tr>
<tr>
<td>$\hat{d}(t)$</td>
<td>Minimum mean-square estimate of $d(t)$</td>
</tr>
<tr>
<td>$\partial$</td>
<td>Partial derivative operator</td>
</tr>
<tr>
<td>$D$</td>
<td>Rate of change of the moving fluid particle velocity</td>
</tr>
<tr>
<td>$D(\omega)$</td>
<td>Fourier transform of desired signal $d(t)$</td>
</tr>
<tr>
<td>$D_{new}(\omega)$</td>
<td>Retrospectively modified received ultrasound signal</td>
</tr>
<tr>
<td>D/A</td>
<td>Digital to analog converter</td>
</tr>
<tr>
<td>$e$</td>
<td>Euler’s number</td>
</tr>
<tr>
<td>$E{.}$</td>
<td>Expected value operator</td>
</tr>
<tr>
<td>$f(x, y, z : t)$, $F(x, y, z : \omega)$</td>
<td>Source distribution forcing function (inhomogeneous Helmholtz equation)</td>
</tr>
<tr>
<td>$f_s$</td>
<td>Sampling frequency [Hz]</td>
</tr>
<tr>
<td>$F$</td>
<td>Focusing depth [m]</td>
</tr>
</tbody>
</table>
Unilateral $z$-transform

Temporal Fourier transform; inverse temporal Fourier transform

Retrospectively modified transducer surface velocity vector

3D Spatial Fourier transform with respect to $(x, y, z)$

Inverse 2D Fourier transform with respect to $(k_x, k_y)$

Frequency modulated

Full-Width-at-Half-Maximum

Green’s function

Transducer impulse response (general form, transmit, receive) [m s$^{-1}$]

Impulse response of attenuating medium

Impulse response of spherically focused concave transducer

Deconvolution filter

Deconvolution filter

Impulse response in a lossless medium

Impulse response of unapodized, focused piston transducer

Temporal Fourier transform of transducer impulse response $h(r : t)$

Zero-order Hankel and inverse Hankel transform

Discrete time Fourier transform of $h_{\text{deconv}}[n]$

Time-averaged wave intensity

Imaginary number

Bessel function of the first kind and zero order

Bessel function of the first kind and first order

Wave number [rad m$^{-1}$]

Unit wave vector

Complex wave number [rad m$^{-1}$]

Discrete time parameter

Cylindrical spatial frequency coordinates
\((k_x, k_y, k_z)\) Cartesian components of wave vector; spatial frequencies

\(L_{x,y,z}[\cdot]\) Multi-dimensional Laplace transform with respect to \((x, y, z)\)

\(L^{-1}_{s_x,s_y,s_z}[\cdot]\) Inverse multi-dimensional Laplace transform with respect to \((s_x, s_y, s_z)\)

LMMSE Linear Minimum Mean Square Error

\(M_{vec}\) Vector representing \(\sum_{r_i} \sum_{z_i} G(r_i, z_i, \omega) \delta [B(r_i, z_i, \omega) \delta (x-r_i, z)]\)

\(n\) Exponential parameter in power-law frequency dependent attenuation

\([n]\) Discrete time

\(N\) Number of elements in vector or array

\(Nr\) Number of radial field points

\(Nz\) Number of axial field points

\(p\) Pressure distribution on the surface of the fluid particle \([\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]\)

\(p_0\) Equilibrium pressure \([\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]\)

\(p_1\) Local pressure variation \([\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]\)

\(P(\omega)\) Retrospective transducer adjustment term

\(P_d(e^{j\omega})\) Power spectral density of desired signal \(d[n]\)

\(\hat{P}_d(e^{j\omega})\) Power spectral density of estimated desired signal \(\hat{d}[n]\)

\(P_w(e^{j\omega})\) Power spectral density of additive noise \(w[n]\)

\(P_y(e^{j\omega})\) Power spectral density of desired signal \(y[n]\)

PSF Point spread function, \(h(t \mid r = r_0)\)

\(r\) Three dimensional field position vector \([\text{m}]\)

\((r, \theta, z)\) Cylindrical spatial coordinates

\(r_d[n]\) Autocorrelation function of desired signal \(d[n]\)

\(r_w[n]\) Autocorrelation function of additive noise \(w[n]\)

\(rx\) Receive transducer

\(R\) Spherical radial distance \([\text{m}]\)

\(R_d\) Toeplitz matrix form of auto-correlation vector \(R_d = [r_d[1] \ldots r_d[N]]^T\)

\(R_p\) Distance between observation point and elementary ring on piston
\( \mathbf{R}_w \)  
Toeplitz matrix form of auto-correlation vector \( \mathbf{r}_w = [r_w[1] \ldots r_w[N]]^T \)

\[ s = \sqrt{F^2 + r^2} \]  
Substitution variable

\( (s_r, s_z) \)  
Complex spatial frequencies in cylindrical coordinates

\( (s_x, s_y, s_z) \)  
Complex spatial frequencies in Cartesian coordinates

\( S \)  
Angular spectrum form of equation

\( S_1 \)  
Transducer surface

\( \text{SNR} \)  
Signal to noise ratio

\( \text{SSF} \)  
Spatial sensitivity function \( h(\mathbf{r} | t = t_0) \)

\( t \)  
Time [s]

\( tx \)  
Transmit transducer

\( ^T \)  
Transpose operator

\( T/R \)  
Transmit/Receive

\( u \)  
Unit step

\( \mathbf{v} \)  
Vector velocity of a fluid particle [m/s]

\( \mathbf{v}_L \)  
Longitudinal wave component of velocity [m/s]

\( v_z \)  
Normal component of transducer velocity

\( v_{z, \text{apod}} \)  
Apodization component of transducer surface velocity

\( v_{z, \text{focus TxFinite}} \)  
Focusing component of transducer surface velocity for finite geometry

\( v_{z, \text{focus TxInfinite}} \)  
Focusing component of transducer surface velocity for infinite geometry

\( v_{z, \text{geom}} \)  
Geometric component of transducer surface velocity

\( v_0(t) \)  
Transducer excitation voltage [V]

\( \mathbf{V} \)  
Circulant matrix form of vector \( \mathbf{v} = [v[1] \ldots v[N]]^T \)

\( V(e^{j\omega}) \)  
Discrete time Fourier transform of \( v[n] \)

\( V_{\text{vec}} \)  
Initial transducer surface velocity vector representing \( v_z(r, 0: \omega) \)

\( w(t) \)  
Additive noise

\( (x, y, z) \)  
Cartesian coordinates [m]

\( (\hat{x}, \hat{y}, \hat{z}) \)  
Cartesian unit basis vectors

\( y(t) \)  
Received voltage signal on transducer [V]
Rayleigh distance [m]

Axial position of transducer [m]

Axial position of observation plane or point [m]

Total amplitude attenuation coefficient

Attenuation coefficient due to absorption

Attenuation coefficient due to scattering

Angle between vectors \( V_{vec} \) and \( M_{vec} \)

Attenuation coefficient at reference angular frequency

Directional components of attenuation coefficient

Cylindrical coordinate components of attenuation coefficient

Viscosity constant [s]

Angle between vectors \( V_{vec} \) and \( F_{vec} \)

Delta impulse function

Gradient operator

Electromechanical response of transducer (general form, transmit, receive)

Angle between vectors \( F_{vec} \) and \( M_{vec} \)

Angle between vectors \( a,b \)

Complex-valued angle

Real-valued Hermitian angle

Adiabatic compressibility of the medium \([m^3kg^{-1}]\)

Wavelength [m]

Coefficient of shear viscosity \([kgm^{-1}s^{-1}]\)

Coefficient of bulk viscosity \([kgm^{-1}s^{-1}]\)

Mean-square error

Pi

Fluid particle density \([kgm^{-3}]\)

Cosine of Hermitian angle

Equilibrium density \([kgm^{-3}]\)

Local density variation \([kgm^{-3}]\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>Gaussian apodization standard deviation</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Focusing phase delay</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>Known signal to be deconvolved</td>
</tr>
<tr>
<td>$\phi, \Phi$</td>
<td>Scalar velocity potential (temporal, temporal angular frequency)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Phase of transducer surface velocity</td>
</tr>
<tr>
<td>$\phi_{PA}$</td>
<td>Kasner’s pseudo-angle</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Phase of infinite geometry, focused transducer on the observation plane</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Temporal angular frequency [rad s$^{-1}$]</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>Reference angular frequency [rad s$^{-1}$]</td>
</tr>
<tr>
<td>$\ast, \ast, \ast$</td>
<td>Convolution operator over time, spatial positions $x, y$</td>
</tr>
<tr>
<td>$\ast, \ast, \ast$</td>
<td>Convolution operator over time, spatial positions $x, y$</td>
</tr>
<tr>
<td>$#, #$</td>
<td>Hankel convolution operator</td>
</tr>
<tr>
<td>$k_r, r$</td>
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</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

B-mode ultrasound has been used for decades to non-invasively image tissues such as gallstones, fetuses, and tumors. These images provide a two-dimensional (2D) representation of the variation in density and compressibility within a planar region of tissue, which when presented as a greyscale image to the clinician, can provide useful information for medical diagnoses, treatment administration, and assessing the efficacy of treatment protocols.

A 2D B-mode ultrasound image generally represents an image of tissue on the insonated axial-lateral plane when generated by a 1D linear array transducer. The distance along the transducer normal is the axial dimension, while the lateral dimension is along the longer transducer axis, as shown in Figure 1.1. The third dimension, elevation, is often assumed to make a negligible contribution to the image. A B-mode image is formed by stacking multiple A-lines signals laterally to create a 2D image, where each A-line is created using a different aperture on a transducer array. In the case where the transducer is a 1D linear array, the aperture is moved along the array before each transducer pulse excitation in order to generate A-lines at different lateral imaging positions. By introducing small time delays between the signals transmitted or received on each of the transducer elements within the aperture, the beam profile shape changes, resulting in transmit focusing, receive focusing, or beam-steering. These strategies can improve the imaging properties of the signal at various angles and insonation depths. The development of signal processing techniques to improve spatial resolution and contrast can significantly improve the presentation of imaging data to the clinician, thereby increasing the likelihood of an accurate medical diagnosis and treatment assessment.
1.1 Measures of image quality

There are three main measures of image quality in ultrasound—echo signal-to-noise ratio (SNR), spatial resolution, and contrast (Cobbold 2007, p.510-517). Echo SNR arises from properties of the transmitted and backscattered signal and noise sources. Factors affecting the signal power include the energy and intensity of the excitation and backscattered signals, and attenuation experienced by the propagating wave due to absorption and scattering. However, the excitation wave energy and intensity are regulated to prevent mechanical and thermal tissue injury. The noise power is predominantly influenced by the imaging equipment, e.g., transducer and preamplifier, and the insonated medium itself. Higher noise levels introduce a speckled appearance in the final image that can mask key diagnostic information.

Both the spatial resolution and contrast (the portion not arising from noise sources) are described by the point spread function (PSF) of the imaging system. The PSF represents the spatial relationship between a point source object and its image. In an ultrasound system, the PSF identifies how a single scatterer located in the insonated field is represented in the resulting
image. A simple graphical representation of this concept is shown in Figure 1.2. Additive noise can be added to the PSF image, however it has not been included in this example.

![Figure 1.2: Sketch showing the relationship between a point source object and its image, the point spread function.](image)

In general, the point spread function is not circularly symmetric, as it depends on the properties of the imaging system. By examining a cross-sectional view of the PSF from Figure 1.2 using the dB scale in Figure 1.3, metrics defining the spatial resolution and contrast can be identified in a particular dimension by examining properties of the mainlobe and sidelobes.

![Figure 1.3: Cross-sectional view of the point spread function to identify the spatial resolution and contrast.](image)

Image spatial resolution is quantitatively described by the Full-Width-at-Half-Maximum (FWHM) of the PSF mainlobe. It quantifies the minimum distance between two scatterers (assuming equal reflectivity) to ensure they are distinguishable as separate objects in the image. The resolution of a B-mode image will vary with measurement position and the observed
dimension—lateral, elevational, and axial—due to the properties of the transducer. A narrower mainlobe results in a higher spatial resolution and higher spatial frequency content.

Image contrast influences the ability to identify object boundaries. It effectively measures the relative mean signal intensities from inside and outside an area of interest and is dependent on both the sidelobe properties of the PSF and the noise power of the imaging system. If regions A and B are adjacent in the image, the PSF sidelobe amplitude will define how significantly region A will leak into B and vice versa. Lower sidelobes can therefore improve contrast. Noise sources will also degrade contrast by adding random speckle to the image. Models that define contrast levels are highly dependent on the relative signal intensities inside and outside the region of interest in ultrasound imaging systems.

1.2 Strategies to improve image quality

There have been many proposed techniques to improve upon the spatial resolution and contrast of B-mode images. The traditional technique involves using fixed transmit focusing, dynamic received focusing, followed by matched filtering and envelope detection of the received signal (Cobbold 2007, Ch.7,8). A frequently used method to overcome this single transmit focus limitation uses a montage of images generated using multiple transmit focal depths to construct a single ultrasound image. However, this also results in a lowered frame rate. An alternative method involves optimally selecting the focusing time delays on each array element to account for wave propagation effects (Zemp and Insana 2007). The main disadvantage of these B-mode imaging techniques is the low frame rate, as it is heavily influenced by the desired insonation depth and number of A-lines representing the lateral span of the image.

Degradation of the final image also occurs due to the spatial variation in tissue compressibility and density, which introduces a speckle characteristic or noise to B-mode images. To overcome this limitation, a variety of techniques have been proposed including compound averaging of images (Bamber 1993, Foster et al. 1981), spatial filtering (Bamber and Daft 1986), and transducer apodization (t'Hoen 1982) to reduce the spatial sidelobes at the cost of increasing the mainlobe width (improving contrast at the expense of lower spatial resolution). The dynamic receive focusing technique also addresses this problem.
Synthetic aperture imaging is another technique designed to overcome the low frame rate and limited resolution of traditional B-mode imaging. The fundamental approach is to transmit from a single transducer element in the array, but receive the backscattered signal on every transducer array element. Using post-processing techniques such as dynamic receive focusing, a low resolution image is generated. This method is then repeated using different transmit elements to generate a set of low resolution images, each representing a slightly different perspective of the insonated tissue. Using image processing techniques, the low resolution images are combined into a single high resolution B-mode image. While synthetic aperture imaging can provide a higher resolution image than that of traditional B-mode imaging, the low SNR arising from the single transmit element excitation, high computational requirements, and tissue motion produce significant challenges with the approach (Jensen et al. 2006).

Coded excitation wave methods have also been proposed using frequency modulated (FM) chirp (O’Donnell 1992) and binary Golay code (Takeuchi 1979a, 1979b) waveforms. These approaches use long duration excitation pulses in order to increase the imaging SNR without increasing the peak transmitted power. A long duration excitation pulse normally results in a reduction in spatial resolution, as two scatterers need to be located at least half a pulse width apart to be distinguishable at the receiver. However, when correlation methods are later applied on the received signal, the coded excitation waveforms are significantly compressed yet maintain the same bandwidth, thereby significantly improving resolution and contrast of the final image.

1.3 Outline of thesis study

1.3.1 Research motivation and hypothesis

The primary motivation for this research is the development of signal processing methods to improve B-mode image quality. Several possible approaches can be used. For example, one option is to use signal (image) processing methods to filter the image. A second is to improve the quality of individual A-lines so that the image reconstructed from these lines is improved in terms of contrast, SNR, and resolution. A third method is to use a synthetic aperture approach. All such approaches have advantages and limitations. It is our intention to use the second
approach in which the effects of a fixed transmit focus can be corrected for by optimization of individual A-lines.

1.3.1.1 Ultrasound hardware circuitry

To understand our research strategy, it is necessary to have a general concept of the underlying ultrasound hardware circuitry. A simple block diagram is shown in Figure 1.4 (Cobbold 2007, p.493). User input at a console instructs the central and local controllers to create a digital transmit signal for each of the $N$ array elements (possibly including the effects of apodization and focusing). The digital signal is converted into an analog waveform using a digital to analog converter (D/A), amplified, and passed through the Transmit/Receive (T/R) switch and matching network (optimizing the transmit SNR) prior to transmission into the tissue by the transducer array.

The backscattered signal received on the same transducer array is switched to the receive beamformer where it is amplified using a low noise amplifier and converted to a digital signal.
The sampling frequency of the analog to digital converter (A/D) must be high enough to account for the delays required for accurate dynamic receive focusing, however higher sampling frequencies limit the dynamic range of the data. At a centre carrier frequency of 5MHz, a sampling frequency of approximately 200MHz is generally required. The corresponding dynamic range is dependent on the characteristics of the hardware circuitry, but is generally in the range of 10-16 bits. The digitized signal is passed through a block to account for receive aperture focusing delays and apodization for each array element signal. The signals are then summed, matched filtered with the transmitted excitation wave, envelope detected, and graphically displayed to the user. Our research strategy involves replacing the summation and matched filter/envelope detection blocks with an alternative approach in order to generate an improved image for display.

1.3.1.2 Hypothesis

We hypothesize that in order to identify methods to improve image quality of a single A-line as described in Figure 1.4, which will subsequently improve the resultant B-mode image, a comprehensive understanding of the fundamental properties of acoustic wave propagation is required using a variety of tools. If we can couple analytical equations describing the acoustic radiation field to simulations, this may offer additional interpretations of field phenomena that might not be identified through numerical simulations alone. This can aid in the identification and exploitation of relationships between different radiated field characteristics for the development of methods for image enhancement.

To test this hypothesis, we will assume a simpler circular, apodized and focused, plane piston transducer instead of the common 1D linear array of rectangular elements, in order to allow for a circularly symmetric field distribution. It is hoped that the insights and solutions developed for the circular, plane piston transducer can be later extrapolated to the linear array transducer by applying the derivations and simulations described in this thesis to a new transducer geometry.

1.3.1.3 Linear model

Consider a linear model representing the ultrasound scattering system,
\[
y(t) = v_{tx}(t)^* v_{rx}(t)^* \sum_r h_{tx}(r : t)^* h_{rx}(r : t)^* \frac{dv_0(t)}{dt} \cdot bsc(r) + w(t),
\]

where \( y(t) \) is the received voltage signal over time \( t \), \( v(t) \) is the electromechanical response of the transducer on transmit \( tx \) and receive \( rx \) (assumed known), \( h_{tx}(r : t) \) and \( h_{rx}(r : t) \) represent the impulse responses of the transmit and receive transducer apertures, respectively, at field position \( r \), \( v_0(t) \) is the excitation voltage of the transducer, \( bsc(r) \) is the backscattering cross-section of a scatterer, and \( w(t) \) represents additive noise.

Part of our investigation includes the use of the spatial sensitivity function (SSF) introduced by Zemp et al. (2003, 2007). The SSF provides an alternative approach to the traditional method of interpreting the impulse response as a point spread function (PSF). The PSF, \( h(t | r_1) \), describes the response at \( r_1 \) as a function of time when the transducer is excited by an impulse of velocity. Thus, the SSF \( h(r | t_1) \) describes the spatial distribution of the field at a given instant of time. As illustrated in Figure 1.5, the PSF and SSF are closely related, each providing a different interpretation of the field caused by a transducer velocity impulse.

![Figure 1.5: Sketch illustrating the point spread function (PSF) and the spatial sensitivity function (SSF). The PSF identifies the impulse response over time for a single spatial location. The SSF describes the spatial distribution of the wave at a single point in time.](image-url)
Zemp and Insana (2007) analyzed the properties of the PSF and SSF for a linear array transducer by making several important assumptions, including the Fresnel approximation. Our analysis is for a much simpler geometry in order to identify insights that may be masked by more complicated configurations, and avoids making any significant assumptions.

Our approach is to examine the impulse response $h(r,t)$ arising from a finite geometry, apodized and focused, plane piston transducer using analytical derivations and simulations. By considering how the impulse response is related to the received ultrasound signal described in (1.1), a new signal processing technique is proposed to minimize the SSF in regions away from the focus. Results are compared to traditional focusing methods in order to evaluate the technique.

1.3.2 Research objectives

This study was divided into three core components:

1. Using the angular spectrum method, an accurate analytical model is developed for the field distribution arising from a finite geometry, apodized and focused, plane piston transducer in a lossless medium. The equations are simulated and analyzed in order to identify insights into the radiation characteristics, specifically those describing the angular spectrum magnitude and phase characteristics, velocity potential, and impulse response. Properties of the PSF and SSF are also identified. The Fresnel approximation of focusing is examined and shown to provide an inaccurate representation of field characteristics. Finally, the on-axis impulse response of the focused, plane piston transducer is derived and compared with the spherically-shaped, concave piston transducer. Key differences are identified.

2. The focused, plane piston transducer model is extended to address the effects of attenuation and dispersion on the radiated field. Following a review of known models, a Green’s function solution of the Navier-Stokes equation for spherical wave propagation in a classical viscous medium is derived using the Laplace transform. Modifications to the angular spectrum method are presented and used to analyze the radiated field of a focused, plane piston transducer. Insights into the radiated field of viscous media are also
identified, specifically related to the angular spectrum characteristics in comparison to the lossless medium.

3. The ultrasound imaging system equation (1.1) is examined in conjunction with the mathematical derivations. Two signal processing methods to minimize the SSF are proposed and analyzed, with results compared with the traditional matched filtering and envelope detection method. The first technique involves removing the effects of the excitation wave and suppression of additive noise using Wiener deconvolution or Linear Minimum Mean Square Error (LMMSE) filtering. While this method has been previously proposed in the literature, the SSF analysis is new and the results provide the necessary inputs to the second signal processing method. The second technique involves the mathematical derivation and simulation of a retrospective method for adjusting the transmit transducer focal depth. Proof of concept simulations are presented and analyzed to assess the efficacy of the techniques in optimizing the spatial resolution and contrast.

1.4 Organization of thesis

This thesis is organized into several chapters as per the research objectives.

- Chapter 2 outlines the fundamental mathematics governing acoustic wave propagation using the Navier-Stokes equation and the angular spectrum method.
- Chapter 3 presents an analytical mathematical derivation describing the pressure field from a finite geometry, apodized and focused, plane piston transducer in a lossless medium. Simulation results are presented and analyzed, and the model is compared with both the spherically-shaped, concave piston transducer and the Fresnel focusing approximation to identify key differences.
- Chapter 4 presents modifications to the focused, piston transducer model to account for attenuation and dispersion.
- In Chapter 5, the two signal processing techniques are described to improve image quality.
- Chapter 6 presents a summary of this thesis, the dominant conclusions, the most significant research contributions, and recommendations for future work.
CHAPTER 2

ACOUSTIC WAVE PROPAGATION THEORY

The underlying physics governing wave propagation in ultrasound systems are well understood and are described by the Navier-Stokes wave equations.

2.1 Navier-Stokes and Helmholtz equations

The Navier-Stokes equation describes the properties of wave propagation in fluids. It is given by (Cobbold 2007, p.21),

$$\rho \frac{Dv}{Dt} = -\nabla p + \left( \mu_B + \frac{1}{3} \mu \right) \nabla (\nabla \cdot v) + \mu \nabla^2 v,$$

(2.1)

where $\rho$ is the fluid particle density [kg m$^{-3}$], $v$ is the vector velocity of a fluid particle [m s$^{-1}$], $D$ refers to the rate of change of the moving fluid particle velocity, $p$ is the pressure distribution on the surface of the fluid particle [kg m$^{-1}$ s$^{-2}$], $\mu_B$ is the coefficient of bulk viscosity [kg m$^{-1}$ s$^{-1}$], which accounts for energy loss in compressible fluids, $\mu$ is the coefficient of shear viscosity, which causes velocity differences and energy loss between fluid layers [kg m$^{-1}$ s$^{-1}$], and $\nabla$ is the gradient operator $\nabla = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}$, where $(\hat{x}, \hat{y}, \hat{z})$ represent the unit vectors in Cartesian coordinates $(x, y, z)$. Details regarding the derivation of this equation can be found in Cobbold (2007, Ch.1).

Because of the limited wave energy transmitted in ultrasound imaging systems, a linearized form of (2.1) can be obtained using the small signal approximation. This approximation assumes that any local variations in pressure $p_1(\mathbf{r}:t)$, density $\rho_1(\mathbf{r}:t)$, and
velocity \( \mathbf{v}(\mathbf{r} : t) \), due to the propagating acoustic wave are much less than the equilibrium values in the fluid, \((p_0, \rho_0, c_0)\), respectively (Cobbold 2007, p.21),

\[
\begin{align*}
\rho &= \rho_0 + \rho_1 \quad |\rho_1| << \rho_0 \\
p &= p_0 + p_1 \quad |p_1| << \rho_0 c_0^2 \\
v &= 0 + \mathbf{v} \quad |\mathbf{v}| << c_0
\end{align*}
\]

Assuming that only longitudinal waves propagate in the insonated medium, \( \mathbf{v} = \mathbf{v}_L \), while noting that \( \kappa \frac{\partial p_1}{\partial t} + \nabla \cdot \mathbf{v} \approx 0 \), we can simplify (2.1) (Cobbold 2007, p.22-23),

\[
\kappa \rho_0 \frac{\partial^2 \mathbf{v}_L}{\partial t^2} = \nabla^2 \mathbf{v}_L + \kappa \left( \mu_B + \frac{4}{3} \mu \right) \frac{\partial}{\partial t} \left( \nabla^2 \mathbf{v}_L \right),
\]

where \( \kappa \) is the adiabatic compressibility of the medium \([\text{m}^2\text{kg}^{-1}]\). By noting that the scalar velocity potential, \( \phi(\mathbf{r} : t) = \phi(x, y, z : t) \) \([\text{m}^2\text{s}^{-1}]\) is defined in terms of the particle velocity,

\[
\mathbf{v}_L(x, y, z : t) = -\nabla \phi(x, y, z : t)
= -\left( \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z} \right),
\]

the linearized Navier-Stokes wave equation can be summarized as,

\[
\kappa \rho_0 \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi + \kappa \left( \mu_B + \frac{4}{3} \mu \right) \frac{\partial}{\partial t} \left( \nabla^2 \phi \right). \tag{2.3}
\]

The velocity potential is a scalar potential, whose gradient is an irrotational, longitudinal velocity vector \( \mathbf{v}_L \) (Cobbold 2007, p.23). For an inviscid medium, \( \mu_B = \mu = 0 \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1} \).

If we consider a traveling harmonic wave of the form, \( \phi(x, y, z : t) = \Phi(x, y, z : \omega) e^{j\omega t} \), it is useful to represent the Navier-Stokes equation (2.3) in terms of the temporal angular frequency \( \omega \) \([\text{rad/s}]\). This form of the Navier-Stokes equation is also known as the Helmholtz equation:

\[
\nabla^2 \Phi + \kappa^2 \Phi = 0
\]

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + \kappa^2 \Phi = 0, \quad \tag{2.4}
\]
where \( \vec{k} = \frac{k}{\sqrt{1 + j \omega \gamma}} \) [rad m\(^{-1}\)] is the complex wave number, \( k = \frac{\omega}{c_0} \) [rad m\(^{-1}\)] is the wave number, \( c_0 = \frac{1}{\sqrt{k \rho_0}} \) [m s\(^{-1}\)] is the wave speed, and \( \gamma = \kappa \left( \mu_B + \frac{4}{3} \mu \right) \) [s] is the viscosity constant.

By ignoring the bulk viscosity term, \( \gamma \) can be expressed as \( \gamma = \frac{4 \mu}{3 \rho_0 c_0^2} \) (Cobbold 2007, p.21,32, Buckingham 2005).

The Navier-Stokes and Helmholtz wave equations describe the fundamental properties of wave propagation in ultrasound systems. Using this information, we can determine the velocity potential and impulse response in the pressure field arising from various ultrasound transducer configurations. One solution in particular for ultrasound systems, which we use in this research, uses the angular spectrum method to determine the velocity potential within the medium.

### 2.2 Angular spectrum method

The angular spectrum method is a mathematical technique to identify the 2D field distribution of plane waves in the spatial frequency domain arising from a plane wave source (Cobbold 2007, p.121-126). This allows for the identification of the velocity potential on any observation plane \( \Phi(x, y, z = z_1 : \omega) \), \( z_1 \) = constant, given a definition of the velocity potential on a source plane \( \Phi(x, y, z = z_0 : \omega) \), \( z_0 \) = constant. These two parallel planes positioned at different axial depths generally require that \( z_0 \leq z_1 \), to ensure stability of the solution due to the presence of rapidly decaying evanescent waves with increasing \( z \). As applied to ultrasound and wave propagation in tissue, the angular spectrum method uses the velocity potential defined on the transducer plane positioned at \( z_0 = 0 \) to determine the wave distribution on a parallel observation plane within the medium of interest. However, this is mathematically equivalent to performing the Rayleigh integral for a single temporal frequency.

The angular spectrum method identifies the relationship between two solutions of the homogeneous Helmholtz equation (2.4) as determined by the velocity potentials, \( \Phi(x, y, z = z_1 : \omega) \) and \( \Phi(x, y, z = z_0 : \omega) \). It is important to note that the homogeneous Helmholtz equation provides the steady-state solution. By taking the 2D spatial Fourier
transform of this equation with respect to \((x, y)\), to obtain the angular spectrum in spatial frequencies \((k_x, k_y)\), where the angular spectrum \(S(k_x, k_y : z, \omega)\) and its inverse are defined as

\[
S(k_x, k_y : z, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x, y, z : \omega) e^{-j(k_x x + k_y y)} \, dx \, dy
\]

\[
\Phi(x, y, z : \omega) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y : z, \omega) e^{j(k_x x + k_y y)} \, dk_x \, dk_y
\]

we obtain,

\[
(jk_x)^2 S(k_x, k_y : z, \omega) + (jk_y)^2 S(k_x, k_y : z, \omega) + \frac{\partial^2 S(k_x, k_y : z, \omega)}{\partial z^2} + k^2 S(k_x, k_y : z, \omega) = 0
\]

\[
\frac{\partial^2 S(k_x, k_y : z, \omega)}{\partial z^2} + \left(k^2 - k_x^2 - k_y^2\right) S(k_x, k_y : z, \omega) = 0
\]  

(2.7)

We have assumed an inviscid medium, to allow for the application of the Fourier transform for real valued spatial frequencies \((k_x, k_y)\) (i.e. \(\vec{k} = k = \frac{\omega}{c_0}\)). If we had assumed a classical viscous medium, the Laplace transform would be required, since the spatial frequencies would be complex to represent both wave propagation and viscous losses. The necessary modifications required for this model to account for viscous loss will be discussed in Chapter 4.

