Magnetic Attitude Control of Microsatellites In Geocentric Orbits

by

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Attitude control of spacecraft in low Earth orbits can be achieved by exploiting the torques generated by the geomagnetic field. Recent research has demonstrated that attitude stability of a spacecraft is possible using a linear combination of Euler parameters and angular velocity feedback. The research carried out in this thesis implements a hybrid scheme consisting of magnetic control using on-board dipole moments and a three-axis actuation scheme such as reaction wheels and thrusters. A stability analysis is formulated and analyzed using Floquet and Lyapunov stability theorems.
If we knew what it was we were doing, it would not be called research, would it?

Albert Einstein
Dedicated to my parents and my amazing little sister
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Chapter 1

Introduction

Over the years many missions to the outer space have been motivated by scientists and researchers around the world and their greed to acquire greater knowledge of the solar system and the planet we live in today. Various satellites from different parts of the world have been injected into orbits around the Earth as well as around other planets in the solar system. When in orbit, a satellite faces disturbance torques due to a variety of external factors. These disturbances affect the attitude of the satellite thereby interfering with the primary mission. In order to counter these effects, various spacecraft attitude control techniques such as gravity-gradient control and spin control have been suggested and employed in the past. Traditionally these techniques used active or passive control systems to control the satellite. Despite their advantages, these methods involve instruments that are bulky, expensive and require a lot of operating power.

In recent years, extensive literature is available on a new concept utilizing magnetic torquers for satellite attitude control. The main motivation behind this concept was derived from the Orsted Satellite Mission and the research carried out by Wiśniewski [5]. The primary goal of this mission was to provide an accurate global mapping of the Earth’s magnetic field. These measurements have been used to improve the existing models of the Earth’s magnetic field and to determine the changes of the field. By combining data from various other satellites and from many ground-based observatories, the measurements from the Orsted satellite were used to study the shift in magnetic field vectors. The variations of both of the strong field from inside the
Earth and of the weaker, rapidly varying, field resulting from the interaction between
the ion/particle streams from the Sun (the solar wind) and the Earth’s magnetosphere
are included in the studies. Furthermore, the transfer of energy from the solar wind
to the magnetosphere and further down to the lower layers of the atmosphere will be
studied. All of these studies will benefit not only from the magnetic field measurements
but also from the measurements of the flow of energetic particles around the satellite.

Recently, micro- and nano-scaled satellite missions have gained tremendous mo-
mmentum due to low cost launch opportunities and the declining costs of microelectronic
circuitry. Magnetic torquing is an attractive alternative for such small satellites in
Low Earth Orbit. Magnetic control systems are relatively lightweight, require low
power to operate and can be constructed for low budget applications. In this thesis,
control system laws are developed to stabilize the spacecraft on its three axes using
a magnetic attitude control system in conjunction with reaction wheels or thrusters.
This method of attitude stabilization shall be referred to as \textit{hybrid magnetic at-
titude control}. The major advantages of using such a hybrid system would be a
reduction of power consumption overall.

In this thesis, the hybrid attitude control system is formulated and stability anal-
ysis using various theorems is performed. The feedback control system is formulated
using a tilted dipole model of the Earth’s magnetic field. Simulation results from
these theories are used to validate the analysis.

1.1 Literature Review

White, Shigemoto, and Bourquin [1] were amongst the first to come up with a control
system using magnetic torques for spacecraft control in 1961. Their primary work
involved the analysis and feasibility study of using the interaction of the Earth’s
magnetic field and current-carrying coils. These analyses led to the development of
control laws to track the spin axis of any given spacecraft. They also determined
that, given changes in the angular velocity of the spacecraft, it is possible to obtain
magnetic torques about the spacecraft’s axes. This change in torque can be further
used to stabilize and re-orient the spacecraft in any desired direction.
Using the above theories, in 1965, Ergin and Wheeler [2] developed control laws for spin stabilization of a spacecraft using a magnetic torque coil. Wheeler determined that theoretical stability is provided by control laws that direct the spin axis of any axially symmetric spinning satellite in a circular Earth orbit to any direction in space.

In 1989, Musser and Ebert [3] were among the first to attempt to use a fully magnetic attitude control system for three-axis stability. They claim that this became possible due to the increase in computer computational power onboard spacecraft. Musser and Ebert developed linear feedback control laws which use a linear quadratic regulator to obtain the value of the magnetic control torque. The control laws as a function of time were replaced with laws that were a function of orbital position. Musser and Ebert performed simulations showing that their technique was a good candidate for onboard attitude control systems.

Wiśniewski [4, 5] further developed the ideas of Musser and Ebert. He used a combination of linear and nonlinear system theory to develop control laws for three-axis stabilization of the spacecraft. Linear theory was used to obtain both time-varying and constant gain controllers for a satellite with a gravity-gradient boom. His analysis used the fact that the geomagnetic field varies nearly periodically at high inclination orbits. In addition, he developed a nonlinear controller for a satellite without appendages based on sliding mode control theory. He showed that three-axis control can be achieved with magnetic torquers only, and implemented this idea on the Danish Orsted satellite.

In 1991, Wen and Kreutz-Delgado [6] established that using a linear combination of Euler parameterers and angular velocity in a full three-axis actuation system one could attain global asymptotic stability of a spacecraft. The most recent work carried out by Lovera and Astolfi [7, 8] addresses the issue of spacecraft attitude control using geomagnetic torques. They examined proportional-derivative (PD) control using Euler parameters and angular velocity.

Arduini and Baiocco [9] examined control laws for magnetic control of a gravity-gradient stabilized spacecraft. They discussed the challenges that exist due to magnetic torques only applied perpendicular to the magnetic field. Their control algorithm was based on first determining the ideal torque, and then generating the actual torque through a series of suggested approaches. They also discussed the relationship
between stability and the change in energy of the system. They discovered a good agreement between the analytical approximations and numerical solutions.

In the recent years, C.J. Damaren established that the attitude of a spacecraft can be controlled via the torques provided by the geomagnetic field. The present work carried out by C.J. Damaren introduces a hybrid scheme consisting of magnetic control using on-board dipole moments and independent three-axis actuation scheme.

1.2 Approach and Results

The primary goal of this thesis is to develop control laws based on the methods provided by Wiśniewski [5], Lovera & Astolfi [7]. Nonlinear equations of motion are developed and then linearized in a time-varying form. Magnetic control laws are developed using the control laws suggested by Lovera and Astolfi.

The control laws are tested by performing simulations with various spacecraft configurations and initial conditions. In these simulations the effects of changing parameters on the attitude motion of the spacecraft are studied and analyzed in detail. Also, conditions in which the controller fails to stabilize the spacecraft are also analyzed. Stability of the spacecraft is studied primarily with the aid of Floquet and Lyapunov stability theorems [10].

Furthermore, critical parameter values for guaranteed stability of the spacecraft for a given set of initial conditions is also established.

1.3 Outline of Thesis

Chapter 2 introduces the magnetic field and discusses specifics of the Earth’s magnetic field. Mathematical models of the geomagnetic field such as the spherical harmonic model and the tilted dipole are presented for use in later simulations. The chapter ends with a discussion of the magnetic fields of other celestial bodies and the importance of accurate magnetic field models for attitude simulation purposes.
Chapter 3 begins with a discussion of spacecraft dynamics. The orbital equations of motions are developed in detail and a general equation of motion for a spacecraft in a geocentric orbit is derived. The attitude motion equations are developed with a brief understanding of reference frames and rotations. General descriptions of prominent environmental torques acting on the spacecraft in orbit are derived.

In Chapter 4, the two stability theories, namely, Floquet’s Theorem and the Lyapunov stability theorems, are described in detail.

Chapter 5 develops control laws to magnetically control the spacecraft in three axes. Equations of motion are first developed and linearized. Magnetic control laws adopted from various sources in derived and adopted for attitude stability of the spacecraft.

Chapter 6 details the results from simulations performed with the control laws. The results are compared for gravity-gradient and non-gravity-gradient stabilized spacecraft. The stability techniques outlined in Chapter 4 are demonstrated with various varying parameters.

Chapter 7 concludes this thesis. It summarizes the results and suggests areas for further research.
Chapter 2

Earth’s Magnetic Field

In order to understand the workings of magnetic control on board a spacecraft, a basic understanding of magnetic field theory and the Earth’s magnetic field is required. These theories are established in the following sections. Furthermore, mathematical models of the Earth’s magnetic field are developed.

2.1 Origins of the Earth’s magnetic field

The first study of the Earth’s magnetic field was conducted by the German mathematician and physicist Karl Gauss in the early 19th Century. Over the years extensive research has been conducted by various organizations and groups to gain a better understanding of this field. In 1819, Hans Christian Orsted discovered that magnetic fields and electric fields are closely related to each other. Furthermore, he discovered that passing current through a closed-loop coil could generate magnetic field. This phenomenon can be explained by the orbital rotations of electrons around the nucleus of an atom and the spin of the electron about its own axis.

Maxwell’s equations form the foundation of modern electromagnetic theory [12] and expand upon the the laws of Ampere, Faraday, and Gauss [13]. These equations
can be represented in differential form as follows:

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]  \hspace{2cm} (2.1)

\[ \nabla \cdot \mathbf{B} = 0 \]  \hspace{2cm} (2.2)

\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \]  \hspace{2cm} (2.3)

\[ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E} + \mu_0 \mathbf{J} \]  \hspace{2cm} (2.4)

where \( \mathbf{E} \) is the Electric Field and \( \mathbf{B} \) is the Magnetic Field. The magnetic field of the Earth, closely resembles the field produced by a simple bar magnet. The magnetic field is often visualized in terms of magnetic field lines, or lines of force, that leave one end of the magnet, called the North Pole, arc through space, and re-enter the magnet at the other end, the South Pole. Such a field is called a dipole field because it has two poles, located at either end of the magnet, where the strength of the field is maximum. At the midpoint between the poles the strength is half of its value at the poles.

\[ \mathbf{H} = \frac{3(m \times \hat{a}_r)\hat{a}_r - m}{r^3} \]  \hspace{2cm} (2.5)

where \( r \) is the distance between the magnetic field source and the measurement point, \( \hat{a}_r \) is a unit vector from the center of the magnet to the measurement point, and \( m \)
is the magnetic dipole moment of the bar magnet and is dependent on the size of the magnet. The strength or intensity of a magnetized object depends on the density of its volume-distributed moments. This intensity is called its magnetization $M$, which is defined as the moments per unit volume:

$$M = \frac{m}{\text{volume}}$$

(2.6)

In free space, the magnetic induction is proportional to the magnetic field by the constant factor, $\mu_0$:

$$B = \mu_0 H$$

(2.7)

The value of $\mu_0$ is equal to 1 in the International System of Units (SI), so the value of the magnetic field and that of the magnetic induction are equal in free space.

