INTERFERENCE MITIGATION FOR CELLULAR NETWORKS: FUNDAMENTAL LIMITS AND APPLICATIONS

by

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Abstract

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Interference is a key limiting factor in modern communication systems. In a wireless cellular network, the performance of cell-edge users is severely limited by the intercell interference. This thesis studies the use of interference-channel and relay-channel techniques to mitigate intercell interference and to improve the throughput and coverage of cellular networks. The aim of this thesis is to demonstrate the benefit of the proposed interference mitigation schemes through both information theoretical studies and applications in the cellular network.

There are three main results in this thesis: First, it is shown that for the $K$-user cyclic Gaussian interference channel, where the $k$th user interferes with only the $(k - 1)$th user (mod $K$) in the network, the Etkin-Tse-Wang power splitting strategy achieves the capacity region to within 2 bits in the weak interference regime. For the special 3-user case, this gap can be sharpened to $1\frac{1}{2}$ bits by the time-sharing technique. Second, it is shown that for a two-user Gaussian interference channel with an in-band-reception and out-of-band-transmission relay, generalized hash-and-forward together with Han-Kobayashi information splitting can achieve the capacity region of this channel to within a constant number of bits in a certain weak-relay regime. A generalized-degrees-of-freedom analysis in the high signal-to-noise ratio regime reveals that in the symmetric channel setting, each common relay bit improves the sum rate up to two bits. The third part of this thesis studies an uplink multicell joint processing model in which the base-stations are connected to a centralized processing server via rate-limited digital backhaul links. This thesis proposes a suboptimal achievability scheme employing the Wyner-Ziv compress-and-forward relaying technique and successive-interference-cancellation decoding. The main advantage of the proposed approach is that it results in achievable rate regions that are easily computable, in contrast to previous schemes in which the rate regions can only be characterized by exponential number of rate constraints.
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Chapter 1

Introduction

A recent technology research shows that by the year of 2013, more than 50% of internet connections will be coming from mobile devices (smart phone, tablet etc.) rather than traditional PCs or laptops [1]. This tremendously growing demand for mobile network access and high-speed data services has perpetuated extensive research aiming at improving the throughput and coverage of cellular networks. The ever growing demand for mobile network connectivity mainly comes from enormous new software applications and the desire for real-time data services, for example, social networking, video conferencing and online gaming etc. However, intercell interference is known to be the main impediment to the system performance, especially in urban cellular systems.

The conventional approach to mitigate intercell interference for cellular networks is done via frequency planning, i.e., the frequency resource is divided into a number of parts, each of which is reused in different clusters of cells. At each frequency tone, several users are arranged to operate at different time slots to avoid interference. A typical cellular system utilizing such frequency reuse scheme is known as the GSM (Global System for Mobile Communications) system developed in Europe in the 1980s, in which 25MHz downlink/uplink bandwidth is divided into a number of 200kHz sub-channels and each sub-channel is shared by eight users in a time-division fashion. Although this approach effectively mitigates the intercell-interference, the price paid is the largely reduced spectral efficiency. With the trend of reducing the frequency reuse ratio in modern communication systems, e.g., CDMA (Code Division Multiple Access) system and 4G LTE (Long Term Evolution) system, the current set of techniques is apparently not enough to handle the intercell interference in a multicell environment with full frequency reuse. As such, improving the performance of cell-edge users becomes extremely demanding. However, this problem is highly nontrivial as simply boosting the transmit power will not only increase the received signal strength at the cell-edge users, but also create stronger intercell interference to the neighbouring cells. This interference-limited communication problem urges the development of new communication techniques aiming at intercell-interference mitigation for the next generation wireless systems.

1.1 Interference Channels

The intercell-interference mitigation problem can be studied through the two-user Gaussian interference channel (Fig. 1.1), which is the simplest and tractable model yet preserves the interference nature of multicell networks. As shown in Fig. 1.1, two transmitter-receiver pairs representing two MS-BS (Mobile
Chapter 1. Introduction

Station-Base Station) pairs communicate while interfering with each other. Specifically, $X_1 \in \mathcal{R}$ and $X_2 \in \mathcal{R}$ are the transmit signals with power constraints $P_1$ and $P_2$ respectively, $h_{ij}$ represents the real-valued channel gain from transmitter $i$ to receiver $j$, where $i, j = 1, 2$, and $Z_1, Z_2$ are the independent additive white Gaussian noises (AWGN) with power $N_0$.

The two-user Gaussian interference channel in Fig. 1.1 is an ideal model in the sense that some features of wireless environment are not captured, for example, frequency selectivity due to multipath propagation and Doppler effect due to moving objects, etc. However, thanks to the OFDM (orthogonal frequency division multiplexing) technique in which the system bandwidth is divided into a number of parts and information is transmitted in each subband, the wireless channel in each subband can be treated as a frequency flat channel that does not change within the channel coherence time. As such, the two-user Gaussian interference channel model is indeed valid and useful.

A natural transmission strategy is to entirely avoid the mutual interference, i.e., $X_1$ and $X_2$ in Fig. 1.1 are not active simultaneously: when $X_1$ is transmitting, $X_2$ transmits the zero signal; each transmitter-receiver pair occupies a fraction of time (frequency). This leads to the same idea of frequency planning in the GSM system. However, the resulting achievable rate region has been proved suboptimal in general for not efficiently utilizing the degrees of freedom [2].

To fully exploit the degrees of freedom in the interference channel, the two transmitter-receiver pairs are required to operate at the same time, in which case mutual interference arises. The most crucial problem of the interference channel lies in how to deal with the mutual interference. Depending on how interference is treated at the receiver, available strategies can be categorized as: 1) treating interference as noise, 2) partially decoding interference, and 3) fully decoding interference. Under different channel conditions, these strategies can be optimal or approximately optimal, i.e. achieve the capacity region to within a constant gap.

Intuitively, when interference links are weak, the receivers get highly distorted interfering signals, which are difficult to decode from the noisy environment. In this case, ignoring the structure of the interfering signals and simply treating them as noise is a good choice. Under this scheme, the effective noise level is $N_0 + |h_{21}|^2P_2$ at receiver 1, and $N_0 + |h_{12}|^2P_1$ at receiver 2. This results in an interference-
limited scenario with the following achievable rate region

\[
\begin{aligned}
&\left\{ (R_1, R_2) \left| \begin{array}{c}
R_1 \leq \frac{1}{2} \log \left( 1 + \frac{|h_{11}|^2 P_1}{|h_{21}|^2 P_2 + N_0} \right) \\
R_2 \leq \frac{1}{2} \log \left( 1 + \frac{|h_{22}|^2 P_2}{|h_{12}|^2 P_1 + N_0} \right)
\end{array} \right. \\
\right\}.
\]

(1.1)

Motivated by this observation, Annapureddy, Veeravalli, Shang, Kramer, Chen, Motahari, and Khandani [3–5] showed that when the interference level is lower than a certain threshold, treating interference as noise at the receiver achieves the sum capacity.

On the other extreme when the interference links are sufficiently strong, interference signals become the dominant components in the received signals. In this case, the best strategy for the receivers is to exploit the interference structure and fully decode the interfering signals. It has been shown that [2,6] in the “strong interference regime” where \(|h_{21}| \geq |h_{22}|\) and \(|h_{12}| \geq |h_{11}|\), the capacity region of the two-user Gaussian interference channel can be achieved by joint decoding of \(X_1\) and \(X_2\) at both receivers, and is given by the following pentagon:

\[
\begin{aligned}
&\left\{ (R_1, R_2) \left| \begin{array}{c}
R_1 \leq \frac{1}{2} \log \left( 1 + \frac{|h_{11}|^2 P_1}{N_0} \right) \\
R_2 \leq \frac{1}{2} \log \left( 1 + \frac{|h_{22}|^2 P_2}{N_0} \right) \\
R_1 + R_2 \leq \min \left\{ \frac{1}{2} \log \left( 1 + \frac{|h_{11}|^2 P_1 + |h_{21}|^2 P_2}{N_0} \right), \frac{1}{2} \log \left( 1 + \frac{|h_{22}|^2 P_2 + |h_{12}|^2 P_1}{N_0} \right) \right\}
\end{array} \right. \\
\right\}.
\]

(1.2)

In general, when interference links are moderate, the codeword structure of interfering signals cannot be fully exploited yet still being beneficial. As such, neither “treating interference as noise” and “fully decoding interference” is optimal in this case. The best known strategy is to partially decode the interfering signals and treat the remaining part as noise. The general form of this strategy is proposed by Han and Kobayashi [2] using a superposition coding scheme, a common-private information splitting strategy, and a time-sharing technique. The resulting rate region, which is often referred to as Han-Kobayashi rate region, is so far the best-known achievable rate region in literature. In this Han-Kobayashi coding strategy, each user’s signal is set to be a superposition of two parts: private message and common message. The private messages can be decoded only at the intended receivers, while the common messages can be decoded by both receivers. Furthermore, Han and Kobayashi introduce a time-sharing factor that naturally incorporates time- or frequency-division multiplex into the information splitting scheme. The optimality of the Han-Kobayashi scheme however, is in general open due to the fact that it is computationally prohibitive to evaluate the rate region.

In a recent breakthrough, Etkin, Tse and Wang [7,8] show that the Han-Kobayashi scheme in fact achieves to within \(\frac{1}{8}\) bits of the capacity region for the two-user Gaussian interference channel for all channel parameters. Toward this end, they fix the time-sharing parameter and set the power of the private message to be such that it is received at the noise level at the nonintended-receiver side. By doing this, the interference caused by the private message has a minor impact on the other receiver. Meanwhile, the private message conveys a substantial amount of information. With this very simple Han-Kobayashi scheme, it is possible to reach within one bit of the capacity region, the theoretical upper
limit. They also show that this simple scheme is asymptotically optimal in certain high signal-to-noise ratio regimes. The Etkin-Tse-Wang scheme makes significant progress on the Gaussian interference channel by showing that a very simple Han-Kobayashi type of input can be good enough in practice. It also shed lights on the future direction of channel capacity studies: approximation.

The above developments suggest a promising way of interference mitigation for the uplink of cellular networks: letting the base-stations “cooperate” at the codebook level, i.e., each base-station knows the codebook information of all other base-stations so as to partially decode and mitigate the mutual interference. However, this interference-mitigation approach applies only to the two-user scenario. Practical communication systems often have more than two transmitter-receiver pairs, yet the generalization of Etkin, Tse and Wang’s work to the interference channels with more than two users has proved difficult for the following reasons. First, it appears that the Han-Kobayashi private-common superposition coding is no longer adequate for the $K$-user interference channel. Interference-alignment types of coding schemes [9] [10] can potentially enlarge the achievable rate region. Second, even within the Han-Kobayashi framework, when more than two receivers are involved, multiple common messages at each transmitter may be needed, making the optimization of the resulting rate region difficult.

Instead of treating the general $K$-user interference channel, the first part of this thesis focuses on a cyclic Gaussian interference channel model, where the $k$th user interferes with only the $(k-1)$th user as shown in Fig. 1.2. In this case, each transmitter interferes with only one other receiver, and each receiver suffers interference from only one other transmitter, thereby avoiding the difficulties mentioned earlier. The $K$-user cyclic interference channel is motivated by the Wyner model [11] which best describes the soft handoff behaviour of cellular network. For the $K$-user cyclic interference channel, the Etkin, Tse and Wang’s coding strategy remains a natural one. One of the main contribution of this part is to show that it indeed achieves to within one bit of the capacity region for this cyclic model in the weak interference regime. For the special 3-user case, this gap can be sharpened to $\frac{3}{4}$ bits by the time-sharing technique.
1.2 Relay Networks and Relaying Strategies

Another line of research directly involved in this thesis includes relay channel and relaying strategies. The relay channel models a communication scenario in which a relay is placed to help the communications between the transmitter and the receiver. Cover and El Gamal [12] introduced two important relaying schemes for the classic three-node relay channel: decode-and-forward and quantize-and-forward. The first relaying scheme applies when the relay has a better observation of the transmit signal than the receiver. In this case, the relay decodes the transmit signal and cooperatively transmits the signal with the transmitter to the destination. When the relay has a lower-quality observation of the transmit signal than the receiver does, quantize-and-forward could be a good choice, where the relay simply compresses the observed signal and forwards the description to the receiver. The essence of the quantize-and-forward relaying scheme is a side information aware coding strategy, known as the Wyner-Ziv coding [13].

Although the optimality of decode-and-forward and quantize-and-forward relaying strategies is open in general, the capacities of several relay networks with simple structures have been approximated to within constant number of bits using these two relaying strategies. For example, for the three-node Gaussian relay channel, Avestimehr and Tse [14] showed that the decode-and-forward strategy achieves to within half a bit of the capacity; Chang, Chung, and Lee [15] proved that the compress-and-forward rate is within half a bit of the capacity, and the amplify-and-forward rate is within one bit.

Progress has also been made in relay networks. In their breakthrough work, Avestimehr and Tse [14] further showed that, the capacity of the single-source single-destination Gaussian relay network in general can be achieved to within constant bits via a universal relaying scheme called quantize-map-and-forward (QMF). They also showed that, the gap to capacity is only related to the number of nodes in the network.

Parallel to Avestimehr and Tse’s work, Lim, Kim, El Gamal, and Chung [16] proposed a noisy network coding strategy that naturally extends the conventional compress-and-forward scheme of Cover and El Gamal [12] and the classic network coding by Ahlswede, Cai, Li, and Yeung [17] to noisy networks. The main idea of noisy network coding is to derive an explicit expression of the achievable rate for each cut-set of the network. Then, by comparing with the cut-set upper bound, noisy network coding can be shown to achieve to within constant gap to the capacity of general multisource multicast Gaussian networks.

Motivated by these results, the second part of the thesis considers a two-cell wireless network with two mobile terminals communicating with their respective base-stations while interfering with each other, as shown in Fig. 1.3. The deployment of a cell-edge relay, which observes a linear combination of the two
transmit signals from the mobile terminals and is capable of independently communicating with the base-stations over a pair of relay links, can significantly help the base-stations mitigate intercell interference. The second part of this thesis studies a generalized hash-and-forward (GHF) relay strategy [18] for the relay-interference channel and characterizes the capacity region to within a constant gap under a weak-relay regime. The GHF relay strategy adopted in this thesis is essentially the same as the noisy network coding [16, 19, 20] and the quantize-map-and-forward relay strategies [14]. The result of this thesis can be thought of as an effort in generalizing these relay strategies to a particular case of the multiple unicast setting, for which constant-gap result continues to hold.

The third part of this thesis studies a very large-scale interference cancellation-technology in a multicell environment with focus on the uplink processing. A hexagonal cellular network is studied as depicted in Fig. 1.4, where a base-station is deployed at the centre of each cell surrounded by six neighbouring cells, and where mobile terminals are uniformly distributed among the cells. Traditional cellular networks often do not involve base-station cooperation except in a soft-handoff scenario when a mobile terminal is migrating from one cell to another while multiple base-stations in the vicinity serve this mobile terminal at the same time. This is often referred to as macro diversity in the literature [21]. This thesis, however, studies an interference mitigation technology where all base-stations are connected to a centralized processor via finite-capacity back-haul links. Upon receiving the signal transmitted from the mobile terminals, the base-station forwards its description to the centralized processor without decoding. Unlike previous studies where the centralized processor jointly decodes all the source messages, this thesis proposes a suboptimal achievability scheme employing the Wyner-Ziv compress-and-forward relaying technique and successive-interference-cancellation (SIC) decoding at the centralized processor. The main advantage of the proposed approach is that it results in an achievable rate region that is easily computable, in contrast to other schemes in which the rate regions can only be characterized by an exponential number of rate constraints. Numerical simulations are also performed in a 19-cell OFDMA network to confirm the gain of the proposed SIC framework.

Figure 1.4: Uplink multicell joint processing via a centralized processor
Chapter 2

K-User Cyclic Gaussian Interference Channel

This chapter studies the capacity region of a $K$-user cyclic Gaussian interference channel, where the $k$th user interferes with only the $(k-1)$th user (mod $K$) in the network. Inspired by the work of Etkin, Tse and Wang, who derived a capacity region outer bound for the two-user Gaussian interference channel and proved that a simple Han-Kobayashi power splitting scheme can achieve to within one bit of the capacity region for all values of channel parameters, this chapter shows that a similar strategy also achieves the capacity region of the $K$-user cyclic interference channel to within a constant gap in the weak interference regime. Specifically, for the $K$-user cyclic Gaussian interference channel, a compact representation of the Han-Kobayashi achievable rate region using Fourier-Motzkin elimination is first derived, a capacity region outer bound is then established. It is shown that the Etkin-Tse-Wang power splitting strategy gives a constant gap of at most one bit in the weak interference regime. For the special 3-user case, this gap can be sharpened to $\frac{3}{4}$ bits by time-sharing of several different strategies. The capacity result of the $K$-user cyclic Gaussian interference channel in the strong interference regime is also given. Further, based on the capacity results, this chapter studies the generalized degrees of freedom (GDoF) of the symmetric cyclic interference channel. It is shown that the GDoF of the symmetric capacity is the same as that of the classic two-user interference channel, no matter how many users are in the network.

2.1 Introduction

The interference channel models a communication scenario where several mutually interfering transmitter-receiver pairs share the same physical medium. The interference channel is a useful model for many practical systems such as the wireless network. The capacity region of the interference channel, however, has not been completely characterized, even for the two-user Gaussian case.

The largest known achievable rate region for the two-user interference channel is given by Han and Kobayashi [2] using a coding scheme involving common-private power splitting. Recently, Chong et al. [22] obtained the same rate region in a simpler form by applying the Fourier-Motzkin algorithm together with a time-sharing technique to the Han and Kobayashi’s rate region characterization. The optimality of the Han-Kobayashi region for the two-user Gaussian interference channel is still an open problem in general, except in the strong interference regime where transmission with common information
only can be shown to achieve the capacity region \([2, 23, 24]\), and in a noisy interference regime where transmission with private information only can be shown to be sum-capacity achieving \([3–5]\).

In a recent breakthrough, Etkin, Tse and Wang \([7]\) showed that the Han-Kobayashi scheme can in fact achieve to within \(\frac{1}{2}\) bits of the capacity region for the two-user Gaussian interference channel for all channel parameters. Their key insight was that the interference-to-noise ratio (INR) of the private message should be chosen to be as close to 1 as possible in the Han-Kobayashi scheme. They also found a new capacity region outer bound using a genie-aided technique. In the rest of this chapter, we refer this particular setting of the private message power as the Etkin-Tse-Wang (ETW) power-splitting strategy.

As mentioned before, Etkin, Tse and Wang’s result applies only to the two-user interference channel. Practical communication systems are often multiple transmitter-receiver in nature yet the generalization of Etkin, Tse and Wang’s work to the interference channels with more than two users is highly nontrivial for the following reasons. First, Han-Kobayashi common-private superposition coding is in fact no longer adequate for the \(K\)-user interference channel, in which case interference-alignment types of coding schemes \([9] [10]\) can potentially enlarge the achievable rate region. Second, even under the Han-Kobayashi framework, when there are more than two transmitter-receiver pairs, multiple common messages at each transmitter may be needed, making the optimization of the resulting rate region difficult.

In the context of \(K\)-user Gaussian interference channels, sum capacity results are available in the noisy interference regime \([3, 25]\). In particular, Annapureddy et al. \([3]\) obtained the sum capacity for the symmetric three-user Gaussian interference channel, the one-to-many, and the many-to-one Gaussian interference channels under the noisy interference criterion. Similarly, Shang et al. \([25]\) studied the fully connected \(K\)-user Gaussian interference channel and showed that treating interference as noise at the receiver is sum-capacity achieving when the transmit power and the cross channel gains are sufficiently weak to satisfy a certain criterion. Further, achievability and outer bounds for the three-user interference channel has also been studied in \([26]\) and \([27]\). Finally, much work has been carried out on the generalized degree of freedom (GDoF as defined in \([7]\)) of the \(K\)-user interference channel and its variations \([9, 28, 29]\).

Instead of treating the general \(K\)-user interference channel, this chapter focuses on a cyclic Gaussian interference channel model, where the \(k\)th user interferes with only the \((k-1)\)th user. In this case, each transmitter interferes with only one other receiver, and each receiver suffers interference from only one other transmitter, thereby avoiding the difficulties mentioned earlier. For the \(K\)-user cyclic interference channel, the Etkin, Tse and Wang’s coding strategy remains a natural one. The main objective of this chapter is to show that it indeed achieves to within a constant gap of the capacity region for this cyclic model in the weak interference regime to be defined later.

The cyclic interference channel model is motivated by the so-called modified Wyner model, which
Chapter 2. K-User Cyclic Gaussian Interference Channel

describes the soft handoff scenario of a cellular network [30]. The original Wyner model [11] assumes that all cells are arranged in a linear array with the base-stations located at the center of each cell, and where intercell interference comes from only the two adjacent cells. In the modified Wyner model [30] cells are arranged in a circular array as shown in Fig. 2.1. The mobile terminals are located along the circular array. If one assumes that the mobile terminals always communicate with the intended base-station to their left (or right), while only suffering from interference due to the base-station to their right (or left), one arrives at the K-user cyclic Gaussian interference channel studied in this chapter. The modified Wyner model has been extensively studied in the literature [30–32], but often either with interference treated as noise or with the assumption of full base-station cooperation. This chapter studies the modified Wyner model without base-station cooperation, in which case the soft-handoff problem becomes that of a cyclic interference channel.

This chapter primarily focuses on the K-user cyclic Gaussian interference channel in the weak interference regime. The main contributions of this chapter are as follows. This chapter first derives a compact characterization of the Han-Kobayashi achievable rate region by applying the Fourier-Motzkin elimination algorithm. A capacity region outer bound is then obtained. It is shown that with the Etkin, Tse and Wang’s coding strategy, one can achieve to within $\frac{3}{4}$ bits of the capacity region when $K = 3$ (with time-sharing), and to within one bit of the capacity region in general in the weak interference regime. Finally, the capacity result for the strong interference regime is also derived.

A key part of the development involves a Fourier-Motzkin elimination procedure on the achievable rate region of the K-user cyclic interference channel. To deal with the large number of inequality constraints, an induction proof is used. It is shown that as compared to the two-user case, where the rate region is defined by constraints on the individual rate $R_i$, the sum rate $R_1 + R_2$, and the sum rate plus an individual rate $2R_i + R_j$ ($i \neq j$), the achievable rate region for the K-user cyclic interference channel is defined by an additional set of constraints on the sum rate of any arbitrary $l$ adjacent users, where $2 \leq l < K$. These four types of rate constraints completely characterize the Han-Kobayashi region for the K-user cyclic interference channel. They give rise to a total of $K^2 + 1$ constraints.

For the symmetric K-user cyclic channel where all direct links share the same channel gain and all cross links share another channel gain, it is shown that the GDoF of the symmetric capacity is not dependent on the number of users in the network. Therefore, adding more users to a K-user cyclic interference channel with symmetric channel parameters, the per-user rate is not affected. This makes sharp contrast with the case where feedback is provided to the transmitters [33].

2.2 Channel Model

The K-user cyclic Gaussian interference channel (as depicted in Fig. 2.2) has K transmitter-receiver pairs. Each transmitter tries to communicate with its intended receiver while causing interference to only one neighboring receiver. Each receiver receives a signal intended for it and an interference signal from only one neighboring sender plus an additive white Gaussian noise (AWGN). As shown in Fig. 2.2, $X_1, X_2, \cdots, X_K \in \mathcal{R}$ and $Y_1, Y_2, \cdots, Y_K \in \mathcal{R}$ are the input and output signals, respectively, and $Z_i \sim \mathcal{N}(0, \sigma^2)$ is the independent and identically distributed (i.i.d) Gaussian noise at receiver $i$. The input-
output model can be written as

\[
\begin{align*}
Y_1 & = h_{1,1}X_1 + h_{2,1}X_2 + Z_1, \\
Y_2 & = h_{2,2}X_2 + h_{3,2}X_3 + Z_2, \\
& \vdots \\
Y_K & = h_{K,K}X_K + h_{1,K}X_1 + Z_K, 
\end{align*}
\] (2.1)

where each \(X_i\) has a power constraint \(P_i\) associated with it, i.e., \(E[|x_i|^2] \leq P_i\). Here, \(h_{i,j}\) is the real-valued channel gain from transmitter \(i\) to receiver \(j\).

Define the signal-to-noise and interference-to-noise ratios for each user as follows\(^1\):

\[
\begin{align*}
\text{SNR}_i = \frac{|h_{i,i}|^2 P_i}{\sigma^2} \\
\text{INR}_i = \frac{|h_{i,i-1}|^2 P_i}{\sigma^2}, \quad i = 1, 2, \ldots, K.
\end{align*}
\] (2.2)

The \(K\)-user cyclic Gaussian interference channel is said to be in the weak interference regime if

\[
\text{INR}_i \leq \text{SNR}_i, \quad \forall i = 1, 2, \ldots, K.
\] (2.3)

and the strong interference regime if

\[
\text{INR}_i \geq \text{SNR}_i, \quad \forall i = 1, 2, \ldots, K.
\] (2.4)

Otherwise, it is said to be in the mixed interference regime. Throughout this chapter, modulo arithmetic is implicitly used on the user indices, e.g., \(K + 1 = 1\) and \(1 - 1 = K\). Note that when \(K = 2\), the cyclic channel reduces to the conventional two-user interference channel.

\(^1\)Note that the definition of INR is slightly different from that of Etkin, Tse and Wang [7].
2.3 Within One Bit of the Capacity Region in the Weak Interference Regime

The generalization of Etkin, Tse and Wang’s result to the capacity region of a general (nonsymmetric) $K$-user cyclic Gaussian interference channel is significantly more complicated. In the two-user case, the shape of the Han-Kobayashi achievable rate region is the union of polyhedrons (each corresponding to a fixed input distribution) with boundaries defined by rate constraints on $R_1$, $R_2$, $R_1 + R_2$, $2R_1 + R_2$ and $2R_2 + R_1$, respectively. In the multiuser case, to extend Etkin, Tse and Wang’s result, one needs to find a similar rate region characterization for the general $K$-user cyclic interference channel first.

A key feature of the cyclic Gaussian interference channel model is that each transmitter sends signal to its intended receiver while causing interference to only one of its neighboring receivers; meanwhile, each receiver receives the intended signal plus the interfering signal from only one of its neighboring transmitters. Using this fact and with the help of Fourier-Motzkin elimination algorithm, this section shows that the achievable rate region of the $K$-user cyclic Gaussian interference channel is the union of polyhedrons with boundaries defined by rate constraints on the individual rates $R_i$, the sum rate $R_{\text{sum}}$, the sum rate plus an individual rate $R_{\text{sum}} + R_i$ ($i = 1, 2, \ldots, K$), and the sum rate for arbitrary $l$ adjacent users ($2 \leq l < K$). This last rate constraint on arbitrary $l$ adjacent users’ rates is new as compared with the two-user case.

The preceding characterization together with outer bounds to be proved later in the section allows us to prove that the capacity region of the $K$-user cyclic Gaussian interference channel can be achieved to within a constant gap using the ETW power-splitting strategy in the weak interference regime. However, instead of the $\frac{1}{2}$-bit result for the two-user interference channel, this section shows that one can achieve to within $\frac{3}{4}$ bits of the capacity region when $K = 3$ (with time-sharing), and within one bit of the capacity region for general $K$. Again, the strong interference regime is treated later.

2.3.1 Achievable Rate Region

**Theorem 1.** Let $\mathcal{P}$ denote the set of probability distributions $P(\cdot)$ that factor as

$$P(q, w_1, x_1, w_2, x_2, \ldots, w_K, x_K) = p(q)p(x_1|q)p(x_2|w_2|q)\cdots p(x_K|w_K|q). \quad (2.5)$$

For a fixed $P \in \mathcal{P}$, let $R_{\text{HK}}^{(K)}(P)$ be the set of all rate tuples $(R_1, R_2, \ldots, R_K)$ satisfying

$$0 \leq R_i \leq \min\{d_i, a_i + e_{i-1}\}, \quad (2.6)$$

$$\sum_{j=m}^{m+l-1} R_j \leq \min \left\{ g_m + \sum_{j=m+1}^{m+l-2} e_j + a_{m+l-1}, \sum_{j=m-1}^{m+l-2} e_j + a_{m+l-1} \right\}, \quad (2.7)$$

$$R_{\text{sum}} = \sum_{j=1}^{K} R_j \leq \min \left\{ \sum_{j=1}^{K} e_j, r_1, r_2, \ldots, r_K \right\}, \quad (2.8)$$

$$\sum_{j=1}^{K} R_j + R_i \leq a_i + g_i + \sum_{j=1, j \neq i}^{K} e_j, \quad (2.9)$$
where \(a_i, d_i, e_i, g_i\) and \(r_i\) are defined as follows:

\[
a_i = I(Y_i; X_i | W_i, W_{i+1}, Q) \quad (2.10)
\]
\[
d_i = I(Y_i; X_i | W_{i+1}, Q) \quad (2.11)
\]
\[
e_i = I(Y_i; W_{i+1}, X_i | W_i, Q) \quad (2.12)
\]
\[
g_i = I(Y_i; W_{i+1}, X_i | Q) \quad (2.13)
\]
\[
r_i = a_{i-1} + g_i + \sum_{j=1, j \neq \{i, i-1\}}^K e_j, \quad (2.14)
\]

and the range of indices are \(i, m = 1, 2, \cdots, K\) in (2.6) and (2.9), \(l = 2, 3, \cdots, K - 1\) in (2.7). Define

\[
\mathcal{R}_{HK}^{(K)} = \bigcup_{P \in \mathcal{P}} \mathcal{R}_{HK}^{(K)}(P). \quad (2.15)
\]

Then \(\mathcal{R}_{HK}^{(K)}\) is an achievable rate region for the \(K\)-user cyclic interference channel.

\[2\]

**Proof.** The achievable rate region can be proved by the Fourier-Motzkin algorithm together with an induction step. The proof follows the Kobayashi and Han’s strategy [36] of eliminating a common message at each step. The details are presented in Appendix A. \(\square\)

In the above achievable rate region, (2.6) is the constraint on the achievable rate of an individual user, (2.7) is the constraint on the achievable sum rate for any \(l\) adjacent users \((2 \leq l < K)\), (2.8) is the constraint on the achievable sum rate of all \(K\) users, and (2.9) is the constraint on the achievable sum rate for all \(K\) users plus a repeated user. We can also think of (2.6)-(2.9) as the sum-rate constraints for arbitrary \(l\) adjacent users, where \(l = 1\) for (2.6), \(2 \leq l < K\) for (2.7), \(l = K\) for (2.8) and \(l = K + 1\) for (2.9).

