SUPPLY CHAIN DESIGN - COMPETITIVE AND FINANCIAL PERSPECTIVES

by

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Abstract

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In this thesis we study problems in the context of inventory control and facility location. In chapter 2 we study the competition among risk averse newsvendors. We showed that the well-known result for the single-product monopoly firm, which states higher risk aversion causes the firm to reduce its order quantity, cease to hold under the competition. We concluded that the higher risk aversion does not necessarily cause both firms to reduce their order quantity. We showed that the impact of risk aversion on equilibrium quantities is a trade-off between two effects: (a) Own risk aversion increment which causes that the firm reduces its order quantity and (b) Effect of spillover demand from competitor which causes that the firm increases its order quantity. We also show which firm raises its order quantity as both firms become more risk averse depending on their attributes: profitability ratio (overstocking to understocking ratio), initial risk aversion level and demand characteristic (distribution and substitution). In Chapter 3, we study how the operational decisions of a firm’s manager depend on her own incentives, the capital structure, and financial decisions in the context of the newsvendor framework. We showed that in contrast to common practices, tying the manager’s compensation to stock price (equity value) may not be optimal for shareholders. We propose to tie the managers’ compensation to the firm value or include a debt-like instrument in the compensation package to mitigate the risk taking behaviour of the managers. We also show how the board of directors can modify the compensation structure based on the
state of the economy and publicly available information about company’s demand. In Chapter 4, we study the effect of risk attitude of decision makers on well-known location problems with uncertain demand. In addition to providing mathematical formulations for those problems, we also discussed how we can solve these problems using linearization techniques. We also shed some light on the importance of considering the volatility and correlation structure. Furthermore, we apply a Bayesian updating method, a useful tool for updating the probability distribution to incorporate the consultants’ view about uncertain factors in location problems.
Dedication

I would like to dedicate this thesis to my wife, my parents and my sister for their love, endless support and encouragement.
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Chapter 1

Introduction

In this thesis, we study three problems in the context of inventory control and facility location. In Chapter 2, titled “Competition among Risk-Averse Newsvendors”, we study single-period inventory competition among risk-averse duopoly firms in the context of the classic newsvendor framework. Each firm chooses its order quantity to maximize its expected utility, taking into account the possibility of spill-over demand from its competitor. We provide existence and uniqueness results for this newsvendor game and characterize the equilibrium order quantities as a function of the problem parameters. Furthermore, we investigate the effect of increasing risk aversion on the equilibrium order quantities. Higher risk aversion is known to reduce the order of a monopoly firm. We identify conditions under which this "order less" result ceases to hold under competition.

In Chapter 3, we study how the operational decisions of the firm’s manager depend on her own incentives, the capital structure, and financial decisions in the context of the newsvendor framework. We establish a relationship between the firm’s cost of raising funds and the riskiness of the inventory decisions of the manager. Initially we consider four types of managers: profit, equity, firm value and profit-equity maximizers, and assume that they may raise funds to increase the inventory level only by issuing debt. We show that the shareholders are indifferent between the different types of managers when
the coefficient of variation (CV) of demand is low. However, this is not the case when the CV of demand is high. Based on the demand and the firm’s specific characteristics such as profitability, leverage, and bankruptcy costs, the shareholders might be better off with the manager whose compensation package is tied to the firm value as opposed to the equity value. We, then, extend our model by allowing the manager to raise the required funds by issuing both debt and equity. For this case we focus on the equity and firm value maximizer managers and show that our earlier results (for the debt only case) still hold subject to the cost of issuing equity. However the benefit of the firm value maximizer manager over the equity maximizer manager for shareholders is considerably less in this case compared to the case where the manager can only issue debt. The Board of Directors can take these factors into consideration when establishing/modifying the right incentive package for the managers. We also incorporate the notion of the asymmetric information to capture its impact on the board of directors’ decision about the managers’ incentive package.

In Chapter 4, we study the effect of risk attitude of decision makers in three well-known location problems with uncertain demand: the Median, the Center and the Generalized Maximal Covering Location problems. We provide mathematical formulation for each problem in the form of quadratic programming and discuss how we can solve the problems using different linearization techniques. In particular, for the median and generalized maximal covering location problems we apply Lawler and Glover methods and compare them in terms of the time it takes to solve various instances numerically. For the center problem we show how this problem reduces to the classical center problem. Furthermore we shed some light on the importance of considering the volatility, the correlation structure and accurate estimation of them for the decision maker who is not risk neutral (a common assumption in location theory). In the spirit of improving the mean, volatility and correlation structure of the demand, we also present a Bayesian updating method, a mathematical tool to incorporate the expert/consultant view about
the uncertain factor in location problems.
Chapter 2

Competition among Risk Averse Newsvendors

2.1 Introduction

It is now understood and experimentally verified that risk aversion may play an important role in inventory decisions. For example, experimental evidence suggests that inventory planners for high profit products exhibit risk-averse behavior, trading off lower expected profits in return for downside protection against possible losses (Schweitzer and Cachon, 2000). The level and nature of risk aversion may further vary depending on various factors and events such as: beliefs about market volatility, the availability of cash/credit and the cost of financing. While the importance of risk aversion for inventory decisions is now recognized, there are to our knowledge no studies that investigate the interaction of risk aversion and competition on inventory decisions.

Literature and positioning. All the early and much of the recent inventory control literature assumes risk neutral decision makers (we refer to Khouja (1999)). However there are several papers study the impact of risk aversion on inventories for a monopoly. Berman and Schnabel (1986) were among the first to study the impact of risk aversion
on the order quantity under a mean-variance utility function. Eeckhoudt et al. (1995) study a monopoly newsvendor under various utility functions. Agrawal and Seshadri (2000) study the impact of uncertainty and risk aversion on price and order quantity. Van Mieghem (2007) studies newsvendor networks with many products and resources under mean variance utility function. The paper discuss how operational strategies (operational hedging) may reduce total risk and hence create value. It shows that compared to a risk neutral newsvendor, a risk averse newsvendor may invest more on resources to reduce the profit variance and mitigate the risk in the network. Chen et al. (2007) study the effect of risk aversion in multiperiod inventory models. Choi et al. (2011) study a multi product risk averse monopoly newsvendor under general law-invariant coherent measures of risk. They show that for identical products with independent demands, increased risk aversion lead to decreased order quantities. Furthermore they numerically show that the same result holds for or a two-product newsvendor with positive correlation. However, for a two-product newsvendor with negatively correlated demand they, show numerically that increased risk aversion may result in a greater order quantity for one of the products, compared to a risk neutral newsvendor. Choi and Ruszczyński (2011) also study the multi product risk averse newsvendor with exponential utility function.

There are also several studies of inventory competition assuming risk-neutral decision-makers. While competition clearly affects risk, existing studies consider risk aversion and competition only in isolation, leaving open many questions on their interaction. Parlar (1988) was first to study a duopoly in the context of the classic newsvendor framework. He showed the existence and uniqueness of Nash equilibrium for risk neutral duopoly newsvendor game. Lippman and McCardle (1997) study the effect initial allocation rule on the equilibrium. Netessine and Rudi (2003) study the centralized and competitive models for multiple products with demand substitution.

Another set of related papers are those that consider both pricing and inventory decisions. Ayden and Porteus (2008) seek optimal inventory levels and prices under
price based substitutions but not stock-out based substitutions. Zhao and Atkins (2008) studied more general case. They obtain optimal inventory levels and price for N vendors selling substitutable products. However, they consider both price based and stock-out based substitutions. Hu et al (2011) extended Zhao and Atkins (2008). They consider two-period game in contrast to Zhao and Atkins (2008) single period problem and also endogenized the consumer’s switching behavior upon a stockout. We refer to Ayden and Porteus (2008), Zhao and Atkins (2008), and Hu et al. (2011) for further references.

The interaction between risk aversion and competition also caught the interest of researchers in economics and finance. However, those papers ignore inventory decisions and supply constraints; see, e.g., Eldor and Zilcha (1990) and Asplund (2002) for studies of risk averse oligopoly in the absence of inventory constraints.

The following table summarizes the existing literature on inventory models when we consider the interaction between risk aversion and competition.

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<th>Monopoly</th>
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<td>Risk Neutral</td>
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We first study the newsvendor game among risk-averse competitors with a general risk preferences who each place a single order for one substitutable product. We assume that the product costs and prices are fixed. The only source of uncertainty is demand. The firms order simultaneously prior to observing demand, aiming to maximize their expected utility. Each firm faces its primary demand plus spillover demand if its competitor stocks out (a deterministic fraction of her unmet demand). We characterize the Nash equilibrium orders as a function of the problem parameters and investigate their sensitivity to risk aversion levels. We show that one of the firms may raise its order if both firms become more risk averse. This suggests that ignoring the presence of competition may lead to wrong recommendations on how to adjust to increasing risk aversion. To put these results in context, it is well known that the optimal order quantity of a single-product monopoly
decreases in the risk aversion level; e.g., see Eeckhoudt et al. (1995). In this case, there is no demand substitution effect among products. Choi et al. (2008, 2011) study a multi-product risk averse monopoly newsvendor without focusing on demand substitution effects. The interaction between the own risk aversion effect and the demand substitution effect in the presence of competition appears not to have been studied so far.

We show that impact of risk aversion on equilibrium quantities is a trade-off between two effects:

1. The direct own risk aversion effect, which causes a firm to reduce its order quantity.
2. The indirect demand substitution or demand spillover effect, which causes the firm to increase its order quantity.

Next, to gain specific insights on how risk aversion affects the equilibrium depending on the model parameters, we assume that firms risk preferences exhibit constant absolute risk aversion (CARA Utility Function) and each firm’s primary demand is low or high. Initially, we consider the case uncorrelated primary demands. Later we relax this assumption.

We specify which firm raises its order quantity as both firms become more risk averse for three scenarios of asymmetric firms that differ in one of the following attributes: the profitability of their product, measured by their under- to overstocking cost ratio, their risk aversion level, and their spillover demand fraction.

Different profitability ratio. We show that when we have uncorrelated demand not only the more profitable firm may increase its order quantity as both firms become more risk averse, but it could be the case that the less profitable firm increases its order quantity as both firms become more risk averse.

Different initial risk aversion. We show that both initially more risk averse and less risk averse firms may increase their order quantity as both firms become more risk averse. Which case applies also depends on the profitability of the firms. In particular, if the profitability ratio is low, only the initially less risk averse firm will increase its order
quantity as both of the firms become more risk averse. If profitability is high, the less risk averse firm increases its order quantity if its competitor's initial risk aversion is low or quite significant, and the more risk averse firm increases its order quantity when its competitor's initial risk aversion is in the intermediate range.

As mentioned before the effect of two counter factors, own's risk aversion increment and spillover demand, cause that one firm increases its order quantity. For example, when the profitability is low, both firms' order quantities are small due to the low profitability. As a result the effect of increasing risk aversion is minimal such that a reduction in overstocking risk could result in increasing the order quantity. It turns out that the initially higher risk averse firm in this case at the equilibrium gets enough spillover demand to sell out and therefore, increasing spillover demand does not change its overstocking risk. However initially less risk averse firm's overstocking risk decreases given the other firm reduces its order quantity. Hence the initially more risk averse firm reduces its order quantity and the initially less risk averse one increases its order quantity.

Different spillover fractions. We show that when both firms are identical except that they have a different spillover rate, the firm with higher or lower spillover rate may increase its order quantity as both firms become more risk averse. Which case applies also depends on the profitability of the firms. When the profitability is very low the firm with less spillover demand will increase its order quantity as both firms become more risk averse. However, when the profitability is in the low to intermediate range, the firm with more spillover demand increases its order quantity as both firms become more risk averse. When both firm are highly profitable neither of the firms increase their order quantity as both become more risk averse.

Correlated demand. We also study the impact of correlated demand. We show that when the demand is perfectly positively correlated, only the more profitable firm or the less risk averse firm may increase its order quantity as both firms become more risk averse. The only case in which a firm increases its order quantity as both firms
become more risk averse, is when it is willing to order the high demand plus the spillover from its competitor. To order such a significant order quantity, the firm should either be significantly more profitable or initially less risk averse. When demand is perfectly negatively correlated, only the less profitable or the more risk averse firm may increase its order quantity. In this case the firm with lower order quantity at the equilibrium increases its order quantity as both firms become more risk averse. To have a lower order quantity at the equilibrium, the firm must have lower profitability or higher initial risk aversion.

We also show that in the two extreme cases of perfect positive and negative correlation, different spillover fractions for otherwise identical firms do not result in an increase in the order quantity of either firm.

2.2 Basic Model

We consider a single-period newsvendor game in stocking decisions between risk-averse duopoly firms which sell substitutable products at fixed prices. Firms are indexed by \( i \in \{A, B\} \), and quantities pertaining to firm \( i \) are denoted by a superscript. Each firm sells a single product. Firms make stocking decisions independently and simultaneously. Firm \( i \) orders \( Q^i \) units by the beginning of the sales period at unit cost \( c^i \), sells its product at unit price \( p^i \) and disposes of leftover inventory at unit salvage value \( s^i \), where \( p^i > c^i > s^i > 0 \). Define the underage cost \( C^i_u = p^i - c^i \), the overage cost \( C^i_o = c^i - s^i \), and let \( K^i = C^i_u + C^i_o \) denote their sum.

Each firm’s total demand equals the sum of its own primary demand plus spillover demand from her competitor. Primary demands do not depend on the order quantities while each firm’s spillover demand equals a deterministic fraction of her competitor’s unsatisfied primary demand. Firm \( i \)’s spillover demand equals a fraction \( b^i > 0 \) of her competitor’s unsatisfied primary demand, where we allow \( b^i \neq b^j \). This model of competition
based on consumer-driven substitution is frequently used in the literature (McGillivray and Silver 1978, Parlar and Goyal 1984, Noonan 1995). It describes situations in which consumers first try to buy the product they prefer, e.g., based on brand loyalty, perceived quality or shopping convenience, and then may substitute a similar product if their first choice is stocked out.

The random variable $D^j(Q^j)$ denotes firm $i$’s total demand as a function of $Q^j$ for $j \neq i$. Let $d_{xy}^i(Q^j)$ denote a total demand realization. We assume that the joint probability distribution of Firm $A$ and $B$’s demand is given, i.e. Firm $A$ and $B$ experience primary demand $d^A_x$ for $x = 1, 2, \ldots, N^A$ and $d^B_y$ for $y = 1, 2, \ldots, N^B$ respectively where $d^A_1 < d^A_2 < \ldots < d^A_{N^A}$ and $d^B_1 < d^B_2 < \ldots < d^B_{N^B}$. Therefore firm $A$ faces a total demand realization $d_{xy}^A(Q^B) = d^A_x + b^A(d^B_y - Q^B)^+$ with probability $q_{xy}$. Note that a given total demand realization may correspond to multiple pairs of primary demand realizations. In this case at most one of these pairs involve no spillover demand, and the other pair(s) involve spillover demand. All of these pairs with spillover continue to yield the same total demand in response to small changes in the rival’s order. Notice that the results of Sections 2.3 and 2.4 generalize in a natural way to the case of arbitrary continuous primary demand distributions. However, we present them for discrete distributions to build on them for the special case of binary demand in Sections 2.5 and 2.6.

**Remark.** The way that the total demand is defined can accommodate different initial demand allocation rules, including but not limited to “Independent Random Demands”, “Deterministic Splitting”, “Simple Random Splitting” and “Incremental Random Splitting” which defined in Lippman and McCardle (1997). See Lippman and McCardle (1997) for a detailed discussion of these initial allocation rules.

Let $\Pi^i(Q^i, d)$ denote firm $i$’s payoff as a function of her order quantity $Q^i$ and the demand realization $d$. It satisfies

$$\Pi^i(Q^i, d) = (C^i_u + C^i_o)d - C^i_o Q^i \mathbb{1}\{d < Q^i\} + C^i_u Q^i \mathbb{1}\{d \geq Q^i\}. \quad (2.1)$$

As noted above we consider risk-averse firms.
Assumption 1. We assume the utility function, $u^i(x)$, has a general form such that $u''(x) > 0$, $u'''(x) < 0$, and $u''''(x)/u''(x)$ is a non-decreasing function of $x$.

Assumption 1 corresponds to non-increasing Arrow-Pratt absolute risk aversion coefficient. Note that Assumption 1 with regard to the utility function is not restrictive and holds for many classes of utility functions such as Constant Absolute Risk Aversion (CARA) utility function

$$U(x) = 1 - \exp(-Rx)$$

Constant Relative Risk Aversion (CRRA) utility function

$$U(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } R > 0, R \neq 1 \\ \ln x & \text{if } R = 1 \end{cases}$$

and Hyperbolic Absolute Risk Aversion (HARA) utility function when $\gamma < 1$

$$U(x) = \frac{1 - \gamma}{\gamma} \left( \frac{ax}{1-\gamma} + b \right)^\gamma, a > 0 \text{ and } \frac{ax}{1-\gamma} + b > 0.$$  

Remark. (1) As $\gamma \to -\infty$ and $b = 1$, HARA utility function converge to CARA. (2) When $b = 0$, and $\gamma \leq 1$, the HARA utility function becomes CRRA. When $b = 0$ and $\gamma < 1$, HARA utility function is equivalent to the following form of CRRA utility function

$$U(x) = \frac{x^{1-R}}{1-R}$$

and when $b = 0$ and $\gamma \to 1$, HARA utility function is equivalent to the logarithmic form of CRRA utility function.

Note that as pointed out before, in case of HARA utility function, we must have:

$$\frac{ax}{1-\gamma} + b > 0 \Leftrightarrow x > -\frac{b(1-\gamma)}{a}.$$  

Since the payoff is:

$$x = ((C_u + C_o)d - C_oQ) \{d < Q\} + C_uQ \{d \geq Q\}$$

we must have:

$$(C_u + C_o)d_{\min} - C_oQ > -\frac{b(1-\gamma)}{a} \Leftrightarrow \left(\frac{C_u}{C_o} + 1\right)d_{\min} + \frac{b(1-\gamma)}{aC_o} > Q$$
and

\[ Q > -\frac{b(1-\gamma)}{aC_u}. \]

By choosing appropriate parameter for HARA utility function the following range is not a restrictive for order quantity:

\[ (\frac{C_u}{C_o} + 1)d_{\text{min}} + \frac{b(1-\gamma)}{aC_o} > Q > -\frac{b(1-\gamma)}{aC_u}. \]

Firm \( i \)'s expected utility function given the order of firm \( j \neq i \) is defined as

\[ U^i(Q^i|Q^j) = E\left[u^i\left(\Pi^i(Q^i, D^i(Q^j))\right)\right] = \sum_x \sum_y q_{xy} u^i(\Pi^i(Q^i, d^i_{xy}(Q^j))). \]

In detail this gives

\[ U^i(Q^i|Q^j) = \sum_x \sum_y q_{xy} \left[u^i(K^i d^i_{xy}(Q^j) - C^i_o Q^j) 1\{d^i_{xy}(Q^j) < Q^i\} + u^i(C^i_u Q^j) 1\{d^i_{xy}(Q^j) \geq Q^i\}\right]. \]

Each firm chooses the order quantity that maximizes the expected utility of its payoff. Let \( f^A(\cdot) \triangleq \arg \max_{Q^A} U^A(Q^A|Q^B) \) denote firm \( A \)'s best response as a function of her competitor’s order quantity \( Q^B \). We denote a Nash equilibrium by \( Q^* = (Q^{A*}, Q^{B*}) \). The Nash equilibrium for a two-player, continuous game is a pair \( (Q^{A*}, Q^{B*}) \) with the property that:

\[ U^A(Q^{A*}|Q^{B*}) \geq U^A(Q^A|Q^{B*}) \quad \text{for all } Q^A \]

\[ U^B(Q^{A*}|Q^{B*}) \geq U^B(Q^{A*}|Q^B) \quad \text{for all } Q^B. \]

### 2.3 Equilibrium Characterization

In this section we initially obtain the unique best response function for a given competitor’s order quantity, and then we show the existence and uniqueness of the equilibrium.

**Best Response.**

The best response problem of firm \( i \) is mathematically equivalent to the problem of a monopoly facing an exogenous random demand \( D^i(Q^i) \). For ease of exposition we first
characterize the solution of this problem for general utility function where \( u' > 0, \quad u'' < 0 \) and \( u''(x)/u'(x) \) is a non-decreasing function of \( x \). We then map it to the duopoly best response. Dropping superscripts, let

\[
U(Q) \triangleq E [u (\Pi (Q, D))] = \sum_{i=1}^{N} q_i \cdot u (\Pi (Q, d_i)) \tag{2.3}
\]

denote the expected utility of a monopoly firm facing exogenous random demand \( D \) where \( d_i > 0 \) and \( q_i \triangleq P(D = d_i) \). Lemma 2.1 summarizes basic properties of \( U(Q) \) and \( Q^* \triangleq \arg \max_{Q \geq 0} U(Q) \).

**Lemma 2.1.** The expected utility function \( U \) and its maximizer \( Q^* \) have the following properties.

1. The expected utility satisfies \( U(Q) = U_k(Q) \) for \( Q \in [d_k, d_{k+1}] \) where \( d_0 \triangleq 0, \quad d_{N+1} \triangleq \infty \) and

\[
U_k(Q) \triangleq \sum_{i=1}^{k} q_i \cdot u((C_u + C_o)d_i - C_o Q) + \left( 1 - \sum_{i=1}^{k} q_i \right) \cdot u(C_u Q), \quad k = 0, 1, 2, \ldots, N. \tag{2.4}
\]

It is continuous and strictly concave. For \( k \in \{1, 2, \ldots, N\} \), \( U \) has left and right derivatives \( U'_- \) and \( U'_+ \) at \( d_k \), is twice continuously differentiable on \( (d_k, d_{k+1}) \) and satisfies \( U''(Q) = U''_k(Q) < 0 \) if \( Q \in (d_k, d_{k+1}) \) and \( U''_-(d_k) = U''_{k-1}(d_k) > U''_+(d_k) \).

2. The optimal order quantity \( Q^* \) is unique. It satisfies \( Q^* \in [d_1, d_N] \) and

\[
Q^* = \max \left\{ Q \geq 0 : \sum_{i=1}^{N} 1\{d_i < Q\} \cdot q_i \cdot \left( \frac{u'((C_u + C_o)d_i - C_o Q)}{u'(C_u Q)} C_o + C_u \right) \leq C_u \right\}. \tag{2.5}
\]

Lemma 2.1 clearly applies for any demand distribution. For given \( Q^i \), the total demand distribution \( D^i(Q^i) \) is induced and hence the best response function of firm \( i \), \( f^i(Q^i) \), is well defined and we can translate the order prescription of Lemma 2.1 into the duopoly best response.
Lemma 2.2. The expected utility function $U^i(Q^i|Q^j)$ and the best response function $f^i(Q^j)$ of firm $i$ are continuous and piecewise differentiable in $Q^j$.

1. If firm $j$’s order quantity is less than its own maximum primary demand i.e. $Q^j < d^j_N$, and firm $i$’s order quantity is larger than or equal to its minimum total demand that involves spillover demand i.e.

$$Q^i \geq \min_x \left( d^i_x + b^i \left( \tilde{d}^i (d^i_x, Q^j) - Q^j \right) \right)$$

where $\tilde{d}^i (d^i_x, Q^j) = \min_z \left[ d^i_z : 1_{\{d^i_z > Q^j, q_x > 0\}} \right]$, then $U^i(Q^i|Q^j)$ is strictly decreasing in $Q^j$. Otherwise, $U^i(Q^i|Q^j)$ is constant in $Q^j$.

2. The left derivative $f^-_i$ and the right derivative $f^+_i$ of the best response function satisfy

$$0 \geq f^-_i, f^+_i \geq -b^i. \quad (2.6)$$

Note that if the primary demands of both firms were uncorrelated, $\tilde{d}^i (d^i_x, Q^j) = \min_z [d^i_z : d^i_z > Q^j]$.

Equilibrium Existence and Uniqueness.

Proposition 2.1. There exists a Nash equilibrium. If $b^A b^B < 1$ then it is unique.

Remark and assumption. If $b^A = b^B = 1$, the equilibrium is not necessarily unique. There may exist a continuum of equilibria whereby both firms order minimum primary demand plus spillover. For simplicity we henceforth assume that $b^A b^B < 1$.

2.4 Impact of Risk Aversion on Equilibrium Orders

Having established existence and uniqueness of the equilibrium, we now turn to the main question of this chapter: how does a change in risk aversion affect the equilibrium order quantities? We address this question by studying the comparative statics of the order equilibrium with respect to an infinitesimal increase in both firms’ risk aversion parameters. Note that based on Pratt Theorem (see Pratt (1964)), increasing risk aversion
is a concave transformation of the utility function which is equivalent to an increasing Arrow-Pratt absolute risk aversion coefficient for given payoff realization. For utility functions with one parameter such as CARA or CRRA to increase Arrow-Pratt absolute risk aversion coefficient, there is only one parameter that can be changed. However for utility function with multiple parameter such as HARA, increasing Arrow-Pratt absolute risk aversion coefficient can be potentially achieved by changing many parameters. In such cases, we assume that all parameters are fixed except for one.

Let \( R = (R_A, R_B) \) denote the vector of initial risk aversion parameters and \( Q^*(R) = (Q_{is}(R), Q_{is}(R)) \) be the corresponding equilibrium. Note that \( Q^*(R) \) is only piecewise differentiable, but it is continuous, and its left and right partial derivatives with respect to \( R \) are well defined.

Our analysis focuses on the question: which firm, if any, increases its equilibrium order quantity as both firms’ risk aversion increases, and under what conditions? Mathematically, under what conditions does the following hold?

\[
\frac{\partial Q_{is}^*(R)}{\partial R_A} + \frac{\partial Q_{is}^*(R)}{\partial R_B} > 0, \quad \text{for } i = A \text{ or } i = B, \tag{2.7}
\]

where \( \frac{\partial Q_{is}^*(R)}{\partial R_A}, \frac{\partial Q_{is}^*(R)}{\partial R_B} \) denote the right partial derivatives of \( Q_{is}^*(R) \).

Theorem 2.1 and Corollary 2.1 establish necessary and sufficient conditions for (2.7) in general form. This analysis formally identifies two countervailing effects that determine the sensitivity of each firm’s equilibrium order to an increase in risk aversion, the own risk aversion and the demand substitution effect.

**Theorem 2.1.** Fix a risk aversion vector \( R > 0 \) and the corresponding equilibrium \( Q^*(R) \). Let the function \( g^i(Q^{-i}, R^i) \) be a piece of the firm \( i \) best response function \( f^i \) that satisfies the equilibrium conditions

\[
Q_{is}^*(R) = g^i\left(Q^{-is}(R), R^i\right) = f^i\left(Q^{-is}(R), R^i\right), \quad i \in \{A, B\}. \tag{2.8}
\]

The firm \( i \) equilibrium order increases as both risk aversion parameters increase, i.e.,

\[
\frac{\partial Q_{is}^*(R)}{\partial R_A} + \frac{\partial Q_{is}^*(R)}{\partial R_B} > 0, \tag{2.9}
\]
if and only if the following two conditions hold:

1. The functions $g^i(Q^i, R^i)$ and $g^{-i}(Q^i, R^{-i})$ agree with the firms’ best response functions for larger risk aversion in some neighborhood of $R$: for some $\tilde{\delta} > 0$ and $\delta \in [0, \tilde{\delta})$,

$$Q^{i*}(R(\delta)) = g^i(Q^{-i*}(R(\delta)), R^i(\delta)) = f^i(Q^{-i*}(R(\delta)), R^i(\delta)), \ i \in \{A, B\},$$

(2.10)

where $R(\delta) = R + \delta e$ and $e = (1, 1)$.

2. The partial derivatives of $g^i(Q^i, R^i)$ and $g^{-i}(Q^i, R^{-i})$ satisfy

$$\frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial R^{-i}} \frac{\partial g^i(Q^{-i*}(R), R^i)}{\partial Q^{-i}} > -\frac{\partial g^i(Q^{-i*}(R), R^i)}{\partial R^i}. \quad (2.11)$$

Condition (2.10) is technical in nature: it is only required since the best response functions are piecewise functions. The important condition is (2.11). It formally identifies two countervailing effects that determine the sensitivity of each firm’s equilibrium order to an increase in both firms’ risk aversion level:

1. The direct own risk aversion effect. An increase in firm $i$’s own risk aversion reduces its order, holding its rival’s order and risk aversion level fixed. The RHS of (2.11) measures the magnitude of this order reduction.

2. The indirect demand substitution effect. An increase in the risk aversion level of firm $i$’s rival increases its own order, holding its own risk aversion level fixed. The LHS of (2.11) measures the magnitude of this effect. It is the product of two factors, the reduction of the rival’s order size in response to its higher risk aversion, and the increase in firm $i$’s order to capture the resulting higher spillover demand.

Firm $i$’s equilibrium order increases in response to higher risk aversion if and only if the demand substitution effect is larger than the own risk aversion effect.

The following Corollary provides two general observations that follow from Theorem 2.1.
Corollary 2.1. Fix a risk aversion vector $\mathbf{R} > \mathbf{0}$ and the corresponding equilibrium $Q^*(\mathbf{R})$.

1. If firms are symmetric, then their equilibrium orders do not increase as both risk aversion parameters increase, i.e.,

$$
\frac{\partial Q^i_+ (\mathbf{R})}{\partial R^A} + \frac{\partial Q^{-i}_+ (\mathbf{R})}{\partial R^B} \leq 0, \ i \in \{A, B\}. \tag{2.12}
$$

2. If the firm $i$ equilibrium order increases as both risk aversion parameters increase, i.e.,

$$
\frac{\partial Q^i_+ (\mathbf{R})}{\partial R^A} + \frac{\partial Q^{-i}_+ (\mathbf{R})}{\partial R^B} > 0, \tag{2.13}
$$

then the equilibrium order of its rival does not increase.

These results are quite intuitive. Since each firm only receives a fraction of its rival’s unmet demand, it finds it optimal to increase its order by less than the reduction in its rival’s order; see Part 2 of Lemma 2.2. Therefore, (2.11) can only hold for a firm if its own risk aversion effect is strictly smaller in magnitude than that of its rival. Clearly, this can only hold for one of the firms. It holds for neither firm if they are symmetric, for then the initial equilibrium is symmetric and the own risk aversion effect is equal for both firms.

To put Theorem 2.1 and Corollary 2.1 in context, it is well known that the optimal order quantity of a single-product monopoly decreases in the risk aversion level; e.g., see Eeckhoudt et al. (1995). In this case, there is no demand substitution effect. Choi et al. (2008, 2010) study a multi-product risk averse monopoly newsvendor without focusing on demand substitution effects. The interaction between the own risk aversion effect and the demand substitution effect in the presence of competition appears not to have been studied so far.
2.5 Uncorrelated Binary Primary Demand with CARA Utility

To obtain and understand specific insights with regard to the impact of risk aversion, in this section we study the problem for a specific demand distribution and utility function. We assume that the primary demand distributions are binary, uncorrelated and identically distributed and each firm has a Constant Absolute Risk aversion (CARA) utility function. The CARA utility function is commonly used in the literature. It offers tractability, which helps us obtain specific insights into the impact of risk aversion on equilibrium orders. In Section 2.5.1, first, in light of Lemma 2.1, we obtain the best response function, and then based on that, we identify 17 distinct cases as possible equilibria, each corresponding to a particular subset of the order space. We, then, in Section 2.5.2, study the impact of risk aversion. In particular, we identify 6 cases out of those 17 cases that one of the firm may increase its order quantity as both firms become more risk averse. In Sections 2.5.3 to 2.5.5, we specify which firm raises its order quantity as both firms become more risk averse for three scenarios of asymmetric firms that differ in one of the following attributes: the profitability of their product, measured by their under- to overstocking cost ratio, their risk aversion level, and their spillover demand fraction. Also note that latter in Section 2.6 we relax the assumption of uncorrelated primary demand and study the impact of primary demand correlation.

2.5.1 Model and Equilibrium Characterization

In this section, we assume that the primary demand of each firm is $d_L$ with probability $q_L$ and $d_H > d_L$ with probability $q_H = 1 - q_L$. The assumption of binary primary demand yields explicit equilibrium characterizations, which allows us to gain specific insights on how risk aversion affects the equilibrium depending on the model parameters.

Let $d_{xy}(Q)$ denote a demand realization and $q_{xy} = q_xq_y$ the corresponding probability
where \( x, y \in \{ L, H \} \), \( x \) is firm \( i \)'s primary demand realization and \( y \) her competitor’s. We assume w.l.o.g. that \( Q^A, Q^B \geq d_L \): firms have no incentive to order less than \( d_L \). Since the primary demands are uncorrelated, the p.d.f. of firm \( A \) total demand \( D^A(Q^B) \) satisfies:

\[
P(D^A(Q^B) = d) = \begin{cases} 
q_{LL} = q_L^2, & d = d^{A}_{LL}(Q^B) = d_L \\
q_{LH} = q_Lq_H = q_L(1-q_L), & d = d^{A}_{LH}(Q^B) = d_L + b^A[d_H - Q^B]^+ \\
q_{HL} = q_Hq_L = (1-q_L)q_L, & d = d^{A}_{HL}(Q^B) = d_H \\
q_{HH} = q_H^2 = (1-q_L)^2, & d = d^{A}_{HH}(Q^B) = d_H + b^A[d_H - Q^B]^+ 
\end{cases} .
\]

(2.14)

If \( Q^B < d_H \), firm \( A \) receives spillover demand if her competitor’s primary demand is high. Otherwise, she adopts a monopoly strategy.

As mentioned before, we model the preferences of each firm by a constant absolute risk aversion (CARA) utility function, i.e. \( u^i(x) = 1 - \exp(-R^i x) \) where \( R^i > 0 \) is the risk aversion rate.

If \( Q^i \geq d_H \) or \( b^i = 0 \) then firm \( i \) receives no spillover demand, \( d^i_{xL}(Q^i) = d^i_{xH}(Q^i) = d_x \) and \( q_{xL} + q_{xH} = q_x \) for \( x \in \{ L, H \} \). In this case

\[
U^i(Q^i | Q^j) = \sum_{x \in \{ L, H \}} q_x \left[ u^i(K^i d_x - C_o Q^i) 1\{d_x < Q^i\} + u^i(C_u Q^i) 1\{d_x \geq Q^i\} \right].
\]

First consider how Lemma 2.1 specializes for CARA utility given by \( u(x) = 1 - \exp(-Rx) \).

**Lemma 2.3.** To compute \( Q^* \) in Lemma 2.1, consider equation (2.4) and let \( Q^*_k \triangleq \arg\max_Q U^i_k(Q) \) for \( k \in \{1, 2, ..., N-1\} \) where \( Q^*_k \) is the unique solution of \( U^i_k(Q) = 0 \) and \( Q^*_0 \triangleq \infty > Q^*_1 > Q^*_2 > ... > Q^*_N-1 \). The optimal order quantity is \( Q^* = \min(Q^*_m, d_{m+1}) \) where \( m \triangleq \max\{0 \leq k \leq N-1 : Q^*_k \geq d_k\} \).

Letting \( K = C_u + C_o \), the maximizers \( Q^*_k \) from above Lemma

\[
U^i_k(Q^*_k) = 0 \iff Q^*_k = \frac{1}{RK} \ln \left( \frac{C_u}{C_o \sum^{k} _{i=1} q_i \exp(-RK d_i)} \right), \quad k \in \{0, 1, 2, ..., N-1\},
\]

(2.15)
where \( Q_0 = \infty \). The optimal order quantity \( Q^* \) equals \( Q^*_k \) if and only if \( d_k \leq Q^*_k \leq d_{k+1} \), or equivalently:

\[
\sum_{i=1}^{k} q_i \exp(-RKd_i) \exp(RKd_k) \leq \frac{C_u}{C_o} \leq \sum_{i=1}^{k} q_i \exp(-RKd_i) \exp(RKd_{k+1}).
\]

(2.16)

The optimal order is \( Q^* = \min(Q^*_m, d_{m+1}) \) where

\[
m = \max \left\{ 0 \leq k \leq N - 1 : \sum_{i=1}^{k} q_i \exp(-RKd_i) \exp(RKd_k) \leq \frac{C_u}{C_o} \right\}. \tag{2.17}
\]

The optimal order is fully specified by the model primitives: underage and overage costs, the demand distribution and the risk aversion parameter. To complete the translation to the best response, add superscripts and replace the generic demand distribution by (2.14), to obtain the following Lemma. For simplicity it suppresses the dependence on \( Q^B \).

**Lemma 2.4.** Firm A’s best response \( f^A(Q^B) \) is unique. Let \( \overline{R}^i = R^i K^i \) for \( i \in \{A, B\} \).

1. If \( Q^B \geq d_H \), then \( f^A(Q^B) \) is constant in \( Q^B \) and satisfies

\[
f^A(Q^B) = \begin{cases} 
   d_L, & \text{if} \quad \frac{C_A}{C_o} \leq t_L^A \exp(\overline{R}^A d_L) \\
   \frac{1}{\overline{R}^A} \ln \left( \frac{C_A}{C_o} \frac{1}{t_L^A} \right), & \text{if} \quad t_L^A \exp(\overline{R}^A d_L) \leq \frac{C_A}{C_o} \leq t_L^A \exp(\overline{R}^A d_H), \\
   d_H, & \text{if} \quad t_L^A \exp(\overline{R}^A d_H) \leq \frac{C_A}{C_o}
\end{cases}
\]

(2.18)

where \( t_L^A \triangleq q_L \exp(-\overline{R}^A d_L) / (1 - q_L) \).
2. If \( d_L \leq Q^B \leq d_H \), then \( f^A(Q^B) \) satisfies

\[
f^A(Q^B) = \begin{cases} 
    d^A_{LL}, & t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \leq C^A \frac{C^A}{C^B} \leq t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \\
    \frac{1}{R^A} \ln\left(\frac{C^A}{C^B} \frac{1}{t^A_{LL}}\right), & t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \leq C^A \frac{C^A}{C^B} \leq t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \\
    d^A_{LH}, & t^A_{LH} \exp(\overline{R}^A d^A_{LH}) \leq C^A \frac{C^A}{C^B} \leq t^A_{LH} \exp(\overline{R}^A d^A_{LH}) \\
    \frac{1}{R^A} \ln\left(\frac{C^A}{C^B} \frac{1}{t^A_{LH}}\right), & t^A_{LH} \exp(\overline{R}^A d^A_{LH}) \leq C^A \frac{C^A}{C^B} \leq t^A_{LH} \exp(\overline{R}^A d^A_{LH}) \\
    d^A_{HH}, & t^A_{HH} \exp(\overline{R}^A d^A_{HH}) \leq C^A \frac{C^A}{C^B} \leq t^A_{HH} \exp(\overline{R}^A d^A_{HH}) \\
\end{cases}
\]

(2.19)

where the demand points are

\[d^A_{LL} = d_L < d^A_{LH} = d_L + b^A(d_H - Q^B) < d^A_{HH} = d_H < d^A_{HH} = d_H + b^A(d_H - Q^B),\]

the thresholds \( t^A_{LL}, t^A_{LH} \) and \( t^A_{HH} \) are

\[
t^A_k \triangleq \frac{\sum_{x,y \in \{L,H\}} q_x q_y \exp(-\overline{R}^A d^A_{xy}) \cdot 1\{d^A_{xy} \leq d^A_k\}}{1 - \sum_{x,y \in \{L,H\}} q_x q_y \cdot 1\{d^A_{xy} \leq d^A_k\}}, k \in \{LL, LH, HL\}, \tag{2.20}
\]

and \( t^A_{LL} \) is constant in \( Q^B \), whereas \( t^A_{LH} \) and \( t^A_{HH} \) increase in \( Q^B \).

Figure 2.1 shows how the primary distribution and the spillover fractions partition the order space. We discuss the figure from firm A’s perspective. If firm B orders at least its maximum primary demand, \( Q^B \geq d_H \), then firm A does not get any spillover demand. It adopts a monopoly strategy based only on its primary demand distribution with mass points \( d_L \) and \( d_H \). However, if firm B orders strictly less than its maximum primary demand, i.e. \( Q^B \in [d_L, d_H) \), then firm A gets \( b^A(d_H - Q^B) \) units of spillover demand from its competitor with probability \( q_H \). For illustration, suppose that firm B orders the quantity indicated by the dashed horizontal line. As a result firm A’s total demand has four possible realizations: low primary demand \( d_L \), shown as point a; low primary plus spillover demand, \( d_L + b^A(d_H - Q^B) \), point b; high primary demand \( d_H \), point c; and high primary plus spillover demand, \( d_H + b^A(d_H - Q^B) \), point d.
The impact of a small change in firm $B$’s order quantity on firm $A$’s marginal underage and overage risks is sensitive to how much firm $A$ orders. If firm $A$ orders less than low primary plus spillover demand, $Q^A < d_L + b^A(d_H - Q^B)$, then $A$ sells out if either firm experiences high demand. In this case a small change in $Q^B$ has no impact on $A$’s profitability since it does not alter the amount of spillover demand that $A$ can satisfy. The profitability of firm $A$ only depends on $Q^B$ if it orders at least low primary plus spillover demand, $Q^A \geq d_L + b^A(d_H - Q^B)$. In this case, if firm $A$ has low primary demand then it leftover inventory ($Q^A - d_L$) exceeds her potential spillover demand. If $Q^A < d_H$, then $A$ sells out only if its own primary demand is high, so it benefits from spillover demand only if its own primary demand is low. If firm $A$ orders more than high primary demand $Q^A \geq d_H$ then it sells out only if both firms’ primary demands are high, so it benefits from spillover demand regardless of her own demand realization. Note that if $Q^A \leq d_H + b^A(d_H - Q^B)$ then the impact of a marginal change in $Q^B$ on $A$’s profitability only comes from higher sales under low primary demand: if firm $A$ has high primary demand then its overstock is $Q^A - d_H$ without spillover demand and zero with spillover demand, independent of $Q^B$.

![Figure 2.1: Primary demand distribution and spillover fractions partition the order space.](image-url)
Remark and assumption. If \( b^A = b^B = 1 \), the equilibrium is unique if \( Q^{A*} + Q^{B*} \neq d_L + d_H \). Otherwise there may exist a continuum of equilibria along the line \( Q^A + Q^B = d_L + d_H \). For simplicity we henceforth assume that \( b^A, b^B < 1 \).

The impact of risk aversion on the equilibrium orders depends on the location of the initial equilibrium, relative to the order space partition portrayed in Figure 2.1. As a stepping stone for the analysis of the risk aversion impact, Proposition 2.2 categorizes the possible equilibria into 17 distinct cases, each corresponding to a particular subset of the order space. For concreteness Figure 2.2 illustrates these cases from the perspective of firm A, and the Proposition specifies the cases accordingly.

![Figure 2.2: Equilibrium cases of Proposition 2.2.](image)

**Proposition 2.2.** The location of the equilibrium orders \((Q^{A*}, Q^{B*})\) can be categorized
into the following cases. For concreteness, let \( i = A \) and \( -i = B \).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Case} & \text{Firm A Equilibrium Order } Q_A^* & \text{Conditions} & \text{Firm B Equilibrium Order } Q_B^* \\
\hline
1 & d_{LH}^A (Q_B^*) & Q_{LH}^A (Q_B^*) \leq d_{LH}^A (Q_B^*) < Q_{LL}^A & Q_{LL}^B \\
& & & d_L < Q_{LL}^B \leq d_{LH}^B (Q_A^*) \\
2 & d_{LH}^A (Q_B^*) & Q_{LH}^A (Q_B^*) \leq d_{LH}^A (Q_B^*) < O_{LL}^A & Q_{LH}^B (Q_A^*) \\
& & & d_{LH}^B (Q_A^*) < Q_{LH}^B (Q_A^*) < d_H \\
3 & Q_{LH}^A (Q_B^*) & d_{LH}^A (Q_B^*) < Q_{LH}^A (Q_B^*) \leq d_H & Q_{LH}^B (Q_A^*) \\
& & & d_{LH}^B (Q_A^*) < Q_{LH}^B (Q_A^*) \leq d_H \\
4 & Q_{LH}^A (Q_B^*) & d_{LH}^A (Q_B^*) < Q_{LH}^A (Q_B^*) \leq d_H & Q_{LL}^B \\
& & & d_L < Q_{LL}^B \leq d_{LH}^B (Q_A^*) \\
5 & Q_{HL}^A (Q_B^*) & d_H \leq Q_{HL}^A (Q_B^*) \leq d_{HH}^A (Q_B^*) & Q_{M}^B \\
& & & d_L < Q_{M}^B < d_H \\
6 & d_{HH}^A (Q_B^*) & d_{HH}^A (Q_B^*) < Q_{HL}^A (Q_B^*) & Q_{M}^B \\
& & & d_L < Q_{M}^B < d_H \\
7 & d_{HH}^A (Q_B^*) & d_{HH}^A (Q_B^*) < Q_{HL}^A (Q_B^*) & d_L \\
& & & Q_{M}^B \leq d_L \\
8 & Q_{HL}^A (Q_B^*) & d_H < Q_{HL}^A (Q_B^*) \leq d_{HH}^A (Q_B^*) & d_L \\
& & & Q_{M}^B \leq d_L \\
9 & d_H & d_H \leq Q_{M}^A & d_H \\
& & & d_H \leq Q_{M}^B \\
10 & d_H & Q_{HL}^A (Q_B^*) < d_H < Q_{LH}^A (Q_B^*) & Q_{M}^B \\
& & & d_L < Q_{M}^B < d_H \\
11 & d_H & Q_{HL}^A (Q_B^*) \leq d_H < Q_{LH}^A (Q_B^*) & d_L \\
& & & Q_{M}^B \leq d_L \\
12 & Q_{LH}^A (Q_B^*) & d_{LH}^A (Q_B^*) < Q_{LH}^A (Q_B^*) \leq d_H & d_L \\
& & & Q_{LL}^B \leq d_L \\
13 & d_{LH}^A (Q_B^*) & Q_{LH}^A (Q_B^*) \leq d_{LH}^A < Q_{LL}^A & d_{LH}^B (Q_A^*) \\
& & & O_{LH}^B (Q_A^*) \leq d_{LH}^B (Q_A^*) < Q_{LL}^B \\
14 & d_{LH}^A (Q_B^*) & Q_{LH}^A \leq d_{LH}^A < Q_{LL}^A & d_L \\
& & & Q_{LL}^B \leq d_L \\
15 & Q_{LL}^A & d_{LL}^A < O_{LL}^A \leq d_{LH}^A & Q_{LL}^B \\
& & & d_L < Q_{LL}^B \leq d_{LH}^B (Q_A^*) \\
16 & Q_{LL}^A & d_{LL}^A < O_{LL}^A \leq d_{LH}^A & d_L \\
& & & Q_{LL}^B \leq d_L \\
17 & d_L & Q_{LL}^A \leq d_L & d_L \\
& & & Q_{LL}^B \leq d_L \\
\hline
\end{array}
\]

The demand points satisfy

\[
d_L < d_{LH}^A (Q) = d_L + b^i (d_H - Q) < d_H < d_{HH}^A (Q) = d_H + b^i (d_H - Q) \text{ for } Q \in [d_L, d_H),
\]

(2.21)
and the functions $Q_M^i$, $Q_{LL}^i$, $Q_{LH}^i(Q)$, and $Q_{HL}^i(Q)$ are defined as

$$Q_M^i \triangleq \frac{1}{R^i} \ln \left( \frac{C_{o}^{i} - 1}{C_{o}^{i} q_{L} \exp(-R d_{L})} \right),$$  \hspace{1cm} (2.22)$$

$$Q_{LL}^i \triangleq \frac{1}{R^i} \ln \left( \frac{C_{o}^{i} - 1}{C_{o}^{i} q_{LL} \exp(-R d_{L})} \right),$$  \hspace{1cm} (2.23)$$

$$Q_{LH}^i(Q) \triangleq \frac{1}{R^i} \ln \left( \frac{1 - q_{LL} - q_{LH}}{C_{o}^{i} q_{LL} \exp(-R d_{L}) + q_{LH} \exp(-R d_{L}^i(Q))} \right),$$  \hspace{1cm} (2.24)$$

$$Q_{HL}^i(Q) \triangleq \frac{1}{R^i} \ln \left( \frac{1 - q_{LL} - q_{LH} - q_{HL}}{C_{o}^{i} q_{LL} \exp(-R d_{L}) + q_{LH} \exp(-R d_{L}^i(Q)) + q_{HL} \exp(-R d_{H})} \right).$$  \hspace{1cm} (2.25)$$

Note that $Q_{LL}^i > Q_{LH}^i(Q) > Q_{HL}^i(Q)$ and $Q_{LL}^i(d_H) = Q_M^i$.