Different units are used to represent magnetic fields. The relationship between these units is listed in Table 2.1. In this thesis, units of Tesla are used to describe magnetic fields.

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B = 10^4$ Gauss (G)</td>
<td>$B = 1$ Weber/meter$^2$ (Wb/m$^2$)</td>
</tr>
<tr>
<td>$B = 10^9$ gamma ($\gamma$)</td>
<td>$B = 1$ Tesla (T)</td>
</tr>
</tbody>
</table>

Table 2.1: Magnetic Field of a Bar Magnet

A comparison of various magnetic field strengths is shown in Table 2.2. The strongest man-made magnetic field sustained in a lab is approximately 400,000 Gauss, or 40 T, but stronger fields have been created momentarily by explosive compression. The strongest naturally occurring field is found on a magnetar, which is a type of neutron star.

Furthermore, in the early 1600s, William Gilbert [14] discovered that the magnetic field of the Earth closely resembles a uniformly magnetized sphere, or a tilted dipole. If we could place a bar magnet inside the Earth as shown in Figure 2.2, inclined approximately 11.5° to the rotational axis and offset about 550 km from the Earth’s centre we could account for 90% of the observed magnetic field. We could
CHAPTER 2. EARTH'S MAGNETIC FIELD

2.2 Origin and Effects of the Earth's Magnetic Field

The geomagnetic field is described in detail in NASA Technical Report SP-8017, as well as by Wertz, Campbell, and Thompson. A description of the geomagnetic field, its characteristics, and variations follows.

2.2.1 Geomagnetic Field

The magnetic field around the Earth resembles that of a uniformly magnetized sphere, or a dipole, which is tilted as shown in Figure 2.2. The fact that it approximates a tilted dipole was discovered in 1600 by William Gilbert, and was published in his treatise *De Magnete*. In 1635, Gellibrand was the first to show that the geomagnetic field is both time and position dependent.

The strength of the magnetic field is approximately 30000 nT at the equator and 60000 nT at the poles on the surface of the Earth, as mentioned in the Geological Survey of Canada. The magnetic dipole axis, designated as $\mathbf{m}$ in Figure 2.2, is located at 79.8° N latitude and 107.0° W longitude, in the year 1999. This location is near the Ellef Rignes Island in Canada, and is approximately 700 miles from the geographic North Pole. The magnetic dipole axis is currently at an inclination angle of 11.5° with the equatorial plane. The axis is drifting westward at about 0.2°/year, and the strength is decreasing by 0.05% per year.

Although this simple description provides a way of conceptualizing the magnetic field it does not represent what really happens inside the Earth. In order to calculate the magnetic field of the Earth, various mathematical models have been developed over the years. The following sections will review the various available theories behind these mathematical models.
2.2 Effects of Earth’s magnetic field on Spacecraft

Over the years, there has been extensive demand for satellites in orbit to carry out vital operations. Vital services such as flight control, weather monitoring and prediction, geological mappings, and telecommunications are completely dependent on satellites in orbits to function accurately and efficiently. As a result examining the magnetic field interactions caused by the Earth’s magnetic field abroad these satellites is of primary importance. The magnetic field in essence protects the Earth from the harmful space environment and plays a very important role in our daily lives. Despite this importance, this field can negatively affect the overall performance of the satellites in orbit. The lifetime of spacecraft on-board computers is based on the amount of radiation damage to their circuitry. In an event of a solar storm, the magnetic disturbances caused can significantly affect the controllability of a spacecraft in orbit. The magnetic field also has the ability to change the orientation of a spacecraft. While this may act to complicate the dynamics of the spacecraft, when used correctly the magnetic field may be used for satellite control.

2.3 Coordinate Systems

The Earth’s magnetic field is a vector quantity, i.e. at a given point in space it has a strength and a direction. The three quantities that completely define the magnetic field vector are given by,

- three orthogonal strength components \((X, Y, \text{ and } Z)\);
- the total field strength and two angles \((F, D, I)\); or
- two strength components and an angle \((H, Z, D)\)

The relationship between these 7 elements is shown in Figure 2.3. These quantities are described in Table 2.3.
Figure 2.8: Magnetic Field Coordinates (Adapted from Ref. 42)

where the geodetic latitude is defined by

\[ \phi = 90 - \psi \]  \hspace{1cm} (2.24)

and \( \psi \) is the co-elevation.

In satellite applications, the geocentric inertial components are often used. The transformation from local tangential to geocentric inertial is:

\[ B_x = (B_r \cos \psi + B_\psi \sin \psi) \cos \nu \]  \hspace{1cm} (2.25)

\[ B_y = (B_r \cos \psi + B_\psi \sin \psi) \sin \nu + B_H \cos \nu \]  \hspace{1cm} (2.26)

\[ B_z = (B_r \sin \psi) \]  \hspace{1cm} (2.27)

The magnetic field is a function of both declination and right ascension, \( \nu \), defined by

\[ \nu = \omega \]  \hspace{1cm} (2.28)

where

\( \omega \) is the longitude and \( \omega_g \) is the sidereal time at Greenwich.

2.3.2 Analytic Models

Analytic models have been created to describe the characteristics of the geomagnetic field. These models provide the local magnetic field at any location around the Earth, assuming

Table 2.3: Magnetic Field Components

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>the total intensity of the magnetic field vector</td>
</tr>
<tr>
<td>( H )</td>
<td>the horizontal component of the magnetic field vector</td>
</tr>
<tr>
<td>( Z )</td>
<td>the vertical component of the magnetic field vector; by convention ( Z ) is positive downward</td>
</tr>
<tr>
<td>( X )</td>
<td>the north component of the magnetic field</td>
</tr>
<tr>
<td>( Y )</td>
<td>the east component of the magnetic field</td>
</tr>
<tr>
<td>( D )</td>
<td>magnetic declination, defined as the angle between true north and the horizontal component of the field measured eastward from true north</td>
</tr>
<tr>
<td>( I )</td>
<td>magnetic inclination, defined as the angle measured from the horizontal plane to the magnetic field vector; downward is positive</td>
</tr>
</tbody>
</table>

\( D \) and \( I \) are measured in degrees. All other elements are measured in nanotesla (nT). The seven elements are related through the following simple expressions.

\[ D = \tan^{-1} \left( \frac{Y}{X} \right) \]  \hspace{1cm} (2.8)

\[ I = \tan^{-1} \left( \frac{Z}{H} \right) \]  \hspace{1cm} (2.9)

\[ H = \sqrt{X^2 + Y^2} \]  \hspace{1cm} (2.10)

\[ X = H \cos(D) \]  \hspace{1cm} (2.11)

\[ Y = H \sin(D) \]  \hspace{1cm} (2.12)

\[ F = X^2 + Y^2 + Z^2 \]  \hspace{1cm} (2.13)
In the literature, the magnetic field is often presented in \( \mathbf{X}, \mathbf{Y}, \mathbf{Z} \) form. The transformation from local tangential coordinates to \( \mathbf{X}, \mathbf{Y}, \mathbf{Z} \) is given by,

\[
\begin{align*}
X &= -B_\theta \cos \epsilon - B_r \sin \epsilon \\
Y &= B_\phi \\
Z &= B_\theta \sin \epsilon - B_r \cos \epsilon
\end{align*}
\] (2.14)

where \( B_r \) is the radial component measured outward positive, \( B_\theta \) is the coelevation measured south positive, and \( B_\phi \) is the azimuthal component measured east positive. The correction term for the oblateness of the Earth, \( \epsilon \), is

\[ \epsilon = \lambda - \delta < 0.2^\circ \] (2.17)

where the geodetic latitude is defined by \( \lambda \), and \( \delta \) is the declination, where,

\[ \delta = 90^\circ - \theta \] (2.18)

and \( \theta \) is the co-elevation. In satellite applications, the geocentric inertial components are often used. The transformation from local tangential to geocentric inertial is given by,

\[
\begin{align*}
B_x &= (B_r \cos \delta + B_\theta \sin \delta) \cos \alpha - B_\phi \sin \alpha \\
B_y &= (B_r \cos \delta + B_\theta \sin \delta) \sin \alpha + B_\phi \cos \alpha \\
B_z &= (B_r \sin \delta - B_\theta \cos \delta)
\end{align*}
\] (2.19-2.21)

The magnetic field is a function of both declination and right ascension, \( \alpha \), defined by \( \phi = \alpha - \theta_g \). Here, \( \phi \) is the longitude and \( \theta_g \) is the sidereal time at Greenwich.

## 2.4 Magnetic Field Models

Analytical models have been created to formulate the geomagnetic field of the Earth. Assuming a steady-state field that has no external sources or localized Earth anomalies, these models provide means of calculating the local magnetic field at any location around the Earth.
Chapter 2. Earth’s Magnetic Field

The magnetic field, $\mathbf{B}$, can be described as the negative gradient of a scalar potential function, $V$ represented by

$$
\mathbf{B} = -\nabla V
$$

(2.22)

where, $V$ is described by a series of spherical harmonics denoted by the function,

$$
V = R_e \sum_{n=1}^{k} \left( \frac{R_e}{r} \right)^{n+1} \sum_{m=0}^{n} \left[ g_{n}^{m} \cos m\phi + h_{n}^{m} \sin m\phi \right] P^{n,m}(\theta)
$$

(2.23)

where $R_e$ is the equatorial radius of the Earth, $g_{n}^{m}$ and $h_{n}^{m}$ are Gaussian coefficients, $r$, $\theta$, and $\phi$ are the geocentric distance, coelevation, and East longitude from Greenwich, and $P^{n,m}$ is the associated Legendre function of degree $n$ and order $m$.

### 2.4.1 International Geomagnetic Reference Field

The International Geomagnetic Reference Field (IGRF) is an established numerical model used to calculate the large scale, internal, part of the Earth’s magnetic field at times between 1900 A.D. and the present, at locations on or above Earth’s surface. It is produced and maintained by a team of geomagnetic field modellers under the auspices of the International Association of Geomagnetism and Aeronomy (IAGA) Working Group V-MOD and is derived from observations collected by satellites, at magnetic observatories, and during magnetic surveys. It is used by scientists (e.g. in studies of space weather or in investigations of local magnetic anomalies) and also by commercial organizations and private individuals who often use the geomagnetic field as a source of orientation information.

The internal part of the geomagnetic field, which is almost entirely core generated, undergoes slow, but noticeable, changes on timescales of years to decades. Consequently the IGRF must be revised, typically every five years, to remain up to date and as accurate as possible. The IGRF models also include a set of secular variation terms that are valid for five years after the model epoch. These values are listed in nT/year, and can be used to extrapolate the coefficients for five years past the IGRF data.
2.5 Magnetic Field Data Collection

The spherical harmonic expansion model uses the Gaussian coefficients. Attempts have been made to calculate these coefficients theoretically, but better results have been obtained by measuring the magnetic field at points around the Earth and using a fitting technique.