From (2.6) to (2.9), there are a total of \(K + K(K - 2) + 1 + K = K^2 + 1\) constraints. Together they describe the shape of the achievable rate region under a fixed input distribution. The quadratic growth in the number of constraints as a function of \(K\) makes the Fourier-Motzkin elimination of the Han-Kobayashi region quite complex. The proof in Appendix A uses induction to deal with the large number of the constraints.

As an example, for the two-user Gaussian interference channel, there are \(2^2 + 1 = 5\) rate constraints, corresponding to that of \(R_1, R_2, R_1 + R_2, 2R_1 + R_2\) and \(2R_2 + R_1\), as in [2, 7, 22, 36]. Specifically, substituting \(K = 2\) in Theorem 1 gives us the following achievable rate region:

\[
0 \leq R_1 \leq \min\{d_1, a_1 + e_2\}, \quad (2.16)
\]
\[
0 \leq R_2 \leq \min\{d_2, a_2 + e_1\}, \quad (2.17)
\]
\[
R_1 + R_2 \leq \min\{e_1 + e_2, a_1 + g_2, a_2 + g_1\}, \quad (2.18)
\]
\[
2R_1 + R_2 \leq a_1 + g_1 + e_2, \quad (2.19)
\]
\[
2R_2 + R_1 \leq a_2 + g_2 + e_1. \quad (2.20)
\]

The above region for the two-user Gaussian interference channel is exactly that of Theorem D in [36].

---

\[2\] This result was first shown in [34], then independently proved in [35]. Both used the same coding strategy. However in [34], only achievability result is proved, outer bounds are not studied.
2.3.2 Capacity Region Outer Bound

**Theorem 2.** For the K-user cyclic Gaussian interference channel in the weak interference regime, the capacity region is included in the set of rate tuples \((R_1, R_2, \cdots, R_K)\) such that

\[
R_i \leq \lambda_i, \quad \sum_{j=m}^{m+l-1} R_j \leq \min \left\{ \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1}, \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} \right\},
\]

\[
R_{sum} = \sum_{j=1}^{K} R_j \leq \min \left\{ \sum_{j=1}^{K} \alpha_j, \rho_1, \rho_2, \cdots, \rho_K \right\},
\]

\[
\sum_{j=1}^{K} R_j + R_i \leq \beta_i + \gamma_i + \sum_{j=1, j \neq i}^{K} \alpha_j,
\]

where the ranges of the indices \(i, m, l\) are as defined in Theorem 1, and

\[
\alpha_i = \frac{1}{2} \log \left( \frac{1 + \text{INR}_{i+1} + \text{SNR}_i}{1 + \text{INR}_i} \right),
\]

\[
\beta_i = \frac{1}{2} \log \left( \frac{1 + \text{SNR}_i}{1 + \text{INR}_i} \right),
\]

\[
\gamma_i = \frac{1}{2} \log \left( 1 + \text{INR}_{i+1} + \text{SNR}_i \right),
\]

\[
\lambda_i = \frac{1}{2} \log(1 + \text{SNR}_i),
\]

\[
\mu_i = \frac{1}{2} \log(1 + \text{INR}_i),
\]

\[
\rho_i = \beta_{i-1} + \gamma_i + \sum_{j=1, j \neq i}^{K} \alpha_j.
\]

**Proof.** See Appendix B. 

2.3.3 Capacity Region to Within One Bit

**Theorem 3.** For the K-user cyclic Gaussian interference channel in the weak interference regime, the fixed ETW power-splitting strategy achieves within one bit of the capacity region\(^3\).

**Proof.** Applying the ETW power-splitting strategy (i.e., \(\text{INR}_{ip} = \min(\text{INR}_i, 1)\)) to Theorem 1, parameters \(a_i, d_i, e_i, g_i\) can be easily calculated as follows:

\[
a_i = \frac{1}{2} \log (2 + \text{SNR}_{ip}) - 1,
\]

\[
d_i = \frac{1}{2} \log (2 + \text{SNR}_i) - 1,
\]

\[
e_i = \frac{1}{2} \log (1 + \text{INR}_{i+1} + \text{SNR}_{ip}) - 1,
\]

\[
g_i = \frac{1}{2} \log (1 + \text{INR}_{i+1} + \text{SNR}_i) - 1,
\]

\(^3\)This thesis follows the definition from [7] that if a rate tuple \((R_1, R_2, \cdots, R_K)\) is achievable and \((R_1 + k, R_2 + k, \cdots, R_K + k)\) is outside the capacity region, then \((R_1, R_2, \cdots, R_K)\) is within \(k\) bits of the capacity region.
where $\text{SNR}_{ip} = |h_{i,i}|^2 P_{ip}/\sigma^2$. To prove that the achievable rate region in Theorem 1 with the above $a_i, d_i, e_i, g_i$ is within one bit of the outer bound in Theorem 2, we show that each of the rate constraints in (2.6)-(2.9) is within one bit of their corresponding outer bound in (2.21)-(2.24) in the weak interference regime, i.e., the following inequalities hold for all $i, m, l$ in the ranges defined in Theorem 1:

\[
\delta R_i < 1, \\
\delta R_{m+\ldots+m+l-1} < l, \\
\delta R_{\text{sum}} < K, \\
\delta R_{\text{sum}+R_i} < K + 1,
\]

where $\delta_{R_i}$ is the difference between the achievable rate in Theorem 1 and its corresponding outer bound in Theorem 2. The proof makes use of a set of inequalities provided in Appendix D. For example, the facts that $\lambda_i - d_i < 1$ and $\lambda_i - (a_i + e_{i-1}) < 2, \forall i$ are used in the proof of $\delta_{R_i}$. Likewise, the facts that $\gamma_i - g_i < 1, \alpha_i - e_i < 1$ and $\beta_i - a_i < 1, \forall i$ are used in the proof involving $\delta_1$ below, etc.

- $\delta_{R_i}$:
  \[
  \delta_{R_i} = \lambda_i - \min\{d_i, a_i + e_{i-1}\} = \max\{\lambda_i - d_i, \lambda_i - (a_i + e_{i-1})\} < 1
  \]

- $\delta_{R_{m+\ldots+m+l-1}}$: First, compare the first term of (2.7) and (2.22):
  \[
  \delta_1 = \left(\gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1}\right) - \left(g_m + \sum_{j=m+1}^{m+l-2} e_j + a_{m+l-1}\right) < \frac{1}{2}
  \]

Similarly, the difference between the second term of (2.7) and (2.22) is bounded by

\[
\delta_2 = \left(\mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1}\right) - \left(\sum_{j=m-1}^{m+l-2} e_j + a_{m+l-1}\right) < \frac{l+1}{2}
\]
we obtain
\[ \delta_{R_m+\ldots+R_{m+l-1}} \leq \max\{\delta_1, \delta_2\} < \frac{l+1}{2}. \] (2.49)

- \( \delta_{R_{\text{sum}}}: \) First, the difference between the first term of (2.8) and (2.23) is bounded by
\[ \sum_{j=1}^{K} \alpha_j - \sum_{j=1}^{K} e_j = \sum_{j=1}^{K} (\alpha_j - e_j) < \frac{K}{2}. \] (2.50)

In addition,
\[
\begin{align*}
\rho_i - r_i &= \left( \beta_{i-1} + \gamma_i + \sum_{j=1,j \neq \{i,i-1\}}^{K} \alpha_j \right) - \left( \alpha_{i-1} + g_i + \sum_{j=1,j \neq \{i,i-1\}}^{K} e_j \right) \\
&= (\beta_{i-1} - a_{i-1}) + (\gamma_i - g_i) + \sum_{j=1,j \neq \{i,i-1\}}^{K} (\alpha_j - e_j) \\
&< \frac{K}{2}
\end{align*}
\] (2.53)
for \( i = 1, 2, \ldots, K \). As a result, the gap on the sum rate is bounded by
\[
\begin{align*}
\delta_{R_{\text{sum}}} &= \min \left\{ \sum_{j=1}^{K} \alpha_j, \rho_1, \rho_2, \ldots, \rho_K \right\} - \min \left\{ \sum_{j=1}^{K} e_j, r_1, r_2, \ldots, r_K \right\} \\
&\leq \max \left\{ \sum_{j=1}^{K} (\alpha_j - e_j), \rho_1 - r_1, \rho_2 - r_2, \ldots, \rho_K - r_K \right\} \\
&< \frac{K}{2}
\end{align*}
\] (2.56)

- \( \mathcal{R}_{\text{sum}} + R_i \):
\[
\begin{align*}
\delta_{R_{\text{sum}}+R_i} &= \left( \beta_i + \gamma_i + \sum_{j=1,j \neq i}^{K} \alpha_j \right) - \left( \alpha_i + g_i + \sum_{j=1,j \neq i}^{K} e_j \right) \\
&= (\beta_i - a_i) + (\gamma_i - g_i) + \sum_{j=1,j \neq i}^{K} (\alpha_j - e_j) \\
&< \frac{K+1}{2}
\end{align*}
\] (2.59)

Since the inequalities in (2.35)-(2.38) hold for all the ranges of \( i, m, \) and \( l \) defined in Theorem 1, this proves that the ETW power-splitting strategy achieves to within one bit of the capacity region in the weak interference regime.

2.3.4 Capacity Region to Within \( \frac{3}{4} \) Bits in the 3-User Case

Chong, Motani and Garg [22] showed that by time-sharing with marginalized versions of the input distribution, the Han-Kobayashi region for the two-user interference channel as stated in (2.16)-(2.20) can be further simplified by removing the \( a_1 + e_2 \) and \( a_2 + e_1 \) terms from (2.16) and (2.17) respectively.
The resulting rate region without the $a_1 + e_2$ and $a_2 + e_1$ terms is proved to be equivalent to the original Han-Kobayashi region (2.16)-(2.20).

This section shows that the aforementioned time-sharing technique can be directly applied to the 3-user cyclic interference channel (but not to $K \geq 4$). By a similar time-sharing strategy, the second rate constraint on $R_1, R_2$ and $R_3$ can be removed, and the achievable rate region can be shown to be within $\frac{3}{4}$ bits of the capacity region in the weak interference regime.

**Theorem 4.** Let $\mathcal{P}_3$ denote the set of probability distributions $P_3(\cdot)$ that factor as

$$P_3(q, w_1, x_1, w_2, x_2, w_3, x_3) = p(q)p(x_1|w_1)p(x_2|w_2)p(x_3|w_3).$$

(2.60)

For a fixed $P_3 \in \mathcal{P}_3$, let $\mathcal{R}^{(3)}_{\text{HK-TS}}(P_3)$ be the set of all rate tuples $(R_1, R_2, R_3)$ satisfying

$$\begin{align*}
R_i &\leq d_i, \quad i = 1, 2, 3, \\
R_1 + R_2 &\leq \min\{g_1 + a_2, e_3 + e_1 + a_2\}, \\
R_2 + R_3 &\leq \min\{g_2 + a_3, e_1 + e_2 + a_3\}, \\
R_3 + R_1 &\leq \min\{g_3 + a_1, e_2 + e_3 + a_1\}, \\
R_1 + R_2 + R_3 &\leq \min\{e_1 + e_2 + e_3, a_3 + g_1 + e_2, a_1 + a_2, a_1 + g_2 + e_3, a_2 + g_3 + e_1\}, \\
2R_1 + R_2 + R_3 &\leq a_1 + g_1 + e_2 + e_3, \\
R_1 + 2R_2 + R_3 &\leq a_2 + g_2 + e_3 + e_1, \\
R_1 + R_2 + 2R_3 &\leq a_3 + g_3 + e_1 + e_2,
\end{align*}$$

(2.61)-(2.68)

where $a_i, d_i, e_i, g_i$ are as defined before. Define

$$\mathcal{R}^{(3)}_{\text{HK-TS}} = \bigcup_{P_3 \in \mathcal{P}_3} \mathcal{R}^{(3)}_{\text{HK-TS}}(P_3).$$

(2.69)

Then, $\mathcal{R}^{(3)}_{\text{HK-TS}}$ is an achievable rate region for the 3-user cyclic Gaussian interference channel. Further, when $P_3$ is set according to the ETW power-splitting strategy, the rate region $\mathcal{R}^{(3)}_{\text{HK-TS}}(P_3)$ is within $\frac{3}{4}$ bits of the capacity region in the weak interference regime.

**Proof.** We prove the achievability of $\mathcal{R}^{(3)}_{\text{HK-TS}}$ by showing that $\mathcal{R}^{(3)}_{\text{HK-TS}}$ is equivalent to $\mathcal{R}^{(3)}_{\text{HK}}$. First, since $\mathcal{R}^{(3)}_{\text{HK}}$ contains an extra constraint on each of $R_1, R_2$ and $R_3$ (see (2.6)), it immediately follows that

$$\mathcal{R}^{(3)}_{\text{HK}} \subseteq \mathcal{R}^{(3)}_{\text{HK-TS}}.$$  

(2.70)

In Appendix C, it is shown that the inclusion also holds the other way around. Therefore, $\mathcal{R}^{(3)}_{\text{HK}} = \mathcal{R}^{(3)}_{\text{HK-TS}}$ and as a result, $\mathcal{R}^{(3)}_{\text{HK-TS}}$ is achievable.

Applying the ETW power-splitting strategy (i.e., $\text{INR}_p = \min\{\text{INR}_i, 1\}$ and $Q$ is fixed) to $\mathcal{R}^{(3)}_{\text{HK-TS}}(P_3)$,
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and following along the same line of the proof of Theorem 3, we obtain

\[ \delta R_i \leq \frac{1}{2}, \quad (2.71) \]
\[ \delta R_i + R_{i+1} \leq \frac{3}{2}, \quad (2.72) \]
\[ \delta R_{\text{sum}} \leq \frac{3}{2}, \quad (2.73) \]
\[ \delta R_{\text{sum}} + R_i \leq 2, \quad (2.74) \]

where \( i = 1, 2, 3 \). It then follows that the gap to the capacity region is at most \( \frac{3}{4} \) bits in the weak interference regime.

As shown in Appendix C, the rate region (2.61)-(2.68) is obtained by taking the union over the achievable rate regions with input distributions \( P_3, P_3^*, P_3^{**} \) and \( P_3^{***} \), where \( P_3^*, P_3^{**} \) and \( P_3^{***} \) are the marginalized versions of \( P_3 \). Thus, to achieve within \( \frac{3}{4} \) bits of the capacity region, one needs to time-share among the ETW power-splitting and its three marginalized variations, rather than using the fixed ETW’s input alone.

The key improvement of \( R_{\text{HK-TS}}^{(3)} \) over \( R_{\text{HK}}^{(3)} \) is the removal of term \( a_i + c_{i-1} \) in (2.6) using a time-sharing technique. However, the results in Appendix C hold only for \( K = 3 \). When \( K \geq 4 \), it is easy to verify that \( R_{\text{HK-TS}}^{(4)}(P_4) \) is not within the union of \( R_{\text{HK}}^{(4)}(P_4) \) and its marginalized variations, i.e., \( R_{\text{HK}}^{(4)} \notin R_{\text{HK-TS}}^{(4)} \). Therefore, the techniques used in this chapter only allow the two-bit result to be sharpened to a \( \frac{3}{4} \)-bit result for the three-user cyclic Gaussian interference channel, but not for \( K \geq 4 \).

2.4 Capacity Region in the Strong Interference Regime

The results so far pertain only to the weak interference regime, where \( \text{SNR}_i \geq \text{INR}_i, \forall i \). In the strong interference regime, where \( \text{SNR}_i < \text{INR}_i, \forall i \), the capacity result in [2] [24] for the two-user Gaussian interference channel can be easily extended to the \( K \)-user cyclic case.

**Theorem 5.** For the \( K \)-user cyclic Gaussian interference channel in the strong interference regime, the capacity region is given by the set of \( (R_1, R_2, \cdots, R_K) \) such that \(^4\)

\[
\begin{align*}
R_i &\leq \frac{1}{2} \log(1 + \text{SNR}_i) \\
R_i + R_{i+1} &\leq \frac{1}{2} \log(1 + \text{SNR}_i + \text{INR}_{i+1}),
\end{align*}
\]

for \( i = 1, 2, \cdots, K \). In the very strong interference regime where \( \text{INR}_i \geq (1 + \text{SNR}_{i-1})\text{SNR}_i, \forall i \), the capacity region is the set of \( (R_1, R_2, \cdots, R_K) \) with

\[ R_i \leq \frac{1}{2} \log(1 + \text{SNR}_i), \quad i = 1, 2, \cdots, K. \quad (2.76) \]

\(^4\)This capacity result was also recently obtained in [37].
The genie-aided Gaussian interference channel is defined by the Gaussian interference channel (see Fig. 2.2) with genie $X_i^n$ given to receiver $i$. The capacity region of the $K$-user cyclic Gaussian interference channel must reside inside the capacity region of the genie-aided one.

Assume that a rate tuple $(R_1, R_2, \cdots, R_K)$ is achievable for the $K$-user cyclic Gaussian interference channel. In this case, after $X_i^n$ is decoded, with the knowledge of the genie $X_{i+2}^n$, receiver $i$ can construct the following signal:

$$
\tilde{Y}_i^n = \frac{h_{i+1,i+1}}{h_{i+1,i}} (Y_i^n - h_{i,i} X_i^n) + h_{i+2,i+1} X_{i+2}^n + \frac{h_{i+1,i+1}}{h_{i+1,i}} Z_i^n,
$$

which contains the signal component of $Y_i^n$ but with less noise since $|h_{i+1,i}| \geq |h_{i+1,i+1}|$ in the strong interference regime. Now, since $X_{i+1}^n$ is decodable at receiver $i+1$, it must also be decodable at receiver $i$ using the constructed $\tilde{Y}_i^n$. Therefore, $X_i^n$ and $X_{i+1}^n$ are both decodable at receiver $i$. As a result, the achievable rate region of $(R_i, R_{i+1})$ is bounded by the capacity region of the multiple-access channel $(X_i^n, X_{i+1}^n, Y_i^n)$, which is shown in (2.77). Since (2.77) reduces to (2.75) in the strong interference regime, we have shown that (2.75) is an outer bound of the $K$-user cyclic Gaussian interference channel in the strong interference regime. This completes the converse proof.

In the very strong interference regime where $\text{INR}_i \geq (1 + \text{SNR}_{i-1}) \text{SNR}_i, \forall i$, it is easy to verify that the second constraint in (2.75) is no longer active. This results in the capacity region (2.76).

### 2.5 Symmetric Channel and Generalized Degrees of Freedom

Consider the symmetric cyclic Gaussian interference channel, where all the direct links from the transmitters to the receivers share a same channel gain and all the cross links share another same channel gain. In addition, all the input signals have the same power constraint $P$, i.e., $\mathbb{E} [X_i^2] \leq P, \forall i$

The symmetric capacity of the $K$-user interference channel is defined as

$$
C_{sym} = \left\{ \text{maximize } \min\{R_1, R_2, \cdots, R_K\} \text{ subject to } (R_1, R_2, \cdots, R_K) \in \mathcal{R} \right\}
$$

(2.78)
where $\mathcal{R}$ is the capacity region of the $K$-user interference channel. For the symmetric interference channel, $C_{sym} = \frac{1}{K} C_{sum}$, where $C_{sum}$ is the sum capacity. As a direct consequence of Theorem 3 and Theorem 5, the generalized degree of freedom of the symmetric capacity for the symmetric cyclic channel can be derived as follows.

**Corollary 1.** For the $K$-user symmetric cyclic Gaussian interference channel,

$$d_{sym} = \begin{cases} \min \{ \max \{ \alpha, 1 - \alpha \}, 1 - \frac{2}{K} \}, & 0 \leq \alpha < 1 \\ \min \{ \frac{K}{2}, 1 \}, & \alpha \geq 1 \end{cases}$$

(2.79)

where $d_{sym}$ is the generalized degrees of freedom of the symmetric capacity.

Note that, the above $d_{sym}$ for the $K$-user cyclic interference channel with symmetric channel parameters is the same as that of the two-user interference channel derived in [7].

### 2.6 Summary

This chapter investigates the capacity and the coding strategy for the $K$-user cyclic Gaussian interference channel. Specifically, this chapter shows that in the weak interference regime, the ETW power-splitting strategy can achieve within one bit of the capacity region. Further, in the special case of $K = 3$ and with the help of a time-sharing technique, one can achieve to within $\frac{3}{4}$ bits of the capacity region in the weak interference regime.

The capacity result for the $K$-user cyclic Gaussian interference channel in the strong interference regime is a straightforward extension of the corresponding two-user case. However, in the mixed interference regime, although the constant gap result may well continue to hold, the proof becomes considerably more complicated, as different mixed scenarios need to be enumerated and the corresponding outer bounds derived.
Chapter 3

Gaussian Interference Channel with a Degraded Broadcasting Relay

This chapter studies incremental relay strategies for a two-user Gaussian relay-interference channel with an in-band-reception and out-of-band-transmission relay, where the link between the relay and the two receivers is modelled as a degraded broadcast channel. It is shown that generalized hash-and-forward (GHF) can achieve the capacity region of this channel to within a constant number of bits in a certain weak relay regime, where the transmitter-to-relay link gains are not unboundedly stronger than the interference links between the transmitters and the receivers. The GHF relaying strategy is ideally suited for the broadcasting relay because it can be implemented in an incremental fashion, i.e., the relay message to one receiver is a degraded version of the message to the other receiver. A generalized-degree-of-freedom (GDoF) analysis in the high signal-to-noise ratio (SNR) regime reveals that in the symmetric channel setting, each common relay bit can improve the sum rate roughly by either one bit or two bits asymptotically depending on the operating regime, and the rate gain can be interpreted as coming solely from the improvement of the common message rates, or alternatively in the very weak interference regime as solely coming from the rate improvement of the private messages. Further, this chapter studies an asymmetric case in which the relay has only a single single link to one of the destinations. It is shown that with only one relay-destination link, the approximate capacity region can be established for a larger regime of channel parameters. Further, from a GDoF point of view, the sum-capacity gain due to the relay can now be thought as coming from either signal relaying only, or interference forwarding only, but not from both at the same time.

3.1 Introduction

Interference is a key limiting factor in modern communication systems. In a wireless cellular network, the performance of cell-edge users is severely limited by intercell interference. This chapter considers the use of relays in cellular networks. The uses of relays to combat channel shadowing and to extend coverage for wireless systems have been widely studied in the literature. The main goal of this chapter is to demonstrate the benefit of relaying for interference mitigation in the interference-limited regime.

Consider a two-cell wireless network with two base-stations each serving their respective receivers while interfering with each other, as shown in Fig. 3.1. The deployment of a cell-edge relay, which
observes a linear combination of the two transmit signals from the base-stations and is capable of independently communicating with the receivers over a pair of relay links, can significantly help the receivers mitigate intercell interference. This model is often referred to as an in-band-reception and out-of-band-transmission relay-interference channel, as the relay-to-receiver transmission can be thought of as taking place on a different frequency band.

A particular feature of the channel model considered in this chapter is that the relay-to-receiver link is modeled as a Gaussian broadcast channel. This is motivated by the fact that the relay’s transmission to the remote receivers often takes place in a wireless medium. Consequently, the same relay message can be heard by both receivers and can potentially help both receivers at the same time. Further, it is convenient (and without loss of generality as shown later) to model the relay-to-receiver links as digital links (i.e., noiseless channels) with capacities $C_1$ and $C_2$ respectively, but where one relay message is required to be a degraded version of the other relay message, as in a Gaussian broadcast channel. The goal of this chapter is to devise an incremental relaying strategy and to quantify its benefit for this particular relay-interference channel.

### 3.1.1 Related Work

The classic two-user interference channel consists of two transmitter-receiver pairs communicating in the presence of interference from each other. Although the capacity region of the two-user Gaussian interference channel is still not known exactly, it can be approximated to within one bit [7] using a Han-Kobayashi power splitting strategy [2].

The use of cooperative communication for interference mitigation has received much attention recently. For example, [38–40] studied the Gaussian Z-interference channel with a unidirectional receiver cooperation link, and [41–44] studied the Gaussian interference channel with bi-directional transmitter/receiver cooperation links. In addition, the Gaussian interference channel with an additional relay node has also been studied extensively in the literature. Depending on the types of the links between the relay and the transmitters/receivers, the relay-interference channel can be categorized as having in-band transmission/reception [45–51], out-of-band transmission/reception [52–54], out-of-band transmission and in-band reception [55], or in-band transmission and out-of-band reception [18, 56–58], the last of which is directly related to the channel model in this chapter. In the following, we review different transmission schemes and relaying strategies that have emerged for each of these cases.

For interference channels equipped with an in-band transmission and reception relay, the relay interacts with both transmitters and receivers in the same frequency band. Relaying strategies that have

![Figure 3.1: A two-cell network with an in-band reception and out-of-band-broadcasting relay for interference mitigation](image-url)
been investigated in the literature include decode-and-forward, compress-and-forward, and amplify-and-forward. For example, [47, 48] show that decoding-and-forwarding either the intended signal or the interfering signal to a receiver can both be beneficial. The former is termed as signal relaying, the latter interference forwarding. Decode-and-forward and half duplex amplify-and-forward strategies are also studied in [49, 51]. When combining decode-and-forward relaying strategy and the Han-Kobayashi rate splitting input scheme, [45] gives an achievable rate region that has a shape similar to the CMG region [22]. The exact capacity for this type of relay-interference channel is in general open, but there is a special potent-relay (i.e. relay with abundant power) case [46] for which the sum capacity is known in some specific regimes.

The difficulty in establishing the capacity of the interference channel with in-band transmission/reception relay is in part due to the fact that the relay’s received and transmit signals intertwine with that of the underlying interference channel. To simplify the matter, the interference channel with an out-of-band transmission/reception relay has been studied in [52–54]. In this channel model, the relay essentially operates on a separate set of parallel channels. Based on signal relaying and interference forwarding strategies, [52] identifies the condition under which the capacity region can be achieved with separable or nonseparable coding between the out-of-band relay and the underlying interference channel. Further, [53] studies this channel model in a symmetric setting and characterizes the sum capacity to within 1.15 bits. The transmission scheme of [53] involves further splitting of the common messages in the Han-Kobayashi scheme and a relay strategy that combines nested lattice coding and Gaussian codes. It is shown that in the strong interference regime, the use of structured codes is optimal.

Another variation of the relay-interference channel involves an out-of-band reception and in-band transmission relay. This channel is studied in [55], in which the transmitter further splits the Han-Kobayashi codewords; the relay decodes only some of the codewords depending the capacity of the transmitter-relay links; the rest of the codewords are transmitted directly from the sources to the destinations without the help of the relay. With this partial decode-and-forward relaying scheme, the sum capacity is found under a so-called strong relay-interference condition.

The interference channel with an in-band reception/out-of-band transmission relay has been briefly discussed in [56], and studied in [18, 57] for a case where the relay-destination links are shared between the two receivers. Conventional decode-and-forward and compress-and-forward relay strategies [59] are not well matched for helping both receivers simultaneously with a common relayed message. Thus, [18, 57] consider a generalized hash-and-forward (GHF) strategy, which generalizes the conventional compress-and-forward scheme, and is shown to achieve the capacity region of this channel model to within a constant number of bits for the special case where the shared relay-destination link rate is sufficiently small. The channel model under consideration in this chapter further extends the shared relay-destination link to degraded broadcasting links. The main objective is similar: how to efficiently use the relay bits to benefit both users simultaneously.

Finally, the GHF relay strategy used in this chapter is essentially the same as the noisy network coding [16, 19, 20] and the quantize-map-and-forward relay strategies [14]. The result of this chapter can be thought of as an effort in generalizing these relay strategies to a particular case of the multiple unicast setting, for which constant-gap result continues to hold for certain channel-parameter regimes. Related works for the multiple unicast problem include [60–62].
3.1.2 Main Contributions

This chapter considers a relay-interference channel with in-band reception and out-of-band degraded broadcasting links from the relay to the receivers. The key features of the transmission strategy and the main results of the chapter are as follows.

Incremental Relaying

This chapter uses a GHF relaying strategy to take advantage of the in-band reception link and the out-of-band broadcasting link from the relay to the receivers. In GHF, the relay quantizes its observation, which is a linear combination of the transmitted signals, using a fixed quantizer, then bins and forwards the quantized observation to the receivers. This strategy of fixing the quantization level is near optimal when a certain weak relay condition is satisfied, and is ideally matched to the degraded broadcasting relay-to-receiver links with capacities $C_1$ and $C_2$, because it allows an incremental binning strategy at the relay. Assuming that $C_1 \leq C_2$, the relay may first bin its quantized observation into $2^n C_1$ bins and send the bin index to both receivers, then further divide each bin into $2^n (C_2 - C_1)$ sub-bins and sends the extra bin index to receiver 2 only. Thus, the relay message to the first receiver is a degraded version of the message to the second receiver.

Oblivious Power Splitting

The transmission scheme used in this chapter consists of a Han-Kobayashi power splitting strategy [2] at the transmitter. The common-private power splitting ratio in such a strategy is crucial. In a study of the interference channel with conferencing links [41], Wang and Tse used the power splitting strategy of Etkin, Tse and Wang [7] where the private power is set at the noise level at the receivers. This is sensible for the conferencing-receiver model considered in [41], but not necessarily so for the interference channel with an independent relay, unless again a certain weak-relay condition is satisfied. This strategy of fixing the power splitting at the transmitter to be independent of the relay is termed oblivious power splitting in [57]. Oblivious power splitting is used in this chapter as well.