Proposition 2.2 serves two purposes for our analysis.

First, it helps us identify the cases in which higher risk aversion may yield a strictly higher equilibrium order for one of the firms: they are the cases 1-6, as we show in Section 2.5.2. In all other cases, higher risk aversion implies (weakly) lower equilibrium orders for both firms.

Second, Proposition 2.2 yields specific parameter conditions that must hold in each equilibrium case. Combined with the forthcoming analysis, these conditions allow us to systematically analyze and identify the impact of higher risk aversion as a function of the model parameters. For illustration, consider Case 1 of Proposition 2.2. Firm $A$ orders its low primary plus spillover demand, and firm $B$ orders more than low primary demand but less than low primary plus spillover demand. Substituting from (2.21) and (2.23) yields the equilibrium order quantities in closed form:

$$Q^A = d_{LH}^A (Q_{LL}^B) = d_L + b^A \left( d_H - \frac{1}{R^B} \ln \left( \frac{C_{o}^{B} - 1}{C_{o}^{B} q_{LL} \exp(-R d_{L})} \right) \right)$$

and

$$Q^B = Q_{LL}^B = \frac{1}{R^B} \ln \left( \frac{C_{o}^{B} - 1}{C_{o}^{B} q_{LL} \exp(-R d_{L})} \right).$$

Substituting these quantities into the equilibrium conditions $Q_{LL}^A (Q^B) \leq d_{LH}^A (Q^B) < Q_{LL}^A$ and $d_L < Q_{LL}^B \leq d_{LH}^A (Q^A)$ yields conditions that only involve the problem parameters. All other cases, except for cases 2 and 3, also yield closed-form expressions for
the equilibrium order quantities. The equilibrium equations for cases 2 and 3 are easily solved numerically.

The following intuitive equilibrium properties are straightforward from Lemmas 2.4-2.2.

**Corollary 2.2.** If both firms are identical except for:

1. Their profitability, measured by the ratio $C^i_u/C^i_o$, while $C^i_u + C^i_o = C^{-i}_u + C^{-i}_o$, then the equilibrium order of the firm with the higher ratio (weakly) exceeds that of its rival.

2. Their risk aversion rate $R^i$, then the equilibrium order of the less risk-averse firm (weakly) exceeds that of its rival.

3. Their spillover fraction $b^i$, then the equilibrium order of the firm with the higher spillover fraction (weakly) exceeds that of its rival.

### 2.5.2 Impact of Risk Aversion on Equilibrium Orders

We turn to the main question of this chapter: how does the order quantity equilibrium of risk-averse duopoly firms change in response to a change in their risk aversion?

Theorem 2.2 identifies, for our demand model with i.i.d. binary primary demand and deterministic spillover fractions, which equilibrium cases of Proposition 2.2 satisfy the conditions of Theorem 2.1, and under what additional conditions. This analysis shows how the impact of increasing risk aversion depends on the location of the initial equilibrium, and it yields specific conditions that can be evaluated numerically to determine the set of model parameters for which (2.7) holds.

We illustrate and discuss the results of Theorem 2.2 with numerical examples for three scenarios of asymmetric firms that differ in exactly one of the following attributes:

(i) the profitability of their product, measured by their under- to overstocking cost ratio $C^i_u/C^i_o$, (ii) their risk aversion parameter $R^i$, and (iii) their spillover demand fraction $b^i$. 
In our model framework, the magnitudes of the own risk aversion and demand substitution effects are closely linked to the location of the initial equilibrium relative to the order space partition. By exploiting these relationships, we can narrow down the set of all possible equilibrium cases identified in Proposition 2.2 to only a handful of relevant ones.

**Theorem 2.2.** Fix $R > 0$ and the corresponding equilibrium $Q^*(R)$. For concreteness, let $i = A$ and $-i = B$. The equilibrium order of firm $A$ increases as both firms become more risk averse,

$$\frac{\partial Q^A_{s} (R)}{\partial R^A} + \frac{\partial Q^B_{s} (R)}{\partial R^B} > 0, \quad (2.26)$$

if and only if $Q^*(R)$ satisfies one of cases 1-6 of Proposition 2.2 and further conditions in cases 3-5.

1. Firm $A$ orders minimum demand plus spillover, firm $B$ between minimum demand and minimum demand plus spillover:

$$Q^A_{LH} (Q^B) \leq Q^A = d^A_{LH} (Q^B) < Q^A_{LL} \quad \text{and} \quad d^B < Q^B = Q^B_{LL} \leq d^B_{LH} (Q^A).$$

2. Firm $A$ orders minimum demand plus spillover, firm $B$ between minimum demand plus spillover and maximum primary demand:

$$Q^A_{LH} (Q^B) \leq Q^A = d^A_{LH} (Q^B) < Q^A_{LL} \quad \text{and} \quad d^B_{LH} (Q^A) < Q^B = Q^B_{LH} (Q^A) < d_H.$$

3. Both firms order between minimum demand plus spillover and maximum primary demand:

$$d^i_{LH} (Q^{-i}) < Q^i = Q^i_{LH} (Q^{-i}) < d_H, \quad i \in \{A, B\},$$

and moreover,

$$\frac{\partial Q^B_{LH} (Q^A, R^B)}{\partial R^B} \frac{\partial Q^A_{LH} (Q^B, R^A)}{\partial Q^B} > - \frac{\partial Q^A_{LH} (Q^B, R^A)}{\partial R^A}. \quad (2.27)$$
4. Firm A orders between minimum demand plus spillover and maximum primary demand, firm B between minimum demand and minimum demand plus spillover:

\[ d_{LH}^A (Q^B*) < Q^A* = Q_{LH}^A (Q^B*) \leq d_H \text{ and } d_L < Q^B* = Q_{LL}^B \leq d_{LH}^B (Q^A), \]

and moreover,

\[ Q_{LL}^B (R^B) \frac{\partial Q_{LH}^A (Q^B*, R^A)}{\partial Q^B} > -\frac{\partial Q_{LH}^A (Q^B*, R^A)}{\partial R^A}. \tag{2.28} \]

5. Firm A orders between maximum primary demand and maximum demand, firm B between minimum and maximum primary demand:

\[ d_H \leq Q^A* = Q_{HL}^A (Q^B*) \leq d_{HH}^A (Q^B*) \text{ and } d_L < Q^B* = Q_{M}^B < d_H, \]

and moreover,

\[ Q_{M}^B (R^B) \frac{\partial Q_{HL}^A (Q^B*, R^A)}{\partial Q^B} > -\frac{\partial Q_{HL}^A (Q^B*, R^A)}{\partial R^A}. \tag{2.29} \]

6. Firm A orders maximum primary demand plus spillover, firm B between minimum and maximum primary demand:

\[ Q^A* = d_{HH}^A (Q^B*) < Q_{HL}^A (Q^B*) \text{ and } d_L < Q^B* = Q_{M}^B < d_H. \]

Remark. Except for the equilibrium conditions for Case 3, all the conditions in the Theorem can be translated into conditions that only involve the model parameters. See Proposition 2.2 for the equilibrium conditions, and the proof of 2.2 for the additional conditions (2.27)-(2.29).

Figure 2.3 illustrates the equilibrium cases identified in Theorem 2.2 for firm A.

In cases 1, 2, and 6, firm A always increases its order in response to higher risk aversion levels. In these cases, firm A’s initial equilibrium equals low or high primary demand plus spillover demand, and it is locally insensitive to higher risk aversion. Its own risk aversion effect is therefore zero, whereas the demand substitution effect is strictly positive.
In cases 3, 4, and 5, firm A may, but need not, increase its equilibrium order. Firm A increases its order if and only if the appropriate additional condition from (2.27)-(2.29) holds, which measure the magnitudes of the own risk aversion and the demand substitution effects for each case.

By Theorem 2.2, neither firm increases its equilibrium order in response to higher risk aversion levels in the remaining eleven equilibrium cases of Proposition 2.2. We briefly explain why.

In seven of these cases, one firm, call it $i$, orders the minimum demand, $d_L$. At this level, firm $i$ has no overstocking risk. An increase in spillover demand from its rival has no impact on firm $i$’s over- and understocking risks, so it does not change firm $i$’s order. Since firm $i$ will not order less than $d_L$, its rival does not increase its order in response to higher risk aversion.

In case 9, firms are symmetric, and Part 1 of Corollary 2.1 applies.

In case 10, firm $A$ initially orders its maximum primary demand, and it is locally insensitive to higher risk aversion. Therefore, it does not change its order in response to higher risk aversion, while the order of its rival decreases.

In cases 13 and 15, both firms initially order less than or equal to their minimum demand plus spillover demand. In these cases, the order quantities of both firms are so low that their spillover demand is enough for them to sell out. A small reduction in one firm’s order quantity therefore does not alter the overstocking risks of its rival, and the demand substitution effect is zero. Each firm maintains or reduces its order, depending on whether its own risk aversion effect is zero or not.

In the next Sections 2.5.3-2.5.5 we illustrate and discuss the cases identified in Theorem 2.2 with numerical examples for three scenarios of firm asymmetries, considering in turn firms that differ in (i) the profitability of their product, measured by their under-to overstocking cost ratio $C_u^i/C_o^i$, (ii) their risk aversion parameter $R^i$, and (iii) their spillover demand fraction $b^i$. 
Figure 2.3: Theorem 2. Equilibrium cases in which one of the firm always increases (cases 1, 2, 6) or may increase (cases 3, 4, 5) its order in response to higher risk aversion levels.

2.5.3 Different Profitability ($C^i_u / C^i_o$)

A key measure in the newsvendor model is the ratio of under- to overstocking cost. In this Section we consider firms that are identical, except for the profitability of their products, measured by their $C^i_u / C^i_o$ ratios. However, we assume that the difference between price and salvage value is the same for both firms: $C^i_u + C^i_o = K^i = K^{-i}$. In settings with equal prices, the firm with the higher $C^i_u / C^i_o$ ratio has the lower unit cost. In settings with equal unit costs, the firm with the higher $C^i_u / C^i_o$ ratio commands a higher price and salvage value. Our results show that the firm that responds with a higher equilibrium order to an increase in risk aversion can be the one with the higher or the one with the lower under- to overstocking cost ratio. We identify which case applies under what conditions and explain why.

Corollary 2.3. Suppose that both firms are identical except for their profitability, measured by their $C^i_u / C^i_o$ ratios, but $C^i_u + C^i_o = K^i = K^{-i}$. By Part 1. of Corollary 2.2, the equilibrium order of the firm with the higher ratio (weakly) exceeds that of its rival.
1. In the following cases of Theorem 2.2, as both firms become more risk averse, the more profitable firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes larger.

(a) Case 1. The more profitable firm orders minimum demand plus spillover, its rival orders between its minimum demand and minimum demand plus spillover.

(b) Case 4. The more profitable firm orders between minimum demand plus spillover and maximum primary demand, its rival orders between minimum demand and minimum demand plus spillover.

(c) Case 6. The more profitable firm orders maximum demand plus spillover, its rival orders between minimum and maximum primary demand.

2. In the following cases of Theorem 2.2, as both firms become more risk averse, the less profitable firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes smaller.

(a) Case 2. The less profitable firm orders minimum demand plus spillover, its rival orders between minimum demand plus spillover and maximum primary demand.

(b) Case 3. Both firms order between minimum demand plus spillover and maximum primary demand.

3. Furthermore, Case 5 of Theorem 2.2 never holds.

Figures 2.4 and 2.5 illustrate Theorem 2.2 and Corollary 2.3. Fixing all the model parameters, except for the $C_i^u/C_i^o$ ratios, at the values indicated in the figures, it identifies the pairs $(C_u^A/C_o^A, C_u^B/C_o^B)$ that yield one of the cases of Theorem 2.2. The shaded areas in figures 2.4 and 2.5 illustrate the regions in $(C_u^A/C_o^A, C_u^B/C_o^B)$-space where one of the firms will increase its order quantity as both firms become more risk averse. In the rest
of the space, neither of the firms will increase their order quantity as both firms become more risk averse. As shown in Figure 2.4, in some cases it is the more profitable firm which increases its order quantity, in other cases it is the less profitable one. Since these regions are symmetric about the diagonal, we focus in our discussion on the cases where firm A increases its order quantity hereafter. Figure 2.5 represents these cases. For the model parameters in this example the conditions of case 4 of Theorem 2.2 are not satisfied. The equilibrium case 5 does not appear in the figures, consistent with Corollary 2.3. This leaves equilibrium cases 1 and 6, in which firm A has the higher $C^u_i/C^o_i$ ratio, and cases 2 and 3, where firm A has the lower $C^u_i/C^o_i$ ratio.

In cases 1, 2 and 6, firm A’s initial equilibrium order is on a demand point that depends on its rival spillover, which has two implications. First, firm A’s own risk aversion effect is zero. Second, its best response strictly increases if her rival’s order decreases: the resulting larger spillover demand increases firm A’s expected marginal utility from ordering an extra unit since this additional unit is more likely to sell than under lower spillover demand. Since firm B’s own risk aversion effect is strictly positive in each of these cases, the demand substitution effect is positive for firm A and it orders more in response to higher risk aversion. The cases 1, 2 and 6 differ in the relative magnitudes of the firms’ $C^u_i/C^o_i$ ratios. In case 6, firm A is much more profitable than firm B, and its initial order equals maximum primary demand plus spillover demand. At the representative point $a$, the ratios are $C^A_u/C^A_o = 4$ and $C^B_u/C^B_o = 1$. In cases 1 and 2, firm A is much less profitable. E.g., $C^A_u/C^A_o = 0.5$ at the representative points $b$ and $c$. The same marginal effects are at play at both points, causing firm A to increase its order in response to higher risk aversion. The only difference between the points is that firm B is the less profitable firm at point $b$ and the more profitable one at point $c$.

In the equilibrium case 3, both firms order more than minimum demand plus spillover but less than maximum primary demand. Unlike in the preceding cases, both firms’ own risk aversion effect is nonzero. As discussed above, only the firm with the strictly lower
Figure 2.4: Illustration of Theorem 2 for both firms. Impact of risk aversion depending on understocking to overstocking ratios $C_u^i/C_o^i$ of both firms. Parameters: $K^i = 0.5, R^i = 0.3, b^i = 0.5, q_L = 0.4, d_L = 5, d_H = 10$.

Figure 2.5: Illustration of Theorem 2 for Firm A. Impact of risk aversion depending on understocking to overstocking ratios $C_u^i/C_o^i$ of both firms. Parameters: $K^i = 0.5, R^i = 0.3, b^i = 0.5, q_L = 0.4, d_L = 5, d_H = 10$. 
own risk aversion effect can increase its equilibrium order in response to higher risk aversion. It can be shown that, holding other factors fixed, the magnitude of a firm’s own risk aversion effect is increasing in its initial order quantity. Intuitively, the larger the order quantity, the larger a firm’s risk exposure and the larger the marginal increase in this risk in response to higher risk aversion. Since the firm with the lower $C_i^u/C_i^o$ ratio orders less than its rival (if firms are otherwise symmetric), it faces a smaller own risk aversion effect. It is therefore the firm that increases its order in response to higher risk aversion. In Figure 2.5, this is the case for firm $A$ for values of $(C_A^u/C_A^o, C_B^u/C_B^o)$ in region 3.

### 2.5.4 Different Risk Aversion ($R_i$)

**Corollary 2.4.** Suppose that both firms are identical, except that they have different initial risk aversion. By Part 2. of Corollary 2.2, the equilibrium order of the initially less risk-averse firm (weakly) exceeds that of its rival.

1. In the following cases of Theorem 2.2, as both firms become more risk averse, the initially less risk averse firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes larger.

   (a) **Case 1.** The initially less risk averse firm orders minimum demand plus spillover, its rival orders between its minimum demand and minimum demand plus spillover.

   (b) **Case 4.** The initially less risk averse firm orders between minimum demand plus spillover and maximum primary demand, its rival orders between minimum demand and minimum demand plus spillover.

   (c) **Case 6.** The initially less risk averse firm orders maximum demand plus spillover, its rival orders between minimum and maximum primary demand.
2. In the following cases of Theorem 2.2, as both firms become more risk averse, the initially more risk averse firm increases its equilibrium order, and the difference between the firms' equilibrium orders becomes smaller.

(a) Case 2. The initially more risk averse firm orders minimum demand plus spillover, its rival orders between minimum demand plus spillover and maximum primary demand.

(b) Case 3. Both firms order between minimum demand plus spillover and maximum primary demand.

3. Furthermore, Case 5 of Theorem 2.2 never holds.

The cases in which the initially less (more) risk averse firm increases its order quantity are comparable to the cases in which the more (less) profitable firm increases its order quantity. Refer to figures 2.6 and 2.7 which illustrate the Corollary numerically. As in Figure 2.4, for the model parameters in this example the conditions of case 4 of Theorem 2.2 are not satisfied. As figures 2.6 and 2.7 show, there is a wide range of initial risk aversion parameters such that one of the firms increases its equilibrium order quantity as both firms become more risk averse. This range is highly dependent on the firms' understocking to overstocking ratios. When $C_u/C_o = 0.6$, in Figure 2.6, if either firm increases its order quantity as both firms become more risk averse, it is the less risk averse firm. However, when $C_u/C_o = 2.5$, in Figure 2.7, it can be either the more or the less risk averse firm that increases its order quantity as both firms become more risk averse.

Further note that certain equilibrium cases only emerge at higher profitability ratios. For example, fix $R^A = 0.7$ and $R^B = 3$. Under low profitability ratios for both firms, it is the less risk averse firm (A) that increases its order as both firms become more risk averse; see Figure 2.6. Under low profitability, the initial equilibrium orders are relatively small and the corresponding equilibrium meets the conditions of case 1 of Theorem 2.2. By contrast, under relatively large profitability ratio for firms, it is the more risk averse
firm (B) that increases its order quantity as both firms become more risk averse; see Figure 2.7. Under high profitability, the initial equilibrium orders are higher and the corresponding equilibrium has the characteristics of case 3 of Theorem 2.2. In this case, both firms order between minimum demand plus spillover and high demand, and their own risk aversion effects are nonzero. Since the more risk averse firm B orders less initially, its own risk aversion effect is smaller than that of its rival.

Observe in Figure 2.7 that for fixed initial risk aversion parameter of firm A \( (R^A = 4) \), as firm B’s initial risk aversion parameter varies from very low to very high, the relative sensitivity of the firms in response to higher risk aversion changes. When firm B’s initial risk aversion rate is very low or quite high, lower than 0.2 or in the 2.3-2.8 range, then it is firm B that increases its order quantity as both firms become more risk averse (the equilibria for very low \( R^B \) are instances of equilibrium case 6 of Theorem 2.2, whereby firm B orders high demand plus spillover; the equilibria for values of \( R^B \) in the 2.3-2.8 range are instances of equilibrium case 1 of Theorem 2.2, whereby firm B orders minimum demand plus spillover.) However, for \( R^B \) in some intermediate range, 0.5-2.3, it is firm A that increases its order quantity as both firms become more risk averse (these are instances of equilibrium cases 2 and 3 of Theorem 2.2).

### 2.5.5 Different Spillover Fractions \((\hat{b}^i)\)

**Corollary 2.5.** Suppose that both firms are identical, except for their spillover fractions. By Part 3. of Corollary 2.2, the equilibrium order of the firm with the higher spillover fraction (weakly) exceeds that of its rival.

1. In the following cases of Theorem 2.2, as both firms become more risk averse, the firm with the lower spillover fraction increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes smaller.

   (a) Case 1. The firm with lower spillover fraction orders minimum demand plus
Figure 2.6: Illustration of Theorem 2. Impact of risk aversion depending on initial risk aversion levels $R^i$ of both firms where $\frac{C_{\text{ti}}}{C_{\text{ho}}} = 0.6$. Parameters: $K^i = 0.5, b^i = 0.5, q_L = 0.4, d_L = 5, D_H = 10$.

Figure 2.7: Illustration of Theorem 2. Impact of risk aversion depending on initial risk aversion levels $R^i$ of both firms where $\frac{C_{\text{ti}}}{C_{\text{ho}}} = 2.5$. Parameters: $K^i = 0.5, b^i = 0.5, q_L = 0.4, d_L = 5, D_H = 10$. 
spillover, its rival orders between its minimum demand and minimum demand plus spillover.

(b) Case 4. The firm with lower spillover fraction orders between minimum demand plus spillover and maximum primary demand, its rival orders between minimum demand and minimum demand plus spillover.

2. In case 2 of Theorem 2.2, as both firms become more risk averse, the firm with the higher spillover fraction increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes larger. The firm with the higher spillover fraction orders minimum demand plus spillover, its rival orders between minimum demand plus spillover and maximum primary demand.

3. Furthermore, Cases 5 and 6 of Theorem 2.2 never hold.

Figures 2.8 and 2.9 show that for a range of spillover fractions, one of the firm may increase its order quantity as both become more risk averse, but this range is highly dependent on profitability ratio. Note that, as in the previous examples, for the model parameters in this example the conditions of case 4 of Theorem 2.2 are not satisfied. For example, when \( b^A = 0.7 \) and \( b^B = 0.3 \), if the firms are relatively less profitable \( (C_{iu}^i/C_{io}^i = 0.25) \), it is the firm with lower spillover rate (Firm B) that increases its order quantity as both firms become more risk averse. This case refers to Part 1 of Theorem 2.2. However if the firms are more profitable \( (C_{iu}^i/C_{io}^i = 0.75) \) the firm with higher spillover rate (Firm A) increases its order quantity as both firms become more risk averse. This case refers to Part 2 of Theorem 2.2.

Unlike in the cases where firms differ in their risk aversion parameters \( (R^i) \), varying the spillover rate \( (b^i) \) has a less significant effect and it highly depends on the profitability ratio \( C_{iu}^i/C_{io}^i \). When both firms are identical except for their spillover fraction, a high profitability ratio leads to equilibrium for both firms, independent of their individual spillover fractions. Similarly, low profitability leads to low equilibrium orders. As a
Figure 2.8: Illustration of Theorem 2. Impact of risk aversion depending on spillover fractions $b^i$ of both firms where $\frac{C_i}{C^i} = 0.25$. Parameters $K^i = 0.5, R^i = 0.3, q_L = 0.4, d_L = 5, d_H = 10$.

Figure 2.9: Illustration of Theorem 2. Impact of risk aversion depending on spillover fractions $b^i$ of both firms where $\frac{C_i}{C^i} = 0.75$. Parameters $K^i = 0.5, R^i = 0.3, q_L = 0.4, d_L = 5, d_H = 10$. 
result, some of the equilibrium cases do not emerge under asymmetric spillover fractions. For example, it cannot be an equilibrium for one firm, say firm A, to order maximum primary demand plus spillover and the firm B to order between minimum demand and maximum primary demand. To order maximum primary demand plus spillover, firm A should be so profitable that it orders more than its own high demand when it adopts monopoly strategy. Since we assume same profitability in this section, firm B must be equally profitable, so that the equilibrium orders of both firms would equal their maximum primary demand. In these cases, one can argue that the profitability is the main drivers of the equilibrium orders, and spillover demand results in limited deviation on top of that.

2.6 Sensitivity Analysis: Primary Demand Correlation

The analysis has so far assumed uncorrelated primary demands. However, since the products offered by the firms are partially substitutable, their primary demands might also be correlated. In this section we relax the independent demand distribution assumption in Section 2.5 to incorporate primary demand correlation.

2.6.1 Model and Equilibrium Characterization

We model correlation by using conditional probability: given that firm A experiences low primary demand $d_L$, firm B will experience the same primary demand with probability $v_1$, and given that firm A experiences high primary demand $d_H$, firm B will also do so
with probability $v_2$. The resulting p.d.f. of firm A total demand $D^A(Q^B)$ is

$$
P(D^A(Q^B) = d) = \begin{cases} 
q_{LL} \triangleq q_L v_1, \\
q_{LH} \triangleq q_L(1 - v_1), \\
q_{HL} \triangleq q_H(1 - v_2) = (1 - q_L)(1 - v_2), \\
q_{HH} \triangleq q_H v_2 = (1 - q_L)v_2,
\end{cases}
\begin{align*}
d &= d_{LL}^A(Q^B) \triangleq d_L \\
d &= d_{LH}^A(Q^B) \triangleq d_L + b^A[d_H - Q^B]^+ \\
d &= d_{HL}^A(Q^B) \triangleq d_H \\
d &= d_{HH}^A(Q^B) \triangleq d_H + b^A[d_H - Q^B]^+. \tag{2.30}
\end{align*}
$$

This p.d.f. is structurally equivalent to (2.14) in the uncorrelated case, it only differs in the probabilities of the various outcomes. If we set $v_1 = q_L$ and $v_2 = 1 - q_L$, then (2.30) specializes to (2.14). Hence, all the previous analysis and results in Section 2.5 can be replicated for this case by using appropriate probabilities. We focus on two extreme cases: perfect positive and perfect negative correlation.

Note that as pointed out in Section 2.2, our total demand model is very generic and it can accommodate different initial demand allocation rules. For example, in our model in Section 2.5, the initial demand allocation is similar in nature to that in Parlar (1988). It is also deterministic analog of the probabilistic splitting rule "Independent Random Demands" in Lippman and McCardle (1997). By incorporating primary demand correlation in our model, we can represent different initial allocation rules that discussed in Lippman and McCardle (1997).

**Perfect positive primary demand correlation.** In this case, $v_1 = v_2 = 1$, and (2.30) specializes to

$$
P(D^A(Q^B) = d) = \begin{cases} 
d &= d_{LL}^A(Q^B) = d_L & \text{with probability } q_L \\
d &= d_{HH}^A(Q^B) = d_H + b^A[d_H - Q^B]^+ & \text{with probability } (1 - q_L).
\end{cases} \tag{2.31}
$$

As a result, the best response function of firm A is:

$$
Q^A = \begin{cases} 
d_{LL}^A = d_L \\
d_{LH}^A = d_L + \frac{1}{R} \ln \left[ \frac{C_A^A}{C_o^A} \left( \frac{1-q_L}{q_L} \right) \right] & C_A^A \leq \frac{C_H^A}{C_o^A} \leq t_{LL}^A \\
d_{HL}^A = d_H & t_{HL}^A < \frac{C_A^A}{C_o^A} \leq t_{LL}^A e^{\frac{t_{HH}^A(d_{HH}^A(Q^B)-d_{LL}^A(Q^B))}{C_A^A}} \leq t_{HH}^A e^{\frac{t_{HH}^A(d_{HH}^A(Q^B)-d_{LL}^A(Q^B))}{C_A^A}} < \frac{C_A^A}{C_o^A} \\
d_{HH}^A = d_H + b^A[d_H - Q^B]^+ & t_{HH}^A e^{\frac{t_{HH}^A(d_{HH}^A(Q^B)-d_{LL}^A(Q^B))}{C_A^A}} < \frac{C_A^A}{C_o^A}
\end{cases}. \tag{2.32}
$$
where
\[ t_{LL}^A = \frac{q_L}{1 - q_L}. \]

**Perfect negative primary demand correlation.** In this case, \( \nu_1 = \nu_2 = 0 \), and (2.30) specializes to
\[
P \left( D^A(Q^B) = d \right) = \begin{cases} 
  d = d_{LH} = d_L + b^A[d_H - Q^B]^+ & \text{with probability } q_L \\
  d = d_{HL} = d_H & \text{with probability } (1 - q_L)
\end{cases}.
\]

As a result, the best response function of firm A is:
\[
Q^A = \begin{cases} 
  d_{LH} = d_L + b^A(d_H - Q^B)^+ & t_{LL}^A < \frac{C_A^A}{C^A_o} \leq t_{LL}^A e^{R^A(d_{HL}^L(Q^B) - d_{LH}^L(Q^B))} \\
  d_{HL} = d_H & t_{LL}^A e^{R^A(d_{HL}^L(Q^B) - d_{LH}^L(Q^B))} < \frac{C_A^A}{C^A_o}
\end{cases}
\]

(2.33)

### 2.6.2 Impact of Risk Aversion: Perfect Correlation

We discuss the impact of risk aversion under perfect primary demand correlation.

**Theorem 2.3.** For concreteness, let \( i = A \) and \( j = B \). Suppose the primary demands of the two firms are perfectly positively correlated. As both firms become more risk averse, firm A increases its order quantity if and only if the initial equilibrium is such that firm A orders maximum primary demand plus spillover, firm B orders between minimum demand \( (d_L) \) and maximum primary demand \( (d_H) \)
\[
Q^A = d_H + b^A(d_H - Q^B) \quad \text{and} \quad Q^B = d_L + \frac{\ln \left[ \frac{C^B_o}{C^A_o} \left( \frac{1 - q_L}{q_L} \right) \right]}{R^B},
\]

This is the unique equilibrium if the following conditions hold
\[
\frac{C^A_o}{C^A_u} > \frac{R^A}{1 - q_L} \frac{(1+b^A)(d_H - d_L) - b^A b^A \left[ \frac{C^B_o}{C^A_o} \left( \frac{1 - q_L}{q_L} \right) \right]}{R^B} = t_{LL}^A e^{R^A(d_{HL}^A(Q^B) - d_{LH}^A(Q^B))}
\]
\[
t_{LL}^B = \frac{q_L}{1 - q_L} \leq \frac{C^B_u}{C^B_o} \leq \frac{q_L e^{R^B(d_H - d_L)}}{1 - q_L} = t_{LL}^B e^{R^B(d_{HL}^B(Q^A) - d_{LH}^B(Q^A))}
\]

Similar statement, as above, is valid for the firm B.
The case in Theorem 2.3 corresponds to the equilibrium case 6 of Proposition 2.2 and Theorem 2.2. The other five equilibrium cases do not apply here, due to perfect positive primary demand correlation.

Under perfect positive primary demand correlation, firm A experiences either low primary demand or high primary demand plus potential spillover \(d_H + b^A[d_H - Q^B]^+\). Therefore, firm A’s order quantity can be its low demand, between its low demand and high demand plus spillover, or equal its high demand plus spillover demand. We investigate the impact of risk aversion under possible initial equilibrium.

1. If both firms order less than high primary demand \(d_H\), both of them order the monopoly order quantity. At this quantity, they do not require any spillover demand to sell out, and a change in spillover demand does not change their overstocking cost. As a result, their own risk aversion effect is the only factor that has an impact on both firms order quantity. Therefore both firms reduce their order quantity as both become more risk averse.

2. One of the firms, say firm A, orders more than its high primary demand, but less than high demand plus spillover, and firm B orders less than high demand. In this case also, both firms choose the monopoly order quantity. Although one of the firms orders more than its high primary demand, the current level of spillover demand is enough to sell out, so that more spillover demand does not reduce its overstocking risk. In this case, since primary demands are perfectly correlated, firm A will sell out if it experiences the high demand. Hence, as in the previous case, only the own risk aversion effect has an impact on both firms order quantity, and both firms reduce their order quantity in response to higher risk aversion.

3. If firm A’s competitor (firm B) orders at least high demand \(d_H\), firm A only experiences the low or high demand as it does not get any spillover demand. Therefore it will adapt monopoly strategy. Furthermore, note that if firm B orders less than the high demand \(d_H\) and firm A orders between its low demand and high demand plus spillover,
firm A’s order quantity will be the the monopoly order quantity (independent of firm B’s order quantity).

As a result, the only case in which firm A’s order quantity depends on that of its competitor is the case in which firm A orders maximum demand plus spillover. In all other cases firm A’s order quantity is independent of its competitor’s order quantity. Therefore, in those cases, as both firms become more risk averse, "own risk aversion increment" is the only factor that has an impact on firm A’s order quantity. Hence firm A reduces its order quantity. However when firm A’s order quantity is maximum demand plus spillover, the impact of spillover demand is influential. Note that in this case firm A should be significantly profitable (and/or significantly less risk averse) to order the maximum demand plus spillover. Such a high order quantity ($Q^A > d_H$) results in no spillover demand for firm B. On the other hand firm B’s profitability should be in the intermediate range (and/or relatively more risk averse), to have an initial order quantity between its own minimum demand ($d_L$) and maximum demand ($d_H$).

First consider how an increase in the risk aversion rate of both firms affects their expected marginal utilities. Firm A’s initial equilibrium order is very large, hence its own risk aversion effect is significant, however its strong profitability (and/or low risk aversion rate) cancels out the effect of risk aversion increment. Hence the effect of own risk aversion for firm A is not going to be materialized. On the other hand firm B will realize the effect of the risk aversion increment.

Next consider the effect of spillover demand. Note that in this case the effect of spillover demand does not exist for firm B, since firm A’s order quantity is significantly large ($Q^A > d_H$). On the other hand the effect of demand spillover is influential for firm A. Given that risk aversion impact is not an issue for firm A hence even slight reduction of order quantity by firm B, results in that firm A increases its order quantity. Aggregating these two effects, firm B reduces its order quantity due to risk aversion increment while firm A increases its order quantity given the reduction of firm B’s order quantity.
Theorem 2.4. For concreteness, let $i = A$ and $j = B$. Suppose the primary demands of the two firms are perfectly negatively correlated. As both firms become more risk averse, Firm A increases its order quantity if and only if the initial equilibrium is one of the following types:

1. Firm A orders minimum demand plus spillover and firm B orders between its minimum demand plus spillover and maximum primary demand

$$Q^A = d_L + b^A(d_H - Q^B) \text{ and } Q^B = d_L + b^B(d_H - Q^A) + \ln \left[ \frac{C^B_u}{C^B_o} \left( \frac{1-q_L}{q_L} \right) \right] \leq d_H.$$  

This is the unique equilibrium if the following conditions hold

$$\frac{C^A_u}{C^A_o} \leq \frac{q_L}{1-q_L} = t^A_{iLL},$$

$$t^B_{iLL} = \frac{q_L}{1-q_L} < \frac{C^B_u}{C^B_o} \leq t^B_{iLL} e^{R^B(i^\dagger_{iHL}(Q^A) - i^\dagger_{iHL}(Q^A))},$$

where

$$t^B_{iLL} e^{R^B(i^\dagger_{iHL}(Q^A) - i^\dagger_{iHL}(Q^A))} = \frac{q_L}{1-q_L} \exp \left( R^B \left( \frac{d_H - d_L}{1-b^B} - \frac{b^A b^B}{1-b^A} \ln \left[ \frac{C^B_u}{C^B_o} \left( \frac{1-q_L}{q_L} \right) \right] \right) \right).$$

Note that $\frac{C^B_u}{C^B_o} < t^B_{iLL} e^{R^B(i^\dagger_{iHL}(Q^A) - i^\dagger_{iHL}(Q^A))}$ is equivalent to

$$\frac{C^B_u}{C^B_o} < \frac{q_L}{1-q_L} e^{R^B(1-b^B)(d_H - d_L)}.$$  

2. Both firms’ order quantities are between their minimum demand plus spillover and their maximum demand

$$Q^A = d_L + b^A(d_H - Q^B) + \ln \left[ \frac{C^A_u}{C^A_o} \left( \frac{1-q_L}{q_L} \right) \right] \text{ and } Q^B = d_L + b^B(d_H - Q^A) + \ln \left[ \frac{C^B_u}{C^B_o} \left( \frac{1-q_L}{q_L} \right) \right].$$  

This is the unique equilibrium if the following conditions hold

$$t^i_{iLL} = \frac{q}{1-q} < \frac{C^i_u}{C^i_o} \leq t^i_{iLL} e^{R^i(i^\dagger_{iHL}(Q^j) - i^\dagger_{iHL}(Q^j))} \text{ for } i \neq j, i, j = \{A, B\}.$$
In addition it must be that $Q^A < d_H$, and the following condition must also hold:

$$\frac{C^B_m}{C^B_o} > \frac{q_L}{1 - q_L} \left[ \frac{C^A_u}{C^A_o} \left( \frac{1 - q_L}{q_L} \right)^{\frac{b^B R^B}{b^A R^A}} \right],$$  \hspace{1cm} (2.34)

where

$$t^i_{LL} e^{\overline{R}(d^i_{HL}(Q^i) - d^i_{LH}(Q^i))} = \frac{q_L}{1 - q_L} e^{-\frac{\overline{R}}{1 - b^L} \left( d_H - d_L - b^L d_H + b^L d_L + b^L \ln \left( \frac{C^j}{C^j_o} \left( \frac{1 - q_L}{q_L} \right) \right) \right)} - b^L \ln \left( \frac{C^j}{C^j_o} \left( \frac{1 - q_L}{q_L} \right) \right) - b^L \ln \left( \frac{C^i}{C^i_o} \left( \frac{1 - q_L}{q_L} \right) \right) \right).$$

Note that $\frac{C^i}{C^i_o} \leq t^i_{LL} e^{\overline{R}(d^i_{HL}(Q^i) - d^i_{LH}(Q^i))}$ is equivalent to

$$\frac{C^i}{C^i_o} \leq \frac{q_L}{1 - q_L} e^{-\frac{\overline{R}}{1 - b^L} \left( d_H - d_L - b^L d_H + b^L d_L + b^L \ln \left( \frac{C^j}{C^j_o} \left( \frac{1 - q_L}{q_L} \right) \right) \right)}.$$

Similar statement, as above, is valid for the firm B.

When the demand is perfectly negatively correlated, firm A only experiences low demand plus potential spillover $(d_L + b^A[d_H - Q^B])$ or high demand $(d_H)$. Therefore firm A’s order quantity can be the low demand plus spillover, between low demand plus spillover and high demand, or High demand.

As long as firm A’s competitor (Firm B) orders less than high demand $(d_H)$, firm A’s order quantity depends on firm B’s order quantity. In two cases we observe such behavior.

In other cases, since firm A’s order quantity is independent of firm B’s order quantity, hence as both firms become more risk averse, firm A does not increase its order quantity. In these cases the only factor that has an impact on firm A’s decision is the own risk aversion increment.

However if the firm A’s order quantity depends on its competitor’s order quantity, then firm A may increase its order quantity as both firms become risk averse. We will investigate these two cases in detail considering different situations. In nutshell, we will argue that when firm A increases its order quantity, the impact of spillover demand
surpass the impact of own risk aversion increment, and as result firm A increases its order quantity. Note that, these two cases are comparable with Case 2 and 3 of Theorem 2.2.

Observe that Cases 1, 4, 5, and 6 of Theorem 2.2 are irrelevant under perfect negative correlation, since both firms at least order the minimum demand plus spillover, and at most order the maximum demand. As a result these equilibrium cases of Theorem 2.4 can not happen under perfect negative correlation.

Now we study Theorem 2.3 and Theorem 2.4 in more detail. In particular, we investigate three important circumstances: 1. different understocking to overstocking ratio \( (C_{iu}/C_{io}) \). 2. Different risk aversion rate \( (R^i) \) 3. Different spillover rate \( (b^i) \).

1. Different understocking to overstocking ratio \( (C_{iu}/C_{io}) \). In this case both firms are identical except that they have different profitability ratio \( (C_{iu}/C_{io}) \). Note that the order quantity of the more profitable firm is more than or equal to the less profitable firm’s order quantity at the equilibrium.

In case of perfect positive correlation, it is straight forward to show that it is the more profitable firm that increases its order quantity if any of them increases its order quantity as both firms become more risk averse. Note that, as we mentioned before, the only case that a firm may increase its order quantity as both firms become more risk averse, is the case that the firm orders the maximum demand plus spillover \( (d_H + b^A(d_H - Q^B)) \) while its competitor orders less than maximum demand \( (d_H) \). To be able to order the maximum demand plus spillover, since both firms are identical except that they have different profitability ratio, the firm must be more profitable while its competitor’s profitability should be in intermediate range to avoid ordering its own maximum demand \( (d_H) \). As mentioned before this is very similar to Case 5 of Theorem 2.2, hence the section 4.1’s discussion about Case 5 is applied here as well.

In case of perfect negative correlation, it is the less profitable firm that increases its order quantity if any of them increases its order quantity as both firms become more risk
Under the equilibrium with initial risk aversion levels, firm A, less profitable firm, orders less and has lower spillover demand than firm B.

First consider how an increase in the risk aversion rate $R$ of both firms affects their expected marginal utilities. Since firm A’s initial equilibrium order is relatively lower, its own risk aversion effect is relatively smaller than firm B. Firm B gets hit harder by its own risk aversion effect since its risk exposure is larger due to its larger initial order.

Next consider the effect of spillover demand. Since firm A’s spillover demand is relatively smaller at the initial equilibrium, its expected marginal utility is more sensitive to a change in its competitor’s order. In particular, a given reduction in firm B’s order significantly reduces firm A’s expected marginal utility loss due to overstocking. By contrast, since firm B’s spillover demand is relatively larger at the initial equilibrium, its expected marginal utility is less sensitive to a change in the order of firm A. Therefore, the magnitude of the spillover demand effect per unit change in the competitor’s order is significantly larger for firm A than for firm B. A relatively small reduction in firm B’s order translates into a spillover demand effect for firm A which is large enough to offset firm A’s relatively small negative own risk aversion effect. Firm B, however, requires a relatively more significant reduction in firm A’s order for firm B’s spillover demand effect to offset its own risk aversion effect, because firm B has both a lower marginal sensitivity to more spillover demand and a more negative own risk aversion effect. Therefore, the net effect is for firm A to increase and for firm B to decrease its equilibrium order.

In particular, in Part 2 of Theorem 2.4, the inequality (2.34) will be simplified as follows:

\[
\frac{C_B}{C_B^o} > q_L \left( \frac{\left( 1 - q_L \right)}{q_L} \right)^{\frac{1}{5\pi}}
\]

Therefore, to have above inequality, firm A should be less profitable compare to firm B to increases its order quantity as both firm become more risk averse.

As mentioned before Part 1 and 2 of Theorem 2.4 are very similar to Case 2 and 3.
of Theorem 2.2 respectively, hence the section 4.1’s discussion about Cases 2 and 3 are applied here as well.

2. **Different risk aversion rate** ($R^i$): In this case both firms are identical except that they have different risk aversion rate ($R^i$). In case of perfect positive correlation, it is straightforward to see that it is less risk averse firm that increases its order quantity if any of them increases its order quantity as both firms become more risk averse. Similar argument to that of different understocking to overstocking ratio ($\frac{C_u}{C_o}$) case will apply here as well, except this time for firm A to order maximum demand plus spillover, it should have initially sufficient low risk aversion rate and its competitor should not have very low risk aversion, to avoid ordering the high demand ($d_H$).

In case of perfect negative correlation, Part 1 of Theorem 2.4 is not applicable under this assumption, since both firms have same profitability ratio. Therefore, either both firms order the minimum demand plus spillover or neither. However conditions of Part 2 of Theorem 2.4 imply that it is more risk averse firm that increases its order quantity if any of them increases its order quantity as both firms become more risk averse. In particular, Inequality (2.34) in Theorem 2.4 will be simplified as follows:

$$\frac{C_u}{C_o} \left( \frac{1 - q_L}{q_L} \right) > \left[ \frac{C_u}{C_o} \left( \frac{1 - q_L}{q_L} \right) \right]^{\frac{R^B}{b(R^A)^2}}$$

hence, firm A must be initially more risk averse than firm B, to increases its order quantity as both firms become more risk averse. Similar argument to that of that of different understocking to overstocking ratio ($\frac{C_u}{C_o}$) case will apply here as well. Under the equilibrium with initial risk aversion levels, firm A therefore orders less and has lower spillover demand than firm B.

Similar to different profitability ratio, first consider how an increase in the risk aversion rate $R$ of both firms affects their expected marginal utilities. Since firm A’s initial equilibrium order is relatively lower, its own risk aversion effect is relatively smaller than firm B. In other words, firm B gets hit harder by its own risk aversion effect since its risk
exposure is larger due to its larger initial order.

Similarly we consider the effect of spillover demand. Since firm A’s spillover demand is relatively smaller at the initial equilibrium, its expected marginal utility is more sensitive to a change in its competitor’s order. In particular, a given reduction in firm B’s order significantly reduces firm A’s expected marginal utility loss due to overstocking. By contrast, since firm B’s spillover demand is relatively larger at the initial equilibrium, its expected marginal utility is less sensitive to a change in the order of firm A. Therefore, the magnitude of the spillover demand effect per unit change in the competitor’s order is significantly larger for firm A than for firm B. A relatively small reduction in firm B’s order translates into a spillover demand effect for firm A which is large enough to offset firm A’s relatively small negative own risk aversion effect. Firm B, however, requires a relatively more significant reduction in firm A’s order for firm B’s spillover demand effect to offset its own risk aversion effect, because firm B has both a lower marginal sensitivity to more spillover demand and a more negative own risk aversion effect. Therefore, aggregate impact is for firm A to increase and for firm B to decrease its equilibrium order.

3. Different spillover rate ($b^i$): In this case both firms are identical except that they have different spillover rate ($b^i$). In such circumstances, firm A will not increase its order quantity given that the stated conditions in Theorem 2.3 and 2.4 never hold.

As we discussed in Section 4.3, varying the spillover rate ($b^i$) has a less significant impact than changing profitability ratio or initial risk aversion ratio. In the perfect correlation case, we observe the extreme situation. Neither of the firms increases its order quantity as both firms become more risk averse when both firms are identical except that they have different spillover rate. As we discussed above firm A increases its order quantity if it has significantly higher order quantity in the perfect positive correlation case or lower order quantity in the perfect negative correlation case at the equilibrium compared to its competitor. However since spillover rate has a secondary effect compared to profitability ratio and initial risk aversion rate, when both firms are identical except
that they have different spillover rate, the difference between two firms order quantity is not significantly high, hence neither of the firm increase its order quantity as both become more risk averse.

