Reach Control Problems on Polytopes

by

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Abstract

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As control systems become more integrated with high-end engineering systems as well as consumer products, they are expected to achieve specifications that may include logic rules, safety constraints, startup procedures, and so forth. Control design for such complex specifications is a relatively unexplored research area. One possible design approach is based on partitioning the state space into polytopic regions, and then formulating a certain control problem on each polytope, with the intention that the set of all controllers so obtained would collectively achieve the specification. The control problem which must be solved for each polytope is called the reach control problem, and it has been identified as turnkey to the further development of this approach. The reach control problem (RCP) is to find a state feedback to make the closed-loop trajectories of an affine (or linear) control system defined on a polytope reach and exit a prescribed facet of the polytope in finite time. This dissertation studies a number of aspects of the reach control problem, and it uses tools from convex analysis, nonsmooth analysis, and computational geometry for this study.

The dissertation has three main themes. First, we formulate and solve a variant of RCP in which trajectories exit the polytope in a monotonic sense; this provides a triangulation-independent solution of RCP. Second, we develop a Lyapunov-like theory for verifying if RCP is solved using a given candidate controller. This involves the introduction of the notion of generalized flow functions, a LaSalle Principle for RCP, and several converse theorems on existence of generalized flow functions. Third, we study the relationship between affine feedbacks and continuous state feedbacks for RCP on simplices. Although the two feedback classes have been shown to be equivalent under an assumption on the triangulation of the state space, we show by a counterexample that the equivalence is no longer true under arbitrary triangulations. Then we provide for single-input systems a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices.
Dedication

To my parents for your love, guidance, and support throughout my life.
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List of Symbols and Acronyms

General:

\( x := y \)  \( x \) is defined as \( y \)

\( \forall \)  for all

\( \exists \)  there exists

\( \Rightarrow \)  implies

\( \Leftrightarrow \)  equivalent

\( C^1 \)  continuously differentiable

\( U \)  a control class such as open-loop controls, continuous state feedback, affine feedback, etc

Sets and Fields:

\( \mathbb{R} \)  field of real numbers

\( \mathbb{R}^n \)  the set of \( n \) tuples of the real numbers

\( \mathbb{R}^{n \times m} \)  the set of \( n \times m \) matrices with real elements

\( \mathbb{R}_+ \)  the set of non-negative real numbers

\( 2^{\mathbb{R}^n} \)  the power set of \( \mathbb{R}^n \) (the set of all subsets of \( \mathbb{R}^n \))

\( \mathbb{C} \)  field of complex numbers

\( \mathbb{C}^- \)  open left-half complex plane

\( \emptyset \)  the empty set

\( \{0\} \)  the subset of \( \mathbb{R}^n \) containing only the zero vector
$x \in \mathcal{K}$  $x$ is an element of the set $\mathcal{K}$

$\dim(\mathcal{K})$ affine dimension of the set $\mathcal{K}$ (dimension of the affine hull of $\mathcal{K}$)

$\mathcal{K}_1 \subset \mathcal{K}_2$ the set $\mathcal{K}_1$ is contained in the set $\mathcal{K}_2$

$\mathcal{K}_1 \subseteq \mathcal{K}_2$ the set $\mathcal{K}_1$ is contained in or it is equal to the set $\mathcal{K}_2$

$\mathcal{K}_1 \cap \mathcal{K}_2$ the intersection of the sets $\mathcal{K}_1$ and $\mathcal{K}_2$

$\mathcal{K}_1 \cup \mathcal{K}_2$ the union of the sets $\mathcal{K}_1$ and $\mathcal{K}_2$

$\mathcal{K}_1 \setminus \mathcal{K}_2$ elements of the set $\mathcal{K}_1$ not contained in the set $\mathcal{K}_2$

$\mathcal{K}^c$ the complement of $\mathcal{K}$ ($\mathcal{K}^c := \mathbb{R}^n \setminus \mathcal{K}$)

$\overline{\mathcal{K}}$ the closure of $\mathcal{K}$

$\mathcal{K}^\circ$ the interior of $\mathcal{K}$

$\partial \mathcal{K}$ the boundary of $\mathcal{K}$ ($\overline{\mathcal{K}} \setminus \mathcal{K}^\circ$)

$\text{ri } (\mathcal{K})$ the relative interior of $\mathcal{K}$

$\text{rb } (\mathcal{K})$ the relative boundary of $\mathcal{K}$ ($\overline{\mathcal{K}} \setminus \text{ri } (\mathcal{K})$)

$\text{co } (\mathcal{K})$ the convex hull of the set $\mathcal{K}$

$\text{co } \{v_1, v_2, \ldots\}$ the convex hull of a set of points $v_i \in \mathbb{R}^n$

$\text{aff } (\mathcal{K})$ the affine hull of the set $\mathcal{K}$

$\text{aff } \{v_1, v_2, \ldots\}$ the affine hull of a set of points $v_i \in \mathbb{R}^n$

$T_{\mathcal{K}}(x)$ the Bouligand tangent cone to a set $\mathcal{K} \subset \mathbb{R}^n$ at point $x$

$\mathcal{B}$ the open ball of radius 1 centered at the origin

$\mathcal{B}_\delta(x)$ the open ball of radius $\delta$ centered at $x$

**Matrices:**

$A : \mathcal{X} \mapsto \mathcal{Y}$ $A$ maps a vector in domain $\mathcal{X}$ to a vector in codomain $\mathcal{Y}$

$A^T$ transpose of $A$

$A^{-1}$ inverse of a square matrix $A$
\( \sigma(A) \) spectrum of a square matrix \( A \)

\( I \) identity

**Subspaces:**

\( \text{Im} \ A \) image or range of \( A \)

\( \text{Ker} \ A \) kernel or null space of \( A \)

**Vectors and Scalars:**

\( x \cdot y \) the dot product of two vectors \( x, y \in \mathbb{R}^n \)

\( x^T \) transpose of the vector \( x \)

\( L_f V(x) \) the Lie derivative of function \( V : \mathbb{R}^n \to \mathbb{R} \) with respect to function \( f : \mathbb{R}^n \to \mathbb{R}^n \)

**Units:**

\( ^\circ C \) degree Celsius

\( s \) seconds

**Acronyms:**

LMI Linear Matrix Inequality

PWA Piecewise Affine

PWL Piecewise Linear

RCP Reach Control Problem

MRCP Monotonic Reach Control Problem

LP Linear Programming problem

NP Nondeterministic Polynomial-time

w.l.o.g. without loss of generality
Chapter 1

Introduction

In reviewing the last forty years of systems control, one observes that most control methods are focused on achieving stabilization or tracking. Often, these methods do not address the complex control tasks that are faced in industry. Real problems require control specifications that can include logic statements, safety constraints, human intervention, startup procedures, etc [10]. For instance, some industrial processes require control specifications that are temporal in nature; do task $A$ two times and then do task $B$ ad infinitum. As another example, some practical systems require that the system states reach a desired region in the state space while avoiding unsafe regions during the transient period. This requirement cannot be addressed by the specification language used in the stabilization problem (percent overshoot, settling time, etc.). This has highlighted the need for the development of control techniques which achieve complex specifications.

In the 1990’s, researchers in computer science and control systems converged to form a field called hybrid systems [10, 9, 44, 25]. A hybrid system is a dynamical system that combines both discrete event and continuous time behavior. In the last decade, hybrid systems have received special interest for the reason that they more accurately capture the complex models and specifications faced in industry [25, 44]. Interesting applications of hybrid systems have been identified in robotics [6], automotive [32], aerospace [7, 38], communication networks [41], power systems [21, 5], chemical reactors [31], power electronics [24], and so forth.

In this dissertation we develop theoretical foundations for the control for complex specifications. In particular, we study a reachability problem for hybrid systems, namely the reach control problem (RCP) for affine systems on polytopes. The problem is to design a state feedback to force closed-loop trajectories starting anywhere in a polytopic state space to leave the polytope from a prescribed exit
Chapter 1. Introduction

Figure 1.1: A piecewise affine hybrid system

\[
\begin{align*}
\dot{x} &= A_1 x + B_1 u + a_1 \\
\dot{x} &= A_2 x + B_2 u + a_2 \\
\dot{x} &= A_3 x + B_3 u + a_3
\end{align*}
\]

The main motivation behind RCP for affine systems on polytopes is the control of a subclass of hybrid systems called piecewise affine hybrid systems [2, 9, 26]. Piecewise affine systems receive special focus [11, 58, 50, 54] since they can approximate nonlinear dynamics with arbitrary accuracy [58]. Also, there exist several techniques for the identification of these systems from experimental data [33]. A piecewise affine hybrid system consists of a discrete automaton such that each discrete mode is equipped with its own continuous-time affine dynamics defined on a polytope. For example, Figure 1.1 represents a piecewise affine hybrid system consisting of three discrete modes. In each discrete mode, the dynamics of the system is described by \( \dot{x} = A_i x + B_i u + a_i, \) where \( P_i \) is an \( n \)-dimensional polytope, \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the input, \( A_i \in \mathbb{R}^{n \times n}, a_i \in \mathbb{R}^n, \) and \( B_i \in \mathbb{R}^{n \times m} \). When the continuous state trajectory crosses a facet (an \( (n-1) \)-dimensional boundary) of the polytope, the system is transferred to a new discrete mode. For instance, in Figure 1.1 crossing the facet \( F_1 \) of \( P_2 \) may trigger the discrete event \( \alpha_1 \) which moves the system from discrete mode 2 to discrete mode 1, while crossing the facet \( F_3 \) of \( P_2 \) may represent the transition to discrete mode 3. Reach control for piecewise affine hybrid systems requires at each discrete mode to prevent transitions to certain discrete modes, and to force a transition...
to a desired discrete mode. This requirement is translated at the continuous level to force all the state trajectories of a continuous-time affine system defined on a polytope to leave the polytope through a prescribed exit facet in finite time - that is, to solve RCP for an affine system on a polytope [29].

Interesting applications specifically of RCP can include motion of robots in complex environments [6], aircraft and underwater vehicles [7], material transfer system [3], anesthesia [23], genetic networks [8], smart buildings, process control [31], among others [25].

In this chapter we give a motivating scenario for studying RCP, then we provide a literature review on the topic, and finally we show the organization and main contributions of this dissertation.

1.1 Motivating Example

In this section we discuss a motivating scenario for control of complex specifications originally suggested by Elham Semsar-Kazerooni. We explain briefly the example, discuss the complex specifications required in it, show why the control problem cannot be solved using traditional control techniques (stabilization, tracking, or open-loop controls), and discuss briefly how to solve the problem using RCP.

Consider the two tank system shown in Figure 1.2 which supplies water to a chain of chemical processes consisting of three distinct stages that repeat sequentially. To make the outflow water useful for the processes, the temperature of the outflow water should be maintained within a certain temperature range at each stage, and any changes in the outflow temperature should occur at a proper rate.

Let $T_0$ and $T_2$ represent the temperatures, in degree Celsius ($^\circ$C), of the inflow and outflow water of the
first and second tanks, respectively. For simplicity we assume that the heater is off, and so the outflow temperature is controlled only by the inflow temperature. The system model for this case is [20]:

\[ T_2(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} T_0(s), \]

where \( \tau_1 \) and \( \tau_2 \) are the time constants of the two tanks, respectively (\( \tau_1 = 10 \) seconds, \( \tau_2 = 50 \) s).

Let \( x_1 = T_2, \ x_2 = \dot{x}_1, \) and \( u = T_0. \) Then the state space model is

\[
\dot{x} = \begin{bmatrix}
0 & 1 \\
-0.002 & -0.12
\end{bmatrix} x + \begin{bmatrix}
0 \\
0.002
\end{bmatrix} u. \tag{1.1}
\]

Next we summarize the control objectives in the following points.

1. The temperature of the outflow water should not exceed \( 80^\circ C. \)
2. The rate of change in the outflow temperature is constrained as follows: \( -0.6 \leq \dot{T}_2 \leq 1.4, \ \dot{T}_2 \leq \frac{T_2}{50} \) if \( \dot{T}_2 \geq 0, \) and \( \dot{T}_2 \geq -\frac{T_2}{100} \) if \( \dot{T}_2 \leq 0. \)
3. At each respective stage, \( T_2 \) should lie within the following temperature regions:
   - Stage 1: \( 50^\circ C < T_2 < 60^\circ C; \)
   - Stage 2: \( 60^\circ C < T_2 < 70^\circ C; \)
   - Stage 3: \( 45^\circ C < T_2 < 50^\circ C. \)

Also, the two tank system has the following specifications:

1. No cooler is available in the system. Both \( T_0 \) and \( T_2 \) are above the ambient temperature \( 25^\circ C. \)
2. The container providing the inflow water can provide water with \( 70^\circ C \) at most.

The intersection of the state constraints described above gives the feasible region of the state space shown in Figure 1.3. First, we try to solve the problem using a traditional control approach. In particular, we define a constant reference signal \( r(t) \) for each stage, then we design a tracking controller for each stage to make the output temperature \( T_2 \) follow the constant setpoint \( r \) of the stage. To that end, we define the error signal:

\[
e(t) = r(t) - T_2(t) = r(t) - x_1(t), \tag{1.2}
\]

where \( r \) is the desired set point for the stage under consideration. It is required to find \( u(x) \) to make \( e(t) \to 0 \) as \( t \to \infty. \) By following a standard procedure for designing tracking controllers (see Chapter 4
Figure 1.3: The feasible region of the state space in the two tank example.

We get the following control law:

\[ u(x) = 500(0.002x_1 + 0.12x_2 - v), \]

where \( v \) is of the form

\[ v = -0.02(r_i - x_1) - 0.3(-x_2), \]

and \( r_i \) is the setpoint for stage \( i = 1, \cdots, 3 \). For our control, we define the desired setpoints to be \( r_1 = 55, \ r_2 = 65, \) and \( r_3 = 47 \), which are roughly the midpoints of the three stages.

The system behavior under \( u(x) \) for stage 1 is shown in Figure 1.4. Although the state trajectories initiated in the feasible region eventually tend to the desired region \( (50^\circ C \leq T_2 \leq 60^\circ C) \), they leave the feasible region during the transient period, and so the state constraints are violated. The same problem appears in stages 2, 3.

Based on the above discussion, the required control specifications cannot be achieved using traditional control techniques (stabilization or tracking) since using these techniques there is no guarantee that for any initial condition in the feasible region, the state trajectory will remain in the feasible region. For open-loop controls, it is extremely difficult to find for each initial condition a corresponding open-loop control sequence.

Instead, we use the reach control problem (RCP) to solve this example. The idea is to divide the feasible region in Figure 1.3 into convex polytopes. Then design controllers, one for each polytope, to satisfy the state constraints and to enforce that trajectories periodically visit the regions corresponding
Figure 1.4: Stage1: System behavior under the traditional tracking controller $u(x)$.

to the three stages. A complete solution is provided in Chapter 6.

1.2 Literature Review

Although reachability problems have been studied since the 1960’s, the reach control problem for affine systems on polytopes was formulated recently. The problem was first introduced by Luc Habets and Jan van Schuppen [27]. The main developments on RCP are illustrated in the following tree. We add a keyword for each paper to help the reader follow the tree.
Chapter 1. Introduction

(HvS01,[27]) Problem Formulation

| (HvS04,[28]) Invariance Conditions
| (HvS06,[29]),(RB06,[52]) Affine Feedback

| (B10,[14]) | (KB11,[34]),(KB12,[35]) | (LB11,[43]) | (HvS12,[30]) |
| Geometric Conditions | Reach Controllability | RCP on Polytopes | Output Feedback |
| (BG12,[15]) |
| Discontinuous PWA Feedback |
| (AB12,[4]) | (KB12,[36]) |
| Time-varying Robustness of RCP Feedback |

Now we explain the contributions of each of these papers and their relationship to each other.

- Habets and van Schuppen, 2001 (HvS01,[27])
  Habets and van Schuppen formulated the reach control problem (also known as the control-to-facet problem).

- Habets and van Schuppen, 2004 (HvS04,[28])
  The objective of the paper was to find a continuous piecewise affine (PWA) feedback solving RCP on a polytope \( \mathcal{P} \). First, the paper introduced the invariance conditions to ensure that trajectories do not exit the polytope from facets which are not designated as exit facets. These conditions are easily solved at each vertex using a linear program. Second, the invariance conditions were shown to be necessary for solvability of the problem by continuous state feedback (reviewed in Theorem 4.1.1). Third, a second sufficient condition was given to enforce trajectories to exit the polytope by imposing that the vector field has a positive component in the direction of \( \mathbf{h}_0 \), the unit normal vector to the exit facet pointing outside the polytope (reviewed in Theorem 4.1.3). Fourth, the paper provided an elegant method to compute an affine feedback on a simplex using only the
control values at the vertices of the simplex (reviewed in Lemma 3.2.1). To find the continuous PWA feedback: first, simultaneously solve the invariance conditions and the sufficient condition imposing a positive velocity in the direction $h_0$; second, triangulate the polytope $\mathcal{P}$ into simplices; finally, construct an affine feedback on each simplex.

- Habets, Collins, and van Schuppen, 2006 (HvS06,[29]), Roszak and Broucke, 2006 (RB06,[52])

It was observed that RCP on polytopes is a very difficult problem to study, and so the focus turned in these two papers to a simpler problem, RCP on simplices. First, they introduced the modern formulation of RCP, relaxing a restriction of [27, 28] that trajectories must exit upon reaching the exit facet for the first time (see Problem 3.1.1). Second, the problem was generalized to reach a set of facets of the simplex (not necessarily a single exit facet). Third, necessary and sufficient conditions were obtained to solve RCP on simplices by affine feedback (reviewed in Theorem 3.2.2). In [52], invariance conditions were combined with a flow condition to guarantee that trajectories exit the simplex. The flow condition is that the vector field at vertices of the simplex has positive components in a common direction (not necessarily $h_0$). It was shown this flow condition is equivalent to the statement that the simplex does not contain closed-loop equilibria. Fourth, the flow condition in [52] was first expressed by bilinear inequalities, but a general algorithm was presented to convert the bilinear inequalities to a series of linear programming (LP) problems whose number increases exponentially with the system dimension.

- Broucke, 2010 (B10,[14])

The shortcoming of the results of [29, 52] was that checking the flow condition is computationally expensive. To solve this problem, a new trend appeared in [14]; it is to study the geometric conditions for solvability of RCP. In particular, the goal of [14] was to replace the flow condition by a condition depending directly on the system data and the simplex data. To make the synthesis method tractable, the paper proposed a preferred triangulation of the polytopic state space with respect to $\mathcal{O}$, the set of possible equilibria of the system. Specifically, for any simplex $\mathcal{S}$ of the triangulation, $\mathcal{S} \cap \mathcal{O}$ is either the empty set or a face of $\mathcal{S}$ (the triangulation procedure is reviewed in Section 2.3.3). The first result of the paper was to provide geometric conditions for RCP on simplices by affine feedback which are stated directly in terms of the problem data (reviewed in Theorems 3.2.3, 3.2.4). The second result of the paper was to prove that under the preferred triangulation, continuous state feedback and affine feedback are equivalent from the point of view of solvability of RCP on simplices.

- Broucke and Ganness, 2012 (BG12,[15])
The paper studied under the preferred triangulation of [14] solvability of RCP on simplices by discontinuous state feedbacks for the case where continuous feedbacks fail to solve the problem. First, the paper introduced the reach control indices, which embody system structure that causes RCP to fail using continuous state feedbacks. Second, it proposed a triangulation algorithm that partitions the given simplex into sub-simplices and an associated discontinuous PWA feedback solving RCP on the simplex when continuous feedbacks fail. Third, it proved that under the preferred triangulation of the polytopic state space in [14], the class of discontinuous PWA feedbacks is the largest class of feedbacks to solve RCP on simplices in the sense that RCP is solvable by discontinuous PWA feedback if and only if it is solvable by open-loop controls.

- **Ashford and Broucke, 2012 (AB12,[4])**
  The paper studied under the preferred triangulation of [14] solvability of RCP on simplices by time-varying feedback for the case where continuous state feedbacks fail to solve the problem. It also relies on the reach control indices of [15]. The main practical reason for studying time-varying feedback is to avoid chattering that may appear using the discontinuous PWA feedback of [15], especially in the presence of measurement errors. This paper concluded the study of RCP on simplices under the preferred triangulation of [14].

- **Kazerooni and Broucke, 2012 (KB12,[36])**
  In this paper a study on the robustness of RCP is considered. It is shown that the affine and PWA feedbacks introduced in [14], [15] for solving RCP on simplices behave well in the presence of small perturbations of system parameters.

- **Kazerooni and Broucke, 2011 (KB11,[34]), 2012 (KB12,[35])**
  These papers studied solvability of RCP on simplices by affine feedback for single-input systems when the triangulation assumption of [14] is not achieved (the set $\mathcal{O}$ intersects the interior of the simplex). First, they discovered that for this case equilibria can only appear on the boundary of the simplex (reviewed in Theorem 8.2.1). Second, they initiated the concept of reach controllability which quantifies how the control inputs affect the dynamics on faces of the simplex. Third, they gave an evidence that reach controllability is a geometric property by providing a decomposition of the dynamics into face dynamics and transversal dynamics (reviewed in Lemma 8.2.3). Fourth, reach controllability was used to obtain novel necessary and sufficient conditions for solvability of RCP by affine feedback. Unlike the flow condition, reach controllability can be verified directly from the problem data.
- Lin and Broucke, 2011 (LB11,[43])

The paper studied RCP for affine hypersurface systems on polytopes (hypersurface systems have \( n - 1 \) inputs, where \( n \) is the system dimension). The objective was to find the largest class of feedbacks needed to solve RCP (RCP is solvable by a feedback of this class if the problem is solvable by open-loop controls). First, the paper proposed methods of subdivision, triangulation, and covers specially tailored to the control problem. Second, it provided a synthesis method for discontinuous PWA feedbacks solving RCP on polytopes. A solution to RCP using discontinuous PWA feedback is obtained if the problem is solvable using open-loop controls.

- Habets, Collins, and van Schuppen, 2012 (HvS12,[30])

In this paper solvability of RCP on polytopes by static output feedbacks is studied. In particular, the paper generalizes the results of [28], [29] to the case of partial state observations.

Based on the above review, one observes the following:

- The most definitive results on RCP are focused on reach control on simplices by affine feedback [29, 52, 14, 34]. Indeed, there is an emerging belief in the literature that the right way to solve RCP on polytopes by PWA feedback is to firstly triangulate the polytope into simplices, then to reformulate the original reachability problem on the polytope to a sequence of reachability problems on the simplices of the triangulation. The latter can be solved efficiently by affine feedbacks using existing simplex methods. We review this indirect approach in Section 3.3. However, we show in Example 3.3.1 that solvability using this approach depends mainly on the choice of triangulation, which may lead in some examples to an iterative method where one tries all possible triangulations until a triangulation works. Also, in Example 3.3.2 we show this approach may fail, yet the original reachability problem on the polytope is solvable. Therefore, we directly study the original problem, RCP on polytopes. In Chapter 5 we formulate the monotonic reach control problem (MRCP), which provides a novel technique for solving RCP on polytopes. One advantage of MRCP over the existing simplex-based approach is that it provides a solution for RCP on polytopes that can be implemented using any choice of triangulation (Section 5.3). The latter is useful to avoid iteration in triangulating the polytope, particularly when triangulation is performed by a standalone software not adapted to control problems.

- Past research on reach control on polytopes has either required strong sufficient conditions [28] or restrictive assumptions on the system dynamics [43]. In this dissertation we initiate a study of RCP on polytopes in which such restrictions are removed; instead geometric properties of
the system are exploited to the best possible extent. In particular, the location of the set of possible equilibria of the affine system, $O$, with respect to the polytope $\mathcal{P}$ plays a key role, and in some geometric situations clear necessary and sufficient conditions for the multi-input systems are obtained (Theorems 5.2.2, 5.2.3).

- Existing numerical algorithms for RCP are for simplices and they require solving a large number of LP problems which increases exponentially with the system dimension [29], [52]. By exploring the geometry of the problem, we are able to provide in Section 5.4 an efficient algorithm for RCP on polytopes for single-input systems that requires solving a small number of LP problems which increases linearly with the number of vertices of $\mathcal{P}$.

- By comparing the literature of RCP on polytopes to the one of stability, one observes that in RCP we lack a verification tool that can be used to analyze solvability of the problem by a given continuous state feedback without calculating the closed-loop trajectories. Indeed, we have found many examples in which a given continuous state feedback is verified to solve the problem via exhaustive simulation of the closed-loop system, yet no existing technique can explain why RCP is solved. Motivated by these examples, we provide in Chapter 7 a verification tool for RCP, analogous to Lyapunov theory for stability, that can be used to analyze whether an instance of RCP is solved, without resorting to exhaustive simulation.

- The equivalence of affine feedbacks and continuous feedbacks for RCP on simplices has been shown under an assumption on the triangulation of the state space [14]. There remains the question of whether this equivalence holds under arbitrary triangulations. In Chapter 8 we show that the answer for this question is negative by constructing a counterexample in which affine feedbacks fail to solve RCP, yet a continuous state feedback solves it (Section 8.1). Then we identify an alternative continuous feedback class to solve RCP when affine feedbacks fail (Section 8.3).

1.3 Organization

The dissertation is organized as follows. The next chapter provides a mathematical background including notations, functions, geometric background (simplices, polytopes, and triangulation), convex analysis, and nonsmooth analysis. In Chapter 3 we review basic results for solvability of RCP on simplices, particularly by affine feedback (Section 3.2). Then we show how to use existing simplex methods to solve RCP on a polytopic state space (Section 3.3). Finally, we show via illustrative examples some drawbacks of existing simplex methods when applied to RCP on polytopes (Examples 3.3.1, 3.3.2).
Motivated by these examples, we start in Chapter 4 a direct study of RCP on polytopes. In particular, we firstly formulate the reach control problem on polytopes (Section 4.1). Then we develop novel necessary conditions for RCP on polytopes by open-loop controls (Theorem 4.2.1) and by continuous state feedback (Lemma 4.2.2 and Theorem 4.2.3).

Armed with this set of necessary conditions, we continue in Chapter 5 the study of RCP on polytopes following the geometric point of view of [14]. In particular, in this chapter we extend the geometric conditions for simplices in [14] to analogous conditions for polytopes (Lemmas 5.1.3, 5.1.4). Then we formulate the monotonic reach control problem (MRCP) (Problem 5.2.1), a problem in which we incorporate the flow condition into the problem statement of RCP. Necessary and sufficient conditions for solvability of MRCP are studied for three geometric cases (Theorems 5.2.2, 5.2.3, 5.2.4). Then we show the relationship between MRCP and solvability of RCP by arbitrary triangulations (Theorem 5.3.1). After that we present in Section 5.4 an algorithm for the solvability of MRCP by continuous PWA feedback. Finally, we provide several illustrative examples including examples that compare MRCP to existing simplex methods (Examples 5.5.1, 5.5.2, 5.5.3) and an example in which RCP is solvable but not by any known technique (Example 5.5.4). Then in Chapter 6 we provide a practical example: the two tank temperature control problem to show some practical advantages of MRCP over existing simplex methods.

Motivated by Example 5.5.4, we propose in Chapter 7 a verification tool for the analysis of RCP, analogous to Lyapunov theorem for stability. Specifically, we introduce the notion of generalized flow conditions, which give a necessary and sufficient condition for closed-loop trajectories to exit the polytope (Theorem 7.1.1). Then we provide a set of results to analyze whether an instance of RCP is solved, without resorting to exhaustive simulation of the closed-loop system (Theorems 7.1.2, 7.1.4). This includes a variant of the LaSalle Principle tailored to RCP (Theorem 7.2.1). After that we propose a suitable class of functions to generate a generalized flow condition when using PWA feedback (Theorems 7.3.1, 7.3.3). Then in Section 7.4 we provide several examples illustrating the novel verification tool.

In the previous chapters, our main focus was on solvability of RCP on polytopes by continuous PWA feedbacks. The question arises of whether continuous PWA feedbacks is the largest continuous feedback class needed to solve RCP on polytopes. As a first step in answering this question, we focus in Chapter 8 on simplices and study the relationship between affine feedbacks and continuous state feedbacks for RCP on simplices under arbitrary triangulations. Using the results presented in Chapter 7, we are able to construct an example for which no solution based on affine feedback exists, yet a continuous state feedback solves the problem (Section 8.1). Then we provide for single-input systems an alternative feedback class for RCP on simplices for the case where affine feedbacks fail to solve the problem. In
particular, we firstly explore in Section 8.2 the geometric properties of the set of open-loop equilibria in the simplex. Then using these properties and the results of Chapter 7, we provide in Section 8.3 a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices. We conclude this chapter by providing illustrative examples of the multi-affine feedback synthesis method (Section 8.4).

Finally, Chapter 9 summarizes the dissertation, and shows the future directions of research in RCP.

1.4 Main Contributions

The dissertation develops control methods for RCP. We summarize the main contributions of the dissertation in the following points:

1. Chapter 4: Novel necessary conditions for RCP on polytopes

   - Invariance conditions are not necessary for RCP on polytopes by open-loop controls, Example 4.2.1. ¹
   - For a special class of polytopes, called simple polytopes (reviewed in Section 2.3.2), invariance conditions are necessary for RCP on polytopes by open-loop controls, Theorem 4.2.1.
   - Geometric necessary conditions for RCP on polytopes by continuous state feedback, Lemma 4.2.2 and Theorem 4.2.3.

2. Chapter 5: Monotonic reach control problem (MRCP)

   - Problem Formulation, Problem 5.2.1.
   - Study MRCP for three geometric situations, Theorems 5.2.2, 5.2.3, 5.2.4.
   - Provide efficient algorithm for solvability of MRCP for single-input systems, Section 5.4.
   - For generic polytopes, MRCP is equivalent to solvability of RCP for any choice of triangulation, Theorem 5.3.1.

3. Chapter 7: Introduce the notion of generalized flow conditions, a verification tool for the analysis of RCP.

   - Conditions for leaving a polytope in finite time, Theorems 7.1.1, 7.1.2, 7.1.4.
   - A LaSalle Principle for RCP, Theorem 7.2.1.

¹ Example 4.2.1 was suggested by Zhiyun Lin. In this dissertation, we develop the example, and analyze it in greater depth.
• A class of functions that gives rise to a generalized flow condition for PWA feedback, Theorem 7.3.1.

• An LP-based computational method for finding a generalized flow function, Corollary 7.3.2.

• Chain of simplices, Theorem 7.3.3.

4. Chapter 8: Study the limits of affine feedbacks for RCP on simplices.

• Counterexample for the equivalence of affine feedbacks and continuous feedbacks for RCP on simplices under arbitrary triangulations, Section 8.1.

• Multi-affine feedback for RCP on simplices, Theorem 8.3.3.
Chapter 2

Background

In this chapter we review the preliminary mathematical definitions and results needed for the reach control problem. In Section 2.1 we introduce the notations used in the dissertation. Section 2.2 reviews functions. Section 2.3 provides a geometric background including simplices, polytopes, and triangulation. In Section 2.4 we review convex analysis. The background concludes with a review of nonsmooth analysis in Section 2.5.

2.1 Notation

We write $x := y$ to denote that $x$ is defined as $y$. The notation $\forall$ denotes for all, while the notation $\exists$ denotes there exists. The notation $A \Rightarrow B$ denotes that statement $A$ implies statement $B$, and the notation $A \Leftrightarrow B$ denotes that statement $A$ is equivalent to statement $B$. The notation $C^1$ denotes continuously differentiable, and the symbol $U$ represents a control class such as open-loop controls, continuous state feedback, affine feedback, etc.

Let $\mathbb{R}$ denote the field of real numbers, and let $\mathbb{R}^n$ denote the set of $n$ tuples of the real numbers. As such it is an $n$-dimensional vector space with the real numbers as the field of scalars. The notation $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ matrices with real numbers, and the notation $\mathbb{R}_+$ denotes the set of nonnegative real numbers. The notation $2^{\mathbb{R}^n}$ denotes the power set of $\mathbb{R}^n$, the set of all subsets of $\mathbb{R}^n$. The notation $\mathbb{C}$ denotes the field of complex numbers, and the notation $\mathbb{C}^-$ denotes the open left-half complex plane. The notation $\emptyset$ denotes the empty set, while the notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $B$ denotes the open ball of radius 1 centered at the origin, and the notation $B_\delta(x)$ denotes the open ball of radius $\delta$ centered at $x$.

Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The notation $x \in \mathcal{K}$ denotes that $x$ is an element of the set $\mathcal{K}$. The notation
dim(K) denotes the affine dimension of the set K, which is the dimension of the affine hull of the set K (the smallest affine set containing K). The notation \( K_1 \subset K_2 \) denotes that the set \( K_1 \) is contained in the set \( K_2 \), and the notation \( K_1 \subseteq K_2 \) denotes that the set \( K_1 \) is contained in or it is equal to the set \( K_2 \). The notation \( K_1 \cap K_2 \) denotes the intersection of the sets \( K_1 \) and \( K_2 \), and \( K_1 \cup K_2 \) denotes the union of the sets \( K_1 \) and \( K_2 \). The notation \( K_1 \setminus K_2 \) denotes the elements of the set \( K_1 \) not contained in the set \( K_2 \). For the set \( K \), the complement of \( K \) is \( K^c := R^n \setminus K \), the closure is \( \overline{K} \), the interior is \( K^\circ \), and the boundary is \( \partial K := K \setminus K^\circ \). The relative interior is denoted \( \text{ri}(K) \) and the relative boundary of \( K \), denoted \( \text{rb}(K) \) is \( K \setminus \text{ri}(K) \). The notation \( \text{co}(K) \) denotes the convex hull of the set \( K \) (the smallest convex set containing \( K \)), and \( \text{aff}(K) \) denotes the affine hull of the set \( K \). The notation \( \text{co}\{v_1, v_2, \ldots\} \) denotes the convex hull of a set of points \( v_i \in R^n \), while \( \text{aff}\{v_1, v_2, \ldots\} \) denotes the affine hull of a set of points \( v_i \in R^n \). The notation \( T_K(x) \) denotes the Bouligand tangent cone to the set \( K \) at point \( x \).

Let \( A : \mathcal{X} \to \mathcal{Y} \) denote a matrix that maps a vector in domain \( \mathcal{X} \) to a vector in codomain \( \mathcal{Y} \). The notation \( A^T \) denotes the transpose of the matrix \( A \), and \( \|A\| \) denotes the induced norm of the matrix \( A \). If \( A \) is a square matrix, then \( A^{-1} \) denotes the inverse of \( A \), and \( \sigma(A) \) denotes the spectrum of \( A \). Let \( \text{Im} \ A \) denote the image or range of \( A \), and let \( \text{Ker} \ A \) denote the kernel or null space of \( A \). The notation \( I \) denotes the identity matrix.

For vectors \( x, y \in R^n \), \( x^T \) denotes the transpose of \( x \), \( \|x\| \) denotes the Euclidean norm of \( x \), and \( x \cdot y \) denotes the dot product of the two vectors. The notation \( L_f V(x) \) denotes the Lie derivative of function \( V : R^n \to R \) with respect to function \( f : R^n \to R^n \).

### 2.2 Functions

In this section we review the definitions of some classes of functions that will be used in the feedback laws solving RCP throughout the dissertation.

**Definition 2.2.1 ([53]).** Consider two sets \( \mathcal{X} \) and \( \mathcal{Y} \). A function \( f : \mathcal{X} \to \mathcal{Y} \) is a unique assignment to each element \( x \in \mathcal{X} \) of an element \( f(x) \in \mathcal{Y} \).

**Definition 2.2.2 ([53]).** A function \( f : \mathcal{X} \to \mathcal{Y} \) is continuous at \( x_0 \in \mathcal{X} \) if

\[
(\forall \epsilon > 0) \ (\exists \delta > 0) \ (\forall x \in \mathcal{X}) \ x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0)).
\]

A function \( f : \mathcal{X} \to \mathcal{Y} \) is continuous if it is continuous at every \( x \in \mathcal{X} \).
Chapter 2. Background

**Definition 2.2.3** ([53]). A function \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is Lipschitz continuous at \( x_0 \in \mathcal{X} \) if

\[
(\exists \delta, L > 0) \ (\forall x \in \mathcal{X}) \ x \in B_\delta(x_0) \Rightarrow \| f(x) - f(x_0) \| \leq L \| x - x_0 \|.
\]

We call \( L \) a Lipschitz constant at \( x_0 \). If \( f \) is Lipschitz continuous at every \( x \in \mathcal{X} \), then \( f \) is said to be locally Lipschitz on \( \mathcal{X} \).

A locally Lipschitz function is a continuous function, but the converse is not always true.

Now we define linear and affine functions, which are used to solve RCP on simplices (Chapter 3).

**Definition 2.2.4** ([17, 53]). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is linear if for every \( x, y \in \mathbb{R}^n \) and \( \alpha, \beta \in \mathbb{R} \), we have:

\[
f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).
\]

**Definition 2.2.5** ([17, 53]). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is affine if for every \( x, y \in \mathbb{R}^n \) and \( \alpha \in \mathbb{R} \), we have:

\[
f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha) f(y).
\]

Any linear function is affine, but the converse is not true.

**Remark 2.2.1.** If \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is affine, then there exist a matrix \( F \in \mathbb{R}^{m \times n} \) and a vector \( g \in \mathbb{R}^m \) such that \( f(x) = Fx + g, \ x \in \mathbb{R}^n \).

Now we review piecewise affine (PWA) functions that are used to solve RCP on polytopes in Chapters 5, 7.

**Definition 2.2.6** ([46]). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a piecewise affine function if there exist finite number of sets \( \Omega_1, \ldots, \Omega_L \) such that \( \bigcup_{i=1}^L \Omega_i = \mathbb{R}^n \), and \( f(x) \) is affine on each \( \Omega_i \). In particular, for each \( i = 1, \ldots, L \), there exist \( F_i \in \mathbb{R}^{m \times n} \) and \( g_i \in \mathbb{R}^m \) such that \( f(x) = F_ix + g_i, \ x \in \Omega_i \). Also, \( f \) is continuous piecewise affine function if additionally:

\[
F_ix + g_i = F_jx + g_j, \ x \in \Omega_i \cap \Omega_j, \ i, j \in \{1, \ldots, L\}.
\]

We review multi-affine functions which are used to solve RCP on simplices in Chapter 8.