Constant Gap to Capacity in the Weak Relay Regime

The main result of this chapter is that when the relay links are not unboundedly stronger than the interfering links, i.e.,

$$\max \left\{ \left| \frac{g_1}{|h_{12}|^2} \right|^2, \left| \frac{g_2}{|h_{21}|^2} \right|^2 \right\} = \rho < \infty,$$

(3.1)

for some fixed $\rho$, the capacity of the relay-interference channel with a broadcast link can be achieved to within a constant gap, where the gap is a function of $\rho$ but otherwise independent of channel parameters. This operating regime is called the weak-relay regime in this chapter.

The main result of this chapter is motivated by the results in [18] and [57], which studied a two-user interference channel augmented with a shared digital relay link to the receivers of rate $R_0$, and obtained a constant-gap-to-capacity result under a certain small-$R_0$ condition using GHF and oblivious power splitting. The relay strategy studied in this chapter goes one step further in that the relay-to-receivers link is modeled as a degraded broadcast channel. Moreover, the weak relay regime studied in this chapter is a counterpart of the small-$R_0$ regime studied in [57], as can be visualized in the practical setup of Fig. 3.1. When the mobiles are close to their respective cell centers, the relay link capacities $C_1$ and $C_2$. 


are small, thereby satisfying the small-$R_0$ condition of [57]. In the more practically important regime where the mobile terminals are close to the cell edge, the channel falls into the weak relay regime of this chapter. An interesting feature of the result in this chapter is that the gap to capacity is a function of $\rho$, the relative channel strength between the interfering channel and the channel to the relay; the gap becomes smaller as $\rho \to 1$. In the limiting case with $\rho = 1$, corresponding to the situation where the mobiles are at the cell edge, the capacity region can be achieved to within $\frac{1}{2} \log \frac{5}{2+\sqrt{33}} = 1.2128$ bits.

A technical contribution of this chapter is a particular set of capacity region outer bounds which are established by giving different combinations of side information (genies) to the receivers and by applying the known outer-bound results of the Gaussian interference channel [7] and the single-input multiple-output (SIMO) Gaussian interference channel [63]. It is shown that there are two constraints for the individual rates $R_1$ and $R_2$, twelve constraints for the sum rate $R_1 + R_2$, six constraints for $2R_1 + R_2$, and six constraints for $R_1 + 2R_2$. Furthermore, the capacity region outer bounds established in this chapter hold for all channel parameters. This set of outer bounds is tight to within a constant gap in the weak-relay regime.

To obtain insights from the performance gain brought by the relay, this chapter further investigates the improvement in the generalized degrees of freedom (GDoF) per user for the relay-interference channel due to a broadcasting link. In the symmetric setting, it is shown that a common broadcast link can improve the sum capacity by two bits per each relay bit in the very weak, moderately weak, and very strong interference regimes, but by one bit per each relay bit in other regimes. This asymptotic behavior can be interpreted by noting that the relay link essentially behaves like a deterministic channel in the high signal-to-noise-ratio (SNR) regime. Further, in the symmetric setting, the sum-capacity gain due to the relay can be thought of as solely coming from the rate improvement of the common messages, or alternatively in a very weak interference regime as solely coming from the rate improvement of the private messages.

In asymmetric settings, the improvement in the sum capacity by the relay can be interpreted in different ways. To illustrate this point, this chapter investigates a special case of the channel model, where the relay link is available to only one but not to both destinations. In this case, the relay may forward information about both the intended signal and the interference, and the capacity can benefit from both signal-relaying and interference-forwarding. This chapter shows that a constant-gap-to-capacity result can be derived for this setting under a more relaxed weak-relay condition that requires only $|g_2| \leq \sqrt{\rho}|h_{21}|$ (and not $|g_1| \leq \sqrt{\rho}|h_{12}|$). Moreover, this chapter shows that in term of GDoF, when the relay link is above a certain threshold, the sum-capacity gain is equivalent to that of that of a single relay link from user 1. When the relay link is below the threshold, the sum-capacity gain is equivalent to that of a single relay link from user 2.

Finally, the results of this chapter show that GHF is sufficient for achieving the approximated capacity region of an in-band reception and out-of-band transmission Gaussian relay-interference channel in the weak-relay regime. Thus, more advanced relay techniques based on compute-and-forward or lattice coding is not necessary at least in this case. The optimal relay strategies outside of this regime remain an open problem.
Figure 3.2: Gaussian relay-interference channel with two independent digital relay links

3.2 Gaussian Relay-Interference Channel: General Case

3.2.1 Channel Model and Definitions

A Gaussian relay-interference channel consists of two transmitter-receiver pairs and an independent relay. Each transmitter communicates with the intended receiver while causing interference to the other transmitter-receiver pair. The relay receives a linear combination of the two transmit signals and helps the transmitter-receiver pairs by forwarding a message to receiver 1 and another message to receiver 2 through rate-limited digital links with capacities $C_1$ and $C_2$ respectively. We start by treating a channel model with independent relay links, and later show that requiring one relay message to be a degraded version of the other is without loss of approximate optimality. As shown in Fig. 3.2, $X_1, X_2$ and $Y_1, Y_2$ are the input and output signals, respectively, and $Y_R$ is the observation of the relay. The receiver noises are assumed to be independent and identically distributed (i.i.d.) Gaussian random variables with variances one, i.e., $Z_i \sim \mathcal{N}(0,1), i = 1, 2$ and $R$. The input-output relationship can be described by

$$
Y_1 = h_{11}X_1 + h_{21}X_2 + Z_1, \quad (3.2a)
$$
$$
Y_2 = h_{22}X_2 + h_{12}X_1 + Z_2, \quad (3.2b)
$$
$$
Y_R = g_1X_1 + g_2X_2 + Z_R, \quad (3.2c)
$$

where $h_{ij}$ is the real-valued channel gain from transmitter $i$ to receiver $j$, and $g_j$ is the channel gain from transmitter $j$ to the relay. The powers of the input signals are normalized to one, i.e., $\mathbb{E}[|X_i|^2] \leq 1, i = 1, 2$.

Define the signal-to-noise ratios and interference-to-noise ratios as follows:

$$
\text{SNR}_i = |h_{ii}|^2, \quad \text{SNR}_{ri} = |g_i|^2, \quad i = 1, 2
$$
$$
\text{INR}_1 = |h_{12}|^2, \quad \text{INR}_2 = |h_{21}|^2.
$$

For a fixed constant $\rho$, define functions $\alpha(\cdot)$ and $\beta(\cdot)$ as

$$
\alpha(x) = \frac{1}{2} \log(2x + 2 + \rho), \quad \beta(x) = \frac{1}{2} + \frac{1}{2} \log \left(1 + \frac{1+\rho}{x}\right), \quad (3.3)
$$
where \(\log(\cdot)\) is base 2 and \(\rho\) is defined as

\[
\rho \triangleq \max \left\{ \frac{|g_1|^2}{|h_{12}|^2}, \frac{|g_2|^2}{|h_{21}|^2} \right\}.
\]  \(3.4\)

This chapter considers the weak relay regime where \(\rho\) is a finite constant.

### 3.2.2 Outer Bounds and Achievable Rate Region

We first present outer bounds and achievability results that are applicable to the relay-interference channel model with two independent digital relay links as shown in Fig. 3.2.

**Theorem 6** (Capacity Region Outer Bounds). The capacity region of the Gaussian relay-interference channel as depicted in Fig. 3.2 is contained in the outer bound \(\mathcal{C}\) for all channel parameters, where \(\mathcal{C}\) is given by the set of \((R_1, R_2)\) for which

\[
R_1 \leq \frac{1}{2} \log(1 + \text{SNR}_1) + \min \left\{ C_1, \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_r}{1 + \text{SNR}_1} \right) \right\}
\]  \(3.5\)

\[
R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2) + \min \left\{ C_2, \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_r}{1 + \text{SNR}_2} \right) \right\},
\]  \(3.6\)
and

\[ R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) + C_1 + C_2 \] (3.7)

\[ R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + C_1 + C_2 \] (3.8)

\[ R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1}) + \frac{1}{2} \log(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}) + C_1 + C_2 \] (3.9)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_r} \right) \\
+ \frac{1}{2} \log(1 + \text{SNR}_2(1 + \phi_2^2 \text{SNR}_r) + \text{SNR}_{r2} + \text{INR}_1 + \text{SNR}_{r1}) + C_1 \] (3.10)

\[ R_1 + R_2 \leq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 + \text{SNR}_{r2}}{1 + \text{INR}_2} \right) + C_1 \] (3.11)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_r} + \text{INR}_2 \right) \\
+ \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_r) + \text{SNR}_{r2}}{1 + \text{INR}_2} + \text{INR}_1 + \text{SNR}_{r1} \right) + C_1 \] (3.12)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) \\
+ \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_{r2}) + \text{SNR}_{r2} + \text{INR}_1}{1 + \text{INR}_2} + \text{SNR}_{r1} \right) + C_2 \] (3.13)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{SNR}_r} + \text{INR}_1 \right) \\
+ \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_r) + \text{SNR}_{r2}}{1 + \text{INR}_2} + \text{SNR}_{r1} \right) + C_2 \] (3.14)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) \\
+ \frac{1}{2} \log(1 + \text{SNR}_2(1 + \phi_1^2 \text{SNR}_r) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) \] (3.16)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 + \text{SNR}_{r2}}{1 + \text{INR}_2 + \text{SNR}_{r2}} \right) \\
+ \frac{1}{2} \log(1 + \text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r2} + \text{INR}_2 + \text{SNR}_{r1}) \] (3.17)

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r1}) + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{SNR}_{r1}} \right) \\
+ \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_r) + \text{SNR}_{r2}}{1 + \text{INR}_2 + \text{SNR}_{r2}} + \text{INR}_1 + \text{SNR}_{r1} \right) \] (3.18)
and

\[ 2R_1 + R_2 \leq \frac{1}{2} \log (1 + SNR_1 + INR_2) + \frac{1}{2} \log \left( 1 + \frac{SNR_2}{1 + INR_2} \right) \]
\[ + \frac{1}{2} \log \left( 1 + \frac{SNR_1 + SNR_{r1}}{1 + INR_1 + SNR_{r1}} \right) + 2C_1 + C_2 \]  
(3.19)

\[ 2R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{SNR_2(1 + \phi_2^2 SNR_{r2}) + SNR_{r2}}{1 + INR_2 + SNR_{r2}} \right) + \frac{1}{2} \log \left( 1 + \frac{SNR_1 + SNR_{r1}}{1 + INR_1 + SNR_{r1}} \right) + C_1 \]  
(3.20)

\[ 2R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{SNR_1 + SNR_{r1}}{1 + INR_1} \right) + \frac{1}{2} \log \left( 1 + \frac{SNR_2}{1 + INR_2} \right) \]
\[ + \frac{1}{2} \log \left( 1 + \frac{SNR_1(1 + \phi_1^2 SNR_{r2}) + SNR_{r1} + INR_2 + SNR_{r2}}{1 + INR_1 + SNR_{r2}} \right) + C_1 \]  
(3.21)

\[ 2R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{SNR_1 + SNR_{r1}}{1 + INR_1} \right) 
+ \frac{1}{2} \log \left( 1 + \frac{SNR_2(1 + \phi_2^2 SNR_{r1}) + SNR_{r2}}{1 + INR_2} \right) + \frac{1}{2} \log \left( 1 + \frac{SNR_1 + SNR_{r1}}{1 + INR_1} \right) + 2C_1 \]  
(3.22)

\[ 2R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{SNR_1 + SNR_{r1}}{1 + INR_1} \right) 
+ \frac{1}{2} \log \left( 1 + \frac{SNR_2}{1 + INR_2} \right) \]
\[ + \frac{1}{2} \log \left( 1 + \frac{SNR_1(1 + \phi_1^2 SNR_{r2}) + SNR_{r1} + INR_2 + SNR_{r2}}{1 + INR_1 + SNR_{r2}} \right) + C_1 + C_2, \]  
(3.24)

and \( R_1 + 2R_2 \) is bounded by (3.19)-(3.24) with indices 1 and 2 switched. Note that, \( \phi_1^2 \) and \( \phi_2^2 \) are defined as

\[ \phi_1^2 = \left| \frac{g_1 h_{21}}{g_2 h_{11}} - 1 \right|^2, \quad \phi_2^2 = \left| \frac{g_2 h_{12}}{g_1 h_{22}} - 1 \right|^2. \]  
(3.25)

**Proof.** The above outer bounds can be proved in a genie-aided approach. See Appendix E for details. \( \square \)

---

**Theorem 7** (Achievable Rate Region). Let \( \mathcal{P} \) denote the set of probability distributions \( P(\cdot) \) that factor as

\[ P(q, w_1, w_2, x_1, x_2, y_1, y_2, y_R, \hat{y}_{R1}, \hat{y}_{R2}) \]
\[ = p(q)p(x_1, w_1|q)p(x_2, w_2|q)p(y_1, y_2, y_R|x_1, x_2, q)p(\hat{y}_{R1}, \hat{y}_{R2}|y_R, q). \]  
(3.26)
For a fixed distribution $P \in \mathcal{P}$, let $\mathcal{R}(P)$ be the set of all rate pairs $(R_1, R_2)$ satisfying

\begin{align}
0 \leq R_1 &\leq d_1 + \min\left\{ (C_1 - \xi_1)^+, \Delta \check{d}_1 \right\}, \\
0 \leq R_2 &\leq d_2 + \min\left\{ (C_2 - \xi_2)^+, \Delta \check{d}_2 \right\}, \\
R_1 + R_2 &\leq a_1 + g_2 + \min\left\{ (C_1 - \xi_1)^+, \Delta \check{a}_1 \right\} + \min\left\{ (C_2 - \xi_2)^+, \Delta \check{g}_2 \right\}, \\
R_1 + R_2 &\leq a_2 + g_1 + \min\left\{ (C_1 - \xi_1)^+, \Delta \check{a}_1 \right\} + \min\left\{ (C_2 - \xi_2)^+, \Delta \check{g}_2 \right\}, \\
R_1 + R_2 &\leq e_1 + e_2 + \min\left\{ (C_1 - \xi_1)^+, \Delta \check{e}_1 \right\} + \min\left\{ (C_2 - \xi_2)^+, \Delta \check{e}_2 \right\}, \\
2R_1 + R_2 &\leq a_1 + g_1 + e_2 + \min\left\{ (C_1 - \xi_1)^+, \Delta \check{a}_1 \right\} + \min\left\{ (C_2 - \xi_2)^+, \Delta \check{g}_2 \right\}, \\
R_1 + 2R_2 &\leq a_2 + g_2 + e_1 + \min\left\{ (C_2 - \xi_2)^+, \Delta \check{a}_2 \right\} + \min\left\{ (C_1 - \xi_1)^+, \Delta \check{g}_2 \right\}, \\
\end{align}

where

\begin{align}
a_1 &= I(X_1; Y_1|W_1, W_2, Q), & \Delta \check{a}_1 &= I(X_1; \hat{Y}_{R_1}|Y_1, W_1, W_2, Q), \\
d_1 &= I(X_1; Y_1|W_2, Q), & \Delta \check{d}_1 &= I(X_1; \hat{Y}_{R_1}|Y_1, W_2, Q), \\
e_1 &= I(X_1, W_2; Y_1|W_1, Q), & \Delta \check{e}_1 &= I(X_1, W_2; \hat{Y}_{R_1}|Y_1, W_1, Q), \\
g_1 &= I(X_1, W_2; Y_1|Q), & \Delta \check{g}_1 &= I(X_1, W_2; \hat{Y}_{R_1}|Y_1, Q), \\
\xi_1 &= I(Y_R; \hat{Y}_{R_1}|Y_1, X_1, W_2, Q),
\end{align}

and $a_2, \Delta \check{a}_2, d_2, \Delta \check{d}_2, e_2, \Delta \check{e}_2, g_2, \Delta \check{g}_2$, and $\xi_2$ are defined by (3.34)-(3.38) with indices 1 and 2 switched. Then

\[
\mathcal{R} = \bigcup_{P \in \mathcal{P}} \mathcal{R}(P)
\]

is an achievable rate region for the Gaussian relay-interference channel as shown in Fig. 3.2.

Proof. The achievable scheme consists of a Han-Kobayashi strategy at the transmitters and a generalized hash-and-forward strategy at the relay. They are the same strategies as adopted in [57] except that unlike the GHF relaying scheme in [57, Theorem 2], where the relay quantizes the received signal and broadcasts its bin index to both receivers through a shared digital link, the relay here quantizes the received signal with two different quantization resolutions, then sends the bin indices of the quantized signals to receivers through separated digital links of rates $C_1$ and $C_2$. The following is a sketch of the encoding/decoding process.

Encoding: Each transmit signal is comprised of a common message of rate $T_1$ and a private message of rate $S_i$. The common message codewords $W^n_i(j)$, $j = 1, 2, \ldots, 2^{nT_1}$, are generated according to $p(w_i|q)$. The private message codewords $X^n_i(j, k)$, $k = 1, 2, \ldots, 2^{nS_i}$, are generated according to $p(x_i|w_i, q)$. At the relay, after receiving $Y^n_R$, the relay quantizes $Y^n_R$ into $\hat{Y}^n_{R_1}$ and $\hat{Y}^n_{R_2}$, then bins $\hat{Y}^n_{R_1}$ to $2^{nC_1}$ bins, and bins $\hat{Y}^n_{R_2}$ to $2^{nC_1}$ bins, and sends the bin indices to the receivers through the digital links.

Decoding: The decoding process follows the Han-Kobayashi framework: $X^n_1$ and $W^n_2$ are decoded by receiver 1 with the help of the index of the relayed message $\hat{Y}^n_{R_1}$; $X^n_2$ and $W^n_1$ are decoded by receiver 2 with the help of the index of the relayed message $\hat{Y}^n_{R_2}$. To decode, receiver 1 first constructs a list of candidates for the relayed message $\hat{Y}^n_{R_1}$, then jointly decodes $X^n_1$, $W^n_2$, and $\hat{Y}^n_{R_1}$ using typicality decoding.
Similarly, receiver 2 jointly decodes $X^n_2$, $W^n_1$ and $\hat{Y}^n_{R2}$. The rate tuple $(S_1, T_1, S_2, T_2)$ satisfying the following constraints is achievable:

Constraints at receiver 1:

$$S_1 \leq \min \{ I(X_1;Y_1|W_1,W_2,Q) + (C_1 - \xi_1)^+, I(X_1;Y_1,\hat{Y}_{R1}|W_1,W_2,Q) \} \quad (3.40)$$

$$S_1 + T_1 \leq \min \{ I(X_1;Y_1|W_2,Q) + (C_1 - \xi_1)^+, I(X_1;Y_1,\hat{Y}_{R1}|W_2,Q) \} \quad (3.41)$$

$$S_1 + T_2 \leq \min \{ I(X_1,W_2;Y_1|W_1,Q) + (C_1 - \xi_1)^+, I(X_1,W_2,Y_1,\hat{Y}_{R1}|W_1,Q) \} \quad (3.42)$$

$$S_1 + T_1 + T_2 \leq \min \{ I(X_1,W_2;Y_1|Q) + (C_1 - \xi_1)^+, I(X_1,W_2,Y_1,\hat{Y}_{R1}|Q) \} \quad (3.43)$$

Constraints at receiver 2:

$$S_2 \leq \min \{ I(X_2;Y_2|W_1,W_2,Q) + (C_2 - \xi_2)^+, I(X_2;Y_2,\hat{Y}_{R2}|W_1,W_2,Q) \} \quad (3.44)$$

$$S_2 + T_2 \leq \min \{ I(X_2;Y_2|W_1,Q) + (C_2 - \xi_2)^+, I(X_2;Y_2,\hat{Y}_{R2}|W_1,Q) \} \quad (3.45)$$

$$S_2 + T_1 \leq \min \{ I(X_2,W_2;Y_2|W_2,Q) + (C_2 - \xi_2)^+, I(X_2,W_2,Y_2,\hat{Y}_{R2}|W_2,Q) \} \quad (3.46)$$

$$S_2 + T_2 + T_1 \leq \min \{ I(X_2,W_1;Y_2|Q) + (C_2 - \xi_2)^+, I(X_2,W_1,Y_2,\hat{Y}_{R2}|Q) \} \quad (3.47)$$

The achievable rate region consists of all rate pairs $(R_1, R_2)$ such that $R_1 = S_1 + T_1$ and $R_2 = S_2 + T_2$. Applying the Fourier-Motzkin elimination procedure [36] gives the achievable rate region (3.27)-(3.33).

### 3.2.3 Constant Gap to Capacity Region

We now specialize to the Gaussian case, and show that under the weak-relay condition (3.1), the achievable rate region and the outer bounds of the Gaussian relay-interference channel with independent relay links can be made to be within a constant gap to each other. The relaying strategy that achieves this capacity to within a constant gap turns out to be naturally suited for the Gaussian relay-interference channel with a degraded broadcasting relay, thus establishing the constant-gap result for the broadcasting-relay case as well.

Assuming Gaussian codebooks and a Gaussian quantization scheme, the key design parameters are the choice of common-private power splitting ratio at the transmitters and the quantization level at the relay. Our choice of design parameters is inspired by that of Wang and Tse [41], where the capacity region of a Gaussian interference channel with rate-limited receiver cooperation is characterized to within a constant gap. Two key observations are made in [41]. First, the Etkin-Tse-Wang strategy ([7]) of setting the private power to be at the noise level at the opposite receiver is used. Second, the relay quantizes its observation at the private signal level of the source in order to preserve all the information of interest to the destinations. At the destinations, a joint decoding (see [14, 18, 64, 65]) is performed to recover the source messages.

Consider now the optimal power splitting in a Gaussian relay-interference channel with independent relay links. The Etkin-Tse-Wang strategy, i.e., setting private powers $P_{1p}$ as

$$P_{1p} = \min \{ 1, h^{-2}_{12} \}, \quad P_{2p} = \min \{ 1, h^{-2}_{21} \}. \quad (3.48)$$

\(^1\)Note that time-sharing is implicitly used to arrive at (3.27) and (3.28).
is near optimal for the Gaussian interference channel with conferencing receivers, but is not necessarily so for relay-interference channel shown in Fig. 3.2 in its most general form. Consider an extreme scenario of $C_1, C_2 \to \infty$. In this case, the relay fully cooperates with both receivers, so the relay-interference channel becomes a single-input multiple-output (SIMO) interference channel with two antennas at the receivers. Thus, the private powers at the transmitters must be set at the effective noise level for the two-antenna output in order to achieve capacity to within constant bits [63] [66], i.e.,

$$P_{1p} = \min \{ 1, (g_1^2 + h_{12}^2)^{-1} \}, \quad P_{2p} = \min \{ 1, (g_2^2 + h_{21}^2)^{-1} \}. \quad (3.49)$$

When $C_1$ and $C_2$ are finite, the optimal power splitting strategy is expected to be a function of not only $h_{12}$ and $h_{21}$ but also $g_1$, $g_2$, $C_1$ and $C_2$, lying somewhere between (3.48) and (3.49).

This complication can be avoided, however, if we focus on the weak-relay regime (3.1), namely $|g_1| \leq \sqrt{\rho} |h_{12}|$ and $|g_2| \leq \sqrt{\rho} |h_{21}|$ for some finite constant $\rho$. In this case, the power splitting (3.48) and (3.49) differ by at most a constant factor. The main result of this section shows that in this weak-relay regime, the Etkin-Tse-Wang’s original power splitting (3.48) is sufficient for achieving the capacity of the Gaussian relay-interference channel to within a constant gap (as a function of $\rho$).

Consider next the optimal quantization level. Applying the insight of [41] to the Gaussian relay-interference channel with independent relay links shown in Fig. 3.2, the quantized messages for receiver 1 and receiver 2 can be expressed as

$$\hat{Y}_{R1} = g_1 U_1 + g_1 W_1 + g_2 W_2 + g_2 U_2 + Z_R + \eta_1 \quad (3.50)$$
$$\hat{Y}_{R2} = g_1 U_1 + g_2 U_2 + g_2 W_2 + g_1 W_1 + Z_R + \eta_2 \quad (3.51)$$

where $W_i$ and $U_i$ are common message and private message respectively, and $\eta_i \sim \mathcal{N}(0, q_i)$ is the quantization noise, $i = 1, 2$. Therefore, a reasonable choice of the quantization levels for receiver 1 and receiver 2 is

$$q_1 = 1 + g_2^2 P_{2p}, \quad q_2 = 1 + g_1^2 P_{1p}. \quad (3.52)$$

Now observe that in the weak-relay regime, i.e., $|g_1| \leq \sqrt{\rho} |h_{12}|$, $|g_2| \leq \sqrt{\rho} |h_{21}|$, the above quantization levels (with Etkin-Tse-Wang power splitting) are between 1 and the constant $\rho + 1$. Thus, we can choose constant quantization levels for both receivers and optimize between 1 and $\rho + 1$.

**Theorem 8 (Constant Gap to the Capacity Region).** For the Gaussian relay-interference channel with independent relay links depicted in Fig. 3.2, in the weak relay regime, using the generalized hash-and-forward relaying scheme with quantization levels $q_1 = q_2 = \frac{\sqrt{\rho^2 + 16\rho + 16} + \rho}{4}$, where $\rho$ is defined in (3.4), and the Han-Kobayashi scheme with Etkin-Tse-Wang power splitting strategy, $X_i = U_i + W_i$, $i = 1, 2$, where $U_i$ and $W_i$ are both Gaussian distributed with the powers of $U_1$ and $U_2$ set according to $P_{1p} = \min \{ 1, h_{12}^2 \}$ and $P_{2p} = \min \{ 1, h_{21}^2 \}$, respectively, the achievable rate region derived in Theorem 7 is within

$$\delta = \frac{1}{2} \log \left( 2 + \frac{\rho + \sqrt{\rho^2 + 16\rho + 16}}{2} \right) \quad (3.53)$$

bits of the capacity region outer bound in Theorem 6.

**Proof.** The main step is to show that using superposition coding $X_i = U_i + W_i$, $i = 1, 2$, where $U_i \sim \mathcal{N}(0,$
\( N(0, P_{tp}) \) and \( W_i \sim N(0, P_{ic}) \) with \( P_{tp} + P_{ic} = 1 \) and the private message powers are set to \( P_{tp} = \min\{1, h_1^2\} \) and \( P_{2p} = \min\{1, h_2^2\} \), each of the achievable rate constraints in (3.27)-(3.33) has a finite gap to the corresponding upper bound in (3.5)-(3.24). Specifically, it is shown in Appendix G that

- Individual rate (3.27) is within
  \[
  \delta_{R_1} = \max\{\alpha(q_1), \beta(q_1)\}
  \]  
  (3.54)
  bits of the upper bound (3.5), where \( \alpha(\cdot) \) and \( \beta(\cdot) \) are as defined in (3.3).

- Individual rate (3.28) is within
  \[
  \delta_{R_2} = \max\{\alpha(q_2), \beta(q_2)\}
  \]  
  (3.55)
  bits of the upper bound (3.6).

- Sum rates (3.29), (3.30), and (3.31) are within
  \[
  \delta_{R_1+R_2} = \max\{\alpha(q_1) + \alpha(q_2), \alpha(q_1) + \beta(q_2), \beta(q_1) + \alpha(q_2), \beta(q_1) + \beta(q_2)\}
  \]  
  (3.56)
  bits of the upper bounds (3.7)-(3.18).

- \( 2R_1 + R_2 \) rate (3.32) is within
  \[
  \delta_{2R_1+R_2} = \max\{2\alpha(q_1) + \alpha(q_2), \alpha(q_1) + \beta(q_1) + \alpha(q_2), 2\beta(q_1) + \alpha(q_2),
  2\alpha(q_1) + \beta(q_2), \alpha(q_1) + \beta(q_1) + \beta(q_2), 2\beta(q_1) + \beta(q_2)\}
  \]  
  (3.57)
  bits of the upper bounds (3.19)-(3.24).

- \( R_1 + 2R_2 \) rate (3.33) is within
  \[
  \delta_{R_1+2R_2} = \max\{\alpha(q_1) + 2\alpha(q_2), \alpha(q_1) + \beta(q_2) + \alpha(q_2), \alpha(q_1) + 2\beta(q_2),
  \beta(q_1) + 2\alpha(q_2), \beta(q_1) + \beta(q_2) + \alpha(q_2), \beta(q_1) + 2\beta(q_2)\}.
  \]  
  (3.58)
  bits of the upper bounds not shown explicitly but can be obtained by switching the indices 1 and 2 of (3.19)-(3.24).

Since \( \alpha(\cdot) \) is a monotonically increasing function and \( \beta(\cdot) \) is a monotonically decreasing function. In order to minimize the above gaps over \( q_1 \) and \( q_2 \), the quantization levels should be set such that

\[
\alpha(q_1^*) = \beta(q_1^*) = \alpha(q_2^*) = \beta(q_2^*),
\]  
(3.59)

which results in \( q_1^* = q_2^* = \sqrt{\rho_2^2 + 16 \rho_2 + 16 - \rho} \). Substituting \( q_1^* \) and \( q_2^* \) into the above gaps, we prove that the constant gap is \( \delta \) bits per dimension, where \( \delta \) is as shown in (3.53).

Note that the finite capacity gap is an increasing function of \( \rho \): smaller \( \rho \) results in a smaller gap. In the case that \( \rho = 1 \), i.e., \( |g_1| \leq |h_{12}| \) and \( |g_2| \leq |h_{21}| \), the optimal quantization levels are \( q_1^* = q_2^* = \sqrt{\frac{33}{4}} - 1 \), and the gap to the capacity is given by \( \frac{1}{2} \log \left( \frac{2 + \sqrt{33}}{4} \right) \approx 1.2128 \) bits.
Chapter 3. Gaussian Interference Channel with a Degraded Broadcasting Relay

3.2.4 Gaussian Relay-Interference Channel with a Broadcasting Relay

The GHF relaying scheme originally stated in Theorem 7 requires independent relay links. As shown in Fig. 3.3(a), the relay observation $Y_R^n$ undergoes two separate quantization and binning processes to obtain the two messages for the two receivers. However, in the weak-relay regime, Theorem 8 shows that using an identical quantization level for the two receivers is without loss of approximate optimality, thus a common quantization process can be shared between the two receivers. Further, since the same $\hat{Y}_R^n$ is binned into bins of sizes $2^{nC_1}$ and $2^{nC_2}$, this is equivalent to first binning $\hat{Y}_R^n$ into $2^{nC_1}$ bins (assuming $C_1 \leq C_2$) then further binning each bin into $2^{n(C_2-C_1)}$ sub-bins, as shown in Fig. 3.3(b). The message sent to receiver 2 can be thought of as the refinement of the message sent to receiver 1. This is exactly the incremental relaying strategy we seek for the Gaussian interference channel with a broadcasting relay, where the message to receiver 1 is a degraded version of the message to receiver 2. Finally, if $C_1 = C_2 = C$, the relay-interference channel reduces to the universal relaying scheme studied in [57], where a digital link is shared between the relay and the receivers, as shown in Fig. 3.3(c).

