NONLINEAR QUANTUM OPTICS IN ARTIFICIALLY STRUCTURED MEDIA

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Physics
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Abstract
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2013

This thesis presents an analysis of photon pairs generated via either spontaneous parametric downconversion or spontaneous four-wave mixing in channel waveguides as well as in microring resonators side-coupled to channel waveguides. The state of photons exiting a particular device is calculated within a general Hamiltonian formalism that simplifies the link between quantum nonlinear optics experiments and classical nonlinear optics experiments. This state contains information regarding photon pair production efficiency as well as modal and spectral correlations between the two photons, characterized by a two-dimensional spectral distribution function called the biphoton wave function.

In the limit of a low probability of pair production, photon pair production efficiencies are cast into forms resembling corresponding well-known classical nonlinear optical frequency conversion efficiencies, making it easy to see what plays the role of a classical “seed” field in an un-seeded (quantum) process. This also allows photon pair production efficiencies to be calculated based on the results of classical nonlinear optical experiments. It is further calculated that, unless generated photons are collected over a very narrow frequency range, their generation efficiency does not scale the same way with device length in a channel waveguide, or resonance quality factor in a microring resonator, as might be expected from the corresponding classical frequency conversion efficiency. Although calculations do not include self- or cross-phase modulation, nor two-photon absorption or free-carrier absorption, it is calculated that their neglect is justified in the low pair production probability limit. Linear (scattering) loss is also neglected, though partially addressed in the final chapter of this thesis.
Biphoton wave functions are calculated explicitly, such that their shape and orientation, including approximate analytic expressions for their widths, can easily be determined. This further allows estimation of the suitability of their associated photon pairs for various quantum information processing applications. As an alternative to dispersion engineering a channel waveguide photon pair source, it is calculated that microring resonators can very naturally produce nearly spectrally uncorrelated photon pairs, which behave very much like idealized single-mode photons and are thus useful for applications involving the interference of photons from multiple sources.
Acknowledgements

Thanks must go to my principal supervisor, John Sipe, for his inspiration, guidance, and support. I will leave the University of Toronto in 2013 with a PhD every bit as excited about and interested in quantum mechanics as I became during his undergraduate lectures back in 2004. I also would like to thank my co-supervisors Amr Helmy and Daniel James for helping me stay on track as well as numerous useful comments and suggestions. This thesis is better for them.

Thanks also goes to the excellent postdoctoral fellows of John’s Marco Liscidini and Sergei Zhukovsky that I worked with at the University of Toronto. You are both hugely responsible for shaping my methodology and outlook towards physics research, as well as many of the successes of my PhD. I look forward to continuing to work with both of you.

Thanks to Mike Steel, for inviting me to come visit Australia and hosting me at Macquarie University for 9 weeks in 2012. It was an incredibly rewarding experience that has led, and will hopefully continue to lead, to many interesting collaborations. Thanks also goes to his PhD student Thomas Meany, who kindly invited me out to experience the city, and all of the MQ Photonics and CUDOS teams, in particular Alex Clark, Chunle Xiong, and Chad Husko, for their hospitality and many interesting discussions.

Thanks to my (largely experimental) collaborators Eric Zhu, Li Qian, Payam Abolghasem, Dongpeng Kang, Rolf Horn, Nathalie Vermeulen, and Daniele Bajoni. I have learned so much while working with you.

Thanks to my officemates throughout the years, Ganesh Ramachandran, Federico Duque-Gomez, Rob Schaffer, Vijay Venkataraman, Aida Delfan, Sylvia Swiecicki, Zaheen Sadeq, Amanda O’Halloran, Ting Wang, and the 10th floor crew, including Eric Lee, and Jeff Rau, for letting me bounce ideas off of you, helping me understand things, and many great discussions and lunches. I don’t think there’s much in life that Ganesh can’t help with.

Thanks to my family, for instilling in me a desire for lifelong learning and encouraging me to follow my dreams.

Thanks finally to my good friends Graham Haines and Mark Norman. You’ve been there throughout the entire process and have played a major role in helping me see this through to the end.
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Chapter 1

Introduction

The field of nonlinear optics, first explored over 50 years ago [1], continues today to see improvements in conversion efficiency and reductions in system size. Advances in materials science research and fabrication techniques are moving bulk crystal experiments onto a chip and into integrated commercial devices. Microring resonators, for instance, have found use in signal processing and optical sensing, as classical frequency conversion processes within them are well-understood [2, 3]. As with the move to integrated circuits from vacuum tubes and discrete elements, microring resonators and other artificially structured media allow for nonlinear optical applications to be realized more compactly and requiring less power than with bulk crystal optics.

Such frequency conversion processes are possible because in general a nonlinear optical material is capable of having an induced polarization $P(r, t)$, and thus emitting radiation [4], proportional to all incident fields as well as any power of each. A simple model that illustrates this is

$$P^i(r, t) = \sum_j \epsilon_0 \chi_1^{ij}(r) E^j(r, t) + \sum_{jk} \epsilon_0 \chi_2^{ijk}(r) E^j(r, t) E^k(r, t) + \sum_{jkl} \epsilon_0 \chi_3^{ijkl}(r) E^j(r, t) E^k(r, t) E^l(r, t) + \cdots,$$

where $\epsilon_0$ is the permittivity of free space, $\chi_1(r)$ is the linear electric susceptibility, $\chi_2(r)$ the second-order nonlinear electric susceptibility, $\chi_3(r)$ the third-order nonlinear electric susceptibility, $E^i(r, t)$ the $i$th component of the total electric field, and superscripts indicate Cartesian components. The $\chi$’s are taken to be real and, while they depend on position and therefore allow the nonlinear optical material to be inhomogeneous, the response is taken to be local in both time and space. That they are real and the response is local implies that absorptive and dispersive effects, respectively, are negligible over the
frequency range of interest. From the viewpoint of an underlying microscopic theory, this results when said frequency range is well below any resonant frequencies of the nonlinear medium. As the total electric field can be composed of many incident fields, each at a different frequency, the emitted radiation can be at any frequency \( n\omega_1 + m\omega_2 + \ldots \), where \( n \) and \( m \) are integers and \( \omega_1 \) and \( \omega_2 \) are the frequencies associated with the incident fields. This allows for a multitude of possible energy exchanging processes involving incident and generated fields.

Yet just because an energy exchanging process is possible in a material does not automatically mean that it is efficient. The conversion efficiency of a given process can be limited because the polarization induced by the incident field(s) is composed of many local polarizations, which may or may not be radiating in phase. That is, fields originating at different points within the material with the nonlinear electric susceptibility will accumulate phase as they travel, which may add constructively or destructively at the back or exit surface of the material. The technique of phase matching ensures that, for at least one energy exchanging process, the sum of these fields at the back surface only increases with increasing path length through the nonlinear optical material. Thus, over substantial path lengths, phase matching can determine which energy exchanging processes are most efficient, as all others will have sums of locally generated fields that at some point decrease with increasing path length.

By way of example, consider the process of second harmonic generation (SHG) in a uniform medium. In this process, there is one incident field, only a second order nonlinear electric susceptibility, \( \chi_2 \), is required, and the generated radiation is at twice the frequency of the incident field, also known as the second harmonic. Note that this process is not possible in a material that has a center of inversion, as symmetry arguments show that the only possible \( \chi_2 \) for such a material is one that is equal to zero. For simplicity, assume a dispersive material, i.e. one where \( k(2\omega) \neq 2k(\omega) \), with a non-zero \( \chi_2 \), and take the propagation to be entirely in the \( z \) direction. Then the phase, \( \phi \), accumulated at the back surface, \( z = z_b \), by a field generated at the front or entrance surface, \( z = z_f \), can be expressed as

\[
\phi_{z_f} (z_b) = k(2\omega)(z_b - z_f) = k(2\omega)(d - z_f) + k(2\omega)(z_b - d). \tag{1.1}
\]

Similarly, the phase accumulated at the back surface by a field generated at some point, \( z = d \), between the front and back surfaces can be expressed as

\[
\phi_d (z_b) = 2k(\omega)(d - z_f) + k(2\omega)(z_b - d). \tag{1.2}
\]
The $2$ appears because the phase is proportional to the square of the field at $\omega$. Subtracting equation (1.2) from equation (1.1) leads to

$$\Delta \phi = \Delta k (d - z_f),$$

with

$$\Delta k = k(2\omega) - 2k(\omega).$$

Thus, the phase difference accumulated at the exit surface by second harmonic fields generated at different points within the nonlinear material is just a function of the point within the material at which they were generated. Since this phase difference can only get as large as $\pi$ before the sum of the locally generated second harmonic fields at the back surface starts to decrease, a coherence length can be defined as

$$L_{coh} = \frac{\pi}{|\Delta k|}.$$  

This coherence length defines the maximum path length over which power can be efficiently produced at the second harmonic. Phase matching consists of engineering the nonlinear material to make (1.3) equal to zero and thus the coherence length (1.4) infinite. For integrated devices, this could consist of exploiting the effect of cross-section on dispersion [5]. A similar technique that achieves the same effect is known as quasi-phase matching. It involves reorienting the nonlinear material every coherence length so that the sum of generated fields at the back surface never decreases with increasing path length, even though (1.3) is not made equal to zero.

Note that the coherence length (1.4) does not depend on whether two photons at frequency $\omega$ are converted into a single photon at frequency $2\omega$, or a single photon at frequency $2\omega$ is converted into two photons at frequency $\omega$. This other process in which, more generally, the incident field is at a frequency $\omega_3$, and the generated radiation is at frequencies $\omega_1$ and $\omega_2$, where $\omega_1 + \omega_2 = \omega_3$, also relies on a $\chi^2$ nonlinearity and is known as spontaneous parametric downconversion (SPDC). At first glance, since the generated radiation is not at a frequency that is some linear combination of the frequencies of the incident fields, it would appear as though SPDC is not possible. However, once the electric field has been quantized and vacuum fluctuations considered, one can think of a second field comprising fluctuations at $\omega_1$ or $\omega_2$ and interacting with the incident field. This makes SPDC an inherently quantum, and weak, process. Nevertheless, if a structure is designed to be phase matched for SHG with an incident field at $\omega$, it will automatically also be phase matched for SPDC with an incident field at $2\omega$, as these will
be the dominant energy exchanging processes over any substantial path length through the nonlinear optical material; all other processes will be completely canceled out every two of their own coherence lengths.

Being weak, SPDC, and its associated \( \chi_3 \) process, spontaneous four-wave mixing (SFWM), are not particularly useful for frequency conversion. However, being quantum, they have attracted interest for enabling the creation of quantum correlated photon pairs. Both correlated photon pairs as well as single photons, obtained from these processes when one photon of a pair is detected to signal the presence of the other, have use in optical implementations of quantum information processing (QIP), including fundamental tests of quantum mechanics [6, 7], quantum cryptography [8], super-dense coding [9], quantum teleportation [10], and quantum computing [11]. In terms of devices, just as with classical nonlinear optics devices, the catalogue of quantum correlated pair sources has expanded from bulk crystals in the 1960s [12] to include integrated devices [13, 14, 15, 16] and even microring resonators [17, 18, 19] today. In terms of theory, early treatments relied on discrete coupled modes in bulk crystals [20, 21]; often, the incident field was assumed to be an entirely classical object [22, 23], or monochromatic [24, 25]. More recent theoretical descriptions address these problems [26] and can deal with arbitrary incident field bandwidths [27] and even account for material dispersion in the normalization of the field modes involved [28]. Yet even these approaches struggle when considering arbitrary artificially structured media.

For example in problems in nonlinear optics where a full quantum treatment is desired, such as SPDC and SFWM, often the first step is the identification of an appropriate Hamiltonian. A natural starting point would be a linear Hamiltonian that involves cavity modes, modes of the electromagnetic field that can carry energy to and away from the cavity, and a coupling Hamiltonian between them. Such a Hamiltonian is often referred to as a Gardiner-Collett Hamiltonian [29], and was used by those authors in their treatment of damped quantum systems [30]. In some structures this approach can be easily implemented. For example, in the coupling of a channel waveguide to a microring resonator the coupling is often idealized as occurring at a single point [31]. Within this approximation, it is straightforward to build an effective Hamiltonian of the Gardiner-Collett type that models the optics of the resonator and channel [32, 33]. However, the problem is nontrivial for other cavity structures, such as even a simple one-dimensional Fabry-Pérot cavity. There is an extensive literature about how quasi-modes can be introduced for a Fabry-Pérot cavity and their coupling with the outside world described (see, e.g. [34]); a review of the history of the subject, and a very careful approach to the problem, has been presented by Dutra [29]. A careful analysis of more complicated
structures even in one dimension, such as a photonic crystal cavity, would certainly be more difficult. In more complex structures the problem would be even worse.

Additionally, while connections between classical and quantum conversion processes were made in the early literature [21], it is not clear how these carry over to integrated structures often designed with the enhancement of a classical nonlinear optical process in mind. While many structures exist today that could potentially be used for the generation of quantum correlated photon pairs (see, e.g., [35]), there are still some open questions. Just how efficient will this photon pair generation be? Do enhancements to the performance of devices in classical experiments scale the same way as in photon pair generation experiments? How does one estimate the efficiency of photon pair generation in a specific device given the results of a classical experiment? Is it possible to identify what plays the role of a classical “seed” field in a process such as SPDC or SFWM in general? What is the output of a general correlated photon pair source? Thus rather than work with correlation functions [36, 37, 38] or Wigner or positive P quasi-probability distributions [39], here the focus is on the Schrödinger picture output state itself. It is calculated analytically rather than numerically to gain as much physical insight as possible, and assist in the design and optimization of integrated photon pair sources.

Note that the output referred to above does not simply refer to the rate of photon pair production. While the output state of a correlated photon pair source is often treated as existing in the “computational basis” [11], a more detailed analysis shows this to be an idealization [40]. Not only do single photons have a frequency spectrum, i.e. they have a state more like

$$|\psi\rangle_{\text{real}} = \int_0^\infty d\omega \phi(\omega) a^{\dagger}_\omega |\text{vac}\rangle,$$

where $a^{\dagger}_\omega$ is a photon creation operator satisfying $[a_\omega, a^{\dagger}_{\omega'}] = \delta(\omega - \omega')$ and $\phi(\omega)$ a spectral distribution function normalized such that $\int_0^\infty d\omega |\phi(\omega)|^2 = 1$, than

$$|\psi\rangle_{\text{ideal}} = a^{\dagger}_{\omega_0} |\text{vac}\rangle,$$

where $a^{\dagger}_{\omega_0}$ is a photon creation operator satisfying $[a_{\omega_0}, a^{\dagger}_{\omega_0'}] = \delta_{\omega_0, \omega_0'}$, but also a two-photon state generated via SPDC or SFWM

$$|\psi\rangle_{\Pi} = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi(\omega_1, \omega_2) a^{\dagger}_{\omega_1} a^{\dagger}_{\omega_2} |\text{vac}\rangle,$$

contains frequency correlations between the two photons such that the biphoton wave function (BWF) $\phi(\omega_1, \omega_2)$, normalized so that $\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi(\omega_1, \omega_2)|^2 = 1$, cannot
be factored into a product of a function of $\omega_1$ and a function of $\omega_2$, i.e. $\phi(\omega_1, \omega_2) \neq f(\omega_1) f(\omega_2)$. These issues become quite important when interacting single photons with e.g. quantum memories [41] and interfering photons produced from multiple sources [42], respectively. However as there is more freedom in the design of artificially structured media than bulk crystals, there is also more freedom over the exact form the BWF, and thus the biphoton probability density $|\phi(\omega_1, \omega_2)|^2$, that they produce takes compared to bulk crystals.

In short, the move from bulk crystal optics to integrated devices has brought with it a number of theoretical challenges and questions. There is a need for theoretical tools that can calculate the output of a wide range of correlated photon pair sources, allowing for easy comparison of the utility of various sources for varied applications as well as the optimization of proposed sources for specific applications before development is started. This thesis presents and applies just that, and is organized as follows.

In Chapter 2 a general Hamiltonian formalism that places quantum and classical wave mixing processes on equal theoretical footing is presented. It correctly accounts for material and modal dispersion in the normalization of the linear modes of the system, and can easily be applied to integrated nonlinear structures. Building upon the backward Heisenberg picture approach of Yang et al. [28], it leverages the solutions of linear scattering problems to attack nonlinear scattering problems, allowing the consideration of complex cavity structures without the need to construct a Hamiltonian for the cavity modes or for coupling between cavity modes and propagating modes. Indeed, this approach makes it very simple to identify the physics of all of the interactions of the fields involved, whether classical or quantum, as well as all approximations involved in a given calculation. This is made possible by taking advantage of the fact that, to first approximation, one expects the output state from a given device to consist of an undepleted pump as well as the state of generated photons; calculating the evolution of the undepleted pump, which is often quite simple, and pulling it aside implies that what remains in the output state must be the generated photons of interest.

In Chapter 3 the formalism is applied to two specific structure types, both when a second-order nonlinearity is dominant and when a third-order nonlinearity is dominant, to answer questions relating to pair generation efficiency, enhancement, and scaling. The formalism introduced in the previous Chapter enables general expressions for the output state to be written down regardless of the source, and comparisons between the generation efficiencies of SPDC and difference frequency generation (DFG) or SHG, as well as between those of SFWM and classical four-wave mixing (FWM), in both channel waveguides and microring resonator structures are made as examples. In all cases it is shown
that, in the continuous wave (CW) limit and under the undepleted pump approximation, the average energy of a generated photon divided by a characteristic time plays the role of the classical “seed” signal in a quantum process, and that extending the length of a structure or taking advantage of a resonant cavity does not enhance spontaneous processes the same way as stimulated processes. This allows one to benchmark the efficiency of a device for generating quantum correlated photons by performing the, often simpler, corresponding classical experiment. An experimental result demonstrating this ability in silicon microring resonators and obtained by colleagues at the University of Pavia is also presented.

However it is not only the generation efficiency that should be calculated, but care should be taken to correctly calculate the state of generated photons as well. As mentioned previously, while these states are often idealized in the literature, they truly contain frequency correlations, the nature of which may not affect experiments relying on single pairs of polarization-encoded qubits, but will certainly affect multiple-source interferometry experiments [42]. Thus, in Chapter 4 the range of possible biphoton probability densities that can be generated in both channel waveguides and microring resonators is investigated. The structure and modal parameters that must be tuned in order to achieve identical generated photon spectra (marginal frequency distributions) and control their correlations are identified.

While losses, quantum fluctuations in stimulated experiments, generated photons seeding stimulated processes in spontaneous experiments, and nonlinear effects such as self- and cross-phase modulation can be important in integrated devices, the focus of the first four Chapters has been on developing simple and intuitive scaling relationships, as well as expressions for the state of generated photons including their associated biphoton probability densities. All forms of loss as well as all nonlinear effects other than the process at hand have been neglected, and results presented are only strictly valid in the undepleted pump approximation. Therefore the final two Chapters begin to address these issues. Chapter 5 presents a hierarchy of the importance of various linear and nonlinear terms that might be included in a nonlinear Schrödinger equation [43]. It is estimated that in many situations self- and cross-phase modulation, as well as two-photon absorption and associated free-carrier absorption, are likely to be negligible at the pump powers required to keep multi-pair generation low. This prediction is tested against two photon pair production experiments from the literature, where again a channel waveguide and a microring resonator are chosen as example structures.

Lastly, in Chapter 6, this thesis also partially addresses the issue of loss in artificially structured media as it relates to quantum states of light. Unlike in bulk crystals, where
usually coupling and absorption losses are the most relevant losses, the dominant loss in integrated devices is often scattering loss due to sidewall roughness from fabrication imperfections. Hamiltonians for systems of statistically independent reservoirs and the coupling of waveguide modes to them are introduced, and quantum Langevin equations are solved, with the reservoir modes eventually traced over, to calculate the propagation of a coherent state, two-photon state, and squeezed-vacuum state in the presence of scattering loss, while preserving the necessary canonical equal-time bosonic commutation relations as the system evolves.

The seventh and eighth paragraphs of this introduction are published in [44] and [45] respectively. Chapter 2 contains excerpts from M. Liscidini, L. G. Helt, and J. E. Sipe, "Asymptotic fields for a Hamiltonian treatment of nonlinear electromagnetic phenomena," Phys. Rev. A, 85(1):013833, (2012). Copyright (2012) by the American Physical Society. Additionally, the first two Sections of Chapter 3 are published in [45] (albeit with a slightly different calculation method) and the experimental Section (3.3) of Chapter 3 is adapted from [18].
Chapter 2

Formalism

In this Chapter I introduce the formalism upon which much of this thesis is based. It is a Hamiltonian formalism for which the field dynamics and normalization condition are well-known [46], but it leverages classical asymptotic-in and asymptotic-out fields to expand the magnetic and electric displacement fields and field operators into sets of modes that are useful for describing both classical and quantum nonlinear optics problems in artificially structured media. These fields are the stationary solutions of the classical linear Maxwell equations, and can be evaluated analytically or numerically.

This strategy avoids the introduction of artificial “cavity modes,” which of course do not truly exist in e.g. a microring resonator or photonic crystal structure, as well as their coupling to input/output channels; the enhancement of nonlinear optical effects due to the time light spends in a cavity-like structure is completely captured by the distribution of the asymptotic-in and asymptotic-out fields. Further, in photon generation problems the formalism takes advantage of the fact that, in the perturbative limit, the output state is expected to consist of an undepleted pump as well as the state of generated photons. This enables the state of the pump to be isolated and pulled aside, for its evolution is well-known and can be easily calculated [43], and allows identification of the remainder of the output state with the state of generated photons. Performing this isolation before the output state is calculated avoids the need to separate the (potentially non-commuting) operators of the pump and generated fields with Baker-Campbell-Hausdorff relations [47], and thus makes the physics of both the pump and generated photons more transparent.

While earlier work has anticipated some aspects of what is presented here in optics [48, 49], and analogous approaches have arisen in the treatment of electron transport in channels and cavity-like structures (see, e.g. [50]), here a general framework for problems in nonlinear quantum optics is developed. The formalism can consider single-particle states, as in elementary quantum scattering theory, as well as more complicated states...
including coherent states, squeezed vacua, and Fock states. Additionally, both the material and modal dispersion of the input and output channels, such as those sketched in Fig. 2.1, can be important in the analysis of real artificially structured materials; this approach can include them very naturally.

2.1 Asymptotic-In and -Out Fields

The asymptotic-in/-out field (operator) expansion harks back to the elementary scattering theory in quantum mechanics. There one can introduce asymptotic-in and asymptotic-out states, as done in the classic paper of Breit and Bethe [51]. These are full solutions of the (linear) Schrödinger equation, including the scattering potential, and exist everywhere in space. But they are built so that a superposition of them corresponds to an incident wave packet at $t = -\infty$ (asymptotic-in), or an exiting wave packet at $t = \infty$ (asymptotic-out) [52]. In elementary quantum mechanics the construction of these states constitutes a solution of the scattering problem itself. In photonics the construction of the corresponding solutions of the linear Maxwell equations for a structure such as that in Fig. 2.1, which can often be found numerically even if not analytically, constitutes a solution of only the linear scattering problem. Indeed, as we show in detail below, that linear scattering problem can be encapsulated in a transformation from the asymptotic-in to the asymptotic-out fields. However, those fields also form the natural basis for the treatment of a nonlinear optics problem, especially if the nonlinearity can be idealized as restricted to an “interaction” region as indicated schematically in Fig. 2.1; in the particular example of a microring resonator side-coupled to a channel waveguide this would correspond to the ring itself.

2.1.1 Structures of Interest

We consider structures of the form indicated schematically in Fig. 2.1, where there are a number of channels connected by an interaction region. The channels could all be of the same type, or of different types, as indicated in the figure. However, it will be useful to adopt the language appropriate for the kinds of channels sketched in Fig. 2.1 to fix the notation. The only really important assumption is that these channels identify all the ways that energy can move towards or away from the interaction region, i.e. all pathways of any significance are taken into account, neglecting absorption and scattering losses for now. We leave the details of the interaction region, indicated only in “cartoon fashion” by the central circular region, unspecified for the moment. When we include nonlinear
Chapter 2. Formalism

Figure 2.1: A sketch of the kind of structure of interest. The origin of the laboratory frame is at the center of the interaction region, which is where the local z coordinates of the channels take the value 0.

effects we will assume they arise only in the interaction region.

Associated with each channel we indicate a local coordinate system, with unit vectors $\hat{x}_n$, $\hat{y}_n$, and $\hat{z}_n$, where $n$ identifies the channel. The orientation of these vectors are restricted only in that each of the $\hat{z}_n$ points towards the interaction region, and the values $z_n = 0$ occur at a point that would lie near the center of the interaction region. We denote the coordinates of a position vector in one of these local frames by $\mathbf{r}_n = (x_n, y_n, z_n)$, reserving $\mathbf{r} = (x, y, z)$ to denote the coordinates in a laboratory frame, placing $\mathbf{r} = 0$ at the center of the interaction region.

It is clear from Fig. 2.1 that we assume each channel must “end” at a certain point, either at the boundary of the interaction region or somewhere within it. So below when we indicate functions of $\mathbf{r}_n$, $f(\mathbf{r}_n)$, what we mean are such functions truncated at the ending point of the indicated channel. That is, we take $f(\mathbf{r}_n)$ to really mean

$$f(\mathbf{r}_n) \to f(\mathbf{r}_n) \theta(D_n - z_n),$$

where $\theta$ is the usual step function, $\theta(z) = 0$ when $z < 0$ and $\theta(z) = 1$ when $z > 0$, and
Figure 2.2: Individual channels of the same type as in the structure of interest. They are assumed infinite in length and separated far enough from each other that they can be considered isolated.

$D_n$ is a (negative) number (recalling the defining of $z_n = 0$) that indicates the end of the channel. Functions of $r$, on the other hand, are taken to be defined everywhere in space.

We also consider individual channels that carry on infinitely in the local propagation direction, but are of the same type as the channels appearing in our actual structure. These are shown schematically in Fig. 2.2, and are to be imagined far enough apart so they can be considered as isolated systems. When considering these isolated channels we simply use $\hat{x}$, $\hat{y}$, and $\hat{z}$ to indicate the unit vectors, with $\hat{z}$ pointing along the propagation direction.

Returning to Fig. 2.1, we see that as $|r|$ increases the distance between any two channels also increases. In writing general functions of $r$, $F(r)$, we use $F(r) \sim \cdots$ to indicate that $|r|$ is large enough so that the channels can be taken to be isolated, with $\cdots$ then a good approximation of $F(r)$. We begin by neglecting the presence of any nonlinearity, and formulate both classical and quantum descriptions of the linear optical properties of our structures.
2.1.2 Classical Formulation

A starting point for our calculations will be the classical optics of the isolated channels sketched in Fig. 2.2. Since the displacement field \( D(\mathbf{r},t) \) and the magnetic field \( B(\mathbf{r},t) \) are divergenceless, it is useful to consider them as our fundamental fields. The fields \( E(\mathbf{r},t) \) and \( H(\mathbf{r},t) \) are then taken as derived fields, and our constitutive relations are strictly taken as conditions that relate \( E(\mathbf{r},t) \) and \( H(\mathbf{r},t) \) to \( D(\mathbf{r},t) \) and \( B(\mathbf{r},t) \). Within such a framework the Maxwell equations

\[
\frac{\partial B(\mathbf{r},t)}{\partial t} = -\nabla \times E(\mathbf{r},t)
\]

and

\[
\frac{\partial D(\mathbf{r},t)}{\partial t} = \nabla \times H(\mathbf{r},t)
\]

guarantee that if \( D(\mathbf{r},t) \) and \( B(\mathbf{r},t) \) are set divergenceless as an initial condition then they will remain so. Alternately, if we seek stationary fields

\[
D(\mathbf{r},t) = D_{nIk}(\mathbf{r}) e^{-i\omega_{nIk} t} + \text{c.c.},
\]

\[
B(\mathbf{r},t) = B_{nIk}(\mathbf{r}) e^{-i\omega_{nIk} t} + \text{c.c.},
\]

(2.2)

where c.c. stands for complex conjugation, those same Maxwell equations guarantee that \( D(\mathbf{r},t) \) and \( B(\mathbf{r},t) \) are divergenceless. Our notation is motivated by the fact that these fields are simply the propagation modes (e.g. “lowest-order TE-like mode” of a ridge waveguide or “first TM-band” of a photonic crystal), \( I \), of the channels, \( n \), for given wave number dependent frequencies \( \omega_{nIk} > 0 \). They can be found starting from the equations for the time derivatives, along with linear constitutive relations. Neglecting magnetic effects and taking the medium to be isotropic, for stationary fields the constitutive relations are of the form

\[
H_{nIk}(\mathbf{r}) = \mu_0^{-1} B_{nIk}(\mathbf{r}) \quad \text{and} \quad E_{nIk}(\mathbf{r}) = \epsilon_0^{-1} n^{-2} (\mathbf{r};\omega_{nIk}) D_{nIk}(\mathbf{r}),
\]

where \( \epsilon_0 n^2 (\mathbf{r};\omega_{nIk}) \) is the dielectric constant at the frequency \( \omega_{nIk} \) to be determined. These lead to time-independent Maxwell equations for the displacement and magnetic mode fields

\[
\nabla \times \left[ \frac{D_{nIk}(\mathbf{r})}{n^2(\mathbf{r};\omega_{nIk})} \right] = i\epsilon_0 \omega_{nIk} B_{nIk}(\mathbf{r}),
\]

\[
\nabla \times B_{nIk}(\mathbf{r}) = -i\mu_0 \omega_{nIk} D_{nIk}(\mathbf{r}).
\]

(2.3)

which can then be combined into a Hermitian eigenvalue problem or so-called “Master equation”

\[
\nabla \times \left[ \frac{\nabla \times B_{nIk}(\mathbf{r})}{n^2(\mathbf{r};\omega_{nIk})} \right] = \frac{\omega_{nIk}^2}{c^2} B_{nIk}(\mathbf{r}),
\]

(2.4)

for \( \omega_{nIk} \neq 0 \) subject to the divergence condition \( \nabla \cdot B_{nIk}(\mathbf{r}) = 0 \); the \( D_{nIk}(\mathbf{r}) \) can then be found from the \( B_{nIk}(\mathbf{r}) \) using the second of (2.3). In practice, (2.4) can be solved using FDTD, local eigensolver, or iterative methods. We can then expand the full fields...
for a set of isolated channels as shown in Fig. 2.1 according to

\[ D(r,t) = \sum_{n,I \in \sigma_n} \int_{-\infty}^{\infty} \mathrm{d}k \, \gamma_nIk e^{-i\omega_nIk t} D_{nIk}(r) + \text{c.c.}, \]

\[ B(r,t) = \sum_{n,I \in \sigma_n} \int_{-\infty}^{\infty} \mathrm{d}k \, \gamma_nIk e^{-i\omega_nIk t} B_{nIk}(r) + \text{c.c.}, \]

with complex mode amplitudes \( \gamma_{nIk} \); the set of modes allowed in channel \( n \) is specified by \( \sigma_n \). Here we are only interested in modes confined to the structure, and in general only over a certain frequency range, and so the integral range from \(-\infty \) to \( \infty \) in (2.5) is schematic; in general there will be mode cut-offs, and if the waveguide structure has periodicity in the \( z \) direction the range of \( k \) will at most be the first Brillouin zone. To take into account possible periodicity along the \( z \) direction of the structure we write

\[ D_{nIk}(r) = d_{nIk}(r) e^{ikz}, \]

\[ B_{nIk}(r) = b_{nIk}(r) e^{ikz}, \]

where the functions \( d_{nIk}(r) \) and \( b_{nIk}(r) \) are periodic in the \( z \) direction with the periodicity, \( \Lambda_n \), of the structure,

\[ d_{nIk}(r) = d_{nIk}(r + \Lambda_n \hat{z}), \]

\[ b_{nIk}(r) = b_{nIk}(r + \Lambda_n \hat{z}). \]

We assume no significant absorption or scattering losses in the frequency range of interest, but material dispersion is taken into account by the normalization condition \([46]\)

\[ \frac{1}{\Lambda_n} \int_{V} \mathrm{d}^3r \, \frac{d_{nIk}^*(r) \cdot d_{nIk}(r)}{\varepsilon_0 n^2(r;\omega_{nIk})} \frac{v_p(r;\omega_{nIk})}{v_g(r;\omega_{nIk})} = 1, \]

where \( v_p \) and \( v_g \) are group and phase velocities, respectively, and the integration is performed over a volume \( V \) corresponding to the entire \((x,y)\) plane as well as a single period \( \Lambda_n \) along the \( z \) axis. This condition is obtained from consideration of the full polariton modes of the system, including electromagnetic and medium components. While the medium component can be found from the electromagnetic component and the response of the system \( n(r;\omega_{nIk}) \) \([46]\), for our purposes it suffices to concentrate on the electromagnetic component. Note that while the full polariton modes form a complete orthonormal set, their electromagnetic component need not be orthogonal. The solutions
at $k$ and $-k$ are not independent, and the relation between them can be fixed by taking
\[
D_{nI(-k)}(r) = D^*_{nIk}(r), \quad B_{nI(-k)}(r) = -B^*_{nIk}(r). \tag{2.10}
\]

We return to Fig. 2.1 now, and look for solutions of the classical Maxwell equations for this structure that oscillate at frequency $\omega_{nIk}$ and have mode fields of the form
\[
D_{nIk}(r) = D_{nIk}(r_n) + D_{nIk}(r) + B_{nIk}(r) + B_{nIk}(r) + \sum_{n',I} T_{out}^{nI-nI'}(k,k') D_{nI'}(-k') (r_n'), \tag{2.12}
\]
for positive $k$. We refer to $(D_{nIk}(r), B_{nIk}(r))$ as an asymptotic-in mode field. Recall from above that the tilde indicates that the right hand side of (2.12) is only a valid approximation to $D_{nIk}(r)$ in the asymptotic regime, i.e. for large $|r|$. For an interaction region with dimensions on the order of tens of wavelengths of a given asymptotic mode field, the asymptotic regime generally corresponds to at least several thousands of wavelengths from the interaction region. This is indeed true for a cavity structure, where the fields within a few wavelengths of the cavity have a more complicated form than in (2.12). However, it should be noted that in some cases this regime may be relaxed. For some effectively one-dimensional structures, for which there is only one output channel directly in line with the input channel, such as a planar one-dimensional photonic crystal, it may be relaxed all the way to the edge of the interaction region itself [44]. Of more importance is that in practice sources and detectors are placed far enough from the interaction region to be in the asymptotic regime, and so it is photons corresponding to the fields on right hand side of (2.12) that are created and/or detected.

In expressions like (2.12) when coordinates such as $r_n$ appear on the right hand side of the equation, the implication is that they should be evaluated at the physical point corresponding to the coordinates $r$ in the laboratory frame appearing on the left hand side of the equation (but recall (2.1)). Since this mode field corresponds to a field oscillating at frequency $\omega_{nIk}$, the only $k'$ for which $T_{out}^{nI-nI'}(k,k')$ can be non-vanishing are those for
which $\omega_{n'I'k'} = \omega_{nIk}$ for positive $k$ and $k'$. If we denote by $s_{nI'n'I'}(k)$ the positive value of $k'$ for which indeed $\omega_{n'I'k'} = \omega_{nIk}$ holds for each given $k$, then $T_{nI'n'I'}^{\text{out}}(k,k')$ must be of the form

$$T_{nI'n'I'}^{\text{out}}(k,k') = \tau_{nI'n'I'}^{\text{out}}(k) \delta(k' - s_{nI'n'I'}(k)),$$

where the Dirac delta function provides the restriction.

The form of $\tau_{nI'n'I'}^{\text{out}}(k)$ is of course determined by the interaction region, but the physics of (2.12) is clear. Far from the interaction region, the exact solution $D_{nIk}^{\text{asy-in}}(r)$ of Maxwell’s equations corresponds to a wave “in-coming” in mode $I$ of channel $n$, and “out-going” waves in general in all modes of all channels. Thus these asymptotic-in mode fields are analogous to the asymptotic-in states that appear in the elementary quantum theory of scattering [51]. They share with them an important feature, which we now recall.

Suppose that we construct a superposition of asymptotic-in mode fields of the form

$$D(r,t) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \, \alpha_{nIk} e^{-i\omega_{nIk}t} D_{nIk}^{\text{asy-in}}(r) + \text{c.c.},$$

$$B(r,t) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \, \alpha_{nIk} e^{-i\omega_{nIk}t} B_{nIk}^{\text{asy-in}}(r) + \text{c.c.}$$

(2.14)

Choosing the complex amplitudes $\alpha_{nIk}$ such that, were we to put $\gamma_{nIk} = \alpha_{nIk}$ for $k > 0$ and vanishing otherwise in (2.5), we would have field profiles propagating in the $+\hat{z}_n$ direction in each channel and, at $t = 0$, the field profiles would be centered at $z_n = 0$; at $t_0 \ll 0$, they would correspond to field profiles centered at $z_n \ll 0$. Following arguments in [51], as $t \to -\infty$ the fields $(D(r,t), B(r,t))$ of (2.14) can be written in the simplified form

$$D(r,t) \to \sum_{n,I \in \sigma_n} \int_0^\infty dk \, \alpha_{nIk} e^{-i\omega_{nIk}t} D_{nIk}(r_n) + \text{c.c.},$$

$$B(r,t) \to \sum_{n,I \in \sigma_n} \int_0^\infty dk \, \alpha_{nIk} e^{-i\omega_{nIk}t} B_{nIk}(r_n) + \text{c.c.}$$

(2.15)

That is, we have field profiles incident on the interaction region from the different channels; at such an early time the fields $D_{nIk}^{\text{out}}(r)$ and $B_{nIk}^{\text{out}}(r)$ in (2.11) can be neglected, not because they are small, but because they are added together with different phases in (2.14) for different $k$ such that their net contribution vanishes. As $t \to \infty$, on the other hand, when with our choice of $\gamma_{nIk}$ in (2.5) the field profiles would be centered at $z_n \gg 0$, only the $D_{nIk}(r)$ and $B_{nIk}(r)$ in the asymptotic-in mode fields will make a contribution
to (2.14). Indeed, the only regions of \( r_n \) for which the part of the field arising from the superposition of the \( D_{nI k} (r_n) \) and \( B_{nI k} (r_n) \) would make a contribution correspond to the region \( D_n > z_n \) in the notation of (2.1), and thus do not correspond to regions in the physical structure. Thus for \( t \to \infty \) there are only out-going field profiles in each channel.

Such a superposition (2.14), with the amplitudes \( \alpha_{nI k} \) properly chosen, corresponds to a scattering experiment with fields incident in general from each channel, as described by (2.15). Two important consequences follow from this. The first is that the asymptotic-in mode fields can be calculated numerically, even if they cannot be found or reasonably modeled analytically. The second is that since the asymptotic-in mode fields can be used to model all scattering experiments, they can be used to describe all electromagnetic fields of the system, as long as there are no modes of the electromagnetic field completely bound to the interaction region. And so (2.14) can be taken as an expansion of all electromagnetic fields of interest.

Similarly we can introduce asymptotic-out mode fields: full solutions of the Maxwell equations labeled by positive \( k \) and of the form

\[
\begin{align*}
D^{\text{asy-out}}_{nI k} (r) &= D_{nI(-k)} (r_n) + D^{\text{in}}_{nI k} (r), \\
B^{\text{asy-out}}_{nI k} (r) &= B_{nI(-k)} (r_n) + B^{\text{in}}_{nI k} (r),
\end{align*}
\]  

(2.16)

where the “in” superscript indicates that far from the interaction region \( D^{\text{in}}_{nI k} (r) \) and \( B^{\text{in}}_{nI k} (r) \) consist of “in-coming” waves, i.e. waves traveling in the \(+\hat{z}_n\) direction, in each channel. We write this as

\[
D^{\text{asy-out}}_{nI k} (r) \sim D_{nI(-k)} (r_n) + \sum_{n', I' \in \sigma_n} \int_0^\infty \! dk' T^{\text{in}}_{nI, n'I'} (k, k') D_{n'I' k'} (r_{n'}),
\]  

(2.17)

and, again using the fact that this field must be oscillating at frequency \( \omega_{nI(-k)} = \omega_{nI k} \), the coefficient \( T^{\text{in}}_{nI, n'I'} (k, k') \) must be of the form

\[
T^{\text{in}}_{nI, n'I'} (k, k') = \tau^{\text{in}}_{nI, n'I'} (k) \delta (k' - s_{nI, n'I'} (k)).
\]

We can identify the physics of these mode fields in analogy with the scenario considered
above for the asymptotic-in mode fields. Here we construct

\[
D(r,t) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \beta_{nIk} e^{-i\omega_{nIkt}} D_{nI}^{\text{asy-out}}(r) + c.c.,
\]

\[
B(r,t) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \beta_{nIk} e^{-i\omega_{nIkt}} B_{nI}^{\text{asy-out}}(r) + c.c.,
\]

(2.18)

and now choose the complex amplitudes \(\beta_{nIk}\) such that, were we to put \(\gamma_{nI}(-k) = \beta_{nIk}\) for \(k > 0\) and vanishing otherwise in (2.5), we would have field profiles propagating in the \(-\hat{z}_n\) direction in each channel and, at \(t = 0\), the field profiles would be centered at \(z_n = 0\); at \(t_1 \gg 0\), they would correspond to field profiles centered at \(z_n \ll 0\). Again following arguments in [51], as \(t \to \infty\) the fields \((D(r,t), B(r,t))\) of (2.18) can be written in the simplified form

\[
D(r,t) \to \sum_{n,I \in \sigma_n} \int_0^\infty dk \beta_{nIk} e^{-i\omega_{nIkt}} D_{nI}(-k)(r_n) + c.c.,
\]

\[
B(r,t) \to \sum_{n,I \in \sigma_n} \int_0^\infty dk \beta_{nIk} e^{-i\omega_{nIkt}} B_{nI}(-k)(r_n) + c.c.
\]

(2.19)

Here we have field profiles moving away from the interaction region in all the channels; as \(t \to -\infty\), on the other hand, there are only in-coming field profiles in each channel.

Again neglecting any possible electromagnetic field modes bound to the interaction region, we can take (2.18) to be an expansion of all electromagnetic fields of interest. Indeed, both of the expansions (2.14) and (2.18) correspond to field profiles incident on the interaction region for \(t \ll 0\), and field profiles moving away from the interaction region for \(t \gg 0\). The difference is that in the asymptotic-in expansion it is easier to identify the amplitudes of the fields incident on the structure (the \(\alpha_{nIk}\), see (2.15)), and in the asymptotic-out expansion it is easier to identify the amplitudes of the fields departing from the structure (the \(\beta_{nIk}\), see (2.19)). Clearly the two sets of mode fields are not independent; looking at the mode fields in (2.11) and (2.16), and recalling the choice (2.10), Maxwell’s equations ensure that we have

\[
D_{nI}^{\text{asy-out}}(r) = \left(D_{nI}^{\text{asy-in}}(r)\right)^*,
\]

\[
B_{nI}^{\text{asy-out}}(r) = \left(-B_{nI}^{\text{asy-in}}(r)\right)^*.
\]

(2.20)

So a determination of the asymptotic-in mode fields suffices to determine the asymptotic-
out mode fields as well. From the asymptotic forms (2.12) and (2.17) we find the relations
\[ T_{nI,n'I'}^{\text{in}}(k,k') = (T_{nI,n'I'}^{\text{out}}(k,k'))^*, \] (2.21)
These expressions allow us to rewrite a number of equations in the following Section that involve the \( T_{nI,n'I'}^{\text{out}}(k,k') \) in equivalent forms that instead involve the \( T_{nI,n'I'}^{\text{in}}(k,k') \).

### 2.1.3 Quantum Formulation

This is a convenient point to quantize the electromagnetic field. For the isolated channels of Fig. 2.2 we can write our field operators as
\[ D(r) = \sum_{n,I \in \sigma_n} \int_{-\infty}^{\infty} dk \sqrt{\frac{\hbar \omega_{nI}}{2}} c_{nI} D_{nI}(r) + \text{H.c.}, \]
\[ B(r) = \sum_{n,I \in \sigma_n} \int_{-\infty}^{\infty} dk \sqrt{\frac{\hbar \omega_{nI}}{2}} c_{nI} B_{nI}(r) + \text{H.c.}, \] (2.22)
where we work in the Schrödinger picture. The factors \( \sqrt{\frac{\hbar \omega_{nI}}{2}} \) are explicitly included in the definitions of \( D(r) \) and \( B(r) \) so that, with the mode fields \( D_{nI}(r) \) and \( B_{nI}(r) \) normalized as above (recall (2.8) and (2.9)), the operators \( c_{nI} \) and their adjoints \( c_{nI}^\dagger \) satisfy the canonical commutation relations,
\[ [c_{nI}, c_{n'I'}^{\dagger}] = 0, \]
\[ [c_{nI}, c_{n'I'}^{\dagger}] = \delta_{nn'} \delta_{II'} \delta(k - k') . \] (2.23)
With these definitions, the linear Hamiltonian for these isolated channels is given by
\[ H_{1C} = \sum_{n,I \in \sigma_n} \int_{-\infty}^{\infty} dk \hbar \omega_{nI} c_{nI}^{\dagger} c_{nI} , \] (2.24)
neglecting the zero-point energy.