**Definition 2.2.7** ([8]). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is multi-affine if it is a polynomial in the indeterminates \( x_1, \ldots, x_n \) with the property that the degree of \( f \) in any of the indeterminates \( x_1, \ldots, x_n \) is either 0 or
1. Equivalently, $f$ has the form:

\[ f(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \in \{0,1\}} c_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \]

with $c_{i_1, \ldots, i_n} \in \mathbb{R}^m$ for all $i_1, \ldots, i_n \in \{0,1\}$ and using the convention that if $i_k = 0$, then $x_k^{i_k} = 1$.

Figure 2.1 shows the relationship between the different classes of functions defined above.

### 2.3 Geometric Background

In this dissertation we study the reach control problem on polytopes (Chapters 4 - 7) and on simplices (Chapters 3, 8). Hence, it is important to review their geometric properties that we use throughout the dissertation.

#### 2.3.1 Simplices

A simplex is a generalization of the notion of a triangle to arbitrary dimension. Let $V := \{v_0, \ldots, v_n\}$ be a set of $n + 1$ points in $\mathbb{R}^n$. We say $\{v_0, \ldots, v_n\}$ are *affinely independent* if they do not lie in an $(n-1)$-dimensional plane in $\mathbb{R}^n$. Equivalently, $\{v_0, \ldots, v_n\}$ are affinely independent if $\{v_1 - v_0, \ldots, v_n - v_0\}$ are linearly independent. An $n$-dimensional simplex is the convex hull of $n + 1$ affinely independent
points in $\mathbb{R}^n$. Suppose $\{v_0, \cdots, v_n\}$ are affinely independent, and define the $n$-dimensional simplex

$$S := \text{co} \{v_0, \cdots, v_n\}.$$

Figure 2.2 shows a simplex in $\mathbb{R}^2$. A face of $S$ is any sub-simplex of $S$ which makes up its boundary. A facet is an $(n-1)$-dimensional face of $S$. We denote the facets of $S$ by $F_0, \cdots, F_n$. Our numbering convention is that each facet is indexed by the vertex it does not contain. Let $h_i$ denote the unit normal vector to $F_i$ pointing outside $S$. It is possible to define the simplex $S$ using these normal vectors. In particular, there exist $\alpha_0, \cdots, \alpha_n \in \mathbb{R}$ such that

$$S = \{ x \in \mathbb{R}^n \mid h_i \cdot x \leq \alpha_i, \forall i \in \{0, \cdots, n\} \}.$$

The following lemma summarizes useful properties of simplices.

**Lemma 2.3.1** ([23]). Let $S$ be an $n$-dimensional simplex. Then the following hold:

(i) If $x \in \text{co} \{v_1, \cdots, v_k\}$, then $x \in F_j$ for $k + 1 \leq j \leq n$;

(ii) $h_j \cdot (v_i - v_0) = 0$, for all $1 \leq i, j \leq n$ and $j \neq i$;

(iii) $h_j \cdot (v_i - v_k) = 0$, for all $0 \leq i, k \leq n$ and $j \neq i, k$;

(iv) $h_i \cdot (v_i - v_0) < 0$, for all $1 \leq i \leq n$;

(v) $h_j \cdot (v_i - x) > 0$, for all $x \in S \setminus F_j$ and $1 \leq i, j \leq n$, $i \neq j$;

(vi) $h_0 \cdot (v_i - v_0) > 0$, for all $1 \leq i \leq n$;

(vii) The vectors $\{v_1 - v_0, \cdots, v_n - v_0\}$ are a basis for $\mathbb{R}^n$. 

(viii) The vectors \( \{ h_1, \cdots, h_n \} \) are a basis for \( \mathbb{R}^n \);

(ix) There exist \( \gamma_1 > 0, \cdots, \gamma_n > 0 \) such that \( h_0 = -\gamma_1 h_1 - \cdots - \gamma_n h_n \).

### 2.3.2 Polytopes

An \( n \)-dimensional polytope is the convex hull of a finite set of points in \( \mathbb{R}^n \) [13]. In particular, let \( \{ v_1, \cdots, v_p \} \) be a set of points in \( \mathbb{R}^n \), where \( p > n \), and suppose that \( \{ v_1, \cdots, v_p \} \) contains \( (n+1) \) affinely independent points. We define the \( n \)-dimensional polytope

\[
\mathcal{P} := \text{co} \ \{ v_1, \cdots, v_p \}.
\]

Clearly, a simplex is a special case of polytopes in which \( p = n + 1 \). Figure 2.3 shows a polytope in \( \mathbb{R}^2 \).

A face of \( \mathcal{P} \) is any sub-polytope of \( \mathcal{P} \) which makes up its boundary. The polytope \( \mathcal{P} \) itself is considered as a trivial face, and all other faces (of dimension less than \( n \)) are called proper faces. An edge of \( \mathcal{P} \) is a 1-dimensional face of \( \mathcal{P} \). A facet of \( \mathcal{P} \) is an \( (n-1) \)-dimensional face of \( \mathcal{P} \). We denote the facets of \( \mathcal{P} \) by \( \mathcal{F}_0, \cdots, \mathcal{F}_r \). Let \( h_i \) denote the unit normal vector to \( \mathcal{F}_i \) pointing outside \( \mathcal{P} \). An implicit description of \( \mathcal{P} \) can be obtained using the normal vectors. Precisely, there exist \( \alpha_1, \cdots, \alpha_r \) such that

\[
\mathcal{P} = \{ x \in \mathbb{R}^n \mid h_i \cdot x \leq \alpha_i, \ \forall i \in \{0, \cdots, r\} \}.
\]

In the following part, we review special types of polytopes that we will use in Chapters 4 and 5. First, a simplicial polytope is a polytope whose proper faces are simplices. Second, we review generic polytopes [16]. A set of \( p > n \) points in \( \mathbb{R}^n \) are in general position if any \( (n+1) \) points of them are affinely independent (form an \( n \)-dimensional simplex). A generic polytope is the convex hull of a set of points in
general position in $\mathbb{R}^n$. For a generic polytope, all proper faces are simplices. Notice that any generic polytope is simplicial, while the converse is not true. Third, an $n$-dimensional polytope $\mathcal{P}$ is said to be *simple* if each $k$-dimensional face of $\mathcal{P}$ is contained in exactly $n-k$ facets.

**Remark 2.3.1.** If $\mathcal{P}$ is an $n$-dimensional simple polytope, then $\mathcal{P}$ has the following properties [13]:

(i) Each vertex of $\mathcal{P}$ is contained in exactly $n$ edges.

(ii) Let $\mathcal{F}$ be a facet of $\mathcal{P}$ and $v$ a vertex of $\mathcal{P}$ in $\mathcal{F}$. Then there are exactly $n-1$ edges in $\mathcal{F}$ containing $v$.

### 2.3.3 Triangulation

Triangulation plays a key role in the control synthesis of PWA feedbacks for RCP on polytopes (Sections 3.3, 5.3, and 5.4).

**Definition 2.3.1** ([40]). A triangulation $\mathcal{T}$ of an $n$-dimensional polytope $\mathcal{P}$ is a finite collection of $n$-dimensional simplices $S_1, \cdots, S_L$ such that (i) $\mathcal{P} = \bigcup_{i=1}^{L} S_i$, (ii) For all $i, j \in \{1, \cdots, L\}$ with $i \neq j$, the intersection $S_i \cap S_j$ is either empty or a common face of $S_i$ and $S_j$.

Let $\mathcal{P}$ be an $n$-dimensional polytope, and let $\mathcal{O}$ be an affine space with dimension less than $n$. Also, suppose $\mathcal{O}_\mathcal{P} := \mathcal{P} \cap \mathcal{O}$ is a polytope with vertices $V_\mathcal{O} = \{o_1, \cdots, o_r\}$. In the following part, we review the placing triangulation, which is used to *triangulate $\mathcal{P}$ with respect to $\mathcal{O}$* - that is, to triangulate $\mathcal{P}$ such that $\mathcal{O}_\mathcal{P}$ is the union of lower dimensional simplices of the triangulation. In the context of RCP, the placing triangulation was used in [14] to triangulate the polytopic state space with respect to the set of possible equilibria, $\mathcal{O}$. In this dissertation we use the preferred triangulation of [14] to construct a continuous PWA feedback excluding equilibria in the polytope (Theorem 5.1.2). Also, we use the placing triangulation in the proof of Theorem 5.3.1. Before we present the placing triangulation method, we review a few definitions.

Suppose $V$ is a finite set of points such that $\mathcal{P} = \text{co} (V)$ is a $k$-dimensional polytope. A subdivision of $V$ is a finite collection $S = \{\mathcal{P}_1, \cdots, \mathcal{P}_q\}$ of $k$-dimensional polytopes such that: (i) The vertices of each $\mathcal{P}_i$ are drawn from $V$ (though not every point in $V$ need be used), (ii) $\mathcal{P} = \bigcup_i \mathcal{P}_i$, (iii) If $i \neq j$, then $\mathcal{P}_i \cap \mathcal{P}_j$ is a common (possibly empty) face of $\mathcal{P}_i$ and $\mathcal{P}_j$.

**Definition 2.3.2** ([40]). Let $x \in \mathbb{R}^n$, $\mathcal{P}$ an $n$-dimensional polytope, and $\mathcal{F}$ a facet of $\mathcal{P}$. The hyperplane $H = \text{aff} (\mathcal{F})$ defines an open half-space containing $\mathcal{P}^o$, the interior of $\mathcal{P}$. If $x$ is contained in the opposite open half-space, then $\mathcal{F}$ is said to be visible from $x$. 

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Figure 2.4: Example for Definition 2.3.2

For instance, in Figure 2.4 the facet $\mathcal{F}$ is visible from $x_1$, but not visible from $x_2$ or $x_3$. Now we describe what it means to place a vertex. Let $\mathcal{S} = \{P_1, \ldots, P_q\}$ be a subdivision of $V$ and $v \in \mathbb{R}^n$ be such that $v \notin V$. Placing the vertex $v$ means that $v$ is adjoined to the point set $V$ after which the following subdivision of $V \cup \{v\}$ is computed.

**Definition 2.3.3 ([40]).** The subdivision $T$ of $V \cup \{v\}$ that results from placing $v$ is obtained as

(i) If $v \notin \text{aff} (V)$, then for each $P_i \in \mathcal{S}$, include $\text{co} (P_i \cup \{v\})$ in $T$.

(ii) If $v \in \text{aff} (V)$, then for each $P_i \in \mathcal{S}$, $P_i \in T$ and if $\mathcal{F}$ is a facet of $P_i$ that is contained in a facet of $\text{co} (V)$ visible from $v$, then $\text{co} (\mathcal{F} \cup \{v\}) \in T$.

For example, in Figure 2.5 let $V = \{o_1, o_2\}$ and $\mathcal{S} = \{P_1\}$, where $P_1 = \text{co} \{o_1, o_2\}$. The subdivision $T$ of $V \cup \{v_4\}$ that results from placing $v_4$ is obtained as follows. Since $v_4 \notin \text{aff} (V)$, then by Definition 2.3.3(i), $T = \{S_1\}$, where $S_1 = \text{co} \{o_1, o_2, v_4\}$. As another example, let $V = \{o_1, o_2, v_4\}$ and $\mathcal{S} = \{P_1\}$, where $P_1 = \text{co} \{o_1, o_2, v_4\}$. The subdivision $T$ of $V \cup \{v_1\}$ that results from placing $v_1$ is obtained as follows. Since $v_1 \in \text{aff} (V)$, then by Definition 2.3.3(ii), we have $P_1 = \text{co} \{o_1, o_2, v_4\} \in T$. Also, since $\text{co} \{o_1, o_2\}$ is a facet of $P_1 = \text{co} (V)$ visible from $v_1$, we include $\text{co} \{o_1, o_2, v_1\}$ in $T$. We conclude that $T = \{S_1, S_2\}$, where $S_1 = \text{co} \{o_1, o_2, v_4\}$ and $S_2 = \text{co} \{o_1, o_2, v_1\}$.

**Theorem 2.3.2 ([40]).** Suppose $V_O$ and $V$ are finite sets of points such that $V_O \subset V$ and $\mathcal{P} := \text{co} (V)$ is an $n$-dimensional polytope. Let $O_{\mathcal{P}} := \text{co} (V_O)$. If the points of $V$ are ordered such that the points of $V_O$ are listed first and if $T$ is the subdivision obtained by placing the points of $V$ in order, then $T$ is a
Figure 2.5: A triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$

A triangulation of $V$ such that for every $n$-dimensional simplex $S \in T$, $S^c \cap \mathcal{O}_\mathcal{P} = \emptyset$ and if $S \cap \mathcal{O}_\mathcal{P} \neq \emptyset$, then $S \cap \mathcal{O}_\mathcal{P}$ is a face of $S$.

Figure 2.5 shows a triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$ that results from placing the points of the set $V = \{o_1, o_2, v_4, v_1, v_3, v_2\}$ in order.

### 2.4 Convex Analysis

Convex analysis is a fundamental tool in the reach control problem. It has been widely used in RCP on simplices by affine feedback [52, 29, 14]. We also find it relevant to the monotonic reach control problem to be developed in Chapter 5. In particular, we use convex analysis in proving the main results of this chapter (Theorems 5.2.2, 5.2.3, 5.2.4). Moreover, in Chapter 8 we use convex analysis in the synthesis of multi-affine feedback for RCP on simplices (Theorem 8.3.3).

The following background is extracted from [49]. A set $C \subset \mathbb{R}^n$ is said to be affine if for every $x, y \in C$ and $\alpha \in \mathbb{R}$, we have $\alpha x + (1 - \alpha)y \in C$. Also, a set $C \subset \mathbb{R}^n$ is said to be convex if for every $x, y \in C$ and $0 < \alpha < 1$, we have $\alpha x + (1 - \alpha)y \in C$. All affine sets are convex, while the converse is not true.

Let $W_1$ and $W_2$ be nonempty sets in $\mathbb{R}^n$. A hyperplane $\mathcal{H}$ is said to separate $W_1$ and $W_2$ if $W_1$ is contained in one of the closed half-spaces associated with $\mathcal{H}$ and $W_2$ lies in the opposite closed half-space. It is said to separate $W_1$ and $W_2$ properly if additionally $W_1$ and $W_2$ are not both contained in $\mathcal{H}$ itself.
It is said to separate $W_1$ and $W_2$ strongly if there exists some $\epsilon > 0$ such that $W_1 + \epsilon B$ is contained in one of the open half-spaces associated with $H$, and $W_2 + \epsilon B$ is contained in the opposite half-space.

The relative interior of a convex set $C$ in $\mathbb{R}^n$, denoted by $\text{ri}(C)$, is defined as the interior of $C$ which results when $C$ is regarded as a subset of its affine hull $\text{aff}(C)$. A convex set $C$ is said to be relatively open if $\text{ri}(C) = C$.

**Theorem 2.4.1.** Let $C$ be a nonempty relatively open convex set in $\mathbb{R}^n$, and let $M$ be a nonempty affine set in $\mathbb{R}^n$ such that $M \cap C = \emptyset$. Then there exists a hyperplane $H$ containing $M$, such that one of the open half-spaces associated with $H$ contains $C$.

**Theorem 2.4.2.** Let $W_1$ and $W_2$ be nonempty convex sets in $\mathbb{R}^n$ whose closures are disjoint. If either set is bounded, then there exists a hyperplane separating $W_1$ and $W_2$ strongly.

A polyhedral convex set in $\mathbb{R}^n$ is by definition a set which can be expressed as the intersection of some finite collection of closed half-spaces.

**Theorem 2.4.3.** If $W_1$ and $W_2$ are nonempty disjoint polyhedral convex sets in $\mathbb{R}^n$, then there exists a hyperplane separating $W_1$ and $W_2$ strongly.

### 2.5 Nonsmooth Analysis

In the context of RCP, nonsmooth analysis is usually used in studying solvability by open-loop controls [43, 23]. In this dissertation we use it in our study on the necessity of the invariance conditions for solvability of RCP on polytopes by open-loop controls (Theorem 4.2.1). Also, we find nonsmooth analysis, particularly Dini derivatives, useful for studying the generalized flow conditions to be developed in Chapter 7. Let $S \subseteq \mathbb{R}^n$ be a closed set. We define the distance function

$$d_S(x) := \inf \{ \|x - y\| \mid y \in S\}.$$

The Bouligand tangent cone (or simply tangent cone) to $S$ at $x$, denoted $T_S(x)$, is defined by

$$T_S(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{t \to 0^+} \frac{d_S(x + tv)}{t} = 0 \right\}.$$

The following lemma provides a useful characterization of the tangent cone $T_S(x)$ that we use in the proof of Theorem 4.2.1.
Lemma 2.5.1 ([18]). Let $S$ be a closed set in $\mathbb{R}^n$. Let $\{x^k\}$ be a sequence such that $x^k \in S$ and $\lim_{k \to \infty} x^k = x$. Also, let $\{t^k\}$ be a sequence such that $t^k > 0$ and $\lim_{k \to \infty} t^k = 0$. Then

$$
\lim_{k \to \infty} \frac{x^k - x}{t^k} \in T_S(x).
$$

A set-valued map $Y : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is said to be upper semicontinuous at $x \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$
\|x - x'\| < \delta \Rightarrow Y(x') \subset Y(x) + \epsilon \mathbb{B}.
$$

Lemma 2.5.2. Let $A \in \mathbb{R}^{n \times n}$, $A \neq 0$, $B \in \mathbb{R}^{n \times m}$, and $a \in \mathbb{R}^n$. The set-valued map $Y(x) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ defined as

$$
Y(x) := \{Ax + Bu + a \mid u \in \mathbb{R}^m\}
$$

is upper semicontinuous.

Proof. Let $L := \|A\|$ and fix $\epsilon > 0$. Select $\delta = \frac{\epsilon}{L}$. Let $x, x' \in \mathbb{R}^n$ be such that $\|x - x'\| < \delta$. Let $y' \in Y(x')$ be arbitrary. There exists $u' \in \mathbb{R}^m$ such that $y' = Ax' + Bu' + a$. Now consider $y := Ax + Bu' + a \in Y(x)$. We have

$$
\|y - y'\| = \|Ax + Bu' + a - Ax' - Bu' - a\| \leq \|A\| \|x - x'\| < L \frac{\epsilon}{L} = \epsilon.
$$

Since $y' \in Y(x')$ is arbitrary, we obtain

$$
Y(x') \subset Y(x) + \epsilon \mathbb{B},
$$

as required.

We conclude this section by reviewing Dini derivatives of locally Lipschitz functions [18, 51]. Consider

$$
\dot{x} = f(x)
$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. Let $\phi(t, x_0)$ denote the unique solution of (2.1) starting at $x_0$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The upper right Dini derivative of $V(\phi(t, x_0))$ with respect to $t$ is

$$
D^+ V(\phi(t, x_0)) := \lim_{\tau \to 0^+} \sup_{t^+} \frac{V(\phi(t + \tau, x_0)) - V(\phi(t, x_0))}{\tau}.
$$
We can also define the upper Dini derivative of $V$ with respect to $f$ given by

$$D^+_f V(x) := \limsup_{\tau \to 0^+} \frac{V(x + \tau f(x)) - V(x)}{\tau}.$$

(2.2)

It was shown by Yoshizawa [59] that for $V$ locally Lipschitz $D^+_f V(\phi(t, x_0)) = D^+_f V(x)$ where $x = \phi(t, x_0)$.
Chapter 3

Reach Control Problem on Simplices

In this chapter we review the reach control problem (RCP) on simplices. The problem is to design a state feedback to force all closed-loop trajectories starting anywhere in a simplex to leave the simplex through a prescribed exit facet in finite time.

We begin this chapter by formulating the reach control problem on simplices in Section 3.1. Then in Section 3.2 we review some basic results on solvability of RCP on simplices, particularly by affine feedback. The results are derived from convexity properties of affine systems and from the geometry of the simplex. Since these results have appeared in previous theses and papers, we suppress the proofs in this section. Then in Section 3.3 we show how to use existing simplex methods to solve RCP on a polytopic state space. Also, we provide two examples illustrating the drawbacks of existing simplex methods when applied to RCP on polytopes.

3.1 Problem Formulation

We consider an $n$-dimensional simplex $S := \text{co} \{v_0, v_1, \cdots, v_n\}$ with vertex set $V := \{v_0, v_1, \cdots, v_n\}$ and facets $F_0, \cdots, F_n$ (the facet is indexed by the vertex it does not contain). Let $h_i$, $i \in \{0, \cdots, n\}$, be the unit normal vector to the facet $F_i$ pointing outside the simplex, and let $F_0$ be the exit facet. We call the other facets $F_1, \cdots, F_n$ the restricted facets. Define the index sets $I := \{1, \cdots, n\}$, and $I_i := I \setminus \{i\}$ (note $I_0 = I$). That is, $I$ is the set of indices of the restricted facets of $S$, and $I_i$ is the set of indices of the restricted facets in which $v_i$ is a point. Given $x \in S$, let $I(x)$ be the minimal index set such that $x \in \text{co} \{v_i \mid i \in I(x)\}$.
Consider the affine control system defined on $\mathcal{S}$:

$$
\dot{x} = Ax + Bu + a, \quad x \in \mathcal{S},
$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\text{rank}(B) = m$. Let $\phi_u(t, x_0)$ denote the trajectory of (3.1) under a control law $u$ starting from $x_0 \in \mathcal{S}$ and evaluated at time $t$. We are interested in studying reachability of the exit facet $F_0$ from $\mathcal{S}$.

**Problem 3.1.1.** (Reach Control Problem (RCP)) Consider system (3.1) defined on a simplex $\mathcal{S}$. Find a state feedback $u(x)$ such that

(i) for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in F_0$, and $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

RCP says that trajectories of (3.1) starting in $\mathcal{S}$ reach and exit $F_0$ in finite time. Unlike previous work [27, 28], it is not required here that trajectories leave the simplex once they reach the exit facet for the first time. For example, RCP may allow a state trajectory starting on the exit facet $F_0$ to enter the interior of the simplex $\mathcal{S}$ for some time as long as the state trajectory eventually leaves $\mathcal{S}$ through $F_0$ in finite time (see Figure 3.1). Notice that in order for the above problem definition to make sense it is assumed that the dynamics (3.1) are extended to a small neighborhood of $\mathcal{S}$. We write $\mathcal{S} \xrightarrow{\mathcal{U}} F_0$ by control type $\mathcal{U}$ if RCP is solvable by a control of type $\mathcal{U}$.

For $v_i \in V$, define the closed, convex cone $\mathcal{C}(v_i)$ by

$$
\mathcal{C}(v_i) := \{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \quad j \in I_i \}.
$$
Figure 3.2: The convex cones $C(v_i)$ in a two-dimensional simplex.

Notice that the cones $C(v_i)$ have their apex at 0. Only in Figure 3.2, $C(v_i)$ are depicted as shaded cones attached at each $v_i$ since they will be used to characterize tangent velocity vectors. Also, notice that for $v_0$, $C(v_0)$ is exactly the Bouligand tangent cone to $S$ at $v_0$, $T_S(v_0)$. Instead, at $v_i \in F_0$, they are different since $C(v_i)$ includes directions pointing out of $S$. Indeed the definition of $C(v_i)$ does not involve $h_0$ because $F_0$ is the exit facet.

**Definition 3.1.1.** We say the invariance conditions are solvable if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that

$$Av_i + Bu_i + a \in C(v_i), \ i \in \{0, \ldots, n\}. \quad (3.2a)$$

Equivalently,

$$h_j \cdot (Av_i + Bu_i + a) \leq 0, \ j \in I_i, \ i \in \{0, \ldots, n\}. \quad (3.2b)$$

The invariance conditions (3.2) can be used to construct affine feedbacks [28]. The procedure is reviewed in the next section (Lemma 3.2.1). For a given affine feedback $u(x) = kx + g$, achieving the invariance conditions (3.2) $(Av_i + Bu(v_i) + a \in C(v_i), \ i \in \{0, \cdots, n\})$ ensures by the convexity of the closed-loop vector field that the restricted facets are blocked (trajectories can leave $S$ only through $F_0$) [28]. However, for a given continuous state feedback $u(x)$, which is not necessarily convex, achieving the invariance conditions (3.2) $(Av_i + Bu(v_i) + a \in C(v_i), \ i \in \{0, \cdots, n\})$ is in general not enough to ensure that. Instead, $u(x)$ must satisfy stronger conditions. To that end, define the closed, convex cone $C(x)$ by

$$C(x) := \{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \ j \in I \setminus J(x) \}.$$
Definition 3.1.2. We say a state feedback $u(x)$ satisfies the invariance conditions if $Ax + Bu(x) + a \in C(x)$, $x \in S$. Equivalently, for all $x \in S$,

$$h_j \cdot (Ax + Bu(x) + a) \leq 0, \quad j \in I \setminus I(x). \quad (3.3)$$

Note that $I \setminus I(x)$ is the set of indices of the restricted facets in which $x$ is a point. Therefore, at points $x \in S^\circ \cup \text{ri} (F_0)$, where $S^\circ$ is the interior of $S$ and $\text{ri} (F_0)$ is the relative interior of $F_0$, the invariance conditions (3.3) are trivially achieved ($C(x) = \mathbb{R}^n$).

3.2 Basic Results for Simplices

In this section we summarize some important results for RCP on simplices. The most definitive results on RCP are focused on solvability of RCP on simplices by affine feedback [29, 52, 14, 34, 35]. Affine feedbacks have been widely used for RCP on simplices because of the following useful property: the input at any point in the simplex can be expressed as a convex combination of the inputs at the vertices in a unique way. This allows to perform the affine feedback design in two independent steps: first, one assigns control inputs at the vertices of the simplex guaranteeing propitious closed-loop behavior; second, one finds the unique affine feedback on the simplex based on the control values selected at the vertices in the first step [28].

We firstly review the elegant method of constructing the unique affine feedback on the simplex from the control inputs selected at the vertices [28]. The method depends mainly on the following lemma.

Lemma 3.2.1. Consider two sets of points $V = \{v_0, \cdots, v_n \mid v_i \in \mathbb{R}^n\}$ and $u_0, \cdots, u_n \mid u_i \in \mathbb{R}^m\}$. Suppose $V$ is affinely independent. Then there exist unique matrices $K \in \mathbb{R}^{m \times n}$ and $g \in \mathbb{R}^m$ such that

$$u_i = Kv_i + g, \quad i \in \{0, \cdots, n\}.$$

Proof. We need to show there exist unique matrices $K$ and $g$ such that

$$
\begin{bmatrix}
v_0^T & 1 \\
\vdots & \vdots \\
v_n^T & 1
\end{bmatrix}
\begin{bmatrix}
K^T \\
g^T
\end{bmatrix}
= 
\begin{bmatrix}
v_0^T \\
\vdots \\
v_n^T
\end{bmatrix}
\begin{bmatrix}
u_0 \\
\vdots \\
u_n
\end{bmatrix}.
$$

If the $(n+1) \times (n+1)$ left-hand matrix is full rank, then multiplying by its inverse from the left yields
the unique solutions \( K \) and \( g \). We have

\[
\begin{bmatrix}
  v_0^T & 1 \\
  \vdots & \vdots \\
  v_n^T & 1 \\
\end{bmatrix}
\begin{bmatrix}
  v_1^T - v_0^T \\
  \vdots \\
  v_n^T - v_0^T \\
\end{bmatrix}
\]

\[
= 1 + \begin{bmatrix}
  v_0^T \\
  \vdots \\
  v_n^T \\
\end{bmatrix}
\begin{bmatrix}
  u_0^T \\
  \vdots \\
  u_n^T \\
\end{bmatrix}
\]

The last equality follows since the points \( \{v_0, \ldots, v_n\} \) are affinely independent if and only if \( \{v_1 - v_0, \ldots, v_n - v_0\} \) are linearly independent.

We conclude that the unique affine feedback can be simply obtained from

\[
\begin{bmatrix}
  K^T \\
  g^T \\
\end{bmatrix}
= \begin{bmatrix}
  v_0^T & 1 \\
  \vdots & \vdots \\
  v_n^T & 1 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
  u_0^T \\
  \vdots \\
  u_n^T \\
\end{bmatrix}.
\]  

(3.4)

Next we review some basic results for solvability by affine feedback. For Problem 3.1.1 the following necessary and sufficient conditions have been established for the case of affine feedback.

**Theorem 3.2.2** (Theorem 4.16 of [29], Theorem 8 of [52]). Given the system (3.1) on an \( n \)-dimensional simplex \( S \) and an affine feedback \( u(x) = Kx + g \), where \( K \in \mathbb{R}^{m \times n} \), \( g \in \mathbb{R}^m \), and \( u_0 = u(v_0), \ldots, u_n = u(v_n) \), the closed-loop system satisfies \( S \xrightarrow{\mathcal{L}} F_0 \) if and only if

(a) The invariance conditions (3.2) hold,

(b) There is no closed-loop equilibrium in \( S \).

Under affine feedback, condition (b) has been shown to be equivalent to a so-called flow condition [52]. The flow condition is that there exists \( \xi \in \mathbb{R}^n \) such that for all \( v_i \in V \), \( \xi \cdot (Av_i + Bu(v_i) + a) < 0 \).

However, Theorem 3.2.2 is not useful in the control synthesis since it depends on having a candidate affine feedback to apply the theorem. Instead, the geometry of the problem should be explored to find constructive necessary and sufficient conditions for RCP. To that end, let \( B = \text{Im} \ (B) \), the image of \( B \). Define

\[
\mathcal{O} := \{ x \in \mathbb{R}^n : Ax + a \in B \}.
\]  

(3.5)

It is easy to show if \( \mathcal{O} \neq \emptyset \), then \( \mathcal{O} \) is an affine space with dimension \( m \leq \kappa \leq n \). Notice that at any \( x \in \mathcal{O} \) the vector field \( Ax + Bu + a \) can vanish for an appropriate choice of \( u \). Indeed, \( \mathcal{O} \) is the set of
all possible equilibrium points of the system. That is, if \( x_0 \) is an equilibrium of (3.1) under feedback control, then \( x_0 \in \mathcal{O} \). Define

\[
\mathcal{O}_S := \mathcal{S} \cap \mathcal{O}.
\]

Since \( \mathcal{O} \) is an affine space, either \( \mathcal{O}_S = \emptyset \) or \( \mathcal{O}_S \) is a convex polytope in \( \mathcal{S} \). The following theorems provide for certain geometric situations clear conditions for solvability by affine feedback, which make us avoid the high computational effort associated with checking condition (b) in Theorem 3.2.2.

**Theorem 3.2.3** (Theorem 6.1 of [14]). Suppose \( \mathcal{O}_S = \emptyset \). If the invariance conditions (3.2) are solvable, then \( \mathcal{S} \rightarrow \mathcal{F}_0 \) by affine feedback.

**Theorem 3.2.4** (Theorem 6.7 of [14]). Suppose \( \mathcal{O}_S \) is a \( \kappa \)-dimensional face of \( \mathcal{S} \) and \( \mathcal{O}_S = \text{co} \{ v_1, \ldots, v_{\kappa+1} \} \), with \( 0 \leq \kappa < m \). Suppose the following conditions hold.

(i) The invariance conditions (3.2) are solvable.

(ii) There exists a linearly independent set of vectors \( \{ b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i) \} \).

Then \( \mathcal{S} \rightarrow \mathcal{F}_0 \) by affine feedback.

Let \( \text{cone}(\mathcal{S}) := \mathcal{C}(v_0) \), the tangent cone to simplex \( \mathcal{S} \) at the vertex not contained in the exit facet \( \mathcal{F}_0 \). Another important result is:

**Theorem 3.2.5** (Theorem 5 of [34], [35]). If the invariance conditions (3.2) are solvable and \( \mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \emptyset \), then \( \mathcal{S} \rightarrow \mathcal{F}_0 \) by affine feedback.

We conclude this section by reviewing some necessary conditions for solvability of RCP on simplices by continuous state feedback.

**Theorem 3.2.6** ([28]). If \( \mathcal{S} \rightarrow \mathcal{F}_0 \) by a continuous state feedback \( u(x) \), then \( u(x) \) satisfies the invariance conditions (3.3).

Let \( V_{\mathcal{O}_S} := \{ o_1, \ldots, o_{\kappa+1} \} \) denote the set of vertices of \( \mathcal{O}_S \) and \( I_{\mathcal{O}_S} := \{ 1, \ldots, \kappa + 1 \} \). We define

\[
\text{cone}(\mathcal{O}_S) := \bigcap_{i \in I_{\mathcal{O}_S}} \mathcal{C}(o_i).
\]

**Theorem 3.2.7** ([34, 35]). Suppose \( m = 1 \). If \( \mathcal{S} \rightarrow \mathcal{F}_0 \) by continuous state feedback, then \( \mathcal{B} \cap \text{cone}(\mathcal{O}_S) \neq \emptyset \).

The above theorem is a special case of one of our results to be presented and discussed in the next chapter. We present Theorem 3.2.7 here only for completeness.
3.3 Simplex Methods

In this section we consider the reach control problem on a polytopic state space. In particular, we present the main idea of the simplex-based approach, which is widely used to solve RCP on polytopes [6, 14, 29, 52]. Since RCP on simplices by affine feedback has been studied deeply, there is a belief in the literature that the right way to solve RCP on polytopes is to firstly triangulate the polytope \( \mathcal{P} \) into simplices, then to reformulate the original reachability problem on \( \mathcal{P} \) to a sequence of reachability problems on the simplices of the triangulation. The latter can be solved efficiently by affine feedbacks [29], [52]. The obtained control law is a (possibly discontinuous) PWA feedback on \( \mathcal{P} \). An illustrative example is shown in Figure 3.3. To solve the original reachability problem on \( \mathcal{P} \) in Figure 3.3(a), we triangulate \( \mathcal{P} \) into two simplices \( T = \{ \mathcal{S}_1, \mathcal{S}_2 \} \) shown in Figure 3.3(b), then we solve two reachability problems: \( \mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F} := \mathcal{S}_1 \cap \mathcal{S}_2 \) by affine feedback and \( \mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0 \) by affine feedback. Notice that the control assignment shown in Figure 3.3(b) solves RCP using simplex methods although it does not have a positive velocity in the direction \( h_0 \).

Now we show via illustrative examples some drawbacks of the simplex-based approach.

**Example 3.3.1.** In the first example we show that using simplex-based methods for reach control, RCP is solvable for one triangulation but not for another. Consider the system

\[
\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix}.
\]
The polytope is shown in Figure 3.4. The vertices of \( \mathcal{P} \) are: \( v_1 = (0,0), v_2 = (1,0), v_3 = (1,1), \) and \( v_4 = (0,1) \). The control objective is to solve RCP on \( \mathcal{P} \) by PWA feedback using existing simplex methods. Suppose we triangulate \( \mathcal{P} \) as in Figure 3.4(a). Then the control objective splits as \( \mathcal{S}_1 \xrightarrow{S_1} \mathcal{F}_0 \) by affine feedback and \( \mathcal{S}_2 \xrightarrow{S_2} \mathcal{F} \) by affine feedback. We study the invariance conditions of \( \mathcal{S}_1 \) at \( v_4 \). We have \( Av_4 + a = (1,0) \), thus for any choice of control the velocity vector \( Av_4 + Bu(v_4) + a \) points outside \( \mathcal{S}_1 \). This implies the invariance conditions of \( \mathcal{S}_1 \) are not solvable at \( v_4 \). Consequently, \( \mathcal{S}_1 \xrightarrow{S_1} \mathcal{F}_0 \) is not achievable. Instead, suppose we triangulate \( \mathcal{P} \) as in Figure 3.4(b). The control objective is again \( \mathcal{S}_1 \xrightarrow{S_1} \mathcal{F}_0 \) by affine feedback and \( \mathcal{S}_2 \xrightarrow{S_2} \mathcal{F} \) by affine feedback. We choose the control values at the vertices to be \( u_1 = 4, u_2 = 0, u_3 = 0, u_4 = 2 \). The corresponding velocity vector, \( y_i \), at each vertex \( v_i \) is shown in Figure 3.4(b). Based on these selected control values at the vertices, one can construct a PWA feedback such that the control objective based on existing simplex methods is achieved [28]. We obtain the following continuous PWA feedback:

\[
u(x) = \begin{cases} 
-4 & 0 \end{cases} \begin{bmatrix} x + 4, & x \in \mathcal{S}_1 \\
-2 & -2 \end{bmatrix} x + 4, & x \in \mathcal{S}_2.
\]

**Example 3.3.2.** In the previous example the simplex-based approach could be used to solve RCP, although it worked only for a particular choice of triangulation. Now we consider an example where the
simplex-based approach fails for any choice of triangulation. Consider the system

\[
\dot{x} = \begin{bmatrix}
1 & 1 & -2 \\
1 & -3 & -2 \\
0 & 0 & -1
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]

The polytope is shown in Figure 3.5. The vertices of \( \mathcal{P} \) are: \( v_1 = (1, 0, 0) \), \( v_2 = (1, 1, 0) \), \( v_3 = (1, 0, 1) \), \( v_4 = (0, 0, 0) \), and \( v_5 = (0, 1, 0) \). We check whether the problem is solvable using simplex methods.