Next we present the effect of perfect correlation on impact of risk aversion graphically. Figures 2.10, 2.11 and 2.12 compare three cases: 1) firms have perfectly positively correlated demand 2) firms have independent demand distribution and 3) firms have perfectly negatively correlated demand. It is interesting to observe that when the demands are perfectly positively correlated, if any firms increases its order quantity as both firms become more risk averse, that is more profitable one. When the demand are perfectly negatively correlated then it is less profitable firm that may increase its order quantity as both firms become more risk averse. This observation is, in fact, to some extent intuitive. When the demand of two firms are perfectly positively correlated it means that if the firm’s competitor faces high (low) demand, firm’s realized demand will be high (low) as well. Furthermore, we know that the firm will only get spillover demand if its competitor faces high demand. But if the competitor faces high demand, it means that the firm also faces high demand. So there is only one way that firm can benefit from spillover demand and that is when firms initial order quantity is more than its own high demand \(d_H\). As we already discussed, firm should be highly profitable to be willing to order more than its own high demand \(d_H\). Therefore the the only situation that "spillover demand" effect may dominate the "increasing own risk aversion" effect and firm increase its order quantity is when the firm is highly profitable or has very low risk aversion.

On the other extreme end, if firms’ demand are perfectly negatively correlated, then when firms competitor faces high (low) demand, the firm would face low (high) demand. Hence the maximum possible demand a firm may realize in this situation, is its own maximum demand \(d_H\), and as a result, firm is never going to order more than \(d_H\). Therefore if the firm is highly profitable it will at most order its own maximum demand and the effect of spillover demand does not exist. Therefore highly profitable firm will
Figure 2.10: Impact of Risk Aversion for perfect positive correlated demand case in understocking to overstocking space \( \frac{C_i}{C_o} \). Parameters: \( K^i = 0.5, R^i = 0.3, b^i = 0.5, q = 0.4, d_L = 5, d_H = 10 \).

Figure 2.11: Impact of Risk Aversion for independent demand case (Base Case) in understocking to overstocking space \( \frac{C_i}{C_o} \). Parameters: \( K^i = 0.5, R^i = 0.3, b^i = 0.5, q = 0.4, d_L = 5, d_H = 10 \).
Figure 2.12: Impact of Risk Aversion for perfect negative correlated demand case in understocking to overstocking space $\frac{C_i}{C_o}$. Parameters: $K^i = 0.5, R^i = 0.3, b^i = 0.5, q = 0.4, d_L = 5, d_H = 10$.

not increases its order quantity as both firm becomes more risk averse.

Furthermore, note that, the minimum possible realized demand is different due to negative correlation compare to the independent demand case. In fact, firm’s minimum realize demand is at least its own low demand plus spillover $(d_L + b^i(d_H - Q^j))$. From the independent demand case, we already know that when the firm orders between its own low demand plus spillover and its high demand, it is the less profitable firm that may increase its order quantity as both firms become more risk averse. It turns out we have similar case here and its less profitable firm that increases its order quantity as both firm becomes more risk averse, if any for the similar intuitive reason that we have discussed in the independent demand case.

2.7 Summary

We have studied the impact of the risk aversion on firms’ order quantities. We have characterized the best response function and proved the existence and uniqueness of
Nash equilibrium under general demand distribution and utility function. We have then shown that under competition, one of the firms may increase its order quantity as both firms become more risk averse. To gain specific insights on how risk aversion affects the equilibrium depending on the model parameters, we chose two point demand distribution and CARA utility function. Table 2.1 summarizes our results. It shows which firm may increase its order quantity as both firms become more risk averse.

<table>
<thead>
<tr>
<th>Perfect Negative Correlation</th>
<th>No Correlation</th>
<th>Perfect Positive Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Different $\frac{C_u}{C_o}$</td>
<td>Lower $\frac{C_u}{C_o}$</td>
<td>Both Higher or Lower $\frac{C_u}{C_o}$</td>
</tr>
<tr>
<td>Different $R$</td>
<td>Initially Higher $R$</td>
<td>Initially Both Higher or Lower $R$</td>
</tr>
<tr>
<td>Different $b$</td>
<td>None</td>
<td>Both Lower or Higher $b$</td>
</tr>
</tbody>
</table>

Table 2.1: Summary of Results

**Future Research.** The analysis in this chapter points to a range of directions for future research. For one it seems called for comparison of our duopoly results to those for a two-product monopoly. Another direction is considering different game settings such as stackelberg, etc. This problem could be also studied considering both pricing and stocking decisions.

### 2.8 Proofs

**Proof of Lemma 2.1.** It is immediate by definition of $U_k(Q)$ that $U(Q) = U_k(Q)$ on $[d_k, d_{k+1}]$.

**Part 1.** That $U$ is continuous in $Q$ follows by continuity of $\Pi(Q, d)$ and continuity of $u$. The first derivative of $U_k$ satisfies

$$U'_k(Q) = -C_o \sum_{i=1}^{k} q_i u'((C_u + C_o)d_i - C_oQ) + C_u \left( 1 - \sum_{i=1}^{k} q_i \right) u'(C_uQ), \quad (2.35)$$
where \( U''_0(Q) > 0 > U''_N(Q) \) for all \( Q \) since \( u' > 0 \). That \( U''_k < 0 \) follows since \( u'' < 0 \).

Noting that \( U'_{k-1}(Q) > U'_k(Q) \) for all \( k \) and \( Q \), it follows that \( U \) is strictly concave as claimed, with \( U''(Q) = U''_k(Q) < 0 \) for \( Q \in (d_k, d_{k+1}) \) and \( U'_-(d_k) = U'_{k-1}(d_k) > U'_k(d_k) = U'_+(d_k) \) for \( Q = d_k \).

**Part 2.** Since \( U \) is strictly concave it has an unique maximum \( Q^\star \). It satisfies \( U'_k(d_k) \geq 0 \geq U'_k(d_k) \) if \( Q^\star = d_k \), and \( U'(Q^\star) = U'_k(Q^\star) = 0 \) if \( Q^\star \in (d_k, d_{k+1}) \) for some \( k \). Therefore it is the largest order quantity at which the left derivative \( U'_- \) is non-negative: \( Q^\star = \max \{ Q \geq 0 : U'_-(Q) \geq 0 \} \), where \( Q^\star \in [d_1, d_N] \) since \( U'_-(d_1) = U'_0(d_1) > 0 \) and \( U'_-(Q) = U'_N(Q) < 0 \) for \( Q > d_N \). Calculating \( U'_-(Q) \) and rearranging terms yields the condition (2.5).

**Proof of Lemma 2.2.** Recall the expected utility function (2.2)

\[
U^i(Q^i|Q^j) = \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} q_{xy}^i u^i \left( \Pi^i \left( Q^i, d_{xy}^i(Q^j) \right) \right).
\]

(2.36)

The payoff function (2.1) satisfies

\[
\Pi^i \left( Q^i, d \right) = \left( (C^i_u + C^i_o) d - C^i_s Q^i \right) 1 \{ d < Q^i \} + C^i_u Q^i 1 \{ d \geq Q^i \}.
\]

(2.37)

Continuity and piecewise differentiability of \( U^i(Q^i|Q^j) \) in \( Q^j \) follow since \( \Pi^i \) and \( u^i \) have these same properties. Continuity and piecewise differentiability of \( Q^i \) in \( Q^j \) follow since \( U^i(Q^i|Q^j) \) is concave in \( Q^i \) for every \( Q^j \), and \( U^i(Q^i|Q^j) \) is continuous and piecewise differentiable in \( (Q^i, Q^j) \).

**Part 1.** Since \( u'' > 0 \), \( \Pi^i \) is non-decreasing in \( d \), and \( d_{xy}^i(Q^j) \) are non-increasing in \( Q^j \), \( U^i(Q^i|Q^j) \) is constant in \( Q^j \) iff \( \Pi^i \left( Q^i, d_{xy}^i(Q^j) \right) \) is constant in \( Q^j \), and strictly decreasing otherwise. For fixed \( Q^j \) the payoff \( \Pi^i \left( Q^i, d \right) \) is strictly increasing in \( d < Q^i \) and constant in \( d \geq Q^i \).

\( d_{xy}^i(Q^j) \) has the following form: \( d_{xy}^i(Q^j) = d_{x}^i + b^i (d_{y}^i - Q^j)^+ \). Therefore \( u^i \left( \Pi^i \left( Q^i, d_{xy}^i(Q^j) \right) \right) \) could be constant or decreasing in \( Q^j \). Consequently \( U^i(Q^i|Q^j) \) is constant or decreasing in \( Q^j \).
Part 2. The claim is: \( 0 \geq \frac{\partial Q^i}{\partial Q^j} \geq -b \). Similar to equation (2.5), define
\[
f(Q^i, Q^j) = \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d_{xy}^i(Q^j) < Q^i\} \cdot q_{x,y} \cdot \left( \frac{u'(C_u + C_o)d_{xy}^i(Q^j) - C_oQ^i}{u'(C_uQ^i)} \right) C_o + C_u - C_u. \tag{2.38}
\]

Based on Lemma 2.1 the optimal order quantity for firm \( i \) for given \( Q^j \) is unique and given by
\[
Q^i(Q^j) = \max\{Q'^i \geq 0 : f(Q^i, Q^j) \leq 0\}. \tag{2.39}
\]

Notice that there are two possibilities: either \( Q^i(Q^j) \) equals one of the total demand realizations, or \( Q^i(Q^j) \) is between two total demand realizations. To prove the claim we consider two cases: Case A. when \( Q^i(Q^j) \) is equal to a total demand realization, and Case B when \( Q^i(Q^j) \) is between two total demand realizations.

Before we prove the claim for Case A, we present four key properties of the function \( f(Q^i, Q^j) \).

**Property 1.** For fixed \( Q^i \), \( f(Q^i, Q^j) \) is non-decreasing in \( Q^j \) with possible jumps in \( f(Q^i, Q^j) \)'s value.

By (2.38), \( f(Q^i, Q^j) \) is equal to the sum of the following summands minus the constant \( C_u \):
\[
1\{d_{xy}^i(Q^j) < Q^i\} \cdot q_{x,y} \cdot \left( \frac{u'(C_u + C_o)d_{xy}^i(Q^j) - C_oQ^i}{u'(C_uQ^i)} \right) C_o + C_u. \tag{2.40}
\]

As \( Q^j \) increases, we observe two effects: (a) potential change in the value of the indicator function in (2.40) (b) potential change in value of
\[
u' \left( (C_u + C_o)d_{xy}^i(Q^j) - C_oQ^i \right).
\]

As \( Q^j \) increases, \( d_{xy}^i(Q^j) \) remains constant or decreases. If the condition of the indicator function is already satisfied, it will remain satisfied since \( d_{xy}^i(Q^j) \) remains constant or decreases. However if the condition of indicator function is not already satisfied, it may become satisfied. For fixed \( Q^i \), if \( d_{xy}^i(Q^j) \) decreases the condition in the indicator function of expression (2.40) might become satisfied (effect (a)). As a result the number
of summands of function \( f(Q^i, Q^j) \) may increase. Notice that as the indicator function becomes 1 for a total demand realization, the value of \( f(Q^i, Q^j) \) jumps up. Also note that

\[
u' \left( (C_u + C_o) d_{xy}^i (Q^j) - C_o Q^i \right)
\]

in expression (2.40) is a non-decreasing function of \( Q^j \) for fixed \( Q^i \) (effect (b)). Therefore, both effects (a) and (b) show that \( f(Q^i, Q^j) \) is a non-decreasing function of \( Q^i \).

**Property 2.** For fixed \( Q^i \), \( f(Q^i, Q^j) \) is increasing in \( Q^i \).

As \( Q^i \) increases, we observe two effects: (a) potential change in the value of the indicator function in (2.40) (b) potential change in value of

\[
u' \left( (C_u + C_o) d_{xy}^i (Q^j) - C_o Q^i \right)
\]

\[
u' (C_u Q^i)
\]

Again consider expression (2.40). As \( Q^i \) increases and \( Q^j \) remains fixed, the condition in the indicator function might be satisfied for more total demand realizations (effect (a)), since \( Q^i \) increases and the total demand realizations remain the same (\( Q^j \) remains fixed). Also note that

\[
u' \left( (C_u + C_o) d_{xy}^i (Q^j) - C_o Q^i \right)
\]

\[
u' (C_u Q^i)
\]

in expression (2.40) is an increasing function of \( Q^i \) for fixed \( Q^j \). As \( Q^i \) increases, the numerator increases and denominator decreases. Therefore, in light of effects (a) and (b) we can conclude \( f(Q^i, Q^j) \) is increasing in \( Q^i \) for fixed \( Q^j \).

**Property 3.** Consider the case that the best response is equal to a total demand realization that involves spillover demand (i.e. \( Q^i (Q^j) = d_{sp}^i (Q^j) = d_s^i + b_i (d_s^j - Q^j) \) where \( d_s^j > Q^j \)) and remains equal to this total demand realization as \( Q^j \) changes, we show that \( \frac{df(Q^i, Q^j)}{dQ^j} < 0 \).

In this case as \( Q^j \) increases infinitesimally the number of summands in (2.38) remains the same. The condition of indicator function in (2.38) remain unsatisfied for those demand points that are larger than or equal to \( d_{sp}^i (Q^j) \). Although those demand points that are larger than or equal to \( d_{sp}^i (Q^j) \) may decrease as \( Q^j \) increases, \( Q^i (Q^j) \) also decreases.
with the same rate so the ranking of those total demand realizations with respect to 
\( d_{sp}^i(Q^j) \) remain the same. Those demand points that are less than \( d_{sp}^i(Q^j) \), also remain
less than \( d_{sp}^i(Q^j) \) as infinitesimal increase in \( Q^j \) keeps the ranking of the total demand
realizations that are smaller than \( d_{sp}^i(Q^j) \) with respect to \( d_{sp}^i(Q^j) \) intact. Therefore the
condition of the indicator function in (2.38) for the total demand realizations smaller
than \( d_{sp}^i(Q^j) \) remain satisfied. As a result, to show Property 3, we only need to focus
on changes in the following ratio for the total demand realizations \( d_{xy}^i(Q^j) \)s such that
\( d_{xy}^i(Q^j) < d_{sp}^i(Q^j) \):

\[
 h(Q^i, Q^j) = \frac{u' ((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^i)}{u' (C_u Q^i)} \]

\[
 = \frac{u' ((C_u + C_o)d_{xy}^i(Q^j) - C_o (d_s^i + b^i(d_s^j - Q^j)))}{u' (C_u (d_s^i + b^i(d_s^j - Q^j)))}
\]

Taking the derivative of \( h(Q^i, Q^j) \) with respect to \( Q^j \), we have:

\[
 \frac{dh(Q^i, Q^j)}{dQ^j} = \frac{u'' ((C_u + C_o)d_{xy}^i(Q^j) - C_o (d_s^i + b^i(d_s^j - Q^j))) (C_o b^i - (C_u + C_o)b^i \cdot 1\{d_y^j > Q^j\})}{u' (C_u (d_s^i + b^i(d_s^j - Q^j)))}
\]

\[
 -u' ((C_u + C_o)d_{xy}^i(Q^j) - C_o (d_s^i + b^i(d_s^j - Q^j))) \frac{(C_u b^i) u'' (C_u (d_s^i + b^i(d_s^j - Q^j)))}{(u' (C_u (d_s^i + b^i(d_s^j - Q^j)))^2}
\]

Replacing \( Q^i = d_s^i + b^i(d_s^j - Q^j) \), we have:

\[
 \frac{dh(Q^i, Q^j)}{dQ^j} = \frac{u'' ((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^i)}{u' (C_u Q^i)} \times
\]

\[
 [(C_o b^i - (C_u + C_o)b^i \cdot 1\{d_y^j > Q^j\}) - (-C_u b^i) \frac{u' ((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^i)}{u'' ((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^i)} \frac{u'' (C_u Q^i)}{u' (C_u Q^i)}]
\]

Define

\[
 g(x) = \frac{u'' (x)}{u' (x)}
\]

Substituting \( g(x) \) in \( \frac{dh(Q^i, Q^j)}{dQ^j} \), we have:

\[
 \frac{dh(Q^i, Q^j)}{dQ^j} = \frac{u'' ((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^i)}{u' (C_u Q^i)} \times
\]

\[
 b^i \left[ \left( C_o + C_u \frac{g (C_u Q^i)}{g ((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^i)} \right) - (C_u + C_o) \cdot 1\{d_y^j > Q^j\} \right]
\]

(2.41)
Since (1) \( g'(x) \geq 0 \) (based on Assumption 1) and (2) 

\[ C_u Q^i \geq (C_u + C_o) d^i_{xy}(Q^j) - C_o Q^i \]

because \( Q^i(Q^j) = d^i_{sp}(Q^j) \geq d^i_{xy}(Q^j) \) as mentioned before, we have 

\[ \frac{g(C_u Q^i)}{g(C_u + C_o) d^i_{xy}(Q^j) - C_o Q^i} > 1 \]

Therefore \( \frac{d h(Q^j, Q^i)}{d Q^j} < 0 \), since inside the bracket in Equation (2.41) is positive and 

\[ \frac{u''((C_u + C_o) d^i_{xy}(Q^j) - C_o Q^i)}{u'(C_u Q^i)} < 0. \]

**Property 4.** \( f(Q^i, Q^j) \) is continuous in \( Q^j \) except for the points that the indicator function in (2.38) become satisfied for a total demand realization. Consider one of the summands in (2.38):

\[ 1 \{ d^i_{xy}(Q^j) < Q^j \} \cdot q_{xy} \cdot \left( \frac{u'((C_u + C_o) d^i_{xy}(Q^j) - C_o Q^i)}{u'(C_u Q^i)} \right) \cdot \frac{C_o + C_u}{C_u Q^i} \]

Since \( u'((C_u + C_o) d^i_{xy}(Q^j) - C_o Q^i) \) is a continuous function in \( Q^j \), if the values of indicator function does not change, the above expression is continuous in \( Q^j \), otherwise we have a discontinuity when the value of the indicator function changes. Based on this argument if the value of the indicator function does not change in (2.38), \( f(Q^i, Q^j) \) is continuous in \( Q^j \), otherwise we have discontinuity when the value of the indicator function changes.

**Case A.** We now prove the claim for the case where the best response is equal to a total demand realization i.e. \( f(Q^i, Q^j) \leq 0 \) for \( Q^i \leq d^i_{sp}(Q^j) \) and \( f(Q^i, Q^j) > 0 \) for \( Q^i > d^i_{sp}(Q^j) \). To prove the claim, we consider two cases: A.1 when the best response is equal to a total demand realization that involves spillover demand, i.e. \( Q^i(Q^j) = d^i_{sp}(Q^j) = d^i_s + b^i (d^j_p - Q^j) \) where \( d^j_p > Q^j \), and A.2 when the best response is only equal to a total demand realization that does not involve spillover demand, i.e. \( Q^i(Q^j) = d^i_{sp}(Q^j) = d^i_s \).

**Case A.1.** Consider the case where the best response is equal to a total demand realization that involves spillover demand (i.e. \( Q^i(\hat{Q}^j) = d^i_{sp}(\hat{Q}^j) = d^i_s + b^i (d^j_s - \hat{Q}^j) \), where \( d^j_s > \hat{Q}^j \) and remains on this demand point as the competitors order quantity changes. As shown in Property 3, \( f(Q^i, Q^j) \) is a decreasing function of \( Q^j \) in this case. Hence for small \( \varepsilon > 0 \), \( f(Q^i, \hat{Q}^j + \varepsilon) < 0 \). Therefore firm \( i \)'s best response should be
larger than or equal to \( d_{sp}^i (\widehat{Q}^j + \varepsilon) \) when the competitor orders \( \widehat{Q}^j + \varepsilon \). If \( Q^i (\widehat{Q}^j + \varepsilon) \) is equal to \( d_{sp}^i (\widehat{Q}^j) = d_s^i + b \left( d_p^i - \widehat{Q}^j - \varepsilon \right) \), we have \( \frac{\partial Q^i}{\partial Q^j} = -b \), otherwise \( Q^i (\widehat{Q}^j + \varepsilon) \) must be larger than \( d_{sp}^i (\widehat{Q}^j + \varepsilon) \). This means that \( Q^i (\widehat{Q}^j + \varepsilon) \) is between two demand points which shows that \(-b < \frac{\partial Q^i}{\partial Q^j} < 0\).

**Case A.2.** Consider the case where firm \( i \)'s best response function is only equal to a total demand realization that does not involve spillover demand (i.e. \( Q^i (\widehat{Q}^j) = d_{sp}^i (\widehat{Q}^j) = d_s^i \)) and remains on this demand point as the competitor’s order quantity changes. We break down this case to two sub cases: Case A.2.a. when \( f (Q^i, \widehat{Q}^j) < 0 \), and Case A.2.b. when \( f (Q^i, \widehat{Q}^j) = 0 \).

**Case A.2.a.** If \( f (Q^i, \widehat{Q}^j) < 0 \) for \( Q^i (\widehat{Q}^j) = d_{sp}^i (\widehat{Q}^j) = d_s^i \) then we show that Firm \( i \)'s best response stays on this demand point for small changes in its rival’s order quantity, i.e., \( Q^i (\widehat{Q}^j + \varepsilon) = d_{sp}^i (\widehat{Q}^j + \varepsilon) \) for small \( \varepsilon > 0 \). If \( d_{sp}^i (\widehat{Q}^j) \) is constant (i.e. \( d_{sp}^i (\widehat{Q}^j) = d_s^i \)), small increases in \( Q^j \), results in either very small increase or no change in \( f(Q^i, Q^j) \) (as shown in Property 1). However since the change in \( f(Q^i, Q^j) \) is infinitesimal and \( f(Q^i, Q^j) \) is continuous in \( Q^j \) based on Property 4 (since the indicator functions in \( f(Q^i, Q^j) \) remain the same as \( Q^i (\widehat{Q}^j) \) and \( d_{sp}^i (\widehat{Q}^j) \) are constant and all other demand points remain the same or decrease infinitesimally as \( Q^j \) increases infinitesimally), \( f (Q^i, \widehat{Q}^j + \varepsilon) \) remains negative. On the other hand based on (2.39) we have \( f (Q^i, \widehat{Q}^j + \varepsilon) > 0 \) for \( Q^i > d_{sp}^i (\widehat{Q}^j + \varepsilon) \). Note that the following holds: \( f (Q^i, \widehat{Q}^j + \varepsilon) > 0 \) for \( Q^i > d_{sp}^i (\widehat{Q}^j + \varepsilon) = d_s^i \) since as \( Q^j \) increases \( d_{sp}^i (\widehat{Q}^j) \) remains the same and based on Property 1, \( f(Q^i, Q^j) \) is an increasing function of \( Q^j \). Therefore, we can conclude that \( f (Q^i, \widehat{Q}^j + \varepsilon) > f (Q^i, \widehat{Q}^j) > 0 \) for \( Q^i > d_{sp}^i (\widehat{Q}^j + \varepsilon) = d_s^i \) which shows that the best response function remains at this demand point and as a result \( \frac{\partial Q^i}{\partial Q^j} = 0 \).

**Case A.2.b.** If \( f (Q^i, \widehat{Q}^j) = 0 \) for \( Q^i = d_{sp}^i (\widehat{Q}^j) = d_s^i \). We show that, as \( Q^j \) increases, \( Q^i \) may decrease, and does not necessarily remain on demand point \( d_{sp}^i (\widehat{Q}^j) \). In the case where \( d_{sp}^i (\widehat{Q}^j) \) is constant in \( Q^j \) (i.e. \( d_{sp}^i (\widehat{Q}^j) = d_s^i \)), as \( Q^j \) increases,
$f(Q^i, Q^j)$ may increase or remain the same (shown in Property 1). If $f(Q^i, Q^j)$ remains the same, the best response remains on this demand point so the claim holds since $\frac{\partial f}{\partial Q^j} = 0$. However, if $f(Q^i, Q^j)$ increases, then we will have $f(Q^i, Q^j + \varepsilon) > 0$ at $Q^i \left( \hat{Q}^j \right) = d^i_{sp} \left( \hat{Q}^j \right)$. As a result we can conclude that firm $i$’s best response can not be equal to $d^i_{sp} \left( \hat{Q}^j \right)$. Based on Property 2, we know that $f(Q^i, Q^j)$ is an increasing function of $Q^i$ for given $Q^j$, hence the new best response must be less than $d^i_{sp}\left(\hat{Q}^j\right)$. Therefore this can be considered as a limiting case of the situation in which $Q^i(\hat{Q}^j)$ is between two total demand realizations, which will be discussed next.

**Case B.** We, now, show $0 \geq \frac{\partial f}{\partial Q^j} \geq -b$ when $Q^i(\hat{Q}^j)$ is between two total demand realizations. Note that in this case $f(Q^i, Q^j)$ is well defined and differentiable with respect to both $Q^i$ and $Q^j$. Taking the total derivative of $f(Q^i, Q^j)$ with respect to $Q^j$, we have:

$$\frac{df(Q^i, Q^j)}{dQ^j} = \frac{\partial f(Q^i, Q^j)}{\partial Q^j} + \frac{\partial f(Q^i, Q^j)}{\partial Q^i} \frac{\partial Q^i}{\partial Q^j},$$

where

$$\frac{\partial f(Q^i, Q^j)}{\partial Q^j} = \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} \{ d_{xy}^i(Q^j) < Q^i \} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^j)} \cdot u'' \left( (C_u + C_o) d_{xy}^i(Q^j) - C_o Q^j \right) \cdot (C_u + C_o) b \cdot 1 \{ d_y^i > Q^j \}$$

and

$$\frac{\partial f(Q^i, Q^j)}{\partial Q^i} = - \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} \{ d_{xy}^i(Q^j) < Q^i \} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^j)} \left[ C_u u'' \left( (C_u + C_o) d_{xy}^i(Q^j) - C_o Q^j \right) \right]$$

$$+ C_u \frac{u''(C_u Q^j)}{u'(C_u Q^j)} u' \left( (C_u + C_o) d_{xy}^i(Q^j) - C_o Q^j \right).$$

To find $\frac{\partial Q^i}{\partial Q^j}$, set $\frac{df(Q^i, Q^j)}{dQ^j} = 0$ which results in:

$$\frac{\partial Q^i}{\partial Q^j} = - \frac{\frac{\partial f(Q^i, Q^j)}{\partial Q^j}}{\frac{\partial f(Q^i, Q^j)}{\partial Q^j}}.$$

We have $\frac{\partial f(Q^i, Q^j)}{\partial Q^j} \geq 0$ and $\frac{\partial f(Q^i, Q^j)}{\partial Q^j} > 0$ since $u''(x) \leq 0$. Therefore $\frac{\partial Q^i}{\partial Q^j} \leq 0$. Next we show that $\frac{\partial Q^i}{\partial Q^j} \geq -b$:

$$- \frac{\frac{df(Q^i, Q^j)}{df(Q^j, Q^j)}}{\frac{df(Q^i, Q^j)}{df(Q^j, Q^j)}} \geq -b \iff \frac{\partial f(Q^i, Q^j)}{\partial Q^j} \leq b \frac{\partial f(Q^i, Q^j)}{\partial Q^j}.$$
which is equivalent to showing
\[
\frac{\partial f(Q^i, Q^j)}{\partial Q^j} - b \frac{\partial f(Q^i, Q^j)}{\partial Q^i} \leq 0
\]

Substituting for \(\frac{\partial f(Q^i, Q^j)}{\partial Q^j}\) and \(\frac{\partial f(Q^i, Q^j)}{\partial Q^i}\) in the above inequality, we have:
\[
\sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d^i_{xy}(Q^j) < Q^i\} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^i)} \cdot u'' \left((C_u + C_o)d^i_{xy}(Q^j) - C_o Q^i\right) \cdot \\
\left[\left(-b (C_u + C_o) \cdot 1\{d^j_y > Q^j\}\right) + b \left( C_o + C_u \cdot \frac{g(C_u Q^i)}{g((C_u + C_o)d^i_{xy}(Q^j) - C_o Q^i)}\right) \right] \leq 0
\] (2.42)

Similar to above, define:
\[
g(x) = \frac{u''(x)}{u'(x)}
\]

Substituting \(g(x)\) in inequality (2.42), we have:
\[
\sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d^i_{xy}(Q^j) < Q^i\} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^i)} \cdot u'' \left((C_u + C_o)d^i_{xy}(Q^j) - C_o Q^i\right) \cdot \\
\left[\left(-b (C_u + C_o) \cdot 1\{d^j_y > Q^j\}\right) + b \left( C_o + C_u \cdot \frac{g(C_u Q^i)}{g((C_u + C_o)d^i_{xy}(Q^j) - C_o Q^i)}\right) \right] \leq 0
\]
The above inequality holds since (a) \(g'(x) \geq 0\) (based on Assumption 1) and (b) \(C_u Q^i \geq (C_u + C_o)d^i_{xy}(Q^j) - C_o Q^i\) because \(Q^i \geq d^i_{xy}(Q^j)\).

Therefore we have shown the claim for the both possibilities: \(Q^i(Q^j)\) is equal to one of the total demand realizations, or \(Q^i(Q^j)\) is between two total demand realizations.

\[
0 \geq \frac{\partial Q^i}{\partial Q^j} \geq -b.
\]

\[\square\]

**Proof of Proposition 2.1. Equilibrium existence.** It is well-known that there exists at least one Nash equilibrium in submodular two-player games. That \(U^i(Q^i|Q^j)\) is
submodular follows because the right and left derivatives of \( u^i \left( \Pi^i \left( Q^i, d_{xy}^i(Q^j) \right) \right) \) with respect to \( Q^j \) are non-increasing in \( Q^j \) for all \( x, y \). The left and right derivatives satisfy, respectively,

\[
\begin{align*}
&u'' \left( \Pi^i \left( Q^i, d_{xy}^i(Q^j) \right) \right) \left( -C^i_a 1 \left\{ d_{xy}^i(Q^j) < Q^i \right\} + C^i_u 1 \left\{ d_{xy}^i(Q^j) \geq Q^i \right\} \right) \\
&u'' \left( \Pi^i \left( Q^i, d_{xy}^i(Q^j) \right) \right) \left( -C^i_a 1 \left\{ d_{xy}^i(Q^j) < Q^i \right\} \right).
\end{align*}
\]

The claim follows since \( u'' < 0 \) and \( d_{xy}^i(Q^j) \) is non-increasing in \( Q^j \).

**Uniqueness.** From Lemma 2.2.2 we have \( \left| f^-_i(Q^j) \right|, \left| f^+_i(Q^j) \right| \leq b^i \leq 1 \) for \( i \neq j \). Therefore if \( b^i < 1 \) then firm \( i \)'s best response function \( f^i(Q^j) \) is a contraction and the equilibrium is unique.

**Proof of Theorem 2.1.** Fix \( \mathbf{R} \) and the equilibrium \( Q^*(\mathbf{R}) \).

First note that the best response function of each firm has exactly one piece that satisfies (2.10). This holds due to two facts that follow from Lemma 2.4. (i) The best response function of each firm has exactly one or two pieces that satisfy (2.8). (ii) If two pieces of firm \( i \)'s best response function satisfy (2.8), i.e., \( g^i_1(Q^{-i}(\mathbf{R}), R^i) = g^i_2(Q^{-i}(\mathbf{R}), R^i) = Q^i(\mathbf{R}) \), then they only agree for \( Q^{-i} = Q^{-i}(\mathbf{R}) \) and at the initial \( \mathbf{R} \) vector, that is \( g^i_1(Q^{-i}, R^i) \neq g^i_2(Q^{-i}, R^i) \) for \( Q^{-i} \neq Q^{-i}(\mathbf{R}) \), and \( g^i_1(Q^{-i}(\mathbf{R}(\delta)), R^i(\delta)) \neq g^i_2(Q^{-i}(\mathbf{R}(\delta)), R^i(\delta)) \) for \( \delta \in (0, \bar{\delta}) \).

Suppose that \( g^A(Q^{-B}, R^A) \) and \( g^B(Q^{-A}, R^B) \) are pieces of the firm \( A \) and \( B \) best response functions, respectively, which satisfy (2.8). It is evident from Lemma 2.4 that \( g^A(Q^{-B}, R^A) \) and \( g^B(Q^{-A}, R^B) \) are continuously differentiable. Let \( \overline{Q}(\mathbf{R}(\delta)) = (Q^A(\mathbf{R}(\delta)), Q^B(\mathbf{R}(\delta))) \) denote the order vector that satisfies

\[
\overline{Q}^i(\mathbf{R}(\delta)) = g^i(\overline{Q}^{-i}(\mathbf{R}(\delta)), R^i(\delta)), \ i \in \{A, B\}, \quad (2.43)
\]

for \( \delta \) in a neighborhood of 0, where \( \overline{Q}(\mathbf{R}(0)) = Q^*(\mathbf{R}(0)) = Q^*(\mathbf{R}) \). As we show next, this vector is well defined, but first notice that for \( \delta \neq 0 \) we have \( \overline{Q}(\mathbf{R}(\delta)) = Q^*(\mathbf{R}(\delta)) \) only if (2.10) holds.
Taking the total derivative of (2.43) with respect to \(y\) yields
\[
\frac{\partial Q_i(R)}{\partial R^A} + \frac{\partial Q_i(R)}{\partial R^B} = \frac{\partial g^i(Q^{-i}_i(R), R^i)}{\partial R^A} \left( \frac{\partial Q^{-i}_i(R)}{\partial R^A} + \frac{\partial Q^{-i}_i(R)}{\partial R^B} \right) + \frac{\partial g^i(Q^{-i}_i(R), R^i)}{\partial R^i}, \; i \in \{A, B\}.
\]

Solving these equations yields
\[
\frac{\partial Q^i(R)}{\partial R^A} + \frac{\partial Q^i(R)}{\partial R^B} = \frac{\partial g^{-i}(Q^{-i}(R), R^{-i}) \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}} + \partial g^i(Q^{-i}(R), R^i)}{1 - \partial g^{-i}(Q^{-i}(R), R^{-i}) \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}}}. \tag{2.44}
\]

By Part 2 of Lemma 2.2, all pieces of the best response functions satisfy
\[
\left| \frac{\partial g^i(Q^{-i}, R^i)}{\partial Q^{-i}} \right| \leq b^i < 1 \quad \text{and} \quad \left| \frac{\partial g^{-i}(Q^{-i}, R^{-i})}{\partial Q^{-i}} \right| \leq b^{-i} < 1,
\]
which implies that the partial derivatives in (2.44) are well defined. Furthermore, the RHS of (2.44) has the same sign as its numerator, and
\[
\frac{\partial Q^i(R)}{\partial R^A} + \frac{\partial Q^i(R)}{\partial R^B} > 0 \iff \frac{\partial g^{-i}(Q^{-i}(R), R^{-i}) \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}} + \partial g^i(Q^{-i}(R), R^i)}{1 - \partial g^{-i}(Q^{-i}(R), R^{-i}) \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}}} > 0.
\]

To complete the proof, note that for the unique pair of best response function pieces \(g^A(Q^B, R^A)\) and \(g^B(Q^A, R^B)\) that satisfy (2.10), we have \(\overline{Q}(R(\delta)) = Q^*(R(\delta))\) for \(\delta \in [0, \delta]\), and
\[
\frac{\partial Q_{+}^{i}(R)}{\partial R^A} + \frac{\partial Q_{+}^{-i}(R)}{\partial R^B} = \frac{\partial Q^{i}(R)}{\partial R^A} + \frac{\partial Q^{i}(R)}{\partial R^B}.
\]

**Proof of Corollary 2.1.** **Part 1.** If firms are symmetric then
\[
\frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial R^{-i}} = \frac{\partial g^{i}(Q^{-i}(R), R^i)}{\partial R^i} \leq 0.
\]
That these partial derivatives are nonpositive follows from the best response function in Lemma 2.4. If they are zero, then (2.11) is clearly violated. If the inequality is strict, then (2.11) is equivalent to
\[
\frac{\partial g^{i}(Q^{-i*}(R), R^i)}{\partial Q^{-i}} < -1,
\]
which cannot hold by Part 2 of Lemma 2.2.
Part 2. Suppose to the contrary that (2.11) holds for $i$ and its rival:

\[
\frac{\partial g^i(Q^{-i} (R), R^i)}{\partial Q^{-i}} \frac{\partial g^{-i}(Q^{i*} (R), R^{-i})}{\partial R^i} + \frac{\partial g^i(Q^{-i} (R), R^i)}{\partial R^i} > 0,
\]

\[
\frac{\partial g^{-i}(Q^i (R), R^{-i})}{\partial Q^i} \frac{\partial g^{-i}(Q^{i*} (R), R^{-i})}{\partial R^{-i}} > 0.
\]

Multiply the first inequality by $-\partial g^{-i}(Q^{i*} (R), R^{-i})/\partial Q^i$ (which is nonnegative), and add the result to the second to obtain:

\[
\left(1 - \frac{\partial g^i(Q^{-i} (R), R^i)}{\partial Q^{-i}} \frac{\partial g^{-i}(Q^{i*} (R), R^{-i})}{\partial Q^i}\right) \frac{\partial g^{-i}(Q^{i*} (R), R^{-i})}{\partial R^{-i}} > 0.
\]

This cannot hold since the bracketed term is positive and its multiplier is nonnegative. □

**Proof of Lemma 2.3** Since $\lim_{x \to \infty} u'(x) = 0$, $\lim_{x \to -\infty} u'(-x) = \infty$, and $U''_k < 0$, for $k \in \{1, 2, \ldots, N - 1\}$ the maximizer $Q^*_k \triangleq \arg \max_Q U'_k(Q)$ is well-defined and the unique solution of the first-order condition of $U'_k(Q) = 0$. The fact that $U'_{k-1}(Q) > U'_k(Q)$ for all $k$ and $Q$, and $U''_k < 0$ implies that $Q^*_1 > Q^*_2 > \ldots > Q^*_N$. That $Q^* = \min(Q^*_m, d_{m+1})$ follows since $Q^*_k \geq d_k \Leftrightarrow U'_k(d_k) \geq 0$. Therefore $U'_m(d_m) \geq 0 > U'_m(d_{m+1})$, which implies the claim. □

**Proof of Lemma 2.4.** By Lemma 2.1 the best response $f^A(Q^B)$ is unique. The expressions in (2.18)-(2.20) are immediate from Lemma 2.3, by using the appropriate demand distribution to calculate the maximizers $Q^*_k$ in (2.15) and the conditions in (2.16)-(2.17). For illustration, in Part 1, $Q^B \geq d_H$ implies that firm $A$’s total demand distribution has $N = 2$ mass points, $d^A_{LL}(Q^B) = d^A_{LH}(Q^B) = d_L$ with probability $q_L = q_{LL} + q_{LH}$, and $d^A_{HLL}(Q^B) = d^A_{HH}(Q^B) = d_H$ with probability $1 - q_L$. Substitute $d_1 = d_L$ and $d_2 = d_H$ with $q_1 = q_L$ and $q_2 = q_H$, in (2.15) to get the maximizer

\[
Q^*_1 = \frac{1}{R^A} \ln \left( \frac{C^A_w}{C^A_o} q_L \frac{1 - q_L}{q_L \exp \left(-\frac{1}{R^A d_L}\right)} \right) = \frac{1}{R^A} \ln \left( \frac{C^A_w}{C^A_o} \frac{1}{q_L} \frac{1}{d_L}\right).
\]

This maximizer is the optimal order quantity if and only if $d_L \leq Q^*_1 \leq d_H$, or equivalently, (2.16) holds for $k = 1$. This yields the condition in (2.18) for $f^A(Q^B) = Q^*_1$. 
In Part 2, \( d_L \leq Q^B < d_H \) implies the total demand distribution (2.14) with \( N = 4 \) mass points. Substituting \( d_1 = d_{LL}^A, d_2 = d_{LH}^A, d_3 = d_{HL}^A \) and \( d_4 = d_{HH}^A \), along with the corresponding probabilities, into (2.15)-(2.17) yields (2.19)-(2.20).

**Proof of Proposition 2.2.** The proof is based on Lemma 2.4.

We illustrate this by proving Case 1. Suppose that firm \( B \) orders \( Q^B < d_H \). We need to show that firm \( A \)'s best response \( f^A (Q^B) = d_{LH}^A (Q^B) \) if the conditions \( Q_{LH}^A (Q^B) \leq d_{LH}^A (Q^B) < Q_{LL}^A \) are satisfied. By (2.19)-(2.20) of Lemma 2.4, this holds if

\[
t_{LL}^A (Q^B) \exp(R^A d_{LH}^A (Q^B)) < \frac{C_u^A}{C_o^A} \leq t_{LH}^A (Q^B) \exp(R^A d_{LH}^A (Q^B)),
\]

(2.45) where (noting that \( d_{LL}^A = d_L \))

\[
t_{LL}^A = \frac{q_{LL} \exp(-R^A d_L)}{1 - q_{LL}},
\]

\[
t_{LH}^A (Q^B) = \frac{q_{LL} \exp(-R^A d_L) + q_{LH} \exp(-R^A d_{LH}^A (Q^B))}{1 - q_{LL} - q_{LH}}.
\]

Substituting the thresholds into (2.45) and rearranging terms yields the conditions \( Q_{LH}^A (Q^B) \leq d_{LH}^A (Q^B) < Q_{LL}^A \). Next, suppose that firm \( A \) orders \( Q^A < d_H \). We need to show that firm \( B \)'s best response \( f^B (Q^A) = Q_{LL}^B \) if the conditions \( d_L < Q_{LL}^B \leq d_{LH}^B (Q^A) \) are satisfied.

By (2.19)-(2.20) of Lemma 2.4, this holds if (noting that \( d_{LL}^B = d_L \))

\[
t_{LL}^B \exp(R^B d_L) < \frac{C_u^B}{C_o^B} \leq t_{LL}^B \exp(R^B d_{LH}^B (Q^A)), \quad \text{where } t_{LL}^B = \frac{q_{LL} \exp(-R^B d_L)}{1 - q_{LL}},
\]

which is equivalent to \( d_L < Q_{LL}^B \leq d_{LH}^B (Q^A) \).

The proofs of the remaining cases follow the same line of argument and are therefore omitted.

**Proof of Corollary 2.2.** First consider identical firms. Then both firms have exactly the same best response function and equal equilibrium order quantities. It follows from (2.18)-(2.20) in Lemma 2.4 that a firm’s best response function weakly increases in \( C_o^i/C_u^i \) and \( b_i \), and weakly decreases in \( R^i \). Since the best response functions are weakly decreasing by Lemma 2.2, it follows that in equilibrium, the firm with higher \( C_o^i/C_u^i \), lower \( R^i \), or higher \( b^i \) orders at least as much as its rival.
**Proof of Theorem 2.2.** The proof consists of checking for each of the 17 equilibrium cases specified in Proposition 2.2 whether the necessary and sufficient conditions (2.10)-(2.11) of Theorem 2.1 can be satisfied, and if so under what conditions. Doing so involves two steps.

Step 1. Identify the pair of best response function pieces, \( g_A(Q^B, R^A), g_B(Q^A, R^B) \), that satisfy (2.10), which we replicate here. Let \( R(\delta) = R + \delta \epsilon \). For some \( \delta > 0 \) and \( \delta \in [0, \delta) \),

\[
Q^{*i}(R(\delta)) = g^i(Q^{-*i}(R(\delta)), R^i(\delta)) = f^i(Q^{-*i}(R(\delta)), R^i(\delta)), \quad i \in \{A, B\}. \tag{2.46}
\]

Step 2. Check whether the functions identified in step 1 satisfy condition (2.11), which we also replicate here, for \( i = A \) and \(-i = A\):

\[
\frac{\partial g^A(Q^{B*}(R), R^A)}{\partial Q^B} \frac{\partial g^B(Q^{A*}(R), R^B)}{\partial R^B} > -\frac{\partial g^A(Q^{B*}(R), R^A)}{\partial R^A}. \tag{2.47}
\]

For reference, we replicate the best response function pieces (2.21)-(2.25) of Proposition 2.2, making their dependence on \( R^i \) and \( Q^{-i} \) explicit. We suppress these arguments unless they are needed for clarity.

\[
\begin{align*}
  d_{LH}^i(Q^{-i}) &= d_L + b^i(d_H - Q^{-i}) \quad \text{and} \quad d_{HH}^i(Q^{-i}) = d_H + b^i(d_H - Q^{-i}), \tag{2.48} \\
  Q_{LM}^i(R^i) &= \frac{1}{R^i} \ln \left( \frac{C_{u}^i}{C_{o}^i} \frac{1 - q_{L}}{q_{L} \exp(-R^i d_{L})} \right), \tag{2.49} \\
  Q_{LL}^i(R^i) &= \frac{1}{R^i} \ln \left( \frac{C_{u}^i}{C_{o}^i} \frac{1 - q_{LL}}{q_{LL} \exp(-R^i d_{L})} \right), \tag{2.50} \\
  Q_{LH}^i(Q^{-i}, R^i) &= \frac{1}{R^i} \ln \left( \frac{C_{u}^i}{C_{o}^i} \frac{1 - q_{LL} - q_{LH}}{q_{LH} \exp(-R^i d_{L}) + q_{LH} \exp(-R^i d_{LH}(Q^{-i}))} \right), \tag{2.51} \\
  Q_{HL}^i(Q^{-i}, R^i) &= \frac{1}{R^i} \ln \left( \frac{C_{u}^i}{C_{o}^i} \frac{1 - q_{LL} - q_{LH} - q_{HL}}{q_{LH} \exp(-R^i d_{L}) + q_{LH} \exp(-R^i d_{LH}(Q^{-i})) + q_{HL} \exp(-R^i d_{H})} \right). \tag{2.52}
\end{align*}
\]
The partial derivatives of (2.49)-(2.52) with respect to $R^i$ are nonnegative and satisfy

$$\frac{\partial Q^i_k}{\partial R^i} = Q''_k (R^i) = \frac{d_L - Q^i_k}{R^i} \leq 0 \text{ for } k \in \{M, LL\},$$  \hspace{1cm} (2.53)

$$\frac{\partial Q^i_{LL}}{\partial R^i} = \frac{1}{R^i} \left[ q_{LL}d_L \exp(-R^i d_L) + q_{LL}d_{LL} \exp(-R^i d_{LL}) - \frac{Q^i_{LL}}{R^i} \right] \leq 0, \hspace{1cm} (2.54)$$

$$\frac{\partial Q^i_{HL}}{\partial R^i} = \frac{1}{R^i} \left[ q_{LL}d_L \exp(-R^i d_L) + q_{HL}d_{HL} \exp(-R^i d_{HL}) + q_{HL}d_H \exp(-R^i d_H) - \frac{Q^i_{HL}}{R^i} \right] < 0. \hspace{1cm} (2.55)$$

The inequality in (2.54) holds since $Q^i_{LL} \geq d^i_{LL} \geq d_L$, the one in (2.55) since $Q^i_{HL} \geq d_H > d_L$.