Turning now to the structure of Fig. 2.1, we will be interested in states \( |\psi(t)\rangle \) such that for \( t = t_0 \ll 0 \) we have energy moving toward the interaction region, and for \( t = t_1 \gg 0 \) energy is departing from the interaction region. We can expand the field operators in...
terms of the asymptotic-in fields

\[ D(\mathbf{r}) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} a_{nI_k} D_{nI_k}^{\text{asy-in}}(\mathbf{r}) + \text{H.c.}, \]

\[ B(\mathbf{r}) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} a_{nI_k} B_{nI_k}^{\text{asy-in}}(\mathbf{r}) + \text{H.c.}, \tag{2.25} \]

or asymptotic-out fields

\[ D(\mathbf{r}) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} b_{nI_k} D_{nI_k}^{\text{asy-out}}(\mathbf{r}) + \text{H.c.}, \]

\[ B(\mathbf{r}) = \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} b_{nI_k} B_{nI_k}^{\text{asy-out}}(\mathbf{r}) + \text{H.c.} \tag{2.26} \]

Looking at the first (2.25) of these, we use the expression (2.12) to write

\[ D(\mathbf{r}) \sim \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} a_{nI_k} D_{nI_k}(\mathbf{r}) + \text{H.c.} \]

\[ + \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} \bar{c}_{nI_k} D_{nI(-k)}(\mathbf{r}) + \text{H.c.}, \tag{2.27} \]

where we have used (2.13) and

\[ \bar{c}_{nI_k} = \sum_{n',I' \in \sigma_n'} \int_0^\infty dk' a_{n'I'k'} T_{n'I'nI}^{\text{out}}(k', k). \tag{2.28} \]

Now as \( t \to -\infty \) only the first term in (2.27) will be relevant, and since in this limit we could either use (2.27) or the isolated channel limit (2.22) to describe the electromagnetic field operators, following (2.23) we must have

\[ [a_{nI_k}, a_{n'I'k'}] = 0, \]

\[ [a_{nI_k}, a_{n'I'k'}^\dagger] = \delta_{nn'} \delta_{II'} \delta(k - k'). \tag{2.29} \]

As \( t \to \infty \) only the second term in (2.27) will be relevant, and again since we could either
use \((2.27)\) or the isolated channel limit \((2.22)\) we must have
\[
\left[\bar{c}_{nIk}, \bar{c}_{n'I'k'}\right] = 0,
\left[\bar{c}_{nIk}, \bar{c}_{n'I'k'}^\dagger\right] = \delta_{nn'}\delta_{II'}\delta(k - k').
\tag{2.30}
\]
The first of \((2.30)\) automatically follows from \((2.28)\), and the second identifies the condition
\[
\sum_{n',I'\in\sigma_{n'}} \int_0^\infty dk' T^{\text{out}}_{n'I';nI} (k', k_1) \left[T^{\text{out}}_{n'I';n_2I_2} (k', k_2)\right]^* = \delta_{n_1n_2}\delta_{I_1I_2}\delta(k_1 - k_2).\tag{2.31}
\]
Corresponding arguments starting from \((2.26)\) and the asymptotic form \((2.17)\) of the asymptotic-out mode fields leads to
\[
\left[\bar{b}_{nIk}, \bar{b}_{n'I'k'}\right] = 0,
\left[\bar{b}_{nIk}, \bar{b}_{n'I'k'}^\dagger\right] = \delta_{nn'}\delta_{II'}\delta(k - k'),
\tag{2.32}
\]
and again \((2.31)\), written in terms of the \(T^{\text{in}}_{n'I';nI} (k', k)\) (recall \((2.21)\)).

We can now identify the relation between the \(D^{\text{asy-out}}_{nIk} (r)\) and the \(D^{\text{asy-in}}_{nIk} (r)\); \((2.20)\) gives one relation involving complex conjugation, but since both asymptotic-in and -out fields are complete it is possible to identify a relation of the form
\[
D^{\text{asy-out}}_{nIk} (r) = \sum_{n',I'\in\sigma_{n'}} \int_0^\infty dk' \mathcal{C} (nIk; n'I'k') \cdot D^{\text{asy-in}}_{n'I'k'} (r). \tag{2.33}
\]
As both \(D^{\text{asy-out}}_{nIk} (r)\) and \(D^{\text{asy-in}}_{nIk} (r)\) correspond to solutions with frequency \(\omega_{nIk}\), we can expect the coefficients \(\mathcal{C} (nIk; n'I'k')\) to contain a factor \(\delta (k' - s_{n'I';nI} (k))\). It suffices to look at the asymptotic form of the two sides of \((2.20)\), and compare the out-going parts; these must be equal, and using \((2.12)\) and \((2.17)\) we find this is satisfied with \(\mathcal{C} (nIk; n'I'k') = \left[T^{\text{out}}_{n'I';nI} (k', k)\right]^* = T^{\text{in}}_{n'I';nI} (k', k)\), where we have used \((2.21)\), and so we have
\[
D^{\text{asy-out}}_{nIk} (r) = \sum_{n',I'\in\sigma_{n'}} \int_0^\infty dk' T^{\text{in}}_{n'I';nI} (k', k) \cdot D^{\text{asy-in}}_{n'I'k'} (r). \tag{2.34}
\]
The factor \(T^{\text{in}}_{n'I';nI} (k', k)\) is restricted to \(k = s_{n'I';nI} (k')\), which is the same as the restriction \(k' = s_{nI;n'I'} (k)\) that was expected. With this in hand we can determine the relation between the lowering operators \(a_{nIk}\) and \(b_{nIk}\) appearing in the expansions \((2.25)\) and \((2.26)\) in terms of asymptotic-in and asymptotic-out fields respectively. Using \((2.34)\)
in (2.26) and comparing with (2.25), we find

\[
a_{nI} = \sum_{n', I' \in \sigma_{n'}} \int_0^\infty dk' \text{T}^{\text{in}}_{nI; n'I'} (k, k') b_{n'I'}. \tag{2.35}
\]

Using the commutation relations for the \(a_{nI}\) and their adjoints (2.29), and those of the \(b_{nI}\) and their adjoints (2.32), along with (2.21), we find

\[
\sum_{n', I' \in \sigma_{n'}} \int_0^\infty dk' \text{T}^{\text{out}}_{n1, n'I'} (k_1, k') [\text{T}^{\text{out}}_{n2, n'I'} (k_2, k')]^* = \delta_{n1n2} \delta_{I1I2} (k_1 - k_2), \tag{2.36}
\]

which complements (2.31). An additional useful property of the \(\text{T}^{\text{in}}_{nI; n'I'} (k, k')\) can be obtained by substituting (2.12) in (2.34). This leads to

\[
\text{D}^{\text{asy-out}}_{nI} (r) \sim \sum_{n', I' \in \sigma_{n'R}} \int_0^\infty dk' \begin{pmatrix} \text{T}^{\text{in}}_{n'I'; nI} (k', k) \text{D}^{\text{asy-out}}_{n'I'k'} (r_{n'}) \\ + \sum_{n'' I'' \in \sigma_{n'R}} \int_0^\infty dk'' \text{T}^{\text{in}}_{n'' I''; nI} (k', k) \text{T}^{\text{out}}_{n'I'; n'' I'} (k', k'') \text{D}^{\text{asy-out}}_{n'' I'k''} (r_{n''}) \end{pmatrix} \\
= \sum_{n', I' \in \sigma_{n'}} \int_0^\infty dk' \text{T}^{\text{in}}_{n'I'; nI} (k', k) \text{D}^{\text{asy-out}}_{n'I'k'} (r_{n'}) + \text{D}_{nI(-k)} (r_n),
\]

where in the last line we have used (2.21) and (2.31). Comparing this result with (2.17) shows that

\[
\text{T}^{\text{in}}_{n'I'; nI} (k', k) = \text{T}^{\text{in}}_{nI; n'I'} (k, k'). \tag{2.37}
\]

Expressions for \(\text{D}^{\text{asy-in}}_{nI} (r)\) in terms of the \(\text{D}^{\text{asy-out}}_{nI} (r)\), the \(a_{nI}\) in terms of the \(a_{nI}\), and the \(\text{T}^{\text{out}}_{nI; n'I'} (k, k')\) can be similarly derived; we find

\[
\text{D}^{\text{asy-in}}_{nI} (r) = \sum_{n', I' \in \sigma_{n'}} \int_0^\infty dk' \text{T}^{\text{out}}_{n'I'; nI} (k', k) \text{D}^{\text{asy-out}}_{n'I'k'} (r), \tag{2.38}
\]

\[
b_{nI} = \sum_{n', I' \in \sigma_{n'}} \int_0^\infty dk' \text{T}^{\text{out}}_{nI; n'I'} (k, k') a_{n'I'}, \tag{2.39}
\]

and

\[
\text{T}^{\text{out}}_{nI; n'I'} (k, k') = \text{T}^{\text{out}}_{nI; n'I'} (k, k'). \tag{2.40}
\]
2.2 Scattering in the Linear Regime

We now treat the quantum optics of scattering from our structure in the linear regime. Because of the form of the isolated channel Hamiltonian, arguments along the lines of those following (2.25) and (2.26) lead to a Hamiltonian for the structure of Fig. 2.1 of

\[ H_L = \sum_{n,I \in \sigma_n} \int_0^{\infty} dk \ h\omega_{nk} a_{nk}^\dagger a_{nk} \]

\[ = \sum_{n,I \in \sigma_n} \int_0^{\infty} dk \ h\omega_{nk} b_{nk}^\dagger b_{nk}. \quad (2.41) \]

We construct an initial state \( |\psi(t_0)\rangle \) for \( t_0 \ll 0 \) that corresponds to energy incident on the interaction region from one or more of the channels. For reasons that will soon become apparent, we write this state in the form

\[ |\psi(t_0)\rangle = e^{-i H_L t_0 / \hbar} K_{in} \left( \left\{ \mu_{nk} a_{nk}^\dagger \right\}, \left\{ \mu_{nk}^* a_{nk} \right\} \right) |\text{vac}\rangle \]

\[ = K_{in} \left( \left\{ \mu_{nk}(t_0) a_{nk}^\dagger \right\}, \left\{ \mu_{nk}^* (t_0) a_{nk} \right\} \right) |\text{vac}\rangle, \quad (2.42) \]

where \( \mu_{nk}(t) = \mu_{nk} e^{-i \omega_{nk} t} \) with \( \{\mu_{nk}\} \) a set of complex numbers having units of the square root of length, \( |\text{vac}\rangle \) is the vacuum state of the system, \( K_{in}(\{\mu_{nk} a_{nk}^\dagger\}, \{\mu_{nk}^* a_{nk}\}) \) is a function of the sets of operators and complex numbers indicated, and in the second line we have used the fact that \( \exp(i H_L t_0 / \hbar) |\text{vac}\rangle = |\text{vac}\rangle \). Because of the nature of the asymptotic-in mode fields, it will be easy to write down expectation values of operators in \( |\psi(t_0)\rangle \), as we illustrate in an example below. Note that with our choice of the form of the initial state (2.42), at all later times we will have

\[ |\psi(t)\rangle = e^{-i H_L t / \hbar} K_{in} \left( \left\{ \mu_{nk} a_{nk}^\dagger \right\}, \left\{ \mu_{nk}^* a_{nk} \right\} \right) |\text{vac}\rangle. \quad (2.43) \]

For times \( t \gg 0 \), however, it will be convenient to write the set \( \{a_{nk}\} \) in terms of the set \( \{b_{nk}\} \), and the set \( \{a_{nk}^\dagger\} \) in terms of the set \( \{b_{nk}^\dagger\} \), using (2.35). The function \( K_{in} \left( \left\{ \mu_{nk} a_{nk}^\dagger \right\}, \left\{ \mu_{nk}^* a_{nk} \right\} \right) \) written in terms of the \( \{b_{nk}\} \) and \( \{b_{nk}^\dagger\} \) will then define a new function according to

\[ K_{out} \left( \left\{ \nu_{nk} b_{nk}^\dagger \right\}, \left\{ \nu_{nk}^* b_{nk} \right\} \right) = K_{in} \left( \left\{ \mu_{nk} a_{nk}^\dagger \right\}, \left\{ \mu_{nk}^* a_{nk} \right\} \right), \]
where the use of (2.35) is implicit and the $\{\nu_{nIk}\}$ are defined by said use. At any time, from (2.42) we can then write

$$|\psi(t)\rangle = e^{-iH_Lt/\hbar}K_{out} \left( \left\{ \nu_{nIk}b_{nIk}^\dagger \right\}, \left\{ \nu_{nIk}^*b_{nIk} \right\} \right) |\text{vac}\rangle,$$

and so at a time $t_1 \gg 0$ we then have

$$|\psi(t_1)\rangle = e^{-iH_Lt_1/\hbar}K_{out} \left( \left\{ \nu_{nIk}(t_1)b_{nIk}^\dagger \right\}, \left\{ \nu_{nIk}^*(t_1)b_{nIk} \right\} \right) |\text{vac}\rangle,$$

where $\nu_{nIk}(t) = \nu_{nIk}e^{-i\omega_{nIk}t}$, and due to the properties of the asymptotic-out mode fields it will be convenient to write down the expectation value of operators at $t_1$ using this form.

As an example, suppose we want to describe coherent states incident on the structure at $t_0$. Then we take

$$K_{in} \left( \left\{ \mu_{nIk}a_{nIk}^\dagger \right\}, \left\{ \mu_{nIk}^*a_{nIk} \right\} \right) = \exp \left( \sum_{n,I \in \sigma_n} \int_0^\infty dk \mu_{nIk}a_{nIk}^\dagger - \text{H.c.} \right),$$

and from (2.42) we have

$$|\psi(t_0)\rangle = \exp \left( \sum_{n,I \in \sigma_n} \int_0^\infty dk \mu_{nIk}(t_0)a_{nIk}^\dagger - \text{H.c.} \right) |\text{vac}\rangle.$$

In evaluating $\langle \psi(t_0)|D(r)|\psi(t_0)\rangle$ we will have

$$\langle \psi(t_0)|D(r)|\psi(t_0)\rangle \rightarrow \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar\omega_{nIk}}{2}} \mu_{nIk}(t_0)D_{nIk}(r_n) + \text{c.c.},$$

for $t_0 \ll 0$ which mimics our classical description (see the first of (2.15)), assuming that the $\mu_{nIk}\sqrt{\hbar\omega_{nIk}/2}$ have the properties of $\alpha_{nIk}$; only the “in-coming” part makes a contribution to the asymptotic-in field expansion of (2.27). Now using (2.35) and (2.21) we can construct $K_{out} \left( \left\{ \nu_{nIk}b_{nIk}^\dagger \right\}, \left\{ \nu_{nIk}^*b_{nIk} \right\} \right)$; we find

$$K_{out} \left( \left\{ \nu_{nIk}b_{nIk}^\dagger \right\}, \left\{ \nu_{nIk}^*b_{nIk} \right\} \right) = \exp \left( \sum_{n,I \in \sigma_n} \int_0^\infty dk \nu_{nIk}b_{nIk}^\dagger - \text{H.c.} \right).$$
where
\[ \nu_{nI} = \sum_{n',I' \in \sigma_n} \int_0^\infty dk'T_{n'\sigma_n}^{\text{out}}(k',k) \mu_{n'I'k'}. \]

Then from (2.44) we have
\[ |\psi(t_1)\rangle = \exp \left( \sum_{n,I \in \sigma_n} \int_0^\infty dk \nu_{nI}(t_1) b_{nI}^\dagger - \text{H.c.} \right), \]

and for \( t_1 \) large enough only the out-going part of \( D_{nI}^{\text{asy-out}}(r) \) (see (2.17)) in the first of (2.26) will give a contribution to quantities such as \( \langle \psi(t_1) | D(r) | \psi(t_1) \rangle \), and we will have

\[ \langle \psi(t_1) | D(r) | \psi(t_1) \rangle \to \sum_{n,I \in \sigma_n} \int_0^\infty dk \sqrt{\frac{\hbar \omega_{nI}}{2}} \nu_{nI}(t_1) D_{nI(-k)}(r_n) + \text{c.c.,} \tag{2.45} \]

for \( t_1 \gg 0 \).

This example is particularly simple because it essentially repeats the classical calculation given earlier. Indeed, following the calculation (2.15) to times \( t \gg 0 \) and using the results of the discussion below that equation, we find the same expression as (2.45) if we replace \( \nu_{nI} \sqrt{\hbar \omega_{nI}/2} \) by \( \alpha_{nI} \) to start. More important is the general strategy, which allows us to move from a representation in terms of the asymptotic-in mode fields to one in terms of the asymptotic-out mode fields, and we now explore with a concrete example.

### 2.2.1 A Concrete Example

As a familiar passive linear optical system, we consider a symmetric, frequency- and mode-independent, beam-splitter with two input channels or “ports”, 1 and 2, and two output ports, 3 and 4 (see Fig. 2.3). We also use these numbers to identify regions of space where we want to specify the field, and thus we can say that we are interested in problems where light is incident on the beam-splitter from regions 1 and 2, and exits into regions 3 and 4.

The usual strategy (see e.g. [53]) starts in the classical regime, with equations that connect the fields in the different numbered regions of space: In region 1 we have a (usually assumed plane-) wave propagating towards the beam-splitter with amplitude \( D_1 \), in region 2 we have a wave propagating towards the beam-splitter with amplitude \( D_2 \), in region 3 we have a way propagating away from the beam-splitter with amplitude \( D_3 \), and in region 4 we have a wave propagating away from the beam-splitter with amplitude \( D_4 \). In a careful discussion the amplitudes are usually taken to be those at the position...
of the beam-splitter, and reflection and transmission coefficients are introduced via

$$
\begin{align*}
D_3 &= R_{31} D_1 + T_{32} D_2, \\
D_4 &= T_{41} D_1 + T_{42} D_2,
\end{align*}
$$

which would follow from a solution of the Maxwell equations. Conservation of energy for all possible incident fields leads to the conditions

$$
|R_{31}|^2 + |T_{41}|^2 = |R_{42}|^2 + |T_{32}|^2 = 1,
$$

and

$$
R_{31} T_{32}^* + T_{41} R_{42}^* = 0.
$$

While the actual values of the Fresnel coefficients depend on how the beam-splitter is constructed, for simplicity we take

$$
R_{31} = R_{42} = i r \\
T_{32} = T_{41} = t
$$

where $r$ and $t$ are real numbers satisfying $r^2 + t^2 = 1$. This is a symmetric beam-splitter where a fraction $t^2$ of an incident field intensity from either port 1 or port 2 is transmitted, and a fraction $r^2$ of such an incident field intensity is reflected.

The simplest generalization of this to the quantum regime is to take $D_i \rightarrow a_i$ in (2.46), where the $a_i$ are taken to be the lowering operators associated with photon modes.

Figure 2.3: A symmetric, frequency- and mode-independent beam-splitter.
in the different channels [54]. But it is not entirely obvious that this makes sense. There are no real “modes” associated with each channel, at least if one takes a mode to be as we have defined it in this thesis: as a solution of Maxwell’s equations at a particular frequency. Light incident from one channel does not stay in the channel; that indeed is the point of the beam-splitter. However, if one thinks of the beam-splitter as an example of a scattering problem one can easily identify asymptotic-in and asymptotic-out fields. Here we assume that the mode fields are the same in each channel, and thus for a single frequency may drop the subscripts $n$, $I$, and $k$ of (2.2) and use $D$ to indicate the channel modes. Recall that these identify what would be the mode fields in the individual channels if those channels extended indefinitely both forward and backward and no interaction region were present. For simplicity we take them to have propagation directions as indicated in Fig. 2.3. Then the asymptotic-in mode field $D^\text{asy-in}_1$ (see Fig. 2.4) would be given by (cf. 2.12)

\[
D^\text{asy-in}_1 = \begin{cases} 
D \text{ in region 1,} \\
=tD \text{ in region 4,} \\
=irD \text{ in region 3,} \\
=0 \text{ in region 2.}
\end{cases}
\] (2.48)

Note that this is truly a mode; the fields identified satisfy the equations (2.46) with (2.47). More precisely (2.48) and its associated magnetic mode field, with their implicit space dependencies made explicit, and multiplied by the appropriate plane-wave time dependence $e^{-i\omega t}$, are solutions of Maxwell’s equations in all of space (within the plane-wave idealizations of the usual beam-splitter problem, of course). Similarly, the asymptotic-in mode field $D^\text{asy-in}_2$ would be given by

\[
D^\text{asy-in}_2 = \begin{cases} 
D \text{ in region 2,} \\
=tD \text{ in region 3,} \\
=irD \text{ in region 4,} \\
=0 \text{ in region 1.}
\end{cases}
\]

The asymptotic-out mode fields relevant here are those associated with channels 3 and
4, and are given by

\[ D_{3}^{\text{asy-out}} = D \text{ in region 3,} \]
\[ = tD \text{ in region 2,} \]
\[ = - irD \text{ in region 1,} \]
\[ = 0 \text{ in region 4,} \]

and

\[ D_{4}^{\text{asy-out}} = D \text{ in region 4,} \]
\[ = tD \text{ in region 1,} \]
\[ = - irD \text{ in region 2,} \]
\[ = 0 \text{ in region 3.} \]

Again, note that each of these mode fields satisfies (2.46) with (2.47).

*Either* \( D_{1}^{\text{asy-in}} \) and \( D_{2}^{\text{asy-in}} \), or \( D_{3}^{\text{asy-out}} \) and \( D_{4}^{\text{asy-out}} \), form a complete set of fields for the problems of interest here, where fields are incident from regions 1 and 2 and exit into regions 3 and 4. If we adopt \( D_{1}^{\text{asy-in}} \) and \( D_{2}^{\text{asy-in}} \) we can quantize by introducing lowering operators \( a_{1} \) and \( a_{2} \), respectively, for those modes, while if we adopt \( D_{3}^{\text{asy-out}} \) and \( D_{4}^{\text{asy-out}} \) we can introduce lowering operators \( b_{3} \) and \( b_{4} \). To treat pulses, of course, these have to be extended to include a range of frequencies. To describe incident pulses the asymptotic-in fields are the most natural, since as \( t \to -\infty \) appropriate superpositions of modes at different frequencies will include only incident pulses from channels 1 and 2 (recall (2.15) and the discussion that follows); to describe exiting pulses the asymptotic-out fields are the most natural, since as \( t \to \infty \) appropriate superpositions of modes at different frequencies will include only exiting pulses from channels 3 and 4 (recall (2.19) and the discussion that follows). The action of the beam-splitter, which is just a special case of a scattering problem, can then be described in terms of the canonical transformation from the asymptotic-in fields to the asymptotic-out fields (recall (2.34) and the subsequent discussion).

We illustrate that here, with a Hong-Ou-Mandel interference calculation [55]. For a two-photon state incident on ports 1 and 2

\[ |\psi_{12}\rangle = a_{1}^{\dagger}a_{2}^{\dagger}|\text{vac}\rangle, \]
the probability of a coincidence measurement in ports 3 and 4 is

\[ P_{34} = \langle \psi_{12} | a_3^\dagger a_4^\dagger a_3 a_4 | \psi_{12} \rangle. \]

We first note that the equations written down in the usual description of the beam-splitter, namely (2.46) and (2.47) suggest a transformation of the operators inside the expectation value, rather than of the input state itself. That is

\[
P_{34} = \langle \psi_{12} | a_3^\dagger a_4^\dagger a_3 a_4 | \psi_{12} \rangle \\
= \langle \psi_{12} | \left( -ira_1^\dagger + ta_2^\dagger \right) \left( ta_1^\dagger - ira_2^\dagger \right) (ira_1 + ta_2) (ta_1 + ira_2) | \psi_{12} \rangle \\
= (t^2 - r^2)^2 \langle \psi_{12} | a_1^\dagger a_2^\dagger a_1 a_2 | \psi_{12} \rangle \\
= (t^2 - r^2)^2.
\]

However, comparing the asymptotic fields description of the beam-splitter with (2.17)
shows that for this beam-splitter (2.47) we have, still in our simplified notation

\[ T^{\text{in}}_{3;1} = T^{\text{in}}_{4;2} = ir \]
\[ T^{\text{in}}_{4;1} = T^{\text{in}}_{3;2} = t. \]

This suggests a transformation of the state as prescribed by (2.35). That is,

\[ |\psi_{12}\rangle = a_1^\dagger a_2^\dagger |\text{vac}\rangle = \left( -irb_3^\dagger + tb_4^\dagger \right) \left( tb_3^\dagger - irb_4^\dagger \right) |\text{vac}\rangle = \left[ -irt \left( b_3^\dagger b_3^\dagger + b_4^\dagger b_4^\dagger \right) + (t^2 - r^2) b_3^\dagger b_4^\dagger \right] |\text{vac}\rangle, \]

where we have used (2.37) and so the probability of a coincidence measurement is

\[ P_{34} = \langle \psi_{12} | b_3^\dagger b_4^\dagger b_3 b_4 | \psi_{12}\rangle = (t^2 - r^2) (t^2 - r^2) = (t^2 - r^2)^2, \]

as expected. Thus it is seen that in linear problems the scattering is described by nothing more than a change of basis from asymptotic-in to asymptotic-out mode fields; the physics of the scattering is contained within the asymptotic mode fields themselves. If nonlinearities are included the situation is more complicated, but the strategy devised here can be extended to aid in the solution of such problems. We turn to that in the next Section.

### 2.3 Scattering With Nonlinearities

We now include a nonlinear term in our Hamiltonian,

\[ H = H_L + H_{\text{NL}}, \]

assuming that only field operators in the interaction region contribute to \( H_{\text{NL}} \). When \( H_{\text{NL}} \) is considered explicitly in subsequent Chapters, we will neglect terms (arising from commutators) that constitute corrections to \( H_L \) or the ground state expectation value, as well as terms containing only creation or only annihilation operators, as they do not approximately conserve energy, a concept that will be more clearly defined at the end of this Section. This leaves us with a normal-ordered nonlinear Hamiltonian, containing
three operators for a second-order nonlinearity or four for a third-order nonlinearity, for which one can clearly see the destruction of photons leading to the creation of other photons.

As there is no nonlinearity in the region of excitation, we can still start with a state $|\psi(t_0)\rangle$ at $t_0 \ll 0$ consisting of energy incident on the interaction region but far from it,

$$|\psi(t_0)\rangle = e^{-iH_{L}t_0/\hbar}K_{in}\left(\left\{\zeta_{nIk}(t_0)a_{nIk}^\dagger\right\},\left\{\zeta^{*}_{nIk}(t_0)a_{nIk}\right\}\right)|\text{vac}\rangle,$$

where we have introduced the set of variables $\{\zeta_{nIk}(t)\}$ in place of $\{\mu_{nIk}(t)\}$ as now that we are considering nonlinearities the temporal evolution of the operators may be more complicated than simple phase accumulation for which $\zeta_{nIk}(t) = \zeta_{nIk}e^{-i\omega_{nIk}(t-t_0)}$ where the $\zeta_{nIk}$ are complex numbers with units of the square root of length. The temporal evolution of the $\{\zeta_{nIk}(t)\}$ will be specified below. We define an asymptotic-in ket $|\psi_{in}\rangle$ as the ket to which this would evolve at $t = 0$ were there no nonlinearity:

$$|\psi_{in}\rangle = e^{-iH_{L}(0-t_0)/\hbar}|\psi(t_0)\rangle$$

$$= K_{in}\left(\left\{\zeta_{nIk}(t_0)a_{nIk}^\dagger\right\},\left\{\zeta^{*}_{nIk}(t_0)a_{nIk}\right\}\right)|\text{vac}\rangle. \quad (2.49)$$

In fact the ket evolves according to the entire Hamiltonian $H$, and at a later time $t_1$ the ket will be

$$|\psi(t_1)\rangle = e^{-iH(t_1-t_0)/\hbar}|\psi(t_0)\rangle.$$

The asymptotic-out ket is the ket that would evolve to this from $t = 0$ were there no nonlinearity,

$$|\psi(t_1)\rangle = e^{-iH_{L}(t_1-0)/\hbar}|\psi_{out}\rangle, \quad (2.50)$$

or

$$|\psi_{out}\rangle = e^{iH_{L}t_1/\hbar}|\psi(t_1)\rangle$$

$$= e^{iH_{L}t_1/\hbar}e^{-iH(t_1-t_0)/\hbar}|\psi(t_0)\rangle$$

$$= e^{iH_{L}t_1/\hbar}e^{-iH(t_1-t_0)/\hbar}e^{-iH_{L}t_0/\hbar}|\psi_{in}\rangle$$

$$= U(t_1,t_0)|\psi_{in}\rangle,$$

where

$$U(t_1,t_0) \equiv e^{iH_{L}t_1/\hbar}e^{-iH(t_1-t_0)/\hbar}e^{-iH_{L}t_0/\hbar}.$$ \quad (2.51)

The entire effect of the nonlinearity is described in the transition $|\psi_{in}\rangle \rightarrow |\psi_{out}\rangle$; the rest of the dynamics in the evolution from $|\psi(t_0)\rangle \rightarrow |\psi(t_1)\rangle$ is purely linear and more easily
described. In the absence of nonlinearity, of course, \( H = H_L \) and clearly \( |\psi_{\text{out}}\rangle = |\psi_{\text{in}}\rangle \).

More generally, although the form (2.49) is the easiest in which to specify the incoming fields, for the ultimate calculation of the out-going fields it is usually more convenient to use the alternate form

\[
|\psi_{\text{in}}\rangle = K_{\text{out}} \left( \left\{ \eta_{nIk}(t_0)b_{nIk}^\dagger \right\}, \{\eta_{nIk}^*(t_0)b_{nIk}\} \right) |\text{vac}\rangle ,
\]

where the variables \( \{\eta_{nIk}(t)\} \) are introduced in place of \( \{\nu_{nIk}(t)\} \) for the same reason \( \{\zeta_{nIk}(t)\} \) were introduced in place of \( \{\mu_{nIk}(t)\} \), and can be found for a particular state in terms of \( \{\zeta_{nIk}(t)\} \) using (2.35). Then the transition to \( |\psi_{\text{out}}\rangle \),

\[
|\psi_{\text{out}}\rangle = U(t_1, t_0) K_{\text{out}} \left( \left\{ \eta_{nIk}(t_0)b_{nIk}^\dagger \right\}, \{\eta_{nIk}^*(t_0)b_{nIk}\} \right) |\text{vac}\rangle
\]

can be calculated or approximated using the asymptotic-out basis, as well as, if desired, the transformation (2.50) to \( |\psi(t_1)\rangle \). However, we will see that this general strategy can be simplified in examples we give later.

As an aid in calculating \( |\psi_{\text{out}}\rangle \) we introduce a formal ket \( |\Psi(t)\rangle \) according to

\[
|\Psi(t)\rangle = U(t, t_0) |\psi_{\text{in}}\rangle ,
\]

where \( t \) varies from \( t_0 \) to \( t_1 \), and so

\[
|\psi_{\text{in}}\rangle = |\Psi(t_0)\rangle ,
\]
\[
|\psi_{\text{out}}\rangle = |\Psi(t_1)\rangle .
\]

From its definition (2.51) we have

\[
i\hbar \frac{dU(t, t_0)}{dt} = e^{iH_{L}t/\hbar} (-H_L + H) e^{-iH(t-t_0)/\hbar} e^{-iH_{L}t_0/\hbar}
\]
\[
= e^{iH_{L}t/\hbar} H_{\text{NL}} e^{-iH(t-t_0)/\hbar} e^{-iH_{L}t_0/\hbar}
\]
\[
= e^{iH_{L}t/\hbar} H_{\text{NL}} e^{-iH_{L}t/\hbar} e^{iH(t-t_0)/\hbar} e^{-iH_{L}t_0/\hbar}
\]
\[
= e^{iH_{L}t/\hbar} H_{\text{NL}} e^{-iH_{L}t/\hbar} U(t, t_0)
\]
\[
= V \left( \left\{ b_{nIk}^\dagger \right\}, \{b_{nIk}\}; t \right) U(t, t_0) ,
\]

where we have defined

\[
V \left( \left\{ b_{nIk}^\dagger \right\}, \{b_{nIk}\}; t \right) = e^{iH_{L}t/\hbar} H_{\text{NL}} e^{-iH_{L}t/\hbar} ,
\]
and so in particular
\[
\frac{\hbar}{i} \frac{d}{dt} |\Psi(t)\rangle = V \left( \left\{ b^\dagger_{nIk} \right\}, \left\{ b_{nIk} \right\} ; t \right) |\Psi(t)\rangle .
\] (2.55)

At times \( t_1 > t > t_0 \) we then seek a \( |\Psi(t)\rangle \) of the form
\[
|\Psi(t)\rangle = K_{\text{out}} \left( \left\{ \eta_{nIk}(t) b^\dagger_{nIk} \right\}, \left\{ \eta^*_n(t) b_n \right\} \right) |\tilde{\Psi}(t)\rangle ,
\] (2.56)
such that the evolution of \( |\tilde{\Psi}(t)\rangle \) then follows from the dynamics of \( |\Psi(t)\rangle \). From (2.53) and (2.49) it is clear that
\[
|\tilde{\Psi}(t_0)\rangle = |\text{vac}\rangle ,
\]
and from (2.52), (2.49), and (2.56) we have
\[
|\psi_{\text{in}}\rangle = |\Psi(t_0)\rangle = K_{\text{out}} \left( \left\{ \eta_{nIk}(t_0) b^\dagger_{nIk} \right\}, \left\{ \eta^*_n(t_0) b_n \right\} \right) |\tilde{\Psi}(t_0)\rangle ,
\]
\[
|\psi_{\text{out}}\rangle = |\Psi(t_1)\rangle = K_{\text{out}} \left( \left\{ \eta_{nIk}(t_1) b^\dagger_{nIk} \right\}, \left\{ \eta^*_n(t_1) b_n \right\} \right) |\tilde{\Psi}(t_1)\rangle ,
\] (2.57)
so the problem of finding \( |\psi_{\text{out}}\rangle \) from \( |\psi_{\text{in}}\rangle \) is reduced to the problem of evolving \( |\tilde{\Psi}(t)\rangle \) from \( t_0 \) to \( t_1 \). This task can be simplified by choosing a proper evolution of the \( \left\{ \eta_{nIk}(t) \right\} \).

At least roughly speaking, it is clear that the “best” \( \left\{ \eta(t) \right\} \) — in the sense that the evolution of the resulting \( |\tilde{\Psi}(t)\rangle \) from \( t_0 \) to \( t_1 \) is best approximated as the state just remaining in the vacuum — identifies the “best” evolution of the asymptotic-in ket in the absence of photon generation.

From (2.56) we have
\[
|\tilde{\Psi}(t)\rangle = K_{\text{out}}^\dagger \left( \left\{ \eta_{nIk}(t) b^\dagger_{nIk} \right\}, \left\{ \eta^*_n(t) b_n \right\} \right) |\Psi(t)\rangle ,
\]
so

\[
\frac{\hbar}{i} \frac{d}{dt} \langle \Psi(t) \rangle = \left( \frac{\hbar}{i} \frac{d}{dt} K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) \right) |\Psi(t)\rangle \\
+ K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) V \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} ; t \right) |\Psi(t)\rangle \\
= \left( \frac{\hbar}{i} \frac{d}{dt} K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) \right) \\
\times K_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) |\tilde{\Psi}(t)\rangle \\
+ K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) V \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} ; t \right) \\
\times K_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) |\tilde{\Psi}(t)\rangle,
\]

(2.58)

where we have used (2.55). Defining

\[
H_b(t) K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) \equiv i\frac{\hbar}{d} K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right),
\]

and

\[
\tilde{V}(t) \equiv K^\dagger_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right) V \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} ; t \right) \\
\times K_{\text{out}} \left( \left\{ \eta_{nI_k}(t) b_{nI_k}^\dagger, \eta^*_{nI_k}(t) b_{nI_k} \right\} \right),
\]

(2.60)

we rewrite (2.58) as

\[
\frac{\hbar}{i} \frac{d}{dt} |\tilde{\Psi}(t)\rangle = \tilde{H}(t) |\tilde{\Psi}(t)\rangle,
\]

where

\[
\tilde{H}(t) = H_b(t) + \tilde{V}(t).
\]

For the particular case of a generalized coherent state as the asymptotic-in state

\[
|\psi_{\text{in}}\rangle = \exp \left( \sum_{n,I} \int_0^\infty dk \zeta_{nI_k}(t_0) a_{nI_k}^\dagger - \text{H.c.} \right) |\tilde{\Psi}(t_0)\rangle
\]

\[
= \exp \left( \sum_{n,I} \int_0^\infty dk \eta_{nI_k}(t_0) b_{nI_k}^\dagger - \text{H.c.} \right) |\tilde{\Psi}(t_0)\rangle,
\]

where

\[
\eta_{nI_k}(t_0) = \sum_{n',I'} \int_0^\infty dk' T_{n'I';nI}(t_0, k') \zeta_{n'I'k'}(t_0),
\]
we have, using a Baker-Campbell-Hausdorff relation [47],

\[
K_{\text{out}}^\dagger \left\{ \eta_{nIk}(t) b_{nIk}^\dagger, \{ \eta^*_{nIk}(t) b_{nIk} \} \right\} = \exp \left( - \sum_{n,I \in \sigma} \int_0^\infty dk \eta_{nIk}(t) b_{nIk}^\dagger + \sum_{n,I \in \sigma} \int_0^\infty dk \eta^*_{nIk}(t) b_{nIk} \right) \\
= \exp \left( - \sum_{n,I \in \sigma} \int_0^\infty dk \eta_{nIk}(t) b_{nIk}^\dagger \right) \exp \left( \sum_{n,I \in \sigma} \int_0^\infty dk \eta^*_{nIk}(t) b_{nIk} \right) \\
\times \exp \left( - \frac{1}{2} \sum_{n,I \in \sigma} \int_0^\infty dk \left| \eta_{nIk}(t) \right|^2 \right).
\]

Furthermore, using

\[
\exp \left( - \sum_{n,I \in \sigma} \int_0^\infty dk \eta_{nIk}(t_0) b_{nIk}^\dagger + \text{H.c.} \right) b_{nIk} \exp \left( \sum_{n,I \in \sigma} \int_0^\infty dk \eta_{nIk}(t_0) b_{nIk}^\dagger - \text{H.c.} \right) = b_{nIk} + \eta_{nIk}(t),
\]

we find (recall (2.59))

\[
H_b(t) = \frac{i\hbar}{2} \sum_{n,I \in \sigma} \int_0^\infty dk \left[ \left( \frac{d\eta^*_{nIk}(t)}{dt} \eta_{nIk}(t) - \eta^*_{nIk}(t) \frac{d\eta_{nIk}(t)}{dt} \right) \right. \\
\left. + 2 \left( \frac{d\eta^*_{nIk}(t)}{dt} b_{nIk} - \frac{d\eta_{nIk}(t)}{dt} b_{nIk}^\dagger \right) \right],
\]

as well as

\[
\hat{V}(t) = V \left( \left\{ b_{nIk}^\dagger + \eta^*_{nIk}(t) \right\}, \{ b_{nIk} + \eta_{nIk}(t) \} ; t \right),
\]

and recognize that by a proper choice of the dynamics of \{\eta_{nIk}(t)\} we can eliminate terms in \(\hat{V}(t)\) that are first order in the operators. The equations the \{\eta_{nIk}(t)\} satisfy are the equations that would follow for the asymptotic-out mode amplitudes in the corresponding classical problem; they include the full classical nonlinear evolution through the material, including the classical generation of any higher harmonics, the classical mixing of any combinations of frequencies, and the like. Of course, they need not be determined by solving those equations; they can be determined with the aid of whatever approximations might be appropriate, or even from a numerical solution of the nonlinear equations for the Maxwell fields, and then extracting the \{\eta_{nIk}(t)\}. But however they are determined, their use in (2.56) describes how the quantum problem would evolve if the modes remained in coherent states with amplitudes determined by these classical solutions. All “quantum
effects” are contained in how the “new” $\tilde{H}(t)$ will cause $\left| \tilde{\Psi}(t) \right\rangle$ to differ from the vacuum as $t$ varies from $t_0$ to $t_1$. As such, the operators $\{b_{nIk}^\dagger\}$ and $\{b_{nIk}\}$ now appearing in $\tilde{H}(t)$ can be understood as operators that describe fluctuations about the classical state described by the $\{\eta_{nIk}\}$. Particular examples will be presented in the next Chapter.

With terms that are first order in the operators eliminated, it is clear that we may split $\tilde{H}(t)$ into a piece that involves a single raising operator followed by a single lowering operator, which we call $H_X(t)$, as well as a piece consisting of terms that involve no operators (and thus have no influence on the dynamics of the fields), or two raising or two lowering operators, or higher powers of operators, which we call $H_{\text{rest}}(t)$. Loosely speaking, the first piece corresponds to self- and cross-modulation-like terms while the second to photon pair generation and squeezing-like terms. The goal is thus to solve

$$i\hbar \frac{d}{dt} \left| \tilde{\Psi}(t) \right\rangle = (H_X(t) + H_{\text{rest}}(t)) \left| \tilde{\Psi}(t) \right\rangle,$$

subject to the initial condition

$$\left| \tilde{\Psi}(t_0) \right\rangle = \left| \text{vac} \right\rangle,$$

so that we may determine

$$\left| \psi_{\text{out}} \right\rangle = \exp \left( \sum_{n,I \in \sigma_n} \int_0^\infty dk \, \eta_{nIk}(t_1) b_{nIk}^\dagger - \text{H.c.} \right) \left| \tilde{\Psi}(t_1) \right\rangle.$$

The coherent state in front of $\left| \tilde{\Psi}(t_1) \right\rangle$ is what $\left| \psi_{\text{in}} \right\rangle$ would evolve to were there no non-linearity. To solve (2.61) we introduce an evolution operator of the form

$$U(t,t') = U_X(t,t') U_{\text{rest}}(t,t'),$$

such that

$$\left| \tilde{\Psi}(t) \right\rangle = U(t,t') \left| \tilde{\Psi}(t') \right\rangle,$$

with

$$i\hbar \frac{dU_X(t,t')}{dt} = H_X(t) U_X(t,t'),$$

and

$$U_X(t,t') \left| \text{vac} \right\rangle = U_X^\dagger(t,t') \left| \text{vac} \right\rangle = \left| \text{vac} \right\rangle.$$
From (2.61) it is seen that the evolution operator satisfies
\[ i\hbar \frac{dU(t, t')}{dt} = (H_X(t) + H_{\text{rest}}(t)) U(t, t'), \]
and thus
\[ i\hbar \frac{dU_{\text{rest}}(t, t')}{dt} = U_X^\dagger(t, t') H_{\text{rest}}(t) U_X(t, t') U_{\text{rest}}(t, t'). \]  \hfill (2.65)

Finally, applying Theorem III from the classic paper of Magnus [56], namely that
\[ \frac{d}{dt} U(t) = A(t) U(t), \]
has solution
\[ U(t) = e^{\Omega(t)}, \]
where
\[ \Omega(t) = \int_0^t A(\tau) d\tau + \frac{1}{2} \int_0^t \left[ A(\tau), \int_0^\tau A(\sigma) d\sigma \right] d\tau + \cdots, \]  \hfill (2.66)
we find the solution for (2.65) to first order in the nonlinearity
\[ \frac{dU_{\text{rest}}(t, t')}{dt} = \exp \left( \frac{1}{i\hbar} \int_{t'}^t U_X^\dagger(\tau, t') H_{\text{rest}}(\tau) U_X(\tau, t') d\tau \right), \]
and are able to solve
\[
\left| \tilde{\Psi}(t_1) \right\rangle = U(t_1, t_0) \left| \tilde{\Psi}(t_0) \right\rangle \\
= U_X(t_1, t_0) U_{\text{rest}}(t_1, t_0) U_X^\dagger(t_1, t_0) \left| \text{vac} \right\rangle \\
= U_X^\dagger(t_0, t_1) \exp \left( \frac{1}{i\hbar} \int_{t_0}^{t_1} U_X^\dagger(\tau, t_0) H_{\text{rest}}(\tau) U_X(\tau, t_0) d\tau \right) U_X(t_0, t_1) \left| \text{vac} \right\rangle \\
= \exp \left( \frac{1}{i\hbar} \int_{t_0}^{t_1} U_X^\dagger(\tau, t_1) H_{\text{rest}}(\tau) U_X(\tau, t_1) d\tau \right) \left| \text{vac} \right\rangle, \]  \hfill (2.67)
where we have used (2.62) and (2.64) in the second line, as well as (A.10) in the third and (A.9) in the fourth. We note that the time integral in (2.67), via the exponential factors introduced through (2.54), determines which terms in $H_{\text{rest}}(\tau)$ are approximately energy conserving. The longer the time from $t_0$ to $t_1$ the stronger the requirement that the exponential factors approximately sum to zero. Terms with exponential factors that do not approximately sum to zero may be neglected with little error. Returning to the discussion at the start of this Section, it is clearly not possible for the exponential factors to sum to zero if all of the factors have the same sign, i.e. if they all result from creation...
operators or all result from annihilation operators.