Solving (2.7) generates waves traveling in both the +z and −z directions,

\[
S(k_x, k_y : z, \omega) = C_1 e^{-j\sqrt{k^2 - k_x^2 - k_y^2}} + C_2 e^{j\sqrt{k^2 - k_x^2 - k_y^2}}.
\]

However, all wave energy propagates in the +z direction due to imposed Neumann boundary conditions, requiring \(C_2 = 0\). For both \(\Phi(x, y, z = z_1 : \omega)\) and \(\Phi(x, y, z = z_0 : \omega)\) to satisfy the solution of the homogeneous Helmholtz formula,

\[
C_1 = S(k_x, k_y : z_0, \omega)
\]

\[
S(k_x, k_y : z_1, \omega) = S(k_x, k_y : z_0, \omega) e^{-j(z_1 - z_0)\sqrt{k^2 - k_x^2 - k_y^2}}.
\]  

(2.8)

Equation (2.8) provides the relationship between the angular spectra of waves traveling between two parallel planes. If the angular spectrum or velocity potential is known on the \(z_0\) plane, it is possible to obtain the corresponding wave distribution on the \(z_1\) plane (Cobbold 2007, p.122-125).
From the angular spectrum relationships (2.5), (2.6), and (2.8), the wave distribution on an observation plane can be predicted from that defined on a parallel transducer source plane. These steps can be summarized as,

1. Identify the velocity potential distribution on the transducer source plane:
   \[ \Phi(x, y, 0 : \omega) \].

2. Take the 2D Fourier transform over \((x, y)\) to obtain the angular spectrum:
   \[ S(k_x, k_y : 0, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x, y, 0 : \omega) e^{-j(k_x x + k_y y)} dx dy. \]  \(2.5\)

3. Shift the angular spectrum to the observation plane \(z\) using the following transformation:
   \[ S(k_x, k_y : z, \omega) = S(k_x, k_y : 0, \omega) e^{-j\sqrt{k_x^2 - k_y^2} z}. \]  \(2.8\)

4. Obtain the velocity potential on the observation plane \(z\) by taking the inverse 2D spatial Fourier transform,
   \[ \Phi(x, y, z : \omega) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y : z, \omega) e^{j(k_x x + k_y y)} dk_x dk_y. \]  \(2.6\)

### 2.2.1 Velocity and velocity potential

It is useful to define the normal component of the transducer surface velocity instead of the velocity potential, since the transducer will vibrate with a velocity normal to its surface when excited by an input voltage waveform. Since the velocity \( \mathbf{v}_L(x, y, z : \omega) \) is the gradient of the velocity potential, \( \mathbf{v}_L(x, y, z : \omega) = -\nabla \Phi(x, y, z : \omega) \), the normal component of the transducer surface velocity (in the \( \hat{z} \) direction) is defined as \( v_z(x, y, z = 0, \omega) = -\frac{\partial \Phi}{\partial z} \bigg|_{z=0} \) (Cobbold 2007, p.124-125). Noting that

\[ S(k_x, k_y : z, \omega) = S(k_x, k_y : 0, \omega) e^{-j\sqrt{k_x^2 - k_y^2} z}, \text{ and} \]

\[ S(k_x, k_y : z, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x, y, z : \omega) e^{-j(k_x x + k_y y)} dx dy, \] \(2.5\)

by taking the derivative of these equations with respect to \(z\),
\[
\frac{\partial S(k_x, k_y : z, \omega)}{\partial z} = S(k_x, k_y : 0, \omega) \left( -j \sqrt{k_x^2 - k_y^2 - k^2} \right)e^{-jz\sqrt{k_x^2 - k_y^2 - k^2}}
\]

\[
\frac{\partial S(k_x, k_y : z, \omega)}{\partial z} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \Phi(x, y, z : \omega)}{\partial z} e^{-j(k_x x + k_y y)} \, dx \, dy
\]  
(2.9)

By equating the results of (2.9),

\[
S(k_x, k_y : 0, \omega) \left( -j \sqrt{k_x^2 - k_y^2 - k^2} \right)e^{-jz\sqrt{k_x^2 - k_y^2 - k^2}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \Phi(x, y, z : \omega)}{\partial z} e^{-j(k_x x + k_y y)} \, dx \, dy
\]

\[
S(k_x, k_y : z, \omega) = \frac{1}{\left( -j \sqrt{k_x^2 - k_y^2 - k^2} \right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x, y, z : \omega) e^{-j(k_x x + k_y y)} \, dx \, dy
\]  
(2.10)

\[
v_z(x, y, z : \omega) = \frac{j \sqrt{k_x^2 - k_y^2 - k^2}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y : z, \omega) e^{j(k_x x + k_y y)} \, dk_x \, dk_y
\]  
(2.11)

### 2.2.2 Velocity impulse response

By assuming that the transducer is excited by an impulse of voltage \( v_0(t) = \delta(t) \), where \( \delta(t) \) is the delta impulse function, the velocity impulse response \( h(x, y, z : t) \) is the inverse temporal Fourier transform of the velocity potential \( \Phi(x, y, z : \omega) \) (Cobbold 2007, p.111),

\[
\phi(x, y, z : t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(x, y, z : \omega) e^{j\omega t} \, d\omega
\]

\[
= v_0(t) * h(x, y, z : t)
\]

\[
= \delta(t) * h(x, y, z : t)
\]

\[
= h(x, y, z : t)
\]

\[
\Phi(x, y, z : \omega) = \int_{-\infty}^{\infty} \phi(x, y, z : t) e^{-j\omega t} \, dt
\]  
(2.12)
2.2.3 Wave number

Assuming that the wave number is real and represents wave propagation in an inviscid medium, where \( k = \frac{\omega}{c_0} \), the wave number can be interpreted as the magnitude of the wave vector \( \hat{k} k \), where \( \hat{k} \) is the unit wave vector. The wave vector can be decomposed into Cartesian spatial frequency components,

\[
k \hat{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z},
\]

where \((\text{Cobbold 2007, p.123})\)

\[
k^2 = k_x^2 + k_y^2 + k_z^2. \tag{2.13}
\]

This relation allows for the substitution of \( k_z = \sqrt{k^2 - k_x^2 - k_y^2} \) in the angular spectrum equations.

Equation (2.13) is also consistent with the solution to the homogeneous Helmholtz equation. By taking the spatial Fourier transform of (2.7) with respect to \( z \),

\[
(j k_z)^2 S(k_x, k_y, k_z : \omega) + (k^2 - k_x^2 - k_y^2) \hat{S}(k_x, k_y, k_z : \omega) = 0
\]

\[
(k^2 - k_x^2 - k_y^2 - k_z^2) S(k_x, k_y, k_z : \omega) = 0. \tag{2.14}
\]

To avoid the trivial solution where \( S(k_x, k_y, k_z : \omega) = 0 \), we require \( k^2 - k_x^2 - k_y^2 - k_z^2 = 0 \). In addition, the Fourier transform in (2.14) implies that the spatial frequencies \( (k_x, k_y, k_z) \) are real-valued. However, evanescent wave propagation requires that \( k_z \) be strictly imaginary \((\text{Cobbold 2007, p.124})\) since \( k^2 - k_x^2 - k_y^2 < 0 \), which is at odds with the Fourier transform requirement. We therefore need a new interpretation of the meaning of \( k_z \), as one not limited to a spatial frequency. While this concept will be further discussed in a following subsection (2.2.3.2 Case 2: Evanescent wave), these ideas are necessary for Chapter 4, which describes an analytical solution of the Navier-Stokes equation in a classical viscous medium using the Laplace transform.

Noting that (2.8) is not explicitly written in terms of \( k_z \) since the result is in terms of the spatial coordinate \( z \), we are not limited to the case where \( k^2 - k_x^2 - k_y^2 \geq 0 \). Since the angular
spectrum equations are valid over all real values of \( (k_x, k_y) \), (2.8) can be separated into three cases:

### 2.2.3.1 Case 1: Homogenous wave

When \( k^2 > k_x^2 + k_y^2 \), \( k_z \) is real-valued. When \( k_z = \sqrt{k^2 - k_x^2 - k_y^2} \) is substituted into (2.8) we obtain,

\[
S(k_x, k_y : z, \omega) = S(k_x, k_y : 0, \omega) e^{-jz\sqrt{k^2 - k_x^2 - k_y^2}}.
\]

The complex exponential term \( e^{-jzk_z} \) implies a phase difference between the angular spectrum of the transducer source plane and observation plane.

### 2.2.3.2 Case 2: Evanescent wave

When \( k^2 < k_x^2 + k_y^2 \), equation (2.14) does not technically hold since \( k_z \) is imaginary. However, since (2.8) is in terms of only \( (k_x, k_y) \), equation (2.14) is irrelevant. We can therefore reinterpret \( k_z \) as a parameter instead of a spatial frequency in an isolated equation defined by \( k_z = \sqrt{k^2 - k_x^2 - k_y^2} \). Equation (2.8) can therefore be modified, while noting that \( \sqrt{-1} = \pm j \) and \( k_z^2 < 0 \), to obtain,

\[
S(k_x, k_y : z, \omega) = S(k_x, k_y : 0, \omega) e^{-jz\sqrt{k^2 - k_x^2 - k_y^2}}
\]

\[
= S(k_x, k_y : 0, \omega) e^{-z\sqrt{k^2 + k_x^2 + k_y^2}}.
\]

This solution describes the evanescent wave component of wave propagation, which decays with increasing axial distance \( z \) and \( |k_z| \).

We have limited our research to the forward projection problem where the response on the observation plane is determined from that described on the transducer source plane \( z > 0 \). The backward projection problem—determining the source plane distribution from the observation plane—generally results in exponential growth of evanescent waves. However, a
backward projection technique that accommodates for this problem, by effectively pivoting the system of equations so the evanescent component is orthogonal to the calculations, is available (Leeman and Healey 1997).

2.2.3.3 Case 3: Singularity

A singularity occurs in the angular spectrum of (2.10) when \( k^2 = k_x^2 + k_y^2 \), since \( k_z = 0 \). Williams and Maynard (1982) reported that this can produce significant simulation difficulties, however we can assume that \( k_z = 0^+ \) or \( k_z = 0^- \) during mathematical analysis.

2.2.4 Green’s function

The Green’s function describes the velocity potential field distribution arising from a delta impulse excitation.

Consider the inhomogeneous form of the inviscid Helmholtz equation (Cobbold 2007, p.98),

\[
\nabla^2 \Phi + k^2 \Phi = -F(x, y, z : \omega),
\]

assuming a source distribution forcing function, \( F(x, y, z : \omega) \). The Green’s function solution to (2.15) assumes delta impulse source function, \( f(x, y, z : t) = \delta(x, y, z : t) \), where \( F(x, y, z : \omega) = F_t[f(x, y, z : t)] \) and \( F_t[] \) is the temporal Fourier transform. Noting that

\[
\delta(x, y, z) = \int_{-\infty}^{\infty} \delta(x, y, z : t) e^{-j\omega t} dt,
\]

and using the Green’s function notation \( \Phi(x, y, z : \omega) = g(x, y, z : \omega) \), equation (2.15) becomes

\[
\nabla^2 g + k^2 g = -\delta(x, y, z)
\]

\[
\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} + k^2 g = -\delta(x, y, z).
\]

The Green’s function can be found using the following method:

1. Perform the 3D spatial Fourier transform of (2.16) and solve for the resultant \( G(k_x, k_y, k_z : \omega) \) function in the spatial frequency domain, where

\[
G(k_x, k_y, k_z : \omega) = F_{x,y,z}[g(x, y, z : \omega)].
\]
\[(jk_x)^2 G(k_x, k_y, k_z : \omega) + (jk_y)^2 G(k_x, k_y, k_z : \omega) + (jk_z)^2 G(k_x, k_y, k_z : \omega) + k^2 G(k_x, k_y, k_z : \omega) = -1\]
\[G(k_x, k_y, k_z : \omega) = \frac{-1}{k^2 - k_x^2 - k_y^2 - k_z^2}\]

(2.17)

2. Assuming Neumann boundary conditions, which limits wave propagation to \(z > 0\), by taking the inverse 3D spatial Fourier transform of (2.17), and solving for the initial conditions, we obtain the commonly known 3D Green’s function:
\[g(x, y, z : \omega) = \frac{1}{2\pi R} e^{-jkR},\]
where \(R = \sqrt{x^2 + y^2 + z^2}\).

3. However, if instead of the inverse 3D Fourier transform, we take the inverse spatial Fourier transform of (2.17) with respect to the \(k_z\) term only, while assuming Neumann boundary conditions,\n\[G(k_x, k_y, z : \omega) = \frac{-e^{-jz\sqrt{k^2 - k_x^2 - k_y^2}}}{j\sqrt{k^2 - k_x^2 - k_y^2}}.\]

The Green’s function solution can be merged with the angular spectrum equations previously described,
\[S(k_x, k_y : z, \omega) = S(k_x, k_y : 0, \omega) e^{-jz\sqrt{k^2 - k_x^2 - k_y^2}}\]
\[= S(k_x, k_y : 0, \omega) G(k_x, k_y, z_1) \left( j\sqrt{k^2 - k_x^2 - k_y^2} \right).\]

2.2.5 Angular spectrum in cylindrical coordinates

This thesis assumes a circularly symmetric transducer geometry in \((r, z)\), where \(r^2 = x^2 + y^2\). It is therefore helpful to represent the velocity, velocity potential, and angular spectrum equations in cylindrical form (Cobbold 2007, p.136-137).
The angular spectrum \( S(k_x, k_y : 0, \omega) \) of the velocity potential \( \Phi(x, y : 0 : \omega) \) or the normal component of the transducer surface velocity \( v_z(x, y : 0 : \omega) \) is defined as,

\[
S(k_x, k_y : 0, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x, y : 0 : \omega) e^{-j(k_x x + k_y y)} \, dx \, dy
\]

\[
= \frac{1}{j\sqrt{k^2 - k_x^2 - k_y^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_z(x, y : 0 : \omega) e^{-j(k_x x + k_y y)} \, dx \, dy
\]

(2.18)

Because the circular piston is circularly symmetric, \( S(k_x, k_y : 0, \omega) \) can be written in cylindrical coordinates, \( S(k_r : 0, \omega) \), using the following relations,

\[
r = x + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta, \quad dx \, dy = r \, dr \, d\theta,
\]

\[
k_r^2 = k_x^2 + k_y^2, \quad k_x = k_r \cos \phi, \quad k_y = k_r \sin \phi
\]

to obtain,

\[
S(k_r : 0, \omega) = \frac{-j}{\sqrt{k^2 - k_r^2}} \int_{0}^{2\pi} \int_{0}^{\infty} v_z(r, 0 : \omega) e^{-j r k_r (\cos \phi \cos \theta + \sin \phi \sin \theta)} r \, dr \, d\theta.
\]

(2.19)

Using the trigonometric identity: \( \cos(A - B) = \cos A \cos B + \sin A \sin B \), (2.19) can be simplified:

\[
S(k_r : 0, \omega) = \frac{-j}{\sqrt{k^2 - k_r^2}} \int_{0}^{2\pi} \int_{0}^{\infty} v_z(r, 0 : \omega) e^{-j r k_r \cos(\theta - \phi)} r \, dr \, d\theta.
\]

(2.20)

Noting the Bessel identity, \( J_0(A) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{\pm j A \cos(\theta - \phi)} d\theta \), where \( J_0(\cdot) \) is the cylindrical Bessel function of the first kind and zero order, the angular spectrum equation (2.20) can be simplified to,

\[
S(k_r : 0, \omega) = \frac{-2\pi j}{\sqrt{k^2 - k_r^2}} \int_{0}^{\infty} v_z(r, 0 : \omega) J_0(r k_r) r \, dr,
\]

(2.21)

\[
= \frac{-j}{\sqrt{k^2 - k_r^2}} H[v_z(r, 0 : \omega)]
\]

where \( H[\cdot] \) denotes the zero-order Hankel transform defined by (Erdelyi 1954, vol 2)
\[
H[q(r)] = 2\pi \int_0^\infty q(r)J_0(rk_r)rdr = Q(k_r),
\]
and whose inverse is
\[
H^{-1}[Q(k_r)] = \frac{1}{2\pi} \int_0^\infty Q(k_r)J_0(rk_r)dk_r = q(r).
\]

The velocity potential can be determined using the inverse zero-order Hankel transform:
\[
\Phi(r, z : \omega) = \frac{1}{2\pi} \int_0^\infty S(k_r : z, \omega)J_0(rk_r)dk_r.
\]
\[
= H^{-1}[S(k_r : z, \omega)]
\]

2.3 Assumptions and simplifications

The angular spectrum method assumptions and simplifications include,

- Small signal approximation of the Navier-Stokes equation. This implies that the local changes in pressure, density, and particle velocity due to the propagating wave are much smaller than equilibrium values.
- Linear model of wave propagation.
- Longitudinal wave propagation only. No shear wave propagation.

While only the relevant items from this chapter are described here, a comprehensive list of all assumptions arising from the entire research study are summarized in Appendix A.

2.4 Conclusions

We have shown the analytical derivation of the angular spectrum method in cylindrical coordinates from the Navier-Stokes equation. This mathematical foundation is necessary for the analytical derivation of the radiation field arising from the circular piston transducer.
A careful analysis of wave propagation in medical ultrasound systems can lead to models that identify new and improved interpretations of the relationships between different radiated field characteristics. Such analyses are useful since they can aid the development of new signal processing techniques to optimize the presentation of imaging data to the user. These methods can also provide additional information than that obtained through numerical solutions alone, since the fundamental mathematical relationships can be coupled with simulations to suggest opportunities for improvement.

The purpose of this chapter is to develop a comprehensive, analytical ultrasound transducer model of the plane piston transducer with focusing and apodization. The plane piston transducer is focused by a continuous variation of the excitation signal phase (or time delay) over the transducer surface. Prior analyses of this scheme used the Fresnel approximation, thereby limiting the validity. Using the angular spectrum method, an accurate radiation model of such a transducer has been developed. The derivation includes the effects of diffraction and evanescent waves without using the Fresnel approximation. Moreover, this model develops insights into radiated field characteristics. The analytical model and associated insights enhance understanding of the radiated field characteristics, which can be of value in the development of signal processing techniques for image enhancement.

3.1 Background

While there is a thorough understanding of the radiated field arising from uniformly excited plane piston transducers (Harris 1981a, Harris 1981b, Oberhettinger 1961, Stepanishen
1971, Hutchins and Hayward 1990, Daly and Rao 2000), the focused equivalent, i.e., one that allows a continuous change in phase (or the time delay) over the plane piston surface, is incomplete (Daly and Rao 2001, Lu and Greenleaf 1990). Moreover, it seems that prior work on such a transducer assumed the Fresnel approximation, thereby limiting the analysis. As will be shown, this approximation introduces significant errors in the radiation field, even in the focal region. Since an accurate model of wave propagation without the Fresnel approximation is not available in the literature, it suggests that this approximation has been previously assumed to sufficiently describe the radiated field without a thorough assessment of its validity.

Most early analytical models of the uniformly excited plane piston transducer used either the Rayleigh integral, the King integral (King 1934) or the Schoch solution (Schoch 1941). The King integral can be considered a precursor to the angular spectrum method, while the Schoch solution converts the Rayleigh integral solution into a line integral calculated over the transducer surface. However, solutions using these methods have been limited to the uniform or apodized, plane piston transducer without any phase change (e.g., Harris 1981b, Stepanishen 1971, Greenspan 1979). Research on the phase apodized (focused) piston transducer is relatively limited, as few analytical models seem to have been developed (Goldstein et al. 1998, Daly and Rao 2001, Cavanagh and Cook 1981, Lucas and Muir 1982). Lu and Greenleaf (1990) developed a numerical simulation model of a non-diffracting piston transducer with Gaussian apodization and focusing and identified the depth of field and lateral resolution under certain geometric constraints. Angular spectrum analytical solutions have also been developed for various transducer geometries (Stepanishen and Benjamin 1982, Williams and Maynard 1982, Guyomar and Powers 1987, Christopher and Parker 1991, Zeng and McGough 2008, Zemp and Insana 2007). But, for the cases where phasing was included, Fresnel approximations were used. Our analysis of such a transducer, hereinafter referred to as a focused plane piston transducer, avoids this restriction.

The literature also suggests that the focused plane piston transducer is equivalent to the spherically-shaped concave piston transducer when they have the same aperture and focal point (Cavanaugh and Cook 1981, Lucas and Muir 1982), but this does not appear to have been theoretically examined. Impulse response expressions for the spherically-shaped concave transducer are well known and have been derived for arbitrary observation points (Pentinnen and Luukkala 1976, Cathignol et al. 1997, Gibson et al. 1993). However, unless an annular array
structure is used, which is generally limited to relatively few array elements (Foster et al. 1989, Brown et al. 2004, Dietz et al. 1979), such a configuration is limited to a single focal depth. In addition, all of these assume that the effects of secondary diffraction are negligible (Penttinen and Luukkala 1976, O’Neil 1949), so that the assumptions under which the Rayleigh integral was derived, remain valid. Using an exact approach, the effects of such an approximation have been examined in the frequency domain by Coulouvrat (1993) who showed that the effects of secondary diffraction can be ignored except when the transducer has very small radius of curvature.

Using the angular spectrum approach, we present an analysis of the apodized focused plane piston transducer shown in Figure 3.1, and from this, identify new features of the radiated field that could be of value for ultrasound image enhancement. However, despite the availability of extensive published literature on the piston transducer, to the best of our knowledge this is the first to provide an analytical solution without making Fresnel approximations. Part of our investigation includes the use of the spatial sensitivity function (SSF) introduced by Zemp et al. (2003, 2007). Zemp and Insana (2007) analyzed the properties of the PSF and SSF for a linear array transducer by making several important assumptions. Our analysis is for a much simpler geometry in order to identify insights that may be masked by more complicated configurations, and avoids making any significant assumptions.
Figure 3.1: Sketch of a focused and apodized plane piston with a focal point $F$, showing the cylindrical geometry used in the analysis. It is assumed that the amplitude $\varphi$ and phase $\psi$ of the velocity vary over the piston surface to minimize diffraction effects and to produce focusing.

In addition, using an analytical approach we derive an expression for the on-axis impulse response of the unapodized, focused, plane piston transducer that is compared with the response from a spherically-shaped concave piston transducer. It is shown that for the same focal length and aperture, the on-axis impulse responses of the two transducers are similar, though not identical. These differences influence the variation in phase as well as the transient waveform response.

In summary, the primary objectives of this chapter are: (a) to analytically derive equations describing the angular spectrum and velocity potential of the finite geometry, apodized and focused, plane piston transducer; (b) to simulate and analyze the model equations in order to identify radiation characteristics; (c) to simulate and analyze the impulse response in terms of the PSF and SSF; (d) to demonstrate why an accurate derivation is required for describing transducer focusing; and (e) to derive the on-axis impulse response of the focused plane piston transducer.
and identify why this configuration is not equivalent to the spherically-shaped concave transducer.

### 3.2 Angular spectrum method

The angular spectrum method enables the field on an observation plane to be predicted from the wave distribution on a parallel plane. Our analysis assumes linear propagation in a homogeneous, inviscid medium and uses cylindrical coordinates \((r, z)\). The following four steps are needed to arrive at the velocity impulse response (Cobbold 2007, Ch.2-3)

1. Obtain the angular spectrum on the transducer plane \((z = 0)\) by performing the zero-order Hankel transform of the normal component of the surface velocity \(v_z\),

\[
S(k_r : z = 0, \omega) = \frac{-j}{\sqrt{k^2 - k_r^2}} H[v_z(r, z = 0 : \omega)], \quad (3.1)
\]

2. Obtain the angular spectrum on an observation plane at \(z > 0\) by using:

\[
S(k_r : z, \omega) = S(k_r : 0, \omega) \left( e^{-jz\sqrt{k^2 - k_r^2}} + e^{-z\sqrt{k_r^2 - k^2}} \right)_{k_r < k}, \quad \left(3.2\right)
\]

where the homogeneous and evanescent components correspond to \(k_r < k\) and \(k_r > k\), respectively.

3. Obtain the velocity potential on the observation plane \(z\) by taking the inverse Hankel transform,

\[
\Phi(r, z : \omega) = \frac{1}{2\pi} \int_0^\infty S(k_r : z, \omega) J_0(rk_r)k_r dk_r. \quad (3.3)
\]

4. If the transducer temporal excitation velocity is an impulse, then the velocity impulse response \(h(r, z : t)\) is given by the inverse temporal Fourier transform \(F^{-1}[\cdot]\) of the velocity potential,

\[
h(r, z : t) = F^{-1}\left[ \frac{1}{2\pi} \int_0^\infty S(k_r : z, \omega) J_0(rk_r)k_r dk_r \right].
\]
3.3 Transducer velocity

There are three components that contribute to the normal component of the transducer surface velocity distribution, $v_z(r,0;\omega)$: the finite geometry, Gaussian apodization, and focusing.

3.3.1 Finite geometry and diffraction

The circular piston transducer selected for this model assumes a finite geometry of radius $a$,

$$v_{z\text{-geom}}(r,0;\omega) = \text{circ}\left(\frac{r}{a}\right)$$

(3.4)

where $\text{circ}\left(\frac{r}{a}\right) = \begin{cases} 1 & r \leq a \\ 0 & r > a \end{cases}$.

An inherent consequence of using a finite geometry is diffraction. Diffraction describes how the edges of the transducer will affect wave propagation. Huygen’s principle provides an interpretation of wave propagation that aids in understanding diffraction: where a wave-front can be decomposed into a compilation of point sources from which spherical wavelets emanate, and whose composite wavelet envelope describes the wavefront after propagation (Born and Wolf 1980).

Using Huygen’s principle, we can define two types of diffraction: primary and secondary. Primary diffraction describes how waves arising from spherical wavelets at the transducer edge will affect wave propagation. Secondary diffraction is caused by wavelets emanating from around the transducer edge, on a different plane than the primary transducer surface (assuming a plane transducer). In our model we have ignored secondary diffraction by assuming the transducer is of negligible thickness. Secondary diffraction effects can become significant in non-planar transducers.

In general, diffraction introduces sidelobes in the radiated field around the desired mainlobe, decreasing imaging contrast. Strategies such as transducer apodization and signal processing are often implemented to mitigate its effects.
3.3.2 Gaussian apodization

Gaussian apodization was modeled in this work using

\[ v_{z\_apod}(r,0; \omega) = e^{2\sigma^2 r^2}, \tag{3.5} \]

where \( \sigma \) is the standard deviation (Cobbold 2007, p.189-195). Although apodization reduces diffractive effects, the maximum pressure and lateral resolution are reduced.

Two metrics designed to analyze the effects of apodization in the radiated field are the depth of field and lateral resolution. The depth of field describes how the magnitude of the radiated field decreases with axial distance. It is characterized by the Rayleigh distance, \( z_R \), which describes the axial depth where the pressure magnitude is half of its maximum. For a flat piston transducer, \( z_R \approx \frac{\pi a^2}{\lambda} \), where \( \lambda \) is the wavelength, while for the Gaussian apodized, unfocused transducer, \( z_R \approx \frac{\pi (2\sigma^2)}{\lambda} \) (Cobbold 2007, p.159, 193).

The lateral resolution or FWHM of the unfocused piston at \( z_R \) increases with axial depth, \( FWHM = 4.41 \frac{z}{ka} \), while for the Gaussian apodized, unfocused piston, \( FWHM \approx 1.67 \frac{\sigma}{\sqrt{2}} \) (Cobbold 2007, p.162,193).

3.3.3 Focusing

The phase delay required to focus on-axis at \( z = F \), assuming a finite geometry transducer of radius \( a \) is,

\[ v_{z\_focus\_TxFinite}(r,0; \omega) = e^{jk\left( \sqrt{F^2 + r^2} - \sqrt{F^2 + a^2} \right)}. \tag{3.6} \]

Alternatively, using the Fresnel approximation of the exponent term,

\[ \sqrt{F^2 + r^2} - \sqrt{F^2 + a^2} \approx F - \sqrt{F^2 + a^2} + \frac{r^2}{2F}, \]

the focusing term can be expressed as,

\[ v_{z\_focus\_TxFinite}(r,0; \omega) = e^{jk\left( \sqrt{F^2 + r^2} - \sqrt{F^2 + a^2} \right)} \approx e^{jk\left( F - \sqrt{F^2 + a^2} + \frac{r^2}{2F} \right)}. \tag{3.7} \]
However, it will become apparent in our analysis that the Fresnel approximation is inadequate and the full equation is required.