There are two possible triangulations of \( \mathcal{P} \), shown in Figure 3.5. For the first triangulation the control objective is \( S_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0 \) by affine feedback and \( S_2 \xrightarrow{\mathcal{S}_2} \mathcal{F} = S_1 \cap S_2 \) by affine feedback. We examine the invariance conditions of \( S_1 \) at \( v_4 \). We have \( Av_4 + Bu_4 + a = (0,1,u_4) \). The normal vectors to facets \( \mathcal{F}_3 \) and \( \mathcal{F}_1 \) in \( S_1 \) are \( h_3 = (0,0,-1) \) and \( h_1 = (-0.5774, 0.5774, 0.5774) \) respectively. The invariance conditions of \( S_1 \) at \( v_4 \) yield \( h_3 \cdot (Av_4 + Bu_4 + a) \leq 0 \) and \( h_1 \cdot (Av_4 + Bu_4 + a) \leq 0 \). That is, \( u_4 \geq 0 \) and \( u_4 \leq -1 \). Thus, RCP is not solvable by simplex methods using this triangulation. Now we try the second triangulation in Figure 3.5. We examine the invariance conditions of \( S_1 \) at \( v_3 \). In this case, we have \( Av_3 + Bu_3 + a = (-1,0,-1+u_3) \), \( h_2 = (-0.7071,-0.7071,0) \), and \( h_2 \cdot (Av_3 + Bu_3 + a) > 0 \) for all \( u_3 \in \mathbb{R} \). Again, RCP is not solvable by simplex methods using this triangulation. Despite the failure of the simplex-based approach, we will show in Chapter 5 that the original reachability problem on \( \mathcal{P} \) is solvable by continuous PWA feedback in this example.

The above examples show some disadvantages of the existing simplex-based approach:

1. Solvability of RCP using this approach depends mainly on the choice of triangulation (Example 3.3.1). This may lead in some examples to an iterative method where one tries all possible triangulations until a triangulation works.
2. Simplex methods require the solvability of the invariance conditions of each simplex in the triangulation, which imposes stronger conditions on the vector field at the vertices of the polytope. Therefore, RCP using simplex methods is a restricted version of the original reachability problem on $\mathcal{P}$ in the sense that simplex methods may fail, yet the original reachability problem on $\mathcal{P}$ is solvable (Example 3.3.2).

To overcome these disadvantages, we study directly RCP on polytopes in the next chapter.
Chapter 4

Reach Control Problem on Polytopes

In the previous chapter we have shown the drawbacks of existing simplex methods when applied to RCP on polytopes. Indeed, we have found many examples in which simplex methods fail to solve RCP on polytopes for any choice of triangulation, yet the original reachability problem on the polytope is solvable (see for instance Example 3.3.2). This motivates us to study directly RCP on polytopes. The main advantage that we obtain from considering the original problem is that we relax the requirement of the solvability of the invariance conditions of each simplex in the triangulation. For instance, in Figure 4.1, we may allow the state trajectory initiated at $x_0 \in S_1$ to leave it through $F$ toward $S_2$ as long as the state trajectory leaves $P$ eventually through $F_0$.

In this chapter we start the direct study of RCP on polytopes by formulating the problem in Section 4.1. Then to carry out this research study, one must first identify necessary conditions for solvability of RCP on polytopes. Hence, we explore in Section 4.2 novel necessary conditions for solvability of RCP on polytopes by open-loop controls and by continuous state feedback. The conditions will be used in the next chapter to develop novel control methods for RCP on polytopes.

4.1 Problem Formulation

Consider an $n$-dimensional polytope

$$P := \text{co} \{v_1, \ldots, v_p\}$$
with vertex set \( V := \{v_1, \ldots, v_p\} \) and facets \( F_0, F_1, \ldots, F_r \). The exit facet is the facet \( F_0 \) of \( P \). We call \( F_1, \ldots, F_r \) the restricted facets of \( P \). Let \( h_i \) be the unit normal vector to each facet \( F_i \) pointing outside the polytope. Define the index sets \( I := \{1, \ldots, p\} \), \( J := \{1, \ldots, r\} \), and \( J(x) := \{j \in J \mid x \in F_j\} \). That is, \( J(x) \) is the set of indices of the restricted facets in which \( x \) is a point. For each \( x \in P \), define the closed, convex cone
\[
\mathcal{C}(x) := \{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in J(x) \}.
\]
Note that the cones \( \mathcal{C}(x) \) have their apex at 0. In our drawings, \( \mathcal{C}(x) \) are depicted as shaded cones attached at each \( x \) since they will be used to characterize tangent velocity vectors (see for instance Figure 4.2). Also, note that 0 never appears in \( J(x) \) since \( F_0 \) is the exit facet. Therefore, for any \( x \in P \setminus F_0 \), \( \mathcal{C}(x) \) is exactly the Bouligand tangent cone to \( P \) at \( x \), \( T_P(x) \). Instead, at \( x \in F_0 \), \( \mathcal{C}(x) \) and \( T_P(x) \) are different since \( \mathcal{C}(x) \) includes directions pointing out of \( P \).

We consider the affine control system defined on \( P \):
\[
\dot{x} = Ax + Bu + a, \quad x \in P,
\]
where \( A \in \mathbb{R}^{n \times n} \), \( a \in \mathbb{R}^n \), \( B \in \mathbb{R}^{n \times m} \), and \( \text{rank}(B) = m \). Let \( \phi_u(t, x_0) \) be the trajectory of (4.1) under a control law \( u \) starting from \( x_0 \in P \), and let \( B = \text{Im}(B) \), the image of \( B \). We are interested in studying reachability of the exit facet \( F_0 \) from \( P \) by feedback control.

**Problem 4.1.1 (Reach Control Problem (RCP)).** Consider system (4.1) defined on \( P \). Find a state feedback \( u(x) \) such that:

(i) for every \( x_0 \in P \) there exist \( T \geq 0 \) and \( \gamma > 0 \) such that \( \phi_u(t, x_0) \in P \) for all \( t \in [0, T] \), \( \phi_u(T, x_0) \in F_0 \), and \( \phi_u(t, x_0) \notin P \) for all \( t \in (T, T + \gamma) \).
RCP says that trajectories of (4.1) starting from initial conditions in $\mathcal{P}$ reach and exit the facet $F_0$ in finite time. Also, RCP does not require that trajectories leave the polytope immediately after they reach the exit facet for the first time. Notice that in order for condition (i) to make sense it is assumed that the dynamics (4.1) are extended to a small neighborhood of $\mathcal{P}$. In the sequel we use the shorthand notation $\mathcal{P} \xrightarrow{P} F_0$ to denote that condition (i) of Problem 4.1.1 holds for some control.

We define the set of possible equilibrium points of (4.1) on $\mathcal{P}$ by

$$\mathcal{O}_P := \mathcal{P} \cap \mathcal{O},$$

where $\mathcal{O}$ is defined in (3.5). Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_P = \emptyset$ or $\mathcal{O}_P$ is a $\kappa$-dimensional polytope in $\mathcal{P}$. If $\mathcal{O}_P \neq \emptyset$, we define the vertex set of $\mathcal{O}_P$ to be $V_{\mathcal{O}} := \{o_1, \ldots, o_q\}$, where $o_i$ are the vertices of $\mathcal{O}_P$ (not necessarily vertices of $\mathcal{P}$). Also define the index set $I_{\mathcal{O}} := \{1, \ldots, q\}$.

In the next part we review the relevance of the invariance conditions to RCP on polytopes.

**Definition 4.1.1.** We say the invariance conditions are solvable if for each $v \in V$ there exists $u \in \mathbb{R}^m$ such that

$$Av + Bu + a \in \mathcal{C}(v).$$

Equivalently,

$$h_j \cdot (Av + Bu + a) \leq 0, \quad j \in J(v).$$
Equation (4.3a) or (4.3b) is referred to as the invariance conditions either for a specific vertex, or collecting all conditions for all vertices, for a polytope.

**Theorem 4.1.1** (Proposition 3.1 of [28]). If $\mathcal{P} \xrightarrow{\mathcal{F}_0} \mathcal{F}_0$ by continuous state feedback, then the invariance conditions (4.3) are solvable.

The invariance conditions (4.3) can be used to construct continuous PWA feedbacks on $\mathcal{P}$ [28]. The procedure is to solve the invariance conditions (4.3), then to triangulate $\mathcal{P}$ into simplices, and finally to use Lemma 3.2.1 to find on each simplex the unique affine feedback based on the control values at the vertices.

For a given continuous PWA feedback $u(x)$ defined on a triangulation $\mathcal{T}$ carried out by vertices of $\mathcal{P}$, achieving the invariance conditions (4.3) $(Av_i + Bu_i + a \in \mathcal{C}(v_i), \ v_i \in V)$ ensures by convexity of the closed-loop vector field on each simplex that restricted facets of $\mathcal{P}$ are blocked (trajectories that leave $\mathcal{P}$ do so only through $\mathcal{F}_0$). However, for a given continuous state feedback $u(x)$, which is not necessarily convex, satisfying the invariance conditions (4.3) $(Av_i + Bu_i + a \in \mathcal{C}(v_i), \ v_i \in V)$ is in general not enough to ensure that. Instead, $u(x)$ must satisfy stronger conditions shown in the next lemma.

**Lemma 4.1.2** ([28]). Consider the system (4.1) on $\mathcal{P}$ and let $u(x)$ be a continuous state feedback on $\mathcal{P}$ such that the closed-loop system has unique solutions. Suppose that the following conditions hold:

$$h_j \cdot (Ax + Bu(x) + a) \leq 0, \ j \in J(x), \ x \in \mathcal{P}. \quad (4.4)$$

Then all trajectories originating in $\mathcal{P}$ that leave $\mathcal{P}$ do so via $\mathcal{F}_0$.

Notice that at points $x \in \mathcal{P}^\circ \cup \text{ri} (\mathcal{F}_0)$, where $\mathcal{P}^\circ$ is the interior of $\mathcal{P}$ and $\text{ri} (\mathcal{F}_0)$ is the relative interior of $\mathcal{F}_0$, $J(x) = \emptyset$, and so the invariance conditions (4.4) are trivially achieved. Therefore, it is enough to check conditions (4.4) only at points on the restricted facets of $\mathcal{P}$. Indeed, (4.4) are precisely the invariance conditions presented in Theorem 4.1(ii) of [28]. Also, it should be noted that the invariance conditions (4.4) are also necessary for solvability of RCP by the given continuous feedback $u(x)$ [28].

In [28], the invariance conditions (4.3) were combined with a second sufficient condition to solve RCP on polytopes by continuous PWA feedback.

**Theorem 4.1.3** (Theorem 4.7 of [28]). Consider the system (4.1) on $\mathcal{P}$. Suppose there exist $u_i \in \mathbb{R}^m, \ i \in I$, such that

(i) $h_j \cdot (Av_i + Bu_i + a) \leq 0, \ j \in J(v_i), \ i \in I.$

(ii) $h_0 \cdot (Av_i + Bu_i + a) > 0, \ i \in I.$
Then $\mathcal{P} \xrightarrow{\mu} \mathcal{F}_0$ by continuous PWA feedback.

However, condition (ii) of the above theorem is very conservative as we will see in the next chapter.

4.2 Necessary Conditions

In this section we explore necessary conditions for solvability of RCP on polytopes under various classes of controls. Particularly, open-loop controls have received relatively less attention than continuous state feedbacks. First, in [28] it was shown that the invariance conditions (4.3) are necessary for solvability of RCP on polytopes by continuous state feedback. Here we study if the invariance conditions are necessary for solvability by open-loop controls; the main result appears in Theorem 4.2.1. Second, we expose a necessary condition, stated in Lemma 4.2.2, associated with removal of equilibria on vertices. This condition arises directly from the geometric relationship between the space $\mathcal{B}$ and the tangent cones at vertices of $\mathcal{O}_P$. Finally, we study a topological necessary condition, stated in Theorem 4.2.3, associated with existence of equilibria using continuous state feedback.

4.2.1 Open-loop Controls

Recently there has been a quest to find the largest class of feedbacks to solve RCP [43, 14]. In particular, [43] aims to find a feedback class that solves RCP if RCP is solvable by open-loop controls. To carry out this research program, one must first identify necessary conditions for solvability by open-loop controls. Then one should use these necessary conditions to find a feedback class that solves RCP whenever open-loop controls do. In this subsection, we study whether invariance conditions are necessary for solvability of RCP by open-loop controls.

We say that a function $\mu : [0, \infty) \to \mathbb{R}^m$ is an open-loop control for (4.1) if it is bounded on any compact interval and it is measurable. By Caratheodory’s theorem [22] solutions of (4.1) using open-loop controls exist and are unique.

Despite the positive results reported in [28] for continuous state feedbacks, unfortunately, when one works with general polytopes and open-loop controls, the invariance conditions are no longer necessary for solvability of RCP. The following counterexample suggested by Zhiyun Lin illustrates this fact.

Example 4.2.1. Consider the polytope $\mathcal{P} = \text{co} \{v_1, \cdots, v_5\} \subset \mathbb{R}^3$, shown in Figure 4.3. The exit facet is $\mathcal{F}_0 = \text{co} \{v_1, v_2, v_3\}$, depicted as a hatched region in the figure. Suppose that the two hyperplanes containing the facet $\mathcal{F}_1 = \text{co} \{v_1, v_2, v_5\}$ and the facet $\mathcal{F}_2 = \text{co} \{v_1, v_3, v_4\}$ intersect at the line through $v_1$ and $v'_1$, and this line is parallel to the horizontal hyperplane containing $\text{co} \{v_2, v_3, v_4, v_5\}$. For the
Figure 4.3: The invariance conditions are not solvable but RCP is solvable by open-loop controls.

system, we suppose $\dim(\mathcal{B}) = 2$, where $\mathcal{B} = \text{Im}(B)$, $\mathcal{B}$ is parallel to the horizontal hyperplane containing $\text{co} \ \{v_2, v_3, v_4, v_5\}$, $(A, B)$ is controllable, $\mathcal{P} \cap \mathcal{O} = \emptyset$, where $\mathcal{O}$ is the set of possible equilibria defined in (3.5), and $Av_1 + a$ has a strictly positive $x_3$ component, as shown in the figure. Now we show that for the polytope and the system described above, RCP is solvable by open-loop controls, even if the invariance conditions are not solvable.

First, we show that the invariance conditions are not solvable at $v_1$. Let $y_1 := Av_1 + a + Bu_1$. Since $Av_1 + a$ has a strictly positive $x_3$ component, $\mathcal{B}$ is horizontal, and $v_1 \notin \mathcal{O}$, for any choice $u_1 \in \mathbb{R}^2$, $y_1$ has a strictly positive $x_3$ component. On the other hand, to achieve the invariance conditions at $v_1$, $y_1$ must lie in the closed cone $\{y \in \mathbb{R}^3 \mid h_1 \cdot y \leq 0, \ h_2 \cdot y \leq 0, \ h_3 \cdot y \leq 0\}$, where $h_3$ is the outward unit normal vector of $F_3 = \text{co} \ \{v_1, v_4, v_5\}$. In particular, to satisfy the invariance conditions of $F_1$ and $F_2$ simultaneously, $y_1$ must have a non-positive $x_3$ component.

Second, we show there exist open-loop controls solving RCP on $\mathcal{P}$ by invoking Theorem 6 of [43]. The conditions to apply this theorem are that (i) $(A, B)$ is controllable; (ii) $m = n - 1$; (iii) if $\mathcal{P} \cap \mathcal{O} \neq \emptyset$, then $\mathcal{P} \cap \mathcal{O}$ is a $\kappa$-dimensional face of $\mathcal{P}$; and (iv) certain sets denoted $\mathcal{A}^-$ and $\mathcal{A}^+$ describing states that cannot reach $F_0$ by open-loop controls are empty. In our case, conditions (i) and (ii) hold by assumption, and (iii) holds trivially since $\mathcal{P} \cap \mathcal{O} = \emptyset$. Only (iv) requires a small computation aided by the fact that $\mathcal{P} \cap \mathcal{O} = \emptyset$. We omit the details. We conclude $\mathcal{P} \xrightarrow{\mathcal{P}} F_0$ by open-loop controls, from which Theorem 9 of [43] gives $\mathcal{P} \xrightarrow{\mathcal{P}} F_0$ by discontinuous piecewise affine feedback.
To make the previous discussion more concrete, consider the system

\[
\dot{x} = \begin{bmatrix}
2 & 0 & -1 \\
0 & 2 & 0 \\
\frac{1}{10} & 0 & 1 \\
\end{bmatrix} x + \begin{bmatrix}
0 \\
-2 \\
1 \\
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} u.
\]

Polytope \( P \subset \mathbb{R}^3 \) is as in Figure 4.3 with \( v_1 = (1/2, 1, 1), v_2 = (0, 1, 0), v_3 = (1, 1, 0), \) and \( v_5 = (0, 0, 0) \). It can be verified that the system and polytope satisfy all the conditions described above, including that the invariance conditions of \( P \) are not solvable at \( v_1 \). Now we follow the procedure in Section 4 of [43] to construct a discontinuous piecewise affine feedback solving RCP. First, using the Basic Triangulation Algorithm of [43], we triangulate \( P \) into \( S_1 = \text{co} \{ v_1, v_2, v_3, v_5 \} \) and \( S_2 = \text{co} \{ v_1, v_3, v_4, v_5 \} \) as shown in Figure 4.3. Let \( F := S_1 \cap S_2 = \text{co} \{ v_1, v_3, v_5 \} \). We split the control objective as \( S_2 \rightarrow F \) by affine feedback and \( S_1 \rightarrow F \) by affine feedback. For simplices, RCP is solvable by affine feedback if and only if the invariance conditions of the simplex are solvable and the unique affine feedback built from one such solution does not admit a closed-loop equilibrium in the simplex (Theorem 3.2.2). Since \( P \cap \mathcal{O} = \emptyset \), neither \( S_1 \) nor \( S_2 \) can have equilibria, so we must only construct affine feedbacks satisfying the invariance conditions of \( S_1 \) and \( S_2 \), respectively. For the vertices of \( S_2 \) we select control values \( u_1 = (-5, 10), u_3 = (-12, 10), u_4 = (-12, 12), \) and \( u_5 = (5, 12) \). Then using (3.4) we construct the unique affine feedback \( u(x) \) on \( S_2 \) satisfying \( u(v_i) = u_i, \) \( v_i \in S_2 \) [28]. Similarly, we construct the affine feedback on \( S_1 \) that achieves \( S_1 \rightarrow F \). We conclude by Theorem 9 of [43] that the following discontinuous piecewise affine feedback solves RCP on \( P \).

\[
u(x) = \begin{cases}
\begin{bmatrix}
-22 & 5 & 6 \\
0 & -2 & 0 \\
-17 & 0 & -\frac{3}{2} \\
0 & -2 & 0 \\
\end{bmatrix} x + \begin{bmatrix}
5 \\
12 \\
5 \\
12 \\
\end{bmatrix}, & x \in S_1 \\
\end{cases}
\]

This feedback has a discontinuity along \( F \), but it does not have sliding modes. Finally, we note that on \( F \) the controller for \( S_1 \) is selected because \( S_1 \) has the shortest path to reach \( F_0 \) (see step 3(e) of Algorithm 2 in [42]).

To overcome the obstacle identified in the previous example, we identify a suitable class of polytopes for which the invariance conditions remain necessary conditions.

**Theorem 4.2.1.** Let \( P \) be an \( n \)-dimensional simple polytope. If \( P \rightarrow F_0 \) by open-loop controls, then
the invariance conditions (4.3) are solvable.

Proof. Define \( \mathcal{Y}(x) := \{ Ax + Bw + a \mid w \in \mathbb{R}^m \} \). Let \( x_0 \in \mathcal{P} \setminus \mathcal{F}_0 \). By assumption there exists \( \mu(t) \) and a time \( T > 0 \) such that \( \phi_\mu(t, x_0) \in \mathcal{P} \) for all \( t \in [0, T] \). Since \( \mu(t) \) is an open-loop control, there exists \( c > 0 \) such that \( \| \mu(t) \| \leq c \), for all \( t \in [0, T] \). Consider the set \( \mathcal{Y}_c(x) := \{ Ax + Bw + a \mid w \in \mathbb{R}^m, \| w \| \leq c \} \).

Using Lemma 2.5.2, both \( x \mapsto \mathcal{Y}_c(x) \) and \( x \mapsto \mathcal{Y}(x) \) are upper semicontinuous. Now take a sequence \( \{ t_i \mid t_i \in (0, T) \} \) with \( t_i \to 0 \). Since \( \{ y \in \mathcal{Y}_c(x) \mid x \in \mathbb{R}^n \} \) is bounded on \( \mathcal{P} \), there exists \( M > 0 \) such that \( \| \phi_\mu(t_i, x_0) - x_0 \| \leq Mt_i \). Therefore \( \frac{\phi_\mu(t_i, x_0) - x_0}{t_i} \) is a bounded sequence and there exists a convergent subsequence (with indices relabeled) such that \( \lim_{i \to \infty} \frac{\phi_\mu(t_i, x_0) - x_0}{t_i} =: v \). Since \( \phi_\mu(t_i, x_0) \in \mathcal{P} \), by Lemma 2.5.1, \( v \in T_\mathcal{P}(x_0) \). Now we show \( v \in \mathcal{Y}(x_0) \).

We have

\[
\frac{\phi_\mu(t_i, x_0) - x_0}{t_i} = \frac{1}{t_i} \int_0^{t_i} [A\phi_\mu(\tau, x_0) + B\mu(\tau) + a] d\tau.
\]

Taking the limit, we get

\[
v = Ax_0 + a + B \lim_{i \to \infty} \frac{1}{t_i} \int_0^{t_i} \mu(\tau) d\tau \in \mathcal{Y}(x_0).
\]

We conclude that \( \mathcal{Y}(x_0) \cap T_\mathcal{P}(x_0) \neq \emptyset \), \( x_0 \in \mathcal{P} \setminus \mathcal{F}_0 \). Since \( T_\mathcal{P}(x_0) = \mathcal{C}(x_0) \), \( x_0 \in \mathcal{P} \setminus \mathcal{F}_0 \), it follows that the invariance conditions are solvable at \( x_0 \in \mathcal{P} \setminus \mathcal{F}_0 \).

Now consider \( v_i \in \mathcal{F}_0 \). If \( v_i \in \mathcal{O} \), then the invariance conditions are solvable by selecting \( u_i \in \mathbb{R}^m \) such that \( Av_i + Bu_i + a = 0 \). Instead suppose \( v_i \notin \mathcal{O} \). Suppose by the way of contradiction that \( \mathcal{Y}(v_i) \cap \mathcal{C}(v_i) = \emptyset \). Then \( \mathcal{Y}(v_i) \) and \( \mathcal{C}(v_i) \) are non-empty disjoint polyhedral convex sets in \( \mathbb{R}^n \). By Theorem 2.4.3, they are strongly separated. So, there exists \( \epsilon > 0 \) such that \( \inf_{y \in \mathcal{Y}(v_i), z \in \mathcal{C}(v_i)} \| y - z \| > \epsilon \). By the upper semicontinuity of \( x \mapsto \mathcal{Y}(x) \), there exists \( \delta > 0 \) such that if \( \| x - v_i \| < \delta \), then \( \mathcal{Y}(x) \subset \mathcal{Y}(v_i) + \frac{\delta}{2} \mathcal{B} \).

As \( \mathcal{P} \) is a simple polytope, by Remark 2.3.1(i) \( v_i \in \mathcal{F}_0 \) is the intersection of exactly \( n \) edges, and by Remark 2.3.1(ii) \( n - 1 \) of these edges are contained in \( \mathcal{F}_0 \). Let \( \overline{v_i v_j} \) be the edge that is not contained in \( \mathcal{F}_0 \). By definition of a simple polytope, \( \overline{v_i v_j} \) is the intersection of exactly \( n - 1 \) facets. Since \( \overline{v_i v_j} \) is not contained in \( \mathcal{F}_0 \), the \( n - 1 \) facets are the restricted facets at \( v_i \). We conclude \( \forall x \in [v_i, v_j], \mathcal{C}(x) = \mathcal{C}(v_i) \).

Let \( \bar{x} \in (v_i, v_j) \cap \{ x \in \mathbb{R}^n \mid \| x - v_i \| < \delta \} \). We have \( \mathcal{C}(\bar{x}) = \mathcal{C}(v_i) \), and \( \mathcal{Y}(\bar{x}) \subset \mathcal{Y}(v_i) + \frac{\delta}{2} \mathcal{B} \). Therefore, \( \mathcal{C}(\bar{x}) \cap \mathcal{Y}(\bar{x}) = \emptyset \), a contradiction.

Suppose that we are given an \( n \)-dimensional polytope \( \mathcal{P} \), which can also be described as \( \mathcal{P} = \{ x \in \mathbb{R}^n \mid h_i \cdot x \leq \alpha_i, \forall i \in \{0, \cdots, r\} \} \). It is easy to check whether \( \mathcal{P} \) is simple by verifying that for each vertex \( v \in V, \ h_i \cdot v = \alpha_i \) for precisely \( n \) indices in \( \{0, \cdots, r\} \).

Example 4.2.2. We return to Example 4.2.1 and identify the defect to be that \( \mathcal{P} \) is not simple - vertex \( v_1 \) is contained in four facets.
4.2.2 Continuous Feedbacks

In this subsection, we explore two other necessary conditions for solvability of RCP, this time for general polytopes and for continuous state feedbacks. These conditions are used in the algorithmic solution for RCP on polytopes, to be developed in Section 5.4. The first necessary condition is a rather intuitive condition that targets the appearance of equilibria at vertices of $\mathcal{O}_P$. We know that equilibria in $P$ can only appear in $\mathcal{O}_P$. Moreover, for a vertex $o_i \in V_{\mathcal{O}}$ and for any $u_i \in \mathbb{R}^m$, $Ao_i + a + Bu_i \in \mathcal{B}$. That is, velocity vectors at any $o_i \in V_{\mathcal{O}}$ are only drawn from $\mathcal{B}$. From [28] it follows that if RCP is solvable by continuous state feedback, then $\mathcal{B} \cap C(o_i) \neq \emptyset$ for $o_i \in V_{\mathcal{O}}$. The next result says that, moreover, the zero vector cannot be the only element of $\mathcal{B} \cap C(o_i)$.

**Lemma 4.2.2.** If $P \overset{P}{\rightarrow} F_0$ by a continuous state feedback, then $\mathcal{B} \cap C(o_i) \neq \emptyset$ for all $o_i \in V_{\mathcal{O}}$.

**Proof.** Let $u(x)$ be a continuous state feedback that achieves $P \overset{P}{\rightarrow} F_0$, and define $y(x) := Ax + Bu(x) + a$. Clearly, $y(x) \neq 0$ for all $x \in P$. Suppose by the way of contradiction that for $o_i \in V_{\mathcal{O}}$, $\mathcal{B} \cap C(o_i) = \emptyset$. Since $y(o_i) \neq 0$ and necessarily $y(o_i) \in \mathcal{B}$, we get $y(o_i) \notin C(o_i)$. That is, $y(x)$ violates the invariance conditions at $o_i$. Then since the invariance conditions (4.4) are necessary for solvability of RCP by the continuous feedback $u(x)$, RCP is not solved using $u(x)$, a contradiction.

Finally, we introduce a third necessary condition for solvability of RCP on $P$ by continuous state feedback. Define

$$\text{cone}(\mathcal{O}_P) := \bigcap_{o \in V_{\mathcal{O}}} C(o).$$

In particular, $\mathcal{B} \cap \text{cone}(\mathcal{O}_P)$ is the cone of directions in $\mathcal{B}$ that simultaneously satisfy the union of all invariance conditions at all vertices of $\mathcal{O}_P$.

In the following theorem it is shown that for single-input systems the condition $\mathcal{B} \cap \text{cone}(\mathcal{O}_P) \neq \emptyset$ is necessary for solvability of RCP by continuous state feedback. This condition regards a topological obstruction to reach control by continuous state feedback. Finding the multi-input version of this condition is challenging, and it is still an open problem in the area. Notice that the following theorem is the general form of Theorem 3.2.7.

**Theorem 4.2.3.** Consider the system (4.1) defined on an $n$-dimensional polytope $P$. Suppose $m = 1$ and $\mathcal{O}_P \neq \emptyset$. If RCP is solvable by continuous state feedback, then $\mathcal{B} \cap \text{cone}(\mathcal{O}_P) \neq \emptyset$.

**Proof.** Let $u(x)$ be a continuous state feedback that achieves $P \overset{P}{\rightarrow} F_0$, and define $y(x) := Ax + Bu(x) + a$. Since the invariance conditions (4.4) are necessary for solvability by the continuous feedback $u(x)$, $u(x)$ satisfies (4.4). Let $\mathcal{O}_P = \text{co} \{o_1, \ldots, o_q\}$ and suppose by way of contradiction $\mathcal{B} \cap \text{cone}(\mathcal{O}_P) = \emptyset$. If
Figure 4.4: $B \cap \text{cone}(O_P) = 0$

$q = 1$, then $B \cap C(o_1) = 0$. This contradicts Lemma 4.2.2. Instead suppose $q > 1$ and without loss of generality (w.l.o.g.) $0 \neq b := Ao_1 + Bu(o_1) + a \in B \cap C(o_1)$. Then there exists $k \in \{2, \ldots, q\}$ such that $b \notin C(o_k)$. Consider the segment $\overline{o_1o_k}$. Since $\overline{o_1o_k} \subset O$, $y(x) \in B$ for all $x \in \overline{o_1o_k}$. Thus there exists a continuous function $c : \mathbb{R}^n \to \mathbb{R}$ such that $y(x) = c(x)b$ for $x \in \overline{o_1o_k}$, with $c(o_1) > 0$ and $c(o_k) \leq 0$. By the Intermediate Value Theorem, there exists $x^* \in \overline{o_1o_k} \subset \mathcal{P}$ such that $c(x^*) = 0$. The closed-loop system has an equilibrium in $\mathcal{P}$, a contradiction. \hfill \square

**Example 4.2.3.** Consider the system

$$
\dot{x} = \begin{bmatrix} 1 & -10 \\ 1 & -10 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

The polytope $\mathcal{P}$, shown in Figure 4.4, has vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, and $v_4 = (0, 1)$. The exit facet is $\mathcal{F}_0 = \text{co} \{v_1, v_2\}$. We find by direct computation that $\mathcal{O} = \{x \mid x_1 - 10x_2 = -1\}$, depicted as a dashed line in Figure 4.4. It can be verified that $\mathcal{O}_P = \text{co} \{o_1, o_2\}$, where $o_1 = (0, 0.1)$ and $o_2 = (1, 0.2)$, and also the invariance conditions are solvable. The subspace $B$ is shown in the figure attached at $v_1$. We know that in $\mathcal{O}_P$ the only velocity vectors available to the closed-loop system are vectors in $B$. To achieve the invariance conditions at $o_1 \in \mathcal{O}_P$, the velocity vector $b \in B$, shown in Figure 4.4, must be selected. On the other hand, $b$ violates the invariance conditions at $o_2$. At $o_2$ the velocity vector $-b \in B$ must be selected to achieve the invariance conditions. It is clear that there does
not exist a non-zero vector in $\mathcal{B}$ that satisfies the invariance conditions at both $o_1$ and $o_2$ simultaneously. That is, $\mathcal{B} \cap \text{cone}(O_P) = \mathbf{0}$. By Theorem 4.2.3, RCP is not solvable by continuous state feedback.
Chapter 5

Monotonic Reach Control on Polytopes

Results for RCP on polytopes come in one of two forms. Either one must perform a triangulation of the polytope and apply simplex-based reach control methods [29, 52, 14] (Section 3.3). Alternatively, one may impose conditions so that the design can be carried out in two independent steps: first, one assigns control inputs at the vertices of the polytope guaranteeing propitious closed-loop behavior; second, one selects any triangulation of the polytope and one forms a (continuous) PWA feedback based on the vertex control values of step one. The distinction between these two approaches is that simplex-based methods depend mainly on the choice of triangulation (Example 3.3.1) and they require stronger conditions on the vector field at the vertices of the polytope. For instance, closed-loop trajectories can only exit from one exit facet of a given simplex (Example 3.3.2). Instead, the second approach imposes weaker conditions at the vertices; for example, trajectories can exit from more than one facet of a given simplex. The penalty for the more relaxed requirements of the second method is that trajectories may not actually achieve the specification to exit the polytope. To guarantee this, an extra, exogenous condition must be added. We study in this chapter the relative merits of the two approaches, and we find both theoretically and via examples that the two methods are complementary.

Armed with the set of necessary conditions presented in the previous chapter, we begin in this chapter a study of the second approach for RCP on polytopes following the geometric point of view of [14]. The invariance conditions are taken as a necessary first computational step for solving RCP by continuous state feedback. Then we ask, given a continuous PWA feedback that satisfies the invariance conditions, what additional conditions are needed to guarantee that either (i) there is no closed-loop equilibrium
in the polytope, or (ii) closed-loop trajectories exit the polytope? Whereas under affine feedbacks for simplices the two questions are the same [29], [52], we show they are distinct under continuous PWA feedbacks for polytopes, and so the no-equilibrium condition is not enough to deduce that RCP is solvable on polytopes. To solve this problem, we formulate the monotonic reach control problem (MRCP), a restricted version of RCP in which we incorporate the flow condition into the problem statement. Then we study necessary and sufficient conditions for solvability of MRCP.

While past research on reach control on polytopes has either required strong sufficient conditions or restrictive assumptions on the system dynamics [28, 43], we initiate in this chapter a study of MRCP in which such restrictions are removed; instead geometric properties of the system are exploited to the best possible extent. In particular, the placement of $\mathcal{O}$, the set of possible equilibria, relative to the polytope $\mathcal{P}$ plays a key role, and in certain cases, clear necessary and sufficient conditions can be obtained which remove the conservativism or restrictiveness of previous work.

Additionally, we show that for generic polytopes, solvability of MRCP by PWA feedback is equivalent to solvability of RCP by PWA feedback using any choice of triangulation of $\mathcal{P}$. The latter is particularly useful when triangulation is performed by a standalone software not adapted to control problems.

This chapter is organized as follows. In the next section we explore the extent to which existing results for simplices carry over to polytopes. In Section 5.2 we formulate the monotonic reach control problem (MRCP) on polytopes. Section 5.3 shows the relationship between MRCP and solvability of RCP by arbitrary triangulations. In Section 5.4 we present an algorithm for the solvability of MRCP by continuous PWA feedback. In Section 5.5 several examples are given to compare the proposed MRCP technique to the existing simplex methods.

5.1 From Simplices to Polytopes

It is known that for simplices, RCP is solvable by affine feedback if and only if two conditions hold: (a) the invariance conditions (3.2) are solvable, and (b) the unique affine feedback built from one such solution does not admit a closed-loop equilibrium in the simplex (Theorem 3.2.2). The no-equilibrium requirement can also be expressed as the flow condition, which gives an equivalent numerical test [52]. We are interested to obtain the most immediate extension of this result for polytopes. First, we restrict our attention to continuous PWA feedback. Assuming PWA feedback, the invariance conditions remain necessary conditions for solvability of RCP on polytopes (Theorem 4.1.1). Instead, the flow condition is no longer necessary for solvability on polytopes. Indeed the statement that there is no closed-loop equilibrium is no longer equivalent to existence of a flow condition when dealing with general polytopes,
Chapter 5. Monotonic Reach Control on Polytopes

because the equivalence relies on the convexity of the closed-loop vector field. Convexity is preserved with affine feedback, but it may not be with PWA feedback. On the other hand, the flow condition affords useful properties; particularly that trajectories exit the polytope in an orderly way. In this section we begin an exploration of the extent to which results for simplices carry over to polytopes. Guided by these insights, we formulate in Section 5.2 a restricted version of RCP: we incorporate the requirement of a flow condition into the problem statement, and we call this restricted problem monotonic reach control.

First, we review some important definitions. Let \( T \) be a triangulation of the \( n \)-dimensional polytope \( P \). A point \( x \in P \) lies in the interior of precisely one simplex \( S_x \) in \( T \) whose vertices are, say, \( v_1, \ldots, v_k \) (note that \( S_x \) is not necessarily an \( n \)-dimensional simplex). Then \( x = \sum_{i=1}^{k} \lambda_i v_i \), where \( \lambda_i > 0 \) and \( \sum_{i} \lambda_i = 1 \). Coefficients \( \lambda_1, \ldots, \lambda_k \) are called the barycentric coordinates of \( x \). While in Definition 2.2.6 we provided a general definition of piecewise affine functions, we provide below a specific definition of piecewise affine feedbacks associated with a triangulation \( T \) in terms of barycentric coordinates. This definition is useful for the proofs of the main results in this chapter. Given a state feedback \( u(x) \) on \( P \), we say it is a piecewise affine feedback associated with \( T \) if for any \( x \in P \),

\[
x = \sum_{i} \lambda_i v_i \quad \implies \quad u(x) = \sum_{i} \lambda_i u(v_i),
\]

where \( \{v_i\} \) are the vertices of \( S_x \) and the \( \lambda_i \) are the barycentric coordinates of \( x \). It is easy to show that \( u(x) \) is a continuous state feedback on \( P \) [46].

Remark 5.1.1. If \( u(x) \) is a piecewise affine feedback on \( P \), then for each \( n \)-dimensional simplex \( S_k \in T \), there exist \( K_k \in \mathbb{R}^{m \times n} \) and \( g_k \in \mathbb{R}^m \) such that \( u(x) \) takes the form

\[
u(x) = K_k x + g_k, \quad x \in S_k.\]

We say \( T \) is a triangulation of \( P \) with respect to \( O \) if \( T \) is a triangulation of \( P \) such that \( O_P = P \cap O \) is a union of simplices of the triangulation. The procedure of triangulating \( P \) with respect to the affine space \( O \) was reviewed in Section 2.3.3.

Example 5.1.1. Consider the polytope in Figure 5.1. In Figure 5.1(a) \( O_P = \text{co} \{o_1, o_2\} = S_1 \cap S_2 \) is a 1-dimensional simplex in \( T \), so we say \( T \) is a triangulation with respect to \( O \). In Figure 5.1(b) \( O_P \) cannot be expressed as a union of simplices in \( T \), so \( T \) is not a triangulation with respect to \( O \).

Second, we provide a condition for excluding closed-loop equilibria in \( P \). Although the no-equilibrium condition (combined with the invariance conditions) is not enough to deduce that RCP is solvable, it is
necessary for solvability of RCP, and so the coming condition can be useful in constructing a candidate continuous feedback for RCP on polytopes. Then one can use the verification tool in Chapter 7 to verify that the candidate continuous feedback solves RCP. Suppose we are given a triangulation $T$ of $\mathcal{P}$ with respect to $O$ and we are given $u(x)$, a piecewise affine feedback defined on $T$ which satisfies the invariance conditions of $\mathcal{P}$. Define

$$b_i := A\alpha_i + Bu(o_i) + a \in B \cap C(o_i), \quad i \in I_O. \quad (5.1)$$

If we want to exclude closed-loop equilibria in $\mathcal{P}$, then we only need to concentrate on the behavior of the closed-loop vector field in $O_\mathcal{P}$.

