PORTFOLIO SELECTION BY SECOND ORDER STOCHASTIC
DOMINANCE BASED ON THE RISK AVERSION DEGREE OF INVESTORS

by

Leili Javanmardi

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Graduate Department of Chemical Engineering and Applied Chemistry
University of Toronto

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Abstract

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Graduate Department of Chemical Engineering and Applied Chemistry

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Second order stochastic dominance is an optimal rule for portfolio selection of risk averse investors when we only know that the investors’ utility function is increasing concave. The main advantage of SSD is that it makes no assumptions regarding the return distributions of investment assets and has been proven to lead to utility maximization for the class of increasing concave utility functions. A number of different SSD models have emerged in the literature for portfolio selection based on SSD. However, current SSD models produce the same SSD efficient portfolio for all risk averse investors, regardless of their risk aversion degree. In this thesis, we have developed a new SSD efficiency model, SSD-DP, which unlike existing SSD efficiency models in the literature, provides an SSD efficient portfolio as a function of investors’ risk aversion degrees. The SSD-DP model is based on the linear programming technique and finds an SSD efficient portfolio by minimizing the dual power transform (DP) of a weighted portfolio of assets for a given risk aversion degree. We show that the optimal portfolio of the proposed model is SSD efficient, i.e. it is not dominated by SSD by any other portfolio, and, through empirical
studies of historical data, we show that the method is a promising tool for constructing trading strategies.
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Nomenclature

List of Abbreviations

**ATMR** above target mean return

**BTSD** below target semi-deviation

**CVaR** Conditional Value at Risk

**DASD** Downside absolute semi-deviation which is the standard deviation of returns below the mean return value

**M-V** mean variance analysis

**MDUA** the difference between mean return MR and downside absolute semi-deviation as safety measure DASD

**MR** Mean return

**QRM** Quadratic risk measure

**SD** Stochastic Dominance
SSD  second order Stochastic Dominance

SSD-DP  our proposed SSD model which is based on SSD and Dual Power Transform (DP) risk measure

TVaR  difference between mean return (MR) and weighted absolute deviation from quantile (WADQ)

WADQ  weighted absolute deviation from quantile

List of Greek Symbols

$\alpha_j$  confidence level

$\Delta g_j = g\left(\frac{M-i+1}{M}\right) - g\left(\frac{M-i}{M}\right)$  distorted probability density function value for the loss level $Z(j)$

$\Phi(p)$  nonincreasing weight function referred to as a spectral (or risk aversion) function

$\Phi_j$  spectral function value at return level $X(j)$

$\kappa(= \frac{R_p-R_l}{\sigma_p})$  the variable in Roy criterion that need to be maximized

$\Lambda$  portfolio possibilities set

$\lambda$  vector of portfolio weights

$\bar{\sigma} (= \frac{1}{M} \sum_t \sigma_t)$  the average of $\sigma_t$ or DASD of the portfolio $X\lambda$

$\Phi^\nu_j$  spectral function value at a given risk aversion degree $\nu$

$\nu$  risk aversion degree parameter of DP risk measure
Π class of permutation matrices

ρ_\_g distortion risk measure with distortion function g

σ_t deviation of the return of the portfolio X_\lambda in period t below its mean return

τ the target return

Ξ class of doubly stochastic matrices

ζ weight vector for a hypothetical portfolio

δ_q the reduction of WADQ

δ_s the increase in mean downside under-achievement MDUA

δ_d the reduction of DASD or the reduction of BTSD

δ_r the increase in MR

σ_t^+ deviation of the return of the portfolio X_\lambda in period t above the pth quantile or above the target return τ

σ_t^- deviation of the return of the portfolio X_\lambda in period t below the pth quantile or below the target return τ

List of Relation Symbols

\nu \rightarrow 0 risk aversion degree parameter of DP risk measure in the limit for an extremely risk averse investor
\( k \to 1 \) risk aversion degree parameter of QRM risk measure in the limit for for an extremely risk averse investor

\( X_1 >_{SSD} X_2 \) portfolio with the random return vector \( X_1 \) dominates a portfolio with the random return vector \( X_2 \) by SSD

**List of Variables**

1. \( 1_{\text{cond}} \) the indicator function that takes value one if the condition is true and zero otherwise

2. \( \lambda_j \) weight of asset \( j \) in a target portfolio \( X_\lambda \)

3. \( \mathbb{E}(U) \) expected utility

4. \( \mathbb{E}(x - X_\lambda)_+ \) expected shortfall for a portfolio with return vector \( X_\lambda \)

5. \( \mathbb{E}(x - Y)_+ \) expected shortfall for a portfolio with return vector \( X_\lambda \)

6. \( \mathbb{W} \) doubly stochastic matrix

7. \( D^*(Y) \) the optimal solution for the SSD model \( C1 \)

8. \( d_J(= \frac{1}{M} \sum_t \text{max}(\bar{y}_t - y_t, 0)) \) the DASD of the return of the assessed portfolio \( Y \)

9. \( d_{r,J}(= \frac{1}{M} \sum_t \text{max}(\tau - y_t, 0)) \) the BTSD of the assessed portfolio \( Y \)

10. \( d_{q,J}(= \frac{1}{M} \sum_t \text{max}(y_t - q_{p,J}, 1-p \cdot (q_{p,J} - y_t))) \) the WADQ for the \( p \text{th} \) quantile of the return distribution of assessed portfolio \( Y \).

11. \( e \) vector of ones
$F_X^{(-1)}(p)$ first quantile function of a portfolio with a random return vector $X$

g    distortion function

$H_\emptyset$    Spectral risk measure with spectral function $\emptyset(p)$

$J$    index for the assessed portfolio $Y$

$j$    index for the available marketed assets

$k$    risk aversion degree parameter of QRM risk measure

$M$    number of the states of nature

$n$    number of available marketed assets

$P$    permutation matrix

$P(W)$    probability of outcome of investment equal to $W$

$p_{ij}$    probability of $i$th outcome for portfolio $j$

$q_{pJ}$    $p$th quantile of the return distribution of assessed portfolio $J$, i.e. $\frac{1}{T} \sum_t 1_{x_{jt} < q_{pJ}} \leq p \leq \frac{1}{M} \sum_t 1_{x_{jt} \leq q_{pJ}}$

$R$    return of portfolio

$R_L$    pre-determined return level

$R_P$    portfolio return

$R_{ij}$    $i$-th possible return of the $j$-th portfolio
\( s^+_{ij} \) the positive part of the matrix \( W - \frac{1}{2} \)

\( s^-_{ij} \) the negative part of the matrix \( W - \frac{1}{2} \)

\( s_k \) mean difference of the returns for the portfolio \( X\lambda \) and the assessed portfolio \( Y \) for the different values of \( k=1, \ldots, M \)

\( s_M \) mean difference of the returns for the portfolio \( X\lambda \) and the assessed portfolio \( Y \) for the number of return observations \( k=M \)

\( t \) index for time period or state of nature

\( U(W) \) investor’s utility function

\( U_2 \) set of VNM utility functions \( (U_2 \equiv \{ u : \mathbb{R} \rightarrow \mathbb{R} \ s.t. \ u_t' \geq 0, \ u_t'' \leq 0, \forall t \}) \)

\( u_t(= \alpha_t + \beta_t x_t) \) linear utility function for each state of nature \( t \) which is characterized by a vector of intercept coefficients \( \alpha \equiv (\alpha_1 \ldots \alpha_M)^M \) and a vector of normalized slope coefficients \( \beta, \ (\beta_1 \geq \beta_2 \geq \ldots \geq \beta_M = 1) \)

\( W \) outcome of an investment in terms of wealth

\( X \) matrix of returns of available marketed assets

\( x \) portfolio return value

\( X\lambda \) portfolio created from the set of assets in matrix \( X \), with weight vector \( \lambda \)

\( X\lambda^* \) the Optimal solution for the optimization model

\( X^m=(x^m_1, x^m_2, \ldots, x^m_N), m=1, \ldots, M \) the \( m \)-th row of matrix \( X \), and represents the \( m \)-th return observation for all \( n \) assets
$X_{(j)}$ vector of sorted portfolio return values from smallest to largest

$x_{jt}$ observed return of asset $j$ in period $t$

$x_{jt}$ observed return of portfolio $Y$ in period $t$

$x_t$ portfolio return value in state $t$

$Y$ return vector for the assessed portfolio

$Z(=-X)$ portfolio’s random loss vector

$Z_{(j)}$ the $j$-th element of the vector of sorted loss values of the portfolio from smallest to largest

$\bar{x}_j(=\frac{1}{M}\sum_t x_{jt})$ mean return of asset $j$

$\bar{y}_t(=\frac{1}{M}\sum_t \max(y_t-\tau,0))$ the ATMR of the assessed portfolio $Y$

$\bar{y}_j(=\frac{1}{M}\sum_t y_t)$ mean return of asset $Y$

$\text{CVaR}_{\alpha_j}(-Y)$ the CVaR for a portfolio with loss vector $-Y$ at confidence level $\alpha_j$