The magnetic field measurements are made at magnetic observatories on the Earth, as well as by orbiting satellites. There are approximately 250 permanent observatories located around the world. These are not evenly spaced, however, so some field estimation is required in areas with sparse magnetic observatories. In recent years, a switch has been made from relying on Earth-based magnetic observatories, to utilizing satellites to do a large portion of the modelling of the main magnetic field. The IGRF 2000 model was strongly based on data from the Orsted satellite.

Whereas satellite measurements offer many advantages over ground magnetic measurements, they also provide a few difficulties. Ionospheric dynamos and equatorial electrojet currents are internal to the satellite orbit, along with high-latitude electrojet and field-aligned currents, which contaminate the measured data. Also, satellite lifetimes are often not long enough to measure a long-term secular change. In order to compensate for this, satellites used for magnetic measurements are put in near circular orbits at altitudes above 500 km, which is as low as possible for field sensitivity, but high enough to avoid the majority of the atmospheric drag. The best magnetic field models are obtained when the ground and space data are combined.

2.5.1 Spherical Harmonic Model

In 1838, the German mathematician and magnetician Karl Gauss [15] developed a method of representing the magnetic field in terms of a converging series whose terms were functions of latitude, longitude and radial distance from the center of the Earth. The spherical harmonic analysis creates a mathematical representation of the entire main field using only a small table of numbers. This resulting sphere is shown in Figure 2.4 [16]. In theory, the components of the magnetic field, \( B \), in spherical
coordinates are given by,

\[
B_r = -\frac{\partial V}{\partial r} = \sum_{n=1}^{k} \left( \frac{R_e}{r} \right) ^{n+2} (n+1) \sum_{m=0}^{n} (g_{n,m} \cos m\phi + h_{n,m} \sin m\phi) P^{n,m}(\theta) \quad (2.24)
\]

\[
B_\theta = -\frac{1}{r} \frac{\partial V}{\partial \theta} = -\sum_{n=1}^{k} \left( \frac{R_e}{r} \right) ^{n+2} \sum_{m=0}^{n} (g_{n,m} \cos m\phi + h_{n,m} \sin m\phi) \frac{P^{n,m}(\theta)}{\partial \theta} \quad (2.25)
\]

\[
B_\phi = -\frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} = -\frac{1}{\sin \theta} \sum_{n=1}^{k} \left( \frac{R_e}{r} \right) ^{n+2} \sum_{m=0}^{n} m (-g_{n,m} \sin m\phi + h_{n,m} \cos m\phi) P^{n,m}(\theta) \quad (2.26)
\]

where \( \phi \) refers to longitude, \( \theta \) refers to latitude, \( r \) is the radial distance, \( n \) is the degree of the term, \( m \) is the order of the term and \( V \) is the scalar potential function. Furthermore, \( P^{n,m} \) is known as the associated Legendre polynomial and \( g^m_n \) and \( h^m_n \) are known as Gauss coefficients and are determined through a least-squares analysis of a worldwide distribution of magnetic observations. These terms as defined as follows,

\[
g_{n,m} \equiv S_{n,m} g^m_n \quad (2.28)
\]

\[
h_{n,m} \equiv S_{n,m} h^m_n \quad (2.29)
\]
and,
\[ S_{n,m} = \left[ (2 - \delta_m^n) (n - m)! \right] \left( \frac{2n - 1}{(n + m)!} \right)^{\frac{1}{2}} \left( \frac{2n}{(n - m)!} \right) \] (2.30)

The Kronocker delta, \( \delta_j^i = 1 \), if \( i = j \) and 0 otherwise. In addition \( P_{n}^m \) is defined by,

\[ P_{0,0}^n = 1 \] (2.31)
\[ P_{n,n}^n = \sin \theta P_{n-1,n-1}^{n-1} \] (2.32)
\[ P_{n,m}^n = \cos \theta P_{n-1,m-1}^{n-1} - K_{n,m}^{n-2} P_{n-2,m}^{n-2} \] (2.33)

where,
\[ K_{n,m}^{n,m} = \begin{cases} \frac{(n-1)^2 - m^2}{(2n-1)(2n-3)} & n > 1 \\ 0 & n = 1 \end{cases} \] (2.34)

The magnetic field values computed above are measured in the body frame and can be rotated into other reference frames depending on the required use.

### 2.5.2 Tilted dipole model

For analytical analysis, a tilted dipole model of the geomagnetic field can be obtained by calculating the spherical harmonic model to the first degree (n=1) and all orders (m = 0,1). Under this assumption, the scalar potential function, \( V \), given by equation 2.23 reduces to,

\[ V(r, \theta, \phi) = \frac{R_0^3}{r^3} \left[ g_0^0 P_1^0(\theta) + (g_1^0 \cos \phi + h_1^0 \sin \phi) P_1^1(\theta) \right] \] (2.35)
\[ = \frac{1}{r^2} \left( g_0^0 R_0^2 \cos \theta + g_1^0 R_0 \cos \phi \sin \theta + h_1^0 R_0 \sin \phi \sin \theta \right) \] (2.36)

The Gaussian coefficients for the year 1995 are calculated as,

\[ g_0^0 = -29682 \text{ nT} \] (2.37)
\[ g_1^0 = -1789 \text{ nT} \] (2.38)
\[ h_1^0 = 5310 \text{ nT} \] (2.39)

The total dipole strength is given by,
\[ H_0 = \sqrt{(g_0^0)^2 + (g_1^0)^2 + (h_1^0)^2} = 30115 \text{ nT} \] (2.40)
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The co-elevation and the east longitude of the dipole are given by,

\[ \theta_m = \cos^{-1} \left( \frac{g_0}{H_0} \right) = 196.54^\circ \]  
(2.41)

\[ \phi_m = \tan^{-1} \left( \frac{h_1}{g_1} \right) = 108.43^\circ \]  
(2.42)

The tangential magnetic components thereby reduce to,

\[ B_r = 2 \left( \frac{R_e}{r} \right)^3 \left[ g_0 \cos \theta + (g_1 \cos \phi + h_1 \sin \phi) \sin \theta \right] \]  
(2.43)

\[ B_\theta = \left( \frac{R_e}{r} \right)^3 \left[ g_0 \sin \theta - (g_1 \cos \phi + h_1 \sin \phi) \cos \theta \right] \]  
(2.44)

\[ B_\phi = \left( \frac{R_e}{r} \right)^3 \left[ g_1 \sin \phi - h_1 \cos \phi \right] \]  
(2.45)

By assuming that the magnetic field of the Earth is due to a vector dipole with strength and pole direction given above, the magnetic field can be calculated in vector form:

\[ \mathbf{B}(\mathbf{R}) = \frac{R_e^3 H_0}{r^3} \left[ 3(\hat{m} \cdot \hat{R}) \hat{R} - \hat{m} \right] \]  
(2.46)

where \( \mathbf{R} \) is the position vector of the desired point in the magnetic field, and \( \hat{m} \) is the dipole direction. \( \mathbf{R} \) and \( \hat{m} \) designate unit vectors. This vector can be calculated in any coordinate system, and this value in the geocentric inertial frame is:

\[ \hat{m} = \begin{bmatrix} \sin \theta_m' \cos \alpha_m \\ \sin \theta_m' \sin \alpha_m \\ \cos \theta_m' \end{bmatrix} \]  
(2.47)

where,

\[ \alpha_m = \theta_{g0} + \omega_e t + \phi_m' \]  
(2.48)

where \( \theta_{g0} \) is the Greenwich sidereal time at some reference time, \( \omega_e \) is the average rotation rate of the Earth equal to \( 7.2921152 \times 10^5 \) rad/sec, \( t \) is the time since reference and \( \theta_m' \) and \( \phi_m' \) are defined by equations 2.41 and 2.42. This calculation leads to
magnetic field components in geocentric inertial components for a tilted dipole model,

\[ B_x = \frac{R^3 H_0}{r^3} \left[ 3(\hat{m} \cdot \hat{R})R_x - \sin \theta'_m \cos \alpha_m \right] \tag{2.49} \]

\[ B_y = \frac{R^3 H_0}{r^3} \left[ 3(\hat{m} \cdot \hat{R})R_y - \sin \theta'_m \sin \alpha_m \right] \tag{2.50} \]

\[ B_z = \frac{R^3 H_0}{r^3} \left[ 3(\hat{m} \cdot \hat{R})R_z - \cos \theta'_m \right] \tag{2.51} \]

### 2.5.3 Other Models

The most popular methods of describing the magnetic field are the spherical harmonic model and the tilted dipole model, but a few other models are occasionally used. These models include the quadrupole and centered dipole models.

**Quadrupole**

The quadrupole model is another way to represent the geomagnetic field using spherical harmonics. With this model, the first eight terms of the spherical harmonic model are expanded, and includes both dipolar and quadrupolar effects.

**Centered Dipole**

A centered dipole model is obtained by using the first term in the spherical harmonic expansion model. This expansion models the magnetic field with the magnetic dipole coincident with the spin-axis of the Earth. This model of the geomagnetic field is the least accurate.

### 2.6 Magnetic Field of other Celestial Bodies

The study of magnetic fields of various celestial bodies in our solar system has grown considerably over the past few decades. While an abundance of information is still unknown, much advancement has been made in the understanding of magnetic fields other than Earth. Various instruments have been built, and observational data has
been examined with specially formulated methods of analysis. The magnetic fields of a few celestial bodies have been discovered, and their strengths and fields have been modelled. An understanding of the role that the magnetic field plays in activities such as the birth of sunspot, flare outbursts, and solar cycles has been acquired. In addition, there has been a beginning into the investigation of the theory behind the origin of cosmic magnetism and its effect on stars, galaxies, and other bodies. Unfortunately, even with these advancements, much is still unknown about the magnetic fields of celestial bodies.

2.7 Summary

The geomagnetic field is well known and modelled, and therefore can be used for attitude determination and control of Earth orbiting spacecraft. However, these same techniques cannot be used when the spacecraft is more than a few Earth radii from the surface. For satellites travelling to other planets, knowledge of the interplanetary magnetic field must be understood. In addition, if a satellite is to orbit other bodies, a complete and accurate model of the individual planetary magnetic model is required. The knowledge of planetary magnetic fields is limited at this time, so using magnetic determination and control is currently only feasible in Earth orbits.
Chapter 3

Spacecraft Dynamics

3.1 Orbit Mechanics

3.1.1 Keplerian Orbits

Orbit dynamics is used to determine the position of the spacecraft orbiting the Earth. Orbital laws are also applicable for other planetary bodies orbiting about the sun. In 1609, Johann Kepler discovered that planetary bodies orbit in elliptical orbits around the Sun. Based on these discoveries, Kepler established a set of three primary laws describing the orbital motion of the planets:

- The orbit of each planet is an ellipse with the Sun at a focus.
- The line joining the planet to the Sun sweeps out equal areas in equal times.
- The square of the period of a planet is proportional to the cube of its mean distance from the Sun.