The partial derivatives of (2.51)-(2.52) with respect to $Q^{-i}$ satisfy

$$\frac{\partial Q^i_{LL}}{\partial Q^{-i}} = -b^i \frac{q_{LL} \exp(-R^i d_{LL})}{q_{LL} \exp(-R^i d_L) + q_{HL} \exp(-R^i d_{HL})},$$  \hspace{1cm} (2.56)

$$\frac{\partial Q^i_{HL}}{\partial Q^{-i}} = -b^i \frac{q_{LL} \exp(-R^i d_{HL})}{q_{LL} \exp(-R^i d_L) + q_{HL} \exp(-R^i d_{HL}) + q_{HL} \exp(-R^i d_H)}. \hspace{1cm} (2.57)$$

**Case 1.** The equilibrium order quantities $Q^{A*} (R)$ and $Q^{B*} (R)$ for case 1 satisfy

$$Q^A_{LL} (Q^{B*} (R), R^A) \leq Q^{A*} (R) = d^A_{LL} (Q^{B*} (R)) < Q^A_{LL} (R^A)$$

$$d_L < Q^{B*} (R) = Q^B_{LL} (R^B) \leq d^B_{LL} (Q^{A*} (R)).$$

Step 1. The best response function pieces that satisfy (2.46) are $g^A = d^A_{LL}$ and $g^B = Q^B_{LL}$. This is obvious if the inequalities in the equilibrium conditions are strict. If $Q^A_{LL} (Q^{B*} (R), R^A) = Q^{A*} (R) = d^A_{LL} (Q^{B*} (R))$ and/or $Q^{B*} (R) = Q^B_{LL} (R^B) = d^B_{LL} (Q^{A*} (R))$, this follows since $Q^B_{LL} (R^B) < 0$ by (2.53) and $\partial Q^A_{LL}/\partial R^A < 0$ by (2.54), whereas $\partial d^i_{LL}/\partial R^i = 0$ for $i \in \{A, B\}$.

Step 2. That (2.47) holds follows since $g^A = d^A_{LL}$ is constant in $R^A$ and

$$\frac{\partial g^B}{\partial R^B} \frac{\partial g^A}{\partial Q^B} = Q^B_{LL} (R^B) \frac{\partial d^A_{LL}}{\partial Q^B} > 0.$$

**Case 2.** The equilibrium order quantities $Q^{A*} (R)$ and $Q^{B*} (R)$ for case 2 satisfy

$$Q^A_{LL} (Q^{B*} (R), R^A) \leq Q^{A*} (R) = d^A_{LL} (Q^{B*} (R)) < Q^A_{LL} (R^A),$$

$$d^B_{LL} (Q^{A*} (R)) < Q^{B*} (R) = Q^B_{LL} (Q^{A*} (R), R^B) < d_H.$$
Step 1. The best response function pieces that satisfy (2.46) are \( g^i = d_{LH}^i \) and \( g^B = Q_{LH}^B \). This follows since \( \partial Q_{LH}^A / \partial R^A < 0 \) by (2.54), whereas \( \partial d_{LH}^A / \partial R^A = 0 \).

Step 2. That (2.47) holds follows since \( g^A = d_{LH}^A \) is constant in \( R^A \) and strictly decreasing in \( Q^B \), and from (2.54) we have

\[
\frac{\partial g^B(Q^A^*(R), R^B)}{\partial R^B} < 0,
\]

because \( Q^{B^*}(R) < d_H \) implies that \( Q^A^*(R) = d_{LH}^A (Q^{B^*}(R)) > d_L \).

**Case 3.** The equilibrium order quantities \( Q^A^*(R) \) and \( Q^{B^*}(R) \) for case 3 satisfy

\[
d_{LH}^i (Q^{-i^*}(R)) < Q^{i^*}(R) = Q_{LH}^i (Q^{-i^*}(R), R^i) \leq d_H, \quad i \in \{A, B\}.
\]

Step 1. We have two cases. If both equilibrium conditions hold with strict inequality, then the best response function pieces that satisfy (2.46) are \( g^i = Q_{LH}^i, \quad i \in \{A, B\} \).

However, if \( Q^{i^*}(R) = Q_{LH}^i (Q^{-i^*}(R), R^i) = d_H \) for \( i = A \) and/or \( i = B \), then the pair \( g^i = Q_{LH}^i, \quad i \in \{A, B\} \) need not satisfy (2.46), in which case we must have \( g^i = d_H \) for \( i = A \) and/or \( i = B \).

Step 2. If \( g^i = d_H \) for \( i = A \) and/or \( i = B \), then (2.47) cannot hold: its LHS is zero, and its RHS is nonnegative for every piece of the best response functions. Therefore, a necessary condition for (2.47) is that \( Q_{LH}^i (Q^{-i^*}(R), R^i) < d_H, \quad i \in \{A, B\} \), so that \( g^i = Q_{LH}^i, \quad i \in \{A, B\} \) satisfy (2.46). For these functions (2.47) is equivalent to

\[
\frac{\partial Q_{LH}^B(Q^A^*(R), R^B)}{\partial R^B} \frac{\partial Q_{LH}^A(Q^{B^*}(R), R^A)}{\partial Q^B} > -\frac{\partial Q_{LH}^A(Q^{B^*}(R), R^A)}{\partial R^A},
\]

where each term is strictly negative. Substituting from (2.54) and (2.56) yields, after some algebra:

\[
R^B Q^A - d_L - \frac{q_{LL} b^A (d_H - Q^{B^*})}{q_{LH} \exp(R^A b^A (d_H - Q^{B^*})) + q_{LL}} < \frac{b^A q_{LH}}{q_{LL} \exp(R^A b^A (d_H - Q^{B^*})) + q_{LL}}.
\]

**Case 4.** The equilibrium order quantities \( Q^A^*(R) \) and \( Q^{B^*}(R) \) satisfy

\[
d_{LH}^A (Q^{B^*}(R)) < Q^A^*(R) = Q_{LH}^A (Q^{B^*}(R), R^A) \leq d_H,
\]

\[
d_L < Q^{B^*}(R) = Q_{LL}^B (R^B) \leq d_{LH}^B (Q^A^*(R)).
\]
Step 1. The best response function pieces that satisfy (2.46) are \( g^A = Q_{HL}^A \) and \( g^B = Q_{LL}^B \). This follows since \( Q_{LL}^B(R^B) < 0 \) by (2.53) and \( \partial Q_{HL}^A/\partial R^A < 0 \) by (2.54), whereas \( d_H \) and \( d_{HL}^B \) are independent of risk aversion.

Step 2. Since \( Q_{LL}^B \) is constant in \( Q^A \), it is clear that firm \( B \) reduces its order quantity as both firms become more risk averse. For \( g^A = Q_{HL}^A \) and \( g^B = Q_{LL}^B \), condition (2.47) is equivalent to

\[
Q_{LL}^B(R^B) \frac{\partial Q_{HL}^A(Q^A*(R), R^A)}{\partial Q^B} > -\frac{\partial Q_{HL}^A(Q^A*(R), R^A)}{\partial R^A},
\]

where each term is strictly negative. Substituting from (2.53) and (2.56) yields, after some algebra:

\[
b^A q_{HL} \exp \left( -R^A b^A (d_H - Q^A*) \right) \left( R^A \frac{Q^B* - d_H - Q^B*}{R^B} + d_H - Q^B* \right) > Q^A* - d_L.
\]

(2.59)

Case 5. The equilibrium order quantities \( Q^A*(R) \) and \( Q^B*(R) \) satisfy

\[
d_H \leq Q^A*(R) = Q_{HL}^A(Q^B*(R), R^A) \leq d_{HH}^A(Q^B*(R)) < Q_{HL}^A(Q^B*(R))
\]

\[
d_L < Q^B*(R) = Q_{M}^B(R^B) < d_H.
\]

Step 1. The best response function pieces that satisfy (2.46) are \( g^B = Q_{M}^B \) for firm \( B \), and for firm \( A \) either \( g^A = d_H \) or \( g^A = Q_{HL}^A \). The function \( d_{HH}^A \) cannot be part of the best response as \( R \) increases, even if

\[
Q_{HL}^A(Q^B*(R), R^A) = d_{HH}^A(Q^B*(R))
\]

this is because \( d_{HH}^A \) is constant in \( R^A \) while \( Q_{HL}^A \) decreases in \( R^A \) and \( Q_{M}^B \) decreases in \( R^B \). As shown for case 3 if \( g^A = d_H \) then the condition (2.47) cannot hold. Therefore, the only pair that satisfies (2.46) and for which (2.47) can possibly hold is \( g^A = Q_{HL}^A \) and \( g^B = Q_{M}^B \).

Step 2. For \( g^A = Q_{HL}^A \) and \( g^B = Q_{M}^B \), condition (2.47) is equivalent to

\[
Q_{M}^B(R^B) \frac{\partial Q_{HL}^A(Q^B*(R), R^A)}{\partial Q^B} > -\frac{\partial Q_{HL}^A(Q^B*(R), R^A)}{\partial R^A},
\]
where each term is strictly negative. Substituting from (2.53) and (2.55) yields, after some algebra the following necessary condition for (2.47) to hold:

\[
q_{LH}b^A(Q^B - d_L)\left(1 - \frac{R^B}{R^A}\right) > \frac{R^B}{R^A}(d_H - d_L)\left[q_{LL}\exp\left(\frac{R^A}{b^A}(d_H - Q^B)\right) + q_{LH}(1 - b^A)\right].
\]

(2.60)

Note that this condition is violated if \(R^A \geq R^B\).

**Case 6.** The equilibrium order quantities \(Q^A^* (R)\) and \(Q^B^* (R)\) satisfy

\[
Q^A^* = d^A_{HH}(Q^B^* (R)) < Q^A_{HL}(Q^B^* (R), R^A) \quad d_L < Q^B^* = Q^B_M (R^B) < d_H.
\]

Step 1. The best response function pieces that satisfy (2.46) are \(g^A = d^A_{HH}\) and \(g^B = Q^B_M\).

Step 2. Since \(d^A_{HH}\) is constant in \(R^A\) but strictly decreasing in \(Q^B\), and \(g^B\) is strictly decreasing in \(R^B\), it follows that (2.47) holds.

**Cases 7-17.** The other equilibrium cases of Proposition 2.2 cannot satisfy (2.46) and (2.47).

In all of these cases, except for case 13 and 15, the best response function pieces that satisfy (2.46) have \(g^i = d^i_{LH}\) or \(g^i = d^i_H\) for at least one of the firms. In any such case, the LHS of (2.47) is zero whereas the RHS is nonnegative.

In case 13, the best response function pieces that satisfy (2.46) are \(g^i = d^i_{LH}\), for \(i = A \) and \(i = B\). Since \(d^i_{LH}\) is constant in \(R^i\), both sides of (2.47) are zero.

In case 15, the best response function pieces that satisfy (2.46) are \(g^i = Q^i_{LL}\), for \(i = A \) and \(i = B\). Since \(Q^i_{LL}\) is constant in \(Q^{-i}\), the LHS of (2.47) is zero whereas the RHS is nonnegative.

**Proof of Corollary 2.3.** Based on Corollary 2.2, when both firms are identical except for their \(C^i_u/C^i_o\) ratio, the firm with the larger ratio orders more at the equilibrium. By inspection of the equilibrium cases specified in Proposition 2.2 and Theorem 2.2, cases 1, 4 and 6 apply for the firm with the larger equilibrium order, and case 2 applies for
the firm with the smaller equilibrium order. This establishes Parts 1 and 2(a) of the Corollary.

Part 2(b). For equilibrium case 3 it is a priori not clear from Theorem 2.2 which of the firms increases its order in response to higher risk aversion. The following argument shows that it cannot be the firm with the higher $C_u^i/C_o^i$ ratio. In particular, the condition (2.27) cannot hold in this case. For concreteness, let $i = A$ be the firm that increases its order in response to higher risk aversion. In light of Corollary 2.2, we need to show that firm $A$ must have the smaller equilibrium order.

From (2.58) in the proof of Theorem 2.2, condition (2.27) for firm $A$ to increase its order from a case 3 equilibrium is equivalent to:

\[
\frac{R^B Q^A - d_L}{R^A Q^B - d_L} - \frac{q_{LL} b^A (d_H - Q^B)}{q_{LL} \exp(R^B b^A (d_H - Q^B)) + q_{LH}} < \frac{b^A q_{LH}}{q_{LL} \exp(R^A b^A (d_H - Q^B)) + q_{LH}}. \tag{2.61}
\]

With $R^i = R$, $K^i = C_u^i + C_o^i = K$ and $b^i = b$, this condition simplifies to

\[
\frac{Q^A - d_L}{Q^B - d_L} - \frac{q_{LL} b (d_H - Q^B)}{q_{LL} \exp(R b (d_H - Q^B)) + q_{LH}} < b \frac{q_{LH}}{q_{LL} \exp(R b (d_H - Q^B)) + q_{LH}}. \tag{2.62}
\]

We show that the above condition does not hold if the order quantity of Firm $A$ is larger than the order quantity of Firm $B$ at the equilibrium, $Q^A > Q^B$. Before attempting to show the claim, note that condition (2.62) does not hold for the case that both firms’ order quantities are identical (i.e. in this case, both firms have the same profitability ratio), since the LHS of condition (2.62) is 1 and the RHS is less than 1, so the condition is violated.

Suppose Firm $A$’s order quantity is larger than Firm $B$’s order quantity at the equilibrium, $Q^A > Q^B$. We know there exist $\epsilon = \sqrt{0.5d}$, where $d$ is the distance between point $(Q^A, Q^B)$ and its orthogonal projection $(Q^A_p, Q^B_p)$ on diagonal line $(Q^A = Q^B)$, such that

1) $Q^A = Q^A_p + \epsilon$, 2) $Q^B = Q^B_p - \epsilon$ and 3) $Q^A_p = Q^B_p$
We know that condition (2.62), does not hold for the following order quantities:

\((Q_p^A, Q_p^B)\) since \(Q_p^A = Q_p^B\). Define the following function:

\[
g(Q^i, Q^j) = Q^i - d_L - \frac{q_{LL}b(d_H - Q^j)}{q_{LL}\exp(Rb(d_H - Q^j)) + q_{LL}}
\]

Note that if \(d_L + b'(d_H - Q^{-}) < Q^i < d_H\), for \(i = A, B\), then \(g(Q^i, Q^{-}) < 1\), but we show that the LHS becomes bigger than 1. The LHS of condition (2.62) can be written as

\[
g(Q^i, Q^j) = Q^i - d_L - \frac{q_{LL}b(d_H - Q^j)}{q_{LL}\exp(Rb(d_H - Q^j)) + q_{LL}} > Q^i - d_L - b(d_H - Q^j) > 0.
\]

Therefore, \(g(Q^i, Q^{-}) > 0\) for \((Q^i, Q^{-}) = (Q_p^A + \epsilon, Q_p^B - \epsilon)\) and \((Q^i, Q^{-}) = (Q_p^B - \epsilon, Q_p^A + \epsilon)\).

We have:

\[
g(Q_p^A + \epsilon, Q_p^B - \epsilon) = Q_p^A + \epsilon - d_L - \frac{q_{LL}b(d_H - Q_p^B + \epsilon)}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LL}}
\]

\[
= Q_p^A - d_L - \frac{q_{LL}b(d_H - Q_p^B)}{q_{LL}\exp(Rb(d_H - Q_p^B)) + q_{LL}} + \epsilon \left(1 - \frac{q_{LL}b}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LL}}\right)
\]

\[
> g(Q_p^A, Q_p^B) + \epsilon \left(1 - \frac{q_{LL}b}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LL}}\right),
\]

and

\[
g(Q_p^B - \epsilon, Q_p^A + \epsilon) = Q_p^B - \epsilon - d_L - \frac{q_{LL}b(d_H - Q_p^A - \epsilon)}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LL}}
\]

\[
= Q_p^B - d_L - \frac{q_{LL}b(d_H - Q_p^A)}{q_{LL}\exp(Rb(d_H - Q_p^A)) + q_{LL}} - \epsilon \left(1 - \frac{q_{LL}b}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LL}}\right)
\]

\[
< g(Q_p^B, Q_p^A) - \epsilon \left(1 - \frac{q_{LL}b}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LL}}\right).
\]

When Firm A orders \(Q_p^A + \epsilon\) and Firm B orders \(Q_p^B - \epsilon\) the RHS of condition (2.62) is still less than 1, but we show that the LHS becomes bigger than 1. The LHS of condition (2.62) can be written as \(g(Q_p^A + \epsilon, Q_p^B - \epsilon)/g(Q_p^B - \epsilon, Q_p^A + \epsilon)\). On the other hand we have:

\[
g(Q_p^A + \epsilon, Q_p^B - \epsilon) - g(Q_p^B - \epsilon, Q_p^A + \epsilon) > \frac{g(Q_p^A, Q_p^B) + \epsilon \left(1 - \frac{q_{LL}b}{q_{LL}\exp(Rb(d_H - Q_p^A + \epsilon)) + q_{LL}}\right)}{g(Q_p^B, Q_p^A) - \epsilon \left(1 - \frac{q_{LL}b}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LL}}\right)} > 1
\]

On the other hand we have:
where the first inequality holds because of inequalities (2.63) and (2.64), the second inequality holds since $Q_A^p = Q_B^p$, and therefore $g(Q_A^p, Q_B^p) = g(Q_B^p, Q_A^p)$.

Therefore, if firm $A$ increases its order in response to higher risk aversion, it must be the firm with the smaller initial equilibrium order, and therefore the firm with the smaller $C_u^i / C_o^i$ ratio.

Part 3. That Case 5 never holds follows since the necessary condition (2.60) is violated when $R^A = R^B$. See the proof of Theorem 2.2.

Proof of Corollary 2.4. Based on Corollary 2.2, when both firms are identical except for their risk aversion parameters, the firm with the lower risk aversion parameter orders more at the equilibrium. By inspection of the equilibrium cases specified in Proposition 2.2 and Theorem 2.2, cases 1, 4 and 6 apply for the firm with the larger equilibrium order, and case 2 applies for the firm with the smaller equilibrium order. This establishes Parts 1 and 2(a) of the Corollary.

Part 2(b). For equilibrium case 3 it is a priori not clear from Theorem 2.2 which of the firms increases its order in response to higher risk aversion. The following argument shows that it cannot be the firm with the initially lower $R^i$. In particular, the condition (2.27) cannot hold in this case. For concreteness, let $i = A$ be the firm that increases its order in response to higher risk aversion. In light of Corollary 2.2, we need to show that firm $A$ must have the larger initial risk aversion parameter. From (2.58) in the proof of Theorem 2.2, with $b^i = b$, condition (2.27) for firm $A$ to increase its order from a case 3 equilibrium is equivalent to:

$$R^B Q^{A^*} - d_L - \frac{q_{LL} b (d_H - Q^{B*})}{q_{LL} \exp(R^A b (d_H - Q^{A^*}))+ q_{LH}} < \frac{b q_{LH}}{q_{LL} \exp(R^B b (d_H - Q^{B*}))+ q_{LH}}.$$  \hspace{1cm} (2.65)

We show that the above condition does not hold if the order quantity of Firm $A$ is larger than the order quantity of Firm $B$ at the equilibrium, $Q^{A^*} > Q^{B^*}$. We know there exist $\epsilon = \sqrt{0.5d}$, where $d$ is the distance between point $(Q^{A^*}, Q^{B^*})$ and its orthogonal
projection \((Q_p^A, Q_p^B)\) on diagonal line \((Q^A = Q^B)\), such that

1) \(Q^{A*} = Q_p^A + \epsilon\), 2) \(Q^{B*} = Q_p^B - \epsilon\) and 3) \(Q_p^A = Q_p^B = Q_p\).

Note that if \(d_L + b^i(d_H - Q^i) < Q^i < d_H\), for \(i = A, B\), then

\[Q^i - d_L - \frac{q_LHb(d_H - Q^i)}{q_{LL} \exp(\bar{R}b(d_H - Q^i)) + q_{LH}} > Q^i - d_L - b(d_H - Q^j) > 0.\]

We show inequality (2.65) does not hold, when Firm A orders \(Q^{A*} = Q_p^A + \epsilon\) and firm B orders \(Q^{B*} = Q_p^B - \epsilon\). In light of Corollary 2.2, this can happen only if \(R^A < R^B\). In this case the RHS of the condition (2.65) is still less than 1, but we claim that the LHS is larger than 1:

\[
\frac{R^B}{R^A} \frac{Q_p^A + \epsilon - d_L - \frac{q_LHb(d_H - Q_p^B + \epsilon)}{q_{LL} \exp(\bar{R}b(d_H - Q_p^B + \epsilon)) + q_{LH}}}{Q_p^B - \epsilon - d_L - \frac{q_LHb(d_H - Q_p^A - \epsilon)}{q_{LL} \exp(\bar{R}b(d_H - Q_p^A - \epsilon)) + q_{LH}}} > 1. \tag{2.66}
\]

for simplicity we replace \(Q_p^A\) and \(Q_p^B\) with \(Q_p\). Therefore, we have:

\[
\frac{R^B}{R^A} \left( Q_p + \epsilon - d_L - \frac{q_LHb(d_H - Q_p + \epsilon)}{q_{LL} \exp(\bar{R}b(d_H - Q_p + \epsilon)) + q_{LH}} \right) > Q_p - \epsilon - d_L - \frac{q_LHb(d_H - Q_p - \epsilon)}{q_{LL} \exp(\bar{R}b(d_H - Q_p - \epsilon)) + q_{LH}}.
\]

Rearranging the above inequality yields:

\[
\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) \epsilon - \frac{R^B}{R^A} \left( \frac{q_LHb(d_H - Q_p)}{q_{LL} \exp(\bar{R}b(d_H - Q_p + \epsilon)) + q_{LH}} \right)
+ \frac{q_LHb(d_H - Q_p)}{q_{LL} \exp(\bar{R}b(d_H - Q_p - \epsilon)) + q_{LH}} - \frac{q_LHb(d_H - Q_p)}{q_{LL} \exp(\bar{R}b(d_H - Q_p - \epsilon)) + q_{LH}}
\]

\[-b \epsilon \left[ \frac{R^B}{R^A} \left( \frac{q_LHb(d_H - Q_p + \epsilon)}{q_{LL} \exp(\bar{R}b(d_H - Q_p + \epsilon)) + q_{LH}} \right) + \frac{q_LHb(d_H - Q_p - \epsilon)}{q_{LL} \exp(\bar{R}b(d_H - Q_p - \epsilon)) + q_{LH}} \right] > 0 \]

To establish (2.67), it suffice to show that the following inequality holds:

\[
\text{LHS of inequality (2.67)} > \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon - \frac{q_LHb(d_H - Q_p)}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right) b(d_H - Q_p). \tag{2.68}
\]
This is because the RHS of inequality (2.68) is larger than 0:

\[
\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon - \frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right) b (d_H - Q_p) > 0,
\]

because \( \frac{R^B}{R^A} - 1 > 0 \) (since \( R^A < R^B \)) and \( Q_p \geq d_L + b (d_H - Q_p) \) (based on Proposition 2.2, both firms order quantities are between their own minimum demand plus spillover and their own maximum demand).

Therefore we now focus on showing that inequality (2.68) holds. We break inequality (2.68) into two inequalities i.e. Inequality A and B where the LHS (RHS) of Inequality A plus the LHS (RHS) of Inequality B equals the LHS (RHS) of inequality (2.68). Therefore, showing inequalities A and B holds, proves that inequality (2.68) holds. We define the inequality A and B as follows:

Inequality A:

\[
-b \epsilon \left( \frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) + \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} \right) > \frac{\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) \epsilon}{\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon},
\]

and Inequality B:

\[
-\frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) + \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} > -\frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right).
\]

First we show that inequality A holds. Since

\[
\left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) < 1,
\]

\[
\frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} < 1.
\]
therefore we have:

\[
-b \left[ \frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) + \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} \right] \right.
\]

\[
\left. \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) \epsilon - b \epsilon \left[ \frac{R^B}{R^A} + 1 \right] \right)
\]

\[
\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon
\]

which proves that inequality A holds.

Next we show that inequality B holds. If \( R^A b (d_H - Q_p + \epsilon) \geq R^B b (d_H - Q_p - \epsilon) \), we have:

\[
\frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) - \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} < \frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right)
\]

which shows that inequality B holds in this case. However, if \( R^A b (d_H - Q_p + \epsilon) < R^B b (d_H - Q_p - \epsilon) \), we have:

\[
\frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) - \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} < \left( \frac{R^B}{R^A} - 1 \right) \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right)
\]

\[
\left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) - \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p + \epsilon)) + q_{LH}} < \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) \left( \frac{R^B}{R^A} - 1 \right)
\]

\[
< \frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right).
\]
which also shows that inequality B holds in this case. Since Inequality A and B hold, it follows that inequality (2.68) holds which establishes (2.66). If firm A increases its order in response to higher risk aversion, it must be the firm with the smaller initial equilibrium order, and therefore the firm with the higher initial risk aversion rate.

Part 3. By inspection of the equilibrium case 5 specified in Proposition 2.2 and Theorem 2.2, case 5 applies for the firm with the larger equilibrium order, which is the firm with the smaller risk aversion parameter by Corollary 2.2. Say it is firm A and $R_B/R_A > 1$. However, the necessary condition (2.60) is violated when $R_B/R_A > 1$. See the proof of Theorem 2.2.  

Proof of Corollary 2.5. Based on Corollary 2.2, when both firms are identical except for their spillover fractions, the firm with the larger spillover fractions orders at least as much as its rival at the equilibrium. By inspection of the equilibrium cases specified in Proposition 2.2 and Theorem 2.2, it is a prior not clear whether these cases can occur, and if so, whether the firm with the lower or the one with the higher spillover fraction increases its order. The proof clarifies this point.

Part 1(a). Case 1. The following argument shows that the equilibrium conditions for case 1 of Proposition 2.2 can only hold for the firm with the lower spillover fraction and equilibrium order. Consider case 1 of Theorem 2.2, where firm A orders the minimum demand plus spillover and firm B orders between the minimum demand and the minimum demand plus spillover. Firm A must have smaller spillover rate to increase its order quantity. So the claim is $b^A < b^B$. We prove it by contradiction. Suppose that $b^A \geq b^B$. Therefore by Corollary 2.2, we have $Q^A \geq Q^B$. Based on Proposition 2.2 the following inequality must hold for Firm B to order between the minimum demand and the minimum demand plus spillover:

$$\frac{C^B_n}{C^B_o} < q_{LL} e^{R_B b^B (d_H-Q^A)}.$$
However since both firms are identical except for spillover rate we must have:

\[
\frac{C_u^A}{C_o^A} = \frac{C_u^B}{C_o^B} < \frac{q_{LL}}{1 - q_{LL}} \frac{1 - q_{LL}}{q_{LL}} e^{R_B b^A(d_H - Q^A)} \leq \frac{q_{LL}}{1 - q_{LL}} e^{R_A b^A(d_H - Q^B)},
\]

where the second inequality holds because \( b^A \geq b^B \) and \( Q^A \geq Q^B \), which mean that \( b^B(d_H - Q^A) \leq b^A(d_H - Q^B) \). However, having the following inequality

\[
\frac{C_u^A}{C_o^A} < \frac{q_{LL}}{1 - q_{LL}} e^{R_A b^A(d_H - Q^B)},
\]

contradicts the condition in Proposition 2.2 for firm A in case 1 to order minimum demand plus spillover.

**Part 1(b).** Case 4. Consider case 4 where firm A orders between the minimum demand plus spillover and the maximum demand and Firm B order between the minimum demand and the minimum demand plus spillover:

\[
Q^A = d_L + b^A(d_H - Q^B) + \ln \left( \frac{C_u^A}{C_o^A} \left( \frac{1 - q_{LL}}{q_{LL}} e^{R_A b^A(d_H - Q^B) + q_{LL}} \right) \right) \quad \text{and} \quad Q^B = d_L + \ln \left( \frac{C_u^B}{C_o^B} \left( \frac{1 - q_{LL}}{q_{LL}} \right) \right).
\]

The following argument shows that the equilibrium conditions for case 4 of Proposition 2.2 can only hold for the firm with the lower spillover fraction and equilibrium order, so the claim is \( b^A < b^B \). We prove the claim by contradiction. Suppose that \( b^A \geq b^B \). Based on Proposition 2.2, for Firm B to order between the minimum demand and the minimum demand plus spillover, we must have:

\[
\frac{q_{LL}}{1 - q_{LL}} \frac{C_u^B}{C_o^B} \leq \frac{q_{LL}}{1 - q_{LL}} e^{R_B b^B(d_H - Q^A)}.
\]

Since both firms are identical except for their spillover rate we must have:

\[
\frac{q_{LL}}{1 - q_{LL}} \frac{C_u^B}{C_o^B} = \frac{C_u^A}{C_o^A} \leq \frac{q_{LL}}{1 - q_{LL}} e^{R_A b^A(d_H - Q^A)} \leq \frac{q_{LL}}{1 - q_{LL}} e^{R_A b^A(d_H - Q^B)},
\]

where the last inequality holds because \( b^A \geq b^B \) and \( Q^A \geq Q^B \), which means that \( b^B(d_H - Q^A) \leq b^A(d_H - Q^B) \). However, based on Lemma 2.4, if the following inequality

\[
\frac{q_{LL}}{1 - q_{LL}} \frac{C_u^A}{C_o^A} \leq \frac{q_{LL}}{1 - q_{LL}} e^{R_A b^A(d_H - Q^B)},
\]
holds, Firm A’s order quantity must be between the minimum demand and the minimum demand plus spillover, which is a contradiction.

**Part 2.** (Case 2). Consider case 2 where firm A orders the minimum demand plus spillover and firm B orders between the minimum demand plus spillover and the maximum demand:

\[ Q^A = d_L + b^A(d_H - Q^B) \] and \[ Q^A < Q^A_X = \frac{(1 - b^A)d_L + b^A(1 - b^B)d_H}{(1 - b^A b^B)} \]

\[ Q^B = d_L + b^B(d_H - Q^A) + \ln \left[ \left( \frac{C^B}{C^A} \right) \left( \frac{1 - q_{LL} - q_{LH}}{q_{LL} e^{R^B d_H (d_H - Q^A) + q_{LH}}} \right) \right] \]

The following argument shows that the equilibrium conditions for case 2 of Proposition 2.2 can only hold for the firm with the higher spillover fraction and equilibrium order. We prove the claim by contradiction. Suppose that \( b^B \geq b^A \). Based on Proposition 2.2, if Firm A orders the minimum demand plus spillover in equilibrium, we must have:

\[ \frac{C^A_u}{C^A_o} \leq \frac{q_{LL} e^{R^A d_H (d_H - d_L)} + q_{LH}}{1 - q_{LL} - q_{LH}} \]

and since both firms are identical except for their spillover rate (based on our assumption; \( b^B \geq b^A \)), we must have:

\[ \frac{C^B_u}{C^B_o} = \frac{C^A_u}{C^A_o} \leq \frac{q_{LL} e^{R^A d_H (d_H - d_L)} + q_{LH}}{1 - q_{LL} - q_{LH}} \leq \frac{q_{LL} e^{R^B d^B (d_H - d_L)} + q_{LH}}{1 - q_{LL} - q_{LH}} \]

where the second inequality holds since \( b^B \geq b^A \). However, based on Lemma 2.4 the above inequality means that Firm B’s order quantity can be at most equal to minimum demand plus spillover, which is a contradiction.

**Part 3.** That Case 5 never holds follows since the necessary condition (2.60) is violated when \( R^A = R^B \). See the proof of Theorem 2.2.

Case 6. Consider case 6, where firm A orders the maximum demand plus spillover and firm B adopts monopoly strategy and orders between the minimum demand and maximum demand:

\[ Q^A = d_H + b^A(d_H - Q^B) \] and \[ Q^B = d_L + \frac{\ln \left[ \left( \frac{C^B}{C^B_o} \left( \frac{1 - q_{LH}}{q_{LH}} \right) \right) \right]}{R^B} \]
The following argument shows that the equilibrium conditions for case 6 of Proposition 2.2 cannot hold. Based on Proposition 2.2 to have such an equilibrium, the following conditions must hold for firms A and B:

\[
\frac{C_u^A}{C_o^A} > \frac{q_{LL} e^{\bar{R}_A(d_H-d_L+b_A(d_H-Q^B))} + q_{LH} e^{\bar{R}_A(d_H-d_L)} + q_{HL} e^{\bar{R}_A b_A(d_H-Q^B)}}{1 - q_{LL} - q_{LH} - q_{HL}}, \tag{2.69}
\]

\[
\frac{C_u^B}{C_o^B} < \frac{q_{L} e^{\bar{R}_B(d_H-d_L)}}{1 - q_{L}}. \tag{2.70}
\]

We show that conditions (2.69) and (2.70) are mutually exclusive. Since both firms are identical except for spillover rate, i.e. \(C_u^A/C_o^A = C_u^B/C_o^B\), and \(\bar{R}_A = \bar{R}_B\), it suffices to show that the RHS of inequality (2.69) is larger than the RHS of inequality (2.70).

Note that the RHS of inequality (2.69) is a decreasing function of \(Q^B\), therefore inequality (2.69) must hold for \(Q^B = d_H\):

\[
\frac{C_u^A}{C_o^A} > \frac{e^{\bar{R}_A(d_H-d_L)}(q_{LL} + q_{LH}) + q_{HL}}{1 - q_{LL} - q_{LH} - q_{HL}}. \tag{2.71}
\]

However, it is straightforward to see that the following inequality holds

\[
\frac{e^{\bar{R}_A(d_H-d_L)}(q_{LL} + q_{LH}) + q_{HL}}{1 - q_{LL} - q_{LH} - q_{HL}} = \frac{e^{\bar{R}_A(d_H-d_L)}q_L + q_{HL}}{1 - q_{L} - q_{HL}} > \frac{q_{L} e^{\bar{R}_A(d_H-d_L)}}{1 - q_{L}},
\]

where \(q_{LL} + q_{LH} = q_{L}\). Therefore we have shown that conditions (2.69) and (2.70) are mutually exclusive. Therefore when both firms are identical except for their spillover rate, case 6 cannot happen.

\[\square\]

**Proof of Theorem 2.3.** Let \(A = i\) and \(B = j\). Suppose In perfect positive correlation case, the total demand of each firm has only two points: minimum demand and maximum demand plus spillover. Based the best response characterization in (2.32),
there are six possibilities for the location of equilibrium relative to the demand points:

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>$Q^i$</th>
<th>$Q^j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$d_L$</td>
<td>$d_L$</td>
</tr>
<tr>
<td>2</td>
<td>$d_L + \frac{1}{R^i} \ln \left( \frac{C^2}{C^1} \left( \frac{1-q_L}{q_L} \right) \right)$</td>
<td>$d_L$</td>
</tr>
<tr>
<td>3</td>
<td>$d_H + b^i(d_H - d_L)$</td>
<td>$d_L$</td>
</tr>
<tr>
<td>4</td>
<td>$d_L + \frac{1}{R^j} \ln \left( \frac{C^2}{C^1} \left( \frac{1-q_L}{q_L} \right) \right)$</td>
<td>$d_L + \frac{1}{R^j} \ln \left( \frac{C^2}{C^1} \left( \frac{1-q_L}{q_L} \right) \right) &lt; d_H$</td>
</tr>
<tr>
<td>5</td>
<td>$d_H$</td>
<td>$d_H$</td>
</tr>
<tr>
<td>6</td>
<td>$d_H + b^j(d_H - Q^j)$</td>
<td>$d_L + \frac{1}{R^j} \ln \left( \frac{C^2}{C^1} \left( \frac{1-q_L}{q_L} \right) \right) &lt; d_H$</td>
</tr>
</tbody>
</table>

Note that except for equilibrium 6, in the other equilibria neither firm’s equilibrium order quantity depends on its competitor’s order quantity. Therefore the own risk aversion effect is the only factor that has an impact on each firm’s order quantity. As a result both firms either reduce or do not change their order quantities as both become more risk averse. For equilibrium 6, it is clear by inspection that firm $i$ increases its order quantity as both firms become more risk averse. In particular firm $j$ reduces its order quantity as both firms become more risk averse since the effect of its own risk aversion is the only factor that has an impact on its order quantity and hence firm $i$ increases its order quantity. Since we have closed form solution for the equilibrium, the conditions for equilibrium type 6 follow immediately.

Proof of Theorem 2.4. Let $A = i$ and $B = j$. In perfect negative correlation case, the total demand of each firm has only two points: minimum demand plus spillover and maximum demand. Based the best response characterization in (2.33), there are six
possibilities for the location of equilibrium relative to the demand points:

\[
\begin{array}{|c|c|c|}
\hline
\text{Equilibrium} & Q^i & Q^j \\
\hline
1 & d_L + b^i(d_H - Q^i) = \frac{d_L(1-\hat{b}^i) + d_Hb^i(1-\hat{b}^i)}{1-\hat{b}^i} & d_L + b^i(d_H - Q^i) = \frac{d_L(1-\hat{b}^i) + d_Hb^i(1-\hat{b}^i)}{1-\hat{b}^i} \\
2 & d_L + b^i(d_H - Q^i) & d_L + b^i(d_H - Q^i) + \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right) < d_H \\
3 & d_L + b^i(d_H - Q^i) + \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right) < d_H & d_L + b^i(d_H - Q^i) + \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right) < d_H \\
4 & d_L & d_H \\
5 & d_L + \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right) & d_H \\
6 & d_H & d_H \\
\hline
\end{array}
\]

Note that except for equilibriums 2 and 3, in the other equilibria neither firm’s equilibrium order quantity depends on its competitor’s order quantity. Therefore the own risk aversion effect is the only factor that has an impact on each firm’s order quantity. As a result both firms either reduce or do not change their order quantities as both become more risk averse. Next we focus on equilibriums 2 and 3 respectively.

Part 1. For concreteness suppose that \( i = A \) and \( j = B \). It is clear by inspection that firm \( A \) increases its order quantity as both firms become more risk averse, if the order quantities of the of firms \( A \) and \( B \) are as follows:

\[
Q^A = d_L + b^A(d_H - Q^B) = \frac{d_L (1 - b^A) + b^A d_H (1 - b^B) - b^A \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right)}{1 - b^A b^B},
\]

\[
Q^B = d_L + b^B(d_H - Q^A) + \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right) = \frac{d_L (1 - b^B) + b^B d_H (1 - b^A) + \ln \left( \frac{C_B}{C_A} \left( \frac{1-q_L}{q_L} \right) \frac{R}{R^2} \right)}{1 - b^A b^B}.
\]

Since we have closed form solution for the equilibrium, the conditions for equilibrium type 2 follow immediately from the best response characterization.

Part 2. For concreteness suppose that \( i = A \) and \( j = B \). If firm \( A \) and \( B \)’s order
quantity at the equilibrium are as follows:

\[ Q^A = d_L + b^A(d_H - Q^B) + \frac{\ln \left[ \frac{C^A}{C^o} \left( \frac{1-q_L}{q_L} \right) \right]}{R^A}, \]

\[ Q^B = d_L + b^B(d_H - Q^A) + \frac{\ln \left[ \frac{C^B}{C^o} \left( \frac{1-q_L}{q_L} \right) \right]}{R^B}, \]

then firm A increases its order quantity as both firms become more risk averse if the following condition holds:

\[ \frac{\partial f^A(Q^B, R^A)}{\partial Q^B} \times \frac{df^B(Q^A, R^B)}{dR^B} + \frac{df^A(Q^B, R^A)}{dR^A} > 0, \]

or

\[ b^A \left( \frac{\ln \left[ \frac{C^B}{C^o} \left( \frac{1-q_L}{q_L} \right) \right]}{R^B R^B} \right) - \frac{\ln \left[ \frac{C^A}{C^o} \left( \frac{1-q_L}{q_L} \right) \right]}{R^A R^A} > 0, \]

or

\[ \frac{C^B_u}{C^B_o} > \frac{q_L}{1-q_L} \left[ \frac{C^A_u}{C^A_o} \left( \frac{1-q_L}{q_L} \right) \right] \cdot \frac{\frac{b^B}{\pi^B} \pi^B}{b^A \pi^A}. \]

To show the desired condition for the equilibrium, again since we have closed form solution for the equilibrium, the conditions for equilibrium type 3 follow immediately from the best response characterization. \( \square \)
Chapter 3

Operational & Financial Decisions: A Newsvendor Case

3.1 Introduction

Operations management and finance are usually studied separately. The operations management literature tends to ignore financial decisions based on the assumption that the firm’s production or inventory decisions can be financed by available funds, and similarly the finance literature has largely ignored the operational decisions of the firm. The separation of operations and capital structure is backed up by the seminal paper of Modigliani and Miller (1958) who show that a firm in a complete and perfect capital market can treat financial and operational decisions independently. However, many firms exist outside the Modigliani and Miller perfect and complete capital market world. Firms often have limited working capital and operate in imperfect and incomplete capital markets. Therefore, the separation of operations and financial decisions of such firms could result in suboptimal decisions.

In reality, many firms face financial constraints; as a result their growth and profit prospects are highly dependent on whether they can raise the required funds through
different instruments in the capital market. Therefore, the firms' operational decisions might be greatly impacted by their financial choices. There are two major ways to raise the required funds in the capital market: (i) Issuing equity, (ii) Issuing bond (raising debt).

As shown in the empirical finance literature (e.g., Barca, 1995; Berglof & Perotti, 1994; European Corporate Governance Network, 1997; Franks & Mayer, 1994; Prose, 1992; and others), many firms around the world have concentrated ownership. In these firms the major stock owners which have significant controlling power may not be interested in issuing equity at all, since they can lose the controlling power due to the dilution effect of issuing equity. Moreover, issuing equity can dilute the earning per share of the firm. In such cases, issuing debt may be the only option for raising funds. It is also known that in most cases the cost of issuing equity is significantly higher than the cost of issuing debt due to the tax shield of interest expenses and the seniority of the debt claim in the event of bankruptcy. As a result, many firms prefer debt financing over equity financing.

We study two models based on the discussion above. First, we focus on the model where issuing debt is the only option to finance the operations and afterwards we consider the model where both debt and equity are sources of financing the operations.

In most of the operations management models, it is assumed that managers always act in the shareholders' best interest by maximizing the profit/revenue or by minimizing the cost (we will discuss this in more detail later). However, in reality this may not be the case and managers may deviate from the actions that are in the shareholders' best interest due to their incentives. Therefore, ignoring financial requirements and management incentives might have a grave negative impact on the shareholders' interest.

Based on the principal-agent theory, shareholders ensure that managers take actions that maximize their return by tying executive compensation to performance measures. Managerial stock or stock options have been proposed in the finance literature to mitigate the agency problem that exists between managers and shareholders, and it has been
widely used across different industries/companies. The following table is a small sample of such compensation packages for the fiscal year of 2008 (available at reuters.com):

<table>
<thead>
<tr>
<th>Name</th>
<th>Position</th>
<th>(Company)</th>
<th>Stock Awards ($)</th>
<th>Options ($)</th>
<th>Total Compensation ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>J. Tom Wyatt</td>
<td>President, Old Navy</td>
<td>(Gap)</td>
<td>1,360,760</td>
<td>449,840</td>
<td>3,698,479</td>
</tr>
<tr>
<td>Leslee K. Herro</td>
<td>EVP, Planning &amp; Allocation</td>
<td>(Abercrombie &amp; Fitch)</td>
<td>2,116,367</td>
<td>925,035</td>
<td>4,417,300</td>
</tr>
<tr>
<td>Diane Chang</td>
<td>EVP, Sourcing</td>
<td>(Abercrombie &amp; Fitch)</td>
<td>2,116,367</td>
<td>925,035</td>
<td>4,386,696</td>
</tr>
</tbody>
</table>

Table 3.1: A small example of the executives’ compensation packages

However, if the shareholders design the compensation package, ignoring the relationship between the managers and other stakeholders, they may end up with a suboptimal result. Specifically, if they ignore the relationship between the bondholders and managers, they may end up with a higher cost of debt. Given that the compensation structure of the managers is available to the bondholders, the bondholders take this into consideration when they assess the riskiness of the company. If the structure promotes excessive risk taking behavior by managers, the bondholders may require a higher interest rate to compensate for the risk that they take. This ultimately affects shareholders’ wealth since they may experience higher cost of debt. Hence, shareholders can design a compensation package to reduce the agency costs of the debt financing. John and John (1993) were the first to suggest such a compensation package by tying executive compensation to the amount of debt in addition to equity value. Furthermore, the debt can mitigate
an agency problem between managers and shareholders through the use of restrictive covenants, close monitoring of the managers’ actions, and the reduction of the free cash flow available to managers. For the purpose of this study we ignore such effects.

The contribution of this chapter, while using the setting of the inventory control problem of a newsvendor with constrained capital, is twofold. First, we show how operational decisions of the manager depend on the capital structure of the firm and her own incentives. Second, we show that, depending on the market demand characteristics and available information to shareholders and bondholders about the state of the demand, the shareholders might be better off with the manager whose compensation package is tied to the enterprise value (firm value) as opposed to the equity value.

Initially, we consider four different types of managers when debt is the only available option to raise the required fund:

(i) Profit maximizer (PM): This type of manager is widely considered in the traditional operations management literature. We can assume that the compensation package of such manager is tied to the net income (profit) of the firm.

(ii) Equity maximizer (EM): This type of manager is very common in the industry since most of the executives’ compensation packages include stock and/or stock options. We assume that the compensation package of this type of manager is tied to the equity value of the firm.

(iii) Firm value maximizer (FVM): This type of manager is not common in the industry because the industry practitioners believe that managers should maximize the equity value, and not the bonds value. Hence it is very hard to find a manager whose compensation package is tied to the firm value.

(iv) Profit-equity maximizer (PEM): The PEM manager’s incentive is a mixture of the PM and EM managers’ incentives. In this case, which is common in the industry, the compensation packages include stock and/or stock options as well as a bonus which is related to the profit/net income of the company.
Chapter 3. Operational & Financial Decisions: A Newsvendor Case

Note that when the firm does not issue any debt all types of manager are effectively become PM manager. This can be verified later when the objective function of different types of managers presented.

Later, when we extend our model to allow the manager to issue both debt and equity, we will focus on the EM and FVM managers.

First, we will establish the relationship between the bondholders’ required interest on the bond and the inventory decisions of the different types of managers. We will use the fact that as a bond becomes riskier, investors require higher interest on the bond. Then, we study how the inventory decisions of the manager depend on her own incentives, the current capital structure of the firm, the demand characteristics, and the interest rate required by the bondholder. This in essence, enables us to investigate which type of manager might be more beneficial to shareholders. Note that depending on the risk taking behavior of the manager, bondholders might require different interest rates, which ultimately can affect the wealth of shareholders.

We observe that there are two counter effects which have an impact on shareholders’ wealth:

(i) The operational decision of managers which affects the level of revenue.

(ii) The cost of borrowing which is implicitly determined by the bondholders and the cost of raising equity.

Therefore, depending on the market demand characteristics and the incentives of the manager, we can determine which of the two effects will dominate. This analysis will illustrate what kind of incentive packages should be offered to the manager by the board of directors representing the shareholders.