$\text{CVaR}_{\alpha}(-X\lambda)$ the CVaR for a portfolio with loss vector $-X\lambda$ at confidence level $\alpha$

$\text{CVaR}_{\alpha}(-Y)$ the CVaR for a portfolio with loss vector $-Y$ at confidence level $\alpha$

$D_j$ the difference of the CVaR values for portfolios $X\lambda$ and $Y$ at each confidence level $\alpha_j$

$F_{X\lambda}(x)$ integral of the cumulative distribution function of a portfolio with random return vector $X\lambda$ and $F_{X\lambda}(x) \equiv \int_{-\infty}^{x} F_{X\lambda}(t) dt = \mathbb{E}(x - X\lambda)_+$
$F_Y^{(2)}(x)$ integral of the cumulative distribution function of a portfolio with random return vector $Y$ and $F_Y^{(2)}(x) \equiv \int_{-\infty}^{x} F_Y(t) \, dt = \mathbb{E}(x - Y)_+$

$S_Z(z) = 1 - F(Z \leq z)$ portfolio’s loss decumulative distribution function
Chapter 1

Introduction
1.1 Background

Expected utility maximization is the main approach for selecting optimal investment portfolios based on investors’ risk aversion degrees. However, this approach is only applicable when we have enough information to properly model each investor’s utility function. Most of the time only partial information regarding investors’ utility functions is available. For this reason, a number of portfolio selection models have been developed that can employ this partial information to select optimal investment portfolios. Among them, mean variance analysis (M-V) of Markowitz (1959) can differentiate among optimal investment portfolios based on investors’ risk aversion degrees. However, M-V uses historical variance as a measure of risk, ignores moments of return distribution higher than mean and variance, and does not guarantee utility maximization. Alternatively, Stochastic Dominance (SD), introduced by Lehmann (1955) and Quirk and Saposnik (1962), is a more universal rule that is often referred to as the basic concept of decision theory. The main advantage of this approach, compared to M-V analysis, is that SD makes no assumptions regarding the return distributions of investment assets and has been proven to lead to utility maximization (Quirk and Saposnik, 1962).

Different orders of SD are developed for different groups of investors’ utility functions. Among them, Second Order Stochastic Dominance (SSD) assumes concave increasing utility functions and is applicable for risk averse investors. Since most investors are risk averse, this study focuses on SSD as the optimal investment rule for risk averse investors. SD orders higher than SSD are excluded from our analysis since they correspond to increasingly smaller subsets of risk averse investors (Linton et al, 2005) and SSD can be
applied in a practical setting compared to higher order methods.

The first SSD empirical models were developed by Lehmann (1955), and Hanoch and Levy (1969). These models were based on pairwise comparisons of portfolios’ return distributions, and could not account for diversification. As a result, a number of models for SSD analysis have emerged to address these limitations. Lizyayev (2012) provides a systematic review of all of the major SSD models in the literature. In Lizyayev’s review, SSD models of Kuosmanen (2004), Dentcheva and Ruszczynski (2004), Post (2003), Kopa and Chovanec (2008), and Lizyayev (2009) are compared. Based on these comparisons, the SSD models of Post (2003), and Lizyayev (2009) can only identify SSD inefficiency of a given portfolio. Some of the other SSD models, including Dentcheva and Ruszczynski (2004), Kuosmanen (2004), and Post (2008), can provide an SSD dominant portfolio for an inefficiently assessed portfolio. But, Lizyayev shows that the SSD dominant portfolios given by these models may still be dominated by another portfolio. The review suggests that the Kopa and Chovanec (2008) SSD model is the best, since it is the only model that can provide an SSD efficient portfolio. An SSD efficient portfolio is a portfolio which is not dominated, by SSD, by any other portfolio. However, the Kopa and Chovanec (2008) model provides the same SSD efficient portfolio for all risk averse investors and therefore cannot handle risk aversion at the quantitative level. As we will explain in Chapter 4, this model cannot be customized for a given risk aversion degree because of its risk measure (Conditional Value at Risk). Moreover, the Kopa and Chovanec (2008) model, like the other aforementioned SSD efficiency models, can only provide an SSD efficient portfolio for a given benchmark and does not provide a mechanism to pick a given benchmark portfolio based on an investor’s risk aversion degree.
1.2 Thesis Objective

The objective of this thesis is to develop a method for determining an optimal investment portfolio, for a given set of assets and an investor’s risk aversion degree. Given the SSD rule as an optimal criterion for portfolio selection for risk averse investors, in this thesis, we want to develop a new SSD efficiency model, which, unlike existing SSD efficiency models in the literature, does not require a benchmark portfolio, and provides an SSD efficient portfolio as a function of quantified risk aversion degree for a given investor. In order to accomplish this, we need to investigate a risk measure that is capable of modeling risk averse preferences and is consistent with SSD. We show that the optimal portfolio of the proposed model is SSD efficient, i.e it is not dominated by SSD by any other portfolio, and, through empirical studies of historical data, we show that the method can be employed in trading strategies.

1.3 Main Contributions

Given the SSD rule as an optimal criterion for portfolio selection for risk averse investors, the main contributions of this thesis are:

- Establishing a method for determining an SSD efficient portfolio, given the investors’ risk aversion degree.
- Developing a risk-return relationship similar to CAPM for SSD.
- Conducting various empirical analyses of the proposed method based on real world data.
1.4 Organization of Thesis

The rest of the thesis is organized as follows. In Chapter 2, we provide a brief introduction to utility theory and portfolio selection methods when partial information on investors’ utility functions is available. In Chapter 3, we discuss the literature review of portfolio selection models based on SSD. We present our proposed model, SSD-DP, in Chapter 4. In Chapters 5 we present a number of empirical studies to analyze the results of the SSD-DP model for investors with various risk aversion degrees. Finally, we present the discussion of results and recommendations for future work in Chapter 6.
Chapter 2

Utility Theory and Portfolio Selection
Expected Utility Maximization is the main approach for selecting optimal investment portfolios based on an investor’s risk aversion degree, in which the investor needs to define his/her preference (utility function) to differentiate the best portfolio among an admissible set of choices. In general, if full information of an investor’s utility function is available, Expected Utility Maximization is an optimal approach for portfolio selection. However, most of the time only partial information of an investor’s utility function is available. Therefore, one can look for an approach that can employ this partial information to select optimal investment portfolios. For this reason, a number of methods have been developed; each has its pros and cons and none are free of major drawbacks. Below is a brief overview of utility theory and major methods of portfolio selection that are applicable when we have partial information about investors’ utility functions.

This chapter is organized as follows. In Section 2.1, we briefly discuss the utility theory, the Expected Utility Maximization method for portfolio selection, properties of utility functions and different forms of utility functions. In Sections 2.2 to 2.8, we briefly review alternative portfolio selection models to the Expected Utility Maximization model that are applicable when we have partial information about investors’ utility functions. We note that Sections 2.1 to 2.7 are summaries taken from Elton and Gruber (1991) and Section 2.8 is a summary taken from Levy (2006).

2.1 Utility Theory

Based on the Expected Utility Maximization method for portfolio selection, an investor needs to define his/her preference, or utility function, to select the best investment port-
folio among an admissible set of choices. Different investors give different weights to the outcomes of investment portfolios, where the weights are defined by an investor’s utility function. Considering the outcome of an investment in terms of wealth, \( W \), the utility function will be denoted by \( U(W) \). In general, investors seek to maximize the expected utility of their investment to find the optimal portfolio. The expression for expected utility is

\[
E(U) = \sum_w U(W) P(W),
\]  

(2.1)

where

- \( E(U) \) is the expected utility;
- \( U(W) \) is the investor’s utility function;
- \( P(W) \) is the probability of an outcome of the investment equal to \( W \).

Now consider Table 2.1 with two hypothetical investment Portfolios, \( A \) and \( B \), and assume that an investor has a utility function of the form \( U(W) = 2W - .45W^2 \). The expected utility of investment \( A \) and \( B \) with respect to Eq. 2.1 are equal to 8.8% and 8.15% respectively. Therefore, investment \( A \) will be preferred to \( B \) by an investor with this form of utility function.

It is interesting to note that the portfolio selected based on the utility function \( U(W) \) will remain the same if the utility function were changed to \( a + bU(W) \), for \( a \geq 0, \ b > 0 \).

The first step in any investment is to determine the utility function that an investor implicitly uses. By applying this utility function to an admissible set, the investor can select his/her optimal portfolio. The common way that financial institutions use to develop utility functions of investors is to develop these functions for a series of simple
Table 2.1: Two hypothetical portfolios with two different outcomes and probabilities (Elton and Gruber, 1991).

<table>
<thead>
<tr>
<th>Outcome (% Return)</th>
<th>Probability</th>
<th>Outcome (% Return)</th>
<th>Probability</th>
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<td>.1</td>
<td>12</td>
<td>.15</td>
</tr>
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</table>

investments and apply them to more complicated ones.

### 2.1.1 Utility Function Properties

The first property of a utility function is a preference for more wealth to less and is called the non-satiation property in economic theory. This property simply means that $U(W + 1)$ is greater than $U(W)$, which implies that the first derivative of a utility function should be positive.