All bodies are under the influence of Newton’s Laws. In particular, Newton’s Second Law states that the rate of change of angular momentum is direct proportional to the force applied in the same direction as that force, or

\[ \sum F = m \ddot{r} \]  (3.1)
where $\sum \mathbf{F}$ is the vector sum of forces acting on mass, $m$, and $\ddot{\mathbf{r}}$ is the acceleration of mass, $m$, relative to an inertial frame. In addition, Newton formulated his Law of Universal Gravitation. This law states that any two bodies attract one another with a force proportional to their masses and inversely proportional to the square of the distance between them, or

$$\mathbf{F} = -\frac{G M m \mathbf{r}}{r^2}$$

(3.2)

where $\mathbf{F}$ is the force on mass $m$ due to mass $M$, $\mathbf{r}$ is the vector from $M$ to $m$, and $G$ is the universal gravitational constant with a value of $6.670 \times 10^{11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$.

When examining the relative motion of two bodies, such as the orbit of a planet around the Sun, or a satellite around a planet, it is convenient to consider the two-body problem. Two assumptions are made to simplify the problem. One is that the bodies are spherically symmetric, and can be modelled with the masses concentrated at the center of the body. The second assumption declares that there are no external or internal forces acting on the system other than the force of the gravitational attractions. The relative motion of two bodies is shown in Figure 3.1.

![Figure 3.1: Relative Motion of two Bodies](image)

The vector $\mathbf{r}$, defining the relative position between the two bodies is denoted by,

$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$$

(3.3)

Applying Newton’s Laws to the system gives,

$$\ddot{\mathbf{r}}_m = -\frac{G M m \mathbf{r}}{r^2}$$

(3.4)
The vector equation of the relative motion for a two body problem can be obtained by subtracting Equation 3.4 from Equation 3.5 and substituting in Equation 3.3:

\[
\ddot{r} = -\frac{G(M + m)}{r^3} r
\]  

(3.6)

In our case, since the mass of the spacecraft is much less than the mass of Earth,

\[
G(M + m) \approx GM = \mu
\]  

(3.7)

where \( \mu \) is defined as the gravitational parameter. For Earth, \( \mu = 3.986032 \times 10^5 \) km\(^3\)/s\(^2\). With the approximation in equation 3.7, the two-body equation of motion becomes,

\[
\ddot{r} + \frac{\mu}{r^3} r = 0
\]  

(3.8)

where \( r \) is the position vector of the second body with respect to the first.

### 3.1.2 Angular Momentum

Angular momentum (per unit mass), \( \vec{h} \), is a constant of motion because gravity is a central force. A force tangential to the center of rotation must be applied to change the angular momentum of a rotational motion system. Since the gravitational force is always directed inward towards the center of mass, the angular momentum of the spacecraft around the center of mass does not change. The specific angular momentum is defined as,

\[
\vec{h} = \vec{r} \times \vec{v}
\]  

(3.9)

The directions of \( \vec{r} \) and \( \vec{v} \) must remain in the plane of the orbit in order for the angular velocity to be constant. The magnitude of the angular velocity is

\[
h = rv \cos \phi_f
\]  

(3.10)

where \( \phi_f \) is the flight path angle, or the angle between the local horizontal and the velocity vector direction.
3.1.3 Orbit Elements

The orbit of a satellite can be described by five numbers, known as orbital elements. A sixth orbital element is added to determine the location of the satellite along the orbit. The orbital elements are shown in Figure 3.2.

The first two elements describe the shape of the orbit.

1. $a$, semi-major axis - a constant defining the size of the orbit.

2. $e$, eccentricity - a constant defining the shape of the orbit.

3. $i$, inclination - angle between the $K$ axis and the angular momentum vector, $\mathbf{h}$.

4. $\Omega$, right ascension of the ascending node - the angle in the fundamental plane, between the $I$ axis and the point where the satellite crosses the fundamental plane in an ascending direction measured counterclockwise when viewed from north of the fundamental plane.
5. \( \omega \), argument of periapsis - the angle, in the plane of the satellites orbit, between the ascending node and the periapsis point, measured in the direction of the satellites motion.

6. \( T \), time of periapsis passage - time when the satellite was at periapsis.

While the above six orbital elements are classically used to describe the position of a satellite, they are not exclusive and can be replace by alternative orbital parameters shown in Figure 3.3. Instead of the argument of periapsis, the longitude of periapsis, \( \Pi \), is occasionally used. This is the angle from the \( I \) direction to periapsis measured eastward.

The time of periapsis passage, \( T \), can be replaced with the true anomaly at epoch, \( \nu_0 \), which is the angle in the plane of the satellites orbit between periapsis and the position of the satellite at a particular epoch, \( t_0 \). Additionally, the argument of latitude at epoch, \( u_0 \) can be used. This is the angle in the plane of the orbit between the ascending node and the radius vector of the satellite at epoch. These values are related by,

\[
u_0 = \omega + \nu_0\]

(3.11)

The true longitude at epoch, \( l_0 \), may also be used to describe the position of the satellite. This angle is measured eastward from the \( I \) axis to the ascending node, and then in the orbital plane to the radius direction at epoch. This relation is defined as

\[
l_0 = \Omega + \omega + \nu_0 = \Pi + \nu_0 = \Omega + u_0\]

(3.12)
Certain orbits cause some of the orbital elements to be undefined. When the orbit is circular, there is no periapsis, and $\omega$, $\Omega$, and $\nu_0$ are undefined. When the orbit is equatorial, there is no ascending node and therefore $\omega$ and $u_0$ are undefined. In these cases, using $l_0$ instead of $\omega$ is useful.

### 3.2 Coordinate Systems

When describing orbital motion, it is necessary to reveal which coordinate system is used. The coordinate systems must be inertial such that the frame is fixed to an outside observer. A fixed coordinate system may also be used to describe satellite motion, but a rotation between the inertial and Earth-fixed coordinate systems must be incorporated into the definition.

**Heliocentric-Ecliptic**

Bodies that orbit around the Sun, such as the Earth and other planets as well as interplanetary space vehicles, are typically described in the Heliocentric-ecliptic frame of reference shown in Figure 3.4. This reference frame is inertial, with the Z direction perpendicular to the plane of the ecliptic, which is the plane of the Earth’s revolution around the Sun. The direction of the X axis is in the vernal equinox direction, and the Y direction is orthogonal.

**Earth-Centered Inertial (ECI)**

The geocentric Equatorial coordinate system has its origin at the Earth’s centre as shown in Figure 3.5. The fundamental plane of reference in this system is the equatorial plane. The positive X-axis points in the vernal equinox direction. The Z-axis points in the direction of the North Pole and the Y-axis is orthogonal. When working in the ECI frame, it is important to make a note that, the frame is not fixed to the Earth, rather the Earth rotates with respect to the ECI coordinate frame.
Earth-Centered Inertial (ECI) coordinate system is centered in the middle of the Earth, as shown in Figure 3.5. The \( Z \) axis points through the geographic North Pole, or the axis of rotation. The \( X \) axis is in the direction of the vernal equinox, and the \( Y \) direction is orthogonal. The Earth rotates with respect to the ECI coordinate frame.

Earth-centered Earth-fixed (ECEF) reference frame also has its origin at the center of the Earth, but it rotates relative to inertial space, shown in Figure 3.6. The \( K \) axis is through the North Pole, and the \( I \) axis points to the Greenwich Meridian. The angle between the vernal equinox direction and the Greenwich Meridian must be defined. This is known as the Greenwich sidereal time, \( \gamma \). Greenwich sidereal time is documented at various epochs and can be extracted from data tables as \( \gamma_0 \). At any time after epoch, \( \gamma \) can be determined from

\[
\gamma = \gamma_0 + \omega (t - t_0) \tag{3.19}
\]

where \( \omega \) is the angular velocity of the Earth. On January 1, 2000 at midnight, the value of \( \gamma_0 \) was equal to 6h39m52.2707s, or 99.96779 according to the Multiyear Interactive Computer Almanac from the U.S. Naval Observatory.

This value changes slightly from year to year, and on January 1, 2001 was equal to 6h42m51.5354s, or 100.71473.

Figure 3.4: Heliocentric-Ecliptic Coordinate System

Figure 3.5: Earth-Centered Inertial Reference Frame

Figure 3.6: Earth-Centered Earth-Fixed Reference Frame
Earth-Centered Earth-Fixed

The Earth-Centered Earth-Fixed (ECEF) reference frame also has its origin at the center of the Earth, but it rotates relative to inertial space as shown in Figure 3.6. The K axis is through the North Pole, and the I axis points to the Greenwich Meridian. The angle between the vernal equinox direction and the Greenwich Meridian must be defined. This is known as the Greenwich sidereal time, $\theta_g$. Greenwich sidereal time is documented at various epochs and can be extracted from data tables as $\theta_{g0}$. At any time after epoch, $\theta_g$ can be determined from $\theta_{g0}$ by

$$\theta_g = \theta_{g0} + \omega_{\oplus}(t - t_0)$$  \hspace{1cm} (3.13)

where $\omega_{\oplus}$ is the angular velocity of the Earth. On January 1, 2000 at midnight, the value of $\theta_{g0}$ was equal to 6h 39m 52.2707s, or 99.96779° according to the Multiyear Interactive Computer Almanac from the U.S. Naval Observatory [17]. This value changes slightly from year to year, and on January 1, 2001 was equal to 6h 42m 51.5354s, or 100.71473°.
3.2.1 Reference frames

Three main reference frames are used to describe the orientation, or attitude, of a spacecraft in orbit. These are the inertial, orbital, and body frames.

Inertial Frame

An inertial frame is used for attitude applications. The $X$ direction points from the focus of the orbit to the vernal equinox, $\Upsilon$, the $Z$ direction is in the orbital angular velocity direction, and $Y$ is perpendicular to $X$ and $Z$.

Orbital Frame

The orbital frame is located at the mass center of the spacecraft, and the motion of the frame depends on the orbit. This frame is non-inertial because of orbital acceleration and the rotation of the frame. The $\hat{o}_3$ axis is in the direction from the spacecraft to the Earth, $\hat{o}_2$ is the direction opposite to the orbit normal, and $\hat{o}_1$ is perpendicular to $\hat{o}_2$ and $\hat{o}_3$. In circular orbits, $\hat{o}_1$ is the direction of the spacecraft velocity. The three directions $\hat{o}_1$, $\hat{o}_2$, and $\hat{o}_3$ are also known as the roll, pitch, and yaw axes, respectively. Figure 3.7 shows a comparison of the inertial and orbital frames in an equatorial orbit.