Assuming a finite geometry, unapodized, focused transducer, the lateral resolution is (Cobbold 2007, p.194),

\[ FWHM \approx 1.4 \frac{\lambda F}{2a}. \]

In the unbounded geometry case, (3.7) becomes,

\[ v_{z, focus, Tx, infinite}(r, 0 : \omega) = e^{-jk \left( \sqrt{F^2 + r^2} - F \right)} \approx e^{-\frac{jk r^2}{2F}}. \]  (3.8)

### 3.3.4 Transducer velocity

The normal component of the transducer surface velocity can be constructed from (3.4), (3.5), (3.7), and (3.8),

\[ v_{z, Tx, finite}(r, 0 : \omega) = \text{circ} \left( \frac{r}{a} \right) \frac{-r^2}{2\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \approx \text{circ} \left( \frac{r}{a} \right) e^{\frac{-r^2}{2\sigma^2}} \left( \frac{r}{2F} \right), \]  (3.9)

\[ v_{z, Tx, infinite}(r, 0 : \omega) = e^{\frac{-r^2}{2\sigma^2}} e^{-\frac{r^2}{2\sigma^2}} \approx e^{\frac{-r^2}{2\sigma^2}} e^{-\frac{jk r^2}{2F}}. \]  (3.10)

Lu and Greenleaf (1990) identified the depth of field and lateral resolution for the special case of a non-diffracting transducer with the following constraints:

\[ v_{z, Tx, finite}(r = a, z = 0 : \omega) = 0 \text{ (apodized transducer)}, \]

\[ F^2 >> 2\sigma^2 \text{ (focused far from transducer)}, \]

\[ \left( \frac{\lambda}{\sqrt{2\pi\sigma}} \right)^2 << \frac{2\sigma^2}{F^2} \text{ (wavelength is small)} \]

to find,

\[ z_R \approx \frac{\lambda F^2}{\pi \sigma^2} \sqrt{1 + \frac{1}{2} \left( \frac{\lambda F}{\pi \sigma^2} \right)^2}, \]

\[ FWHM = \frac{1.4 \lambda F}{(2\sqrt{2})^2}. \]

It is important to note several characteristics of the transducer velocity equation (3.9):
1. Both apodization and focusing are assumed to be continuously distributed over the transducer surface.

2. Both continuous and pulsed wave systems can be easily accommodated with the appropriate selection of a temporal excitation velocity and a temporal convolution with the velocity impulse response.

3. The magnitude of \( v_z(r,0: \omega) \) is temporal frequency independent and describes the finite transducer geometry (diffraction) and apodization terms.

4. The phase of \( v_z(r,0: \omega) \) is dependent on the wave number \( k \) and is influenced solely by the focusing term.

5. Neumann boundary conditions have been assumed with a rigid baffle surrounding the transducer so that the velocity is zero on the surface outside the transducer geometric boundary.

3.4 Angular spectrum solution

The normal component of the transducer surface velocity (3.9) is used to determine the radiated field on any observation plane \( z > 0 \) using the derivation described in Section 2.2 However, two different solutions exist and will be presented:

3.4.1 Case A: Accurate focusing equation

The angular spectrum on the transducer plane, can now be written as,

\[
S(k_r : 0, \omega) = \frac{-2\pi j}{\sqrt{k^2 - k_r^2}} \int_0^\infty \text{circ}\left(\frac{r}{a}\right) e^{2r^2} e^{jk\left(\sqrt{r^2 + a^2} - \sqrt{F^2 + a^2}\right)} J_0(rk_r) r dr .
\]  

A direct solution of (3.11) does not exist in the extensive Hankel transform tables of Erdelyi (1954, vol.2) or Oberhettinger (1972). However, it is possible to determine a solution to this equation in terms of the known Hankel convolution \#-operator, and the known identity, \( H[f(r)g(r)] = H[f(r)] \# H[g(r)] \). The definition and properties of the Hankel convolution can be found in Appendix B. By expressing the angular spectrum on the transducer plane in terms of the Hankel convolution, each component in the velocity distribution can be dealt with separately. This enables a more intuitive understanding of the resultant equation.
Consider the following Hankel transform pairs, the first two of which are well known (Cobbold 2007, p.137 eq.3.3, Erdelyi 1954, vol.2, p.29 eq.10), while the third was analytically determined as described in Appendix C and verified in simulation:

\[
H\left[\text{circ} \left( \frac{r}{a} \right) \right] = \pi a^2 \left\{ \frac{2 J_1(ak_r)}{ak_r} \right\} = \pi a^2 \text{jinc}(ak_r)
\]

\[
H \left[ e^{\frac{-r^2}{2\sigma^2}} \right] = -2\pi\sigma^2 e^{-\frac{r^2}{2}} \tag{3.12}
\]

\[
H \left[ e^{jk\sqrt{k^2 - k_r^2}} \right] = \begin{cases} 
-2\pi k e^{jF\sqrt{k^2 - k_r^2}} \left\{ \left( k^2 - k_r^2 \right)^{3/2} - jF \left( k^2 - k_r^2 \right)^{1/2} \right\} & 0 < k_r < k \\
-2\pi j k e^{-F\sqrt{k_r^2 - k^2}} \left\{ \left( k_r^2 - k^2 \right)^{3/2} + F \left( k_r^2 - k^2 \right)^{1/2} \right\} & k < k_r < \infty
\end{cases}
\]

where \( J_1(\cdot) \) is the Bessel function of the first kind and first order. With the help of these transform pairs, the angular spectrum on the transducer plane can now be written as,

\[
S(k_r : 0, \omega) = \frac{-\left(2\pi\right)^3}{\sqrt{k^2 - k_r^2}} \left\{ \frac{a^2 J_1(ak_r)}{ak_r} \right\} \# \left\{ -\sigma^2 e^{-\frac{\sigma^2 k_r^2}{2}} \right\} \cdot \begin{cases} 
-ke^{jF\sqrt{k^2 - k_r^2}} \left\{ \left( k^2 - k_r^2 \right)^{3/2} - jF \left( k^2 - k_r^2 \right)^{1/2} \right\} & 0 < k_r < k \\
-jk e^{-F\sqrt{k_r^2 - k^2}} \left\{ \left( k_r^2 - k^2 \right)^{3/2} + F \left( k_r^2 - k^2 \right)^{1/2} \right\} & k < k_r < \infty
\end{cases} \tag{3.13}
\]

From (3.2), (3.3) and (3.13) the angular spectrum arising from the homogeneous and evanescent components on any observation plane \( z \) can be written as

\[
\Phi(r, z : \omega) = \frac{1}{2\pi} \int_0^k \left(2\pi\right)^3 \left\{ \frac{a^2 J_1(ak_r)}{ak_r} \right\} \# \left\{ -\sigma^2 e^{-\frac{\sigma^2 k_r^2}{2}} \right\} \cdot \begin{cases} 
-ke^{jF\sqrt{k^2 - k_r^2}} \left\{ \left( k^2 - k_r^2 \right)^{3/2} - jF \left( k^2 - k_r^2 \right)^{1/2} \right\} & 0 < k_r < k \\
-jk e^{-F\sqrt{k_r^2 - k^2}} \left\{ \left( k_r^2 - k^2 \right)^{3/2} + F \left( k_r^2 - k^2 \right)^{1/2} \right\} & k < k_r < \infty
\end{cases} J_0(r k_r) k_r dk_r + \frac{1}{2\pi} \int_k^\infty \left(2\pi\right)^3 \left\{ \frac{a^2 J_1(ak_r)}{ak_r} \right\} \# \left\{ -\sigma^2 e^{-\frac{\sigma^2 k_r^2}{2}} \right\} \cdot \begin{cases} 
-ke^{jF\sqrt{k^2 - k_r^2}} \left\{ \left( k^2 - k_r^2 \right)^{3/2} - jF \left( k^2 - k_r^2 \right)^{1/2} \right\} & 0 < k_r < k \\
-jk e^{-F\sqrt{k_r^2 - k^2}} \left\{ \left( k_r^2 - k^2 \right)^{3/2} + F \left( k_r^2 - k^2 \right)^{1/2} \right\} & k < k_r < \infty
\end{cases} J_0(r k_r) k_r dk_r \tag{3.14}
\]

This expression can be simplified by making use of the following result from Erdelyi (1954, vol.2, p.9 eq.26)
\[
H^{-1} \begin{bmatrix}
\begin{array}{c}
\pm j \\
\frac{e^{\pm j z \sqrt{k_r^2 - k^2}}}{\sqrt{k_r^2 - k^2}} \\
\frac{1}{\sqrt{k_r^2 - k^2}} e^{-z \sqrt{k_r^2 - k^2}}
\end{array}
\end{bmatrix} = \begin{cases}
\frac{1}{2\pi \sqrt{r^2 + z^2}} e^{\pm j k \sqrt{r^2 + z^2}}, & \text{if } k < k_r < \infty
\end{cases}, \tag{3.15}
\]

enabling the velocity potential to be expressed as
\[
\Phi(r, z; \omega) = \left\{ \begin{array}{c}
\text{circ} \left( \frac{r}{a} \right) e^{\frac{-r^2}{2a^2}} e^{jk \sqrt{F^2 + r^2 - \sqrt{F^2 + a^2}}} \\
\frac{1}{2\pi \sqrt{r^2 + z^2}} e^{-jk \sqrt{r^2 + z^2}}
\end{array} \right\}.
\tag{3.16}
\]

Thus, the velocity potential on any observation plane is simply the Hankel convolution of the source velocity distribution with the Green’s function assuming Neumann boundary conditions (Waag et al. 1985, Liu and Waag 1997).

The homogeneous and evanescent wave portions of the Green’s function converge to the same general result in the spatial domain. Not only do evanescent waves decay rapidly with axial distance \( z \), to negligible values a few wavelengths from the transducer, but they also decay rapidly with increasing \( k_r \) (for \( k_r > k \)) due to the \( \sqrt{k_r^2 - k^2} \) terms in (3.15). Recent experimental research (Sukhovich et al. 2009) has shown that subwavelength imaging resolution can be achieved through amplification of the evanescent waves generated in a phononic crystal lens. However, evanescent wave decay with increasing spatial frequency \( k_r \) may limit the maximum achievable spatial resolution.

Several notable characteristics of the three Hankel transforms shown in (3.12) include:

1. The Hankel transform of the Gaussian apodization is also Gaussian.
2. Both the diffraction and apodization terms decay with increasing spatial frequency \( k_r \), independent of the relative value of \( k \).
3. The focusing term has two solutions depending on \( k_r \). The homogeneous wave solution exhibits a significant complex exponential component related to the focusing depth \( F \). However, the evanescent wave component decays rapidly with \( z \) and \( k_r \) due to the negative exponential in (3.12).
3.4.2 Case B: Fresnel approximation

The objective in this case is to determine the velocity potential using the Fresnel approximation of the focusing term in (3.9). The normal component of the transducer surface velocity is,

$$v_{z,TxFinite}(r,0; \omega) \approx \text{circ}\left(\frac{r}{a}\right) e^{\frac{-r^2}{2\sigma^2}} e^{jk \left( F - \sqrt{F^2 + a^2} + \frac{r^2}{2F} \right)}.$$  

The transducer angular spectrum can be determined using the substitution $b = -\frac{1}{2\sigma^2} + \frac{jk}{2F}$,

$$S(k_r : 0; \omega) = -\frac{2\pi j \cdot e^{jk \left( F - \sqrt{F^2 + a^2} \right)}}{\sqrt{k^2 - k_r^2}} \int_0^{\infty} \text{circ}\left(\frac{r}{a}\right) e^{br^2} J_0(rk_r) r dr,$$

and the following Hankel transform pair (Erdelyi 1954, vol.2, p.29 eq.10),

$$H[e^{br^2}] = \frac{2\pi}{2b} e^{\frac{k_r^2}{4b}}.$$

The homogeneous wave solution ($k_r < k$) of (3.17) yields,

$$S(k_r : 0; \omega) = -\frac{2\pi j \cdot e^{jk \left( F - \sqrt{F^2 + a^2} \right)}}{\sqrt{k^2 - k_r^2}} \int_0^{\infty} \text{circ}\left(\frac{r}{a}\right) e^{br^2} J_0(rk_r) r dr$$

$$= -\left(2\pi\right)^2 j \cdot e^{jk \left( F - \sqrt{F^2 + a^2} \right)} \left[ \frac{a^2 J_1(ak_r)}{ak_r} \right] k_r \left[ \frac{k_r^2}{e^{\frac{4b}{2b}}} \right]$$

$$= -\left(2\pi\right)^2 j \cdot e^{jk \left( F - \sqrt{F^2 + a^2} \right)} \left[ \frac{a^2 J_1(ak_r)}{ak_r} \right] k_r \left[ \frac{1}{\sigma^2 + \frac{2 \sigma^2}{F}} e^{-\frac{2 \sigma^2}{F} jk} \right]$$

The velocity potential on the observation plane is eventually determined,

$$\Phi(r, z; \omega) = \left[ \text{circ}\left(\frac{r}{a}\right) e^{-\frac{r^2}{2\sigma^2}} e^{jk \left( F - \sqrt{F^2 + a^2} + \frac{r^2}{2F} \right)} \right] r \left[ \frac{1}{2\pi \sqrt{r^2 + z^2}} e^{-jkr\sqrt{r^2 + z^2}} \right].$$
Simulation results will show that the Fresnel approximation introduces significant errors that can be avoided by using the results of Case A.

3.5 Comparison with spherically-shaped concave piston transducer

It has sometimes been assumed that the focused plane piston transducer and the spherically-shaped concave piston transducer are equivalent structures when both have the same aperture and focal point. This assumption has not been previously examined, nor has an expression for the on-axis impulse response of the focused plane piston transducer been derived. It is shown here how such an expression can be obtained in the time domain directly from the Rayleigh integral.

Consider a focused plane piston transducer of radius $a$ with a focal depth $F$ and a spherically-shaped concave piston transducer with a radius of curvature $A = \sqrt{F^2 + a^2}$. As shown in Figure 3.2, both transducers are assumed to have the same apertures, $2a$ and focal depth $F$. In the analysis that follows, we assume propagation into an inviscid medium and, as noted above, for the concave transducer we assume secondary diffraction to be negligible.

![Figure 3.2: Sketch of a spherically-shaped concave piston and a focused plane piston transducer that have the same aperture and focal point.](image-url)
3.5.1 **Focused plane piston**

For the plane piston, focusing is achieved by using time delays that vary in a continuous manner as a function of the transducer radial position $r$. These delays are given by

$$
\tau(r) = \frac{A - \sqrt{F^2 + r^2}}{c_0},
$$

(3.20)

where $c_0$ is the wave propagation speed. Consequently, the delay at the edge of the transducer has been taken to be zero, while it is maximum at the centre, i.e., $\tau(0) = (A - F)/c_0$. If the excitation surface velocity impulse is assumed to emit from the transducer edge at time $t=0$, the on-axis impulse response can be obtained by considering how these time delays will influence the response of the plane piston transducer.

Based on the Rayleigh integral, the impulse response of an unapodized transducer at the observation point $(z, r = 0)$ can be written as (see Cobbold 2007, eq. 2.16)

$$
h_p(z,0:t) = \int_{S_1} \frac{\delta(t - \tau(r) - R_p/c_0)}{2\pi R_p} dS_1 = \int_{0}^{a} \frac{\delta(t - \tau(r) - R_p/c_0)}{2\pi R_p} 2\pi R_p dr,
$$

(3.21)

where $\delta(t)$ is the delta impulse function, and $R_p = \sqrt{z^2 + r^2}$ is the distance between the observation point and an elementary transducer ring at radial position $r$ on the surface $S_1$. Substituting (3.20) into the above equation and rewriting it in terms of $R_p$ yields

$$
h_p(z,0:t) = \int_{z}^{\sqrt{z^2 + a^2}} \delta\left( t - \frac{A + R_p - \sqrt{F^2 + R_p^2 - z^2}}{c_0} \right) dR_p.
$$

(3.22)

The presence of $R_p$ in the argument of the delta function makes evaluation of this integral a non-trivial task. One approach is to examine the response at the time point $t_n$ over a small time increment $dt_n$ using the relations,

$$
t_n = \frac{A + R_p - \sqrt{F^2 + R_p^2 - z^2}}{c_0}
$$

(3.23a)

$$
\frac{dt_n}{dR_p} = \frac{1}{c_0} \left( 1 - \frac{R_p}{\sqrt{F^2 + R_p^2 - z^2}} \right) = \frac{1}{c_0} \left( 1 - \frac{\sqrt{z^2 + r^2}}{\sqrt{F^2 + r^2}} \right)
$$

(3.23b)

so that (3.22) simplifies to,
where the absolute value operator accounts for the $z < F$ and $z > F$ regions and $r$ is determined from

$$c_0t_n = \sqrt{z^2 + r^2} - \sqrt{F^2 + r^2} + A$$

using (3.23a) and $R_p = \sqrt{z^2 + r^2}$. The analytical solution of the above expression can be shown to be

$$r = \frac{1}{2c_0t_n - A} \sqrt{\left[(c_0t_n - A)^2 + 2F^2 - z^2\right]^2 + F^2\left[-6(c_0t_n - A)^2 - 3F^2 + 2z^2\right]} \quad (3.24b)$$

It should be noted that times for which the impulse response is non-zero on the axis are given by

$$t_{\min} = \begin{cases} \frac{z - F + A}{c_0} & 0 < z < F \\ \frac{\sqrt{z^2 + a^2}}{c_0} & z > F \end{cases} \quad , \quad t_{\max} = \begin{cases} \frac{\sqrt{z^2 + a^2}}{c_0} & 0 < z < F \\ \frac{z - F + A}{c_0} & z > F \end{cases} \quad . \quad (3.25)$$

### 3.5.2 Spherically-shaped concave piston transducer

The on-axis impulse response arising from a spherically-shaped concave piston transducer is well known (Penttinen and Luukkala 1976) and, with the help of the definitions given in Figure 3.2, it can be expressed as (see Cobbold 2007, eq. 3.42)

$$h_c(z,0:t) = \frac{c_0 A}{z - F} \text{rect}\left[\frac{t - (t_{\max} + t_{\min})/2}{t_{\max} - t_{\min}}\right],$$
where \( t_{\text{min}} \) and \( t_{\text{max}} \) are also given by (3.25) and \( \text{rect}(X) \) is defined to be equal to 0 if \( |X| > 0.5 \), and equal to 1 if \( |X| \leq 0.5 \). Alternatively, the above equation can be written as

\[
h_c(z,0:t) = \begin{cases} \frac{c_0 A}{|z - F|} & \text{for } t_{\text{min}} < t < t_{\text{max}} \\ 0 & \text{elsewhere} \end{cases}
\]

(3.26)

### 3.6 Assumptions and simplifications

Assumptions and simplifications used in the analytical piston models include,

- Planar piston transducer model of finite geometry with continuously varying apodization and focusing.
- Wave propagation in an homogeneous, inviscid medium in the absence of attenuation and dispersion.
- Neumann boundary conditions for a rigid baffle. The normal component of the transducer surface velocity is zero outside the transducer boundary.
- The transducer is of negligible thickness.
- The effects of primary diffraction arising from the transducer edge were included. Secondary diffraction effects were ignored.
- Evanescent waves are present and included in the model.

A summary of these items are also listed in Appendix A.

### 3.7 Simulation results

Comprehensive simulations of the piston transducer model were performed to validate the analytical derivation and to identify insights into wave propagation. Coupling these equations to simulations provides an opportunity to identify propagation phenomena that may not be possible through numerical simulations alone.

All equations were validated by comparing the results obtained in MATLAB (MathWorks, USA) with those obtained from Field II (Jensen 1996). In particular, the analytical model was studied using the parameter values listed in Table 3.1 for the four transducer
configurations given in Table 3.2. Additional validation was performed by comparing the velocity potential results for the two unfocused piston transducers of Table 3.2 to well established theory.

Figure 3.3 shows the amplitude and phase variation of the normal component of the surface velocity distribution over the transducer radius for the assumed focusing and apodization values. Except where noted, we have plotted the simulation results for Section 3.4.1 Case A: Accurate focusing equation.

Assuming a transducer radius of 3 cm, the focal depth was selected to be in the near-field at 4.5 cm. The analysed temporal frequency was 5 MHz, consistent with many ultrasound imaging systems. The standard deviation of the Gaussian apodization was one quarter of the transducer radius to minimize effects of diffraction arising from the edge of the transducer. The radiated field was simulated at a 100 MHz sampling frequency and analyzed over a region twice the transducer radius (6 cm) in the radial direction and up to 6 cm in the axial direction to maintain simulation accuracy. Four cases were analysed in simulation: Case 1 without apodization or focusing, Case 2 with apodization, Case 3 with focusing, and Case 4 with both apodization and focusing.

<table>
<thead>
<tr>
<th>Configuration Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temporal frequency</td>
<td>$f$</td>
<td>5 MHz</td>
</tr>
<tr>
<td>Wave speed</td>
<td>$c_0$</td>
<td>1500 m/s</td>
</tr>
<tr>
<td>Transducer radius</td>
<td>$a$</td>
<td>3 cm</td>
</tr>
<tr>
<td>Apodization standard deviation</td>
<td>$\sigma$</td>
<td>$0.25a = 0.75$ cm</td>
</tr>
<tr>
<td>Focusing depth</td>
<td>$F$</td>
<td>4.5 cm</td>
</tr>
</tbody>
</table>

Table 3.2: Simulation test cases for a finite geometry piston transducer.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Properties</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No apodization or focusing</td>
<td>$\sigma \sim \infty$  $F \sim \infty$</td>
</tr>
<tr>
<td>2</td>
<td>With apodization but no focusing</td>
<td>$\sigma = 0.25a$ cm  $F \sim \infty$</td>
</tr>
<tr>
<td>3</td>
<td>With focusing but no apodization</td>
<td>$\sigma \sim \infty$  $F = 4.5$ cm</td>
</tr>
<tr>
<td>4</td>
<td>With apodization and focusing</td>
<td>$\sigma = 0.25a$ cm  $F = 4.5$ cm</td>
</tr>
</tbody>
</table>
Figure 3.3: Transducer velocity magnitude (apodization, $a(r)$) and phase (focusing, $\phi(r)$) distributions for the assumed values of $F$ and $\sigma$ given in Table 3.2.

### 3.7.1 Transducer angular spectrum

The angular spectrum responses (3.13) on the transducer plane are shown in Figure 3.4 for all four cases. Two numerical methods were used for validation purposes: (a) by simulating the Hankel transform of (3.9); and (b), by calculating (3.13) directly using a numerical implementation of the Hankel convolution. The results were consistent.

Figure 3.4a shows the unfocused response both without and with apodization (Cases 1 and 2). Without apodization, the response depends only on the piston geometry through the jinc function, and demonstrates a -3 dB spatial frequency bandwidth of $\sim 0.53$ rad/cm. Because the spatial frequency bandwidth is a measure of the best possible spatial resolution (Zemp 2004), such a low bandwidth implies poor spatial resolution. It will be later shown that both the spatial frequency bandwidth and phase determine the spatial resolution. The bandwidth determines the best possible spatial resolution that can be achieved for a transducer configuration, while the
phase determines where in the radiated field such resolution is attained. With Gaussian apodization (Case 2, $\sigma=0.75$ cm), the bandwidth was almost double that of Case 1 yet, as expected, it approaches Case 1 for $\sigma \to \infty$. For the two focusing cases shown in Figure 3.4b, our simulations indicate that the spatial frequency bandwidth is increased, up to $\sim 120$ rad/cm for the unapodized case, corresponding to a much greater possible spatial resolution. In addition it can be seen that singularities are present at $k_r = k = 209.4$ rad/cm, which arise from the multiple terms in the denominator of (3.13). Though not shown in Figure 3.4a, this same singularity is also present for the unfocused transducer cases.

Figure 3.4b also shows the response of the focusing term in (3.13) but in isolation, corresponding to an unapodized, focused transducer of infinite radius. This term aids in understanding the Hankel convolution. The Hankel convolution of the geometric and apodization terms with the focusing term in (3.13) directly affects both the magnitude and phase of the angular spectrum. This is effectively achieved by Hankel convolving the Case 1 or 2 angular spectrum results with the focusing term, in order to obtain the Case 3 or 4 results, respectively.

The angular spectra of Cases 1 and 2 benefit from inclusion of focusing by increasing the spatial frequency bandwidth and altering the phase of the response; however neither the magnitude nor the phase are completely identical to that of the focusing term for all spatial frequencies. However, as the effective transducer radius is increased (by increasing $a$ and $\sigma$), the angular spectrum rapidly approaches that of the focusing term alone, as seen in Figure 3.4b,c. In fact it becomes nearly equivalent to the spatial convolution of a spatial delta impulse function with the focusing term. At the start of the evanescent region ($k_r > k$), corresponding to the vertical arrow in Figure 3.4b, the $e^{-F \sqrt{k_r^2-k^2}}$ term in (3.13) causes the spectral magnitude to rapidly decay with increasing $k_r$. 

Figure 3.4: Angular spectra results on the transducer plane ($z=0$). The normalized magnitudes are shown for (a) Cases 1 and 2 (unfocused) and for (b) Cases 3 and 4 (focused) together with the response for the focusing term in equation (3.13). (c) Phase characteristics on the transducer plane.
3.7.2 Observation plane angular spectrum—Phase

It does not seem possible to analytically determine the phase of the angular spectrum on any plane for the focused, finite geometry transducer since the solutions described in (3.13) and (3.14) are in terms of Hankel convolutions. However, the numerically determined phase for Cases 3 and 4 are shown in Figure 3.4c where it is noted that the phase characteristics are coincident with the isolated focusing term in (3.13) for \( k_r < \sim 120 \) rad/cm. This range corresponds to the most significant spectral magnitudes. Because of this phase relationship, it is shown below that analysis of the focusing term phase characteristics in isolation provides significant insights into the field propagation characteristics for a focused transducer. Not shown, are the phase variations for Cases 1 and 2 which oscillate between 0 and \( \pi \) due to the properties of the jinc function.

The angular spectrum phase \( \psi(k_r, z) \), for an infinite radius, unapodized, focused transducer on any observation plane, can be determined by eliminating the integral, geometry, and apodization terms in (3.14), leading to

\[
\psi(k_r, z) = \begin{cases} 
\pi - k \sqrt{F^2 + a^2} + (F - z) \sqrt{k^2 - k_r^2} + \arctan \left( - F \left( k^2 - k_r^2 \right)^{1/2} \right) & 0 < k_r < k \\
- k \sqrt{F^2 + a^2} - z \sqrt{k^2 - k_r^2} & k < k_r < \infty 
\end{cases} \quad (3.27)
\]

In the homogenous wave region, the fourth term remains nearly constant (within 1%) for \( k_r < 0.998k \). In addition, it can be seen that at the focus \( (z=F) \) the third term is zero resulting in a nearly constant phase over \( k_r \). In interpreting this result, it should be noted that the ideal impulse response is a delta function at the focal point in the spatial domain. In the spatial frequency domain, this corresponds to a constant phase on the focal plane. As noted above, this is nearly achieved at the focus of an infinite radius, unapodized, focused transducer.

The above phase analysis can also be applied to the focused, finite transducer configurations of Cases 3 and 4. As observed in Figure 3.4c, the phase is nearly coincident with the focusing term on the transducer plane for spatial frequencies corresponding to the most significant spectral magnitudes \( (k_r < \sim 120 \) rad/cm). Because the magnitude outside this region is negligible (Figure 3.4b), the phase deviations from the focusing term above 120 rad/cm are not significant. The maximum achievable spatial resolution is therefore obtained on the focal plane since the phase corresponding to the most significant spectral magnitudes is constant.
Figure 3.5 shows the spectral phase distribution for Case 4 (apodization and focusing) at several observation depths around the focus. These curves arise from the addition of the spectral phases on the transducer plane and the Green’s function phase on various observation planes between 1 cm and 6 cm. On the focal plane at 4.5 cm and up to $k_r < \sim 120 \text{ rad/cm}$, the two phases are equal in magnitude, but opposite in sign, resulting in a phase of zero. As expected, the result using the focusing term in isolation from (3.13) is zero over all spatial frequencies on the focal plane.

Because the angular spectrum magnitude is independent of $z$, it follows that the spatial frequency bandwidth is independent of the observation plane depth. The spatial resolution is therefore dependent on both the spatial frequency bandwidth and phase, as the bandwidth will determine the best possible spatial resolution, while the phase determines where in the radiated field this resolution is attained. This suggests that an optimal spatial resolution is potentially achievable on any observation plane if signal processing techniques can be developed to modify the phase of the signal.

Figure 3.5: Phase characteristics for Case 4, in addition to the focusing term from (3.13) on several observation planes. Note that on the focal plane ($z = 4.5 \text{ cm}$) the phase is almost constant for $k_r < 120 \text{ rad/cm}$. 
3.7.3 Velocity potential

Figure 3.6 shows the cylindrical velocity potential distributions in the focal region for the two focused cases, as obtained from (3.16). A 3D visualization of the results would therefore require an axisymmetric rotation of the results about the axial axis (the angular component of the cylindrical coordinate system). The velocity potential was calculated at a temporal frequency of 5 MHz using the numerical inverse Hankel transform of the angular spectrum results calculated for multiple observation planes between 3 cm and 6 cm. These results were also validated against the numerical Hankel convolution results of (3.16). The magnitude of the velocity potential responses was then plotted in Figure 3.6.

As expected due to its relatively high spatial bandwidth, Cases 3 and 4 in Figure 3.6c,d produce a narrow velocity potential beamwidth at the focus, which broadens as the distance from the focus increases. This is consistent with the constant angular spectrum phase discussion at $z = F$ in the previous section. As noted in Figure 3.5, at axial depths away from the focus, the angular spectrum phase varies with $k_r$ and the velocity potential will not yield the best spatial resolution. The high sidelobes caused by the effects of diffraction are clearly seen in (c): their suppression by apodization and the broadened mainlobe are seen in (d).

Several velocity potential phase distributions are shown in Figure 3.7 for Case 4. Phase values are only shown when the magnitude of the velocity potential is greater than -40dB of that at the focus. On the focal plane the phase is constant. However, it deviates significantly with radial distance from the focus and varies as $r^2$ on nearby observation planes. The choice of -40 dB was governed by the fact that the effects of diffraction causes the phase to oscillate between 0 and $\pi$ on the focal plane with increasing radial distance.