Corollary 2. The constant-gap-to-capacity result stated in Theorem 8 holds also for the Gaussian relay-interference channel with degraded broadcasting relay links, where (assuming $C_1 \leq C_2$) the message sent through the link with capacity $C_1$ must be a degraded version of the message sent through the link with capacity $C_2$. 

Figure 3.3: Evolution of the generalized hash-and-forward relay scheme
3.2.5 Comments on the Strong Relay Regime

The constant-gap result in this chapter holds only in the weak-relay regime of $|g_1| \leq \sqrt{\rho}|h_{12}|$ and $|g_2| \leq \sqrt{\rho}|h_{21}|$, where $\rho$ is finite. The main difficulty in extending this result to the general case stems from both the optimal choice of the Han-Kobayashi power splitting, and in the GHF relaying strategy. As mentioned earlier, the Etkin-Tse-Wang power splitting is no longer optimal when the relay links $g_i$, $i = 1, 2$ grow unboundedly stronger than the interference links $h_{12}$ and $h_{21}$. Further, GHF may not be an appropriate relay strategy. To see this, assume a channel model with separate relay links, and consider an extreme scenario where the relay links $g_i$, $i = 1, 2$ go to infinity while all other channel parameters are kept constant. This special case is known as the cognitive relay-interference channel. The capacity region outer bound of Theorem 6 for this case reduces to

$$R_1 \leq \frac{1}{2} \log (1 + \text{SNR}_1) + C_1$$

$$R_2 \leq \frac{1}{2} \log (1 + \text{SNR}_2) + C_2$$

$$R_1 + R_2 \leq \frac{1}{2} \log (1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) + C_1 + C_2$$

$$R_1 + R_2 \leq \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}\right) + C_1 + C_2$$

$$R_1 + R_2 \leq \frac{1}{2} \log (1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1}) + \frac{1}{2} \log (1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}) + C_1 + C_2$$

$$2R_1 + 2R_2 \leq \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) + \frac{1}{2} \log \left(1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}\right)$$

$$\quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) + 2C_1 + C_2$$

$$R_1 + 2R_2 \leq \frac{1}{2} \log (1 + \text{SNR}_2 + \text{INR}_1) + \frac{1}{2} \log \left(1 + \text{INR}_2 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right)$$

$$\quad + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}\right) + C_1 + 2C_2,$$  \hspace{1cm} (3.66)

which is in fact the outer bound of the underlying interference channel expanded by $C_1$ bits in the $R_1$ direction and $C_2$ in the $R_2$ direction. In this special case, a decode-and-forward strategy can easily achieve the outer bounds to within a constant gap. This is because the relay is capable of decoding all the source messages, so it can simply forward the bin indices of the privates messages to achieve $(R_1 + C_1, R_2 + C_2)$ for any achievable rate pair $(R_1, R_2)$ in the absence of the relay. Etkin-Tse-Wang power splitting with decode-and-forward then achieves the outer bound to within a constant gap.

3.2.6 Generalized Degrees of Freedom

We can gain further insights into the effect of relaying on the Gaussian interference channel by analyzing the GDoF of the sum rate in the symmetric channel setting. Consider the case where $\text{INR}_1 = \text{INR}_2 = \text{INR}$, $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$, $\text{SNR}_{r1} = \text{SNR}_{r2} = \text{SNR}_r$, and $C_1 = C_2 = C$. In the high SNR regime, similar
to [7, 41], define

\[
\alpha := \lim_{\text{SNR} \to \infty} \frac{\log \text{INR}}{\log \text{SNR}},
\]
\[
\beta := \lim_{\text{SNR} \to \infty} \frac{\log \text{SNR}}{\log \text{SNR}},
\]
\[
\kappa := \lim_{\text{SNR} \to \infty} \frac{C}{\frac{1}{2} \log \text{SNR}}.
\]

The GDoF of the sum capacity is defined as

\[
d_{\text{sum}} = \lim_{\text{SNR} \to \infty} \frac{C_{\text{sum}}}{\frac{1}{2} \log \text{SNR}}\bigg|_{\text{fixed } \alpha, \beta, \kappa}
\]

As a direct consequence of the constant-gap result, \(d_{\text{sum}}\) can be characterized in the weak relay regime as follows.

**Corollary 3.** For the symmetric Gaussian relay-interference channel in the weak relay regime (i.e., \(\beta \leq \alpha\)), the GDoF of the sum capacity is given by the following. When \(0 \leq \alpha < 1\)

\[
d_{\text{sum}} = \min \{2 - \alpha + \min \{\beta, \kappa\}, 2 \max \{\alpha, 1 - \alpha\} + 2\kappa, 2 \max \{\alpha, 1 + \beta - \alpha\}\}.
\]  

When \(\alpha \geq 1\)

\[
d_{\text{sum}} = \min \{\alpha + \kappa, \alpha + \beta, 2(1 + \kappa), 2 \max \{1, \beta\}\}.
\]

Note that when \(\alpha = 1\), the GDoF of the sum capacity is in fact not well defined. This is because both \(\text{INR} = \gamma \text{SNR}\) (where \(\gamma \neq 1\) is finite) and \(\text{INR} = \text{SNR}\) result in the same \(\alpha = 1\). However, in the case of \(\text{INR} = \text{SNR}\), the channel becomes ill conditioned, i.e. \(\phi_1 = \phi_2 = 0\), which results in a \(d_{\text{sum}}\) other than the one in (3.72). In other words, multiple values of \(d_{\text{sum}}\) correspond to the same \(\alpha = 1\). This is similar to the situation of [41, Theorem 7.3]. Applying the similar argument that the event \(\{\text{INR} = \text{SNR}\}\) is of zero measure, we have the GDoF of the sum capacity as shown in (3.72) almost surely.

When the relay links and the interference links share the same channel gain, i.e. \(\alpha = \beta\), the GDoF of the sum capacity reduces to

\[
d_{\text{sum}} = \min \{2 + \kappa - \alpha, 2 \max \{\alpha, 1 - \alpha\} + 2\kappa, 2\}
\]

for \(0 \leq \alpha < 1\), and

\[
d_{\text{sum}} = \min \{\alpha + \kappa, 2(1 + \kappa), 2\alpha\},
\]

for \(\alpha \geq 1\). Interestingly, this is the same as the sum capacity (in GDoF) of the Gaussian interference channel with rate-limited receiver cooperation [41]. Therefore, the same sum capacity gain can be achieved with either receiver cooperation or with an independent in-band-reception and out-of-band-transmission relay assuming that the source-relay links are the same as the interfering links of the underlying interference channel (i.e. \(\alpha = \beta\)).

Fig. 3.4 shows the GDoF gain due to the relay for the \(\alpha = \beta\) case. There are several interesting features. When \(\kappa \leq \frac{1}{2}\), the GDoF curve remains the “W” shape for the conventional Gaussian interference
3.2.7 Interpretation via the Deterministic Relay Channel

In the Han-Kobayashi framework, each input signal of the interference channel consists of both a common message and a private message. The sum-capacity gain due to the relay in the relay-interference channel therefore in general includes improvements in both the common and the private message rates. This section illustrates that in the asymptotic high SNR regime, the rate improvement can often be interpreted as either a private rate gain alone, or a common rate gain alone. Further, the one-bit-per-relay-bit or the two-bits-per-relay-bits GDoF improvement shown in the previous section can be interpreted using a deterministic relay model. The rest of this section illustrates this point for the symmetric Gaussian relay-interference channel in the \( \alpha = \beta \) and \( \kappa \leq \frac{1}{2} \) case as an example.

Very Weak Interference Regime

For the symmetric Gaussian interference channel, in the very weak interference regime of \( 0 \leq \alpha \leq \frac{1}{2} \), common messages do not carry any information (although it can be assigned nonzero powers as in the
At receiver 1, $X_2$ is treated as noise.

At receiver 2, $X_1$ is treated as noise.

Figure 3.5: Asymptotic deterministic relay channels in the very weak interference regime $\kappa \leq \alpha \leq \frac{1}{2}$.

Etkin-Tse-Wang power splitting strategy). Setting $X_1$ and $X_2$ to be private messages only is capacity achieving in terms of GDoF [3–5, 7].

Assigning $X_1$ and $X_2$ to be private only is also optimal for GDoF for the symmetric Gaussian relay-interference channel in the very weak interference regime. This is because when $X_1$ and $X_2$ are both private messages and are treated as noises at $Y_2$ and $Y_1$ respectively, the relay-interference channel asymptotically becomes two deterministic relay channels in the high SNR regime. Consider the relay operation for $Y_1$ as illustrated in Fig. 3.5(a). When noise variances of $Z_1$ and $Z_R$ go down to zero, the observation at the relay becomes $Y_R = gX_1 + gX_2$ and the received signal at receiver 1 becomes $Y_1 = h_dX_1 + h_cX_2$. In this case, the relay’s observation is a deterministic function of $X_1$ and $Y_1$, i.e. $Y_R = gX_1 + \frac{g}{h_c}(Y_1 - h_dX_1)$. Thus $X_1$ and $Y_1$, along with the relay $Y_R$ form a deterministic relay channel of the type studied in [67]. According to [67], the achievable rate of user 1 is given by

$$R_1 = \min \left\{ I(X_1; Y_1, Y_R), I(X_1; Y_1) + C \right\} = \min \left\{ \frac{1}{2} \log(1 + h_d^2), \frac{1}{2} \log \left( 1 + \frac{h_d^2}{h_c^2} \right) + C \right\} \rightarrow \min \{1, 1 - \alpha + \kappa\},$$

resulting in one-bit improvement for each relay bit in the regime $\kappa \leq \alpha \leq \frac{1}{2}$. Similarly, as illustrated in Fig. 3.5(b), $X_2$, $Y_2$, and $Y_R$ form another deterministic relay channel with $X_2$ as the input, $Y_2$ as the output, and $Y_R$ as the relay. Thus, the achievable rate of user 2 is the same as user 1, resulting in the same one-bit-per-relay-bit improvement. Further, as shown in [67], a hash-and-forward relay strategy achieves the capacity for deterministic relay channels. As the hashing operation is the same for both case, the same relay bit can therefore benefit both receivers at the same time, resulting in two-bit increase in sum capacity for one relay bit, as first pointed out in [18].

Moderately Weak and Strong Interference Regimes

The above interpretation, which states that the GDoF improvement in the very weak interference regime comes solely from the private rate gain, is not the only possible interpretation. The rate gain can also be interpreted as improvement in common information rate — an interpretation that applies not only to the very weak interference regime, but in fact to all regimes (for the symmetric rate with symmetric channels). In the following, we illustrate this point by focusing on a two-stage Han-Kobayashi strategy,
where common messages are decoded first, then the private messages. This is the same two-stage Han-Kobayashi scheme used in [7] for the Gaussian interference channel without the relay.

Specifically, the relay uses the same GHF relaying strategy as in Theorem 8, but it is now designed to help the common messages only. Here, both common messages $W^n_1$ and $W^n_2$ are decoded and subtracted at both receivers with the help of the GHF relay (while treating private messages as noise) first, the private messages are then decoded at each receiver treating each other as noise. The decoding of the private message at the second stage results in

$$R_u = \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_p}{1 + \text{INR}_p} \right)$$

$$\rightarrow \max\{0, 1 - \alpha\}$$

(3.76)

Note that the relay does not help the private rate.

In the common-message decoding stage, $W^n_1$ and $W^n_2$ are jointly decoded at both receiver 1 and receiver 2 with the help of the GHF relay. As a result, $(W^n_1, W^n_2, Y^n_1, Y^n_R)$ forms a multiple-access relay channel at receiver 1 with $W^n_1, W^n_2$ as the inputs, $Y^n_1$ as the output and $Y^n_R$ as the relay. The achievable rate region of such a multiple-access channel with a GHF relay is given by

$$R_{w1} \leq I(W_1; Y_1 | W_2) + \min\left\{ (C - \xi)^+, I(W_1; \hat{Y}_R | Y_1, W_2) \right\}$$

$$R_{w2} \leq I(W_2; Y_1 | W_1) + \min\left\{ (C - \xi)^+, I(W_2; \hat{Y}_R | Y_1, W_1) \right\}$$

$$R_{w1} + R_{w2} \leq I(W_1, W_2; Y_1) + \min\left\{ (C - \xi)^+, I(W_1, W_2; \hat{Y}_R | Y_1) \right\}$$

With the Etkin-Tse-Wang input strategy (i.e. $P_{1p} = \min\{1, h_{12}^{-2}\}, P_{2p} = \min\{1, h_{21}^{-2}\}$) and the GHF relaying scheme with $q_1 = q_2 = \sqrt{\frac{\rho^2 + 16\rho + 16 - \rho}{4}}$, it can be shown that the common-message rate region for the receiver 1 in the high SNR regime in term of GDoF is given as follows. When $0 \leq \alpha \leq 1$

$$R_{w1} \leq \alpha$$

$$R_{w2} \leq \min\{\alpha, \kappa + \max\{2\alpha - 1, 0\}\}$$

$$R_{w1} + R_{w2} \leq \alpha + \min\{\alpha, \kappa\}$$

When $\alpha \geq 1$

$$R_{w1} \leq \min\{\alpha, 1 + \kappa\}$$

$$R_{w2} \leq \alpha$$

$$R_{w1} + R_{w2} \leq \alpha + \kappa$$

Due to symmetry, the rate region for the multiple-access relay channel at receiver 2 can be obtained by switching the indices 1 and 2.

Note that in suitable interference regimes, both the individual rate and the sum rate can potentially be increased by one bit for each relay bit. This is again a consequence of the fact that the relay operation has a deterministic relay channel interpretation in the high SNR regime. For example, in the strong interference regime where $1 \leq \alpha \leq 2 + \kappa$, the sum rate of the multiple-access relay channel benefits by one bit for each relay bit in the high SNR regime as shown in Fig. 3.6(a). In the very strong interference
Asymptotic deterministic relay channels in the strong and very strong interference regimes.

Figure 3.6: Asymptotic deterministic relay channels in the strong and very strong interference regimes.

Now, the achievable rates of common messages can be obtained by intersecting the two rate regions. Taking the achievable rates of private messages in (3.76) into account, it is easy to verify that this two-stage Han-Kobayashi scheme achieves the sum capacity in (3.73) and (3.74). As depicted in Fig. 3.4, the sum-capacity gain due to the relay can be one-bit-per-bit or two-bits-per-bit. In the following, we demonstrate in Fig. 4.4 how these gains are obtained by pictorially showing the intersection of the two common-message rate regions for different values of $\alpha$.

- When $\alpha \leq \kappa$, as can be seen from Fig. 3.7(a), the two rate regions are identical and are both given by $\{(R_{w1}, R_{w2}) : R_{w1} \leq \alpha, R_{w2} \leq \alpha\}$. The intersection of the two is the same rectangle with the top-right corner located at $(\alpha, \alpha)$. This gives a $2\alpha$-bit gain over the baseline, which is located at the origin.

- As $\alpha$ increases to $\kappa \leq \alpha \leq \frac{1}{2}$, the baseline rate pair is still at the origin. With the help of the relay, the two common-message rate regions become rectangles $\{(R_{w1}, R_{w2}) : R_{w1} \leq \alpha, R_{w2} \leq \kappa\}$ and $\{(R_{w1}, R_{w2}) : R_{w1} \leq \kappa, R_{w2} \leq \alpha\}$ respectively. As shown in Fig. 3.7(b), the intersection of the two gives a square with the top-right corner located at $(\kappa, \kappa)$. As a result, the sum-capacity gain is $2\kappa$ bits.

- As $\alpha$ increases to $\frac{1}{2} \leq \alpha \leq 1$, the common-message rate regions at receivers 1 and 2 become pentagons. However, depending on the value of $\alpha$, the sum rate improves by different amounts. When $\alpha \leq \frac{2\kappa}{3}$, as shown in Fig. 3.7(c), the intersection of the two pentagon regions gives a square with the top-right corner located at $(2\alpha - 1 + \kappa, 2\alpha - 1 + \kappa)$. Compared with $(2\alpha - 1, 2\alpha - 1)$ achieved without the relay, a sum-capacity gain of $2\kappa$ bits is obtained. However, when $\alpha \geq \frac{2\kappa}{3}$, as depicted in Fig. 3.7(d), the intersection of the two rate regions is still a pentagon with the sum-capacity limited by $R_{w1} + R_{w2} \leq 2 - \alpha + \kappa$. In this case, depending on the value of $\alpha$, the sum-rate gain is $2 - 3\alpha + \kappa$ bits when $\frac{2\kappa}{3} \leq \alpha \leq \frac{2}{3}$, and is $\kappa$ bits when $\frac{2}{3} \leq \alpha \leq 1$. (The latter case is shown in Fig. 3.7(d).)

- When $1 \leq \alpha \leq 2 + \kappa$, the common-message rate regions are again pentagons and the interpretation is similar to the $\frac{2\kappa}{3} \leq \alpha \leq 1$ case. Fig. 3.7(e) shows an example of $1 \leq \alpha \leq 1 + \kappa$. In this case, the two rate regions are identical pentagons with the sum capacity limited by $R_{w1} + R_{w2} \leq \alpha + \kappa$. 

Figure 3.7: Generalized-degree-of-freedom gain due to relaying is roughly $\kappa$ or $2\kappa$ depending on how the two common-message multiple-access regions are intersected.

Compared with the baseline sum capacity, a $\kappa$-bits gain is obtained. When $1 + \kappa \leq \alpha \leq 2 + \kappa$, the intersection of the two common-message rate regions again gives a sum-capacity of $\alpha + \kappa$. However, since the baseline sum capacity becomes saturated when $\alpha \geq 2$ [7, 23, 24], the sum-capacity gain over the baseline is $\kappa$ bits when $1 \leq \alpha \leq 2$, and is $\alpha + \kappa - 2$ bits when $2 \leq \alpha \leq 2 + \kappa$.

- Finally, $\alpha \geq 2 + \kappa$ falls into the very strong interference regime. The two common-message rate regions are identical in this case. The intersection is a rectangle with the top-right corner located at $(1 + \kappa, 1 + \kappa)$ as shown in Fig. 3.7(f). The sum-capacity gain is thus $2\kappa$ bits in the very strong interference regime.

## 3.3 Gaussian Relay-Interference Channel With a Single Digital Link

The result of the previous section shows that for the symmetric channel, the sum-capacity improvement can be thought as coming solely from the improvement of the common message rate, or in a very weak interference regime as coming solely from the improvement of the private message rates. Thus, the function of the relay for the symmetric rate in symmetric channel is solely in forwarding useful signals. This interpretation does not necessarily hold for the asymmetric cases. In this section, we study a particular asymmetric channel to illustrate the composition of the sum-capacity gain. We are motivated
by the fact that the relay’s observation in a relay-interference channel is a linear combination of the intended signal and the interfering signal. Clearly, forwarding the intended signal and the interfering signal can both be beneficial (e.g. [47]). This section illustrates that depending on the different channel parameters, the sum-rate gain from forwarding both intended signal and interference signal happens to be the same as that of forwarding intended signal only or forwarding interference signal only.

Specifically, we focus on a particular asymmetric model as shown in Fig. 3.8, where the digital relay link exists only for receiver 1, and not for receiver 2, i.e., $C_2 = 0$. This section first derives a constant-gap-to-capacity result for this channel. Note that this channel is a special case of the general channel model studied in the previous section, but the constant-gap-to-capacity result can be established in this special case for a broader set of channels. Unlike the weak-relay assumption $|g_1| \leq \sqrt{\rho} |h_{12}|$ and $|g_2| \leq \sqrt{\rho} |h_{21}|$ made in the previous section, this section assumes that $|g_2| \leq \sqrt{\rho} |h_{21}|$ only with no constraints on $g_1$ or $h_{12}$. Under this channel setup, it can be shown that in the high SNR regime, the sum capacity improvement can also be obtained by forwarding intended signal only or forwarding interference signal only.

### 3.3.1 Capacity Region to within Constant Gap

Since the channel model studied in Fig. 3.8 is a special case of the general Gaussian relay-interference channel, we first simplify the achievable rate region in Theorem 7 to the following corollary by setting $C_2 = 0$. The only difference in the coding scheme is that instead of performing two quantizations as in the general relay-interference channel, the relay in Fig. 3.8 does one quantization of the received signal $Y_R$ into $\hat{Y}_{R1}$ and sends the bin index of $\hat{Y}_{R1}$ to receiver 1 through the digital link $C_1$.

**Corollary 4.** For the Gaussian relay-interference channel with a single digital link as shown in Fig. 3.8,
the following rate region is achievable:

\[
0 \leq R_1 \leq d_1 + \min \left\{ (C_1 - \xi_1)^+, \Delta \tilde{a}_1 \right\} \tag{3.77}
\]

\[
0 \leq R_2 \leq d_2 \tag{3.78}
\]

\[
R_1 + R_2 \leq a_1 + g_2 + \min \left\{ (C_1 - \xi_1)^+, \Delta \tilde{a}_1 \right\} \tag{3.79}
\]

\[
R_1 + R_2 \leq a_2 + g_1 + \min \left\{ (C_1 - \xi_1)^+, \Delta \tilde{g}_1 \right\} \tag{3.80}
\]

\[
R_1 + R_2 \leq e_1 + e_2 + \min \left\{ (C_1 - \xi_1)^+, \Delta \tilde{e}_1 \right\} \tag{3.81}
\]

\[
2R_1 + R_2 \leq a_1 + g_1 + e_2 + \min \left\{ 2(C_1 - \xi_1)^+, (C_1 - \xi_1)^+ + \Delta \tilde{a}_1, \Delta \tilde{a}_1 + \Delta \tilde{g}_1 \right\} \tag{3.82}
\]

\[
R_1 + 2R_2 \leq a_2 + g_2 + e_1 + \min \left\{ (C_1 - \xi_1)^+, \Delta \tilde{e}_1 \right\} \tag{3.83}
\]

where all the parameters are as defined in Theorem 7.

The proof follows directly from Theorem 7. Note that in (3.82), we apply the fact that \( \Delta \tilde{a}_1 \leq \Delta \tilde{g}_1 \). Likewise, the capacity region outer bound in Theorem 6 also simplifies when \( C_2 = 0 \). We can now prove the following constant-gap theorem for the Gaussian relay-interference channel with a single digital link.

**Theorem 9.** For the Gaussian relay-interference channel with a single digital link as depicted in Fig. 3.8, with the same signaling strategy as in Theorem 8, i.e., a combination of the Han-Kobayashi scheme with Etkin-Tse-Wang power splitting strategy and the GHF relaying scheme with the fixed quantization level \( q_1 = \sqrt{\rho^2 + 16\rho + 16 - \rho} \), in the weak-relay regime of \( |g_2| \leq \sqrt{|h_{21}|} \), the achievable rate region in Corollary 4 is within \( \delta \) bits of the capacity region outer bound in Theorem 6 (with \( C_2 \) set to zero), where \( \delta \) is defined in Theorem 8.

**Proof.** Although the signalling scheme and the constant gap result resemble those of Theorem 8, Theorem 9 is not simply obtained by setting \( C_2 = 0 \) in Theorem 8, since the weak-relay condition has been relaxed. In the following, we prove the constant-gap result by directly comparing each achievable rate expression with its corresponding upper bound.

Applying the inequalities of Lemma 1 and following along the same lines of the proof of Theorem 8 in Appendix G, it is easy to show that each of the achievable rates in (3.77)-(3.83) achieves to within a constant gap of its corresponding upper bound in Theorem 6 (with \( C_2 \) set to zero). The constant gaps are shown as follows:

- Individual rate (3.77) is within
  \[
  \delta_{R_1} = \max \{ \alpha(q_1), \beta(q_1) \} \tag{3.84}
  \]
  bits of (3.5).

- Individual rate (3.78) is within
  \[
  \delta_{R_2} = \frac{1}{2} \tag{3.85}
  \]
  bits of (3.6).

- Sum rates (3.79), (3.80) and (3.81) are within
  \[
  \delta_{R_1 + R_2} = \frac{1}{2} + \max \{ \alpha(q_1), \beta(q_1) \} \tag{3.86}
  \]
bits of their upper bounds (3.7), (3.14), (3.8), (3.13), (3.9), and (3.15). Specifically,

- The first term of (3.79) is within $\frac{1}{2} + \beta(q_1)$ bits of (3.7). The second term is within $\frac{1}{2} + \alpha(q_1)$ bits of (3.14).
- The first term of (3.80) is within $\frac{1}{2} + \beta(q_1)$ bits of (3.8). The second term is within $\frac{1}{2} + \alpha(q_1)$ bits of (3.13).
- The first term of (3.81) is within $\frac{1}{2} + \beta(q_1)$ bits of (3.9). The second term is within $\frac{1}{2} + \alpha(q_1)$ bits of (3.15).

Therefore, the achievable sum rate in (3.79)-(3.81) is within a constant gap of the sum-rate upper bound specified by (3.7)-(3.18).

- $2R_1 + R_2$ rate (3.82) is within
  \[
  \delta_{2R_1+R_2} = \frac{1}{2} \max \{ \alpha(q_1), \beta(q_1) \} 
  \]  
  bits of the upper bounds (3.19), (3.24), and (3.22). Specifically, the first term of (3.82) is within $\frac{1}{2} + 2\beta(q_1)$ bits of (3.19). The second term is within $\frac{1}{2} + \alpha(q_1) + \beta(q_1)$ bits of (3.24). The third term is within $\frac{1}{2} + 2\alpha(q_1)$ bits of (3.22).

- $R_1 + 2R_2$ rate (3.83) is within
  \[
  \delta_{R_1+2R_2} = 1 + \max \{ \alpha(q_1), \beta(q_1) \} 
  \]  
  bits of the upper bounds

  \[
  2R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 + \text{INR}_1}{1} \right) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{\text{INR}_1} \right) + C_1 
  \]

  \[
  2R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1(1 + \phi^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) + \text{INR}_2 + \text{SNR}_r 
  \]  
  (3.90)

which are not shown explicitly in Theorem 6 but can be obtained by switching the indices 1 and 2 of (3.19) and (3.23) followed by setting $C_2 = 0$.

Since $\alpha(\cdot)$ is an increasing function and $\beta(\cdot)$ is a decreasing function, to minimize the gaps above, we need

\[
\alpha(q_1^*) = \beta(q_1^*) 
\]  
(3.91)

which results in the quantization level $q_1^* = \sqrt{\rho^2 + 16\rho + 16 - \rho}$. With this optimal quantization level applied to the gaps above, we prove that the achievable rate region (3.77)-(3.83) is within

\[
\max \left\{ \frac{1}{2} \cdot \frac{1}{2} \log \left( 2 + \frac{\rho + \sqrt{\rho^2 + 16\rho + 16}}{2} \right) \right\} = \frac{1}{2} \log \left( 2 + \frac{\rho + \sqrt{\rho^2 + 16\rho + 16}}{2} \right) 
\]  
(3.92)

bits of the capacity region.
Table 3.2: Sum-capacity GDoF gain due to the relay for the symmetric Gaussian relay-interference channel with a single digital link for $\alpha = \beta_1 = \beta_2$

![Figure 3.9: Impact of the relay-destination link on sum capacity](image)

**3.3.2 Generalized Degree of Freedom**

We now derive the GDoF of the channel depicted in Fig. 3.8, for the case where the underlying interference channel is symmetric, i.e., INR$_1 = INR_2 = INR$ and SNR$_1 = SNR_2 = SNR$. In the high SNR regime, define

$$\beta_i := \lim_{\text{SNR} \to \infty} \frac{\log \text{SNR}_1}{\log \text{SNR}}, \quad i = 1, 2,$$

$$\kappa_1 := \lim_{\text{SNR} \to \infty} \frac{C_1}{\log \text{SNR}}.$$  

(3.93)  

(3.94)

Applying Theorem 9, we have the following result on the GDoF:

**Corollary 5.** In the weak-relay regime where $\beta_2 \leq \alpha$, the GDoF sum capacity of the symmetric relay-interference channel with a single digital link is given by the following. For $0 \leq \alpha < 1$

$$d_{\text{sum}} = \begin{cases} \min\{2 - \alpha, \max(\alpha, 1 - \alpha) + \kappa_1, \max(\alpha, 1 - \alpha) + \max(\beta_1, 1 + \beta_2 - \alpha, \alpha)\}, & \beta_1 \leq 1 \\ \min\{2 - \alpha + \kappa_1, 2 \max(\alpha, 1 - \alpha) + \kappa_1, \alpha + 1 \beta_1 - \alpha\}, & \beta_1 \geq 1 \end{cases}$$

(3.95)

For $\alpha \geq 1$

$$d_{\text{sum}} = \min\{\alpha, 2 + \kappa\}.$$  

(3.96)

Table 3.2 and Fig. 3.9 illustrate the GDoF gain due to the relay where the direct links, the interference links and the links to the relay are symmetric for both users, and where $\alpha = \beta_1 = \beta_2$. The main feature
here is that there is no gain in sum capacity for $\frac{2}{3} \leq \alpha \leq 2$. In other regimes of $\alpha$, the sum-capacity gain is roughly one bit per relay bit.

### 3.3.3 Signal Relaying vs. Interference Forwarding

In the relay-interference channel, the relay observes a corrupted version of the weighted sum of two source signals $X_1$ and $X_2$, and forwards a description to the receiver. Intuitively, the observations about both source signals are helpful. For the receiver 1, the observation about $X_1$ helps receiver 1 reinforce the signal intended for it; the observation about $X_2$ helps receiver 1 mitigate the interference. The former is referred to as signal relaying, the latter interference forwarding.