Body Frame

Like the orbital frame, the body frame has its origin at the spacecraft’s mass center. This frame is fixed in the body, and therefore is non-inertial. The relative orientation between the inertial and the body frames is the basis of attitude dynamics and control.

Principal Axis

Principal axes are a specific body-fixed reference frame. This axis system has its origin at the mass center, and is oriented such that the moment of inertia matrix is diagonal. The diagonal entries are known as the principal moments of inertia.
3.2.2 Rotations

Rotations and transformations have to be performed to obtain the required vector in an alternate reference frame. Two notations commonly used to describe this rotation are Euler angles and quaternions.

Suppose we have a vector \( \mathbf{v} \), and we know its components in \( \mathcal{F}_b \), denoted \( \mathbf{v}_b \), and we need to determine its components in \( \mathcal{F}_i \), denoted \( \mathbf{v}_i \). Since,

\[
\mathbf{v} = \mathbf{v}_i^T \mathcal{F}_i = \mathbf{v}_b^T \mathcal{F}_b \tag{3.14}
\]

we need a way to express \( \mathcal{F}_i \) in terms of \( \mathcal{F}_b \), say,

\[
\mathcal{F}_b = \mathbf{C}_{bi} \mathcal{F}_i \tag{3.15}
\]

where \( \mathbf{C}_{bi} \) is a \( 3 \times 3 \) rotation matrix. Then we can write,

\[
\mathbf{v} = \mathbf{v}_i^T \mathcal{F}_i = \mathbf{v}_i^T \mathbf{C}_{bi}^T \mathcal{F}_b = \mathbf{v}_b^T \mathcal{F}_b \tag{3.16}
\]

Comparing Equations 3.14 and 3.16, we see that,

\[
\mathbf{v}_b = \mathbf{C}_{bi} \mathbf{v}_i \tag{3.17}
\]
Thus, to compute $v_i$, we just need to determine $C_{bi}$. In general, the components of the rotation matrix are the direction cosines of the two sets of reference axes given by,

$$C_{bi} = \begin{bmatrix} \cos \theta_{xx} & \cos \theta_{xy} & \cos \theta_{xz} \\ \cos \theta_{yx} & \cos \theta_{yy} & \cos \theta_{yz} \\ \cos \theta_{zx} & \cos \theta_{zy} & \cos \theta_{zz} \end{bmatrix}$$

(3.18)

where $\cos \theta_{xy}$ is the cosine of the angle between the $x$-axis of the first frame and the $y$-axis of the second frame. Thus $C_{bi}$ is a matrix of direction cosines, and is frequently referred to as the DCM or Direction Cosine Matrix.

### Euler Angles

Computing the direction cosines, DCM, is one way to construct a rotation matrix. One of the easiest to visualize is the Euler angle approach [18, 19]. In 1783, Euler reasoned that any rotation from one frame to another can be visualized as a sequence of three simple rotations about base vectors. Consider the case of a circular orbit, with right ascension, $\Omega$, inclination, $i$, and argument of latitude, $u$. The rotation matrix relating the inertial frame to the orbital frame can be determined by using a 3-1-3 Euler sequence as follows,

1. a 3-rotation about the inertial $\hat{i}_3$ axis through the right ascension, $\Omega$
2. a 1-rotation about the $\hat{i}_1$ axis through the inclination, $i$
3. another 3-rotation about the $\hat{i}_3$ axis through the argument of latitude, $u$

Following the above sequence, the rotation matrix that takes vectors from the inertial frame to the orbital frame is given by,

$$C_{oi} = C_3(u) \cdot C_1(i) \cdot C_3(\Omega)$$

(3.19)
which when expanded, gives,

\[
C_{oi} = \begin{bmatrix}
-\sin u \cos \Omega - \cos u \cos i \sin \Omega & -\sin u \sin \Omega + \cos u \cos i \cos \Omega & \cos u \sin i \\
-\sin i \sin \Omega & \sin i \cos \Omega & -\cos i \\
-\cos u \cos \Omega + \sin u \cos i \sin \Omega & -\cos u \sin \Omega - \sin u \cos i \cos \Omega & -\sin u \sin i 
\end{bmatrix}
\] (3.20)

Given a Direction Cosine Matrix, we can extract the angle associated with \( C \) as developed above as follows,

\[
i = \cos^{-1} \left( -C_{23} \right)
\]

\[
u = \tan^{-1} \left( \frac{-C_{33}}{C_{13}} \right)
\]

\[
\Omega = \tan^{-1} \left( \frac{-C_{21}}{C_{22}} \right)
\]

(3.21) \hspace{1cm} (3.22) \hspace{1cm} (3.23)

**Quaternions**

The Euler angle sequence approach to describing the relative orientation of two frames is reasonably easy to develop and to visualize, but it is not the most useful approach for spacecraft dynamics. Euler also states that the most general motion of a rigid body with a fixed point is a rotation about a fixed axis. This is formally known as Euler’s Theorem. Thus, instead of using three simple rotations (and three angles) to keep track of rotational motion, we only need to use a single rotation (and a single angle) about the fixed axis mentioned in the theorem. This axis, denoted by \( \mathbf{a} \), is called the Euler axis, or the eigenaxis, and the angle, denoted by \( \Phi \), is called the Euler angle.

Based on Euler’s theorem, we now define four variables in terms of \( \mathbf{a} \) and \( \Phi \) known as *quaternions*. The quaternion set \( \mathbf{\tilde{q}} \), is a \( 4 \times 1 \) matrix consisting of a vector portion denoted by \( \mathbf{q} \) and a scalar portion denoted by \( q_4 \)

\[
\mathbf{\tilde{q}} = \begin{bmatrix}
\mathbf{q} \\
q_4
\end{bmatrix} = \begin{bmatrix}
\mathbf{a} \sin \frac{\Phi}{2} \\
\cos \frac{\Phi}{2}
\end{bmatrix}
\] (3.24)
Also, it is relatively easy to show that the rotation matrix relating the inertial frame to the body frame can be computed from the quaternion vectors using

\[ C_{bi} = (q_4^2 - q^T q) 1 + 2qq^T - 2q_4q^\times \]  

(3.25)

where \( q^\times \) is the skew-symmetric matrix defined by,

\[ q^\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \]  

(3.26)

In addition, if the rotation matrix is known, the quaternion vectors can be directly computed from the rotation matrix as follows,

\[ q_4 = \pm \frac{1}{2} \sqrt{1 + tr(C_{bi})} \]  

(3.27)

\[ q = \frac{1}{4q_4} \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix}, q_4 \neq 0 \]  

(3.28)

\subsection{Equations of motion}

Attitude dynamics of any spacecraft can be divided into two sets of equations: the kinematics equations of motions and the dynamic equations of motion. Kinematics seeks to describe the motion of a spacecraft in orbit. These equations can be formulated irrespective of any forces acting on the spacecraft. The kinematic equations are a set of first order differential equations describing the time evolution of the spacecraft’s angular velocity. On the other hand, dynamics is the study of a spacecraft’s motion with change in time.

\subsection{Kinematic equations of motion}

The kinematic equations can be modelled using the Euler parameters \cite{18}. The kinematic equations of motion are obtained by beginning with the definition of a quaternion. Let the quaternion, \( q \) represent the orientation of the rigid body with respect
to the reference system at time \( t \):

\[
\tilde{q}(t) = \begin{bmatrix} a \sin \frac{\Phi}{2} \\ \cos \frac{\Phi}{2} \end{bmatrix}
\]  

(3.29)

At time \( t + \delta t \), the quaternion is equal to,

\[
\tilde{q}(t + \Delta t) = \left( \cos \frac{\Delta \Phi}{2} \mathbf{1} + \sin \frac{\Delta \Phi}{2} \begin{bmatrix} 0 & a_c & -a_2 & a_1 \\ -a_3 & 0 & a_1 & a_2 \\ a_2 & -a_1 & 0 & a_3 \\ -a_1 & a_2 & -a_3 & 0 \end{bmatrix} \right) \tilde{q}(t)
\]  

(3.30)

Since \( \Delta t \) is small and \( \Delta \Phi = \omega \Delta t \), where \( \Omega \) is the magnitude of the instantaneous angular velocity of the spacecraft, the following assumptions can be made,

\[
\cos \frac{\Delta \Phi}{2} \approx 1 \quad \sin \frac{\Delta \Phi}{2} \approx \frac{1}{2} \omega \Delta t
\]  

(3.31)

From equations 3.30 and 3.31, we get,

\[
\tilde{q}(t + \Delta t) = \left[ 1 + \frac{1}{2} \Omega \Delta t \right] \tilde{q}(t)
\]  

(3.32)

where \( \Omega \) is a skew-symmetric matrix denoted by,

\[
\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & -\omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & \omega_2 & -\omega_3 & 0 \end{bmatrix}
\]  

(3.33)

The time derivative of the quaternion is given by,

\[
\dot{\tilde{q}} = \lim_{\Delta t \to 0} \frac{\tilde{q}(t + \Delta t) - \tilde{q}(t)}{\Delta t} = \frac{1}{2} \Omega \tilde{q}
\]  

(3.34)
by using equations 3.33 and 3.34, the kinematical equations of motion can be rearranged to satisfy,

\[
\dot{q} = \frac{1}{2} \begin{bmatrix}
q_4 & -q_3 & q_2 \\
q_3 & q_4 & -q_1 \\
-q_2 & q_1 & q_4 \\
-q_1 & -q_2 & -q_3
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
q^\times + q_4 \mathbf{1} \\
-q^T
\end{bmatrix} \omega
\] (3.35)

### 3.3.2 Dynamic Equations of Motion

We develop the rotational equations of motion for a rigid body using Euler’s Law. The rotational equations for a spacecraft can be express by,

\[
\dot{h} = N
\] (3.36)

where \( h \) is the angular momentum vector and \( N \) is the total moment generated by the disturbance torques acting on the spacecraft about its mass center. They are given by

\[
N = n_{gg} + n_m + n_u
\] (3.37)

where \( n_{gg} \), \( n_m \) and \( n_u \) are the torques acting on the spacecraft due to gravity-gradient, magnetic moment, and non-magnetic control respectively. Furthermore, given that the body frame is fixed to the mass center of the spacecraft, the angular momentum can be represented by

\[
h = I\omega_b
\] (3.38)

where \( \omega_b \) is the angular velocity and \( I \) is the moment of inertia of the body. Substituting Equation 3.37 and 3.38 in Equation 3.36, we get,

\[
I\dot{\omega}_b + \omega_b \times I\omega_b = n_{gg} + n_m + n_u
\] (3.39)

If principal axes are used as a reference, then \( I \) is diagonal, i.e. \( I = \text{diag}(I_1, I_2, I_3) \). These equations are known as Euler’s Equations of motion.
3.4 Disturbance Torques

Spacecraft in orbit encounter small disturbance torques from various environmental sources. These torques are either secular, which accumulate over time, or cyclic, which vary sinusoidally over an orbit. Both types are discussed by Hughes [18]. Different environmental torques are more prevalent at different altitudes. The relationship between altitude and disturbance torque strength is shown in Figure 3.8.