The velocity potential results of the unfocused transducer in Cases 1 and 2, corresponding to Figure 3.6a,b, were significantly worse than that of the focused transducers, as the velocity potential magnitude was significantly higher over the entire examined spatial field. These trends are consistent with expectations due to the low spectral bandwidth in Figure 3.4a.

The implication that the velocity potential on the observation plane can be obtained through the Hankel convolution of the normal component of the transducer velocity and the Green’s function offers the potential for the development of signal processing techniques to retrospectively modify the transducer normal velocity distribution, since it remains constant over the pressure field and may be separable from the Green’s function.
Figure 3.6: Normalized velocity potential [dB] around the focus. (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4.

Figure 3.7: Velocity potential phase for Case 4 on three observation planes around the focus. The radial range results were shown corresponding to normalized velocity potential magnitudes to -40dB.
All simulations have been performed in the transducer near-field. However, by expanding the field of view in Figure 3.8a, the velocity potential results of Case 1 are in fact consistent with that of Cobbold (2007, p.149). Case 2 results are shown in Figure 3.8b. Despite Case 2 demonstrating a higher spatial frequency bandwidth than that of Case 1, the Case 2 FWHM is larger than the unapodized case in the far-field. This result can be attributed to the angular spectrum phase discussion for the focused piston cases. We have shown that the optimal spatial resolution is obtained when the angular spectrum phase is constant. In Cases 1 and 2, this corresponds to the transducer plane. The mainlobe of the apodized Case 2 is in fact narrower than that of the unapodized Case 1 on the transducer plane, which is consistent with the spectral bandwidth analysis. However, other observation planes demonstrate the commonly known assertion that apodization increases the mainlobe width.

Figure 3.8: Magnitude of velocity potential [dB] using an expanded field of view. (a) Case 1 and (b) Case 2.

Table 3.3 summarizes the Rayleigh distance and FWHM results from the velocity potential, as previously described in Section 3.3. The FWHM results were obtained at the Rayleigh distance for Cases 1 and 2 and at the focus for Cases 3 and 4 at 5 MHz. The Rayleigh
distance for Cases 3 and 4 were obtained at an axial position in the far-field region of the focal
distance \((z_R > F)\).

<table>
<thead>
<tr>
<th>Case #</th>
<th>Rayleigh Distance ((z_R))</th>
<th>FWHM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>942 cm</td>
<td>6.6 cm</td>
</tr>
<tr>
<td>2</td>
<td>118 cm</td>
<td>1.75 cm</td>
</tr>
<tr>
<td>3</td>
<td>4.58 cm</td>
<td>0.027 cm</td>
</tr>
<tr>
<td>4</td>
<td>4.68 cm</td>
<td>0.050 cm</td>
</tr>
</tbody>
</table>

### 3.7.4 The point spread function and spatial sensitivity function

The SSF responses of the transducer impulse response shown in Figure 3.9 were obtained
by numerically calculating the inverse temporal Fourier transform of the velocity potential. The
SSF at three time points were superimposed in each of the plots of Figure 3.9c,d for the two
focused transducer configurations. Notable characteristics include the magnitude, axial span
(over \(z\)), lateral span (over \(r\)), and spatial curvature.

For the unfocused transducer configurations shown in Figure 3.9a,b, the propagating
wave characteristics did not deviate significantly over the shown depth range since the velocity
potentials were relatively uniform over axial depth. In addition, the response was not
bandlimited in temporal frequency. The sampling frequency was 100 MHz, minimizing error in
the impulse response. The lateral span of the SSFs was bounded by the magnitude of the
velocity potentials. The SSF curvature was influenced by the phase of the Green’s function in
the velocity potential. This curvature changed minimally with axial depth since the phase
changes slowly over \(z\) in the Green’s function.

The SSF results for the focused transducers are shown in Figure 3.9c,d. Unlike the
unfocused transducers, these SSF characteristics varied significantly over the pressure field. At
the focus and as predicted from the velocity potential results, the magnitude is maximized, axial
and lateral spans minimized, and the curvature is zero. Significant variations were observed as
the distance from the focus increased. The curvature of the SSF is highly influenced by the
velocity potential phase at the corresponding observational plane depth.
The SSF curvatures of Cases 3 and 4 are different—this is due to the relative differences in the spatial frequency bandwidth in the angular spectrum plots of Figure 3.4b. Despite the phase distributions being nearly identical, the spatial frequency bandwidth of Case 3 (focusing without apodization) is much larger than that of Case 4 (focusing with apodization). This difference results in Case 3 having a smaller lateral span due to the larger bandwidth, as well as more curvature since a wider range of phases are included in the higher bandwidth response.

The PSF results provide a different interpretation of the impulse response as it represents the velocity impulse response at a given position over all time. The responses are highly position dependent with the optimal response given the particular transducer normal surface velocity located on-axis at the focus for Cases 3 and 4. All other responses are suboptimal as has been widely reported in the literature for the unfocused piston (Cobbold 2007).

Figure 3.9: SSF results [dB] for (a) Case 1, (b) Case 2, (c) Case 3, and (d) Case 4, displaying an overlay of three time responses around the focus.
3.7.5  Fresnel approximation

Second-order Fresnel approximations of the transducer focusing term in (3.9) are often assumed in the literature in order to simplify analytical derivations. However, this approximation restricts the focal depth to a particular range and introduces significant errors in the solution, as it shifts the focus away from its expected position and creates sidelobes in the near-field.

Using a second-order binomial expansion, the Fresnel approximation of the focusing exponential term in (3.9) is,

\[
\sqrt{F^2 + r^2} - \sqrt{F^2 + a^2} = F \left[ 1 + \left( \frac{r}{F} \right)^2 \right]^{1/2} - \sqrt{F^2 + a^2}.
\]

\[
= F - \sqrt{F^2 + a^2} + \frac{r^2}{2F}.
\]

To ensure convergence of the series, \( \frac{r}{F} < 1 \). However, since \( r \leq a, \ F > a \) is required. If the Fresnel approximation is not used, \( F \) is not restricted. Figure 3.10 shows the velocity potential of the apodized, focused piston transducer using the Fresnel approximation.
51

3.7.6 Spherically-shaped concave transducer comparison

The two analytical models described by equations (3.24a) and (3.26) were simulated in MATLAB for the assumed parameters given in Table 3.4.

Figure 3.11 shows the impulse responses for the focused plane piston and spherically-shaped concave piston transducers for on-axis observation points that differ from the 45 mm focal point by 10 mm and 2 mm. It should be noted that because of the identical $t_{\text{min}}$ and $t_{\text{max}}$ values, the width of the responses are identical but, while the amplitude of the concave transducer is constant, the focused plane piston transducer varies. Closer to the focus (Figure 3.11c) it can be seen that the difference remains much the same. Very near the focus the
response approaches a delta function and, as a result, it can be expected that the response at the focus to a wideband signal for both transducer types will be nearly identical.

Table 3.4: Focused plane piston and concave piston simulation parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aperture radius</td>
<td>( a )</td>
<td>30 mm</td>
</tr>
<tr>
<td>Focal point</td>
<td>( F )</td>
<td>45 mm</td>
</tr>
<tr>
<td>Radius of curvature</td>
<td>( A )</td>
<td>54 mm</td>
</tr>
<tr>
<td>Frequency</td>
<td>( f )</td>
<td>5 MHz</td>
</tr>
<tr>
<td>Wave speed</td>
<td>( c_0 )</td>
<td>1500 m/s</td>
</tr>
</tbody>
</table>
Figure 3.11: Showing the impulse responses at various axial locations for the spherically-shaped concave piston and focused plane piston transducer at: (a) 35 mm and (b) 55 mm, (c) closer to the focus: at 43 mm and 47 mm. Note that the vertical scale is in km/s and that the focus for both transducers is at 45 mm.

To illustrate possible differences seen for transients close to the focal point, a wideband Gaussian modulated pulse at a center frequency of 5 MHz was used for axial points 2 mm on either side of the focus. Figure 3.12 shows the pressure waveforms obtained by convolving the impulse response with the derivative of this single cycle excitation pulse. Although differences are present, it is unlikely that they would have practical significance.
Figure 3.12: Comparing the transient response of spherically-shaped concave piston and focused, plane piston transducer. The excitation was a 5 MHz Gaussian modulated pulse with a 100% bandwidth. The pulse shape is shown at the top right. (a) Response at 43 mm (focus is at 45 mm), (b) Response at 47 mm.
To investigate possible differences between the two transducers in the frequency domain, the on-axis velocity potentials were calculated at 5MHz. This was achieved by first calculating the on-axis impulse responses for multiple positions between 30 mm and 60 mm, as obtained from (3.24a) for the focused, plane piston transducer and from (3.26) for the concave transducer. Fourier transforms of these results were then performed to determine the responses at 5MHz and the results are shown in Figure 3.13. The magnitudes shown in Figure 3.13a are nearly equivalent since the concave and focused, plane piston impulse responses have identical $t_{\text{min}}$ and $t_{\text{max}}$ values. However, as indicated in Figure 3.13b, the phase differences between the two transducers are significant even in the focal region, as the phase difference is not zero. This phase discrepancy describes how the differing impulse responses result in a phase or time delay of each frequency component when represented in the temporal frequency domain.

It is worthwhile looking at the physical reason for the differences in the impulse response, and for this purpose, a sketch is shown in Figure 3.14. Over the time interval from $t_n$ to $t_n + \Delta t$, where $\Delta t$ is a small and constant time increment, the spherically-shaped concave piston transducer incremental area contributing to the impulse response corresponds to an incremental ring on the concave surface whose edges are distant $R_c(t_n)$ and $R_c(t_n) + \Delta R_c(t_n + \Delta t)$ from the observation point. Now $R_c(t) = c_o t$, so that for a constant incremental time $\Delta t$, $\Delta R_c(t)$ is constant over the interval $t_{\text{min}} < t < t_{\text{max}}$, and it can be shown that this results in the constant amplitude impulse response given by (3.26). However, this is not true for the focused plane piston shown in Figure 3.14b. Over the time interval from $t_n$ to $t_n + \Delta t$, the transducer area contributing to the impulse response is determined by both the distance from the observation point $R_p$ and the focusing phase delay, as given by (3.20). Because the phase delay varies with the radial location on the piston, $\Delta R_p(t)$ will be a function of time, causing the impulse response to vary as shown in Figure 3.11.
Figure 3.13: On-axis velocity potential for 5MHz excitation illustrating the differences between the two transducers in the frequency domain. (a) Normalized magnitude, (b) Phase difference between the two transducers.

The impulse response arising from the spherically-shaped concave transducer is also highly influenced by the direct wave arising from the transducer centre at \( r = 0 \) and the edge wave arising from the transducer edge at \( r = a \). While these two waves arrive at the focal point at the same time, for \( z < F \) the direct wave arrives before the edge wave, while the reverse is true for \( z > F \) (Penttinen and Luukkala 1976). The times at which these two waves reach the observation point \( z \) defines the temporal extent of the impulse response.
The same direct and edge wave relationship is observed for the focused, plane piston transducer. However, for time points between the arrival of these two waves, the response differs markedly. For a constant $\Delta t$, the area of the transducer surface that contributes to the impulse response increases as one moves from the transducer centre to the outer edge. This results in the increasing impulse response amplitude for $z < F$, and the decreasing amplitude for $z > F$.

Figure 3.14: Sketches showing the incremental sections of (a) spherically-shaped concave piston and (b) focused, plane piston transducer that contribute to the impulse response over a small increment of time. The dashed lines in (b) represent the duration of the delay times.

### 3.8 Discussion

The equation derivation and simulation analysis presented for the apodized, focused plane piston transducer suggest that many of these results are also applicable to other plane transducer geometries such as the 1D linear array transducer. Since the Hankel transform is equivalent to the 2D spatial Fourier transform over Cartesian coordinates $(x,y)$, by replacing the transducer surface velocity equation with that corresponding to one with independent $x$ and $y$ coordinates, similar results are achievable. In addition, since an accurate spectral representation of the transducer focusing term has not been previously identified in the literature without using the Fresnel approximation, by following the derivation outlined in Appendix C, an equivalent solution for a transducer where focusing is limited to the $x$ or $y$ direction can be found.
The radiation characteristic that describes how the spectral bandwidth is constant over axial depth is applicable to other plane transducer geometries, as the response on any observation plane is equivalent to the 2D spatial convolution of the transducer surface velocity with the Green’s function. The transducer surface velocity is arbitrary and can represent any plane transducer.

Our analysis has identified several radiation characteristics that can be used for the development of new signal processing techniques for the improvement of image quality in ultrasound. The most notable of which is the spectral phase analysis. Since the angular spectrum bandwidth remains constant over all observation planes in a lossless medium, new techniques may be developed to account for the predictable phase discrepancy—perhaps using retrospective transmit focusing, transmit pulse shaping, or other post-processing methods. While our analysis was limited to the focused, plane piston transducer, the focusing component of the transducer surface velocity and its interaction with the Green’s function should be applicable to other transducer geometries, such as the linear 1D array transducer, since the basic forms of the equations are relatively consistent for plane transducer geometries. However, this will require a solution for focusing in the lateral and/or elevation planes for the 1D linear array transducer, which may be derived using Appendix C.

The SSF and PSF analysis found that signal processing techniques that can minimize the size of the SSF, which represents the spatial extent of the propagating wave, is highly affected by apodization and focusing. By considering both PSF and SSF analysis methods during signal processing algorithm development, a comprehensive picture of the radiated field can be obtained. One can then identify how changes to an impulse response at even one field position can have a larger impact on the surrounding radiated field. Although not analysed in this chapter, if for example, a process (such as perhaps attenuation or an anechoic region) was included that reduced the magnitude of the focal point response in isolation, this could have significant effects on the SSF as the relative contributions of nearby unaffected field locations would be amplified in the response. By limiting analysis to the PSF, these effects on the ultrasound imaging system may have been masked.
3.9 Conclusions

An accurate analytical model was developed using the angular spectrum method to describe the radiated field from a finite geometry, apodized, and focused piston transducer without the use of approximations. The model includes the effects of diffraction and evanescent waves and is valid at all near- and far-field observation positions. Subsequent analysis of the angular spectrum, velocity potential, and impulse response has identified that:

1. The spatial frequency bandwidth is constant over observation plane depth. The bandwidth determines the best possible spatial resolution that can be achieved for a particular transducer configuration.
2. On any observation plane, a constant angular spectrum phase is required to achieve the best spatial resolution of a particular transducer configuration. As expected, this is achieved on the focal plane.
3. Focusing the transducer can increase the spatial frequency bandwidth, which will improve the focal plane spatial resolution.
4. The velocity potential on any plane parallel to the transducer is the Hankel convolution of the normal component of the transducer surface velocity with the Green’s function.
5. Evanescent waves decay both with increasing spatial frequency $k_r$ and axial depth $z$.
6. The velocity potential magnitude determines the lateral span of the SSF, while the phase predicts the curvature.
7. Fresnel approximations of the transducer focusing term are inadequate for accurate representation of the velocity potential.
8. The impulse responses of the focused, plane piston and spherically-shaped concave piston transducers are not equivalent, despite demonstrating comparable effects at the focal point. Attenuation, dispersion, and secondary diffraction will also further differentiate the results, although these were not included in the analysis.

The main limitation of this work involves assumptions regarding the nature of the medium in which the wave propagates, as a lossless medium in the absence of attenuation and dispersion has been assumed. However, these effects will be addressed in the next chapter using various methods.
CHAPTER 4
ATTENUATION AND DISPERSION

An acoustic wave will experience attenuation and dispersion as it propagates in biological tissue or viscous fluid media. Attenuation, which quantifies a decrease in propagating wave intensity, arises from two mechanisms: absorption and scattering. Absorption is used to quantify the developed phase delay between the density of a medium and the propagating pressure wave, i.e., to what degree does a viscous medium delay a compression/rarefaction response due to a cyclic, longitudinal, incident pressure wave. The portion of the wave energy that does not produce an immediate compression/rarefaction response is converted into other forms such as heat, as energy must be conserved according to thermodynamics laws. Scattering describes how the wave energy will be redirected due to refraction, reflection, and variations in density and compressibility within the medium. Dispersion quantifies the changes in phase velocity during wave propagation, resulting in an altered wave shape. While the underlying mechanisms of attenuation and dispersion are not fully understood and experimental measurements of wave propagation in such media are limited, the development of models that can suggest or predict the pressure field in such mediums is helpful. These models can improve knowledge about the subtleties of ultrasound imaging and aid the development of solutions for image quality enhancement.

The purpose of this chapter is to extend the focused, plane piston transducer model developed in the previous chapter to account for the effects of energy loss due to attenuation and dispersion. The five primary objectives of this chapter are: a) to provide an overview of known attenuation and dispersion models, including both non-causal and causal solutions; b) to derive a Green’s function solution to the Navier-Stokes equation assuming spherical wave propagation in
a classical viscous medium using the Laplace transform; c) to propose modifications to the angular spectrum method to predict the radiated field of a focused, plane piston transducer in the near- and far-field of a viscous medium; d) to simulate and analyze the model equations in order to identify radiated field characteristics; and e) to simulate and analyze the attenuated impulse response in terms of the PSF and SSF.

4.1 Background

Attenuation models quantify the decrease in plane wave intensity as a wave propagates through a medium. It is defined by (Cobbold 2007, p.71),

$$I(z) = I(0)e^{-2\alpha z} = I(0)e^{-2(\alpha_a + \alpha_s)z}$$

(4.1)

where $I(z)$ is the time-averaged wave intensity at distance $z$ from the origin, $\alpha$ is the total amplitude attenuation coefficient, and $\alpha_a$ and $\alpha_s$ represent the components arising from absorption and scattering, respectively. In most biological fluids and non-porous tissue, such as blood and muscle, the effects of absorption dominate over scattering. In classical viscous fluids, attenuation is caused by absorption since the fluid is homogenous and does not scatter wave energy.

The total amplitude attenuation coefficient is generally determined experimentally and is modeled as (Cobbold 2007),

$$\alpha(\omega) = \alpha_0|\omega|^n.$$  

(4.2)

Note that for simplification purposes, we chose to use the notation $\alpha_0$ to represent the angular frequency form of the reference attenuation rate, which is different than that of Cobbold (2007), $\alpha_0'$. While certain fluid media and biological tissue have shown frequency dependencies in the range of $1 < n < 2$, in classical viscous media, $n = 2$ is often assumed (Bamber 1998).

A comprehensive, accurate, and fully verified model of attenuation is not possible at this time since both experimental and theoretical research remains incomplete. Moreover, attenuation modeling is limited by the plane wave assumption of attenuation in (4.1), which limits radiated field analyses to the far-field. In the near-field of the transducer, the wave is more accurately represented as a set of spherical waves using Huygen’s principle. Spherical waves only become quasi-planar far from the transducer origin. However, since the main principle of attenuation assumes a dependency on the wave propagation distance, a spherical wave model can
be seen as a reasonable extension of the plane wave attenuation model. We will therefore assume an intensity model of \( I(R) = \tilde{I}(0)e^{-2\alpha R} \), where \( R \) represents the radial distance from the origin. We will also assume in the following analysis that attenuation is limited to absorption processes.

### 4.1.1 Complex wave number

Dispersion is caused by absorption and quantifies the change in phase velocity of a propagating wave. Since absorption quantifies the loss in wave energy when an incident pressure wave experiences a delayed compression/rarefaction response, this delay inherently defines a change in phase velocity or dispersion. A quantitative relationship therefore exists between these two concepts, which is described by the Kramers-Kronig relations,

\[
\tilde{k}(\omega) = \frac{\omega}{c(\omega)} - j\alpha(\omega) \tag{4.3}
\]

\[
\alpha(\omega) = -\frac{2\omega^2}{\pi} \int_0^\infty \left[ \frac{1}{c(\omega')} - \frac{1}{c(\omega)} \right] \frac{d\omega'}{\omega'^2 - \omega^2}
\]

\[
c(\omega) = \frac{2}{\pi} \int_0^\infty [\alpha(\omega') - \alpha(\omega)] \frac{d\omega'}{\omega'^2 - \omega^2},
\]

The Kramers-Kronig equations are linear and causal, and describe a complex wave number using a frequency dependent phase velocity \( c(\omega) \) and attenuation coefficient \( \alpha(\omega) \geq 0 \) (Ginzberg 1955).

Szabo (1995) identified a specific solution to (4.3) for biological tissue that is consistent with (4.2). Assuming that the absorption rate is much smaller than the wave number, \((\alpha(\omega) << k)\), the solution simplifies to,

\[
\frac{1}{c(\omega)} = \begin{cases} 
\frac{1}{c_0} + \alpha_0 \tan \left( \frac{\pi n}{2} \right) \left| \omega \right|^{n-1} - \left| \omega_0 \right|^{n-1} & 0 < n < 1; \quad 1 < n < 3 \\
\frac{1}{c_0} - \frac{2}{\pi} \alpha_0 \ln \left| \frac{\omega}{\omega_0} \right| & n = 1
\end{cases}
\tag{4.4}
\]
where \( c_0 \) is the wave speed at a reference frequency of \( \omega_0 \). For classical viscous media \((n = 2)\), there is no dispersion since \( c(\omega) = c_0 \). When \( n > 2 \), the phase velocity decreases with increasing frequency, which is known as normal dispersion. However, for \( n < 2 \), anomalous dispersion causes the phase velocity to increase with frequency. Dispersion is absent at the cross-over point \((n = 2)\).

By examining the Navier-Stokes equation for classical viscous media, Buckingham (2005) came to a slightly different definition of attenuation and dispersion:

\[
\begin{align*}
\bar{k}(\omega) &= \frac{\omega}{c_0 \sqrt{1 + j\omega\gamma}} = \frac{\omega}{c(\omega)} - j\alpha(\omega) \\
c(\omega) &= \frac{c_0}{\text{Re}[\sqrt{1 + j\omega\gamma}]} \\
\alpha(\omega) &= -\frac{\omega \text{Im}[\sqrt{1 - j\omega\gamma}]}{c_0 \sqrt{1 + \omega^2\gamma^2}}
\end{align*}
\]

where \( \gamma = \kappa \left( \mu_B + \frac{4}{3} \mu_t \right) \) is the viscosity constant. While (4.5) satisfies the Kramers-Kronig relations for causality and linearity, they are consistent with the Szabo (1995) attenuation model for square-law frequency dependence when \( \omega\gamma \ll 1 \):

\[
\begin{align*}
c(\omega) &\sim c_0 \\
\alpha(\omega) &\sim \frac{\omega^2 \gamma}{2c_0}
\end{align*}
\]

Figure 4.1a shows several \( \alpha(\omega) \) curves from (4.2) and (4.5) for the parameters listed in Table 4.1.
Table 4.1: Attenuating medium simulation parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temporal frequency</td>
<td>( f = 5 ) MHz</td>
<td></td>
</tr>
<tr>
<td>Wave number</td>
<td>( k = 209.4 ) rad/cm</td>
<td></td>
</tr>
<tr>
<td>Wave propagation speed</td>
<td>( c_0 = 1500 ) m/s</td>
<td></td>
</tr>
<tr>
<td>Viscosity constant</td>
<td>( \gamma = 10^{-12.3} ) s</td>
<td>low – water</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 10^{-9.35} ) s</td>
<td>medium – glycerin (20°C)</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 10^{-7} ) s</td>
<td>high – peanut butter (although example is non-Newtonian)</td>
</tr>
<tr>
<td>Power-law frequency dependent attenuation</td>
<td>( n=1; \alpha = 0.5 ) dB/cm at 1MHz</td>
<td>low – human spleen</td>
</tr>
<tr>
<td></td>
<td>( n=2; \alpha = 0.5 ) dB/cm at 1MHz</td>
<td>medium</td>
</tr>
<tr>
<td></td>
<td>( n=2; \alpha = 5 ) dB/cm at 1MHz</td>
<td>high – skull bone</td>
</tr>
</tbody>
</table>

Three classical viscous media cases were included using (4.5): low viscosity (water) \( \gamma = 10^{-12.3} \) s, medium viscosity \( \gamma = 10^{-9.35} \) s, which corresponds to \( \alpha = 0.5 \) dB/cm at 1MHz (similar to human spleen and lies between blood and skeletal muscle (Bamber 1998)), and a high viscosity case \( \gamma = 10^{-7} \) s, which is similar to peanut butter (although peanut butter is not a classical viscous fluid).

Figure 4.1b,c shows the corresponding fractional change in wave speed, \( \frac{c(\omega) - c_0}{c_0} \), corresponding to (4.4) and (4.5). The low viscosity \( \gamma = 10^{-12.3} \) s and the \( n = 2 \) cases for power-law frequency dependent attenuation are coincident and have no dispersion since \( c(\omega) \sim c_0 \).

Figure 4.2 shows the \( c(\omega) \) and \( \alpha(\omega) \) relationship over \( \omega \gamma \) for the viscous medium. The ‘knee’ of the curve at \( \omega \gamma \approx 1 \) is quite significant. For \( \omega \gamma << 1 \), both \( \alpha(\omega) \) and \( c(\omega) \) are consistent with the square-frequency law from Szabo (1995). For \( \omega \gamma \geq 1 \), dispersive effects become significant as the fractional change in speed is quite large. While increasing viscosities can affect the rate of attenuation and dispersion, this can also be achieved using higher frequencies due to the dependency on \( \omega \gamma \). This can be an important consideration for high-frequency ultrasound systems, where frequencies in the range of 50 MHz are common.
Two representations of the attenuation coefficient are shown, $\alpha(\omega)$ at $\gamma = 10^{-7}$ s and $\frac{\alpha(\omega)}{\omega}$ from (4.5). The $\frac{\alpha(\omega)}{\omega}$ curve removes the dependency on $\omega$ and displays the result as a function of parameter $\omega\gamma$. There are two clear slopes in this curve, one corresponding to $\alpha(\omega) \propto \omega^2$ for $\omega\gamma << 1$ and a second at $\alpha(\omega) \propto \omega^{1/2}$ for $\omega\gamma >> 1$. 
Figure 4.1: (a) Attenuation coefficient $\alpha(f)$ for power-law frequency dependent attenuation and classical viscous media from (4.2) and (4.5). (b,c) Corresponding fractional change in wave speed from (4.4) and (4.5). The red and magenta dashed curves are coincident.
Figure 4.2: Frequency dependent attenuation coefficient and fractional change in wave speed over $\omega \gamma$ in a viscous medium for $\gamma = 10^{-7}$ s (Buckingham 2005). Since the curves refer to different metrics, as specified in the descriptions adjacent to each curve, the vertical axis remains dimensionless. The horizontal axis is in terms of $\omega \gamma$ since these two parameters are paired in the attenuation and wave speed equations.
4.2 Models of attenuation

Attenuation models have been predominantly concerned with identifying causal solutions for the impulse response assuming power-law frequency dependent attenuation \((4.2)\). However, simplified attenuation models often ignore the effects of dispersion, resulting in a non-causal solution. To ensure causality, the Kramers-Kronig relations of attenuation and dispersion must be considered. The complex wave number solutions presented by Szabo (1995) and Buckingham (2005) satisfy this requirement.

Developing an accurate radiation model of attenuation and dispersion in biological or classical viscous media is very difficult at this time since little experimental verification has been performed outside of a few highly studied fluids such as water. In addition, while frequency dependent attenuation has been characterized for a variety of media, minimal dispersion data is available as it is not easily experimentally measured. Existing attenuation models also often assume plane wave propagation, which may be inappropriate for modeling spherical wave propagation in the near-field of the transducer. These analytical models are therefore estimates of the radiated field in attenuating media, without a complete set of supporting experimental validation data. Nevertheless, attenuation models are useful as they can provide insights into how a propagating wave may be distorted in the medium. A mathematical analysis of these effects using the fundamentals of wave physics can suggest how attenuation may affect wave propagation. In addition, mathematical derivations remain useful as they have often preceded experimental verification since the latter is limited by technological advancements.

Two primary types of attenuation models will be examined: power-law frequency dependent attenuation using \((4.2)\); and absorption in a classical viscous medium using the Navier-Stokes equation.

4.2.1 Power-law frequency dependent attenuation

Both causal and non-causal transducer impulse responses in an attenuating medium have been proposed assuming power-law frequency dependent attenuation,

\[
\alpha(\omega) = \alpha_0 |\omega|^n.
\] (4.2)
These models are often expressed as a temporal convolution between an impulse response \( h_\alpha(r : t) \) describing the attenuating medium, and the transducer impulse response assuming a lossless medium \( h_{\text{lossless}}(r : t) \),

\[
h(r : t) = h_\alpha(r : t)^* h_{\text{lossless}}(r : t). \tag{4.6}
\]

The simplest model of attenuation assumes plane waves that decay with \( z \) (Cobbold 2007, p.208),

\[
h_\alpha(z : \omega) = e^{-\alpha(\omega)z} = e^{-\alpha_0|\omega|^n z}. \tag{4.7}
\]

However, this model is non-causal when \( n \neq 2 \) since dispersion is not addressed. Gurumurthy and Arthur (1982) determined a minimum phase equivalent of (4.7) assuming \( n = 1 \) and negligible dispersion. Sushilov et al. (2002) modified the Rayleigh-Sommerfeld diffraction equations to account for attenuation using a complex wave number \( \bar{k}(\omega) \) and assuming spherical wave propagation,

\[
\Phi(r : \omega) = \frac{1}{2\pi} \int_{S_1} \frac{v_z(r : \omega)e^{-j\bar{k}(\omega)r}}{R} dS_1,
\]

where \( S_1 \) represents the transducer surface, and \( v_z(r : \omega) \) is the normal component of the transducer surface velocity. The complex wave number is defined using (4.4) while assuming \( n = 1 \),

\[
\bar{k}(\omega) = \frac{\omega}{c_0} - \frac{2}{\pi} \alpha_0 \omega \ln \left| \frac{\omega}{\omega_0} \right| - j\alpha_0|\omega|. \tag{4.9}
\]

The on-axis impulse response \( h(r : t) \) of the Gaussian apodized piston transducer was also identified.