**Literature Review and Positioning**

Recently, several studies in the operations management community have tried to address the interface between finance and operations management. Xu and Birge (2004) use a single period newsvendor model to study the inventory financing decisions. First,
they replicate the Modigliani-Miller (1958) theory that in a perfect market the firm’s production decision is independent of its capital structure and the value of a firm does not depend on its capital structure whenever its optimal investment level can be fully financed. Then, they study the investment and capital structure in an imperfect market and show that the value of the firm may be affected by considering debt financing along with production quantities. Similar to our model, the paper starts with the assumption that the firm has a limited initial capital (wealth) and therefore the amount of required debt should be equal to the total cost of ordering less the initial capital (wealth). However, later in the paper when investment and capital structure are studied in an imperfect market, the initial capital is effectively ignored; this means that the optimal debt is not necessarily equal to the total cost of ordering minus the initial capital (wealth). Hence, it might be the case that the firm still requires more funds due to the gap between total cost of ordering and the amount of debt that is raised through some other sources. However, the authors do not explain how this possible gap is financed. One may assume that this gap is financed by raising external equity. However in that case the cost of issuing equity is ignored; this contradicts the common market understanding that issuing equity is even more expensive than issuing debt for most firms that are not near bankruptcy. In contrast, in our second model where the manager can issue both debt and equity, the cost associated with raising equity is taken into consideration.

The most closely related papers to this chapter in the operations management literature are Xu and Birge (2008) and Dada and Hu (2008). Xu and Birge (2008) study the interactive mechanism between a firm’s production decision and its capital structure and they also suggest a compensation package to mitigate the agency cost between shareholders and managers while using the same model as in Xu and Birge (2004). Although their model seems to be similar to ours, as discussed above, it is not clear how in their model the gap between the total cost of ordering and the amount of debt is financed or why there is no cost for issuing equity. Consequently, their modeling assumptions have direct
impact on their conclusions and results. In particular, they consider the compensation packages which are similar to our EM and PEM managers’ compensation and show that if the manager’s compensation is 100% equity (in our notation the EM manager), then the manager’s optimal decisions also maximize the benefit of shareholders. This is in contrast with our results that the PEM manager is more beneficial for shareholders than the EM manager when debt is the only option to finance the operations.

Dada and Hu (2008) consider a similar model to Xu and Birge (2008), except that they assume that the total amount of debt is equal to the total cost of ordering minus the initial wealth and use a Stackelberg game to obtain the optimal order quantity decision and the interest rate charged by the bank. They additionally propose a coordination mechanism between the manager and the bank which results in an optimal solution that maximizes the supply chain expected contribution. In contrast to their models, our more realistic approach assumes that the coordination between different entities is not possible, and each entity makes an independent decision. We determine the principal and interest of the debt via a simultaneous game between the manager and the bondholders. Moreover we consider the impact of asymmetric information of different parities (i.e. shareholders, manager, and bondholders) on the shareholders’ decision of which type of compensation package to offer to their manager.

Xu and Birge (2006) also extend their single period model to a multiperiod model using a stochastic integer programming model with nonlinear constraints and conclude that the production and financial decisions should be made simultaneously.

In a related paper, Rao and Gutierrez (2010) study how to maximize shareholders’ wealth in the context of operational decisions and financial hedging. However, they do not rely on the usual market frictions such as tax and bankruptcy issues.

In another related paper, Subramaniam (1998) studies how debt financing can mitigate the inefficiencies which arise in a monopolist’s relation with its supplier due to the lack of complete contracts. In particular it is shown that a small amount of debt can be
used to mimic an optimal contract and increase the shareholders’ equity value. To study the problem, Subramaniam (1998) considers the case that the manager of the firm is a shareholder value maximizer (in our notation the EM manager) and models the conflict of interest between the bondholders and managers.


A vast body of the finance literature studies the capital structure and executive compensation packages. In one of the earliest studies, Jensen and Meckling (1976) introduce the agency cost approach and identify two types of conflicts: those between the shareholders and managers, and those between the equity holders and debt holders. Haugan and Senbet (1981) suggest managerial stock options as one way to ameliorate the agency problem between shareholders and managers. We refer to Ortiz-Molina (2005) and the references therein for more detail on compensation structure of executives and capital structure.

The most closely related theoretical papers in the finance literature are John and John (1993) and Edmans and Liu (2010). John and John obtain optimal compensation packages in the framework where a firm can invest either in a risky project with binomial payoff or in a riskless project. They suggest, as mentioned earlier, optimal compensation packages that are tied to the amount of debt in addition to the equity value. Edmans and Liu find the optimal compensation package in a similar framework of John and John, however they consider debt-like instrument (pensions) instead of company’s debt. In contrast to their model, our model, using significantly different settings, involves operational decisions, demand characteristics, and endogenous determination of principal and interest of risky debt via a simultaneous game between the manager and bondholders.

A recent empirical study by Sundaram and Yermack (2007), in fact, suggests that
CEOs in firms with large market capital (share price × number of shares outstanding) exhibit a balance between debt-based and equity-based incentives where the portion of the debt-based incentive increases as the CEOs grow older. More recently after the 2007-2009 credit crisis, it was suggested by many regulators, academics, and even some industry practitioners to change the compensation structure of top executives to mitigate risk-taking behaviors similar to those that contributed to the financial crises. In light of these theoretical papers and recent empirical evidences, in this chapter we propose to consider the FVM manager whose compensation is tied to the firm value (value of the equity and debt) as opposed to the EM manager, which is a common practice in industry. We suggest that, depending on the state of the market, the compensation structure of the top executives should also depend on the bond performance in addition to the stock performance of the firm. This kind of compensation structure may appear as an option which is not fully in alignment with the shareholders’ interest. However we will show that, under some conditions, shareholders might be better off with the FVM managers rather than with the EM managers.

Although in practice managers usually do not maximize debt value or have the firm’s debt in their compensation, as the empirical study by Sundaram and Yermack (2007) suggests, they may have debt-like instruments such as pensions or deferred compensations which essentially behave like the firm’s debt with regards to mitigating risk shifting behavior. Hence the results of this chapter are still applicable to the cases where debt-like instruments are part of the compensation package.

The rest of this chapter is organized as follows. First, we present the model where issuing debt is the only option for the manager to finance the operations. In the section after, we discuss what interest rate is required by the bondholders and analyze and compare the inventory decisions of the four types of managers assuming that they can only raise debt. We study the game between the manager and bondholders and establish a relationship between the inventory decisions of the managers and the bondholders’
required interest rate. Then, we argue which type of manager increases shareholders’
expected wealth the most. Next, we extend our analysis to the case where the managers
can issue both debt and equity. In the last section, we provide conclusions and outlook.

3.2 Model: The Case of Issuing Debt

We consider the classical newsvendor problem with a common cost structure. The man-
ger of the newsvendor firm orders $Q$ units at a cost of $c$ per unit at the beginning of the
period, and sells the product at price $p$ ($c < p$) per unit during the period. If the realized
demand is less than the quantity ordered, the manager can liquidate the excess inventory
at price $s$ ($s < c$) per unit. The seasonal demand $X$, has a cumulative probability dis-
tribution $F(\cdot)$, and a density probability distribution $f(\cdot)$ defined on the interval $[a, b]$, where $0 \leq a < b \leq \infty$. If the firm’s initial wealth (capital), $W$, is sufficient to cover the
cost of the products (i.e. $cQ \leq W$), the classical newsvendor model will be applied to
find the optimal order quantity. If the initial wealth (capital) of the firm is not sufficient,
that is $cQ > W$, the firm can issue a bond at the beginning of the period with a principal
value of $B$ and an interest rate (yield) of $r$, where $r \geq r_f$ ($r_f$ is the risk free rate).

We assume that the manager of the firm behaves rationally. Specifically, she only
issues sufficient amount of debt to purchase the desired order quantity. Therefore, in the
case that she issues a bond (i.e. $cQ > W$):

$$B = cQ - W.$$ 

If the firm decides to borrow money to finance its inventory, it will face bankruptcy if
it is unable to pay back the principal and interest obligations to the bondholders. Similar
to Leland (1994), it is assumed that the cost of bankruptcy (court, lawyers, accountants’
fees) is linear in revenue. Furthermore, the model reflects the tax advantage of financing
with debt, where the firm’s marginal income tax rate is denoted by $\tau$. 
Depending on her incentives, the manager of the firm might behave differently. Traditionally in the newsvendor setting, it is assumed that the manager is a PM manager. However, as mentioned earlier, in addition to the PM manager, we study the behavior of the EM, FVM, and PEM managers. The PM manager maximizes the net income ignoring the possibility of bankruptcy, while the EM manager maximizes the net income considering the possibility of bankruptcy. The FVM manager acts as if she maximizes both the expected value of the debt and equity while considering the possibility of bankruptcy. The PEM manager has a compensations structure that depends on both the equity value and the profit of the firm.

In the rest of the chapter we use the notation in Table 3.2.

3.3 Analyzing the Behaviour of the Different Types of Managers

In this section we analyze the behavior of the four types of managers when they have only the debt financing option. Recall that the manager’s actions depend on her incentives and moreover, based on these incentives, she needs to evaluate whether or not to issue a bond. Therefore, first she solves a traditional newsvendor problem without any constraint (no debt), and finds the optimal order quantity, $Q_0$. If $Q_0 \leq \frac{W}{c}$, then the optimal order quantity is $Q^* = Q_0$. However if $Q_0 > \frac{W}{c}$, the initial wealth (capital) of the firm is not sufficient, the manager might be better off by raising debt. In this case, the manager solves the newsvendor problem with her desired objective while considering raising sufficient debt to purchase the desired order quantity, $Q_1$. If $Q_1 \leq \frac{W}{c}$, then the optimal order quantity is $Q^* = \frac{W}{c}$ and it is not beneficial to issue any bond (i.e. $B^* = 0$). However if $Q_1 > \frac{W}{c}$, then the optimal order quantity is $Q^* = Q_1$, and the optimal debt level is $B^* = cQ_1 - W$.

Given a demand $X$, we define the revenue of the firm as follows: $R(Q, X) = p \min(Q, X) +$
\( p \): Price of the product
\( c \): Cost of the product
\( s \): Salvage value
\( F(\cdot) \): Cumulative probability distribution
\( f(\cdot) \): Density probability distribution
\( W \): Initial wealth (capital/equity)
\( B \): Amount of debt issued by the manager
\( E \): Amount of external equity issued by the manager
\( r \): Interest on the debt
\( r_f \): Risk free rate
\( \tau \): Marginal tax rate
\( R(Q, X) \): Firm’s revenue given order quantity \( Q \) and demand \( X \)
\( \Omega \): Bankruptcy cost
\( 1 - k \): The asset recovery rate after bankruptcy
\( Q \): Order quantity

\( Q_0 \): Optimal order quantity of the traditional newsvendor (no debt)
\( Q_{PM} \): Optimal order quantity of the PM manager
\( Q_{EM} \): Optimal order quantity of the EM manager
\( Q_{FVM} \): Optimal order quantity of the FVM manager
\( Q_{PEM} \): Optimal order quantity of the PEM manager
\( x_P \): The demand level for which the net income is zero
\( x_B \): The minimum required demand for the firm to be in a bankruptcy remote state
\( \theta \): Weight of profit incentive in the PEM manager’s incentive structure

\textbf{Table 3.2: Notation.}
s \max(0, Q - X)$. Hence, the amount of profit (after tax) at the end of the period after paying the principal and interest of the bond is:

\[
\begin{cases}
[R(Q, X) - B(1 + r) - W] (1 - \tau) & \text{if the net income is positive } (X > x_P) \\
R(Q, X) - B(1 + r) - W & \text{if the net income is negative } (X \leq x_P)
\end{cases}
\] (3.1)

where $x_P$, the demand level for which the net income is zero, is given by:

\[
x_P = \frac{Q c(1 + r) - s}{p - s} - \frac{W r}{p - s},
\] (3.2)

which is obtained by solving following equation:

\[
px_P + s (Q - x_P) - (cQ - W)(1 + r) - W = 0
\]

Note that when demand is below $x_P$, the net income of the firm is negative, hence, the firm is not required to pay any taxes. Therefore the total amount of the profit at the end of the period is $R(Q, X) - B(1 + r) - W$.

We assume that each type of manager has two options: she can use the available funds either to order the product, or to invest in a risk-free rate bearing account. For each case we define the incremental value denoted by $\Pi(Q)$ as the difference between these two options, or in other words it is the value after taking into consideration the opportunity cost of capital.

### 3.3.1 Interest Rate

In this section, we obtain the optimal interest rate required by the bondholders for a fixed order quantity. This enable us to explore the relationship between the required interest rate and the manager’s order quantity. Although bondholders are not directly involved in the determination of the interest rate on the bond, they may indirectly imply their required interest rate by not purchasing the bond. Therefore we can model the problem as if the bondholders are responsible for choosing the interest rate.
Following Dotan and Ravid (1985), we consider an equivalent risk neutral measure of demand and assume that the bond market is very competitive and efficient. Therefore the interest rate required by the bondholder must guarantee the following equality:

\[ E(V_B) = B(1 + rf), \]  

(3.3)

where the expected value of debt \( E(V_B) \) is given by:

\[ E(V_B) = \int_a^{x_B} [R(Q, x) - \Omega] f(x)dx + \int_{x_B}^b B(1 + r) f(x)dx. \]  

(3.4)

In Equation (3.4), \( x_B \) is the minimum required demand for the firm to be in a bankruptcy remote state. The bankruptcy remote state of demand (BRSD) is a state of demand in which the firm has sufficient funds at the end of the period to pay back the principal and interest on the bond and avoid bankruptcy. Therefore \( x_B \) for the firm that orders \( Q \) units of product is defined as follows:

\[ px_B + s(Q - x_B) = B(1 + r), \]

substituting \( B \) with \( cQ - W \), we have

\[ x_B = Q \left[ \frac{c(1 + r) - s}{p - s} \right] - \frac{W(1 + r)}{(p - s)}. \]  

(3.5)

Note that \( F(x_B) \) is the probability that the firm will declare bankruptcy. Furthermore, from Equations (3.2) and (3.5), it is straightforward to show \( x_P > x_B \).

Also in Equation (3.4), \( \Omega = kR(Q) \) represents the bankruptcy cost \((0 \leq k \leq 1)\). The bankruptcy cost refers to the unavoidable fees charged by lawyers, accountants, court and other associate entities.

Substituting \( B \) with \( cQ - W \) and simplifying Equation (3.4), the expected value of debt can be written as follows:

\[ E(V_B) = (1 - k) \int_a^{x_B} [px + s(Q - x)] f(x)dx + [cQ - W] [1 - F(x_B)](1 + r). \]

The optimal interest rate, then, can be found by solving Equation (3.3). The following lemma expresses this result rigorously:
Lemma 3.1. The optimal interest rate, $r^*$, for this fixed order quantity, $Q$, can be obtained by solving the following equation:

$$(1-k) \int_a^{x_B} [(p-s)x+sQ]f(x)dx + [cQ-W][1-F(x_B)](1+r^*) = (cQ-W)(1+r_f), \tag{3.6}$$

Proof. Since $\frac{dr}{dQ} > 0$ (see the proof of Proposition 3.2 of Xu and Birge (2004)), it is straightforward to show that $r^*$ exists. \qed

Note that the interest rate on the bond should in principal be larger than the risk free rate unless the input data is not consistent. If the interest rate on the bond is smaller than the risk free rate, the bondholder requires a lower interest rate on a risky firm’s bond compared to the government bond (or a bond without any risk). This clearly contradicts the definition of a risk free rate and shows that the input data is inconsistent and there exists an arbitrage opportunity. The firm can issue the bond with a lower interest rate than the risk free rate and then invest the proceeds of the issued bond in the government bond at a risk free rate.

Before we study the different types of managers, we briefly review the traditional newsvendor problem.

### 3.3.2 The Traditional Newsvendor Problem Review

Using the objective function of the traditional newsvendor, we define the incremental profit, $\Pi_T(Q)$, as follows:

$$\Pi_T(Q) = \int_a^{x_P} [R(Q,x) - B(1+r) - W] f(x)dx + (1-\tau) \left( \int_{x_P}^{b} [R(Q,x) - B(1+r) - W] f(x)dx \right) - (cr_fQ)(1-\tau).$$

Note that since in the traditional newsvendor problem $B = 0$, and $W = cQ$. Therefore $\Pi_T(Q)$ and $x_P$ will be simplified as follows:
\[ \Pi_T(Q) = \int_a^{x_p} [(p - s)x - (c - s)Q] f(x)dx + (1 - \tau) \left( \int_{x_p}^{Q} [(p - s)x - (c - s)Q] f(x)dx \right) + (1 - \tau)(p - c)Q(1 - F(Q)) - (cr_f)(1 - \tau), \]  

while

\[ x_p = \left( \frac{c - s}{p - s} \right) Q. \]

Maximizing the incremental profit results in obtaining the optimal order quantity \( Q_0 \) as shown in the next lemma:

**Lemma 3.2.** The optimal order quantity, \( Q_0 \), which maximizes \( \Pi_T(Q) \), can be found by solving the following equation

\[ F(Q_0) = \frac{p - c(1 + r_f)}{(p - s)} - \left( \frac{\tau}{1 - \tau} \right) \left( \frac{c - s}{p - s} \right) F(x_p). \]  

**Proof.** Taking the derivative of \( \Pi_T(Q) \) in Equation (3.7) with respect to \( Q \), using the Leibniz integral rule, we have:

\[
\frac{d\Pi_T(Q)}{dQ} = -(1 - \tau)(c - s)F(Q) - \tau(c - s)F(x_p) + (1 - \tau)(p - c) - (1 - \tau)(p - c)F(Q) - (cr_f)(1 - \tau).
\]

If we set \( \frac{d\Pi_T(Q)}{dQ} = 0 \), we obtain the optimal order quantity, \( Q_0 \), by solving Equation (3.8).

Note that \( \Pi_T(Q) \) is concave since

\[
\frac{d^2\Pi_T(Q)}{dQ^2} = -(1 - \tau)(c - s)f(Q) - \tau(c - s)f(x_p) \left( \frac{c - s}{p - s} \right) - (1 - \tau)(p - c)f(Q) < 0.
\]
Note that ignoring taxes ($\tau = 0$) will result in the well known traditional newsvendor problem solution.

When the firm’s initial capital (wealth) is sufficient to purchase the desired order quantity, the manager of the firm does not need to issue any bond. Since there is no bankruptcy concern, the objective functions of the four types of the managers that we study become identical. In fact by maximizing the net income of the firm, the manager also maximizes the equity value and the firm value. Therefore, solving the traditional newsvendor problem (maximizing the net income) gives us the optimal order quantity for the four types of managers.

Next we investigate the case in which the manager decides to issue a bond. Due to the possibility of bankruptcy, the four types of managers may choose different order quantities based on their incentives. We find the optimal order quantity for each type and then compare their behavior under various conditions.

### 3.3.3 Profit Maximizer

The PM manager is interested in maximizing the incremental profit, $\Pi_{PM}(Q)$, at the end of the period, ignoring the possibility of bankruptcy. Hence the profit function will be the same as Equation (3.1). Therefore, the objective function of the PM manager is as follows:

\[
\Pi_{PM}(Q) = \int_{a}^{x_P} [R(Q, x) - B(1 + r) - W] f(x) dx + \int_{x_P}^{b} [R(Q, x) - B(1 + r) - W] (1 - \tau) f(x) dx - Wr_f (1 - \tau)
\]

\[
= \int_{a}^{b} [R(Q, x) - cQ(1 + r) + Wr] f(x) dx - \tau \int_{x_P}^{b} [R(Q, x) - cQ(1 + r) + Wr] f(x) dx - Wr_f (1 - \tau).
\]

subject to Equation (3.6) and $B = cQ - W$.

Maximizing the incremental profit subject to Equation (3.6) and $B = cQ - W$ results
in obtaining the optimal order quantity, $Q_{1}^{PM}$, and interest rate as shown in the next lemma:

**Lemma 3.3.** The optimal order quantity, $Q_{1}^{PM}$, which maximizes $\Pi_{PM}(Q)$ and optimal interest rate simultaneously can be found by solving the following equation:

$$F(Q_{1}^{PM}) = \frac{p - c(1 + r)}{p - s} - \left( \frac{\tau}{1 - \tau} \right) \left( \frac{c(1 + r) - s}{p - s} \right) F(x_{P}),$$

(3.10)

together with

$$(1 - k) \int_{a}^{x_{B}} [(p - s)x + sQ] f(x) dx + [cQ - W][1 - F(x_{B})](1 + r) = (cQ - W)(1 + r_{f}),$$

where $x_{P}$ and $x_{B}$ are given by Equations (3.2) and (3.5).

**Proof.** The proof is very similar to that of Lemma 3.2, hence it is omitted. \qed

Note that in this case, if the tax rate was zero then we would have had

$$F(Q_{1}^{PM}) = \frac{p - c(1 + r)}{p - s},$$

which is identical to the well-known traditional newsvendor optimality condition (without incorporating the tax effect), except that $r_{f}$ is replaced by $r$.

Comparing the traditional newsvendor problem (incorporating the tax effect) with the PM manager, it is straightforward to show that since $r_{f} \leq r$, the following corollary holds:

**Corollary 3.1.** $Q_{1}^{PM} \leq Q_{0}$.

### 3.3.4 Equity Maximizer

The EM manager is essentially interested in maximizing the incremental residual cash after paying the debt and taxes in a bankruptcy remote state of demand.
The incremental residual cash after paying the debt and taxes in a BRSD is:

\[
\begin{cases}
[R(Q, X) - B(1 + r) - W](1 - \tau) + W & \text{if the net income is positive } (x_P < X) \\
[R(Q, X) - B(1 + r) - W] + W & \text{if the net income is negative in a BRSD } (x_B \leq X \leq x_P) \\
0 & \text{if the firm is bankrupt } (X < x_B)
\end{cases}
\] (3.11)

Hence the objective function of the EM manager, which is maximizing the incremental value of equity, is defined as follows:

\[\Pi_{EM}(Q) = E(V_E) - W_r f(1 - \tau) - W,\] (3.12)

where \(E(V_E)\) is the expected value of the equity given by:

\[
E(V_E) = \int_{x_B}^{x_P} [(R(Q, x) - B(1 + r) - W) + W] f(x)dx \\
+ \int_{x_P}^{b} [(R(Q, x) - B(1 + r) - W) (1 - \tau) + W) f(x)dx \\
= \int_{x_B}^{x_P} [R(Q, x) - B(1 + r)] f(x)dx \\
+ \int_{x_P}^{b} [(R(Q, x) - B(1 + r)](1 - \tau) + W\tau) f(x)dx.
\] (3.13)

subject to equation (3.6) and \(B = cQ - W\).

Note that since the equity value is zero in the state of the demand that the firm is bankrupt, then, in contrast to the formulation of the traditional newsvendor, the lower limit of the integral captures the revenue which is in excess of the amount pledged to the bondholders.

Maximizing the incremental equity value subject to Equation (3.6) and \(B = cQ - W\) results in obtaining the optimal order quantity, \(Q_{EM}^1\) and interest rate as shown in the next lemma:

**Lemma 3.4.** The optimal order quantity, \(Q_{EM}^1\), which maximizes \(\Pi_{EM}(Q)\) and optimal interest rate simultaneously can be found by solving the following equation:

\[
F(Q_{EM}^1) = \frac{p - c(1 + r)}{p - s} + \left( \frac{c(1 + r) - s}{(p - s)(1 - \tau)} \right) (F(x_B) - \tau F(x_P)).
\] (3.14)
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together with
\[(1 - k) \int_a^{x_B} [(p - s) x + sQ] f(x) dx + [cQ - W] [1 - F(x_B)](1 + r) = (cQ - W)(1 + r_f),\]
if the following condition holds:
\[(c(1 + r) - s) \left[ \frac{c(1 + r) - s}{p - s} \right] (f(x_B) - \tau f(x_P)) - (p - s)(1 - \tau) f(Q) < 0\]

Proof. From Equation (3.12) we have:
\[
\Pi_{EM}(Q) = \int_{x_B}^{x_P} [(p - s)x - (c(1 + r) - s)Q + W(1 + r)] f(x) dx
\]
\[+ \int_{x_B}^{Q} \left[ [(p - s)x - (c(1 + r) - s)Q + W(1 + r)](1 - \tau) + W\tau \right] f(x) dx
\]
\[+ \int_{Q}^{b} \left[ [(p - c(1 + r))Q + W(1 + r)](1 - \tau) + W\tau \right] f(x) dx - Wr_f(1 - \tau) - W,
\]
taking the derivative with respect to \(Q\), using the Leibniz integral rule, we have:
\[
\frac{d\Pi_{EM}(Q)}{dQ} = (c(1 + r) - s) (F(x_B) - \tau F(x_P)) + (p - c(1 + r))(1 - \tau) - (p - s)(1 - \tau) F(Q).
\]
Solving \(\frac{d\Pi_{EM}(Q)}{dQ} = 0\) together with Equation (3.6), we obtain the optimal order quantity, \(Q_{1EM}\) given by (3.14) and the required interest rate. Also note that if the following condition holds:
\[
\frac{d^2 \Pi_{EM}(Q)}{dQ^2} = (c(1 + r) - s) \left[ \frac{c(1 + r) - s}{p - s} \right] (f(x_B) - \tau f(x_P))
\]
\[-(p - s)(1 - \tau) f(Q) < 0\]
\(\Pi_{EM}(Q)\) is a concave function.

3.3.5 Firm Value Maximizer

The FVM manager is interested in maximizing the incremental value of the firm.

The expected value of the firm, is the sum of the expected value of the equity \(E(V_E)\) obtained from Equation (3.12) and the expected value of the debt \(E(V_B)\) given by Equation (3.4).
The incremental value of the firm, $\Pi_{FVM}(Q)$, can be written as:

$$
\Pi_{FVM}(Q) = E(V_B) + E(V_E) - W r_f (1 - \tau) - W - B (1 + r_f)
$$

(3.15)

subject to Equation (3.6) and $B = cQ - W$.

Maximizing the incremental value of the firm subject to Equation (3.6) and $B = cQ - W$ results in the optimal order quantity, $Q_1^{FVM}$ and interest rate presented in the following lemma:

**Lemma 3.5.** The optimal order quantity, $Q_1^{FVM}$, which maximizes $\Pi_{FVM}(Q)$ and optimal interest rate simultaneously can be found by solving the following equation:

$$
F(Q_1^{FVM}) = \left( \frac{p - c(1 + r)}{p - s} \right) + \frac{c(r - r_f)}{(p - s)(1 - \tau)} - \left( \frac{c(1 + r) - s}{p - s} \right) \frac{\tau F(x_P)}{1 - \tau} - \frac{k}{(p - s)(1 - \tau)} \left( s F(x_B) + \left[ \frac{c(1 + r) - s}{p - s} \right] (cQ_1^{FVM} - W) (1 + r) f(x_B) \right).
$$

(3.16)

**Proof.** From Equation (3.15)

$$
\Pi_{FVM}(Q) = (1 - k) \int_a^b [p x + s (Q - x)] f(x)dx + (cQ - W) (1 - F(x_B))(1 + r) = (cQ - W)(1 + r_f).
$$

Taking the derivative with respect to order quantity, $Q$, while using the Leibniz integral rule, we have:

$$
\frac{d\Pi_{FVM}(Q)}{dQ} = -k s F(x_B) - k \left[ \frac{c(1 + r) - s}{p - s} \right] (cQ - W) (1 + r) f(x_B) + c(r - r_f) - (c(1 + r) - s) \tau F(x_P) + (p - c(1 + r))(1 - \tau) - (p - s)(1 - \tau) F(Q).
$$
Solving \( \frac{d\Pi_{FVM}(Q)}{dQ} = 0 \) together with Equation (3.6), one can obtain the optimal order quantity given by Equation (3.16) and the required interest rate. Also note that \( \Pi_{FVM}(Q) \) is a concave function since:

\[
\frac{d^2 \Pi_{FVM}(Q)}{dQ^2} = -k\left[ \frac{c(1+r) - s}{p-s} \right] f(x_B) \\
- k\left[ \frac{c(1+r) - s}{p-s} \right] (1+r)f(x_B) \\
- k\left[ \frac{c(1+r) - s}{p-s} \right]^2 (cQ - W)(1+r)f'(x_B) \\
- \left[ \frac{c(1+r) - s}{p-s} \right] (c(1+r) - s)\tau f(x_P) - (p-s)(1-\tau)f(Q) < 0.
\]

\[\square\]

### 3.3.6 Profit and Equity Maximizer

The manager’s compensation can also be related to a combination of profit, equity value, and firm value. In this section we study a common compensation structure which ties the compensation of the manager to both the profit and equity value of the firm. Therefore the PEM manager aims to maximize the incremental profit and value of equity.

\[
\Pi_{PEM}(Q) = \theta \Pi_{PM}(Q) + (1 - \theta) \Pi_{EM}(Q),
\]

subject to Equation (3.6) and \( B = cQ - W \), where \( \theta \) (\( 0 \leq \theta \leq 1 \)) is the weight of profit incentive in the PEM manager’s compensation structure. Note that when \( \theta = 0 \), the PEM manager becomes the EM manager and when \( \theta = 1 \), the PEM manager becomes the PM manager.

Maximizing the incremental profit and value of equity subject to Equation (3.6) and \( B = cQ - W \), results in the optimal order quantity, \( Q_{PEM}^1 \) and interest rate presented in the following lemma:

**Lemma 3.6.** The optimal order quantity, \( Q_{PEM}^1 \), which maximizes \( \Pi_{PEM}(Q) \) and opti-
mal interest rate can be found by solving the following equation:

\[ F(Q_{1}^{PEM}) = \frac{p - c(1 + r)}{(p - s)} + \left( \frac{c(1 + r) - s}{(p - s)(1 - \tau)} \right) ((1 - \theta) F(x_B) - \tau F(x_P)) \]

together with

\[(1 - k) \int_{a}^{x_B} [(p - s)x + sQ]f(x)dx + [cQ - W] [1 - F(x_B)](1 + r) = (cQ - W)(1 + r_f),\]

if the following condition holds:

\[ \left[ \frac{c(1 + r) - s}{p - s} \right]^2 (1 - \theta) f(x_B) - \tau f(x_P) < (1 - \tau)f(Q) \]

**Proof.** The proof is similar to the proofs of Lemmas 3.3 and 3.4:

\[
\frac{d\Pi_{PEM}(Q)}{dQ} = (1 - \theta) \left[ (c(1 + r) - s) (F(x_B) - \tau F(x_P)) \right] + \\
(1 - \theta) \left[ (p - c(1 + r))(1 - \tau) - (p - s)(1 - \tau) F(Q) \right] + \\
[(p - c(1 + r))(1 - \tau) - \tau (c(1 + r) - s) F(x_P) - F(Q)(p - s)(1 - \tau)] \theta
\]

Solving \( \frac{d\Pi_{PEM}(Q)}{dQ} = 0 \) together with (3.6), we obtain the optimal order quantity, \( Q_{1}^{PEM} \) and required interest rate. We also have:

\[
\frac{d^2\Pi_{PEM}(Q)}{dQ^2} = (1 - \theta) \left[ (c(1 + r) - s) (f(x_B) - \tau f(x_P)) \left[ \frac{c(1 + r) - s}{p - s} \right] - (p - s)(1 - \tau)f(Q) \right] \\
- \left[ \tau (c(1 + r) - s) \left[ \frac{c(1 + r) - s}{p - s} \right] f(x_P) + f(Q)(p - s)(1 - \tau) \right] \theta \\
= (1 - \theta) \left[ (c(1 + r) - s) (f(x_B) - \tau f(x_P)) \left[ \frac{c(1 + r) - s}{p - s} \right] \\
- (1 - \theta) (p - s)(1 - \tau)f(Q) \\
- \theta \tau (c(1 + r) - s) \left[ \frac{c(1 + r) - s}{p - s} \right] f(x_P) - \theta f(Q)(p - s)(1 - \tau) \right]
\]

\[
= (1 - \theta) (c(1 + r) - s)f(x_B) \left[ \frac{c(1 + r) - s}{p - s} \right] - (c(1 + r) - s)\tau f(x_P) \left[ \frac{c(1 + r) - s}{p - s} \right] \\
- (p - s)(1 - \tau)f(Q)
\]

If the following condition holds:

\[ \left[ \frac{c(1 + r) - s}{p - s} \right]^2 (1 - \theta) f(x_B) - \tau f(x_P) < (1 - \tau)f(Q) \]

\( \Pi_{PEM}(Q) \) is a concave function.
3.3.7 Comparing the Managers’ Inventory Level Decisions

**Corollary 3.2.** For a fixed interest rate required by the bondholder, the order quantity of the EM manager is larger than or equal to the order quantity of the PM manager. The PEM manager’s order quantity is larger than or equal to that of the PM manager and less than or equal to that of the EM manager.

**Proof.** We define the following functions corresponding to the optimal order quantity for the PM manager \( G_{PM}(Q) \) based on Lemma 3.3), and the EM manager \( G_{EM}(Q) \) based on Lemma 3.4) respectively:

\[
G_{PM}(Q) = \frac{p - c(1+r)}{p-s} - \left( \frac{\tau}{1-\tau} \right) \left( \frac{c(1+r) - s}{p-s} \right) F(x_P) - F(Q),
\]

\[
G_{EM}(Q) = \frac{p - c(1+r)}{p-s} + \left( \frac{c(1+r) - s}{(p-s)(1-\tau)} \right) \left( F(x_B) - \tau F(x_P) \right) - F(Q).
\]

We compare the optimal order quantity of the EM manager with that of the PM manager. Consider the following:

\[
G_{EM}(Q) - G_{PM}(Q) = \left( \frac{c(1+r) - s}{(p-s)(1-\tau)} \right) F(x_B).
\]

Since \( \frac{c(1+r) - s}{(p-s)(1-\tau)} \) \( F(x_B) \geq 0 \), for \( Q \in [a,b] \), and the cumulative probability distribution \( F(.) \), is an increasing function, we have:

\[
Q_{1EM}^* \geq Q_{1PM}^*.
\]

Proof of the second claim is similar to that of the first one and hence it is omitted. \( \Box \)

Obviously, the larger the value of \( \frac{c(1+r) - s}{(p-s)(1-\tau)} \) \( F(x_B) \), the larger the quantity ordered by the EM manager compared to that ordered by the PM manager. However if the probability of bankruptcy becomes negligible \( F(x_B) \) goes to zero), both types of managers order the same quantity:

\[
Q_{1EM}^* = Q_{1PM}^*.
\]
Note that the equity value can be seen as the value of a call option on the assets of the company as is discussed in the seminal paper by Black and Scholes (1973). The EM manager’s goal is to maximize the call option and she only cares about the upside for the company. Therefore, in case of high volatility, the EM manager orders a large quantity to capture the potential upside at the expense of the bondholder, since she does not care about the potential bankruptcy (downside for the company). However the other types of managers, due to their incentive structure, do care about the downside for the company. As the volatility increases, the possibility of bankruptcy for a given order quantity increases, hence these managers reduce their order quantity. This implies that the EM manager’s decision is much riskier compared to the other types of managers.

### 3.4 Which Type of Manager is Better for the Shareholders?

In this section we investigate the effect of the different types of managers on the shareholders’ wealth. We consider two cases: symmetric information and asymmetric information. First we assume that all stakeholders of the firm (i.e. shareholders, bondholders, and manager) have perfect information about the demand characteristics. However, this might not be an accurate representation of reality. Usually bondholders and shareholders do not have the same information about the demand characteristics as does the manager. Therefore, we will modify our model to incorporate the notion of asymmetric information and analyze its impact on our results.

Note that the shareholders’ goal is to maximize the incremental value of the equity, hence their objective function is similar to that of the EM manager. However, the shareholders do not have any authority directly over operational and financial decisions of the firm. They can only offer an appropriate compensation package to their manager. As a result, the objective function of the shareholders who have a type $M$ manager is as
follows:

$$\Pi^M_S = \int_{x^b_M}^{x^b_P} [R(Q^*_M) - B^M(1 + r^*_M)] f(x) dx$$

$$+ \int_{x^b_P}^b (R(Q^*_M) - B^M(1 + r^*_M)(1 - \tau) + W\tau) f(x) dx - Wr_f(1 - \tau) - W.$$

To be able to compare the different types of managers from the shareholders’ point of view for the given parameters, we first obtain the optimal order quantity, the amount of debt required, and the optimal interest rate for each type of manager, based on the equilibrium analysis which was discussed in the previous sections. Then for each set of decisions, we calculate the expected shareholder’s equity value and compare them. Hence, we can find the type of manager that increases the expected equity value the most.

### 3.4.1 The Symmetric Information Case

Although the shareholders and EM manager have the same objective function, given that the bondholders may require different interest rates based on the type of manager, it might be more profitable for the shareholders not to have an EM manager! Shareholders can act appropriately by aligning their manager’s interest such that she acts based on their desired fashion. To find which type of manager potentially increases the wealth of the shareholders the most, shareholders can compare their expected equity value under the assumption of the different types of managers. Assume that the optimal order quantity for a type $M$ ($N$) manager and the optimal interest rate required by the bondholders are respectively $Q^*_M$, and $r^*_M$ ($Q^*_N$, and $r^*_N$). Shareholders are better off with a type $M$
Table 3.3: Optimal inventory level and required interest rate for the symmetric information case.

<table>
<thead>
<tr>
<th>α</th>
<th>CV</th>
<th>Q₀</th>
<th>(Qₐ, r)</th>
<th>(Qₐ, r)</th>
<th>(Qₑ, r)</th>
<th>(Qₑ, r)</th>
<th>Mₛ⁻⁻⁻⁻</th>
<th>Mₛ⁻⁻⁻⁻</th>
<th>E WHATSOEVER</th>
<th>F��⁻⁻⁻⁻</th>
<th>Mₛ⁻⁻⁻⁻</th>
<th>Fﬁ⁻⁻⁻⁻</th>
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</thead>
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<td>(288.44, 0.0525)</td>
<td>(288.24, 0.0525)</td>
<td>(288.24, 0.0525)</td>
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<td>4182.98</td>
<td>4183.63</td>
<td>4183.16</td>
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<tr>
<td>5</td>
<td>0.2290</td>
<td>270.52</td>
<td>(264.63, 0.0776)</td>
<td>(272.76, 0.0837)</td>
<td>(268.37, 0.0883)</td>
<td>(268.37, 0.0883)</td>
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<td>(231.62, 0.1273)</td>
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<td>156.44</td>
<td>(123.42, 0.1457)</td>
<td>(183.45, 0.3013)</td>
<td>(141.90, 0.1809)</td>
<td>(141.90, 0.1809)</td>
<td>1240.34</td>
<td>759.58</td>
<td>1228.82</td>
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<td>(49.85, 0.1807)</td>
<td>(49.85, 0.1807)</td>
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<td>(35.64, 0.1633)</td>
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<td>144.20</td>
<td>232.97</td>
<td>235.92</td>
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</table>

There are two major effects that dominate in the determination of the type of manager which is most beneficial for the shareholders: potential revenue and cost of debt. Clearly as the manager increases the order quantity, the potential revenue also increases. However, as discussed before, as the manager increases her order quantity, the probability of bankruptcy increases accordingly, hence the bondholders require a higher interest rate. This causes the shareholders to experience a higher cost of debt, which might ultimately negate the effect of potential higher revenue. Next we discuss in detail the effect of these two opposite forces via an example.

Suppose that the demand distribution is Weibull with a mean of 300 and a shape parameter of α. Table 3.3, shows a case that we hereafter refer to as the base case. The parameters for our base case are: p = 90, c = 60, s = 10, W = 1000, k = 0.7, τ = 0.4, rₐ = 5%, θ = 0.5.
Figure 3.1: The base case.

Figure 3.1 partially summarizes the above results. The solid line represents \( \Pi_S^{PM} - \Pi_S^{EM} \), the dashed line represents \( \Pi_S^{FVM} - \Pi_S^{EM} \), and the dotted line represents \( \Pi_S^{PEM} - \Pi_S^{EM} \) as a function of \( CV \).

We observe the following properties:

- From the shareholders’ point of view, all types of managers have similar performances when the volatility of demand is low (i.e. in Figure 3.1, when \( CV \leq 0.12 \)). When the volatility of the demand is low, the different types of managers tend to order similarly, hence the bondholders require a similar interest rate. Therefore, the shareholders are indifferent among the different types of managers.

- When the volatility of the demand is not too low (when \( 0.12 < CV \)), the FVM, PM, and PEM managers are more beneficial for the shareholders than the EM manager. The FVM, PM, and PEM managers tend to order less than the EM manager, hence the bondholders require a lower interest rate from them. Although the potential
revenue with the FVM, PM, and PEM managers is reduced due to a lower order quantity, the shareholders will benefit from the lower cost of the debt. Clearly, in this case, the effect of the lower cost of the debt dominates the potential revenue increment. Hence, it is more beneficial for shareholders to have a manager that acts as an FVM, PM, or PEM manager.

- As the demand volatility increases further, the benefit of the FVM, PM, or PEM manager becomes pronounced compared to the EM manager, up to some threshold (i.e. $CV = 0.68$). After that, as the volatility of the demand increases, the benefit of these types of managers over the EM manager becomes smaller.

- As we expected, the bondholders require a higher interest rate from the firm whose manager’s compensation is tied to the equity value due to higher risk appetite of the EM manager. When the demand exhibits high volatility, the EM manager shows the riskier behavior compared to the other types. It is worth mentioning that even with the higher interest rate that the EM manager experiences, her order quantity is still higher than that of the other types of managers, since she still tries to capture the upside of the equity (similar to a call option).

- In general, from the shareholders’ point of view, the PM and PEM managers’ performances are very similar to that of the FVM manager.

- Note that Table 3.3 and Figure 3.1 also present the sensitivity analysis on $\theta$, since when $\theta = 0$, the PEM manager becomes the EM manager and when $\theta = 1$, the PEM manager becomes the PM manager.

We now investigate whether the behavior we have observed in the base case depends on the demand distribution. For this purpose, we consider the Weibull, lognormal, gamma, and uniform demand distributions (note that for the uniform distribution we are restricted to a limited range of $CV$ values if we fix the mean) using the same para-
meters as in our base case. As shown in Figures 3.2, 3.3, and 3.4, it turns out that the outperformance of the PM, FVM, and PEM managers over the EM manager for a range of CV is robust and does not depend on the distribution.

Next, we study when the FVM manager outperforms the EM manager based on the problem parameters. We perform the analysis in two stages. First, we keep everything the same except one of the parameters to check whether the FVM manager exhibits similar performance as in the base case. Next for a fixed CV, we study the influence level of the different parameters on how the performance of the FVM manager surpasses the EM manager’s performance.

![Figure 3.2: Sensitivity of $\Pi_S^{FVM} - \Pi_S^{EM}$ to the demand distribution.](image)

**Stage 1**

We observed that with a change in any parameter, the PM, FVM, and PEM managers outperform the EM manager similar to our base case. Furthermore, from the shareholders’ point of view, the performance of the PM manager, PEM manager, and FVM manager are very close. Therefore, hereafter, our focus will be on the difference...
Figure 3.3: Sensitivity of $\Pi^{PEM}_S - \Pi^{EM}_S$ to the demand distribution.

Figure 3.4: Sensitivity of $\Pi^{PEM}_S - \Pi^{EM}_S$ to the demand distribution.
between the FVM manager and the EM manager.

Figure 3.5 shows the effect of several parameters on \((\Pi_S^{FVM} - \Pi_S^{EM})\). Specifically, it shows the effect of increasing the leverage (reducing initial wealth), cost, salvage value, and the cost of bankruptcy. As seen in Figure 3.5 we observe a similar behavior to that in the base case. However, there are ranges of demand volatility where the importance of having the FVM manager as opposed to the EM manager becomes more pronounced for the shareholders.

![Figure 3.5: Sensitivity of \(\Pi_S^{FVM} - \Pi_S^{EM}\) to the problem parameters.](image)

**Initial wealth (leverage):** As shown in Figure 3.5, higher leverage (lower initial wealth/equity) intensifies the benefit of the FVM manager for shareholders since the risk shifting behavior becomes more important when the firm is highly levered. The FVM manager’s incentive is such that it mitigates risk shifting behavior (reduces manager’s appetite for taking risky decisions), hence the bondholders’ required interest rate is lower than that of the EM manager. Therefore, shareholders experience a lower cost of debt. This is also observed empirically by Ortiz-Molina (2005), who suggests that highly levered
firms set lower pay-performance sensitivities to mitigate risk-shifting incentives.

**Cost of the product:** For certain demand volatility, the higher cost of product \((c)\) causes the FVM manager to become more valuable for the shareholders as compared to the base case. As the cost of the product increases while the other parameters stay the same, the potential profitability of the product decreases. For example, in this case, the profitability of the firm is reduced by almost half (the under-stocking to over-stocking, \(\frac{p-c}{c-s}\), ratio reduces from 60% to 33.33%). Therefore, because the potential loss due to overstocking becomes larger, mitigating risk shifting behavior becomes more important.

**Salvage value:** The effect of the salvage value \((s)\) may appear counter intuitive. One might expect that as the salvage value increases, the potential loss due to overstocking decreases, hence the benefit of the mitigating risk shifting behavior by tying the manager’s compensation to the firm’s value declines. This is not always true, and we will examine this fact in more detail later. In this case the potential loss of overstocking decreases, therefore both types of managers increase their order quantity. However, since the EM manager’s order quantity in the base case is relatively higher than the FVM manager’s, the impact of the reduction of overstocking risk is relatively lower for the EM manager compared to the FVM manager. As a result, the FVM manager tends to increase her order quantity more than the EM manager, which results in the FVM manager experiencing higher revenue increments. This results in the slightly higher benefit to the shareholders for some range of CV.

**Cost of bankruptcy:** The cost of bankruptcy might not be directly important for the shareholders. However, it has a significant impact on the bondholders’ required interest rate. Therefore, as the cost of bankruptcy increases, bondholders require a higher interest rate to compensate for the higher risk they are absorbing. Hence the cost of debt increases for shareholders. In such a case, mitigating risk shifting behavior becomes even more important. This makes the FVM manager’s benefit for shareholders even more significant.
Stage 2

Although the above discussion sheds some light on the impact of the major parameters, we need to look at their effect in more detail.

In Figure 3.6, for a fixed $CV = 0.68$, where the FVM manager is more profitable for shareholders than the EM manager in the base case (Table 3.3), we investigate the impact of different parameters (i.e. $W, c, s$, and $k$). For example the initial wealth ($W$) is varied from $0$ to $7000$. (note that 0% corresponds to the base case of $1000$.)

**Initial wealth (leverage):** As the firm’s leverage increases (initial wealth decreases), the benefit of the FVM manager for shareholders intensifies (Figure 3.6). As the leverage of the firm increases, the risk shifting behavior becomes more important for bondholders. Hence if the compensation package mitigates such behavior, the cost of debt for shareholders will be relatively lower compared to the case in which the compensation package promotes risk shifting behavior. In Figure 3.6, when the change is -100% the firm becomes highly levered, so we observe that the FVM manager becomes even more
profitable (Figure 3.6) compared to the base case (0% change).

As the leverage of the firm decreases and the firm moves toward 100% equity financing the benefit of the FVM manager for shareholders diminishes. When most of the funds of the firm come from equity, the impact of the cost of debt is minimal and all types of managers tend to make similar decisions. This can be seen in Figure 3.6; when the change of initial wealth is around 400%, we observe that all types of managers make similar decisions. Hence, the difference between the EM manager and the FVM manager is insignificant.