The second property of utility functions defines the preference of investors towards risk. Risk averse, risk neutral, and risk seeking are the three different groups of investors with different tastes for risk. The second derivative of the utility function with respect to wealth for risk averse investors is always negative. This property can be explained by a simple example. Consider the data in Table 2.2, which shows the outcome of investing or not investing $1 in a certain risky project. The data for the investment opportunity of Table 2.2 is considered as a fair gamble, since the expected value of investing is equal to the expected value of not investing in the risky project. Since investment in a fair gamble
would always be rejected by a risk averse investor, the expected utility of not investing should be greater than investing, for this case. This statement can be written as:

\[ U(1) > .5U(2) + .5U(0) . \]  

(2.2)

which is equivalent to:

\[ U(1) - U(0) > U(2) - U(1) . \]  

(2.3)

Eq. (2.3) means that one unit of change in outcome from 0 to 1 is more valuable than one unit of change in outcome from 1 to 2 for a risk averse investor. A function with this property is a function with a negative second derivative with respect to outcome of investment \( U''(W) < 0 \). A risk neutral investor is indifferent to take or not to take the fair gamble. This preference leads to a zero second derivative of the utility function with respect to wealth. Considering the example in Table 2.2, an indifference to invest can be written as:

\[ U(1) = .5U(2) + .5U(0) . \]  

(2.4)

which is equivalent to:

\[ U(1) - U(0) = U(2) - U(1) . \]  

(2.5)
Eq. (2.5) means that the investor is indifferent to one unit of change in the outcome from 0 to 1 and 1 to 2. This is associated with a function with a zero second derivative, i.e. \( U''(W) = 0. \)

A risk seeking investor would choose to invest in a fair gamble and therefore the second derivative of the utility function is positive with respect to return or wealth. In Table 2.2 this property means that the expected utility of investing must be greater than not investing, or:

\[
U(1) < 0.5U(2) + 0.5U(0). \tag{2.6}
\]

which is equivalent to:

\[
U(1) - U(0) < U(2) - U(1). \tag{2.7}
\]

Eq. (2.7) shows that one unit of change from 1 to 2 is more appealing than one unit of change from 0 to 1. A function with this property has a positive second derivative or \( U''(W) > 0. \) Therefore, the second property of the utility function implies that different preferences of investors toward risk can result in a different sign of the second derivative of the utility function.

The investor’s degree of risk aversion changes as his wealth varies through time. The third property of the utility function shows the changes of risk aversion as a function of wealth changes. The investor’s absolute risk aversion can be measured in terms of the first and second derivative of the utility function. Arrow (1971) and Pratt (1964) defined the absolute measure of risk aversion degree of an investor with utility function \( U(W) \)
as:

\[ A(W) = \frac{-U''(W)}{U'(W)}. \]  

(2.8)

There are three different ways that an investor may react as his wealth increases. If he invests more in risky assets, his absolute risk aversion decreases as his wealth increases. If his investment in risky assets remains constant or becomes less, then he is said to have constant or decreasing absolute risk aversion. The derivative of measure in Eq. (2.8) with respect to wealth can show how absolute risk aversion changes as wealth changes. Table 2.3 shows absolute risk aversion changes in terms of \( A'(w) \). The last property

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
  \hline
  Condition & Definition & \( A'(w) \) & Example of utility function \\
  \hline
  Increasing absolute risk aversion & As wealth increases hold fewer dollars in risky assets & \( A'(w) > 0 \) & \( U(W) = W - bW^2 \) \\
  \hline
  Constant absolute risk aversion & As wealth increases hold same dollars in risky assets & \( A'(w) = 0 \) & \( U(W) = \ln W \) \\
  \hline
  Decreasing absolute risk aversion & As wealth increases hold more dollars in risky assets & \( A'(w) < 0 \) & \( U(W) = -e^{2W-1/2} \) \\
  \hline
\end{tabular}
\end{table}

of the utility function is related to the degree of relative risk aversion. While absolute risk aversion represents the change in the dollar amount of investment in risky assets as wealth changes, relative risk aversion represents the percentage change of investment in risky assets as wealth changes. For example, an investor may hold 40% of his money in risky assets and this percentage may increase, decrease or remain the same as his wealth
increases. The measure of relative risk aversion (see Arrow, 1971 and Pratt, 1964) is:

\[ R(W) = \frac{-WU''(W)}{U'(W)} = WA(W). \tag{2.9} \]

Table 2.4 shows the different degrees of relative risk aversion and their properties. Arrow (1971) and Pratt (1964) have shown that in general, most investors exhibit decreasing absolute risk aversion.

### Table 2.4: Changes in relative risk aversion with wealth (Elton and Gruber, 1991).

<table>
<thead>
<tr>
<th>Condition</th>
<th>Definition</th>
<th>( R'(w) )</th>
<th>Example of utility function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increasing relative risk aversion</td>
<td>Percentage invested in risky asset declines as wealth increases</td>
<td>( R'(w) &gt; 0 )</td>
<td>( U(W) = W - bW^2 )</td>
</tr>
<tr>
<td>Constant relative risk aversion</td>
<td>Percentage invested in risky asset remains unchanged as wealth increases</td>
<td>( R'(w) = 0 )</td>
<td>( U(W) = \ln W )</td>
</tr>
<tr>
<td>Decreasing relative risk aversion</td>
<td>Percentage invested in risky asset increases as wealth increases</td>
<td>( R'(w) &lt; 0 )</td>
<td>( U(W) = -e^{2W-1/2} )</td>
</tr>
</tbody>
</table>

2.1.2 Different Kinds of Utility Functions

There are different classes of utility functions that are used to model investors’ preferences. Quadratic and log utility functions are the most popular ones. The quadratic utility function that can be expressed in terms of mean and variance of an investment and is used to maximize the expected utility in the Mean-Variance (M-V) analysis. The
general form of this function can be defined as

\[ U(W) = W - bW^2, \]  
(2.10)

where its first and second derivatives are

\[ U'(W) = 1 - 2bW. \]  
(2.11)

\[ U''(W) = -2b. \]  
(2.12)

As mentioned before, for rational investors who prefer more to less, the first derivative of the utility function is positive, and for investors who are risk averse, the second derivative of the utility function is negative. Therefore, being risk averse, restricts the values of \( b \) to be positive. The problem with the quadratic utility function is that for positive values of \( b \) in Eq. 2.10, there are always some values of \( W \) which make the first derivative of this function negative, which means that an investor will prefer less wealth to more. The absolute and relative risk aversion of the quadratic utility function and their first derivatives are as follows,

\[ A(w) = \frac{-U''(W)}{U'(W)} = \frac{2b}{1 - 2bW} \text{ with } A'(w) = \frac{4b^2}{(1 - 2bW)^2} > 0. \]  
(2.13)

\[ R(w) = \frac{2bw}{1 - 2bW} \text{ with } R'(w) = \frac{2b}{(1 - 2bW)^2} > 0. \]  
(2.14)

Both of the first derivatives of absolute and relative risk aversion for the quadratic utility function are positive. This positivity implies increasing absolute and relative risk aversion
which is not consistent with the majority of investors who exhibit decreasing absolute and relative risk aversion as their wealth levels increase.

Logarithmic utility is the second example of a class of utility functions. This function, in general, can be expressed as,

\[
U(W) = \ln(W).
\]  
(2.15)

The first and second derivatives of this function are

\[
U'(W) = \frac{1}{W} > 0.
\]  
(2.16)

\[
U''(W) = -\frac{1}{W^2} < 0.
\]  
(2.17)

The absolute and relative risk aversions along with their first derivatives are

\[
A(w) = \frac{1}{W} \quad \text{with} \quad A'(w) = -\frac{1}{W^2} < 0.
\]  
(2.18)

\[
R(w) = 1 \quad \text{with} \quad R'(w) = 0.
\]  
(2.19)

Based on all these measures, the log utility function is appropriate for risk averse investors who prefer more to less and exhibit decreasing absolute and constant relative risk aversion as their wealth changes.

Expected Utility Maximization is the optimal portfolio selection approach since it results in utility maximization for investors. However, this approach is not applicable in the majority of cases since we only have partial information about the investors’
utility functions. Therefore, one can look for an approach that can employ this partial information to select optimal investment portfolios. For this reason, a number of methods have been developed; each has its pros and cons and none are free of major drawbacks. Below is a brief overview of these methods.