Lastly, if we consider a general operator, $\mathcal{O}$, residing in $H_\text{rest}(\tau)$ and ask how it evolves under the unitary transformation $U_X^\dagger(t,t')\mathcal{O}U_X(t,t') \equiv \overline{\mathcal{O}}(t)$ seen in (2.67), we can use (2.63) to show

$$i\hbar \frac{d}{dt} \overline{\mathcal{O}}(t) = \left[ \overline{\mathcal{O}}(t), H_X(t) \right],$$

subject to the initial condition

$$\overline{\mathcal{O}}(t') = U_X^\dagger(t',t')\mathcal{O}U_X(t',t') = \mathcal{O}.$$

This completes the specification of $\overline{\Psi}(t_1)$ and thus $|\psi_{\text{out}}\rangle$.

### 2.4 Discussion

In this Chapter I have presented a Hamiltonian formalism useful for calculations in nonlinear classical and quantum optics which is applicable to a large variety of photonic systems, from integrated circuits to layered structures. It exploits asymptotic-in and asymptotic-out fields, in close analogy with scattering theory in quantum mechanics. These fields are the stationary solutions of the classical linear Maxwell equations, and can be evaluated analytically or numerically. In Section 2.1 it was demonstrated that such solutions can be used to describe the electromagnetic field in either a classical or a quantum framework, and so from the results of classical calculations it is possible to construct an appropriate Hamiltonian associated with the problem of interest. In particular, in Section 2.2 it was shown that the linear scattering problem can be viewed as a basis transformation from that of asymptotic-in fields to that of asymptotic-out fields. In Section 2.3 this strategy was applied to the description of scattering in the presence of nonlinearities.

Examples in Chapter 3 illustrate the versatility of this approach. They show how this strategy leverages solutions of the classical linear Maxwell equations, found either analytically or numerically, to simplify the solution of nonlinear classical or quantum problems. They also show how it highlights the physics of the nonlinear interaction and its enhancement in cavity structures.
Chapter 3

Power Scaling Relationships

With the formalism established in the previous Chapter, I now employ it to calculate the average number of photons generated in integrated devices under various pumping situations. The formalism places quantum and corresponding classical wave mixing processes in both straight waveguides and cavity structures on equal theoretical footing, making it easy to compare how one process scales relative to another while identifying common terms.

Note that there is a difference between what is meant by corresponding processes, and reverse processes. For the purposes of this Chapter, corresponding processes are pumped at the same frequency whereas reverse processes have opposite pump and generated frequencies (see Fig. 3.1). For example, SPDC is a quantum $\chi_2$ process in which a pump photon at $\omega_P$ is converted into a photon at each $\omega_S$ and $\omega_I$, with $\omega_P = \omega_S + \omega_I$ (see Fig. 3.2). The classical $\chi_2$ process corresponding to SPDC is DFG, in which a pump photon at $\omega_P$ as well as a seed photon at $\omega_S$ create a photon at $\omega_I$. SPDC’s reverse process, on the other hand, is sum frequency generation (SFG) in which pump photons at each $\omega_S$ and $\omega_I$ are converted into a photon at $\omega_P$. Similarly the classical $\chi_3$ process corresponding to SFWM, where two pump photons at $\omega_P$ are converted into a photon at each $\omega_S$ and $\omega_I$, with $2\omega_P = \omega_S + \omega_I$ (see Fig. 3.2), is FWM, where two pump photons at $\omega_P$ as well as a seed photon at $\omega_S$ lead to a photon at $\omega_I$. The $\chi_3$ process that is the reverse of SFWM is dual-pump SFWM, in which photons at each $\omega_S$ and $\omega_I$ are converted into two photons at $\omega_P$. While it is true that knowledge of the efficiency of the classical reverse process of a quantum process in a given device - or indeed that of any nonlinear optical process of the same order as the quantum process - will allow one to infer something about the efficiency of said quantum process in the same device, it is convenient to compare quantum and classical processes each pumped at the same frequency. Therefore this Chapter focuses on corresponding processes.
The remainder of this Chapter is organized as follows. In Section 3.1 formulae derived for corresponding quantum and classical efficiencies for both $\chi_2$ and $\chi_3$ processes in channel waveguides are presented and compared. In Section 3.2 the same is done for a resonant structure, with a microring resonator side-coupled to a single channel waveguide chosen as an example. These structures are chosen to contrast the nonlinear optics of a non-resonant structure with that of a resonant structure in a particularly simple way. Both structures are effectively one-dimensional, having just a single input and output channel, and thus represent simple examples to consider. Furthermore, the microring resonator side-coupled to a single channel waveguide can be idealized as an all-pass device, coupling to the waveguide at a single point and causing no reflection, and is therefore even simpler to consider than a Fabry-Pérot cavity. Despite the fact that the physics of corresponding spontaneous and stimulated processes is most easily related, it is also generally true that it is experimentally easier to determine the efficiency of SHG than that of DFG. So as an aid in estimating the effectiveness of optical devices as SPDC
sources based on the results of classical experiments, a short discussion of SHG efficiency within the general framework is also given. In Section 3.3, the results of an experiment, performed by colleagues at the University of Pavia, Italy, that compares FWM and SFWM in silicon microring resonators is presented. A discussion and summary of the theoretical and experimental results follows in Section 3.4.

While losses, quantum fluctuations in stimulated experiments, generated photons seeding stimulated processes in spontaneous experiments, and nonlinear effects such as self- and cross-phase modulation (SPM and XPM, respectively) can be important in integrated devices, the focus here is on developing simple and intuitive scaling relationships, and so we defer their inclusion at present. In this Chapter all forms of loss as well as all nonlinear effects other than the process at hand are neglected, and the results presented are only strictly valid in the undepleted pump approximation.

### 3.1 Non-Resonant Structures

Here we consider spontaneous processes and their corresponding stimulated processes in a device such as a channel waveguide [57, 58, 59], nonlinear fiber [60, 61], or Bragg reflection waveguide [16], where propagation can be treated as effectively one-dimensional (see Fig. 3.3). We first consider $\chi_2$ processes.
3.1.1 Second-Order Processes

SPDC in such a structure was considered in an earlier work [28], but without taking advantage of the full formalism presented in the previous Chapter. For a generic integrated device with just two channels connecting to the nonlinear region, left (L) and right (R), for a pump incident on the structure from the left and generated photons exiting to the right we take as our Hamiltonian (recall (2.41), (2.26), and the discussion at the start of Section 2.3)

\[ H = H_L + H_{NL}, \]

\[ H_L = \int_0^\infty dk \hbar \omega_{Rk} b_R^\dagger b_R + \int_0^\infty dk \hbar \omega_{Lk} b_L^\dagger b_L, \]

\[ H_{NL} = -\frac{1}{3\varepsilon_0} \int d^3r \Gamma_{ij}^{jk}(r) D^i(r) D^j(r) D^k(r) \]

\[ = -\int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk \sum_{A,B,C=L,R} S_{AB;C}(k_1, k_2, k) b_{Ak_1}^\dagger b_{Bk_2}^\dagger b_{Ck} + \text{H.c.,} \]

where, for simplicity, we have assumed that all of the photons in each channel have the same mode-type (e.g. lowest-order TE-like mode) and therefore suppressed the mode-type index \( I \). We note that generalizations are straightforward [62]. Here the \( \omega_{nk} \) are eigenfrequencies of the photon modes in each channel, \( \Gamma_{ij}^{jk}(r) \) characterizes the nonlinearity, and is related to the usual material second-order optical nonlinearity \( \chi_{ij}^{jk}(r) \), neglecting dispersion and magneto-optic effects [28], and

\[ S_{AB;C}(k_1, k_2, k) = \frac{1}{\varepsilon_0} \sqrt{\frac{\hbar \omega_{Ak_1} \hbar \omega_{Bk_2} \hbar \omega_{Ck}}{8}} \]

\[ \times \int d^3r \left\{ \frac{\chi_{ij}^{jk}(r)}{\varepsilon_0 n^2(r, \omega_{Ak_1}) n^2(r, \omega_{Bk_2}) n^2(r, \omega_{Ck})} \right\} \]

\[ \times \left[ D_{A_{k_1}}^{\text{asy-out}}(r) \right]^* \left[ D_{B_{k_2}}^{\text{asy-out}}(r) \right]^* \left[ D_{C_k}^{\text{asy-out}}(r) \right]. \]

If we so chose, we could have expanded the full fields in terms of asymptotic-in mode fields (2.26) instead of asymptotic-out mode fields, or perhaps just one full field as an asymptotic-in field and the other two as asymptotic-out fields, as such an expression is most illustrative of the physics. However we expand all full fields in terms of asymptotic-out fields to simplify the following calculations.
Note that we can also write an asymptotic-in ket of the form

$$|\psi_{in}\rangle = \exp \left( \int_0^\infty \mathrm{d}k \zeta_{Lk} (t_0) a_{Lk}^\dagger - \mathrm{H.c.} \right) |\text{vac}\rangle,$$

in terms of $b_{nk}^\dagger$ operators instead of $a_{nk}^\dagger$ operators, that is

$$|\psi_{in}\rangle = \exp \left( \sum_{C=L,R} \int_0^\infty \mathrm{d}k \eta_{Ck} (t_0) b_{Ck}^\dagger - \mathrm{H.c.} \right) |\text{vac}\rangle,$$

where

$$\eta_{Ck} (t_0) = \int_0^\infty \mathrm{d}k' T_{L,C}^{\text{out}} (k', k) \zeta_{Lk'} (t_0).$$

However, restricting ourselves to devices that are all-pass filters (i.e. devices that do not reflect the pump pulse back along the same channel) we can simplify this as

$$|\psi_{in}\rangle = \exp \left( \int_0^\infty \mathrm{d}k \eta_{Rk} (t_0) b_{Rk}^\dagger - \mathrm{H.c.} \right) |\text{vac}\rangle,$$

where

$$\eta_{Rk} (t_0) = \int_0^\infty \mathrm{d}k' T_{L,R}^{\text{out}} (k', k) \zeta_{Lk'} (t_0),$$

and find (recall (2.59), (2.54), and (2.60))

$$H_b (t) = \frac{i\hbar}{2} \int_0^\infty \mathrm{d}k \left[ \left( \frac{\mathrm{d}\eta_{Rk}^*(t)}{\mathrm{d}t} \eta_{Rk} (t) - \eta_{Rk}^*(t) \frac{\mathrm{d}\eta_{Rk} (t)}{\mathrm{d}t} \right) 
+ 2 \left( \frac{\mathrm{d}\eta_{Rk}^*(t)}{\mathrm{d}t} b_{Rk} - \frac{\mathrm{d}\eta_{Rk} (t)}{\mathrm{d}t} b_{Rk}^\dagger \right) \right],$$

$$\hat{V} (t) = - \int_0^\infty \mathrm{d}k_1 \int_0^\infty \mathrm{d}k_2 \int_0^\infty \mathrm{d}k \sum_{A,B,C=L,R} \left[ S_{AB,C} (k_1, k_2, k; t) \left( \eta_{Ak_1}^* + \eta_{Rk_1} (t) \delta_{AR} \right) 
\times \left( \eta_{Bk_2} + \eta_{Rk_2} (t) \delta_{BR} \right) \left( \eta_{Ck}^* + \eta_{Rk} (t) \delta_{CR} \right) + \mathrm{H.c.} \right],$$

where

$$S_{AB,C} (k_1, k_2, k; t) = S_{AB,C} (k_1, k_2, k) e^{i(\omega_{Ak_1} + \omega_{Bk_2} - \omega_{Ck})t}. \quad (3.3)$$

This expression identifies approximately energy conserving terms as those for which
\(\omega_{Ak_1} + \omega_{Bk_2} - \omega_{Cl} \approx 0\). Furthermore, for such a simple structure we have (recall (2.6))

\[
D_{\text{Lk}}^{i,\text{asy-in}} (r) = \frac{d_{\text{Lk}}^{i,\text{asy-in}} (r)}{\sqrt{2\pi}} e^{ikz},
\]

\[
D_{\text{Rk}}^{i,\text{asy-out}} (r) = \frac{d_{\text{Rk}}^{i,\text{asy-out}} (r)}{\sqrt{2\pi}} e^{ikz},
\]

and so (3.4) also identifies approximately momentum conserving terms as those containing \(|k - k_1 - k_2|\). This simplifies (3.3) to

\[
\hat{V} (t) = -\int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{\text{RR},R} (k_1, k_2, k; t) \eta_{Rk_1}^* (t) \eta_{Rk_2} (t) \eta_{Rk} (t) + \text{H.c.}
\]

\[
-\int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{\text{LL},L} (k_1, k_2, k; t) b_{Lk_1}^\dagger b_{Lk_2} b_{Lk} + \text{H.c.}
\]

\[
-\int_0^\infty dk_S \int_0^\infty dk S_{\text{RR},R} (k_1, k_2, k; t) b_{\text{Rk_1}}^\dagger b_{\text{Rk_2}} b_{\text{Rk}} + \text{H.c.}
\]

\[
-2 \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{\text{RR},R} (k_1, k_2, k; t) b_{\text{Rk_1}}^\dagger \eta_{\text{Rk_2}} (t) b_{\text{Rk}} + \text{H.c.}
\]

\[
-\int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{\text{RR},R} (k_1, k_2, k; t) b_{\text{Rk_1}}^\dagger b_{\text{Rk_2}}^\dagger \eta_{\text{Rk}} (t) + \text{H.c.}
\]

\[
-2 \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{\text{RR},R} (k_1, k_2, k; t) b_{\text{Rk_1}}^\dagger \eta_{\text{Rk_2}}^* (t) \eta_{\text{Rk}} (t) + \text{H.c.}
\]

\[
-\int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{\text{RR},R} (k_1, k_2, k; t) \eta_{\text{Rk_1}}^* (t) \eta_{\text{Rk_2}}^* (t) b_{\text{Rk}} + \text{H.c.}
\]

Here the first term is a classical function of time and does not contribute to the state of generated photons, the second, third, and fourth terms correspond to SPDC-like terms driven by fluctuations about the pump, the fifth to SPDC itself, the sixth to DFG, and the seventh term corresponds to SFG. We note that we can eliminate the second to last term by setting

\[
\frac{d\eta_{\text{Rk}} (t)}{dt} = \frac{2i}{\hbar} \int_0^\infty dk_1 \int_0^\infty dk_2 S_{\text{RR},R} (k, k_1, k_2; t) \eta_{\text{Rk_1}}^* (t) \eta_{\text{Rk_2}} (t),
\]

the right hand side of which, for a device phase matched for SPDC and for a single pump, is unlikely to be non-zero. Therefore as a first approximation we assume an unmodified pump, and set \(\eta_{\text{Rk}} (t) = \alpha \phi_p (k)\) which in turn sets

\[H_b (t) = 0.\]

We also neglect the constants, fluctuations, and back-action on the pump, and imagine
the pump to contain photons at roughly twice the frequency of the generated photons, leading us to keep only

$$H_b(t) + \hat{V}(t) = -\alpha \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk S_{RR;R}(k_1, k_2, k; t) b^\dagger_{Rk_1} b^\dagger_{Rk_2} \phi_P(k) + \text{H.c.}$$

This greatly simplifies the calculation of the output state, as here $U_X(t_1, t_0) = I$, (recall (2.63)) and we find (recall (2.57) and (2.67))

$$|\psi_{\text{out}}\rangle = \exp \left( \alpha \int_0^\infty dk \phi_P(k) b^\dagger_{Rk} - \text{H.c.} \right) |\psi_{\text{gen}}\rangle,$$

where

$$|\psi_{\text{gen}}\rangle = \exp \left( \frac{i\alpha}{\hbar} \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk \phi_P(k) \right. \left. \times \left[ \int_{t_0}^{t_1} d\tau S_{RR;R}(k_1, k_2, \tau) \right] b^\dagger_{Rk_1} b^\dagger_{Rk_2} - \text{H.c.} \right) |\text{vac}\rangle.$$  

With the nonlinearity centered at $r = 0$ such that the pump pulse only interacts with it for times near $t = 0$, we extend $t_0 \to -\infty$, and $t_1 \to \infty$, and find that the state of SPDC generated photons in a two-channel device due to an exciting pump pulse can be written as the single-mode squeezed vacuum

$$|\psi_{\text{gen}}\rangle = \exp \left( \frac{\beta}{\sqrt{2}} \int_0^\infty dk_1 \int_0^\infty dk_2 \phi(k_1, k_2) b^\dagger_{Rk_1} b^\dagger_{Rk_2} - \text{H.c.} \right) |\text{vac}\rangle,$$

where $|\beta|^2$ is proportional to the average number of photons in the pump pulse $N_P = |\alpha|^2$ and

$$\phi(k_1, k_2) = \frac{2\sqrt{2}\pi \alpha}{\beta} \frac{i}{\hbar} \int_0^\infty dk \phi_P(k) S_{RR;R}(k_1, k_2, k) \delta(\omega_{Rk_1} + \omega_{Rk_2} - \omega_{Rk}).$$  

(3.6)

Alternately, switching to a description in terms of frequency rather than wavenumber [28],

$$|\psi_{\text{gen}}\rangle = \exp \left( \beta C^\dagger_{II} - \text{H.c.} \right) |\text{vac}\rangle,$$

(3.7)

where

$$C^\dagger_{II} = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi(\omega_1, \omega_2) b^\dagger_{R\omega_1} b^\dagger_{R\omega_2},$$  

(3.8)
such that $C_{II}^I \ket{\text{vac}}$ is a normalized two-photon state characterized by the BWF

$$
\phi (\omega_1,\omega_2) = i\pi \sqrt{\frac{dk (\omega_1)}{d\omega_1}} \sqrt{\frac{dk (\omega_2)}{d\omega_2}} \left( \sqrt{\frac{dk (\omega)}{d\omega}} \right)_{\omega=\omega_1+\omega_2} \times \sqrt{\frac{h (\omega_1+\omega_2) \omega_1 \omega_2 (\chi_2)^2 N_P}{\varepsilon_0 |\beta|^2 \pi^6}} \phi_p (\omega_1+\omega_2) J (\omega_1,\omega_2), \quad (3.9)
$$

with

$$
J (\omega_1,\omega_2) = \frac{\pi^3}{X_2} \int d^3 r \left\{ \frac{\chi^{ijk} (r)}{\varepsilon_0^{3/2} n^2 (r,\omega_1) n^2 (r,\omega_2) n^2 (r,\omega_1+\omega_2)} \times \left[ D_{Rk(\omega_1)}^{i,\text{asy-out}} (r) \right]^* \left[ D_{Rk(\omega_2)}^{i,\text{asy-out}} (r) \right]^* D_{Rk(\omega_1+\omega_2)}^{k,\text{asy-out}} (r) \right\}, \quad (3.10)
$$

and the $\pi$ and $X_2$ are, respectively, a typical value of refractive index and a typical value of second-order nonlinearity introduced solely for convenience. We note that the BWF contains information about the correlations between generated photons that can be expressed in terms of frequency (3.9), wavenumber

$$
\phi (k_1, k_2) = \left[ \sqrt{\frac{dk (\omega_1)}{d\omega_1}} \sqrt{\frac{dk (\omega_2)}{d\omega_2}} \right]^{-1} \phi (\omega_1,\omega_2),
$$
or even time

$$
\phi (t_1, t_2) = \frac{1}{2\pi} \int d\omega_1 d\omega_2 \phi (\omega_1,\omega_2) e^{i(\omega_1 t_1+\omega_2 t_2)}.
$$

However, as we view said correlations to be the result of energy (and momentum) conservation, in this thesis we prefer to work with BWFs represented in the frequency domain.

While everything above has been quite general, for the kind of device considered here, a channel waveguide in which all propagation, from channel $L$ through the nonlinearity to channel $R$, occurs along the $z$ direction

$$
D_{Lk(\omega)}^{i,\text{asy-in}} (r) = D_{Rk(\omega)}^{i,\text{asy-out}} (r) = d_{k(\omega)}^i (x,y) \frac{e^{ikz}}{\sqrt{2\pi}}, \quad (3.11)
$$

and so we drop the redundant label $R$ for the remainder of this Section. This simplifies
\[
\phi(\omega_1, \omega_2) = \ii L \sqrt{\frac{d k(\omega_1)}{d \omega_1}} \sqrt{\frac{d k(\omega_2)}{d \omega_2}} \left( \sqrt{\frac{d k(\omega)}{d \omega}} \right)_{\omega = \omega_1 + \omega_2} \times \frac{\sqrt{\omega_1 \omega_2 (\chi^2 / n)^2 N_p}}{8 \pi \varepsilon_0 |\beta|^2 n^6} \frac{h(\omega_1 + \omega_2)}{A[k(\omega_1), k(\omega_2), k(\omega_1 + \omega_2)]} \times \phi_P(\omega_1 + \omega_2) \text{sinc} \left\{ [k(\omega_1) + k(\omega_2) - k(\omega_1 + \omega_2)] L/2 \right\} ,
\]

where \(L\) is the length of the nonlinear structure, and
\[
A[k(\omega_1), k(\omega_2), k(\omega_1 + \omega_2)]
= \int dxdy \frac{n^3 \chi^{ijk}(x,y) \left[ d_{k(\omega_1)}^i(x,y) \right]^* \left[ d_{k(\omega_2)}^j(x,y) \right]^* d_{k(\omega_1+\omega_2)}^k(x,y)}{\chi \varepsilon_0^{3/2} n^2(x,y;\omega_1) n^2(x,y;\omega_2) n^2(x,y;\omega_1+\omega_2)} ,
\]
is an effective area with \(d_{k(\omega)}^i(x,y)\) the \(i\)-th component of the displacement field at frequency \(\omega\), and \(n(x,y;\omega)\) the material refractive index at frequency \(\omega\), both at waveguide cross-sectional position \((x,y)\), and we have chosen the mode amplitudes such that we can take the phase associated with the effective area to be zero [28].

That \(C_{II}^{\dagger} |\text{vac}\rangle\) is normalized requires
\[
\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi(\omega_1, \omega_2)|^2 = 1,
\]
which in turn determines \(|\beta|\). In the limit \(|\beta| \ll 1\),
\[
|\psi_{\text{gen}}\rangle \approx |\text{vac}\rangle + \beta C_{II}^{\dagger} |\text{vac}\rangle ,
\]
and \(2|\beta|^2 \equiv N_D\) can be thought of as the average number of generated photons per pump pulse. Indeed this line of thinking holds even in the limit of a pulse long enough that its intensity spectrum that can be approximated as a Dirac delta function, \(|\phi_P(\omega_1 + \omega_2)|^2 \approx \delta(\omega_1 + \omega_2 - \omega_P)\), in comparison to the phase matching sinc function of the BWF of (3.12); the result for \(N_D\) is then
\[
N_D = \frac{2\hbar \omega_P N_p L^2}{P \Delta \tau} ,
\]
provided \(|\beta| \ll 1\). Note that in deriving (3.14) we have integrated over \(\omega_2\), and taken the effective area, \(A\), and group velocities, \(v_{SH(F)}^{-1} = (d k/d \omega)|_{\omega = \omega_P(\omega_P/2)}\), as constant over the
generation bandwidth time

\[ T = \frac{2\pi}{\int_0^{\omega_p^2/2} d\Omega \left[ 1 - (2\Omega/\omega_p)^2 \right] \sin^2 \left\{ [k (\omega_p/2 + \Omega) + k (\omega_p/2 - \Omega) - k (\omega_p)] L/2 \right\}}, \]  

(3.15)

where \( \Omega = \omega_1 - \omega_p/2 \) represents positive detuning from \( \omega_p/2 \), and \( \mathcal{P} = 8\varepsilon_0\pi^2 v_p^2 v_{SH} / (\chi_2^2 \omega_p) \) is a characteristic power for a \( \chi_2 \) material [2]. Yet it is also true that if we consider a constant rate of pump photons or average pump power, \( P_p = h\omega_p N_p / T \), where \( T \) is the time that the pump is on, \( N_p / T = 2P_p L^2 / (\mathcal{P} \mathcal{A} T) \) can be thought of as the average rate of generated photons provided that \( |\beta| \sqrt{p_1} \ll 1 \), where \( \sqrt{p_1} \) is the largest Schmidt coefficient in a Schmidt decomposition [63] of the BWF. For if one writes (3.8) in the basis of the Schmidt modes, and calculates the expectation value of the number of photons in (3.7), one sees that (3.14) remains true in the continuous wave (CW) limit, provided \( P_p \ll \mathcal{P} \mathcal{A} / L^2 \) (see (B.6)). Defining \( P_1 = h\omega_p N_D / (2T) \) as the average generated power associated with one photon of each pair, we rewrite (3.14) as

\[ P_1 = \frac{h\omega_p P_p L^2}{T} \frac{\mathcal{P} \mathcal{A}}{P_A}. \]  

(3.16)

To derive expressions for classical second-order nonlinear processes we begin with the same Hamiltonian as above (3.1) and proceed more or less as above, yet keeping different terms. For a DFG calculation, we take the pump and signal as both incident from the left and, while they may or may not exist in two distinct propagation modes of the waveguide, take their center frequencies separated enough that we can still give them distinct labels \( a_{L,S}k \equiv a_Sk \), and \( a_{L,P}k \equiv a_Pk \), with \( \left[ a_Sk, a_P^\dagger k \right] = 0 \). This is certainly true for a CW calculation. We take as our input state

\[ |\psi_{in}\rangle = \exp \left( \sum_{I=S,P} \int_0^\infty dk \eta_{Ik} (t_0) b_{Ik}^\dagger - \text{H.c.} \right) |\text{vac}\rangle, \]  

(3.17)

and, making approximations similar to above find

\[ H_b (t) + \hat{V} (t) = -2\alpha_0^* \alpha P \int_0^\infty dk \int_0^\infty dk_1 \int_0^\infty dk_2 S_{RR,R} (k, k_1, k_2; t) b_{Rk}^\dagger \phi_k^S (k_1) \phi_p (k_2) + \text{H.c.}, \]

and thus

\[ |\psi_{gen}\rangle = \exp \left( \beta \int_0^\infty d\omega \phi (\omega) b_{R\omega}^\dagger - \text{H.c.} \right) |\text{vac}\rangle, \]
with the single photon wave function

\[
\phi (\omega) = \sqrt{2i\pi} \int_0^\infty d\omega_1 \left[ \sqrt{\frac{d\kappa (\omega)}{d\omega}} \sqrt{\frac{d\kappa (\omega_1)}{d\omega_1}} \left( \sqrt{\frac{d\kappa (\omega_2)}{d\omega_2}} \right) \right]_{\omega_2 = \omega + \omega_1}
\]

\times \sqrt{\frac{\hbar \omega_1 (\omega + \omega_1) (\alpha_\Sigma^2 |\alpha_P|^2 \phi_\Sigma^* (\omega_1) \phi_P (\omega + \omega_1)}{\varepsilon_0 |\beta|^2 \pi^6}}

\times \omega_1 (\alpha_\Sigma^2 |\alpha_P|^2 \phi_\Sigma^* (\omega_1) \phi_P (\omega + \omega_1) J (\omega, \omega_1)) \right], \quad (3.18)

where

\[
\phi_l (\omega) = \frac{\sin [(\omega - \omega_1) T/2]}{(\omega - \omega_1) \sqrt{\pi T/2}},
\]

and \(J (\omega_1, \omega_2)\) is as above (3.10). Note that here, under the undepleted pump approximation, \(|\beta|^2\) corresponds to the number of generated idler photons.

For a channel waveguide (3.11)

\[
\phi (\omega) = iL \int_0^\infty d\omega_1 \left[ \sqrt{\frac{d\kappa (\omega)}{d\omega}} \sqrt{\frac{d\kappa (\omega_1)}{d\omega_1}} \left( \sqrt{\frac{d\kappa (\omega_2)}{d\omega_2}} \right) \right]_{\omega_2 = \omega + \omega_1}
\]

\times \sqrt{\frac{\omega_1 (\alpha_\Sigma^2 |\alpha_P|^2 \phi_\Sigma^* (\omega_1) \phi_P (\omega + \omega_1)}{4\pi \varepsilon_0 |\beta|^2 \pi^6}} \sqrt{\frac{\hbar (\omega + \omega_1)}{A [k (\omega), k (\omega_1), k (\omega + \omega_1)]}}

\times \phi_\Sigma^* (\omega_1) \phi_P (\omega + \omega_1) \text{sinc} \{[k (\omega) + k (\omega_1) - k (\omega + \omega_1)] L/2\}.

If we make the approximation

\[
\left| \int_0^\infty d\omega_1 \sqrt{\omega_1 (\omega + \omega_1) \phi_\Sigma^* (\omega) \phi_P (\omega + \omega_1) \text{sinc} \{[k (\omega) + k (\omega_1) - k (\omega + \omega_1)] L/2\} \right|^2
\]

\approx \omega_S (\omega + \omega_S) \text{sinc}^2 \{[k (\omega) + k (\omega_1) - k (\omega + \omega_1)] L/2\} \left| \int_{-\infty}^\infty d\omega_1 \phi_\Sigma^* (\omega_1) \phi_P (\omega + \omega_1) \right|^2

= \omega_S (\omega + \omega_S) \text{sinc}^2 \{[k (\omega) + k (\omega_1) - k (\omega + \omega_1)] L/2\} \frac{2\pi}{T} |\phi_P (\omega + \omega_S)|^2, \quad (3.20)

then, as the normalization \(\int_0^\infty d\omega |\phi (\omega)|^2 = 1\) determines \(|\beta|\), in the CW limit, we can calculate

\[
|\beta|^2 = \frac{\hbar (\omega_P - \omega_S) |\alpha_S|^2 |\alpha_P|^2 L^2}{T P A} \text{sinc}^2 \{[k (\omega_P - \omega_S) + k (\omega_S) - k (\omega_P)] L/2\},
\]

where we have taken \(\omega_P = \omega_{SH}, v_P = v_{SH}\), and assumed that the signal frequency is close
enough to half the pump frequency that, outside the sinc function, \( \omega_S \approx \omega_P - \omega_S \approx \omega_P / 2 \) as well as \( (\partial k / \partial \omega)_{\omega=\omega_S} \approx (\partial k / \partial \omega)_{\omega=\omega_P-\omega_S} \approx \nu_P^{-1} \) and \( d_k^{(\omega)}(x,y) \approx d_k^{(\omega_P-\omega_S)}(x,y) \approx d_k^{(\omega_P/2)}(x,y) \). Defining \( P_1 = \hbar (\omega_P - \omega_S) |\beta|^2 / T \approx \hbar \omega_S |\beta|^2 / T \), \( P_S = \hbar \omega_S |\alpha_S|^2 / T \approx h (\omega_P - \omega_S) |\alpha_S|^2 / T \), \( P_P = \hbar \omega_P |\alpha_P|^2 / T \approx 2 \hbar \omega_S |\alpha_P|^2 / T \), we can write this in the more usual form

\[
P_1 = P_5 \frac{P_P L^2}{\mathcal{P} \mathcal{A}} \sin^2 \left\{ \left[ k (\omega_P - \omega_S) + k (\omega_S) - k (\omega_P) \right] \frac{L}{2} \right\}.
\]

If we had instead followed the same steps and assumptions for a CW SHG calculation, starting from the seventh of (3.5), we would have found

\[
P_{\text{SH}} = \frac{P_P^2 L^2}{\mathcal{P} \mathcal{A}} \sin^2 \left\{ \left[ k (2\omega_F) - 2k (\omega_F) \right] \frac{L}{2} \right\},
\]

which agrees with the well-known result [64]: exact agreement is achieved if we set our \( \mathcal{A} \) equal to the effective area presented there, and then, for we are free to do so, set \( \pi = n \) and \( \chi_2 = 2d_{ijk}^{(\omega)} / \varepsilon_0 \), as well as approximate \( v_m \approx c/n \).

Comparing our quantum (3.16) and classical (3.21), (3.22) expressions, we first note that SPDC, DFG, and SHG all scale the same way with the characteristic power, \( \mathcal{P} \), and effective area, \( \mathcal{A} \), of the device. But an important difference is the way in which energy conservation enters in these expressions. For CW pumps in classical (stimulated) experiments the generated light appears at a single frequency that is tunable by varying the pump (and signal, in DFG) frequency, as determined by energy conservation. The efficiency of the process is determined by how well phase matching is achieved, in DFG for the so-determined idler frequency \( \omega_I = \omega_P - \omega_S \), as expressed in the \( \sin^2 \) function appearing in (3.21). On the contrary, in quantum (spontaneous) experiments, since only the energy sum of the signal and idler photons is fixed in the approximately CW experiments described by (3.16), there is a probability of photons being generated over a wide frequency range, centered about \( \omega_P / 2 \). This "left spectral linewidth of emission" [65] is determined by \( \mathcal{T}^{-1} \). Comparing (3.16) in the limit of perfect phase matching with (3.21), we see that in the quantum case the role of the classical "seed" power \( P_S \) is played by \( \hbar \omega_F / \mathcal{T} \), the average energy of one downconverted photon in a time \( \mathcal{T} \), and that the wider the bandwidth of possible emission, the larger the fluctuating power available to drive the process.

Turning now to an evaluation of \( \mathcal{T} \), if the phase matching condition is satisfied over a range that is small compared to the range of integration, i.e. when \( 1 - (2\Omega/\omega_P)^2 \) remains essentially constant over the frequency range within which the squared sinc function is
significantly nonzero, we have

\[ T \approx \frac{2\pi}{\int_0^{\omega_p/2} d\Omega \, \sin^2 \{ [k (\omega_P/2 - \Omega) + k (\omega_P/2 + \Omega) - k (\omega_P)] L/2 \}}. \]  
(3.23)

Taylor expanding dispersion relations to quadratic order about the phase matched case i.e.

\[ k (\omega) = k (\omega_P/2) + \frac{1}{v_F} (\omega - \omega_P/2) + \frac{\beta_2 (\omega_P/2)}{2} (\omega - \omega_P/2)^2, \]  
(3.24)

for \( 2k (\omega_P/2) - k (\omega_P) = 0 \), we find

\[ T \approx \frac{3}{2} \sqrt{2\pi} |\beta_2 (\omega_P/2)| L, \]  
(3.25)

from (3.23) for a pump centered at the phase matched frequency, and thus, inserting (3.25) in (3.16) we obtain

\[ P_I = \frac{\hbar \omega_F}{\frac{3}{2} \sqrt{2\pi} |\beta_2 (\omega_P/2)| L \, \mathcal{PA}} \frac{P_P L^2}{\mathcal{PA}}. \]  
(3.26)

Since the time \( T \) is determined by material dispersion, it is proportional to \( L^{1/2} \). Therefore the generated power scales as \( L^{3/2} \) in an SPDC experiment, and not with the square of the device length as it does in DFG. We note that the spontaneous expression (3.16), with \( T \) given by (3.23) agrees with a previously published result [65]: exact agreement is achieved if we set our \( \mathcal{A} \) equal to the effective area presented there, and then, for we are free to do so, set \( \bar{n} = (n_S n_I n_P)^{1/3} \) and \( \bar{\chi}_2 = 2d_{\text{eff}}/\varepsilon_0 \), and approximate \( v_m \approx c/n_m \), as well as \( \omega_1 \approx \omega_S = \omega_F \), similar to above.

So far we have assumed that the SPDC is unfiltered. If a bandpass filter with bandwidth \( \Delta \omega = 2\pi B \) is used, narrow enough that the sinc function varies negligibly over that range, the time \( T \) of (3.23) becomes

\[ T (\Omega) \approx (B \sin^2 \{ [k (\omega_P/2 - \Omega) + k (\omega_P/2 + \Omega) - k (\omega_P)] L/2 \})^{-1}, \]

where \( \Omega + \omega_P/2 \) identifies the center frequency of the bandpass filter, and the generated power

\[ P_I (\Omega) = \frac{\hbar \omega_F}{B^{-1}} \frac{P_P L^2}{\mathcal{PA}} \sin^2 \left\{ [k (\omega_P/2 - \Omega) + k (\omega_P/2 + \Omega) - k (\omega_P)] \frac{L}{2} \right\}, \]  
(3.27)

scales with \( L^2 \) as in the classical expression. Note that when comparing DFG (3.21) to this expression (3.27) it is not just the efficiency of the process that has changed - reduced
by a factor of $\hbar \omega_F / (B^{-1} P_S)$ - but also the frequency of generated light has gone from being determined by $\omega_1 = \omega_P - \omega_S$, to being characterized by a spectral density. And so here it is by tuning $\Omega$ that the idler frequency is selected, rather than by tuning $\omega_S$ as in the stimulated (DFG) scenario.

Finally, it is interesting to consider the artificial scenario in which there is no dispersion and the interacting fields are perfectly phase matched. In practice this assumption may not be so unrealistic, for if the system length $L$ is much shorter than the coherence length $L_{\text{coh}} = \pi / |k (\omega_P/2 - \Omega) + k (\omega_P/2 + \Omega) - k (\omega_P)|$ over the integration range of (3.15) where the modes exist, then the sinc function is essentially unity everywhere there. Of course, an actual calculation would have to take into account mode cut-offs in limiting the range of integration [28]. But if we further neglect those cut-offs and assume that the modes exist over the entire integration range of (3.15), then if there is no filtering we find that the generation bandwidth time is on the order of the period of a generated photon $T = 3\pi / (\omega_P/2)$, so that

$$P_1 = \frac{\hbar \omega_F P_P L^2}{\frac{3}{2} \tau_F \mathcal{P} A},$$

(3.28)

where we have identified $\tau_m = 2\pi / \omega_m$ as a typical photon period. Unlike in (3.26), here the output also scales as $L^2$, for there is no length dependent bandwidth over which the output is generated.

With these considerations, determination of the material parameters $L^2 / (\mathcal{P} A)$ in a classical experiment (here, DFG or SHG) allows for an accurate prediction of the average power of photons generated via SPDC.

### 3.1.2 Third-Order Processes

The method and results of this Section closely parallel that of the second-order case. We begin with the Hamiltonian

$$H = H_L + H_{NL},$$

(3.29)

$$H_L = \int_0^\infty \! dk \hbar \omega_L b_L^\dagger b_L + \int_0^\infty \! dk \hbar \omega_R b_R^\dagger b_R,$$
\[ H_{NL} = -\frac{1}{4\varepsilon_0} \int d^3 r \Gamma^{ijkl}_3 (r) D^i (r) D^j (r) D^k (r) D^l (r) \]
\[ = -\int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk_3 \int_0^\infty dk_4 \]
\[ \times \sum_{A,B,C,D=L,R} S_{AB;CD} (k_1, k_2, k_3, k_4) b^{\dagger}_{Ak_1} b^{\dagger}_{Bk_2} b_{Ck_3} b_{Dk_4} \]

where

\[ S_{AB;CD} (k_1, k_2, k_3, k_4) = \frac{3}{2\varepsilon_0} \sqrt{\frac{\hbar \omega_{Ak_1}}{\hbar \omega_{Bk_2} (\hbar \omega_{Ck_3}) (\hbar \omega_{Dk_4})}} \]
\[ \times \int \! dr \left\{ \frac{\chi^{ijkl}_3 (r)}{\varepsilon_0 n^2 (r, \omega_{Ak_1}) n^2 (r, \omega_{Bk_2}) n^2 (r, \omega_{Ck_3}) n^2 (r, \omega_{Dk_4})} \right\} \]
\[ \times \left[ D_{Ak_1}^{i,\text{asy-out}} (r) \right]^* \left[ D_{Bk_2}^{j,\text{asy-out}} (r) \right]^* D_{Ck_3}^{k,\text{asy-out}} (r) D_{Dk_4}^{l,\text{asy-out}} (r) \right\}, \]

and we have neglected terms corresponding to three-photon generation or third harmonic generation as these processes are not phase matched for the devices considered here. Again, we have chosen to expand the full fields in terms of asymptotic-out fields only.

For an input coherent state of the form

\[ |\psi_{\text{in}}\rangle = \exp \left( \int_0^\infty \! dk \zeta_{Lk} (t_0) a_{Lk}^{\dagger} - \text{H.c.} \right) |\text{vac}\rangle \]
\[ = \exp \left( \int_0^\infty \! dk \eta_{Rk} (t_0) b_{Rk}^{\dagger} - \text{H.c.} \right) |\text{vac}\rangle , \]

where

\[ \eta_{Rk} (t_0) = \int_0^\infty \! dk' T_{L;R}^{\text{out}} (k', k) \zeta_{Lk'} (t_0) , \]

as again we restrict ourselves to all-pass filters, we find (recall (2.59), (2.54), and (2.60))

\[ H_b (t) = \frac{ih}{2} \int_0^\infty \! dk \left[ \left( \frac{d\eta_{Rk}^* (t)}{dt} - \eta_{Rk}^* (t) \frac{d\eta_{Rk} (t)}{dt} \right) \right. \]
\[ + 2 \left( \frac{d\eta_{Rk}^* (t)}{dt} b_{Rk} - \frac{d\eta_{Rk} (t)}{dt} b_{Rk}^{\dagger} \right) \]},
This expression identifies approximately energy conserving terms as those for which
\[ S_{AB,CD} (k_1, k_2, k_3, k_4) \]
where
\[ S_{AB,CD} (k_1, k_2, k_3, k_4; t) = S_{AB,CD} (k_1, k_2, k_3, k_4) e^{i(\omega_{Ak_1} + \omega_{Bk_2} - \omega_{Ck_3} - \omega_{Dk_4})t} \]
This expression identifies approximately energy conserving terms as those for which \( \omega_{Ak_1} + \omega_{Bk_2} - \omega_{Ck_3} - \omega_{Dk_4} \approx 0 \), and approximately momentum conserving terms as those containing \( |k_3 + k_4 - k_1 - k_2| \). This simplifies \( \hat{V} (t) \) to
\[ \hat{V} (t) = - \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk_3 \int_0^\infty dk_4 \sum_{A,B,C,D=L,R} \left[ S_{AB,CD} (k_1, k_2, k_3, k_4) \right] \]
where the first terms is a classical function of time and does not contribute to the state of generated photons, the second and third terms correspond to squeezing of pump, the fourth and fifth to SFWM-like terms that result in back action on the pump, the sixth and seventh to SFWM itself, the eighth to XPM of the generated photons caused by the
pump, and the final two terms also to SFWM-like terms that result in back action on
the pump. We note that we can eliminate the final two terms by setting
\[
\frac{d\eta_{R_k}(t)}{dt} = \frac{2i}{\hbar} \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk_3 S_{RR,RR}(k, k_2, k_3, k_4; t) \eta_{R_k_2}(t) \eta_{R_k_3}(t) \eta_{R_k_4}(t),
\]
however, as a first approximation we assume an unmodified pump, and neglect the con-
stant term as well as the squeezing, back action on the pump, and SPM or XPM-like
terms, leaving only
\[
H_b(t) + \hat{V}(t) = -\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d_\omega_1 d_\omega_2 d_\omega_3 d_\omega_4 S_{RR,RR}(k_1, k_2, k_3, k_4; t) \eta_{R_k_1}(t) \eta_{R_k_2}(t) \eta_{R_k_3}(t) \eta_{R_k_4}(t),
\]
with \(\eta_{R_k}(t) = \alpha\phi_P(k)\).