An interesting observation is that as the initial wealth of the firm increases, the EM manager reduces her order quantity. However, both the FVM and PM managers tend to increase their order quantity with a decreasing rate (i.e. when the initial wealth is $200, the FVM manager orders 107 units. However, when the initial wealth becomes $4200, the FVM manager only orders 135 units). Therefore, the greater the initial wealth, the less risky the firm with the EM manager becomes. Hence, the bondholders tend to require a lower interest rate. As a result, the FVM manager’s advantage on the cost of debt over the EM manager tends to vanish.

**Cost of product:** First we discuss two extreme cases: when the cost of the product is close to the salvage value and when it is close to the selling price. If the cost of the product is very close to the salvage value (left of 0%), the advantage of the FVM manager for the shareholders vanishes. When the cost of the product is low and close to the salvage value the potential loss due to overstocking is very low and the product is highly profitable. The low product cost causes the manager to require a smaller amount of debt. Moreover, since the product is highly profitable, the bondholders are highly confident that the firm will be able to pay back its debt even if it cannot sell all the ordered products. Hence, the risk of bankruptcy of the firm is very low and the bondholders are not concerned about the risk shifting behavior of the manager.

If the cost of product is very close to the selling price, profitability of the product
is very low and the potential loss due to overstocking is very high. Hence all types of managers reduce their order quantity significantly compared to the base case, such that their order quantities become closer to each other. This results in a similar interest rate being charged by the bondholders for the different types of manager. Therefore, from the shareholders' point of view the benefit of the FVM manager compared to the EM manager is significantly reduced.

As the cost of the product goes from very low to very high, we observe that the benefit of the FVM manager for shareholders initially increases and then decreases. As the cost of product increases from its lowest amount, the profitability of the product decreases and the potential loss due to overstocking increases, hence the risk shifting behavior becomes more important for bondholders. However, beyond a threshold, due to the significant overstocking cost and low profitability, different types of managers start to reduce their order quantity and tend to order in similar quantities to each other. As a result, the benefit of the FVM manager for shareholders gradually vanishes.

Salvage value: As mentioned earlier, the effect of the salvage value is to some extent counter intuitive. For $CV = 0.68$ (Figure 3.6), we observed that as the salvage value increases, the potential loss due to the overstocking decreases. Hence, both types of manager increase their order quantity. However, the FVM manager increases her order quantity more than the EM manager does. Therefore, we observe that up to a certain threshold, as the salvage value increases the benefit of the FVM manager increases with an increasing rate. Beyond that threshold, the benefit of the FVM manager increases at a decreasing rate. As the salvage value becomes closer to the cost of the product, the potential loss due to overstocking becomes negligible. Hence, the FVM manager's order quantity becomes close to that of the EM manager, and as a result the benefit of the FVM manager for the shareholders compared to the EM manager starts to flatten.

Cost of bankruptcy: As mentioned earlier, the cost of bankruptcy does not have direct impact on the shareholders. However, since a higher cost of bankruptcy means
that bondholders incur greater expenses to claim the residual assets of the firm in the case of bankruptcy, their realized loss could potentially become higher compared to the case where the cost of bankruptcy is low. Hence, bondholders require a higher interest rate on the debt in order to be compensated for the higher risk they are taking. As we observe in Figure 3.6, as the cost of bankruptcy increases the benefit of the FVM manager increases for the shareholders. This is because, as the cost of bankruptcy increases, (due to higher expenses realized in case of bankruptcy) bondholders become more concerned about the risk shifting behavior of the manager and require a higher interest rate on the debt. Therefore the compensation package that mitigates the risk shifting behavior helps shareholders to reduce the cost of debt.

3.4.2 The Asymmetric Information Case

This case is similar to the previous one, except that we assume that the bondholders and shareholders might not have access to the same information about demand as the manager does. It is natural to assume that shareholders and bondholders have access to information about demand that is available to the public. However, the manager of the firm usually has access to a more accurate forecast of the demand. We assume that the demand distribution considered by the shareholders and bondholders has the same mean and is defined on the same interval as the demand distribution considered by the manager, however, it has a different standard deviation. Given that the manager has access to a more accurate forecast, the estimated standard deviation of the demand by the manager is smaller than the one that is estimated by the shareholders/bondholders. Therefore, we assume that the demand considered by the shareholders and bondholders has a cumulative probability distribution $G(.)$ and a density probability distribution $g(.)$.

We consider two circumstances:

(i) The shareholders and bondholders consider the demand to be more volatile than does the manager.
(ii) The shareholders and bondholders consider the demand to be less volatile than does the manager.

For illustration, let’s look at the bondholders and the EM manager problem. In this case, similar to Lemma 3.4, we have:

\[(cQ - W)(1 + r_f) = (1 - k) \int_a^{x_B} [(p - s)x + sQ]g(x)dx + [cQ - W][1 - G(x_B)](1 + r),\]

and

\[F(Q) = \frac{p - c(1 + r)}{p - s} + \left(\frac{c(1 + r) - s}{(p - s)(1 - \tau)}\right)(F(x_B) - \tau F(x_P)).\]

Similarly, we can find the optimal order quantity and interest rate for the other types of manager.

To find out which type of manager is preferred by the shareholders in a case of information asymmetry, we use a similar approach to the one we used in the symmetric information case. Particularly, the shareholders are better off with a type \(M\) manager than a type \(N\) manager, if:

\[\begin{align*}
\int_{x_B^M}^{x_B} [R(Q_M) - B^M(1 + r^*_M)]g(x)dx + \int_{x_P^M}^{x_B^M} ([R(Q_M^*) - B^M(1 + r^*_M)](1 - \tau) + W\tau) g(x)dx > \\
\int_{x_B^N}^{x_B} [R(Q_N) - B^N(1 + r^*_N)]g(x)dx + \int_{x_P^N}^{x_B^N} ([R(Q_N^*) - B^N(1 + r^*_N)](1 - \tau) + W\tau) g(x)dx.
\end{align*}\]

Next, we consider the same example used for the symmetric information case, except that the shareholders’ and bondholders’ assessment of volatility is not as accurate as that of the manager. In Table 3.4, \(\alpha_E\) and \(CV_E\) are the shape parameter and CV considered by the shareholders and bondholders. Furthermore, we assume that the CV that is accurately estimated by the manager is 0.6790. Table 3.4 presents the results.

Figure 3.7 is a partial summarization of Table 3.4. It indicates that if the shareholders and bondholders only slightly overestimate or underestimate the volatility of the demand market, the shareholders’ decision will not be affected compared to the symmetric information case. In other words, if the shareholders and bondholders slightly overestimate or underestimate the volatility of demand, the shareholders still prefer the FVM manager.
### Table 3.4: Optimal inventory level and required interest rate for the asymmetric information case.

<table>
<thead>
<tr>
<th>$\alpha_{E}$</th>
<th>$CV_{E}$</th>
<th>$Q_{0}$</th>
<th>$(Q_{PM}^{P}, r)$</th>
<th>$(Q_{EM}^{P}, r)$</th>
<th>$(Q_{FVM}^{P}, r)$</th>
<th>$(Q_{PEM}^{P}, r)$</th>
<th>$\pi_{S_{PM}}^{P}$</th>
<th>$\pi_{S_{EM}}^{P}$</th>
<th>$\pi_{S_{FVM}}^{P}$</th>
<th>$\pi_{S_{PEM}}^{P}$</th>
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<tr>
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<td>(96.46, 0.2353)</td>
<td>(139.35, 0.3641)</td>
<td>(114.11, 0.2851)</td>
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<td>(106.34, 0.1809)</td>
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<td>402.89</td>
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<td>(183.45, 0.3013)</td>
<td>(113.70, 0.1380)</td>
<td>(141.90, 0.1809)</td>
<td>1240.34</td>
<td>759.58</td>
<td>1228.82</td>
<td>1283.56</td>
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<td>2305.77</td>
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<td>2450.97</td>
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<td>(153.00, 0.0511)</td>
<td>(265.21, 0.0870)</td>
<td>(112.29, 0.0581)</td>
<td>(186.84, 0.0534)</td>
<td>2447.24</td>
<td>3394.57</td>
<td>1814.19</td>
<td>2986.79</td>
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</table>

Note that the FVM manager’s order quantity decision has a low sensitivity to the shareholders’ and bondholders’ estimation of the demand volatility. When the sharehold-
ers and bondholders significantly overestimate the demand volatility (and as a result the bondholders tend to require a higher interest rate), in contrast to the EM manager, the FVM manager does not significantly reduce her order quantity. This imposes a relatively higher cost of debt on the shareholders, hence they tend to prefer the EM manager. Similarly, when the shareholders and bondholders underestimate the demand volatility, the FVM manager does not increase her order quantity, hence the potential revenue is significantly lower under this type of manager compared to the EM manager. As a result, the benefit of the FVM manager for shareholders declines compared to that of the EM manager.

Furthermore, as we expected, if the shareholders and bondholders overestimate the volatility of the demand, the expected equity value will be lower compared to the case where they have perfect information. When the bondholders think the demand is very volatile, they tend to require a higher interest rate due to the uncertainty. The higher cost of debt causes the shareholders’ expected profit to be lower. In this case, the manager may actually have an incentive to share her own accurate market data with bondholders and shareholders in order to get a better interest rate on the bond. However, if the shareholders and bondholders underestimate the volatility of the demand, the expected equity value will be higher compared to the case in which they have perfect information. When the bondholders underestimate the volatility of the market demand, they tend to require a lower interest rate. Therefore, shareholders incur a lower cost of debt compared to the case in which the bondholders have perfect information, and this results in a higher profit. Hence, the manager may not have an incentive to share her market assessment with bondholders and shareholders.

The above discussion shows that unlike the perfect information case, the bondholders and shareholders are usually better off with the EM manager when they do not have the same information about the demand as the manager. This may help to explain why we observe that many firms offer only stock or stock options and avoid debt-like
instruments. Perhaps one of the reasons that we observe such a practice among the firms is simply that the board of directors is not certain about the accuracy of the information for different projects’ payoff distributions and believes that the potential bondholders do not have accurate information. Therefore, by not including debt-like instruments they set themselves to be better off.

3.5 Analyzing the Behaviour of the EM and FVM Managers: The Case of Issuing Debt and Equity

In this section we extend our original model by allowing the manager to issue both debt and equity to raise the required funds. In particular, we would like to investigate whether our earlier results still hold when we allow the manager to issue equity as well as debt. Following Hennessy and Whited (2007), we assume that the cost of external equity, \( \Lambda(E) \), is as follows:

\[
\Lambda(E) = A_0 1_{(E>0)} + A_1 E + A_2 E^2, \quad A_i \geq 0 \text{ for } i = 0, 1, 2
\]

where \( 1_{(E>0)} \) is the indicator function that is equal to 1 if the equity is issued and 0 otherwise.

We define the new value of the wealth (equity) as the initial wealth (capital) plus the amount of external equity minus the cost of issuing the equity:

\[
\widehat{W}(E) = W + E - \Lambda(E)
\]

and similarly we assume that the manager only issues sufficient amount of debt to be able to cover the cost, hence:

\[
B = cQ - \widehat{W}(E)
\]

The profit (after tax) at the end of the period after paying the principal and interest
of the bond is (modified version of Equation (3.1)):

\[
\begin{align*}
\begin{cases}
R(Q, X) - B(1 + r) - \widetilde{W}(E) & \text{if the net income is positive } (X > x'_p) \\
R(Q, X) - B(1 + r) - \widetilde{W}(E) & \text{if the net income is negative } (X \leq x'_p)
\end{cases}
\end{align*}
\]

(3.17)

where \(x'_p\), the demand level for which the net income is zero, is given by:

\[
x'_p = Q \frac{c(1 + r) - s}{(p - s)} - \frac{(W + E - A_01_{E>0} - A_1E - A_2E^2)}{(p - s)}(1 + r).
\]

(3.18)

In essence the board of directors (on behalf of the shareholders) should decide whether the manager can use external equity as a source of financing. In our model this means that the objective function of the shareholders must be compared in two cases: (i) debt is the only source of financing, (ii) both debt and equity are the sources of financing. Case (i) was studied in the previous sections, therefore our focus will be on case (ii) in which \(B \geq 0, E > 0\). Since \(E > 0\), we have \(\Lambda(E) = A_0 + A_1E + A_2E^2\).

In the rest of the section we focus on two types of managers: EM and FVM managers.

### 3.5.1 Equity Maximizer

Given that the manager can issue equity, the minimum required demand, \(x'_B\), to be bankruptcy remote for the firm that orders \(Q\) units of product is modified (compared to equation (3.5)) as follow:

\[
x'_B = Q \frac{c(1 + r) - s}{(p - s)} - \frac{(W + E - A_01_{E>0} - A_1E - A_2E^2)}{(p - s)}(1 + r).
\]

(3.19)

Note that comparing Equation (3.18) and (3.19), we still have \(x'_p > x'_B\).

The incremental residual cash after paying the debt and taxes in a BRSD is also modified as follows:

\[
\begin{align*}
\begin{cases}
\left[ R(Q, X) - B(1 + r) - \widetilde{W}(E) \right] (1 - \tau) + \widetilde{W}(E) & \text{if the NI is positive } (x'_p < X) \\
\left[ R(Q, X) - B(1 + r) - \widetilde{W}(E) \right] + \widetilde{W}(E) & \text{if the NI is negative in a BRSD } (x'_B \leq X \leq x'_p) \\
0 & \text{if the firm is bankrupt } (X < x'_B)
\end{cases}
\end{align*}
\]
where NI stands for Net Income.

As a result the objective function of the EM manager is as follows:

$$\Pi_{EM}(Q, E) = E(V_E) - \tilde{W}(E)r_f(1 - \tau) - \tilde{W}(E),$$

where $E(V_E)$ is:

$$E(V_E) = \int_{x_B}^{x_P} \left[ (R(Q, x) - B(1 + r) - \tilde{W}(E)) + \tilde{W}(E) \right] f(x) dx$$

subject to equation (3.6) where $\tilde{W}(E)$ replace $W$ and $B = cQ - \tilde{W}(E)$.

Maximizing the incremental equity value subject to Equation (3.6) and $B = cQ - \tilde{W}(E)$ results in obtaining the optimal order quantity, $Q_{EM}^1$, external equity level, $E$, and interest rate as shown in the next lemma:

**Lemma 3.7.** The optimal order quantity, $Q_{EM}^1$, and required external equity, $E_{EM}$, which maximizes $\Pi_{EM}(Q, E)$ and optimal interest rate simultaneously can be found by solving the following equations:

$$F(Q_{EM}^1) = \frac{p - c(1 + r)}{p - s} + \left( \frac{c(1 + r) - s}{(p - s)(1 - \tau)} \right) \left( F(x_B') - \tau F(x_P') \right),$$

$$F(x_B') = \frac{r\tau F(x_P') + (r - r_f)(1 - \tau)}{1 + r}$$

together with

$$(1 - k) \int_{a}^{x_B'} [(p - s) x + sQ_{EM}^1] f(x) dx + \left[ cQ_{EM}^1 - \tilde{W}(E_{EM}) \right] [1 - F(x_B')](1 + r)$$

$$= (cQ_{EM}^1 - \tilde{W}(E_{EM}))(1 + r_f),$$
Taking the derivative with respect to \( Q \) if the following conditions are satisfied

\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} + \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} \pm \sqrt{4 \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E \partial Q} \right)^2 + \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} - \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} \right)^2} \leq 0.
\]

where

\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} = (c(1 + r) - s) \left( \frac{c(1 + r) - s}{p - s} \right) \left( f(x_B') - \tau f(x_p') \right) - (p - s)(1 - \tau)f(Q),
\]

\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} = -2A_2 \left( r\tau F(x_p') - (1 + r)F(x_B') + (r - r_f)(1 - \tau) \right)
- (1 - A_1 - 2A_2E) \left( r^2 \tau f(x_p') - (1 + r)^2\tau f(x_B') \right) \left( \frac{1 - A_1 - 2A_2E}{p - s} \right),
\]

\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E \partial Q} = (1 - A_1 - 2A_2E) \left( c(1 + r) - s \right) \left( r\tau f(x_p') - (1 + r)f(x_B') \right).
\]

**Proof.** We maximize:

\[
\Pi_{EM}(Q, E) = \int_{x_B'}^{x_p} \left[ (p - s)x - (c(1 + r) - s)Q + \tilde{W}(E)(1 + r) \right] f(x)dx
+ \int_{x_p}^{Q} \left[ (p - s)x - (c(1 + r) - s)Q + \tilde{W}(E)(1 + r) \right] f(x)dx
+ \int_{Q}^{b} \left[ (p - c(1 + r))Q + \tilde{W}(E)(1 + r) \right] f(x)dx
- \tilde{W}(E)r_f(1 - \tau) - \tilde{W}(E),
\]

where \( \tilde{W}(E) = W + E - \Lambda(E) \) and \( B = cQ - \tilde{W}(E) \) is subject to Equation (3.6).

Taking the derivative with respect to \( Q \), using the Leibniz integral rule, we have:

\[
\frac{\partial \Pi_{EM}(Q, E)}{\partial Q} = (c(1 + r) - s) \left( F(x_B') - \tau F(x_p') \right) + (p - c(1 + r))(1 - \tau) - (p - s)(1 - \tau)F(Q).
\]

Taking the derivative with respect to \( E \):

\[
\frac{\partial \Pi_{EM}(Q, E)}{\partial E} = (1 - A_1 - 2A_2E) \left( r\tau F(x_p') - (1 + r)F(x_B') + (r - r_f)(1 - \tau) \right)
\]

Solving \( \frac{d\Pi_{EM}(Q, E)}{dQ} = 0 \) and \( \frac{d\Pi_{EM}(Q, E)}{dE} = 0 \) together with equation (3.6) where \( \tilde{W}(E) \) replaces \( W \) and \( B = cQ - \tilde{W}(E) \), we obtain the optimal order quantity, \( Q_{1EM} \), required external equity, \( E_{EM} \), and bondholder required interest rate. We have the followings:

\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} = (c(1 + r) - s) \left( \frac{c(1 + r) - s}{p - s} \right) \left( f(x_B') - \tau f(x_p') \right) - (p - s)(1 - \tau)f(Q),
\]
\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} = -2A_2 \left( r\tau F(x'_p) - (1 + r)F(x'_B) + (r - r_f)(1 - \tau) \right) \\
- (1 - A_1 - 2A_2E) \left( r^2 F(x'_p) - (1 + r)^2 F(x'_B) \right) \left( \frac{1 - A_1 - 2A_2E}{p - s} \right),
\]
and
\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E \partial Q} = (1 - A_1 - 2A_2E) \left( \frac{c(1 + r) - s}{p - s} \right) \left( r\tau f(x'_p) - (1 + r)f(x'_B) \right).
\]

The following conditions must hold to have a negative semi-definite Hessian matrix:
\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} + \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} \pm \sqrt{4 \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E \partial Q} \right)^2 + \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} - \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} \right)^2} \leq 0.
\]

\section*{3.5.2 Firm Value Maximizer}

Given that the manager can issue equity, the incremental value of the firm, \( \Pi_{FVM}(Q) \), is modified as follows:
\[
\Pi_{FVM}(Q, E) = E(V_B) + E(V_E) - \widetilde{W}(E)r_f(1 - \tau) - \widetilde{W}(E) - B(1 + r_f)
\]
\[
= \int_a^{x'_B} [R(Q) - \Omega] f(x)dx + \int_{x'_B}^b B(1 + r) f(x)dx
\]
\[
+ \int_{x'_B}^{x'_p} [R(Q) - B(1 + r)] f(x)dx
\]
\[
+ \int_{x'_p}^b \left( [R(Q) - B(1 + r)](1 - \tau) + \widetilde{W}(E)\tau \right) f(x)dx
\]
\[
- \widetilde{W}(E)r_f(1 - \tau) - \widetilde{W}(E) - B(1 + r_f).
\]

subject to equation (3.6) where \( \widetilde{W}(E) \) replace \( W \) and \( B = cQ - \widetilde{W}(E) \).

Maximizing the incremental value of the firm subject to Equation (3.6) and \( B = cQ - \widetilde{W}(E) \) results in the optimal order quantity, \( Q_{FVM} \), external equity level, \( E \), and interest rate presented in the following lemma:

\begin{lemma}
The optimal order quantity, \( Q_{FVM} \), and required external equity, \( E_{FVM} \), which maximizes \( \Pi_{FVM}(Q, E) \) and optimal interest rate simultaneously can be found by
\end{lemma}
solving the following equations:

\[ F(Q_1^{FV,M}) = \left( \frac{p - c(1 + r)}{p - s} \right) + \frac{c(r - r_f)}{(p - s)(1 - \tau)} - \left( \frac{c(1 + r) - s}{p - s} \right) \frac{\tau F(x'_p)}{(1 - \tau)} \]

\[ - \frac{k}{(p - s)(1 - \tau)} \left( s F(x'_B) + \left[ \frac{c(1 + r) - s}{p - s} \right] \left( cQ_1^{FV,M} - \bar{W}(E^{FV,M}) \right)(1 + r) f(x'_B) \right) \]

\[ F(x'_p) = \left( \frac{r - r_f}{r} \right) \]

\[ + \left( \frac{k (1 + r)}{r \tau (p - s)} \right) \left[ cQ - (W + E^{FV,M} - A_0 - A_1 E^{FV,M} - A_2 (E^{FV,M})^2) \right] (1 + r) f(x'_B) \]

together with

\[ (1-k) \int_a^{x'_B} [(p - s)x + sQ] f(x) dx + \left[ cQ - \bar{W}(E^{FV,M}) \right] \left[ 1 - F(x'_B) \right] (1 + r) = (cQ - \bar{W}(E^{FV,M}))(1 + r_f), \]

if the following conditions are satisfied

\[ \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} + \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} \pm \sqrt{4 \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E \partial Q} \right)^2 + \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} - \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} \right)^2} \leq 0, \]

where

\[ \frac{\partial^2 \Pi_{FVM}(Q, E)}{\partial Q^2} = -ks \left[ \frac{c(1 + r) - s}{p - s} \right] f(x'_B) - kc \left[ \frac{c(1 + r) - s}{p - s} \right] (1 + r) f(x'_B) \]

\[ - k \left[ \frac{c(1 + r) - s}{p - s} \right]^2 (cQ - W)(1 + r) f'(x'_B) \]

\[ - (c(1 + r) - s) \tau f(x'_p) \left[ \frac{c(1 + r) - s}{p - s} \right] - (p - s)(1 - \tau) f(Q) < 0, \]

\[ \frac{\partial^2 \Pi_{FVM}(Q, E)}{\partial E^2} = k \left( \frac{2A_2}{p - s} (1 + r) \right) \left[ cQ - (W + E - A_0 - A_1 E - A_2 E^2) \right] (1 + r) f(x'_B) \]

\[ + k \left( \frac{1 - A_1 - 2A_2 E}{p - s} \right) (1 + r) (1 - A_1 - 2A_2 E) (1 + r) f(x'_B) \]

\[ + k \left( \frac{1 - A_1 - 2A_2 E}{p - s} \right)^2 (1 + r) \times \]

\[ \left[ cQ - (W + E - A_0 - A_1 E - A_2 E^2) \right] (1 + r) f'(x'_B) \]

\[ - 2A_2 \tau F(x'_p) - (1 - A_1 - 2A_2 E) \tau f(x'_p) \left( \frac{1 - A_1 - 2A_2 E}{p - s} \right) r \]

\[ + 2A_2 (r - r_f) \tau, \]
Proof. We maximize:

\[
\frac{\partial^2 \Pi_{FVM}(Q, E)}{\partial E \partial Q} = -kc \left( \frac{1 - A_1 - 2A_2E}{p - s} \right) (1 + r) f(x_B') \\
- k \left( \frac{1 - A_1 - 2A_2E}{p - s} \right) (1 + r) \left[ \frac{c(1 + r) - s}{p - s} \right] \times \\
\left[ cQ - (W + E - A_0 - A_1E - A_2E^2) \right] (1 + r) f'(x_B') \\
+ \left[ \frac{c(1 + r) - s}{p - s} \right] (1 - A_1 - 2A_2E) r\tau f(x_p').
\]

Taking the derivative with respect to order quantity, \(Q\), while using the Leibniz integral rule, we have:

\[
\Pi_{FVM}(Q, E) = (1 - k) \int_{x_B'}^{x_B} [px + s(Q - x)] f(x)dx + (cQ - \bar{W}(E))(1 + r)(1 - F(x_B')) \\
+ \int_{x_B'}^{x_p} [R(Q) - B(1 + r)] f(x)dx \\
+ \int_{x_B'}^{x_p} (R(Q) - B(1 + r))(1 - \tau) + \bar{W}(E)\tau f(x)dx \\
- \bar{W}(E)r_f(1 - \tau) - \bar{W}(E) - (cQ - \bar{W}(E))(1 + r_f),
\]

where \(\bar{W}(E) = W + E - \Lambda(E)\) and \(B = cQ - \bar{W}(E)\) is subject to Equation (3.6).

Taking the derivative with respect to order quantity, \(Q\), while using the Leibniz integral rule, we have:

\[
\frac{\partial \Pi_{FVM}(Q, E)}{\partial Q} = -kSF(x_B') - k \left[ \frac{c(1 + r) - s}{p - s} \right] (cQ - W)(1 + r)f(x_B') + c(r - r_f) \\
- (c(1 + r) - s)\tau F(x_p') + (p - c(1 + r))(1 - \tau) - (p - s)(1 - \tau)F(Q).
\]

We also have:

\[
\frac{\partial \Pi_{FVM}(Q, E)}{\partial E} = -k \left( \frac{1 - A_1 - 2A_2E}{p - s} \right) (1 + r) \times \\
\left[ cQ - (W + E - A_0 - A_1E - A_2E^2) \right] (1 + r)f(x_B') \\
+ (1 - A_1 - 2A_2E) r\tau F(x_p') - (1 - A_1 - 2A_2E)(r - r_f)\tau
\]

Solving \(\frac{d\Pi_{FVM}(Q, E)}{dQ} = 0\) and \(\frac{\partial \Pi_{FVM}(Q, E)}{\partial E} = 0\) together with Equation (3.6) where \(\bar{W}(E)\) replaces \(W\) and \(B = cQ - \bar{W}(E)\), we obtain the optimal order quantity, \(Q_1^{FVM}\), required external equity, \(E^{FVM}\), and bondholder required interest rate.
We have the followings:

\[
\frac{\partial^2 \Pi_{FVM}(Q, E)}{\partial Q^2} = -ks \left[ \frac{c(1 + r) - s}{p - s} \right] f(x_B') - kc \left[ \frac{c(1 + r) - s}{p - s} \right] (1 + r) f(x_B') \\
- k \left[ \frac{c(1 + r) - s}{p - s} \right]^2 (cQ - W) (1 + r) f'(x_B') \\
-(c(1 + r) - s) \tau f(x_p') \left[ \frac{c(1 + r) - s}{p - s} \right] - (p - s)(1 - \tau) f(Q) < 0,
\]

\[
\frac{\partial^2 \Pi_{FVM}(Q, E)}{\partial E^2} = k \left( \frac{2A_2}{(p - s)} (1 + r) \right) [cQ - (W + E - A_0 - A_1E - A_2E^2)] (1 + r) f(x_B') \\
+k \left( \frac{1 - A_1 - 2A_2E}{(p - s)} \right) (1 + r) (1 - A_1 - 2A_2E)(1 + r) f(x_B') \\
+k \left( \frac{1 - A_1 - 2A_2E}{(p - s)} \right)^2 \times \\
[cQ - (W + E - A_0 - A_1E - A_2E^2)] (1 + r) f'(x_B') \\
-2A_2 r \tau F(x_p') - (1 - A_1 - 2A_2E) r \tau f(x_p') \left( \frac{1 - A_1 - 2A_2E}{p - s} \right) r \\
+2A_2 (r - r_f) \tau,
\]

and

\[
\frac{\partial^2 \Pi_{FVM}(Q, E)}{\partial E \partial Q} = -kc \left( \frac{1 - A_1 - 2A_2E}{(p - s)} \right) (1 + r) f(x_B') \\
-k \left( \frac{1 - A_1 - 2A_2E}{(p - s)} \right) (1 + r) \left[ \frac{c(1 + r) - s}{p - s} \right] \times \\
[cQ - (W + E - A_0 - A_1E - A_2E^2)] (1 + r) f'(x_B') \\
+ \left[ \frac{c(1 + r) - s}{p - s} \right] (1 - A_1 - 2A_2E) r \tau f(x_p').
\]

The following conditions must hold to have a negative semi-definite Hessian matrix:

\[
\frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} + \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} \pm \sqrt{4 \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E \partial Q} \right)^2 + \left( \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial Q^2} - \frac{\partial^2 \Pi_{EM}(Q, E)}{\partial E^2} \right)^2} \leq 0.
\]

3.5.3 Which Type of Manager is Better for the Shareholders?

In this section, we conduct similar analysis to that of Section 4. Specifically, we compare the EM and FVM managers. We again consider two cases: symmetric and asymmetric
The Symmetric Information Case

We assume again that all stakeholders have the same information with regard to uncertain demand. We perform similar analysis to that of Table 3.3, assuming the manager issues both debt and equity. We assume that the demand distribution is Weibull with a mean of 300 and a shape parameter of $\alpha$. We also assume the following parameters for this numerical example: $p = 9, c = 6, s = 1, W = 100, k = 0.7, \tau = 0.4, r_f = 5\%, A_0 = 0.39, A_1 = 0.05, A_2 = 0.0002$. In Table 3.5, in addition to the case in which the manager is able to issue both debt and equity, we also present the case in which the manager is only allowed to issue debt. Note that in Table 3.5, since we present the required interest rate with only four significant digits, the interest rate required by the bondholder for the FVM manager looks the same in almost all cases, but each is actually slightly different.

Figures 3.8 and 3.9 partially summarize Table 3.5. As can be seen from Figure 3.9 the FVM manager remains more beneficial than the EM manager for shareholders, however the benefit is much less significant for the case in which the manager can issue both debt and equity (Figure 3.9) than for the case in which the manager can issue only debt (Figure 3.8). This happens because by issuing equity the manager can reduce the risk of bankruptcy significantly, causing both types of manager to become very similar to each other from the shareholders’ point of view. It is important to note that due to the simplicity of our debt-equity model, which ignores some of the adverse effects of issuing equity such as the dilution effect and downward price pressure from increased supply of shares, the amount of external equity raised is overestimated. We expect that in practice the manager (either type) issues less equity and more debt. As pointed out by Fama and French (2005), the secondary equity offerings (SEOs) are rare and the aggregate level of net new equity issues is relatively small in any given year.
Only debt is allowed  Both debt and external equity are allowed

<table>
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<th>α</th>
<th>CV</th>
<th>(Q^EM, r)</th>
<th>(Q^FV_M, r)</th>
<th>H^EM</th>
<th>H^FV_M</th>
<th>(Q^EM, r, E)</th>
<th>(Q^FV_M, r, E)</th>
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<th>H^FV_M</th>
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<td>463.46</td>
<td>(296.64, 0.0500, 73.03)</td>
<td>(296.64, 0.0580, 260.22)</td>
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<td>463.46</td>
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<tr>
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<td>452.47</td>
<td>(294.76, 0.0500, 155.63)</td>
<td>(294.76, 0.0580, 399.06)</td>
<td>452.48</td>
<td>452.49</td>
</tr>
<tr>
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<td>418.29</td>
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<td>335.09</td>
<td>341.32</td>
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</tr>
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<td>(197.65, 0.0980)</td>
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<td>253.42</td>
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Table 3.5: Optimal Inventory level, external equity, and required interest rate for the symmetric information case.

Figure 3.8: $\Pi_S^{EM} - \Pi_S^{FV_M}$ for the case that the manager can only issue bond.
The Asymmetric Information Case

Similar to Section 4.2, we consider the case where the shareholders’ and bondholders’ assessment of volatility is not as accurate as of the managers. We repeat the analysis of Table 3.4 for the case in which the manager can issue both debt and equity. We assume that the demand distribution for the bondholders and shareholders is Weibull with a mean of 300, a shape parameter of \( \alpha_E \), and the following parameters: \( p = 9, c = 6, s = 1, W = 100, k = 0.7, \tau = 0.4, r_f = 5\%, A_0 = 0.39, A_1 = 0.05, A_2 = 0.0002 \). We also assume that the CV that is accurately estimated by the manager is 0.6790. Table 3.6 presents the results.

Figures 3.10 and 3.11 partially summarize Table 3.6. Figure 3.11 shows that our results from Section 4.2 still hold. However, when the manager can issue both debt and equity the difference between the EM manager and FVM manager is less significant (Figure 3.11) than the case in which the manager is only allowed to issue debt (Figure...
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As mentioned, in the symmetric case, due to the simplicity of the equity model (ignoring adverse impacts such as the equity dilution effect), the amount of external equity raised is overestimated to some extent.

3.6 Summary and Outlook

In this chapter we studied the effect of capital structure and manager’s incentive on both operational and financial decisions of the firm in the context of the newsvendor framework. Although our models contain several simplifications of reality such as a single period and product, they capture the effect of uncertainty and its impact on decision making. We believe that they provide a tractable platform to study the relationship between the capital structure, the operational decisions, and the compensation structure of the manager.

We considered two models: the case in which the firm considers debt as the only option to finance its operation and when it considers both debt and equity as sources of financing. In both models, we showed that when all the stakeholders of the firm have the same information about the demand, the shareholders are better off having a manager whose compensation is tied to the value of the firm as opposed to a manager whose compensation is tied to the equity value. This may partially explain why there are compensation packages with debt-like instruments as reported by Sundaram and Yermack (2007). When the manager’s compensation is tied to the debt or debt-like instrument, it

<table>
<thead>
<tr>
<th>αE</th>
<th>CV_E</th>
<th>(Q^{EM}_{EM}, v)</th>
<th>(Q^{EM}_{EM}, v)</th>
<th>π^{EM}_{EM}</th>
<th>π^{EM}_{EM}</th>
<th>(Q^{EM}_{EM}, r, E)</th>
<th>(Q^{EM}_{EM}, r, E)</th>
<th>π^{EM}_{EM}</th>
<th>π^{EM}_{EM}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.9102</td>
<td>(142.73, 0.3681)</td>
<td>(114.23, 0.2505)</td>
<td>31.17</td>
<td>65.76</td>
<td>(140.90, 0.2681,118.50)</td>
<td>(153.92, 0.0507,866.85)</td>
<td>48.13</td>
<td>71.27</td>
</tr>
<tr>
<td>1.5</td>
<td>0.6790</td>
<td>(183.45, 0.3013)</td>
<td>(113.70, 0.1300)</td>
<td>75.95</td>
<td>122.88</td>
<td>(155.55, 0.0543,760.63)</td>
<td>(153.75, 0.0500,864.13)</td>
<td>145.33</td>
<td>145.98</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5227</td>
<td>(213.89, 0.2375)</td>
<td>(112.93, 0.0794)</td>
<td>133.50</td>
<td>155.17</td>
<td>(155.82, 0.0503,802.70)</td>
<td>(153.73, 0.0500,863.87)</td>
<td>194.05</td>
<td>193.01</td>
</tr>
<tr>
<td>3.0</td>
<td>0.3634</td>
<td>(244.57, 0.1499)</td>
<td>(112.40, 0.0549)</td>
<td>230.57</td>
<td>175.51</td>
<td>(155.84, 0.0500,806.54)</td>
<td>(153.73, 0.0500,863.86)</td>
<td>232.33</td>
<td>229.98</td>
</tr>
<tr>
<td>4.0</td>
<td>0.2805</td>
<td>(258.41, 0.1027)</td>
<td>(112.31, 0.0509)</td>
<td>296.75</td>
<td>180.19</td>
<td>(155.84, 0.0500,806.61)</td>
<td>(153.73, 0.0500,863.86)</td>
<td>244.71</td>
<td>241.81</td>
</tr>
<tr>
<td>5.0</td>
<td>0.2291</td>
<td>(265.21, 0.0789)</td>
<td>(112.29, 0.0501)</td>
<td>339.45</td>
<td>181.41</td>
<td>(155.84, 0.0500,806.61)</td>
<td>(153.73, 0.0500,863.86)</td>
<td>249.28</td>
<td>246.11</td>
</tr>
</tbody>
</table>

Table 3.6: Optimal Inventory level, external equity, and required interest rate for the asymmetric information case.
Figure 3.10: $\Pi_S^{EM} - \Pi_S^{FM}$ for the case that the manager can only issue bond.

Figure 3.11: $\Pi_S^{EM} - \Pi_S^{FM}$ for the case that the manager can issue both bond and equity.
mitigates the excessive risk taking behavior of the manager which results in a lower cost of the debt for shareholders compared to the case when the manager’s compensation is based only on stock or stock options. It is also important to mention that when the firm considers both debt and equity as sources of funding, the benefit of the manager whose compensation is tied to the firm value over the manager whose compensation is tied to equity, is less significant compared the case where the firm only considers debt as the source of financing its operation.

When managers have more accurate information about the demand volatility than the other stakeholders of the firm, shareholders are usually (but not always) better off having an EM manager. This may help justify why many firms have managers that are compensated by stock or stock options. In fact, the board of directors may feel that they do not have as accurate information about the future demand/projects’ payoff as do the managers. Therefore, as shown in this chapter, they may be better off offering a compensation package that encourages risk taking behavior by providing incentives such as stock or stock options.

Our analysis suggests that the design of a compensation package has strategic implications on both operational and financial decisions. To avoid mathematical complexity and to gain better insight about the dynamics of the problem, we did not attempt to obtain the optimal compensation structure. However, one possible future research could be conducting such an analysis.

As mentioned above, we studied only the single period and single product case. The more realistic multi-period, multi-product setting which involves taking into consideration the future stream of cash flows, can be a subject for future research. To study such a problem, one can use the single period model as a building block for the multi-period problem.

Furthermore, our equity-debt model involves significant simplification, specifically ignoring the dilution effect due to issuing equity. Future research could attempt to study
a more complex equity model in this context.
Chapter 4

Location Problems with Uncertain Demand

4.1 Introduction

Decisions with regard to the location of facilities such as plants, warehouses and shopping malls are very important and are often classified as strategic decisions. They usually result in significant fixed costs and more importantly they have considerable impact on growth prospects of the firm. Moreover, relocating the facilities is usually not easy and very costly. As a result firms executives make these strategic decisions carefully as they are aware of their significant economical importance.

Given the substantial impact of the facility location decision, it is arguable that the decision maker may not always act as a risk neutral decision maker, a common assumption in the facility location literature. In fact there are many different factors that encourage a decision maker to be risk averse or risk lover. In particular, the level and nature of risk attitude may vary based on incentives of executives, the availability of cash/credit, the cost of financing and many other related events/factors. In this chapter we consider the decision maker’s risk attitude and investigate how the risk attitude can change the
optimal facility location for several well studied facility location problem. Specifically, we study three facility location problems with uncertain demand: the median problem, the center problem and the generalized maximal covering location problem. Our main objective is to show how the decision maker’s risk attitude can affect the optimal facility location and how the risk neutral assumption may result in a suboptimal facility location for the risk averse or risk lover decision makers. We also try to shed some light on the important role that demand variability and correlation structure play for the risk averse or risk lover decision makers. In particular, we show that ignoring the volatility and correlation could result in a suboptimal facility location.

We analyze each problem and provide a mathematical programming formulation in the form of a quadratic program. We discuss how we can solve the problem using linearization techniques which are shown to be quite effective for problem with quadratic objective function (see Burkard et. al. (1998)). Specifically for the median problem and generalized maximal covering location problem we adopt two well-known linearization techniques: Lawler’s (1963) and Glover’s (1975) methods. For the center problem, we show how the problem is reduced to the classical center problem.

Using a simple example we provide intuitive insights on how the risk averse or risk lover decision maker tends to choose the optimal location of the facilities and how the demand’s mean, volatility and correlation affect the optimal facility location.

Furthermore, the importance of the facility location problem may encourage firms to make this kind of decision carefully and possibly consult with experts to improve the firms’ knowledge with regard to uncertain factors. The naive way to incorporate the information provided by experts is by simply changing the input data, which may not necessarily be an appropriate approach. The alternative way for incorporating the expert provided information is using a Bayesian updating approach. We show how we can take advantage of the Bayesian updating approach as a useful tool to improve the decision making process.
Literature Review and Positioning

All the early and much of the recent facility location literature, in particular the Median, Center and Maximal Covering problems assume risk neutral decision makers. However there are a few papers that either point out that risk neutral objective function might not be appropriate or try to introduce the notion of volatility to these classical problems. Frank (1966) is among the first researchers who studied the minimum variance absolute median problem with uncertain demand on a network considering independent demand distributions at the nodes. Frank (1967) extends his earlier work and studies the minimum variance absolute median problem with correlated demands. Jucker and Carlson (1976) is among the first papers that considered decision makers that are not risk neutral in a facility location problem. They assumed that the decision maker has a mean-variance objective where the decision maker’s objective function is a linear function of the mean and variance of payoff. They studied the uncapacitated facility location problem with uncertain but uncorrelated demand or price. They developed an efficient procedure that finds a dominant set of locations under several conditions which often help to solve the problem with ease. Their problem is similar to our p-median problem, however the main difference is that they ignore correlation which significantly increases the complexity of the problem. Martinich and Hurter (1982) and Hurter and Martinich (1984) study the production location problems under price uncertainty assuming that the firm has Von Neumann-Morgenstern utility function on profit. They examine the effects of change in risk aversion, uncertainty and production structure on the firm’s optimal location and production level decisions. Hodder and Jucker (1985) study the uncapacitated facility location problem under uncertainty. They extend the earlier work by Jucker and Carlson (1976) and assume that the prices in various markets are correlated with certain structure. Similar to Jucker and Carlson (1976), they assumed that the decision maker has a mean-variance criterion. Hodder and Jucker’s problem is to some extent similar to our p-median problem. A major difference is that they chose a very specific correlation structure for the
price charged by facilities (which is the random variable in their model) whereas we use a general correlation structure for the demand weights. Consequently our optimization problem is much harder than theirs.

There are also a few papers that introduce the notion of variance to the classical facility location problem with deterministic demand, i.e. the Median problem with deterministic demand. Halperen and Maimon (1983), Kincaid and Maimon (1987) and Maimon (1986), study the problem with the objective function of minimizing the variance of distance traveled by all customers to the facility. Berman (1990) extends this problem by combining the mean and variance measures.

The rest of the chapter is organized as follows. In Sections 4.2, 4.3 and 4.4, we analyze the optimal facility location respectively for the Median, Center and Generalized Maximal Covering problems. In Section 4.5, we try to provide insights for the specific behavior of the decision maker given her risk attitude via an example. In Section 4.6, we present the Bayesian updating approach in order to incorporate the experts' view. In Section 4.7, we investigate numerically the effect of linearization for solving the median problems with uncertain demand (Since the linearization method is very similar for the Median problem and Generalized Maximal Covering Location problem, we only consider the p-Median problem in our computational experience). Finally in Section 4.8 we provide a brief summary and outlook.

4.2 Median Problem

In this section we study the median problem with uncertain demand (MPUD). We first analyze the case where the decision maker wants to locate one facility, 1-MPUD and afterwards we consider the p facility case, p-MPUD.
4.2.1 Model

Let $G = (N, L)$ be an undirected network with a set of nodes $N (|N| = n)$ and a set of links $L$. The shortest distance between any node $j$ and a point $X \in G$ is denoted by $d(j, X)$. The demand at node $j$, $h_j$, is assumed to be a random variable with mean $\mu_j$ and standard deviation $\sigma_j$ and its distribution may be correlated with that of any other demand point of $G$.

Following Jucker and Carlson (1976), we assume that the decision maker has a mean-variance objective function as follows:

$$U(Y) = E(Y) - \lambda Var(Y)$$  \hspace{1cm} (4.1)

where $Y$ is a random variable and $\lambda$ is a risk attitude coefficient. Note that when $\lambda > 0$, the decision maker is risk averse, when $\lambda = 0$, the decision maker is risk neutral, and when $\lambda < 0$, the decision maker is risk lover. The use of the mean-variance objective is approximately consistent with the principle of maximizing expected utility if the decision maker’s utility function can be represented by a quadratic function of payoff or the subjective probability distribution of payoff is a two-parameter distribution such as the normal distribution (see Hanoch and Levy (1969), Philippatos and Gressis (1975) and Porter and Gaumntz (1972) for more detail).

1-Median Problem with Uncertain Demand (1-MPUD): The decision maker plans to locate a facility which maximizes her objective function. Note that the decision maker’s objective function increases as the distance traveled by customers in the system decreases. Hence, the objective function of the decision maker for the 1-median problem
with uncertain demand is (here $Y$ in Equation (4.1) is $\sum_{j=1}^{n} h_j d(x, j)$):

$$\max_{X \in \mathcal{G}} U \left(- \sum_{j=1}^{n} h_j d(X, j) \right) = \max \left[ E \left(- \sum_{j=1}^{n} h_j d(X, j) \right) - \lambda Var \left(- \sum_{j=1}^{n} h_j d(X, j) \right) \right]$$

$$= \max \left[ -E \left( \sum_{j=1}^{n} h_j d(X, j) \right) - \lambda Var \left( \sum_{j=1}^{n} h_j d(X, j) \right) \right]$$

$$= - \min \left[ E \left( \sum_{j=1}^{n} h_j d(X, j) \right) + \lambda Var \left( \sum_{j=1}^{n} h_j d(X, j) \right) \right]$$

$$= - \min \left[ \sum_{j=1}^{n} \mu_j d(X, j) + \lambda \sum_{j=1}^{n} [d(X, j)]^2 \sigma_j^2 + \lambda \sum_{j=1}^{n} \sum_{j\neq k} d(X, j)d(X, k)\sigma_j \sigma_k \rho_{jk} \right]$$

(4.3)

where $\rho_{jk}$ is the correlation coefficient of $h_j$ and $h_k$. Let us denote $M(X) = - \sum_{j=1}^{n} h_j d(X, j)$. Next we find the optimal solution for 1-MPUD.

### 4.2.2 Analysis

Define a breakpoint $X_i$ on link $(a, b)$ as a point such that the distance from $X_i$ to node $i$ through node $a$ is equal to the distance from $X_i$ to node $i$ through node $b$. Following Berman et. al. [2003], we refer to the region between two consecutive breakpoints on link $(a, b)$ as a primary region. Note that based on the definition, for any primary region on link $(a, b)$ the sets of nodes optimally reachable from nodes $a$ and $b$ respectively are unchanged. Let us denote these sets by $A$ and $B$ respectively for nodes $a$ and $b$ and the length of link $(a, b)$ by $l$.