Kelly and McGough (2008a) proposed a fractional partial differential wave equation assuming a complex wave number for \( 0 < n < 2 \), \( n \neq 1 \) and spherical wave propagation,

\[
\bar{k}(\omega) = \frac{\omega}{c_0} - \frac{\alpha_0 (-j)^{n+1}\omega^n}{\cos \left( \frac{\pi n}{2} \right)}, \tag{4.10}
\]

where the real and imaginary parts of the complex wave number simplify to the attenuation coefficient and wave velocity as obtained by Szabo (1995) and as given in (4.4),
Imag$[\bar{k}(\omega)] = \alpha_0|\omega|^n$
\[\text{Real}[\bar{k}(\omega)] = \frac{\omega}{c(\omega)} = \frac{\omega}{c_0} + \alpha_0\omega\tan\left(\frac{\pi n}{2}\right)|\omega|^{n-1} \quad 0 < n < 2; \quad n \neq 1,
\]

where it has been assumed that $\omega_0 = 0$. The exact, unbounded 3D Green’s function solution was identified as,

$$g(R: \omega) = \left\{ e^{\frac{j\omega R}{c_0}} \right\} \left\{ e^{-\alpha_0 R|\omega|^{n-1}} - \alpha_0 R|\omega|^{n-1}\tan\left(\frac{\pi n}{2}\right)|\omega|^{n-1} \right\}, \quad (4.11)$$

However, it seems that Kelly and McGough (2008a) used a variant of the Fourier transform definition (Kilbas et al. 2006) in their paper, where

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt,$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-j\omega t} d\omega.$$

This resulted in the unbounded Green’s function solution converging to $g(R: \omega) = \frac{e^{\frac{j\omega R}{c_0}}}{4\pi R}$ instead of the form presented in this thesis, $g(R: \omega) = \frac{e^{\frac{-j\omega R}{c_0}}}{4\pi R}$ for the lossless case ($\alpha_0 = 0$).

Suitable modifications to use the common engineering Fourier transform definitions of

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt,$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega,$$

would therefore require replacing (4.10) and (4.11) with

$$\bar{k}(\omega) = \frac{\omega}{c_0} - \frac{\alpha_0(j)^{n+1}\omega^n}{\cos\left(\frac{\pi n}{2}\right)}, \quad (4.12)$$
\[ \text{Im} \{ \mathcal{K}(\omega) \} = -\alpha_0 |\omega|^n \]
\[ \text{Re} \{ \mathcal{K}(\omega) \} = \frac{\omega}{c_0(\omega)} + \alpha_0 \omega \tan \left( \frac{\pi n}{2} \right) |\omega|^{n-1} \quad 0 < n < 2; \quad n \neq 1 \]
\[
g(R : \omega) = \begin{cases} 
\frac{e^{-j\omega R}}{4\pi R} & 
\frac{e^{\alpha_0 R \left[ |\omega|^n + j\omega \tan \left( \frac{\pi n}{2} \right) \right] |\omega|^{n-1}}} 
\end{cases} 
\]
(4.13)

The two analytical Green's function models identified by Sushilov et al. (2002) and the modified equations of Kelly and McGough (2008a) are consistent. Both papers assumed, with some justification, that the Green's function in an attenuating and dispersive medium can be found by replacing the lossless wave number with a complex value. Kelly and McGough (2008a) identified the solution using partial fraction differential equations where \( n \neq 1 \), while Sushilov et al. (2002) identified the solution for \( n = 1 \).

### 4.2.2 Classical viscous loss

While steady state solutions of the Navier-Stokes equation for classical viscous media are well known, transient solutions have been more difficult to obtain as numerical approximations are often required. Several one-dimensional (1D) transient solutions have been proposed including, the approximate transient response from an abruptly switched sinusoidal source (Blackstock 1967), power series expansions (Hanin 1957, Norwood 1968), solutions in integral form (Ludwig and Levin 1995), and numerical approaches using the FFT (Christopher and Parker 1991). However, ensuring the transient response remains causal has been a significant challenge.

Buckingham (2005) identified that non-causal transient solutions arise when series expansion approximations truncate higher temporal frequency terms. Since an analytical Green's function solution of the Navier-Stokes equation requires both spatial and temporal Fourier transforms, the order of the inverse transforms may predict if the solution remains causal. If the inverse temporal frequency transform is approximated using a series expansion, which is generally required when the inverse spatial transform is performed first, the result will become non-causal. However, when the inverse temporal frequency transform is evaluated first, this
approximation is not required and a causal solution will result. Thus, Buckingham (2005) maintained that the exact solution to the Navier-Stokes equation is in fact causal.

Buckingham (2005) obtained a causal, Green’s function solution for the Navier-Stokes equation assuming plane wave propagation, and suggested an integral form solution for spherical waves in the time domain. Kelly and McGough (2008b) later approximated and simplified the spherical wave results of Buckingham (2005) using a far-field assumption and minimal viscosity. To the best of our knowledge, spherical wave Green’s function solutions for viscous media have only been proposed by Buckingham (2005) and Kelly and McGough (2008b) in the time domain. No temporal frequency domain solutions seem to exist, which is required for our focused, plane piston transducer model. Since the angular spectrum method was used for the transducer model derivation in Chapter 3, a Green’s function solution to the Navier-Stokes equation assuming a classical viscous medium is required in the temporal frequency domain.

### 4.2.2.1 Spherical wave propagation in a classical viscous medium

Our purpose is to derive a causal, spherical wave, Green’s function in the temporal frequency domain \( g(r : \omega) \) for the Navier-Stokes equation (2.3) in classical viscous media.

Assuming a delta impulse excitation \( \delta(t) \), the inhomogeneous Navier-Stokes equation is,

\[
\nabla^2 g(r : t) + \frac{\gamma}{\partial t} [\nabla^2 g(r : t)] - \frac{1}{c_0^2} \frac{\partial^2 g(r : t)}{\partial t^2} = -\delta(r) \delta(t) , \tag{4.14}
\]

The temporal Fourier transform of (4.14) yields,

\[
\nabla^2 g(r : \omega) + \frac{\omega^2}{c_0^2 (1 + j\omega \gamma)} g(r : \omega) = \frac{-1}{(1 + j\omega \gamma)} \delta(r). \tag{4.15}
\]

The Green’s function will be solved in the spatial frequency domain using the multi-dimensional Laplace transform, since the spatial frequencies must be complex to represent both wave propagation and attenuation in the viscous medium,

\[
\left(s_x, s_y, s_z\right) = \left(\alpha_x + jk_x, \alpha_y + jk_y, \alpha_z + jk_z\right) , \tag{4.16}
\]

assuming the directional components of spatial attenuation \( \left(\alpha_x, \alpha_y, \alpha_z\right) \) and spatial frequencies \( \left(k_x, k_y, k_z\right) \) in Cartesian coordinates \( (x, y, z) \). The multi-dimensional Laplace transform \( L_{x,y,z}[\cdot] \) (Brychkov et al. 1992, eq.2.6, p.85) is defined as,
while its inverse is (Brychkov et al. 1992, eq.2.41, p.101),

\[ f(x, y, z) = L_{x,y,z}^{-1}[F(s_x, s_y, s_z)] = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{s_x x} e^{s_y y} e^{s_z z} F(s_x, s_y, s_z) ds_x ds_y ds_z , \]  

(4.18)

where these latter integrals are defined over the individual regions of convergence for \((\alpha_x, \alpha_y, \alpha_z)\), and \(-\infty < (k_x, k_y, k_z) < \infty\).

Using (4.17), the Laplace transform of (4.15) is,

\[ G(s_x, s_y, s_z : \omega) = \frac{-1}{(1 + j\omega\gamma)^2 \left( s_x^2 + s_y^2 + s_z^2 + \frac{\omega^2}{c_0^2 (1 + j\omega\gamma)} \right)} , \]  

(4.19)

provided the region of convergence is satisfied. The region of convergence will be soon identified.

By taking the inverse Laplace transform of (4.19) with respect to \(s_z\), \(G(s_x, s_y, z : \omega)\) can be identified. This solution is useful as it can suggest necessary modifications to the angular spectrum method, which predicts the radiation field on an observation plane from that described on a parallel source plane.

By expanding (4.19) into partial fractions,

\[ G(s_x, s_y, s_z : \omega) = \frac{1}{(1 + j\omega\gamma)^2 j \sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega\gamma)}}} \left[ \frac{1}{s_z + j \sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega\gamma)}}} \right] \left[ -\frac{1}{s_z - j \sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega\gamma)}}} \right] . \]  

(4.20)

Noting the following bilateral 1D Laplace transform pairs (Lathi 1992, p.251-252 eq.4.7b):
\[ L_z \left[ e^{-\alpha z} u(z) \right] = \frac{1}{s + a}, \quad s > -a \]
\[ L_z \left[ -e^{-\alpha z} u(-z) \right] = \frac{1}{s + a}, \quad s < -a \]

where \( u(z) \) is the unit step function, and that Neumann boundary conditions will limit wave propagation to the \(+z\) direction, thereby eliminating the second fraction in (4.19) (which can be interpreted as reflecting the wave into the \(+z\) direction (Cobbold 2007, p.107)),

\[
G(s_x, s_y, s_z : \omega) = \frac{1}{(1 + j\omega \gamma) \sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega \gamma)}}} \left[ \frac{1}{s_z + j\sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega \gamma)}}} \right].
\]

(4.22)

The region of convergence, as determined by setting the denominator of the above equation to greater than zero, is,

\[
\alpha_x^2 + \alpha_y^2 < \left\{ \text{Real} \left[ \frac{-j\omega}{c_0 \sqrt{1 + j\omega \gamma}} \right] \right\}^2
\]

(4.23)

\[
\alpha_z > -\sqrt{-\alpha_x^2 - \alpha_y^2 + \left\{ \text{Real} \left[ \frac{-j\omega}{c_0 \sqrt{1 + j\omega \gamma}} \right] \right\}^2}
\]

(4.24)

Using (4.21), the Green’s function on the axial \( z \) plane is,

\[
G(s_x, s_y : z, \omega) = \frac{u(z)}{j(1 + j\omega \gamma) \sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega \gamma)}}} \exp \left[ -j\sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2 (1 + j\omega \gamma)}} \right],
\]

(4.25)

which converges to the known Green’s function solution in an inviscid medium,

\[
G(k_r : z, \omega) = \frac{1}{j\sqrt{k^2 - k_r^2}} e^{-j\sqrt{k^2 - k_r^2} k_r} \quad 0 < k_r < \infty; \quad z \geq 0
\]

(4.26)

for \( \gamma = 0 \), \( k = \frac{\omega}{c_0} \), and \( s_x^2 + s_y^2 = (jk_r)^2 = -k_x^2 - k_y^2 \).
Both homogeneous and evanescent wave propagation is described by (4.25) since 
\(-\infty < k_x, k_y < \infty\) and the real (non-attenuating) component of the wave number is finite,

\[
\text{Real} \left( \frac{\omega}{c_0 \sqrt{1 + j \omega \gamma}} \right) < \infty .
\]

Homogeneous waves arise from spatial frequencies

\[- \text{Real} \left( \frac{\omega}{c_0 \sqrt{1 + j \omega \gamma}} \right) < k_x, k_y < \text{Real} \left( \frac{\omega}{c_0 \sqrt{1 + j \omega \gamma}} \right),\]

while evanescent waves correspond to

\[
\text{Real} \left( \frac{\omega}{c_0 \sqrt{1 + j \omega \gamma}} \right) < |k_x, k_y| .
\]

In addition, since the attenuation components \((\alpha_x, \alpha_y)\) of the complex spatial frequencies are orthogonal to spatial frequencies \((k_x, k_y)\) since \((s_x, s_y) = (\alpha_x + j k_x, \alpha_y + j k_y)\), both homogeneous and evanescent waves will experience attenuation and dispersion in the medium. However, this conclusion is at odds with Christopher and Parker (1991), who suggested that evanescent waves are not attenuated, despite providing neither an explanation nor a mathematical derivation proving their assertion.

To determine the Green’s function solution in the spatial domain, \(g(\mathbf{r} : \omega)\), the inverse 2D Laplace transform of (4.25) is required. While this solution is not available in multi-dimensional Laplace transform tables (Brychkov et al. 1992, Ditkin and Prudnikov 1962), the inverse Hankel transform of (3.15) is well known,

\[
H^{-1} \left[ \begin{array}{c} \frac{\pm j}{\sqrt{k^2 - k_r^2}} e^{\pm j z \sqrt{k^2 - k_r^2}} \quad 0 < k_r < k \\
1 \quad e^{-z \sqrt{k^2 - k_r^2}} \quad k < k_r < \infty \end{array} \right] = \frac{1}{2\pi \sqrt{r^2 + z^2}} e^{\pm j k \sqrt{r^2 + z^2}}, \quad (3.15)
\]

which is equivalent to the following inverse 2D spatial Fourier transform in Cartesian coordinates:

\[
F_{k_x,k_y}^{-1} \left[ \begin{array}{c} \frac{1}{j \sqrt{k^2 - k_x^2 - k_y^2}} e^{-j z \sqrt{k^2 - k_x^2 - k_y^2}} \quad -\infty < k_x, k_y < \infty \\
1 \quad 2\pi \sqrt{x^2 + y^2 + z^2} e^{-j k \sqrt{x^2 + y^2 + z^2}} \end{array} \right] = \frac{1}{2\pi \sqrt{x^2 + y^2 + z^2}} e^{-j k \sqrt{x^2 + y^2 + z^2}}, \quad (4.27)
\]
where \( F_{k_x,k_y}^{-1} \) is the inverse 2D spatial Fourier transform in \((k_x,k_y)\), and \( k = \frac{\omega}{c_0} \) is the wave number. It is known that the Fourier transform can be identified from the Laplace transform using the substitution \( s \to jk \), provided the Fourier transform exists and the signal is asymptotically stable and absolutely integrable (Lathi 1992, p.482-483). By performing the reverse substitution \( jk \to s \), the equivalent Laplace transform pair can be found, provided the Laplace transform convergence requirements are satisfied. Using (4.27) and the substitution \((jk_x, jk_y) \to (s_x, s_y)\), the inverse 2D Laplace transform of (4.25) is,

\[
g(R: \omega) = \frac{1}{2\pi(1+j\omega)R} \exp \left[ -j \frac{\omega}{c_0 \sqrt{1+j\omega}} R \right] u(z), \tag{4.28}
\]

where \( R = \sqrt{x^2 + y^2 + z^2} \). This result is causal since no approximations have been made in the derivation of the Green’s function (Buckingham 2005). In an inviscid medium, (4.28) converges to the known Green’s function solution,

\[
g(R: \omega) = \frac{1}{2\pi R} e^{-jkR} u(z). \tag{4.29}
\]

Equation (4.28) includes an extra multiplicative term in the denominator, \( (1+j\omega\gamma) \), which is not identified in the Green’s function definitions obtained for power-law frequency dependent attenuation proposed by Sushilov et al. (2002) using (4.8) or the modified equation of Kelly and McGough (2008a) as given by (4.13),

\[
g(r, z: \omega) = \frac{e^{-jk(\omega)\sqrt{r^2 + z^2}}}{2\pi \sqrt{r^2 + z^2}}, \tag{4.30}
\]

where \( r = \sqrt{x^2 + y^2} \). Generally, the attenuation and dispersion relations described by square-law frequency dependent attenuation are equivalent to that of a viscous medium when \( \omega\gamma \ll 1 \). In this limiting case, the Green’s functions will also be equivalent since for the denominator term of (4.28), \( (1+j\omega\gamma) \to 1 \). However, while the Szabo (1995) relations require that attenuation be much smaller than the wave number, the viscous medium solution is not restricted in viscosity or frequency.
4.3 Modified angular spectrum method

To determine the impulse response $\Phi(x, y, z : \omega)$ at an observation point arising from a planar transducer excitation on the $z = 0$ plane, the Green’s function should be spatially convolved with the transducer surface excitation velocity. The Green’s function can correspond to either the viscous version (4.28) or the inviscid version (4.29), depending on the properties of the medium.

Assuming a classical viscous medium, the angular spectrum method must be modified to account for the complex wave number by using the 2D spatial Laplace transform instead of the 2D spatial Fourier transform to find the spectral representations on any observation plane. The transformation required to find the angular spectrum on the observation plane $S(s_x, s_y : z, \omega)$ from the transducer surface velocity can be obtained from (4.25),

$$ S(s_x, s_y : z, \omega) = \frac{L_{xy}[v_z(x, y, 0 : \omega)]}{j(1 + j\omega\gamma)\sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2(1 + j\omega\gamma)}}} \exp\left[ -jz\sqrt{s_x^2 + s_y^2 + \frac{\omega^2}{c_0^2(1 + j\omega\gamma)}} \right]. \quad (4.31) $$

The velocity potential on the observation plane follows an equivalent solution as the inviscid case,

$$ \Phi(r, z : \omega) = v_z(r, 0 : \omega)\frac{u(z)}{r} \left\{ \frac{u(z)}{2\pi(1 + j\omega\gamma)\sqrt{r^2 + z^2}} \exp\left[ -j\frac{\omega v\sqrt{r^2 + z^2}}{c_0(1 + j\omega\gamma)} \right] \right\}. \quad (4.32) $$

For a power-law frequency dependent attenuation, the equivalent angular spectrum derivation is,

$$ S(s_x, s_y : z, \omega) = \frac{L_{xy}[v_z(x, y, 0 : \omega)]}{j\sqrt{s_x^2 + s_y^2 + k^2}} \exp\left[ -j\sqrt{s_x^2 + s_y^2 + k^2} \right]. \quad (4.33) $$

$$ \Phi(r, z : \omega) = v_z(r, 0 : \omega)\frac{1}{2\pi\sqrt{r^2 + z^2}} \exp\left[ -j\sqrt{r^2 + z^2} \right]. $$
4.4 Assumptions and simplifications

The primary assumptions and simplifications used in the attenuation and dispersion model include,

- A spherical wave model of attenuation is a reasonable extension of the plane wave model in the transducer near-field.
- The attenuation coefficient is much smaller than the wave number (Szabo 1995) for power-law frequency dependent attenuation.
- Attenuation is limited to absorption processes.
- Attenuation is represented using a complex spatial frequency that decays with distance, instead of a complex temporal frequency that decays with propagation time.
- The cylindrical form of the 3D Laplace transform exists.

A summary of these items are also listed in Appendix A.

4.5 Simulation results

We have simulated and compared the attenuation and dispersion models for spherical wave propagation with power-law frequency dependent attenuation and classical viscous media, assuming the simulation parameters listed in Table 3.1, Table 3.2 for Case 3 (focusing without apodization), and Table 4.1, which describes attenuation and viscosity parameters.

4.5.1 Green’s function

The Green’s function in the time domain was found numerically using the inverse temporal Fourier transform of (4.28) and (4.30) and is shown in Figure 4.3 in normalized time \((R/c_0)\) at a distance \(R\) from the source. The results are consistent with that obtained from numerically solving the integral, spherical wave solution proposed by Buckingham (2005). While the results are causal, dispersion temporally skews the waveform so it is no longer symmetrical about \(R/c_0\) when viscosity is very high or \(n=1\). However, in high bandwidth systems, this effect is almost negligible since higher temporal frequencies that experience high dispersive effects are also highly attenuated. The remaining low frequency content is dominant in the Green’s function solutions, resulting in the broadened signal centered very close to the
inviscid solution. As expected, when the viscous medium \( (\gamma = 10^{-0.35} \text{s}) \) has comparable attenuation and dispersion rates with power-law frequency dependent attenuation \( (n = 2, \alpha = 0.5 \text{dB/cm at } 1\text{MHz}) \), the Green’s functions are coincident.

Figure 4.3: Normalized time Green’s function results at \( R \) for both classical viscous media and frequency dependent attenuation.
4.5.2  Focused, plane piston transducer

The radiation field of the circularly symmetric, unapodized and focused, plane piston transducer was examined in simulation assuming a normal surface velocity of, \( v_z(r,0) = \circ \left( \frac{r}{a} \right) e^{j \omega \sqrt{F^2 + r^2 - \sqrt{F^2 + a^2}}}, \)

where \( r = \sqrt{x^2 + y^2}, \) \( \circ \left( \frac{r}{a} \right) = \begin{cases} 1 & r \leq a \\ 0 & r > a \end{cases}, \) and focusing is defined at an axial depth of \( z = F \).

4.5.2.1 Angular spectrum

For graphical purposes only (not derivation), the following circularly symmetric 2D Laplace transform relation has been assumed,

\[
(s_r, s_z) = (\alpha_r + jk_r, \alpha_z + jk_z)
\]

\[
\alpha_r = \sqrt{\alpha_x^2 + \alpha_y^2}
\]

\[
k_r = \sqrt{k_x^2 + k_y^2}
\]

These relations cannot be explicitly supported by a mathematical derivation since the cylindrical form of the 2D Laplace transform does not seem to exist in the literature, and such a derivation is outside the scope of this thesis. Nevertheless, these relations will be assumed for graphical purposes as we will interpret this orthogonal 6-dimensional system using a grouping of two orthogonal and cylindrical 3-dimensional systems: one describing attenuation and the other describing spatial frequency.

The angular spectrum on the transducer and observation planes were obtained using (4.31) for the viscous medium and (4.33) for power-law frequency dependent attenuation. In a lossless medium, the angular spectrum magnitude and spatial frequency dependent bandwidth remain constant over observation plane depth since the exponential wave propagation term in (3.2) is solely imaginary for homogeneous waves \((k_r < k)\). However, this is not the case in an attenuating medium. Figure 4.4 shows the angular spectrum magnitude on the imaginary \( s_r \) axis \((\alpha_r = 0, k_r)\) for several observation plane depths assuming power-law frequency dependent attenuation \((n = 1, \alpha = 0.5 \text{ dB/cm at 1MHz})\). The results are normalized to that on the transducer plane. The spectral magnitude decays as the observation plane depth increases, as
expected. Since \( \alpha = \sqrt{\alpha_z^2 + \alpha_r^2} \), by plotting the result on the imaginary \( s_r \) axis \( (\alpha_r = 0) \), the attenuation coefficient is equal to \( \alpha = \alpha_z \) and will therefore cause the spectral magnitude to decay with \( e^{-z\alpha} \) in Figure 4.4. However, if the results are plotted on a different \( \alpha_r \) axis, the spectral magnitude will decay with \( e^{-z\sqrt{\alpha_z^2 - \alpha_r^2}} \) due to the properties of spherical wave attenuation in the complex spatial frequency domain.

The effect of a high viscosity \( (\gamma = 10^{-7} \text{ s}) \) on the angular spectrum response is shown in Figure 4.5. The angular spectrum magnitude for this classical viscous case is shown on the 4.5 cm focal plane in Figure 4.5a, while the phase on several planes around the focus (3 cm to 4.8 cm) is shown in Figure 4.5b. The high rate of attenuation reduces the spectral bandwidth considerably in Figure 4.5a, as compared to the corresponding response on the transducer plane shown in Figure 4.4. This reduced bandwidth will significantly reduce the spatial resolution in the final image. High spatial frequency content is required to resolve fine imaging detail. Only the results to -800dB were shown.

The phase results in Figure 4.5b reflect the highly dispersive properties of the medium. In Chapter 3, we found that the angular spectrum phase is constant on the focal plane for spatial frequencies corresponding to the most significant spectral magnitudes of a focused transducer. By examining the limiting case of a focused transducer of infinite radius, the impulse response converges to a spatial delta impulse function at the focal point. The phase of the corresponding spatial frequency response is therefore constant on the focal plane. For a finite geometry, focused transducer, while the optimal impulse response is also achieved at the focus due to the constant phase, the response is limited by the spectral bandwidth.

In a non-dispersive medium, the phases arising from the wave propagation term in (3.2) and the angular spectrum of the transducer surface velocity focusing term (4.34) cancel on the focal plane, resulting in a nearly constant result. However, dispersion will prevent the phase from being constant on the focal plane since the frequency dependent wave speed in (3.2) will not match the constant wave speed assumed by the transducer focusing term. The propagating wave will therefore focus on a different observation plane for each temporal frequency since the wave speed is frequency dependent. However, this assertion is approximate since the terms will not completely cancel due to the nature of the waveforms. As shown in Figure 4.5b at 5 MHz
and for the most significant spectral magnitudes \( k_r < 85 \text{ rad/cm} \), the angular spectrum phase is constant near 3cm, causing the 5 MHz signal to focus around this depth.

Figure 4.4: Angular spectrum magnitude on the imaginary axis \( s_r = \alpha_r + jk_r = jk_r \) for several observation planes at 5MHz. While the shape of the curve is consistent regardless of observation plane depth, the entire observation plane curve decays with \( z \). The results are normalized to that on the transducer plane. Results are shown for frequency dependent attenuation using \( n = 1, \alpha = 0.5 \text{ dB/cm} \) at 1MHz.
Figure 4.5: Angular spectrum for a classical viscous medium ($\gamma = 10^{-7}$ s) (a) Normalized magnitude on focal plane at 5MHz. (b) Phase on various observation planes. On the focal plane, the phase is not constant due to dispersion. However, the 5 MHz signal will focus near $z = 3$ cm due to the nearly constant phase over $k_r < 40$ rad/cm.

4.5.2.2 Velocity potential

The velocity potential results are shown in Figure 4.6 in cylindrical coordinates. While the on-axis focal point is distinguishable for low and medium viscosities and power-law frequency dependent attenuation cases in Figure 4.6a,b,d,e, the magnitude of the responses decay with axial depth. For the highly viscous medium in Figure 4.6c, the radiated field is attenuated so significantly that the focus is no longer distinguishable. However, the effect of spherical wave attenuation is evident by observing the curvature of the constant dB contours. The results are normalized in Figure 4.6c,f to the maximum shown at 3cm since the magnitude decays rapidly over distance.
Figure 4.6: Velocity potential results [dB] at 5 MHz for the three viscosity and power-law frequency dependent attenuation cases assuming an unapodized, focused piston transducer.
4.5.2.3  **Point spread function and spatial sensitivity function**

The velocity impulse response was found by numerically simulating the inverse temporal Fourier transform of (4.32). Several PSF results around the focus are shown in Figure 4.7 and Figure 4.8. All results are normalized to the focal point magnitude in the lossless medium. Since the medium viscosity ($\gamma = 10^{-9.35} \text{s}$) case had comparable attenuation and dispersion rates to the square-law frequency dependent attenuation case ($n = 2$, $\alpha = 0.5 \text{dB/cm at 1MHz}$), only the viscosity case was shown in Figure 4.7. It is interesting to note that while the amplitude of the impulse response is significantly attenuated at the focus due to the loss of high spatial frequency content, this same viscosity may minimally affect the PSF in regions around the focus. For the high viscosity and power-law frequency dependent attenuation cases in Figure 4.8, the focal response is not distinguishable from the PSF signals at nearby observation points.

The SSF results are shown in Figure 4.9 for three superimposed time points. Results were shown normalized to the peak attenuated response over the shown time range. This corresponded to the magnitude of the focal point response for all but the highest viscosities and power-law frequency dependent attenuation. Notable characteristics include the magnitude, axial span (over $z$), lateral span (over $r$), and spatial curvature. It is apparent that medium viscosities and power-law frequency dependent attenuation increase the significance of a larger portion of the radiated field. Ideally, when the propagating wave travels through the focal point, corresponding to near $t = 36 \mu\text{s}$ in Figure 4.9, the SSF would be localized around the focus with very little spread in either the lateral or axial directions, as shown in Figure 4.9a. This is due to the significantly higher PSF response at the focus than is observed at nearby observation points. However, since medium viscosity and power-law frequency dependent attenuation was seen to significantly affect the magnitude of the PSF in Figure 4.7, this will have the effect of increasing the relative significance of nearby regions in the SSF of Figure 4.9b. For highly viscous media, the lower spatial frequency bandwidth of Figure 4.5a significantly increases the SSF field distributions due to the high rates of attenuation and dispersion. The SSF lateral and axial spans are minimized closer to the transducer due to the effects of dispersion on the angular spectrum phase previously discussed in Figure 4.5b. The phase was relatively constant near the 3 cm axial plane for the analyzed 5MHz velocity potential.
Figure 4.7: PSF results at several positions around the focus for the low and medium viscosity, and power-law frequency dependent attenuation cases. Note that while the magnitude of the response at the focus can be significantly attenuated due to the loss of high spatial frequency content, its effect further from the focus can be minimal.
Figure 4.8: PSF results for high viscosity and power-law frequency dependent attenuation. Note that the focal response is not distinguishable in the responses, unlike that observed in Figure 4.7.
Figure 4.9: SSF results [dB] for three time points as the wave passes through the 4.5 cm focal point. Results were normalized to the peak magnitude during the time range shown. Since attenuation was observed to significantly affect the focal point PSF response in Figure 4.7, the relative SSF magnitudes at positions near the focus become more significant, thereby increasing the lateral and axial span in subplots (b,d,e).
4.6 Discussion

It is anticipated that the derivation and results obtained in this chapter for the focused, plane piston transducer are readily applicable to more standard, focused transducer geometries, such as the focused, 1D linear array transducer. Since the modified angular spectrum derivation was presented for a generic transducer surface velocity, the selection of the focused, plane piston transducer was arbitrary.