**Lemma 5.1.1.** Let $\{b_1, \ldots, b_q \mid b_i \in B\}$ be such that $0 \notin \text{co} \{b_1, \ldots, b_q\}$. Then there exists $\beta \in B$ such that

$$\beta \cdot b_i < 0, \quad i = 1, \ldots, q.$$

**Proof.** Let $\mathcal{W}_1 := 0$ and $\mathcal{W}_2 := \text{co} \{b_1, \ldots, b_q\}$. Note that both $\mathcal{W}_1$ and $\mathcal{W}_2$ are compact, convex sets, and by assumption $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. By Theorem 2.4.2, there exists a hyperplane $H$ separating $\mathcal{W}_1$ and $\mathcal{W}_2$ strongly. Let $\xi$ be the normal vector to $H$ pointing to the side containing $\mathcal{W}_1$. Then, $\xi \cdot b_i < 0$ for $i = 1, \ldots, q$. Now let $\xi = \beta + \eta$, where $\beta \in B$ and $\eta \in \text{Ker} (B^T)$. This is always possible since the subspace $\text{Ker} (B^T)$ is the orthogonal complement of the subspace $B$ in $\mathbb{R}^n$. Then we have

$$\xi \cdot b_i = \beta \cdot b_i < 0, \quad i = 1, \ldots, q.$$
as desired.

The condition that \(0 \notin \text{co}\{b_1, \ldots, b_q\}\) can be related to the existence of closed-loop equilibria in \(\mathcal{P}\).

**Theorem 5.1.2.** Consider the system (4.1) defined on a polytope \(\mathcal{P}\). Let \(\mathbb{T}\) be a triangulation of \(\mathcal{P}\) with respect to \(\mathcal{O}\), \(u(x)\) be a piecewise affine feedback defined on \(\mathbb{T}\), and \(b_i\) be as in (5.1). If \(0 \notin \text{co}\{b_1, \ldots, b_q\}\), then the closed-loop system has no equilibrium in \(\mathcal{P}\).

**Proof.** Let \(x \in \mathcal{O}_\mathcal{P}\), and w.l.o.g. suppose \(x = \sum_{i=1}^{k} \lambda_i o_i\), where \(\lambda_i\) are the barycentric coordinates of \(x\) such that \(\lambda_i > 0\) and \(\sum_{i=1}^{k} \lambda_i = 1\). Let \(\beta \in \mathcal{B}\) be as in Lemma 5.1.1. Since \(\mathcal{O}_\mathcal{P}\) is a union of simplices in \(\mathbb{T}\), and \(y(x) := Ax + Bu(x) + a\) is affine on each simplex, we have

\[
\beta \cdot y(x) = \beta \cdot \left( \sum_{i=1}^{k} \lambda_i y(o_i) \right) = \sum_{i=1}^{k} \lambda_i (\beta \cdot b_i) < 0.
\]

Thus, \(y(x) \neq 0\) for all \(x \in \mathcal{O}_\mathcal{P}\). Since \(y(x) \neq 0\) for all \(x \in \mathcal{P} \setminus \mathcal{O}_\mathcal{P}\), the result is obtained.

The previous theorem gives a general condition in order that the closed-loop system has no equilibrium in \(\mathcal{P}\).

Third, we extend geometric conditions for simplices in [14] to analogous conditions for polytopes. In [14], two geometric sufficient conditions were presented to guarantee that there are no closed-loop equilibria in a given simplex. The first condition was that \(\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq 0\), where, as mentioned before, \(\text{cone}(\mathcal{S})\) is the tangent cone to simplex \(\mathcal{S}\) at the vertex not contained in the exit facet \(\mathcal{F}_0\). The second condition was that there is a set of linearly independent vectors \(\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}\), where it is assumed that \(v_1, \ldots, v_q\) are the vertices of \(\mathcal{S} \cap \mathcal{O}\). We would like to translate these two geometric conditions for simplices to the more general setting of polytopes. This is a straightforward exercise whose outcome is Lemmas 5.1.3 and 5.1.4 below.

First, the condition \(\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) \neq 0\), introduced in Section 4.2, has analogies with the statement for a simplex \(\mathcal{S}\) that \(\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq 0\).

**Lemma 5.1.3.** Suppose \(\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) \neq 0\). Then there exists \(\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}\) such that \(0 \notin \text{co}\{b_1, \ldots, b_q\}\).

**Proof.** Select any \(0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})\) and set \(b_i = b\) for \(i \in I_\mathcal{O}\). Since \(\text{cone}(\mathcal{O}_\mathcal{P}) \subset \mathcal{C}(o_i)\), we have \(b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\) for all \(o_i \in V_\mathcal{O}\). Clearly \(0 \notin \text{co}\{b_1, \ldots, b_q\}\).

Next, consider the condition for a simplex \(\mathcal{S}\) that there is a linearly independent set of vectors \(\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}\). Removing the restriction that vertices of \(\mathcal{O}_\mathcal{P}\) are vertices of \(\mathcal{P}\), we have the following analogous condition for polytopes.
Lemma 5.1.4. Suppose there exists a linearly independent set of vectors \( \{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i) \} \). Then \( 0 \notin \text{co} \{b_1, \ldots, b_q \} \).

Proof. If \( 0 \in \text{co} \{b_1, \ldots, b_q \} \), then \( \{b_1, \ldots, b_q \} \) are linearly dependent. \( \square \)

The previous two conditions provide the analogy for polytopes of the related geometric conditions for simplices. Then based on Theorem 5.1.2, both of the previous conditions imply there is no closed-loop equilibrium in \( \mathcal{P} \), assuming \( \mathcal{P} \) is triangulated with respect to \( \mathcal{O} \). Also, we will show in Section 5.2 that for the geometric situation when \( \mathcal{O}_P \) is a face of \( \mathcal{P} \), both of the previous conditions are sufficient that all closed-loop trajectories initiated in \( \mathcal{P} \) exit it in finite time (Theorem 5.2.3).

We conclude this section by showing that solvability of RCP on polytopes by any class of controls is inextricably linked to what happens on \( \mathcal{O}_P \). In particular, if RCP is not solved by some control strategy, it is because some trajectory encircles \( \mathcal{O}_P \), approaches \( \mathcal{O}_P \), or remains on \( \mathcal{O}_P \). The result shows that the placement of the set \( \mathcal{O} \) with respect to the polytope \( \mathcal{P} \) plays a key role in solvability of RCP on polytopes, and so in Section 5.2 we classify the study of the problem into three geometric cases based on the location of \( \mathcal{O} \) with respect to \( \mathcal{P} \).

Lemma 5.1.5. Suppose there exists \( x_0 \in \mathcal{P} \) and an open-loop control \( u(t) \) such that the associated (unique) solution \( \phi_u(t, x_0) \) of (4.1) satisfies \( \phi_u(t, x_0) \in \mathcal{P} \) for all \( t \geq 0 \). Define \( \mathcal{A} = \text{co} \{\phi_u(t, x_0) \mid t \geq 0\} \). Then \( \mathcal{A} \cap \mathcal{O}_P \neq \emptyset \).

Proof. The set \( \mathcal{A} \) is by construction convex and compact. Suppose by way of contradiction that \( \mathcal{A} \cap \mathcal{O}_P = \emptyset \). Since \( \mathcal{A} \) is convex and compact, the image of \( \mathcal{A} \) under the affine map \( x \mapsto Ax + a \), denoted \( \mathcal{W}_1 \), is also convex and compact. Also, \( \mathcal{W}_1 \cap \mathcal{B} = \emptyset \). For suppose there is a point \( x \in \mathcal{A} \) such that \( Ax + a \in \mathcal{B} \). Then \( x \in \mathcal{O}_P \), a contradiction. Thus, \( \mathcal{W}_1 \cap \mathcal{B} = \emptyset \). Note that both \( \mathcal{W}_1 \) and \( \mathcal{B} \) are convex sets, and that \( \mathcal{W}_1 \) is bounded. By Theorem 2.4.2, there exists a hyperplane \( \mathcal{H} \) separating \( \mathcal{B} \) and \( \mathcal{W}_1 \) strongly. This implies \( \mathcal{B} \) is parallel to \( \mathcal{H} \) since \( \mathcal{B} \) is a subspace. Let \( \xi \in \mathbb{R}^n \) be the normal vector to \( \mathcal{H} \) pointing to the side containing \( \mathcal{B} \). Then \( \xi \in \text{Ker} (B^T) \) and

\[
\xi \cdot (Ax + Bu + a) = \xi \cdot (Ax + a) < 0, \quad x \in \mathcal{A}, \; u \in \mathbb{R}^m.
\]

As \( \mathcal{A} \) is compact and \( \xi \cdot (Ax + a) \) is continuous in \( x \), there is \( \epsilon > 0 \) such that \( \xi \cdot (Ax + Bu + a) = \xi \cdot (Ax + a) < -\epsilon \) for all \( x \in \mathcal{A}, \; u \in \mathbb{R}^m \). In particular,

\[
\xi \cdot (A\phi_u(t, x_0) + Bu(t) + a) < -\epsilon, \quad \forall \; t \geq 0.
\]
By integrating both sides of this equation, we get

$$\xi \cdot \phi_u(t, x_0) < \xi \cdot x_0 - \epsilon t, \quad \forall \ t \geq 0.$$  \hspace{1cm} (5.2)

This contradicts the compactness of $\mathcal{P}$. We conclude $\mathcal{A} \cap \mathcal{O}_\mathcal{P} \neq \emptyset$. 

\section{5.2 Monotonic Reach Control Problem}

The previous section identified issues concerning existence of equilibria on $\mathcal{O}_\mathcal{P}$ and the relationship between failure to solve RCP and behavior of trajectories with respect to $\mathcal{O}_\mathcal{P}$. However, clear necessary and sufficient conditions for solvability are not obtained. This is because a no-equilibrium condition (in addition to solvability of invariance conditions) is not known to be sufficient to solve RCP on polytopes. Instead, we study a more restrictive form of the problem which does lead to the natural analog of results for simplices. These necessary and sufficient conditions for solvability are examined under various assumptions on the placement of $\mathcal{O}_\mathcal{P}$. We also make comparisons with the main results for simplices to better understand the limits of those results when dealing with polytopes.

\textit{Problem 5.2.1 (Monotonic Reach Control Problem (MRCP)).} Consider system (4.1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

(ii) There exists $\xi \in \mathbb{R}^n$ such that for all $x \in \mathcal{P}$, $\xi \cdot (Ax + Bu(x) + a) < 0$.

The new condition (ii) is called a \textit{flow condition}, and intuitively it represents imposing positive velocity in the direction $(-\xi)$. The problem is called “monotonic” because trajectories flow through the polytope in a common sense with respect to a foliation of parallel hyperplanes with normal vector $\xi$. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically if properties (i)-(ii) of Problem 5.2.1 hold.

Although MRCP is a restricted version of RCP, it is more general than the existing technique that depends on imposing positive velocity in the direction $h_0$ (Theorem 4.1.3). Clearly, this strong sufficient condition is a special case of the flow condition in which $\xi = -h_0$.

\textit{Remark 5.2.1.} It can be easily recognized that algebraic necessary and sufficient conditions for solvability of MRCP by continuous PWA feedback can be obtained directly based on the control values at the vertices of $\mathcal{P}$. In particular, MRCP is solvable by continuous PWA feedback if and only if there exist
\( \xi \in \mathbb{R}^n \) and \( u_i \in \mathbb{R}^m \), \( i = 1, \cdots, p \), such that

\[
\xi_i \cdot (A v_i + B u_i + a) < 0, \quad i = 1, \cdots, p,
\]

and

\[
h_j \cdot (A v_i + B u_i + a) \leq 0, \quad j \in J(v_i), \quad i = 1, \cdots, p.
\]

However, the inequalities (5.3) are bilinear inequalities whose solving is NP hard \cite{57}. In \cite{29} and \cite{52}, numerical algorithms were proposed for RCP on simplices to convert the bilinear inequalities associated with the flow condition to a series of LP problems whose number increases exponentially with the system dimension. Instead of these computationally expensive techniques, we explore the geometric conditions for solvability of MRCP in this section, which will lead to efficient synthesis methods.

Now we investigate geometric necessary and sufficient conditions for solvability of MRCP under assumptions on the placement of \( O \) with respect to \( P \). The first result when \( O_P = \emptyset \) is based on the following technical lemma.

**Lemma 5.2.1.** Consider the system (4.1) defined on a compact, convex set \( A \). If \( A \cap O = \emptyset \), then there exists \( \beta \in \ker (B^T) \) such that

\[
\beta \cdot (A x + Bu + a) < 0, \quad \forall x \in A, \quad \forall u \in \mathbb{R}^m.
\]

**Proof.** By assumption, the set \( A \) is compact, convex, and \( A \cap O = \emptyset \). Then by the same argument used in the first part of the proof of Lemma 5.1.5, there exists \( \beta \in \ker (B^T) \) such that \( \beta \cdot (A x + a) < 0 \), for all \( x \in A \). Since \( \beta \cdot B = 0 \), the results follows. \( \square \)

**Theorem 5.2.2.** Consider the system (4.1) defined on a polytope \( P \), and suppose \( O_P = \emptyset \). Then \( P \xrightarrow{\rho} F_0 \) monotonically by continuous piecewise affine feedback if and only if the invariance conditions (4.3) are solvable.

**Proof.** \( (\Rightarrow) \) Follows from the necessity of the invariance conditions (Theorem 4.1.1). \( (\Leftarrow) \) Select the control \( u_i \in \mathbb{R}^m \) for each vertex \( v_i \in V \) to satisfy the invariance conditions (4.3). Form a triangulation \( T \) of \( P \). Using the method of \cite{28} (Lemma 3.2.1), one can find unique \( K_j \) and \( g_j \) corresponding to the affine feedback \( u(x) = K_j x + g_j \) on each simplex \( S_j \in T \) such that \( u(v_i) = u_i, \quad i = 1, \ldots, p \). We obtain the piecewise affine closed-loop system \( \dot{x} = (A + BK_j)x + (a + B g_j) \). (Note that since \( P \cap O = \emptyset \), the
closed-loop system has no equilibria in $\mathcal{P}$.) By Lemma 5.2.1 there exists $\beta \in \text{Ker}(B^T)$ such that

$$\beta \cdot (Ax + Bu(x) + a) = \beta \cdot (Ax + a) < 0, \quad \forall x \in \mathcal{P}.$$ 

By the same argument expressed in (5.2) all trajectories exit $\mathcal{P}$ in finite time. Moreover, by the convexity of $Ax + Bu(x) + a$ on each simplex of $\mathcal{T}$, conditions (4.3) imply (4.4). By Lemma 4.1.2 trajectories in $\mathcal{P}$ exit through $\mathcal{F}_0$. Thus, $\mathcal{P} \xrightarrow{P} \mathcal{F}_0$ monotonically by continuous piecewise affine feedback.

Suppose we have found $\xi \in \mathbb{R}^n$ and $\epsilon > 0$ such that $\xi \cdot (Ax + Bu(x) + a) < -\epsilon, \ x \in \mathcal{P}$. Then, it can be easily shown that an upper bound on the time to leave $\mathcal{P}$ is

$$T_u = \max_{x \in \mathcal{P}} \xi \cdot x - \min_{x \in \mathcal{P}} \xi \cdot x \over \epsilon.$$ 

For more details, refer to Remark 4.9 of [28].

In [14] necessary and sufficient conditions for solvability of RCP on simplices were obtained based on the assumption that $\mathcal{O}_S$ is a face of the simplex. The same assumption for polytopes makes possible a straightforward generalization to polytopes for solvability of MRCP.

**Assumption 5.2.1.** Polytope $\mathcal{P}$ and system (4.1) satisfy the following condition: $\mathcal{O}_\mathcal{P}$ is a $\kappa$-dimensional face of $\mathcal{P}$, where $0 \leq \kappa \leq n$. In particular,

$$\mathcal{O}_\mathcal{P} = \text{co} \{v_1, \ldots, v_q\},$$

where $v_i$ is a vertex of $\mathcal{P}$, and let $V_\mathcal{O} := \{v_1, \ldots, v_q\}$.

**Theorem 5.2.3.** Consider the system (4.1) defined on $\mathcal{P}$ and suppose Assumption 5.2.1 holds. Then $\mathcal{P} \xrightarrow{P} \mathcal{F}_0$ monotonically by continuous piecewise affine feedback if and only if

(i) The invariance conditions (4.3) are solvable.

(ii) There exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap C(v_i)\}$ such that $0 \not\in \text{co} \{b_1, \ldots, b_q\}$.

**Proof.** ($\Rightarrow$) Let $y(x) := Ax + Bu(x) + a$, where $u(x)$ is a PWA feedback achieving $\mathcal{P} \xrightarrow{P} \mathcal{F}_0$ monotonically. Since $u(x)$ is a continuous state feedback, the invariance conditions are solvable [28]. Now suppose that condition (ii) does not hold. This implies $0 \not\in \text{co} \{y(v_1), \ldots, y(v_p)\}$. On the other hand, by the assumption that $\mathcal{P} \xrightarrow{P} \mathcal{F}_0$ monotonically, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y(v_i) < 0$ for $i \in I$. This implies $0$ and $\text{co} \{y(v_1), \ldots, y(v_p)\}$ are strongly separated, a contradiction.
(\Leftrightarrow) For each vertex \( v_i \in V \setminus \mathcal{O}_P \), select a control \( u_i \in \mathbb{R}^m \) to satisfy the invariance conditions (4.3). For \( v_i \in V_o \), select \( u_i \in \mathbb{R}^m \) such that \( A v_i + Bu_i + a = b_i \in \mathcal{B} \cap \mathcal{C}(v_i) \). Form a triangulation \( T \) of \( \mathcal{P} \). Using the method of [28] (Lemma 3.2.1), one can find unique \( K_j \) and \( g_j \) corresponding to the affine feedback \( u(x) = K_j x + g_j \) on each \( n \)-dimensional simplex \( S_j \in T \) such that \( u(v_i) = u_i \), \( i = 1, \ldots, p \) and \( y(v_i) = b_i \), \( i = 1, \ldots, q \). We obtain the piecewise affine closed-loop system

\[
\dot{x} = (A + BK_j)x + (a + Bg_j) =: y(x), \quad x \in \mathcal{P}.
\]

We show a flow condition holds on \( \mathcal{P} \). First, by Lemma 5.1.1, a flow condition holds for the closed loop vector field \( y(x) := (A + BK_i)x + Bg_i + a \) at vertices of \( \mathcal{O}_P \). That is, there exists \( \beta_1 \in \mathcal{B} \) such that

\[
\beta_1 \cdot y(v_i) = \beta_1 \cdot b_i < 0, \quad i = 1, \ldots, q.
\]

Next let \( \mathcal{P}' := \text{co} \{ v_i \mid v_i \in V \setminus V_o \} \). Note that because \( \mathcal{O}_P \) is a face of \( \mathcal{P} \), \( \mathcal{P}' \cap \mathcal{O} = \emptyset \). According to Lemma 5.2.1, there exists \( \beta_2 \in \ker (B^T) \) such that for all \( x \in \mathcal{P}' \), \( \beta_2 \cdot (Ax + Bu(x) + a) < 0 \). Define

\[
\beta = \alpha \beta_1 + (1 - \alpha) \beta_2
\]

for some \( \alpha \in (0, 1) \). Consider \( v_i \in V_o \). Using the fact that \( \beta_2 \cdot b_i = 0 \), we have

\[
\beta \cdot y(v_i) = \alpha \beta_1 \cdot y(v_i) < 0.
\]

Next consider \( v_i \in V \setminus V_o \). We have

\[
\beta \cdot (Av_i + Bu_i + a) = \alpha \beta_1 \cdot (Av_i + Bu_i + a) + (1 - \alpha) \beta_2 \cdot (Av_i + a).
\]

The term \( \beta_1 \cdot (Av_i + Bu_i + a) \) is a constant of unknown sign, whereas we know \( \beta_2 \cdot (Av_i + a) < 0 \). Therefore it is possible to select \( \alpha \) sufficiently small so that \( \beta \cdot (Av_i + Bu_i + a) < 0 \) for all \( v_i \in V \setminus V_o \).

We conclude that for all \( v_i \in V, \beta \cdot y(v_i) < 0 \).

Now let \( x \in \mathcal{P} \), and w.l.o.g. suppose \( x = \sum_{i=1}^{k} \lambda_i v_i \), where \( \lambda_i \) are the barycentric coordinates of \( x \) such that \( \lambda_i > 0 \) and \( \sum_{i=1}^{k} \lambda_i = 1 \). Since \( y(x) \) is affine on simplices of \( T \), we have \( y(x) = \sum_{i=1}^{k} \lambda_i y(v_i) \). Therefore, for \( x \in \mathcal{P} \),

\[
\beta \cdot y(x) = \sum_{i=1}^{k} \lambda_i \beta \cdot y(v_i) < 0.
\]

By the same argument expressed in (5.2) all trajectories exit \( \mathcal{P} \) in finite time, and by Lemma 4.1.2 they
do so through $\mathcal{F}_0$. Thus, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by continuous piecewise affine feedback.

\[ \square \]

\textbf{Remark 5.2.2.} Lemmas 5.1.3 and 5.1.4 provide sufficient geometric conditions for condition (ii) of Theorem 5.2.3. These provide the analog to the results for simplices appearing in [14].

Finally, we consider the general case when $\mathcal{O}_P \cap P^o \neq \emptyset$. This case is considerably more difficult; indeed a complete solution is not known even for simplices. Therefore, we study only single-input systems. Starting from Theorem 4.2.3, we create a monotonic flow by “pushing” a vector $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_P)$ onto each of the vertices of $\mathcal{P}$ while preserving the invariance conditions. We show that if MRCP is solvable, then it is solvable by this $b$-extremal solution. This then leads to a design procedure for constructing the appropriate controls, to be developed in Section 5.4.

Let $y \in \mathbb{R}^n$ and define the index set

$$ I_y := \{ i \in I \mid y \in \mathcal{C}(v_i) \}. $$

That is, $I_y$ is the index set of vertices for which the velocity vector $y$ satisfies the invariance conditions of that vertex. By Theorem 4.2.3, $\mathcal{B} \cap \text{cone}(\mathcal{O}_P) \neq \emptyset$ is a necessary condition for solvability of RCP when $m = 1$, so we assume we have such a $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_P)$. For the indices $i \notin I_b$, let $\pi_i$ be such that $\bar{y}_i := Av_i + B\pi_i + a \in \mathcal{C}(v_i)$ contains the maximal $b$-component. In particular, $\pi_i$ is the solution of the following LP

\begin{align}
\max_{u \in \mathbb{R}} & \quad b \cdot (Av_i + Bu + a) \\
\text{subject to:} & \quad Av_i + Bu + a \in \mathcal{C}(v_i).
\end{align}

(5.5a)

(5.5b)

Since $b \notin \mathcal{C}(v_i)$ and $m = 1$, the maximum exists and is unique, and it corresponds to one or more invariance conditions evaluating to zero at $v_i$. A rigorous proof of this claim is provided in the proof of the next theorem. Given a triangulation $\mathcal{T}$ of $\mathcal{P}$, let $\overline{u}(x)$ denote any PWA feedback associated with $\mathcal{T}$ such that $\overline{u}(v_i) = \pi_i$, $i \notin I_b$.

The following result tells us that a $b$-extremal controller, in the sense just described, can always be selected to solve MRCP, if it is solvable by PWA feedback. The second condition (ii) below, presently less meaningful, will be seen to provide a useful tool in the algorithmic solution of MRCP, to be developed in Section 5.4.

\textbf{Theorem 5.2.4.} Consider the system (4.1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_P \neq \emptyset$. Suppose $\mathcal{T}$ is a triangulation and $u(x)$ an associated continuous piecewise affine control such that $\mathcal{P} \xrightarrow{\mathcal{P}}$
\( \mathcal{F}_0 \) monotonically using \( u(x) \). Then there exist \( 0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_P) \), \( \overline{\eta}_i = Av_i + B\overline{\pi}_i + a, i \notin I_b \), where \( \overline{\pi}_i \) is the solution of the LP (5.5), and \( \overline{\pi}(x) \) as above such that:

(i) \( P \xrightarrow{P} \mathcal{F}_0 \) monotonically using \( \overline{\pi}(x) \),

(ii) \( 0 \notin \text{co} \{ b, \overline{\eta}_i \mid i \notin I_b \} \).

**Proof.** Since \( u(x) \) is a continuous state feedback solving \( P \xrightarrow{P} \mathcal{F}_0, u(x) \) satisfies the invariance conditions [28]. Let \( \mathcal{O}_P = \text{co} \{ o_1, \ldots, o_q \} \). Define \( b = Ao_1 + Bu(o_1) + a \in \mathcal{B} \cap \mathcal{C}(o_1) \). Then we must have \( Ao_i + Bu(o_i) + a = \alpha_ib \) with \( \alpha_i > 0 \) for \( i = 2, \ldots, q \). Otherwise, by the same argument as in the proof of Theorem 4.2.3, there is an equilibrium in \( P \) using \( u(x) \). We conclude \( 0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_P) \).

Secondly, we show that the solution of the LP (5.5), \( \overline{\pi}_i, i \notin I_b \), exists. Because the invariance conditions are solvable, a feasible solution of the constraint (5.5b) exists. Suppose, by contradiction, that for some \( i \notin I_b \) the maximum does not exist under the constraint (5.5b). Because \( i \notin I_b, b \notin \mathcal{C}(v_i) \) so there exists \( j \in J(v_i) \) such that \( h_j \cdot b > 0 \). Then we have that the \( b \)-component in \( Bu \) can be made arbitrary large, while also \( h_j \cdot (Av_i + Bu + a) \leq 0 \). This is clearly impossible since \( h_j \cdot b > 0 \). Because \( m = 1 \), it can be easily shown that the solution \( \overline{\pi}_i \) is also unique.

Now we show that (i)-(ii) are achieved. Since \( P \xrightarrow{P} \mathcal{F}_0 \) monotonically using \( u(x) \), there exists \( \xi \in \mathbb{R}^n \) such that

\[
\xi \cdot (Ax + Bu(x) + a) < 0, \quad x \in P.
\]

In particular, \( \xi \cdot (Ao_1 + Bu(o_1) + a) = \xi \cdot b < 0 \). Now define \( \overline{\pi}(x) := u(x) + w(x) \), where \( w(x) \) is determined by \( Bw(x) = c(x)b \), such that the PWA function \( c(x) \geq 0 \) associated with \( \mathbb{T} \) arises from assigning \( \overline{\pi}_i, i \notin I_b \), setting \( c(v_i) = 0, i \notin I_b \), and then following Lemma 3.2.1 on each simplex in \( \mathbb{T} \).

Then the invariance conditions still hold, and for \( x \in P \),

\[
\xi \cdot (Ax + B\overline{\pi}(x) + a) = \xi \cdot (Ax + Bu(x) + a) + \xi \cdot c(x)b < 0. \tag{5.6}
\]

We conclude \( P \xrightarrow{P} \mathcal{F}_0 \) monotonically using \( \overline{\pi}(x) \). By equation (5.6) and the fact that \( \xi \cdot b < 0 \), we obtain (ii).

**Remark 5.2.3.** Although the proposed results in this section are presented for the unconstrained control
problem \((u \in \mathbb{R}^m)\), they can also be extended to the constrained control problem case \((u \in \mathcal{U} \subset \mathbb{R}^m,\) where \(\mathcal{U}\) is a polytope). As a sample of this extension, we show how to extend Theorem 5.2.2. For the constrained control problem case, we say the invariance conditions are solvable if for each \(v \in V\), there exists \(u \in \mathcal{U}\) such that \(Av + Bu + a \in C(v)\). We keep the same mathematical definitions of \(B\), \(O\), and \(O_P\). Using the new definition of the invariance conditions, Theorem 5.2.2 is true in its current form. Here is the explanation. Since we have not changed the definitions of \(B\) and \(O\), it can be easily verified that Lemma 5.2.1 holds. Therefore, if \(O_P = \emptyset\), then there exists \(\beta \in \ker (B^T)\) such that \(\beta \cdot (Ax + Bu(x) + a) < 0, \forall x \in P, \forall u \in \mathbb{R}^m\). In particular, \(\beta \cdot (Ax + Bu(x) + a) < 0, \forall x \in P, \forall u \in \mathcal{U}\).

Then by using the same proof of Theorem 5.2.2 (with the new definition of the invariance conditions, and by replacing \(\mathbb{R}^m\) with \(\mathcal{U}\)), the result follows.

5.3 Arbitrary Triangulation and MRCP

In the previous section we studied MRCP and, while the results are centered on PWA feedback, they did not depend on the particular choice of triangulation. Indeed, the effect of the flow condition is to allow a solution that does not depend on the choice of triangulation. This is a useful feature if the triangulation is performed by a standalone software not adapted to control problems. An intuition emerges that the role of the flow condition is precisely to provide this invariance to triangulation. In this section we explore the extent to which this intuition is correct. For this we formulate a version of RCP under arbitrary triangulations. By arbitrary triangulation of \(P\) we mean any triangulation of \(P\) with the property that if \(v \in P\) is a vertex of a simplex belonging to \(T\), then \(v\) is a vertex of \(P\). Then we show that for generic polytopes, MRCP by PWA feedback and RCP by arbitrary triangulations are equivalent.

**Problem 5.3.1 (RCP by Arbitrary Triangulations).** Consider the system (4.1) defined on \(P\). Find a control assignment \(u_i, i \in I\), such that for an arbitrary triangulation \(T\) of \(P\), the associated PWA feedback \(u(x)\) with \(u(v_i) = u_i, i \in I\), achieves \(P \rightarrow F_0\).

Recall that a generic polytope is the convex hull of a set of points in general position in \(\mathbb{R}^n\).

**Theorem 5.3.1.** Consider the system (4.1) defined on a generic polytope \(P\). MRCP by continuous PWA feedback is solvable if and only if RCP by arbitrary triangulations is solvable.

**Proof.** \((\Rightarrow)\) By the same argument as at the end of the proof of Theorem 5.2.3, for any choice of triangulation and associated PWA feedback, \(P \rightarrow F_0\).

\((\Leftarrow)\) Suppose \(u_i, i \in I\), is a control assignment such that for any triangulation \(T\) of \(P\), the associated PWA feedback \(u(x)\) with \(u(v_i) = u_i\) achieves \(P \rightarrow F_0\). Let \(y_i := Av_i + Bu_i + a, i \in I\). We claim
0 ∉ co \{y_1, \ldots, y_p\}. Suppose not. By Caratheodory’s Theorem [49] and w.l.o.g. there exist \(\alpha_1, \ldots, \alpha_k\) with \(1 \leq k \leq n + 1\) such that \(0 = \sum_{i=1}^{k} \alpha_i y_i\) with \(\alpha_i > 0\) and \(\sum_i \alpha_i = 1\). Let \(\pi = \sum_{i=1}^{k} \alpha_i v_i \in \mathcal{P}\). Since \(\{v_1, \ldots, v_k\}\) are in general position, one can apply the placing triangulation (reviewed in Section 2.3.3) to the ordered point set \(V = \{v_1, \ldots, v_p\}\) such that \(S := \text{co} \{v_1, \ldots, v_k\}\) is a simplex of the resulting triangulation \(T\). Let \(u(x)\) be the PWA feedback associated with \(T\) such that \(u(v_i) = u_i\). Since \(u(x)\) is affine on \(S\),
\[
A\pi + Bu(\pi) + a = \sum_{i=1}^{k} \alpha_i (Av_i + Bu(v_i) + a) = \sum_{i=1}^{k} \alpha_i y_i = 0.
\]
That is \(\pi\) is an equilibrium of the closed-loop system, so RCP is not solved using this triangulation, a contradiction. We conclude \(0 ∉ \text{co} \{y_1, \ldots, y_p\}\). By the same argument as in the proof of Lemma 5.1.1, there exists \(\xi ∈ \mathbb{R}^n\) such that \(\xi \cdot y_i < 0, i ∈ I\). By the same argument as at the end of the proof of Theorem 5.2.3, we have \(\mathcal{P} \rightarrow \mathcal{F}_0\) monotonically by PWA feedback.

We remark that the sufficiency part of Theorem 5.3.1 (MRCP by continuous PWA feedback implies RCP by arbitrary triangulations) is also true for non-generic polytopes.

### 5.4 Algorithm for MRCP

In this section we present an algorithm for solving MRCP by PWA feedback for single-input systems. It is assumed that \(O_\mathcal{P} \neq \emptyset\), for if \(O_\mathcal{P} = \emptyset\), then Theorem 5.2.2 provides a solution. Also, if \(O_\mathcal{P}\) is a face of \(\mathcal{P}\), then Theorem 5.2.3 provides a solution. The algorithm, inspired by Theorem 5.2.4, is easily explained in words: for a single-input system, there are only two control directions \(b, -b \in B\). Choose \(b ∈ B \cap \text{cone}(O_\mathcal{P})\) (step 1). At all those vertices \(v_i\) where \(b ∉ C(v_i)\), we inject a maximal \(b\)-component into the vector field by choice of control \(u_i\) (step 2). If MRCP is solvable, then Theorem 5.2.4 tells us that such an extremal solution exists. A flow condition must hold with extremal control values; that is, we can find a candidate \(\xi ∈ \mathbb{R}^n\) for Problem 5.2.1 (step 3). Then we use \(\xi\) to select control values at the remaining vertices \(v_i\) where \(b ∈ C(v_i)\) (step 4). If \(\xi\) cannot be found, then the procedure is repeated with \(-b ∈ B \cap \text{cone}(O_\mathcal{P})\) (step 5). Theorem 5.4.1 shows that this procedure is sound and complete.

**Algorithm 1:**

1. Select \(0 ≠ b ∈ B \cap \text{cone}(O_\mathcal{P})\).
2. For each $i \notin I_b$, solve the LP for $\pi_i \in \mathbb{R}$:

$$\max_{u \in \mathbb{R}} \quad b \cdot (Av_i + a + Bu)$$

subject to: $Av_i + a + Bu \in \mathcal{C}(v_i)$

(5.7)

3. Solve the LP for $\xi \in \mathbb{R}^n$:

$$\xi \cdot (Av_i + a + B\pi_i) < 0, \quad i \notin I_b$$

(5.8a)

$$\xi \cdot b < 0.$$  

(5.8b)

4. If (5.8) is solvable, then for each $i \in I_b$, solve the LP for $\pi_i \in \mathbb{R}$:

$$\xi \cdot (Av_i + a + B\pi_i) < 0$$

(5.9a)

$$Av_i + a + B\pi_i \in \mathcal{C}(v_i).$$

(5.9b)

5. If (5.8) is not solvable, select $-b \in B \cap \text{cone}(\mathcal{O}_P)$ and repeat steps 2 - 4 after replacing $b$ by $-b$.

6. Form a triangulation $T$ of $P$ using only vertices of $P$. Construct an affine feedback $u(x) = K_j x + g_j$

for each $n$-dimensional simplex $S_j \in T$ such that $u(v_i) = \pi_i$, $i = 1, \ldots, p$.

**Theorem 5.4.1.** Consider the system (4.1) defined on a polytope $P$. Suppose $m = 1$ and $\mathcal{O}_P \neq \emptyset$. MRCP is solvable by continuous PWA feedback if and only if Algorithm 1 terminates successfully.

**Proof.** ($\Leftarrow$) Suppose the algorithm terminates successfully. It is required to show that the PWA feedback $u(x)$ calculated in step 6 solves MRCP on $P$. From (5.7) and (5.9b), $u(x)$ satisfies the invariance conditions (4.3). From (5.8a) and (5.9a), a flow condition holds at the vertices of $P$. By the same argument as at the end of the proof of Theorem 5.2.3 (with $\beta$ replaced by $\xi$), $P \xrightarrow{\beta} F_0$ monotonically by the PWA feedback $u(x)$.

($\Rightarrow$) Suppose that MRCP is solvable by continuous PWA feedback. By way of contradiction, we show that if Algorithm 1 does not terminate successfully, then MRCP is not solvable by continuous PWA feedback. Let's consider all the cases where the algorithm does not terminate successfully. Let $\eta_i := Av_i + B\pi_i + a, i = 1, \ldots, p$.

1. The algorithm terminates in step 1 if $B \cap \text{cone}(\mathcal{O}_P) = \emptyset$. By Theorem 4.2.3, MRCP is not solvable by continuous state feedback.
2. The algorithm terminates in step 2 if either (5.7) is not solvable, but then the invariance conditions (4.3) are not solvable. By Theorem 4.1.1, MRCP is not solvable by continuous state feedback. Alternatively, for some \( i \notin I_b \), the maximum does not exist under (5.7). But, this is impossible as shown in the second part of the proof of Theorem 5.2.4.

3. The algorithm terminates in step 4 if the LP is not feasible. As above, if (5.9b) is not solvable, then MRCP is not solvable by continuous state feedback. Instead, suppose (5.9a) is not achievable simultaneously with (5.9b). This can’t happen because \( \xi \cdot b < 0 \) and \( b \in C(v_i), i \in I_b \), so any sufficiently large \( b \)-component added to a velocity vector already satisfying (5.9b) solves the LP (see Remark 5.4.1 below).

4. The algorithm terminates in step 5 if either \(-b \notin B \cap \text{cone}(O_P)\), or one of the LP problems in steps 2–4 is not solvable (for \(-b\)). First, consider the cases where \(-b \notin B \cap \text{cone}(O_P)\), or the LP in step 3 is not solvable (for \(-b\)). For these cases, for every \( 0 \neq b \in B \cap \text{cone}(O_P) \) the LP problem in step 3 is not solvable. Equivalently, by a result analogous to Lemma 5.1.1, for every \( 0 \neq b \in B \cap \text{cone}(O_P) \), \( 0 \in \text{co} \{ b, \overline{y}_i | i \notin I_b \} \). Then by Theorem 5.2.4(ii), MRCP is not solvable by continuous PWA feedback. Secondly, consider the cases where the LP problem in step 2, or 4 is not solvable (using \(-b\)). By a similar argument to the previous two points, MRCP is not solvable by continuous state feedback.