2.2 Mean-Variance Analysis

The most established and widely accepted method of optimizing the return on an investment is the classic Mean-Variance (M-V) analysis of Markowitz (1952). M-V analysis assumes that investors only care about the mean and variance of the distribution of the returns of a portfolio. In M-V analysis, Portfolio $A$ is preferred over $B$ if and only if $A$ has a larger or equal mean and a smaller or equal variance. The main disadvantage of the M-V approach is that it only takes into account mean and variance of the return distribution and relies on the assumption of quadratic utility functions, an assumption that is not realistic Xiong and Idzorek (2011). Therefore, M-V analysis can only maximize quadratic forms of utility functions and does not guarantee utility maximization for investors with utility functions of other forms. Consider, for example, two Portfolios $A$ and $B$ with the following return distributions:

$$P(A = 0.01) = 0.8, P(A = 1) = 0.2.$$  

$$P(B = 0.1) = 0.99, P(B = 10) = 0.01.$$  

The mean and variance of the distributions are

$$\mathbb{E}(A) = 0.208 > 0.199 = \mathbb{E}(B) \text{ And } \text{Var}(A) = 0.1468 < 0.9703 = \text{Var}(B).$$

Therefore $A$ dominates $B$ by the M-V criterion. However, if a risk averse investor has
the utility function $u(x) = \log_{10} x$, this results in $\mathbb{E}(u(A)) = -1.6 < -0.98 = \mathbb{E}(u(B))$. Hence this investor will select Portfolio $B$, which differs from that of the M-V criterion. Furthermore, sometimes the M-V criterion cannot determine in favor of any investment although one investment is clearly better than the other. For example, consider the case where the distributions of two investments are as follows,

$$P(A = 0) = 1 - p \quad P(A = a) = p \quad 0 < p < 1, \quad a > 0.$$  
$$P(B = 0) = 1.$$  

The mean and variance of the distributions are

$$\mathbb{E}(A) = pa > 0 = \mathbb{E}(B) \quad \text{and} \quad \text{Var}(A) = p(1 - p)a^2 > 0 = \text{Var}(B).$$

In this case M-V analysis dose not result in the selection of either portfolio although the outcomes of $A$ are always greater or equal to those of $B$, which means any rational investor who prefers more wealth to less would choose Portfolio $A$ over $B$. These examples show that a criterion which only considers mean and variance of returns instead of the whole distribution of the returns does not always lead to an economically meaningful decision.

### 2.3 Geometric Mean Return

One reasonable criterion for portfolio selection is choosing a portfolio with the highest expected value of terminal wealth. Brieman (1960) have shown that maximizing Geometric Mean Return (GMR) results in the highest expected value of terminal wealth. The GMR can be written as:

$$R_{Gj} = \left(1 + R_{ij}\right)^{p_{ij}} - 1,$$

(2.20)
where

- \( R_{ij} \) is the \( i \)-th possible return of the \( j \)-th portfolio;

- \( p_{ij} \) is the probability of \( ith \) outcome for Portfolio \( j \).

A maximum Geometric Mean Return criterion in portfolio selection can result in a well diversified portfolio. In fact, GMR increases with portfolio diversification. However, GMR does not guarantee utility maximization. While utility theory requires investors to maximize the expected utility of terminal wealth, the GMR method seeks to maximize the expected value of terminal wealth, and these two objectives are not identical. A log utility function is the only exception, resulting in the same efficient set of portfolios both in utility theory and GMR.

2.4 Safety First

Safety First is another proposed criteria for portfolio selection in the literature. The Safety First portfolio selection method tries to first prevent unpleasant investment outcomes associated with large negative returns. There are different proposed safety criteria. Roy (1952) have put forth the safety criteria of minimizing the possibility of having a return less than a pre-determined value and, at the same time, maximizing the expected return of the portfolio. The expression can be written as

\[
\min \ P \left( R_P < R_L \right), \tag{2.21}
\]

where

- \( R_P \) is the portfolio return;
\( R_L \) is the pre-determined return level (assuming the mean return is above \( R_L \)).

For instance, for \( R_L \) equal to 5\%, the best portfolio would be the one with the smallest probability of having a return below 5\%. It is also possible to define the \( R_L \) value in terms of the number of standard deviations below the portfolio mean value. For example, if 5\% is equivalent to one standard deviation from Portfolio A’s mean value and two standard deviations from Portfolio B’s mean value then, assuming the same distribution, Roy’s criterion would select Portfolio B since the probability of having a return below two standard deviations is less than having a return below one standard deviation. For a portfolio with a normal distribution, the number of standard deviations from the portfolio mean value would be \( \frac{R_L - R_P}{\sigma_p} \). Therefore the Roy criterion can be defined as

\[
\min \frac{R_L - R_P}{\sigma_p} = \min \{-\kappa\};
\]

Or

\[
\max \frac{R_P - R_L}{\sigma_p} = \max \{\kappa\};
\]

(2.22)

where \( \kappa = \frac{R_p - R_l}{\sigma_p} \). The higher the value of the \( \kappa \), the more appealing the portfolio would be under the Roy criterion. By defining \( R_l \) as the risk-free rate of return, the ratio \( \frac{R_P - R_l}{\sigma_p} \) would be the same as the Sharpe ratio in the M-V analysis. Therefore, the Roy criterion for a normal distribution of returns with the value of \( R_l \) set to the risk-free rate would result in the same efficient set as M-V analysis with unlimited risk free borrowing and lending opportunity.
2.5 Skewness Analysis

The other approach for portfolio selection is based on analysis of other features of the return distribution. While M-V analysis measures the first two moments of the return distribution, authors like Arditti (1967) have proposed analyses based on the first three moments: mean, variance and skewness. Skewness is a measure of asymmetry of a return distribution. Symmetric distributions, like a normal distribution, have zero skewness while other distributions like lognormal have positive skewness. Distributions with positive skewness imply a low probability of obtaining a large negative return and there is less chance of large negative deviations as indicated in Figure 2.1. Therefore, it appears to be logical for investors to prefer portfolios with maximum positive skewness, maximum return and minimum variance of the return distribution. However, measuring skewness, like variance, requires estimation of the correlation structure of the return of assets within the portfolio and this can be difficult for large portfolios.

![Figure 2.1: Positively skewed return distribution.](image-url)
2.6 Downside Risk Analysis Based on Semi-Variance

The variance of returns has been questioned as an appropriate measure of risk since it treats positive and negative returns of portfolios in the same way (see for e.g. Mao, 1970). For this reason, downside risk is one of the proposed measures that can be considered instead of variance. One indicator of downside risk is the downside Semi-Variance of the return distribution (see for e.g. Mao, 1970). Downside Semi-Variance is the variance of returns below the mean value. This risk measure focuses on minimizing losses as opposed to variance, which considers gains as well as losses. Hogan and Warren (1972) proposed the mean-semi variance portfolio selection model, which is similar to the M-V analysis in terms of the optimal selection rule for risk averse investors who prefer more to less. However, downside Semi-Variance, like all of the other downside measures of risk, ignores the information of returns higher than the mean value and results in the omission of potentially better investment portfolios.

2.7 Value at Risk

Value at Risk, or VaR, is an alternative criterion for measuring downside risk of an investment portfolio. VaR is the worst amount of loss at a certain confidence level that may happen in an investment within a given time period. The VaR of a portfolio with an initial investment value of $W_0$ at a confidence level $\alpha$ is defined as:

$$
\text{VaR} = -W_0 \cdot (1 + R_{\text{min}}),
$$

(2.23)
where $R_{\min}$ is defined by $P(R \geq R_{\min}) \geq \alpha$ and $R_{\min}$ is the minimum or worst amount of return at a confidence level $\alpha$.

Portfolio selection based on VaR can be thought of as a way to maximize the expected return of a portfolio with the VaR being no greater than a pre-determined level. Although VaR is an attractive measure that is easy to understand and offers a single number that summarizes the total risk of a portfolio within a certain confidence level, it is not a valid measure of downside risk. Based on the properties of a coherent risk measure proposed by Artzner et al (1999), VaR has been criticized as being a non-coherent risk measure because it does not satisfy the subadditivity\(^1\) condition. Instead, Conditional Value at Risk (CVaR)\(^2\), which measures the expected loss conditional on the loss being greater than VaR, is a coherent risk measure that can be used instead of VaR, as a proper measure of downside risk. However, as we mentioned before, CVaR, like all the other downside measures of risk, ignores the information of returns higher than VaR and results in the omission of potentially better investment portfolios.

### 2.8 Stochastic Dominance

Stochastic Dominance (SD) is another portfolio selection model that is applicable when we have partial information about investors’ utility functions. This approach is based on the relations between pairs of cumulative distribution functions (CDF) of portfolios’ return distributions. An SD efficient portfolio is not dominated by other portfolios and

---

\(^1\)The subadditivity condition requires the risk measure of a merged portfolio to be not greater than the sum of the risk measures of individual portfolios.

\(^2\)Also called expected shortfall or tail loss.
an SD inefficient portfolio is at least dominated by one other portfolio. What makes
the SD approach appealing to investors is the development of efficient sets based on the
genral property of an investor’s utility function.