Following the approach above, we find the state of generated photons to be
\[
|\psi_{gen}\rangle = \exp(\beta C_{II}^\dagger - \text{H.c.}) |\text{vac}\rangle,
\]
where \(|\beta|^2\) is proportional to the average number of photons in the pump pulse \(N_P = |\alpha|^2\) and
\[
C_{II}^\dagger = \frac{1}{\sqrt{2}} \int_0^\infty d_\omega_1 \int_0^\infty d_\omega_2 \phi(\omega_1, \omega_2) b_{R_{\omega_1}}^\dagger b_{R_{\omega_2}}^\dagger,
\]
such that \(C_{II}^\dagger |\text{vac}\rangle\) is a normalized two-photon state characterized by the BWF
\[
\phi(\omega_1, \omega_2) = 3i\pi \sqrt{\frac{\partial k(\omega_1)}{\partial \omega_1}} \sqrt{\frac{\partial k(\omega_2)}{\partial \omega_2}} \int_0^{\omega_1 + \omega_2} d_\omega \left( \sqrt{\frac{dk(\omega)}{d_\omega}} \sqrt{\frac{dk(\omega')}{d_\omega'}} \right)_{\omega' = \omega_1 + \omega_2 - \omega} \times \sqrt{\frac{\hbar^2 \omega_1 \omega_2 (\omega_1 + \omega_2 - \omega)}{8c_0^2 |\beta|^2 \pi^8}} \phi_P(\omega) \phi_P(\omega_1 + \omega_2 - \omega) J(\omega_1, \omega_2, \omega),
\]
(3.30)
with

\[
J (\omega_1, \omega_2, \omega) = \frac{n_1}{\chi_3} \int d^3r \frac{\chi_{ijkl} (r)}{\epsilon_0 n^2 (r, \omega_1) n^2 (r, \omega_2) n^2 (r, \omega) n^2 (r, \omega_1 + \omega_2 - \omega)} \\
\times \left\{ D_{Rk(\omega_1)}^{i_1, \text{asy-out}} (r) \right\}^* \left\{ D_{Rk(\omega_2)}^{i_2, \text{asy-out}} (r) \right\}^* D_{Rk(\omega_1+\omega_2-\omega)}^{k, \text{asy-out}} (r) \left\{ D_{Rk(\omega)}^{k_1, \text{asy-out}} (r) \right\}.
\] (3.31)

For a channel waveguide (3.11) the BWF can simply be written

\[
\phi (\omega_1, \omega_2) = 3iL \sqrt{\frac{d^2 (\omega_1) \sqrt{d^2 (\omega_2)}}{d\omega_1 \, d\omega_2}} \int_0^{\omega_1 + \omega_2} d\omega \left( \sqrt{\frac{d^2 (\omega) \sqrt{d^2 (\omega')}}{d\omega \, d\omega'}} \right)_{\omega' = \omega_1 + \omega_2 - \omega} \\
\times \sqrt{\frac{\omega_1 \omega_2 (\chi_3)^2 N_\beta^2}{128 \pi^2 \epsilon_0 \beta^2 n^8}} \sqrt{\frac{\epsilon \omega (\omega_1 + \omega_2 - \omega)}{\mathcal{A}^2 [k (\omega_1), k (\omega_2), k (\omega), k (\omega_1 + \omega_2 - \omega)]}} \phi_p (\omega) \\
\times \phi_p (\omega_1 + \omega_2 - \omega) \text{sinc} \left\{ [k (\omega_1) + k (\omega_2) - k (\omega) - k (\omega_1 + \omega_2 - \omega)] L/2 \right\},
\] (3.32)

where

\[
\mathcal{A} [k (\omega_1), k (\omega_2), k (\omega), k (\omega_1 + \omega_2 - \omega)] = \left| \int dx dy \frac{n^{3/2} \chi_{2}^{ijkl} (x, y) [d^i_{k(\omega_1)} (x, y)]^* [d^j_{k(\omega_2)} (x, y)]^* d^k_{k(\omega)} (x, y) d^l_{k(\omega_1+\omega_2-\omega)} (x, y)}{\chi_3 \epsilon_0 n^2 (x, y; \omega_1) n^2 (x, y; \omega_2) n^2 (x, y; \omega) n^2 (x, y; \omega_1 + \omega_2 - \omega)} \right|^{-1}.
\] (3.33)

is an effective area and, as above, we have chosen the mode amplitudes such that we can take its phase to be zero. We calculate the average power associated with one photon of each pair generated due to a CW SFWM process in a channel waveguide pumped at \(\omega_p\), following the same arguments as above including the low-power limit in which the result is valid (see (B.7)). Making the additional approximation cf. (3.20)

\[
\left| \int_0^{\omega_1 + \omega_2} d\omega \sqrt{\omega (\omega_1 + \omega_2 - \omega)} \phi_p (\omega) \phi_p (\omega_1 + \omega_2 - \omega) \text{sinc} \left\{ [k (\omega_1) + k (\omega_2) - k (\omega) - k (\omega_1 + \omega_2 - \omega)] L/2 \right\} \right|^2 \\
\approx \omega_p^2 \text{sinc}^2 \left\{ [k (\omega_1) + k (\omega_2) - 2k (\omega_p)] L/2 \right\} \frac{2\pi}{T} \left| \phi_p (\omega_1 + \omega_2 - \omega_p) \right|^2,
\] (3.34)
we find

\[ P_I = \frac{\hbar \omega_P}{T} (\gamma P_L)^2, \] (3.35)

where \( L \) is the length of the nonlinear structure, \( \gamma = \frac{3 \chi_3 \omega_P}{(4 \varepsilon_0 v^2 \pi l A)} \) is the usual nonlinear parameter \[43\], \( P_T = \hbar \omega_P N_P / T \) and \( P_I = \hbar \omega_P N_D / (2T) \), analogous to the expressions preceding (3.16), with the generation bandwidth time

\[ T = \frac{2\pi}{\int_0^{\omega_P} d\Omega \left[ 1 - (\Omega/\omega_P)^2 \right] \text{sinc}^2 \left\{ [2k(\omega_P) - k(\omega_P + \Omega) - k(\omega_P - \Omega)] L/2 \right\}, \] (3.36)

and \( \Omega = \omega_1 - \omega_P \) representing positive detuning from \( \omega_P \). This result is in a form that anticipates the classical FWM result, namely

\[ P_I = P_S (\gamma P_L)^2 \text{sinc}^2 \left\{ [2k(\omega_P) - k(\omega_S) - k(2\omega_P - \omega_S)] L/2 \right\}, \] (3.37)

The classical equation is derived from the same Hamiltonian as the quantum equation (3.29), in much the same fashion as the expressions for classical second-order processes were. Using the same input state (3.17) and assumptions as above we find

\[ H_b (t) + \tilde{V} (t) = -2\alpha^* S \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk_3 \int_0^\infty dk_4 \left[ S_{RR:RR} (k_1, k_2, k_3, k_4; t) \times b_{Rk_1}^\dagger \phi^* S (k_2) \phi_P (k_3) \phi_P (k_4) + \text{H.c.} \right], \]

and thus

\[ |\psi_{\text{gen}}\rangle = \exp \left( \int_0^\infty d\omega \phi (\omega) b_{R\omega}^\dagger - \text{H.c.} \right) |\text{vac}\rangle, \]

with the single photon wave function

\[ \phi (\omega) = 3i\pi \sqrt{\frac{dk (\omega)}{d\omega}} \int_0^\infty d\omega_1 \int_0^{\omega - \omega_1} d\omega_2 \left\{ \sqrt{\frac{dk (\omega_1)}{d\omega_1}} \sqrt{\frac{dk (\omega_2)}{d\omega_2}} \left( \sqrt{\frac{dk (\omega')}{d\omega'}} \right)_{\omega' = \omega + \omega_1 - \omega_2} \times \sqrt{\frac{h^2 \omega_1 \omega_2 (\omega + \omega_1 - \omega_2) (\chi_3)^2 |\alpha_S|^2 |\alpha_P|^4}{2\varepsilon_0^2 |\beta|^2 n_s}} \times \phi^* S (\omega_1) \phi_P (\omega_2) \phi_P (\omega + \omega_1 - \omega_2) J (\omega, \omega_1, \omega_2) \right\}, \] (3.38)
where \( J(\omega_1, \omega_2, \omega) \) is as above (3.31). For a channel waveguide (3.11)

\[
\phi(\omega) = 3iL \sqrt{\frac{d k(\omega)}{d \omega}} \int_0^\infty d\omega_1 \int_{\omega-\omega_1}^{\omega_1} d\omega_2 \left( \sqrt{\frac{d k(\omega_1)}{d \omega_1}} \sqrt{\frac{d k(\omega_2)}{d \omega_2}} \left( \sqrt{\frac{d k(\omega')}{d \omega'}} \right)_{\omega' = \omega + \omega_1 - \omega_2} \right)
\times \sqrt{\frac{\omega (\chi_3)^2 |\alpha_S|^2 |\alpha_P|^4}{32\pi^2 \varepsilon_0^2 |\beta|^2 n^8}} \sqrt{h^2 \omega_1 \omega_2 (\omega + \omega_1 - \omega_2)}
\times \phi_s^*(\omega_1) \phi_P(\omega_2) \phi_P(\omega + \omega_1 - \omega_2)
\times \text{sinc} \left\{ [k(\omega) + k(\omega_1) - k(\omega_2) - k(\omega + \omega_1 - \omega_2)] L/2 \right\},
\]

and, as the normalization \( \int_0^\infty d\omega |\phi(\omega)|^2 = 1 \) determines \( |\beta| \), in the CW limit we calculate

\[
|\beta|^2 = |\alpha_S|^2 \left( \gamma P_P L \right)^2 \text{sinc}^2 \left\{ [k(\omega_S) + k(\omega_P - \omega_S) - 2k(\omega_P)] L/2 \right\},
\]

where we have taken \( \omega_P \approx \omega_S \approx \omega_1 \). Defining \( P_1 = h(\omega_P - \omega_S) |\beta|^2 / T \approx h \omega_S |\beta|^2 / T \), \( P_S = h \omega_S |\alpha_S|^2 / T \) we can write this in the more usual form (3.37).

Comparing the corresponding results here, we see that there is a clear analogy with the comparison of second-order processes in that the classical (stimulated) process leads to the generation of a fixed idler frequency, for set pump and signal frequencies, with an efficiency determined by the familiar phase matching function. Similarly, in the quantum (spontaneous) process, the signal and idler photons can be generated over a range of frequencies determined by a bandwidth \( T^{-1} \). Comparing (3.37) in the limit of perfect phase matching with (3.35), we see that the role of the seed power \( P_S \) in the classical (stimulated) process is played by \( h \omega_P / T \) in the quantum (spontaneous) process, the average energy of one generated photon in a time \( T \), and that the wider the bandwidth of possible emission, the larger the fluctuating power available to drive the process.

We now turn to an evaluation of \( T \) (3.36). If the phase matching condition is satisfied over a range that is small compared to the range of integration, we note that \( 1 - (\Omega/\omega_P)^2 \) stays very close to 1 over the range of frequencies in which the squared sinc function is significantly nonzero, and a Taylor expansion of the dispersion relation, similar to (3.24) but about \( \omega_P \),

\[
k(\omega) = k(\omega_P) + \frac{1}{v} (\omega - \omega_P) + \frac{\beta_2(\omega_P)}{2} (\omega - \omega_P)^2,
\]

in (3.36) leads to

\[
T \approx \frac{3}{2} \sqrt{2\pi |\beta_2(\omega_P)|} L. \tag{3.39}
\]
This is essentially the same expression found above for the SPDC generation bandwidth time (3.25) when generated photons are collected over the entire available bandwidth. Substituting (3.39) in (3.35) we obtain

\[ P_I = \frac{\hbar \omega_P}{\frac{3}{2} \sqrt{2\pi |\beta_2(\omega_P)|} L} \left( \gamma P_P L \right)^2. \] \hspace{1cm} (3.40)

Since the time \( T \) is determined by material dispersion it is proportional to \( L^{1/2} \), and therefore the generated power scales as \( L^{3/2} \) in a SFWM experiment, and not with the square of the device length as it does in FWM. We note that (3.35), (3.39), and (3.37) agree with well-known results [26, 43].

If instead of looking at the entire SFWM spectrum we choose to focus on a narrow spectral region of bandwidth \( 2\pi B \), we find

\[ T(\Omega) \approx \left( B \text{sinc}^2 \left\{ 2k(\omega_P) - k(\omega_P + \Omega) - k(\omega_P - \Omega) \right\} / 2 \right) L/2 \right) ^{-1}. \]

Under this assumption, for a particular detuning \( 0 \leq \Omega \leq \omega_P \), from \( \omega_P \), the generated power can be written

\[ P_I(\Omega) = \frac{\hbar \omega_P}{B^{-1}} \left( \gamma P_P L \right)^2 \text{sinc}^2 \left\{ 2k(\omega_P) - k(\omega_P + \Omega) - k(\omega_P - \Omega) \right\} L/2 \right). \] \hspace{1cm} (3.41)

and scales with \( L^2 \) as in the classical expression.

In the somewhat artificial scenario in which the system length is much smaller than the coherence length \( L_{\text{coh}} = \pi / |2k(\omega_P) - k(\omega_P + \Omega) - k(\omega_P - \Omega)| \) over the relevant range of integration (3.36), where again we neglect the fact that in a real calculation some of this integration range will not contribute due to mode cut-offs, we find \( T = 3\pi/\omega_P \) if there is no filtering, and the generated power becomes (cf. (3.28))

\[ P_I = \frac{\hbar \omega_P}{\frac{3}{2} \tau_P} \left( \gamma P_P L \right)^2, \] \hspace{1cm} (3.42)

where \( \tau_P = 2\pi/\omega_P \) is a typical photon period. Unlike in (3.40), here the output also scales as \( L^2 \), for there is no length dependent bandwidth over which the output is generated.

We conclude this Section by summarizing all of the results for spontaneous and stimulated processes in a channel waveguide. If one wishes to make a quick estimate of the average generated photon power at a certain input power, in either an SPDC or SFWM experiment, one can simply perform the corresponding classical experiment, extract the relevant material and structure parameters, and lastly replace \( P_S \) with the energy of a
typical generated photon ($\hbar \omega_P / 2$ or $\hbar \omega_P$, for SPDC or SFWM respectively) divided by the generation bandwidth time $T$ and adjusting accordingly for any filtering before detection. If the power can be collected over the entire generation bandwidth, then (3.25) or (3.39) should be used. It is interesting to note that for the spontaneous processes in the short system length limit $L \ll L_{coh}$, if mode cut-offs are neglected a power of $2\hbar \omega / (3\tau)$ (about 26 $\mu$W for a 1 eV photon) plays the role of the classical “seed” power of the corresponding classical calculation for both SPDC (3.28) and SFWM (3.42).

The expressions required for such estimates (the “idler” powers generated in second- and third-order nonlinear processes) are shown in Tab. 3.1. In this table, we have chosen to rewrite the expressions to highlight the dependence on quantities that might be of interest in guiding the design of a device or in helping the interpretation of experimental results. Interestingly, for a channel waveguide, the scaling behavior of either stimulated or spontaneous processes with the length $L$ of the channel is independent of the order of the nonlinearity. In particular, the efficiency of phase matched stimulated generation scales with $L^2$ whether we are considering DFG or FWM. The same scaling is obtained for the generation rate of spontaneous processes within a sufficiently narrow spectral region around the phase matching condition. Similarly, the generated idler power, integrated over the entire spectrum, scales with $L^{3/2}$ for both SPDC and SFWM, as in the absence of filtering the generation bandwidth time scales as $L^{1/2}$. Of the quantum expressions, we note that the bandpass filtered expression is most often investigated experimentally, especially for third-order nonlinear devices where strong pump suppression is required and Raman noise should be avoided [15], and so $L^2$ scaling is typically observed. Finally, considering the field enhancement due to the light confinement in the channel, which is inversely proportional to the effective area, we observe that the generated power is independent of the nature of the process, stimulated or spontaneous, while it depends upon the order of the nonlinearity. Indeed, for DFG and SPDC, the generation rate is proportional to $1/A$, while for FWM and SFWM it scales with $\gamma^2 \propto 1/A^2$.

3.2 Resonant Structures

In this Section we consider corresponding processes in a resonant structure, in particular a microring resonator side-coupled to a channel waveguide [66, 67, 31] (see Fig. 3.4). Nonlinear processes can be largely amplified near particular frequencies when large field enhancements are achieved due to constructive interference of the electromagnetic field inside the ring. Thus, compared to processes in channel waveguides, we expect to be able
Table 3.1: Expressions for the “Idler” Power Generated via Corresponding Nonlinear Optical Processes in Channel Waveguides

<table>
<thead>
<tr>
<th>Process</th>
<th>Idler Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>DFG</td>
<td>( P_I = P_S \frac{P_p L^2}{P_A} \text{sinc}^2 \left{ \left[ k (\omega_P - \omega_S) + k (\omega_S) - k (\omega_P) \right] \frac{L}{2} \right} ) (Eq. (3.21))</td>
</tr>
<tr>
<td>SPDC</td>
<td>( P_I (\Omega) = h\omega_F B \frac{P_p L^2}{P_A} \times \text{sinc}^2 \left{ \left[ k (\omega_P/2 - \Omega) + k (\omega_P/2 + \Omega) - k (\omega_P) \right] \frac{L}{2} \right} ) (filtered; Eq. (3.27))</td>
</tr>
<tr>
<td>FWM</td>
<td>( P_I = P_S (\gamma P_P L)^2 \text{sinc}^2 \left{ \left[ 2k (\omega_P) - k (\omega_S) - k (2\omega_P - \omega_S) \right] \frac{L}{2} \right} ) (Eq. (3.37))</td>
</tr>
<tr>
<td>SFWM</td>
<td>( P_I (\Omega) = h\omega_F B (\gamma P_P L)^2 \times \text{sinc}^2 \left{ \left[ 2k (\omega_P) - k (\omega_P + \Omega) - k (\omega_P - \Omega) \right] \frac{L}{2} \right} ) (filtered; Eq. (3.41))</td>
</tr>
<tr>
<td></td>
<td>( P_I = \frac{h\omega_F}{\sqrt{2\pi</td>
</tr>
</tbody>
</table>
to generate the same idler power with reduced pump (and signal) power(s) at particular frequencies. Furthermore, in a high-Q ring resonator we expect the generation bandwidth time to reflect the fact that this enhancement occurs over a series of narrow frequency ranges associated with the ring resonances, separated by the free spectral range (FSR) of the ring. As above, we first consider SPDC, DFG, and SHG before moving on to $\chi_3$ processes.

### 3.2.1 Second-Order Processes

We treat optical dynamics in a microring resonator side-coupled to a channel waveguide when only the second-order optical nonlinearity is relevant with the same Hamiltonian as above (3.1), leading to the same BWF as above (3.9), the only difference now being the definition of the asymptotic fields (cf. (3.11))

$$D_{Rk}^{i,\text{asy-out}}(r) = \left\{ \begin{array}{ll}
T^{-1}(k) \frac{d_k^+(r_L)}{\sqrt{2\pi}} e^{ikz}, & \text{when } z < 0 \\
\frac{d_k^+(r_L)}{\sqrt{2\pi}} e^{ikz}, & \text{when } z > 0 \\
D_k^{i,\text{ring}}(r), & \text{in the ring} \end{array} \right., \quad (3.43)
$$

where $r_\perp$ indicates the coordinates in the plane normal to the waveguide (whether in a channel or in the ring) and the coupling between the channels and the ring is treated in the usual way [31], with self-, $\sigma(k)$, and cross-, $\kappa(k)$, coupling coefficients satisfying $\sigma(k)^2 + \kappa(k)^2 = 1$, and where

$$T(k) = \sigma(k) + \frac{(i\kappa(k))^2}{1 - \sigma(k)} e^{ikL} = \exp \left\{ i \left[ \pi + kL + 2 \tan^{-1} \left( \frac{\sigma(k) \sin(kL)}{1 - \sigma(k) \cos(kL)} \right) \right] \right\}.$$
The field in the ring is written as

$$D_{k}^{i, \text{ring}} (r) = \frac{i\kappa(k)}{1 - \sigma(k) e^{ikL} / \sqrt{2\pi}} \frac{d_{k}^{i}(r_{\perp}, \zeta / R)}{\sqrt{\sqrt{2\pi} e^{ik\zeta}}},$$

where $\zeta$ is the coordinate in the counterclockwise direction along the ring circumference of length $L = 2\pi R$, and ring resonances are described by $k(\omega_{m}) R = N_{m}$ with $N_{m} \in \mathbb{Z}$.

While our formalism can treat nonlinear susceptibility tensors of general symmetry, for instance rings made of III-V semiconductors in which there are unusual quasi-phase matching conditions [2], for the sake of simplicity we here assume that the nonlinearity experienced by the field of a ring mode remains constant as that field circles the ring. We also assume that the fields involved are at frequencies near resonances, leading to strong field enhancement due to constructive interference in the ring resonator and allowing us to safely neglect any nonlinear effects outside the ring in the waveguide, where such enhancement does not take place. To quantify how close the fields are to ring resonances, consider first the example of SHG where we assume that $2k(\omega_{F_{0}}) - k(\omega_{SH_{0}}) R = 2N_{F} - N_{SH} = 0$. For the frequencies $\omega_{F_{0}}$ and $\omega_{SH_{0}}$, in general we will have $\omega_{SH_{0}} \neq 2\omega_{F_{0}}$ because of dispersion, but we will assume that the fundamental and second harmonic frequencies $\omega_{F}$ and $\omega_{SH}$ respectively, with $\omega_{SH} = 2\omega_{F}$, are characterized by detunings $\delta_{F(SH)} = \omega_{F(SH)} - \omega_{F_{0}(SH_{0})}$ so small that $L\delta_{F(SH)}/v_{F(SH)} \ll 1$. Thus

$$\phi(\omega_{1}, \omega_{2}) = iL \sqrt{\frac{d_{k}^{i}(\omega_{1})}{\omega_{1}} \sqrt{\frac{d_{k}^{i}(\omega_{2})}{\omega_{2}} \sqrt{\frac{d_{k}^{i}(\omega)}{\omega = \omega_{1} + \omega_{2}}}} \sqrt{\frac{\hbar(\omega_{1} + \omega_{2})}{8\pi \varepsilon_{0} |\beta|^{2} \pi^{6} \sqrt{A[k(\omega_{1}), k(\omega_{2}), k(\omega_{1} + \omega_{2})]}}} \times \phi_{P}(\omega_{1} + \omega_{2}) \text{sinc} \{[k(\omega_{1}) + k(\omega_{2}) - k(\omega_{1} + \omega_{2})] L / 2 \} \times \frac{-i\kappa(k(\omega_{1}))}{1 - \sigma(k(\omega_{1})) e^{-ik(\omega_{1})L}} \frac{-i\kappa(k(\omega_{2}))}{1 - \sigma(k(\omega_{2})) e^{-ik(\omega_{2})L}} \frac{i\kappa(k(\omega_{1} + \omega_{2}))}{1 - \sigma(k(\omega_{1} + \omega_{2})) e^{ik(\omega_{1} + \omega_{2})L}},$$

and, for a CW pump of frequency $\omega_{p} \equiv \omega_{SH} = \omega_{SH_{0}}$, at a ring resonance, taking the effective area, group velocities, and coupling terms $\kappa_{SH} = \kappa(\omega_{p})$, $\kappa_{F} = \kappa(\omega_{p} / 2)$, as constant, we find that the average generated power associated with one photon of each pair is

$$P_{1} = \frac{\hbar \omega_{F} P_{P} L^{2}}{T \cdot P \cdot A} |F_{P}(\omega_{p})|^{2},$$

(3.45)
with the generation bandwidth time being

\[ \mathcal{T} = 2\pi \left( \int_0^{\omega_P/2} d\Omega \left[ 1 - (2\Omega/\omega_P)^2 \right] \text{sinc} \left\{ [k(\omega_P/2 - \Omega) + k(\omega_P/2 + \Omega) - k(\omega_P)]L/2 \right\} \times |F_F(\omega_P/2 - \Omega)|^2 |F_F(\omega_P/2 + \Omega)|^2 \right), \]

provided \( P_P \ll \mathcal{P} A / (L^2 |F_F(\omega_{F_0})|^3 |F_P(\omega_P)|^2) \) (see (B.8)). Here \( \mathcal{P}, A, P_I, P_P, \) and \( \Omega \) are as above (3.16), and the

\[ F_m(\omega) = \frac{i\kappa_m}{1 - \sigma_m e^{ik(\omega)L}}, \]

are field enhancement factors. In the limit of no loss, these field enhancement factors are equivalent to those defined in [3]. In particular, when the frequency is at a ring resonance frequency, \( \omega = \omega_{m_0} \), we can write the enhancement factors in terms of the quality factor(s) of the ring

\[ F_m(\omega_{m_0}) = 2i \sqrt{\frac{Q_m v_m}{\omega_{m_0} L}}. \] (3.46)

Classical expressions follow easily from (3.18) using (3.43) for the asymptotic fields. Employing a similar approach to the channel calculation above, a CW DFG calculation yields

\[ P_I = P_S \frac{P_P L^2}{\mathcal{P} A} |F_S(\omega_S)|^2 |F_I(\omega_P - \omega_S)|^2 |F_P(\omega_P)|^2, \] (3.47)

whereas the SHG result is

\[ P_{SH} = \frac{P_F^2 L^2}{\mathcal{P} A} |F_F(\omega_F)|^4 |F_F(2\omega_F)|^2. \] (3.48)

We note that (3.48) was presented earlier [2], with consideration for arbitrary detuning from resonance, for a GaAs ring resonator in which quasi-phase matching must be taken into account.

We now turn to a comparison of the classical (stimulated) processes described by (3.47) and (3.48), and the quantum (spontaneous) process described by (3.45). Note first that, setting aside intensity enhancement factors, there is a quantum power \( h\omega_F/\mathcal{T} \) that plays the role of the seed power \( P_S \) in (3.47), analogous to what was seen in comparing SPDC (3.16) with DFG (3.21) in a channel waveguide. For the structures considered here there is an intensity enhancement factor for the pump \( |F_P(\omega_P)|^2 \) (two, resulting in \( |F_F(\omega_F)|^4 \) in (3.48)) appearing in all three expressions (3.45), (3.47), and (3.48). In
the stimulated processes we also have intensity enhancement factors associated with
the signal input and idler output in (3.47), and with the second harmonic output in
(3.48). In (3.45), which describes an unseeded process, the frequency enhancement factors
associated with the generated fields reside in the generation bandwidth time $T$, and reflect
the fact that there is a range of frequencies over which the generated photons can appear.
Unlike in the channel calculation, looking only at photons generated within only a single
resonance near $\omega_{ISH}/2$, so that we may take
\[ T \approx \frac{2\pi}{\int_0^{\omega_p/2} d\Omega |F_F(\omega_p/2 - \Omega)|^2 |F_F(\omega_p/2 + \Omega)|^2}, \]
that time can be calculated here exactly, in the limit of small detuning and weak cou-
pling ($\sigma_m \approx 1$), so that the intensity enhancement factors (i.e. absolute value of field
enhancement factors squared) become Lorentzians
\[ |F_m(\omega)|^2 = \frac{\kappa_m^2}{(1 - \sigma_m)^2 + (\omega - \omega_m)^2 (L/v_m)^2}, \]
and under the mild assumption that the integration range over the Lorentzians can be
extended to infinity without serious error. The result across a single resonance is
\[ T = \frac{|F_F(\omega_{F_0})|^4 L^3 (2\omega_{F_0} - \omega_p)^2 + 16v_F^2 L}{8v_F^3 |F_F(\omega_{F_0})|^2}, \] (3.49)
where recall $\omega_p = \omega_{ISH}$ is the center frequency of the ring resonance at which the pump
frequency $\omega_p$ is also centered, and $\omega_{F_0}$ is the center frequency of the ring resonance in
which $\omega_p/2$ resides. We see that generation efficiency is worse the farther $\omega_{F_0}$ is from
$\omega_{ISH}/2$. For $\omega_{F_0}$ sufficiently close to $\omega_{ISH}/2$ we have
\[ T = \frac{2L}{\nu_F |F_F(\omega_{F_0})|^2} = \frac{\omega_{F_0}L^2}{2v_F^2 Q_F}, \] (3.50)
where we have used (3.46). We note that a similar result, albeit in a slightly different
form and calculated for a GaAs add/drop ring resonator instead of a ring resonator side-
coupled to a single channel waveguide, was previously presented [32]. Substituting (3.50)
in (3.45) we find

\[
P_I = \frac{\hbar \omega_F v_F}{2L} \frac{P_P L^2}{\mathcal{P} \mathcal{A}} |F_P(\omega_F)|^2 |F_F(\omega_F)|^4
\]
\[
= \frac{\hbar \omega_F}{Q_F \tau_{F_0}} \frac{P_P L^2 Q_F v_F}{\mathcal{P} \mathcal{A}} \left( \frac{Q_F v_F}{\omega_F L} \right)^2
\]

(3.51)

where we have introduced \( \tau_{F_0} = \frac{2\pi}{\omega_{F_0}} \). On the basis of (3.47) and (3.48), and the fact the intensity enhancement factors \( |F_m(\omega_m)|^2 \) are proportional to the \( Q_m \) (3.46), one might have expected to see three quality factors in (3.51). However, because of the resonance, the generation bandwidth time is also increased by a factor of \( Q_F \) and so there are, in the end, only two that result.

Recall that we have assumed that all of the light is collected over a full ring resonance linewidth. If one had a narrow enough filter with bandwidth \( 2\pi B \) that would select out photons generated within just a small window of a ring resonance, the associated calculation would yield

\[
P_I(\Omega) = \frac{\hbar \omega_F}{B^{-1} \mathcal{P} \mathcal{A}} \frac{P_P L^2}{|F_P(\omega_F)|^2 |F_F(\omega_F)|^2 |F_F(\omega_F - \Omega)|^2 |F_F(\omega_F + \Omega)|^2}
\]

(3.52)

where \( \Omega + \omega_F/2 \) now represents the center frequency of the accepted light.

It is worth noting that there is an entire frequency “comb” of resonances on either side of half of the resonance being pumped that essentially satisfy both energy and momentum conservation for the generated correlated photon pairs. Generation is significant up until \( \omega_{S_0} + \omega_{I_0} - \omega_{F_0} = 2\omega_{F_0} - \omega_F \) for the case considered above) becomes too large. Quantifying this a bit more, the generation rate will be reduced by a factor of two when \( \omega_{S_0} + \omega_{I_0} - \omega_F = 4v_F / \left( |F_F(\omega_{F_0})|^2 L \right) = \omega_{F_0} / Q_F \) (recall (3.49)). Taylor expanding \( \omega_{S_0, I_0} = \omega_{F_0} + v_F (N - N_F) / R + \Xi_F (N - N_F)^2 / R^2 \), still taking \( 2\omega_{F_0} - \omega_F \approx 0 \), and remembering that if \( \omega_{S_0} \) is associated with resonance order \( N \) that \( \omega_{I_0} \) is associated with resonance order \( N_F - N \), we find \( \omega_{S_0} + \omega_{I_0} - \omega_{F_0} = 2\Xi_F (N - N_F)^2 / R^2 \). Thus, when

\[
N - N_F = R \left[ \frac{\omega_{F_0}}{2 |\Xi_F| Q_F} \right]^{1/2},
\]

the generation rate drops by a factor of two.

Armed with (3.47),(3.45), and (3.52), a classical (namely, DFG or SHG) experiment that determines \( L^2 / (\mathcal{P} \mathcal{A}) \) as well as the strength of a typical enhancement factor, \( |F_F(\omega_{F_0})|^2 \), allows for an accurate prediction of the average power of photons generated in a corresponding quantum experiment.
3.2.2 Third-Order Processes

As with the second order ring calculations, we can use the general expressions derived in the Section on non-resonant structures to greatly simplify the calculations for a microring resonator side-coupled to a channel waveguide. Starting with the same BWF as above (3.30), and using the asymptotic fields appropriate for a microring resonator (3.43) we find

\[ P_{\omega} = \frac{\hbar \omega \omega_0}{2 \pi} \left( \gamma P \right)^2 |F_{\omega}(\omega_0)|^4. \]  

(3.54)

We have also assumed that we are looking at photons generated within a pair of resonances near enough to \( N_\omega \) that we may take all group velocities, cross-sectional displacement mode fields, resonant frequencies, and thus field enhancement factors to be those of the pump, as well as the appropriate low-power limit (see (B.9)). The generation bandwidth time is

\[ \tau = \frac{2\pi}{\int_{-\omega_P}^{\omega_P} d\Omega |F_{\omega}(\omega_0 - \Omega)|^2 |F_{\omega}(\omega_0 + \Omega)|^2}. \]

Starting from (3.38), the corresponding CW FWM calculation yields

\[ P_{1} = P_{S} (\gamma P_{L})^2 |F_{\omega}(\omega_0)|^{4} |F_{\omega}(\omega_0)|^{2} |F_{1}(2\omega_0 - \omega_0)|^{2}, \]

(3.55)

where \( \gamma \) is as above. We note that this expression agrees with a well-known result [3], in the limit of no loss, small detuning, and weak coupling (\( \sigma \approx 1 \)). The comparison of the classical (stimulated) process described by (3.55) with the quantum (spontaneous)
Chapter 3. Power Scaling Relationships

process described by (3.54) largely follows the comparison between (3.47) and (3.45) respectively for second-order processes. Again we find that the generation bandwidth time $\mathcal{T}$ can be calculated, with the result here given by

$$
\mathcal{T} = \frac{2L}{\nu_p |F_p(\omega_p)|^2} = \frac{\omega_p L^2}{2\nu_p^2 Q_p},
$$
(3.56)

and thus, if photons are collected across an entire ring resonance linewidth, substituting (3.56) in (3.54) we find

$$
P_I = \frac{h\omega_p \nu_p}{2L |F_p(\omega_p)|^2} (\gamma P P L)^2 |F_p(\omega_p)|^8 = \frac{h\omega_p}{Q P_r \tau_p} (\gamma P P L)^2 \left(\frac{Q P \nu_p}{\omega_p L}\right)^4,
$$
(3.57)

where we have introduced $\tau_p = 2\pi/\omega_p$. Note that the appearance of $Q_P$ in the generation bandwidth time leads to a net dependence of $P_I$ on three powers of $Q_P$, rather than the four that might be expected from (3.55). If one had a filter with bandwidth $2\pi B$, and collected photons generated within just a small window of a ring resonance, one would find instead

$$
P_I(\Omega) = \frac{h\omega_p \nu_p}{B^{-1}} (\gamma P P L)^2 |F_p(\omega_p)|^4 |F_p(\omega_p - \Omega)|^2 |F_p(\omega_p + \Omega)|^2,
$$
(3.58)

where $\Omega + \omega_p$ is the center of the narrow frequency window selected by the filter.

Once more there is an entire frequency comb of generated photons. Here the generation rate is reduced by a factor of two when $\omega_{S_0} + \omega_I - 2\omega_p = 4\nu_p / (|F_p(\omega_p)|^2 L) = \omega_P/Q_P$. Taylor expanding as above while remembering that if $\omega_{S_0}$ is associated with resonance order $N$ that $\omega_I$ is associated with resonance order $2N_P - N$, we find that, quite similar to above, when

$$
N - N_P = R [\omega_P / (2|\Xi_{P}| Q_P)]^{1/2},
$$

the generation rate drops by a factor of two.

We conclude our analysis by summarizing in Tab. 3.2 all of the results for stimulated and spontaneous processes in microring resonators. Here we report the “idler” powers as a function of the quality factors $Q$ and ring radius $R$, which are parameters that can be easily measured and thus are experimentally used to characterize a device. These quan-
tities are directly related to the on-resonant intensity enhancement factors \(|F_p(\omega_p)|^2\), as such enhancement depends on the effective mode volume \((\propto AR)\) as well as constructive interference of the field in the ring, which is proportional to the time that light spends in the resonator (proportional to \(Q\)).

Looking at the expressions reported in Tab. 3.2 we see that for DFG and SPDC, the generated powers are inversely proportional to \(AR\). Similarly, the generated idler powers in FWM and SFWM scale with the inverse of \((AR)^2\). In other words, the intensity enhancement associated with the volume of the region in which light is confined has the same weight, whether the process is spontaneous or stimulated. The situation is different if we focus on the scaling of the generated powers with \(Q\). In this case, unfiltered SPDC and SFWM “idler” powers are proportional to \(Q^2\) and \(Q^3\), respectively, while the corresponding processes DFG and FWM scale with \(Q^3\) and \(Q^4\), respectively. Finally, as in channel waveguides, stimulated and spontaneous powers scale the same when we consider the spontaneous generated power in a sufficiently narrow spectral region centered at the idler/signal resonance.

### 3.3 Experimental Verification

In response to our theoretical work, our experimental colleagues at the University of Pavia, Italy, investigated the efficiencies of FWM and SFWM in silicon microring resonators as functions of pump power for various ring radii \(R\) and resonance quality factors \(Q\) \[18\]. While not involved in the experiment I was involved in its analysis, and connected some of the scaling expressions developed earlier in this Chapter with their results. This Section summarizes that collaborative work. In particular the expressions (3.55) and (3.57) are investigated, as is the ratio between them.

#### 3.3.1 Set Up

The resonators under investigation were fabricated on a silicon-on-insulator (SOI) wafer via e-beam lithography and inductively coupled plasma etching. Rings with radii of 5, 10, 20, and 30 \(\mu\)m were side-coupled to a bus waveguide. The waveguides and rings have a \(220 \times 500\ \text{nm}^2\) rectangular cross section. Spot-size converters were used for efficient coupling \[68\]. The silicon chip was finally coated with a protective polymethyl methacrylate (PMMA) layer.

The samples’ linear properties were characterized using a swept laser. As an example, the transmission curve for the \(R = 5\ \mu\)m ring is shown in Fig. 3.5. The spectrum shows
Table 3.2: Expressions for the “Idler” Power Generated via Corresponding Nonlinear Optical Processes in Microring Resonators

<table>
<thead>
<tr>
<th>Process</th>
<th>Idler Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>DFG</td>
<td>( P_1 = P_S \frac{P_0^{32\nu_S\nu_P}}{\pi \omega_S \omega_I \omega_P^3 \omega_{P0}^4 A} \left( \frac{Q_{S} Q_{I}}{R} \right) ) (Eq. (3.47))</td>
</tr>
<tr>
<td>SPDC</td>
<td>( P_1 (\Omega) = hB \frac{P_0^{32\nu_P^2 \nu_S^2}}{\pi \omega_P A \omega_{P0}^4} \left( \frac{Q_{I}^2}{R} \right) ) (filtered; Eq. (3.52))</td>
</tr>
<tr>
<td></td>
<td>( P_1 = h \frac{P_0^{4\nu_P^2 \nu_S^2}}{\pi \omega_P A} \left( \frac{Q_{I}^2}{R} \right) ) (unfiltered; Eq. (3.51))</td>
</tr>
<tr>
<td>FWM</td>
<td>( P_1 = P_S (\gamma P_P)^2 \frac{64\nu_P^2 \nu_S^2}{\pi^2 \omega_P^2 \omega_{S0} \omega_{I0}} \left( \frac{Q_{I}^2 Q_{S} Q_{I}}{R} \right) ) (Eq. (3.55))</td>
</tr>
<tr>
<td>SFWM</td>
<td>( P_1 (\Omega) = hB (\gamma P_P)^2 \frac{64\nu_P^4}{\pi^2 \omega_P^2} \left( \frac{Q_{I}^4}{R^2} \right) ) (filtered; Eq. (3.58))</td>
</tr>
<tr>
<td></td>
<td>( P_1 = h (\gamma P_P)^2 \frac{8 \nu_P^4}{\pi^2 \omega_P^2} \left( \frac{Q_{I}^2}{R^2} \right) ) (unfiltered; Eq. (3.57))</td>
</tr>
</tbody>
</table>
distinct resonances ($Q \sim 7900$) with the transmission falling to less than 1%, a sign that the ring is critically coupled with the bus waveguide. In the other samples, the $Q$ factors were determined to be $Q \sim 8400$ for $R = 10 \ \mu m$, $Q \sim 12000$ for $R = 20 \ \mu m$ and $Q \sim 15000$ for $R = 30 \ \mu m$ [69].

For the nonlinear experiments, a tunable laser (Santec TSL-510) centered at the pump frequency and, for the FWM experiments, another centered at the signal frequency, were injected into a bus waveguide with a tapered lensed fiber. The emitted light was spectrally filtered and sent to a CCD detector. The power coupled into a ring and generated within a ring were estimated by measuring the losses of each component of the experimental setup and consequently rescaling the output power from the laser or the power measured by the CCD (which was calibrated before the experiment). Coupling losses to and from the sample were taken as half its total insertion loss (measured to be about 7 dB). This results in a total error of about 10% for each estimated power.

Dividing by $\hbar \omega_p$ to get photon rates instead of powers, we rewrite Eqs. (3.55) and
(3.57) in the case of triply resonant operation in terms of the easily determined parameters $Q$, $R$, and $v$ as

$$\eta_{\text{ST}} = \left(\frac{2\gamma}{\pi R}\right)^2 \left(\frac{Qv_p}{\omega_p}\right)^4 \frac{P_S P_p^2}{\hbar \omega_p},$$

(3.59)

and

$$\eta_{\text{SP}} = \left(\frac{2\gamma}{\pi R}\right)^2 \left(\frac{Qv_p}{\omega_p}\right)^3 \frac{v_p P_p^2}{4},$$

(3.60)

respectively, where $\eta_{\text{ST}}$ is the idler photon generation rate for FWM and $\eta_{\text{SP}}$ is the idler (or signal) photon generation rate for SFWM. While distributed losses are not included in the calculations presented earlier in this Chapter, we expect their inclusion to have the largest effect on the field enhancement factors. Therefore, as a first approximation, for critically coupled rings with distributed losses we take $|F_m(\omega_0)|^2 \approx Q v_p / (\omega_p \pi R)$ rather than (3.46). We also note that the ratio between the two expressions above is

$$\frac{\eta_{\text{SP}}}{\eta_{\text{ST}}} = \frac{1}{4Q} \frac{\hbar \omega_p^2}{P_S},$$

(3.61)

and is independent of ring size but depends on the resonance quality factor, signal power in the stimulated experiment, and a characteristic power $\hbar \omega_p^2$. For example, at $\hbar \omega_p = 0.8$ eV, we have $\hbar \omega_p^2 \approx 160 \, \mu W$.

### 3.3.2 Results

Figure 3.6 shows the results of both FWM and SFWM experiments on the $R = 5\mu m$ ring resonator, the sample with the smallest footprint, using the resonances marked in Fig. 3.5. Figure 3.6(a) shows an example of spontaneously generated signal and idler beams. The widths of the peaks are identical to those measured in the transmission experiments, indicating that there is little degradation from free carrier absorption. The integrated intensities for the signal and idler beams for SFWM, as well as the integrated intensity of the idler beam for FWM at fixed signal input power ($P_S = 47 \, \mu W$), are shown in Fig. 3.6(b) as a function of $P_p$. In all cases, the intensities follow the expected quadratic dependence of Eqs. (3.59) and (3.60). Note that the saturation observed for $P_p > 2$ mW is due to the thermo-optic effect induced by two-photon absorption, which produces a small redshift of the ring resonances [70, 71]. Furthermore it is also verified that, in FWM, the idler intensity scales linearly with $P_S$.

A best fit of the FWM data [black dashed curve in Fig. 3.6(b)] was performed using $\gamma$ as the only fit parameter. The group velocity was taken as $v_p = c/n_{\text{eff}}$, where $n_{\text{eff}} = 2.47$ was estimated by numerical simulation and is a typical mode effective index for this kind
of waveguide [67], yielding $\gamma = 190 \text{ W}^{-1}\text{m}^{-1}$, a value consistent with what was previously reported in [67]. This value was then used to compute the expected rate of generated photon pairs via SFWM using (3.60). The result is shown as a short dashed blue line in Fig. 3.6(b), which is in good agreement with the experiments.

In addition, FWM and SFWM experiments were carried out on rings with radii of 5, 10, 20 and 30 $\mu$m to verify the scaling of (3.59), (3.60), and (3.61) with $R$. In all cases, rings near the critical coupling condition were chosen and the same separation in energy between the signal, pump, and idler resonances was always used: i.e. the signal and idler resonances were the first neighbors of the pumped resonance for the $R = 5 \mu$m ring [Fig. 3.5(b)], second neighbors for $R = 10 \mu$m, fourth neighbors for $R = 20 \mu$m, and sixth neighbors for $R = 30 \mu$m. In Fig. 3.7(a), FWM- and SFWM-generated idler powers are shown as a function of $R$: the data were averaged over a set of measurements in which $P_P$ and $P_S$ were varied. Note that for all data presented in Fig. 3.7, it was ensured that $P_P < 1 \text{ mW}$ to avoid the thermo-optic effect. As expected from (3.59) and (3.60), both of the generated powers scale as $R^{-2}$. Finally, in Fig. 3.7(b) the ratio between the measured generated idler powers in SFWM and FWM experiments performed using the same pump power is shown as a function of $Q \cdot P_S$. The data shows an excellent agreement with (3.61) and confirms that the ratio is independent of ring radius. More importantly, these results
prove that data from FWM can be used to precisely predict SFWM generation rates. For clarity, in Fig. 3.7(b) only data for $P_p = 0.12$ mW is shown, but the same results were obtained for all investigated pump powers from 60 µW to 1 mW.