Remark 5.4.1. Indeed if (5.9b) is satisfied using \( u'_i \), then it can be easily verified that a control that satisfies (5.9) is calculated as follows. For \( i \in I_b \), let \( c_i > \max(0, -\frac{\xi(Av_i + \alpha + Bu'_i)}{\xi \cdot b}) \). Select \( \overline{u}_i = u'_i + w(v_i) \), where \( w(v_i) \) is determined by \( Bw(v_i) = c_i b \).

Remark 5.4.2. One advantage of Algorithm 1 is that it can be used for general polytopes. Also, in Algorithm 1 the number of LP problems does not exceed \( 2p + 2 \).

We conclude this section by presenting an illustrative example for Algorithm 1.

Example 5.4.1. In this example we study a case where \( O_P \cap P^o \neq \emptyset \), and we use Algorithm 1 to solve MRCP. Consider the system

\[
\dot{x} = \begin{bmatrix} 2.1 & 2.5 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

The polytope is shown in Figure 5.2. The vertices of \( P \) are: \( v_1 = (0, 0) \), \( v_2 = (1, 0) \), \( v_3 = (1, 1) \), and \( v_4 = (0, 1) \). First, we check if RCP is solvable by the existing technique that depends on imposing the
sufficient condition $h_0 \cdot (Ax + Bu(x) + a) > 0$, $x \in \mathcal{P}$ [28]. It can be easily verified that there does not exist a control input $u(v_1)$ that achieves invariance conditions of $\mathcal{P}$ at $v_1$, and simultaneously satisfies $h_0 \cdot (Av_1 + Bu(v_1) + a) > 0$. Hence, it is impossible in this example to solve RCP by verifying the sufficient condition $h_0 \cdot (Ax + Bu(x) + a) > 0$, $x \in \mathcal{P}$. Instead, we show how to use Algorithm 1 to solve MRCP in this example. We find by direct computation that $\mathcal{O} = \{x \in \mathbb{R}^2 | 1.1x_1 + 1.5x_2 = 1\}$, depicted as a dashed line in Figure 5.2. Clearly, $\mathcal{P}^o \cap \mathcal{O} \neq \emptyset$. It can be easily verified that $\mathcal{O}_\mathcal{P} = \text{co} \{o_1, o_2\}$, where $o_1 = (0.2/3)$ and $o_2 = (0.90909, 0)$. By definition, cone($\mathcal{O}_\mathcal{P}$) is the cone of directions that simultaneously satisfy the union of the invariance conditions at all the points $o_i \in \mathcal{O}_\mathcal{P}$. So, in this example cone($\mathcal{O}_\mathcal{P}$) = $\{y \in \mathbb{R}^2 | h_1 \cdot y \leq 0\}$, where $h_1 = (-1, 0)$ as shown in Figure 5.2.

Now we follow the steps of Algorithm 1 to solve MRCP by PWA feedback.

**Step 1:** Let $b := (1, 1) \in \mathcal{B}$. It can be easily verified that $h_1 \cdot b < 0$. So, $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$.

Note: It can be verified that $I_b = \{1\}$. As seen in Figure 5.2, $b$ dips into $\mathcal{C}(v_1)$.

**Step 2:** For the vertices $v_2$, $v_3$, and $v_4$: we solve at each vertex the LP problem to obtain $\bar{\pi}_2$, $\bar{\pi}_3$, and $\bar{\pi}_4$ respectively. We get $\bar{\pi}_2 = -2.1$, $\bar{\pi}_3 = -4.6$, and $\bar{\pi}_4 = -2$. As seen in the figure, certain invariance conditions evaluate to zero at these vertices.

**Step 3:** We solve the LP problem. The problem is feasible, and we get $\xi = (-2.01, 0.01)$.

**Step 4:** For $v_1$, we solve the LP. We obtain $\bar{\pi}_1 = 0.0075$.

Note: The control value $\bar{\pi}_1$ can also be calculated as shown in Remark 5.4.1. Let $u'_1 = 0$. It can be easily verified that $u'_1$ satisfies the invariance conditions at $v_1$. We must select $c_1 > \max(0, -\frac{\xi(Av_1 + a + Bu'_1)}{\xi b}) = 0.005$. Let $c_1 = 0.0075$. Then, we find $\bar{\pi}_1 = 0.0075$.

**Step 6:** We form a triangulation $\mathcal{T}$ of $\mathcal{P}$ consisting of two simplices $S_1 = \text{co} \{v_1, v_2, v_3\}$ and $S_2 =
co \{v_1, v_3, v_4\}. The corresponding piecewise affine feedback is:

\[ u(x) = \begin{cases} 
-2.1075 & -2.5 \ x + 0.0075, \ x \in S_1 \\
-2.6 & -2.0075 \ x + 0.0075, \ x \in S_2.
\end{cases} \]

The example shows that the method of pushing \( b \) works even if \( b \) does not point to the exit facet \( F_0 \). Also, we did not need to push a large amount of \( b \) at \( v_1 \). It turns out that a small push (\( c_1 b, c_1 > 0.005 \)) is enough to construct a flow condition on \( P \). This small push is important. If we select \( c_1 = 0 \), it can be verified that 0.090909(\( Av_1 + Bu_1 + a \)) + 0.90909(\( Av_2 + Bu_2 + a \)) = 0, so a flow condition cannot be achieved on \( P \).

5.5 Examples

5.5.1 MRCP vs Simplex Methods

Example 5.5.1. Consider again Example 3.3.1. We have shown that using simplex-based approach for reach control, RCP is solvable for one triangulation but not for another. Now we show that in this example MRCP is solvable using any triangulation, thereby illustrating Theorem 5.3.1.

We choose the same control values used in Example 3.3.1 at the vertices. In particular, we have \( u_1 = 4, u_2 = 0, u_3 = 0, \) and \( u_4 = 2 \). Let \( \xi = (0,1) \). It can be verified that \( \xi \cdot (Av_i + Bu_i + a) < 0 \), for \( i = 1, \ldots, 4 \). Triangulate \( P \) using any triangulation \( T \) and construct the associated PWA feedback \( u(x) \) based on the control values at the vertices (Lemma 3.2.1). For instance, if the second triangulation in Figure 3.4 is selected, then we obtain the following PWA control law:

\[ u(x) = \begin{cases} 
-4 & 0 \ x + 4, \ x \in S_1 \\
-2 & -2 \ x + 4, \ x \in S_2.
\end{cases} \]

Since the invariance conditions of \( P \) are satisfied, we get \( P \rightarrow F_0 \) monotonically. Notice the result holds even if the first triangulation were selected.

Example 5.5.2. In the previous example simplex methods could be used to solve RCP, although there was an advantage to the solution via MRCP since it was valid for any triangulation. Now we consider an example where simplex methods fail for any choice of triangulation, but MRCP is solvable.

Consider again Example 3.3.2. We have shown that RCP is not solvable using simplex methods for any choice of triangulation. Now we study solvability of MRCP. Select control values \( u_1 = 0, u_2 = 0, \)
$u_3 = -20$, $u_4 = 0$, and $u_5 = 0$. It can be verified they satisfy the invariance conditions of $P$. Let \( \xi = (-1, -0.01, 1) \). One can then verify that \( \xi \cdot (Av_i + Bu_i + a) < 0 \), for all \( i = 1, \ldots, 5 \). Triangulate $P$ using any triangulation $T$ and construct the associated PWA feedback $u(x)$ based on the control values at the vertices (Lemma 3.2.1). For instance, if the first triangulation in Figure 3.5 is selected, then we obtain the following control law:

$$
\begin{align*}
\mathbf{u}(x) = \begin{cases} \\
\begin{bmatrix} 0 & 0 & -20 \end{bmatrix} x, & x \in S_1 \\
\begin{bmatrix} 0 & 0 & -20 \end{bmatrix} x, & x \in S_2.
\end{cases}
\end{align*}
$$

Since the invariance conditions of $P$ are satisfied, we get $P \xrightarrow{F_0} \mathcal{F}_0$ monotonically.

In the above examples we see that an advantage of MRCP over simplex-based methods is that MRCP does not require the solvability of the invariance conditions of each simplex in the triangulation. It only requires the solvability of the invariance conditions for the overall polytope, and the latter are less restrictive. Also, MRCP allows to use an arbitrary triangulation of the polytope to implement the controller.

**Example 5.5.3.** The previous examples have shown the advantages of MRCP over existing simplex-based methods. Now we consider an example where MRCP is not solvable by continuous state feedback, while RCP is solvable using simplex-based methods.
Consider the system
\[ \dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \]

The polytope is shown in Figure 5.3. The vertices of \( \mathcal{P} \) are \( v_1 = (0, 0), v_2 = (1, 0), v_3 = (1, 1), \) and \( v_4 = (0, 1). \) We find \( \mathcal{O} = \{ x \in \mathbb{R}^2 \mid x_2 = 0.5 x_1 \} \), depicted as a dashed line in Figure 5.3. First, we study if \( \mathcal{P} \rightarrow \mathcal{F}_0 \) monotonically by continuous PWA feedback. It can be verified that the invariance conditions of \( \mathcal{P} \) are solvable. However, it is clear from Figure 5.3 that there does not exist a non-zero vector in \( \mathcal{B} \) that simultaneously satisfies the union of the invariance conditions at all vertices of \( \mathcal{O}_\mathcal{P}. \) That is to say, \( \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) = 0. \) According to Theorem 4.2.3, RCP is not solvable by continuous state feedback.

Now we study solvability of RCP by discontinuous PWA feedback using simplex-based methods. Consider the triangulation shown in Figure 5.4. The control objective is \( \mathcal{S}_1 \rightarrow \mathcal{F} \) by affine feedback and \( \mathcal{S}_2 \rightarrow \mathcal{F}_0 \) by affine feedback. For the simplex \( \mathcal{S}_1 \), choose the control values \( u_1 = 1, u_3 = 0, \) and \( u_4 = 0. \) Let \( \xi_1 = (-1.9, 1). \) It can be verified that \( \xi_1 \cdot (Av_i + Bu_i + a) < 0, \) for \( i = 1, 3, 4. \) Also, invariance conditions of \( \mathcal{S}_1 \) are achieved. So, \( \mathcal{S}_1 \rightarrow \mathcal{F} \) by affine feedback. For the simplex \( \mathcal{S}_2 \), choose the control values \( u_1 = -1, u_2 = 0, \) and \( u_3 = 0. \) Let \( \xi_2 = (0, 1). \) One can verify that \( \xi_2 \cdot (Av_i + Bu_i + a) < 0, \) for \( i = 1, \ldots, 3. \) Moreover, invariance conditions of \( \mathcal{S}_2 \) are achieved. As a result, \( \mathcal{S}_2 \rightarrow \mathcal{F}_0 \) by affine feedback. So, RCP is solvable by discontinuous PWA feedback using simplex-based methods. Finally, we construct the affine feedback on each simplex (Lemma 3.2.1). The final implementation is:
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Figure 5.5: $\mathcal{P}$ for Example 5.5.4

$$u(x) = \begin{cases} 
0 & x + 1, \quad x \in S_1 \setminus S_2 \\
1 & x - 1, \quad x \in S_2.
\end{cases}$$

Notice the control law is discontinuous at the vertex $v_1$.

The above examples show that the two existing techniques are complementary. MRCP may work when simplex methods fail (Example 5.5.2). Conversely, simplex methods may work when MRCP fails (Example 5.5.3).

5.5.2 Example: Both Techniques Fail

Example 5.5.4. In this example RCP is solvable but not by the two methods discussed above. Consider the system

$$\dot{x} = \begin{bmatrix} 1.25 & 3 & 0 \\ -1 & -11.5 & -1 \\ -1.25 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ -1 \\ -10 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(5.10)

defined on a polytope $\mathcal{P}$. The polytope is shown in Figure 5.5. The vertices of $\mathcal{P}$ are $v_0 = (0,0,0)$, $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$, and $v_4 = (1,1,1)$. The exit facet is $\mathcal{F}_0 = co \{v_1, v_3, v_4\}$. Let $f(x) := Ax + Bu(x) + a$. The control objective in this example is to solve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by continuous PWA feedback.
First, we study if MRCP is solvable. We find by direct computation that
\[ \mathcal{O} = \{ x \in \mathbb{R}^3 \mid 4x_2 - 2x_3 = 0, \ 0.875x_1 + 11.2x_2 + x_3 = 1 \}, \]
depicted as a dashed line in Figure 5.5. It is clear from the figure that \( \mathcal{O} \) intersects \( \mathcal{P}^o \). We find \( \mathcal{O}_\mathcal{P} = \text{co} \{ o_1, o_2 \} \) where:
\[
\begin{align*}
o_1 &= 0.7727v_0 + 0.07576v_2 + 0.15152v_3, \\
o_2 &= 0.97972v_1 + 0.01014v_3 + 0.01014v_4.
\end{align*}
\]
It can be verified that \( \text{cone}(\mathcal{O}_\mathcal{P}) = \{ y \in \mathbb{R}^3 \mid [-1 0 0]y \leq 0 \} \). Now we follow the steps of Algorithm 1 to solve MRCP.

**Step 1:** Let \( b := (10, -1, -10) \in \mathcal{B} \). It can be easily verified that \([-1 0 0]b < 0\). So, \( 0 \notin \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) \).

Note: For each \( v_i \in \mathcal{P}, b \notin \mathcal{C}(v_i) \).

**Step 2:** We solve at each vertex of \( \mathcal{P} \) the LP problem to obtain the \( b \)-extremal control values. We get \( u(v_0) = 0, u(v_1) = -0.13158, u(v_2) = 0.1, u(v_3) = 0, \) and \( u(v_4) = 0.3158 \).

**Step 3:** We solve the LP problem. The problem is not feasible. So, we go to step 5.

**Step 5:** We have \(-b \notin \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})\), and so MRCP is not solvable by continuous PWA feedback.

Secondly, we study if RCP is solvable using simplex methods. There are two possible triangulations of \( \mathcal{P} \) shown in Figure 5.6. For the first triangulation, we examine the invariance conditions of \( \mathcal{S}_1 = \text{co} \{ v_1, v_2, v_3, v_4 \} \) at \( v_3 \). We have \( f(v_3) = Av_3 + Bu_3 + a = (10u_3, -u_3, -2 - 10u_3) \). The normal vectors to facets \( \mathcal{F}_4 = \text{co} \{ v_1, v_2, v_3 \} \) and \( \mathcal{F}_1 = \text{co} \{ v_2, v_3, v_4 \} \) in \( \mathcal{S}_1 \) are \( h_4 = (-1, -1, -1) \) and \( h_1 = (-1, 1, 1) \) respectively. The invariance conditions of \( \mathcal{S}_1 \) at \( v_3 \) yield \( h_4 \cdot f(v_3) \leq 0 \) and \( h_1 \cdot f(v_3) \leq 0 \). That is, \( u_3 \leq -2 \).
and \( u_3 \geq -0.09524 \). Thus, \( S_1 \xrightarrow{S_t} F_0 \) is not solvable by affine feedback, and RCP is not solvable using simplex methods for this triangulation. Now we try the second triangulation in Figure 5.6. The simplices of the triangulation are \( S_1 = \text{co} \{ v_0, v_1, v_3, v_4 \} \), \( S_2 = \text{co} \{ v_0, v_2, v_3, v_4 \} \), and \( S_3 = \text{co} \{ v_0, v_1, v_2, v_4 \} \). We examine the invariance conditions of \( S_1 \) at \( v_0 \). We have \( f(v_0) = Ax_0 + Bu_0 + a = (10u_0, 1 - u_0, -10u_0) \).

In this case, the normal vectors to the facets \( F_3 = \text{co} \{ v_0, v_1, v_4 \} \) and \( F_1 = \text{co} \{ v_0, v_3, v_4 \} \) in \( S_1 \) are \( h_3 = (0, 1, -1) \) and \( h_1 = (-1, 1, 0) \) respectively. The invariance conditions of \( S_1 \) at \( v_0 \) require \( h_3 \cdot f(v_0) \leq 0 \) and \( h_1 \cdot f(v_0) \leq 0 \). Equivalently, \( u_0 \leq -0.1111 \) and \( u_0 \geq 0.0909 \). Again, RCP is not solvable using simplex methods for this triangulation.

Based on the above, RCP is not solvable by continuous PWA feedback using any known technique. Since MRCP is not solvable, we know from Theorem 5.3.1 that there is no hope to solve RCP by continuous PWA feedback using arbitrary triangulations. Now we show there exists a triangulation for which RCP is solvable by continuous PWA feedback. We select \( u(v_0) = 0 \), \( u(v_1) = -0.13158 \), \( u(v_2) = 0.1 \), \( u(v_3) = 0 \), and \( u(v_4) = 0.3158 \), which is the \( b \)-extremal control assignment used above for MRCP. This assignment achieves the invariance conditions of \( P \). If we triangulate \( P \) as shown in Figure 5.6(a), then \( P \) will contain a closed-loop equilibrium point since \( 0 \in \text{co} \{ f(v_1), \cdots, f(v_4) \} \). Instead, we triangulate \( P \) as shown in Figure 5.6(b), and construct the affine feedback on each simplex (Lemma 3.2.1). We get

\[
    u(x) = \begin{cases} 
    K_1 x + g_1, & x \in S_1 \\
    K_2 x + g_2, & x \in S_2 \\
    K_3 x + g_3, & x \in S_3 
    \end{cases}
\]

where \( K_1 = [-0.1316 \ 0.4474 \ 0] \), \( g_1 = 0 \), \( K_2 = [0.2158 \ 0.1 \ 0] \), \( g_2 = 0 \), \( K_3 = [-0.1316 \ 0.1 \ 0.3474] \), \( g_3 = 0 \). It can be checked that \( 0 \notin \text{co} \{ f(v_0), f(v_1), f(v_3), f(v_4) \} \), \( 0 \notin \text{co} \{ f(v_0), f(v_2), f(v_3), f(v_4) \} \), and \( 0 \notin \text{co} \{ f(v_0), f(v_1), f(v_2), f(v_4) \} \). So, using \( u(x) \), \( P \) does not contain closed-loop equilibrium points. Indeed, we have found through exhaustive simulation of the closed-loop system that for each \( x_o \in P \), \( \phi_u(t, x_o) \) leaves \( P \) through \( F_0 \) in finite time. So, RCP is solvable by continuous PWA feedback for this triangulation.

In the above example the two known techniques for RCP on polytopes fail to explain why \( u(x) \) solves RCP; nevertheless, simulation results show that it solves RCP on \( P \). The investigation highlights that new research is needed to find a more general test for leaving \( P \) in finite time. In Chapter 7 we propose a tool for analysis of controllers for solving RCP on polytopes; specifically, a tool that tells us if closed-loop trajectories exit the polytope without resorting to exhaustive simulation of the closed-loop system.
Chapter 6

Practical Example: The Two Tank Temperature Control Problem

We present a practical example that shows how RCP can efficiently accommodate for complex specifications. Also, it shows how to use our proposed results on MRCP, presented in the previous chapter, for the constrained control problem case ($u \in \mathcal{U} \subset \mathbb{R}^m$, where $\mathcal{U}$ is a polytope). Moreover, it shows some practical advantages of MRCP over existing simplex methods.

Consider again the two tank temperature control example, presented in Section 1.1. We have shown that the traditional control techniques (stabilization and tracking) fail to achieve the complex specifications required in this example since using these techniques there is no guarantee that for any initial condition in the feasible region shown in Figure 1.3, the state trajectory will remain in the feasible region. Instead, we use the RCP approach to solve the example.

In [45] the simplex-based approach, reviewed in Section 3.3, was used to solve the two tank temperature control example. However, the simplex-based approach requires a proper triangulation of the feasible region with respect to $O$. Also, solvability using this approach depends mainly on the choice of triangulation, which leads to an iterative design method when one tries all possible triangulations until a triangulation works. Moreover, the control law for each stage is a discontinuous PWA feedback, and this requires a sudden change in the inflow temperature on the boundary between some simplices, which is not practical to implement. In the next part we show how the proposed technique, MRCP, overcomes these drawbacks.

We firstly calculate the set $O := \{ x \in \mathbb{R}^n \mid Ax + a \in B \} = \{ x \in \mathbb{R}^2 \mid x_2 = 0 \}$ (the $x_1$ axis). Then since the problem has three distinct stages with different requirements, we treat each stage as a
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Figure 6.1: Stage 1: The polytopic feasible region.

separate problem and design for it a continuous PWA feedback solving the problem. We assume that
the switch between the different stages is carried out by a supervisory controller for the discrete event
system (DES).

Stage 1:
The polytopic feasible region is shown in Figure 6.1. It can be noted that the feasible region is a slightly
modified version of the region generated by the state constraints, shown in Figure 1.3. This adjustment
is used to ensure that the invariance conditions, required for RCP, are solvable at all the vertices of the
polytopic region. Also, notice this adjustment imposes stricter constraints on the system states, and so
if the trajectories remain in the feasible region in Figure 6.1, then they will automatically remain in the
feasible region in Figure 1.3.

We divide the feasible region in Figure 6.1 into three polytopes: $P_1 = \text{co} \{v_1, \ldots, v_5\}$, $P_2 =
\text{co} \{v_3, v_4, v_6, v_7\}$, and $P_3 = \text{co} \{v_9, \ldots, v_{11}\}$, where $v_1 = (25, 0)$, $v_2 = (25, 0.5)$, $v_3 = (50, 1)$, $v_4 =
(50, -0.5)$, $v_5 = (30, -0.3)$, $v_6 = (60, -0.6)$, $v_7 = (60, 1.2)$, $v_8 = (70, 1.4)$, $v_9 = (73, 1.3)$, $v_{10} = (80, 0)$,
and $v_{11} = (80, -0.6)$. Then we solve two reachability problems on $P_1$ and $P_3$ to force all the state
trajectories to enter the desired region, $P_2$, without leaving the polytopic feasible region.

First for $P_1$, we solve $P_1 \xrightarrow{F_1} F_1$ monotonically, where $F_1 = \text{co} \{v_3, v_4\}$. Clearly, $P_1 \cap O \neq \emptyset$. It
can be easily verified that $b = (0, 1) \in B \cap \text{cone}(O_{F_1})$. To solve MRCP, Algorithm 1 proposes that we
inject the largest possible $b$-component at vertices of $P_1$, without violating the invariance conditions.
Since $u \leq 70$, the maximal $b$-component is defined at all the vertices, and we find $\pi_1 = 70$, $\pi_2 =
60$, $\pi_3 = 70$, $\pi_4 = 70$, and $\pi_5 = 70$. The corresponding closed-loop velocity vectors are: $y_1 = (0, 0.09)$,
\( \bar{y}_2 = (0.5, 0.01), \bar{y}_3 = (1, -0.08), \bar{y}_4 = (-0.5, 0.1), \) and \( \bar{y}_5 = (-0.3, 0.116) \). Then it can be verified that 
\( 0 \notin \co \{ \bar{y}_1, \cdots, \bar{y}_5 \} \), and so \( \mathcal{P}_1 \overset{P_1}{\rightarrow} \mathcal{F}_1 \) monotonically by continuous PWA feedback.

Then for \( \mathcal{P}_3 \), we solve \( \mathcal{P}_3 \overset{P_3}{\rightarrow} \mathcal{F}_3 \) monotonically, where \( \mathcal{F}_3 = \co \{ v_6, v_7 \} \). Again \( \mathcal{P}_3^* \cap \mathcal{O} \neq \emptyset \). It can be verified that \( -b = (0, -1) \in B \cap \text{cone}(\mathcal{O}_P) \). To solve MRCP, we inject the largest \( (-b) \) component at vertices of \( \mathcal{P}_3 \), without violating the invariance conditions. Since \( u \geq 25 \), the maximal \( (-b) \) component is defined at all the vertices, and we find \( \bar{u}_6 = 27, \bar{u}_7 = 25, \bar{u}_8 = 25, \bar{u}_9 = 25, \bar{u}_{10} = 25, \) and \( \bar{u}_{11} = 44 \). The corresponding closed-loop velocity vectors are: \( \bar{y}_6 = (-0.6, 0.006), \bar{y}_7 = (1.2, -0.214), \bar{y}_8 = (1.4, -0.258), \bar{y}_9 = (1.3, -0.252), \bar{y}_{10} = (0, -0.11), \) and \( \bar{y}_{11} = (-0.6, 0) \). It can be verified that 
\( 0 \notin \co \{ \bar{y}_6, \cdots, \bar{y}_{11} \} \), and so \( \mathcal{P}_3 \overset{P_3}{\rightarrow} \mathcal{F}_3 \) monotonically by continuous PWA feedback.

After we have calculated the control values at the vertices of \( \mathcal{P}_1, \mathcal{P}_2, \) and \( \mathcal{P}_3 \), we perform an arbitrary triangulation of each polytope, then we find on each simplex the unique affine feedback based on the selected values at the vertices (Lemma 3.2.1). A possible continuous PWA feedback that solves the problem is:

\[
\begin{align*}
\text{On } \mathcal{S}_1 &= \co \{ v_1, v_2, v_5 \}, u = [-1.2 - 20]x + 100, \\
\text{On } \mathcal{S}_2 &= \co \{ v_2, v_3, v_5 \}, u = [0.5778 - 8.8889]x + 50, \\
\text{On } \mathcal{S}_3 &= \co \{ v_3, v_4, v_5 \}, u = 70, \\
\text{On } \mathcal{S}_4 &= \co \{ v_3, v_4, v_6 \}, u = [-4.3 0]x + 285, \\
\text{On } \mathcal{S}_5 &= \co \{ v_3, v_6, v_7 \}, u = [-4.4778 - 1.111]x + 295, \\
\text{On } \mathcal{S}_6 &= \co \{ v_6, v_7, v_8 \}, u = [0.0222 - 1.111]x + 25, \\
\text{On } \mathcal{S}_7 &= \co \{ v_6, v_8, v_9 \}, u = [-0.0286 - 0.8571]x + 28.2, \\
\text{On } \mathcal{S}_8 &= \co \{ v_6, v_9, v_{10} \}, u = [-0.0861 - 0.4636]x + 31.8874, \\
\text{On } \mathcal{S}_9 &= \co \{ v_6, v_{10}, v_{11} \}, u = [0.85 - 31.667]x - 43.
\end{align*}
\]

The system behavior under this continuous PWA feedback is shown in Figure 6.2. The state trajectories remain in the feasible region, and they reach the polytope \( \mathcal{P}_2 \) in finite time as required.

**Stage 2:**

The polytopic feasible region is shown in Figure 6.3. Again we divide the region into three polytopes: \( \mathcal{P}_1 = \co \{ v_1, \cdots, v_5 \}, \mathcal{P}_2 = \co \{ v_3, v_4, v_6, v_7 \}, \) and \( \mathcal{P}_3 = \co \{ v_6, \cdots, v_{10} \} \), where \( v_1 = (25, 0), v_2 = (25, 0.5), v_3 = (60, 1.2), v_4 = (60, -0.6), v_5 = (30, -0.3), v_6 = (70, -0.6), v_7 = (70, 1.4), v_8 = (73, 1.3), v_9 = (80, 0), \) and \( v_{10} = (80, -0.6) \). Then we solve two reachability problems on \( \mathcal{P}_1 \) and \( \mathcal{P}_3 \) to force all the state trajectories to enter the desired region, \( \mathcal{P}_2 \), without leaving the polytopic feasible region. By a similar argument to the one used in the first stage, it can be shown that for \( \mathcal{P}_1 \) and \( \mathcal{P}_3 \), MRCP is solvable by continuous PWA feedback. A possible continuous PWA feedback that solves the problem is:

\[
\begin{align*}
\text{On } \mathcal{S}_1 &= \co \{ v_1, v_2, v_5 \}, u = [-1.2 - 20]x + 100,
\end{align*}
\]
Figure 6.2: Stage 1: System behavior under the continuous PWA feedback.

On $\mathcal{S}_2 = \text{co } \{v_2, v_3, v_5\}$, $u = [0.4762 \quad -9.5238]x + 52.8571$,
On $\mathcal{S}_3 = \text{co } \{v_3, v_4, v_5\}$, $u = 70$,
On $\mathcal{S}_4 = \text{co } \{v_3, v_4, v_6\}$, $u = [-3.6 \quad 0]x + 286$,
On $\mathcal{S}_5 = \text{co } \{v_3, v_6, v_7\}$, $u = [-4.41 \quad -4.5]x + 340$,
On $\mathcal{S}_6 = \text{co } \{v_6, v_7, v_8\}$, $u = [-0.15 \quad -4.5]x + 41.8$,
On $\mathcal{S}_7 = \text{co } \{v_6, v_8, v_9\}$, $u = [-0.6802 \quad -3.6628]x + 79.4186$,
On $\mathcal{S}_8 = \text{co } \{v_6, v_9, v_{10}\}$, $u = [1 \quad -31.6667]x - 55$.

The system behavior under the above continuous PWA feedback is shown in Figure 6.4. It is easy to observe that the control objectives for stage 2 are achieved.

**Stage 3:**

The polytopic feasible region is shown in Figure 6.5. Again we divide the region into three polytopes: $\mathcal{P}_1 = \text{co } \{v_1, \ldots, v_5\}$, $\mathcal{P}_2 = \text{co } \{v_3, v_4, v_6, v_7\}$, and $\mathcal{P}_3 = \text{co } \{v_6, \ldots, v_{12}\}$, where $v_1 = (25, 0)$, $v_2 = (25, 0.5)$, $v_3 = (45, 0.9)$, $v_4 = (45, -0.45)$, $v_5 = (30, -0.3)$, $v_6 = (50, -0.5)$, $v_7 = (50, 1)$, $v_8 = (70, 1.4)$, $v_9 = (73, 1.3)$, $v_{10} = (80, 0)$, $v_{11} = (80, -0.6)$, and $v_{12} = (60, -0.6)$. By the same argument used above for stages 1 and 2, we get:

On $\mathcal{S}_1 = \text{co } \{v_1, v_2, v_5\}$, $u = [-1.2 \quad -20]x + 100$,
On $\mathcal{S}_2 = \text{co } \{v_2, v_3, v_5\}$, $u = [0.6667 \quad -8.333]x + 47.5$,
On $\mathcal{S}_3 = \text{co } \{v_3, v_4, v_5\}$, $u = 70$,
On $\mathcal{S}_4 = \text{co } \{v_3, v_4, v_6\}$, $u = [-9 \quad 0]x + 475$,
On $\mathcal{S}_5 = \text{co } \{v_3, v_6, v_7\}$, $u = [-9 \quad 0]x + 475$,
On $\mathcal{S}_6 = \text{co } \{v_6, v_7, v_8\}$, $u = 25$. 

Figure 6.3: Stage 2: The polytopic feasible region.

On $S_7 = \text{co} \{v_6, v_8, v_9\}$, $u = 25$,

On $S_8 = \text{co} \{v_6, v_9, v_{10}\}$, $u = 25$,

On $S_9 = \text{co} \{v_6, v_{10}, v_{11}\}$, $u = [0.5278 \ - \ 31.667]x - 17.2222$,

On $S_{10} = \text{co} \{v_6, v_{11}, v_{12}\}$, $u = [0.85 \ 65]x + 15$.

The system behavior under the above continuous PWA feedback is shown in Figure 6.6.

We conclude that the MRCP approach, presented in the previous chapter, can be used to solve the two tank temperature control problem. Unlike the simplex-based approach used in [45], the control law for each stage is continuous. Another advantage of MRCP over the simplex-based approach is that for MRCP, an arbitrary triangulation can be used to implement the continuous PWA feedback, which simplifies the design procedure.
Figure 6.4: Stage 2: System behavior under the continuous PWA feedback.

Figure 6.5: Stage 3: The polytopic feasible region.
Figure 6.6: Stage3: System behavior under the continuous PWA feedback.
Chapter 7

Generalized Flow Conditions

In Chapter 5 we have shown that for continuous PWA feedbacks, the flow condition is no longer necessary for leaving a polytope $\mathcal{P}$ in finite time. Indeed, we have found many examples in which a given continuous control law does not yield a flow condition on $\mathcal{P}$; nevertheless, simulation results show that it solves RCP on $\mathcal{P}$ (see for instance Example 5.5.4). This investigation highlights that new research is needed to find a more general test for leaving $\mathcal{P}$ in finite time.

In this chapter we propose a verification tool for the the analysis of RCP, analogous to Lyapunov theorem for stability. In particular, we introduce the notion of generalized flow conditions, which give a necessary and sufficient condition for closed-loop trajectories to exit the polytope. In analogy with Lyapunov stability theory, the generalized flow condition comprises a functional that decreases along closed-loop trajectories. We provide a set of results to analyze whether an instance of RCP is solved, without resorting to exhaustive simulation of the closed-loop system. This includes a variant of the LaSalle Principle tailored to RCP. An open problem is to identify suitable classes of functionals that give rise to a generalized flow condition. We explore functions of the form $V(x) = \max\{V_i(x)\}$, and we give evidence that these functions arise naturally when RCP is solved using continuous PWA feedbacks.

The chapter is organized as follows. In the next section we introduce the generalized flow condition. In Section 7.2 we present a LaSalle Principle for RCP on polytopes. In Section 7.3 we propose a suitable class of functions to generate a generalized flow condition when using continuous PWA feedback. In Section 7.4 several examples are given illustrating the novel verification tool.
Chapter 7. Generalized Flow Conditions

7.1 Generalized Flow Conditions

In this section we introduce the main ideas of the chapter. These ideas are simple, yet, like Lyapunov theory, they have the potential to be far-reaching. We propose a tool for analysis of controllers for solving RCP on polytopes; specifically, a tool that tells us if closed-loop trajectories exit the polytope.

Suppose we are presented with an instance of RCP on a polytope and we have in hand a continuous feedback $u(x)$ as a candidate feedback solution. Clearly, this candidate should satisfy the necessary conditions for solvability of RCP by continuous feedback. Since the invariance conditions (4.4) are necessary for solvability of RCP by continuous feedback [28], we assume that $u(x)$ already achieves (4.4). Immediately we conclude that trajectories can only exit $P$ through $F_0$. Then to verify if $u(x)$ actually solves RCP on $P$, we only have to verify whether all trajectories initiated in $P$ leave it in finite time. Like Lyapunov theory, we hope to avoid a verification by exhaustive simulation.

In the literature on RCP for simplices and affine feedbacks this verification is performed using a flow condition comprising a linear functional that strictly decreases along closed-loop trajectories. Since the simplex is compact, the strictly decreasing condition means closed-loop trajectories must exit. Such a linear functional always exists if RCP is solved on a simplex by a given affine feedback [29, 52]. On the other hand, linear functionals are too restrictive as a class when verifying feedback solutions on polytopes. Indeed, we have many examples where a continuous feedback $u(x)$ is verified to solve RCP via exhaustive simulation, but no linear functional exists. These examples highlight the need for a more general tool to verify that trajectories leave $P$ in finite time.

At the same time there are well-known results in the literature providing general tests for trajectories to leave compact sets. For example, Proposition 3.5, Chapter 7, of [51] gives the following condition: let $P$ be a compact set and $V$ a continuously differentiable ($C^1$) function defined on a neighborhood of $P$. If $\dot{V}(\phi(t, x_0)) \neq 0$, then all trajectories $\phi(t, x_0)$ starting in $P$ leave it in finite time.

In sum, on the one hand, we have specific forms of the flow condition matching specific forms of the feedback, in the same way that quadratic Lyapunov functions fit with linear systems and feedbacks. On the other hand, we have general forms of the flow condition requiring only certain differentiability assumptions. A **generalized flow condition** will comprise a general functional that strictly decreases along closed-loop trajectories. An open problem is to identify the most useful classes of functionals for RCP. In this chapter we make some headway on this open problem.

We begin with the most general context. Suppose we have a feedback $u(x)$ such that the closed-loop vector field is locally Lipschitz. Suppose we have a functional $V(x)$ bounded from below on $P$ and
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satisfying

\[ V(\phi_u(t, x_0)) \leq V(x_0) - t \]  

(7.1)

for all \( x_0 \in \mathcal{P} \) and \( t \geq 0 \) such that \( \phi_u(\tau, x_0) \in \mathcal{P}, \ \tau \in [0, t] \). It is obvious that trajectories must exit \( \mathcal{P} \) in finite time. Conversely, suppose that using \( u(x) \), all trajectories leave \( \mathcal{P} \) in finite time. Then for each \( x_0 \in \mathcal{P} \), there exist \( T_{x_0} \geq 0 \) and \( \gamma_{x_0} > 0 \) such that \( \phi_u(t, x_0) \in \mathcal{P} \) for all \( t \in [0, T_{x_0}] \), and \( \phi_u(t, x_0) \notin \mathcal{P} \) for all \( t \in (T_{x_0}, T_{x_0} + \gamma_{x_0}) \). Define the map \( T: \mathcal{P} \to \mathbb{R}_+ \) by \( T(x) = T_x, \ x \in \mathcal{P} \). By uniqueness of solutions, \( T \) is a well-defined (i.e. single-valued) function. Also \( T(x) \geq 0 \) on \( \mathcal{P} \).

Thus, we have proved the following straightforward but fundamental result showing that existence of a generalized flow condition satisfying (7.1) is a necessary and sufficient condition.

**Theorem 7.1.1.** Consider the system (4.1) defined on a polytope \( \mathcal{P} \). Let \( u(x) \) be a continuous state feedback such that the closed-loop vector field is locally Lipschitz. All trajectories starting in \( \mathcal{P} \) leave it in finite time if and only if there exists \( V: \mathcal{P} \to \mathbb{R} \) such that \( V(x) \) is bounded from below on \( \mathcal{P} \) and (7.1) holds.

Next consider the case when \( V \) is locally Lipschitz; here only sufficient conditions can be obtained.