Three different orders of SD are developed for three different properties of investor
utility functions. Quirk and Saposnik (1962), Hadar and Russell (1969), and Hanoch and
Levy (1969) developed first order SD rules for the class of increasing utility functions. First
order Stochastic Dominance (FSD) refers to investors who prefer more to less. Therefore,
either a conservative investor or gambler prefers a FSD efficient portfolio to a
FSD inefficient one. For example, assume that the cumulative distribution functions of
the returns for Portfolios A and B are denoted by $F_A$ and $F_B$ respectively. Portfolio A
will be preferred over Portfolio B by FSD if and only if

$$F_B(R) \geq F_A(R), \ \forall R. \tag{2.24}$$

with at least one strict inequality. Eq. (2.24) means that Portfolio A will be preferred over
Portfolio B by FSD, if for all levels of returns, $R$, the probability of a return less than $R$ is
higher for Portfolio B than Portfolio A. Table 2.5 shows an example of two hypothetical
Portfolios A and B. If an investor prefers more to less, Portfolio A is preferred to Portfolio
B by FSD since its returns are always greater. In general the relationship between the
cumulative distribution of returns of portfolios for higher orders of SD can be defined as
follows,

\[ F = \text{CDF of the portfolio’s return distribution}; \]

\[ F = \text{CDF of the portfolio’s return distribution}; \]

\[ F = \text{CDF of the portfolio’s return distribution}; \]

\[ F = \text{CDF of the portfolio’s return distribution}; \]
Table 2.5: Returns of Portfolios $A$ and $B$ based on the different market conditions (Levy, 2006).

<table>
<thead>
<tr>
<th>Market condition</th>
<th>Portfolio $A$</th>
<th>Portfolio $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Return(%)</td>
<td>Probability</td>
</tr>
<tr>
<td>Very good</td>
<td>11</td>
<td>0.25</td>
</tr>
<tr>
<td>Good</td>
<td>10</td>
<td>0.25</td>
</tr>
<tr>
<td>Average</td>
<td>9</td>
<td>0.25</td>
</tr>
<tr>
<td>Poor</td>
<td>8</td>
<td>0.25</td>
</tr>
<tr>
<td>Very poor</td>
<td>7</td>
<td>0.25</td>
</tr>
</tbody>
</table>

$- F^{2} = \text{Integral of } F \text{ from } -\infty \text{ to } R$;

$- F^{3} = \text{Integral of } F^{2} \text{ from } -\infty \text{ to } R \text{ and so on, or, generally,}$

$$F^{S+1}(R) = \int_{0}^{R} F^{S}(z) \, dz \quad S = 1, 2, 3, \ldots, \tag{2.25}$$

where $R$ is the return of the portfolio.

Portfolio $A$ dominates Portfolio $B$ by SD of order $S$ if and only if

$$F^{S}_{B}(R) \geq F^{S}_{A}(R) \quad \forall R, \tag{2.26}$$

with at least one strict inequality. Based on Eq. (2.26), Portfolio $A$ will be preferred to Portfolio $B$ by Second order Stochastic Dominance (SSD) if and only if

$$F^{2}_{B}(R) \geq F^{2}_{A}(R) \quad \forall R, \tag{2.27}$$

with at least one strict inequality. Also Portfolio $A$ will be preferred to Portfolio $B$ by
third order Stochastic Dominance if and only if

\[
F^3_B (R) \geq F^3_A (R) \quad \forall R,
\]

with at least one strict inequality. Second Order Stochastic Dominance (SSD) was first developed by Lehmann (1955) and Quirk and Saposnik (1962), and refers to investors who prefer more to less and are risk averse. Third order Stochastic Dominance (TSD), was first developed by Whitmore (1970) and refers to investors who prefer more to less, are risk averse, and have declining risk aversion with increasing wealth\(^4\).

Table 2.6 shows an example of two hypothetical Portfolios, \(A\) and \(B\). The investor cannot select between these two portfolios using the FSD criteria of Eq. (2.24) since the CDF of either portfolios is not always greater than or equal to the other one. For example, at a return of 4\%, Portfolio \(B\) has a higher cumulative probability (or higher probability of return less than 4\%) than Portfolio \(A\) and at a return of 9\%, Portfolio \(A\) has a higher cumulative probability than \(B\). Therefore an investor should decide between a higher probability of a return less than 4\% and a higher probability of a return less than 9\%.

Assuming that the investor is risk averse, he/she would value each increment in return less than the previous increment. Therefore, in this example, an investor would select Portfolio \(A\) over \(B\) because at low levels of returns, Portfolio \(B\) has higher cumulative probability. This example shows that the SSD efficiency of Portfolio \(A\) over \(B\) can be alternatively stated by saying the sum of cumulative probability of Portfolio \(A\) is always less than or equal to that of Portfolio \(B\) with at least one strict inequality. In general,

\(^4\)This property is referred to the utility function with positive third derivative. The other name of this property is decreasing absolute risk aversion.
Table 2.6: Sum of the cumulative probability distributions for the two Portfolios A and B (Levy, 2006).

<table>
<thead>
<tr>
<th>Return(%)</th>
<th>Cumulative probability</th>
<th>Sum of cumulative probability</th>
<th>Sum of cumulative probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.25</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>7</td>
<td>0.25</td>
<td>0.25</td>
<td>0.5</td>
</tr>
<tr>
<td>9</td>
<td>0.5</td>
<td>.25</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>0.5</td>
<td>0.5</td>
<td>1.5</td>
</tr>
<tr>
<td>12</td>
<td>0.75</td>
<td>0.75</td>
<td>2.25</td>
</tr>
<tr>
<td>12.5</td>
<td>.75</td>
<td>.75</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

dominance at order $S$ always implies dominance at all higher orders. For example in Table 2.6, Portfolio A, which is SSD dominant over $B$, is also TSD dominant (the sum of cumulative probabilities is always not greater than those of Portfolio B).

Since most investors are risk averse, this study focuses on SSD as the optimal investment rule for risk averse investors when we have partial information about the investor’s utility function, i.e. concave increasing utility functions. The main advantage of this approach, compared to previously mentioned portfolio section approaches in the literature, such as M-V, is that SSD makes no assumptions regarding the return distributions of investment assets and has been proven to lead to the utility maximization (Quirk and Saposnik, 1962). However, current SSD models for portfolio selection only take into account risk aversion at a qualitative level and cannot differentiate for risk aversion degree. In the next section we briefly review major SSD models with their limitations. We will show that none of these model are capable of finding optimal SSD efficient portfolio as
a function of risk aversion degree of an investor.
Chapter 3

Literature Review of Portfolio Selection Models based on SSD
In this thesis, we are interested in investigating models to derive SSD efficient portfolios based on quantified risk aversion degrees. Based on our review and the review done by Lizyayev (2012), all of the major SSD models of the peer-reviewed publications can be classified in five main categories:

1. Majorization SSD Models.
2. Revealed-Preference SSD Models.
3. Distribution-Based SSD Models.
4. CVaR-Based SSD Models.
5. DEA-Based SSD Models.

Below is the brief discussion of the SSD models within each category.

### 3.1 Majorization SSD Models

The Majorization SSD models were first developed by Kuosmanen (2004) and further enhanced in Kuosmanen (2007) and J. (2008). Kuosmanen (2004) splits SSD efficiency-based models into necessary and sufficient models. The following subsections characterize the dynamics of this group of models.

#### 3.1.1 Primal Necessary Model for SSD Efficiency

The first Majorization primal necessary SSD model was developed by Kuosmanen (2004). The model looks for a weighted portfolio $X\lambda$ with less risk and more or equal return than those of the assessed portfolio $Y$. If the portfolio $X\lambda$ exists then portfolio $Y$ is not SSD efficient. The model is based on the dominance definition as follows,
Dominance Definition (Kuosmanen, 2004)

Portfolio $X\lambda$ dominates portfolio $Y$ by SSD if and only if

$$X\lambda \geq WY \quad \exists W \in \Xi,$$

Or

$$X\lambda \geq PY \quad \exists P \in \Pi, \quad (3.1)$$

where

- $Y$ is the return vector for the assessed portfolio ($Y \in \mathbb{R}^{M \times 1}$);
- $X$ is the matrix of returns of available marketed assets including portfolio $Y$ ($X \in \mathbb{R}^{M \times n}$);
- $P$ is the permutation matrix ($P \in \Pi$);
- $\Pi$ is the class of permutation matrices where
  $$\Pi = \left\{ P \in \mathbb{R}^{M \times M} \mid 0 \leq p_{ij} \leq 1, \sum_{i=1}^{M} p_{ij} = \sum_{j=1}^{M} p_{ij} = 1, \quad i, j = 1, \ldots, M \right\};$$
- $W$ is the doubly stochastic matrix ($W \in \Xi$);
- $\Xi$ is the class of doubly stochastic matrices where
  $$\Xi = \left\{ W \in \mathbb{R}^{M \times M} \mid 0 \leq w_{ij} \leq 1, \sum_{i=1}^{M} w_{ij} = \sum_{j=1}^{M} w_{ij} = 1, \quad i, j = 1, \ldots, M \right\};$$
- $\lambda$ is the vector of portfolio weights ($\lambda \in \Lambda$);
- $\Lambda$ is the portfolio possibilities set ($\Lambda = \left\{ \lambda \in \mathbb{R}^{n \times 1} \mid \lambda^T e = 1, \quad \lambda \geq 0 \right\}$);
- $e$ is the vector of ones ($e \in \mathbb{R}^{n \times 1}$);
- $n$ is the number of available marketed assets;
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\(- M \) is the number of the states of nature (or the number of return observations).