![Figure 3.8: Disturbance Torques experienced by Spacecraft in Orbit](image)

Accurate attitude determination requires accurate mathematical models of various disturbance torques acting on the spacecraft. The torques acting on the spacecraft can be modelled as a function of time and position in the orbit around Earth. For spacecraft orbits in the vicinity of Earth the major disturbance torques are:

- Gravity-Gradient Torque
• Magnetic Torque
• Aerodynamic Torque
• Solar-Radiation Pressure Torque

3.4.1 Gravity-Gradient Torque

Any spacecraft in orbit can be subject to a gravitational torque because of the variation in the Earth’s gravitational force over the object. This gravity-gradient torque results from the inverse square gravitational force field acting on the spacecraft. The gravitational force, $\mathbf{dF}$, acting on a spacecraft mass element, $dm$, located at position, $\mathbf{R}_0$, relative to the center of the Earth is given by,

$$d\mathbf{F} = -\frac{\mu \, dm}{R_e^3} \mathbf{R}_0$$

(3.40)

where $\mu = GM = 3.986 \times 10^{14}$ m$^3$/s$^2$ is the Earth’s Gravitational constant. The torque about the spacecraft’s geometric center is given by,

$$d\mathbf{N} = \mathbf{R}_0 \times d\mathbf{F} = (\mathbf{R}_0 + \rho) \times d\mathbf{F}$$

(3.41)

where, the vector $\rho$ is measured from the geometric center to the center of mass, $\mathbf{R}_0$ is measured from the center of mass to the mass element, $dm$ as shown in Figure 3.9.

It can be shown that, the gravity-gradient torque acting on the entire spacecraft is
given by,

\[ G = \int_V \left( \rho \times d\mathbf{F} \right) \]  

(3.42)

\[ G = \frac{3\mu}{R_0^5} \mathbf{R}_0 \times \mathbf{I} \mathbf{R}_0 \]  

(3.43)

### 3.4.2 Magnetic Torque

A magnetic torque is caused by the interaction between the spacecraft magnetic dipole moment and the geomagnetic field. The primary sources of this torque are the spacecraft magnetic moment, eddy currents, and hysteresis, with the magnetic moment being dominant. The magnetic torque is cyclic throughout an orbit and is influenced the most by orbit altitude, the residual spacecraft magnetic field and geomagnetic field. This value, \( n_m \), is represented by

\[ n_m = \mathbf{m} \times \mathbf{B} \]  

(3.44)

where \( \mathbf{m} \) is the sum of the individual magnetic moments caused by permanent and induced magnetism and spacecraft generated current loops, and \( \mathbf{B} \) is the magnetic field.
Chapter 4

Formulation And Control Laws

In the previous chapters, the dynamic equations of a spacecraft were discussed in detail and formulation for a spacecraft in orbit subject to external forces was derived. In this chapter, previously derived equations are arranged as linear and nonlinear time-varying equations of motion. These equations set the basis for further simulation carried out in this thesis. Furthermore, the periodic nature of these equations are discussed and implemented.

Magnetic control laws based on stability analysis techniques such as Floquet theory and Lyapunov stability are derived and studied in detail. These techniques are applied to linearized equations of motions derived in the previous chapters and studied concurrently for a given scenario.

4.1 Desired Equilibrium Attitude

For three-axis stability, the body frame should be aligned with the inertial frame, or

\[ \mathbf{\omega} = 0 \]  \hspace{1cm} (4.1)
\[ \mathbf{q} = 0 \]  \hspace{1cm} (4.2)

where \( \mathbf{\omega} \) is the spacecraft angular velocity relative to an inertial frame and \( \mathbf{q} \) is the corresponding vector part of the quaternion.
4.2 Nonlinear Equations of Motion

For computational purposes, the nonlinear equations of motion have to be linearized about the desired equilibrium point. This section describes the method involved to linearized the equations of motion derived earlier.

A nonlinear model is used to calculate the motion of the spacecraft with applied magnetic control. As derived in Section 3.3.2, the nonlinear dynamic equations of motion are given by,

\[
\dot{\tilde{q}} = \frac{1}{2} \begin{bmatrix} q^x + q_4 I \\ -q^T \end{bmatrix} \omega
\]

(4.3)

\[
I \ddot{\omega} + \omega \times I \omega = n_{gg} + n_m + n_u
\]

(4.4)

where, \( n_{gg} \), \( n_m \) and \( n_u \) are the torques generated due to gravity-gradient, magnetic moment, and control system respectively. The total magnetic torque, \( n_m \), acting on the spacecraft is given by,

\[
n_m = m^x B
\]

(4.5)

Furthermore, assuming that forces due to the gravity-gradient are negligible and the only forces acting on the spacecraft are due to the onboard magnetic controller and an additional source of control torque, we get,

\[
I \ddot{\omega} + \omega \times I \omega = m^x B_b + u
\]

(4.6)

where \( n_u = u \) is the control torque due to the non-magnetics onboard the spacecraft such as the reaction wheels or thrusters.

4.3 Linearized Equations of Motion

Recall that the nonlinear equations of motion for a spacecraft are given by,

\[
I \ddot{\omega} + \omega \times I \omega = m^x B_b + u
\]

The equations of motion will be linearized in terms of Euler angles, \( \theta \). Let \( x = \begin{bmatrix} \theta^T \\ \dot{\theta}^T \end{bmatrix}^T \) be the state vector containing the Euler angles and their rates of change.
For the three axis system, it is assumed that the torques are generated according to the control law,

\[
  u(t) = -\gamma I^{-1} \left[ \epsilon K_d \omega(t) + 2\epsilon^2 K_p q(t) \right]
\]  

(4.7)

where \( \gamma \) is a dimensionless scaling factor and is a rough measure of the relative size of the three axis magnetic torques and the torques generated by magnetic coils on board and \( K_p \) and \( K_d \) are gain parameters. Furthermore, following references [7, 10], the following control law for magnetic dipole moment is adopted,

\[
  m(t) = \frac{B_i^x}{\|B_i\|^2} v(t)
\]  

(4.8)

\[
  v(t) = -I^{-1} \left[ \epsilon K_d \omega(t) + 2\epsilon^2 K_p q(t) \right]
\]  

(4.9)

Assuming that the body frame differs from the inertial frame by small angles, i.e \( \theta = 2q \), we can make the following assumptions,

\[
  \omega = \dot{\theta}
\]  

(4.10)

\[
  C_{bi} \approx 1 - \theta^x = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}
\]  

(4.11)

Making these substitutions in equation 4.6 above, and neglecting products of small terms leads to a linearized equation of motion given by,

\[
  \ddot{\theta} + (\gamma I^{-2} - I^{-1} \tilde{B}_i^x \tilde{B}_i^x I^{-1}) (\epsilon K_d \theta + \epsilon^2 K_p \theta) = 0
\]  

(4.12)

where \( \tilde{B}_i = \|B_i\|^{-1} B_i \) and \( B_i \) is defined by

\[
  B_i = \begin{bmatrix} (B_r \cos \delta + B_\theta \sin \delta) \cos \alpha - B_\phi \sin \alpha \\ (B_r \cos \delta + B_\theta \sin \delta) \sin \alpha + B_\phi \cos \alpha \\ B_r \sin \delta - B_\theta \cos \delta \end{bmatrix}
\]  

(4.13)
Comparing equation 4.14 and 4.12, we can show that the equations of motion are
\[ \dot{x} = Ax \]
where,
\[ A = \begin{bmatrix} 0 & 1 \\ -\epsilon^2 K_p \left( \gamma I^{-2} - I^{-1} \tilde{B} \times \tilde{B} \times I^{-1} \right) & -\epsilon K_d \left( \gamma I^{-2} - I^{-1} \tilde{B} \times \tilde{B} \times I^{-1} \right) \end{bmatrix} \quad (4.14) \]

### 4.4 Floquet Stability Analysis

Floquet theory is the mathematical theory of linear, periodic systems of ordinary differential equations (ODEs). We will utilize Floquet theory to analyze the stability of our linearized model in Equation 4.14. The stability of a periodic system is to be considered over time, therefore the state transition matrix calculated for one period, plays a significant role in the stability analysis. The placement of the eigenvalues of the transition matrix in the open unit disk, is equivalent to the stability of the system.

Consider the linear time-varying system
\[ \dot{x}(t) = A(t)x(t) \quad (4.15) \]
where \( A(t) \) is a periodic function. Let \( \Phi(t) \) be a fundamental solution of the above equation, i.e.,
\[ \dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0) = 1 \quad (4.16) \]
where \( \Phi \) is nonsingular. Then from the periodicity of \( A(\cdot) \), \( \Phi(t + T) \) again satisfies a fundamental solution. Since there are \( n \) independent solutions of Equation 4.15, thus there exists a constant nonsingular matrix \( C \) such that
\[ \Phi(t + T) = \Phi(t) C \quad (4.17) \]

Since any solution of \( x \) can be expressed as \( \Phi(t)c \) for some constant vector \( c \), we get,
\[ x(t + T) = \Phi(t)Cc \quad (4.18) \]

Now if we choose \( c \) to be an eigenvector of \( C \) with eigenvalue \( \rho \). Then the corresponding \( x(t) := \Phi(t)c \) satisfies
\[ x(t + T) = \rho x(t) \quad (4.19) \]
Let $C$ has eigenvalues $\rho_1, \ldots, \rho_n$ with eigenvectors $c_1, \ldots, c_n$. Then the solution

$$x_i(t) := \Phi(t)c_i$$

satisfies

$$x_i(t + T) = \rho_i x_i(t)$$

The above definition of $\rho_i$ are independent of the choice of a particular choice of the fundamental solution. Furthermore, It is convenient to express $\rho_i = e^{T \mu_i}$. The constants $\mu_i, i = 1, \ldots, n$ are called the Floquet exponents of Equation 4.15. If we define $\Phi_i(t) = x(t) e^{-\mu_i t}$, then $\Phi_i$ is a periodic function.

$$\Phi_i(t) = x(t + T) e^{-\mu_i (t+T)}$$

$$= e^{-\mu_i T} x_i(t + T) e^{-\mu_i (t+T)}$$

$$= x_i(t) e^{-\mu_i T}$$

$$= \Phi_i(t)$$

Thus, we may express,

$$x_i(t) = e^{\mu_i t} \Phi_i(t)$$

If the real part of $\mu_i \leq 0$ or $|\rho_i| \leq 1$, then the corresponding $x_i(t)$ is bounded for $t \geq 0$.