**Theorem 7.1.2.** Consider the system (4.1) defined on a polytope \( \mathcal{P} \). Let \( u(x) \) be a continuous state feedback such that the closed-loop vector field \( f(x) \) is locally Lipschitz on a neighborhood of \( \mathcal{P} \). All trajectories starting in \( \mathcal{P} \) leave it in finite time if there exists a function \( V: \mathbb{R}^n \to \mathbb{R} \) that is locally Lipschitz on a neighborhood of \( \mathcal{P} \) and satisfies

\[ D_f^+ V(x) \leq -1, \quad x \in \mathcal{P}. \]  

(7.2)

This result is a minor variant of Theorem 7.1.1. Since \( V(x) \) is continuous and \( \mathcal{P} \) is compact, \( V(x) \) is bounded from below on \( \mathcal{P} \). Then since \( V \) is locally Lipschitz, \( D_f^+ V(\phi_u(t, x_0)) = D_f^+ V(x) \) with \( x = \phi_u(t, x_0) \). By the Comparison Lemma [37], (7.2) gives

\[ V(\phi_u(t, x_0)) \leq V(x_0) - t \]  

(7.3)

for all \( x_0 \in \mathcal{P} \) and \( t \geq 0 \) such that \( \phi_u(\tau, x_0) \in \mathcal{P}, \ \tau \in [0, t] \). Then we can apply Theorem 7.1.1.

Now we focus on a particular form of \( V \) that appears to have special relevance to RCP. Let \( \mathcal{I}_0 = \)
\{1, 2, \cdots, L\}$, and suppose for each $i \in I_0$, $V_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^1$ function. Define

$$V(x) := \max_{i \in I_0} V_i(x). \quad (7.4)$$

**Lemma 7.1.3** ([19]). Consider the system (2.1) and let $V(x)$ be as in (7.4). Then $V(x)$ is locally Lipschitz and

$$D_f^+ V(x) = \max_{i \in I(x)} L_f V_i(x)$$

where $I(x) = \{i \in I_0 \mid V_i(x) = V(x)\}$.

With this choice of $V$ the condition (7.2) can be further relaxed.

**Theorem 7.1.4.** Consider the system (4.1) defined on a polytope $P$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $P$. Let $V$ be as in (7.4). All trajectories starting in $P$ leave it in finite time if

$$D_f^+ V(x) < 0, \quad x \in P.$$ 

**Proof.** Consider the sets $\Gamma_i := \{x \in P \mid i \in I(x)\}$, $i \in I_0$. Since each $V_i$ is continuous and $\Gamma_i$ can be expressed as $\Gamma_i = \bigcap_{j \neq i} \{x \in P \mid V_i(x) - V_j(x) \geq 0\}$, each $\Gamma_i$ is compact. By assumption and Lemma 7.1.3, $L_f V_i(x) < 0$ for $x \in \Gamma_i$, $i \in I_0$. Since $L_f V_i(x)$ is continuous and $\Gamma_i$ is compact, there exists $\epsilon_i > 0$ such that $L_f V_i(x) < -\epsilon_i$ for $x \in \Gamma_i, i \in I_0$. Let $\epsilon := \min_{i \in I_0} \epsilon_i$. Define $\tilde{V}(x) = \frac{V(x)}{\epsilon}$. Then for any $x \in P$

$$D_f^+ \tilde{V}(x) = \frac{1}{\epsilon} \max_{i \in I(x)} L_f V_i(x) < \frac{1}{\epsilon} \max_{i \in I(x)} -\epsilon_i \leq -1.$$ 

By Lemma 7.1.3, $\tilde{V}(x)$ is locally Lipschitz. The result follows from Theorem 7.1.2.

Note that the above results are generic in the sense that they are also true for compact non-convex sets. This fact is useful in solving examples (see Example 7.4.4). In Section 7.3, we explore in greater depth the properties of polytopes, and we provide a suitable class of functions that generates generalized flow conditions when using continuous PWA feedbacks.

**Remark 7.1.1.** It is clear from the above results that generalized flow conditions have strong analogies with Lyapunov stability theory. However, the control objective in RCP is very different, and for generalized flow conditions, the function $V$ need not be positive definite. Also, generalized flow conditions are related to barrier certificates [47], [48], which are mainly used in the verification of the safety of hybrid systems. But again the considered problems are different, and unlike barrier certificates, our function $V$
does not encode a safe set.

7.2 LaSalle Principle for RCP

In this section we study the case where a generalized flow condition has not been found, but we have identified a locally Lipschitz function \( V \) satisfying \( D^+_f V(x) \leq 0 \) for all \( x \in \mathcal{P} \). The question is whether this information is enough to deduce that closed-loop trajectories exit \( \mathcal{P} \). For this we use an argument similar to LaSalle Theorem, but we use it in the opposite way to how LaSalle Principle is normally applied. The LaSalle Principle is used in Lyapunov theory in the case when a positive definite Lyapunov function is not available, but some function that is non-increasing along solutions is available. It allows to show that trajectories tend to an invariant set. Instead, we use the LaSalle principle in the case when a generalized flow condition is not available, but some function that is non-increasing along solutions is available. We use this information to show that trajectories exit from \( \mathcal{P} \) if there is no invariant set in a particular subset of \( \mathcal{P} \). Two examples are given in Section 7.4. Thus, the novelty and the contribution is in showing that a LaSalle principle is meaningful in the context of RCP, despite RCP imposing a completely different requirement from Lyapunov stability. As such, the proof method is almost identical to the standard LaSalle principle.

**Theorem 7.2.1** (LaSalle). Consider the system (4.1) defined on a polytope \( \mathcal{P} \). Let \( u(x) \) be a continuous state feedback such that the closed-loop vector field \( f(x) \) is locally Lipschitz on a neighborhood of \( \mathcal{P} \). Suppose there exists \( V : \mathbb{R}^n \to \mathbb{R} \) that is locally Lipschitz on a neighborhood of \( \mathcal{P} \) and satisfies

\[
D^+_f V(x) \leq 0, \quad x \in \mathcal{P}.
\]

Let

\[
\mathcal{M} := \{ x \in \mathcal{P} \mid D^+_f V(x) = 0 \}.
\]

If \( \mathcal{M} \) does not contain an invariant set, then all trajectories starting in \( \mathcal{P} \) leave it in finite time.

**Proof.** Suppose \( \mathcal{M} \) does not contain invariant sets. We must show for all \( x_0 \in \mathcal{P} \), \( \phi_u(t,x_0) \) leaves \( \mathcal{P} \) in finite time. Suppose by the way of contradiction there exists \( x_0 \in \mathcal{P} \) such that \( \phi_u(t,x_0) \in \mathcal{P} \) for all \( t \geq 0 \).

First, we show that \( \lim_{t \to \infty} V(\phi_u(t,x_0)) = a \). By assumption, \( D^+_f V(x) \leq 0, \quad x \in \mathcal{P} \). In particular, \( D^+ V(\phi_u(t,x_0)) \leq 0, \quad t \geq 0 \). This implies the function \( t \mapsto V(\phi_u(t,x_0)) \) is non-increasing [51]. Also, since \( V(x) \) is a continuous function and \( \mathcal{P} \) is a compact set, then \( V(x) \) has a minimum on \( \mathcal{P} \). So,
$V(\phi_u(t,x_0))$ is bounded from below. Now since $t \mapsto V(\phi_u(t,x_0))$ is continuous, non-increasing function bounded from below, then $\lim_{t \to \infty} V(\phi_u(t,x_0)) = a$ [53].

Second, let $L^+(x_0)$ denote the positive limit set for $x_0$. We show that $L^+(x_0) \neq \emptyset$, and $V(x) = a$, $x \in L^+(x_0)$. By assumption, $\phi_u(t,x_0) \in P$ for all $t \geq 0$. Since $P$ is bounded, $\phi_u(t,x_0)$ is bounded. Using Birkhoff’s Theorem, $L^+(x_0)$ is non-empty, invariant, and compact. By definition of a limit set, for each $x \in L^+(x_0)$ there exists a sequence $\{t_i\}$ with $t_i \to \infty$ such that $\phi_u(t_i,x_0) \to x$. By continuity of $V$, we have $V(\phi_u(t_i,x_0)) \to V(x)$. However, we know that $V(\phi_u(t_i,x_0)) \to a$. Therefore, $V(x) = a$ [39].

Now we show that $L^+(x_0) \subset M$. Let $p_1 \in L^+(x_0)$ be arbitrary. Since $L^+(x_0)$ is positive invariant, then $\phi_u(t,p_1) \in L^+(x_0)$ for all $t \geq 0$. So, $V(\phi_u(t,p_1)) = a$, $t \geq 0$. This implies $D^+V(\phi_u(t,p_1)) = 0$, $t \geq 0$. In particular, we have $D^+_f V(p_1) = 0$. So, $D^+_f V(x) = 0$, $x \in L^+(x_0)$. Also, since $\phi_u(t,x_0) \in P$ for all $t \geq 0$ and $P$ is closed, then $L^+(x_0) \subset P$. We conclude that $L^+(x_0) \subset \{x \in P \mid D^+_f V(x) = 0\} = M$.

But, $L^+(x_0)$ is a non-empty invariant set in $M$. Contradiction. We conclude that for each $x_0 \in P$, $\phi_u(t,x_0)$ leaves $P$ in finite time.

Remark 7.2.1. A more direct statement of our version of the LaSalle Principle could be as follows: if $P$ contains no invariant set of the closed loop system, then all closed-loop trajectories exit $P$. The proof would be a straightforward application of Birkhoff’s theorem. This simpler statement is far less useful than Theorem 7.2.1. This is because searching for invariant sets is difficult, and Theorem 7.2.1 restricts that search to $M$. We will see by way of examples this can simplify the analysis problem.

### 7.3 PWA Feedback

In this section we focus on (continuous) PWA feedback which is widely used to solve RCP on polytopes [28, 29]. As we have shown in Chapters 3 and 5, there are currently two techniques to solve RCP on polytopes by PWA feedback: MRCP and simplex methods [29, 52]. MRCP requires the existence of a linear flow function, like the case of simplices with affine feedback, but it does not require that all the invariance conditions of individual simplices of the triangulation are satisfied by the feedback. On the other hand, simplex methods relax the requirement for a linear flow function, but they require that the invariance conditions of each simplex in the triangulation be satisfied. We have found through examples that these two techniques are complementary: one technique may work when other one fails, and conversely. Also, we have found examples in which both techniques fail; nevertheless via exhaustive simulation we have verified that a PWA feedback solves RCP on a polyope. Evidently existing techniques are not general enough to explain why a given continuous PWA feedback solves RCP.
In comparing MRCP with the proposed approach, it is clear that MRCP is merely a special case when the generalized flow function is linear. More interesting is the question of the relationship between generalized flow conditions and simplex methods. That is, what class of generalized flow functions emerges when RCP is solved by simplex methods? The answer may give clues about the preferred classes of generalized flow functions for PWA feedback. Because this problem has not been formulated before, we firstly study it in the simplest possible context.

### 7.3.1 Case A: Two Simplices

We consider the case when \( \mathcal{P} \subset \mathbb{R}^n \) consists of two simplices \( S_1 \) and \( S_2 \). Let \( \mathcal{T} = \{S_1, S_2\} \) denote the triangulation of \( \mathcal{P} \). See Figure 7.1. Saying that RCP is solved by simplex methods using a PWA feedback \( u(x) \) on \( \mathcal{T} \) means that

- \( S_1 \rightarrow_{S_1} F_0 \) using \( u_1(x) = K_1 x + g_1 \), and
- \( S_2 \rightarrow_{S_2} F \) using \( u_2(x) = K_2 x + g_2 \), where \( F = S_1 \cap S_2 \).

Moreover, the controller

\[
u(x) = \begin{cases} 
  u_1(x), & x \in S_1 \\
  u_2(x), & x \in S_2 \setminus S_1 
\end{cases}
\]

is continuous. What form does the generalized flow function take in this case?

**Theorem 7.3.1.** Consider an \( n \)-dimensional polytope \( \mathcal{P} \), a triangulation \( \mathcal{T} = \{S_1, S_2\} \) of \( \mathcal{P} \), and a continuous PWA feedback \( u(x) \) defined on \( \mathcal{T} \). If \( S_1 \rightarrow_{S_1} F_0 \) and \( S_2 \rightarrow_{S_2} F \) using \( u(x) \), then there exist affine functions \( V_1 \) and \( V_2 \) such that

\[
V(x) = \max\{V_1(x), V_2(x)\} \tag{7.5}
\]
satisfies $D^+_f V(x) < 0$ for all $x \in \mathcal{P}$.

Proof. Let $h$ be the unit normal vector to $\mathcal{F}$ pointing out of $S_2$, and define $\alpha := h \cdot x$, where $x$ is any point in $\mathcal{F}$. Also, let $S_1 = \text{co} \{v_1, \cdots, v_{n+1}\}$ and $S_2 = \text{co} \{v_2, \cdots, v_{n+2}\}$. First, by the results of [52] there exists $\xi_1 \in \mathbb{R}^n$ such that

$$\xi_1 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S_1. \quad (7.6)$$

We choose $V_1(x) = \xi_1 \cdot x$. Second, because $S_2 \xrightarrow{S_2} \mathcal{F}$, the invariance conditions hold at $v_{n+2}$. By the geometry of the simplex

$$(-h) \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0.$$  

Also, because $u(x)$ is continuous and $S_1 \xrightarrow{S_1} \mathcal{F}$, we get

$$(-h) \cdot (Av_j + Bu(v_j) + a) \leq 0, \quad v_j \in \mathcal{F}. \quad (7.7)$$

Now define

$$\xi_2 := \xi_1 - ch \quad (7.8)$$

where $c > 0$ is selected sufficiently large such that $\xi_2 \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0$. Using (7.6), (7.7), and (7.8), for $v_j \in \mathcal{F}$ we get

$$\xi_2 \cdot (Av_j + Bu(v_j) + a) < 0.$$  

Since $u(x)$ is affine on $S_2$, we get $\xi_2 \cdot (Ax + Bu(x) + a) < 0, x \in S_2$. We choose $V_2(x) = \xi_2 \cdot x + c\alpha$ and let $V(x)$ be as in (7.5). It remains to show $D^+_f V(x) < 0$ for $x \in \mathcal{P}$. Our results above give

$$L_f V_1(x) = \xi_1 \cdot (Ax + Bu(x) + a) < 0, x \in S_1$$

$$L_f V_2(x) = \xi_2 \cdot (Ax + Bu(x) + a) < 0, x \in S_2.$$  

Recall that $ch \cdot x = c\alpha$ for any $x \in S_1 \cap S_2$ and $h$ points outside of $S_2$. Thus

$$ch \cdot x = (\xi_1 - \xi_2) \cdot x \leq c\alpha, \quad x \in S_2$$

$$ch \cdot x = (\xi_1 - \xi_2) \cdot x \geq c\alpha, \quad x \in S_1.$$  

Define the sets $\Gamma_1 = \{x \in \mathcal{P} \mid \xi_1 \cdot x \geq \xi_2 \cdot x + c\alpha\}$ and $\Gamma_2 = \{x \in \mathcal{P} \mid \xi_1 \cdot x \leq \xi_2 \cdot x + c\alpha\}$. Clearly we have $\Gamma_i = S_i, i = 1, 2$. Finally, we apply Lemma 7.1.3 to conclude $D^+_f V(x) < 0$ for all $x \in \mathcal{P}$.  

\qed
Figure 7.2: Invariance conditions of $S_1$ are not solvable at $v_3$.

The goal of the previous result was to discover a form of the generalized flow function that naturally arises from solving RCP via simplex methods. The result appears to be primarily of theoretical interest because if we know that $S_1 \xrightarrow{S_1} F_0$ and $S_2 \xrightarrow{S_2} F$, then we know that RCP is solved. However, the result is of practical interest when simplex methods fail, yet a generalized flow function of the form (7.5) may still be relevant. A typical scenario is when simplex methods fail because the invariance conditions of $S_1$ are not solvable at some vertices on $F$. For instance, in Figure 7.2 the invariance conditions of $P$ are solvable at $v_3$. However, for any $u(v_3)$ that we select, the velocity vector $Av_3 + Bu(v_3) + a$ will point outside $S_1$, and so $S_1 \xrightarrow{S_1} F_0$ always fails for any PWA feedback $u(x)$ on $T$. Despite this failure, the overall problem to exit the polytope may still be solved by the same $u(x)$ and by verifying the existence of a generalized flow function of the form (7.5). The next result gives a computational test that explicitly depends on the form of generalized flow function given in the proof of Theorem 7.3.1.

**Corollary 7.3.2.** Consider an $n$-dimensional polytope $P$ and a triangulation $T = \{S_1, S_2\}$ of $P$, where $S_1 = \text{co} \{v_1, \cdots, v_{n+1}\}$ and $S_2 = \text{co} \{v_2, \cdots, v_{n+2}\}$. Let $u(x)$ be a continuous PWA feedback on $T$ that satisfies invariance conditions of $P$, and does not achieve invariance conditions of $S_1$ at vertices $v_k, \cdots, v_{n+1} \in F$, where $2 < k \leq n + 1$. Suppose that the following linear programming (LP) problem is solvable

$$
\begin{bmatrix}
 f(v_1)^T & 0 \\
 \vdots & \vdots \\
 f(v_{n+1})^T & 0 \\
 f(v_k)^T & -h \cdot f(v_k) \\
 \vdots & \vdots \\
 f(v_{n+2})^T & -h \cdot f(v_{n+2}) \\
 0 & -1
\end{bmatrix}
\begin{bmatrix}
 \xi_1 \\
 c
\end{bmatrix} < 0
$$

(7.9)
Then there exists a function $V(x)$ of the form (7.5) such that $D^+ f V(x) < 0$ for all $x \in \mathcal{P}$.

Notice that $D^+ f V(x) < 0$ for all $x \in \mathcal{P}$ implies from Theorem 7.1.4 that all trajectories initiated in $\mathcal{P}$ leave it in finite time. Then since the given continuous PWA feedback $u(x)$ satisfies the invariance conditions of $\mathcal{P}$, trajectories can leave $\mathcal{P}$ only through $\mathcal{F}_0$ (Lemma 4.1.2). Therefore, $u(x)$ solves RCP.

The idea of Corollary 7.3.2 is easy to explain. The first $n+1$ inequalities in (7.9) impose (7.6). For the vertices $v_j$ of $\mathcal{F}$ at which the invariance conditions of $\mathcal{S}_1$ are satisfied, the proof of Theorem 7.3.1 gives $\xi_2 \cdot (Av_j + Bu(v_j) + a) < 0$, so these conditions need not appear in the LP. Instead, for the remaining vertices in $\mathcal{F}$, the inequality $\xi_2 \cdot (Av_j + Bu(v_j) + a) < 0$ is explicitly imposed by the LP. The second to last inequality ensures that $c > 0$ is selected sufficiently large as in the proof of Theorem 7.3.1.

Corollary 7.3.2 provides a simple tool for verifying that all closed-loop trajectories initiated in $\mathcal{P}$ leave it in finite time for the case where existing techniques fail. An illustrative example is given in Section 7.4.

### 7.3.2 Case B: Chain of Simplices

In this subsection we study if the above results can be generalized to the case where we have a connected, not necessarily convex, chain of $L$ simplices. See Figure 7.3(b). This case is of high practical importance because of its applications in reach-avoid control problems [29], [48] such as motion of robots in complex environments [6] and anesthesia [23].

Let $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_L$ be a connected chain of simplices such that if $\mathcal{S}_i \cap \mathcal{S}_j \neq \emptyset$, then $\mathcal{S}_i \cap \mathcal{S}_j$ is a common face of $\mathcal{S}_i$ and $\mathcal{S}_j$. Suppose that for $i \in \{1, \cdots, L-1\}$, $\mathcal{F}'_i := \mathcal{S}_i \cap \mathcal{S}_{i+1}$ is an $(n-1)$-dimensional simplex. Let $h'_i$ be the unit normal vector to $\mathcal{F}'_i$ pointing toward $\mathcal{S}_i$, and suppose $h'_i \cdot x = \alpha_i$, where $x$ is any point in $\mathcal{F}'_i$. Assume that the exit facet $\mathcal{F}_0 \subset \mathcal{S}_1$. Define $\mathcal{P} := \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_L$, and $\mathcal{F}_0' := \mathcal{F}_0$.

In our study we focus on the chains of simplices that satisfy the following reasonable assumption.

**Assumption 7.3.1.** For all $i \in \{1, \cdots, L-1\}$, we have

1. $\{x \in \mathbb{R}^n \mid h'_i \cdot x = \alpha_i\} \cap \mathcal{P} = \mathcal{F}'_i$;
2. $\mathcal{S}_j \subset \{x \in \mathbb{R}^n \mid h'_i \cdot x \geq \alpha_i\}$, for all $j \in \{1, \cdots, i\}$, and $\mathcal{S}_j \subset \{x \in \mathbb{R}^n \mid h'_i \cdot x \leq \alpha_i\}$, for all $j \in \{i + 1, \cdots, L\}$.

Figure 7.3 shows examples in which Assumption 7.3.1 is satisfied, while Figure 7.4 shows an example in which Assumption 7.3.1 is not satisfied. Intuitively, Assumption 7.3.1 requires that the given chain of simplices does not make a circulation in the state space. As shown in Figure 7.4, the chain of simplices starting from $\mathcal{S}_8$ does not follow the shortest path in the state space to reach $\mathcal{F}_0$. Instead, a circulation
happens in the state space before reaching $F_0$. This may be required in some examples as a control objective, but in this case this chain can be divided into two or more chains satisfying Assumption 7.3.1, so that the next results can be applied.

**Theorem 7.3.3.** Suppose that Assumption 7.3.1 holds, and RCP is solvable on $\mathcal{P} = S_1 \cup \cdots \cup S_L$ by a continuous PWA feedback $u(x)$ using simplex methods ($S_i \xrightarrow{S} F'_{i-1}$ by affine feedback for all $i \in \{1, \cdots, L\}$). Then there exist affine functions $V_1, V_2, \cdots, V_L$ such that

$$V(x) = \max \{V_1(x), \cdots, V_L(x)\}$$

satisfies $D_f^+V(x) < 0$ for all $x \in \mathcal{P}$.

**Proof.** Since $u(x)$ solves RCP on $\mathcal{P}$ using simplex methods, then by the results of [52] there exists $\xi_1 \in \mathbb{R}^n$ such that

$$\xi_1 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S_1.$$  \hfill (7.11)

We choose $V_1(x) = \xi_1 \cdot x$. Then we study simplex $S_i, i \in \{2, \cdots, L\}$. Let $v_{n+i}$ be the vertex in $S_i$ that satisfies $v_{n+i} \notin F'_{i-1} \subseteq S_{i-1}$. Since $S_i \xrightarrow{S} F'_{i-1}$, we have

$$-h'_{i-1} \cdot (Av_{n+i} + Bu(v_{n+i}) + a) < 0, \quad i \in \{2, \cdots, L\}.$$  \hfill (7.12)
We define
\[ \xi_i = \xi_{i-1} - c_{i-1} h_i', \quad i \in \{2, \cdots, L\}, \tag{7.13} \]
where \( c_{i-1} > 0 \) is selected sufficiently large such that \( \xi_i \cdot (A v_{n+i} + B u(v_{n+i}) + a) < 0 \). Now we use mathematical induction to show that for \( i \in \{1, \cdots, L\} \), \( \xi_i \cdot (A x + B u(x) + a) < 0 \), \( x \in S_i \). For the base, the relation is achieved by (7.11). Then for the induction step, we assume that \( \xi_k \cdot (A x + B u(x) + a) < 0 \), \( x \in S_k \). We must show \( \xi_{k+1} \cdot (A x + B u(x) + a) < 0 \), \( x \in S_{k+1} \), where \( \xi_{k+1} = \xi_k - c_k h_k' \). Since \( S_k \xrightarrow{S_k} F_{k-1} \) and \( u(x) \) is continuous
\[ -h_k' \cdot (A v_j + B u(v_j) + a) \leq 0, \quad v_j \in F_k. \tag{7.14} \]
Then by assumption and (7.14), we get
\[ \xi_{k+1} \cdot (A v_j + B u(v_j) + a) < 0, \quad v_j \in F_k. \]
Since \( u(x) \) is affine on \( S_{k+1} \), we have \( \xi_{k+1} \cdot (A x + B u(x) + a) < 0, \quad x \in S_{k+1} \).

Now we choose \( V_i(x) = \xi_i \cdot x + \sum_{j=1}^{i-1} c_j \alpha_j, \) \( i \in \{2, \cdots, L\} \), and let \( V(x) \) be as in (7.10). It remains to show \( D^+_f V(x) < 0 \) for all \( x \in \mathcal{P} \). Our results above give for \( i \in \{1, \cdots, L\} \)
\[ L_f V_i(x) = \xi_i \cdot (A x + B u(x) + a) < 0, \quad x \in S_i. \]
Then for \( i \in \{1, \cdots, L-1\} \), we have
\[ V_i(x) \geq V_{i+1}(x) \Leftrightarrow (\xi_i - \xi_{i+1}) \cdot x = c_i h_i' \cdot x \geq c_i \alpha_i. \tag{7.15} \]
Table 7.1: The analogy between the proposed verification tool for RCP and Lyapunov theory for stability.

<table>
<thead>
<tr>
<th>Asymptotic Stability</th>
<th>RCP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lyapunov Function: $V : \mathcal{D} \to \mathbb{R}$ is a positive definite $C^1$ function, where $\mathcal{D}$ is a domain containing the equilibrium point $x = 0$.</td>
<td>Generalized Flow Function: $V : \mathcal{D} \to \mathbb{R}$ is a locally Lipschitz function, where $\mathcal{D}$ is a neighborhood of $\mathcal{P}$.</td>
</tr>
<tr>
<td>Negative definite condition: $V(x) &lt; 0$, $x \in \mathcal{D} \setminus {0}$</td>
<td>Generalized Flow Condition: $D_f^+ V(x) \leq -1$, $x \in \mathcal{P}$.</td>
</tr>
<tr>
<td>LaSalle theorem: $\dot{V}(x) \leq 0$, trajectories tend to an invariant set.</td>
<td>LaSalle theorem: $D_f^+ V(x) \leq 0$, trajectories leave $\mathcal{P}$ if there are no invariant sets in a particular subset of $\mathcal{P}$.</td>
</tr>
<tr>
<td>Linear systems: $V(x) = x^T P x$</td>
<td>Affine Feedbacks: $V(x) = \xi \cdot x$.</td>
</tr>
<tr>
<td>Piecewise linear (PWL) systems: PWL Lyapunov functions [12]</td>
<td>PWA Feedbacks: $V(x) = max{V_i(x)}$, where $V_i$ are affine functions.</td>
</tr>
</tbody>
</table>

Clearly, (7.15) holds for $(\leq)$, $(>)$, and $(<)$.

Define the sets $\Gamma_i = \bigcap_{j \neq i} \{x \in \mathcal{P} \mid V_i(x) \geq V_j(x)\}$, $i \in \{1, \cdots, L\}$. Now we show that $\Gamma_i \subseteq \mathcal{S}_i$, $i \in \{1, \cdots, L\}$. First, for any $x \in \Gamma_1$, we have by definition $V_1(x) \geq V_2(x)$, which is equivalent from (7.15) to $h'_1 \cdot x \geq \alpha_1$. Also, by Assumption 7.3.1, it can be easily shown that $(\mathcal{S}_2 \cup \cdots \cup \mathcal{S}_L) \setminus \mathcal{S}_1 \subseteq \{x \in \mathbb{R}^n \mid h'_1 \cdot x < \alpha_1\}$. Therefore, $\Gamma_1 \subseteq \mathcal{S}_1$. Then, we study $\Gamma_i$, $i \in \{2, \cdots, L-1\}$. By a similar argument to the case where $i = 1$, it can be shown that $\Gamma_i \subseteq (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_i)$. By Assumption 7.3.1, we have for any $x \in (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{i-1}) \setminus \mathcal{S}_i$, $h'_{i-1} \cdot x > \alpha_{i-1}$, which is equivalent from (7.15) to $V_{i-1}(x) > V_i(x)$. Therefore, it must be $\Gamma_i \subseteq \mathcal{S}_i$, $i \in \{2, \cdots, L-1\}$. Finally, for $x \in \Gamma_L$, we have by definition $V_{L-1}(x) \leq V_L(x)$, which is equivalent from (7.15) to $h'_{L-1} \cdot x \leq \alpha_{L-1}$. Again by Assumption 7.3.1, we have $(\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{L-1}) \setminus \mathcal{S}_L \subseteq \{x \in \mathbb{R}^n \mid h'_{L-1} \cdot x > \alpha_{L-1}\}$, and so $\Gamma_L \subseteq \mathcal{S}_L$.

Now we study $L_f V_i(x)$ on $\Gamma_i$, $i \in \{1, \cdots, L\}$. We have $L_f V_i(x) = \xi_i \cdot (Ax + Bu(x) + a) < 0$, $x \in \Gamma_i \subseteq \mathcal{S}_i$. This implies, from Lemma 7.1.3, that $D_f^+ V(x) < 0$, $x \in \mathcal{P}$.

Theorem 7.3.3 appears to be primarily of theoretical interest. However, it has also practical importance when simplex methods fail because invariance conditions of some simplices in the chain are not solvable, yet a generalized flow function of the form (7.10) may still be relevant. Similarly to Corollary 7.3.2, we can check the existence of a generalized flow condition based on a function $V(x)$ in the form (7.10) by solving an LP problem in the decision variables $\xi_1, c_1, \cdots, c_{L-1}$. An illustrative example is given in the next section.

To sum up the results of this chapter, we present Table 7.1 which shows the analogy between the proposed verification tool for RCP and Lyapunov theory for stability.
7.4 Examples

Example 7.4.1. In this example we show how to use the generalized flow condition to check if a given locally Lipschitz control law \( u(x) \) solves RCP on \( P \). Consider the system

\[
\dot{x} = \begin{bmatrix}
0 & -1 & -1 \\
-1 & -2 & -1 \\
1 & 0 & -2
\end{bmatrix} x + \begin{bmatrix}
0 \\
-1 \\
0.5
\end{bmatrix} u + \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]  (7.16)

defined on a polytope \( P \). The polytope is shown in Figure 7.5. The vertices of \( P \) are \( v_0 = (0,0,0) \), \( v_1 = (1,0,0) \), \( v_2 = (0,1,0) \), \( v_3 = (0,0,1) \), and \( v_4 = (1,1,1) \). The exit facet is \( F_0 = \text{co} \{ v_1, v_3, v_4 \} \).

Let \( S_1 := \text{co} \{ v_1, v_2, v_3, v_4 \} \) and \( S_2 := \text{co} \{ v_0, v_1, v_2, v_3 \} \). Suppose that the following continuous PWA feedback is used on \( P \)

\[
u(x) = \begin{cases}
-1 & -1 & -1 \\
0 & 0 & 0
\end{cases} x + 1, \quad x \in S_1 \\
0 & 0 & 0 \\
0.5 & 1 & 1
\]

It is required to verify that \( u(x) \) solves RCP on \( P \). Let \( f(x) := Ax + Bu(x) + a \). Then \( f(v_0) = (1,1,1) \), \( f(v_1) = (1,0,2) \), \( f(v_2) = (0,-1,1) \), \( f(v_3) = (0,0,-1) \), and \( f(v_4) = (-1,-1,-1) \). It is easily verified that the \( f(v_j) \) satisfy the invariance conditions (4.3). As continuous PWA feedback is used, invariance conditions (4.3) imply (4.4), and so trajectories that leave \( P \) do so only through \( F_0 \). Then, we check that \( P \) does not contain closed-loop equilibria. It can be verified that \( 0 \notin \text{co} \{ f(v_1), f(v_2), f(v_3), f(v_4) \} \).
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and \( 0 \notin \text{co} \{ f(v_0), f(v_1), f(v_2), f(v_3) \} \). Therefore, there exists a flow condition on each simplex \( S_i \) \[52\]. However, the no-equilibrium condition is not sufficient to conclude that all trajectories initiated in \( P \) leave it in finite time, as discussed in Chapter 5.

To verify that closed-loop trajectories leave \( P \), we first check if MRCP is satisfied. We compute
\[
0.5f(v_0) + 0.5f(v_4) = 0 \text{ or } 0 \in \text{co} \{ f(v_0), f(v_1), f(v_2), f(v_3) \},
\]
and so there does not exist \( \xi \in \mathbb{R}^n \) such that \( \xi \cdot f(x) < 0, x \in P \). So, \( u(x) \) does not solve MRCP.

Next we check if \( u(x) \) solves RCP using simplex methods. Let \( F := S_1 \cap S_2 \), and \( h \) be the unit normal vector to \( F \) pointing toward \( S_1 \). We compute \( h = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \) and \( h \cdot f(v_3) < 0 \); hence, the velocity vector \( f(v_3) \) dips inside \( S_2 \). Therefore, \( u(x) \) does not achieve \( S_1 \rightarrow F_0 \), and \( u(x) \) does not solve RCP by simplex methods.

We conclude that RCP is not solved by \( u(x) \) via any existing technique. Now we check if a generalized flow condition exists. It is verified that \( v_3 \) is the only vertex on \( F \) at which the velocity vector \( f(v_3) \) does not satisfy the invariance conditions of \( S_1 \). Based on Corollary 7.3.2 we check the existence of a generalized flow condition with \( V \) of the form (7.5) by solving the LP

\[
\begin{bmatrix}
  f(v_1)^T & 0 \\
  f(v_2)^T & 0 \\
  f(v_3)^T & 0 \\
  f(v_4)^T & 0 \\
  f(v_3)^T & -h \cdot f(v_3) \\
  f(v_0)^T & -h \cdot f(v_0) \\
  0 & -1
\end{bmatrix}
\begin{bmatrix}
  \xi_1 \\
  c
\end{bmatrix} < 0.
\]

A solution of this LP is \( \xi_1 = (-167.855, 150.962, 66.055) \), and \( c = 71.303 \). Then, we calculate \( \xi_2 = \xi_1 - ch = (-209.023, 109.795, 24.888) \). We define \( V(x) = \max(\xi_1 \cdot x, \xi_2 \cdot x + c) \). By Corollary 7.3.2, \( D^+_V(x) < 0 \), \( x \in P \). By Theorem 7.1.4, all closed-loop trajectories exit \( P \) in finite time. Since the invariance conditions of \( P \) hold, they do so only through \( F_0 \). We conclude that \( u(x) \) solves RCP on \( P \).

Example 7.4.2. In this example we show how to use our proposed results in this chapter to analyze reachability problems of multi-affine systems on rectangles, which receive special interest because of their applications in the control of multi-affine hybrid systems \[7\], \[8\]. Consider the multi-affine vector field

\[
\dot{x} = f(x) = (-x_1 - x_1x_2 + 1, -3x_1 - 3x_2 + 2x_1x_2 + 2)
\]

(7.17)
defined on a rectangle $\mathcal{P}$. The rectangle is shown in Figure 7.6. The vertices of $\mathcal{P}$ are $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, and $v_4 = (0, 1)$. The exit facet is $\mathcal{F}_0 = \text{co} \{v_1, v_2\}$. It is required to check if $\mathcal{P} \xrightarrow{P} \mathcal{F}_0$.

By direct computation, we get $f(v_1) = (1, 2)$, $f(v_2) = (0, -1)$, $f(v_3) = (-1, -2)$, and $f(v_4) = (1, -1)$.

It can be verified that $f(v_i)$, $i = 1, \cdots, 4$, satisfy the invariance conditions (4.3). By Corollary 1 of [8], this implies that invariance conditions (4.4) are achieved. Therefore, trajectories that leave $\mathcal{P}$ do so only via $\mathcal{F}_0$.

Now it remains to check if all trajectories initiated in $\mathcal{P}$ leave it in finite time. First, in [8] the sufficient condition $h_0 \cdot f(x) > 0$, $x \in \mathcal{P}$, was used to check leaving $\mathcal{P}$ in finite time. However, in this example it can be easily verified that $h_0 \cdot f(v_1) < 0$.

Secondly, we check if $f(x)$ constructs a linear flow function on $\mathcal{P}$. It can be verified that $0.5f(v_1) + 0.5f(v_3) = 0$, or $0 \in \text{co} \{f(v_1), \cdots, f(v_4)\}$. This implies there does not exist $\xi \in \mathbb{R}^n$ such that $\xi \cdot f(x) < 0$, $x \in \mathcal{P}$.

Now we show using LaSalle Principle for RCP that all trajectories initiated in $\mathcal{P}$ leave it in finite time. Let $V(x) = \xi \cdot x$, where $\xi = (-1, 0.5)$. It can be verified that $\xi \cdot f(v_i) \leq 0$, $v_i \in \mathcal{P}$. The equality holds only at $v_1$ and $v_3$. By the convexity of the multi-affine vector field on $\mathcal{P}$ (Proposition 2 of [8]), we get $L_f V(x) = \xi \cdot f(x) \leq 0$, $x \in \mathcal{P}$.
Then, we identify the set $M$. The vector field $f(x)$ can be expressed as [8]:

$$f(x) = \sum_{i=1}^{4} \alpha_i f(v_i),$$

where $\alpha_1 = (1 - x_1)(1 - x_2)$, $\alpha_2 = x_1(1 - x_2)$, $\alpha_3 = x_1x_2$, and $\alpha_4 = (1 - x_1)x_2$. In particular, we have

$$\xi \cdot f(x) = \sum_{i=1}^{4} \alpha_i \xi \cdot f(v_i).$$

So, $\xi \cdot f(x) = 0$ is achieved only at $x \in P$ satisfying $\alpha_2 = 0$ and $\alpha_4 = 0$ simultaneously. But, this happens only at $v_1$ and $v_3$. We conclude $M = \{v_1\} \cup \{v_3\}$. Since $f(v_1) \neq 0$ and $f(v_3) \neq 0$, $M$ does not contain an invariant set.

By Theorem 7.2.1, all trajectories initiated in $P$ leave it in finite time. Since invariance conditions of $P$ are achieved, we have $P \rightarrow F_0$.

It should be noted that in this example another method can be used to show that all trajectories initiated in $P$ leave it in finite time; the method is to prove that $P$ does not contain equilibrium points or closed orbits. However, our proposed method is simpler, and can be applied to examples of higher dimension ($n > 2$).