It is important to note the extra \((1 + j\omega\gamma)\) term in the Green’s function response of the classical viscous medium derivation. Since previous Green’s function analyses assumed that the attenuating medium simply required replacement of the wave number with a complex equivalent, the presence of this extra denominator term is notable. However, this term is not necessary when the rate of attenuation is much less than the wave number, as defined by Szabo (1995). Nevertheless, since the attenuation rate is frequency dependent, it can increase rapidly with frequency (4.2). New high frequency ultrasound technologies for imaging biological tissue, will therefore require consideration of this term for modeling attenuating wave propagation effects in order to ensure the responses remain causal. The power-law frequency dependent attenuation models may require revisiting to accommodate for these effects in cases where the Szabo (1995) requirements are not met.

The effects of attenuation on the simulation results around the focal point are very interesting. The PSF responses identified that attenuation seems to highly affect the focal point response more significantly than that observed nearby. This has a significant effect on the SSF and the resulting received transducer responses in a full transducer modeling system. Since the SSF identifies the relative spatial distribution of the propagating wave in the medium, this corresponds to how various spatial positions contribute to the received transducer signal at various times, and their relative magnitudes in the response. In a lossless medium, the received transducer signal at a particular time corresponding to the wave passing through the focus, will be highly biased towards a region tightly around the focal point. A minimal proportion of the received signal will arise from positions away from the focus since the focal response is so large compared to nearby positions (see SSF plot in Figure 3.9c). However, in an attenuating medium, the focal point response is highly attenuated, while regions away from the focus are not
comparably affected. The SSF plots corresponding to the time when the wave passes through the focal point have a significantly wider span (see SSF plots in Figure 4.9). As a result, a much larger spatial region will contribute to the received transducer response at the time when the wave propagates through the focal point. Attenuation can therefore significantly degrade the spatial resolution of an imaging system. This analysis should be also applicable to more standard, focused transducer geometries, as the selection of the focused, plane piston transducer was arbitrary in our mathematical derivation.

4.7 Conclusions

An overview of existing non-causal and causal attenuation models for both plane and spherical wave propagation has been presented, and a causal, Green’s function solution for spherical wave propagation in a classical viscous medium has been derived from the Navier-Stokes equation. Modifications to the angular spectrum method were also proposed to account for the transient response of a viscous medium. The radiated field characteristics of the Green’s function as well as that arising from a focused, plane piston transducer were then examined in simulation. Subsequent analysis has identified that:

1. The Green’s function solution for a classical viscous medium includes a multiplicative term \((1 + j\omega\gamma)\) in the denominator that is not present in the Green’s function solutions for lossless media or power-law frequency dependent attenuation. While this term can become significant in highly viscous media or at high frequencies, its effect is negligible when \(\omega\gamma << 1\).

2. Dispersion can degrade the focusing ability of the transducer, since the frequency dependent wave speed will alter the depth at which each frequency will focus.

3. While medium viscosities and power-law frequency dependent attenuation can significantly decrease the magnitude of the PSF at the focus, it may minimally affect the response away from the focal point.

4. Since medium viscosities significantly attenuate the PSF at the focal point, the relative significance of nearby regions increase considerably in the SSF.
5. For high temporal bandwidth systems, dispersion will not significantly shift the time of the Green’s function peak response, as the highest rates of dispersion correspond to frequencies experiencing the highest rates of attenuation.

6. Both homogeneous and evanescent waves experience attenuation and dispersion.

While the developed models can provide a reasonable estimation of the effects of attenuation and dispersion in biological tissue and classical viscous fluids, there is minimal experimental validation data available in the literature. Nevertheless, such models are useful as they offer a means to analyze energy loss in the radiated field, aiding the development of new algorithms to accommodate for these effects.
CHAPTER 5
RETROSPECTIVE TRANSMIT FOCUSING

The primary motivation for this research is the development of signal processing methods to improve B-mode image quality. However, since a B-mode image is formed by stacking multiple independent A-lines generated at different lateral positions, image quality improvements can be achieved by optimizing the spatial resolution and contrast of an individual A-line. Each A-line remains independent in the final B-mode image as signal processing techniques are not usually applied between them.

We initially hypothesized that in order to improve a single A-line, a comprehensive understanding of wave propagation is required using a variety of tools. The purpose of this chapter is to use the mathematical derivations and insights previously developed in Chapters 3 and 4, which describe the radiated field from a finite geometry, apodized and focused, plane piston transducer, to improve the spatial resolution and contrast of an A-line.

5.1 Strategy

The ultrasound hardware circuitry was identified in Chapter 1, Figure 1.4. On receipt of the backscattered ultrasound signal by the transducer, the signal is switched to the receive beamformer where it is amplified and converted to a digital signal. The digitized signal is passed through a block to account for receive aperture focusing delays and apodization for each array element signal. The signals are then summed, matched filtered with the excitation wave, envelope detected, and graphically displayed to the user. Our research strategy involves replacing the summation and matched filter/envelope detection blocks with an alternative approach in order to generate an improved image for display.
An A-line is a time domain signal, whose amplitude reflects the variation in compressibility and density of a medium. An incrementally small volume of changing compressibility and density is called a scatterer. At time \( t = t_1 \), the magnitude of the A-line response will depend on a distribution of scatterers, as described by the SSF.

By assuming that the aperture is centered about the origin on the \( z = 0 \) plane, an ideal A-line response \( t = t_1 \) would result from a single on-axis scatterer, located at \( z = t_1 / c_0 \). This case would maximize the spatial resolution and contrast of the A-line image as nearby scatterers would not contribute to the response at \( t = t_1 \), and the signal arising from the scatterer at \( z = t_1 / c_0 \) would not leak into nearby time samples. However, both the lateral and axial spans of the SSF will affect the response at \( t = t_1 \), decreasing both the spatial resolution and contrast. A worthwhile goal is to therefore develop signal processing strategies to reduce the spatial extent of the SSF.

The SSF of the focused, plane piston transducer impulse response was previously identified in Chapters 3 and 4. However, the SSF of the entire ultrasound imaging system is considerably larger, as it consists of the temporal convolution of significantly more terms. Increasing the SSF spatial distribution will degrade spatial resolution since more scatterers will contribute to the response at a particular time. Consider the following linear model representing the ultrasound scattering system previously described in (1.1),

\[
y(t) = e_{tx}(t) * e_{rx}(t) * \sum_{r} h_{tx}(r : t) * h_{rx}(r : t) * \frac{dv_0(t)}{dt} * bsc(r) + w(t),
\]

(1.1)

where \( y(t) \) is the received voltage signal over time \( t \), \( e(t) \) is the electromechanical response of the transducer on transmit \( tx \) and receive \( rx \) (assumed known), \( h_{tx}(r : t) \) and \( h_{rx}(r : t) \) represent the impulse responses of the transmit and receive transducer apertures, respectively, at field position \( r \), \( v_0(t) \) is the temporal excitation voltage of the transducer, \( bsc(r) \) is the backscattering cross-section of a scatterer, and \( w(t) \) represents additive noise.

The temporal convolutions in (1.1) will significantly broaden the PSF since the temporal convolution of two waveforms, e.g., \( X(t) \), \( 0 < t < t_N \) and \( Y(t) \), \( 0 < t < t_M \), will result in a waveform of duration equal to the sum of the duration of the original waveforms, \( X(t) * Y(t) \), \( 0 < t < t_N + t_M \). Each temporal convolution increases the time span of the PSF, thereby
increasing the SSF, as more spatial positions will have a non-zero response at a particular time. Consider that if the temporal length of the PSF at a particular position is increased, then the range of times that the SSF is non-zero at this same position will also increase. When applied to all scatterers in the insonated field, this would result in a broadened SSF since more scatterers will have a non-zero response at a particular time.

However, if we can identify methods to either remove or reduce the temporal duration of individual terms in (1.1), then the SSF can be reduced, which will also improve resolution and contrast. Our primary strategy is to identify which terms in (1.1) are common to all scatterer positions and develop methods to either remove or modify them. A simple sketch illustrating this concept is shown in Figure 5.1. Figure 5.1a shows a sketch of the SSF of a generalized transducer impulse response on transmission into the medium. Figure 5.1b shows a sketch of the significantly broader received ultrasound signal after the multiple temporal convolutions identified in (1.1).

![Figure 5.1: Spatial sensitivity functions (SSF) a) Transducer impulse response on transmission into the medium. b) Received ultrasound signal after temporal convolving several terms in (1.1).](image)

**5.2 Excitation voltage and noise suppression**

An initial examination of (1.1) reveals that both the electromechanical response of the transducer $\varepsilon(t)$ and the transducer excitation voltage $v_0(t)$ are known and common among all scatterer positions in the insonation field. We will assume here that the electromechanical
response is a delta impulse function, although alternatively this signal can be considered lumped with the excitation voltage.

Assuming that the statistics of the remaining terms are known and the system is wide-sense stationary, Wiener deconvolution, or equivalently Linear Minimum Mean Squared Error (LMMSE) filtering techniques can be used to generate an estimate of the received signals with the effects of the excitation wave removed and additive noise suppressed. These methods require that the power spectral density or autocorrelation functions be known, however the additive noise need not be white.

5.2.1 Wiener deconvolution

Wiener deconvolution (Hayes 1996, p.369-371) is a frequency domain deconvolution method that estimates the desired signal \( d(t) \) from

\[
y(t) = d(t)^* v(t) + w(t)
\]

where \( y(t) \) is the known received signal, \( v(t) \) is the known signal to be deconvolved, and \( w(t) \) represents additive noise. If the statistics of \( d(t) \) and \( w(t) \) are also known, a minimum mean-square estimate \( \hat{d}(t) \) of the desired signal \( d(t) \) can be obtained. The Wiener deconvolution method identifies a deconvolution filter, \( h_{\text{deconv}}(t) \), such that,

\[
\hat{d}(t) = y(t)^* h_{\text{deconv}}(t).
\]

Since the ultrasound system is in discrete time \([n]\), the deconvolution method will be presented in its time sampled version. Assuming a system described by,

\[
y[n] = d[n]^* v[n] + w[n]
\]

\[
\hat{d}[n] = y[n]^* h_{\text{deconv}}[n],
\]

\( \hat{d}[n] \) can be identified by minimizing the mean-squared error \( \xi \),

\[
\xi = E \left\{ |d[n] - \hat{d}[n]|^2 \right\},
\]
where $E[d[n]] = \frac{1}{N} \sum_{n=0}^{N-1} d[n]$ is the expected value operator. If auto-correlation functions, $r_d[n]$ and $r_w[n]$, of $d[n]$ and $w[n]$ are known, respectively, their power spectral densities $P_d(e^{j\omega})$ and $P_w(e^{j\omega})$ can be determined using the discrete time Fourier transform:

$$r_d[k] = E[d[n+k]d^*[k]]$$

$$P_d(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_d[k]e^{-jk\omega}$$

$$r_d[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_d(e^{j\omega})e^{jk\omega}d\omega$$

where $^*$ indicates the conjugate operator.

The Wiener filter is defined as,

$$H(e^{j\omega}) = \frac{P_d(e^{j\omega})V^*(e^{j\omega})}{P_d(e^{j\omega})V(e^{j\omega})^T + P_w(e^{j\omega})},$$

where $H(e^{j\omega})$ and $V(e^{j\omega})$ are the discrete time Fourier transforms of $h_{deconv}[n]$ and $v[n]$, respectively.

The deconvolved signal is the best estimate of $d[n]$ in the absence of the excitation wave and with additive noise suppressed.

### 5.2.2 Linear Minimum Mean Squared Error (LMMSE)

A comparable deconvolution approach in the time domain is the Linear Minimum Mean Squared Error solution (Lingvall and Olofsson 2007). The LMMSE deconvolution filter is defined as,

$$h_{deconv} = R_d V^T (VR_d V^T + R_w)^{-1},$$

where $h_{deconv} = [h_{deconv}[1] \ldots h_{deconv}[N]]^T$, $R_d$ and $R_w$ are the Toeplitz matrix form of auto-correlation vectors $r_d = [r_d[1] \ldots r_d[N]]^T$ and $r_w = [r_w[1] \ldots r_w[N]]^T$, respectively, $V$ is the circulant matrix form of vector $v = [v[1] \ldots v[N]]^T$, and $^T$ is the transpose operator.
Although excitation wave removal and additive noise suppression methods using deconvolution methods were originally proposed by Haider et al. (1998), the concept of SSF minimization using these techniques is new.

5.3 Retrospective transmit focusing method

After removing the excitation wave and suppressing additive noise from the ultrasound signal using the deconvolution approaches described in Section 5.2, the best estimate of the remaining terms in our ultrasound imaging system is,

\[
d(t) = \sum_r h_{tx}(r : t) * h_{rx}(r : t) \cdot bsc(r)
\] (5.5)

where the superscript ^ has been dropped to simplify the notation.

The SSF will be minimized if the temporal convolutions and/or the individual SSFs of the remaining terms in (5.5) can be eliminated or reduced. This can potentially be achieved by identifying the terms that are common to all scatterers and implementing mathematical techniques to shift these terms to outside the scatterer summation. It may then be possible to either modify or completely remove the term from the equation, thereby improving the SSF.

Consider the following form of (5.5),

\[
d(t) = \sum_r h(r : t) * B(r : t),
\] (5.6)

where \( h(r : t) = h_{tx}(r : t) \) and \( B(r : t) = h_{rx}(r : t) \cdot bsc(r) \). The impulse response terms can represent the radiated field in either a lossless or attenuating medium, the latter described by power-law frequency dependent attenuation or classical viscous loss models.

By taking the temporal Fourier transform of (5.6),

\[
D(\omega) = \sum_r H(r : \omega) \cdot B(r : \omega) = \sum_{r_1} \sum_{z_1} H(r_1, z_1 : \omega) \cdot B(r_1, z_1 : \omega).
\] (5.7)

This summation has been expressed in 2D cylindrical coordinates over radial position \( r_1 \) and axial depth \( z_1 \), however the insonated medium should be considered a 3D construct. We have therefore assumed that the cylindrical angle \( \theta \) is irrelevant as \( B(r_1, z_1 : \omega) \) will contain the sum of all scattered signals located at a distance \( (r_1, z_1) \) from the transducer for \( 0 \leq \theta < 2\pi \),
Since our goal is to recover the signal arising from on-axis scatterers, the angle is irrelevant since there will only be one scatterer at \((r_1 = 0, z_1)\). The signal arising from all off-axis scatterers \((r_1 \neq 0, z_1)\) can be accounted for by a single lumped term.

It was shown in Chapter 3 that the velocity potential on an observation plane is the Hankel convolution of the normal component of the transducer surface velocity with the Green’s function defined on the observation plane. Assuming a lossless medium,

\[
\Phi(r, z_1 : \omega) = v_z(r, 0 : \omega) \# \left\{ \frac{1}{2\pi \sqrt{r^2 + z_1^2}} e^{-jk \sqrt{r^2 + z_1^2}} \right\}.
\] (5.9)

Since \(\Phi(r_1, z_1, \omega) = H(r_1, z_1, \omega)\) in (5.7),

\[
D(\omega) = \sum_{r_1} \sum_{z_1} v_z(r, 0 : \omega) \# \left\{ \frac{1}{2\pi \sqrt{r^2 + z_1^2}} e^{-jk \sqrt{r^2 + z_1^2}} \right\} \cdot B(r_1, z_1 : \omega)
\]

\[
= \sum_{r_1} \sum_{z_1} \left\{ v_z(r, 0 : \omega) \# G(r, z : \omega) \right\} \cdot B(r_1, z_1, \omega),
\] (5.10)

where \(G(r, z : \omega)\) is the Green’s function.

The Hankel convolution \# is calculated over radial position \(r\) and is equivalent to the 2D linear convolution over \((x, y)\), where \(r = \sqrt{x^2 + y^2}\). This can be mathematically expressed as,

\[
\left\{ v_z(r, 0 : \omega) \# G(r, z : \omega) \right\}_{r_1, z_1} = \sum_{x} \sum_{y} v_z(r, 0 : \omega) \cdot G(\sqrt{(r_1 - x)^2 + y^2}, z_1, \omega)
\] (5.11)

A graphical representation of (5.11) is described in Figure 5.2.
In Figure 5.2, \( v_z(r,0: \omega) \) is a 2D circularly symmetric function centred about the origin, representing the normal component of the transducer surface velocity. The Green’s function \( G\left( \sqrt{(r_1 - x)^2 + y^2}, z_1, \omega \right) \) has been spatially shifted by \( r_1 \) due to the Hankel convolution. The centre of the circularly symmetric Green’s function is indicated by the star. The spatial shift was arbitrarily selected along the \( x \)-axis, although since the functions are circularly symmetric, it could have been on any 2D axis on the \( z_1 \) plane. The Hankel convolution is therefore equivalent to the element-wise multiplication of the 2D \( v_z(r,0: \omega) \) distribution with \( G\left( \sqrt{(r_1 - x)^2 + y^2}, z_1, \omega \right) \), which is then summed over all 2D spatial positions. While the Green’s function is defined over the entire plane at \( z = z_1 \), it is effectively limited by the dimensions of \( v_z(r,0: \omega) \) due to the element-wise multiplication in the convolution calculation. The scalar result of the convolution is then multiplied by the scalar \( B(r_1, z_1 : \omega) \) to determine the received ultrasound signal arising from a scatterer at \( (r_1, z_1) \).

The received ultrasound signal in (5.10) is determined by evaluating the convolution for each scatterer and summing the result over the entire radiated field. However, since \( v_z(r,0: \omega) \) is the same for all scatterer positions, perhaps we can rearrange the equation to move this term outside of the summation. The term can then be potentially removed or modified—a
retrospective approach since this operation acts on the received ultrasound signal. Using (5.11), equation (5.10) can be rearranged as follows,

\[
D(\omega) = \sum_{n} \sum_{z} \left\{ v_z(r,0:\omega) \# G(r,z:\omega) \right\}_{n,z} \cdot B(r_1,z_1:\omega)
\]

\[
= \sum_{n} \sum_{z} \left\{ \sum_{x} \sum_{y} v_z(r,0:\omega) \cdot G\left(\sqrt{(r_1-x)^2 + y^2}, z_1, \omega \right) \right\}_{n,z} \cdot B(r_1,z_1:\omega)
\]

\[
= \sum_{n} \sum_{z} \sum_{x} \sum_{y} v_z(r,0:\omega) \cdot \left\{ G(r,z_1:\omega) \# \left[ B(r_1,z_1:\omega) \delta\left(\sqrt{(r_1-x)^2 + y^2}, z-z_1, \omega \right) \right] \right\}
\]

\[
= \sum_{x} \sum_{y} v_z(r,0:\omega) \sum_{n} \sum_{z} G(r,z_1:\omega) \# \left[ B(r_1,z_1:\omega) \delta\left(\sqrt{(r_1-x)^2 + y^2}, z-z_1, \omega \right) \right]
\]

(5.12)

The implications of (5.12) are significant. Firstly, the normal component of the transducer surface velocity \( v_z(r,0:\omega) \) can in fact be moved outside the scatterer summation, and secondly, the resultant solution,

\[
D(\omega) = \sum_{x} \sum_{y} v_z(r,0:\omega) \sum_{n} \sum_{z} G(r,z_1:\omega) \# \left[ B(r_1,z_1:\omega) \delta\left(\sqrt{(r_1-x)^2 + y^2}, z-z_1, \omega \right) \right]
\]

(5.13)

can be seen as a dot product equation between two vectors: \( v_z(r,0:\omega) \) and \( \sum_{n} \sum_{z} G(r,z_1:\omega) \# \left[ B(r_1,z_1:\omega) \delta\left(\sqrt{(r_1-x)^2 + y^2}, z-z_1, \omega \right) \right] \). A graphical representation of the latter Hankel convolution in (5.13) is shown in Figure 5.3.
Figure 5.3: Graphical representation of \( G(r, z_1, \omega) \# B(r_1, z_1 : \omega) \delta \left( \sqrt{(r_1 - x)^2 + y^2}, z - z_1, \omega \right) \) from (5.13).

Figure 5.3 describes the 2D planes of \( G(r_1, z_1, \omega) \), \( B(r_1, z_1 : \omega) \delta \left( \sqrt{(r_1 - x)^2 + y^2}, z - z_1, \omega \right) \), and the resultant convolution. The Green’s function \( G(r_1, z_1, \omega) \) is centred about the origin as indicated by the star, which is then spatially convolved with a delta impulse function located at the scatterer position \( (r_1, z_1) \) and with amplitude \( B(r_1, z_1 : \omega) \). Because we have ignored the effect of scatterers over cylindrical angle, \( \theta \), a single scatterer located at \( (r_1, z_1) \) is used to represent all scatterers at a distance \( (r_1, z_1) \) over \( 0 \leq \theta < 2\pi \) from the origin, as described in (5.8). As a result, the delta impulse function has been arbitrarily shown as shifted along the \( x \)-axis. We could have shifted the delta impulse function with respect to the \( y \)-axis or any other 2D axis on the \( z_1 \) plane and produced the same result. After spatially convolving these two functions, the resultant distribution effectively shifts the Green’s function by \( r_1 \) and multiplies the entire Green’s function plane by the scalar, \( B(r_1, z_1 : \omega) \), as represented by the grey coloured sections in Figure 5.3.

Figure 5.4 shows the complete results generated by (5.13). By summing all the resultant \( G(r_1, z_1, \omega) \# B(r_1, z_1 : \omega) \delta \left( \sqrt{(r_1 - x)^2 + y^2}, z - z_1, \omega \right) \) planes arising from each scatterer in the radiated field, the second plane in Figure 5.4 is obtained. Element-wise multiplication of this
second plane with the normal component of the transducer surface velocity \( v_z(r, 0: \omega) \), followed by summation over \( x \) and \( y \), generates the resultant \( D(\omega) \) scalar.

\[
\sum_x \sum_y v_z(r, 0: \omega) \cdot \sum_{r_1} \sum_{z_1} G(r_1, z_1, \omega) \cdot B(r_1, z_1, \omega) \delta\left( \sqrt{(r_1 - x)^2 + y^2}, z - z_1, \omega \right)
\]

Figure 5.4: Final graphical representation of (5.13). The summation of all scatterer planes followed by element wise multiplication and summation of the transducer surface velocity generates the scalar received signal, \( D(\omega) \).

Figure 5.4 is mathematically equivalent to the dot product operation. The 2D functions are element-wise multiplied together and summed over \( x \) and \( y \). By representing these two 2D functions as 1D vectors, \( V_{vec} \) and \( M_{vec} \), the following equation is obtained,

\[
D(\omega) = \sum_x \sum_y v_z(r, 0: \omega) \cdot \sum_{r_1} \sum_{z_1} G(r, z_1, \omega) \cdot B(r_1, z_1, \omega) \delta(x - r_1, z) = V_{vec}(r: \omega) \cdot M_{vec}(r: \omega) \tag{5.14}
\]

The best estimate of the received ultrasound signal \( D(\omega) \) is therefore the dot product between a known vector \( V_{vec} \) and an unknown vector \( M_{vec} \). However, the dot product operation is also equivalent to,

\[
D(\omega) = \langle V_{vec}(r: \omega), M_{vec}(r: \omega) \rangle = |V_{vec}(r: \omega)| \cdot |M_{vec}(r: \omega)| \cos \alpha_{VM} \tag{5.15}
\]
where $|.|$ is the vector magnitude and $\alpha_{VM}$ is the angle between the two vectors. This suggests that by multiplying $D(\omega)$ by a scalar constant (at a particular angular frequency), the normal component of the transducer surface velocity can be retrospectively altered to potentially anything, including a delta impulse function or retrospective dynamic transmit focusing. This can be achieved by introducing a new vector, $F_{vec}$ to replace $V_{vec}$, where $F_{vec}$ is a 1D vector with a different amplitude and phase representing a different transducer surface velocity distribution, e.g., different geometry, apodization, or focusing depth.

\[
D(\omega) = |V_{vec}(r:\omega)| |M_{vec}(r:\omega)| \cos \alpha_{VM} \\
D_{new}(\omega) = P(\omega)D(\omega) \\
= P(\omega)|V_{vec}(r:\omega)| |M_{vec}(r:\omega)| \cos \alpha_{VM}, \quad (5.16) \\
P(\omega) = \frac{|F_{vec}(r:\omega)| \cos \zeta_{FM}}{|V_{vec}(r:\omega)| \cos \alpha_{VM}}
\]

For a particular angular frequency, $P(\omega)$ is a scalar constant. The angle between the new transducer vector $F_{vec}$ and $M_{vec}$ is $\zeta_{FM}$, while the newly calculated ultrasound backscattered signal is indicated by $D_{new}(\omega)$. Now, only $D(\omega)$, $|F_{vec}(r:\omega)|$, $|V_{vec}(r:\omega)|$, and $V_{vec}(r:\omega)$ are known, whereas $|M_{vec}(r:\omega)|$, $\alpha_{VM}$, and $\zeta_{FM}$ are unknown. We need to determine $D_{new}(\omega)$ using the new transducer surface velocity vector $F_{vec}$. Equation (5.16) is therefore a single equation in three unknowns. In addition, since $V_{vec}$, $M_{vec}$, and $F_{vec}$ are complex-valued vectors, the existence of the dot product relations for complex vector spaces must be verified since standard dot product definitions assume real-valued vectors.

### 5.3.1 Complex dot product

The 1D vectors described in (5.15) should be assumed complex-valued as the normal component of the transducer surface velocity is complex when the transducer is focused (3.9), and $M_{vec}$ consists of the complex-valued Green’s function (5.10). The identification of dot product relations for complex-valued vectors are therefore necessary, as they may be different than the commonly known dot product definitions that assume real-valued vectors.
Galántai and Hegedűs (2006) and Scharnhorst (2001) identified that the dot product is in fact defined for complex-valued vector space, and provided a definition for the complex-valued angle $\theta_C$ between two complex-valued vectors, $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}\rangle \langle \mathbf{b}| \cos \theta_C$$

(5.17)

The dot product between complex-valued vectors is defined as,

$$\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{N} a^*_k b_k,$$

(5.18)

where $\mathbf{a} = [a_1 \ a_2 \ \ldots \ a_N]$ and $\mathbf{b} = [b_1 \ b_2 \ \ldots \ b_N]$, and $^*$ is the complex conjugate. The complex-valued angle $\theta_C$ is,

$$\cos \theta_C = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \rho_H e^{i\phi_{PA}}$$

(5.19)

where $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The magnitude of complex-valued $\cos \theta_C$ provides the real-valued Hermitian angle $\theta_H$,

$$\rho_H = \cos \theta_H = |\cos \theta_C|,$$

(5.20)

where $\rho_H \leq 1$, $0 \leq \theta_H \leq \frac{\pi}{2}$. The phase of (5.19) is defined as the real-valued Kasner’s pseudo-angle $\phi_{PA}$, where $-\pi \leq \phi_{PA} \leq \pi$.

The angle between two lines in the complex vector space describes the Hermitian angle, while the pseudo-angle describes the orientation or rotation of the Hermitian angle. Multiplying vector $\mathbf{a}$ or $\mathbf{b}$ by a complex constant significantly affects the pseudo-angle but does not affect the Hermitian angle.

Since the complex dot product (5.18) is defined as the multiplication of the complex conjugate of one vector with a second vector, the definition of equations (5.14) and (5.15) should be adjusted to account for this complex conjugate. Equation (5.14) should therefore be redefined as,
\[
D(\omega) = \sum_{x} \sum_{y} v_z (r, 0: \omega) \cdot \sum_{x} \sum_{y} G(r, z_1, \omega) \cdot \delta(x - r_1, z)
\]

\[
= V_{vec} * (r: \omega) \cdot M_{vec} (r: \omega)
\]

\[
= |V_{vec} (r: \omega)| |M_{vec} (r: \omega)| \cos \alpha_{VM}
\]

where \(V_{vec} * (r: \omega) = [v_z * (r_0, 0: \omega) \ v_z * (r_1, 0: \omega) \ ... \ v_z * (r_N, 0: \omega)]\). This will ensure that using the complex dot product relations defined in (5.18), equation (5.14) is accurately represented.

The angle \(\gamma_{VF} \) between \(V_{vec} \) and \(F_{vec} \) can be determined since both vectors are known and \(V_{vec} (r, \omega) \cdot F_{vec} (r, \omega) = |V_{vec} (r, \omega)| |F_{vec} (r, \omega)| \cos \gamma_{VF} \). However, knowledge of \(\gamma_{VF} \) does not directly determine \(P(\omega) \). The angle, \(\gamma_{VF} \), can however narrow the possible range of angles of \(\zeta_{FM} \). The maximum and minimum angle \(\zeta_{FM} \) are observed when vectors \(M_{vec}, F_{vec}, \) and \(V_{vec} \) are in the same plane. This corresponds to \(\zeta_{FM} = \alpha_{VM} \pm \gamma_{VF} \). Therefore \(\frac{\cos \zeta_{FM}}{\cos \alpha_{VM}} \) from \(P(\omega)\) lies between,

\[
\frac{\cos \zeta_{FM}}{\cos \alpha_{VM}} = \left[ \frac{\cos(\alpha_{VM} - \gamma_{VF})}{\cos \alpha_{VM}}, \frac{\cos(\alpha_{VM} + \gamma_{VF})}{\cos \alpha_{VM}} \right]
\]

\[
= \left[ \frac{\cos \alpha_{VM} \cos \gamma_{VF} + \sin \alpha_{VM} \sin \gamma_{VF}}{\cos \alpha_{VM}}, \frac{\cos \alpha_{VM} \cos \gamma_{VF} - \sin \alpha_{VM} \sin \gamma_{VF}}{\cos \alpha_{VM}} \right]
\]

\[
= \left[ \cos \gamma_{VF} + \sin \gamma_{VF} \tan \alpha_{VM}, \cos \gamma_{VF} - \sin \gamma_{VF} \tan \alpha_{VM} \right]
\]

A graphical representation of (5.22) is shown in Figure 5.5. In the sideview (Figure 5.5a), all three vectors, \(F_{vec}, M_{vec}, \) and \(V_{vec}\) are shown, however they are not necessarily coplanar. The angles between \(M_{vec} \) and \(F_{vec}, \) as well as \(M_{vec} \) and \(V_{vec} \) are \(\zeta_{FM} \) and \(\alpha_{VM} \), respectively. The angle \(\gamma_{VF} \) between transducer vectors, \(F_{vec} \) and \(V_{vec} \) is also specified. Our goal is to determine either angles \(\zeta_{FM} \) and \(\alpha_{VM} \) or the ratio of the cosines of these two angles. In a frontview representation (Figure 5.5b), the viable range of \(\zeta_{FM} \) for a single \(\gamma_{VF} \) angle is shown. Despite knowing angle \(\gamma_{VF} \), which will cause the \(V_{vec} \) vector to lie anywhere on the dotted circle around \(F_{vec}, \) this will correspond to a range of possible \(\zeta_{FM} \) angles, whose minimum and maximum are determined by (5.22).
A significant challenge with this approach is the difficulty in identifying the unknown variables. Numerous analytical approaches were considered, however, they proved unsuccessful in determining an analytical solution to the derivation since there are more unknowns than equations. We will therefore propose a solution based on trends in the simulation results. This dot product technique, developed through examination of the SSF, is a new contribution and provides a framework for future research.