## 4.5 Lyapunov Stability Analysis

Lyapunov stability is named after Aleksandr Lyapunov, a Russian mathematician who published his book "The General Problem of Stability of Motion" in 1892. Lyapunov was the first to consider the modifications necessary in nonlinear systems to the linear theory of stability based on linearizing near a point of equilibrium. Lyapunov, in his original work in 1892, proposed two methods for demonstrating stability [20]. The second method, which is almost universally used nowadays, makes use of a Lyapunov function $V(x)$ which has an analogy to the potential function of classical dynamics. It is introduced as follows for a system having a point of equilibrium at $x = 0$.

Lyapunov stability makes use of a function $V(x) : \mathbb{R}^n \to R$. If
• $V(x) > 0$ ($x \neq 0$) is positive definite;
• $\dot{V}(t) = \frac{\partial}{\partial t} V(x) < 0$ ($x \neq 0$) is negative definite.

then $V(x)$ is called a Lyapunov function and the system is asymptotically stable.

The above definition gives sufficient conditions for the stability of the origin of a system. It does not, however, give a prescription for determining the Lyapunov function $V(t)$. Since the theorem only gives sufficient conditions, the search for a Lyapunov function establishing stability of an equilibrium point could be tedious. Using this theorem, it can be shown that for a linear system, the Lyapunov function can be chosen to be quadratic, that is,

$$V(x) = x^T P x, \quad P = P^T > 0$$  \hspace{1cm} (4.27)

According to the definition, taking the derivative of equation 4.27, we get,

$$\dot{V}(x) = x^T (A^T P + PA) x$$  \hspace{1cm} (4.28)

The system is said to be asymptotically stable if the following condition is satisfied,

$$A^T P + PA < 0$$  \hspace{1cm} (4.29)

Based on the above given theory, we will now formulate a mathematical equivalent of the Lyapunov Stability Theory for analysis.

Recall the system defined by equation 4.12,

$$\ddot{\theta} + (\gamma I^{-2} - I^{-1}\tilde{B}_i^\times \tilde{B}_i^\times I^{-1}) (\epsilon K_d \dot{\theta} + \epsilon^2 K_p \theta) = 0$$

This system resembles a general $2^{nd}$ order differential system in the form of,

$$\ddot{\theta}(t) + D(t) \dot{\theta}(t) + K(t) \theta(t) = 0$$  \hspace{1cm} (4.30)

where $D(t)$ and $K(t)$ are symmetric matrices ($D = D^T$ and $K = K^T$) given by,

$$D(t) = \epsilon K_d [\gamma I^{-2} - I^{-1}\tilde{B}_i^\times \tilde{B}_i^\times I^{-1}]$$  \hspace{1cm} (4.31)
\[ K(t) = \epsilon^2 K_p \left[ \gamma I^{-2} - I^{-1} \tilde{B}_i \tilde{B}_i^T I^{-1} \right] \]  \hspace{1cm} (4.32)

Comparing equations 4.31 and 4.32, we can also show that,

\[ D(t) = \alpha K(t) \]  \hspace{1cm} (4.33)

\[ \alpha = \frac{K_d}{\epsilon K_p} \]  \hspace{1cm} (4.34)

We now try to apply Lyapunov’s direct method to determine stability. Following reference [11], we adopt the following Lyapunov function to use in this context,

\[ V(t) = \frac{1}{2} q^T q + \frac{1}{2} \dot{q}^T K^{-1} \dot{q} \]  \hspace{1cm} (4.35)

Taking the derivative of \( V(t) \) along trajectories of the system gives,

\[ \dot{V}(t) = -\dot{q}^T (K^{-1} D + \frac{1}{2} K^{-1} \dot{K} K^{-1}) \dot{q} \]  \hspace{1cm} (4.36)

In the above equation, the value of \( \dot{K} \) can be calculated as follows. Beginning with

\[ K K^{-1} = 1 \]  \hspace{1cm} (4.37)

and taking the derivative and solving for \( \dot{K}^{-1} \), we get,

\[ \dot{K}^{-1} = -K^{-1} \dot{K} K^{-1} \]  \hspace{1cm} (4.38)

Substituting equation 4.38 in 4.36, we get,

\[ \dot{V}(t) = -\dot{q}^T (K^{-1} D + \frac{1}{2} K^{-1} \dot{K} K^{-1}) \dot{q} \]  \hspace{1cm} (4.39)

Thus, from the above equation and Lyapunov’s Stability theorem, we can conclude that, \( V(t) \leq 0 \) if and only if,

\[ K^{-1} D + \frac{1}{2} K^{-1} \dot{K} K^{-1} \geq 0 \]  \hspace{1cm} (4.40)

It is evident that equation 4.12 is stable if equation 4.40 holds. Mathematically, we can show that for the system to be stable, the eigenvalues of the matrix on the left hand side of equation 4.40 must always be non-negative. It is important to realize that, Lyapunov stability analysis allows us to check for stability based on a desired
equilibrium. However, this method does not provide any means of predicting the value of $\epsilon$. 
Chapter 5

Simulation Results

A computer code has been implemented based on the theories discussed in the previous sections. This code provides us the means of altering the governing parameters to better understand the behaviour of the spacecraft. In the end the simulation results are plotted and discussed in further detail.

5.1 Simulation Parameters

For Simulation purposes, let us consider a rigid spacecraft. The inputs to the simulation include the moments of inertia of the spacecraft, initial attitude and orbital parameters of the spacecraft. The orbital position is governed by:

- Altitude: $a = 450\ km$
- Eccentricity: $e = 0$
- Inclination: $i = 87^\circ$
- Right Ascension: $\Omega_{t0} = 0$
- Argument of Latitude: $u_0 = 0$
- True Anomaly: $\theta_0 = 0$
For simulation purposes, we will assume that the spacecraft is orbiting in a circular orbit around Earth. The initial attitude parameters of the spacecraft at \( t = 0 \) are,

\[
\mathbf{q}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T
\]

\[
\mathbf{\omega}_0 = \begin{bmatrix} 0.2 & 0.2 & 0.2 \end{bmatrix}^T \text{ rad/s}
\]

The moment of inertia matrix of the spacecraft is,

\[
\mathbf{I} = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 25 \end{bmatrix} \text{ kg} \cdot \text{m}^2
\] (5.1)

The initial gain parameters \( K_p \) and \( K_d \) are,

\[
K_d = 625 \text{ kg}^2 \cdot \text{m}^4/\text{s}
\]

\[
K_p = 625 \text{ kg}^2 \cdot \text{m}^4/\text{s}^2
\]

Based on the input conditions, the simulation determines the attitude and angular velocities of the spacecraft. Furthermore, the spacecraft is also checked for stability based on Floquet and Lyapunov stability theories. The results from these simulations will be studied in further detail in the following sections.

### 5.2 System with Magnetic Control Alone

We begin by simulating the linear system in orbit with control input generated exclusively by the magnetic moment. This simulation also assumes that the spacecraft does not experience any disturbance torques while in orbit. Lack of three-axis control was simulated by setting the value of \( \gamma \) equal to zero. The results of the simulation for \( \epsilon = 0.005 \) are given in Figure 5.1. From the figure it is evident that with sufficiently large \( \epsilon \), the system is unstable throughout the total orbit cycle.
Reducing the value of the gain factor to $\epsilon = 0.001$, gives figure 5.2 where we observe that the spacecraft achieves attitude stability. However, recall that, this simulation was carried out with the assumption that the spacecraft does not experience any disturbance torque.

Examining the stability of the spacecraft under the influence of the gravity-gradient disturbance torque, we notice that the attitude of the spacecraft does not tend to zero, but remains in a bounded motion (see Figure 5.3). Although the motion of the spacecraft is negligible and bounded, additional disturbance torques can easily destabilize the spacecraft. In space, there are considerable number of forces acting on the spacecraft in orbit which will lead to attitude instability under the given conditions.
Figure 5.2: Hybrid Control Simulation $\epsilon = 0.001$, $\gamma = 0$ and No Gravity Disturbance

Figure 5.3: Hybrid Control Simulation $\epsilon = 0.001$, $\gamma = 0$ with Gravity Disturbance
5.3 System with Magnetic Control and Additional Three-axis Actuation

The magnetic control law as suggested in section 4.3 is implemented. The system behaviour is analyzed with magnetic control and three-axis actuation in effect. To test the controller, we implement the hybrid magnetic controller on the system with the values of $\epsilon = 0.005$ and $\gamma = 1.2$. The results from the simulation are shown in Figure 5.4. We notice that, the case which was previously unstable using magnetic actuation alone, can be stabilized using a sufficiently large additional non-magnetic control input.

Reducing the value of the gain factor to $\epsilon = 0.001$ and $\gamma = 1.2$, from Figure 5.5, we observe that the system achieves stability. From the figure, it is evident that the controller along with three-axis actuation provides sufficient control input to achieve attitude stability. Further analysis is carried out with gravity-gradient disturbance implemented. From figure 5.6, we observe that, the controller can easily stabilize the spacecraft under the given disturbance conditions.
Figure 5.5: Hybrid Control Simulation $\epsilon = 0.001$, $\gamma = 1.2$ and No Gravity Disturbance

Figure 5.6: Hybrid Control Simulation $\epsilon = 0.001$, $\gamma = 1.2$ with Gravity Disturbance
Based on the simulations carried out, it is clearly evident that the spacecraft can be stabilized even in scenarios when the spacecraft is subject to disturbance torques. However, these simulations do not offer any method of determining the value of the gain scaling factor, $\epsilon$, or input gain factor, $\gamma$.

Further simulating the given system by varying the amount of control input gain, we observe that for a given value of $\epsilon = 0.005$, stability is achieved for all values of $\gamma > \gamma^* = 0.16$. The results for the case $\gamma = \gamma^*$ are shown in Figure 5.7.

From the figure, we see that control input generated by the onboard controller is effective to stabilize the spacecraft. However, in space, the spacecraft is subjected to a variety of unexpected disturbances which will lead to instability. The controller is ineffective for any value of $\gamma < \gamma^*$ as the control input generated is not sufficient to stabilize the spacecraft.
5.4 Floquet Stability Analysis

Floquet’s theory describes dynamic systems in which the coefficients are periodic [21]. This theory has applications that are directly applicable to the magnetic control problem. It can be demonstrated with the aid of simulations, whether the system is stable or unstable. However, we need to understand how the stability of the spacecraft is affected under a given set of conditions. This can be analyzed using the Floquet theory presented in section 4.4.