### 3.4 Discussion

In this Chapter I have employed a general Hamiltonian formalism, which places quantum and classical wave mixing processes in integrated devices on equal theoretical footing, to calculate expressions for SPDC, DFG, SHG, SFWM and FWM in both channel waveguides and microring resonators side-coupled to channel waveguides as two examples. The end result is two simple tables that clearly demonstrate that spontaneous photon generation efficiencies can be estimated from the results of stimulated experiments. Not only is this demonstrated through theory, in Sections 3.1 and 3.2, but also through experiment, for FWM and SFWM in silicon microring resonators in Section 3.3. Given any photonic structure, the application of this formalism allows the derivation of the number of photon pairs generated in a spontaneous process through a simple measurement of the corresponding stimulated process, greatly simplifying the characterization of any structure designed for the generation of quantum photonic states.

In channel waveguides the scaling behavior of both stimulated and spontaneous processes with the length $L$ of the channel is independent of the order of the nonlinearity. In particular, the efficiency of phase matched stimulated generation scales as $L^2$, whether DFG or FWM is being considered. The same scaling is obtained for the generation rate
of spontaneous processes within a sufficiently narrow spectral region around the phase matching condition. Similarly, the generated idler power, integrated over the entire spectrum, scales with $L^{3/2}$ for both SPDC and SFWM in a channel, as in the absence of filtering the generation bandwidth time scales as $L^{1/2}$. This is in contrast with the results for a microring resonator, in which all second order processes scale with ring radius $R$ as $R^{-1}$ and all third-order processes as $R^{-2}$ regardless of filtering. Considering enhancement due to light confinement in the channel, which is inversely proportional to the effective area, one sees that this effect on the generated power is independent of the nature of the process, stimulated or spontaneous, while it depends upon the order of the nonlinearity. Indeed, for DFG and SPDC, the generation rate is proportional to $1/A$, while for FWM and SFWM it scales with $\gamma^2 \propto 1/A^2$. This result is consistent with the fact that for the example of a channel the scaling with $L$ is due solely to the interference of the fields generated at different positions in the channel. That is, there is no $L$-dependent field enhancement. Again this is in contrast with the results for a microring side-coupled to a channel waveguide, in which the additional confinement in the direction of propagation due to the ring plays a role. DFG and FWM scale with $Q^3$ and $Q^4$, respectively, in a ring resonator as do the corresponding spontaneous processes within a sufficiently narrow spectral region centered at the idler/signal resonance. Unfiltered SPDC and SFWM generated powers, on the other hand, are proportional to $Q^2$ and $Q^3$, respectively.
Chapter 4

Biphoton Engineering

Not all photons are created equal, nor should they be. Different applications demand quantum correlated photon pairs with different properties. For example, maximal violation of Bell’s inequality [6, 7] using polarization entangled photons requires only that both photons of a given pair have the same marginal frequency distribution, whereas entanglement swapping of polarization entangled photon pairs may be spoiled by any sort of frequency correlations [42]. Further, interaction with various other systems (e.g. quantum memories) necessitates the generation of photons with specific frequency bandwidths. Thus, in addition to predicting the rate of pair generation, as in the previous Chapter, it is also important to understand the range of possible quantum correlated states of photon pairs that can be generated in various artificially structured media.

In this Chapter I examine the general form of the biphoton probability density

$$\left| \phi_{\mu \nu}^{(C(R))2(3)}(\omega_1, \omega_2) \right|^2,$$

associated with photons generated in each a channel waveguide, C, and a microring resonator side-coupled to a channel waveguide, R, via a second- or third-order nonlinearity, denoted by 2 and 3 respectively, as well as the parameters that need to be adjusted in order to achieve a particular shape of the biphoton probability density for each nonlinearity in each structure. The Greek subscripts, while not used in the previous Chapter, label the generated photon mode-types (i.e. they are the $I$’s in (2.26)), and will be discussed further in the following Section. While similar idealized theoretical investigations have been carried out previously, such as how to create frequency uncorrelated photon pairs via type-II (defined in Section 4.1) SPDC in bulk crystals [72] as well as in Bragg reflection waveguides [73], how to create maximally polarization-entangled photons via SPDC in Bragg reflection waveguides [74], an analysis of the performance of bulk crystal SPDC-
generated photon pairs in entanglement swapping gates and type-I fusion gates [42], and the optimization of SPDC-generated photons for quantum information processing [40], here the scope is grander. The widths of biphoton probability densities associated with photon pairs generated via not only type-I or type-II SPDC, but also type-I or type-II SFWM, and not only generated in channel waveguides, but also generated in microring resonators, are catalogued such that approximate analytic expressions for the Schmidt numbers [63] of the associated BWFs can be easily obtained, and suitability of the various photon pairs for two specific applications can be easily quantified. It is demonstrated that the shape of the biphoton probability density depends strongly on dispersion for a channel waveguide, and strongly on coupling between the ring and the waveguide for a microring resonator. This second fact presents a novel way to realize the efficient generation of uncorrelated photons on a chip.

In Section 4.1 general expressions for the state of generated photons from a SPDC or SFWM process in either a channel waveguide or microring resonator as well as their associated BWFs are presented. We do not consider any phase accumulated by the generated photons on their way to a detector, or even while propagating in the nonlinear material itself, as it can, at least in principle, be compensated for by an opposite-signed dispersive element. As application examples, in Section 4.2 we discuss CHSH inequality violation [7] and entanglement swapping [42], corresponding to a classic single photon pair source application and multi-source application respectively, with consideration given to how the BWF influences each. In Section 4.3 possible associated biphoton probability densities, centered at the origin, are explored, along with approximate analytic expressions for their widths. These expressions are in terms of pump and photon pair source parameters, providing a method to engineer the correlations in the biphoton probability density necessary for successful implementation of one of the aforementioned applications through careful waveguide design and choice of pump pulse bandwidth. A more quantitative analysis focusing on the Schmidt number is presented in Section 4.4, where a comparison is made with double-Gaussian BWFs, for which the Schmidt number can be calculated analytically, and in Section 4.5 a discussion of these results follows.

As in the previous Chapter, here all nonlinear effects other than SPDC or SFWM are neglected, the results presented are only strictly valid in the undepleted pump approximation at low enough pump powers, and all forms of loss are neglected as well. While it is true that the BWF associated with a photon pair source in the lab may not meet all of these idealizations, it is also true that if an idealized source is not suitable for a particular application then any non-idealizations will certainly not improve its suitability. Therefore the results of this Chapter can be thought of as source design guidelines, if not
4.1 General Expressions

In this Section we write the state of generated photons in a more general form than that of the previous Chapter. In Chapter 3 we always implicitly assumed a type-I process, in which all generated photons are created in the same mode and with the same polarization, for simplicity. However, it is possible that they are created in different modes, as in a type-II process or for two coherent type-I processes, one immediately after the other [75]. Generally speaking, a type-I nonlinear optical process is one that is phase matched for both signal and idler fields having the same polarization, and a type-II nonlinear optical process is one that is phase matched for the signal field having a polarization orthogonal to the idler field. While more complicated naming conventions exist in which the polarization of the pump field is also taken into account, here we adopt the simpler definition involving only signal and idler fields, and extend it beyond polarization to the more general concept of a mode of a channel. Thus for type-I processes we have $\mu = \nu$ whereas for type-II processes $\mu \neq \nu$. We note that the calculations of the previous Chapter can be easily extended to consider type-II phase matching [62], or multiple processes that are phase matched concurrently - indeed the formalism of Chapter 1 is written with this in mind - but here we will not perform such calculations explicitly. As the focus of this Chapter is on BWFs, we simply present the states of generated photons, BWFs, and biphoton probability densities that result from such calculations. Additionally, we examine the symmetries of the BWFs associated with type-I and type-II processes and discuss their implications for violating the CHSH inequality and for entanglement swapping. For ease of comparison, we also rewrite the BWFs resulting from SPDC and SFWM in a channel waveguide and a microring resonator in a form that makes it easy to see the terms responsible for their shapes.

4.1.1 The State of Generated Photons

The most general state of generated photons considered in the previous Chapter was the single-mode squeezed vacuum (recall (3.7), (3.8))

$$|\psi_{gen}\rangle = \exp \left( \beta C_{II}^\dagger - \text{H.c.} \right) |\text{vac}\rangle,$$
Figure 4.1: Symmetry of a type-I BWF. We refer to the dashed red line as the 45° line, and the solid blue line as the pump line.

where

\[
C_{\Pi}^{\dagger} = \frac{1}{\sqrt{2}} \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \phi(\omega_1, \omega_2) b_{\omega_1}^{\dagger} b_{\omega_2}^{\dagger},
\]

Note that here and throughout the rest of this Chapter we have dropped the output (right) channel index, R, to simplify the notation. Although the BWF \( \phi(\omega_1, \omega_2) \) is different for each of the four cases considered previously (SPDC in a channel (3.12), SPDC in a microring resonator (3.44), SFWM in a channel (3.32), and SFWM in a microring resonator (3.53)), the form of the state of generated photons is always the same. The BWF associated with a type-I process is naturally symmetric about a diagonal line 45° between the \( \omega_1 \) and \( \omega_2 \) axes, i.e. \( \phi(\omega_1, \omega_2) = \phi(\omega_2, \omega_1) \) (see Fig. 4.1).

However, if we allow for the generated photons to be created in different modes, we find that the state can instead be written as

\[
|\psi_{\text{gen}}\rangle = \exp \left( \beta C_{\Pi}^{\dagger} - \text{H.c.} \right) |\text{vac}\rangle,
\]

where

\[
C_{\Pi}^{\dagger} = \frac{1}{\sqrt{2}} \sum_{\mu, \nu} \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \phi_{\mu \nu}(\omega_1, \omega_2) b_{\omega_1}^{\dagger} b_{\omega_2}^{\dagger},
\]

the normalization of which sets

\[
\sum_{\mu, \nu} \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 |\phi_{\mu \nu}(\omega_1, \omega_2)|^2 = 1.
\]

The more general BWF \( \phi_{\mu \nu}(\omega_1, \omega_2) \) is symmetric under exchange of both mode and frequency \( \phi_{\mu \nu}(\omega_1, \omega_2) = \phi_{\nu \mu}(\omega_2, \omega_1) \), but in general has \( \phi_{\mu \nu}(\omega_1, \omega_2) \neq \phi_{\nu \mu}(\omega_1, \omega_2) \) (and
thus $\phi_{\mu\nu}(\omega_1, \omega_2) \neq \phi_{\mu\nu}(\omega_2, \omega_1)$. More specifically the dispersion relation, or dependence of $k$ on $\omega$, is not necessarily the same for each of the fields involved, meaning that this type of BWF is not, in general, symmetric about a diagonal line $45^\circ$ between the $\omega_1$ and $\omega_2$ axes (see Fig. 4.1). We refer to this line as the $45^\circ$ line, and a line perpendicular to it and passing through the center of the BWF as the pump line.

Focusing on polarization modes (horizontal, H, and vertical, V) only, note that while

$$|\psi_{\text{HVVH}}\rangle \propto \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \left[ \phi_{\text{HV}}(\omega_1, \omega_2) \right] b_{\text{H}H_{\omega_1}}^\dagger b_{\text{V}V_{\omega_2}}^\dagger |\text{vac}\rangle, \tag{4.1}$$

should be written as such to make its symmetric nature explicit, it could just as easily be written as

$$|\psi_{\text{HVHV}}\rangle \propto \sqrt{2} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi_{\text{HV}}(\omega_1, \omega_2) b_{\text{H}H_{\omega_1}}^\dagger b_{\text{V}V_{\omega_2}}^\dagger |\text{vac}\rangle,$$

i.e. all of the information about the correlations between the two photons is contained within a single function with fixed polarization indices $\phi_{\text{HV}}(\omega_1, \omega_2)$, and the other piece of the total BWF $\phi_{\text{VH}}(\omega_1, \omega_2)$ can be calculated from symmetry arguments. This is why often only a single function with fixed polarization indices is plotted in the literature [27]. However the same is not true for a state such as

$$|\psi_{\text{HHVV}}\rangle \propto \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \left[ \phi_{\text{HH}}(\omega_1, \omega_2) \right] b_{\text{H}H_{\omega_1}}^\dagger b_{\text{H}H_{\omega_2}}^\dagger + \phi_{\text{VV}}(\omega_1, \omega_2) b_{\text{V}V_{\omega_1}}^\dagger b_{\text{V}V_{\omega_2}}^\dagger |\text{vac}\rangle \tag{4.2},$$

Furthermore, symmetry arguments ensure that, for the state $|\psi_{\text{HVHV}}\rangle$ (4.1),

$$\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{\text{HV}}(\omega_1, \omega_2)|^2 = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{\text{HV}}(\omega_1, \omega_2)|^2 = 1/2,$$

whereas while

$$\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{\text{HH}}(\omega_1, \omega_2)|^2 + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{\text{VV}}(\omega_1, \omega_2)|^2 = 1,$$

for the state $|\psi_{\text{HHVV}}\rangle$ (4.2), it is not necessary that they are both equal.

### 4.1.2 The Biphoton Wave Function

As we are concerned with the shape of the absolute value squared of each of these two-dimensional functions here, we rewrite each in a form that highlights the pieces with the greatest influence on their shape. Whether photon pair generation occurs due to a SPDC
process in a channel waveguide or in a microring resonator, the resulting BWF has the same general form (cf. (3.12), (3.44))

\[
\phi_{C}^{\mu\nu}(\omega_1,\omega_2) = B_{C}^{\mu\nu}(\omega_1,\omega_2) \phi_P(\omega_1+\omega_2) \theta_{C}^{\mu\nu}(\omega_1,\omega_2),
\]

\[
\phi_{R}^{\mu\nu}(\omega_1,\omega_2) = B_{R}^{\mu\nu}(\omega_1,\omega_2) \phi_P(\omega_1+\omega_2) \theta_{R}^{\mu\nu}(\omega_1,\omega_2).
\]

(4.3)

Here

\[
B_{C}^{\mu\nu}(\omega_1,\omega_2) = iL \sqrt{\frac{d k_\mu(\omega_1)}{d \omega_1} \frac{d k_\nu(\omega_2)}{d \omega_2} \left(\sqrt{\frac{d k_P(\omega)}{d \omega}}\right)_{\omega=\omega_1+\omega_2}} \times \sqrt{\frac{\omega_1 \omega_2}{8 \pi \varepsilon_0 |\beta|^2 \pi^6}} \frac{h(\omega_1+\omega_2)}{A[k_\mu(\omega_1), k_\nu(\omega_2), k_P(\omega_1+\omega_2)]^3},
\]

and

\[
B_{R}^{\mu\nu}(\omega_1,\omega_2) = -i B_{C}^{\mu\nu}(\omega_1,\omega_2) \text{sinc}\{[k_\mu(\omega_1) + k_\nu(\omega_2) - k_P(\omega_1+\omega_2)]L/2\}
\]

\[
\times \kappa_\mu \kappa_\nu \kappa_P v_\mu v_\nu v_P / L^3
\]

are benign functions of generated photon frequencies, or functions which have little influence over the shape of the biphoton probability density, i.e. they are essentially constant over the range over which the product of the other two functions is significantly nonzero.

Additionally \(A\) is an effective area for the nonlinear process (see (3.13)), \(\phi_P(\omega)\) the pump pulse waveform, and \(\kappa\) (with \(\kappa^2 + \sigma^2 = 1\)) the cross-coupling coefficient between bus waveguide and ring, with \(\sigma\) the self-coupling coefficient. Note that the phase matching sinc function is included in the benign function for SPDC in a microring resonator \(B_{R}^{\mu\nu}(\omega_1,\omega_2)\), as here we are assuming that the center frequencies of the three fields involved are on resonance and that these resonances are narrow enough that the squared sinc function appearing in the biphoton probability density is approximately unity over a resonance linewidth. That is, for some \(\omega_\mu + \omega_\nu = \omega_P\), set so that \(k_m(\omega_m)L = 2\pi N_m\), with \(N_m\) an integer for \(m = \mu, \nu\) or \(P\), we have \(k_\mu(\omega_\mu) + k_\nu(\omega_\nu) = k_P(\omega_P)\). Thus when

\[
F_m(\omega) = \frac{i \kappa(k_m(\omega))}{1 - \sigma(k_m(\omega))e^{ik_m(\omega)L}} \approx \frac{i \kappa_m v_m / L}{\Omega_m - i(\omega - \omega_m)},
\]

is a reasonable approximation for all three resonances, where we have defined \(\kappa(k_m(\omega_m)) \equiv \kappa_m, \sigma(k_m(\omega_m)) \equiv \sigma_m, \Omega_m = (1 - \sigma_m) v_m / L\) and assumed weak coupling, \(\sigma_m \approx 1\), we find
and is related to the quality factor \( Q \) via \( \Omega_m = \omega_m / (2Q_m) \). Furthermore we take
\[
\phi_p (\omega) = \exp \left( -\frac{(\omega - \omega_p)^2}{(2\Delta_\omega)^2} \right),
\]
as the pump waveform - a function of the sum of generated photon frequencies - and
\[
\theta_{\mu \nu}^{C_2} (\omega_1, \omega_2) = \text{sinc} \left\{ [k_\mu (\omega_1) + k_\nu (\omega_2) - k_P (\omega_1 + \omega_2)] L/2 \right\},
\]
\[
\theta_{\mu \nu}^{R_2} (\omega_1, \omega_2) = [\Omega_\mu + i (\omega_1 - \omega_\mu)]^{-1} [\Omega_\nu + i (\omega_2 - \omega_\nu)]^{-1} [\Omega_P - i (\omega_1 + \omega_2 - \omega_P)]^{-1},
\]
are more complicated functions of the generated photon frequencies.

Similarly, for SFWM, taking \( \omega_\mu + \omega_\nu = 2\omega_P \), so that \( k_\mu (\omega_\mu) + k_\nu (\omega_\nu) = 2k_P (\omega_P) \), we have (cf. (3.32), (3.53))
\[
\phi_{\mu \nu}^{C_3} (\omega_1, \omega_2) = B_{\mu \nu}^{C_3} (\omega_1, \omega_2) \theta_{\mu \nu}^{C_3} (\omega_1, \omega_2),
\]
\[
\phi_{\mu \nu}^{R_3} (\omega_1, \omega_2) = B_{\mu \nu}^{R_3} (\omega_1, \omega_2) \theta_{\mu \nu}^{R_3} (\omega_1, \omega_2),
\]
(4.4)
with
\[
B_{\mu \nu}^{C_3} (\omega_1, \omega_2) \approx 3i L \sqrt{\frac{\partial k_\mu (\omega_1)}{\partial \omega_1}} \sqrt{\frac{\partial k_\nu (\omega_2)}{\partial \omega_2}} \sqrt{\frac{\omega_1 \omega_2 (\chi_3)^2 N_p^2}{128 \pi^2 \varepsilon_0^2 |\beta|^2 n^8}} \times \left( \frac{\partial k_P (\omega)}{\partial \omega} \right)_{\omega=\omega_p}^2 \sqrt{\frac{\hbar^2 \omega_p^2}{A^2 [k_\mu (\omega_1), k_\nu (\omega_2), k_P (\omega_P), k_P (\omega_P)]}},
\]
\[
B_{\mu \nu}^{R_3} (\omega_1, \omega_2) \approx B_{\mu \nu}^{C_3} (\omega_1, \omega_2) \text{sinc} \left\{ [k_\mu (\omega_1) + k_\nu (\omega_2) - 2k_P (\omega_P)] L/2 \right\} \times \kappa_\mu \kappa_\nu \kappa_P^2 \nu_\mu \nu_\nu \nu_P^2 / L^4
\]
as well as
\[
\theta_{\mu \nu}^{C_3} (\omega_1, \omega_2) = \int_{0}^{\omega_1 + \omega_2} d\omega \left( \phi_P (\omega) \phi_P (\omega_1 + \omega_2 - \omega) \times \text{sinc} \left\{ [k_\mu (\omega_1) + k_\nu (\omega_2) - k_P (\omega) - k_P (\omega_1 + \omega_2 - \omega)] L/2 \right\} \right),
\]
\[ \theta_{\mu \nu}^{R_3} (\omega_1, \omega_2) = \int_0^{\omega_1 + \omega_2} d\omega \left\{ \phi_P (\omega) \phi_P (\omega_1 + \omega_2 - \omega) [\Omega_{\mu} + i (\omega_1 - \omega_{\mu})]^{-1} [\Omega_\nu + i (\omega_2 - \omega_\nu)]^{-1} \times [\Omega_P - i (\omega - \omega_P)]^{-1} [\Omega_P - i (\omega_1 + \omega_2 - \omega - \omega_P)]^{-1} \right\}, \]

if we assume that the group velocities, coupling constants, effective area (see (3.33)), and sinc function are also essentially constant over the integral over \( \omega \). Note that, unlike for SPDC, \( \theta_{\mu \nu}^{C_3} (\omega_1, \omega_2) \) and \( \theta_{\mu \nu}^{R_3} (\omega_1, \omega_2) \) contain integrals over the pump waveform. This is because two pump photons are annihilated in a SFWM process, and energy conservation alone, expressed by a Dirac delta function in our formalism (recall (3.6)), is not enough to set the frequency of each generated photon in relation to more than one pump photon.

### 4.2 Single-Source vs. Multi-Source Experiments

Again focusing on just polarization modes for simplicity, we imagine an idealized state preparation in which a single pair of photons of the type (4.1) or (4.2) is separated with a 50/50 beam-splitter, and then directed to an appropriate set of polarization beam-splitters and phase delays to become one of the four states [76]

\[ |\Psi^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \left[ \phi_{HV} (\omega_1, \omega_2) |H(\omega_1) V(\omega_2)\rangle_{12} \right. \]

\[ \pm \phi_{VH} (\omega_1, \omega_2) |V(\omega_1) H(\omega_2)\rangle_{12}, \quad (4.5) \]

\[ |\Phi^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \left[ \phi_{HH} (\omega_1, \omega_2) |H(\omega_1) H(\omega_2)\rangle_{12} \right. \]

\[ \pm \phi_{VV} (\omega_1, \omega_2) |V(\omega_1) V(\omega_2)\rangle_{12}, \quad (4.6) \]

where \( |A(\omega_1) B(\omega_2)\rangle_{12} = b_{A_{\omega_1:1}}^\dagger b_{B_{\omega_2:2}}^\dagger |\text{vac}\rangle \), the additional labels 1 and 2 indicating which beam-splitter port the photon exited. While half of the time the photons will both end up exiting the same 50/50 beam-splitter port, these events are simply ignored here in our idealized picture of possible photon pair sources. Alternatively, a similar effect could be obtained with a dichroic beam-splitter without taking a 3dB penalty [74]. As the two photons now exist in separate channels and some have been neglected, the BWFs should be renormalized such that \( \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{HV}(\omega_1, \omega_2)|^2 = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{VH}(\omega_1, \omega_2)|^2 = 1 \), and \( \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{HH}(\omega_1, \omega_2)|^2 + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{VV}(\omega_1, \omega_2)|^2 = 2 \). Note that such a renormalization does not affect any of the \( \theta_{\mu \nu}^{C(R)_{2(1)}} (\omega_1, \omega_2) \) or \( \phi_P (\omega) \) functions that determine the shape of the various BWFs, but simply introduces a multiplicative constant.
that can be absorbed into the benign functions $B^{R}_{\mu \nu} (\omega_1, \omega_2)$.

We now consider the suitability of these states for the two classic applications mentioned in the introduction to this Chapter, namely CHSH inequality violation and entanglement swapping. CHSH inequality violation requires just a single correlated photon pair source, whereas entanglement swapping requires two such sources. In a CHSH inequality violation experiment relying on a polarization-entangled photon pair, one photon of the pair is directed to a polarizer placed before a detector, and the other to a second polarizer placed before a second detector. If such photon pair states can be used to violate [77]

$$S = |E(\theta_A, \theta_B) - E(\theta_A, \theta'_B) + E(\theta'_A, \theta_B) + E(\theta'_A, \theta'_B)| \leq 2,$$

where

$$E(\theta_A, \theta_B) = \frac{C(\theta_A, \theta_B) + C(\theta_A, \theta'_B) - C(\theta_A, \theta_B) - C(\theta_A, \theta'_B)}{C(\theta_A, \theta_B) + C(\theta_A, \theta'_B) + C(\theta_A, \theta_B) + C(\theta_A, \theta'_B)},$$

and $C(\theta_A, \theta_B)$ is the number of coincidences for coincidence detection of the photons in the experiment mentioned above with linear polarizers set at an angle $\theta_i$ from the horizontal H axis, and $\theta'_i = \theta_i + \pi/2$, then, provided the fair sampling hypothesis is correct, at least one of the assumptions of realism and locality that were used to derive the inequality is incorrect [11]. As is shown in Appendix C, $S$ cannot be maximally violated unless $G = G^* = 1$, for $|\Psi^\pm\rangle$, or $H = V = M = M^* = 1$, for $|\Phi^\pm\rangle$ (see (C.3)) where

$$G = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi_{HV}(\omega_1, \omega_2) \phi^*_{HV}(\omega_1, \omega_2),$$

$$H = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{HH}(\omega_1, \omega_2)|^2,$$

$$M = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi_{HH}(\omega_1, \omega_2) \phi^*_{VV}(\omega_1, \omega_2),$$

$$V = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{VV}(\omega_1, \omega_2)|^2.$$

This demonstrates that while the BWF does not need to be factorable in order to be able to experimentally realize a violation of Bell’s inequality with polarization-encoded qubits, it does need to possess certain symmetries. Indeed, $G = G^* = 1$ corresponds to $\phi_{HV}(\omega_1, \omega_2) = \phi_{VH}(\omega_1, \omega_2)$ (see (C.4)) or $\phi_{HV}(\omega_1, \omega_2)$ being symmetric about the 45° line. Similarly $H = M = M^* = V = 1$ corresponds to $\phi_{HH}(\omega_1, \omega_2) = \phi_{VV}(\omega_1, \omega_2)$, and HH and VV pairs being produced with equal probability.
In an entanglement swapping experiment relying on two polarization-entangled photon pairs, we imagine directing one photon of each pair to each input port of a 50/50 beam-splitter, and each output port being directed to a subsequent polarization beam-splitter before detection (see Fig. 4.2). When constrained to linear optical elements and classical feed-forward, interferometric set-ups followed by perfect number-resolving photodetection can distinguish between only two of the four Bell states [76]. This leads to use of the device shown in Fig. 4.2 with the states (4.5) for entanglement swapping. In particular, beginning with the four-photon pure state \( |\psi^+\rangle_{01} |\psi^+\rangle_{23} \), successful detection at detectors placed at output ports 6 and 7 could project the remaining photons onto the state \( |\psi^+\rangle_{03} \) provided certain conditions on the BWF \( \phi_{HV}(\omega_1, \omega_2) \). When this occurs the entanglement between the photons in channels 0 and 1, as well as 2 and 3, is said to have been “swapped” to the photons in channels 0 and 3. As shown in Appendix C (see (C.7)), the necessary condition for successful entanglement swapping of the states (4.5) is that \( \phi_{HV}(\omega_1, \omega_2) \) be factorable into a function of \( \omega_1 \) and a function of \( \omega_2 \). While it is sometimes obvious to see if this is possible, a more quantitative approach involves performing a Schmidt decomposition such that

\[
\phi_{\mu\nu}(\omega_1, \omega_2) = \sum_{\lambda} \sqrt{p_\lambda} \Phi_{\lambda}(\omega_1) \Theta_{\lambda}(\omega_2),
\]

where the functions \( \Phi_{\lambda}(\omega) \) and \( \Theta_{\lambda}(\omega) \) are complete and orthonormal, and \( \sum_{\lambda} p_\lambda = 1 \). If there is only one non-zero Schmidt coefficient \( p_\lambda \), i.e. \( p_\lambda = 1 \) with \( p_i = 0 \), \( \forall i \neq \lambda \), then the BWF is clearly factorable. This can also be quantified with the Schmidt number...
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\[ K = \left( \sum_{\lambda} p_{\lambda}^2 \right)^{-1} \] [63], which reaches a minimum of 1 when the BWF is factorable, and is larger otherwise. We note that the more well-known version of the Schmidt number as defined in e.g. Nielsen and Chuang [11] is simply the number of non-zero Schmidt coefficients. However, as we are not interested in doing any calculations with the Schmidt number but simply using it to characterize the frequency correlations in a given BWF, we prefer Law and Eberly’s definition \( K \) [63]. Theirs appears more reasonable from a physical point of view, for if there were 1000 Schmidt coefficients and 999 of them were only as large as \( 10^{-5} \), Nielsen and Chuang’s definition would give a Schmidt number of 1000, whereas \( K \approx 1.02 \). As mentioned in the introduction to this Chapter, approximate methods of determining the Schmidt number \( K \), and thus whether a BWF is factorable and the photons from a given source suitable for, say, entanglement swapping, will be presented in Section 4.4.

4.3 Shaping

For ease of comparing the shapes of the four biphoton probability densities introduced above, we center all of them at the origin. Defining \( \Omega_1 = \omega_1 - \omega_\mu \), \( \Omega_2 = \omega_2 - \omega_\nu \), \( \Omega = \omega - \omega_P \), with \( \omega_\mu + \omega_\nu = \omega_P \) for SPDC and \( \omega_\mu + \omega_\nu = 2\omega_P \) for SFWM, as well as Taylor expanding dispersion relations to second order

\[
k_m(\omega) = k_m(\omega_m) + v_{m-1}(\omega - \omega_m) + \Lambda_m(\omega - \omega_m)^2,
\]

we have

\[
|\phi_P(\Omega)|^2 = \exp\left(-\frac{\Omega^2}{\Delta_\omega^2}\right) \sqrt{\frac{2}{\pi \Delta_\omega}}, \quad (4.7)
\]

with

\[
|\theta^{C_2}_{\mu \nu}(\Omega_1, \Omega_2)|^2 = \text{sinc}^2 \left\{ \left[ v_{\mu-1} \Omega_1 + v_{\nu-1} \Omega_2 - v_{P-1} (\Omega_1 + \Omega_2) \right] \right.
\]

\[
+ \left. \Lambda_\mu \Omega_1^2 + \Lambda_\nu \Omega_2^2 - \Lambda_P (\Omega_1 + \Omega_2)^2 \right\} L/2 \right\}, \quad (4.8)
\]

\[
|\theta^{R_2}_{\mu \nu}(\Omega_1, \Omega_2)|^2 = \left[ \Omega_1^2 + \Omega_2^2 \right]^{-1} \left[ \Omega_\nu^2 + \Omega_2^2 \right]^{-1} \left[ \Omega_P^2 + (\Omega_1 + \Omega_2)^2 \right]^{-1}, \quad (4.9)
\]

for SPDC and

\[
|\theta^{C_3}_{\mu \nu}(\Omega_1, \Omega_2)|^2 = \left| \int_{-\omega_P}^{\Omega_1 + \Omega_2} d\Omega \phi_P(\Omega) \phi_P(\Omega_1 + \Omega_2 - \Omega) \text{sinc} \left\{ \left[ v_{\mu-1} \Omega_1 + v_{\nu-1} \Omega_2 - v_{P-1} (\Omega_1 + \Omega_2) \right] \right.
\]

\[
+ \left. \Lambda_\mu \Omega_1^2 + \Lambda_\nu \Omega_2^2 - \Lambda_P \Omega^2 - \Lambda_P (\Omega_1 + \Omega_2)^2 \right\} L/2 \right\}^2, \quad (4.10)
\]
Figure 4.3: Sketch of potential pump waveform contributions to the SPDC biphoton probability density (4.7). A representative pump pulse bandwidth $\Delta_\omega$ is increased as panels move from left to right.

$$
\left| \theta^{R3}_{\mu\nu} (\Omega_1, \Omega_2) \right|^2 = \left| \int_{-\omega_p}^{\Omega_1+\Omega_2} \phi_P (\Omega) \phi_P (\Omega_1 + \Omega_2 - \Omega) \times \left[ \Omega_\mu + i\Omega_1 \right]^{-1} \left[ \Omega_\nu + i\Omega_2 \right]^{-1} \left[ \Omega_P - i(\Omega_1 + \Omega_2 - \Omega) \right]^{-1} \right|^2,
$$

(4.11)

for SFWM. We take a closer look at SPDC first.

### 4.3.1 SPDC Biphoto Probability Densities

In $(\Omega_1, \Omega_2)$ space, the pump waveform contribution to the biphoton probability density for SPDC generated photons in either a channel waveguide or a microring resonator, $|\phi_P (\Omega_1 + \Omega_2)|^2$, is simply a two-dimensional Gaussian centered on line of constant energy that we refer to as the pump line (see Fig. 4.1). As we have centered the biphoton probability density at the origin, this line is defined by $\Omega_1 + \Omega_2 = 0$. It is perpendicular to the $45^\circ$ line. The function $|\phi_P (\Omega_1 + \Omega_2)|^2$ has an infinite FWHM along the pump line and a FWHM along the $45^\circ$ line of

$$
\Delta \phi_{P_2} = \sqrt{2\ln(2)} \Delta_\omega,
$$

(4.12)

(see Fig. 4.3). Note that any two SPDC photon pair sources pumped with Gaussian pulses of the same bandwidth $\Delta_\omega$ will have the same pump waveform contribution to the biphoton probability density.

There is far more freedom in the possible shapes of the $\theta_{\mu\nu}^{C(R)2(3)} (\Omega_1, \Omega_2)$ functions. For photons generated in a channel waveguide, their shape is largely determined by the group velocities of the various fields involved in the nonlinear interaction. In fact, for type-II SPDC, in which $\mu \neq \nu$, it is sufficient to neglect the group velocity dispersion
terms, \( \Lambda_m \). Doing so means that \( |\theta^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \) has an infinite FWHM along one line, which we refer to as the phase matching line (defined below) and a finite FWHM along a line perpendicular to the phase matching line. For a type-II SPDC process, if \( v_p > v_\mu, v_\nu \) or \( v_p < v_\mu, v_\nu \) then the phase matching line runs from the lower right quadrant to the upper left quadrant (along \( \Omega_1 + \Omega_2 = 0 \) for type-I SPDC, when \( v_\mu = v_\nu \)). However, if \( v_p = v_\mu, v_\nu \) it runs along \( \Omega_2 = 0 \) (i.e. it is horizontal, or vertical), and if \( v_\mu > v_p > v_\nu \) or \( v_\nu > v_p > v_\mu \) it runs from the lower left quadrant to the upper right quadrant (perpendicular to \( \Omega_1 + \Omega_2 = 0 \) for 2\( v_p - v_\mu = v_\mu - v_\nu \)). Thus, any orientation in the \( \Omega_1, \Omega_2 \) plane is conceivable; in general the phase matching line makes an angle

\[
\theta = \tan^{-1} \left( \frac{(v_\mu^{-1} - v_p^{-1})}{(v_\nu^{-1} - v_p^{-1})} \right),
\]

with the horizontal, as measured from the horizontal in the clockwise direction (see 4.4). The angles \( \theta = 45^\circ \) and \( \theta = 135^\circ \) are most useful for generating BWFs with symmetry about the 45\(^\circ\) line, and hence polarization-encoded qubits that can provide a maximal violation of Bell’s inequality. Perpendicular to the phase matching line \( |\theta^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \) has a FWHM of

\[
\Delta \theta^{C_2} = 4s/\sqrt{\left[ (v_\mu^{-1} - v_p^{-1})^2 + (v_\nu^{-1} - v_p^{-1})^2 \right] L^2},
\]

where \( s \approx 1.391557 \) is the positive root of \( \text{sinc}^2(x) = 0.5 \). As mentioned above \( |\theta^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \) does not have a finite FWHM along the phase matching line if we do not include group velocity dispersion terms. However, as the shape of the biphoton probability density, \( |\phi^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \), is determined by the product of \( |\phi_p (\Omega_1 + \Omega_2)|^2 \) and \( |\theta^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \), the biphoton probability density itself is still localized in \( (\Omega_1, \Omega_2) \) space provided that \( |\theta^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \) is not oriented parallel to \( |\phi_p (\Omega_1 + \Omega_2)|^2 \). This is often the case for type-II SPDC. For type-I SPDC, \( |\phi_p (\Omega_1 + \Omega_2)|^2 \) and \( |\theta^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \) are parallel, and so the FWHM of \( |\phi^{C_2}_{\mu\nu} (\Omega_1, \Omega_2)|^2 \) along the 45\(^\circ\) line is [78]

\[
\Delta \phi^{C_2}_{+} = \min \left( \sqrt{2 \ln(2) \Delta \omega}, 2\sqrt{2}s/ \left( |v_\mu^{-1} - v_p^{-1}| L \right) \right),
\]

whereas the FWHM along the pump line (in this case, also the phase matching line) is

\[
\Delta \phi^{C_2}_{-} = 2\sqrt{2}s/ (|\Lambda_\mu| L).
\]

Note that such a situation, while not particularly useful for producing an uncorrelated BWF, often produces a BWF with the symmetries required for successful violation of a
Figure 4.4: Sketch of potential phase matching function contributions to the channel SPDC biphoton probability density (4.8). A representative value of the angle it makes with the horizontal $\theta$ (see (4.13)) is increased from $45^\circ$ to $180^\circ$ as panels move from left to right across the top row followed by the bottom row.

CHSH inequality (i.e. $G = G^* = 1$) rather naturally. Interestingly, if $2v_\nu^{-1} = v_\mu^{-1} + v_\nu^{-1}$,

$$
|\theta_{\mu\nu}^{C_2}(\Omega_1, \Omega_2)|^2 \approx \text{sinc}^2 \left[ (v_\mu^{-1} - v_\nu^{-1}) (\Omega_1 - \Omega_2) L/2 \right],
$$

and an appropriate choice of $\Delta \omega$ can then vary $|\phi_{\mu\nu}^{C_2}(\Omega_1, \Omega_2)|^2$ from representing essentially anti-correlated pairs $\left( \Delta \omega \ll 2s/\left( \sqrt{\ln(2)} |v_\mu^{-1} - v_\nu^{-1}| L \right) \right)$, to uncorrelated pairs $\left( \Delta \omega \approx 2s/\left( \sqrt{\ln(2)} |v_\mu^{-1} - v_\nu^{-1}| L \right) \right)$, to correlated pairs $\left( \Delta \omega \gg 2s/\left( \sqrt{\ln(2)} |v_\mu^{-1} - v_\nu^{-1}| L \right) \right)$ [73]. Note that, regardless of the actual values of the group velocities or GVD parameters, any two channel waveguide SPDC photon pair sources with the same $\theta$ and $\Delta \theta_{C_2}$ for type-II SPDC, or $\Delta \phi_{\mu}^{C_2}$ and $\Delta \phi_{\nu}^{C_2}$ for type-I SPDC, will make essentially the same phase matching function contribution to the biphoton probability density.

For photons generated via SPDC in a microring resonator, if $\Omega_p \approx \Omega_\mu \approx \Omega_\nu$, then $|\theta_{\mu\nu}^{R_2}(\Omega_1, \Omega_2)|^2$ is nearly circular, but somewhat “squished” along the $45^\circ$ line due to the linewidth of the pumped resonance being essentially the same as the linewidth of a resonance in which a downconverted photon is generated [33]. That is, the biphoton probability density has a FWHM along the $45^\circ$ line of

$$
\Delta \phi_{\mu}^{R_2} = \min \left( \sqrt{2 \ln(2)} \Delta \omega, \sqrt{2} \Omega_p \right),
$$

(4.17)
Figure 4.5: Sketch of potential resonance function contributions to the microring ring resonator SPDC biphoton probability density \((4.9)\), for various resonance linewidths. From left to right: \(\Omega_p \approx \Omega_\mu \approx \Omega_\nu\); \(\Omega_p \gg \Omega_\mu, \Omega_\nu\) with \(\Omega_\mu \approx \Omega_\nu\); and \(\Omega_p \gg \Omega_\mu, \Omega_\nu\) with \(\Omega_\mu \gg \Omega_\nu\)

and a FWHM along the pump line of

\[
\Delta \phi_{R^2} = 2 \sqrt{\Omega_\mu^4 + 6\Omega_\mu^2\Omega_\nu^2 + \Omega_\nu^4 - \Omega_\mu^2 - \Omega_\nu^2}. \tag{4.18}
\]

Alternatively, if \(\Omega_p \gg \Omega_\mu, \Omega_\nu\), \(|\theta_{R^2}^{\mu\nu}(\Omega_1, \Omega_2)|^2\) is essentially factorizable and thus, for a large enough \(\Delta_\omega\), represents nearly uncorrelated photons, with a FWHM along the horizontal of \(2\Omega_\mu\) and along the vertical of \(2\Omega_\nu\), with a more horizontal or vertical shape of \(|\theta_{R^2}^{\mu\nu}(\Omega_1, \Omega_2)|^2\) for \(\Omega_\mu \gg \Omega_\nu\) or \(\Omega_\nu \gg \Omega_\mu\) respectively (see Fig. 4.5). Note that for two microring resonator SPDC photon pair sources to make the same phase resonance function contribution to the biphoton probability density, they must have the same resonance linewidths \(\Omega_\mu\), \(\Omega_\nu\), and \(\Omega_p\).

### 4.3.2 SFWM Biphoto Probability Densities

For type-II SFWM, if we neglect group velocity dispersion terms and make the same approximation about integrating over the pump functions as in Chapter 3 \((3.34)\), the integral in \((4.10)\) can be performed analytically. The resulting expression is

\[
|\theta_{C^2}^{\mu\nu}(\Omega_1, \Omega_2)|^2 = \sqrt{2\pi} \Delta_\omega \left| \phi_p \left[ \left( \Omega_1 + \Omega_2 \right) / \sqrt{2} \right] \right|^2 \times \text{sinc}^2 \left\{ \left[ v_\mu^{-1}\Omega_1 + v_\nu^{-1}\Omega_2 - v_p^{-1}(\Omega_1 + \Omega_2) \right] L / 2 \right\}.
\]

This is essentially the same form as for type-II SPDC in a channel waveguide, albeit with a different pump pulse bandwidth and evaluated for different group velocities. That is, \(|\phi_p \left[ \left( \Omega_1 + \Omega_2 \right) / \sqrt{2} \right] |^2\) still has an infinite FWHM along the pump line, but instead of
a FWHM along the 45° line of $\sqrt{2 \ln (2)} \Delta \omega$, as in (4.12), here the pump has a FWHM along the 45° line of

$$\Delta \phi_{P_3} = 2 \sqrt{\ln 2} \Delta \omega = \sqrt{2} \Delta \phi_{P_2}. \tag{4.19}$$

Furthermore, as

$$|\theta^{C_2}_{\mu \nu} (\Omega_1, \Omega_2)|^2 = \text{sinc}^2 \left\{ \left[ (v_{\mu}^{-1} \Omega_1 + v_{\nu}^{-1} \Omega_2 - v_p^{-1} (\Omega_1 + \Omega_2)) L / 2 \right] \right\},$$

when group velocity dispersion terms can be neglected, the sinc squared function in $|\theta^{C_3}_{\mu \mu} (\Omega_1, \Omega_2)|^2$ has an infinite FWHM along the phase matching line, which is rotated clockwise through an angle $\theta$ from the horizontal, as above (see (4.13)), and a FWHM in the direction perpendicular to the phase matching line of $\Delta \theta^{C_2}$, also as above (see(4.14)). Note that again we only make a connection between the forms of the SPDC and SFWM functions - their arguments need not be the same.