**Example 7.4.3.** In this example we show how to use the generalized flow condition to check if a given continuous PWA feedback solves RCP on a connected chain of simplices. Consider the system

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

defined on the chain of simplices shown in Figure 7.7. The vertices are $v_1 = (6, 6)$, $v_2 = (4, 6)$, $v_3 = (6, 4)$, $v_4 = (3, 4)$, $v_5 = (2, 5)$, $v_6 = (2, 3)$, $v_7 = (1, 4)$, $v_8 = (-1, 3)$, $v_9 = (1, 1)$, and $v_{10} = (0, 0)$. Let $P = S_1 \cup \cdots \cup S_8$, and the exit facet $F_0 = \text{co} \{v_1, v_2\}$. It is required to solve $P \rightarrow F_0$ by continuous PWA feedback.

We check solvability of the problem by simplex-based methods [29], [52], [6]. In particular, it is required to achieve $S_i \rightarrow F_{i-1}$ by affine feedback, $i \in \{1, \cdots, 8\}$. We examine invariance conditions of the simplex $S_6$ at the vertex $v_8$. We have $f(v_8) = Av_8 + Bu_8 + a = (-8 + u_8, -2)$. The normal vector to the facet $F_7 = \text{co} \{v_6, v_8\}$ in $S_6$ is $h_7 = (0, -1)$. The invariance conditions of $S_6$ at $v_8$ yield $h_7 \cdot f(v_8) \leq 0$. Equivalently, $2 \leq 0$, which is impossible. Thus, $S_6 \rightarrow F_5$ is not solvable by affine feedback, and RCP is not solvable by simplex-based methods.
Despite this failure, we show that a continuous PWA feedback on the same triangulation solves RCP on $\mathcal{P}$ by verifying a generalized flow condition on $\mathcal{P}$. Suppose that the following control assignment is selected: $u(v_1) = -6$, $u(v_2) = 8$, $u(v_3) = -6$, $u(v_4) = 1$, $u(v_5) = 2.5$, $u(v_7) = 4.25$, $u(v_8) = 18$, $u(v_9) = 5$, and $u(v_{10}) = 5.5$. Then, the unique affine feedback is constructed on each simplex (Lemma 3.2.1). Now it is required to check if this continuous PWA feedback $u(x)$ solves RCP on $\mathcal{P}$. Let $f(x) := Ax + Bu(x) + a$. Then $f(v_1) = (0, 2)$, $f(v_2) = (10, 0)$, $f(v_3) = (0, 4)$, $f(v_4) = (1, 1)$, $f(v_5) = (10, -1)$, $f(v_6) = (0.5, 1)$, $f(v_7) = (0.25, -1)$, $f(v_8) = (10, -2)$, $f(v_9) = (1, 2)$, and $f(v_{10}) = (-0.5, 2)$. It is easy to verify that $f(v_i)$ satisfy the invariance conditions associated to the facets shown in red in Figure 7.7, and so trajectories that leave $\mathcal{P}$ do so via $\mathcal{F}_0$. Hence, it remains to check if all closed-loop trajectories initiated in $\mathcal{P}$ leave it in finite time.

First, we check if $u(x)$ constructs a linear flow function on $\mathcal{P}$. It can be easily verified that $\frac{2}{3}f(v_7) + \frac{1}{3}f(v_{10}) = 0$. Therefore, there does not exist $\xi \in \mathbb{R}^n$ such that $\xi \cdot f(x) < 0$, $x \in \mathcal{P}$.

Then we check the existence of a generalized flow condition based on a function $V(x)$ in the form (7.10). It can be verified that $h'_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\alpha_1 = \frac{10}{\sqrt{2}}$, $h'_2 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\alpha_2 = \frac{3}{\sqrt{5}}$, $h'_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\alpha_3 = \frac{7}{\sqrt{2}}$, $h'_4 = (1, 0)$, $\alpha_4 = 2$, $h'_5 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\alpha_5 = \frac{5}{\sqrt{2}}$, $h'_6 = (0, 1)$, $\alpha_6 = 3$, $h'_7 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $\alpha_7 = \sqrt{2}$. We solve the following LP to verify the existence of a generalized flow condition in the form (7.10).
Find $\xi_1, c_1, \cdots, c_7$ such that
\begin{equation}
\xi_1 \cdot f(v_i) < 0, \ v_i \in S_1
\end{equation}
\begin{equation}
(\xi_1 - \sum_{j=1}^{i-1} c_j h'_j) \cdot f(v_{n+i}) < 0, \ i = 2, \cdots, 8
\end{equation}
\begin{equation}
(\xi_1 - \sum_{j=1}^{5} c_j h'_j) \cdot f(v_7) < 0
\end{equation}
\begin{equation}
(\xi_1 - \sum_{j=1}^{6} c_j h'_j) \cdot f(v_8) < 0
\end{equation}
\begin{equation}
-c_i < 0, \ i = 1, \cdots, 7.
\end{equation}

The inequalities (7.19b) ensure that $c_i > 0, \ i = 1, \cdots, 7$, are selected sufficiently large as in the proof of Theorem 7.3.3. The inequalities (7.19c) and (7.19d) are added because $f(v_7)$ and $f(v_8)$ violate the invariance conditions of the simplices $S_5$ and $S_6$ respectively. The LP is solvable, and we get $\xi_1 = (-7.1782, -9.4655), c_1 = 3.2771, c_2 = 24.9878, c_3 = 2.6109, c_4 = 5.2658, c_5 = 1.6826, c_6 = 79.2398,$ and $c_7 = 73.3324$. Then by (7.13), we find $\xi_2 = (-9.4955, -11.7828), \xi_3 = (-31.8452, -0.6079), \xi_4 = (-33.6914, -2.4541), \xi_5 = (-38.9572, -2.4541), \xi_6 = (-40.1469, -3.6439), \xi_7 = (-40.1469, -82.8837),$ and $\xi_8 = (-92.008, -134.7375).$ We define
\begin{equation}
V(x) = \max_{i \in \{1, \cdots, 8\}} \xi_i \cdot x + \sum_{j=1}^{i-1} c_j \alpha_j.
\end{equation}

It can be verified that for $i \in \{1, \cdots, 8\}$, we have $\Gamma_i \subseteq S_i$ and $L_f V_i(x) < 0, \ x \in S_i$. This implies from Lemma 7.1.3 that $D_f^+ V(x) < 0, \ x \in \mathcal{P}$. Then by Theorem 7.1.4 (which also applies to nonconvex sets), all closed-loop trajectories exit $\mathcal{P}$ in finite time. Since the invariance conditions hold, they do so only through $\mathcal{F}_0$. We conclude that $u(x)$ solves RCP on $\mathcal{P}$.

**Example 7.4.4.** Consider again Example 5.5.4. As discussed before, this example is not solvable by continuous PWA feedback using any known technique. However, simulation results show that the following continuous PWA feedback solves RCP.

\begin{align*}
u(x) = \begin{cases} 
K_1 x + g_1, & x \in S_1 \\
K_2 x + g_2, & x \in S_2 \\
K_3 x + g_3, & x \in S_3
\end{cases}
\end{align*}

where $K_1 = [-0.1316 \ 0.4474 \ 0], \ g_1 = 0, \ K_2 = [0.2158 \ 0.1 \ 0], \ g_2 = 0, \ K_3 = [-0.1316 \ 0.1 \ 0.3474], \ g_3 = 0.$
$g_3 = 0$. Now we show, using the results obtained in this chapter, that the above continuous PWA feedback $u(x)$ solves RCP on $\mathcal{P}$.

**Proposition 7.4.1.** Given the polytope $\mathcal{P}$ and system (5.10), $\mathcal{P} \xrightarrow{\rho} \mathcal{F}_0$ using $u(x)$.

**Proof.** First, we show that all closed-loop trajectories initiated in $\mathcal{P}$ leave it in finite time. Let $f(x) := Ax + Bu(x) + a$, $V(x) := \max(V_1(x), V_2(x))$, where $V_1(x) = x_3 - x_1$ and $V_2(x) = 0$. We study $D_f^TV(x)$ on $\mathcal{P}$. In this example we have $\Gamma_1 = \{x \in \mathcal{P} \mid x_3 - x_1 \geq 0\}$ and $\Gamma_2 = \{x \in \mathcal{P} \mid x_3 - x_1 \leq 0\}$. Then, we study $L_fV_1(x)$ on $\Gamma_1$. We have $L_fV_1(x) = -2.5x_1 - 2x_2 - 2x_3 - 20u(x)$. It can be verified that $L_fV_1(x) \leq 0$, $x \in \Gamma_1$, with equality holding only at $v_0$. Also, it is clear that $L_fV_2(x) = 0$, $x \in \Gamma_2$. So, for all $i = 1, 2$ we have $L_fV_i(x) \leq 0$, $x \in \Gamma_i$. Using Lemma 7.1.3, it follows that $D_f^TV(x) \leq 0$, $x \in \mathcal{P}$. Also, equality holds for all $x \in \mathcal{P}$ satisfying $x_3 - x_1 \leq 0$. This gives $\mathcal{M} = \{x \in \mathcal{P} \mid x_3 - x_1 \leq 0\}$.

Now to apply Theorem 7.2.1, it remains to show $\mathcal{M}$ does not contain invariant sets. The set $\mathcal{M}$ is compact, and it can be verified that $\mathcal{M} \subset (\mathcal{S}_1 \cup \mathcal{S}_3)$. Consider the function $W(x) := f(x)^TPf(x)$ defined on $\mathcal{M}$, where $P$ is a symmetric matrix determined by solving the following set of linear matrix inequalities (LMIs)

$$(A + BK_1)^TP + P(A + BK_1) < 0,$$

$$(A + BK_3)^TP + P(A + BK_3) < 0.$$ 

The problem is feasible, and we get

$$P = \begin{bmatrix} 0.2066 & 0.0941 & 0.0791 \\ 0.0941 & 0.086 & 0.0001 \\ 0.0791 & 0.0001 & 0.1538 \end{bmatrix}.$$ 

The function $W(x)$ is locally Lipschitz, but not $C^1$ on $\mathcal{M}$. Then, for $x \in (\mathcal{M} \cap \mathcal{S}_i) \setminus (\mathcal{S}_1 \cap \mathcal{S}_3)$, $i = 1, 3$, we have

$$L_fW(x) = f(x)^T((A + BK_1)^TP + P(A + BK_1))f(x).$$

Since $f(x) \neq 0$ on $\mathcal{S}_1$ and $\mathcal{S}_1$ is compact, there exists $\epsilon_1 > 0$ such that $\|f(x)\| > \epsilon_1$, $x \in \mathcal{S}_1$. Then, as $(A + BK_1)^TP + P(A + BK_1)$ is a negative definite matrix, there exists $\epsilon_1 > 0$ such that $f(x)^T((A + BK_1)^TP + P(A + BK_1))f(x) < -\epsilon_1$, $x \in \mathcal{S}_1$. Similarly, there exists $\epsilon_3 > 0$ such that $f(x)^T((A + BK_3)^TP + P(A + BK_3))f(x) < -\epsilon_3$, $x \in \mathcal{S}_3$. Let $\epsilon = \min(\epsilon_1, \epsilon_3)$. Then it can be shown using Proposition 1.5, Chapter 2 of [18] that $D_f^TW(x) < -\epsilon$, $x \in \mathcal{M} \subset (\mathcal{S}_1 \cup \mathcal{S}_3)$. By rescaling $W$ we can apply Theorem 7.1.2 (which also applies to nonconvex sets) to obtain that all trajectories initiated in
$\mathcal{M}$ leave it in finite time. Therefore, $\mathcal{M}$ does not contain invariant sets.

Then by Theorem 7.2.1, all trajectories initiated in $\mathcal{P}$ leave it in finite time. As $u(x)$ satisfies the invariance conditions of $\mathcal{P}$, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ using $u(x)$.

We conclude that in this example our results on generalized flow conditions enabled us to verify that RCP is solvable by $u(x)$, without the need for calculating the state trajectories of the closed-loop system.
Chapter 8

On the Limits of Affine Feedbacks

In the previous chapters, our main focus was on solvability of RCP on polytopes by continuous PWA feedbacks. The question arises of whether continuous PWA feedbacks is the largest continuous feedback class needed to solve RCP on polytopes. As a first step in answering this question, we focus in this chapter on simplices and study the relationship between affine feedbacks and continuous state feedbacks for RCP on simplices.

There is a belief in the literature that affine feedback and continuous state feedback are equivalent from the point of view of solvability of RCP on simplices. This result has been proved in [14] for the case where the state space has been triangulated properly with respect to $\mathcal{O}$; specifically, for any simplex $\mathcal{S}$ of the triangulation, $\mathcal{O}_S$ is either empty or a $\kappa$-dimensional face of $\mathcal{S}$. There remains the question of whether this result can also be proved under arbitrary triangulations. In this chapter we show using the results presented in the previous chapter that the answer for this question is negative by constructing an example for which no solution based on affine feedback exists, yet a continuous state feedback solves the problem. Then for single-input affine systems, we provide a constructive method for the synthesis of multi-affine feedbacks to solve RCP on simplices when affine feedbacks fail to solve the problem.

This chapter is organized as follows. The next section is the counterexample for the equivalence of affine feedback and continuous state feedback under arbitrary triangulations. In Section 8.2 we explore the geometric properties of the set of open-loop equilibria in the simplex. In Section 8.3 we provide a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices. In Section 8.4 two examples are given illustrating the synthesis method.
8.1 The Example

Consider a simplex $S = \text{co}\{v_0, \cdots, v_4\}$, where $v_0 = e_0 = 0$ and $v_i = e_i$, $i \in \{1, \cdots, 4\}$ ($e_i$ is the $i$th Euclidean coordinate vector), and consider the following affine system on $S$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -3 & -6 & -3 & -2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ -5 \\ 8 \\ 4 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$  (8.1)

It can be easily verified that the unit normal vector to the facet $F_j$ is $h_j = -e_j$, $j \in \{1, \cdots, 4\}$. Let $b := (-3, -5, 8, 4)$. We make several observations. First, $B \cap \text{cone}(S) = 0$ since $h_1 \cdot b = 3 > 0$ and $h_3 \cdot (-b) = 8 > 0$, so Theorem 3.2.5 cannot be applied. Secondly, it can be verified that

$$\mathcal{O} = \{ x \in \mathbb{R}^4 \mid x_1 = x_4 + \frac{1}{4}, x_2 = x_4 + \frac{1}{4}, x_3 = -2x_4 + \frac{1}{4} \} .$$

Setting $x_4 = 0$ in the defining equations for $\mathcal{O}$, we get $o_1 := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$. Setting $x_3 = 0$, we get $o_2 := \left(\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{8}\right)$. Thus, $\mathcal{O}_S = \text{co}\{o_1, o_2\}$ where

$$o_1 = \frac{1}{4}v_0 + \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 \in F_4$$

$$o_2 = \frac{1}{8}v_0 + \frac{3}{8}v_1 + \frac{3}{8}v_2 + \frac{1}{8}v_4 \in F_3 .$$

Clearly, $\mathcal{O}_S \cap S^o \neq \emptyset$. Because $o_1 \in F_4$ and $o_2 \in F_3$, we have

$$\text{cone}(\mathcal{O}_S) = \{ y \in \mathbb{R}^4 : h_3 \cdot y \leq 0, h_4 \cdot y \leq 0 \} .$$

Since $h_3 \cdot b < 0$ and $h_4 \cdot b < 0$, $b \in B \cap \text{cone}(\mathcal{O}_S)$, so solvability of RCP by continuous state feedback cannot be ruled out by Theorem 3.2.7. Also, it can be verified that $u = 0$ satisfies the invariance conditions (3.3), so solvability of RCP by continuous state feedback cannot be ruled out by Theorem 3.2.6. Nevertheless, it has been shown in [34] that RCP is not solvable by affine feedback in this example. In the proof of the following proposition, we show the reason behind this failure.

**Proposition 8.1.1.** Given simplex $S$ and system (8.1), RCP is not solvable by affine feedback.

**Proof.** Let $u(x) = Kx + g$ be such that $S \xrightarrow{\mathcal{K}} F_0$. Define $y(x) := Ax + Bu(x) + a$, $u_i := Kv_i + g$, and
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\[ y_i := A v_i + B u_i + a, \ i \in \{0, \cdots, 4\}. \]

Using (8.1) we have

\[
\begin{align*}
y_0 &= (-3u_0, -5u_0 + 3, 8u_0 + 1, 4u_0), \\
y_1 &= (-3u_1 - 1, -5u_1, 1 + 8u_1, 4u_1), \\
y_2 &= (-3u_2, -3 - 5u_2, 1 + 8u_2, 4u_2), \\
y_3 &= (1 - 3u_3, -5u_3, -3 + 8u_3, 4u_3).
\end{align*}
\]

Since \( u(x) \) satisfies the invariance conditions, we get

\[
\begin{align*}
h_1 \cdot y_0 &= 3u_0 \leq 0 \quad \text{and} \quad h_4 \cdot y_0 = -4u_0 \leq 0, \ \text{so} \ u_0 = 0. \\
\text{Similarly,} \ h_2 \cdot y_1 &= 5u_1 \leq 0 \ \text{and} \ h_4 \cdot y_1 = -4u_1 \leq 0, \ \text{so} \ u_1 = 0; \ h_1 \cdot y_2 = 3u_2 \leq 0 \ \text{and} \ h_4 \cdot y_2 = -4u_2 \leq 0, \ \text{so} \ u_2 = 0; \ \text{and} \ h_2 \cdot y_3 &= 5u_3 \leq 0 \ \text{and} \ h_4 \cdot y_3 = -4u_3 \leq 0, \ \text{so} \ u_3 = 0; \ \text{By convexity,} \ u(x) = 0 \ \text{for all} \ x \in \text{co} \ \{v_0, \cdots, v_3\}. \\
\text{Now we observe that} \ y(o_1) &= \frac{1}{4}y_0 + \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 = 0.
\end{align*}
\]

This contradicts \( S \xrightarrow{\mathcal{S}} \mathcal{F}_0. \)

So, the failure happens since the affine feedback cannot achieve the invariance conditions at \( v_i, \ i \in I(o_1), \) without having an equilibrium at \( o_1. \) To solve this problem, we should find a continuous state feedback \( u(x) \) such that (i) \( u(v_i) = 0, \ i \in I(o_1), \) to satisfy the invariance conditions at the vertices, (ii) \( u(o_1) > 0 \) to get rid of the equilibrium point at \( o_1. \) To that end, consider the multi-affine state feedback \( u(x) = x_1x_2x_3. \) It is easy to verify \( u(x) \) achieves (i)-(ii). Now we show that the candidate continuous state feedback \( u(x) \) solves RCP on \( S. \) Note that \( u(x) \geq 0, \ x \in \mathcal{S}, \) with equality holding for \( x \in (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3). \)

Lemma 8.1.2. The closed-loop vector field \( y(x) \) satisfies the invariance conditions (3.3).

Proof. Suppose \( x \in \mathcal{F}_1. \) Since \( h_1 = (-1, 0, 0, 0) \) and \( x_1 = 0, \) we have

\[
h_1 \cdot y(x) = x_1 - x_3 + 3x_1x_2x_3 = -x_3 \leq 0.
\]
Suppose $x \in F_2$. Since $h_2 = (0, -1, 0, 0), x_2 = 0$, and $x_1 + x_3 + x_4 \leq 1$, we have

$$h_2 \cdot y(x) = 3x_1 + 6x_2 + 3x_3 + 2x_4 + 5x_1x_2x_3 - 3 = 3x_1 + 3x_3 + 2x_4 - 3 \leq 3(x_1 + x_3 + x_4) - 3 \leq 0.$$ 

Suppose $x \in F_3$. Since $h_3 = (0, 0, -1, 0)$ and $x_3 = 0$, we have

$$h_3 \cdot y(x) = 4x_3 - 8x_1x_2x_3 - 1 = -1 \leq 0.$$ 

Suppose $x \in F_4$. Since $h_4 = (0, 0, 0, -1)$ and $x_4 = 0$, we have

$$h_4 \cdot y(x) = -4x_4 - 4x_1x_2x_3 = -4x_1x_2x_3 \leq 0.$$ 

Combining these facts, we conclude (3.3). \qed

**Proposition 8.1.3.** Given simplex $\mathcal{S}$ and system (8.1), $\mathcal{S} \rightarrow F_0$ using $u(x)$.

**Proof.** Let $V(x) := h_4 \cdot x$. We compute $\dot{V}(x) = h_4 \cdot y(x) = -4x_4 - 4x_1x_2x_3 \leq 0$, $x \in \mathcal{S}$. Also, $\dot{V}(x) = 0$ when $x_4 = 0$ and $x_1x_2x_3 = 0$. To apply LaSalle Principle for RCP (Theorem 7.2.1), it remains to show $\mathcal{M} := \{x \in \mathcal{S} \mid \dot{V}(x) = 0\}$ does not contain invariant sets. In this example, we have $\mathcal{M} = F_4 \cap (F_1 \cup F_2 \cup F_3)$. On $\mathcal{M}$, $x_4 = 0$ and $x_1x_2x_3 = 0$, so the dynamics reduce to:

$$\dot{x} = \tilde{A}x + \tilde{a} = \begin{bmatrix} -1 & 0 & 1 \\ -3 & -6 & -3 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$ 

Define $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. It is easily verified that $\tilde{A}\pi + \tilde{a} = 0$. Consider the function $\tilde{V}(x) := (x-\pi)^T P(x-\pi)$, and consider the Lyapunov equation

$$\tilde{A}^T P + P \tilde{A} = -I,$$

where $P$ is a symmetric matrix and $I$ is the identity matrix. Because $\sigma(\tilde{A}) \subset \mathbb{C}^-$, a unique solution exists, given by

$$P = \begin{bmatrix} 17/28 & -1/28 & 4/25 \\ -1/28 & 1/12 & -1/35 \\ 4/25 & -1/35 & 261/1400 \end{bmatrix}.$$
Then one can verify $\dot{\tilde{V}}(x) = -\|x - \overline{x}\|^2 < 0$ for all $x \in \mathcal{M}$. Since $\mathcal{M}$ is compact, by a standard argument, all trajectories must leave $\mathcal{M}$. In particular, there is no invariant set in $\mathcal{M}$. By LaSalle Principle for RCP (Theorem 7.2.1), all trajectories initiated in $\mathcal{S}$ leave it in finite time. Then, since the invariance conditions (3.3) are achieved (Lemma 8.1.2), $\mathcal{S} \rightarrow F_0$ using $u(x)$.

The above example shows that under arbitrary triangulations, affine feedback and continuous state feedback are not equivalent from the point of view of solving RCP on simplices. Also, it gives a hint that multi-affine feedbacks can be a suitable class to use in solving RCP on simplices for the case where affine feedbacks fail to solve the problem. One disadvantage of the above analysis is that it is very oriented to this specific example. To avoid that, we aim to find a general methodology of constructing multi-affine feedbacks for RCP on simplices.

8.2 The Equilibrium Set

Our objective in the rest of this chapter is to synthesize multi-affine feedbacks for RCP on simplices for the case where affine feedbacks fail to solve the problem. Since the multi-affine feedback synthesis for RCP on simplices has not been studied before, we focus our attention in this chapter on single-input affine systems. We define the set of open-loop equilibrium points (when $u = 0$)

$$\mathcal{E} := \{x \in \mathbb{R}^n \mid Ax + a = 0\}.$$ 

Also, define $\mathcal{E}_\mathcal{S} := \mathcal{S} \cap \mathcal{E}$. Clearly, $\mathcal{E} \subseteq \mathcal{O}$ and $\mathcal{E}_\mathcal{S} \subseteq \mathcal{O}_\mathcal{S}$. We have found that the multi-affine feedback synthesis depends mainly on the characterization of the set of open-loop equilibrium points in $\mathcal{S}$, $\mathcal{E}_\mathcal{S}$, and so we start by exploring the properties of $\mathcal{E}_\mathcal{S}$. In particular, in this section we present our main assumptions, review some technical results of [34] on $\mathcal{E}_\mathcal{S}$, and finally investigate further properties of $\mathcal{E}_\mathcal{S}$.

First, we present the main assumptions used in the rest of the chapter. As stated above, we assume $m = 1$. In [14] it was assumed that if $\mathcal{O}_\mathcal{S} \neq \emptyset$, then $\mathcal{O}_\mathcal{S}$ is a $\kappa$-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$. Here, we consider the general case where $\mathcal{O}$ intersects the interior of $\mathcal{S}$. In general, this intersection is a convex polytope. However, in this chapter we assume $\mathcal{O}_\mathcal{S}$ is a simplex, and restrict $\mathcal{O}_\mathcal{S}$ so that it does not intersect $\mathcal{F}_0$.

**Assumption 8.2.1.**

(A1) $\mathcal{O}_\mathcal{S} = \text{co} \{o_1, \cdots, o_{\kappa+1}\}$, a $\kappa$-dimensional simplex with $1 \leq \kappa \leq n$.

(A2) If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} = \text{co} \{\epsilon_1, \cdots, \epsilon_{\kappa_0+1}\}$, a $\kappa_0$-dimensional simplex with $0 \leq \kappa_0 \leq \kappa$. 
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(A3) \( O_S \cap S^o \neq \emptyset \).

(A4) \( O_S \cap F_0 = \emptyset \).

Suppose w.l.o.g. that \( v_0 = 0 \). From Theorem 3.2.6, we know that the invariance conditions are necessary for solvability of RCP by continuous state feedback. Here, we assume w.l.o.g. that the invariance conditions are solvable using \( u = 0 \). If the invariance conditions are solvable using \( u(x) = Fx + g \neq 0 \), then we consider the new affine system \( \dot{x} = \dot{A}x + \dot{a} + Bu = (A + BF)x + (a + Bg) + Bu \), and apply the proposed design procedure to it to obtain the multi-affine feedback. The final control law is \( u(x) = Fx + g + w(x) \), where \( w(x) \) is the obtained multi-affine feedback.

In [34] the same assumptions were used to find geometric necessary and sufficient conditions for solvability of RCP by affine feedback. Hence, Assumption 8.2.1 enables us to study an interesting geometric case which is more general than the one studied in [14], and to make use of some results of [34].

In the following part we summarize some important results of [34] on \( E_S \).

**Theorem 8.2.1** ([34]). Suppose that Assumption 8.2.1 holds and \( m = 1 \). Also, suppose \( Av_i + a \in C(v_i) \) for \( i \in \{0, \cdots, n\} \) and \( B \cap \text{cone}(O_S) \neq \emptyset \). If \( E_S \neq \emptyset \), then \( E_S \subset \text{rb}(O_S) \subset \partial S \).

Theorem 8.2.1 says that if the necessary conditions for solvability of RCP on \( S \) by continuous state feedback are achieved, then equilibrium points can appear only on the boundary of \( S \).

**Lemma 8.2.2** ([34]). Suppose that Assumption 8.2.1 holds and \( m = 1 \). Also, suppose \( Av_i + a \in C(v_i) \) for \( i \in \{0, \cdots, n\} \) and \( B \cap \text{cone}(O_S) \neq \emptyset \). If \( E_S \neq \emptyset \), then \( E_S \) is a \( \kappa_0 \)-dimensional face of \( O_S \), where \( 0 \leq \kappa_0 < \kappa \).

**Lemma 8.2.3** ([34]). Suppose \( m = 1 \) and \( Av_i + a \in C(v_i) \) for \( i \in \{0, \cdots, n\} \). Also, suppose there exists \( \pi \in E_S \) such that \( I(\pi) = \{0, \cdots, q\} \), where \( 1 \leq q < n \). Then there exists a coordinate transformation \( z = T^{-1}x \) such that the transformed system has the form

\[
\dot{z} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} z + \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u.
\]

where \( A_1 \in \mathbb{R}^{q \times q} \), \( A_{12} \in \mathbb{R}^{q \times (n-q)} \), \( a_1 \in \mathbb{R}^q \), \( b_1 \in \mathbb{R}^q \), \( A_2 \in \mathbb{R}^{(n-q) \times (n-q)} \), and \( b_2 \in \mathbb{R}^{n-q} \) for \( q > 0 \).

In [34] it was also shown that \( T = [v_1 \cdots v_n] \). The above lemma says that we can always decompose the dynamics into those contributing to equilibria and transversal dynamics.
In the last part of this section, we go beyond [34] and investigate an important property of the equilibrium set, $\mathcal{E}_S$. In particular, we show under Assumption 8.2.1 that if the necessary conditions for solvability of RCP by continuous state feedback are achieved, then it is impossible to have $\dim(\mathcal{E}_S) > 0$.

**Theorem 8.2.4.** Consider the system (3.1) defined on a simplex $S$. Suppose that $m = 1$, Assumption 8.2.1 holds, $Av_i + a \in \mathcal{C}(v_i), \ i \in \{0, \cdots, n\}$, and $B \cap \text{cone}(\mathcal{O}_S) \neq \emptyset$. If $\mathcal{E}_S \neq \emptyset$, then $\dim(\mathcal{E}_S) = 0$.

**Proof.** Suppose, by the way of contradiction, that $\dim(\mathcal{E}_S) = \kappa_0 > 0$. By Assumption 8.2.1 (A2), $\mathcal{E}_S$ is a simplex. In particular, $\mathcal{E}_S = \text{co} \{o_1, \cdots, o_{\kappa_0 + 1}\}$. From Theorem 8.2.1, we have $\mathcal{E}_S \subset \text{rb} (\mathcal{O}_S) \subset \partial S$.

Also, by Assumption 8.2.1 (A4), $\mathcal{E}_S \cap F_0 = \emptyset$. So, suppose w.l.o.g. that $\mathcal{E}_S \subset \text{co} \{v_0, v_1, \cdots, v_q\}$, where $q < n$ is the smallest index satisfying that. Let $S' := \text{co} \{v_0, \cdots, v_q\}$ and $\mathcal{E}_{S'} := \mathcal{E} \cap S'$. Clearly, $\mathcal{E}_{S'} \subset S$ and $\mathcal{E}_{S'} \cap \text{ri} (S') \neq \emptyset$. Note that $\mathcal{E}_{S'}$ can be expressed as the intersection of the affine space $S \cap \text{aff} (F_{q+1}) \cap \cdots \cap \text{aff} (F_n)$ in $\mathbb{R}^q$ with $S'$, a simplex in $\mathbb{R}^q$. So, we have $\text{rb} (\mathcal{E}_{S'}) \subset \text{rb} (S')$ [34, 35].

Since $\mathcal{E}_{S'} \subset S'$, $\text{rb} (\mathcal{E}_{S'}) \subset \text{rb} (S')$, $\mathcal{E}_{S'} \cap \text{ri} (S') \neq \emptyset$, and by Assumption 8.2.1 (A4) $\mathcal{E}_{S'} \cap F_0 = \emptyset$, then each index set $I(o_i), i \in \{1, \cdots, \kappa_0 + 1\}$, has an exclusive member in $\{1, \cdots, q\}$ [34, 35]. That is, there exists $k \in I(o_i), k \neq 0$, and $k \notin I(o_j), j \in \{1, \cdots, \kappa_0 + 1\} \setminus \{i\}$. Suppose w.l.o.g. that we reorder the vertex labeling $\{1, \cdots, q\}$ such that the indices belonging to more than one set $I(o_i), i \in \{1, \cdots, \kappa_0 + 1\}$, come first. Then we bring the indices corresponding to the exclusive members of $I(o_1), \cdots, I(o_{\kappa_0 + 1})$ respectively.

Now we study a vertex $o_k \in \mathcal{E}_{S'}, k \in \{1, \cdots, \kappa_0 + 1\}$. We have

$$h_j \cdot (A o_k + a) = 0, \ j \in I.$$  

We know $o_k = \sum_{i \in I(o_k)} \alpha_i v_i$, where $\alpha_i > 0$ and $\sum_{i \in I(o_k)} \alpha_i = 1$. So,

$$h_j \cdot (A \sum_{i \in I(o_k)} \alpha_i v_i + a) = 0, \ j \in I.$$  

Since $\sum_{i \in I(o_k)} \alpha_i = 1$, this implies

$$\sum_{i \in I(o_k)} \alpha_i h_j \cdot (A v_i + a) = 0, \ j \in I. \quad (8.3)$$  

Since $A v_i + a \in \mathcal{C}(v_i), i \in \{0, \cdots, n\}$, we must have

$$h_j \cdot (A v_i + a) \leq 0, \ i \in I(o_k), \ j \in I \setminus I(o_k). \quad (8.4)$$
Since $\alpha_i > 0$, (8.3) and (8.4) imply that

$$h_j \cdot (Av_i + a) = 0, \quad i \in I(o_k), \quad j \in I \setminus I(o_k), \quad k \in \{1, \ldots, \kappa_0 + 1\}.$$ 

Since $0 \in I(o_k)$ and $v_0 = 0$, we have

$$h_j \cdot a = 0, \quad j \in I \setminus I(o_k), \quad (8.5a)$$

$$h_j \cdot Av_i = 0, \quad i \in I(o_k), \quad j \in I \setminus I(o_k), \quad (8.5b)$$

for all $k \in \{1, \ldots, \kappa_0 + 1\}$.

Then we define the coordinate transformation $z = T^{-1}x$, where $T = [v_1 \cdots v_n]$. Notice that since by definition $\{v_0, v_1, \ldots, v_n\}$ are affinely independent and $v_0 = 0$, $T$ is non-singular [28]. It is easy to verify that the transformed vertices are $e_i = T^{-1}v_i$, $i \in \{0, \ldots, n\}$, and the transformed unit normal vectors are $-e_j = c_jT^Th_j$, $j \in \{1, \ldots, n\}$, where $c_j$ are positive scalars. Also, using (8.5) it can be verified that the dynamics in the new coordinates is

$$\dot{z} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} & \cdots & \Gamma_{0(\kappa_0+1)} & \Gamma'_{02} \\ \Gamma_{11} \\ \vdots \\ \Gamma_{(\kappa_0+1)(\kappa_0+1)} & \Gamma'_{(\kappa_0+1)2} \\ A_2 \end{bmatrix} \begin{bmatrix} Z_{10} \\ \Gamma'_{11} \\ \vdots \\ \Gamma'_{(\kappa_0+1)(\kappa_0+1)} \\ A_2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{1(\kappa_0+1)} \end{bmatrix} + \begin{bmatrix} b_{10} \\ b_{11} \\ \vdots \\ b_{1(\kappa_0+1)} \end{bmatrix} u, \quad (8.6)$$

where empty entries are zeros, $Z_{10}$ is a partition of $z$ corresponding to the indices appearing in more than one set $I(o_i)$, $i \in \{1, \ldots, \kappa_0 + 1\}$, and $Z_{11}, \ldots, Z_{1(\kappa_0+1)}$ are partitions corresponding to the exclusive members of $I(o_1), \ldots, I(o_{\kappa_0+1})$ respectively. Based on the above discussion, $\dim(Z_{1i}) \geq 1$, $i \in \{1, \ldots, \kappa_0 + 1\}$. Finally, $Z_2$ corresponds to the indices $\{q + 1, \ldots, n\}$.

Then the affine set $E \cap \text{aff} (F_{q+1}) \cap \cdots \cap \text{aff} (F_n)$ can be characterized by the equations characterizing $E$ and $Z_2 = 0$, which reduce to the following set of equations

$$\Gamma_{00}Z_{10} + \Gamma_{01}Z_{11} + \cdots + \Gamma_{0(\kappa_0+1)}Z_{1(\kappa_0+1)} + \gamma_1 = 0,$$

$$\Gamma_{11}Z_{11} = 0,$$

$$\vdots$$
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Figure 8.1: The set $E \subset S$ in Example 8.2.1

$$\Gamma_{(\kappa_0+1)(\kappa_0+1)} Z_{1(\kappa_0+1)} = 0.$$ (8.7)

Suppose that $\dim(\Gamma_{ii}) = P_i$, $i \in \{1, \cdots, \kappa_0 + 1\}$. If $\text{rank}(\Gamma_{11}) = P_1$, then any $z \in E \cap \text{aff}(F_q+1) \cap \cdots \cap \text{aff}(F_n)$ must have $Z_{11} = 0$, a contradiction to $E \cap S' \cap \text{ri}(S') \neq \emptyset$. So, $\text{rank}(\Gamma_{11}) < P_1$. Similarly, $\text{rank}(\Gamma_{ii}) < P_i$, $i \in \{2, \cdots, \kappa_0 + 1\}$. Therefore, (8.7) provides at most $q - (\kappa_0 + 1)$ independent equations to characterize $E \cap \text{aff}(F_q+1) \cap \cdots \cap \text{aff}(F_n)$. So, $\dim(E \cap \text{aff}(F_q+1) \cap \cdots \cap \text{aff}(F_n)) \geq \kappa_0 + 1$. Finally, $E \cap \text{aff}(F_q+1) \cap \cdots \cap \text{aff}(F_n)$ is an affine space in $\mathbb{R}^q$ that intersects the interior of $S'$, a simplex in $\mathbb{R}^q$. So, $\dim(E_S') = \dim(E \cap \text{aff}(F_q+1) \cap \cdots \cap \text{aff}(F_n)) \geq \kappa_0 + 1$, a contradiction.

Now we present an illustrative example for the proof of Theorem 8.2.4.

Example 8.2.1. Consider a simplex $S = \text{co} \{v_0, \cdots, v_4\} \subset \mathbb{R}^4$, where $v_0 = 0$, and an affine system (3.1) defined on $S$. Suppose that $m = 1$, Assumption 8.2.1 holds, $Av_i + a \in C(v_i)$, $i \in \{0, \cdots, 4\}$, and $B \cap \text{cone}(O_S) \neq \emptyset$. Also, suppose $E_S = \text{co} \{o_1, o_2\}$ where $I(o_1) = \{0, 1, 2\}$ and $I(o_2) = \{0, 1, 3\}$ as shown in Figure 8.1. Clearly, $E_S \subset \text{co} \{v_0, \cdots, v_3\} \subset \partial S$ as in Theorem 8.2.1. By studying $o_1 \in F_3 \cap F_4$, it is easy to verify

$$h_j \cdot a = 0, \quad h_j \cdot Av_i = 0, \quad i \in \{1, 2\}, \quad j \in \{3, 4\}.$$ (8.8)

Similarly, by studying $o_2$ we get

$$h_j \cdot a = 0, \quad h_j \cdot Av_i = 0, \quad i \in \{1, 3\}, \quad j \in \{2, 4\}.$$ (8.9)
Let $T = [v_1, \cdots, v_4]$, and define $z = T^{-1}x$. The transformed vertices are $e_i = T^{-1}v_i$, $i \in \{0, \cdots, 4\}$, and the transformed unit normal vectors are $-e_j = c_j T^T h_j$, $j \in \{1, \cdots, 4\}$, where $c_j$ are positive scalars.