Let $X$ denote both $d(a, X)$ and the location of $X$ on link $(a, b)$. Using Equation (4.2), the function $U(M(X))$ where $X$ is a point on a primary region in link $(a, b)$, can be
expressed as follows:

\[ U(M(X)) = \sum_{j \in A} \mu_j (X + d(a, j)) + \sum_{j \in B} \mu_j (l - X + d(b, j)) \]

\[ + \lambda \left( \sum_{j \in A} \{ X + d(a, j) \}^2 \sigma_j^2 + \sum_{j \in B} \{ l - X + d(b, j) \}^2 \sigma_j^2 \right) \]

\[ + 2\lambda \sum_{j \in A} \sum_{j \neq k \in A} (X + d(a, j)) (X + d(a, k)) \sigma_j \sigma_k \rho_{jk} \]

\[ + 2\lambda \sum_{j \in A} \sum_{j \neq k \in B} (X + d(a, j)) (l - X + d(b, k)) \sigma_j \sigma_k \rho_{jk} \]

\[ + 2\lambda \sum_{j \in B} \sum_{j \neq k \in B} (l - X + d(b, j)) (l - X + d(b, k)) \sigma_j \sigma_k \rho_{jk} \]

or

\[ U(M(X)) = \sum_{j \in A} \mu_j (X + d(a, j)) + \sum_{j \in B} \mu_j (l - X + d(b, j)) \]

\[ + \lambda \left( \sum_{j \in A} \{ X^2 + 2Xd(a, j) + (d(a, j))^2 \} \sigma_j^2 \right) \]

\[ + \lambda \left( \sum_{j \in B} \{ (l + d(b, j))^2 - 2(l + d(b, j)) X + X^2 \} \sigma_j^2 \right) \]

\[ + 2\lambda \sum_{j \in A} \sum_{j \neq k \in A} \left[ X^2 + Xd(a, j) + Xd(a, k) + d(a, j)d(a, k) \right] \sigma_j \sigma_k \rho_{jk} \]

\[ + 2\lambda \sum_{j \in A} \sum_{j \neq k \in B} \left( -X^2 + X (l + d(b, k)) - Xd(a, j) + d(a, j) (l + d(b, k)) \right) \sigma_j \sigma_k \rho_{jk} \]

\[ + 2\lambda \sum_{j \in B} \sum_{j \neq k \in B} \left( X^2 - X(l + d(b, j)) - X(l + d(b, k)) + (l + d(b, k)) (l + d(b, j)) \right) \sigma_j \sigma_k \rho_{jk} \]
With a little bit manipulation we can rewrite $U(M(X))$ as follows:

\[
U(M(X)) = X \left( \sum_{j \in A} \mu_j - \sum_{j \in B} \mu_j \right) \\
+ \lambda X^2 \left( \sum_{j \in A} \sum_{j \in B} \sigma_j^2 \right) \\
+ 2\lambda X^2 \left( \sum_{j \in A} \sum_{j \in B} \sigma_j \sigma_k \rho_{jk} - \sum_{j \in A} \sum_{j \in B} \sigma_j \sigma_k \rho_{jk} + \sum_{j \in B} \sum_{j \in B} \sigma_j \sigma_k \rho_{jk} \right) \\
+ \lambda X \left( \sum_{j \in A} 2d(a, j) \sigma_j^2 - \sum_{j \in B} 2(l + d(b, j)) \sigma_j^2 \right) \\
+ 2\lambda X \left( \sum_{j \in A} \sum_{j \in B} [d(a, j) + d(a, k)] \sigma_j \sigma_k \rho_{jk} \right) \\
+ 2\lambda X \left( \sum_{j \in A} \sum_{j \in B} (l + d(b, k)) - d(a, j) \right) \sigma_j \sigma_k \rho_{jk} \\
- 2\lambda X \left( \sum_{j \in B} \sum_{j \in B} (l + d(b, j)) + (l + d(b, k)) \right) \sigma_j \sigma_k \rho_{jk} + C
\]

where $C$ is constant with respect to $X$.

In the following lemma, we prove a structural property for $U(M(X))$ which helps us to find the optimal facility location for 1-MPUD.

**Lemma 4.1.** $U(M(X))$ is convex on any primary region if $\lambda > 0$ and it is concave if $\lambda < 0$.

**Proof.** $U(M(X))$ can be written as:

\[
U(M(X)) = \lambda C_1 X^2 + \lambda C_2 X + C
\]
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where

\[ C_1 = \sum_{j \in A} \sigma_j^2 + \sum_{j \in B} \sigma_j^2 + 2 \left( \sum_{j \in A} \sum_{j \neq k \in A} \sigma_j \sigma_k \rho_{jk} - \sum_{j \in A} \sum_{j \neq k \in B} \sigma_j \sigma_k \rho_{jk} + \sum_{j \in B} \sum_{j \neq k \in B} \sigma_j \sigma_k \rho_{jk} \right) \]

\[ C_2 = \frac{\left( \sum_{j \in A} \mu_j - \sum_{j \in B} \mu_j \right)}{\lambda} + \left( \sum_{j \in A} 2d(a, j)\sigma_j^2 - \sum_{j \in B} 2(l + d(b, j))\sigma_j^2 \right) + 2 \left( \sum_{j \in A} \sum_{j \neq k \in A} (d(a, j) + d(a, k))\sigma_j \sigma_k \rho_{jk} \right) + 2 \left( \sum_{j \in A} \sum_{j \neq k \in B} (l + d(b, k))\sigma_j \sigma_k \rho_{jk} \right) - 2 \left( \sum_{j \in B} \sum_{j \neq k \in B} ((l + d(b, j)) + (l + d(b, k)))\sigma_j \sigma_k \rho_{jk} \right) \]

Notice that \( C_1 \geq 0 \) since it can always be written as a variance of summation of random variables \( a_j h_{j}\)'s where \( a_j \) is a constant that equals 1 or -1. Therefore, if \( \lambda > 0 \) convexity of \( U(M(X)) \) is established and if \( \lambda < 0 \) concavity is established.

Based on the above lemma we can present the following theorem, which helps us to come up with an algorithm to solve 1-MPUD.

**Theorem 4.1.** An optimal solution of 1-MPUD on a primary region \([X_1, X_2]\) for:

1. \( \lambda > 0 \) is,

\[ X_1 \quad \text{if} \quad U'(M(X_1)) \geq 0 \]

\[ X_2 \quad \text{if} \quad U'(M(X_2)) \leq 0 \]

\[ X^* = -\frac{C_2}{2C_1} \quad \text{Otherwise} \]

2. for \( \lambda < 0 \) is

\[ X_1 \quad \text{if} \quad U(M(X_1)) \leq U(M(X_2)) \]

\[ X_2 \quad \text{Otherwise} \]

3. for \( \lambda = 0 \), we have the classical 1-median problem with nodal optimal solution.

**Proof.** For parts 1 and 2 of the Theorem, since the objective problem of MPUD is either convex \((\lambda > 0)\) or concave \((\lambda < 0)\) on any primary region, it is straightforward to show
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Now we can present Algorithm 1 to solve 1-MPUD on any network.

**Algorithm 1.**

Step 0. Set $U = -\infty$, all links are initially unlabeled.

Step 1. Choose any unlabeled link $(a, b) \in L$ and calculate the set of breakpoints.

Step 2. Choose any primary region $[X_1, X_2]$ not considered yet on link $(a, b)$ and find the optimal solution on that primary region $[X_1, X_2], Y$, based on Theorem 4.1.

Step 3. If $U(Y) > U$ go to Step 5, otherwise go to Step 4.

Step 4. If there are primary regions not yet considered on $(a, b)$ go to Step 2. Otherwise label link $(a, b)$. If there are unlabeled links in $L$, go to Step 1. Otherwise stop. The optimal solution is $X^0$ on link $(a, b)$.

Step 5. The new incumbent solution is $X^0 = Y$, $(a, b)^0 = (a, b)$ and $U^0 = U(Y)$. Go to Step 4.

4.2.3 The p-Median Problem

**Formulation**

In this section we extend our analysis to the p-median problem, p-MPUD. For $\lambda < 0$, we can show that the finite dominating set (FDS) for p-MPUD is the same as the FDS for 1-MPUD based on Theorem 2 of Hooker et al (1991). Specifically since $U(X)$ is concave when $\lambda < 0$, Theorem 2 of Hooker et al (1991) applies and FDS includes the set of all nodes and breakpoints. For $\lambda = 0$ Hakimi’s (1964, 1965) showed that the FDS for p-median problem is the set of all nodes. For $\lambda > 0$, the FDS for p-MPUD can be obtained by extending the analysis discussed earlier for 1-MPUD. For the sake of simplicity, we present the FDS for 2-MPUD. However, one can use the same analysis to obtain the FDS for p-MPUD. Note that the FDS for p-MPUD is not the same as the FDS for 1-MPUD.
but it can be viewed as the expanded FDS for 1-MPUD.

Suppose we want to locate facility 1 at primary region \((Y_{11}, Y_{21})\) and facility 2 at primary region \((Y_{12}, Y_{22})\). Let \(X_i\) (for \(i = 1, 2\)) denote both the location of \(X_i\) on \((Y_{1i}, Y_{2i})\), and \(d(Y_{1i}, X_i)\). \(U(M(X_1, X_2))\) can be written as follows:

\[
U(M(X_1, X_2)) = \lambda \sum_{i=1}^{2} C_{1i} X_i^2 + \lambda \sum_{i=1}^{2} C_{2i} X_i + \lambda DX_1 X_2
\]

where

\[
C_{1g} = \sum_{j \in A_g} \sigma_j^2 + \sum_{j \in B_g} \sigma_j^2 + 2 \left( \sum_{j \in A_g} \sum_{j \neq k \in A_g} \sigma_j \sigma_k \rho_{jk} - \sum_{j \in A_g} \sum_{j \neq k \in B_g} \sigma_j \sigma_k \rho_{jk} + \sum_{j \in B_g} \sum_{j \neq k \in B_g} \sigma_j \sigma_k \rho_{jk} \right)
\]

\[
C_{2g} = \frac{\left( \sum_{j \in A_g} \mu_j - \sum_{j \in B_g} \mu_j \right)}{\lambda} + \left( \sum_{j \in A_g} 2d(a_g, j) \sigma_j^2 - \sum_{j \in B_g} 2 (l_g + d(b_g, j)) \sigma_j^2 \right) + 2 \left( \sum_{j \in A_g} \sum_{j \neq k \in A_g} [d(a_g, j) + d(a_g, k)] \sigma_j \sigma_k \rho_{jk} \right)
\]

\[
+ 2 \left( \sum_{j \in A_g} \sum_{j \neq k \in A_g} [(l_g + d(b_g, k)) - d(a_g, j)] \sigma_j \sigma_k \rho_{jk} \right)
\]

\[
- 2 \left( \sum_{j \in B_g} \sum_{j \neq k \in B_g} [(l_g + d(b_g, j)) + (l_g + d(b_g, k))] \sigma_j \sigma_k \rho_{jk} \right) + 2 \sum_{j \in A_g} \sum_{k \in A_f} \sigma_j \sigma_k \rho_{jk}
\]

\[
+ 2 \left( \sum_{j \in A_g} \sum_{k \in B_f} (l_f + d(b_f, k)) \sigma_j \sigma_k \rho_{jk} \right) - 2 \left( \sum_{j \in B_g} \sum_{k \in B_f} (l_f + d(b_f, k)) \sigma_j \sigma_k \rho_{jk} \right)
\]

\[
- 2 \left( \sum_{j \in B_g} \sum_{k \in A_f} d(a_f, k) \sigma_j \sigma_k \rho_{jk} \right)
\]

\[
D = 2 \left( \sum_{j \in A_1} \sum_{k \in A_2} \sigma_j \sigma_k \rho_{jk} + \sum_{j \in B_1} \sum_{k \in B_2} \sigma_j \sigma_k \rho_{jk} - \sum_{j \in A_1} \sum_{k \in B_2} \sigma_j \sigma_k \rho_{jk} - \sum_{j \in B_1} \sum_{k \in A_2} \sigma_j \sigma_k \rho_{jk} \right)
\]

and \(g \neq f = 1, 2\).

Similar to Lemma 4.1, we can show that \(U(M(X_1, X_2))\) is convex in \((X_1, X_2)\) on any primary region for \(\lambda > 0\). This is shown formally in the next lemma:

**Lemma 4.2.** \(U(M(X_1, X_2))\) is convex in \((X_1, X_2)\) on any primary region for \(\lambda > 0\).
Chapter 4. Location Problems with Uncertain Demand

Proof. To show that $U(M(X_1, X_2))$ is convex on any primary region for $\lambda > 0$, we use the following theorem (see Boyd Vandenberghe (2009)), $f : \mathbb{R}^2 \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$, $g(t) = f(X + tV)$, $\text{dom } g = \{ t \mid X + tV \in \text{dom } f \}$ is convex in $t$ for any $X \in \text{dom } f, V \in \mathbb{R}^2$. In this case we have:

$$g(t) = U(M(X + tY)) = \lambda C_{11}(x_1 + ty_1)^2 + \lambda C_{12}(x_2 + ty_2)^2 + \lambda C_{21}(x_1 + ty_1)$$
$$+ \lambda C_{22}(x_2 + ty_2) + \lambda D(x_1 + ty_1)(x_2 + ty_2)$$

taking second derivative of function $g(t)$, we have:

$$\frac{d^2 g(t)}{dt^2} = 2\lambda [C_{11}y_1^2 + C_{12}y_2^2 + Dy_1y_2]$$

the above statement can be seen as the variance of summation of random variables $a_j h_j s$ where $a_j$ is a constant that equals 1 or $-1$. Therefore, when $\lambda > 0$, $g(t)$ is a convex function and as a result $U(M(X_1, X_2))$ is convex. 

Therefore similar to Theorem 4.1, the FDS for 2-MPUD can be obtained as follows:

$$\begin{cases} 
X_1 = Y_{k1}, X_2 = Y_{k2} & \text{for } k = 1, 2 \\
X_1 = Y_{k1}, X_2 = Y_{j2} & \text{for } k, j = 1, 2 \text{ and } k \neq j \\
X_i = X_i^*(X_j), X_j = Y_{kj} & \text{for } i, k, j = 1, 2 \text{ and } i \neq j \\
X_1 = X_1^*(X_2), X_2 = X_2^*(X_1) 
\end{cases}$$

where $X_i^*(X_j) = -\frac{DX_j + C_{2i}}{2C_{1i}}$ for $i, j = 1, 2$ and $i \neq j$.

To illustrate the possible solutions for 2-MPUD consider the network presented in Figure 4.1 where the number near the links are lengths.

Let’s assume that we want to locate two facilities: one facility on primary region $(1, B)$ (between node 1 and $B$) where $B$ is the breakpoint on link $(1, 2)$ with respect to node 3 which is 3 units away from Node 1, and another facility on link $(3, 2)$ which is a primary region. The six feasible solutions (out of 9 possible ones) based on the above
Facility 1 location on primary region $(1, B)$ | Facility 2 location on primary region $(3, 2)$
---|---
1 | at 1.403 units away from Node 1 | at 0.685 units away from Node 3
2 | at 2.461 units away from Node 1 | at Node 3
3 | at Node 1 | at Node 2
4 | at Node 1 | at 2 units away from Node 3
5 | at Node 1 | at 1.211 units away from Node 3
6 | At breakpoint $B$ | at Node 3

Note that the optimal solution in this example is the sixth case.

We formulate the p-MPUD as a mathematical programming. We define the following binary variables:

\[
Y_j = \begin{cases} 
1 & \text{if a facility located at node } j, \\
0 & \text{otherwise}
\end{cases}
\]

\[
X_{ij} = \begin{cases} 
1 & \text{if node } i \text{ is covered by facility located at node } j, \\
0 & \text{otherwise}
\end{cases}
\]

To formulate the problem, for convenience without loss of generality, let assume that
$N$ is FDS. The problem which we refer to as $P_M$ can now be formulated as follows:

$$-\min Z = \sum_{j=1}^{n} \sum_{i=1}^{n} \mu_i d(i,j) X_{ij} + \lambda \sum_{j=1}^{n} \sum_{i=1}^{n} [d(i,j)]^2 \sigma_i^2 X_{ij}^2$$

$$+ \lambda \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k \neq i}^{n} d(i,j) d(k,l) \sigma_i \sigma_k \rho_{ik} X_{ij} X_{kl}$$

s.t.

$$\sum_{j=1}^{n} X_{ij} = 1, \text{ for } i = 1, 2, 3, ..., n$$

$$\sum_{j=1}^{n} Y_{j} = p$$

$$Y_{j} - X_{ij} \geq 0 \text{ for } i \neq j$$

$$Y_{j} = \{0, 1\}, X_{ij} = \{0, 1\}, i = 1, 2, ..., n, j = 1, 2, ..., n$$

Note that in problem $P_M$ (and in the other models discussed in this chapter) we assume the firm assign the customers to the facilities (direct assignment) which is not necessarily the closest facility to the customer. However we also consider the case that the customer chooses the closest facility by adding the following constraints to $P_M$:

$$\sum_{j=1}^{n} X_{ij} d(i,j) + (M - d(i,k)) Y_{k} \leq M \text{ for } i, k = 1, 2, ..., n$$

where $M$ is a very large number. One can define $M = \max \{d(i,j), i, j = 1, 2, ..., n\}$.

The above formulation is a quadratic programming formulation which is similar to the Quadratic Assignment Problem (QAP) formulation. For detailed discussion about QAP, we refer readers to the literature review paper by Burkard et. al. (1998). However, in QAP, there is no location decisions. For a given set of facilities’ location, MPUD can be reduced to QAP. It is well known that the quadratic assignment problem is a strongly NP-Hard problem. Since MPUD for a given set of location reduces to QAP, MPUD is also strongly NP-Hard.
Linearization of the Problem

To solve the QAP researchers applied different linearization techniques. The linearization techniques relatively work well for QAP because of its structure and the fact that the decision variables are binary. Given the similarity of our problem to QAP, we also take the same approach and use linearization techniques to solve our problem. Specifically, in this chapter, we discuss two well-known linearization techniques: Lawler’s (1963) technique and Glover’s (1975) method.

**Linearization Using Lawler’s Method.** We introduce a set of new decision variables, $T_{ikjl}$, to replace the product $X_{ij}X_{kl}$ in the objective function. The following three constraints guarantee that $T_{ikjl}$ is 1 when both $X_{ij}$ and $X_{kl}$ are 1 and 0 otherwise

\[
X_{ij} \geq T_{ikjl} \quad \text{for } i, j, k, l = 1, 2, \ldots, n \text{ and } i \neq k
\]

\[
X_{kl} \geq T_{ikjl} \quad \text{for } i, j, k, l = 1, 2, \ldots, n \text{ and } i \neq k
\]

\[
X_{ij} + X_{kl} \leq 1 + T_{ikjl} \quad \text{for } i, j, k = 1, 2, \ldots, n \text{ and } i \neq k
\]

Based on above discussion, the problem could be formulated as follows:

\[
- \min Z_L = \sum_{j=1}^{n} \sum_{i=1}^{n} \mu_i d(i, j) X_{ij} + \lambda \left( \sum_{j=1}^{n} \sum_{i=1}^{n} [d(i, j)]^2 \sigma_i^2 X_{ij} + \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{i \neq k} d(i, j)d(k, j)\sigma_i\sigma_k\rho_{ik}T_{ikj} \right)
\]

s.t.

\[
\sum_{j=1}^{n} X_{ij} = 1, \text{ for } i = 1, 2, 3, \ldots, n
\]

\[
\sum_{j=1}^{n} Y_j = p
\]

\[
Y_j - X_{ij} \geq 0 \text{ for } i \neq j
\]

\[
X_{ij} \geq T_{ikj} \quad \text{for } i, j, k = 1, 2, \ldots, n \text{ and } i \neq k
\]

\[
X_{kj} \geq T_{ikj} \quad \text{for } i, j, k = 1, 2, \ldots, n \text{ and } i \neq k
\]

\[
X_{ij} + X_{kj} \leq 1 + T_{ikj} \quad \text{for } i, j, k = 1, 2, \ldots, n \text{ and } i \neq k
\]

\[
Y_j = \{0, 1\}, X_{ij} = \{0, 1\}, \ i = 1, 2, \ldots, n, j = 1, 2, \ldots, n
\]

\[
T_{ikj} = [0, 1], \ \text{for } i, j, k = 1, 2, \ldots, n \text{ and } i \neq k
\]
The major drawback with this formulation is the very large number of new dummy variables that are required for large size problems. We will see how this issue affects the time that it takes to solve large size problems in our numerical analysis section.

**Linearization Using Glover’s Method.** Define

\[
W_{ij} = \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl}
\]

\[
C_{ij}^+ = \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} \mathbf{1}_{\{\rho_{ik} > 0\}}
\]

\[
C_{ij}^- = \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} \mathbf{1}_{\{\rho_{ik} < 0\}}
\]

We can replace \( X_{ij} \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl} \) in the objective function of the problem with \( W_{ij} \). Notice that the following constraints

\[
W_{ij} \geq \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl} - C_{ij}^+ (1 - X_{ij})
\]

\[
W_{ij} \leq \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl} - C_{ij}^- (1 - X_{ij})
\]

\[
C_{ij}^- X_{ij} \leq W_{ij} \leq C_{ij}^+ X_{ij} \text{ for } i, j
\]

guarantee that when \( X_{ij} \) is 1, \( W_{ij} \) is equal to \( \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl} \) since we have:

\[
W_{ij} \geq \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl}
\]

\[
W_{ij} \leq \sum_{l=1}^{n} \sum_{k \neq i} d(i, j) d(k, l) \sigma_i \sigma_k \rho_{ik} X_{kl}
\]

and when \( X_{ij} \) is 0, \( W_{ij} \) is equal to 0.

Therefore, we can rewrite the p-MPUD as follows:
- \min Z_G = \sum_{j=1}^{n} \sum_{i=1}^{n} [\mu_i d(i, j) + \lambda [d(i, j)]^2 \sigma_i^2] X_{ij} + \lambda \sum_{j=1}^{n} \sum_{i=1}^{n} W_{ij}

\text{s.t.}
\sum_{j=1}^{n} X_{ij} = 1, \text{ for } i = 1, 2, 3, \ldots, n
\sum_{j=1}^{n} Y_j = p
Y_j - X_{ij} \geq 0 \text{ for } i \neq j
C^{-i}_j X_{ij} \leq W_{ij} \leq C^{+i}_j X_{ij} \text{ for } i, j
W_{ij} \geq \sum_{l=1}^{n} \sum_{k \neq i} d(i, j)d(k, l)\sigma_i \sigma_k \rho_{ik} X_{kl} - C^{+i}_j (1 - X_{ij})
W_{ij} \leq \sum_{l=1}^{n} \sum_{k \neq i} d(i, j)d(k, l)\sigma_i \sigma_k \rho_{ik} X_{kl} - C^{-i}_j (1 - X_{ij})
Y_j, X_{ij} = \{0, 1\}, W_{ij} \in \mathbb{R}, \ i = 1, 2, \ldots, n, j = 1, 2, \ldots, n

Glover’s method uses fewer number of dummy variables than the Lawler’s method. The fewer number of dummy variables significantly improves the time that it takes to solve large size problems. This will be shown later in the numerical analysis section.

\textbf{Special Case: No Correlation}

In this section, we investigate the special case: when demands are not correlated. Note that if the demands are not correlated, Problem \( P_M \) can be rewritten as follows:

\[- \min Z_{LNC} = \left[ \left( \sum_{j=1}^{n} \sum_{i=1}^{n} (\lambda (d(i, j))^2 \sigma_i^2 + d(i, j)\mu_i) X_{ij} \right) \right]
\text{s.t.} \]
\[
\sum_{j=1}^{n} X_{ij} = 1, \text{ for } i = 1, 2, 3, ..., n
\]
\[
\sum_{j=1}^{n} Y_j = p
\]
\[
Y_j - X_{ij} \geq 0 \text{ for } i \neq j
\]
\[
Y_j = \{0, 1\}, X_{ij} = \{0, 1\}, i = 1, 2, ..., n, j = 1, 2, ..., n
\]

Clearly, the problem is the classical p-median problem where the coefficient \(\lambda (d(i, j))^2 \sigma_i^2 + d(i, j)\mu_i\) replaces \(\mu_i d(i, j)\) in the objective function.

## 4.3 Center Problem

### 4.3.1 Model

In this section we analyze the weighted center problem with uncertain demand (CPUD). We first, consider a decision makers with mean-variance objective function who plans to locate a single facility which maximizes her minimum possible objective function, 1-CPUD. Later we will extend our analysis to the p-facility location problem, p-CPUD.

In the weighted 1-center problem, similar to the median problem, the decision makers’ objective function increases when the distance traveled by customers in the system decreases. Hence, the objective function of the decision maker for the weighted 1-center problem with uncertain demand is (here \(y\) in Equation (4.1) is \(-h_j d(j, X)\)):

\[
\max_{X \in G} \min_{j \in N} U(-h_j d(j, X)) = \max_{X \in G} \min_{j \in N} (E(-h_j d(j, X)) - \lambda Var(-h_j d(j, X)))
\]
\[
= -\min_{X \in G} \max_{j \in N} (E(h_j d(j, X)) + \lambda Var(h_j d(j, X)))
\]
\[
= -\min_{X \in G} \max_{j \in N} (\mu_j d(j, X) + \lambda (d(j, X))^2 \sigma_j^2)
\]

Note that in this model the fact that the demand of node \(i\) is correlated with the demand of node \(j\), does not have any impact on the optimal solution, since for a given location we can identify which node causes the maximum undesired value independently
of the demand’s correlation structure. Therefore, the fact that this market’s demand is
correlated with any other market demand, does not change the status of the node which
causes the maximum undesired value.

Next we will find the optimal solution for 1-CPUD.

4.3.2 Analysis

Let \( X \) denotes both \( d(a, X) \) and the location of \( X \) on a link \((a, b)\). The function \( U_j(C_j(X)) \) where \( X \) is a point on a link \((a, b)\) and \( C_j(X) = -h_j d(j, X) \), can be expressed as follows:

\[
U_j(C_j(X)) = \mu_j d(j, X) + \lambda (d(j, X))^2 \sigma_j^2
\]

Note that when \( \lambda > 0 \), \( U_j(C_j(X)) \) is a convex function for any \( j \) on any primary region, and since the maximum of convex functions is convex,

\[
C(X) = \max_{j \in N} \left( \mu_j d(j, X) + \lambda (d(j, X))^2 \sigma_j^2 \right),
\]

is convex on any primary region of link \((a, b)\). Hence \( C(X) \) must have a unique global minimum on any primary region. Suppose that the link \((a, b)\), consists of \( m \) primary regions. Note that the minimum of \( U_j(C_j(X)) \) on any primary region \((X_i, X_{i+1})\) of the link \((a, b)\) is

\[
\begin{cases}
- \left( \frac{\mu_j}{2\lambda \sigma_j^2} + d(a, j) \right) & \text{if } j \in A \\
\frac{\mu_j}{2\lambda \sigma_j^2} + d(a, b) + d(b, j) & \text{if } j \in B
\end{cases}
\]

which are both outside the primary region \((X_i, X_{i+1})\) (in fact outside the link \((a, b)\)). Hence the minimum of \( U_j(C_j(X)) \) might be at the end points of the primary region. However, if the end points are breakpoints they cannot be the global minimum since a breakpoint is a local maximum for one of \( U_j(C_j(X)) \) functions. Therefore we can ignore the primary region’s end points that are breakpoints. Furthermore \( U_j(C_j(X)) \) and \( U_i(C_i(X)) \) may have an intersection on the primary region \((X_i, X_{i+1})\). The intersection may be a local minimum of the maximum of \( U_j(C_j(X)) \) and \( U_i(C_i(X)) \). Therefore
the intersection is a candidate for the global minimum of \( C(X) \) on the primary region \((X_i, X_{i+1})\).

The above discussion shows that the only points that might be the global minimum of \( C(X) \) on any primary region are the intersections and possibly nodes \( \{a, b\} \), if the nodes happen to be the end point of the primary region. Therefore on any link \((a, b)\), to find the global minimum of \( C(X) \), we only need to check the intersections and the nodes \( \{a, b\} \). As an example, consider again the network with three markets shown in Figure 4.1.

The mean and standard deviation of each market is as follows: \( \mu_1 = 2, \mu_2 = 2, \mu_3 = 5, \sigma_1 = 2, \sigma_2 = 1.5 \) and \( \sigma_3 = 1 \). We also assume that \( \lambda = 1 \). We would like to find the minimum of \( C(X) \) on the link \((1, 2)\).

We assume \( X \) is the distance from market 1 on link \((1, 2)\). Note that link \((1, 2)\) consists of two primary regions \((1, a)\) and \((a, 2)\) where \( a \) is the breakpoint on link \((1, 2)\), three units away from node 1. The \( U_j(C_j(X)) \)s can be written as follows for \( j = 1, 2, 3 \):

\[
U_1(C_1(X)) = 2X + 4X^2
\]
\[
U_2(C_2(X)) = 2(5 - X) + 2.25(5 - X)^2
\]
\[
U_3(C_3(X)) = \begin{cases} 
5(2 + X) + (2 + X)^2 & 0 \leq X \leq 3 \\
5(8 - X) + (8 - X)^2 & 3 \leq X \leq 5
\end{cases}
\]

A plot of the \( U_j(C_j(X)) \)s is given in Figure 4.2.

As discussed above, to find the minimum of \( C(X) \) on link \((1, 2)\), in addition to the end points of the link (node 1 and 2), it is also required to check the intersection of \( U_j(C_j(X)) \)s. Indeed the minimum of \( C(X) \) on link \((1, 2)\) happens to be at the intersection of \( U_2(C_2(X)) \) and \( U_3(C_3(X)) \). Also notice that the breakpoint "a" cannot be the global minimum because it is a local maximum for \( U_3(C_3(X)) \).

When \( \lambda < 0 \), \( U_j(C_j(X)) \) is a concave function for any \( j \). Based on Zangwill’s (1967) Theorems 1 and 4, the maximum of concave functions is a piecewise concave function.
and furthermore, the minimum of maximum of concave functions is either achieved at the end points of the primary regions or at the intersection of the functions. Note that in contrast to the case that \( \lambda > 0 \), in this case the minimum of the \( C(X) \) might be at breakpoints, since breakpoints can be a local minimum. We consider again the above example for this case. The \( U_j(C_j(X)) \)s can be written as follows for \( j = 1, 2, 3 \):

\[
U_1(C_1(X)) = 2x - 4x^2 \\
U_2(C_2(X)) = 2(5 - x) - 2.25(5 - x)^2 \\
U_3(C_3(X)) = \begin{cases} 
5(2 + x) - (2 + x)^2 & 0 \leq x \leq 3 \\
5(8 - x) - (8 - x)^2 & 3 \leq x \leq 5
\end{cases}
\]

A plot of the \( U_j(C_j(X)) \)s is shown in Figure 4.3.

In this example the minimum of \( C(X) \) is at the breakpoint "a".

The following theorem formally states how one can find the optimal location for the 1-center problem on a link.

**Theorem 4.2.** Let \( N \) be the set of all the intersection of \( U_j(C_j(X)) \)s on a link \((a, b)\) and...
nodes \{a, b\}. Let \(\tilde{N}\) be the set that includes all the points in \(\overline{N}\) and the breakpoints on the link \((a, b)\). An optimal solution of 1-CPUD on the link \((a, b)\):

1. for \(\lambda > 0\), is

\[
\begin{align*}
\text{in } \overline{N} & \quad \text{if } \overline{N} \neq \emptyset \\
\quad a & \quad \text{if } U(a) \leq U(b) \text{ and } \overline{N} = \emptyset \\
\quad b & \quad \text{if } U(a) \geq U(b) \text{ and } \overline{N} = \emptyset
\end{align*}
\]

2. for \(\lambda < 0\), is in Set \(\tilde{N}\)

3. for \(\lambda = 0\), is the optimal solution of the classical 1-center problem which is in \(\overline{N}\).

\textbf{Proof.} For Part 1 and 2, based on the above argument the proof is straightforward, hence omitted. For Part 3 refer to Hakimi (1964).

Notice that in Theorem 4.2, the implicit step is to check whether points \(a, b\) and the points in \(\overline{N}\) and \(\tilde{N}\) are the maximum of \(U_j(X)\)s, before computing the value of \(m(x)\) at each point.

Now we are ready to present Algorithm 2 to solve 1-CPUD based on Theorem 4.2.
Algorithm 2.

Step 0. Set $U = -\infty$, all links are initially unlabeled.

Step 1. Choose any unlabeled link $(a, b) \in L$ and obtain the set $\overline{N}$ if $\lambda > 0$ or $\overline{N}$ if $\lambda < 0$. Find the optimal solution on the link $(a, b), Y$, based on Theorem 4.2.

Step 2. If $U(Y) > U$ go to Step 4, otherwise go to Step 3.

Step 3. Label link $(a, b)$. If there are links not yet considered go to Step 1. Otherwise stop, the optimal solution is $X^0$ on link $(a, b)^0$.

Step 4. The new incumbent solution is $X^0 = Y, (a, b)^0 = (a, b)$ and $U = U(Y)$. Go to Step 3.

Note that the set $\overline{N}$ on any link has at most $\frac{n(n-1)}{2} + 2$ points. Hence finding the set $\overline{N}$ is polynomial $O(n^2)$. Finding the set $\overline{N}$, consists of all the points in $\overline{N}$ and the breakpoints, is also polynomial $O(n^2)$ since finding the breakpoints and set $\overline{N}$ are polynomial $O(n^2)$. As a result the above algorithm is polynomial $O(n^3)$ for both $\lambda > 0$ and $\lambda < 0$.

4.3.3 Mathematical Formulation for CPUD

In this section we extend our analysis to $p$ facility case. Similar to Garfinkel et al’s (1977), we can show that the FDS for $p$-CPUD is the same as the FDS for 1-CPUD by breaking down the $p$-CPUD to $p$ 1-CPUD problems. In particular, we can define a set $S(X) = \{S_j(X) | j = 1, 2, \ldots\}$ where $S_j(X) = \{i | \text{node } i \text{ assigned to the facility located at } j\}$. Then, following the proof of Theorem 3.1 in Garfinkel et al (1977), we can show that the $p$-CPUD reduces to solving the 1-CPUD $p$ times.

We formulate $p$-CPUD as a quadratic mathematical programming for a given FDS. We define the following binary variables:
\[ X_{ij} = \begin{cases} 1 & \text{if node } i \text{ is assigned to the facility located at } j, \\ 0 & \text{otherwise} \end{cases} \]

\[ Y_j = \begin{cases} 1 & \text{if a facility located at } j, \\ 0 & \text{otherwise} \end{cases} \]

Without loss of generality let us assume that \( N \) is the FDS. The problem, which we call \( P_C \), can now be formulated as follows:

\[ \min Z \]

s.t.

\[ \sum_{j=1}^{n} X_{ij} = 1, \text{ for } i = 1, 2, 3, ..., n \]

\[ \sum_{j=1}^{n} Y_j \leq p \]

\[ X_{ij} \leq Y_j \text{ for } i \neq j \]

\[ Z \geq \sum_{j=1}^{n} \mu_i d(i,j) X_{ij} + \lambda (d(i,j))^2 \sigma_i^2 X_{ij}^2 \text{ for } i = 1, 2, ..., n \]

\[ X_{ij}, Y_j = (0, 1), i = 1, 2, ..., n, j = 1, 2, ..., n, Z \geq 0 \]

Note that since \( X_{ij} \) is either 0 or 1, we can easily convert the above quadratic programming formulation to a linear programming formulation. The only change is in the fourth set of constraints, which can be rewritten as follows:

\[ Z \geq \sum_{j=1}^{n} \left( \mu_i d(i,j) + \lambda (d(i,j))^2 \sigma_i^2 \right) X_{ij} \text{ for } i = 1, 2, ..., n \]

It is interesting to point out that, now the problem becomes the classical weighted \( p \)-center problem where \( \mu_i d(i,j) \) is replaced by \( \mu_i d(i,j) + \lambda (d(i,j))^2 \sigma_i^2 \).
4.4 Generalized Maximal Covering Location Problem

In this section we study the generalized maximal covering location problem (GMCLP) with uncertain weight/demand (GMCLPUD) where the decision maker has a mean-variance objective function. Since the problem with 1 facility is not an interesting problem, we only analyze the problem with p facilities. We assume that node \( i \) can be covered at different levels of coverage, depending on its distance from the closest facility location. Moreover similar to Berman and Krass (2002), we assume that the coverage level decreases as a step function of the distance to the closest facility. Therefore, for each node \( i \in N \) we define \( k \) coverage radii \( r_i^0 = 0 < r_i^1 < r_i^2 < \cdots < r_i^k = \infty \) with associate coverage levels \( a_i^1 = 1 > a_i^2 > \cdots > a_i^k \geq 0 \). For a given set of facility location \( S \) and level \( l \leq k \), let \( N(S, l) \) be the set of all nodes whose shortest distance to \( S \) is in the range \([r_i^{l-1}, r_i^l] \), formally:

\[
N(S, l) = \{ i \in N \mid r_i^{l-1} \leq d(S, i) < r_i^l \}
\]

The GMCLPUD now can be written as:

\[
\max_{S \subset X, |S| = p} U \left( \sum_{l=1}^{k} \sum_{j \in N(S, l)} h_j a_j^l \right) = E \left( \sum_{l=1}^{k} \sum_{j \in N(S, l)} h_j a_j^l \right) - \lambda \text{Var} \left( \sum_{l=1}^{k} \sum_{j \in N(S, l)} h_j a_j^l \right)
\]

\[
= \sum_{l=1}^{k} \sum_{j \in N(S, l)} \mu_j a_j^l - \lambda \sum_{l=1}^{k} \sum_{j=1}^{n} (a_j^l)^2 \sigma_j^2
\]

\[
- \lambda \sum_{l=1}^{k} \sum_{j=1}^{n} \sum_{j \neq m} a_j^l a_m^l \sigma_j \sigma_m \rho_{jm}
\]

4.4.1 Analysis

Note that as discussed in Church and Meadows (1979) for the MCLP and Berman and Krass (2002) for the GMCLP, in general, the optimal set of facility locations for both MCLP and GMCLP need not to be nodal. Let us denote by \( NIPS \) the set containing all nodes, and for each node \( i \), all points that are at distance \( r_i^1, r_i^2, \ldots, r_i^k \) from node \( i \).
Define the region between two consecutive points in $NIPS$ as a primary region. Note that moving from one end of the primary region to the other end will not change the coverage levels ($a_i^l$) for any $i$, hence the value of objective function does not change. As a result it is only sufficient to check all the points in $NIPS$. This has been formally stated in the following theorem:

**Theorem 4.3.** An optimal set of locations for GMCLPUD exists in $NIPS$.

**Proof.** Proof is similar to the proof in Berman and Krass (2002). \qed

### 4.4.2 Mathematical Formulation for GMCLPUD

In this section we formulate the problem as a mathematical programming. It is worth mentioning that the GMCLP with deterministic weights (demands) is a NP-Hard problem. Therefore, the GMCLP with uncertain weights/demands (GMCLPUD) is also NP-Hard.

Our mathematical programming formulation is similar to the mathematical formulation in Berman and Krass (2002). We first define two sets: $X_i(l)$ which contains all potential facility locations that could provide level $l$ coverage for customer at node $i$, and $L(i)$ contains all possible coverage levels that can occur for node $i$. Formally these two sets can be defined as follows: for node $i \in N$ and $l \in \{1, \ldots, k\}$, let $X_i(l) = \{x \in X \mid r_i^{l-1} \leq d(i, x) < r_i^l\}$ for $l \leq k$ and $L(i) = \{l \leq k \mid \exists x \in X, \text{ such that } r_i^{l-1} \leq d(i, x) < r_i^l\}$.

We use the following binary decision variables:

$$f_x = \begin{cases} 1 & \text{if facility is located at } x, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i^l = \begin{cases} 1 & \text{if node } i \text{ is covered by some facility } x \in X_i(l) \text{ and there is} \\ 0 & \text{otherwise} \end{cases}$$

$$y_i^l = \begin{cases} 1 & \text{no other facility within radius less than or equal to } r_i^{l-1} \text{ from } i \\ 0 & \text{otherwise} \end{cases}$$
The problem can now be formulated as follows:

$$\max Z = \left[ \sum_{i=1}^{n} \mu_i \left( \sum_{l \in L(i)} a_i^l y_i^l \right) - \lambda \sum_{i=1}^{n} \sigma_i^2 \left( \sum_{l \in L(i)} a_i^l y_i^l \right)^2 \right. $$

$$\left. - \lambda \sum_{i=1}^{n} \sum_{j \neq i} \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(i)} a_i^l y_i^l \right) \left( \sum_{l \in L(j)} a_j^l y_j^l \right) \right]$$

s.t.

$$\sum_{x \in X} f_x = p, $$

$$\sum_{x \in X(i)} f_x > y_i^l, \ i \in N, l \in L(i) $$

$$\sum_{l \in L(i)} y_i^l \leq 1, \ i \in N $$

$$f_x, y_i^l \in (0, 1), \ i \in N, l \in L(i), x \in X$$

where $X$ is a given set of finite locations which could include all the dominant locations.

Similar to the Median problem with uncertain demand, the above formulation is also a quadratic programming formulation which is very similar to QAP. In this case we can also present the two methods of linearization discussed earlier, to solve the problem: Lawler’s and Glover’s methods.

**Linearization Using Lawler’s Method.** We define a new set of variables, $t_{ij}^{lk}$, and add the following constraints to be able to replace $\left( \sum_{l \in L(i)} a_i^l y_i^l \right) \left( \sum_{l \in L(j)} a_j^l y_j^l \right)$ with

$$\sum_{l \in L(i)} \sum_{k \in L(j)} a_i^l a_j^k t_{ij}^{lk}$$

in the objective function:

$$y_i^l \geq t_{ij}^{lk} \text{ for } i, j = 1, 2, ..., n \text{ and } l \in L(i), \ k \in L(j)$$

$$y_j^k \geq t_{ij}^{lk} \text{ for } i, j = 1, 2, ..., n \text{ and } l \in L(i), \ k \in L(j)$$

$$y_i^l + y_j^k \leq 1 + t_{ij}^{lk} \text{ for } i, j = 1, 2, ..., n \text{ and } l \in L(i), \ k \in L(j)$$

The above constraints guarantee that when $y_i^l = 1$ and $y_j^k = 1$, the $t_{ij}^{lk}$ is equal to 1, and 0 otherwise.
As a result we could formulate the problem as follows:

$$\max Z_L = \left[ \sum_{i=1}^{n} \mu_i \left( \sum_{l \in L(i)} a^l_i y_i^l \right) - \lambda \sum_{i=1}^{n} \sigma_i^2 \left( \sum_{l \in L(i)} (a^l_i)^2 y_i^l \right) \right]$$

$$-\lambda \sum_{i=1}^{n} \sum_{j \neq i} \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(i)} \sum_{k \in L(j)} a^l_i a^k_j t_{ij}^{lk} \right)$$

s.t.

$$\sum_{x \in X} f_x = p,$$

$$\sum_{x \in X_L(l)} f_x > y_i^l, \quad i \in N, l \in L(i)$$

$$\sum_{l \in L(i)} y_i^l \leq 1, \quad i \in N$$

$$y_i^l \geq t_{ij}^{lk} \text{ for } i, j \in N \text{ and } l \in L(i), \quad k \in L(j)$$

$$y_j^k \geq t_{ij}^{lk} \text{ for } i, j \in N \text{ and } l \in L(i), \quad k \in L(j)$$

$$y_i^l + y_j^k \leq 1 + t_{ij}^{lk} \text{ for } i, j \in N \text{ and } l \in L(i), \quad k \in L(j)$$

$$f_x, y_i^l \in (0, 1), \quad i \in N, l \in L(i), x \in X$$

$$t_{ij}^{lk} \in [0, 1], \quad i, j \in N, l, k \in L(i)$$

Note that in the objective function the reason that we were able to convert $$\left( \sum_{l \in L(i)} a^l_i y_i^l \right)^2$$
to $$\left( \sum_{l \in L(i)} (a^l_i)^2 y_i^l \right)$$ and replace $$\left( \sum_{l \in L(i)} a^l_i y_i^l \right) \left( \sum_{l \in L(j)} a^l_j y_j^l \right)$$ with $$\sum_{l \in L(i)} \sum_{k \in L(j)} a^l_i a^k_j t_{ij}^{lk}$$ is that for a given $$i$$, there exist at most one $$l^* \in L(i)$$ that $$y_i^{l^*} = 1$$, and for $$l \in L(i)$$ where $$l \neq l^*$$, $$y_i^l = 0$$.

Similar to the median problem, the major drawback for using Lawler’s method to linearize the GMCLPUD is the number of new dummy variables that are required for large size problems.

**Linearization Using Glover’s Method.** We apply Glover’s method to GM-
CLPUD. Define

\[ W_i^h = a_i^h y_i^h \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(j)} a_j^l y_j^l \right) \]

\[ C_i^+ = \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} 1_{\{\rho_{ij} > 0\}} \]

\[ C_i^- = \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} 1_{\{\rho_{ij} < 0\}} \]

Note that, the following set of constraints guarantee that when \( y_i^h = 0 \), \( W_i^h \) is equal to 0, and when \( y_i^h = 1 \), \( W_i^h = a_i^h \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(j)} a_j^l y_j^l \right) \):

\[ C_i^- a_i^h y_i^h \leq W_i^h \leq C_i^+ a_i^h y_i^h \text{ for } i \text{ and } h \in L(i) \]

\[ W_i^h \geq a_i^h \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(j)} a_j^l y_j^l \right) - C_i^+ a_i^h (1 - y_i^h) \]

\[ W_i^h \leq a_i^h \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(j)} a_j^l y_j^l \right) - C_i^- a_i^h (1 - y_i^h) \]

Therefore we can rewrite the GMCLPUD problem as follows:

\[
\max Z = \sum_{i=1}^n \mu_i \left( \sum_{l \in L(i)} a_i^l y_i^l \right) - \lambda \sum_{i=1}^n \sigma_i^2 \left( \sum_{l \in L(i)} (a_i^l)^2 y_i^l \right) - \lambda \sum_{i=1}^n \sum_{h \in L(i)} W_i^h
\]

s.t.

\[
\sum_{x \in X} f_x = p,
\]

\[
\sum_{x \in X_i(l)} f_x > y_i^l, \ i \in N, l \in L(i)
\]

\[
\sum_{l \in L(i)} y_i^l \leq 1, \ i \in N
\]

\[ C_i^- a_i^h y_i^h \leq W_i^h \leq C_i^+ a_i^h y_i^h \text{ for } i \text{ and } h \in L(i) \]

\[ W_i^h \geq a_i^h \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(j)} a_j^l y_j^l \right) - C_i^+ a_i^h (1 - y_i^h) \text{ for } i \in N \text{ and } h \in L(i) \]

\[ W_i^h \leq a_i^h \sum_{j \neq i}^n \sigma_i \sigma_j \rho_{ij} \left( \sum_{l \in L(j)} a_j^l y_j^l \right) - C_i^- a_i^h (1 - y_i^h) \text{ for } i \in N \text{ and } h \in L(i) \]

\[ f_x, y_i^l = \{0, 1\}, \ W_i^h \in \mathbb{R}, \ i \in N, l, h \in L(i), x \in X \]
The main advantage of this method compared to Lawler’s method is that, Glover’s method requires fewer number of new dummy variables for large size problems which significantly helps to improve the time that it takes to solve the problem.