When $\mu = \nu$ instead, group velocity terms must be kept, and an analytic result becomes possible only in the limit of a long pulse

$$|\theta^{C_3}_{\mu \mu} (\Omega_1, \Omega_2)|^2 \propto \phi_p \left[ (\Omega_1 + \Omega_2) / \sqrt{2} \right]^2 \times \text{sinc}^2 \left\{ \left[ (v_{\mu}^{-1} - v_p^{-1}) (\Omega_1 + \Omega_2) + (\Lambda_{\mu} / 2 - \Lambda_p) (\Omega_1 + \Omega_2)^2 + \Lambda_{\mu} / 2 (\Omega_1 - \Omega_2)^2 \right] L / 2 \right\}^2 = |\phi_p \left[ (\Omega_1 + \Omega_2) / \sqrt{2} \right]^2 | \theta^{C_2}_{\mu \mu} (\Omega_1, \Omega_2)|^2$$

such that, as above, the FWHM of $|\phi^{C_3}_{\mu \mu} (\Omega_1, \Omega_2)|^2$ along the 45° line is

$$\Delta \phi^{C_3}_+ = \min \left( 2 \sqrt{\ln (2)} \Delta \omega, 2 \sqrt{2} s / (|v_{\mu}^{-1} - v_p^{-1}| L) \right),$$

and the FWHM along the pump line (in this case, also the phase matching line) is (4.16)

$$\Delta \phi^{C_3}_- = \Delta \phi^{C_2}_- = 2 \sqrt{2} s / (|\Lambda_{\mu}| L). \tag{4.20}$$

Thus we see that it is mathematically possible to achieve identical biphoton probability densities from channel resonator SPDC and SFWM pair sources, provided that the relevant pump waveform contributions, group velocities and/or GVD parameters can be made identical. In practice, however, it is likely that this is only achievable over a given frequency range and not over the entire phase matching bandwidth. Furthermore, note that while channel SFWM biphoton probability densities associated with shorter dura-
tion, larger bandwidth, pump pulses can be calculated by numerical methods, in practice
a larger pump bandwidth will simply cover a larger bandwidth that needs to be filtered
away before generated pairs can be detected.

For photons generated via SFWM in a microring resonator, again an analytic expres-
sion is possible in the limit of a long pulse

\[ |\theta_{\mu\nu}^{R_3}(\Omega_1, \Omega_2)|^2 \propto \left| \phi_P \left[ \frac{(\Omega_1 + \Omega_2)}{\sqrt{2}} \right] \right|^2 \]

\[ \times \left( \frac{\Omega_\mu^2 + \Omega_\nu^2 + \Omega_P^2}{\Omega_\mu^2 + \Omega_\nu^2 + (\Omega_1 + \Omega_2)^2} \right) \]

\[ = \left| \phi_P \left[ \frac{(\Omega_1 + \Omega_2)}{\sqrt{2}} \right] \right|^2 |\theta_{\mu\nu}^{R_2}(\Omega_1, \Omega_2)|^2, \]

such that, when \( \Omega_P \approx \Omega_\mu \approx \Omega_\nu \), the FWHM of \( |\phi_{\mu\nu}^{R_3}(\Omega_1, \Omega_2)|^2 \) along the 45° line is

\[ \Delta \phi_{R_3}^{+} = \min \left( 2 \sqrt{\ln(2)} \Delta_{\omega}, \sqrt{2} \Omega_P \right), \quad (4.21) \]

and along the pump line is

\[ \Delta \phi_{R_3}^{-} = \Delta \phi_{R_3}^{+} = 2 \sqrt{\Omega_\mu^4 + 6 \Omega_\mu^2 \Omega_\nu^2 + \Omega_\nu^4 - \Omega_\mu^2 - \Omega_\nu^2}, \]

quite similar to (4.17) and (4.18) above, though note that the arguments of these functions
will typically be different for different order processes in different structures. Alterna-
tively, in the limit of a short pulse

\[ |\theta_{\mu\nu}^{R_3}(\Omega_1, \Omega_2)|^2 \propto \left| \int_{-\omega_p}^{\Omega_1+\Omega_2} d\Omega \left[ \Omega_P - i\Omega \right]^{-1} \left[ \Omega_P - i(\Omega_1 + \Omega_2 - \Omega) \right]^{-1} \right|^2 \]

\[ \times \left( \frac{\Omega_\mu^2 + \Omega_\nu^2}{\Omega_\mu^2 + \Omega_\nu^2 + \Omega_1^2 + \Omega_2^2} \right) \]

\[ \times \left( \frac{\Omega_\mu^2 + \Omega_\nu^2 + (\Omega_1 + \Omega_2 - 2\Omega_P)^2}{\Omega_\mu^2 + \Omega_\nu^2 + \Omega_1^2 + \Omega_2^2} \right)^{-1}, \quad (4.22) \]

although here, as in the case of a BWF associated with photons generated via SPDC with
a short pump pulse in a microring resonator, the pump function does not play a role in
the shape of the biphoton probability density. Thus we see that it is possible for biphoto
probability densities associated with specific resonances from microring resonator SPDC
and SFWM pair sources to be identical, provided that the relevant linewidths and pump
waveform contributions can be made identical. However in practice it is likely that the
pump linewidth \( \Omega_P \) is smaller than either collection linewidth \( \Omega_\mu \) or \( \Omega_\nu \) of a SPDC source,
and that it lies between the collection linewidths of a SFWM source - in neither case is
it much greater than both collection linewidths. Looking at (4.9) and (4.11) in the
short pulse limit, i.e. (4.22), we see that both are products of three Lorentzians: one with a finite width along the 45° line, one with a finite width along the vertical axis, and one with a finite width along the horizontal axis. Thus while it is possible to achieve \( \Omega_\mu \approx \Omega_\nu \), unless \( \Omega_P \) is large enough, the generated photons will still not be completely uncorrelated. We explore how uncorrelated they might be in the next Section.

### 4.4 Connection to the Schmidt Number

While it has been suggested that type-II processes in channel waveguides can result in an uncorrelated BWFs under appropriate conditions, that processes in microring resonators can do so even more easily, and that type-I processes in channel waveguides naturally have certain desirable symmetries compared to their type-II counterparts, thus far statements such as these have not been very quantitative. Therefore in this Section we use the results of Appendix D to explore approximate analytic expressions for the Schmidt number \( K \) associated with a given BWF, and put the statements above on more solid ground.

We first approximate the BWF associated with type-I SPDC in a channel waveguide as

\[
\phi^{C_2}_{\mu\mu}(\Omega_1, \Omega_2) \approx \sqrt{\frac{2}{\pi \sigma_+ \sigma_-}} \exp \left[ -\frac{(\Omega_1 + \Omega_2)^2}{2\sigma_+^2} \right] \exp \left[ -\frac{(\Omega_1 - \Omega_2)^2}{2\sigma_-^2} \right],
\]

where

\[
\sigma_+ = \Delta \phi^{C_2}_+ / \sqrt{2 \ln(2)},
\]
\[
\sigma_- = \Delta \phi^{C_2}_- / \sqrt{2 \ln(2)}.
\]

Then, knowing that (4.23) has Schmidt number of (D.5)

\[
K = \frac{\sigma_+^2 + \sigma_-^2}{2\sigma_+ \sigma_-}
\]

we find that \( \phi^{C_2}_{\mu\mu}(\Omega_1, \Omega_2) \) has a Schmidt number of approximately

\[
K \approx \frac{(\Delta \phi^{C_2}_+)^2 + (\Delta \phi^{C_2}_-)^2}{2 \left( \Delta \phi^{C_2}_+ \right) \left( \Delta \phi^{C_2}_- \right)}
\]

\[
= \frac{\ln(2) \Delta_\omega^2 + 4s/ (|\Lambda_\mu| L)}{4 \sqrt{2 \ln(2)} \sqrt{s/ (|\Lambda_\mu| L) \Delta_\omega}},
\]

when \( \sqrt{2 \ln(2)} \Delta_\omega < 2\sqrt{2}s/ (|v^{-1}_\mu - v^{-1}_p| L) \). As the pump bandwidth \( \Delta_\omega \) is typically
much narrower than the inverse of the square root of the product of the group velocity dispersion parameter \( \Lambda_\mu \) and the length of the structure \( L \), (4.24) implies that it is essentially impossible to create uncorrelated photon pairs via type-I SPDC in a channel waveguide without drastic dispersion engineering or quasi-phase matching measures. A similar result holds for type-I SFWM in a channel waveguide in the limit of a long pulse, with \( \Delta \phi^C_+ \) and \( \Delta \phi^C_- \) in place of \( \Delta \phi^C_+ \) and \( \Delta \phi^C_- \) respectively.

We also approximate the BWF associated with type-II SPDC for which \( 2v_p^{-1} = v_\mu^{-1} + v_\nu^{-1} \) in a channel waveguide as

\[
\phi^{C_2}_{\mu\nu}(\Omega_1, \Omega_2) \approx \sqrt{2 \frac{\left| \cos \theta_1 \sin \theta_2 - \sin \theta_1 \cos \theta_2 \right|}{\pi \sigma_1 \sigma_2}} \exp \left[ -\frac{(\Omega_1 \sin \theta_1 + \Omega_2 \cos \theta_1)^2}{\sigma_1^2} \right] \times \exp \left[ -\frac{(\Omega_1 \sin \theta_2 + \Omega_2 \cos \theta_2)^2}{\sigma_2^2} \right],
\]

where

\[
\sigma_1 = \frac{\Delta \phi^{C_2}_+}{\sqrt{2 \ln (2)}}, \quad \theta_1 = \frac{\pi}{4}, \quad \sigma_2 = \frac{\Delta \theta^{C_2}}{\sqrt{2 \ln (2)}}, \quad \theta_2 = -\frac{\pi}{4},
\]

and again knowledge of the exact form (D.8) of the Schmidt number of (4.25) allows us to calculate that \( \phi^{C_2}_{\mu\nu}(\Omega_1, \Omega_2) \) has a Schmidt number of approximately

\[
K \approx \frac{(\Delta \phi^{C_2}_+)^2 + (\Delta \theta^{C_2})^2}{2 (\Delta \phi^{C_2}_+) (\Delta \theta^{C_2})} = \frac{\ln (2) \Delta^2_\omega + 4s^2 / \left[ (v_\mu^{-1} - v_p^{-1})^2 L^2 \right]}{4\sqrt{\ln (2)}s \Delta_\omega / \left( |v_\mu^{-1} - v_p^{-1}| L \right)},
\]

when \( \sqrt{2 \ln (2)} \Delta_\omega < 2\sqrt{2s} / \left( |v_\mu^{-1} - v_p^{-1}| L \right) \). We can achieve \( K \approx 1 \), and thus uncorrelated photons suitable for entanglement swapping, here if it is possible to tune the pump bandwidth and group velocities of the modes involved such that

\[
2 \ln (2) \Delta^2_\omega = 8s^2 / \left[ (v_\mu^{-1} - v_p^{-1})^2 L^2 \right],
\]

which is exactly the same condition derived via geometric arguments above (following
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\(\Delta \omega \approx 2 s / \left(\sqrt{\ln (2)} |v_\mu^{-1} - v_P^{-1}| L\right)\).

Additionally, we note that it is only possible to achieve \(G = G^* = 1\) when the \(\theta\) of (4.13) is equal to 45° or 135°.

Turning our attention now to microring resonators, we expect a nearly uncorrelated BWF if \(\Omega_\mu \approx \Omega_\nu \approx \Omega_P\) in the limit of a short pulse. Therefore we consider SFWM and, for the widths and angles required by (D.8), approximate the BWF as

\[
\phi_{\mu \nu}^{R_3}(\Omega_1, \Omega_2) \approx \sqrt{2} \frac{|\cos \theta_1 \sin \theta_2 - \sin \theta_1 \cos \theta_2|}{\pi \sigma_1 \sigma_2} \exp \left[ -\frac{(\Omega_1 \sin \theta_1 + \Omega_2 \cos \theta_1)^2}{\sigma_1^2} \right] \times \exp \left[ -\frac{(\Omega_1 \sin \theta_2 + \Omega_2 \cos \theta_2)^2}{\sigma_2^2} \right],
\]

with

\[
\sigma_1 = \frac{\Omega_P}{\sqrt{\ln 2}}, \\
\theta_1 = \pi/4, \\
\sigma_2 = 2 \sqrt{\frac{\sqrt{2} - 1}{\ln (2)}} \Omega_P, \\
\theta_2 = -\pi/4,
\]

leading to a prediction of the Schmidt number of \(K \approx 1.03\).

We confirm these results with two examples. First, we consider the type-II SPDC source of [79] with a BWF plotted in Fig. 2(c) there. This source has [80]

\[
v_\mu = \frac{1}{6.05736 \times 10^{-9}} \text{ m} \cdot \text{s}^{-1}, \quad v_\nu = \frac{1}{6.41026 \times 10^{-9}} \text{ m} \cdot \text{s}^{-1}, \quad v_P = \frac{1}{7.1603 \times 10^{-9}} \text{ m} \cdot \text{s}^{-1}, \quad L = 3.6 \times 10^{-3} \text{ m},
\]

meaning that \(\theta_{\mu \nu}^{C_2}(\omega_1, \omega_2)\) has an orientation and FWHM given by (4.13) and (4.14) respectively. Calculating

\[
\theta \approx 55.78^\circ,
\]

we can expect this source to have a Schmidt number larger than 1 and a \(G = G^*\) less than 1. We first calculate the Schmidt number numerically on an 800 × 800 point grid
spanning from \(-1 \times 10^{14} \text{ s}^{-1}\) to \(1 \times 10^{14} \text{ s}^{-1}\) along both axes, and find \(K = 18.5\). This corresponds to an entropy of entanglement

\[
E = -\sum_{\lambda} p_{\lambda} \log_2 p_{\lambda},
\]

of \(E \approx 4.55\) which is quite close to the value published in [79] of \(E = 4.6\). Calculating \(K\) instead using the analytic result (D.8) for the Gaussian approximation (4.25), we have

\[
\begin{align*}
\sigma_1 &= \Delta_\omega \\
&\approx 5 \times 10^{12} \text{ s}^{-1} \\
\theta_1 &= \pi/4,
\end{align*}
\]

\[
\sigma_2 = \frac{2\sqrt{2}s}{\sqrt{\ln 2 \left[ (v_\mu^{-1} - v_P^{-1})^2 + (v_\nu^{-1} - v_P^{-1})^2 \right] L^2}}
\]

\[
\approx 9.8455 \times 10^{11} \text{ s}^{-1},
\]

\[
\theta_2 = \tan^{-1} \left( \frac{v_\mu^{-1} - v_P^{-1}}{v_\nu^{-1} - v_P^{-1}} \right)
\]

\[
\approx 55.78^\circ,
\]

and thus find \(K \approx 13.2\). This demonstrates that while the double Gaussian approximation of Appendix D is useful for obtaining an approximation to the Schmidt number, it cannot capture the oscillations of the squared sinc function and therefore underestimates the true Schmidt number. Indeed, when the double-Gaussian form of the BWF is calculated numerically, we also find \(K \approx 13.2\). Both values suggest that this source is not suitable as a source for an entanglement swapping protocol. This source also has \(G = G^* \approx 0.1\), and so fringe visibility will be reduced in correlated polarization measurements.

Second, we consider the SFWM ring resonator source of [33] with a biphoton wave function plotted in Fig. 2(b) there. This source has

\[
\begin{align*}
\Delta_\omega &= \frac{\sqrt{2 \ln 2}}{vT},
\quad v = 1.71 \times 10^8 \text{ m} \cdot \text{s}^{-1}, \quad T = 5 \text{ ns},
\quad \Omega_\mu \approx \Omega_\nu \approx \Omega_P = 2\pi (0.74 \text{ GHz}),
\end{align*}
\]

meaning that it has a biphoton probability density that looks like (4.22), with widths along the 45° line and pump line of (4.21) and (4.18) respectively. We thus expect this
source to have $K \approx 1.03$. While not exact, this value is also close to the published (numerically calculated) value of $K = 1.09$. Again, the double Gaussian approximation has slightly simplified things and therefore caused an underestimation of the true Schmidt number. Nevertheless, the Schmidt number is nearly unity, demonstrating that this source is much more suitable than the other for creating photon pairs to be used for entanglement swapping.

### 4.5 Discussion

In this Chapter I have examined the biphoton probability densities associated with photon pairs generated via SPDC or SFWM in either a channel waveguide or microring resonator side-coupled to a channel waveguide under idealized conditions. In Section 4.1 they were written in a general form that made it easy to see the terms most responsible for their shape. While many terms are rather benign, the pump function and phase matching sinc function in a channel waveguide, and pump function and Lorentzian linewidth functions in a microring resonator have a great deal of influence over the shape of the biphoton probability density. In Section 4.2 I considered two potential uses of photon pairs, violation of Bell’s inequality, which involves just a single source and destroys both photons, and entanglement swapping, which involves two sources and uses two photons to fix the state of the other two. While the polarization-entangled qubits used in these applications are often thought of as being single-mode, a more general theory using the states introduced in Section 4.1 suggests that they are frequency correlated. These correlations are characterized by a biphoton probability density that, as was shown here, must possess certain properties in order for the application to be successful. It must have certain symmetries to maximally violate Bell’s inequality, and must be factorable for perfect entanglement swapping. In Section 4.3 parameters that govern these properties are identified, and in Section 4.4 analytic expressions for the Schmidt number are presented and put to the test. In a channel waveguide, there are many ways to use dispersion engineering to achieve the group velocities necessary to allow control of the correlations of the generated photons by tuning only the spectral width of the pump. In a microring resonator, provided resonances can be found that satisfy both energy conservation and phase matching, this control essentially happens for free. However, to get truly uncorrelated photons, and not a Schmidt number simply “near” 1, one needs to engineer the FWHMs of the resonance linewidths associated with the fields involved in the photon generation process. In practice, it may be difficult to achieve a pump resonance linewidth that is much larger than either of the other two resonance linewidths, $\Omega_p \gg \Omega_\mu, \Omega_\nu$, as
resonance linewidths are generally going to be narrower at higher frequencies due to the evanescent field outside the waveguide dropping off more quickly at higher frequencies. Still, in a device in which the pump and generated photons are in different modes, it may be possible to work around this issue.
Chapter 5

Powers at Which Nonlinear Processes Become Important

Up until now calculations have largely ignored nonlinear effects other than the photon pair generation process under consideration. This makes for clean and intuitive expressions, but leaves much to worry about. Two-photon absorption (TPA) and associated free-carrier absorption (FCA) could cause generated photons to become lost, self-phase modulation (SPM) of the pump and cross-phase modulation (XPM) of the generated photons could cause the optimal pump and collection frequencies to shift, multiple pairs could be produced simultaneously instead of a single pair, and dispersion of the pump pulse could become important if shorter and shorter pulses are used to increase the peak pump power. Yet even without taking any of these effects into account, the scaling relationships developed in Chapter 3 do a decent job of predicting the photon pair generation efficiency of a device [81, 18]. To explain this, in this chapter I clarify the relationship between the various nonlinear processes mentioned above in photon pair sources made of a material with only a third-order nonlinearity $\chi_3$, and demonstrate that if one works with pump powers low enough to suppress multi-pair generation, one is often automatically constraining the strength of many other nonlinear processes as well.

In Section 5.1 we derive expressions for when multi-pair production becomes relevant for a SFWM process both in a channel waveguide and a microring resonator. Written as inequalities, they allow for easy comparison with known conditions for the pump powers at which other nonlinear processes become relevant. In Section 5.2 these inequalities are put to the test for devices in the literature, taking a channel waveguide and a microring resonator as specific examples, and a discussion of these results follows in Section 5.3. Linear (scattering) loss is addressed in the following Chapter.
5.1 A Series of Inequalities

As discussed in Chapter 2 following Eq. (3.12) for a state of generated photons of the form

\[ |\psi_{\text{gen}}\rangle = \exp \left( \beta C_I^\dagger - \text{H.c.} \right) |\text{vac}\rangle, \]

 exiting the right R channel of a given device, where

\[ C_I^\dagger = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi(\omega_1,\omega_2) b_{R\omega_1}^\dagger b_{R\omega_2}^\dagger, \]

 when \(|\beta| \ll 1\)

\[ |\psi_{\text{gen}}\rangle \approx |\text{vac}\rangle + \beta C_I^\dagger |\text{vac}\rangle, \]

 and \(|\beta|^2\) can be thought of as the average number of generated photon pairs per pump pulse. Dividing (3.40) and (3.57) by \(T/(\sim \omega_P)\) where \(T\) is the duration of the pump pulse, in the long-pulse limit we find that the total pair production probability is

\[ N_{\text{pairs}}^{C_{3,\text{ideal}}} = |\beta_{C_3}|^2 = \frac{2}{3} \left( \gamma P_P L \right)^2 \sqrt{\frac{T^2}{2\pi |\beta_2(\omega_P)| L}}, \quad (5.1) \]

 in a channel waveguide and

\[ N_{\text{pairs}}^{R_{3,\text{ideal}}} = |\beta_{R_3}|^2 = \frac{v_P L}{2} \left( \gamma P_P \right)^2 |F_P(\omega_P)|^6 T, \quad (5.2) \]

 in a microring resonator side-coupled to a channel waveguide, just as in Appendix B. Here \(\gamma = 3\chi_3 \omega_P / (4\varepsilon_0 v^2 \pi^4 A)\) is the usual nonlinear parameter [43] with \(A\) an effective area (3.33), the GVD parameter \(\beta_2 = d^2 k / d\omega^2|_{\omega_P}\), \(P_P = h\omega_P N_P / T\) is the average pump power, \(|F_P(\omega_P)|\) is a resonant field enhancement factor (3.46), and \(L\) is the length of the device. Furthermore, it has been assumed that the channel waveguide is pumped at its phase-matching frequency, and the ring resonator at a resonance frequency with adjacent generated photon collection resonances that are phase-matched and conserve energy (i.e. pumped near the zero dispersion wavelength). While TPA, FCA, SPM, and XPM have been neglected in these calculations, we note that they represent the maximum possible number of generated photon pairs per pump pulse, i.e. for a phase matched process the inclusion of additional nonlinear effects would only lead to a lower number of generated photon pairs, and so we refer to (5.1) and (5.2) as the ideal number of generated photon pairs in each type of structure.
5.1.1 Pump Dispersion

For a channel waveguide, the long-pulse limit corresponds to the pump pulse bandwidth \( \Delta_P \approx 4a/T \), where \( a \approx 1.8955 \) is the positive root of \( \text{sinc} (x) = 0.5 \) being smaller than the phase-matching bandwidth \( \Delta_M = 4\sqrt{a/|\beta_2(\omega_P)|L} \) whereas for a microring resonator it corresponds to the pump pulse bandwidth being smaller than a resonance linewidth \( \Delta_R = \omega_P/Q \) where \( Q \) is the quality factor of the ring. Note that we have assumed a pump pulse of the form

\[
\phi_P(\omega) = \sqrt{\frac{T}{2\pi}} \text{sinc} \left( \frac{(\omega - \omega_P) T}{2} \right),
\]

for these approximations, as in Appendix B, and that \( \Delta_M \), with \( \beta_2(\omega_P) = 2\Lambda_\mu \), is closely related to \( \Delta_{C_4} \) (4.20), as the former is a width of the BWF and the latter a width of the biphoton probability density. The long-pulse limit ensures efficient photon pair generation, as no pump bandwidth is wasted outside the phase-matching bandwidth or resonance linewidth. Also, for systems in which \( \Delta_R \ll \Delta_M \), the long-pulse limit ensures

\[
\frac{\Delta_P}{4a/T} \ll 4 \sqrt{\frac{a}{|\beta_2(\omega_P)|L}}, \quad \frac{L}{L_D} \ll \frac{a}{L_D}, \quad (5.3)
\]

where \( L_D = T^2/|\beta_2| \) is the usual dispersion length [43]. Thus we see that if the pump pulse is long enough to ensure that photons are generated efficiently [28], dispersion of its waveform may be neglected during propagation through the device.

5.1.2 Self- and Cross-Phase Modulation

With pump pulse bandwidth concerns dispensed with, we now turn to considerations regarding the pump pulse power. Introducing the nonlinear length \( L_{\text{NL}} = (\gamma P_P)^{-1} [43] \), we can rewrite (5.1) as

\[
N_{\text{pairs}}^{C_3, \text{ideal}} = \frac{2}{3} \left( \frac{L}{L_{\text{NL}}} \right)^2 \sqrt{\frac{L_D}{2\pi L}},
\]

Then, if one demands that

\[
N_{\text{pairs}}^{C_3, \text{ideal}} \ll 1, \quad (5.4)
\]

because we have \( L \ll L_D \) (5.3), in order for the average number of generated pairs per pump pulse in a channel waveguide to remain much less than one, and thus the average
probability of 4-photon generation much much less than one, we see that, qualitatively, we must demand $L \ll L_{NL}$. We note that this is the condition required for one to be able to neglect self- and cross-phase modulation effects. Thus simply by working with pump pulses long enough and weak enough to ensure efficient pair generation per pump pulse without any multi-pair generation, one also ensures that said pulses do not undergo significant dispersion, undergo significant SPM, or affect significant XPM as they propagate through the nonlinear device. Being a bit more precise we find that (5.4) in fact demands

$$L \ll \left( \frac{9\pi L_{NL}^4}{2L_D} \right)^{1/3},$$

or

$$P_P \ll \left( \frac{9\pi L}{2L_D} \right)^{1/4} (\gamma L)^{-1}. \quad (5.6)$$

In the true CW limit, these become (see (B.7))

$$L \ll \left( \frac{9\pi}{64s} \right)^{1/4} L_{NL}, \quad (5.7)$$

and

$$P_P \ll \left( \frac{9\pi}{64s} \right)^{1/4} (\gamma L)^{-1}, \quad (5.8)$$

where $s \approx 1.3916$ is the positive root of sinc$^2(x) = 0.5$.

The equations are similar for the ring resonator. We first rewrite (5.2) as

$$N_{pairs}^{R_{3,\text{ideal}}} = \frac{1}{2} \left( \frac{L}{L_{NL}} \right)^2 \sqrt{\left| \frac{L_D v_P^2}{L} \right|^2 |F_P(\omega_P)|^6},$$

and, because $L \ll L_D$ demanding that

$$N_{pairs}^{R_{3,\text{ideal}}} \ll 1, \quad (5.9)$$

also qualitatively requires $L \ll L_{NL}$. However, being more careful, we find that (5.9) in fact requires

$$L \ll \left( \frac{4L_{NL}^2}{|\beta_2| L_D v_P^2} \right)^{1/2} |F_P(\omega_P)|^{-6}, \quad (5.10)$$

or

$$P_P \ll \left( \frac{4L^2}{|\beta_2| L_D v_P^2} \right)^{1/4} (\gamma L)^{-1} |F_P(\omega_P)|^{-3}. \quad (5.11)$$
In the true CW limit (see (B.9)) these become

\[ L \ll \left( \frac{\sqrt{2} - 1}{16s^2} \right)^{1/4} L_{NL} |F_P(\omega_P)|^{-7/2}, \quad (5.12) \]

and

\[ P_P \ll \left( \frac{\sqrt{2} - 1}{16s^2} \right)^{1/4} (\gamma L)^{-1} |F_P(\omega_P)|^{-7/2}. \quad (5.13) \]

### 5.1.3 Two-Photon and Free-Carrier Absorption

We also consider FCA and TPA. Looking at Eq. (1) of [82], and based on the grounds that self- and cross-phase modulation can be neglected for \( P_P \ll (\gamma L)^{-1} \), two-photon absorption can be neglected in a channel waveguide provided

\[ P_P \ll \frac{2A}{\beta_{TPA} L}, \quad (5.14) \]

where \( \beta_{TPA} \) is the TPA coefficient [82]. If we are working in the limit \( L \ll L_{NL} \) then if one can show that

\[ L_{NL} \lesssim \frac{2A}{\beta_{TPA} P_P}, \quad (5.15) \]

or, written another way, that

\[ \frac{\beta_{TPA}}{2A} \lesssim \gamma, \]

then (5.14) is assuredly satisfied and two-photon absorption can safely be neglected.

Similarly, free-carrier absorption can be neglected in a channel waveguide provided [83]

\[ P_P \ll \frac{3\hbar\omega_P A}{\sigma T}, \quad (5.16) \]

where \( \sigma \) is the free-carrier absorption coefficient [83]. Again, given \( L \ll L_{NL} \), if one is able to show that

\[ \frac{T\sigma}{3\hbar\omega_P A} \lesssim L, \]

or

\[ \frac{T\sigma}{3\hbar\omega_P A L} \lesssim \gamma, \]

then (5.16) is guaranteed to be satisfied. In the CW limit, from Eq. (2) of [83] (or Eq.
(2) of [82]), the free carrier density will reach steady-state at
\[ \frac{\beta_{TPA} P^2 \tau_c}{2 \hbar \omega_T A^2}, \]
where \( \tau_c \) is the free-carrier lifetime, and so again referring to Eq. (1) of [82], we expect FCA to be negligible in a CW experiment in a channel waveguide if
\[ \frac{\beta_{TPA} P^2 \tau_c \sigma}{2 \hbar \omega_P A^2 \sigma L} \ll 1, \]
or
\[ P_P \ll \left( \frac{4 \hbar \omega_P A^2}{\beta_{TPA} \tau_c \sigma L} \right)^{1/2}. \]  
(5.17)
Thus, knowing \( L \ll L_{NL} \), if one is able to show that
\[ \left( \frac{\beta_{TPA} \tau_c \sigma}{4 \hbar \omega_P A^2 L} \right)^{1/2} \ll \gamma, \]
then (5.17) is guaranteed to be satisfied.

For pulsed SFWM in a channel waveguide, for instance, we see that if the photon pair source is such that
\[ \left( \frac{9 \pi L}{2 L_D} \right)^{1/4} \frac{\beta_{TPA}}{2 A} \ll \gamma, \]
and
\[ \left( \frac{9 \pi L}{2 L_D} \right)^{1/4} \frac{T \sigma}{3 \hbar \omega_P A L} \ll \gamma, \]
then by working in the long-pulse limit and keeping the pump power low enough to suppress multi-pair production (5.6) one automatically also avoids pump pulse dispersion, SPM, XPM, TPA, and FCA.

Note that if considering these processes in a microring resonator instead of a channel waveguide, the pump powers at which TPA and FCA become relevant (the right hand sides of (5.14), (5.16), and (5.17)) will be reduced by a factor of \( |F_P (\omega_P)|^2 \). Also, rather than simply taking \( L \ll L_{NL} \), a more precise calculation would involve substituting one of (5.5), (5.7), (5.10), or (5.12), (5.6), (5.8), (5.11), or (5.13) for \( L \) or \( P_P \) as appropriate.


### 5.2 A First Check

In this section we look at SFWM experiments in a silicon microring resonator [18] as well as a silicon slow light photonic crystal [81]. The photon generation experiment in the ring used a CW laser while the photonic crystal experiment used a pulsed laser with a duration of $T = 14$ ps, both centered near $\lambda_p = 1550$ nm. In silicon, it is known that $n_2 = 5 \times 10^{-18} \text{ m}^2 \cdot \text{W}^{-1}$, $\beta_{\text{TPA}} = 1 \times 10^{-11} \text{ m} \cdot \text{W}^{-1}$, $\sigma = 1.45 \times 10^{-21} \text{ m}^2$ [82], and $\tau_c = 1 \text{ ns}$ [83]. The slow light structure has a length of $96 \mu\text{m}$, while the microring resonator has radius of $5 \mu\text{m}$. In addition, the slow light structure has a nonlinear parameter of $\gamma \times S^2 \approx 4000 \text{ m}^{-1} \cdot \text{W}^{-1}$ while the ring has $\gamma \approx 190 \text{ m}^{-1} \cdot \text{W}^{-1}$, where $S \approx 10$ is a slow-down factor [82]. Furthermore, the GVD parameter in the slow light structure is $\beta_2(\omega_P) = 1.3 \text{ ps}^2 \text{km}^{-1}$, and the enhancement factor in the ring is $|F_P(\omega_P)|^2 \approx 390$.

Knowing $\gamma = 2\pi n_2 / (\lambda_p \mathcal{A})$ means that in the ring $\mathcal{A} \approx 0.11 \text{ m}^2$ in the ring and $\mathcal{A} \approx 0.51 \text{ m}^2$ in the slow-light structure. Thus, in the ring

$$\left(\frac{\sqrt{2} - 1}{16s^2}\right)^{1/4} |F_P(\omega_P)|^{-3/2} \frac{\beta_{\text{TPA}} 2}{2\mathcal{A}} \approx 1.8 \times 10^{-1} \text{ m}^{-1} \cdot \text{W}^{-1},$$

and

$$\left(\frac{\sqrt{2} - 1}{16s^2}\right)^{1/4} |F_P(\omega_P)|^{-3/2} \left(\frac{\beta_{\text{TPA}} \tau_c \sigma}{4\hbar \omega_p \mathcal{A}^2 L}\right)^{1/2} \approx 3.1 \times 10^5 \text{ m}^{-1} \cdot \text{W}^{-1},$$

whereas in the photonic crystal

$$\left(\frac{9\pi L}{2L_D}\right)^{\frac{1}{4}} \frac{\beta_{\text{TPA}} \times S^2}{2\mathcal{A}} \approx 3.0 \times 10^2 \text{ m}^{-1} \cdot \text{W}^{-1},$$

and

$$\left(\frac{9\pi L}{2L_D}\right)^{\frac{1}{2}} \frac{T \sigma \times S}{3\hbar \omega_p \mathcal{A} L} \approx 3.3 \times 10^3 \text{ m}^{-1} \cdot \text{W}^{-1},$$

It is clear that in the photonic crystal, multi-pair production will occur for lower pump powers than TPA, or FCA. In the ring, we still need to check if FCA will occur for lower pump powers than other processes. More specifically, multi pair-production will become an issue in the slow light structure unless (5.6) $P_p \ll 0.80 \text{ W}$, FCA unless (5.16) $P_p \ll 0.96 \text{ W}$, SPM and XPM unless $P_p \ll [(\gamma L)^{-1}] = 2.6 \text{ W}$, and TPA unless (5.14) $P_p \ll 10.6 \text{ W}$. In the ring resonator the hierarchy is much the same, multi-pair production will become an issue unless (5.13) $P_p \ll 1.7 \text{ mW}$, FCA unless (5.17) $P_p \ll 9.2 \text{ mW}$, SPM and XPM unless $P_p \ll [(\gamma L |F_P(\omega_P)|^2)^{-1}] = 0.34 \text{ W}$, and TPA unless $P_p \ll 1.7 \text{ W}$. Perhaps not surprisingly, in the ring experiment, pump powers did
not go much higher than 1 mW, whereas in the slow-light structure experiment they were kept under 0.7 W. This ensured that the pump powers were always under the lowest, namely the multi-pair production power, inequality, and therefore allowed a theory that did not take SPM, XPM, FCA, nor TPA into account to still make accurate predictions.

5.3 Discussion

In many photon pair generation experiments with integrated devices, one may worry about the strength of nonlinear processes other than the one responsible for the generation of photon pairs. In Section 5.1 of this Chapter, I have presented a number of equations that suggest that, for an appropriately designed source, many of these processes need not become important if one works with powers low enough to suppress multi-pair production. These equations were confirmed by looking at the parameters associated with two photon pair sources in the literature in Section 5.2. In both experiments, pump powers were indeed kept below powers at which multi-pair production would become significant, with powers at which XPM, SPM, FCA, or TPA would become significant all being higher.
Chapter 6

Loss

In this chapter I consider the effects of scattering loss on a given state of generated photons. This certainly does not solve the problem of how to best account for photon loss and generation simultaneously within the same medium, but is an important first step that serves to demonstrate that there is nothing pathological about such loss when introduced within a particular model.

To include loss in the kinds of systems introduced in Chapter 2, one can imagine connecting the channel modes to systems of statistically independent reservoirs at temperature $T$. This phenomenological model allows for light to couple out of the guided waveguide modes to unguided modes, and the temperature of the reservoirs is eventually taken to be 0 so that no light can ever scatter back into the waveguide modes. We note that this assumption, which is quite reasonable here as room temperature blackbody radiation at the frequencies of interest is negligible, greatly simplifies the calculation of how a given state of generated photons decays.

Indeed from knowledge of how a coherent state decays when coupled to a zero-temperature reservoir, one could perhaps try to include scattering loss “by hand” for a given state of generated photons with a decaying exponential term that is fit with data extracted from a classical experiment. However it is not entirely clear whether a two-photon state should decay according to the same exponential term, or perhaps the same exponential term squared, or how the single photon probability of a density operator that originally represented a two-photon state grows and then falls, not to mention how a squeezed state behaves. Thus, loss is introduced here within the general Hamiltonian formalism of Chapter 2, preserving the necessary canonical equal-time bosonic commutation relations as the system evolves, with a view to considering photon loss and generation simultaneously. This approach enables the evolution of all of the pieces of a given state of generated photons, expressed as a density operator, to be understood.
Chapter 6. Loss

The full Hamiltonian is now

\[ H = H_L + H_{NL} + H_R + H_I, \]

where

\[ H_R = \sum_{n,I \in \sigma_n} \int dk \, d\mu \Omega_{nI\mu k} B_{nI\mu}^\dagger B_{nI\mu k}, \]

\[ H_I = \sum_{n,I \in \sigma_n} \int dk \, d\mu \left( c_{nI\mu k} b_{nI\mu k}^\dagger + c_{nI\mu k}^* B_{nI\mu k}^\dagger b_{nI\mu k} \right), \]  

(6.1)

with

\[ [b_{nI\mu k}, b_{n'I\mu'k'}^\dagger] = \delta_{n'n} \delta_{II'} \delta (k-k') , \]

\[ [B_{nI\mu k}, B_{n'I\mu'k'}^\dagger] = \delta_{n'n} \delta_{II'} \delta (\mu-\mu') \delta (k-k') , \]  

(6.2)

and all other commutators zero; the \( \mu \) describes the magnitude of the reservoir wavevectors in directions transverse to the propagation direction of the guided waveguide mode.

From here, one could derive an expression for the evolution of the reduced density operator for an asymptotic-in state composed of waveguide operators. Written as a differential equation, this is known as the Master Equation, and can be put into Lindblad form [84]. Alternatively, one could work with the Glauber-Sudarshan \( P \) function representation for the density operator [85, 86] and arrive at a Fokker-Planck type equation [87, 88]. However, we find that the approach that most clearly captures the physics and appears to lend itself to integration with the formalism introduced in Chapter 2 is a quantum Langevin formalism in which equations of motion are derived for waveguide operators in terms of “fluctuating” reservoir operators, which are later traced out of the appropriate density operator. In what follows, we consider loss in the absence of any nonlinearity and how it affects different “already generated” states. In Section 6.1 we introduce the general formalism, deriving expressions for the evolution of waveguide operators, and in Section 6.2 we apply the formalism to a coherent state, a two-photon state, and a squeezed vacuum state, before discussing these results in Section 6.3.
6.1 Quantum Langevin Formalism

The Heisenberg equations of motion for the $b^\dagger_{nIk}(t)$ and $B^\dagger_{nI\mu k}(t)$ are (recall (2.41))

$$\frac{d b^\dagger_{nIk}(t)}{dt} = -\frac{i}{\hbar} \left[ b^\dagger_{nIk}(t), H_L + H_R + H_I \right] = i\omega_{nIk} b^\dagger_{nIk}(t) + i \int d\mu c^*_{nI\mu k} B^\dagger_{nI\mu k}(t),$$

$$\frac{dB^\dagger_{nI\mu k}(t)}{dt} = -\frac{i}{\hbar} \left[ B^\dagger_{nI\mu k}(t), H_L + H_R + H_I \right] = i\Omega_{nI\mu k} B_{nI\mu k}(t) + i c_{nI\mu k} b^\dagger_{nIk}(t), \quad \text{(6.3)}$$

the second of which can be solved immediately

$$B^\dagger_{nI\mu k}(t) = i \int_{t_0}^t \! b_{nIk}(\tau) c_{nI\mu k} e^{i\Omega_{nI\mu k}(t-\tau)} d\tau + B^\dagger_{nI\mu k} e^{i\Omega_{nI\mu k}(t-t_0)}, \quad \text{(6.4)}$$

where the lack of a time variable associated with the reservoir operator indicates that it is a Schrödinger operator. Substituting (6.4) in the first of (6.3), we find

$$\frac{db^\dagger_{nIk}(t)}{dt} = \left. \right|_{t_0}^t \! \left[ b_{nIk}(\tau), c_{nI\mu k} e^{i\Omega_{nI\mu k}(t-\tau)} d\tau \right] + B^\dagger_{nI\mu k} e^{i\Omega_{nI\mu k}(t-t_0)}.$$

To evaluate the first integral, we note that the guided modes will only interact strongly with reservoir modes that approximately conserve energy and momentum. If scattering losses are small, we expect that when a guided mode is scattered out of the waveguide it will keep essentially the same value of $k$ and $\mu$ ($\mu = 0$). Thus, we expect only a very small range of $\mu$ values, centered about $\mu = 0$, to be relevant to the problem. With this in mind, we approximate $|c_{nI\mu k}|^2$ as constant over the $\mu$ of interest ($|c_{nI\mu k}|^2_{\mu=0} \equiv c_{nIk}$) such that

$$\int d\mu \int_{t_0}^t \! a^\dagger_{nIk}(\tau) |c_{nI\mu k}|^2 e^{i\Omega_{nI\mu k}(t-\tau)} d\tau = c_{nIk} \int d\mu \int_{t_0}^t \! a^\dagger_{nIk}(\tau) e^{i\Omega_{nI\mu k}(t-\tau)} d\tau.$$

Further, we write

$$d\mu = \frac{d\mu}{d\Omega_{nI\mu k}} d\Omega_{nI\mu k},$$

also approximating $\frac{d\mu}{d\Omega_{nI\mu k}}$ as constant, $(\frac{d\mu}{d\Omega_{nI\mu k}})_{\mu=0} \equiv v_{nIk}$, over the $\mu$ of interest, and
arrive at a quantum mechanical Langevin equation

\[
\frac{db_{nI_k}^\dagger (t)}{dt} = i\omega_{nI_k} b_{nI_k}^\dagger (t) - \frac{c_{nI_k}}{v_{nI_k}} \int d\Omega_{nI\mu_k} \int_{t_0}^t b_{nI_k}^\dagger (\tau) e^{i\Omega_{nI\mu_k}(t-\tau)} d\tau \\
- i \int d\mu B_{nI\mu_k}^\dagger c_{nI\mu_k}^* e^{i\Omega_{nI\mu_k}(t-t_0)} \\
= (i\omega_{nI_k} - \beta_{nI_k}) b_{nI_k}^\dagger (t) - F_{nI_k}^\dagger (t),
\] (6.5)

where we have defined the loss parameter

\[
\beta_{nI_k} \equiv \frac{\pi c_{nI_k}}{v_{nI_k}},
\] (6.6)

and the fluctuation term

\[
F_{nI_k}^\dagger (t) \equiv i \int d\mu B_{nI\mu_k}^\dagger c_{nI\mu_k}^* e^{i\Omega_{nI\mu_k}(t-t_0)}.
\]

Although an exponential decay of \(b_{nI_k}^\dagger (t)\) is implied by (6.5) even without the fluctuation term, the fluctuation term’s presence is necessary to ensure that the equal-time commutation relations of \(b_{nI_k}^\dagger (t)\) do not vary with time, a property that will be verified below. While it is possible, in principle, to determine the \(c_{nI_k}\) and thus the \(\beta_{nI_k}\) from first principles, an ad hoc approach would be to determine the \(\beta_{nI_k}\) directly from the results of classical experiments. Indeed \(\beta_{nI_k}\) is the object, not \(c_{nI_k}\), that appears in equations describing how a given state of generated photons decays.

The Langevin equation (6.5) has solution

\[
b_{nI_k}^\dagger (t) = \Lambda_{nI_k}^\dagger (t) + \Gamma_{nI_k}^\dagger (t),
\]

where, setting \(t_0 = 0\) for convenience, we have defined

\[
\Gamma_{nI_k}^\dagger (t) \equiv - \int_0^t F_{nI_k}^\dagger (\tau) e^{(i\omega_{nI_k} - \beta_{nI_k})(t-\tau)} d\tau \\
\Lambda_{nI_k}^\dagger (t) \equiv b_{nI_k}^\dagger e^{(i\omega_{nI_k} - \beta_{nI_k})t},
\]
with commutation relations (recall (6.2) and (6.6))

\[
\begin{align*}
\left[ F_{nI} (t), F^\dagger_{n'I'k'} (t') \right] &= 2\beta_{nIk} \delta_{nn'} \delta_{II'} \delta (t - t') \delta (k - k') , \\
\left[ \Lambda_{nI} (t), \Lambda^\dagger_{n'I'k'} (t) \right] &= e^{-2\beta_{nIk} t} \delta_{nn'} \delta_{II'} \delta (k - k') , \\
\left[ \Gamma_{nI} (t), \Gamma^\dagger_{n'I'k'} (t) \right] &= (1 - e^{-2\beta_{nIk} t}) \delta_{nn'} \delta_{II'} \delta (k - k') , \\
\left[ \Lambda^\dagger_{nI} (t), \Gamma^\dagger_{n'I'k'} (t) \right] &= 0 .
\end{align*}
\]

(6.7)

Note that, as required, the equal-time commutation relation for \( b^\dagger_{nIk} (t) \) is the same as that for \( b^\dagger_{nIk} \)

\[
\begin{align*}
\left[ b_{nI} (t), b^\dagger_{n'I'k'} (t) \right] &= e^{-2\beta_{nIk} t} \delta_{nn'} \delta_{II'} \delta (k - k') + (1 - e^{-2\beta_{nIk} t}) \delta_{nn'} \delta_{II'} \delta (k - k') \\
&= \delta_{nn'} \delta_{II'} \delta (k - k') .
\end{align*}
\]

### 6.2 Propagation Examples

We describe the evolution of the state incident on the lossy region with a reduced density operator, found by forming the linear system density operator and tracing over the reservoir oscillators

\[
\rho^{\text{red}} (t) = \text{Tr}_R \left( \rho^{\text{tot}} (t) \right) = \text{Tr}_R \left( |\psi (t) \rangle \langle \psi (t) | \right) ,
\]

at temperature \( T = 0 \). In all of the following examples we consider a single mode state propagating in a single channel, and so drop the unnecessary labels \( n \) and \( I \).