Also, the dynamics in the new coordinates is

$$
\dot{z} = \begin{bmatrix}
-c_1 h_1 \cdot Av_1 & -c_1 h_1 \cdot Av_2 & -c_1 h_1 \cdot Av_3 & -c_1 h_1 \cdot Av_4 \\
-c_2 h_2 \cdot Av_1 & -c_2 h_2 \cdot Av_2 & -c_2 h_2 \cdot Av_3 & -c_2 h_2 \cdot Av_4 \\
-c_3 h_3 \cdot Av_1 & -c_3 h_3 \cdot Av_2 & -c_3 h_3 \cdot Av_3 & -c_3 h_3 \cdot Av_4 \\
-c_4 h_4 \cdot Av_1 & -c_4 h_4 \cdot Av_2 & -c_4 h_4 \cdot Av_3 & -c_4 h_4 \cdot Av_4
\end{bmatrix} z 
+ \begin{bmatrix}
-c_1 h_1 \cdot a \\
-c_2 h_2 \cdot a \\
-c_3 h_3 \cdot a \\
-c_4 h_4 \cdot a
\end{bmatrix} + \begin{bmatrix}
b_{10} \\
b_{11} \\
b_{12} \\
b_2
\end{bmatrix} u. \quad (8.10)
$$

Then (8.8) and (8.9) imply

$$
\dot{z} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
0 & a_{22} & 0 & a_{24} \\
0 & 0 & a_{33} & a_{34} \\
0 & 0 & 0 & a_{44}
\end{bmatrix} z 
+ \begin{bmatrix}
\gamma_1 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
b_{10} \\
b_{11} \\
b_{12} \\
b_2
\end{bmatrix} u. \quad (8.11)
$$

Now the intersection of $E$ with the hyperplane $H_4 := \text{aff}(F_4) = \{ z \in \mathbb{R}^4 \mid z_4 = 0 \}$ is characterized by the following set of equations

$$
a_{11}z_1 + a_{12}z_2 + a_{13}z_3 + \gamma_1 = 0,
$$

$$
a_{22}z_2 = 0,
$$

$$
a_{33}z_3 = 0. \quad (8.12)
$$

If $a_{22} \neq 0$, then any $z \in E \cap H_4$ must have $z_2 = 0$ to satisfy the second equation in (8.12), a contradiction to the assumption $E_S \cap \text{ri}(\text{co} \{v_0, \cdots, v_3\}) \neq \emptyset$. Therefore, $a_{22} = 0$. Similarly, $a_{33} = 0$. But, this implies (8.12) provides at most 1 independent equation to characterize $E \cap H_4$. So, dim($E \cap H_4$) $\geq 2$. Then since $E \cap H_4$ is an affine space in $\mathbb{R}^3$ intersecting the interior of $\text{co} \{v_0, \cdots, v_3\}$, a simplex in $\mathbb{R}^3$, we have dim($E_S$) = dim($E \cap H_4$) $\geq 2$, a contradiction to $E_S = \text{co} \{o_1, o_2\}$.

### 8.3 Multi-affine Feedback Synthesis

#### 8.3.1 Main Result

In this section we present a general method for synthesis of multi-affine feedbacks to solve RCP on simplices. We assume throughout the section that (possibly after an affine coordinate transformation)
the simplex is in canonical form:

\[ v_0 = 0, v_i = e_i, \quad i = 1, \ldots, n, \]

where \( e_i \) is the \( i \)th Euclidean coordinate vector. If \( \mathcal{E}_S = \emptyset \), then the conditions \( Av_i + a \in C(v_i) \), \( i \in \{0, \ldots, n\} \), imply from Theorem 3.2.2 that \( u = 0 \) solves RCP on \( S \). Therefore, we focus our attention on the case when \( \mathcal{E}_S \neq \emptyset \). Following Theorem 8.2.4, let \( \mathcal{E}_S = \{o_1\} \) where \( I(o_1) = \{0, \cdots, q\} \). Since \( \mathcal{E}_S \subset \partial S \) (Theorem 8.2.1), we have \( q < n \). We consider a multi-affine feedback of the form

\[ u(x) = x_1x_2 \cdots x_q. \]

We show that this multi-affine feedback solves RCP on \( S \). First we show that \( u(x) \) satisfies the invariance conditions.

**Lemma 8.3.1.** Consider the system (3.1) defined on a simplex \( S \). Suppose Assumption 8.2.1 holds, \( \mathcal{E}_S = \{o_1\} \) with \( I(o_1) = \{0, \cdots, q\} \), and \( m = 1 \). Also suppose

1. \( (N1) \quad Av_i + a \in C(v_i), \ i = 0, \ldots, n. \)
2. \( (N2) \quad B \in B \cap \text{cone} (\mathcal{O}_S) \neq 0. \)

Then the multi-affine feedback \( u(x) = x_1 \cdots x_q \) satisfies the invariance conditions (3.3).

**Proof.** Due to the assignment of vertices, \( x_j = 0 \) when \( x \in \mathcal{F}_j \) for \( j \in \{1, \ldots, n\} \). First consider \( j = 1, \ldots, q \). We have \( u(x) = 0 \) for \( x \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_q \). By (N1) and convexity

\[ h_j \cdot (Ax + Bu(x) + a) = h_j \cdot (Ax + a) \leq 0, \quad x \in \mathcal{F}_j, \quad j = 1, \ldots, q. \]

Next consider \( j = q + 1, \ldots, n \). By (N2), \( B \in C(o_1) \). That is,

\[ h_j \cdot B \leq 0, \quad j \in I \setminus I(o_1) = \{q + 1, \ldots, n\}. \quad (8.13) \]

Notice that with our assignment of vertices of \( S \), \( u(x) \geq 0 \) for all \( x \in S \). Then by combining (N1), convexity, and (8.13) we get

\[ h_j \cdot (Ax + Bu(x) + a) \leq 0, \quad x \in \mathcal{F}_j, \quad j = q + 1, \ldots, n. \]

We conclude \( u(x) \) satisfies (3.3). \( \square \)
Now we show that using our proposed multi-affine feedback \( u(x) = x_1 \cdots x_q \), all closed-loop trajectories initiated in \( S \) leave it in finite time, and so \( u(x) \) solves RCP on \( S \). For this we require a technical lemma.

**Lemma 8.3.2.** Let \( \mathcal{H} := \{ x \in \mathbb{R}^n \mid \rho_1 \cdot x = c_1, \ldots, \rho_r \cdot x = c_r \} \) be an affine space, where \( \rho_i \in \mathbb{R}^n \) and \( c_i \in \mathbb{R} \). Consider the system \( \dot{x} = Ax + a \) and suppose there exist \( x_0 \in \mathcal{H} \) and \( T > 0 \) such that \( \phi(t, x_0) \in \mathcal{H} \) for all \( t \in [0, T) \). Then \( \phi(t, x_0) \in \mathcal{H} \) for all \( t \).

**Proof.** Let \( \mathcal{H}_1 \) be the smallest affine set containing \( \{ \phi(t, x_0) \mid t \in (0, T) \} \). Then \( \mathcal{H}_1 \subseteq \mathcal{H} \), so it can be expressed as

\[
\mathcal{H}_1 = \{ x \in \mathbb{R}^n \mid \rho_1 \cdot x = c_1, \ldots, \rho_s \cdot x = c_s \},
\]

with \( r \leq s \). Now we have \( \rho_j \cdot \phi(t, x_0) = c_j \) for \( t \in (0, T) \) and \( j \in \{1, \ldots, s\} \). Taking the derivative, \( \rho_j \cdot (A\phi(t, x_0) + a) = 0 \) for \( t \in (0, T) \) and \( j \in \{1, \ldots, s\} \). By definition \( \mathcal{H}_1 \) is the affine hull of \( \{ \phi(t, x_0) \mid t \in (0, T) \} \), and so there exist \( t_i \in (0, T) \), \( i = 1, \ldots, z \), such that \( \mathcal{H}_1 = \text{aff} \{ \phi(t_1, x_0), \ldots, \phi(t_z, x_0) \} \). That is, for each \( x \in \mathcal{H}_1 \), \( x = \sum_{i=1}^z \alpha_i^r \phi(t_i, x_0) \) with \( \sum_{i=1}^z \alpha_i^r = 1 \). Then we have

\[
\rho_j \cdot (Ax + a) = \sum_{i=1}^z \alpha_i^r \rho_j \cdot (A\phi(t_i, x_0) + a) = 0, \quad x \in \mathcal{H}_1, \quad j = 1, \ldots, s.
\]

By Nagumo’s Theorem [18], \( \mathcal{H}_1 \) is an invariant set. Hence, \( \phi(t, x_0) \in \mathcal{H}_1 \subseteq \mathcal{H} \) for all \( t \). \qed

The following is the main result of this section.

**Theorem 8.3.3.** Consider the system (3.1) defined on a simplex \( S \). Suppose Assumption 8.2.1 holds, \( E_S = \{ a_1 \} \) with \( I(a_1) = \{0, \cdots, q\} \), and \( m = 1 \). Also suppose

\[ (N1) \quad Av_i + a \in \mathcal{C}(v_i), \quad i = 0, \ldots, n. \]

\[ (N2) \quad B = Ao_{n+1} + a \in \mathcal{B} \cap \text{cone}(O_S) \neq 0. \]

Then \( S \xrightarrow{S} F_0 \) by \( u(x) = x_1 \cdots x_q \).

**Proof.** The first step of the proof is to show there exists \( \xi \in \mathbb{R}^n \) such that \( \xi \cdot (Ax + Bu(x) + a) \leq 0 \) for all \( x \in S \). The image of \( S^o \) under the affine map \( x \mapsto Ax + a \), denoted \( C \), is convex and relatively open by Theorems 3.4 and 6.6 of [49]. By assumption \( E_S = \{ a_1 \} \subset \partial S \). Therefore, \( Ax + a \neq 0 \) for all \( x \in S^o \). Thus, the nonempty relatively open convex set \( C \) and the nonempty affine set \( \{0\} \) do not intersect. By Theorem 2.4.1, there exists a hyperplane \( \mathcal{H} \) containing \( \{0\} \) such that one of the open half-spaces associated with \( \mathcal{H} \) contains \( C \). That is, there exists \( \xi \in \mathbb{R}^n \) such that

\[
\xi \cdot (Ax + a) < 0, \quad x \in S^o.
\]  

(8.14)
By continuity of \( x \mapsto (Ax + a) \) we get

\[
\xi \cdot (Ax + a) \leq 0, \quad x \in S. \tag{8.15}
\]

Next we claim that \( \xi \cdot B < 0 \). For this we first show that \( Ao_i + a = \gamma_i B \) with \( \gamma_i \geq 0 \), for all \( i \in I_{Os} \). First, \( Ao_1 + a = 0 \) so \( \gamma_1 = 0 \). Second, \( \gamma_2, \ldots, \gamma_{k+1} \) must all have the same sign. Otherwise, by convexity there exists \( x \in \text{co} \{ o_2, \ldots, o_{k+1} \} \) such that \( Ax + a = 0 \), which contradicts that \( E_S = \{ o_1 \} \). Since by (N2), \( \gamma_{k+1} = 1 \), we have \( \gamma_i > 0, i = 2, \ldots, k+1 \). Now let \( x_o \in S^o \cap O_S \). By Theorem 8.2.1, \( x_o \in \text{ri} (O_S) \). That is, \( x_o = \sum_{i=1}^{k+1} \beta_i o_i \) with \( \beta_i > 0 \) and \( \sum_{i=1}^{k+1} \beta_i = 1 \). Then \( Ax_o + a = \sum_{i=1}^{k+1} \beta_i (\gamma_i B) = \gamma B \) with \( \gamma > 0 \). Finally, by (8.14) we have

\[
\xi \cdot B = \frac{1}{\gamma} \xi \cdot (Ax_o + a) < 0. \tag{8.16}
\]

By the assignment of vertices, \( u(x) \geq 0 \) for all \( x \in S \). Combining (8.15) with (8.16) we have

\[
\xi \cdot (Ax + Bu(x) + a) \leq 0, \quad x \in S. \tag{8.17}
\]

Moreover,

\[
\xi \cdot (Ax + Bu(x) + a) = 0 \iff \xi \cdot (Ax + a) = 0 \quad \text{and} \quad u(x) = 0. \tag{8.18}
\]

The second step of the proof is to use the flow-like condition (8.17) in the LaSalle Principle for RCP (Theorem 7.2.1) to show that all closed-loop trajectories exit \( S \). For this we identify the set

\[\mathcal{M} = \{ x \in S \mid \xi \cdot (Ax + a) = 0, u(x) = 0 \}.\]

According to the LaSalle Principle for RCP, we must show that \( \mathcal{M} \) does not contain any invariant set. Since \( \mathcal{M} \subset \{ x \in S \mid u(x) = 0 \} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_q \), the dynamics on \( \mathcal{M} \) reduce to \( \dot{x} = Ax + a \). Suppose by the way of contradiction that there exists \( x_0 \in \mathcal{M} \) such that \( \phi(t, x_0) \in \mathcal{M} \) for all \( t \geq 0 \). First we claim there exists a facet \( \mathcal{F}_i, i \in \{ 1, \ldots, q \} \), and an open time interval \( (t_1, t_2) \), \( t_1 < t_2 \), such that \( \phi(t, x_0) \in \mathcal{F}_i \) for \( t \in (t_1, t_2) \). Let \( x_0 \in (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k) \) with \( k \leq q \). Since \( \mathcal{F}_{k+1} \cup \cdots \cup \mathcal{F}_q \) is a compact set and \( x \mapsto Ax + a \) is locally Lipschitz, there exists \( T > 0 \) such that \( \phi(t, x_0) \notin (\mathcal{F}_{k+1} \cup \cdots \cup \mathcal{F}_q) \) for \( t \in [0, T] \). Now \( \phi(t, x_0) \) cannot leave \( \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k \) instantaneously (for if so, there exists \( t' \in (0, T] \) such that \( \phi(t', x_0) \notin \bigcup_{i=1}^q \mathcal{F}_i \) so \( \phi(t', x_0) \notin \mathcal{M} \), a contradiction). Hence, there exists \( \mathcal{F}_j, j \in \{ 1, \cdots, k \} \), and a time \( T_0 > 0 \) such that \( \phi(t, x_0) \in \mathcal{F}_j \) for all \( t \in (0, T_0) \). Applying Lemma 8.3.2, \( \phi(t, x_0) \in \text{aff} (\mathcal{F}_j) \) for all \( t \). But by assumption \( \phi(t, x_0) \in \mathcal{M} \) for \( t \geq 0 \), so we have \( \phi(t, x_0) \in \mathcal{F}_j \) for \( t \geq 0 \). Now we show this is impossible.

Let \( \mathcal{C} \) be the image of \( \mathcal{F}_j \) under the mapping \( x \mapsto Ax + a \). Since \( \mathcal{F}_j \) is convex and compact, \( \mathcal{C} \) is also
convex and compact [49]. Since \( E_S = \{ o_1 \} \), \( I(o_1) = \{ 0, \ldots, q \} \), and \( j \in \{ 1, \ldots, q \} \), we have \( E_S \cap F_j = \emptyset \). Therefore, \( Ax + a \neq 0 \), \( x \in F_j \). Thus, the compact convex sets \( C \) and \( \{ 0 \} \) do not intersect. By Theorem 2.4.2, there exists a hyperplane \( H \) that strongly separates \( C \) and \( \{ 0 \} \). In particular, there exists \( \rho \in \mathbb{R}^n \) such that \( \rho \cdot (Ax + a) < 0 \) for \( x \in F_j \). This implies \( \phi(t, x_0) \) leaves \( F_j \) in finite time, a contradiction. We conclude \( M \) does not contain invariant sets. Then by LaSalle Principle for RCP, all closed-loop trajectories initiated in \( S \) leave it in finite time. Since the invariance conditions hold (Lemma 8.3.1), the trajectories can do so only via \( F_0 \). We conclude \( S \rightarrow F_0 \) by \( u(x) \).

Note that the above theorem is still true for the case when in condition \((N2)\), \( B = c(Ao_{n+1} + a) \), where \( c > 0 \), \( c \neq 1 \).

**Remark 8.3.1.** If the given simplex is not in canonical form, then the obtained multi-affine feedback \( u(x) \) should be converted back to original coordinates. The control law in the original coordinates is continuous but not necessarily multi-affine (see Example 8.4.2).

### 8.3.2 Relation to Affine Feedbacks

In this subsection we study the relation between our main result (Theorem 8.3.3) and some known results for RCP on simplices by affine feedbacks. In particular, if \( q = 1 \) in Theorem 8.3.3 \( (o_1 \in (v_0, v_1)) \), then the theorem says that the affine feedback \( u(x) = x_1 \) solves RCP on \( S \). This may seem strange at the first sight. However, the next lemma shows using existing results for RCP on simplices by affine feedback that if \( E_S = \{ o_1 \} \) where \( o_1 \in (v_0, v_1) \), then RCP is solvable by affine feedback.

**Lemma 8.3.4.** Consider the system (3.1) defined on a simplex \( S \), where \( v_0 = 0 \), \( v_i = e_i \), \( i \in \{ 1, \cdots, n \} \), and \( E_S = \{ o_1 \} \) with \( o_1 \in (v_0, v_1) \). Suppose Assumption 8.2.1 holds, \( m = 1 \), \( Av_i + a \in C(v_i) \), \( i \in \{ 0, \cdots, n \} \), and there exists \( b \in B \cap \text{cone}(O_S) \neq \emptyset \). Then \( S \rightarrow F_0 \) by affine feedback.

**Proof.** First, we show there exists \( o_i, i \in I_{O_S} \), such that \( o_i \in F_1 \). By assumption, we have \( o_1 = (c, 0, \cdots, 0) \), where \( 0 < c < 1 \). Also, by Assumption 8.2.1 we know that \( O_S \cap S^o \neq \emptyset \). Let \( x_o \in O_S \cap S^o \). Then we have \( x_o = (c_1, c_2, \cdots, c_n) \), where \( c_i > 0 \). We study \( \{ o_1, x_o \} \). The set \( O \) is affine, and so \( \text{aff} \{ o_1, x_o \} \subseteq O \). Then \( \{ o_1, x_o \} \) must intersect \( \partial S \) at another point in \( O_S \), call it \( \sigma \). It can be easily recognized that \( \sigma \) will have the form \( \sigma = (k_1, k_2, \cdots, k_n) \), where \( k_i > 0 \), \( i \in \{ 2, \cdots, n \} \). Therefore, we have one of two cases

1. If \( k_1 \neq 0 \), then \( \sigma \in \partial S \) implies \( \sigma \in F_0 \). Thus, \( O_S \cap F_0 \neq \emptyset \), a contradiction to Assumption 8.2.1.

2. \( k_1 = 0 \), and so \( \sigma \in \text{co} \{ v_0, v_2, \cdots, v_n \} = F_1 \).
Now we show that at least one of the vertices of $O_S$ lies on $F_1$. We have $ar{o} = \sum \alpha_i o_i$, where $o_i$ are some vertices of $O_S$, $\alpha_i > 0$, and $\sum \alpha_i = 1$. Then $\bar{o} \in F_1$ implies $e_1 \cdot \bar{o} = 0$, and so $e_1 \cdot \sum \alpha_i o_i = \sum \alpha_i e_1 \cdot o_i = 0$.

But, since $O_S \subset S$, we have $e_1 \cdot o_i \geq 0$. Since $\alpha_i > 0$, we must have $e_1 \cdot o_i = 0$, which implies that there exists a vertex $o_i \in O_S \cap F_1$. W.l.o.g. let $o_2 \in O_S \cap F_1$.

Secondly, we show that $b \in B \cap \text{cone}(S)$. Since $b \in B \cap \text{cone}(O_S)$, we have $b \in C(o_1)$ and $b \in C(o_2)$. As $b \in C(o_1)$ and $o_1 \in (v_0, v_1)$, we have $h_j \cdot b \leq 0$, $j \in \{2, \cdots, n\}$. Also, since $b \in C(o_2)$ and $o_2 \in F_1$, we have $h_1 \cdot b \leq 0$. We conclude $h_j \cdot b \leq 0$, $j \in \{1, \cdots, n\}$, which implies $b \in \text{cone}(S)$.

This together with $Av_i + a \in C(v_i)$, $i \in \{0, \cdots, n\}$, implies from Theorem 3.2.5 that $S \rightarrow F_0$ by affine feedback.

Lemma 8.3.4 provides the analogy between our results and some known results for solvability of RCP on simplices by affine feedback. This lemma can be considered as a special case of Theorem 8.3.3 in which $q = 1$. Also, the proposed theorem is more constructive since it does not only tell us that RCP is solvable, but also provides us with the feedback that solves the problem.

8.4 Examples

Example 8.4.1. Consider again the example presented in Section 8.1. The objective is to show how our results in the previous section can be used to systematically synthesize multi-affine feedback that solves RCP on $S$.

First, we check whether the conditions of Theorem 8.3.3 are achieved. As shown before, $O_S = \text{co} \{o_1, o_2\}$, where $o_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0) \in F_4$ and $o_2 = (\frac{3}{5}, \frac{3}{5}, 0, \frac{1}{5}) \in F_3$. Also, we have $E_S = \{o_1\} \subset F_4$. Hence, in this example $\dim(O_S) = 1$, $\dim(E_S) = 0$, $O_S \cap S^c \neq \emptyset$, and $O_S \cap F_0 = \emptyset$. So, Assumption 8.2.1 holds. Also, it can be verified that $u = 0$ satisfies the invariance conditions at the vertices of $S$. Then as shown before in Section 8.1, $B = (-3, -5, 8, 4) \in B \cap \text{cone}(O_S)$. We conclude that the conditions of Theorem 8.3.3 are achieved, and so $S \rightarrow F_0$ by multi-affine feedback. In this example $q = 3$, and so $u(x) = x_1 x_2 x_3$, which is the same feedback used in the example in Section 8.1.

The advantage here is that Theorem 8.3.3 enables us to find the multi-affine feedback systematically. Also, we don’t need to have a separate proof for each example as we did in Lemma 8.1.2 and Proposition 8.1.3.
Example 8.4.2. Consider the following affine system

\[
\dot{x} = \begin{bmatrix}
-1 & 0 & 0 & 1 & 3 \\
-3 & -6 & 0 & -1 & 13 \\
0 & 0 & 5 & 0 & -1 \\
0 & 0 & 9 & -4 & -1 \\
0 & 0 & 1 & 0 & 2
\end{bmatrix} x + \begin{bmatrix}
2 \\
8 \\
9 \\
17 \\
5
\end{bmatrix} u + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

(8.19)

defined on a simplex \( S = \text{co} \{v_0, \cdots, v_3\} \), where \( v_0 = 0 \), \( v_1 = e_1 \), \( v_2 = e_2 \), \( v_3 = (0,1,0,1,0) \), \( v_4 = (0,0,1,1,0) \), and \( v_5 = (1,1,1,1,1) \). The objective is to find a continuous state feedback that solves RCP on \( S \).

First, it can be verified that \( u = 0 \) solves the invariance conditions at the vertices of \( S \) (\( Av_i + a \in \mathcal{C}(v_i) \), \( i \in \{0, \cdots, 5\} \)). Secondly, we compute the sets \( O_S \) and \( E_S \). It can be verified that \( O_S = \text{co} \{o_1, o_2\} \), where \( o_1 = (0.25, 0.5, 0, 0.25, 0) \in (\mathcal{F}_4 \cap \mathcal{F}_5) \) and \( o_2 = (0.5568, 0.572, 0.2614, 0.2614, 0.1818) \in \mathcal{F}_3 \). Also, \( Ao_1 + a = 0 \), while \( Ao_2 + a \neq 0 \). Therefore, in this example \( \text{dim}(O_S) = 1 \), \( \text{dim}(E_S) = 0 \), \( O_S \cap S^c \neq \emptyset \), and \( O_S \cap F_0 = \emptyset \). So, Assumption 8.2.1 holds. Then we check whether there exists \( 0 \neq b \in B \cap \text{cone}(O_S) \).

Since \( o_1 \in (\mathcal{F}_4 \cap \mathcal{F}_5) \) and \( o_2 \in \mathcal{F}_3 \), we have

\[
\text{cone}(O_S) = \{ y \in \mathbb{R}^5 \mid h_3 \cdot y \leq 0, h_4 \cdot y \leq 0, h_5 \cdot y \leq 0 \}.
\]

It can be verified that \( h_3 = (0,0,1,-1,0) \), \( h_4 = (0,0,-1,0,1) \), \( h_5 = (0,0,0,0,-1) \), and \( B = (2,8,9,17,5) \in B \cap \text{cone}(O_S) \).

Although the necessary conditions for solvability of RCP by continuous state feedback are achieved, it can be shown using an argument similar to the one used in Proposition 8.1.1 that RCP is not solvable by affine feedback. Instead, based on the above, the conditions of Theorem 8.3.3 are achieved, and so RCP is solvable by continuous state feedback.

To find this feedback, we firstly construct the coordinate transformation matrix \( T = [v_1 \cdots v_5] \), and define \( z = T^{-1} x \). In the new coordinates, the transformed vertices are \( e_i = T^{-1} v_i, i \in \{0, \cdots, 5\} \), and the transformed unit normal vectors are \( -e_j, j \in \{1, \cdots, 5\} \). In this example \( o_1 \in (\mathcal{F}_4 \cap \mathcal{F}_5) \), and so \( I(o_1) = \{0, \cdots, 3\} \). Theorem 8.3.3 tells us that the multi-affine feedback \( u(z) = z_1 z_2 z_3 \) solves RCP on \( S \). In the original coordinates the continuous control law is

\[
u(x) = (x_1 - x_5)(x_2 + x_3 - x_4 - x_5)(-x_3 + x_4).
\]
Note that another method of solving this example is to firstly transform the system to the new coordinates, then check the conditions of Theorem 8.3.3 in the new coordinates, and finally return the feedback to the original coordinates.
Chapter 9

Conclusions

Nowadays the control demands on modern industrial systems are becoming more complex; the systems require specifications that can include safety constraints, startup procedures, human intervention, temporal logic statements, etc. On the other hand, traditional control methods such as stabilization and tracking may not be able to satisfy these complex specifications. Therefore, there is an urgent need for the development of novel control techniques that can address the complex specifications.

In this dissertation we develop theoretical foundations for the control for complex specifications. Specifically, we study the reach control problem (RCP) for affine systems on polytopes, which can efficiently accommodate for complex specifications. The dissertation presents a panoply of techniques and ideas for tackling RCP. First, it provides a complete survey of the main developments in RCP. Second, it shows the drawbacks of existing simplex methods when applied to polytopes, which highlights the need for a direct study of RCP on polytopes. Third, it initiates a geometric study of RCP on polytopes that extends the results of [14] on simplices. Fourth, it provides novel necessary conditions for solvability of RCP on polytopes by open-loop controls and by continuous state feedbacks. Fifth, the monotonic reach control problem (MRCP) is formulated and studied for different geometric situations. The relationship between MRCP and RCP by arbitrary triangulations is also studied. Sixth, the dissertation provides a novel verification tool for the analysis of RCP. Seventh, it provides a novel study on the limits of affine feedbacks for RCP on simplices. Finally, it provides several examples that catalog the different reasons for the failure of RCP and illustrate the findings of the dissertation. This includes a practical example: the two tank temperature control problem.

In this chapter we summarize the main results of the dissertation. Then we provide some promising directions for future research.
9.1 Summary of Results

In Chapter 2 we have presented the mathematical background needed to deeply understand the reach control problem. In Chapter 3 we have reviewed RCP on simplices. In particular, we have formulated the problem and reviewed the invariance conditions in Section 3.1. Then in Section 3.2 we have presented basic results for solvability of RCP on simplices, particularly by affine feedback. In Section 3.3 we have shown how to use existing simplex methods to solve RCP on a polytopic state space. After that we have provided two examples (Examples 3.3.1, 3.3.2) showing the drawbacks of existing simplex methods when applied to RCP on polytopes. To overcome these drawbacks, we have started in Chapter 4 a direct study of RCP on polytopes following the geometric point of view of [14] for RCP on simplices. In particular, in Section 4.1 we have formulated the reach control problem on polytopes and reviewed the relevance of the invariance conditions to RCP on polytopes. Then in Section 4.2 we have investigated novel necessary conditions for solvability of RCP on polytopes by open-loop controls and by continuous state feedback.

Then in Chapter 5 we have studied the extent to which existing results for simplices carry over to polytopes. In particular, in Section 5.1 we have provided conditions to exclude closed-loop equilibria in polytopes (Theorem 5.1.2). Also, we have translated the geometric sufficient conditions of [14] for constructing a flow condition on simplices to analogous conditions for polytopes. However, in contrast with the situation for affine feedbacks on simplices, we have shown that for continuous PWA feedbacks, the flow condition is no longer equivalent to the no-equilibrium condition, and so the no-equilibrium condition is not enough to deduce that RCP is solvable on polytopes. To solve this problem, we have formulated in Section 5.2 the monotonic reach control problem (MRCP), in which we incorporate the flow condition into the problem statement of RCP. Then we have studied MRCP for three geometric cases: (i) $\mathcal{O}_P = \emptyset$, (ii) $\mathcal{O}_P$ is a face of $\mathcal{P}$, and (iii) $\mathcal{P} \cap \mathcal{O} \neq \emptyset$. For the first two cases, we have found necessary and sufficient conditions for solvability of MRCP for the multi-input systems (Theorems 5.2.2 and 5.2.3). This extends the results of [14] for simplices to polytopes. Then for the last geometric case, we have focused on single-input systems, and we have proposed a method for solving MRCP using a $b$-extremal controller, where $b$ is a control direction of the single-input system (Theorem 5.2.4). Also, we have provided for this geometric case an algorithmic solution of MRCP (Algorithm 1). Moreover, the soundness of the algorithm has been proved (Theorem 5.4.1). Our algorithm is preferred to existing algorithms in the literature for the single-input systems since it is for polytopes and requires solving a small number of LP problems which increases linearly with the number of vertices of $\mathcal{P}$. Then in Section 5.3 we have shown that for generic polytopes, MRCP and RCP by arbitrary triangulations are equivalent (Theorem 5.3.1), and so MRCP is no loss of generality if invariance to the choice of
triangulation is a requirement in the design. Finally, in Section 5.5 we have provided several illustrative examples. The examples show that MRCP and existing simplex methods are complementary. MRCP may work when simplex methods fail (Example 5.5.2). Conversely, simplex methods may work when MRCP fails (Example 5.5.3), and so forth. Also, in Example 5.5.4 a continuous PWA feedback has been verified to solve RCP via exhaustive simulation of the closed-loop system, yet no existing technique can explain why RCP is solved. This has highlighted the need for a verification tool for the analysis of RCP. Then in Chapter 6 we have provided the two tank temperature control example which shows some practical advantages of MRCP over existing simplex methods.

Chapter 7 has provided a verification tool, analogous to Lyapunov theory for stability, that can be used to analyze whether an instance of RCP is solved, without resorting to exhaustive simulation. In particular, we have introduced the notion of generalized flow conditions, which give a necessary and sufficient condition for closed-loop trajectories to exit the polytope (Theorem 7.1.1). In analogy with Lyapunov stability theory, the generalized flow condition comprises a functional that decreases along closed-loop trajectories. However, the generalized flow function $V$ need not be positive definite. Then we have provided a set of results to analyze whether closed-loop trajectories exit the polytope, without resorting to exhaustive simulation (Theorems 7.1.2, 7.1.4). This includes a variant of the LaSalle Principle tailored to RCP (Theorem 7.2.1). After that we have shown that for continuous PWA feedback, a suitable generalized flow condition is based on a functional of the form $V(x) = \max\{V_i(x)\}$, where $V_i(x)$ are affine functions (Theorems 7.3.1, 7.3.3). Additionally, we have provided an LP-based computational method for finding a generalized flow function of this form (Corollary 7.3.2). Several illustrative examples have been given in Section 7.4.

In Chapter 8 we have studied the relationship between affine feedbacks and continuous state feedbacks for RCP on simplices. Although in [14] it has been shown under an assumption on the triangulation of the polytopic state space that the two feedback classes are equivalent, we have shown in this chapter that this equivalence is no longer true if the triangulation assumption of [14] is removed. In particular, using the results presented in Chapter 7, we were able to construct an example for which no solution based on affine feedback exists, yet a continuous state feedback solves the problem (Section 8.1). Then we have identified for single-input systems an alternative feedback class for RCP on simplices for the case where affine feedbacks fail to solve the problem. Specifically, we have firstly explored the geometric properties of the set of open-loop equilibria in Section 8.2. Using these properties and the results of Chapter 7, we have presented in Section 8.3 a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices. Finally, we have provided in Section 8.4 two examples illustrating the synthesis method.
9.2 Future Directions

In the following points we summarize the most promising directions for future research in RCP.

• In this dissertation we have studied MRCP for three geometric cases. For the case where \( \mathcal{P} \cap \mathcal{O} \neq \emptyset \), the dissertation has presented a solution of MRCP only for single-input systems (Algorithm 1). It remains the question of whether this algorithmic solution can be extended to the multi-input systems. Algorithm 1 depends on pushing a non-zero vector \( b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_P) \), without violating the invariance conditions. Since for single-input systems the condition \( \mathcal{B} \cap \text{cone}(\mathcal{O}_P) \neq 0 \) is necessary for solvability of RCP by continuous state feedback (Theorem 4.2.3), we can assume that such a vector \( b \) always exists. The main difficulty for the multi-input case is that \( \mathcal{B} \cap \text{cone}(\mathcal{O}_P) \neq 0 \) is not known to be necessary for solvability of RCP by continuous state feedback, and so we do not know if such a vector \( b \) exists. Recently it is shown in [45] that for simplices and under the assumption \( \mathcal{O}_S = \mathcal{F}_0, \mathcal{B} \cap \text{cone}(\mathcal{O}_S) \neq 0 \) is a necessary condition for solvability of RCP by continuous state feedback for the multi-input systems. This result is the first step in showing that \( \mathcal{B} \cap \text{cone}(\mathcal{O}_P) \neq 0 \) is also a necessary condition for solvability of RCP by continuous feedback for the multi-input systems, which solves the main difficulty in extending Algorithm 1.

• We have shown that for generic polytopes, MRCP and RCP by arbitrary triangulations are equivalent, and so if MRCP is not solvable, then we cannot find a control assignment at the vertices of \( \mathcal{P} \) such that for any choice of triangulation, the associated continuous PWA feedback solves RCP. Instead, we hope to find a triangulation of \( \mathcal{P} \) and an associated continuous PWA feedback solving RCP. Therefore, we pose the following question for future research: What are the (checkable) necessary and sufficient conditions for the existence of a triangulation such that its associated continuous PWA feedback solves RCP?

• We have developed the idea of generalized flow conditions, which provides a verification tool for the analysis of RCP on polytopes. An open problem is to identify suitable classes of functionals that give rise to a generalized flow condition. In the dissertation we have shown that for continuous PWA feedbacks, a suitable generalized flow condition is based on a functional of the form \( V(x) = \max \{ V_i(x) \} \), where \( V_i(x) \) are affine functions. Similar studies for other feedback classes are needed. For instance, reach control for multi-affine systems on rectangles has been widely studied in the literature [7, 8], and it would be important to identify for multi-affine vector field on rectangles a suitable class of functionals that gives rise to a generalized flow condition.

• Chapter 7 has provided a set of results that can be used to analyze whether an instance of RCP
is solved, without resorting to exhaustive simulation of the closed-loop system (Theorems 7.1.2, 7.1.4, 7.2.1). However, the conditions of these theorems should be checked at each $x \in \mathcal{P}$. It would be important to study for which classes of generalized flow functions and state feedbacks it is sufficient to check these conditions only at vertices of $\mathcal{P}$.

- As discussed in Chapter 7, there is a strong analogy between generalized flow conditions for RCP and Lyapunov theory for stability. In the literature of Lyapunov stability, control Lyapunov functions have been widely used since they allow us to use Lyapunov technique in the synthesis of continuous stabilizers \cite{1, 55, 56}. The existence of a control Lyapunov function does not only tell us that the stabilization problem is solvable, but also it provides us with a continuous feedback stabilizer \cite{56}. In the context of RCP, it would be important to introduce a new notion, control generalized flow function, analogous to control Lyapunov functions for stabilization. This will allow us to use the generalized flow condition in the control synthesis as well.

- In Chapter 8 we have provided a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices. However, the synthesis method is based on Assumption 8.2.1. A possible direction for future research is to extend the synthesis method for the case where some items of Assumption 8.2.1 are not achieved (for instance, $\mathcal{O}_S \cap \mathcal{F}_0 \neq \emptyset$).

- The multi-affine feedback synthesis in Chapter 8 is for single-input systems, and it would be important to extend it to the multi-input case.

- The dissertation focuses mainly on solvability of RCP by continuous state feedbacks. However, we have shown examples where continuous feedbacks fail to solve RCP, yet a discontinuous state feedback solves the problem (Examples 4.2.1, 5.5.3). This highlights the need for studying solvability of RCP by discontinuous state feedbacks. The main objective is to find the largest class of feedbacks to solve RCP. To carry out this research program, one must first identify necessary conditions for solvability by open-loop controls. In this dissertation we initiated this study by showing that for simple polytopes, invariance conditions are necessary for solvability of RCP by open-loop controls. Other necessary conditions for solvability by open-loop controls should be identified. Then one should use these necessary conditions to find a feedback class that solves RCP whenever open-loop controls do.
Bibliography