Kuosmanen (2004)’s necessary SSD efficiency model based on the dominance definition in Eq. (3.1) is as follows,

\[ \theta_2^N(Y) = \max_{\lambda, \mathbb{W}} (X\lambda - Y)^T \cdot e \]
\[ \text{s.t.} \]
\[ X\lambda \geq \mathbb{W}Y \]
\[ \mathbb{W} \in \Xi \quad \lambda \in \Lambda. \]

(3.2)

Given the doubly stochastic matrix, \( \mathbb{W} \in \Xi \), portfolio \( \mathbb{W}Y \), compared to portfolio \( Y \), exhibits more evenly distributed returns across all states of nature. Therefore, \( \mathbb{W}Y \) has equal mean return but less risk compared to \( Y \). If these conditions are met, \( \mathbb{W}Y \) is called the mean-preserving spread of the assessed portfolio \( Y \). In the SSD model \( M1 \), if we can find a portfolio \( X\lambda \) with higher returns than those of portfolio \( \mathbb{W}Y \) then the assessed portfolio \( Y \) would be SSD inefficient and the difference between the mean returns of portfolio \( X\lambda \) and \( Y \) would be positive (\( \theta_2^N(Y) > 0 \)). If \( \theta_2^N(Y) = 0 \), no conclusion can be made about SSD efficiency of the assessed portfolio \( Y \), since a portfolio \( X\lambda \) with equal mean but lower risk may still dominate the assessed portfolio \( Y \) by SSD. Therefore, the SSD model \( M1 \) only examines for the necessary condition of SSD efficiency.

For example, consider a two state scenario where each asset’s return is equally likely. We have asset \( Y \) that has return of 1\% if state one happens and 4\% otherwise. Consider, also, that we have three more assets \( A, B, \) and \( C \) with return vectors \( r_A = (0.5\%, 4.5\%)^T \), \( r_B = (2.5\%, 1.5\%)^T \), \( r_C = (3\%, 0.5\%)^T \) respectively. We want to test
the SSD efficiency of asset \( Y \) against any combination of the four assets \( A, B, C \) and \( Y \). According to the dominance definition in Eq. (3.1), all dominating portfolios of asset \( Y \) are located between the red lines in Figure 3.1, that are created from portfolio \( Y \) permuted returns, i.e. \((1%, 4%)^T \) and \((4%, 1%)^T \). If we create any combination of assets \( A, B, C \) and \( Y \) with the assumption of no short selling then we cannot find a portfolio with returns greater than any convex combination of portfolio \( Y \) permuted returns that are located between the red lines in Figure 3.1. Therefore, portfolio \( Y \) is SSD efficient since no convex combination of assets \( A, B, C \) is located in the region above the red lines.

![Figure 3.1: Illustration of the SSD model M1.](image)

The SSD efficiency condition of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD model \( M1 \) is as follows,

**SSD Efficiency Condition \( M1 \)**

\[ \theta_2^{N^*} = 0 \] is the necessary condition for the SSD efficiency of portfolio \( Y \).
The SSD inefficiency measure of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD model \( M1 \) is as follows,

**Inefficiency Measure \( M1 \)**

The inefficiency measure of Model \( M1 \) is \( \frac{\theta_N}{M} \) which is the difference between the mean return of the optimal portfolio \( X\lambda^* \) and the mean return of portfolio \( Y \).

### 3.1.2 Dual Necessary Model for SSD Efficiency

The dual necessary model of Majorization SSD models, proposed by Kuosmanen (2007), is less intuitive, more complex and is difficult to expand to a sufficiency model. Hence, we skip the explanation here.

### 3.1.3 Primal Sufficient Model for SSD Efficiency

The first Majorization primal sufficient SSD model was also developed by Kuosmanen (2004). The Kuosmanen (2004) model looks for a weighted portfolio \( X\lambda \) which is a mean-preserving spread of portfolio \( Y \). If this portfolio does not exist then portfolio \( Y \) is SSD efficient.

**SSD Model M2\( )**

\[
\theta_2^S(Y) = \min_{\lambda, W, S} \sum_{i=1}^{M} \sum_{j=1}^{M} (s_{ij}^+ + s_{ij}^-)
\]

s.t.

\[X\lambda = WY\]
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\[ s_{ij}^+ + s_{ij}^- = w_{ij} - \frac{1}{2} \quad i, j = 1, \ldots, M \]

\[ s_{ij}^+ + s_{ij}^- \geq 0 \quad i, j = 1, \ldots, M \]  \hspace{1cm} (3.4)

\[ \mathbb{W} \in \Xi \quad \lambda \in \Lambda, \]

where the surplus and slack variables \( s_{ij}^+, s_{ij}^- \) are the positive and negative part of the matrix \( W - \frac{1}{2} \), respectively.

It is easy to verify that any mean-preserving spread \( \mathbb{W}Y \) dominates the assessed portfolio \( Y \) by SSD because it exhibits more evenly distributed returns across the states of nature. The SSD model \( M1 \) measures inefficiency in terms of finding a portfolio with higher mean return but it cannot identify dominance by a mean-preserving spread \( \mathbb{W}Y \).

The SSD model \( M2 \) helps with finding a mean-preserving spread of the assessed portfolio \( Y \). The sum of variables \( s_{ij}^+, s_{ij}^- \) is minimized when the returns are distributed across the states as evenly as possible. Therefore, by minimizing the sum of these variables, the risk for \( \mathbb{W}Y \) will be lower than that of portfolio \( Y \). The SSD efficiency condition of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD model \( M2 \) is as follows,

**SSD Efficiency Condition \( M2 \)**

If \( \theta^* (Y) = \frac{M^2}{2} - \sum_{k=2}^{M} kd_{0k} \), then a mean-preserving spread of portfolio \( Y \) does not exist and portfolio \( Y \) is SSD efficient. Note that \( d_{0k} \) is the number of \( k \)-way ties in the returns of the assessed portfolio \( Y \). For instance, a two-way tie occurs in states of nature \( i \) and \( j \) when \( y_i = y_j \) and a three-way tie happens when \( y_i = y_j = y_k \) and so on.

The SSD inefficiency measure of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD model \( M2 \) is as follows,
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Inefficiency Measure $M2$

If the optimal portfolio $X\lambda^*$ has less risk than portfolio $Y$ and the same return, then the inefficiency measure $\theta^2_S$ can be interpreted as the minimum risk-free premium that needs to be added to portfolio $Y$ to make it SSD efficient.

3.1.4 Majorization SSD Models’ Limitation

In some cases the sufficient model $M2$ may result in the optimal portfolio $X\lambda^*$, which dominates portfolio $Y$ by SSD, but itself is not SSD efficient overall. Lizyayev (2012) has shown this problem by a simple example. Suppose we want to test the SSD efficiency of portfolio $Y$ which has a return vector $[.02, 0, .1]^T$. Both Portfolios $X\lambda_1^* = [.06, .05, .01]^T$ and $X\lambda_2^* = [.04, .04, .04]^T$ are mean-preserving spreads of portfolio $Y$ and dominate $Y$ by SSD with $W_1^* = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $W_2^* = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and have a minimum objective function value $\theta^2_S = \frac{3}{2}$. However, the optimal portfolio $X\lambda_1^*$ is not SSD efficient and is dominated by SSD by portfolio $X\lambda_2^*$, which represents a risk-free portfolio. Therefore, this model can only determine the SSD efficiency of a given portfolio and cannot determine the optimal portfolio that is SSD efficient all of the time. To resolve this issue, Lizyayev (2012) has proposed the SSD model $M3$. The SSD model $M3$ is similar to the SSD model $M2$ but the objective function is replaced to minimize the second moment of $X\lambda$. The SSD model $M3$ selects the mean-preserving spread dominant portfolio with the least risk; that is, portfolios for which the returns are distributed across the states of nature as evenly as possible. The SSD model $M3$ is as follows,
3.2 Revealed-Preference SSD Models

The model proposed by Post (2003) best represents Revealed-Preference SSD models. Below is a brief summary of Revealed-Preference models.

3.2.1 Primal Necessary Model for SSD Efficiency

The primal necessary model of Revealed-Preference models looks for a von Neumann-Morgenstern (VNM) utility function for which the assessed portfolio $Y$ is optimal (Post, 2003). If this utility function does not exist then portfolio $Y$ is not SSD efficient. The model is based on the dominance definition as follows,

**Dominance Definition 2 (Post, 2003)**

Portfolio $X\lambda$ dominates portfolio $Y$ by SSD if and only if

$$\sum_{i=1}^{M} u((X\lambda)_i) \geq \sum_{i=1}^{M} u(Y_i) \quad \exists \lambda \in \Lambda \quad \forall u \in U_2$$

(3.6)
with a strict inequality for at least one \( u \in U_2 \). where

- \( U_2 \) is the set of VNM utility functions \( (U_2 \equiv \{ u : \mathbb{R} \to \mathbb{R} \hspace{1em} s.t. \hspace{1em} u_t' \geq 0, u_t'' \leq 0, \forall t \}) \);
- \( Y \) is the return vector for the assessed portfolio \( (Y \in \mathbb{R}^{M \times 1}) \);
- \( X \) is the matrix of returns of available marketed assets including portfolio \( Y \) \( (X \in \mathbb{R}^{M \times n}) \);
- \( \lambda \) is the vector of portfolio weights \( (\lambda \in \Lambda) \);
- \( \Lambda \) is the portfolio possibilities set \( (\Lambda = \{ \lambda \in \mathbb{R}^{n \times 1} : \lambda^T e = 1, \lambda \geq 0 \}) \);
- \( e \) is the vector of ones \( (e \in \mathbb{R}^{n \times 1}) \);
- \( t \) is the index for state of nature (or return observation);
- \( n \) is the number of available marketed assets;
- \( M \) is the number of the states of nature (or the number of return observations).