**Special Case: No Correlation**

In this part we study the case that demands are not correlated. In such a case, the formulation becomes as follows:

\[
\begin{align*}
\max Z &= \sum_{i=1}^{n} \sum_{l \in L(i)} \left( \mu_i a^l_i - \lambda \sigma_i^2 (a^l_i)^2 \right) y^l_i \\
\text{s.t.} \\
\sum_{x \in X} f_x &= p, \\
\sum_{x \in X(l)} f_x &> y^l_i, \ i \in N, l \in L(i) \\
\sum_{l \in L(i)} y^l_i &\leq 1, \ i \in N \\
f_x, y^l_i &\in (0, 1), \ i \in N, l \in L(i), x \in X
\end{align*}
\]

This is essentially the classical generalized maximal covering problem for a given set of finite locations where the weight at any node \( i \) is \( \mu_i a^l_i - \lambda \sigma_i^2 (a^l_i)^2 \).

### 4.5 Example

In this section we discuss the problems that were introduced in previous sections via a simple example. Consider the link network shown in Figure 4.4 with two markets (nodes) which are two units away from each other.

We consider four cases of demand characteristics and for each case we analyze how sensitive the optimal facility location is to the risk attitude of the decision maker. The
four cases are shown in Table 4.1, where M1 and M2 refers to Market 1 and 2 respectively and \( \rho(\ldots) \) refers to the correlation between the demands of the two markets.

Note that, in Table 4.2, 4.3 and 4.4 we use the notation \((U, 1, x, 2)\) to indicate the expected utility value \((U)\) and the optimal facility location at a distance \(x\) from node 1 on link \((1, 2)\).

### 4.5.1 Median Problem

Table 4.2 presents the objective function value and optimal location for each of the four cases of the demand characteristics. For each case we vary the risk attitude coefficient from \(\lambda = -100\) to \(\lambda = 100\). Note that when the decision maker is risk neutral \((\lambda = 0)\), the problem reduces to the classical 1-median problem. We also consider two other extreme types of decision makers: an extreme risk lover and an extreme risk averse decision makers who only care about the variance of the demand which are referred by "Max-Var" and "Min-Var" respectively.


<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Base Case</th>
<th>Correlated Demand</th>
<th>Higher Volatility</th>
<th>Lower volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-100$</td>
<td>(9970,1,0,2)</td>
<td>(9970,1,0,2)</td>
<td>(25580,1,2,2)</td>
<td>(9970,1,0,2)</td>
</tr>
<tr>
<td>$-10$</td>
<td>(970,1,0,2)</td>
<td>(970,1,0,2)</td>
<td>(2540,1,2,2)</td>
<td>(970,1,0,2)</td>
</tr>
<tr>
<td>$-5$</td>
<td>(470,1,0,2)</td>
<td>(470,1,0,2)</td>
<td>(1260,1,2,2)</td>
<td>(470,1,0,2)</td>
</tr>
<tr>
<td>$-3$</td>
<td>(270,1,0,2)</td>
<td>(270,1,0,2)</td>
<td>(748,1,2,2)</td>
<td>(270,1,0,2)</td>
</tr>
<tr>
<td>$-1$</td>
<td>(70,1,0,2)</td>
<td>(70,1,0,2)</td>
<td>(236,1,2,2)</td>
<td>(70,1,0,2)</td>
</tr>
<tr>
<td>$0$</td>
<td>(−20,1,2,2)</td>
<td>(−20,1,2,2)</td>
<td>(−20,1,2,2)</td>
<td>(−20,1,2,2)</td>
</tr>
<tr>
<td>$1$</td>
<td>(−62.8,1,1.280,2)</td>
<td>(−47.7,1,1.202,2)</td>
<td>(−71.0,1,0.698,2)</td>
<td>(−31.1,1,1.635,2)</td>
</tr>
<tr>
<td>$3$</td>
<td>(−140.9,1,1.240,2)</td>
<td>(−94.9,1,1.172,2)</td>
<td>(−159.9,1,0.685,2)</td>
<td>(−49.3,1,1.590,2)</td>
</tr>
<tr>
<td>$5$</td>
<td>(−219.0,1,1.232,2)</td>
<td>(−142.1,1,1.167,2)</td>
<td>(−248.7,1,0.682,2)</td>
<td>(−67.5,1,1.581,2)</td>
</tr>
<tr>
<td>$10$</td>
<td>(−414.1,1,1.226,2)</td>
<td>(−260.0,1,1.162,2)</td>
<td>(−470.9,1,0.680,2)</td>
<td>(−112.9,1,1.574,2)</td>
</tr>
<tr>
<td>$100$</td>
<td>(−3926.3,1,1.220,2)</td>
<td>(−2382.1,1,1.158,2)</td>
<td>(−4469.6,1,0.678,2)</td>
<td>(−930.3,1,1.568,2)</td>
</tr>
<tr>
<td>Max-Var</td>
<td>(100,1,0,2)</td>
<td>(100,1,0,2)</td>
<td>(256,1,2,2)</td>
<td>(100,1,0,2)</td>
</tr>
<tr>
<td>Min-Var</td>
<td>(−39.0,1,1.219,2)</td>
<td>(−23.6,1,1.158,2)</td>
<td>(−44.4,1,0.678,2)</td>
<td>(−9.1,1,1.568,2)</td>
</tr>
</tbody>
</table>

|Table 4.2: Optimal expected utility value and location for 1-median problem with uncertain demand.|

The "Max-Var" decision maker’s objective function is

$$\max \left[ \sum_{j=1}^{n} [d(x,j)]^2 \sigma_j^2 + \sum_{j=1}^{n} \sum_{j \neq k} d(x,j)d(x,k)\sigma_j \sigma_k \rho_{jk} \right]$$

and the "Min-Var" decision maker’s objective function is

$$- \min \left[ \sum_{j=1}^{n} [d(x,j)]^2 \sigma_j^2 + \sum_{j=1}^{n} \sum_{j \neq k} d(x,j)d(x,k)\sigma_j \sigma_k \rho_{jk} \right]$$

We observe that:

1. Since the risk lover ($\lambda < 0$) decision maker’s objective function is such that as variability of Market 1 increases her objective function value increases (compare columns 2,4 and 5). In the Base Case, she locates the facility at Market 1, farthest from Market
2 which has the highest demand variance. This causes customers from Market 2 with the highest variance to travel to Market 1. In the Higher Volatility Case, when Market 1 becomes the most variable demand market, she locates the facility at Market 2.

2. Since the risk averse decision maker’s objective function is such that as variability of Market 1 decreases (compare columns 2, 4 and 5) her objective function value increases, we observe that in the Base Case, she locates the facility closer to Market 2. By doing so, both markets’ customers will not have to travel too long, in particular Market 2’s customers who have the highest variance. In the Higher Volatility Case, when Market 1 becomes the most variable demand market, she locates the facility closer to Market 1 and in the Lower Volatility Case, when the variance of Market 1 is reduced, she locates the facility even closer to Market 2.

3. As we expected the risk neutral decision maker only cares about the mean of the demand (1-median location problem), hence she locates the facility at Market 2 with the highest mean. However as the decision maker becomes more risk averse or risk lover the effect of the mean vanishes and the effect of variance of the demand distribution becomes more pronounced. For example in the Base Case, as the decision maker becomes more risk averse the optimal location of the facility becomes closer to that of the "Min-Var" decision maker (1, 1.219, 2).

4. Ignoring the demand correlation when comparing the Base Case and the Correlated Demand Case (comparing column 2 and 3 in Table 2) could result in a suboptimal decision. The optimal location in the Correlated Case is closer to Market 1 compared to that of the Base Case. It is because of the negative correlation between the demand of Markets 1 and 2 that the risk averse decision maker locates the facility closer to Market 1. The negative correlation in fact, to some extent mitigate the higher variance in Market 2, so the optimal facility location moves closer to Market 1.

5. It is worth noting that for the problem with two markets, if the markets’ demands are independent the "Min-Var" problem is equivalent to the 1-center problem (discussed
next) with weights at each node equal to the variance of that market.

### 4.5.2 Center Problem

Table 4.3 presents the optimal location and objective function value for each scenario. As mentioned earlier, since the correlation between demands do not have any impact on the optimal solution we do not consider the correlated demand case. Similar to the median problem, for each case we vary the risk attitude coefficient from the extremely risk lover ($\lambda = -100$) to the extremely risk averse ($\lambda = 100$). Note that when the decision maker is risk neutral ($\lambda = 0$), the problem reduces to the classical 1-center problem. As mentioned before we also consider two extreme cases which are referred by "Max-Var" and "Min-Var".

The "Max-Var" decision maker’s objective function is

$$\max \min_{x \in G} \min_{j \in N} \left( (d(j, x))^2 \sigma_j^2 \right)$$

and the "Min-Var" decision maker’s objective function is

$$- \min \max_{x \in G} \max_{j \in N} \left( (d(j, x))^2 \sigma_j^2 \right)$$

We observe that:

1. As expected the risk neutral decision maker (classical center problem) only cares about the mean of the demand distributions. Hence she locates the facility along the link $(1, 2)$, 1.2 units away from Market 1 (solving the equation $10x = 15(2 - x)$). However, like for the median problem, as the decision maker becomes more risk averse or risk lover the effect of the mean diminishes and the effect of the variance of the demand distribution becomes more pronounced. For example in the Base Case, as the decision maker becomes more risk averse the optimal location of the facility approaches the optimal location of the "Min-Var" decision maker and as the decision maker becomes more risk lover the
<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Base Case</th>
<th>Higher Volatility</th>
<th>Lower volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>-100</td>
<td>(1963.2, 1, 1.111, 2)</td>
<td>(3772.6, 1, 0.769, 2)</td>
<td>(803.7, 1, 1.430, 2)</td>
</tr>
<tr>
<td>-10</td>
<td>(185.4, 1, 1.108, 2)</td>
<td>(364.4, 1, 0.762, 2)</td>
<td>(69.0, 1, 1.444, 2)</td>
</tr>
<tr>
<td>-5</td>
<td>(86.7, 1, 1.105, 2)</td>
<td>(175.0, 1, 0.755, 2)</td>
<td>(28.1, 1, 1.462, 2)</td>
</tr>
<tr>
<td>-3</td>
<td>(47.2, 1, 1.101, 2)</td>
<td>(99.2, 1, 0.745, 2)</td>
<td>(11.8, 1, 1.491, 2)</td>
</tr>
<tr>
<td>-1</td>
<td>(7.6, 1, 1.071, 2)</td>
<td>(23.4, 1, 0.687, 2)</td>
<td>(0, 1, 0, 2)</td>
</tr>
<tr>
<td>0</td>
<td>(-12, 1, 1.2, 2)</td>
<td>(-12, 1, 1.2, 2)</td>
<td>(-12, 1, 1.2, 2)</td>
</tr>
<tr>
<td>1</td>
<td>(-31.8, 1, 1.132, 2)</td>
<td>(-52.0, 1, 0.827, 2)</td>
<td>(-20.6, 1, 1.343, 2)</td>
</tr>
<tr>
<td>3</td>
<td>(-71.3, 1, 1.119, 2)</td>
<td>(-127.9, 1, 0.790, 2)</td>
<td>(-37.1, 1, 1.390, 2)</td>
</tr>
<tr>
<td>5</td>
<td>(-110.9, 1, 1.116, 2)</td>
<td>(-203.6, 1, 0.782, 2)</td>
<td>(-53.4, 1, 1.403, 2)</td>
</tr>
<tr>
<td>10</td>
<td>(-209.6, 1, 1.114, 2)</td>
<td>(-393.0, 1, 0.776, 2)</td>
<td>(-94.3, 1, 1.415, 2)</td>
</tr>
<tr>
<td>100</td>
<td>(-1987.4, 1, 1.111, 2)</td>
<td>(-3801.3, 1, 0.770, 2)</td>
<td>(-829.0, 1, 1.427, 2)</td>
</tr>
<tr>
<td>Max-Var</td>
<td>(19.7, 1, 1.111, 2)</td>
<td>(37.9, 1, 0.769, 2)</td>
<td>(8.2, 1, 1.429, 2)</td>
</tr>
<tr>
<td>Min-Var</td>
<td>(-19.7, 1, 1.111, 2)</td>
<td>(-37.9, 1, 0.769, 2)</td>
<td>(-8.2, 1, 1.429, 2)</td>
</tr>
</tbody>
</table>

Table 4.3: Optimal expected utility value and location for 1-Center problem with uncertain demand.
optimal location of the facility approaches the optimal location of the "Max-Var".

2. Comparing the Base, Higher Volatility and Lower Volatility cases, we observe that, in the Base Case, the risk lover decision maker locates the facility closer to Market 2 (for the extreme risk lover decision maker, solving the equation $16x^2 = 25(2 - x)^2$ gives the optimal location at $(1, 1.111, 2)$) which has the highest demand variance. However, in the Higher Volatility Case, when Market 1 becomes the most variable demand market, she locates the facility closer to Market 1 and in the Lower Volatility Case, when the variance of Market 1 was reduced, she locates the facility even closer to Market 2 which has higher variance than Market 1.

3. Comparing the Base, Higher Volatility and Lower Volatility cases, we observe that, in the Base Case, the risk averse decision maker locates the facility closer to Market 2 which has the highest demand variance to reduce the volatility effect. In the Higher Volatility Case, when Market 1 becomes the most variable demand market, she locates the facility closer to Market 1 and in the Lower Volatility Case, when the variance of Market 1 is reduced, she locates the facility even closer to Market 2 than for the base case.

4. For the problem on a link, the optimal facility locations of the "Max-Var" and "Min-Var" are the same. The extreme risk averse and risk lover decision makers will do the same. This result is due to the simple topology of the example and does not necessary holds in general.

### 4.5.3 Generalized Maximal Covering Location Problem

Similar to the median problem, we consider the same four cases of demand characteristics that were introduced in the median problem analysis. Note that when the decision maker is risk neutral ($\lambda = 0$), the problem reduces to the classical generalized maximal covering location problem. Similar to the previous problems we also consider the extreme cases "Max-Var" and "Min-Var".
The "Max-Var" decision maker’s objective function is
\[
\max \left[ \sum_{l=1}^{k} \sum_{j=1}^{n} (a_j^l)^2 \sigma_j^2 + \sum_{l=1}^{k} \sum_{j=1}^{n} \sum_{j \neq m} a_j^l a_m^l \sigma_j \sigma_m \rho_{jm} \right]
\]
and the "Min-Var" decision maker’s objective function is
\[
-\min \left[ \sum_{l=1}^{k} \sum_{j=1}^{n} (a_j^l)^2 \sigma_j^2 + \sum_{l=1}^{k} \sum_{j=1}^{n} \sum_{j \neq m} a_j^l a_m^l \sigma_j \sigma_m \rho_{jm} \right]
\]

We consider the following structure for the coverage (identical for both of the markets):

<table>
<thead>
<tr>
<th>Coverage Radius</th>
<th>Coverage Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^0 = 0 )</td>
<td>( a^1 = 1 )</td>
</tr>
<tr>
<td>( r^1 = 1 )</td>
<td>( a^2 = 0.85 )</td>
</tr>
<tr>
<td>( r^2 = 2 )</td>
<td>( a^1 = 0.7 )</td>
</tr>
</tbody>
</table>

Table 4.4 presents the objective function value and optimal location for each scenario.

Note that due to Theorem 4.3, it is only required to focus on \( NIPS = \{x = 0, x = 1, x = 2\} \) as potential optimal locations.

We observe that:

1. As we expected the risk neutral decision maker only cares about the mean of demand distributions hence she locates the facility at Market 2 (node 2 with the objective function value of \( 15 + 0.7(10) = 22 \)) which has the highest mean. However as the decision maker becomes more risk averse or risk lover the effect of the mean vanishes and the effect of variance of demand distribution becomes more pronounced. Hence the optimal facility location becomes closer to that of the "Min-Var" or "Max-Var" decision maker.

2. Since the risk lover decision maker’s utility is such that as the variability increases her utility increases, in the Base Case, she locates the facility at Market 2 which has the highest demand variance. In the Higher Volatility Case, when Market 1 has the highest demand variance, she locates the facility at Market 1.

3. Since the risk averse decision maker’s utility is such that as the variability decreases
### Table 4.4: Optimal expected utility value and location for generalized maximal covering location problem with uncertain demand.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Base Case</th>
<th>Correlated Demand</th>
<th>Higher Volatility</th>
<th>Lower volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-100$</td>
<td>$(3306.0, 1, 2, 2)$</td>
<td>$(2186.0, 1, 2, 2)$</td>
<td>$(7645.5, 1, 0, 2)$</td>
<td>$(2718.0, 1, 2, 2)$</td>
</tr>
<tr>
<td>$-10$</td>
<td>$(350.4, 1, 2, 2)$</td>
<td>$(238.4, 1, 2, 2)$</td>
<td>$(783.0, 1, 0, 2)$</td>
<td>$(291.6, 1, 2, 2)$</td>
</tr>
<tr>
<td>$-5$</td>
<td>$(186.2, 1, 2, 2)$</td>
<td>$(130.2, 1, 2, 2)$</td>
<td>$(401.8, 1, 0, 2)$</td>
<td>$(156.8, 1, 2, 2)$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$(120.5, 1, 2, 2)$</td>
<td>$(86.9, 1, 2, 2)$</td>
<td>$(249.3, 1, 0, 2)$</td>
<td>$(102.9, 1, 2, 2)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$(54.8, 1, 2, 2)$</td>
<td>$(43.6, 1, 2, 2)$</td>
<td>$(96.8, 1, 0, 2)$</td>
<td>$(49.0, 1, 2, 2)$</td>
</tr>
<tr>
<td>$0$</td>
<td>$(22.0, 1, 2, 2)$</td>
<td>$(22.0, 1, 2, 2)$</td>
<td>$(22.0, 1, 2, 2)$</td>
<td>$(22.0, 1, 2, 2)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$(-7.8, 1, 0, 2)$</td>
<td>$(3.5, 1, 0, 2)$</td>
<td>$(-34.4, 1, 2, 2)$</td>
<td>$(4.3, 1, 0, 2)$</td>
</tr>
<tr>
<td>$3$</td>
<td>$(-64.3, 1, 0, 2)$</td>
<td>$(-30.7, 1, 0, 2)$</td>
<td>$(-147.1, 1, 2, 2)$</td>
<td>$(-28.3, 1, 0, 2)$</td>
</tr>
<tr>
<td>$5$</td>
<td>$(-120.8, 1, 0, 2)$</td>
<td>$(-64.8, 1, 0, 2)$</td>
<td>$(-259.8, 1, 2, 2)$</td>
<td>$(-60.8, 1, 0, 2)$</td>
</tr>
<tr>
<td>$10$</td>
<td>$(-262.0, 1, 0, 2)$</td>
<td>$(-150.0, 1, 0, 2)$</td>
<td>$(-541.6, 1, 2, 2)$</td>
<td>$(-142.0, 1, 0, 2)$</td>
</tr>
<tr>
<td>$100$</td>
<td>$(-2804.5, 1, 0, 2)$</td>
<td>$(-1684.5, 1, 0, 2)$</td>
<td>$(-5614.0, 1, 2, 2)$</td>
<td>$(-1604.5, 1, 0, 2)$</td>
</tr>
<tr>
<td>Max-Var</td>
<td>$(32.8, 1, 2, 2)$</td>
<td>$(21.6, 1, 2, 2)$</td>
<td>$(76.3, 1, 0, 2)$</td>
<td>$(27.0, 1, 2, 2)$</td>
</tr>
<tr>
<td>Min-Var</td>
<td>$(-28.3, 1, 0, 2)$</td>
<td>$(-17.1, 1, 0, 2)$</td>
<td>$(-56.4, 1, 2, 2)$</td>
<td>$(-16.3, 1, 0, 2)$</td>
</tr>
</tbody>
</table>
her utility increases, we observe that in the Base Case, she locates the facility at Market 1 which has the lowest demand variance. However in the Higher Volatility Case, when Market 1 becomes the most variable demand market, she locates the facility at Market 2 and in the Lower Volatility Case, when the variance of Market 1 is reduced, she locates the facility again at Market 1.

4. Although it is not clear from Table 4.4, ignoring the demand correlation could result in a suboptimal decision.

### 4.6 Incorporation of Expert View

Since the facility location is a strategic decision and relocating the facility can be very costly, the firm may consult with experts and/or utilize the expertise that has been developed within the firm for such a decision. The firm may develop a view on the uncertain demand using within the firm or outside resources. In many cases this view may be different compared to the current demand or the forecasted one. Hence incorporating such a view and mixing it with the current demand estimation/forecast using a quantitative tool can be important. Moreover, it will help to avoid making changes subjectively to the solution of the optimization problem. The approach that we are proposing here to facilitate the incorporation of the specific view(s), is using the Bayesian updating method. Black and Litterman (1992) were the first to introduce this concept in the context of portfolio management.

We consider a networks with $N$ nodes, whose demands are normally distributed:

$$D \sim N(\beta, \Lambda).$$

(4.6)

where $\beta$ and $\Lambda$ are, respectively, the mean vector and covariance matrix of random vector $D$. 
As an example, consider a three node network where

\[
D = \begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix}, \beta = \begin{bmatrix}
10 \\
6 \\
10
\end{bmatrix}, \Lambda = \begin{bmatrix}
2 & 0.7071 & -1.1313 \\
0.7071 & 1 & -0.4 \\
-1.1313 & -0.4 & 1
\end{bmatrix}.
\]

Furthermore we assume that there is a set of \(M\) linear uncertain views on the expectations of the demands which is represented by vector \(V\). The random vector \(V\), with dimension \(M\), can be seen as a conditional random vector since the expert’s view can be expressed as a realization of random vector \(D\). We assume that the expert’s view vector is normally distributed:

\[
V|d \sim N(v, \Omega)
\]

where \(v\) and \(\Omega\) are, respectively, the mean vector and covariance matrix of the conditional random vector \(V\) and \(d\) is a realization of random vector \(D\). Note that the expert does not necessarily need to know the realization. Alternatively, we can present the vector \(V\) as follows:

\[
V|d \sim N(Qd, \Omega).
\]

where \(Q\) is an auxiliary matrix that represents the view, whose k-th row is the relative weight of each uncertain demand in the respective view.

The linear view can be expressed with regard to a single market demand or combination of them. As mentioned earlier the expert does not necessarily need to know the realization \(d\). She only has information about the vector \(v\) and matrix \(Q\). Next, we clarify the discussion with a few examples.

In the three node network in Figure 1, assume that the expert has the following views:

1. The mean demand of market 1 is 30 and its variance is 10%,
2. The mean demand of market 3 is 5 and its variance is 10%.

\[
Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, v = \begin{bmatrix}
30 \\
5
\end{bmatrix}, \Omega = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}.
\]
This is the case where the expert has views about the single market demands. However, it can be the case that the expert provides relative views such as follows: the difference between mean demand of market 1 and market 2 is 15 and its variance is 20%.

In this case $Q, v$ and $\Omega$ are as follows:

$$Q = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 15 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & 0 & 0.2 \end{bmatrix}.$$ 

Notice that in this case the view can be expressed via many realizations of the vector $D$. It is worth mentioning that a view does not necessarily require to be in agreement with the current or estimated demand, as a matter of fact it can potentially clash with the current or estimated demand.

Next we show that how to obtain the distribution of $D$ given the views using the Bayesian approach. The theorem and its proof are based on Black and Litterman’s (1992) and Meucci’s (2008). The reader can refer to them for detailed discussion.

**Theorem 4.4.** The distribution of $P_{up}$ ("up" for updated) given the views using Bayesian updating approach is:

$$P_{up} \sim N(\mu_{up}, \Lambda_{up})$$

where

$$\mu_{up} = \beta + \Lambda Q^T (Q\Lambda Q^T + \Omega)^{-1} (v - Q\beta)$$

and

$$\Lambda_{up} = \Lambda - \Lambda Q^T (Q\Lambda Q^T + \Omega)^{-1} QA$$

**Proof.** This proof is modified version of Black and Litterman’s (1992) global asset allocation model, the well-known model in asset allocation literature, and Meucci’s (2008) extensions of the Black-Litterman approach. We have (4.6), therefore:

$$f_P(p) = \frac{(\Lambda)^{-0.5}}{(2\pi)^{0.5N}} e^{-0.5(p-\beta)^T \Lambda^{-1} (p-\beta)}$$

also from (4.7) we obtain:

$$f_{V|p}(v) = \frac{(\Omega)^{-0.5}}{(2\pi)^{0.5K}} e^{-0.5(v-Qp)^T \Omega^{-1} (v-Qp)}.$$
To find the posterior distribution of the price given the views, we can apply Bayesian approach:

\[
f_{P|\nu, \Omega}(p) = \frac{f_{P|\nu}(p, \nu)}{f_{\nu}(\nu)} = \frac{f_{P|\nu}(p) f_P(p)}{\int f_{V|\nu}(v) f_P(p) dp}.
\]

We will show that the numerator can be written as a conditional probability distribution function \(f_{P|\nu}\) and a function that only depends on \(\nu\). We have:

\[
f_{P|\nu}(p, \nu) = \frac{f_{V|\nu}(v) f_{\nu}(\nu) e^{-0.5(p-\beta)^T \Lambda^{-1} (p-\beta)}}{f_{\nu}(\nu)}
\]

Expanding the following expression we have:

\[
(v - Qp)^T \Omega^{-1} (v - Qp) + (p-\beta)^T \Lambda^{-1} (p-\beta)
\]

(4.8)

= \(p^T \Lambda^{-1} p - 2p^T \Lambda^{-1} \beta + \beta^T \Lambda^{-1} \beta + v^T \Omega^{-1} v - 2p^T Q^T \Omega^{-1} v + p^T Q^T \Omega^{-1} Qp\)

= \(p^T (\Lambda^{-1} + Q^T \Omega^{-1} Q) p - 2p^T (\Lambda^{-1} \beta + Q^T \Omega^{-1} v) + \beta^T \Lambda^{-1} \beta + v^T \Omega^{-1} v\)

Define:

\[
\mu_{up} = \left(\Lambda^{-1} + Q^T \Omega^{-1} Q\right)^{-1} (\Lambda^{-1} \beta + Q^T \Omega^{-1} v)
\]

or

\[
\mu_{up} \left(\Lambda^{-1} + Q^T \Omega^{-1} Q\right) = \left(\Lambda^{-1} \beta + Q^T \Omega^{-1} v\right)
\]

Using above definition we can rewrite (4.8) as follows:

\[
= p^T (\Lambda^{-1} + Q^T \Omega^{-1} Q) p - 2p^T (\Lambda^{-1} + Q^T \Omega^{-1} Q) \mu_{up}
\]

(4.9)

+ \mu_{up}^T (\Lambda^{-1} + Q^T \Omega^{-1} Q) \mu_{up} + \beta^T \Lambda^{-1} \beta + v^T \Omega^{-1} v - \mu_{up}^T (\Lambda^{-1} + Q^T \Omega^{-1} Q) \mu_{up}

= \(p - \mu_{up})^T (\Lambda^{-1} + Q^T \Omega^{-1} Q) (p - \mu_{up}) + \theta
where
\[
\theta = \beta^T \Lambda^{-1} \beta + v^T \Omega^{-1} v - \mu_{up}^T \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right) \mu_{up}
\]

Simplifying \( \theta \), we have:
\[
\theta = \beta^T \Lambda^{-1} \beta + v^T \Omega^{-1} v - \beta^T \Lambda^{-1} \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} \Lambda^{-1} \beta \\
- v^T \Omega^{-1} Q \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} Q^T \Omega^{-1} v + 2 \beta^T \Lambda^{-1} \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} Q^T \Omega^{-1} v \\
= v^T \left[ \Omega^{-1} - \Omega^{-1} Q \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} Q^T \Omega^{-1} \right] v \\
+ \beta^T \left( \Lambda^{-1} - \Lambda^{-1} \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right) \Lambda^{-1} \right) \beta \\
\]

Define
\[
\tilde{v} = - \left( \Omega + Q \Lambda Q^T \right) \Omega^{-1} Q \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} \Lambda^{-1} \beta
\]
or
\[
\left( \Omega + Q \Lambda Q^T \right)^{-1} \tilde{v} = - \Omega^{-1} Q \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} \Lambda^{-1} \beta
\]
Furthermore note that we have:
\[
\Omega^{-1} - \Omega^{-1} Q \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right)^{-1} Q^T \Omega^{-1} = \left( \Omega + Q \Lambda Q^T \right)^{-1}
\]
Therefore:
\[
\theta = v^T \left( \Omega + Q \Lambda Q^T \right)^{-1} v - 2 v^T \left( \Omega + Q \Lambda Q^T \right)^{-1} \tilde{v} \\
+ \tilde{v}^T \left( \Omega + Q \Lambda Q^T \right)^{-1} \tilde{v} + \beta^T \left( \Lambda^{-1} - \Lambda^{-1} \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right) \Lambda^{-1} \right) \beta \\
- \tilde{v}^T \left( \Omega + Q \Lambda Q^T \right)^{-1} \tilde{v} \\
= \left( v - \tilde{v} \right)^T \left( \Omega + Q \Lambda Q^T \right)^{-1} \left( v - \tilde{v} \right) + \kappa,
\]
where
\[
\kappa = \beta^T \left( \Lambda^{-1} - \Lambda^{-1} \left( \Lambda^{-1} + Q^T \Omega^{-1} Q \right) \Lambda^{-1} \right) \beta - \tilde{v}^T \left( \Omega + Q \Lambda Q^T \right)^{-1} \tilde{v}
\]
Chapter 4. Location Problems with Uncertain Demand

Note that $\kappa$ does not depend on $v$ or $p$. Therefore we can rewrite (4.9) as follows:

$$= (p - \mu_{up})^T (\Lambda^{-1} + Q^T\Omega^{-1}Q) (p - \mu_{up}) + (v - \bar{v})^T (\Omega + QAQ^T)^{-1} (v - \bar{v}) + \kappa$$

As a result the joint probability distribution becomes:

$$f_{P,V}(p, v) = \frac{(\Omega)^{-0.5} \Lambda^{-0.5} e^{-0.5[(v-Qp)^T\Omega^{-1}(v-Qp)+(p-\beta)^T\Lambda^{-1}(p-\beta)]}}{(2\pi)^{0.5(K+N)}}$$

$$\propto (\Omega)^{-0.5} \Lambda^{-0.5} e^{-0.5[(p-\mu_{up})^T(\Lambda^{-1}+Q^T\Omega^{-1}Q)(p-\mu_{up})]e^{-0.5[(v-\bar{v})^T(\Omega+QAQ^T)^{-1}(v-\bar{v})]}}$$

Note that since we have:

$$\frac{|\Omega + QAQ^T|}{|\Lambda|\Omega|\Lambda^{-1} + Q^T\Omega^{-1}Q|} = 1$$

or

$$|\Lambda|\Omega|\Lambda^{-1} + Q^T\Omega^{-1}Q| = |\Lambda (\Lambda^{-1} + Q^T\Omega^{-1}Q)\Omega|$$

$$= |I + \Lambda Q^T\Omega^{-1}Q|\Omega|$$

$$= |\Omega + QAQ^T|$$

Therefore:

$$f_{P,V}(p, v) = |\Lambda^{-1} + Q^T\Omega^{-1}Q|^{0.5} e^{-0.5[(p-\mu_{up})^T(\Lambda^{-1}+Q^T\Omega^{-1}Q)(p-\mu_{up})]}$$

$$|\Omega + QAQ^T|^{-0.5} e^{-0.5[(v-\bar{v})^T(\Omega+QAQ^T)^{-1}(v-\bar{v})]}.$$

where

$$\mu_{up} = (\Lambda^{-1} + Q^T\Omega^{-1}Q)^{-1} (\Lambda^{-1}\beta + Q^T\Omega^{-1}v)$$

and $\bar{v}$ does not depend on $v$ or $p$. Therefore we showed that $f_{P,V}(p, v)$ can be written as a conditional probability distribution and a function of $v$:

$$f_{P,V}(p, v) \propto f_{P|V}(p \mid v)h(v),$$

where the conditional probability distribution of the price given the view is:

$$f_{P|V}(p \mid v) \propto |\Lambda^{-1} + Q^T\Omega^{-1}Q|^{0.5} e^{-0.5[(p-\mu_{up})^T(\Lambda^{-1}+Q^T\Omega^{-1}Q)(p-\mu_{up})]} \tag{4.10}$$
Note that, since (4.10) is a normal probability density function, therefore the posterior distribution of the price given the view is:

\[(P \mid v; \Omega) \sim N(\mu_{up}, \Lambda_{up})\]

where

\[\mu_{up} = (\Lambda^{-1} + Q^T \Omega^{-1} Q)^{-1} (\Lambda^{-1} \beta + Q^T \Omega^{-1} v)\]

and

\[\Lambda_{up} = (\Lambda^{-1} + Q^T \Omega^{-1} Q)^{-1}\]

Moreover, we can rewrite

\[
\mu_{up} = (\Lambda^{-1} + Q^T \Omega^{-1} Q)^{-1} (\Lambda^{-1} \beta + Q^T \Omega^{-1} v) \\
= \left(\Lambda - \Lambda Q^T (QAQ^T + \Omega)^{-1} QA\right) (\Lambda^{-1} \beta + Q^T \Omega^{-1} v) \\
= \beta + \Lambda Q^T \left(\Omega^{-1} - (QAQ^T + \Omega)^{-1} QAQ^T \Omega^{-1}\right) v - \Lambda Q^T \left((QAQ^T + \Omega)^{-1}\right) Q \beta
\]

Note that

\[\Omega^{-1} - (QAQ^T + \Omega)^{-1} QAQ^T \Omega^{-1} = (QAQ^T + \Omega)^{-1}\]

therefore, we have:

\[\mu_{up} = \beta + \Lambda Q^T (QAQ^T + \Omega)^{-1} (v - Q \beta)\]

Similarly we have:

\[\Lambda_{up} = \Lambda - \Lambda Q^T (QAQ^T + \Omega)^{-1} QA\]

Applying Theorem 4.4 to our example, where the view expressed as follows:

\[Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 30 \\ 5 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.\]
resulted in:

\[
\begin{align*}
\mu_{up} &= \begin{bmatrix} 24.291 \\ 11.053 \\ 15.062 \end{bmatrix}, \quad 
\Lambda_{up} = \begin{bmatrix} 0.0893 & 0.0315 & -0.0109 \\ 0.0315 & 0.7611 & -0.0038 \\ -0.0109 & -0.0038 & 0.0796 \end{bmatrix}.
\end{align*}
\]

Note that now we can use the updated mean, variance and covariance matrix in the problems that we discussed previously.

Next, we present an example to show how sensitive the optimal location of the facility is with regard to the updating process when incorporating the expert views. In particular, we compare three cases:

a) Ignoring the expert view

b) Considering the expert view without using Bayesian updating in a subjective way (ignoring the covariance matrix associated with the expert’s view)

c) Using Bayesian updating to incorporate the expert view.

Consider the network shown in Figure 4.5.

We focus on a particular example to show the difference between the three cases mentioned above. Assume that the historical demand distribution at each node is normal and its characteristic is given in case a in Table 4.5. The decision maker gets the following information after consulting with an expert:
1. The mean demand of market 1 is 30 and its variance is 10%.
2. The mean demand of market 3 is 5 and its variance is 10%.

In Table 4.5 we also present the demand characteristic for Cases b and c. In case b, it was assumed that decision maker ignores the covariance matrix associated to the expert view and only updates the mean of the demand distribution and in Case c, the update is based on Theorem 4.4 (similar to the example we have discussed earlier).

It is clear from Table 4.5 that demand characteristics are very different comparing cases a, b and c. As discussed in previous examples the optimal facility location highly depends on the demand characteristics. Therefore we expect the optimal facility location to be quite different in cases a, b and c. We, now, show the optimal facility location for the different cases presented in Table 4.5 for the generalized maximal covering location problem.
Table 4.6: *Expected utility value and location for generalized maximal covering location problem with uncertain demand for Cases a, b and c*

We consider the following structure for coverage:

<table>
<thead>
<tr>
<th>Coverage radius</th>
<th>Coverage Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^0 = 0$</td>
<td>$a^1 = 1.00$</td>
</tr>
<tr>
<td>$r^1 = 1$</td>
<td>$a^2 = 0.85$</td>
</tr>
<tr>
<td>$r^2 = 2$</td>
<td>$a^3 = 0.70$</td>
</tr>
<tr>
<td>$r^3 = 3$</td>
<td>$a^4 = 0.55$</td>
</tr>
<tr>
<td>$r^4 = 4$</td>
<td>$a^5 = 0.40$</td>
</tr>
<tr>
<td>$r^5 = 5$</td>
<td>$a^6 = 0.25$</td>
</tr>
</tbody>
</table>

Table 4.6 presents the optimal facility location and expected utility for cases a, b and c. Although it has not been shown in Table 4.6, the optimal facility location for the risk lover decision maker in Case c is different than that of Case a and b for $\lambda \leq -30$.

From Table 4.6, it is clear that the objective function value of the decision maker
would be affected if she decides to ignore the expert view (Case a) or only updates the mean ignoring the covariance matrix associate with the expert’s view (Case b).

We investigate what the objective function of the decision maker would be if she uses the optimal location she obtained for Case a and b using the objective function of Case c. This is presented in Table 4.7. In particular, we use the notation \((U^c_i, k, x, z)\) where \(U^c_i\) is the objective function value of the decision maker using the objective function of Case c for the optimal location of Case \(i (i = a, b)\), \(k\) and \(z\) are the end points (nodes) of the link that the optimal location is located on and \(x\) is the distance of the optimal facility location from node \(k\) on link \((k, z)\).

In Table 4.7, we estimate the benefit of information provided by the expert. Table 4.7 shows how much the objective function value of the decision maker would improve, if she had the updated information compared to the cases a and b. The objective function value improvement is defined as follows

\[
\text{Objective Function (OF) Value Improvement} = 100 \left| \frac{U^c_i - U^a_i}{U^c_i} \right|, \quad i = a, b
\]

where \(U^c_i\) is obtained from Table 4.6.

For example when \(\lambda = 5\) the OF Value improvement for "Case a" location is \(100 \left| \frac{37.7 - 30.4}{30.4} \right| = 24.0\%\).

Note that in this example, we observe how updating the demand information using the expert view can improve the decision maker’s objective function. Both risk averse and risk lover decision makers can improve their objective function considerably (although it is now shown in Table 4.7 the improvement for risk lover decision maker starts from \(\lambda = -30\)). In particular in Table 4.7, when the risk averse decision maker ignores the expert view (Case a), her objective function value deteriorates significantly compared to the case that the risk averse decision maker incorporates the expert view (Case c).

It is worth mentioning that sometimes it is very difficult to use a subjective approach to incorporate the expert view. For example, suppose the expert view is that market A’s demand will be higher by five units than market B’s demand, then the decision maker
have several ways to adjust the demand in both market A and B. However it is not clear which way is actually better.

### 4.7 Numerical Analysis

In this section, we investigate the effect of linearizing MPUD to solve large size problems numerically. Tables 8, 9 and 10 compare the three different approaches for solving MPUD: (1) Using the original quadratic programming formulation, (2) Using Lawler’s method and (3) Using Glover’s method.

The random instances for each p-median location problem with less than 100 nodes are created by setting random number of links and random length for each link such that the connectivity of the network is guaranteed and for each node there is at least one link emanating from it. For problems with 100 and 200 nodes, the instances are taken from

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$U^a_c$</th>
<th>Utility Improvement</th>
<th>$U^b_c$</th>
<th>Utility Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>-100</td>
<td>(91.8, 1, 0, 2)</td>
<td>26.9%</td>
<td>(291.4, 1, 0, 2)</td>
<td>26.9%</td>
</tr>
<tr>
<td>-10</td>
<td>(45.5, 1, 0, 2)</td>
<td>0%</td>
<td>(45.5, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>-5</td>
<td>(42.9, 1, 0, 2)</td>
<td>0%</td>
<td>(42.9, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>-3</td>
<td>(41.9, 1, 0, 2)</td>
<td>0%</td>
<td>(41.9, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>-1</td>
<td>(40.8, 1, 0, 2)</td>
<td>0%</td>
<td>(40.8, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>0</td>
<td>(40.3, 1, 0, 2)</td>
<td>0%</td>
<td>(40.3, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>1</td>
<td>(36.9, 1, 1, 3)</td>
<td>7.8%</td>
<td>(39.8, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>3</td>
<td>(33.5, 1, 2, 3)</td>
<td>15.5%</td>
<td>(38.8, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>5</td>
<td>(30.4, 1, 3, 3)</td>
<td>24.0%</td>
<td>(37.7, 1, 0, 2)</td>
<td>0%</td>
</tr>
<tr>
<td>10</td>
<td>(25.0, 2, 4, 3)</td>
<td>40.6%</td>
<td>(32.0, 1, 2, 3)</td>
<td>9.9%</td>
</tr>
<tr>
<td>100</td>
<td>(−4.9, 2, 3, 3)</td>
<td>430.0%</td>
<td>(−4.9, 2, 3, 3)</td>
<td>430.0%</td>
</tr>
</tbody>
</table>

Table 4.7: Updating information value for the decision maker.
Beasley’s (1990) library. For networks with less than 100 nodes, the random length of each link (for relevant cases) is chosen according to the uniform distribution in $[1, 11]$. The mean and standard deviation for each market are also generated randomly according to a uniform distribution in $[0, 20]$ for the mean and in $[0, 10]$ for the standard deviation.

We consider two sets of numerical analysis. In the first set, Tables 4.8 and 4.9, we assume that the correlation among nodes depends on the distance between two nodes as follows:

<table>
<thead>
<tr>
<th>$\rho_{ij}$</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0 \leq d(i, j) &lt; \frac{1}{6} M$</td>
</tr>
<tr>
<td>0.6</td>
<td>$\frac{1}{6} M \leq d(i, j) &lt; \frac{1}{3} M$</td>
</tr>
<tr>
<td>0.3</td>
<td>$\frac{1}{3} M \leq d(i, j) &lt; \frac{1}{2} M$</td>
</tr>
<tr>
<td></td>
<td>$d(i, j) \geq \frac{1}{2} M$</td>
</tr>
</tbody>
</table>

where $M = Median(d(i, j))$ for all $i, j$. In the second set, Table 4.10, we consider a special case of the problem where only those nodes that are covered by the same facility are correlated and the correlation between markets is assumed to be fixed, $\rho_{ij} = 0.3 \forall i, j$. Also note that in Tables 8 and 4.10 we assume that the firm assign customers to the facilities, however in Table 4.9 we assume customers choose the closest facility.

For solving each instance we used the academic version of IBM ILOG CPLEX Optimization Studio installed in a PC with Intel Core i7 CPU and 8 MB RAM. Notice that if the time exceeds 180 minutes we stop solver and report the gap between the lower bound and upper bound.

Tables 4.8 and 4.9 present the results for the first set of numerical analysis for both direct and non-direct assignment. It shows that Glover’s method tends to be much better than the other two methods. In particular, the performance of Glover’s method is better than Lawler’s method mainly due to the smaller dimensions (the number of dummy variables introduced to linearize the problem). In this case the largest instances solved by the Glover’s Method is 100 nodes with 30, 50 and 70 facilities, whereas the Lawler’s method ran out of memory in Table 4.8 with 25 nodes and 6 facilities and in Table 4.9
<table>
<thead>
<tr>
<th>Number of Nodes (Facilities)</th>
<th>Original Formulation (min:sec)</th>
<th>Lawler’s Method (min:sec)</th>
<th>Glover’s Method (min:sec)</th>
</tr>
</thead>
<tbody>
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<td>00:06</td>
<td>&lt;00:01</td>
</tr>
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<td>00:01</td>
<td>00:06</td>
<td>&lt;00:01</td>
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<tr>
<td>10 (7)</td>
<td>&lt;00:01</td>
<td>00:04</td>
<td>&lt;00:01</td>
</tr>
<tr>
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<td>00:54</td>
<td>&lt;00:01</td>
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<td>&lt;00:01</td>
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<td>00:04</td>
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<td>Exceeded Time Limit (42%)</td>
</tr>
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<td>100 (70)</td>
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<td>Ran Out of Memory</td>
<td>Exceeded Time Limit (30%)</td>
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</table>

Table 4.8: The effect of linearization technique on solving MPUD with direct assignment assumption
Chapter 4. Location Problems with Uncertain Demand

<table>
<thead>
<tr>
<th>Number of Nodes (Facilities)</th>
<th>Original Formulation (min:sec)</th>
<th>Lawler’s Method (min:sec)</th>
<th>Glover’s Method (min:sec)</th>
</tr>
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</tr>
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</tr>
</tbody>
</table>

Table 4.9: The effect of linearization technique on solving the MPUD with enforcing shortest distance constraint.
with 50 nodes and 20 facilities. The original formulation ran out of memory with 15
nodes and 7 facilities in Table 4.8 and with 25 nodes and 13 facilities in Table 4.9.

Table 4.10 presents the results for the second set of numerical analysis. It shows that
Glover’s method tends to be much better than the other two methods. In particular,
the performance of Glover’s method is better than Lawler’s method mainly due to the
smaller dimensions (the number of dummy variables introduced to linearize the problem).
In this case the largest instances solved by the Glover’s Method is 200 nodes with 100,
120 and 150 facilities, whereas the lawler method ran out of memory with 90 nodes and 30
facilities and the original formulation ran out of memory with 25 nodes and 13 facilities.

Also comparing Tables 4.8 and 4.10, we observe that it takes more time to solve the
instances in Table 4.8 compared to Table 4.10. This is due to the fact that the size of
the instances in Table 8 is significantly larger than those in Table 4.10.

4.8 Summary and Outlook

In this chapter we studied the Median, Center and Generalized Maximal Covering prob-
lems from the point of view of a decision maker who has a mean-variance objective. We
showed how the optimal facility location depends on the risk preferences of the decision
maker and provided some insights about the reasons why the decision maker with dif-
ferent attitudes towards risk, locates the facility in a specific way. We also shed some
light on the importance of considering the volatility and correlation structure, and having
accurate estimates for both of them for the decision maker who is not risk neutral. The
analysis also suggests that ignoring the risk preference of the decision maker in facility
location problem could potentially result in suboptimal facility locations.

Furthermore we have argued that since finding the optimal location for the facility
is classified as a strategic decision, the firm may want to consult with experts within or
outside the firm and potentially incorporate their view in the estimation of the uncertain
<table>
<thead>
<tr>
<th>Number of Nodes</th>
<th>Original Formulation</th>
<th>Lawler’s Method</th>
<th>Glover’s Method</th>
</tr>
</thead>
<tbody>
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<td>(Facilities)</td>
<td>(min:sec)</td>
<td>(min:sec)</td>
<td>(min:sec)</td>
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</tr>
</tbody>
</table>

**Table 4.10:** The effect of linearization technique on solving the MPUD with direct assignment assumption
demand. We proposed to use Bayesian updating, a well known approach for updating
the probability distribution. Through an example we showed how much the objective
function of decision maker can improve using Bayesian updating compared to other pos-
sible approaches. For future research, one can extend the analysis for the median, center
and generalized maximal covering location problems to more realistic cases where both
demand and price at each market are correlated random variables.
Bibliography