In this subsection, we show that the sum-capacity gain is equivalent to that of a single relay link from user 2 ($g_2$), or a single relay link from user 1 ($g_1$), depending on the channel parameters. Toward this end, first, we set the relay link of user 2 to zero, i.e., $g_2 = 0$, and compute the GDoF of the sum capacity. In this case, the sum-capacity gain must be due to forwarding intended signal $X_1$. Similarly, we can also set $g_1 = 0$, and compute the GDoF of the sum-capacity gain due solely to forwarding interference signal $X_2$. By comparing these rates we show that interestingly when the relay link of user 1 is under certain threshold, i.e., $\beta_1 \leq 1 - \alpha + \beta_2$, the sum-capacity gain is equivalent to that by interference forwarding. When $\beta_1 \geq 1 - \alpha + \beta_2$, the sum-capacity gain is equivalent to that by signal relaying.

More specifically, with $g_2 = 0$, the sum-capacity can be computed as

$$d_{SR} = \begin{cases} \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha) + \kappa_1, \max(\alpha, 1 - \alpha) + \max(\beta_1, 1 - \alpha, \alpha)\}, & \beta_1 \leq 1 \\ \min\{2 - \alpha + \kappa_1, 2 \max(\alpha, 1 - \alpha) + \kappa_1, 1 + \beta_1 - \alpha\}, & \beta_1 \geq 1 \end{cases}$$  

(3.97)

Similarly, let $g_1 = 0$. The sum-capacity GDoF obtained by forwarding interference signal is

$$d_{IF} = \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha) + \kappa_1, \max(\alpha, 1 - \alpha) + \max(1 + \beta_2 - \alpha, \alpha)\}.$$  

(3.98)

Comparing (3.95), (3.97), and (3.98), it is easy to verify that

$$d_{\text{sum}} = \begin{cases} d_{IF} & \text{when } \beta_1 \leq 1 + \beta_2 - \alpha \\ d_{SR} & \text{when } \beta_1 \geq 1 + \beta_2 - \alpha \end{cases}$$  

(3.99)

Therefore, when the relay link from user 1 is weak, the sum-capacity gain is equivalent to that of a single relay link from user 2. As the relay link from user 1 grows stronger and crosses a threshold $\beta_1 \geq 1 + \beta_2 - \alpha \triangleq \beta^*_1$, the sum-capacity gain becomes equivalent to that of a single relay link from user 1. Note that this is a GDoF phenomenon in the high SNR regime. In the general SNR regime, the sum-capacity gain contains contributions from both signal relaying and interference forwarding.

To visualize the interaction of signal relaying and interference forwarding, a numerical example is provided in Fig. 3.10. The channel parameters are set to $\alpha = 0.5$, $\beta_2 = 0.2$, and $\kappa_1 = 0.5$. The GDoF of the sum capacity is plotted as a function of $\beta_1$. The sum capacity of the interference channel without the relay serves as the baseline:

$$d_{BL} = \min\{2 - \alpha, 2 \max(\alpha, 1 - \alpha)\}.$$  

(3.100)

Fig. 3.10 shows the sum-capacity gain due to the relay. When $\beta_1 \leq \beta^*_1 = 0.7$, the gain (labeled as $R_1$) is equivalent to that by forwarding interference signal only. When $\beta \geq 0.7$, the gain (labeled as $R_2$) is
equivalent to that by forwarding intended signal only.

3.4 Summary

This chapter investigates GHF as an incremental relay strategy for a Gaussian interference channel augmented with an out-of-band broadcasting relay, in which the relay message to one receiver is a degraded version of the message to the other receiver. We focus on a weak-relay regime, where the transmitter-to-relay links are not unboundedly stronger than the interfering links of the interference channel, and show that GHF achieves to within a constant gap to the capacity region outer bound. Further, in a symmetric setting, each common relay bit can be worth either one or two bits in the sum capacity gain, illustrating the potential for a cell-edge relay in improving the system throughput of a wireless cellular network.

Furthermore, the Gaussian relay-interference channels with a single relay link is also studied. The capacity region is characterized to within a constant-gap for a larger range of channel parameters. It is shown that in the high SNR regime, the sum-capacity improvement is either equivalent to that of a single relay link from user 1 or equivalent to that of a single relay link from user 2.
Chapter 4

Uplink Multicell Processing with Limited Backhaul

This chapter studies an uplink multicell joint processing model in which the base-stations are connected to a centralized processing server via rate-limited digital backhaul links. Unlike previous studies where the centralized processor jointly decodes all the source messages from all base-stations, this chapter proposes a suboptimal achievability scheme in which the Wyner-Ziv compress-and-forward relaying technique is employed on a per-base-station basis, but successive interference cancellation (SIC) is used at the central processor to mitigate multicell interference. This results in an achievable rate region that is easily computable, in contrast to the joint processing schemes in which the rate regions can only be characterized by exponential number of rate constraints. Under the per-base-station SIC framework, this chapter further studies the impact of the limited-capacity backhaul links on the achievable rates and establishes that in order to achieve to within constant number of bits to the maximal SIC rate with infinite-capacity backhaul, the backhaul capacity must scale logarithmically with the signal-to-interference-and-noise ratio (SINR) at each base-station. Finally, this chapter studies the optimal backhaul rate allocation problem for an uplink multicell joint processing model with a total backhaul capacity constraint. The analysis reveals that the optimal strategy that maximizes the overall sum rate should also scale with the log of the SINR at each base-station.

This chapter also proposes a simplified noisy-network-coding scheme where only three base-stations are involved in the decoding of each user, thus reducing the complexity from exponential level to a constant. Numerical experiments are carried out in a 19-cell OFDMA network with 3 sectors per cell, 10 users per sector and 10MHz bandwidth shared among all the cells/sectors. It is shown that when each base-station is equipped with a backhaul of rate 540Mbps, the average per-cell sum rate can be improved by 189% by the SIC implementation of noisy-network-coding, and 76% by the Wyner-Ziv compress-and-forward technique with SIC structure.

4.1 Introduction

In traditional cellular topologies, out-of-cell interference is treated as part of the noise. When base-stations are densely deployed, the cellular network becomes interference limited. Because of this, in current cellular deployment, the per-cell achievable rate is typically much smaller than that of a single
isolated cell. To address this issue, joint multicell processing has been proposed as a viable approach for intercell interference mitigation in future cellular systems. When base-stations share the transmitted and received signals, the codebooks, and the channel state information with each other, it is theoretically possible to perform joint transmission in the downlink and joint reception in the uplink to eliminate out-of-cell interference entirely.

One way to implement multicell joint processing is to deploy a centralized processing server that connects to all the base-stations via backhaul links. When the capacity of the backhaul links is infinite (or sufficiently large), the uplink joint processing problem becomes that of a multiple-access channel, and the downlink becomes a broadcast channel for which the capacity regions can be easily computed. In the uplink, for example, the centralized processor can jointly decode the source messages for all users in different cells, thus eliminating intercell interference completely. This gives rise to the concept of network MIMO [68, 69].

The practical implementation of a network MIMO system, however, must also consider the effect of finite capacity in the backhaul. In this realm, the information theoretical capacity analysis of the multicell cooperation model becomes more involved. This chapter focuses on the uplink of a network MIMO model with limited backhaul. This uplink model, shown in Fig. 4.1, can be thought of as a combination of a multiple-access channel (with remote terminals acting as the transmitters and the centralized processor as the receiver) and a relay channel (with the base-stations acting as the relay).

Although the information theoretical capacity of this uplink model with limited backhaul is still an open problem, considerable progress has been made for the case of the circular Wyner model, in which all the base-stations are placed along a circular array and each mobile terminal transmits only to two neighbouring base-stations. This channel model is comprehensively studied in [70–72]. In [70], two different types of base-station operation are considered. When the base-stations are not capable of decoding, they quantize the received signals and forward to the centralized processor, which then performs joint decoding of both the source messages and quantized codewords. Alternatively, to reduce the burden on the centralized processor and to more efficiently utilize the backhaul links, base-stations can also decode part of the messages of users of their own cell, then forward the decoded data along with the remaining part to the centralized processor, thus shifting the computational burden using decentralized processing [71]. A comprehensive review of these results is available in [69].

The application of the above results to practical systems, however, poses additional challenges. In particular, the achievability rate region of [70, Proposition IV.1], involves $2^L - 1$ rate constraints, each requiring a minimization of $2^L$ terms, where $L$ is the number of users in the uplink multicell model. This complexity makes the evaluation of the achievable rate region computationally prohibitive, when the number of users is large. We remark that the same achievable rate region can also be derived using the technique of noisy network coding [16]. Further, [16] shows that the rate derived in [70] is in fact within constant gap to the outer bound for this channel model if the quantization levels at the base-stations are chosen appropriately. Nevertheless, the same exponential complexity in the evaluation of the achievable rate region remains.

This chapter aims to derive a computationally feasible achievable rate region for the uplink multicell joint processing problem. Toward this end, this chapter focuses on the fully connected multicell model with finite backhaul to the centralized server, and proposes a suboptimal achievability scheme based on successive decoding. In particular, instead of performing joint decoding of the source messages and quantized messages, this chapter applies the Wyner-Ziv compress-and-forward relaying scheme on a per-
base-station basis and performs single-user decoding with successive interference cancellation (SIC) at the centralized server. Although the resulting rate regions are no longer the best achievable, they are more easily computed, and they lead to receiver architectures that are more amenable to implementation.

Under the proposed per-base-station SIC framework, we also ask the following question: How much backhaul capacity is needed to approximately achieve the theoretical successive-decoding rate with infinite backhaul? As the result of this chapter shows, the backhaul rates need to scale logarithmically with the received signal-to-interference-and-noise ratio in order to achieve to within \( \frac{1}{2} \) bit of the successive decoding rates attainable with unlimited backhaul capacity.

Further, this chapter addresses the question of how should the backhaul rates be allocated across the different backhaul links. Under the proposed SIC framework, in order to maximize the sum rate over the entire network under a sum rate constraint on the backhaul capacity, the individual backhaul links should again have rates allocated according to the log of the SINR.

### 4.2 Channel Model

Consider the uplink of a multicell network with joint processing. Assuming that there is only one user operating in each time-frequency resource block in each cell, the multicell network can be modelled by \( L \) users each sending a message to their corresponding base-station. Base-stations essentially serve as intermediate relays for the centralized server, which eventually decodes all the transmit messages. Equivalently, the uplink multicell joint processing model can be thought of as a multiple access channel with \( L \) users sending messages to the destination, i.e., the centralized processor.

As depicted in Fig. 4.1, the uplink joint processing model consists of two parts. The left half is an \( L \)-user interference channel with \( X_i \) as the input signal from the \( i^{th} \) user, \( Y_i \) as the output signal, \( Z_i \) as the additive white Gaussian noise (AWGN), and \( h_{ij} \) as the channel gain from user \( i \) to user \( j \), where \( i, j = 1, 2, \cdots, L \). The right half can be seen as a digital multiple-access channel, where the received signal \( Y_i \) is quantized to \( \hat{Y}_i \) which is then sent to the centralized processor through the digital link of capacity \( C_i, i = 1, 2, \cdots, L \). Without loss of generality, it is assumed that the power of the input signal \( X_i \) is limited by \( P_i \) and that the variances of the receiver noises are identical, i.e., \( E[|X_i|^2] \leq P_i \) and
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Z_i \sim \mathcal{N}(0, N_0), \ i = 1, 2, \cdots, L.

4.3 Wyner-Ziv Compress-and-Forward with Successive Interference Cancellation

This chapter focuses on a compress-and-forward strategy for the uplink joint processing model, i.e., each of the transmitted signal at the input of the digital backhaul links represents a compression index of \( Y_i \).

We are motivated to adopt this the same scheme as in [16, 70], because it can be proved that with joint decoding at the centralized server, this compress-and-forward scheme can achieve the capacity region of a Gaussian relay network to within a constant gap (which is dependent on the size of the network).

Unfortunately, the evaluation of the achievable rate for joint decoding can be difficult. For example, for the uplink joint processing model studied in this chapter, the achievable rate region using noisy network coding [16] or joint decoding [70] requires a minimization of \( 2^L \) terms for each rate constraint, and there are \( 2^L - 1 \) different rate constraints describing the rate region. Even when the size of the network is in a reasonable range, for example as in a 19-cell topology, it is computationally prohibitive to minimize over \( 2^{19} \) terms for \( 2^{19} - 1 \) different rate constraints.

In order to render the study of the performance of multicell joint processing more tractable, in [70] the fully connected uplink channel is simplified to a modified Wyner model (see [11]), where each transmitter-receiver pair only interferes with one neighbouring transmitter-receiver pair, and is subject to interference from only one neighbouring transmitter-receiver pair. Further, certain symmetry is introduced so that all the direct channels are identical, and so are all interfering channels. With this symmetric and less complex cyclic structure, the computation of the sum rate becomes tractable [70].

In this chapter, instead of studying the symmetric Wyner model with joint decoding, we focus on the general multicell model and propose a suboptimal achievability scheme based on the successive decoding of source messages. Based on the observation that the exponential complexity of noisy network coding is introduced by the joint decoding step at the destination, this chapter proposes to apply the Wyner-Ziv compress-and-forward relaying technique [12] at each base-station independently, but use a SIC decoding scheme at the centralized processor, resulting in much simpler rate expressions.

Specifically, assuming a fixed decoding order of decoding first \( X_1 \), then \( X_2, X_3, \cdots, X_L \). The \( k \)th decoding stage for decoding \( X_k \) at the centralized processor works as follows: Upon receiving \( Y_k \), the base-station \( k \) quantizes \( Y_k \) into \( \hat{Y}_k \) using the Wyner-Ziv compress-and-forward technique and sends the description to the destination via digital link \( C_k \). Note that the quantization process at the base-station \( k \) treats interference from all other users as noise. To decode user \( k \)'s message \( X_k \), the centralized processor first decodes the quantization message \( \hat{Y}_k \) upon receiving its description from the digital link \( C_k \), and then decodes the message of user \( k \) using joint typicality between the quantized message \( \hat{Y}_k \) and \( X_k \). Both the decoding of \( \hat{Y}_k \) and \( X_k \) assume the knowledge of previously decoded messages \( X_1, X_2, \cdots, X_{k-1} \) at the centralized processor. In this way, the impact of interference from \( X_1, \cdots, X_{k-1} \) eventually disappears and the effective interference is only due to users not yet decoded, i.e., \( X_j \), for \( j > k \). After decoding \( X_k \), the central processor moves to the next decoding stage treating \( X_k \) as known side information. The following theorem gives the achievable rate using the proposed per-base-station Wyner-Ziv compress-and-forward relaying scheme and SIC decoding scheme.

**Theorem 10.** For the uplink multicell joint processing channel depicted in Fig. 4.1, the following rate
is achievable using Wyner-Ziv compress-and-forward relaying at the base-stations followed by successive interference cancellation at the centralized processor with a fixed decoding order:

$$R_k = \frac{1}{2} \log \frac{1 + \text{SINR}_k}{1 + 2^{-2C_k \text{SINR}_k}},$$  \hfill (4.1)$$

where

$$\text{SINR}_k = \frac{h_{kk}^2 P_k}{N_0 + \sum_{j > k} h_{jk}^2 P_j}.$$  \hfill (4.2)

Proof. In the $k$th stage of the successive-interference-cancellation decoder, $X_1, \ldots, X_{k-1}$ decoded in the previous decoding stages serve as side information for stage $k$. The equivalent channel of user $k$ is depicted in Fig. 4.2. This is a three-node relay channel without the direct source-destination link. Specifically, source signal $X_k$ is sent from the transmitter to the relay, which receives $Y_k$, quantizes into $\hat{Y}_k$ and forwards its description to the centralized processor via the noiseless digital link of capacity $C_k$. At the centralized processor, $X_1, \ldots, X_{k-1}$ serve as side information and facilitate the decoding of $\hat{Y}_k$ and $X_k$. According to [12, Theorem 6], the achievable rate of user $k$ using Wyner-Ziv compress-and-forward can be written as

$$R_k = I(X_k; \hat{Y}_k | X_1, \ldots, X_{k-1})$$  \hfill (4.3)$$

subject to the constraint

$$I(Y_k; \hat{Y}_k | X_1, \ldots, X_{k-1}) \leq C_k.$$  \hfill (4.4)$$

We constrain ourselves to Gaussian input signals and the Gaussian quantization scheme, i.e., $X_k \sim \mathcal{N}(0, P_k)$ and

$$\hat{Y}_k = Y_k + e_k,$$  \hfill (4.5)$$

where $e_k$ is the Gaussian quantization noise following $\mathcal{N}(0, q_k)$, and is independent of everything else. To fully utilize the digital link, it is natural to set

$$I(Y_k; \hat{Y}_k | X_1, \ldots, X_{k-1}) = C_k.$$  \hfill (4.6)$$

Now, substituting $Y_k = \sum_{j=1}^{L} h_{jk} X_j + Z_k$ and $\hat{Y}_k = Y_k + e_k$ into (4.6), we have

$$C_k = \frac{1}{2} \log \left( 1 + \frac{N_0 + \sum_{j > k} h_{jk}^2 P_j}{q_k} \right).$$  \hfill (4.7)$$
Figure 4.3: Achievable rate of user $k$ versus the backhaul capacity $C_k$

which results in the following quantization level that fully utilizes the digital links $C_k$:

$$q^*_k = \frac{N_0 + \sum_{j > k} h^2_{jk} P_j}{2^{2C_k} - 1}.$$  

(4.8)

With the above $q^*_k$, the achievable rate of user $k$ can be calculated as

$$R_k = I(X_k; \hat{Y}_k | X_1, \cdots, X_{k-1})$$

$$= h(\hat{Y}_k | X_1, \cdots, X_{k-1}) - h(\hat{Y}_k | X_1, \cdots, X_k)$$

$$= \frac{1}{2} \log \frac{q^*_k + N_0 + \sum_{j > k} h^2_{jk} P_j}{q^*_k + N_0 + \sum_{j > k} h^2_{jk} P_j}$$

$$= \frac{1}{2} \log \frac{N_0 + \sum_{j > k} h^2_{jk} P_j + h^2_{kk} P_k}{N_0 + \sum_{j > k} h^2_{jk} P_j + 2^{-2C_k} h^2_{kk} P_k}$$

$$= \frac{1}{2} \log \frac{1 + \text{SINR}_k}{1 + 2^{-2C_k} \text{SINR}_k},$$  

(4.9)

which completes the proof. \hfill \square

Note that the above proposed SIC scheme is not the only possibility for simplifying the joint decoding of $\{X_k, \hat{Y}_k\}_{k=1}^L$. The above SIC scheme essentially imposes a decoding order of $\hat{Y}_1$, then $X_1$, then $\hat{Y}_2$, then $X_2$, etc, with previously decoded $X_k$ serving as side information. Alternatively, one may proceed in a two-stage process of decoding all of $\{\hat{Y}_k\}_{k=1}^L$ first, then $\{X_k\}_{k=1}^L$. Each of these two stages can be accomplished in an SIC fashion. The resulting rate can be obtained from expressions in [73] and also
from independent work [74], as
\begin{equation}
I(Y_k; \hat{Y}_k|\hat{Y}_1, \cdots, \hat{Y}_{k-1}) \leq C_k, \quad k = 1, \cdots, L
\end{equation}
and
\begin{equation}
R_k = I(X_k; \hat{Y}_1, \cdots, \hat{Y}_L|X_1, \cdots, X_{k-1}), \quad k = 1, \cdots, L
\end{equation}

The above rate expression can potentially outperform the achievable rate (4.1), because in the above expression each \(X_k\) is decoded based on the quantized observations of all base-stations, rather than just the \(k\)th base-station. For the same reason, the implementation of the above scheme is also expected to be more involved. For the rest of this chapter, we will only focus on the per-base-station SIC decoding of (4.1).

Now back to Theorem 1, the rate expression (4.1) shows how the achievable rates are affected by the limited capacities of the digital backhaul links under the proposed per-base-station SIC decoding framework. Fig. 4.3 plots the achievable rate of \(R_k\) as a function of the backhaul link capacity \(C_k\) with \(\text{SINR}_k\) equal to 20dB. When \(C_k\) is small, \(R_k\) grows almost linearly with \(C_k\), which means that each bit of the backhaul link provides approximately one bit increase in the achievable rate for user \(k\). The digital backhaul is efficiently exploited in this regime. However, as \(C_k\) grows larger, each bit of the backhaul link returns increasingly less achievable rate. On the extreme scenario where the capacity of the digital link is unlimited, i.e. \(C_k = \infty\), \(R_k\) is saturated and approaches \(\frac{1}{2} \log(1 + \text{SINR}_k) \triangleq \bar{R}_k\), which can be thought of as the upper limit for the rate of user \(k\) when the SIC decoder is employed.

Since backhaul links do not come for free, it is natural to ask how large does \(C_k\) need to be to achieve a rate \(R_k\) that is close to the maximal SIC rate with unlimited backhaul? It is easy to see that when \(C_k = \frac{1}{2} \log(1 + \text{SINR}_k)\), \(\bar{R}_k - R_k\) is upper bounded by one half, i.e.,
\begin{align}
\bar{R}_k - R_k &= \frac{1}{2} \log(1 + \text{SINR}_k) - \frac{1}{2} \log \left( \frac{1 + \text{SINR}_k}{1 + 2^{-2C_k \text{SINR}_k}} \right)_{C_k=\frac{1}{2} \log(1+\text{SINR}_k)} \\
&= \frac{1}{2} \log \left( 1 + \frac{\text{SINR}_k}{1 + \text{SINR}_k} \right) \\
&\leq \frac{1}{2}.
\end{align}

Therefore, when the digital link \(C_k = \frac{1}{2} \log(1 + \text{SINR}_k)\), the achievable rate is half a bit away from the SIC upper limit. This is also the point under which the utilization of \(C_k\) is most efficient, as shown in Fig. 4.3.

### 4.4 Optimal Rate Allocation with a Total Backhaul Capacity Constraint

A practical system may have a constraint on the sum capacity of all digital backhaul links. So, it may be of interest to optimize the allocation of backhaul capabilities among the base-stations in order to achieve an overall maximum sum rate under a total backhaul capacity constraint. This optimization problem
can be formulated as the following:

\[
\begin{align*}
\text{maximize} & \quad \sum_{k=1}^{L} R_k \\
\text{subject to} & \quad C_k \geq 0, \quad k = 1, 2, \cdots, L. \quad (P1) \\
& \quad \sum_{k=1}^{L} C_k \leq C
\end{align*}
\]

where \( R_k, k = 1, 2, \cdots, L \) are functions of \( C_k \) as derived in Theorem 10, and \( C > 0 \) is the total available backhaul capacity. The following theorem provides an optimal solution to the above optimization problem.

**Theorem 11.** For the uplink multicell joint processing model shown in Fig. 4.1, with Wyner-Ziv compress-and-forward relaying and successive interference cancellation at the centralized processor, the optimal allocation of backhaul link capacities subject to a total backhaul capacity constraint \( C \) is given by

\[
C_k^* = \max \left\{ \frac{1}{2} \log(\text{SINR}_k) - \alpha, 0 \right\},
\]

where \( \alpha \) is chosen such that \( \sum_{k=1}^{L} C_k^* = C \).

**Proof.** Substituting the rate expression (4.1) for \( R_k \) into the optimization problem (P1), we obtain the following equivalent minimization problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=1}^{L} \frac{1}{2} \log \left( 1 + 2^{-2C_k \text{SINR}_k} \right) \\
\text{subject to} & \quad C_k \geq 0, \quad k = 1, 2, \cdots, L. \quad (P2) \\
& \quad \sum_{k=1}^{L} C_k \leq C
\end{align*}
\]

It can be easily seen that (P2) is a convex optimization problem, as the constraints are linear and the objective function is the sum of convex functions, as can be verified by taking their second derivatives.

Now introducing Lagrange multipliers \( \nu \in \mathbb{R}_{+}^{L} \) for the positivity constraints \( C_k \geq 0, k = 1, 2, \cdots, L \), and \( \lambda \in \mathbb{R}_{+} \) for the backhaul sum-capacity constraint \( \sum_{k=1}^{L} C_k \leq C \), we form the Lagrangian

\[
L(C_k, \nu, \lambda) = \sum_{k=1}^{L} \frac{1}{2} \log \left( 1 + 2^{-2C_k \text{SINR}_k} \right) - \sum_{k=1}^{L} \nu_k C_k + \lambda \left( \sum_{k=1}^{L} C_k - C \right) \quad (4.14)
\]

Taking the derivative of the above with respect to \( C_k \), we obtain the following Karush-Kuhn-Tucker (KKT) condition

\[
- \frac{2^{-2C_k^* \text{SINR}_k}}{1 + 2^{-2C_k^* \text{SINR}_k}} - \nu_k + \lambda = 0,
\]

for the optimal \( C_k^* \), where \( k = 1, 2, \cdots, L \). Note that \( \nu_k = 0 \) whenever \( C_k > 0 \). Now, the optimal \( C_k^* \) must satisfy the backhaul sum-capacity constraint \( \sum_{k=1}^{L} C_k^* \leq C \) with equality, because the objective of the minimization \( R_k \) monotonically increases with \( C_k \). Solving the condition (4.15) together with the
fact that \( \sum_{k=1}^{L} C_k^* = C \) gives the following optimal \( C_k^* \):

\[
C_k^* = \max \left\{ \frac{1}{2} \log \frac{\text{SINR}_k}{\beta}, 0 \right\},
\]

(4.16)

where \( \beta \) is chosen such that \( \sum_{k=1}^{L} C_k^* = C \). This is equivalent to (4.13).

An interpretation of (4.16) is that whenever the SINR of user \( k \) is above a threshold \( \beta \), \( \frac{1}{2} \log \frac{\text{SINR}_k}{\beta} \) bits of the backhaul link should be allocated to user \( k \). Otherwise, this user is not being used in the uplink transmission. This optimal rate allocation is in fact quite similar to the classic water-filling solution for the sum-capacity maximization problem for a parallel set of Gaussian channels, in which more power (backhaul capacity in this case) is assigned to users with a better channel.

When written as (4.13), the optimal backhaul capacity allocation can be interpreted as follows: \( C_k = \frac{1}{2} \log (\text{SINR}_k) \) can be thought of as the nominal optimal backhaul link capacity. If the total backhaul rate is above (or below) the nominal \( \sum_k \frac{1}{2} \log (\text{SINR}_k) \), then the extra capacity must be distributed (or taken away) from each base-station equally. In other words, all base-stations should nominally operate at the point 1/2 bits away from the SIC limit (as shown in Fig. 4.3). If more (or less) backhaul capacity is available than the nominal value, all base-stations should move above (or below) that operating point in the same manner.

Finally, we remark that the decoding order at the centralized processor plays an important role in the optimal rate allocation. Different decoding orders result in different rate expressions in Theorem 10 and thus different rate allocations in Theorem 11, and as a consequence different achievable sum rates. In order to determine the best decoding order that results in the largest sum rate, we need to exhaustively search over \( K! \) different decoding orders. This is a fairly complex and nontrivial problem that is also encountered in other contexts involving successive decoding.

### 4.5 Numerical Results

#### 4.5.1 A Two-User Symmetric Scenario

To obtain further insights on the SIC-based scheme proposed in this chapter, the achievable rate region of Theorem 10 is now compared with that obtained by three other schemes: 1) single-user decoding without joint processing, 2) noisy network coding, 3) joint base-station processing SIC, for a two-user symmetric scenario where \( L = 2 \), \( P_1 = P_2 = N_0 = 1 \), \( h_{11}^2 = h_{22}^2 = h^2 \), \( h_{12}^2 = h_{21}^2 = g^2 \), and \( C_1 = C_2 = C \). Under the symmetric setting, Theorem 10 gives two symmetric achievable rate pairs depending on the decoding order. Time-sharing of the two achievable rate pairs gives a pentagon shaped achievable rate region.

In single-user decoding without joint processing, each receiver decodes its own signal while treating the other user’s signal as noise. This gives the following achievable rate pair

\[
\begin{align*}
R_1 &= \min \left\{ \frac{1}{2} \log \left( 1 + \frac{h_{11}^2}{1 + h_{21}^2} \right), C_1 \right\} \\
R_2 &= \min \left\{ \frac{1}{2} \log \left( 1 + \frac{h_{22}^2}{1 + h_{12}^2} \right), C_2 \right\}
\end{align*}
\]

(4.17)

which in the symmetric setting results in a square shaped achievable rate region with \( (R_1, R_2) \) as the
top-right corner.

When employing joint base-station SIC [74], assume that the quantization levels at both base-stations are identical, i.e., $q_1 = q_2 = q$. We choose a $q$ such that the resulting average backhaul links is $C$. To find such quantization level, first, according to $I(\hat{Y}_1; Y_1) = C_1$, we have

$$C_1 = \frac{1}{2} \log \left( 1 + \frac{1 + h^2 + g^2}{q} \right).$$

(4.18)

Then, from $I(\hat{Y}_2; Y_2|\hat{Y}_1) = C_2$, we obtain

$$C_2 = I(\hat{Y}_2; Y_2|\hat{Y}_1)$$
$$= h(\hat{Y}_2|\hat{Y}_1) - h(\hat{Y}_2|Y_2)$$
$$= \frac{1}{2} \log \frac{\sigma^2_{\hat{Y}_2|\hat{Y}_1}}{q},$$

(4.19)

According the covariance matrix of $(\hat{Y}_2, \hat{Y}_1)$ which is shown as follows:

$$\text{Cov}\left(\hat{Y}_2, \hat{Y}_1\right) = \begin{pmatrix}
1 + h^2 + g^2 + q & 2hg \\
2hg & 1 + h^2 + g^2 + q
\end{pmatrix},$$

(4.20)

the conditional variance of $\sigma^2_{\hat{Y}_2|Y_1}$ can be calculated as

$$\sigma^2_{\hat{Y}_2|Y_1} = 1 + h^2 + g^2 + q - \frac{4h^2g^2}{1 + h^2 + g^2 + q}.$$

(4.21)

Substituting the above conditional covariance into (4.19), we have

$$C_2 = \frac{1}{2} \log \left( 1 + \frac{1 + h^2 + g^2}{q} - \frac{4h^2g^2}{q(1 + h^2 + g^2 + q)} \right).$$

(4.22)

Since it is required that the average backhaul is $C$, applying (4.18) and the above expression of $C_2$, we have the following equation

$$2C = \frac{1}{2} \log \left( 1 + \frac{1 + h^2 + g^2}{q} \right) + \frac{1}{2} \log \left( 1 + \frac{1 + h^2 + g^2}{q} - \frac{4h^2g^2}{q(1 + h^2 + g^2 + q)} \right).$$

(4.23)

Solving the above equation gives the quantization level as follows:

$$q = \frac{1 + h^2 + g^2 + \sqrt{4h^2g^2 + 24C[(1 + h^2 + g^2)^2 - 4h^2g^2]}}{24C - 1}.$$

(4.24)

With this quantization level, the achievable rate using joint base-station SIC can be calculated as:

$$R_1 = I(X_1; \hat{Y}_1, \hat{Y}_2)$$
$$= \frac{1}{2} \log \left( \frac{(1 + h^2 + g^2 + q)^2 - 4h^2g^2}{(1 + h^2 + q)(1 + g^2 + q) - h^2g^2} \right).$$

(4.25)
and

\[ R_2 = I(X_2; \tilde{Y}_1, \tilde{Y}_2|\tilde{Y}_1) = \frac{1}{2} \log \frac{(1 + h^2 + q)(1 + g^2 + q) - h^2 g^2}{(1 + q)^2}. \]  

(4.26)

Switching the order \( R_1 \) and \( R_2 \), we can have the other corner point of the pentagon.