5.4 Assumptions and simplifications

The primary assumptions and simplifications used in the signal processing algorithms include,

- The excitation wave can be either a pulsed wave or continuous wave.
- The statistics of the noise are known and assumed to be wide sense stationary.
- Additive noise is statistically independent of the desired signal after deconvolution.
- The electromechanical impulse response of the transducer is known.
- The backscattering cross-section is time-independent and the scatterers are immobile.
- The system can be considered linear and the nonlinear effects are negligible.
• The Born approximation has been assumed (Cobbold 2007, p.287), where scattering is sufficiently weak so the effects of multiple scattering can be ignored.
• To allow for dynamic receive focusing, the apodized and focused, plane piston transducer is formed using a set of infinitely thin concentric rings.
A summary of these items are also listed in Appendix A.

5.5 Results

The primary transducer configuration used for the following signal processing algorithm simulations was the apodized and focused piston transducer described in Chapter 3. We have assumed an ideal, lossless medium in these simulations in order to simplify the analysis, however the effects of attenuation and dispersion using either of the two models described in Chapter 4 can be easily applied to both the derivation and simulations. In addition, a variable number of scatterers positioned both on- and off-axis have been simulated depending on the test case, in order to demonstrate both the simplified and complete effect of the simulated techniques. By examining the effect of these methods using a small number of on-axis scatterers, we can attain a concrete understanding of the performance and drawbacks of different techniques. The techniques can then be repeated using a more realistic scatterer distribution throughout the entire radiated field in order to assess the practical efficacy of the method.

Three concepts are analysed in this section: the effect of the transducer impulse response and ultrasound received signal on the SSF; the efficacy of the LMMSE filtering method for excitation wave removal and additive noise suppression in comparison to the traditional matched filtering and envelope detection technique; and the retrospective transmit focusing method. Each case uses a slightly different scatterer distribution to allow for an effective analysis of each technique.

5.5.1 Ultrasound simulation system

As previously described using Figure 5.1, the effect of the temporal convolutions in the linear ultrasound model (1.1) broadens both the PSF and SSF from that found using the single transmit transducer impulse response. The effect of these convolutions on the SSF is examined
in this section. Using the simulation parameters described in Table 5.1 and the 5-cycle, tapered cosine, transducer excitation pulse shown in Figure 5.6, several SSF responses were examined. All SSF simulations in this section were obtained using Field II simulations to minimize simulation runtimes. However, the results were compared to the derivations of Chapter 3 to ensure the results were consistent.

Table 5.1: Apodized and focused plane piston transducer simulation parameters

<table>
<thead>
<tr>
<th>Configuration Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excitation wave centre frequency</td>
<td>$f$</td>
<td>5 MHz</td>
</tr>
<tr>
<td>Wave speed</td>
<td>$c_0$</td>
<td>1500 m/s</td>
</tr>
<tr>
<td>Transducer radius</td>
<td>$a$</td>
<td>3 cm</td>
</tr>
<tr>
<td>Apodization standard deviation</td>
<td>$\sigma$</td>
<td>$0.25a = 0.75$ cm</td>
</tr>
<tr>
<td>Focusing depth (transmit and receive)</td>
<td>$F$</td>
<td>4.5 cm</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>$f_s$</td>
<td>200 MHz</td>
</tr>
<tr>
<td>Number of axial field points</td>
<td>$N_z$</td>
<td>600</td>
</tr>
<tr>
<td>Number of radial field points</td>
<td>$N_r$</td>
<td>50</td>
</tr>
</tbody>
</table>
Three configurations were studied to demonstrate how the temporal convolutions from (1.1) affect the SSF: that of the transmit transducer impulse response, its convolution with the received transducer impulse response, and the final convolution with the excitation wave. Each SSF configuration builds on the previous through the incorporation of an additional temporal convolution.

Figure 5.7 shows the SSF arising from the transmit transducer impulse response, $h_{tx}(r,t)$ for three time points surrounding wave propagation through the 4.5 cm focal point using the apodized and focused, plane piston transducer from Case 4, Chapter 3. As expected, the results are consistent with that of Figure 3.9d. However, in this case, each individual SSF was normalized and superimposed on the same plot in cylindrical coordinates. The transducer is
located at \( z = 0 \) cm and only the axial portion between 3.8 cm and 5.2 cm is shown up to \( r = 0.3 \) cm.

The second SSF configuration describing the convolution of the transmit and receive transducer impulse responses, \( h_t(\mathbf{r} : t) * h_r(\mathbf{r} : t) \), is shown in Figure 5.8. The configuration of both transducers was identical. Again, each individual SSF was normalized and superimposed on the same plot in cylindrical coordinates. The SSF results are significantly broader than that of Figure 5.7 due to the temporal convolution.

The final SSF configuration, involving the convolution of the transducer impulse responses with the excitation waveform of Figure 5.6, \( h_t(\mathbf{r} : t) * h_r(\mathbf{r} : t) * \frac{dv_0(t)}{dt} \), is shown in Figure 5.9. Again, the three individual SSF plots were normalized and superimposed in the same plot. Results were shown in grayscale to more clearly delineate the local minima and maxima arising from the excitation wave oscillations. It has been assumed that the transducer electromechanical response is a delta impulse in these simulations. It is apparent that the excitation wave has a significant broadening effect on the SSF. A worthwhile goal, therefore, is to develop signal processing techniques to reduce the spatial extent of the SSFs, as this will have a direct effect on the spatial resolution and contrast of the resultant A-line. One strategy used in this research is the development of techniques to remove or reduce the effects of the temporal convolutions. The first approach uses Wiener deconvolution or LMMSE filtering to remove the excitation wave convolution and with additive noise suppressed from the SSF in Figure 5.9 in order to obtain that described in Figure 5.8. The second method uses the results of the excitation wave deconvolution to retrospectively alter the transmit transducer focal depth, thereby minimizing the PSF at any on-axis location.
Figure 5.7: SSF [dB] of transmit transducer impulse response for three times: 32.7 µs, 36.0 µs, and 39.3 µs, surrounding the 4.5 cm focal point. Results are shown in cylindrical coordinates over the axial z and radial r axes.
Figure 5.8: SSF [dB] of total transducer impulse response (transmit and receive) for three times: 65.5 µs, 72.1 µs, and 78.7 µs, surrounding wave propagation through the 4.5 cm focal point.
Figure 5.9: SSF of total transducer transmit/receive impulse response with excitation wave for three times: 65.5 µs, 72.1 µs, and 78.7 µs, corresponding to wave propagation through the focal point. Results are normalized.

5.5.2 Excitation voltage and noise suppression

To examine the effects of the Wiener deconvolution and LMMSE filtering methods at removing the excitation wave and suppressing additive noise, a complete ultrasound system was simulated. We initially limited the system to 150 on-axis scatterers evenly distributed between $[z_{\text{min}}, z_{\text{max}}] = [4.35\, \text{cm}, 4.69\, \text{cm}]$ in order to validate the method, then repeated the simulation using both on- and off-axis scatterers representing a more realistic scatterer distribution to determine the SSF response. The backscattering cross-section was assumed to be a uniformly
distributed random variable between 0 and 1, $U(0,1)$, and additive white Gaussian noise was added using an SNR of 20 dB. These parameters are described in Table 5.2 and Table 5.1. The choice of backscattering cross-section probability distribution and range were selected to simplify the system, since normalized results are shown and the choice of backscattering cross-section is arbitrary in our signal processing methods. While the statistics describing the desired received ultrasound signal, $d(t)$, from (5.1) are required, the algorithms are not dependent on the choice of backscattering cross-section probability distribution. Additive white Gaussian noise was also added to simplify the simulation system, despite ultrasound imaging systems being band-limited by the transducer response, which may be limited to a fractional bandwidth of 100% the carrier frequency. In addition, while the 5 MHz wavelength was 0.03 cm, the distance between adjacent on-axis scatterers was 0.0023 cm.

Table 5.2: LMMSE filtering simulation parameters

<table>
<thead>
<tr>
<th>Configuration Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of axial field points</td>
<td>$N_z$</td>
<td>150</td>
</tr>
<tr>
<td>Minimum axial position</td>
<td>$z_{\text{min}}$</td>
<td>4.35 cm</td>
</tr>
<tr>
<td>Maximum axial position</td>
<td>$z_{\text{max}}$</td>
<td>4.69 cm</td>
</tr>
<tr>
<td>Number of radial field points</td>
<td>$N_r$</td>
<td>1</td>
</tr>
<tr>
<td>Radial position</td>
<td>$r_{\text{min}}=r_{\text{max}}$</td>
<td>0 cm</td>
</tr>
<tr>
<td>Backscattering cross-section</td>
<td>$b_{\text{sc}}(r)$</td>
<td>$U(0,1)$</td>
</tr>
<tr>
<td>Signal to noise ratio</td>
<td>$\text{SNR}$</td>
<td>20 dB</td>
</tr>
</tbody>
</table>

The normalized received ultrasound signal in the absence of noise is shown in Figure 5.10 for two cases: (a) with and (b) without the excitation wave. Our goal is to remove the effects of the excitation wave and recover the signal described in Figure 5.10b. Results are also compared with the traditional matched filtering method, which convolves the received signal with the time-reversed excitation wave signal, followed by envelope detection (Cobbold 2007, p.494).
Figure 5.10: Normalized received ultrasound signal (a) with excitation wave; (b) without excitation wave.

The received ultrasound signal, prior to filtering, and including the effects of additive white Gaussian noise (AWGN) at an SNR of 20 dB is shown in Figure 5.11. The response with and without additive noise is shown, however, the results are nearly coincident.
Figure 5.11: Received ultrasound signal including the effects of additive white Gaussian noise (AWGN) at an SNR of 20 dB. (a) Response with the excitation wave, (b) Response without the excitation wave. The legend applies to both plots.

Since the Wiener deconvolution and LMMSE methods are equivalent as described in Section 5.2, only the LMMSE results are shown. This was also verified in simulation. Assuming that the statistical characteristics of the ultrasound signal and additive noise are
wide-sense stationary, an estimate of the received ultrasound signal in the absence of the excitation wave and with additive noise suppressed can be obtained. When the statistics of the ultrasound signal in the absence of the excitation wave and additive noise are not known, Haider et al. (1998) proposed assuming a white Gaussian power spectral density distribution when the backscattering cross-section distribution was also a white Gaussian random process. However, a thorough examination of the effects of estimating the statistical properties on the filtered results is recommended for future research.

Simulation results comparing the LMMSE and matched filtering techniques are shown in Figure 5.12 in addition to the ideal ultrasound transducer response without either the excitation wave or additive noise (see Figure 5.10b). Note that while the ability of the LMMSE method to recover the desired signal is excellent, as the results are nearly equivalent to the ideal response, the matched filtering method is not capable of recovering the more detailed response due to the excitation pulse length. As described earlier, the maximum spatial resolution in the matched filtering case is dependent on the excitation pulse width, however this is not the case for the LMMSE method. Alternative approaches to improve the spatial resolution of the matched filter method include using either a shorter duration excitation pulse or a longer duration frequency modulated pulse (Cobbold 2007, p.519-525).
Figure 5.12: Simulation results for the LMMSE and matched filtering techniques. While the signal recovery capabilities of the LMMSE method are excellent as the results are nearly coincident with the ideal ultrasound response without the excitation wave and noise, the matched filter results cannot recover such high resolution.

However, there are drawbacks and assumptions required of the Wiener deconvolution and LMMSE approaches that are not required by the matched filtering method. The primary limitation is that the statistics of the signals are known and wide-sense stationary. This requires a constant mean and that the auto-correlation function does not change with an alteration of the time origin (Hayes 1996). Both the excitation wave and additive noise statistics arising from the imaging electronics should be known a priori and satisfy the wide-sense stationary requirement. However, properties of the tissue medium may not be completely known and may change over time within the received ultrasound signal A-line. Nevertheless, these can perhaps be accommodated by using a priori knowledge of similar tissue media and using temporal window segmentation of the received signal.

It is also to be noted that at low SNR levels, the Wiener deconvolution and LMMSE methods more closely resemble the results of the matched filtering method. However, at the
SNR level chosen for these simulations, these filtering methods are capable of recovering the desired signal with significant accuracy.

5.5.2.1 SSF

The LMMSE filtering method was also studied in relation to the SSF response. In this case, the received transducer signal arising from a matrix of scatterers, 600 in the axial direction and 50 in the radial direction were simulated in Field II. Again, the 5 cycle, tapered cosine excitation wave, a backscattering cross-section assuming a uniformly distributed random variable, and additive white Gaussian noise with an SNR of 20 dB was used.

To obtain the SSF results, the single received ultrasound signal was simulated in Field II for the case with the excitation wave and additive noise, and the ideal response without these two factors. The LMMSE filtering method was then applied to obtain a single set of filter coefficients. The Field II simulations were then repeated, except the received response arising from each individual scatterer was kept separate. The LMMSE filter previously obtained was then applied on each scattered signal independently to obtain a set of filtered estimates in the absence of the excitation wave and with additive noise suppressed. The matched filtering and envelope detection method was also directly applied to the received transducer response from each individual scatterer to obtain the SSF response. The SSF results were then analyzed, normalized, and plotted at three separate time points, corresponding to the wave propagating near the focal point.

Figure 5.13 shows the original SSF results prior to LMMSE filtering. The alternating black and white bands correspond to the local minima and maxima of the excitation wave. The speckle pattern within each SSF corresponds to the backscattering cross-section for each scatterer, modeled here using a uniformly distributed random variable between 0 and 1.

The SSF results after LMMSE or matched filtering and envelope detection are shown in Figure 5.14 and Figure 5.15, respectively. Each of the individual SSF plots were normalized before superimposing on the plots, where applicable. While the LMMSE method was able to recover a SSF response comparable to the ideal response shown in Figure 5.8 (while considering the additional speckle arising from the backscattering cross-section distribution in Figure 5.14), there is a minor amount of noise added to the resultant signal. For this reason, these three SSF plots were shown separately. The matched filtering and envelope detection method SSF results
shown in Figure 5.15 demonstrate the reduced spatial resolution of the method as the axial and lateral spans of the responses are much larger than that of the LMMSE method. This is consistent with the results shown in Figure 5.12.

Figure 5.13: SSF plots for three time points around the 4.5 cm focal point. The oscillating black and white bars correspond to the local minima and maxima of the excitation wave, while the speckle pattern represents the variation in backscattering cross-section over the insonated medium. Results are normalized.
Figure 5.14: SSF results [dB] for three time points around the 4.5 cm focal point after filtering the excitation wave and additive noise using the LMMSE method. (a) SSF at 65.5 µs, (b) SSF at 72.1 µs, (c) SSF at 78.7 µs.
Figure 5.15: SSF results [dB] for three time points around the 4.5 cm focal point after matched filtering and envelope detection.
5.5.3 Validation of complex dot product mathematics

The complex dot product relations outlined in (5.16) and (5.17) were validated using simple numerical simulations using random real- and complex-valued vectors. Two numerical simulation tests were performed in Matlab: one using real vectors of 40 elements and a second using complex-valued vectors of 40 elements. In each case, three random-valued vectors, \( a, b, c \) were created. The objective was to prove that given \( a \cdot b = |a||b|\cos \theta_{a,b} \) and \( a \cdot c = |a||c|\cos \theta_{a,c} \), we can find the value of scalar \( P \) such that \( a \cdot c = Pa \cdot b \) where

\[
P = \frac{|c|\cos \theta_{a,c}}{|b|\cos \theta_{a,b}}.
\]

The relations were proven in simulation for real-valued vectors of any length. There are no figures necessary since we numerically validated that \( a \cdot c = Pa \cdot b \).

If \( a, b, c \) are complex-valued vectors, the Hermitian angle, \( \theta_H \), can be determined by taking the arc cos of the normalized magnitude of \( a \cdot c \) or \( a \cdot b \). Mathematically, this is expressed as

\[
\cos \theta_{a,b} = \rho_H e^{j\rho_H},
\]

where \( \rho_H = \cos \theta_H = |\cos \theta_{a,b}| \), \( \rho_H \leq 1 \), \( 0 \leq \theta_H \leq \frac{\pi}{2} \). It is also important to note that \( \theta_H \neq |\theta_{a,b}| \). In simulation, modifications to the dot product using the Hermitian angle and pseudo-angles are valid for arbitrary length, complex-valued vectors.

5.5.4 Retrospective transmit focusing method

The retrospective transmit focusing technique was initially validated assuming all relevant parameters are known a priori, then a solution for identifying \( P(\omega) \) was found by examining simulation trends. The solution was then applied to more realistic imaging conditions where parameters describing the tissue medium from (5.16) are not known.

The apodized and focused, plane piston transducer was assumed for all retrospective transmit focusing simulations as described in Table 5.1. In addition, the mathematical derivation described in Chapter 3 was used for proof of concept simulations, in order to accurately verify the approach. Additional simulation parameters specific to the retrospective focusing method are listed in Table 5.3.
Table 5.3: Retrospective focusing simulation parameters

<table>
<thead>
<tr>
<th>Configuration Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of axial field points</td>
<td>Nz</td>
<td>5</td>
</tr>
<tr>
<td>Minimum axial position</td>
<td>z(_{\text{min}})</td>
<td>4.0 cm</td>
</tr>
<tr>
<td>Maximum axial position</td>
<td>z(_{\text{max}})</td>
<td>5.0 cm</td>
</tr>
<tr>
<td>Number of radial field points</td>
<td>Nr</td>
<td>1</td>
</tr>
<tr>
<td>Radial position</td>
<td>(r_{\text{min}}=r_{\text{max}})</td>
<td>0 cm</td>
</tr>
</tbody>
</table>

The equation derivation of (5.16) was verified in simulation using five on-axis scatterers, equally spaced between 4 cm and 5 cm in the absence of both the excitation wave and additive noise, as we have assumed these terms have been accurately removed/suppressed using the Wiener deconvolution or LMMSE filtering methods. These simulations used the same uniform random variable describing the backscattering cross-section as the previous simulations. The received transducer response assuming both the transmit and receive transducers are focused at 3.5 cm is shown in Figure 5.16a. Using the retrospective transmit focusing technique, with all vectors known in (5.16), \(P(\omega)\) was determined in order to adjust the transmit focal depth to 5.5 cm.

In these simulations, it is important to consider the available data from an ultrasound imaging system. Assuming a single transmit focal depth of \(F_{\text{tx}}\), the receive transducer can dynamically change its focal depth \(F_{\text{rx}}\) to any value. We have therefore assumed that multiple received signals can be obtained from a single transmit pulse into the medium. Dynamic receive focusing has therefore been simulated assuming the apodized and focused piston transducer is formed using infinitely thin concentric rings. In particular, this simulation determined the response from three received signals in an identical medium: one using a transmit focal depth of \(F_{\text{tx}}=3.5\) cm and a receive focal depth of \(F_{\text{rx}}=3.5\) cm, a second using \(F_{\text{tx}}=3.5\) cm and \(F_{\text{rx}}=5.5\) cm, and a final configuration of \(F_{\text{tx}}=5.5\) cm and \(F_{\text{rx}}=5.5\) cm. Using data obtained from each configuration, we identified \(P(\omega)\) in order to retrospectively focus the second signal from \(F_{\text{tx}}=3.5\) cm to \(F_{\text{tx}}=5.5\) cm. The results shown in Figure 5.16b are consistent with those obtained using Field II assuming \(F_{\text{tx}}=F_{\text{rx}}=3.5\) cm and \(F_{\text{tx}}=F_{\text{rx}}=5.5\) cm, thereby providing evidence for the validity of the approach. Minor oscillations in a portion of the signal are attributed to the Gibbs effect since these results were obtained using the temporal frequency domain derivation from Chapter 3.
Figure 5.16: Received transducer response without excitation wave or additive noise. (a) Identical transmit and receive transducer configurations focused at 3.5 cm, (b) Results after retrospective transmit focusing $F_{tx}$ to 5.5 cm. All relevant vector information was assumed known *a priori*. 
5.5.4.1 Retrospective transmit focusing – Solution

The simulation system shown in Figure 5.16 was reperformed while anticipating that only the transducer surface velocity vector would be known in an ultrasound imaging system. Properties of the complex $P(\omega)$ parameter were then examined over temporal frequency and focal depth in order to identify patterns that could be estimated using limited ultrasound imaging data. The normalized magnitude of the temporal Fourier transform of the previously described 5 on-axis scatterer system using 40 focal depths between $F_{tx}=F_{rx}=3.5$ cm and $F_{tx}=F_{rx}=5.5$ cm are shown in Figure 5.17a in dB. The curves generally move upwards with increasing focal depth. Only normalized results up to 3.3 MHz are shown in dB. Like the simulations described in the previous section, we have assumed a dynamic receive focusing model, where the retrospective method is applied to change the transmit focal depth, i.e., multiple receive signals are obtained from a single ultrasound transmit pulse, each with a different receive focal depth. The retrospective transmit focusing method is applied to ensure that the transmit and receive focal depths are equal. For example, $P(\omega)$ is used to determine the response for $F_{tx}=F_{rx}=F_a$ from $F_{tx}=F_b, F_{rx}=F_a$.

Using all relevant data, the $P(\omega)$ parameter was obtained and analysed. This parameter was determined by comparing the received ultrasound signal frequency responses between $F_{tx}=3.5$ cm, $F_{rx}=F_a$ and $F_{tx}=F_{rx}=F_a$, where $F_a=[3.5\text{ cm}, 5.5\text{ cm}]$. The magnitude of $P(\omega)$ in dB is shown in Figure 5.17b, while the unwrapped phase (pseudo-angle) is shown in Figure 5.17c. In Figure 5.17b, the magnitude of $P(\omega)$ is shown over temporal frequency for multiple focal depths ranging from $F_{tx}=3.5$ cm to $F_{tx}=5.5$ cm. Each curve represents a different transmit focal depth, which generally increases in magnitude with increasing $F_{tx}$. However, we were unable to identify any distinctive pattern in the response in order to suggest possible values for $P(\omega)$ from the data. Spikes in magnitude corresponded to nulls in the ultrasound frequency plots in Figure 5.17a, which are not overly informative. With exception of these localized spikes, the remainder of the signal was fairly constant.

However, the unwrapped phase or pseudo-angle of $P(\omega)$, shown in Figure 5.17c, suggested a clear relationship between the transmit focal depth and temporal frequency. This
plot shows the pseudo-angle response of $P(\omega)$ over 40 simulated focal depths from 3.5 cm to 5.5 cm. Each curve indicates a separate temporal frequency. Figure 5.17c was limited for clarity to show the first 20 temporal frequencies up to 500 kHz. We found that for a single temporal frequency, the unwrapped phase was consistent with a second-order polynomial that increases with focal depth. In addition, these phase curves increase in amplitude with temporal frequency. We also found through simulation experiments that this phase trend was independent of the number and location of on- or off-axis scatterers, whether the backscattering cross-section was constant or a uniformly distributed random variable, and whether the scatterers were positioned using a uniform or random spacing.

A summary of the key features of the unwrapped phase (pseudo-angle) of $P(\omega)$ include:

1. Each of the unwrapped phases or pseudo-angles of $P(\omega)$ increase monotonically with focal depth for a single temporal frequency.
2. The unwrapped phases of $P(\omega)$ generally increase with temporal frequency, i.e., the phase curve at 10 kHz is higher than that at 5 kHz.
3. The unwrapped phase of $P(\omega)$ can be fit to a second-order polynomial using regression methods.
4. The generalized phase responses identified in features 1-3 are independent of the number and position of on- or off-axis scatterers, the backscattering cross-section, and the relative position of scatterers.
5. Any phase deviations from the monotonic, second-order polynomial trend corresponds to nulls in the received signal frequency response shown in Figure 5.17a.

The features identified through examination of the unwrapped phase of $P(\omega)$ suggest that the phase of this signal can be predicted using at least 3 points on this curve. Assuming that neither the backscattering cross-section nor the scatterer positions change over time, using three independent ultrasound pulses with transmit focusing at, for example, 3.5 cm, 4.5 cm, and 5.5 cm, we can predict the $P(\omega)$ phase required to retrospectively modify the transmit transducer focal depth to any point between 3.5 cm and 5.5 cm. Using this approach, second-order polynomial regression was employed to determine coefficients for each temporal frequency in our 5 on-axis scatterer simulation. Results are shown for several frequencies spaced at 250 kHz.
from 0 Hz to 3.5 MHz in Figure 5.18. The red curve shows the exact \( P(\omega) \) phase required to retrospectively focus at the specified focal depth for a particular temporal frequency curve. The blue curve corresponds to that assuming all 40 simulated focal depths were used in the second-order polynomial regression, while the black curve shows the results of the second-order polynomial regression assuming received ultrasound signals at 3 transmit focal depths were used (3.5 cm, 4.5 cm, and 5.5 cm). The red, exact \( P(\omega) \) phase deviates from the second-order polynomial at temporal frequencies corresponding to nulls in the received ultrasound response of Figure 5.17a, however, they have little effect on the overall response since the corresponding signal magnitudes are small.
Figure 5.17: Temporal frequency response of 5 on-axis scatterers using retrospective transmit focusing. (a) Magnitude of received signal over frequency assuming $F_{tx} = F_{rx}$ for 3.5 cm to 5.5 cm. (b) Magnitude of $P(\omega)$ required to retrospectively modify the received signal. (c) Phase of $P(\omega)$ required to retrospectively modify the received signal.
Figure 5.18: Unwrapped phase of $P(\omega)$ over focal depth in 250 kHz increments up to 3.25 MHz. Since the phase increases fairly monotonically with angular frequency, frequencies in between marked curves should be assumed at 250 kHz between adjacent labels.

The retrospectively modified transmit focusing responses were then analyzed in the time domain for several cases corresponding to transmit focal depths at 3.5, 4.0, 4.5, 5.0, and 5.5 cm. Since the simulations were performed using the temporal frequency domain derivation described in Chapter 3, the Gibbs effect may be evident in part of each simulation. Analysed cases include:

(a) Figure 5.19: Retrospectively modified transducer using both the exact magnitude and phase of $P(\omega)$. Results show the normalized received ultrasound responses after retrospectively focusing to the specified focal depths. Since the transmit focal depth is retrospectively adjusted, the constant phase portion of the focusing equation (3.7) causes the curves to shift over time.
Scatterers are positioned on-axis consistent with that described in Figure 5.16. In addition, the received ultrasound responses assuming dynamic receive focusing at the specified focal depths (3.5, 4.0, 4.5, 5.0, and 5.5 cm) with the transmit focal depth set to the original 3.5 cm is also shown superimposed using darkened curves. These second curves were shown in order to demonstrate the effectiveness of the retrospective transmit focusing technique, as the impulse responses of scatterers near the desired focal depth show an improved spatial resolution than those nearby.

(b) Figure 5.20: Retrospectively modified transducer using only the exact phase of $P(\omega)$. The results are fairly consistent with Figure 5.19, suggesting that a retrospectively modified transmit transducer focal depth is highly dependent on the pseudo-angle of $P(\omega)$. Since there is little discernible pattern in the magnitude of $P(\omega)$ over focal depth and temporal frequency, it is very difficult to estimate an appropriate value for this parameter. However, these plots have shown that the magnitude of $P(\omega)$ is not overly necessary.

(c) Figure 5.21: Retrospectively modified transducer using all 40 simulated transmit focal depths in the second-order polynomial regression of the $P(\omega)$ pseudo-angle. The algorithm was able to successfully retrospectively focus the transmit transducer.

(d) Figure 5.22: Retrospectively modified transducer using 3 simulated transmit focal depths at 3.5 cm, 4.5 cm, and 5.5 cm in the second-order polynomial regression of the $P(\omega)$ pseudo-angle. Although this method introduces noise in the results, a retrospectively modified transmit focal depth is achievable using this approach.