Applying Floquet’s theories, we know that for stability, all the eigenvalues of the system \( \Phi(T) \), or the monodromy matrix, at time \( t = T \), must lie within the unit circle where,

\[
\Phi(T) = \begin{bmatrix} \Phi_1(T) & \Phi_2(T) & \cdots & \Phi_6(T) \end{bmatrix}
\]

(5.2)

using initial values,

\[
\Phi(0) = \begin{bmatrix} \Phi_1(0) & \Phi_2(0) & \cdots & \Phi_6(0) \end{bmatrix}
\]

(5.3)

Recall, the state-space control matrix for the linear system given by Equation 4.14,

\[
A = \begin{bmatrix} 0 & -\epsilon^2 K_p \left( \gamma I^{-2} - I^{-1} \tilde{B} \times \tilde{B} \times I^{-1} \right) & -\epsilon K_d \left( \gamma I^{-2} - I^{-1} \tilde{B} \times \tilde{B} \times I^{-1} \right) \\
-\epsilon^2 K_p \left( \gamma I^{-2} - I^{-1} \tilde{B} \times \tilde{B} \times I^{-1} \right) & 1 & -\epsilon K_d \left( \gamma I^{-2} - I^{-1} \tilde{B} \times \tilde{B} \times I^{-1} \right) \\
\end{bmatrix}
\]

To check for stability, the fundamental solution given by Equation 4.16 can be evaluated using the 4th order Runge-Kutta algorithm. The eigenvalues of the system when \( \epsilon = 0.005 \) and \( \gamma = 1.2 \) are shown in Figure 5.8.

From the figure, it is evident that the system is stable with all the eigenvalues centered at the origin in the unit circle. This state of stability is also evident from the simulation plots shown in Figure 5.4. This leads us to believe that the control input is more than sufficient to stabilize the system. On reducing the control input factor to \( \gamma = 0.1 \), from Figure 5.9, we notice that the eigenvalues of the system lie outside the unit circle and hence, the system is unstable. This instability is due to
the inability of the controller to provide sufficient torque inputs to counteract the spacecraft’s motion.

Furthermore, checking for stability at $\gamma = \gamma^*$, from Figure 5.10, we notice that the system is stable. However, one pair of eigenvalues lies close to the boundary of the unit circle. Further reducing the value of $\gamma$ leads to instability. Thus, we can conclude that the critical value of the control input gain factor is accurate. This reinforces the observations in the previous section.

**Figure 5.8:** Eigenvalues of the system for $\epsilon = 0.005$ and $\gamma = 1.2$

**Figure 5.9:** Eigenvalues of the system for $\epsilon = 0.005$ and $\gamma = 0.1$
We further examine the effects on Floquet stability by varying $\epsilon$ and $\gamma$. This is again studied by the aid of eigenvalue analysis with all the other parameters the same. The system is studied with the values of $\epsilon$ and $\gamma$ ranging from 0.0005 to 0.04 and 0 to 0.3 respectively. The results of this study are given in Figure 5.11.
From the figure, we can see that, for a chosen $\gamma$ and for a range of values of $\epsilon$, the spacecraft enters a zone of instability. For increasing values of $\epsilon$ the spacecraft achieves stability and remains stable. This minimum critical value of $\epsilon$, at which the spacecraft transitions into a state of instability is given by $\epsilon^*$. For the current simulation, the value of $\epsilon^*$ was determined to be 0.0025.

### 5.5 Lyapunov Stability Analysis Analysis

The Lyapunov stability method specialized for linear time-varying systems is studied and applied based on the theories presented in the previous section. The method has more theoretical importance than practical value and can be used to derive and prove accurate stability results. The results for the system with given input conditions are tested and analyzed.

Recall that, for the system to be stable, the eigenvalues of the system as given by equation 4.12 must satisfy the equation,

$$
eig \left( K^{-1}D + \frac{1}{2}K^{-1}KK^{-1} \right) > 0, \forall t$$

Upon simulation, the minimum eigenvalues of the system when $\epsilon = 0.005$ and $\gamma = 1.2$ are found to be,

$$\lambda_{min} = 0.0355 \times 10^{-6}$$

As predicted, the system is stable in the sense of Lyapunov with all the eigenvalues being greater than 0. Similarly, the eigenvalues of the system when $\epsilon = 0.005$ and $\gamma = 0.1$ was found to be,

$$\lambda_{min} = -0.0020 \times 10^{-6}$$

One of the eigenvalues obtained is less than 0 and thus does not satisfy the criterion given by Equation 4.40. Hence, we are unable to prove stability. Furthermore, upon simulating the critical case, the minimum eigenvalue of the system when $\epsilon = 0.005$ and $\gamma = 0.16$ is found to be,

$$\lambda_{min} = -0.014 \times 10^{-6}$$
We observe that the system does not satisfy the stability condition, i.e. the system has eigenvalues less than 0. Even though the Lyapunov Stability theories can be used to predict the stability of a given spacecraft, the test fails in this case and we are unable to prove that the spacecraft is stable. Recall that, though the conditions of stability are not satisfied by the Lyapunov test, the system is not actually unstable. Lyapunov theory is a sufficient test for stability but not a necessary test. From Figure 5.7, it is clearly evident that the system with $\epsilon = 0.005$ and $\gamma^* = 0.16$ is stable as the angular rotations are bounded and decreasing over time.

We further examine the effects on Lyapunov stability with varying $\epsilon$ and $\gamma$. This is again studied by the aid of eigenvalue study with all of the other parameters the same. The results of this study are given in Figure 5.12. From the figure it is evident that, by using Lyapunov stability theories, for a given set of initial conditions, we can establish a zone of stability in which the spacecraft will always be stable in the sense of Lyapunov. In order to determine attitude instability, the Lyapunov stability test must be conducted in conjunction with the Floquet stability test. The Lyapunov stability test provides a means of predicting stability of the spacecraft, but does not aid in predicting instability.
5.6 Summary

Upon superimposing the results obtained by Lyapunov stability and Floquet stability, we observe that, for a given set of conditions, we can establish zones of stability and instability of the spacecraft under a given set of conditions. However, this numerical method is tedious and requires vast computational resources. From Figure 5.13, by varying $\epsilon$ and $\gamma$, we can observe that, for a sufficiently large values of $\epsilon$ and $\gamma$, the system is predicted as stable by either stability theories. Whereas, by decreasing the values of $\epsilon$ and $\gamma$, Lyapunov stability theory cannot predict that the system is be unstable while Floquet stability predicts the system to be entering a zone of instability and stability.

![Floquet Plot](image)

**Figure 5.13:** Superimposed Lyapunov and Floquet Results for Varying $\epsilon$ and $\gamma$

Further simulations are carried out by varying other parameters such as $K_p$, $K_d$ and inclination, $i$. The results of the simulations by varying $K_p$, $K_d$ and $i$ for a given set of initial conditions are shown in following sections.
5.6.1 Variation in $K_p$

Figures 5.14, 5.15, 5.16 and 5.17 demonstrate the behavior of the two different stability theories with changes in the values of $K_p$. From these figures, we observe that for the system given by Equation 4.12, the zone in which the Lyapunov theorem predicts the system to be stable remains unchanged by the changes in the values of $K_p$. Similarly, from this figure we also observe that with increasing values of $K_p$, the zone of instability predicted by the Floquet theory increases with increasing values of $K_p$.

![Figure 5.14: Superimposed Stability Results for $K_p = 100$, $K_d = 625$, $i = 87^\circ$](image)

**Figure 5.14:** Superimposed Stability Results for $K_p = 100$, $K_d = 625$, $i = 87^\circ$
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**Figure 5.15:** Superimposed Stability Results for $K_p = 200$, $K_d = 625$, $i = 87^\circ$

**Figure 5.16:** Superimposed Stability Results for $K_p = 400$, $K_d = 625$, $i = 87^\circ$
Figure 5.17: Superimposed Stability Results for $K_p = 625$, $K_d = 625$, $i = 87^\circ$

5.6.2 Variation in $K_d$

Figures 5.18, 5.19, 5.20 and 5.21 demonstrates the behaviour of the two different stability theories with changes in the values of $K_d$. From these figures, we can observe that for the system given by Equation 4.12, the Lyapunov function derived is independent of changes in the value of $K_d$. Similarly, from this figure we also observe that with increasing values of $K_d$, the zone of instability predicted by Floquet theory increases with increasing values of $K_d$. However, notice that for a given value of $\gamma$, as $\epsilon$ decreases the system goes from a stable to an unstable state and back.
Figure 5.18: Superimposed Stability Results for $K_p = 625$, $K_d = 100$, $i = 87^\circ$

Figure 5.19: Superimposed Stability Results for $K_p = 625$, $K_d = 200$, $i = 87^\circ$
Figure 5.20: Superimposed Stability Results for $K_p = 625$, $K_d = 400$, $i = 87^\circ$

Figure 5.21: Superimposed Stability Results for $K_p = 625$, $K_d = 625$, $i = 87^\circ$
5.6.3 Variation in Inclination

Figures 5.22, 5.23, 5.24 and 5.25 demonstrate the behaviour of the two different stability theories with changes in the values of inclination, $i$. From these figures, we can observe that for the system given by Equation 4.12, the Lyapunov function derived varies nominally with the changes in the value of $i$. Similarly, from this figure we also observe that with increasing values of $i$, the zone of instability predicted by Floquet theory increases with increasing values of $i$.

![Lyapunov Plot](image)

**Figure 5.22**: Superimposed Stability Results for $K_p = 625$, $K_d = 625$, $i = 25^\circ$
Figure 5.23: Superimposed Stability Results for $K_p = 625$, $K_d = 625$, $i = 45^\circ$

Figure 5.24: Superimposed Stability Results for $K_p = 625$, $K_d = 625$, $i = 65^\circ$
Figure 5.25: Superimposed Stability Results for $K_p = 625$, $K_d = 625$, $i = 87^\circ$
Chapter 6

Conclusion And Future Work

In this thesis, it was successfully demonstrated by means of numerical simulations, that a hybrid attitude control scheme consisting of an onboard magnetic controller and an additional three-axis actuation scheme can be implemented to achieve attitude stability. Simulations were performed with different spacecraft configurations and initial conditions. Furthermore, it was determined that spacecraft can be stabilized with magnetic control, whether the configuration is gravity-gradient stable or not. The gains were varied to obtain a solution for each case which damped to the desired equilibrium.

The linear equations are only a good approximation of the nonlinear system when the initial conditions of the system are close to the desired equilibrium. In addition, the check for stability using Floquet theory is only valid for the linear equations; linear equation stability does not necessarily imply nonlinear equation stability.

Further research on magnetic control would be beneficial in the selection of system gains. An optimization technique to determine the values of $K_p$ and $K_d$ can be developed to suit the requirements of the mission. In addition, a method to quantify how stable a set of gains is for the nonlinear equations would make choosing nonlinear gain values easier.
References