This section also plots the noisy-network-coding rate with the quantization levels at the two base-stations set to the noise variance level \( N_0 \), resulting in an achievable rate region which is within a constant gap to capacity. The achievable rate region of the uplink joint processing model using noisy network coding is given in Lemma 2 in Appendix H.

When specialized to the two-user scenario with \( P_1 = P_2 = N_1 = N_2 = 1, q = N_0 \), there are four cuts for each of the rate constraint. For example, when computing the sum rate, we have \( S = \{1, 2\} \) and four choices of \( T: T = \{\emptyset, \{1\}, \{2\}, \{1, 2\} \} \). When \( T = \{\emptyset\} \), we have

\[ H_{ST^c} = \begin{bmatrix} h & g \\ g & h \end{bmatrix} \]  

(4.27)

Substitute this matrix into Lemma 2, we have

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + h^2 + g^2 + \frac{(h^2 - g^2)^2}{4} \right). \]  

(4.28)

Similarly, when \( T = \{1\} \) or \( T = \{2\} \), \( H_{ST^c} = [h, g]^T \) or \( [g, h]^T \) respectively, the sum rate is bounded by

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{h^2 + g^2}{2} \right) + C - \frac{1}{2}. \]  

(4.29)

When \( T = \{1, 2\}, T^c = \emptyset \), the sum rate is bounded by

\[ R_1 + R_2 \leq 2C - 1. \]  

(4.30)

Similarly, individual achievable rates can also be obtained using Lemma 2. Thus, noisy network coding gives the following achievable rate region:

\[
\left\{ \begin{array}{l}
R_i \leq \min \left\{ \frac{1}{2} \log \left( 1 + \frac{h^2 + g^2}{2} \right), \frac{1}{2} \log \left( 1 + \frac{\min\{h^2, g^2\}}{2} \right) + C - \frac{1}{2} \cdot 2C - 1 \right\}, \ i = 1, 2 \\
R_1 + R_2 \leq \min \left\{ \frac{1}{2} \log \left( 1 + h^2 + g^2 + \frac{(h^2 - g^2)^2}{4} \right), \frac{1}{2} \log \left( 1 + \frac{h^2 + g^2}{2} \right) + C - \frac{1}{2} \cdot 2C - 1 \right\}
\end{array} \right.
\]  

(4.31)

In fact, the quantization level can be further optimized, for example, as in the two-stage process (4.10)-(4.11). We restrict ourselves to symmetric quantization levels here, and refer this as the joint base-station processing SIC region in the plots.

The achievable rate regions obtained above are compared for the following channel settings:

- \( h_{ii}^2 = 30dB, \ h_{ij}^2 = 20dB, \ C_i = 5 \) bits;
- \( h_{ii}^2 = 30dB, \ h_{ij}^2 = 5dB, \ C_i = 5 \) bits;
- \( h_{ii}^2 = 30dB, \ h_{ij}^2 = 20dB, \ C_i = 10 \) bits;
Chapter 4. Uplink Multicell Processing with Limited Backhaul

- \( h_{ii}^2 = 30 \text{dB}, h_{ij}^2 = 20 \text{dB}, C_i = 2 \text{ bits}. \)

Fig. 4.4(a) shows the achievable rate regions in the setting where the direct links are 30dB, the cross links are 20dB, and the backhaul links are 5 bits per channel use. As can be seen from the figure, our proposed SIC scheme expands the baseline achievable rate region by about 2.8 bits on both the individual rates and the sum rate. The noisy-network-coding and the joint base-station processing regions further outperform the proposed scheme in sum rate by about 2.5 bits due to the benefits of joint decoding.

However, when the interfering links are weak, as shown in Fig. 4.4(b) where \( h_{12}^2 = 5 \text{dB} \), all four achievable rate regions are close to each other. This is the regime where treating interference as noise is close to optimal, so multicell processing does not provide significant benefits.

In the above two examples, the capacities of the backhaul links are already quite abundant, since they are set to be the rate supported by the direct links: \( \frac{1}{2} \log(h_{11}^2) \approx 5 \). In Fig. 4.4(c), we further increase the backhaul capacity to 10 bits, and show that doing so does not significantly improve the achievable rate region for either SIC or noisy network coding. Note that in this case, SIC may have higher individual rate than noisy network coding. But, this is because the noisy network coding scheme sets the quantization level to be \( N_0 \). The joint base-station processing scheme with appropriate quantization setting ultimately outperforms both the noisy network coding and the proposed per-base-station SIC schemes.

Finally, we decrease the backhaul capacity from 5 bits to 2 bits in Fig. 4.4(d). Interestingly, this is a situation in which the base-line scheme can outperform per-base-station SIC. But the largest sum rate is still obtained with joint base-station processing.

4.5.2 Multicell OFDMA Network

To further understand the proposed Wyner-Ziv SIC scheme in practical systems, in this section, the performance of the Wyner-Ziv SIC scheme is evaluated in a wireless cellular network with 19 cells, 3 sectors per cell, and 10 users per sector, where an OFDMA multiplexing scheme with 64 tones over a fixed 10Mbps bandwidth (see Fig. 4.5) is employed. The assignments of frequency tones for users within each cell are non-overlapping. As a result, users experience only intercell interference and no intracell interference. Both base-stations and mobile users are equipped with single transmit or receive antenna. Each of the 19 base-stations is connected to the centralized processor via a rate-limited backhaul link. Perfect channel estimation is assumed and is made available to all base-stations and the centralized processor. In the simulation, uniform power allocation of \(-27 \text{dBm/Hz}\) is assumed for all the mobile users. A random scheduler is used for user assignment. The base-station-to-base-station distance is set to be 1.4km corresponding to a typical WiMax or LTE topology. Detailed system parameters are outlined in Table 4.1.

In the numerical simulation, the bandwidth of backhaul links is equially assigned to frequency tones, e.g., if the capacity of the backhaul link is 64Mbps, each frequency tone is then assumed to have a backhaul link of rate \( 64 \text{Mbps}/64 = 1 \text{Mbps} \). CDF plot of per-user rate is used to visualize the performance gain of the proposed schemes over a baseline system, in which each base-station decodes the user belongs to it without joint processing at the centralized processor.

This section also tries to evaluate the noisy-network-coding scheme in the 19-cell topology. However, due to the high complexity of noisy-network-coding scheme, this section proposes a suboptimal implementation of noisy network coding in an SIC fashion, which is referred to as NNC SIC in the following of this chapter. As depicted in Fig. 4.6, three base-stations \( Y_i, Y_k \) and \( Y_m \) are chosen by the centralized
process to decode user k’s message. Three quantization codewords $\hat{Y}_l, \hat{Y}_k$ and $\hat{Y}_m$ are conveyed to the centralized processor through noiseless links of capacity $C_l, C_k$ and $C_m$ respectively. In the decoding stage $k$, the centralized processor makes a decision on user k’s message based on $V_l, V_k, V_m$ and previously decoded messages $X_1, X_2, \ldots, X_{k-1}$, while other messages $X_{k+1}, \ldots, X_L$ not yet decoded are treated as interference. Note that accounting for the effect of those interferences, the noises at $Y_l, Y_k$ and $Y_m$ are in fact correlated.

Again, according to Lemma 2 in Appendix H, it is straightforward to show that the following rate is achievable for user k:

$$R_k = \min_{T \subseteq \{l, k, m\}} \left\{ I \left( X_k; \hat{Y}(T) | X_1, \ldots, X_{k-1} \right) + \sum_{i \in T} C_i - I \left( Y(T); \hat{Y}(T) | X_1, \ldots, X_k, \hat{Y}(T^c) \right) \right\}$$

$$= \min_{T \subseteq \{l, k, m\}} \left\{ \frac{1}{2} \log \left| I + P_k h_k, T h_k, T, \Sigma_1 \right| + \sum_{i \in T} C_i - \frac{1}{2} \log \left| \Sigma_1 - \sum_{i \in T} \Sigma_i q_i \right| \right\}, \quad (4.32)$$

which minimizes over 8 different choices of subset $T$: $\{\phi\}, \{l\}, \{k\}, \{m\}, \{l, k\}, \{l, m\}, \{k, m\},$ and $\{l, k, m\}$.
Figure 4.5: A cellular topology with 19 cells, 3 sectors per cell, and 10 users per sector placed randomly.

Figure 4.6: Noisy network coding with successive interference cancellation.

Note that $T^c$ is the complement of $T$, $q_i$ is the quantization level of base-station $i$, and

\[
\Sigma_{11} = \sum_{j > k} h_{j,T}^T h_{j,T} P_j + \text{diag}(N_0 + q_i)_{i \in T}
\]
\[
\Sigma_{22} = \sum_{j > k} h_{j,T^c}^T h_{j,T^c} P_j + \text{diag}(N_0 + q_i)_{i \in T^c}
\]
\[
\Sigma_{12} = \sum_{j > k} h_{j,T}^T h_{j,T^c} P_j = \Sigma_{21}^T,
\]

and $h_{j,T}$ ($h_{j,T^c}$) is a $|T| \times 1$ ($|T^c| \times 1$) vector representing the channel gains from user $j$ to the base-stations in $T$ ($T^c$). For example, if $T = \{k, m\}$, then $T^c = \{l\}$, $h_{j,T} = [h_{j,k}, h_{j,m}]^T$ and $h_{j,T^c} = [h_{j,l}]^T$.

One of the most crucial parameters in the noisy-network-coding technique is the quantization level, which determines how fine the quantization procedure is performed and how much information is conveyed to the destination. A bad choice of quantization levels may result in a performance far away from optimal. For the Gaussian relay network with uncorrelated receiver noises, setting the quantization level to be the noise power has been proved to be approximately optimal. However in the proposed NNC SIC scheme, as it has been mentioned, the effective noise at the three chosen base-stations are in fact corre-
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cellular Layout</td>
<td>Hexagonal, 19 cells, 3 sectors per cell</td>
</tr>
<tr>
<td>BS-to-BS Distance</td>
<td>1.4 km</td>
</tr>
<tr>
<td>Frequency Reuse</td>
<td>1</td>
</tr>
<tr>
<td>Number of User per Sector</td>
<td>10</td>
</tr>
<tr>
<td>Channel Bandwidth</td>
<td>10 MHz</td>
</tr>
<tr>
<td>MS Max PSD</td>
<td>$-27$ dBm/Hz</td>
</tr>
<tr>
<td>Antenna Gain</td>
<td>15 dBi</td>
</tr>
<tr>
<td>Background Noise</td>
<td>$-169$ dBm/Hz</td>
</tr>
<tr>
<td>Noise Figure</td>
<td>7 dB</td>
</tr>
<tr>
<td>BS Rx Antenna Number</td>
<td>1</td>
</tr>
<tr>
<td>MS Tx Antenna Number</td>
<td>1</td>
</tr>
<tr>
<td>Multipath Time Delay Profile</td>
<td>ITU-R M.1225 PedA</td>
</tr>
<tr>
<td>Distance-dependent Path Loss</td>
<td>$128.1 + 37.6 \log_{10}(d)$</td>
</tr>
<tr>
<td>Number of Tones</td>
<td>64</td>
</tr>
<tr>
<td>User Assignment</td>
<td>Random</td>
</tr>
</tbody>
</table>

Table 4.1: Wireless Cellular Model Parameters

lated due to the common interference contributed from users not yet decoded. In this case, noise power may not be the best choice for the quantization level. Although the optimal choice of quantization levels has been studied for the classic three-node relay channel [75] and the four-node line relay network [76], it remains an open problem for general relay networks with correlated receiver noises.

In this section, a brute-force search on the quantization level is performed to locate the most suitable value. Specifically, assume that all the base-stations share the same quantization level $q$, the sum rate is then evaluated for different choices of $q$ ranging from $N_0$ to $N_0 + I_{max}$, where $N_0$ is the noise power and $I_{max}$ is the maximum interference power at the base-stations. To reduce complexity, quantization level $q$ is swept from $N_0$ with a step of 3dB, i.e., $N_0, 2N_0, 4N_0, 8N_0, \cdots$, until $N_0 + I_{max}$ is met. Using this heuristic method, the quantization level that results in the best sum-rate performance can be identified. For example, from Fig. 4.7 it is easy to see that the optimal quantization level is $q^* = 128N_0$ when the backhaul rate is 180Mbps (60Mbps per sector), $q^* = 8N_0$ when the backhaul rate is 360Mbps, and $q^* = N_0$ when the backhaul rate is 540Mbps. In the following numerical simulations, “optimal” quantization levels found through this brute-force search are adopted.

In this section, CDF plots of users’ rates are generated in the 19-cell OFDMA cellular network with different rates of backhaul links. As shown in Fig. 4.8, when the capacity of the backhaul link at each base-station is 180Mbps (60Mbps for each sector), both Wyner-Ziv SIC scheme and NNC SIC scheme perform a bit better than the baseline but still far away from the corresponding CDF curve (dash-dotted and dashed curves respectively) obtained with infinite backhaul rates. One major reason is that the limited backhaul capacity (180Mbps per base-station) becomes the bottleneck for both schemes. Another reason for the limited performance is that backhaul rate is equally assigned to frequency tones. This is obviously not optimal, as we have shown that for the proposed Wyner-Ziv SIC scheme with a total backhaul constraint, adapting the backhaul link with $\text{SINR}_k$ is near optimal. For the NNC SIC scheme, adaptive rate assignment is expected to perform better as well. However, the solution to this problem is nontrivial and is out of the scope of this thesis.

As the capacity of backhaul links increases to 360Mbps per base-station, Fig. 4.9 shows that both Wyner-Ziv SIC scheme and NNC SIC scheme outperform the baseline quite a bit. The solid curve (Wyner-Ziv SIC scheme) is now very close to the dash-dotted curve which is obtained with infinite
backhaul rates. This means that Wyner-Ziv SIC scheme saturates when the rate of the backhaul links is around 360Mbps. However, the NNC SIC scheme, which outperforms both the baseline and the Wyner-Ziv SIC scheme, still has room for improvement. When the rate of backhaul links is increased to 540Mbps, as can be seen from Fig. 4.10, both Wyner-Ziv SIC and NNC SIC schemes saturate.

In order to quantitatively evaluate the performance gain brought by the centralized processor, Table 4.2 shows the average per-cell sum rate obtained by different schemes. As we can see, when the rate of backhaul links is 540Mbps, NNC SIC scheme achieves the most significant rate improvement of 189.35% while Wyner-Ziv SIC produces a rate improvement of 76.27%. When the cellular network is rate-limited on backhaul, i.e., 180Mbps per base-station, Wyner-Ziv SIC scheme and NNC SIC scheme produce a sum-rate gain of 30.39% and 57.70% respectively. When the backhaul rate is around 200Mbps to 500Mbps, NNC SIC scheme is expected to double the sum-rate offered by the baseline system.

<table>
<thead>
<tr>
<th>Per-cell Backhaul (Mbps)</th>
<th>Wyner-Ziv SIC (Mbps)</th>
<th>Improvement</th>
<th>NNC SIC (Mbps)</th>
<th>Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>72.38</td>
<td>30.39%</td>
<td>87.54</td>
<td>57.70%</td>
</tr>
<tr>
<td>360</td>
<td>92.97</td>
<td>67.48%</td>
<td>138.61</td>
<td>149.70%</td>
</tr>
<tr>
<td>540</td>
<td>97.85</td>
<td>76.27%</td>
<td>160.62</td>
<td>189.35%</td>
</tr>
</tbody>
</table>

Table 4.2: Improvement in per-cell sum rate over 19 cells, 3 sectors per cell, 10 users per sector. Cell diameter is 1.4km. Baseline per-cell sum rate is 55.51Mbps
4.6 Summary

This chapter proposes a novel achievability scheme employing the Wyner-Ziv compress-and-forward and the SIC receiver structure on a per-base-station basis for the uplink of the multicell processing system in which the base-stations are connected to a centralized processor with finite capacity backhaul links. The main advantage of the proposed scheme is that the resulting achievable rate region is easily computable, and it leads to an architecture that is more amendable to practical implementation. Under the per-base-station SIC framework, this chapter shows that the capacities of the backhaul links should scale with the logarithm of the SINR in each base-station, both from a point of view of approaching the theoretical maximal SIC rate with unlimited backhaul, as well as for maximizing the overall sum rate subject to a total backhaul rate constraint.
Figure 4.9: Backhaul rate: 360Mbps per base-station

Figure 4.10: Backhaul rate: 540Mbps per base-station
Chapter 5

Conclusion

This thesis studies several interference mitigation techniques for cellular networks. The benefit of the proposed schemes are demonstrated in the interference-limited regime. First, it is shown that for the $K$-user cyclic Gaussian interference channel, Etkin-Tse-Wang power splitting strategy is able to achieve the approximate capacity region. The suggests a promising way of interference mitigation in the multicell network. Second, it is shown that for a two-user Gaussian relay-interference channel with an in-band-reception and out-of-band-transmission relay, generalized hash-and-forward (GHF) together with Han-Kobayashi information splitting can achieve the capacity region of this channel to within a constant number of bits in a certain weak-relay regime.

Therefore, GHF is sufficient for achieving the approximated capacity region of an in-band reception and out-of-band transmission Gaussian relay-interference channel. Other relay techniques based on compute-and-forward or lattice coding is not necessary at least in this case. Third, it is shown that for the uplink multicell joint processing model in which the base-stations are connected to a centralized processing server via rate-limited digital backhaul links, the proposed achievability scheme using Wyner-Ziv compression and SIC decoding results in an achievable rate region that is easily computable. The impact of the limited-capacity backhaul links on the achievable rates is also investigated. It is shown that in order to achieve to within constant number of bits to the maximal SIC rate with infinite-capacity backhaul, the backhaul capacity must scale logarithmically with the signal-to-interference-and-noise ratio at each base-station.
Appendix A

K-user Cyclic Interference Channel: Achievable Rate Region

For the two-user interference channel, Kobayashi and Han [36] gave a detailed Fourier-Motzkin elimination procedure for the achievable rate region. The Fourier-Motzkin elimination for the $K$-user cyclic interference channel involves $K$ elimination steps. The complexity of the process increases with each step. Instead of manually writing down all the inequalities step by step, this appendix uses mathematical induction to derive the final result.

This achievability proof is based on the application of coding scheme in [22] (also referred as the multi-level coding in [77]) to the multi-user setting. Instead of using the original code construction of [2], the following strategy is used in which each common message $W_i, i = 1, 2, \ldots, K$ serves to generate $2^{nT_i}$ cloud centers $W_i(j), j = 1, 2, \ldots, 2^{nT_i}$, each of which is surrounded by $2^{nS_i}$ codewords $X_i(j, k), k = 1, 2, \ldots, 2^{nS_i}$. This results in achievable rate region expressions expressed in terms of $(W_i, X_i, Y_i)$ instead of $(U_i, W_i, Y_i)$. For the two-user interference channel, Chong, Motani and Garg [22, Lemma 3] made a further simplification to the achievable rate region expression. They observed that in the Han-Kobayashi scheme, the common message $W_i$ is only required to be correctly decoded at the intended receiver $Y_i$ and an incorrectly decoded $W_i$ at receiver $Y_{i-1}$ does not cause an error event. Based on this observation, they concluded that for the multiple-access channel with input $(U_i, W_i, W_{i+1})$ and output $Y_i$, the rate constraints on common messages $T_i, T_{i+1}$ and $T_i + T_{i+1}$ are in fact irrelevant to the decoding error probabilities and can be removed, i.e., the rates $(S_i, T_i, T_{i+1})$ are constrained by only the following set of inequalities:

\[
\begin{align*}
S_i & \leq a_i = I(Y_i; X_i|W_i, W_{i+1}, Q) \\
S_i + T_i & \leq d_i = I(Y_i; X_i|W_{i+1}, Q) \\
S_i + T_{i+1} & \leq e_i = I(Y_i; W_{i+1}, X_i|W_i, Q) \\
S_i + T_i + T_{i+1} & \leq g_i = I(Y_i; W_{i+1}, X_i|Q) \\
S_i, T_i, T_{i+1} & \geq 0
\end{align*}
\]

(A.1) (A.2) (A.3) (A.4) (A.5)

Now, compare the $K$-user cyclic interference channel with the two-user interference channel, it is easy to see that in both channel models, each receiver only sees interference from one neighboring transmitter.
This makes the decoding error probability analysis for both channel models the same. Therefore, the set of rates $R(R_1, R_2, \cdots, R_K)$, where $R_i = S_i + T_i$, with $(S_i, T_i)$ satisfy (A.1)-(A.5) for $i = 1, 2, \cdots, K$, characterizes an achievable rate region for the $K$-user cyclic interference channel.

The first step of using the Fourier-Motzkin algorithm is to eliminate all private messages $S_i$ by substituting $S_i = R_i - T_i$ into the $K$ polymatroids (A.1)-(A.5). This results in the following $K$ polymatroids without $S_i$:

$$R_i - T_i \leq a_i,$$  
(A.6)

$$R_i \leq d_i,$$  
(A.7)

$$R_i - T_i + T_{i+1} \leq e_i,$$  
(A.8)

$$R_i + T_{i+1} \leq g_i,$$  
(A.9)

$$-R_i \leq 0,$$  
(A.10)

where $i = 1, 2, \cdots, K$.

Next, use Fourier-Motzkin algorithm to eliminate common message rates $T_1, T_2, \cdots, T_K$ in a step-by-step process so that after $n$ steps, common variables $(T_1, \cdots, T_n)$ are eliminated. The induction hypothesis is the following set of inequalities, which is assumed to be obtained at the end of the $n$th elimination step:

- Inequalities not including private or common variables $S_i$ and $T_i$, $i = 1, 2, \cdots, K$:

$$R_i \leq d_i, \quad i = 1, 2, \cdots, K$$  
(A.11)

$$-R_i \leq 0, \quad i = 1, 2, \cdots, n$$  
(A.12)

$$R_K + R_1 \leq g_K + a_1,$$  
(A.13)

$$R_m \leq a_m + e_{m-1},$$  
(A.14)

$$\sum_{j=l}^{m} R_j \leq \min \left\{ g_l + \sum_{i=l+1}^{m-1} e_j + a_m, \sum_{j=l-1}^{m-1} e_j + a_m \right\},$$  
(A.15)

$$\sum_{j=1}^{m} R_j \leq g_1 + \sum_{j=2}^{m-1} e_j + a_m,$$  
(A.16)

$$\sum_{j=K}^{m} R_j \leq g_K + \sum_{j=1}^{m-1} e_j + a_m,$$  
(A.17)

where $m = 2, 3, \cdots, n$ and $l = 2, 3, \cdots, m - 1$.

- Inequalities including $T_K$ but not including $T_{n+1}$:

$$R_K - T_K \leq a_K,$$  
(A.18)

$$-R_K - T_K \leq 0,$$  
(A.19)

$$-T_K \leq 0,$$  
(A.20)

$$\sum_{j=K}^{p} R_j - T_K \leq \sum_{j=K}^{p-1} e_j + a_p,$$  
(A.21)
where \( p = 1, 2, \ldots, n \).

- All other inequalities not including \( T_{n+1} \):

\[
R_{n+1} + T_{n+2} \leq g_{n+1}, \tag{A.22}
\]
and all the polymatroids in (A.6)-(A.10) indexed from \( n + 2 \) to \( K - 1 \).

- Inequalities including \( T_{n+1} \) with a plus sign:

\[
T_{n+1} \leq e_n, \tag{A.23}
\]
\[
-R_{n+1} + T_{n+1} \leq 0, \tag{A.24}
\]
\[
\sum_{j=l}^{n} R_j + T_{n+1} \leq \min \left\{ \sum_{j=l-1}^{n} e_j, g_l + \sum_{j=l+1}^{n} e_j \right\}, \tag{A.25}
\]
\[
\sum_{j=1}^{n} R_j + T_{n+1} \leq g_1 + \sum_{j=2}^{n} e_j, \tag{A.26}
\]
\[
\sum_{j=K}^{n} R_j + T_{n+1} \leq g_K + \sum_{j=1}^{n} e_j, \tag{A.27}
\]
\[
\sum_{j=K}^{n} R_j + T_{n+1} - T_K \leq \sum_{j=K}^{n} e_j, \tag{A.28}
\]
where \( l \) goes from 2 to \( n \).

- Inequalities including \( T_{n+1} \) with a minus sign:

\[
R_{n+1} - T_{n+1} \leq a_{n+1}, \tag{A.29}
\]
\[
R_{n+1} - T_{n+1} + T_{n+2} \leq e_{n+1}, \tag{A.30}
\]
\[
-T_{n+1} \leq 0. \tag{A.31}
\]

It is easy to verify the correctness of inequalities (A.11)-(A.31) for \( n = 2 \). We next show that for \( n < K - 2 \), if at the end of step \( n \), the inequalities in (A.11)-(A.31) are true, then they must also be true at the end of step \( n + 1 \). Towards this end, we follow the Fourier-Motzkin algorithm [36] by first adding up all the inequalities in (A.23)-(A.28) with each of the inequalities in (A.29)-(A.31) to eliminate \( T_{n+1} \). This results in the following three groups of inequalities:

- Inequalities due to (A.29):
Appendix A. *K*-user Cyclic Interference Channel: Achievable Rate Region

\[ R_{n+1} \leq a_{n+1} + e_n, \quad (A.32) \]
\[ 0 \leq a_{n+1}, \quad (A.33) \]
\[ \sum_{j=l}^{n+1} R_j \leq \min \left\{ \sum_{j=l}^{n} e_j + a_{n+1}, g_l + \sum_{j=l+1}^{n} e_j + a_{n+1} \right\}, \quad (A.34) \]
\[ \sum_{j=1}^{n+1} R_j \leq g_1 + \sum_{j=2}^{n} e_j + a_{n+1}, \quad (A.35) \]
\[ \sum_{j=K}^{n+1} R_j \leq g_K + \sum_{j=1}^{n} e_j + a_{n+1}, \quad (A.36) \]
\[ \sum_{j=K}^{n+1} R_j - T_K \leq \sum_{j=K}^{n} e_j + a_{n+1}, \quad (A.37) \]

where \( l = 2, 3, \cdots, n \).

- Inequalities due to (A.30):

\[ R_{n+1} + T_{n+2} \leq e_n + e_{n+1}, \quad (A.38) \]
\[ T_{n+2} \leq e_{n+1}, \quad (A.39) \]
\[ \sum_{j=l}^{n+1} R_j + T_{n+2} \leq \min \left\{ \sum_{j=l}^{n+1} e_j, g_l + \sum_{j=l+1}^{n+1} e_j \right\}, \quad (A.40) \]
\[ \sum_{j=1}^{n+1} R_j + T_{n+2} \leq g_1 + \sum_{j=2}^{n+1} e_j, \quad (A.41) \]
\[ \sum_{j=K}^{n+1} R_j + T_{n+2} \leq g_K + \sum_{j=1}^{n+1} e_j, \quad (A.42) \]
\[ \sum_{j=K}^{n+1} R_j + T_{n+2} - T_K \leq \sum_{j=K}^{n+1} e_j, \quad (A.43) \]

where \( l = 2, 3, \cdots, n \).
Appendix A.  

K-user Cyclic Interference Channel: Achievable Rate Region

• Inequalities due to (A.31):

\[ 0 \leq e_n, \quad (A.44) \]
\[ -R_{n+1} \leq 0, \quad (A.45) \]
\[ \sum_{j=l}^{n} R_j \leq \min \left\{ \sum_{j=l-1}^{n} e_j, g_l + \sum_{j=l+1}^{n} e_j \right\}, \quad (A.46) \]
\[ \sum_{j=1}^{n} R_j \leq g_1 + \sum_{j=2}^{n} e_j, \quad (A.47) \]
\[ \sum_{j=K}^{n} R_j \leq g_K + \sum_{j=1}^{n} e_j, \quad (A.48) \]
\[ \sum_{j=K}^{n} R_j - T_K \leq \sum_{j=K}^{n} e_j, \quad (A.49) \]

where \( l = 2, 3, \ldots, n \).

Inspecting the above three groups of inequalities, we can see that (A.33) and (A.44) are obviously redundant. Also, (A.46) is redundant due to (A.15), (A.47) is redundant due to (A.16), (A.48) is redundant due to (A.17), and (A.49) is redundant due to (A.21). Now, with these six redundant inequalities removed, the above three groups of inequalities in (A.32)-(A.45) together with (A.11)-(A.22) form the set of inequalities at the end of step \( n+1 \). It can be verified that this new set of inequalities is exactly (A.11)-(A.31) with \( n \) replaced by \( n+1 \). This completes the induction part.