It is especially important to realize that responses from each on-axis scatterer are considerably modified with the retrospective transmit focusing method. For example, the spatial resolution of scatterers positioned at 4.75 cm and 5 cm (two right-most curves in Figure 5.22) have a significantly narrower response using a retrospective focal depth of 5 cm, than that observed in other curves (either the superimposed darkened curve with a transmit focal depth of 3.5 cm, or all other focused responses). The retrospective transmit focusing method is therefore capable and effective of improving the spatial resolution through modifications to the $P(\omega)$ pseudo-angle.
Figure 5.19: Retrospectively modified transducer using the exact magnitude and phase of $P(\omega)$. The vertical axis is dimensionless since each individual curve is normalized and multiple curves are superimposed in the figure. For each curve, the constant baseline response is zero, while the maximum peak is normalized to one.
Figure 5.20: Retrospectively modified transducer using only the exact phase of $P(\omega)$. The vertical axis is dimensionless as described in the caption of Figure 5.19.
Figure 5.21: Retrospectively modified transducer using all 40 simulated transmit focal depths in the second-order polynomial regression for the phase of $P(\omega)$. The vertical axis is dimensionless as described in the caption of Figure 5.19.
Figure 5.22: Retrospectively modified transducer using only 3 points in the second-order polynomial regression for the phase of $P(\omega)$. The vertical axis is dimensionless as described in the caption of Figure 5.19.
Figure 5.23 shows the PSF results in dB for a single, on-axis scatterer at 5 cm that has been retrospectively transmit focused from the original 3.5 cm (black curve) to 5 cm (red curve), assuming $F_{rx} = 5$ cm. The figure corresponds to the 3 point, second-order polynomial regression method from Figure 5.22. This analysis effectively re-examines the PSF from the right-most scatterer in the bright blue curve of Figure 5.22, in order to more clearly identify the spatial resolution (mainlobe) and contrast (sidelobe) characteristics. In addition, the two transmit focal depths have been shown artificially superimposed on the same time point to more clearly identify the differences between the PSFs. While the retrospective transmit focusing method demonstrates improved spatial resolution (mainlobe width of red curve is narrower than black curve), it does not appear to improve contrast.

The signal processing method for retrospective transmit focusing described in this chapter can be summarized as follows:

1. Obtain 3 received ultrasound signals from imaging system using 3 different transmit focal depths (assuming $F_{tx} = F_{rx}$).
2. Identify the power spectral density estimates of noise and the ultrasound signal (in the absence of noise and the excitation wave).
3. Filter excitation wave and suppress additive noise from received ultrasound signals using either Wiener deconvolution or LMMSE filtering.
4. Calculate temporal Fourier transform of remaining 3 ultrasound signals.
5. Determine pseudo-angles of $P(\omega)$ required to perform retrospective transmit focusing of the received ultrasound signal in order to shift the transmit focal depth from its minimum $F_{tx}$ to the two larger values of $F_{tx}$.
6. Perform a second-order polynomial regression to estimate the pseudo-angle of $P(\omega)$ for transmit focal depths between the minimum and maximum values of $F_{tx}$.
7. Select a new desired transmit focal depth for retrospective focusing $F_{tx,new}$.
8. Determine the received ultrasound signal using dynamic receive focusing at $F_{rx} = F_{tx,new}$ and the original minimum transmit focus $F_{tx}$ using post-processing techniques.
9. Adjust the frequency transform results of step 8. by the pseudo-angle of $P(\omega)$ to retrospectively focus the ultrasound system at the new transmit focal depth $F_{tx,new}$. 
Figure 5.23: PSF for a single scatterer at 5 cm before and after retrospectively transmit focusing at 5 cm. Results are shown in dB to identify mainlobe and sidelobe details. The different transmit focal depths will cause the two responses to occur at different times, however they have been shown superimposed on the red plot here. Correct timing is shown in Figure 5.22.

5.6 Discussion

The SSF analysis demonstrated the effect of multiple temporal convolutions on the received ultrasound signal. We have shown that the individual transducer impulse responses and excitation wave have a significant effect on increasing the spatial distribution of the SSF both inside and near the focal region. Since a larger SSF spatial distribution and PSF temporal response correspond to a degraded spatial resolution, a worthwhile goal is the development of
signal processing methods to reduce or eliminate the effect of these temporal convolutions. In addition, we have shown that modifications to any of the individual terms within the received ultrasound signal may also be used to reduce the SSF distribution.

A primary strategy used in this chapter was to mathematically shift individual terms of the received ultrasound signal (1.1), to outside of the scatterer summation, where they could be modified or removed using signal processing methods. The first strategy involved the removal of the excitation wave and suppression of additive noise using Wiener deconvolution and LMMSE filtering methods. Although it was assumed that the electromechanical transducer response was a delta function, this function can be easily lumped with the excitation wave term. The deconvolution method is a statistical signal processing technique that estimates the desired signal in the absence of the excitation wave and with additive noise suppressed using a minimum mean square error method. Its performance is highly dependent on the SNR, producing results in between that of the exact desired signal and a matched filter solution.

Although the simplest deconvolution filter involves simply dividing the received ultrasound signal by the excitation wave in the frequency domain using a linear least squares method, any zeros in the excitation wave frequency response will significantly amplify additive noise in the resultant signal. The Wiener deconvolution and LMMSE filtering methods, however, are able to accommodate for variable SNR levels by minimizing the mean square error between the desired signal and its estimate after deconvolution. In low noise (high SNR) environments, the deconvolution filter is heavily biased towards excitation wave removal, much like the linear least squares method. However, high noise (low SNR) cases converge to a matched filter response.

Summarizing the received ultrasound signal described in (5.2),

\[ y[n] = d[n] + w[n], \]  

the deconvolution method was designed to remove the effects of both \( v[n] \) and \( w[n] \), by defining a deconvolution filter that would estimate the desired signal \( d[n] \) from the received ultrasound signal \( y[n] \). This filter was defined in the temporal frequency domain as,

\[ H(e^{j\omega}) = \frac{P_d(e^{j\omega})v^*(e^{j\omega})}{P_d(e^{j\omega})v(e^{j\omega})^2 + P_w(e^{j\omega})}, \]  

(5.3)
In the absence of noise \( P_w(e^{j\omega}) = 0 \), the deconvolution filter simplifies to
\[
H(e^{j\omega}) = \frac{1}{V(e^{j\omega})},
\]
which is the reciprocal of the excitation wave frequency response. This results in a perfect reproduction of the desired signal since
\[
\hat{D}(e^{j\omega}) = Y(e^{j\omega})H(e^{j\omega}) = \frac{Y(e^{j\omega})}{V(e^{j\omega})} = D(e^{j\omega}),
\]
where, \( Y(e^{j\omega}) \), \( \hat{D}(e^{j\omega}) \) and \( D(e^{j\omega}) \), are the discrete time Fourier transforms of \( y[n] \), \( d[n] \), and \( d[n] \), respectively.

In the case of low SNR or high levels of additive noise, by simplifying (5.3), we obtain,
\[
H(e^{j\omega}) = \left[ \frac{V^*(e^{j\omega})}{\left| V(e^{j\omega}) \right|^2 + \frac{P_w(e^{j\omega})}{P_d(e^{j\omega})}} \right],
\]
which is also known as the matched filter,
\[
\hat{D}(e^{j\omega}) = Y(e^{j\omega})H(e^{j\omega}) = \frac{Y(e^{j\omega})V^*(e^{j\omega})P_d(e^{j\omega})}{P_w(e^{j\omega})},
\]
since it involves the temporal convolution with the time reversed excitation wave.

The primary limitation of the deconvolution method requires that both the desired signal and additive noise are wide-sense stationary. However, this may not be satisfied for the desired signal since biological tissue consists of a distribution of different tissues within the medium. We have also assumed that additive noise is uncorrelated with the desired signal, however this is expected since additive noise is often due to electronic noise sources in the imaging equipment. Nevertheless, by using temporal windowing of the received ultrasound signal and assuming the statistics of similar tissue, the deconvolution technique can be effective in estimating the received ultrasound signal in the absence of the excitation wave and with additive noise suppressed.

Retrospective transmit transducer focusing has been studied by few authors (Hu et al. 2010, Zhou and Hossack 2003, Freeman et al. 1995). Hu et al. (2010) implemented a
retrospective 2D filter to focus the transmit transducer after using a plane wave excitation. Their algorithm was designed to optimize the PSF by maintaining the common mainlobe properties of the plane wave-dynamic receive focused system and that of the focused transmit and receive transducer system, while minimizing the sidelobes present in the former. Zhou and Hossack (2003) used a wideband excitation pulse, with different frequency ranges, each focused at a different depth. Post-processing methods later separated the appropriate focal depth responses. Freeman et al. (1995) implemented an aperture filtering technique that deconvolved the effects of the transmit transducer from the received channel data, while reconvolving a new aperture weighting function to retrospectively focus the transmit aperture. However, implementation of their method requires an extremely large filter bank to describe the adjusted aperture functions for each transmit focal depth and temporal frequency. Determining the appropriate aperture filter length was also a challenge as larger filters were shown to significantly increase computation time and storage requirements. Our method only requires the storage of 3 parameters describing the second-order pseudo-angle polynomial for each temporal frequency and it acts on the entire, composite received signal, not individual channel data, thereby improving both computation time and storage requirements.

Alternative methods for performing transmit focusing at multiple, simultaneous focal depths include: coded excitation schemes that use a different excitation signal on each transducer channel, followed by post-processing techniques that extract the focused signal at different depths (Shen et al. 1994); synthetic aperture systems that recombine multiple transmit and receive apertures to dynamically focus on transmit and receive (Jensen et al. 2006); and X-wave excitations that focus simultaneously over a range of depths (Lu and Greenleaf 1992).

We found during the course of this research that a fully analytical solution to the retrospective transmit focusing equation was not possible using the tools and mathematical manipulations that were implemented. Our primary objective was the identification of either the vector $\mathbf{M}_{\text{rec}}$ or parameter $P(\omega)$ from the available ultrasound data. Approaches that were considered included using various combinations of transmit and receive transducer excitations, the use of matrix methods for solving for either of the two variables, using the receive focused signal to estimate $P(\omega)$, reinterpretation of the relevant parameters of the retrospective method in the temporal frequency and spatial frequency domains, and using graphical interpretations of the vector relationships. Nevertheless, we were able to identify notable trends in the parameter
$P(\omega)$ by examining the effects of the technique in simulation, and were thus able to identify a method to identify the pseudo-angle required to retrospectively modify the transmit focal depth. Since the $P(\omega)$ pseudo-angle was shown to reliably conform to a second-order polynomial over focal depth, perhaps an analytical approach can be identified in future research to more accurately predict the appropriate value of this parameter. In addition, we were able to identify that these trends were independent of the number and location of scatterers and the distribution of backscattering cross-section values.

A limitation of the approach is the assumption that additive noise has been adequately suppressed in the system, in order to enable the mathematical modifications required for the retrospective focusing solution. However, the excitation wave may remain in the equation and grouped with the $B(r:t)$ term in (5.6), although this configuration was not analysed in simulation so possible effects on $P(\omega)$ are unknown. Attenuating media can be accommodated in a similar manner, however this also requires further analysis. Other recommendations for future research include using $P(\omega)$ to retrospectively alter additional transducer surface velocity properties such as the transducer geometry or apodization functions as well as further verification of the method using 2D B-mode simulations and experimental measurements.

5.7 Conclusions

Two signal processing methods were proposed to reduce the spatial extent of the SSF functions of the received ultrasound signals: the Wiener deconvolution and LMMSE filtering methods, which are equivalent, removed effects of the excitation wave and with additive noise suppressed, and the retrospective transmit focusing method, that modified the transmit focal depth using a post-processing technique. Both were shown effective at improving spatial resolution from the traditional matched filtering and envelope detection technique. Although the LMMSE excitation wave removal method has been previously proposed by Haider et al. (1998), the analysis of its effects on the SSF is new.

A new, retrospective transmit focusing technique was mathematically derived and simulated. However, we were unable to identify an analytical solution for the $P(\omega)$ term. Nevertheless, we have shown in simulation that the technique can reliably produce a
retrospectively transmit focused signal and that there are notable trends in the $P(\omega)$ pseudo-angle that are consistent over focal depth and temporal frequency, which can be used to reliably estimate an appropriate value. These trends are also independent of the number or location of on- or off-axis scatterers, the backscattering cross-section (constant or uniform random variable), and the relative position of these scatterers—whether their locations are uniformly or randomly distributed. These proof of concept simulations have shown that the retrospective transmit focusing method can improve the spatial resolution, compared to traditional ultrasound focusing techniques.
CHAPTER 6
CONCLUSIONS

6.1 Summary and conclusions

The primary research objective of this thesis was the development of a new signal processing algorithm to improve image quality of ultrasound A-lines. To achieve this goal, we hypothesized that a comprehensive understanding of acoustic wave propagation is necessary using various tools such as mathematical derivations, simulation, and analyses. By interpreting the physical system from different perspectives, new insights into wave propagation may be identified that can lead to novel solutions.

Using the angular spectrum method, an accurate analytical model has been developed for the field distribution arising from a finite geometry, apodized, and focused piston transducer. The derivation includes the effects of diffraction and evanescent waves without the use of the Fresnel approximation and is applicable at all near- and far-field observation positions in a lossless medium. In addition, the on-axis impulse response of the focused piston transducer is derived and compared with the concave transducer. Moreover, this model develops insights into radiated field characteristics, including: a) the spatial frequency bandwidth is constant over axial depth, suggesting that spatial resolution can be improved away from the focus; b) the phase of the angular spectrum determines the spatial resolution for a given transducer configuration—a constant phase is optimal on any observation plane; c) focusing can significantly increase the spatial frequency bandwidth; d) the velocity potential on a plane parallel to the transducer is the Hankel convolution of the transducer surface velocity with the Green’s function; e) evanescent waves decay both with increasing spatial frequency and axial depth; f) significant errors are introduced when Fresnel focusing approximations are assumed; and g) the concave transducer is
not equivalent to the focused planar piston transducer. The analytical model and associated insights enhance understanding of the radiated field characteristics, which can be of value in the development of signal processing techniques for image enhancement.

The piston transducer model was then extended to the case of an attenuating and dispersive medium. After analysing existing spherical wave Green’s function models of power-law frequency dependent attenuation, a causal, spherical wave Green’s function was derived from the Navier-Stokes equation for a classical viscous medium using the multi-dimensional Laplace transform. The solution includes the effects of homogeneous and evanescent wave propagation in the near- and far-field. Modifications to the angular spectrum method were also presented and used to analyze the radiated field of a focused, planar piston transducer. Moreover, insights into the radiated field of viscous media have identified that: a) the Green’s function includes a multiplicative term that can become significant in highly viscous media or at high frequencies; c) while moderate viscosities can significantly attenuate the impulse response at the focus, it may minimally affect the response away from the focal point; d) dispersion does not significantly alter the time of the Green’s function response, as frequencies experiencing high rates of dispersion are also highly attenuated, yet its effect on transducer focusing can be considerable; and e) both homogeneous and evanescent waves experience attenuation and dispersion. The Green’s function and associated insights enhance the understanding of the radiated field in viscous media, which can be of value for predicting the effects of attenuation and dispersion.

By examining the variables contributing to the received ultrasound signal, including the electromechanical response of the transducer, excitation wave, transducer impulse responses, tissue medium, and backscattering cross-section, we proposed a strategy to minimize the size of the SSF in order to optimize the spatial resolution of the ultrasound imaging system. Since the SSF provides details regarding the spatial distribution of scatterers contributing to the received ultrasound response at a single point in time, a reduction in its axial and lateral span will reduce the number of scatterers contributing to a particular response. This can improve the spatial resolution and contrast of the resulting image. Two signal processing methods were implemented and tested in simulation: Wiener deconvolution or LMMSE filtering to remove the excitation wave and suppress additive noise from the received signal, and retrospective transmit transducer focusing. Using mathematical analysis and extensive simulations, proof of concept
simulations were presented using a variable number of scatterers. Results were also compared with the traditional matched filtering and envelope detection technique. The retrospective transmit focusing method identified that an estimate of the \( P(\omega) \) pseudo-angle could be obtained using second-order polynomial regression. The method is not sensitive to the number or location of scatterers, the backscattering cross-section, nor the uniformity of the scatterer location distribution.

6.2 **Primary research contributions**

The most significant contributions of this thesis are related to the mathematical derivation, simulation, and analysis of the transducer model. In addition, a new signal processing technique for retrospective transmit focusing based on the insights developed from the model was derived and simulated.

The specific contributions of this thesis are:

1. The derivation of an accurate analytical radiation model of a finite geometry, apodized, focused piston transducer in lossless media using the angular spectrum method.
2. The identification of new insights into radiation field characteristics using simulation and analysis of the piston transducer model in lossless media.
3. Analysis of SSF characteristics in lossless media.
4. The derivation of the on-axis, impulse response from an unapodized and focused, plane piston transducer and comparison with the concave transducer to identify key differences.
5. The identification of velocity potential estimation errors introduced by the Fresnel focusing approximation.
6. Analysis of existing causal, spherical wave Green’s functions for power-law frequency dependent attenuation.
7. The derivation of a causal, Green’s function for spherical wave propagation in a classical viscous medium from the Navier-Stokes equation using the Laplace transform.
8. The identification of angular spectrum method modifications to account for transient wave propagation in a classical viscous medium.
9. The identification of new insights using simulation and analysis of the spherical wave Green’s functions for power-law frequency dependent attenuation and viscous media.
10. Analysis of PSF and SSF characteristics in attenuating media.

11. An SSF analysis comparing LMMSE filtering of the excitation wave and suppression of additive noise to the traditional matched filtering and envelope detection method.

12. The mathematical development and simulation of a new signal processing technique to retrospectively modify the transducer transmit focal depth. Proof of concept simulations were performed to validate the approach.

6.3 Suggestions for future work

6.3.1 Transducer model

Our focused, piston transducer model was selected to identify radiation field characteristics that may have been masked by more complex geometries. Based on the mathematical derivation and conclusions developed from this thesis, the model can be extended to more standard geometries such as the 1D linear array transducer.

In addition, since all focused transducer models in the literature seem to have used the Fresnel approximation, we have provided a more accurate solution and have identified why such approximations are inadequate. The mathematical foundation for describing the radiated field of a focused piston transducer in this work can be applied to models of various focused transducer geometries.

6.3.2 Attenuation and dispersion model

It is very difficult at this time to fully validate any spherical wave propagation model of attenuation and dispersion as experimental measurements are limited to plane waves and knowledge of the underlying mechanisms of attenuation and dispersion are not fully understood. More in-depth experimental validation of these models are therefore required.

We identified that attenuation predominately alters the impulse response at the focus for wideband systems. Perhaps its relatively minimal effect on the impulse response in regions away from the focus can be used to develop signal processing algorithms to accommodate for these losses or to increase or study the depth of field associated with a single focused transmit pulse. Currently, multiple A-lines are obtained at a single lateral position, each using a different
transmit focusing depth in order to improve spatial resolution. If the SSF is relatively constant over a larger time range around the focus for attenuating media, it may be possible to reduce the number of required transmit pulses, thereby increasing the imaging frame rate.

The fundamental approach to solving the Navier-Stokes equation for classical viscous media involved modeling attenuation and dispersion using a complex spatial frequency. In solving the equation assuming spherical wave propagation, the 3D Laplace transform was required. However, while there is limited research on the multi-dimensional Laplace transform in Cartesian coordinates, there does not seem to be any mathematical relationships available that describe the cylindrical or spherical equivalents of either the Laplace transform or complex numbers. This form of the Laplace transform would be very helpful for examining spherical wave attenuation for any application.

A fundamental assumption of the attenuation and dispersion model was that these properties are uniformly distributed throughout the medium. However, this assumption is not reasonable when considering a tissue medium, as a single A-line will intersect various organs and tissue structures, each with different properties. Nevertheless, the Green’s function derivation provided in this thesis can provide a basis for further research on the variability of attenuation and dispersion within a medium.

6.3.3 Signal processing solutions

The fundamental purpose of this thesis study was the development of new signal processing solutions to improve ultrasound imaging resolution and contrast. Our method was to initially understand the characteristics of the radiated field both mathematically and intuitively, then to use these newly developed insights to identify new signal processing solutions. The radiated field was examined from several perspectives, including mathematical derivation, angular spectrum, velocity potential, and impulse response using the PSF and SSF. The field was also examined assuming lossless and attenuating media due to power-law frequency dependent attenuation and viscous losses.

Some initial ideas that could be explored using the insights developed in this research include:

1. When using a focused transducer, dispersion will cause the transmitted signal to have a different focusing depth for each temporal frequency. Perhaps by using a chirp function
and cross-correlation of the received signal with signals of different frequencies, it may be possible to recover an A-line with multiple transmit focal depths from a single transmit pulse.

2. The development of methods to improve the retrospective transmit focusing technique by the identification of analytical approaches to estimate the relevant parameters instead of using the method presented in this thesis.

3. Methods to produce a constant angular spectrum phase on any observation plane. This will optimize the spatial resolution in regions away from the focus.

4. By interpreting the impulse response from different perspectives using the PSF and SSF, strategies to improve image quality can be achieved by optimizing either the duration or amplitude of the PSF or by reducing the spatial extent of the SSF.

5. The retrospective transmit focusing method can be extended to adjust other transducer surface velocity parameters such as the geometry or apodization using a different selection of parameter $P(\omega)$.

6. Simulation and analysis of the complete 2D B-mode image.

7. It may also be possible to estimate the ultrasound system statistical parameters from multiple laterally adjacent A-lines. These signals may be correlated since the medium may change slowly with position.

8. An analysis of the effects of attenuation and dispersion on $P(\omega)$ estimation, as a lossless model has been assumed for our simulations.

9. An analysis of the sensitivity of the Wiener deconvolution filtering method to inaccuracies in the statistical estimates of the ultrasound signal

6.4 Final remarks

To develop new signal processing methods to improve ultrasound imaging quality, an intuitive understanding of the radiated field can be helpful. By understanding field characteristics from different perspectives—using tools such as mathematical derivations, simulations, and analysis in the time, space, and their frequency domain counterparts—new ways of thinking about problems and potential solutions are sometimes revealed. We hope that the fundamental ultrasound modeling research provided by this work offers such perspectives and
identifies new ideas that can be used for the improvement of imaging systems. Ultimately, such improvements can advance the usefulness and clarity of ultrasound images that can improve its medical diagnostic and assessment ability.
REFERENCES


The primary assumptions and simplifications in this research are summarized as follows.

A.1 Angular spectrum method

- Small signal approximation of the Navier-Stokes equation. This implies that the local changes in pressure, density, and particle velocity due to the propagating wave are much smaller than equilibrium values.
- Linear model of wave propagation.
- Longitudinal wave propagation only. No shear wave propagation.

A.2 Piston transducer model in lossless medium

- Planar piston transducer model of finite geometry with continuously varying apodization and focusing.
- Wave propagation in an homogeneous, inviscid medium in the absence of attenuation and dispersion.
- Neumann boundary conditions for a rigid baffle. The normal component of the transducer surface velocity is zero outside the transducer boundary.
- The transducer is of negligible thickness.
- The effects of primary diffusion arising from the transducer edge were included. Secondary diffusion effects were ignored.
- Evanescent waves are present and included in the model.
A.3 Attenuation and dispersion model

- A spherical wave model of attenuation is a reasonable extension of the plane wave model in the transducer near-field.
- The attenuation coefficient is much smaller than the wave number (Szabo 1995) for power-law frequency dependent attenuation.
- Attenuation is limited to absorption processes for the viscous medium.
- Attenuation is represented using a complex spatial frequency that decays with distance, instead of a complex temporal frequency that decays with propagation time.
- The cylindrical form of the 3D Laplace transform exists.

A.4 Signal processing algorithms

- The excitation wave can be either a pulsed wave or continuous wave.
- The statistics of the noise are known and assumed to be wide sense stationary.
- Additive noise is statistically independent of the desired signal after deconvolution.
- The electromechanical impulse response of the transducer is known.
- The backscattering cross-section is time-independent and the scatterers are immobile.
- The system can be considered linear and the nonlinear effects are negligible.
- The Born approximation has been assumed (Cobbold 2007, p.287), where scattering is sufficiently weak so the effects of multiple scattering can be ignored.
- To allow for dynamic receive focusing, the apodized and focused, plane piston transducer is formed using a set of infinitely thin concentric rings.
APPENDIX B

HANKEL CONVOLUTION

The Hankel convolution, represented by the #-operator, is the circularly symmetric equivalent of the 2D convolution in Cartesian coordinates.

Two known Hankel convolution identities are (Haimo 1965, Betancor et al. 2008):

\[
H[f(r)]H[g(r)] = H[f(r)\# g(r)]
\]

\[
H[f(r)g(r)] = H[f(r)]\# H[g(r)].
\]

Note that the Hankel #-convolution differs from the standard *-convolution, as it is defined by,

\[
f(\eta)\# g(\eta) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty 2\pi f(r_2) J_0(r_2 k_r) r_2 \cdot dr_2 \int_0^\infty 2\pi g(r_3) J_0(r_3 k_r) r_3 \cdot dr_3 \left[ J_0(\eta k_r) k_r \cdot dk_r \right],
\]

\[
= H^{-1}[H[f(r_2)]H[g(r_3)]],
\]

in which the \((\eta_1, r_2, r_3)\) variables are placeholders to ensure the parameters in the integration remain separate, i.e., \(f(\eta_1) = f(r_2)\) with only the substitution of the variable \(\eta_1 \rightarrow r_2\).

The Hankel convolution is equivalent to the 2D convolution in Cartesian coordinates,

\[
f(r)\# g(r) = f(x, y)^{* *} g(x, y),
\]

where \(* *\) represents convolution in \((x, y)\) coordinates.
APPENDIX C
HANKEL TRANSFORM OF FOCUSING TERM

The Hankel transform of the transducer focusing term (3.12) was determined as described below and verified in simulation. By definition,

$$H \left[ e^{jk\sqrt{F^2+r^2}} \right] = 2\pi \int_{0}^{\infty} e^{jk\sqrt{F^2+r^2}} J_0(rk_F) rdr. \quad (C.1)$$

Substituting $s = \sqrt{F^2 + r^2}$, (C.1) can be expressed as,

$$H \left[ e^{jk\sqrt{F^2+r^2}} \right] = 2\pi \int_{F}^{\infty} e^{jks} J_0 \left( k_F \sqrt{s^2 - F^2} \right) s \cdot ds, \quad (C.2)$$

which is simply the inverse Fourier transform of $J_0 \left( k_F \sqrt{s^2 - F^2} \right) s$ with respect to $s$. Euler’s formula enables (C.2) to be expanded as,

$$H \left[ e^{jk\sqrt{F^2+r^2}} \right] = 2\pi \int_{F}^{\infty} s \cdot J_0 \left( k_F \sqrt{s^2 - F^2} \right) \cos(ks) ds + 2\pi j \int_{F}^{\infty} s \cdot J_0 \left( k_F \sqrt{s^2 - F^2} \right) \sin(ks) ds. \quad (C.3)$$

Now without the $s$ in each integral, the tables in Erdelyi (1954, vol. 1, p.57 eq.48; p.113 eq.47; and p.117 eq.10) list the following transform pairs:
\[
\int_{F}^{\infty} J_0 \left( k_r \sqrt{s^2 - F^2} \right) \cos (ks) ds = \begin{cases} 
\left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} e^{-F \sqrt{k_r^2 - k^2}} & 0 < k < k_r \\
- \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} \sin \left( F \sqrt{k_r^2 - k^2} \right) & k_r < k < \infty 
\end{cases} 
\]

\[
\int_{F}^{\infty} J_0 \left( k_r \sqrt{s^2 - F^2} \right) \sin (ks) ds = \begin{cases} 
\left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} e^{-F \sqrt{k_r^2 - k^2}} & 0 < k < k_r \\
\left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} \cos \left( F \sqrt{k_r^2 - k^2} \right) & k_r < k < \infty 
\end{cases} 
\]

These equations enable the expression for the Hankel transform (C.2), but without the \( s \), to be written as,

\[
2\pi \int_{F}^{\infty} e^{jk_s} (-j) J_0 \left( k_r \sqrt{s^2 - F^2} \right) ds = \begin{cases} 
(-2\pi j) \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} e^{-F \sqrt{k_r^2 - k^2}} & 0 < k < k_r \\
2\pi \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} e^{jF \sqrt{k_r^2 - k^2}} & k_r < k < \infty 
\end{cases} 
\]

By letting \( g(k) = \int_{F}^{\infty} e^{jk_s} (-j) J_0 \left( k_r \sqrt{s^2 - F^2} \right) ds \) in (C.5), we can reincorporate the \( s \) term into the integral by taking the derivative of \( g(k) \), since

\[
\frac{dg(k)}{dk} = 2\pi \int_{F}^{\infty} e^{jk_s} (-j) J_0 \left( k_r \sqrt{s^2 - F^2} \right) js \cdot ds .
\]

With the help of (C.5), the solution to (C.2) for \( 0 < k < k_r \) can be found,

\[
2\pi \int_{F}^{\infty} e^{jk_s} (-j) J_0 \left( k_r \sqrt{s^2 - F^2} \right) js \cdot ds = \frac{d}{dk} \left\{ (-2\pi j) \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{1/2} e^{-F \sqrt{k_r^2 - k^2}} \right\} 
\]

\[
= -2\pi j k e^{-F \sqrt{k_r^2 - k^2}} \left[ \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{3/2} + F \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{-1} \right] 
\]

Similarly, for \( k_r < k < \infty \),

\[
2\pi \int_{F}^{\infty} e^{jk_s} (-j) J_0 \left( k_r \sqrt{s^2 - F^2} \right) js \cdot ds = -2\pi k e^{jF \sqrt{k_r^2 - k^2}} \left[ \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{3/2} - jF \left( \frac{k_r^2 - k^2}{k_r^2 - k^2} \right)^{-1} \right] .
\]

In summary,
\[ H \left[ e^{jk\sqrt{F^2 + r^2}} \right] = \begin{cases} -ke^{jF\sqrt{k^2 - k_r^2}} \left[ (k^2 - k_r^2)^{-3/2} - jF(k^2 - k_r^2)^{-1} \right] & 0 < k_r < k \\ -jke^{-F\sqrt{k_r^2 - k^2}} \left[ (k_r^2 - k^2)^{-3/2} + F(k_r^2 - k^2)^{-1} \right] & k < k_r < \infty \end{cases} \]