The Revealed-Preference primal necessary model developed by Post (2003) is as follows,

**SSD Model \( R1 \)**

\[
\xi(Y) = \min_{\lambda, \theta} \theta
\]

s.t.

\[
\frac{1}{M} \sum_{t=1}^{M} (\alpha_t + \beta_t y_t) - (\alpha_t + \beta_t (X\lambda)_{ti}) + \theta \geq 0 \quad i = 1, \ldots, n \tag{3.7}
\]

\[
\beta_1 \geq \beta_2 \geq \ldots \geq \beta_M = 1
\]

\[
\lambda \in \Lambda,
\]

where

- \( x_t \) is the portfolio return value in state \( t \);
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\(- u_t(= \alpha_t + \beta_t x_t)\) is a series of linear utility functions for each state of nature \(t\) which is characterized by a vector of intercept coefficients \(\alpha \equiv (\alpha_1 \ldots \alpha_M)^M\) and a vector of normalized slope coefficients \(\beta, (\beta_1 \geq \beta_2 \geq \ldots \geq \beta_M = 1)\).

VNM is a class of utility functions with positive first derivative and negative second derivative. This class of increasing concave utility functions are suitable for risk averse investors. The SSD model \(R1\) assumes that the VNM utility functions for a portfolio can be described by a series of \(M\) linear utility functions, i.e. \(u(x_t) = \alpha_t + \beta_t x_t\), for each state of nature. The SSD model \(R1\) is designed to find a portfolio \(X\lambda\) in such a way that the expected utility of portfolio \(X\lambda\) is higher than the expected utility of portfolio \(Y\). The difference between the expected utility of the assessed portfolio \(Y\) and the portfolio \(X\lambda\) is denoted by \(\xi(Y)^* = \theta\) in this model. The SSD efficiency condition of the assessed portfolio \(Y\) based on the optimal value of the objective function in the SSD model \(R1\) is as follows,

**SSD Efficiency Condition R1**

\(\xi(Y)^* = 0\) is the necessary condition for SSD efficiency of portfolio \(Y\).

The SSD inefficiency measure of the assessed portfolio \(Y\) based on the optimal value of the objective function in the SSD model \(R2\) is as follows,

**Inefficiency Measure R1**

\(\xi(Y)^*\) or \(\theta\) can be interpreted as the inefficiency measure.
3.2.2 Dual Necessary Model for SSD Efficiency

The SSD model \( R2 \) is a dual necessary model of Revealed-Preferences models proposed by Post (2003). Note that by assumption, the timing of the portfolio returns are unimportant, therefore this model works with the ranked return observation where, below, \( i \) represents the ranking. The dual necessary model of Revealed-Preferences models is as follows,

**SSD Model \( R2 \)**

\[
\Psi(Y) = \max_{\lambda} \ s_M
\]

s.t.

\[
\frac{1}{M} \sum_{i=1}^{k} (x^i\lambda - y_i) = s_k \quad \quad k = 1, \ldots, M
\]

\[
\lambda \in \Lambda
\]

\[
s \in R^+_M,
\]

where

\(- s_M \) is the mean difference of the returns for the portfolio \( X\lambda \) and the assessed portfolio \( Y \) for the number of return observations \( k=M \).

The SSD model \( R2 \) is an equivalent dual SSD model for the primal model \( R1 \). It measures the mean difference of the returns for the portfolio \( X\lambda \) and the assessed portfolio \( Y \) in terms of slack variables \( s_k \). The model tries to maximize the difference between the expected return for the two portfolios \( X\lambda \) and \( Y \).

3.2.3 Dual Sufficient Model for SSD Efficiency

The SSD model \( R3 \) proposed by Post (2008) is a dual SSD sufficient model as follows,
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SSD Model \( R3 \)

\[
\Psi(Y) = \max_{\lambda} \ s^M \cdot e
\]

s.t.

\[
\frac{1}{M} \sum_{i=1}^{k} (x^i \lambda - y_i) = s_k \quad k = 1, \ldots, M
\]

\[
\lambda \in \Lambda
\]

\[
s \in R^M_+,
\]

where

- \( s_k \) is the mean difference of the returns for the portfolio \( X\lambda \) and the assessed portfolio \( Y \) for the different values of \( k=1, \ldots, M \).

The SSD model \( R3 \) is similar to the SSD model \( R2 \), but the objective function is replaced by the summation of slack variables that are calculated for different numbers of return observations, i.e. \( k=1, \ldots, M \). The SSD efficiency condition of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD model \( R3 \) is as follows,

SSD Efficiency Condition \( R3 \)

\( s^M e = 0 \) is the necessary and sufficient condition for the SSD efficiency of portfolio \( Y \).

The SSD inefficiency measure of the assessed portfolio \( Y \) based on the the optimal value of the objective function in the SSD model \( R3 \) is as follows,

Inefficiency Measure \( R3 \)

The inefficiency measure can be interpreted as the mean difference of portfolio \( X\lambda^* \) and portfolio \( Y \).
3.2.4 Revealed-Preference Models’ Limitation

In some cases the SSD models $R1$ and $R2$ may result in the optimal portfolio $X\lambda^*$ which does not necessarily dominate portfolio $Y$ by SSD. Lizyayev (2012) has shown this limitation by a simple example. Consider a two states scenario. We have asset $Y$ that has a return of 0.01 if state one happens and 0.04 otherwise, which we denote as $r_Y = (0.01, 0.04)^T$. We also have two more assets with return vectors, $r_A = (0.09, 0)^T$, and $r_B = (0, 0.02)^T$. We want to test the SSD efficiency of asset $Y$ against any combination of the three assets $A$, $B$, and $Y$. Using the SSD model $R1$ or the SSD model $R2$, the optimal value of the objective functions are $\xi(Y)^* = \Psi(Y)^* = 2$, and the optimal solution is $X\lambda^* = r_A$. Even though $R1$ and $R2$ identify SSD inefficiency of portfolio $Y$ correctly, portfolio $X\lambda^*$ does not dominate portfolio $Y$ by SSD. Therefore, these models can only determine the SSD inefficiency of a given portfolio, and cannot determine its optimal dominating portfolio. Moreover, the Revealed-Preference models of $R3$ may provide the dominating portfolio for the assessed portfolio $Y$, which itself is not SSD efficient overall (Lizyayev, 2012).

3.3 Distribution-Based SSD Models

3.3.1 Primal Necessary Model for SSD Efficiency

The Distribution-Based primal necessary SSD model looks for a portfolio \( X\lambda \) that dominates portfolio \( Y \) by SSD (Dentcheva and Ruszczyski, 2003). The model considers the SSD dominance relation as a constraint and is based on the dominance definition as follows,

**Dominance Definition 3 (Dentcheva and Ruszczyski, 2003)**

Portfolio \( X\lambda \), with cumulative distribution function \( F_{X\lambda}(x) \), dominates portfolio \( Y \) with cumulative distribution function \( F_Y(x) \) by SSD if and only if

\[
F_{X\lambda}^{(2)}(x) \leq F_Y^{(2)}(x), \forall x \in \mathbb{R} \tag{3.10}
\]

with at least one strict inequality, where

- \( F_{X\lambda}^{(2)}(x) \) is the integral of the cumulative distribution function of a portfolio with random return vector \( X\lambda \) and \( F_{X\lambda}^{(2)}(x) \equiv \int_{-\infty}^{x} F_{X\lambda}(t) \, dt = \mathbb{E}(x - X\lambda)_+ \);

- \( F_Y^{(2)}(x) \) is the integral of the cumulative distribution function of a portfolio with random return vector \( Y \) and \( F_Y^{(2)}(x) \equiv \int_{-\infty}^{x} F_Y(t) \, dt = \mathbb{E}(x - Y)_+ \);

- \( Y \) is the return vector for the assessed portfolio \( Y \in \mathbb{R}^{M \times 1} \);

- \( X \) is the matrix of returns of available marketed assets including portfolio \( Y \) \( (X \in \mathbb{R}^{M \times n}) \);

- \( \lambda \) is the vector of portfolio weights \( \lambda \in \Lambda \);

- \( \Lambda \) is the portfolio possibilities set \( \Lambda = \{ \lambda \in \mathbb{R}^{n \times 1} : \lambda^T e = 1, \lambda \geq 0 \} \);
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- $n$ is the number of available marketed assets;
- $M$ is the number of the states of nature (or the number of return observations).