#### 6.2.1 Coherent State

As a first example, we consider a coherent state

\[
|\psi\rangle_{\text{coh}} = |z\rangle = e^{z A^\dagger_0 - \text{H.c.}} |\text{vac}\rangle ,
\]

(6.8)

where \( z \) is a complex number,

\[
A^\dagger_0 = \int dk \phi_0 (k) b^\dagger_k ,
\]
and

\[ \int dk \, |\phi_0(k)|^2 = 1. \]

In writing \(|\text{vac}\rangle\) we have implicitly assumed that it consists of both a system part and a reservoir part \(|\text{vac}\rangle = |\text{vac}\rangle^S \otimes |\text{vac}\rangle^R\), and while the eventual trace will be taken over only the reservoir part, leaving the system part behind, we will still write just \(|\text{vac}\rangle\) for simplicity of notation. The temporal evolution of this state can be expressed as a product of system and reservoir states

\[ |\psi(t)\rangle_{\text{coh}} = e^{z f_0(t) A_0^\dagger(t) - \text{H.c.}} |\text{vac}\rangle \]

where we have used the fourth of (6.7).

We assume \(|\phi_P(k)|^2\) to be peaked strongly enough for some \(k \equiv k_0\) that in integrals over it involving the loss parameter \(\beta_k\) we may approximate the loss parameter by its value at \(k_0\), e.g.

\[ \int dk \, |\phi_0(k)|^2 e^{-2\beta_k t} \approx e^{-2\beta_0 t}, \]

where we have defined \(\beta_0 \equiv \beta_{k_0}\). This allows us to write \(|\psi(t)\rangle_{\text{coh}}\) in the following form

\[ |\psi(t)\rangle_{\text{coh}} = e^{z f_0(t) A_0^\dagger(t) - \text{H.c.}} e^{z g_0(t) B_0^\dagger(t) - \text{H.c.}} |\text{vac}\rangle, \]

where

\[ f_0(t) = \sqrt{\int dk \phi_0^*(k) \Lambda_k(t), \int dk' \phi_0(k') \Lambda_k^\dagger(t)} = e^{-\beta_0 t}, \]

\[ g_0(t) = \sqrt{\int dk \phi_0^*(k) \Gamma_k(t), \int dk' \phi_0(k') \Gamma_k^\dagger(t)} = \sqrt{1 - e^{-2\beta_0 t}}, \]

and the operators

\[ A_0^\dagger(t) = \frac{\int dk \phi_0(k) \Lambda_k^\dagger(t)}{f_0(t)}, \]

\[ B_0^\dagger(t) = \frac{\int dk \phi_0(k) \Gamma_k^\dagger(t)}{g_0(t)}, \]

(6.10)
are boson creation operators satisfying equal-time canonical commutation relations
\[ [A_0(t), A_0^\dagger(t)] = [B_0(t), B_0^\dagger(t)] = 1. \]

Writing
\[ |\psi(t)\rangle_{coh} = |f_0(t)\rangle |g_0(t)\rangle, \]
where
\[ |f_0(t)\rangle = e^{z f_0(t) A_0^\dagger(t) - \text{H.c.}} |\text{vac}\rangle, \]
\[ |g_0(t)\rangle = e^{z g_0(t) B_0^\dagger(t) - \text{H.c.}} |\text{vac}\rangle, \]
it is seen that
\[
\rho_{coh}^{\text{red}}(t) = |f_0(t)\rangle \langle f_0(t)| \text{Tr}_R \left( \int \frac{d^2 g_0(t)}{\pi} |g_0(t)\rangle \langle g_0(t)| g_0^\prime(t)\rangle \langle g_0^\prime(t)| \right)
\]
\[
= |f_0(t)\rangle \langle f_0(t)| \int \frac{d^2 g_0(t)}{\pi} |\langle g_0(t)| g_0^\prime(t)\rangle|^2
\]
\[
= |f_0(t)\rangle \langle f_0(t)| \int \frac{d^2 g_0(t)}{\pi} e^{\mid g_0(t) - g_0^\prime(t)\mid^2}
\]
\[
= |f_0(t)\rangle \langle f_0(t)|, \quad (6.11)
\]
and, as expected, the coherent input state (6.8) decays exponentially and remains coherent.

6.2.2 Two-Photon State

Let us instead consider a different state incident on the lossy region, the two-photon input state
\[ |\psi\rangle_{II} = \frac{1}{\sqrt{2}} \int dk_1 dk_2 \phi(k_1, k_2) b^\dagger_{k_1} b^\dagger_{k_2} |\text{vac}\rangle, \quad (6.12) \]
where the BWF is normalized
\[
\int dk_1 dk_2 |\phi(k_1, k_2)|^2 = 1,
\]
and symmetric \( \phi(k_1, k_2) = \phi(k_2, k_1) \), propagating along the waveguide and able to scatter out. The calculations that follow are most easily understood if a Schmidt decomposition
Chapter 6. Loss

is performed on the BWF

$$\phi(k_1, k_2) = \sum_{\lambda} \sqrt{p_{\lambda}} \Phi_{\lambda}(k_1) \Phi_{\lambda}(k_2),$$

where \{\Phi_{\lambda}\} is a set of orthonormal functions, and the probabilities sum to 1 i.e.

$$\int dk \Phi_{\lambda}^*(k) \Phi_{\lambda'}(k) = \delta_{\lambda \lambda'},$$

$$\sum_{\lambda} p_{\lambda} = 1.$$

Following the calculations above, the two-photon state evolves according to

$$|\psi(t)\rangle_{II} = \frac{1}{\sqrt{2}} \int dk_1 dk_2 \phi(k_1, k_2) \left[ \Lambda_{k_1}^\dagger(t) + \Gamma_{k_1}^\dagger(t) \right] \left[ \Lambda_{k_2}^\dagger(t) + \Gamma_{k_2}^\dagger(t) \right] |\text{vac}\rangle$$

$$= \frac{1}{\sqrt{2}} \int dk_1 dk_2 \sum_{\lambda} \sqrt{p_{\lambda}} \Phi_{\lambda}(k_1) \Phi_{\lambda}(k_2) \left[ \Lambda_{k_1}^\dagger(t) + \Gamma_{k_1}^\dagger(t) \right] \left[ \Lambda_{k_2}^\dagger(t) + \Gamma_{k_2}^\dagger(t) \right] |\text{vac}\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{\lambda} \sqrt{p_{\lambda}} \left[ f_{\lambda}(t) A_{\lambda}^\dagger(t) + g_{\lambda}(t) B_{\lambda}^\dagger(t) \right] \left[ f_{\lambda}(t) A_{\lambda}^\dagger(t) + g_{\lambda}(t) B_{\lambda}^\dagger(t) \right] |\text{vac}\rangle,$$

where, assuming \(|\Phi_{\lambda}(k)|^2\) to be peaked at some \(k \equiv k_{\lambda}\) similar to above (6.9), we have defined

$$f_{\lambda}(t) = \sqrt{\left[ \int dk \Phi_{\lambda}^*(k') \Lambda_{k'}(t) \right] \left[ \int dk \Phi_{\lambda}^*(k') \Lambda_{k'}^\dagger(t) \right]} = e^{-\beta_{\lambda} t},$$

$$g_{\lambda}(t) = \sqrt{\left[ \int dk \Phi_{\lambda}^*(k') \Gamma_{k'}(t) \right] \left[ \int dk \Phi_{\lambda}^*(k') \Gamma_{k'}^\dagger(t) \right]} = \sqrt{1 - e^{-2\beta_{\lambda} t}}, \quad (6.13)$$

where \(\beta_{\lambda} \equiv \beta_{k_{\lambda}}\), and (6.10)

$$A_{\lambda}^\dagger(t) = \frac{\int dk \Phi_{\lambda}(k) \Lambda_{k}(t)}{f_{\lambda}(t)},$$

$$B_{\lambda}^\dagger(t) = \frac{\int dk \Phi_{\lambda}(k) \Gamma_{k}(t)}{g_{\lambda}(t)}, \quad (6.14)$$

so that

$$[A_{\lambda}(t), B_{\lambda'}^\dagger(t)] = 0,$$

$$[A_{\lambda}(t), A_{\lambda'}^\dagger(t)] = [B_{\lambda}(t), B_{\lambda'}^\dagger(t)] = \delta_{\lambda \lambda'}.$$
We now construct the density operator for this state

\[
\rho_{\text{II}}^{\text{tot}} (t) = |\psi (t)\rangle_{\text{II}} \langle \psi (t)|_{\text{II}}
\]

\[
= \frac{1}{2} \sum_{\lambda, \lambda'} \left\{ \sqrt{p_{\lambda} p_{\lambda'}} \left[ f_{\lambda} (t) A_{\lambda}^\dagger (t) + g_{\lambda} (t) B_{\lambda}^\dagger (t) \right] \left[ f_{\lambda} (t) A_{\lambda}^\dagger (t) + g_{\lambda} (t) B_{\lambda}^\dagger (t) \right] |\text{vac}\rangle \times \langle \text{vac} | f_{\lambda'} (t) A_{\lambda'} (t) + g_{\lambda'} (t) B_{\lambda'} (t) \left[ f_{\lambda'} (t) A_{\lambda'} (t) + g_{\lambda'} (t) B_{\lambda'} (t) \right] \right\},
\]

and trace over the reservoir oscillators

\[
\rho_{\text{II}}^{\text{red}} (t) = \text{Tr}_R (|\psi (t)\rangle_{\text{II}} \langle \psi (t)|_{\text{II}})
\]

\[
= \frac{1}{2} \sum_{\lambda, \lambda'} \left\{ \sqrt{p_{\lambda} p_{\lambda'}} \left[ [f_{\lambda} (t) f_{\lambda'} (t)]^2 A_{\lambda}^\dagger (t) A_{\lambda}^\dagger (t) |\text{vac}\rangle \langle \text{vac} | A_{\lambda'} (t) A_{\lambda'} (t)
\right.
\]

\[
+ 4 f_{\lambda} (t) f_{\lambda'} (t) g_{\lambda} (t) g_{\lambda'} (t) A_{\lambda}^\dagger (t) |\text{vac}\rangle \langle \text{vac} | A_{\lambda'} (t) \delta_{\lambda \lambda'}
\]

\[
+ 2 [g_{\lambda} (t) g_{\lambda'} (t)]^2 |\text{vac}\rangle \langle \text{vac} | \delta_{\lambda \lambda'} \left\}
\]

\[
= \frac{1}{2} \sum_{\lambda, \lambda'} \sqrt{p_{\lambda} p_{\lambda'}} [f_{\lambda} (t) f_{\lambda'} (t)]^2 A_{\lambda}^\dagger (t) A_{\lambda}^\dagger (t) |\text{vac}\rangle \langle \text{vac} | A_{\lambda'} (t) A_{\lambda'} (t)
\]

\[
+ 2 \sum_{\lambda} p_{\lambda} [f_{\lambda} (t) g_{\lambda} (t)]^2 A_{\lambda}^\dagger (t) |\text{vac}\rangle \langle \text{vac} | A_{\lambda} (t) + \sum_{\lambda} p_{\lambda} [g_{\lambda} (t)]^4 |\text{vac}\rangle \langle \text{vac} |.
\]

This final expression for the reduced density operator suggests that the input state now has a probability of being either a two-photon state, a one-photon state, or a zero-photon state, which we write as

\[
\rho_{\text{II}}^{\text{red}} (t) = \sum_{\lambda, \lambda'} \sqrt{p_{\lambda} p_{\lambda'}} \mathcal{P}_{\Pi_{\lambda, \lambda'} (t)} |\Pi_{\lambda'} (t)\rangle \langle \Pi_{\lambda} (t)| + \sum_{\lambda} p_{\lambda} \mathcal{P}_{I_{\lambda, \lambda} (t)} |I_{\lambda} (t)\rangle \langle I_{\lambda} (t)|
\]

\[
+ \sum_{\lambda} p_{\lambda} \mathcal{P}_{0_{\lambda, \lambda} (t)} |\text{vac}\rangle \langle \text{vac} |,
\]

where

\[
\mathcal{P}_{\Pi_{\lambda, \lambda'} (t)} = [f_{\lambda} (t) f_{\lambda'} (t)]^2 = e^{-2(\beta_{\lambda} + \beta_{\lambda'}) t},
\]

\[
\mathcal{P}_{I_{\lambda, \lambda} (t)} = 2 [f_{\lambda} (t) f_{\lambda'} (t)] [g_{\lambda} (t) g_{\lambda'} (t)] = 2 e^{-2(\beta_{\lambda} + \beta_{\lambda'}) t} \sqrt{1 - e^{-2\beta_{\lambda} t}} \sqrt{1 - e^{-2\beta_{\lambda'} t}},
\]

\[
\mathcal{P}_{0_{\lambda, \lambda} (t)} = [g_{\lambda} (t) g_{\lambda'} (t)]^2 = 1 - e^{-2\beta_{\lambda} t} - e^{-2\beta_{\lambda'} t} + e^{-2(\beta_{\lambda} + \beta_{\lambda'}) t},
\]
\[ |\Pi_{\lambda}(t)\rangle = \frac{1}{\sqrt{2}} A^\dagger_{\lambda}(t) A^\dagger_{\lambda}(t) |\text{vac}\rangle, \]
\[ |I_{\lambda}(t)\rangle = A^\dagger_{\lambda}(t) |\text{vac}\rangle, \]

and
\[ \langle \Pi_{\lambda}(t) | \Pi_{\lambda'}(t) \rangle = \delta_{\lambda\lambda'}, \]
\[ \langle I_{\lambda}(t) | I_{\lambda'}(t) \rangle = \delta_{\lambda\lambda'}. \]

Note that the probability for a particular Schmidt mode \( \lambda \) to contain two, one, or zero photons sums to 1, i.e.
\[ \mathcal{P}_{\Pi_{\lambda\lambda}}(t) + \mathcal{P}_{I_{\lambda\lambda}}(t) + \mathcal{P}_{0_{\lambda\lambda}}(t) = 1, \]
for all times. Also note that decoherence does not occur any faster than the decay of the lossiest Schmidt mode, i.e.
\[ \mathcal{P}_{\Pi_{\lambda\lambda}}(t) = e^{-2(\beta_{\lambda_{f}} + \beta_{\lambda})t} > \mathcal{P}_{\Pi_{\lambda_{f}\lambda_{f}}}(t) = e^{-4\beta_{\lambda_{f}}t} \forall t, \lambda \neq \lambda_{f}, \]
where \( \beta_{\lambda_{f}} \) is the largest loss parameter.

Finally, we plot the dynamics of the three probabilities for a specific \( \lambda \) below in units of \( \tau = \beta_{\lambda} t \), where it is seen that \( \mathcal{P}_{I_{\lambda\lambda}}(\tau) \) reaches a maximum value of 1/2, and \( \mathcal{P}_{\Pi_{\lambda\lambda}}(\tau) \) intersects with \( \mathcal{P}_{0_{\lambda\lambda}}(\tau) \) at a value of 1/4, when \( \tau = \ln(2)/2 \). Note also that \( \mathcal{P}_{I_{\lambda\lambda}}(\tau) \) intersects with \( \mathcal{P}_{\Pi_{\lambda\lambda}}(\tau) \) and, later, \( \mathcal{P}_{0_{\lambda\lambda}}(\tau) \) at a value of 4/9, when \( \tau = \ln(3/2)/2 \) and \( \tau = \ln(3/2) \) respectively. Thus, while each Schmidt mode potentially decays at a different rate, each always has a conserved probability of first being a two-photon state, then later a one photon state, and finally a vacuum state, with the probability of the photon pair being found within a given Schmidt mode given by \( p_{\lambda} \).

### 6.2.3 Squeezed Vacuum State

Lastly, let us consider the “squeezed vacuum” state which would be the result of a SPDC (recall (3.7), (3.8) or SFWM process
\[ |\psi(t)\rangle_{sq} = e^{\int \frac{\phi}{\sqrt{2}} dk_{1}dk_{2}\phi(k_{1},k_{2})b_{k_{1}}^{\dagger}b_{k_{2}}^{\dagger} - \text{H.c.}} |\text{vac}\rangle. \]
Again, following the calculations above, and performing a Schmidt decomposition on the biphoton wave function, we can write the propagating state as [89]

$$\left| \psi(t) \right\rangle_{\text{sq}} = \prod_{\lambda} e^{\frac{i}{2} \beta \sqrt{2 p_{\lambda}}} \left[ f_\lambda(t) A_{\lambda}^\dagger(t) + g_\lambda(t) B_{\lambda}^\dagger(t) \right]^2 - \text{H.c.} \left| \text{vac} \right\rangle$$

$$= \prod_{\lambda} \sum_{n=0}^{\infty} \operatorname{tanh}^n \left( |\beta| \sqrt{2 p_{\lambda}} \right) \frac{\operatorname{e}^{in \arg(\beta)}}{2^n n! \sqrt{\cosh (|\beta| \sqrt{2 p_{\lambda}})}} \left[ f_\lambda(t) A_{\lambda}^\dagger(t) + g_\lambda(t) B_{\lambda}^\dagger(t) \right]^{2n} \left| \text{vac} \right\rangle,$$

where $f_\lambda(t)$, $g_\lambda(t)$, $A_{\lambda}^\dagger(t)$, and $B_{\lambda}^\dagger(t)$ are as above (see (6.13) and (6.14) respectively). Constructing the density operator for this state and tracing over the reservoir oscillators,
we find

\[ \rho_{sq}^{\text{red}}(t) = \Tr_R \left( |\psi(t)\rangle_{sq} \langle \psi(t)|_{sq} \right) \]

\[ = \sum_{\lambda,\lambda'} \frac{1}{\sqrt{\cosh (|\beta| \sqrt{2p_\lambda}) \cosh (|\beta| \sqrt{2p_{\lambda'}})}} \times \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\tanh^n (|\beta| \sqrt{2p_\lambda}) \tanh^{n'} (|\beta| \sqrt{2p_{\lambda'}}) e^{i(n-n') \arg(\beta)}}{2^{n+n'} n! n'!} \times \left[ P_{0,\lambda,\lambda'}^{n,n'}(t) |(2n)_{\lambda}(t)\rangle \langle (2n')_{\lambda'}(t)| + \sum_{k=1}^{\min(2n,2n')} P_{k,\lambda,\lambda'}^{n,n'}(t) |(2n-k)_{\lambda}(t)\rangle \langle (2n'-k)_{\lambda'}(t)| \delta_{\lambda\lambda'} \right], \]

where

\[ P_{k,\lambda,\lambda'}^{n,n'}(t) = \binom{2n}{k} \binom{2n'}{k} \sqrt{(2n-k)! (2n'-k)!} f^{(2n-k)}_{\lambda}(t) f^{(2n'-k)}_{\lambda'}(t) g_{k,\lambda}(t) g_{k,\lambda'}(t), \]

and

\[ |k_{\lambda}(t)\rangle = \frac{1}{\sqrt{k!}} \left[ A^+_{\lambda}(t) \right]^k |\text{vac}\rangle. \]

While this is a closed-form solution, it is not necessarily clear or useful. However, for the quantity \(|\beta| \sqrt{2p_\lambda}\) small enough, implying that there is either a small total probability of generating a photon pair or at least a small probability of generating a pair of photons in each Schmidt mode (see Appendix B), we can truncate the sums over \(n\) and \(n'\) at some
small number. Let us consider max \( (2n, 2n') = 4 \).

\[
\rho_{2\text{-pairs}}^{\text{red}}(t) = \sum_{\lambda, \lambda'} \frac{1}{\sqrt{\cosh (|\beta| \sqrt{2p_\lambda})} \sqrt{\cosh (|\beta| \sqrt{2p_{\lambda'}})}} |\text{vac}\rangle \langle \text{vac}| \\
+ \sum_{\lambda, \lambda'} \frac{\tanh (|\beta| \sqrt{2p_\lambda}) \tanh' (|\beta| \sqrt{2p_{\lambda'}})}{4 \sqrt{\cosh (|\beta| \sqrt{2p_\lambda})} \sqrt{\cosh (|\beta| \sqrt{2p_{\lambda'}})}} [2f_\lambda^2(t) f_{\lambda'}^2(t) |2_\lambda(t)\rangle \langle 2_{\lambda'}(t)| \\
+ 4f_\lambda^2(t) g_\lambda^2(t) |1_\lambda(t)\rangle \langle 1_{\lambda}(t)| \delta_{\lambda\lambda'} + 2g_\lambda^4(t) |\text{vac}\rangle \langle \text{vac}| \delta_{\lambda\lambda'}] \\
+ \sum_{\lambda, \lambda'} \frac{\tanh^2 (|\beta| \sqrt{2p_\lambda}) \tanh (|\beta| \sqrt{2p_{\lambda'}}) e^{i \arg(\beta)}}{16 \sqrt{\cosh (|\beta| \sqrt{2p_\lambda})} \sqrt{\cosh (|\beta| \sqrt{2p_{\lambda'}})}} [4\sqrt{3} f_\lambda^6(t) |4_\lambda(t)\rangle \langle 2_{\lambda}(t)| \\
+ 8\sqrt{6} f_\lambda^4(t) g_\lambda^2(t) |3_\lambda(t)\rangle \langle 1_{\lambda}(t)| \delta_{\lambda\lambda'} + 12\sqrt{2} f_\lambda^2(t) g_\lambda^4(t) |2_{\lambda}(t)\rangle \langle \text{vac}| \delta_{\lambda\lambda'}] \\
+ \sum_{\lambda, \lambda'} \frac{\tanh (|\beta| \sqrt{2p_\lambda}) \tanh^2 (|\beta| \sqrt{2p_{\lambda'}}) e^{-i \arg(\beta)}}{16 \sqrt{\cosh (|\beta| \sqrt{2p_\lambda})} \sqrt{\cosh (|\beta| \sqrt{2p_{\lambda'}})}} [4\sqrt{3} f_\lambda^6(t) |2_{\lambda}(t)\rangle \langle 4_{\lambda}(t)| \\
+ 8\sqrt{6} f_\lambda^4(t) g_\lambda^2(t) |1_\lambda(t)\rangle \langle 3_{\lambda}(t)| \delta_{\lambda\lambda'} + 12\sqrt{2} f_\lambda^2(t) g_\lambda^4(t) |\text{vac}\rangle \langle 2_{\lambda}(t)| \delta_{\lambda\lambda'}] \\
+ \sum_{\lambda, \lambda'} \frac{\tanh^2 (|\beta| \sqrt{2p_\lambda}) \tanh (|\beta| \sqrt{2p_{\lambda'}})}{64 \sqrt{\cosh (|\beta| \sqrt{2p_\lambda})} \sqrt{\cosh (|\beta| \sqrt{2p_{\lambda'}})}} [24 f_\lambda^4(t) f_{\lambda'}^4(t) |4_\lambda(t)\rangle \langle 4_{\lambda'}(t)| \\
+ 96 f_\lambda^6(t) g_\lambda^2(t) |3_\lambda(t)\rangle \langle 3_{\lambda}(t)| \delta_{\lambda\lambda'} + 144 f_\lambda^4(t) g_\lambda^4(t) |2_{\lambda}(t)\rangle \langle 2_{\lambda}(t)| \delta_{\lambda\lambda'} \\
+ 96 f_\lambda^2(t) g_\lambda^6(t) |1_\lambda(t)\rangle \langle 1_{\lambda}(t)| \delta_{\lambda\lambda'} + 24g_\lambda^8(t) |\text{vac}\rangle \langle \text{vac}| \delta_{\lambda\lambda'}] ,
\]
which, if $|\beta| \sqrt{2p_x} \ll 1$, $|\beta| \sqrt{2p_N} \ll 1$ can be approximated as

$$
\rho_{\text{2-pairs}}^{\text{red}}(t) \approx \sum_{\lambda,\lambda'} \left[ 1 - \frac{1}{2} |\beta|^2 (p_\lambda + p_\lambda') + \frac{7}{24} |\beta|^4 \left( p_\lambda^2 + \frac{6}{7} p_\lambda p_\lambda' + p_{\lambda'}^2 \right) \right] |\text{vac}\rangle \langle \text{vac}| \\
+ \sum_{\lambda,\lambda'} \left| \beta \right|^2 \sqrt{p_\lambda p_{\lambda'}} - \frac{7}{6} |\beta|^4 \left( \sqrt{p_\lambda^3 p_{\lambda'}} + \sqrt{p_{\lambda'}^3 p_\lambda} \right) \right] \\
\times \left[ f^3_\lambda(t) f^2_{\lambda'}(t) |2_\lambda(t)\rangle \langle 2_{\lambda'}(t)| + 2 f^3_\lambda(t) g^2_\lambda(t) |1_\lambda(t)\rangle \langle 1_{\lambda}(t)| \delta_{\lambda\lambda'} + g^4_\lambda(t) |\text{vac}\rangle \langle \text{vac}| \delta_{\lambda\lambda'} \right] \\
+ \sum_{\lambda,\lambda'} \frac{|\beta|^3}{2} \sqrt{2p_\lambda^2 p_{\lambda'}} e^{i \arg(\beta)} \left[ \sqrt{3} f^6_\lambda(t) |4_\lambda(t)\rangle \langle 2_{\lambda'}(t)| + 2 \sqrt{6} f^4_\lambda(t) g^2_\lambda(t) |3_\lambda(t)\rangle \langle 1_{\lambda}(t)| \delta_{\lambda\lambda'} + 3 \sqrt{2} f^6_\lambda(t) g^4_\lambda(t) |2_\lambda(t)\rangle \langle 2_{\lambda}(t)| \delta_{\lambda\lambda'} \right] \\
+ \sum_{\lambda,\lambda'} \frac{3p_\lambda p_{\lambda'}}{2} \left[ f^4_\lambda(t) f^4_{\lambda'}(t) |4_{\lambda'}(t)\rangle \langle 4_{\lambda'}(t)| + 4 f^4_\lambda(t) g^2_\lambda(t) |3_{\lambda}(t)\rangle \langle 3_{\lambda}(t)| \delta_{\lambda\lambda'} + 6 f^4_\lambda(t) g^4_\lambda(t) |2_{\lambda}(t)\rangle \langle 2_{\lambda}(t)| \delta_{\lambda\lambda'} + 4 f^6_\lambda(t) g^6_\lambda(t) |1_{\lambda}(t)\rangle \langle 1_{\lambda}(t)| \delta_{\lambda\lambda'} + g^8_\lambda(t) |\text{vac}\rangle \langle \text{vac}| \delta_{\lambda\lambda'} \right].
$$

Note that, ignoring the initial vacuum term, the $|\beta|^2$ term is identical to (6.15).

### 6.3 Discussion

In this Chapter I have introduced a quantum Langevin formalism that enables one to calculate how the density operator associated with a given state of generated photons evolves in the presence of scattering loss. Such loss, while not very relevant for beams focused inside bulk crystals, is often necessary when considering propagation within integrated devices due to sidewall surface roughness inherent in their fabrication processes.

In Section 5.1 the formalism is presented, the main result being that a given waveguide mode operator $b_{n,k}^\dagger$ becomes the sum of a decaying exponential term and a fluctuation term when coupled to a continuum of unguided modes having essentially the same $k$ and a small transverse component of their total wavevector. In Section 5.2 the propagation of three different photon states is calculated, by tracing over the fluctuation terms assuming that the reservoir is at zero temperature. This assumption is justified as the frequencies
of interest in the waveguide are at a much higher blackbody radiation temperature than room temperature. It also ensures that mode mixing is limited, such that there is no photon decoherence time that is shorter than any photon loss time.

It is shown that a coherent state decays exponentially and stays coherent, and that the “two-photon” portion of a two-photon density operator decays twice as fast as a coherent state. The evolution of the single-photon and vacuum portion of a two-photon density operator was also calculated and plotted in Fig. 6.1. For a given Schmidt mode, $\lambda$, the single-photon probability reaches a maximum value of $1/2$ when $\beta_\lambda t = \ln(2)/2$, at which point it is more probable than either its associated two-photon or vacuum density operator. For all times beyond $\beta_\lambda t = \ln(3/2)$, the associated vacuum density operator is most probable, rising from $4/9$ to $1$. Lastly, the density operator for a squeezed vacuum state experiencing loss, expanded in the number state basis for the Schmidt modes in the limit of a low probability of pair production, is shown to exhibit exactly the same statistics as a two-photon density operator. Thus it it clear that there is nothing particularly abnormal about a quantum Langevin formalism for loss as applied to general states of generated photons propagating in waveguide structures, and introduced within the framework presented in Chapter 2 of this thesis. This is an important first step towards a calculation that takes into account photon generation and scattering loss simultaneously.
Chapter 7

Conclusion

This thesis has presented an analysis of quantum nonlinear optical phenomena in artificially structured media, specifically photon pair production via either SPDC or SFWM in channel waveguides as well as microring resonators side-coupled to channel waveguides. Not only are photon pair production efficiencies calculated, and cast in a form resembling corresponding well-known classical nonlinear optical frequency conversion efficiencies, but the states of generated photons themselves, characterized by two-dimensional spectral distribution functions called biphoton wave functions, are also calculated. This allows for easy comparison of the utility of various sources for varied applications as well as the optimization of proposed sources for specific applications before development is started.

In Chapter 2 I presented a Hamiltonian formalism that leverages classical asymptotic-in and asymptotic-out fields, in close analogy with scattering theory in quantum mechanics, to expand the magnetic and electric displacement fields and field operators into sets of modes that are useful for describing both classical and quantum nonlinear optics problems in artificially structured media. These fields are the stationary solutions of the classical linear Maxwell equations, and can be evaluated analytically or numerically. This strategy places classical and corresponding quantum nonlinear optical processes on equal theoretical footing, and simplifies problems involving cavity-like structures as it avoids the construction of Hamiltonians for the cavity modes as well as coupling between cavity modes and propagating modes. When considering quantum nonlinear optics problems, the formalism takes advantage of the fact that, in the perturbative limit, the output state is expected to consist of an undepleted pump as well as the state of generated photons; calculating the evolution of the undepleted pump, which is often quite simple, and pulling it aside implies that what remains in the output state must be the generated photons of interest. Additionally, both the material and modal dispersion of the input and output
channels are correctly accounted for.

While the formalism can consider quite general input states of light, such as Fock states and squeezed states, of arbitrary bandwidth and duration, this thesis only considered coherent input states described by Gaussian spectral distributions. Furthermore, evolution of the input coherent state pump was not considered using the full apparatus of the theory and instead, as a first approximation, the pump was left unmodified. Although it is argued in Chapter 5 that it is reasonable to neglect SPM of the pump field, XPM of the generated fields induced by the pump field, and TPA and associated FCA caused by the pump field when pump powers are kept low enough to ensure a low probability of photon pair production, future work should consider what happens at higher pump powers. Whether or not analytic forms of the pair production efficiency and biphoton wave function could still be found remains an open question, but the formalism allows for such quantities to be calculated numerically if not analytically in the presence of self-and cross-phase modulation. Additionally, it would be fruitful to look beyond the undepleted pump approximation and examine the quantum to classical transition that occurs as generated fields begin to seed classical frequency conversion processes.

I then applied the formalism of Chapter 2 to channel waveguides and microring resonators to calculate DFG, SHG, and SPDC efficiencies when a second-order nonlinearity is dominant, and FWM and SFWM efficiencies when a third-order nonlinearity is dominant. Use of the same formalism to make both quantum and classical calculations enabled links between corresponding processes to be readily apparent; I was able to show that photon pair generation efficiencies could be cast into forms resembling well-known classical frequency conversion efficiencies. I showed that, in the long pump pulse limit and under the undepleted pump approximation, the average energy of a generated photon divided by a characteristic time related to the photon generation bandwidth plays the role of the classical “seed” signal in a quantum process: the larger the possible emission bandwidth, the larger the fluctuating power available to drive the process. This close link between quantum and classical nonlinear optical processes also allows photon pair production efficiencies to be calculated based on the results of classical nonlinear optical experiments, a result that was recently experimentally confirmed in silicon microring resonators by colleagues at the University of Pavia in Italy [18]. Furthermore, I calculated that the scaling of these efficiencies as functions of device length $L$ in a channel waveguide and resonance quality factor $Q$ in a microring resonator are different than what might be expected from classical expressions. The generated idler power, integrated over the entire spectrum, scales with $L^{3/2}$ for both SPDC and SFWM in a channel, unless filtered to within a narrow spectral region around the phase matching condition, in which
case it scales as $L^2$ for both SDC and SFWM in a channel waveguide, much like DFG, SHG, and FWM in a channel waveguide. In a ring resonator DFG and FWM scale with $Q^3$ and $Q^4$, respectively, as SPDC and SFWM, respectively within a sufficiently narrow spectral region centered at the idler/signal resonance. On the other hand, SPDC generated power collected across an entire ring resonance is proportional to $Q^2$ and SFWM generated power to $Q^3$, as not only are each the fields involved in a nonlinear process enhanced by $Q$ but the characteristic time related to the photon generation bandwidth is also enhanced by $Q$.

While other results listed above have confirmed existing results in the literature, my results for photon pair production efficiencies in microring resonators are entirely new. The expressions, and experimental confirmation, highlight the strength and versatility of the formalism introduced in Chapter 2. Furthermore, there is no reason why similar comparisons between quantum and classical processes could not be derived for other structures. It would be useful to know if the scaling with $Q$ is peculiar to microring resonators, or a general feature of all resonant photonic structures. The formalism could also be used in the future to calculate efficiencies for three-photon generation, dual-pump SFWM, or even “translation” of single photon states from one center frequency to another with two appropriate input fields [90] in resonant and non-resonant structures.

In Chapter 4, I catalogued the shapes and orientations of possible biphoton probability densities associated with pairs of SPDC- and SFWM-generated photons in both channel waveguides and microring resonators. Again, to the best of my knowledge, the microring resonator results are completely new. Knowledge of these functions enables knowledge of the spectral correlations that exist between pairs of generated photons, and thus their suitability for various quantum information processing applications. By examining two specific applications, namely Bell’s inequality violation and entanglement swapping, requiring a single photon pair and two photon pairs respectively, in detail and without “tracing over the frequency degree of freedom” as in [42] I have determined conditions sufficient for each application to be successful: the biphoton wave function must possess certain symmetries to maximally violate Bell’s inequality, and must be factorable for perfect entanglement swapping, confirming results for bulk crystal SPDC-generated states [40]. While there are many methods have been proposed to control these correlations in channel waveguides [72, 73, 74], I have shown microring resonators to be more naturally suited to generating essentially frequency uncorrelated photon pairs, both pictorially and through the development of analytic expressions for the Schmidt number [63] of any biphoton wave function that can be approximated as a product of two general Gaussian functions. This property of microring resonators is due to their photon gen-
eration bandwidth tending to be (locally) limited by a resonance linewidth and not by phase matching.

Unfortunately, while broad enough pump pulses incident on microring resonators can generate nearly spectrally uncorrelated photons, a small degree of correlation still often exists unless spectral filtering is employed. As with the scaling relationships developed in Chapter 3, it is not known if this is a general feature of all resonant structures, or a property specific to microring resonators. A useful future investigation would be to look at other types of resonant structures and the biphoton probability densities that they can produce, in addition to the power scaling relationships that hold for them.

As mentioned above, in all of the calculations of Chapters 2 and 3 I have neglected the effects of SPM, XPM, TPA, and FCA. Terms associated with SPM and XPM arose naturally from the Hamiltonian formalism in calculations involving a third-order nonlinearity, but, as a first approximation, were discarded to keep the pump unmodified as it evolves and to keep expressions for generation efficiencies simple and intuitive. Despite this, it was also shown in Chapter 3 that the theoretical predictions worked quite well under a specific set of experimental conditions [18]. The worry that something important might have been left out of the theory and the reality that such an omission did not seem to matter in practice was reconciled in Chapter 5, where I determined that the pump powers used in the experiment described in Chapter 3 were too low to cause significant SPM, XPM, TPA, or FCA. Indeed, I showed that, for an appropriately designed channel waveguide or microring resonator, many of these processes need not become important if one works with powers low enough to suppress multi-pair production. As this is typically the regime of interest, it seems reasonable that such processes may rarely need to be included in theoretical treatments of realistic devices.

Scattering loss was also neglected in the calculations of Chapters 2 and 3, and so in the final Chapter of this thesis I partially addressed how it might be included in future work. Rather than directly tackle the problem of simultaneous photon generation and loss, I investigated the linear loss of “already generated” states within a quantum Langevin formalism for loss. This approach guarantees that the cannonical equal-time bosonic commutation relations of the system operators are conserved for all times. Rather than attempt to include such loss by hand, from say, knowledge of how a coherent state decays when coupled to a zero-temperature reservoir, with this method I was able to calculate the temporal evolution of the two-photon, single-photon, and vacuum pieces of a density operator for an initial two-photon state experiencing linear scattering loss. For a given Schmidt mode, \( \lambda \), of the two-photon state, the single-photon probability reaches a maximum value of \( 1/2 \) when \( \beta_{\lambda} t = \ln(2)/2 \), where \( \beta_{\lambda} \) is the loss parameter for the given
Schmidt mode, at which point it is more probable than either its associated two-photon or vacuum density operator. For all times beyond $\beta_{\lambda} t = \ln(3/2)$, the associated vacuum density operator is most probable, rising from $4/9$ to $1$. Furthermore, I demonstrated that such terms can be recovered from a full squeezed-vacuum state undergoing linear scattering loss in the limit of a small probability of pair production, and calculated that, within such a model, decoherence does not occur any faster than the decay of the lossiest Schmidt mode. This can be understood by recalling that a zero-temperature reservoir limits mode mixing.

This suggests that there is nothing pathological about linear scattering loss on general photonic states, and serves as a first step to the introduction of such loss within the formalism introduced at the start of this thesis. A challenge will be how to reconcile the instantaneous nature of the scattering theory approach to photon generation of Chapter 2 with the continuous nature of the loss experienced by photon states of Chapter 6, but a possible work-around could involve the use of reservoir oscillators that are defined with respect to spatial coordinates rather than $k$-space coordinates.
Appendix A

Evolution Operator Identities

Here we consider evolution operators that specify the formal solution

$$|\psi(t)\rangle = U(t, t') |\psi(t')\rangle,$$  \hspace{1cm} (A.1)

to the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle,$$  \hspace{1cm} (A.2)

defined such that

$$U(t_0, t_0) = I,$$  \hspace{1cm} (A.3)

for some initial $t_0$. By taking appropriate partial derivatives of (A.1) and substituting (A.2) we find

$$i\hbar \frac{\partial U(t, t')}{\partial t} = H(t) U(t, t'),$$
$$i\hbar \frac{\partial U(t, t')}{\partial t'} = - U(t, t') H(t'),$$  \hspace{1cm} (A.4)

or

$$dU(t, t') = \left( \frac{1}{i\hbar} H(t) U(t, t') \right) dt + \left( - \frac{1}{i\hbar} U(t, t') H(t') \right) dt'.$$

For $U(t, t')$ to be really a function of just the two variables $(t, t')$, and not also on the path in the $(\tau_1, \tau_2)$ plane used to get from $(\tau, \tau)$ to the $(t, t')$ of interest, this must be a total derivative. That is, we must have

$$\frac{\partial}{\partial t'} \left( \frac{1}{i\hbar} H(t) U(t, t') \right) = \frac{\partial}{\partial t} \left( - \frac{1}{i\hbar} U(t, t') H(t') \right),$$  \hspace{1cm} (A.5)
and using (A.4) it is easy to see that this is satisfied. So this evolution (A.4) can be used to together with (A.3) to map out a unique \( U(t, t') \) for each \((t, t')\). Note that, viewing (A.5) as a fundamental, required property of \( U(t, t') \), we could have just as easily started our discussion assuming \( U(t, t') \) to have the properties (A.4) in the absence of any kets, instead of assuming (A.1) and (A.2).

Other properties of \( U(t, t') \) follow from (A.3) and (A.4). First note that

\[
\frac{d}{d\tau} U(\tau, \tau) = \left( \frac{\partial U(t, t')}{\partial t} \right)_{t=t'=\tau} + \left( \frac{\partial U(t, t')}{\partial t'} \right)_{t=t'=\tau} = \left( \frac{1}{i\hbar} H(t) U(t, t') \right)_{t=t'=\tau} + \left( -\frac{1}{i\hbar} U(t, t') H(t_2) \right)_{t=t'=\tau} = \frac{1}{i\hbar} [H(\tau), U(\tau, \tau)].
\]

(A.6)

From (A.3) we have that

\[
\left( \frac{d}{d\tau} U(\tau, \tau) \right)_{t=t_0} = 0,
\]

and taking the derivative of (A.6) and using that expression itself we find

\[
\frac{d^2}{dt^2} U(t, t) = \frac{1}{i\hbar} \left[ \frac{dH(t)}{dt}, U(t, t) \right] - \frac{1}{\hbar^2} [H(t), [H(t), U(t, t)]] ,
\]

and so

\[
\left( \frac{d^2}{dt^2} U(t, t) \right)_{t=t_0} = 0,
\]

and so on for all higher derivatives. So we have

\[
U(t, t) = I, \quad (A.7)
\]

for all \( t \), not just \( t_0 \).

Next we look at

\[
\frac{\partial}{\partial t} \left( U(t, t') U(t, t') \right) = \frac{1}{i\hbar} U(t, t') H(t) U(t, t') - \frac{1}{i\hbar} U(t, t') H(t) U(t, t') = 0,
\]

where we have used the adjoint of the first of (A.4) and the fact that \( H(t) \) is Hermitian. All higher derivatives vanish, and since \( U(t', t') U(t', t') = I \) by virtue of (A.7), we see that

\[
U(t, t') U(t, t') = U(t, t') U(t, t') = I, \quad (A.8)
\]

for all \( t \) and \( t' \), where the second of these follows from an argument similar to the first.
Appendix A. Evolution Operator Identities

So the operator $U(t, t')$ is unitary for all $(t, t')$.