Now, we proceed with the \((K-1)\)th step. At the end of this step, \( T_1, T_2, \ldots, T_{K-1} \) would all be removed and only \( T_K \) would remain. Because of the cyclic nature of the channel, the set of inequalities (A.11)-(A.31) needs to be modified for this \( n = K-1 \) case. It can be verified that at the end of the \((K-1)\)th step of Fourier-Motzkin algorithm, we obtain the following set of inequalities:

• Inequalities not including \( T_K \): (A.11)-(A.17) with \( n \) replaced by \( K-1 \) and

\[ \sum_{j=1}^{K} R_j \leq \sum_{j=1}^{K} e_j, \quad (A.50) \]

• Inequalities including \( T_K \) with a plus sign: (A.23)-(A.27) with \( n \) replace by \( K-1 \). Note that, (A.28) becomes (A.50) when \( n = K-1 \).

• Inequalities including \( T_K \) with a minus sign:

\[ R_K - T_K \leq a_K, \quad (A.51) \]
\[ \sum_{j=K}^{l} R_j - T_K \leq \sum_{j=K}^{l-1} e_j + a_l, \quad (A.52) \]
\[ -T_K \leq 0, \quad (A.53) \]

where \( l = 1, 2, \ldots, K-1 \).

In the \( K \)th step (final step) of the Fourier-Motzkin algorithm, \( T_K \) is eliminated by adding each of the inequalities involving \( T_K \) with a plus sign and each of the inequalities involving \( T_K \) with a minus
sign to obtain new inequalities not involving $T_K$. (This is quite similar to the procedure of obtaining (A.32)-(A.49).) Finally, after removing all the redundant inequalities, we obtain the set of inequalities in Theorem 1.
Appendix B

\textbf{K-user Cyclic Interference Channel: Outer Bounds}

To obtain the capacity region outer bound, first define the following genies:

\begin{align*}
S_1^n &= \sqrt{g_{c}} X_1^n + Z_2^n \\
S_2^n &= \sqrt{g_{c}} X_2^n + Z_1^n \\
&\vdots \\
S_K^n &= \sqrt{g_{c}} X_K^n + Z_1^n
\end{align*}

Next, we prove the outer bounds from (2.21) to (2.24) one by one.

- (2.21) is simply the cut-set upper bound for user $i$.

- (2.22) is the bound on the sum-rate of $l$ adjacent users starting from $m$. According to Fano's
inequality, for a block of length $n$, we have

$$n \left( \sum_{j=m}^{m+l-1} R_j - \epsilon_n \right) \leq \sum_{j=m}^{m+l-1} I(X_j^n; Y_j^n)$$

$$(a) \leq h(Y_m^n) - h(Y_m^n | X_m^n) + \sum_{j=m+1}^{m+l-2} I(X_j^n; Y_j^n S_j^n) + I(X_{m+l-1}^n; Y_{m+l-1}^n | X_{m+l})$$

$$= h(Y_m^n) - h(S_{m+1}^n) + \sum_{j=m+1}^{m+l-2} \left[ h(S_j^n) - h(Z_{j-1}^n) + h(Y_j^n | S_j^n) - h(S_{j+1}^n) \right]$$

$$+ h(h_{m+l-1,m+l-1} X_{m+l-1}^n + Z_{m+l-1}^n - h(Z_{m+l-1}^n)$$

$$\leq h(Y_m^n) - h(Z_{m+l-1}^n) + \sum_{j=m+1}^{m+l-2} \left[ h(Y_j^n | S_j^n) - h(Z_{j-1}^n) \right]$$

$$+ h(h_{m+l-1,m+l-2} X_{m+l-1}^n + Z_{m+l-2}^n - h(h_{m+l-1,m+l-2} X_{m+l-1}^n + Z_{m+l-2}^n)$$

$$(b) \leq n \left( \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1} \right), \quad (B.1)$$

where in (a) we give genie $S_j^n$ to $Y_j^n$ for $m+1 \leq j \leq m+l-2$ and $X_{m+l}$ to $Y_{m+l-1}^n$ (genies $S_j^n$ are as defined in [78, Theorem 2]), and (b) comes from the fact [7] that Gaussian inputs maximize 1) entropy $h(Y_m^n)$, 2) conditional entropy $h(Y_j^n | S_j^n)$ for any $j$, and 3) entropy difference $h(h_{m+l-1,m+l-1} X_{m+l-1}^n + Z_{m+l-1}^n) - h(h_{m+l-1,m+l-2} X_{m+l-1}^n + Z_{m+l-2}^n)$. This proves the first bound in (2.22).

Similarly, the second upper bound of (2.22) can be obtained by giving genie $S_j^n$ to $Y_j^n$ for $m \leq j \leq m+l-1$...
m + l - 2 and $X^l_m$ to $Y^l_{m+l-1}$:

\[ n \left( \sum_{j=m}^{m+l-1} R_j - \epsilon_n \right) \leq \sum_{j=m}^{m+l-1} I(X^n_j; Y^n_j) \]

\[ \leq \sum_{j=m}^{m+l-2} I(X^n_j; Y^n_j S^n_j) + I(X^n_{m+l-1}; Y^n_{m+l-1}|X^n_{m+1}) \]

\[ = \sum_{j=m}^{m+l-2} \left[ h(S^n_j) - h(Z^n_{j-1}) + h(Y^n_j|S^n_j) - h(S^n_{j+1}) \right] \]

\[ + h(h_{m+l-1,m+l-1} X^n_{m+l-1} + Z^n_{m+l-1}) - h(Z^n_{m+l-1}) \]

\[ = h(S^n_m) - h(Z^n_{m+l-1}) + \sum_{j=m}^{m+l-2} \left[ h(Y^n_j|S^n_j) - h(Z^n_{j-1}) \right] \]

\[ + h(h_{m+l-1,m+l-1} X^n_{m+l-1} + Z^n_{m+l-1}) \]

\[ - h(h_{m+l-1,m+l-2} X^n_{m+l-1} + Z^n_{m+l-2}) \]

\[ \leq n \left( \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} \right). \] (B.2)

Combining (B.1) and (B.2) gives the upper bound in (2.22).

- The first upper bound in (2.23) is in fact the non-symmetric version of [78, Theorem 2], from which we have

\[ R_{sum} - n\epsilon_n \leq \sum_{k=1}^{K} \{ h(Y^n_{k|S_{ki}}) - h(Z^n_{ki}) \} \]

\[ \leq n \sum_{j=1}^{K} \alpha_j. \] (B.3)

The other sum-rate upper bounds (i.e., $\rho_l$) can be derived by giving genies $X^n_l$ to $Y^n_{l-1}$ and $S^n_j$ to
\[ Y^n_j \text{ for } j = 1, 2, \ldots, K, j \neq l, l - 1: \]

\[ n(R_{\text{sum}} - \epsilon_n) \leq I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) + \cdots + I(X^n_K; Y^n_K) \]

\[ = I(X^n_{l-1}; Y^n_{l-1} | X^n_l) + I(X^n_l; Y^n_l) + K \sum_{j=1, j \neq l, l-1} I(X^n_j; Y^n_j S^n_j) \]

\[ = h(h_{l-1,l-1}X^n_{l-1} + Z^n_{l-1}) - h(Z^n_{l-1}) + h(Y^n_l) - h(S^n_{l+1}) \]

\[ + \sum_{j=1, j \neq l, l-1} [h(S^n_j) - h(Z^n_{j-1}) + h(Y^n_j | S^n_j) - h(S^n_{j+1})] \]

\[ = h(Y^n_l) - h(Z^n_{l-1}) + h(h_{l-1,l-1}X^n_{l-1} + Z^n_{l-1}) \]

\[ - h(h_{l-1,l-2}X^n_{l-2} + Z^n_{l-2}) \]

\[ + \sum_{j=1, j \neq l, l-1} [h(Y^n_j | S^n_j) - h(Z^n_{j-1})] \]

\[ \leq n \left( \beta_{l-1} + \gamma_l + \sum_{j=1, j \neq l, l-1} \alpha_j \right) \]

\[ = n \rho_l \quad \text{(B.4)} \]

where \( l = 1, 2, \ldots, K \).

- For the bound in (2.24), from Fano’s inequality, we have

\[ n(R_{\text{sum}} + R_i - \epsilon_n) \leq \sum_{j=1}^K I(X^n_j; Y^n_j) + I(X^n_i; Y^n_i) \]

\[ \leq I(X^n_i; Y^n_i) + I(X^n_i; Y^n_i | X^n_{i+1}) + \sum_{j=1, j \neq i}^K I(X^n_j; Y^n_j | S^n_j) \]

\[ = h(Y^n_i) - h(S^n_{i+1}) + h(h_{i,i}X^n_i + Z^n_i) - h(Z^n_i) \]

\[ + \sum_{j=1, j \neq i}^K [h(S^n_j) - h(Z^n_{j-1}) + h(Y^n_j | S^n_j) - h(S^n_{j+1})] \]

\[ = h(Y^n_i) - h(Z^n_i) + h(h_{i,i}X^n_i + Z^n_i) - h(h_{i,i-1}X^n_i + Z^n_i) \]

\[ + \sum_{j=1, j \neq i}^K [h(Y^n_j | S^n_j) - h(Z^n_{j-1})] \]

\[ \leq n \left( \beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j \right) \quad \text{(B.5)} \]

where in (a) we give genie \( X^n_{i+1} \) to \( Y^n_i \) and \( S^n_j \) to \( Y^n_j \) for \( j = 1, 2, \ldots, K, j \neq i \).
Appendix C

Proof of $\mathcal{R}_{\text{HK-}TS}^{(3)} \subseteq \mathcal{R}_{\text{HK}}^{(3)}$

For a fixed $P_3 \subseteq \mathcal{P}_3$, define

$$
P^*_3 = \sum_{w_1} P_3, \quad P^{**}_3 = \sum_{w_2} P_3, \quad P^{***}_3 = \sum_{w_3} P_3. \quad (C.1)
$$

We will show that

$$
\mathcal{R}_{\text{HK-}TS}^{(3)}(P_3) \subseteq \mathcal{R}_{\text{HK}}^{(3)}(P^*_3) \cup \mathcal{R}_{\text{HK}}^{(3)}(P^{**}_3) \cup \mathcal{R}_{\text{HK}}^{(3)}(P^{***}_3). \quad (C.2)
$$

Suppose that rate triple $(R_1, R_2, R_3)$ is in $\mathcal{R}_{\text{HK-}TS}^{(3)}(P_3)$ but not in $\mathcal{R}_{\text{HK}}^{(3)}(P_3)$. Then at least one of the following inequalities is true:

$$
a_1 + e_3 \leq R_1 \leq d_1, \quad (C.3)
$$
$$
a_2 + e_1 \leq R_2 \leq d_2, \quad (C.4)
$$
$$
a_3 + e_2 \leq R_3 \leq d_3. \quad (C.5)
$$

Without loss of generality, assume that $(C.3)$ holds.

Substituting $W_1 = \emptyset$ into $\mathcal{R}_{\text{HK}}^{(3)}(P_3)$, we obtain $\mathcal{R}_{\text{HK}}^{(3)}(P^*_3)$ as follows:

$$
R_1 \leq d_1, \quad (C.6)
$$
$$
R_2 \leq \min\{d_2, a_2 + g_1\}, \quad (C.7)
$$
$$
R_3 \leq \min\{I(Y_3; X_3|Q), e_2 + I(Y_3; X_3|W_3, Q)\}, \quad (C.8)
$$
$$
R_1 + R_2 \leq a_2 + g_1, \quad (C.9)
$$
$$
R_2 + R_3 \leq \min\{g_2 + I(Y_3; X_3|W_3, Q), g_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \quad (C.10)
$$
$$
R_3 + R_1 \leq \min\{d_1 + I(Y_3; X_3|Q), d_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \quad (C.11)
$$
$$
R_1 + R_2 + R_3 \leq g_1 + e_2 + I(Y_3; X_3|W_3, Q) \quad (C.12)
$$

We will show that whenever $(C.3)$ is true, we have $\mathcal{R}_{\text{HK-}TS}^{(3)}(P_3) \subseteq \mathcal{R}_{\text{HK}}^{(3)}(P^*_3)$. To this end, inspect
Appendix C. Proof of \( \mathcal{R}_{\text{HK-TS}}^{(3)} \subseteq \mathcal{R}_{\text{HK}}^{(3)} \)

\( \mathcal{R}_{\text{HK-TS}}^{(3)}(P_3) \) in (2.61)-(2.68). From (2.61), we have

\[
R_1 \leq d_1,
\]

and from (2.61) and (C.3) and (2.62), we have

\[
R_2 \leq \min\{d_2, a_2 + e_1 - a_1\} \leq \min\{d_2, a_2 + g_1\},
\]

and from (C.3) and (2.64), we have

\[
R_3 \leq \min\{g_3 - e_3, e_2\} \leq \min\{I(Y_3; X_3|Q), e_2 + I(Y_3; X_3|W_3, Q)\},
\]

and from (2.62), we have

\[
R_1 + R_2 \leq a_2 + g_1,
\]

and from (C.3) and (2.65), we have

\[
R_2 + R_3 \leq \min\{g_2, e_1 + e_2 - a_1\} \leq \min\{g_2 + I(Y_3; X_3|W_3, Q), g_1 + e_2 + I(Y_3; X_3|W_3, Q)\},
\]

and from (C.3) and (2.64), we have

\[
R_3 + R_1 \leq \min\{d_1 + g_3 - a_3, e_2 + d_1\} \leq \min\{d_1 + I(Y_3; X_3|Q), d_3 + e_2 + I(Y_3; X_3|W_3, Q)\},
\]

and from (C.3) and (2.66), we have

\[
R_1 + R_2 + R_3 \leq g_1 + e_2 \leq g_1 + e_2 + I(Y_3; X_3|W_3, Q).
\]

It is easy to see that \((R_1, R_2, R_3)\) satisfying the above constrains (C.13)-(C.24) is within the rate region \( \mathcal{R}_{\text{HK}}^{(3)}(P_{3*}) \). In the same way, we can prove the cases for when (C.4) holds and when (C.5) holds.

Therefore, (C.2) is true, and it immediately follows that

\[
\mathcal{R}_{\text{HK-TS}}^{(3)} \subseteq \mathcal{R}_{\text{HK}}^{(3)}.
\]
Appendix D

*K*-user Cyclic Interference Channel: Useful Inequalities

Keep in mind that, with the ETW’s power splitting strategy, i.e., \( \text{SNR}_{ip} = \min\{\text{SNR}_i, \frac{\text{SNR}_i}{1 + \text{INR}_i}\} \), we always have \( \text{SNR}_{ip} > \frac{\text{SNR}_i}{1 + \text{INR}_i} \). This appendix presents several useful inequalities as follows. For all \( i = 1, 2, \cdots, K \),

- \( \lambda_i - d_i < \frac{1}{2} \), because

\[
\lambda_i - d_i = \frac{1}{2} \log(1 + \text{SNR}_i) - \frac{1}{2} \log(2 + \text{SNR}_i) + \frac{1}{2} \\
= \frac{1}{2} - \frac{1}{2} \log \left( \frac{2 + \text{SNR}_i}{1 + \text{SNR}_i} \right) \\
< \frac{1}{2}
\]  
(D.1)

- \( \lambda_i - (a_i + e_{i-1}) < 1 \), because

\[
\lambda_i - (a_i + e_{i-1}) = \frac{1}{2} \log(1 + \text{SNR}_i) - \frac{1}{2} \log (2 + \text{SNR}_{ip}) + 1 - \frac{1}{2} \log (1 + \text{INR}_i + \text{SNR}_{i-1,p}) + \frac{1}{2} \\
< 1 + \frac{1}{2} \log(1 + \text{SNR}_i) - \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_i}{1 + \text{INR}_i} \right) - \frac{1}{2} \log (1 + \text{INR}_i) \\
= 1 - \frac{1}{2} \log \left( 1 + \frac{\text{INR}_i}{1 + \text{SNR}_i} \right) \\
< 1
\]  
(D.2)
Appendix D. \textit{K}-user Cyclic Interference Channel: Useful Inequalities

- $\beta_i - a_i < \frac{1}{2}$, because

\[
\beta_i - a_i = \frac{1}{2} \log \left( \frac{1 + \text{SNR}_i}{1 + \text{INR}_i} \right) - \frac{1}{2} \log \left( 2 + \text{SNR}_{ip} \right) + \frac{1}{2} < \frac{1}{2} - \frac{1}{2} \log \left( 1 + \frac{\text{INR}_i}{1 + \text{SNR}_i} \right)
\]

\[
< \frac{1}{1}
\]

(D.3)

- $\alpha_i - e_i < \frac{1}{2}$, because

\[
\alpha_i - e_i = \frac{1}{2} \log \left( 1 + \text{INR}_{i+1} + \frac{\text{SNR}_i}{1 + \text{INR}_i} \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{i+1} + \text{SNR}_i + \text{SNR}_{ip} \right) + \frac{1}{2}
\]

\[
< \frac{1}{2}
\]

(D.4)

- $\gamma_i - g_i = \frac{1}{2}$, because

\[
\gamma_i - g_i = \frac{1}{2} \log \left( 1 + \text{INR}_{i+1} + \text{SNR}_i \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{i+1} + \text{SNR}_i \right) + \frac{1}{2}
\]

\[
= \frac{1}{2}
\]

(D.5)

- $\mu_i - e_{i-1} < \frac{1}{2}$, because

\[
\mu_i - e_{i-1} = \frac{1}{2} \log \left( 1 + \text{INR}_i \right) - \frac{1}{2} \log \left( 1 + \text{INR}_i + \text{SNR}_{i-1,p} \right) + \frac{1}{2}
\]

\[
< \frac{1}{2}
\]

(D.6)
Appendix E

Gaussian Relay-Interference Channel: Outer Bounds

Define $V_1^n$ as the output of the digital link $C_1$, and $V_2^n$ as the output of the digital link $C_2$. The outer bounds are proved as follows:

- **Individual-rate bounds:** First, the first term of (3.5) is the simple cut-set upper bound for $R_1$. For the second term, starting from Fano’s inequality, we have

  \[ n(R_1 - \epsilon_n) \leq I(X_1^n; Y_1^n, V_1^n) \]
  \[ = I(X_1^n; Y_1^n, Y_R^n) \]
  \[ \leq \frac{n}{2} \log(1 + \text{SNR}_1 + \text{SNR}_{-1}) + nC_1. \]  

(E.1)

The outer bound of $R_2$ in (3.6) can be proved in the same way.

- **Sum-rate bounds:**

  - First, (3.7)-(3.9) are simply cut-set outer bounds, i.e.,

    \[ n(R_1 + R_2 - \epsilon_n) \leq I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \]
    \[ = I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + I(X_1^n; V_1^n|Y_1^n) + I(X_2^n; V_2^n|Y_2^n) \]
    \[ \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + h(V_1^n) + h(V_2^n) \]
    \[ \leq nC_{\sum}(0) + nC_1 + nC_2, \]  

    (E.2)

where $C_{\sum}(0)$ is the sum capacity of the interference channel without relay. Clearly, the sum-rate gain due to the digital relay is upper bounded by the rates of digital links. Although the sum-rate capacity $C_{\sum}(0)$ is not known in general, its upper bound has been studied in literature [3–5,7,63,79]. Applying the sum-rate outer bounds in [63], we obtain (3.7)-(3.9).

  - Second, (3.10)-(3.12) can be obtained by the following steps:

    \[ n(R_1 + R_2 - \epsilon_n) \leq I(X_1^n; Y_1^n, V_1^n) + I(X_2^n; Y_2^n, V_2^n) \]
    \[ \leq I(X_1^n; Y_1^n) + h(V_1^n) + I(X_2^n; Y_2^n, Y_R^n), \]  

    (E.3)
Appendix E. Gaussian Relay-Interference Channel: Outer Bounds

where in (a) we give genie \( Y_R^n \) to receiver 2 and apply the fact that \( \hat{Y}_R \) is a function of \( Y_R \).

Note that \( I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2, Y^n_2) \) is upper bounded by the sum capacity of the SIMO interference channel with \( X^n_1 \) and \( X^n_2 \) as the input, and \( Y^n_1 \) and \( (Y^n_2, Y^n_R) \) as the output. The sum-rate outer bound of such a SIMO interference channel has been studied in [63], which along with \( h(V^n_1) \leq nC_1 \) gives the outer bounds of (3.10)-(3.12).

- Third, (3.13)-(3.15) can be similarly derived following the same steps of (3.10)-(3.12) with indices 1 and 2 switched.

- Fourth, (3.16)-(3.18) can be obtained by giving \( Y_R^n \) as a genie to both receivers, i.e.,

\[
\begin{align*}
n(R_1 + R_2 - \epsilon_n) & \leq I(X^n_1; Y^n_1, V^n_1) + I(X^n_2; Y^n_2, V^n_2) \\
& \leq I(X^n_1; Y^n_1, Y^n_R) + I(X^n_2; Y^n_2, Y^n_R),
\end{align*}
\]

which is upper bounded by the sum capacity of the SIMO interference channel with \( X^n_1 \) and \( X^n_2 \) as input, and \( (Y^n_1, Y^n_R) \) and \( (Y^n_2, Y^n_R) \) as output. Applying the result in [63], we have (3.16)-(3.18).

- \( 2R_1 + R_2 \) bounds: There are six upper bounds on \( 2R_1 + R_2 \).

  - First, (3.19) is simply the cut-set bound, i.e.,

\[
\begin{align*}
n(2R_1 + R_2 - \epsilon_n) & \leq 2I(X^n_1; Y^n_1, V^n_1) + I(X^n_2; Y^n_2, V^n_2) \\
& \leq 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) + 2h(V^n_1) + h(V^n_2),
\end{align*}
\]

where \( 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) \) is upper bounded by the \( 2R_1 + R_2 \) bound of the interference channel with \( X^n_1 \) and \( X^n_2 \) as the input, and \( Y^n_1 \) and \( Y^n_2 \) as the output, which together with \( h(V^n_1) \leq nC_1 \) and \( h(V^n_2) \leq nC_2 \) gives the upper bound in (3.19).

  - Second, (3.20) can be derived by giving genie \( Y_R^n \) to both receivers:

\[
\begin{align*}
n(2R_1 + R_2 - \epsilon_n) & \leq 2I(X^n_1; Y^n_1, V^n_1) + I(X^n_2; Y^n_2, V^n_2) \\
& \leq 2I(X^n_2; Y^n_1, Y^n_R) + I(X^n_2; Y^n_2, Y^n_R),
\end{align*}
\]

which is upper bounded by the \( 2R_1 + R_2 \) bound of the SIMO interference channel with \( X^n_1 \) and \( X^n_2 \) as the input, and \( (Y^n_1, Y^n_R) \) and \( (Y^n_2, Y^n_R) \) as the output. Applying the result of [63], we obtain (3.20).

  - Third, (3.21) can be obtained by giving genies \( (X^n_2, Y^n_R, S^n_1) \) to \( Y^n_1 \) in one of the two \( R_1 \) expressions and \( (S^n_2, Y^n_R) \) to \( Y^n_2 \), where genies \( S^n_1 \) and \( S^n_2 \) are defined as

\[
S^n_1 = h_{12}X^n_1 + Z_2, \quad S^n_2 = h_{21}X^n_2 + Z_1.
\]
According to Fano's inequality, we have

\[ n(2R_1 + R_2 - \epsilon_n) \leq 2I(X^n_1; V^n_1) + I(X_2^n; Y^n_2) \]
\[ \leq I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) + h(V^n_1) + I(X_2^n; Y^n_2, S^n_2) \]
\[ \leq I(X^n_1; Y^n_1) + h(Y^n_1) - h(S^n_2) + nC_1 \]
\[ + I(X^n_2; Y^n_2, S^n_2) + I(X^n_2; Y^n_2 | S^n_2) \]
\[ = I(X^n_1; S^n_2) + I(X^n_2; Y^n_1, Y^n_2 | S^n_2, Z^n_2) + h(Y^n_1) - h(S^n_2) \]
\[ + nC_1 + h(S^n_2) - h(Z^n_1) + h(Y^n_2, Y^n_2 | S^n_2) - h(S^n_2) - h(Y^n_2 | Y^n_2, X^n_2) \]
\[ = h(Y^n_1) - h(Z^n_1) + h(Y^n_1, Y^n_2 | S^n_2, Z^n_2) - h(Z^n_1, Z^n_2) + nC_1 \]
\[ + h(Y^n_2, Y^n_2 | S^n_2) - h(Z^n_2, Z^n_2) - I(Y^n_2, X^n_2 | X^n_2, Y^n_2) \]
\[ \leq h(Y^n_1) - h(Z^n_1) + h(Y^n_1, Y^n_2 | S^n_2, X^n_2) - h(Z^n_1, Z^n_2) + nC_1 \]
\[ + h(Y^n_2, Y^n_2 | S^n_2) - h(Z^n_2, Z^n_2), \]
\[ \text{(E.8)} \]

where in (a) we use the fact that \( X^n_1 \) is independent with \( X^n_2 \). Note that, (E.8) is maximized by Gaussian inputs \( X^n_1 \) and \( X^n_2 \) with i.i.d \( \mathcal{N}(0, 1) \) entries, because (i) \( h(Y^n_1) \) is maximized by Gaussian distributions, and (ii) \( h(Y^n_1, Y^n_2 | S^n_2, X^n_2) \) and \( h(Y^n_2, Y^n_2 | S^n_2) \) are both maximized by Gaussian inputs since the conditional entropy under a power constraint is maximized by Gaussian distributions. Applying Gaussian distributions to (E.8), we have (3.21).

- Fourth, (3.22) can be obtained by giving genie \( Y^n_R \) to \( Y^n_1 \), i.e.,

\[ n(2R_1 + R_2 - \epsilon_n) \leq 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) \]
\[ \leq 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) + h(V^n_1), \]
\[ \text{(E.9)} \]

where \( 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) \) is upper bounded by the \( 2R_1 + R_2 \) bound of the SIMO interference channel with \( X^n_1 \) and \( X^n_2 \) as the input, and \( (Y^n_1, Y^n_2) \) and \( Y^n_2 \) as the output. Applying the result of [63] and the fact that \( h(V^n_1) \leq nC_2 \), we obtain (3.22).

- Fifth, (3.23) can be obtained by giving genie \( Y^n_R \) to \( Y^n_2 \), i.e.,

\[ n(2R_1 + R_2 - \epsilon_n) \leq 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) \]
\[ \leq 2I(X^n_1; Y^n_1) + 2h(V^n_1) + I(X^n_2; Y^n_2, Y^n_R), \]
\[ \text{(E.10)} \]

where \( 2I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2, Y^n_R) \) is upper bounded by the \( 2R_1 + R_2 \) bound of the SIMO interference channel with \( X^n_1 \) and \( X^n_2 \) as the input, and \( Y^n_1 \) and \( (Y^n_2, Y^n_R) \) as the output. Applying the result of [63] and the fact that \( h(V^n_1) \leq nC_1 \), we obtain (3.23).

- Sixth, (3.24) can be obtained by giving genies \((X^n_2, Y^n_R, S^n_1)\) to \( Y^n_1 \) in one of the two \( R_1 \)
expressions, and $S_n^2$ to $Y_n^2$, i.e.,

\[
\begin{align*}
    n(2R_1 + R_2 - \epsilon_n) &\leq 2I(X_1^n; Y_1^n, V_n^n) + I(X_2^n; Y_2^n, V_2^n) \\
    &\leq I(X_1^n; Y_1^n, Y_R^n, S_1^n, X_2^n) + I(X_1^n; Y_1^n) + h(V_1^n) \\
    &\quad + I(X_2^n; Y_2^n, S_2^n) + h(V_2^n) \\
    &\leq I(X_1^n; S_1^n) + I(X_1^n; Y_1^n Y_R^n | S_1^n, X_2^n) + h(Y_1^n) - h(S_1^n) \\
    &\quad + I(X_2^n; S_2^n) + I(X_2^n; Y_2^n | S_2^n) + nC_1 + nC_2 \\
    &\leq h(S_1^n) - h(Z_2^n) + h(Y_1^n Y_R^n | S_1^n, X_2^n) - h(Z_1^n, Z_R^n) + h(Y_1^n) \\
    &\quad - h(S_2^n) + h(S_2^n) + h(Y_2^n | S_2^n) - h(S_1^n) + nC_1 + nC_2 \\
    &\overset{\text{(E.11)}}{=} h(Y_1^n) - h(Z_1^n) + h(Y_1^n, Y_R^n | S_1^n, X_2^n) - h(Z_1^n, Z_R^n) \\
    &\quad + h(Y_2^n | S_2^n) - h(Z_2^n) + nC_1 + nC_2,
\end{align*}
\]

which is maximized by Gaussian distributions of $X_1^n$ and $X_2^n$ with i.i.d entries following $\mathcal{N}(0,1)$. Applying Gaussian distributions to (E.11), we obtain (3.24).
Appendix F

Gaussian Relay-Interference Channel: A Useful Lemma

This appendix provides several inequalities that are useful to prove the constant-gap theorems.