Finally, we look at

$$\frac{\partial}{\partial t'} (U(t, t') U(t', t'')) = -\frac{1}{i\hbar} U(t, t') H(t') U(t', t'') + \frac{1}{i\hbar} U(t, t') H(t') U(t', t'') = 0,$$

where we have used (A.4), and since all higher derivatives also vanish we can write that $U(t, t') U(t', t'') = U(t, t'') U(t', t'').$ The latter is just $U(t, t'')$ by virtue of (A.7), and so we have

$$U(t, t') U(t', t'') = U(t, t'')$$

(A.9)

for all $t_2$. A special case of this is

$$U(t, t') U(t', t) = \mathbb{I},$$

and right-multiplying by $U^\dagger(t', t)$ and using (A.8) we find

$$U(t, t') = U^\dagger(t', t).$$

(A.10)
Appendix B

The CW Limit

We consider a state of generated photons of the form

$$|\psi_{\text{gen}}\rangle = \exp \left( \beta C_{II}^{\dagger} - \text{H.c.} \right) |\text{vac}\rangle,$$  \hspace{1cm} (B.1) 

where, neglecting the output (right) channel index R,

$$C_{II}^{\dagger} |\text{vac}\rangle = \frac{1}{\sqrt{2}} \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \phi(\omega_1, \omega_2) b_{\omega_1}^{\dagger} b_{\omega_2}^{\dagger} |\text{vac}\rangle,$$

is a normalized two-photon state characterized by the (normalized) BWF $\phi(\omega_1, \omega_2)$. We perform a Schmidt decomposition of the BWF [63], so that

$$\phi(\omega_1, \omega_2) = \sum_{\lambda} \sqrt{p_{\lambda}} \Phi_{\lambda}(\omega_1) \Phi_{\lambda}(\omega_2),$$

where $\{\Phi_{\lambda}\}$ is a complete set of orthonormal functions, and the probabilities sum to 1 i.e.

$$\int d\omega \Phi_{\lambda}^{*}(\omega) \Phi_{\lambda'}(\omega) = \delta_{\lambda\lambda'},$$

$$\sum_{\lambda} \Phi_{\lambda}(\omega) \Phi_{\lambda}^{*}(\omega') = \delta(\omega - \omega'),$$

$$\sum_{\lambda} p_{\lambda} = 1,$$
and we also have

\[ A^\dagger_\lambda = \int d\omega \Phi_\lambda (\omega) b^\dagger_\omega, \]

\[ \sum_\lambda \Phi^*_\lambda (\omega') A^\dagger_\lambda = \int d\omega \sum_\lambda \Phi^*_\lambda (\omega') \Phi_\lambda (\omega) b^\dagger_\omega \]

\[ = b^\dagger_\omega, \]

\[ b^\dagger_\omega = \sum_\lambda \Phi^*_\lambda (\omega) A^\dagger_\lambda. \]  

\hspace{2cm} (B.2)

As a quick aside, note that for a double Gaussian BWF

\[ \phi (\omega_1, \omega_2) = \sqrt{\frac{2}{\pi \sigma_+ \sigma_-}} e^{-\frac{(\omega_1 - \omega_2)^2}{2\sigma_-^2}} e^{-\frac{-(\omega_1 + \omega_2)^2}{2\sigma_+^2}}, \]  

\hspace{2cm} (B.3)

the Schmidt decomposition can be found exactly (see Appendix D)

\[ \phi (\omega_1, \omega_2) = \sum_{n=0}^{\infty} \sqrt{p_n} \Phi_n (\omega_1) \Phi_n (\omega_2), \]

for \( \sigma_- \geq \sigma_+ \), where

\[ p_n = \frac{n^n}{(n + 1)^{n+1}}, \]

with

\[ \bar{n} = \frac{(\sigma_+ - \sigma_-)^2}{4\sigma_+ \sigma_-}. \]

The Schmidt decomposition allows us to write the generated state (B.1) as

\[ |\psi_{\text{gen}}\rangle = \exp \left( \frac{\beta}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \sum_\lambda \sqrt{p_\lambda} \Phi_\lambda (\omega_1) \Phi_\lambda (\omega_2) b^\dagger_{\omega_1} b^\dagger_{\omega_2} - \text{H.c.} \right) |\text{vac}\rangle \]

\[ = \prod_\lambda \exp \left( \frac{\beta}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \Phi_\lambda (\omega_1) \Phi_\lambda (\omega_2) b^\dagger_{\omega_1} b^\dagger_{\omega_2} - \text{H.c.} \right) |\text{vac}\rangle \]

\[ = \prod_\lambda \exp \left( \beta \sqrt{p_\lambda} C^\dagger_{\Pi_\lambda} - \text{H.c.} \right) |\text{vac}\rangle, \]
where

\[ C_{\Pi,\lambda}^\dagger |\text{vac}\rangle = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \Phi_\lambda(\omega_1) \Phi_\lambda(\omega_2) b_\omega^\dagger \omega b_\omega^\dagger |\text{vac}\rangle = \frac{1}{\sqrt{2}} A_\lambda^\dagger A_\lambda^\dagger |\text{vac}\rangle , \]

and with this established we can now easily calculate the expectation value of the number of photons in the output state. The number operator is, remembering (B.2),

\[ N = \int d\omega b_\omega^\dagger b_\omega = \sum_\lambda A_\lambda^\dagger A_\lambda , \]

and thus, recalling the transformations

\[ S(z) a S^\dagger (z) = a \cosh(|z|) + a^\dagger e^{i \arg(z)} \sinh(|z|) , \]

\[ S(z) a^\dagger S^\dagger (z) = a^\dagger \cosh(|z|) + ae^{-i \arg(z)} \sinh(|z|) , \]

where

\[ S^\dagger (z) = S^{-1} (z) = e^{\frac{i}{2} (za^\dagger a - \text{H.c.})} , \]

we find the expectation value of the number operator to be

\[ \mathcal{N}_D = \langle \psi_{\text{gen}} | N | \psi_{\text{gen}} \rangle = \sum_\lambda \sinh^2 \left( |\beta| \sqrt{2p_\lambda} \right) . \tag{B.4} \]

Reasonable approximations lead to simple analytic forms of the BWFs associated with various photon generation processes in various structures, from which we can approximate a $\sigma_+$ and a $\sigma_-$. The form of the BWF used here for a SPDC or SFWM process in a
channel waveguide is (see (3.12))

\[
\phi^{C_2}(\omega_1, \omega_2) = B^{C_2}(\omega_1, \omega_2) \phi_p(\omega_1 + \omega_2) \\
\times \text{sinc} \left[ (\omega_1 + \omega_2 - \omega_p) \left( \frac{1}{v_F} - \frac{1}{v_{SH}} \right) L \right] + \left( \frac{\Lambda_{SH} L}{2} - \frac{\Lambda_F L}{4} \right) (\omega_1 + \omega_2 - \omega_p)^2 - \frac{\Lambda_F}{4} L (\omega_1 - \omega_2)^2 
\]

and (see (3.32))

\[
\phi^{C_3}(\omega_1, \omega_2) = B^{C_3}(\omega_1, \omega_2) \int_{0}^{\omega_1 + \omega_2} d\omega \left( \phi_p(\omega) \phi_p(\omega_1 + \omega_2 - \omega) \right) \\
\times \text{sinc} \left\{ \Lambda_p \left[ \left( \omega - \frac{\omega_1 + \omega_2}{2} \right)^2 - \left( \frac{\omega_1 - \omega_2}{2} \right)^2 \right] L \right\}
\]

respectively, where the \( B^{C_2,3}(\omega_1, \omega_2) \) are benign functions of \( \omega_1 \) and \( \omega_2 \), and \( \phi_p(\omega) \) is the pump pulse waveform. In a microring resonator they have a similar form, namely (see (3.44), and (3.53))

\[
\phi^{R_2}(\omega_1, \omega_2) = B^{R_2}(\omega_1, \omega_2) \phi_p(\omega_1 + \omega_2) \left[ i (\omega_1 + \omega_2 - \omega_p) - \frac{v_{SH}}{|F_p(\omega_p)| L} \right]^{-1} \\
\times \frac{B^{R_3}(\omega_1, \omega_2) \int_{0}^{\omega_1 + \omega_2} d\omega \phi_p(\omega) \phi_p(\omega_1 + \omega_2 - \omega) \left[ i (\omega - \omega_p) - \frac{v_p}{|F_p(\omega_p)| L} \right]^{-1}}{\left[ i (\omega_1 - \omega_p) + \frac{v_p}{|F_p(\omega_p)| L} \right] \left[ i (\omega_2 - \omega_p) + \frac{v_p}{|F_p(\omega_p)| L} \right]^{-1}}
\]

We consider a “top hat” pulse in time, with width \( T \) and center frequency \( \omega_p \)

\[
\phi_p(\omega) = \sqrt{\frac{T}{2\pi}} \text{sinc} \left( (\omega - \omega_p) \frac{T}{2} \right),
\]

(B.5)
and, in the limit of a long pulse (see (3.34)), find

\[
\phi_{C3} (\omega_1, \omega_2) \approx B_{C3} (\omega_1, \omega_2) \sqrt{\frac{2\pi}{T}} \phi_P (\omega_1 + \omega_2 - \omega_P) \\
\times \text{sinc} \left\{ \Lambda_P \left[ \left( \omega_p - \frac{\omega_1 + \omega_2}{2} \right)^2 - \left( \frac{\omega_1 - \omega_2}{2} \right)^2 \right] \right\} L \},
\]

\[
\phi_{R3} (\omega_1, \omega_2) \approx \frac{|F_P (\omega_P)|^2 L^2 B_{R3} (\omega_1, \omega_2) \sqrt{\frac{2\pi}{T}} \phi_P (\omega_1 + \omega_2 - \omega_P)}{v_P^2 \left[ i (\omega_1 - \omega_P) + \frac{v_p}{|F_P(\omega_P)|L} \right] \left[ i (\omega_2 - \omega_P) + \frac{v_p}{|F_P(\omega_P)|L} \right]},
\]

as well as (see (3.16), (3.35), (3.45), and (3.54))

\[
|\beta_{C2}|^2 = \frac{P_P L^2}{PA} \times \frac{T}{3\sqrt{\pi |\Lambda_F| L}},
\]

\[
|\beta_{C3}|^2 = (\gamma P_P L)^2 \times \frac{T}{3\sqrt{\pi |\Lambda_P| L}},
\]

\[
|\beta_{R2}|^2 = \frac{v_P L}{2} \frac{P_P}{PA} \left| F_P (\omega_{F0}) \right|^2 \left| F_P (\omega_P) \right|^2 T,
\]

\[
|\beta_{R3}|^2 = \frac{v_P L}{2} (\gamma P_P)^2 \left| F_P (\omega_P) \right|^6 T,
\]

for each of the four BWFs, respectively. Setting the FWHM of the absolute value squared of the \(\omega_1 + \omega_2\) part of (B.3) to equal the FWHM of the absolute value squared of (B.5) we find

\[
\sigma_+ \approx \frac{2s}{\sqrt{\ln(2)T}},
\]

and similarly find

\[
\sigma_{-2} \approx \sqrt{\frac{4s}{\ln(2) |\Lambda_F| L}},
\]

\[
\sigma_{-3} \approx \sqrt{\frac{4s}{\ln(2) |\Lambda_P| L}},
\]

\[
\sigma_{-2} \approx 2 \sqrt{\frac{(\sqrt{2} - 1) v_F^2}{\ln(2) \left| F_P (\omega_{F0}) \right|^2 L^2}},
\]

\[
\sigma_{-3} \approx 2 \sqrt{\frac{(\sqrt{2} - 1) v_p^2}{\ln(2) \left| F_P (\omega_P) \right|^2 L^2}},
\]

where \(s \approx 1.3916\) is the positive root of \(\text{sinc}^2(x) = 0.5\). In each of these four cases, as
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$T \to \infty$, $\sigma_- >\sigma_+$ and $\bar{\pi} \approx \sigma_- / (4\sigma_+)$, with $p_n \approx \bar{\pi}^{-1}$, or, more specifically,

$$\pi^{C_2} \approx \frac{T}{4 \sqrt{s |\Lambda| L}} \approx (p_n^{C_2})^{-1},$$

$$\bar{\pi}^{C_3} \approx \frac{T}{4 \sqrt{s |\Lambda| L}} \approx (p_n^{C_2})^{-1},$$

$$n^{R_2} \approx \frac{T v_F \sqrt{2 - 1}}{4 s |F_F (\omega_{F_0})| L} \approx (p_n^{R_2})^{-1},$$

$$\pi^{R_3} \approx \frac{T v_P \sqrt{2 - 1}}{4 s |F_F (\omega_P)| L} \approx (p_n^{R_3})^{-1}.$$ 

Thus, we see that, provided $\beta \sqrt{2 \pi \lambda} \ll 1$, or

$$\left( \frac{P_P L^2}{\mathcal{P} \mathcal{A}} \frac{T}{3 \sqrt{\pi |\Lambda| L}} \right)^{1/2} \left( \frac{2 4 \sqrt{s |\Lambda| L}}{T} \right)^{1/2} \ll 1,$$

$$\Rightarrow P_P \ll \left( \frac{9 \pi}{64 s} \right)^{1/2} \frac{\mathcal{P} \mathcal{A}}{L^2}, \quad (B.6)$$

$$\left( \frac{\gamma P_P L^2}{\mathcal{P} \mathcal{A}} \right)^{1/2} \left( \frac{T}{3 \sqrt{\pi |\Lambda| L}} \right)^{1/2} \left( \frac{2 4 \sqrt{s |\Lambda| L}}{T} \right)^{1/2} \ll 1,$$

$$\Rightarrow P_P \ll \left( \frac{9 \pi}{64 s} \right)^{1/4} (\gamma L)^{-1}, \quad (B.7)$$

$$\left( \frac{v_F L P_P}{2 \mathcal{P} \mathcal{A}} |F_F (\omega_{F_0})|^2 |F_F (\omega_P)|^2 T \right)^{1/2} \left( \frac{2 4 s |F_F (\omega_{F_0})| L}{T v_F \sqrt{2 - 1}} \right)^{1/2} \ll 1,$$

$$\Rightarrow P_P \ll \left( \sqrt{2 - 1} \frac{1}{16 s^2} \right)^{1/2} \frac{\mathcal{P} \mathcal{A}}{L^2 |F_F (\omega_{F_0})|^3 |F_F (\omega_P)|^2}, \quad (B.8)$$

$$\left( \frac{v_P L}{2 (\gamma P_P)^2} |F_F (\omega_P)|^6 T \right)^{1/2} \left( \frac{2 4 s |F_F (\omega_P)| L}{T v_P \sqrt{2 - 1}} \right)^{1/2} \ll 1,$$

$$\Rightarrow P_P \ll \left( \sqrt{2 - 1} \frac{1}{16 s^2} \right)^{1/4} \frac{(\gamma L)^{-1}}{|F_F (\omega_P)|^{7/2}.} \quad (B.9)$$

which are valid even as $T \to \infty$. If the inequality is satisfied in one of the four cases above, then the associated number of photons (B.4) can be approximated as

$$N_D \approx \sum_{\lambda} |\beta \sqrt{2 \pi \lambda}|^2$$

$$= 2 |\beta|^2.$$
Appendix C

How The Biphoton Wave Function Affects Two Specific Experiments

If a number of identical photon pair states can be used to violate \[77\]

\[
S = |E(\theta_A, \theta_B) - E(\theta_A, \theta'_B) + E(\theta'_A, \theta_B) + E(\theta'_A, \theta'_B)| \leq 2,
\]

where

\[
E(\theta_A, \theta_B) = \frac{C(\theta_A, \theta_B) + C(\theta_A^\perp, \theta_B^\perp) - C(\theta_A^\perp, \theta_B) - C(\theta_A, \theta_B^\perp)}{C(\theta_A, \theta_B) + C(\theta_A^\perp, \theta_B) + C(\theta_A^\perp, \theta_B^\perp) + C(\theta_A, \theta_B^\perp)},
\]

and \(C(\theta_A, \theta_B)\) is the number of coincidences for coincidence detection of photons passing through linear polarizers set at an angle \(\theta_i\) from the horizontal, H, axis, and \(\theta_i^\perp = \theta_i + \pi/2\), then, provided the fair sampling hypothesis is correct, at least one of the assumptions of realism and locality that were used to derive the inequality is incorrect \([11]\). While the coincidences in the preceding equations are meant to be determined experimentally, they may be theoretically approximated as the expectation value of the linear polarization projection operator

\[
P_{\theta_A, \theta_B} \equiv \int d\omega d\omega' \langle \theta_A(\omega) \theta_B(\omega') \rangle_{12} \langle \theta_A(\omega) \theta_B(\omega') \rangle_{12},
\]

where

\[
|\theta_A(\omega) \theta_B(\omega')\rangle_{12} = \cos \theta_A \cos \theta_B |H(\omega) H(\omega')\rangle_{12} + \cos \theta_A \sin \theta_B |H(\omega) V(\omega')\rangle_{12} + \sin \theta_A \cos \theta_B |V(\omega) H(\omega')\rangle_{12} + \sin \theta_A \sin \theta_B |V(\omega) V(\omega')\rangle_{12},
\]
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or

\[ C (\theta_A, \theta_B) = \text{Tr} (P_{\theta_A \theta_B} \rho) , \]

where \( \rho \) is a density operator. Defining the four generalized Bell states

\[ |\Psi^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \left[ \phi_{HV} (\omega_1, \omega_2) |H (\omega_1) V (\omega_2)\rangle_{12} \right. \]
\[ \left. \pm \phi_{VH} (\omega_1, \omega_2) |V (\omega_1) H (\omega_2)\rangle_{12} \right], \]

\[ |\Phi^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \left[ \phi_{HH} (\omega_1, \omega_2) |H (\omega_1) H (\omega_2)\rangle_{12} \right. \]
\[ \left. \pm \phi_{VV} (\omega_1, \omega_2) |V (\omega_1) V (\omega_2)\rangle_{12} \right], \]

normalized according to

\[ \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{HV} (\omega_1, \omega_2)|^2 = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{VH} (\omega_1, \omega_2)|^2 = 1, \quad (C.1) \]

\[ \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{HH} (\omega_1, \omega_2)|^2 + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{VV} (\omega_1, \omega_2)|^2 = 2, \]

\( \rho \) takes the form \(|\Psi^+\rangle \langle \Psi^+|, |\Psi^-\rangle \langle \Psi^-|, |\Phi^+\rangle \langle \Phi^+|, \) or \(|\Phi^-\rangle \langle \Phi^-|\). Thus, we find

\[ C (\theta_A, \theta_B) \]
\[ = \text{Tr} (P_{\theta_A \theta_B} \rho) \]
\[ = \sum_{\sigma_1, \sigma_2} \int d\Omega_1 d\Omega_2 \langle \sigma_1 (\Omega_1) \sigma_2 (\Omega_2) |_{12} P_{\theta_A \theta_B} \rho |\sigma_1 (\Omega_1) \sigma_2 (\Omega_2)\rangle_{12} \]
\[ = \begin{cases} \frac{1}{2} \left[ \cos^2 \theta_A \sin^2 \theta_B \pm (G + G^*) \cos \theta_A \sin \theta_B \cos \theta_B \\ + \sin^2 \theta_A \cos^2 \theta_B \right] & \text{for } |\Psi^{\pm}\rangle_{12} \\
\frac{1}{2} \left[ H \cos^2 \theta_A \cos^2 \theta_B \pm (M + M^*) \cos \theta_A \sin \theta_B \sin \theta_A \cos \theta_B \\ + V \sin^2 \theta_A \sin^2 \theta_B \right] & \text{for } |\Phi^{\pm}\rangle_{12} \end{cases}, \quad (C.2) \]
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where we have used (C.1) and defined
\[
G = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi_{HV}(\omega_1, \omega_2) \phi_{VH}^*(\omega_1, \omega_2),
\]
\[
H = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{HH}(\omega_1, \omega_2)|^2,
\]
\[
M = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \phi_{HH}(\omega_1, \omega_2) \phi_{VV}^*(\omega_1, \omega_2),
\]
\[
V = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 |\phi_{VV}(\omega_1, \omega_2)|^2.
\]
(C.3)

Note that (C.2) reduces to the more familiar
\[
C(\theta_A, \theta_B) = \frac{1}{2} \sin^2 (\theta_A \pm \theta_B),
\]
if and only if \(G + G^* = 2\) for \(|\Psi^\pm\rangle_{12}\), and reduces to the more familiar
\[
C(\theta_A, \theta_B) = \frac{1}{2} \cos^2 (\theta_A \mp \theta_B),
\]
if and only if \(H = V = 1\), as well as \(M + M^* = 2\) for \(|\Phi^\pm\rangle_{12}\). If these conditions are satisfied, maximal violation of the CHSH inequality, \(S = 2\sqrt{2}\), can be achieved for a proper choice of \(\theta_A, \theta_B, \theta'_A, \) and \(\theta'_B\). If they are not, \(S\) will be less than \(2\sqrt{2}\), and the inequality may not even be violated. As an example, setting \(\theta'_A - \theta_A = \pi/4, \theta_B - \theta_A = \pi/8,\) and \(\theta'_B - \theta_A = 3\pi/8\) yields \(S = 2\sqrt{2}\) for the state \(|\Psi^\pm\rangle_{12}\) when \(G = G^* = 1\).

The conditions on (C.3) above imply that the biphoton wave function posesses certain symmetries. To see this, recall the Schwarz inequality
\[
\left| \int d\omega_1 d\omega_2 A(\omega_1, \omega_2) B^*(\omega_1, \omega_2) \right|^2 \leq \int d\omega_1 d\omega_2 |A(\omega_1, \omega_2)|^2 \int d\omega_1 d\omega_2 |B(\omega_1, \omega_2)|^2,
\]
the equality holding for \(A(\omega_1, \omega_2) \propto B(\omega_1, \omega_2)\). This allows us to write
\[
|G|^2 \leq 1,
\]
as well as
\[
|M|^2 \leq HV.
\]
The first of these inequalities, combined with \(G + G^* = 2\), implies that \(G\) is real and thus
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\(|\Psi^\pm\rangle_{12}\) behaves like the idealized single-mode Bell states

\[
\frac{1}{\sqrt{2}} (|HV\rangle_{12} \pm |VH\rangle_{12}),
\]

in general (rank-1) local polarization measurements if and only if

\[
\phi_{HV}(\omega_1, \omega_2) = \phi_{VH}(\omega_1, \omega_2). \tag{C.4}
\]

Similarly, \(|M|^2 \leq HV\) combined with \(H = V = 1\) and \(M + M^* = 2\), implies that \(M\) is real and thus \(|\Phi^\pm\rangle_{12}\) behaves like the idealized single-mode Bell states

\[
\frac{1}{\sqrt{2}} (|HH\rangle_{12} \pm |VV\rangle_{12}),
\]

in general (rank-1) local polarization measurements if and only if

\[
\phi_{HH}(\omega_1, \omega_2) = \phi_{VV}(\omega_1, \omega_2). \tag{C.5}
\]

Alternatively, if either of the states

\[
|\Psi^\pm\rangle_{01} |\Psi^\pm\rangle_{23}, \tag{C.6}
\]

is passed through the series of beam-splitters shown in Fig. 4.2, one might try to trigger on detectors in specific channels such that the entanglement that existed between channels 0 and 1 (or 2 and 3) is “swapped” to channels 0 and 3. In particular, if photons are detected in channels 6 and 7 or 8 and 9 then \(|\Psi^+\rangle_{01} |\Psi^+\rangle_{23}\) is “swapped” to \(|\Psi^+\rangle_{03}\), and if photons are detected in channels 6 and 8 or 7 and 9 then \(|\Psi^-\rangle_{01} |\Psi^-\rangle_{23}\) is “swapped” to \(|\Psi^-\rangle_{03}\), provided the biphoton wave function satisfies a condition shown below. Note that with the set up of Fig. 4.2 only these two states can be successfully swapped. In fact, when constrained to linear optical elements and classical feed-forward, assuming perfect number-resolving photodetection, perfect distinguishability of only two of the four Bell states is possible \([76]\), and thus even with a different interferometric set-up only two of the four Bell states could be successfully swapped. Therefore we focus our attention here on the states (C.6). To pass the photons in channels 1 and 2 through the input arms of
Thus, dropping the integration limits for ease of notation, the states (C.6) become

\[
\begin{align*}
    b_{\omega_1} &\rightarrow \frac{1}{\sqrt{2}} (b_{\omega_5} - ib_{\omega_4}) \\
    b_{\omega_2} &\rightarrow \frac{1}{\sqrt{2}} (b_{\omega_4} - ib_{\omega_5}) ,
\end{align*}
\]

and to pass each output from the 50/50 beam-splitter through a polarization beam-splitter is to make the transformation

\[
\begin{align*}
    b_{V_{\omega_1}} &\rightarrow -ib_{V_{\omega_7}}, b_{H_{\omega_4}} \rightarrow b_{H_{\omega_6}}, \\
    b_{V_{\omega_5}} &\rightarrow -ib_{V_{\omega_8}}, b_{H_{\omega_5}} \rightarrow b_{H_{\omega_9}}.
\end{align*}
\]

Thus, dropping the integration limits for ease of notation, the states (C.6) become

\[
\begin{align*}
\left| \Psi^{\pm} \right\rangle_{01} &\mid \Psi^{\pm} \rangle_{23} \\
&\rightarrow \frac{1}{4} \int d\omega_1 d\omega_2 d\omega'_1 d\omega'_2 \{ \phi_{HV} (\omega_1, \omega_2) \phi_{HV} (\omega'_1, \omega'_2) \\
&\times [ -i | H (\omega_1) V (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0863} - | H (\omega_1) V (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0893} \\
&- | H (\omega_1) V (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0763} + i | H (\omega_1) V (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0793} ] \\
&\pm \phi_{HV} (\omega_1, \omega_2) \phi_{HV} (\omega'_1, \omega'_2) \\
&\times [ -i | V (\omega_1) H (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0873} + i | H (\omega_1) V (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0883} \\
+i | H (\omega_1) V (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0773} + | H (\omega_1) V (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0783} ] \\
&\pm \phi_{HV} (\omega_1, \omega_2) \phi_{HV} (\omega'_1, \omega'_2) \\
&\times [ i | V (\omega_1) H (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0963} - i | V (\omega_1) H (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0993} \\
+i | V (\omega_1) H (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0663} - | V (\omega_1) H (\omega_2) H (\omega'_1) V (\omega'_2) \rangle_{0693} ] \\
&\pm \phi_{HV} (\omega_1, \omega_2) \phi_{HV} (\omega'_1, \omega'_2) \\
&\times [ -i | V (\omega_1) H (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0973} - | V (\omega_1) H (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0983} \\
- | V (\omega_1) H (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0673} + i | V (\omega_1) H (\omega_2) V (\omega'_1) H (\omega'_2) \rangle_{0683} ] \\
\equiv &\mid \Psi^{\pm} \rangle.
\end{align*}
\]

Then two photons can be detected to project the photons in channels 0 and 3 onto the desired state. The state that remains after such detection can be written as a reduced density operator. For instance, if photons from \( \mid \Psi^{\pm} \rangle \) are detected in channels 6 and 7.
we find
\[
\rho_{67} = \frac{1}{2} \int d\Omega_1 d\Omega_2 d\omega_1 d\omega_2 d\omega_3 d\omega_4 \\
\times [\phi_{HV}(\omega_1, \Omega_2) \phi_{HV}(\Omega_1, \omega_2) \phi_{HV}^*(\omega_3, \Omega_2) \phi_{HV}^*(\Omega_1, \omega_4) \\
\times |H(\omega_1) V(\omega_2)\rangle_{03} \langle H(\omega_3) V(\omega_4)|_{03} \\
+ \phi_{HV}(\omega_1, \Omega_2) \phi_{HV}(\Omega_1, \omega_2) \phi_{HV}^*(\omega_3, \Omega_1) \phi_{HV}^*(\Omega_2, \omega_4) \\
\times |H(\omega_1) V(\omega_2)\rangle_{03} \langle V(\omega_3) H(\omega_4)|_{03} \\
+ \phi_{VH}(\omega_1, \Omega_1) \phi_{VH}(\Omega_2, \omega_2) \phi_{VH}^*(\omega_3, \Omega_2) \phi_{VH}^*(\Omega_1, \omega_4) \\
\times |V(\omega_1) H(\omega_2)\rangle_{03} \langle H(\omega_3) V(\omega_4)|_{03} \\
+ \phi_{VH}(\omega_1, \Omega_1) \phi_{VH}(\Omega_2, \omega_2) \phi_{VH}^*(\omega_3, \Omega_1) \phi_{VH}^*(\Omega_2, \omega_4) \\
\times |V(\omega_1) H(\omega_2)\rangle_{03} \langle V(\omega_3) H(\omega_4)|_{03} ],
\]

where
\[
\rho_{nm} = \frac{\text{Tr}_{nm} (P_{nm} |\Psi^\pm\rangle \langle \Psi^\pm|)}{\langle \Psi^\pm| P_{nm} |\Psi^\pm\rangle},
\]

and
\[
P_{nm} = \int d\Omega_1 d\Omega_2 |H(\Omega_1) V(\Omega_2)\rangle_{nm} \langle H(\Omega_1) V(\Omega_2)|_{nm},
\]

Similarly
\[
\rho_{98} = \rho_{67} \equiv \rho_{\Psi^+},
\]

and
\[
\rho_{68} = \rho_{97} = \frac{1}{2} \int d\Omega_1 d\Omega_2 d\omega_1 d\omega_2 d\omega_3 d\omega_4 \\
\times [\phi_{HV}(\omega_1, \Omega_2) \phi_{HV}(\Omega_1, \omega_2) \phi_{HV}^*(\omega_3, \Omega_2) \phi_{HV}^*(\Omega_1, \omega_4) \\
\times |H(\omega_1) V(\omega_2)\rangle_{03} \langle H(\omega_3) V(\omega_4)|_{03} \\
- \phi_{HV}(\omega_1, \Omega_2) \phi_{HV}(\Omega_1, \omega_2) \phi_{HV}^*(\omega_3, \Omega_1) \phi_{HV}^*(\Omega_2, \omega_4) \\
\times |H(\omega_1) V(\omega_2)\rangle_{03} \langle V(\omega_3) H(\omega_4)|_{03} \\
- \phi_{VH}(\omega_1, \Omega_1) \phi_{VH}(\Omega_2, \omega_2) \phi_{VH}^*(\omega_3, \Omega_2) \phi_{VH}^*(\Omega_1, \omega_4) \\
\times |V(\omega_1) H(\omega_2)\rangle_{03} \langle H(\omega_3) V(\omega_4)|_{03} \\
+ \phi_{VH}(\omega_1, \Omega_1) \phi_{VH}(\Omega_2, \omega_2) \phi_{VH}^*(\omega_3, \Omega_1) \phi_{VH}^*(\Omega_2, \omega_4) \\
\times |V(\omega_1) H(\omega_2)\rangle_{03} \langle V(\omega_3) H(\omega_4)|_{03} ]
\equiv \rho_{\Psi^-}
\]

A successful entanglement swapping operation would mean that we could write the
density matrices $\rho_{\Psi^+}$ and $\rho_{\Psi^-}$ as $|\Psi^+\rangle_{03} \langle \Psi^+|_{03}$ and $|\Psi^-\rangle_{03} \langle \Psi^-|_{03}$, respectively. To examine the conditions necessary for this to be possible, we perform the Schmidt decomposition

$$\phi_{HV} (\omega_1, \omega_2) = \sum_{\lambda} \sqrt{p_{\lambda}} \Phi_{\lambda} (\omega_1) \Theta_{\lambda} (\omega_2),$$

Note that the Schmidt functions are orthonormal, satisfying

$$\int d\omega \Phi_{\lambda} (\omega) \Phi^*_{\lambda'} (\omega) = \int d\omega \Theta_{\lambda} (\omega) \Theta^*_{\lambda'} (\omega) = \delta_{\lambda\lambda'},$$

and that $\sqrt{p_{\lambda}} > 0 \forall \lambda$ with $\sum_{\lambda} p_{\lambda} = 1$. Thus, we may simplify the density matrices above to

$$\rho_{\Psi^\pm} = \frac{1}{2} \sum_{\lambda\lambda'} p_{\lambda} p_{\lambda'} \int d\omega_1 d\omega_2 d\omega_3 d\omega_4$$

$$\times [\Phi_{\lambda} (\omega_1) \Theta_{\lambda'} (\omega_2) \Phi^*_{\lambda'} (\omega_3) \Theta^*_{\lambda} (\omega_4) |H (\omega_1) V (\omega_2)\rangle_{03} \langle H (\omega_3) V (\omega_4)|_{03}$$

$$\pm \Phi_{\lambda} (\omega_1) \Theta_{\lambda'} (\omega_2) \Phi^*_{\lambda} (\omega_3) \Theta^*_{\lambda'} (\omega_4) |H (\omega_1) V (\omega_2)\rangle_{03} \langle V (\omega_3) H (\omega_4)|_{03}$$

$$\pm \Theta_{\lambda} (\omega_1) \Phi_{\lambda'} (\omega_2) \Phi^*_{\lambda} (\omega_3) \Theta^*_{\lambda'} (\omega_4) |V (\omega_1) H (\omega_2)\rangle_{03} \langle V (\omega_3) H (\omega_4)|_{03}$$

$$+ \Theta_{\lambda} (\omega_1) \Phi_{\lambda'} (\omega_2) \Theta^*_{\lambda'} (\omega_3) \Phi^*_{\lambda} (\omega_4) |V (\omega_1) H (\omega_2)\rangle_{03} \langle V (\omega_3) H (\omega_4)|_{03},$$

and calculate their fidelity [11] with $|\Psi^\pm\rangle_{03} \langle \Psi^\pm|_{03}$

$$F_{\Psi^\pm} = \sqrt{\langle \Psi^\pm|_{03} \rho_{\Psi^\pm} |\Psi^\pm\rangle_{03}}.$$

We find

$$F_{\Psi^\pm} = \sqrt{\sum_{\lambda} p_{\lambda}^2},$$

a quantity which is less than 1 unless there is only a single Schmidt coefficient $\sqrt{p_1} = 1$. Thus entanglement swapping can be successful for the states $|\Psi^\pm\rangle_{01} |\Psi^\pm\rangle_{23}$ if and only if the biphoto wave function is factorable

$$\phi_{HV} (\omega_1, \omega_2) = \Phi (\omega_1) \Theta (\omega_2). \quad (C.7)$$
Appendix D

Analytic Expressions for the Schmidt Number

Here we make a connection between the reduced density operator of two-dimensional functions that are the product of two Gaussian functions, which in many cases serve as a good approximation to a particular BWF, and the momentum representation of the density operator of a harmonic oscillator, as the latter has well-known eigenvalues and eigenfunctions, permitting an analytic expression for the Schmidt decomposition [63] of said two-dimensional functions. That an analytic expression for the Schmidt decomposition of a double-Gaussian function exists is well-known [72, 40, 91], but unlike previous works here we provide a full and more physical derivation for very general products of two Gaussian functions. The derivation is based on a set of unpublished notes of John Sipe’s [92].

A harmonic oscillator of mass \( m \) and resonant frequency \( \omega \) at temperature \( T \), has density operator

\[
\langle p | \rho_{\text{HO}} | p' \rangle = \frac{x_0}{\hbar \sqrt{\pi}} \frac{1}{\sqrt{2\pi n + 1}} \exp \left[ -\frac{(2\pi + 1) x_0^2}{4\hbar^2} (p - p')^2 \right] \exp \left[ -\frac{x_0^2}{4\hbar^2 (2\pi + 1)} (p + p')^2 \right],
\]

(D.1)

where \( x_0 = \sqrt{\hbar/\langle m\omega \rangle} \), and \( n = \{ \exp [\hbar \omega / (k_B T)] - 1 \}^{-1} \), is the expected number of excitations at temperature \( T \) where \( k_B \) is Boltzmann’s constant. The eigenfunctions of (D.1) are

\[
\int dp' \langle p | \rho_{\text{HO}} | p' \rangle \psi_n (p') = p_n \psi_n (p'),
\]

where

\[
\psi_n (p) = (-i)^n \sqrt{\frac{x_0}{2^n n! \hbar \pi^{1/2}}} \exp \left( \frac{-x_0^2 p^2}{4\hbar^2} \right) H_n \left( \frac{x_0 p}{\hbar} \right),
\]

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and
\[ p_n = \frac{n^n}{(n+1)^{n+1}}, \]
with \( H_n(x) \) the usual Hermite polynomials. Note that
\[ K = \left( \sum_{n=0}^{\infty} p_n^2 \right)^{-1} = 2n + 1, \]
and thus it can be said that the Schmidt number, \( K \), of \( \langle p | \rho_{\text{HO}} | p' \rangle \) is \( 2n + 1 \). The task is now to massage the reduced density operator associated with various products of Gaussian functions into the form of (D.1).

One such product is
\[ \phi(\Omega_1, \Omega_2) = \sqrt{\frac{2}{\pi \sigma_{+} \sigma_{-}}} \exp \left[ -\frac{(\Omega_1 \cos \theta - \Omega_2 \sin \theta)^2}{\sigma_{-}^2} \right] \exp \left[ -\frac{(\Omega_1 \sin \theta + \Omega_2 \cos \theta)^2}{\sigma_{+}^2} \right], \]
(D.2)
for which

\[
\rho_{\text{red}}(\Omega, \Omega') = \int d\Omega_2 \phi(\Omega_1, \Omega_2) \phi^*(\Omega'_1, \Omega_2)
\]

\[
= \frac{2}{\pi \sigma_+ \sigma_-} \int d\Omega_2 \left\{ \exp \left[ -\frac{(\Omega_1 \cos \theta - \Omega_2 \sin \theta)^2 + (\Omega_1' \cos \theta - \Omega_2 \sin \theta)^2}{\sigma_-^2} \right] \times \exp \left[ -\frac{(\Omega_1 \sin \theta + \Omega_2 \cos \theta)^2 + (\Omega_1' \sin \theta + \Omega_2 \cos \theta)^2}{\sigma_+^2} \right] \right\}
\]

\[
= 2 \exp \left[ -\left( \frac{\cos^2 \theta + \sin^2 \theta}{2 \sigma_+^2} \right)(\Omega_1 - \Omega_1')^2 \right]
\]

\[
\times \int d\Omega_2 \left\{ \exp \left[ -\frac{2 \left( \Omega_2 \sin \theta - \frac{(\Omega_1 + \Omega_1')}{2} \cos \theta \right)^2}{\sigma_-^2} \right] \times \exp \left[ -\frac{2 \left( \Omega_2 \cos \theta + \frac{(\Omega_1 + \Omega_1')}{2} \sin \theta \right)^2}{\sigma_+^2} \right] \right\}
\]

\[
= 2 \exp \left[ -\frac{\sigma_+^2 \cos^2 \theta + \sigma_-^2 \sin^2 \theta}{2 \sigma_+^2 \sigma_-^2} (\Omega_1 - \Omega_1')^2 \right]
\]

\[
\times \int d\Omega_2 \exp \left[ -\left( a \Omega_2^2 + b \Omega_2 + c \right) \right] = \sqrt{\frac{2}{\pi \left( \sigma_+^2 \sin^2 \theta + \sigma_-^2 \cos^2 \theta \right)}} \exp \left[ -\frac{\sigma_+^2 \cos^2 \theta + \sigma_-^2 \sin^2 \theta}{2 \sigma_+^2 \sigma_-^2} (\Omega_1 - \Omega_1')^2 \right]
\]

\[
\times \exp \left[ -\frac{1}{2 \left( \sigma_+^2 \sin^2 \theta + \sigma_-^2 \cos^2 \theta \right)} (\Omega_1 + \Omega_1')^2 \right], \quad (D.3)
\]

where the final equality follows from the identity

\[
\int dx \exp \left( -\left( ax^2 + bx + c \right) \right) = \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2 - 4ac}{4a} \right), \quad (D.4)
\]

with

\[
a = 2 \left( \frac{\sigma_+^2 \sin^2 \theta + \sigma_-^2 \cos^2 \theta}{\sigma_+^2 \sigma_-^2} \right)
\]

\[
b = 2 (\Omega_1 + \Omega_1') \sin \theta \cos \theta \left( \frac{\sigma_-^2 - \sigma_+^2}{\sigma_+^2 \sigma_-^2} \right)
\]

\[
c = (\Omega_1 + \Omega_1')^2 \left( \frac{\cos^2 \theta}{2 \sigma_-^2} + \frac{\sin^2 \theta}{2 \sigma_+^2} \right). 
\]
Comparing (D.3) with (D.1), we see that if we set
\[ x_0^4 = \frac{4 \left( \sigma_+^2 \cos^2 \theta + \sigma_-^2 \sin^2 \theta \right)}{\sigma_+^2 \sigma_-^2 \left( \sigma_+^2 \sin^2 \theta + \sigma_-^2 \cos^2 \theta \right)} h^4, \]
and
\[ 2\pi + 1 = \sqrt{\frac{(\sigma_+^2 \cos^2 \theta + \sigma_-^2 \sin^2 \theta) (\sigma_+^2 \sin^2 \theta + \sigma_-^2 \cos^2 \theta)}{\sigma_+ \sigma_-}}, \]
the two expressions agree, and thus (D.2) has a Schmidt number of
\[ K = \sqrt{\frac{(\sigma_+^2 \cos^2 \theta + \sigma_-^2 \sin^2 \theta) (\sigma_+^2 \sin^2 \theta + \sigma_-^2 \cos^2 \theta)}{\sigma_+ \sigma_-}}. \] (D.5)

Note that (D.5) reduces to the more familiar [78]
\[ K = \frac{\sigma_+^2 + \sigma_-^2}{2\sigma_+ \sigma_-}, \]
when \( \theta = \pi/4. \)

Similarly
\[ \phi (\Omega_1, \Omega_2) = \sqrt{\frac{2 \left| \cos \theta_1 \sin \theta_2 - \sin \theta_1 \cos \theta_2 \right|}{\pi \sigma_1 \sigma_2}} \times \exp \left[ -\frac{\left( \Omega_1 \sin \theta_1 + \Omega_2 \cos \theta_1 \right)^2}{\sigma_1^2} \right] \exp \left[ -\frac{\left( \Omega_1 \sin \theta_2 + \Omega_2 \cos \theta_2 \right)^2}{\sigma_2^2} \right], \] (D.6)
has reduced density operator

\[
\rho_{\text{red}}(\Omega_1, \Omega'_1) = \int d\Omega_2 \phi(\Omega_1, \Omega_2) \phi^*(\Omega'_1, \Omega_2)
\]

\[
\frac{2 |\cos \theta_1 \sin \theta_2 - \sin \theta_2 \cos \theta_1|}{\pi \sigma_1 \sigma_2} \times \int d\Omega_2 \exp \left\{ - \frac{(\Omega_1 \sin \theta_1 + \Omega_2 \cos \theta_1)^2 + (\Omega'_1 \cos \theta_1 + \Omega_2 \sin \theta_1)^2}{\sigma_1^2} \right\} \times \exp \left\{ - \frac{(\Omega_1 \sin \theta_2 + \Omega_2 \cos \theta_2)^2 + (\Omega'_1 \sin \theta_2 + \Omega_2 \cos \theta_2)^2}{\sigma_2^2} \right\}
\]

\[
= \frac{2 |\cos \theta_1 \sin \theta_2 - \sin \theta_2 \cos \theta_1|}{\pi \sigma_1 \sigma_2} \exp \left\{ - \frac{\sin^2 \theta_1}{2\sigma_1^2} + \frac{\sin^2 \theta_2}{2\sigma_2^2} \right\} (\Omega_1 - \Omega'_1)^2
\]

\[
\times \int d\Omega_2 \exp \left\{ - \left( a\Omega_2^2 + b\Omega_2 + c \right) \right\}
\]

\[
= \sqrt{2 \frac{(\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1)^2}{\pi (\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1)}} \exp \left\{ - \frac{\sigma_1^2 \sin^2 \theta_2 + \sigma_2^2 \sin^2 \theta_1}{2\sigma_1^2 \sigma_2^2} (\Omega_1 - \Omega'_1)^2 \right\}
\]

\[
\times \exp \left\{ - \frac{(\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1)^2}{2 (\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1)} (\Omega_1 + \Omega'_1)^2 \right\}, \tag{D.7}
\]

where again the last equality follows from (D.4), this time with

\[
a = 2 \left( \frac{\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1}{\sigma_1^2 \sigma_2^2} \right)
\]

\[
b = 2 (\Omega_1 + \Omega'_1) \left( \frac{\sigma_1^2 \sin \theta_2 \cos \theta_1 + \sigma_2^2 \sin \theta_1 \cos \theta_1}{\sigma_1^2 \sigma_2^2} \right)
\]

\[
c = (\Omega_1 + \Omega'_1)^2 \left( \frac{\sin^2 \theta_1}{2\sigma_1^2} + \frac{\sin^2 \theta_2}{2\sigma_2^2} \right).
\]
Comparing (D.7) and (D.1) we see that the two expressions agree if we set

\[ x_0^4 = \frac{4 (\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1)^2 (\sigma_1^2 \sin^2 \theta_2 + \sigma_2^2 \sin^2 \theta_1) (\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1) h^4}{(\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1)^2 \sigma_1^2 \sigma_2^2}, \]

and

\[ 2\pi + 1 = \sqrt{\frac{(\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1) (\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1)}{\sigma_1^2 \sigma_2^2 (\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1)^2}}, \]

making the Schmidt number for (D.6)

\[ K = \sqrt{\frac{(\sigma_1^2 \sin^2 \theta_2 + \sigma_2^2 \sin^2 \theta_1) (\sigma_1^2 \cos^2 \theta_2 + \sigma_2^2 \cos^2 \theta_1)}{\sigma_1^2 \sigma_2^2 (\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1)^2}}. \quad (D.8) \]
Bibliography


[69] The variation of the quality factor between the signal, pump, and idler resonances is below 2% for all four rings. The value we have taken for the quality factor in the equations is an average of the $Q$s of the three resonances.


[71] It was verified that the quadratic trend is maintained up to the maximum available pump power if the pump wavelength is retuned to compensate for the thermo-optic redshift of the resonances for $P_p > 2$ mW. This implies that $Q$ degradation due to free carrier absorption can be neglected for all the investigated $P_p$. The data of Fig. 2 are taken with a fixed value of the pump energy.


[80] Private communication with Dr. Agata Branczyk.