Lemma 1. For $\Delta a_i, a_i, \Delta d_i, d_i, \Delta e_i, e_i, \Delta g_i, g_i$ and $\xi_i, i = 1, 2$ as defined in (3.34)-(3.38), with $Q$ set as a constant, when $W_i, X_i$ are generated from a superposition coding of $X_i = U_i + W_i$ with $U_i \sim \mathcal{N}(0, P_{ip})$ and $W_i \sim \mathcal{N}(0, P_{ic})$, where $P_{ip} + P_{ic} = 1$ and $P_{ip} = \min\{1, h_{12}^{-2}\}, P_{ic} = \min\{1, h_{21}^{-2}\}$, and when the GHF quantization variables are set to $\hat{Y}_{R1} = Y_R + e_1, \hat{Y}_{R2} = Y_R + e_2$, where $e_1 \sim \mathcal{N}(0, q_1)$ and $e_2 \sim \mathcal{N}(0, q_2)$, in the weak relay regime of $|g_1| \leq \sqrt{p|h_{12}|}, |g_2| \leq \sqrt{p|h_{21}|}$, the mutual-information terms in (3.34)-(3.38) can be bounded as follows:

\[
a_1 \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2}, \tag{F.1}
\]
\[
a_1 + \Delta \tilde{a}_1 \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1 + \text{SNR}_r}{1 + \text{INR}_1} \right) - \alpha(q_1), \tag{F.2}
\]
\[
d_1 \geq \frac{1}{2} \log (1 + \text{SNR}_1) - \frac{1}{2}, \tag{F.3}
\]
\[
d_1 + \Delta \tilde{d}_1 \geq \frac{1}{2} \log (1 + \text{SNR}_1 + \text{SNR}_r) - \alpha(q_1), \tag{F.4}
\]
\[
e_1 \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1 + \text{INR}_2} \right) - \frac{1}{2}, \tag{F.5}
\]
\[
e_1 + \Delta \tilde{e}_1 \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{INR}_2 + \text{SNR}_r} \right) - \alpha(q_1), \tag{F.6}
\]
\[
g_1 \geq \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2}, \tag{F.7}
\]
\[
g_1 + \Delta \tilde{g}_1 \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1(1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{SNR}_r + \text{INR}_2}{1 + \text{SNR}_2} \right) - \alpha(q_1), \tag{F.8}
\]
\[
\xi_1 \leq \frac{1}{2} \log \left( 1 + \frac{1 + \rho}{q_1} \right) \leq \beta(q_1) - \frac{1}{2}, \tag{F.9}
\]

and the lower bounds of $a_2, a_2 + \Delta \tilde{a}_2, d_2, d_2 + \Delta \tilde{d}_2, e_2, e_2 + \Delta \tilde{e}_2, g_2, g_2 + \Delta \tilde{g}_2$ and the upper bound of $\xi_2$ can be obtained by switching the indices of 1 and 2 in (F.1)-(F.9).
Appendix F. Gaussian Relay-Interference Channel: A Useful Lemma

Proof. First, define the signal-to-noise and interference-to-noise ratios of the private messages as

\[
\begin{align*}
\text{SNR}_{1p} &= |h_{11}|^2 P_{1p}, \\
\text{INR}_{1p} &= |h_{12}|^2 P_{1p}, \\
\text{SNR}_{r1p} &= |g_1|^2 P_{1p}, \\
\text{SNR}_{2p} &= |h_{22}|^2 P_{2p}, \\
\text{INR}_{2p} &= |h_{21}|^2 P_{2p}, \\
\text{SNR}_{r2p} &= |g_2|^2 P_{2p},
\end{align*}
\]

(F.10)

which can be lower bounded or upper bonded as follows:

\[
\begin{align*}
\text{SNR}_{1p} &= |h_{11}|^2 P_{1p} \\
&= \min \left\{ |h_{11}|^2, \frac{|h_{11}|^2}{|h_{12}|^2} \right\} \\
&= \min \left\{ \text{SNR}_{1}, \frac{\text{SNR}_{r1}}{\text{INR}_{1}} \right\} \\
&\geq \frac{\text{SNR}_{1}}{1 + \text{INR}_{1}},
\end{align*}
\]

(F.12)

and

\[
0 \leq \text{INR}_{1p} = \min \{1, \text{INR}_{1}\} \leq 1,
\]

(F.13)

and

\[
\begin{align*}
\text{SNR}_{r1p} &= |g_1|^2 P_{1p} \\
&= \min \left\{ |g_1|^2, \frac{|g_1|^2}{|h_{12}|^2} \right\} \\
&= \min \left\{ \text{SNR}_{r1}, \frac{\text{SNR}_{r1}}{\text{INR}_{1}} \right\} \\
&\geq \frac{\text{SNR}_{r1}}{1 + \text{INR}_{1}}.
\end{align*}
\]

(F.14)

Since \( |g_1| \leq \sqrt{\text{INR}_{1}} \), \( \text{SNR}_{r1p} \) is upper bounded by \( \rho \). Therefore

\[
\rho \geq \text{SNR}_{r1p} \geq \frac{\text{SNR}_{r1}}{1 + \text{INR}_{1}}.
\]

(F.15)

Switching the indices of 1 and 2, we have

\[
\begin{align*}
\text{SNR}_{2p} &\geq \frac{\text{SNR}_{2}}{1 + \text{INR}_{2}}, \\
1 &\geq \text{INR}_{2p} \geq 0, \\
\rho &\geq \text{SNR}_{r2p} \geq \frac{\text{SNR}_{r2}}{1 + \text{INR}_{2}}.
\end{align*}
\]

(F.16)

(F.17)

(F.18)

Now, starting from (F.1), we prove the inequalities one by one.
Appendix F. Gaussian Relay-Interference Channel: A Useful Lemma

• First, (F.1) is lower bounded by

\[
\alpha_1 = I(X_1; Y_1 | W_1, W_2) = \frac{1}{2} \log \left( \frac{1 + \text{SNR}_{1p} + \text{INR}_{2p}}{1 + \text{INR}_{2p}} \right)
\geq \frac{1}{2} \log(1 + \text{SNR}_{1p}) - \frac{1}{2},
\]  

where (a) holds because \(0 \leq \text{INR}_{2p} \leq 1\) and (b) is due to the fact that \(\text{SNR}_{1p} \geq \frac{\text{SNR}_{1p}}{1 + \text{INR}_{1p}}\).

• Second, (F.2) is lower bounded by

\[
\alpha_1 + \Delta \tilde{\alpha}_1 = I(X_1; Y_1 | W_1, W_2) + I(X_1; \hat{Y}_{R1} | Y_1, W_1, W_2)
\geq \frac{1}{2} \log \left( \frac{(q_1 + 1)(1 + \text{SNR}_{1p} + \text{INR}_{2p}) + \text{SNR}_{r1p} + \text{SNR}_{r2p}(1 + \phi_1^2 \text{SNR}_{1p})}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}} \right)
\geq \frac{1}{2} \log(1 + \text{SNR}_{1p} + \text{SNR}_{r1p}) - \frac{1}{2} \log((q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p})
\geq \frac{1}{2} \log \left( \frac{1 + \text{SNR}_{1} + \text{SNR}_{r1}}{1 + \text{INR}_{1}} \right) - \alpha(q_1),
\]  

where (a) holds because \(\text{SNR}_{1p} \geq \frac{\text{SNR}_{1p}}{1 + \text{INR}_{1}}, \text{SNR}_{r1p} \geq \frac{\text{SNR}_{r1p}}{1 + \text{INR}_{1}},\) and

\[
\frac{1}{2} \log((q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p})
\leq \frac{1}{2} \log((q_1 + 1)(1 + 1) + \rho)
= \alpha(q_1).
\]  

• Third, (F.3) is lower bounded by

\[
\delta_1 = I(X_1; Y_1 | W_2)
\geq \frac{1}{2} \log(1 + \text{SNR}_{1}) - \frac{1}{2},
\]  

• Fourth, (F.4) is lower bounded by

\[
\delta_1 + \Delta \tilde{\delta}_1 = I(X_1; Y_1 | W_2) + I(X_1; \hat{Y}_{R1} | Y_1, W_2)
\geq \frac{1}{2} \log \left( \frac{(q_1 + 1)(1 + \text{SNR}_{1} + \text{INR}_{2p}) + \text{SNR}_{r1} + \text{SNR}_{r2p}(1 + \phi_1^2 \text{SNR}_{1})}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2p}} \right)
\geq \frac{1}{2} \log(1 + \text{SNR}_{1} + \text{SNR}_{r1}) - \alpha(q_1).
\]
• Fifth, (F.5) is lower bounded by
\[
e_1 = I(X_1, W_2; Y_1 | W_1) \\
= \frac{1}{2} \log \left( \frac{1 + \text{SNR}_{1p} + \text{INR}_2}{1 + \text{INR}_{2p}} \right) \\
\geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} + \text{INR}_2 \right) - \frac{1}{2}
\] (F.24)

• Sixth, (F.6) is lower bounded by
\[
e_1 + \Delta \tilde{e}_1 = I(X_1, W_2; Y_1 | W_1) + I(X_1, W_2; \hat{Y}_R | Y_1, W_1) \\
= \frac{1}{2} \log \left( \frac{(q_1 + 1)(1 + \text{SNR}_{1p} + \text{INR}_2) + \text{SNR}_{r1p} + \text{SNR}_{r2}(1 + \phi_1^2 \text{SNR}_{1p})}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2}} \right) \\
\geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1 (1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1}}{1 + \text{INR}_1 + \text{INR}_2 + \text{SNR}_{r2}} \right) - \alpha(q_1). 
\] (F.25)

• Seventh, (F.7) is lower bounded by
\[
g_1 = I(X_1, W_2; Y_1) \\
= \frac{1}{2} \log \left( \frac{1 + \text{SNR}_1 + \text{INR}_2}{1 + \text{INR}_{2p}} \right) \\
\geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2}. 
\] (F.26)

• Eighth, (F.8) is lower bounded
\[
g_1 + \Delta \tilde{g}_1 = I(X_1, W_2; Y_1) + I(X_1, W_2; \hat{Y}_R | Y_1) \\
= \frac{1}{2} \log \left( \frac{(q_1 + 1)(1 + \text{SNR}_1 + \text{INR}_2) + \text{SNR}_{r1} + \text{SNR}_{r2}(1 + \phi_1^2 \text{SNR}_1)}{(q_1 + 1)(1 + \text{INR}_{2p}) + \text{SNR}_{r2}} \right) \\
\geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1 (1 + \phi_1^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{SNR}_{r2}}{1 + \text{INR}_1 + \text{INR}_2 + \text{SNR}_{r2}} \right) - \alpha(q_1). 
\] (F.27)

• Ninth, (F.9) is upper bounded by
\[
\xi_1 = I(Y_R : \hat{Y}_R | Y_1, X_1, W_2) \\
= \frac{1}{2} \log \left( 1 + \frac{1}{q_1} \left( 1 + \frac{\text{SNR}_{r2p}}{1 + \text{INR}_{2p}} \right) \right) \\
\leq \frac{1}{2} \log \left( 1 + \frac{1 + \rho}{q_1} \right) 
\] (F.28)
Appendix G

Gaussian Relay-Interference Channel: Constant-Gap Result

In this appendix, we will show that using the Han-Kobayashi power splitting strategy with the private message power set to $P_{1p} = \min\{1, h_{12}^2\}$ and $P_{2p} = \min\{1, h_{21}^2\}$, all the achievable rates in (3.27)-(3.33) are within constant bits of their corresponding outer bounds in Theorem 6. Note that, in the following proof, inequalities in Appendix F are implicitly used without being mentioned.

- First, (3.27) is within constant bits of (3.5), and (3.28) is within constant bits of (3.6). To see this, the first term of (3.27) is lower bounded by

$$d_1 + (C_1 - \xi_1)^+ \geq \frac{1}{2} \log(1 + \text{SNR}_1) - \frac{1}{2} + C_1 - \xi_1$$

$$\geq \frac{1}{2} \log(1 + \text{SNR}_1) + C_1 - \left(\frac{1}{2} + \frac{1}{2} \log \left(1 + \frac{1 + \rho}{q_1}\right)\right) \tag{G.1}$$

which is within $\beta(q_1)$ bits of the first term of (3.5).

According to Lemma 1, the second term of (3.27) is lower bounded by

$$d_1 + \Delta \tilde{d}_1 \geq \frac{1}{2} \log(1 + \text{SNR}_1 + \text{SNR}_r) - \alpha(q_1), \tag{G.2}$$

which is within $\alpha(q_1)$ bits of the second term of (3.5). As a result, the gap between (3.27) and (3.5) is bounded by

$$\delta_{R_1} = \max \{\alpha(q_1), \beta(q_1)\}. \tag{G.3}$$

Due to symmetry, (3.28) is within

$$\delta_{R_2} = \max \{\alpha(q_2), \beta(q_2)\} \tag{G.4}$$

bits of the upper bound (3.6).

- Second, (3.29)-(3.31) are within constant bits of their upper bounds (3.7)-(3.18). To see this, inspecting the expressions of the achievable sum rates, it is easy to see that each of (3.29)-(3.31) has four possible combinations: having both $C_1$ and $C_2$, having $C_1$ only, having $C_2$ only, and
Appendix G. Gaussian Relay-Interference Channel: Constant-Gap Result

having none of $C_1$ and $C_2$. In the following, we show that, when specialized into the above four combinations, (3.29)-(3.31) are within constant gap to the upper bounds (3.7)-(3.18). The constant gaps are given by $\delta^{(C_1,C_2)}_{R_1+R_2}$, $\delta^{(C_1,0)}_{R_1+R_2}$, $\delta^{(0,C_2)}_{R_1+R_2}$, and $\delta^{(0,0)}_{R_1+R_2}$ (to be defined later) respectively, each corresponding to a specific combination.

- First, when having both $C_1$ and $C_2$, (3.29)-(3.31) become

\[
\begin{align*}
R_1 + R_2 & \leq a_1 + g_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+, \\
R_1 + R_2 & \leq a_2 + g_1 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+, \\
R_1 + R_2 & \leq e_1 + e_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+,
\end{align*}
\tag{G.5, G.6, G.7}
\]

which are within constant bits of (3.7)-(3.9) respectively. To show this, first, according to Lemma 1, (G.5) is lower bounded by

\[
a_1 + g_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+ \\
\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) - \frac{1}{2} + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2 + \text{INR}_1}{\text{SNR}_1}\right) - \frac{1}{2}
+ C_1 - \xi_1 + C_2 - \xi_2,
\tag{G.8}
\]

which is within $\delta^{(C_1,C_2)}_{R_1+R_2} = \beta(q_1) + \beta(q_2)$

bits of the upper bound (3.7). Due to symmetry, (G.6) is within $\delta^{(C_1,C_2)}_{R_1+R_2}$ bits of the upper bound (3.8) as well. Now applying Lemma 1, (G.7) is lower bounded by

\[
e_1 + e_2 + (C_1 - \xi_1)^+ + (C_2 - \xi_2)^+ \\
\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2}{1 + \text{INR}_2}\right) - \frac{1}{2} \\
+ C_1 - \xi_1 + C_2 - \xi_2
\tag{G.10}
\]

which is within $\delta^{(C_1,C_2)}_{R_1+R_2}$ bits of the upper bound (3.9). Therefore, when specialized to the form with both $C_1$ and $C_2$ as shown in (G.5)-(G.7), (3.29)-(3.31) have a gap of $\delta^{(C_1,C_2)}_{R_1+R_2}$ bits to their upper bounds (3.7)-(3.9).

- Second, when having $C_1$ only, (3.29)-(3.31) become

\[
\begin{align*}
R_1 + R_2 & \leq a_1 + g_2 + \Delta \tilde{g}_2 + (C_1 - \xi_1)^+, \\
R_1 + R_2 & \leq a_2 + \Delta \tilde{a}_2 + g_1 + (C_1 - \xi_1)^+, \\
R_1 + R_2 & \leq e_1 + e_2 + \Delta \tilde{e}_2 + (C_1 - \xi_1)^+,
\end{align*}
\tag{G.11, G.12, G.13}
\]

where (G.11) is lower bounded by

\[
a_1 + g_2 + \Delta \tilde{g}_2 + (C_1 - \xi_1)^+ \\
\geq \frac{1}{2} \log \left(1 + \frac{\text{SNR}_1}{1 + \text{INR}_1}\right) - \frac{1}{2} + C_1 - \xi_1
+ \frac{1}{2} \log \left(1 + \frac{\rho_2^2 \text{SNR}_1}{\text{SNR}_1}\right) + \frac{1}{2} \log \left(1 + \frac{\text{SNR}_2 + \text{INR}_1 + \text{SNR}_1}{\text{SNR}_1}\right) - \alpha(q_2),
\tag{G.14}
\]

with $\alpha(q_2)$ to be defined later.
which is within
\[ \delta_{R_1+R_2}^{(C_1,0)} = \alpha(q_2) + \beta(q_1) \] (G.15)
bits of the upper bound (3.10), and (G.12) is lower bounded by
\[ a_2 + \Delta \tilde{a}_2 + g_1 + (C_1 - \xi_1)^+ \]
\[ \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 + \text{SNR}_r}{1 + \text{INR}_2} \right) - \alpha(q_2) \]
\[ + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} C_1 + \xi_1, \] (G.16)
which is within \( \delta_{R_1+R_2}^{(C_1,0)} \) bits of the upper bound (3.11), and (G.13) can be lower bounded by
\[ e_1 + e_2 + \Delta \tilde{e}_2 + (C_1 - \xi_1)^+ \]
\[ \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2}{1 + \text{INR}_1} + \text{INR}_2 \right) - \frac{1}{2} C_1 - \xi_1 \]
\[ + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \phi_2^2 \text{SNR}_r)}{1 + \text{INR}_2} \right) + \text{INR}_1 + \text{SNR}_r - \alpha(q_2), \] (G.17)
which is within \( \delta_{R_1+R_2}^{(C_1,0)} \) bits of the upper bound (3.12).

III. Third, when having \( C_2 \) only, (3.29)-(3.31) become
\[ R_1 + R_2 \leq a_1 + \Delta \tilde{a}_1 + g_2 + (C_2 - \xi_2)^+, \] (G.18)
\[ R_1 + R_2 \leq a_2 + g_1 + \Delta \tilde{g}_1 + (C_2 - \xi_2)^+, \] (G.19)
\[ R_1 + R_2 \leq e_1 + \Delta \tilde{e}_1 + e_2 + (C_2 - \xi_2)^+. \] (G.20)
Due to the symmetry between (G.18)-(G.20) and (G.11)-(G.13), and the symmetry between their upper bounds, we can see that (G.18), (G.19) and (G.20) are within
\[ \delta_{R_1+R_2}^{(0,C_2)} = \alpha(q_1) + \beta(q_2) \] (G.21)
bits of the upper bounds (3.13), (3.14), and (3.15) respectively.

IV. Fourth, when having none of \( C_1 \) and \( C_2 \), (3.29)-(3.31) become
\[ R_1 + R_2 \leq a_1 + \Delta \tilde{a}_1 + g_2 + \Delta \tilde{g}_2, \] (G.22)
\[ R_1 + R_2 \leq a_2 + \Delta \tilde{a}_2 + g_1 + \Delta \tilde{g}_1, \] (G.23)
\[ R_1 + R_2 \leq e_1 + \Delta \tilde{e}_1 + e_2 + \Delta \tilde{e}_2, \] (G.24)
where (G.22) is lower bounded by
\[ a_1 + \Delta \tilde{a}_1 + g_2 + \Delta \tilde{g}_2 \]
\[ \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 + \text{SNR}_r}{1 + \text{INR}_1} \right) - \alpha(q_1) \]
\[ + \frac{1}{2} \log (1 + \text{SNR}_2(1 + \phi_2^2 \text{SNR}_r) + \text{SNR}_r + \text{INR}_1 + \text{SNR}_r) - \alpha(q_2), \] (G.25)
which is within
\[ \delta_{R_1 + R_2}^{(0,0)} = \alpha(q_1) + \alpha(q_2) \]  
(G.26)

bits of the upper bound (3.16). Due to symmetry, (G.23) is within \( \delta_{R_1 + R_2}^{(0,0)} \) bits of the upper bound (3.17) as well. Further, (G.24) can be lower bounded by
\[
e_1 + \Delta \hat{c}_1 + e_2 + \Delta \hat{c}_2 \\
\geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1(1 + \phi_1 \text{SNR}_{r_2}) + \text{SNR}_{r_1}}{1 + \text{INR}_1} + \text{INR}_2 + \text{SNR}_{r_2} \right) - \alpha(q_1) \\
+ \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \phi_2 \text{SNR}_{r_1}) + \text{SNR}_{r_2}}{1 + \text{INR}_2} + \text{INR}_1 + \text{SNR}_{r_1} \right) - \alpha(q_2),
\]  
(G.27)

which is within \( \delta_{R_1 + R_2}^{(0,0)} \) bits of the upper bound (3.18). Therefore, when specialized into the form with none of \( C_1 \) and \( C_2 \), (3.29)-(3.31) is within \( \delta_{R_1 + R_2}^{(0,0)} \) bits of their upper bounds (3.16)-(3.18).

Overall, the gap between the achievable sum-rates (3.29)-(3.31) and the upper bounds in (3.7)-(3.18) is upper bounded as follows:
\[
\delta_{R_1 + R_2} = \max \left\{ \delta_{R_1 + R_2}^{(C_1,0)}, \delta_{R_1 + R_2}^{(C_1,2)}, \delta_{R_1 + R_2}^{(0,C_2)}, \delta_{R_1 + R_2}^{(0,0)} \right\},
\]  
(G.28)

- Third, the achievable rate (3.32) is within constant bits of upper bounds (3.19)-(3.24). To see this, note that (3.32) has 8 different forms as follows:
\[
a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+, \]  
(G.29)
\[
a_1 + \Delta \tilde{a}_1 + g_1 + \Delta \tilde{g}_1 + e_2 + \Delta \tilde{c}_2, \]  
(G.30)
\[
a_1 + \Delta \tilde{a}_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta \tilde{c}_2, \]  
(G.31)
\[
a_1 + \Delta \tilde{a}_1 + g_1 + \Delta \tilde{g}_1 + e_2 + (C_2 - \xi_2)^+, \]  
(G.32)
\[
a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta \tilde{c}_2, \]  
(G.33)
\[
a_1 + \Delta \tilde{a}_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+, \]  
(G.34)
\[
a_1 + (C_1 - \xi_1)^+ + g_1 + \Delta \tilde{g}_1 + e_2 + \Delta \tilde{c}_2, \]  
(G.35)
\[
a_1 + (C_1 - \xi_1)^+ + g_1 + \Delta \tilde{g}_1 + e_2 + (C_2 - \xi_2)^+, \]  
(G.36)

where (G.35) is redundant compared with (G.31) and (G.36) is redundant compared with (G.34) due to the fact that \( \Delta \tilde{g}_1 \geq \Delta \tilde{a}_1 \). Therefore, there are six active rate constraints in total. In the following, we prove that all active achievable rates of \( 2R_1 + R_2 \) in (G.29)-(G.34) are within constant bits of their corresponding upper bounds in (3.19)-(3.24).
Appendix G. Gaussian Relay-Interference Channel: Constant-Gap Result

First, (G.29) is lower bounded by

\[ a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+ \]
\[ \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2} C_1 - \xi_1 \]
\[ + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} C_1 - \xi_1 \]
\[ + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) - \frac{1}{2} C_2 - \xi_2, \]

which is within

\[ \delta_{2R_1+R_2}^{(2C_1,C_2)} = 2\beta(q_1) + \beta(q_2) \]  \hspace{1cm} (G.37)

bits of the upper bound (3.19).

Second, (G.30) is lower bounded by

\[ a_1 + \Delta \tilde{a}_1 + g_1 + g_1 + \Delta \tilde{g}_1 + e_2 + \Delta \tilde{e}_2 \]
\[ \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \]
\[ + \frac{1}{2} \log (1 + \text{SNR}_1 (1 + \phi_2^2 \text{SNR}_{r2}) + \text{SNR}_{r1} + \text{INR}_2 + \text{SNR}_{r2}) - \alpha(q_1) \]
\[ + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 (1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2), \]

which is within

\[ \delta_{2R_1+R_2}^{(0,0)} = 2\alpha(q_1) + \alpha(q_2) \]  \hspace{1cm} (G.39)

bits of the upper bound (3.20).

Third, (G.31) is lower bounded by

\[ a_1 + \Delta \tilde{a}_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta \tilde{e}_2 \]
\[ \geq \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) \]
\[ + \frac{1}{2} \log (1 + \text{SNR}_1 + \text{INR}_2) - \frac{1}{2} C_1 - \xi_1 \]
\[ + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2 (1 + \phi_2^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2), \]

which is within

\[ \delta_{2R_1+R_2}^{(C_1,0)} = \alpha(q_1) + \alpha(q_2) + \beta(q_1) \]  \hspace{1cm} (G.41)

bits of the upper bound (3.21).
Fourth, (G.32) is lower bounded by

\[
\frac{a_1 + \Delta \bar{a}_1 + g_1 + \Delta \bar{g}_1 + e_2 + (C_2 - \xi_2)^+}{2} \geq \log \left( 1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) - \frac{1}{2} + C_2 - \xi_2,
\]

which is within

\[
\delta^{(0, C_2)}_{2R_1 + R_2} = 2\alpha(q_1) + \beta(q_2)
\]

bits of the upper bound (3.22).

Fifth, (G.33) is lower bounded by

\[
\frac{a_1 + (C_1 - \xi_1)^+ + g_1 + (C_1 - \xi_1)^+ + e_2 + \Delta \bar{e}_2}{2} \geq \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_1} \right) - \frac{1}{2} + C_1 - \xi_1 + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2(1 + \varphi^2 \text{SNR}_{r1}) + \text{SNR}_{r2}}{1 + \text{INR}_2} + \text{INR}_1 + \text{SNR}_{r1} \right) - \alpha(q_2),
\]

which is within

\[
\delta^{(2C_1, 0)}_{2R_1 + R_2} = \alpha(q_2) + 2\beta(q_1)
\]

bits of the upper bound (3.23).

Sixth, (G.34) is lower bounded by

\[
\frac{a_1 + \Delta \bar{a}_1 + g_1 + (C_1 - \xi_1)^+ + e_2 + (C_2 - \xi_2)^+}{2} \geq \log \left( 1 + \frac{\text{SNR}_1 + \text{SNR}_{r1}}{1 + \text{INR}_1} \right) - \alpha(q_1) + \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} + \text{INR}_1 \right) - \frac{1}{2} + C_2 - \xi_2,
\]

which is within

\[
\delta^{(C_1, C_2)}_{2R_1 + R_2} = \alpha(q_1) + \beta(q_1) + \beta(q_2)
\]

bits of the upper bound (3.24).

Therefore, the gap between the achievable rate (3.32) and the corresponding upper bounds (3.19)-(3.24) is bounded by the following constant

\[
\delta_{2R_1 + R_2} = \max \left\{ \delta^{(2C_1, C_2)}_{2R_1 + R_2}, \delta^{(0, 0)}_{2R_1 + R_2}, \delta^{(C_1, 0)}_{2R_1 + R_2}, \delta^{(0, C_2)}_{2R_1 + R_2}, \delta^{(2C_1, 0)}_{2R_1 + R_2}, \delta^{(C_1, C_2)}_{2R_1 + R_2}, \delta^{(2C_1, C_2)}_{2R_1 + R_2} \right\}.
\]

Due the the symmetry between (3.33) and (3.32), and the symmetry between their corresponding
upper bounds, it is easy to see that (3.33) is also within constant gap to the upper bounds. The constant gap $\delta_{R_1+2R_2}$ can be obtained by simply switching indices of 1 and 2 in $\delta_{2R_1+R_2}$. 
Appendix H

Noisy-Network-Coding Rate Region of the Uplink Joint Processing Model

Lemma 2. For the multicell joint processing model as depicted in Fig. 4.1, the following rate region is achievable and is within a constant gap to the capacity region:

$$\forall S \subseteq [0, \cdots, L] = \mathcal{L} :$$

$$R(S) \leq \min_{T \subseteq \mathcal{L}} \left\{ \frac{1}{2} \log |I + PA^{-1}(N_0 + q_i)H_{ST}^cH_{ST}^{T^c}| + \sum_{i \in T} (C_i - \frac{1}{2} \log \left(1 + \frac{N_0}{q_i}\right)) \right\}, \quad (H.1)$$

where $q_i$ is a positive real number, $i = 1, 2, \cdots, L$, $\Lambda(1 + q_i)$ is a diagonal matrix with $(i, i)$ entry as $1 + q_i$, and $H_{ST}^c$ denotes for the transfer matrix from input $X(S)$ to output $Y(T^c)$.

Proof. Inspecting the structure of the channel model in Fig. 4.1, it is easy to see that it is a 2-layer multiple unicast Gaussian relay network. For each subset of users $S \subseteq [0, \cdots, L]$, there are in total $2^L$ cuts that separate this subset of users and the destination $D$. Each of these cuts, denoted as $T$, corresponds to a binary combination of $L$ base-stations.

Applying the noisy-network-coding theorem in [16, Theorem 1] to users in the subset $S$, we have the following achievable rate

$$R(S) \leq \min_{T \subseteq \mathcal{L}} \left\{ I(X(S), U(T); \hat{Y}(T^c), V(L)|X(S^c), U(T^c)) - I(Y(T); \hat{Y}(T)|X(L), U(L), V(L), \hat{Y}(T^c)) \right\}$$

$$= \min_{T \subseteq \mathcal{L}} \left\{ \sum_{i \in T} C_i + I(X(S); \hat{Y}(T^c)|X(S^c)) - I(Y(T); \hat{Y}(T)|X(L)) \right\}, \quad (H.2)$$

where $\hat{Y}(L) = Y(L) + \hat{Z}(L)$ stands for the quantization signal conveyed from base-station $i$ to the destination. In (a) the following facts are applied

- $U_i = V_i$, $i \in \mathcal{L}$, and they are independent of anything else,
• $I(U_i; V_i) = h(U_i) \leq C_i$,

• Quantization noise $Z_i$ is independent of anything else.

Now, notice that the channel matrix can be written as

$$
\begin{bmatrix}
Y(T) \\
Y(T^c)
\end{bmatrix} =
\begin{bmatrix}
H_{ST} & H_{S^c T} \\
H_{ST^c} & H_{S^c T^c}
\end{bmatrix}
\begin{bmatrix}
X(S) \\
X(S^c)
\end{bmatrix}
+ \begin{bmatrix}
Z(T) \\
Z(T^c)
\end{bmatrix},
\tag{H.3}
$$

employ Gaussian inputs and the Gaussian quantization scheme, i.e., $X_i \sim \mathcal{N}(0, P)$ and $\hat{Z}_i \sim \mathcal{N}(0, q_i)$, where $q_i$ is the quantization level identical for all base-stations, we have

$$
I(X(S); \hat{Y}(T^c)|X(S^c)) = I(X(S); H_{ST^c}X(S) + Z(T^c) + \hat{Z}(T^c)) \\
= \frac{1}{2} \log \left| I + P\Lambda^{-1}(N_0 + q_i)H_{ST^c}H_{ST^c}^T \right|, \tag{H.4}
$$

and

$$
I(Y(T); \hat{Y}(T)|X(L)) = \sum_{i \in \mathcal{T}} \frac{1}{2} \log \left( 1 + \frac{N_0}{q_i} \right). \tag{H.5}
$$

Proof is completed by substituting (H.4) and (H.5) into (H.2). \qed
Bibliography