The Distribution-Based primal necessary SSD model is as follows,

**SSD Model $D1$**

\[
\begin{align*}
\max_{\lambda} \mathbb{E}(X\lambda) \\
\text{s.t.} \\
\mathbb{E}(x - X\lambda)_+ &\leq \mathbb{E}(x - Y)_+ \forall x 
\end{align*}
\]

(3.11)

where

- $x$ is the portfolio return value;
- $\mathbb{E}(x - X\lambda)_+$ is the expected shortfall for a portfolio with return vector $X\lambda$;
- $\mathbb{E}(x - Y)_+$ is the expected shortfall for a portfolio with return vector $Y$;

The SSD model $D1$ compares the expected shortfalls of the portfolio $X\lambda$ and the assessed portfolio $Y$. If at all the possible return levels, the expected shortfall for portfolio $X\lambda$ is less than or equal to that of assessed portfolio $Y$ with at least one strict inequality, then portfolio $Y$ would be SSD inefficient. The SSD efficiency condition of the assessed portfolio $Y$ based on the optimal value of the objective function in the SSD model $D1$ is as follows,

**SSD Efficiency Condition $D1$**

If there is no feasible solution to problem $D1$ then the SSD efficiency of portfolio $Y$ cannot be rejected. Otherwise, portfolio $Y$ is not SSD efficient.

The SSD inefficiency measure of the assessed portfolio $Y$ based on the the optimal value
of objective function in the SSD model \( D1 \) is as follows,

**Inefficiency Measure \( D1 \)**

the difference of \( E(Y) \) from \( E(X^\lambda) \) is the inefficiency measure, which is the difference of expected return of assessed portfolio \( Y \) from expected return of optimal portfolio \( X^\lambda \).

### 3.3.2 Dual Necessary Model for SSD Efficiency

The dual necessary model was first developed by Dentcheva and Ruszczyski (2003) and then elaborated later by Dentcheva and Ruszczynski (2006–a), Dentcheva and Ruszczynski (2006–b), and Rudolf and Ruszczynski (2008). These models, compared to the primal model, are more computationally demanding. Hence, we skip the explanation here.

### 3.3.3 Distribution-Based Models’ Limitation

The Distribution-Based models only provide necessary models of SSD efficiency. The primal model may only provide portfolio \( X^\lambda \), which dominates the assessed portfolio \( Y \) by SSD but itself is not SSD efficient overall (Lizyayev, 2012).

### 3.4 CVaR-Based Models (Kopa and Chovanec, 2008)

Some authors, including Krokhmal (2007), Kopa and Chovanec (2008), and Fabian and Veszpremi (2008), tried to test SSD efficiency of a given assessed portfolio, with the Conditional Value at Risk (CVaR) constraints. CVaR is a popular risk measure for its unique representation of the optimum of a special minimization problem. This representation
was first introduced by Rockafellar and Uryasev (2002) and made CVaR a tractable risk measure in optimization modeling. They define CVaR at confidence level $\alpha_j$ for the assessed portfolio $Y$ as

$$\text{CVaR}_{\alpha_j}(-Y) = \min_{\varphi} -\varphi + \frac{1}{M(1-\alpha_j)} \sum_{i=1}^{M} (\varphi - (y)_i)_+, \quad (3.12)$$

where

- $\text{CVaR}_{\alpha_j}(-Y)$ is the CVaR for a portfolio with loss vector $-Y$ at confidence level $\alpha_j$;
- $y_i$ is the $i$th observed return of portfolio $Y$;

Providing that this minimization problem has a unique solution, the optimum value of $\varphi$ would be VaR.

### 3.4.1 Primal Necessary and Sufficient Model for SSD Efficiency

Kopa and Chovanec (2008) have proposed a necessary and sufficient SSD model to assess the SSD efficiency of a given portfolio $Y$. This model is based on the dominance definition as follows,

**Dominance Definition 4**

Portfolio $X\lambda$ dominates portfolio $Y$ by SSD if and only if

$$\text{CVaR}_\alpha(-X\lambda) \leq \text{CVAR}_\alpha(-Y), \quad \forall \alpha \in [0,1], \quad (3.13)$$

with at least one strict inequality, where
CVaR_α (−Xλ) is the CVaR for a portfolio with loss vector −Xλ at confidence level α;

CVaR_α (−Y) is the CVaR for a portfolio with loss vector −Y at confidence level α;

Y is the return vector for the assessed portfolio (Y ∈ ℝ^M×1);

X is the matrix of returns of available marketed assets including portfolio Y (X ∈ ℝ^M×n);

λ is the vector of portfolio weights (λ ∈ Λ);

Λ is the portfolio possibilities set (Λ = \{λ ∈ ℝ^{n×1} : λ^T e = 1, λ ≥ 0\});

n is the number of available marketed assets;

M is the number of the states of nature (or the number of return observations).

The Kopa and Chovanec (2008) necessary and sufficient SSD model is as follows,

**SSD Model C1**

\[
\max_{\lambda,D} \sum_{j=1}^{m} D_j,
\]

s. t.

\[
\text{CVaR}_{\alpha_j} (−Y) − \text{CVaR}_{\alpha_j} (−Xλ) \geq D_j \quad j = 1, \ldots, m,
\]

\[
D_j \geq 0 \quad j = 1, \ldots, m,
\]

where D_j is the difference of the CVaR values for portfolios Xλ and Y at each confidence level \(\alpha_j\).

The SSD model C1 compares the value of CVaR for the two portfolios Xλ and Y, at all confidence levels, \(\alpha_j, j = 1, 2, \ldots, M\). The objective function in the SSD model C1
seeks to maximize the sum of the slack variables $D_j$. In other words, the optimization model $C1$ seeks to find a portfolio with return vector $X\lambda$ which holds smaller or equal CVaR values at all confidence levels, $\alpha_j$, than those of the assessed portfolio $Y$, with at least one strict inequality. The SSD efficiency condition of the assessed portfolio $Y$ based on the optimal value of the objective function in the SSD model $C1$ is as follows,

**SSD Efficiency Condition C1**

If $D^*(Y) = 0$, then portfolio $Y$ is SSD efficient. Otherwise, if $D^*(Y) > 0$ then portfolio $Y$ is SSD inefficient.

The SSD inefficiency measure of the assessed portfolio $Y$ based on the optimal value of objective function in the SSD model $C1$ is as follows,

**Inefficiency Measure C1**

In case of portfolio $Y$ inefficiency, $D^*(Y)$ is the inefficiency measure.

### 3.4.2 CVaR-Based SSD Model’s Limitation

Based on the work done recently by Lizyayev (2012), the Kopa and Chovanec SSD model (Kopa and Chovanec, 2008), compared to all other SSD efficiency models, is ranked as the best, since it is the only model that can provide an SSD efficient portfolio, i.e. an optimal dominating portfolio, given any SSD inefficient portfolio $Y$. However, this model cannot differentiate for investors’ risk aversion degrees.

### 3.5 DEA-Based SSD Models

Data Envelopment Analysis (DEA) has been found to be very useful for evaluating the relative efficiency of a given portfolio to a set of portfolios (Lozano and Gutirrez, 2007).
DEA can assess the relative efficiency of portfolios in terms of efficiency scores, considering different inputs and outputs that a portfolio consumes and produces. Different authors have used different approaches for selecting DEA inputs and outputs. Most of the approaches used a measure of risk as one of the inputs and mean return as one of the outputs. Since SSD efficiency is the desirable condition that all risk averse, non-satiated investors look for, Lozano and Guitirrez (2007) have proposed six DEA models with inputs and outputs based on necessary conditions of SSD efficiency. Inputs and outputs in the Lozano and Guitirrez (2007) DEA models are selected from risk, return and safety measures that show consistency with the measures that should be tested for SSD efficiency. Four of the six DEA models of Lozano and Guitirrez (2007) use risk measures as input and return measures as output while two models use return and safety measures as output with a constant input. All of the Lozano and Guitirrez (2007) models are classified as additive\footnote{The objective function is the sum of the attainable input reductions and output increases w.r.t the observed inputs and outputs of fund being assessed.}, variable returns to scale\footnote{Variable returns to scale in general means that the changes in the outputs do not result in proportional changes in the inputs.} DEA models that seek an optimal portfolio with smaller input and larger output. Additive models are questioned by some authors as they add non-homogeneous slacks, but this is not the case for risk, return and safety measures that are all homogeneous and can be safely added. The general notation that is used in the DEA models is as follows,

- $j \ (= 1, 2 \ldots n)$ is the index for the available marketed assets; 
- $J \ (= 1, 2 \ldots n)$ is the index for the assessed portfolio $Y$; 
- $t \ (= 1, 2 \ldots M)$ is the index for time period or state of nature;
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- $Y$ is the return vector for the assessed portfolio ($Y \in \mathbb{R}^{M \times 1}$);

- $X$ is the matrix of returns of available marketed assets including portfolio $Y$ ($X \in \mathbb{R}^{M \times n}$);

- $y_t$ is the observed return of portfolio $Y$ in period $t$;

- $x_{jt}$ is the observed return of asset $j$ in period $t$;

- $\bar{x}_j(=\frac{1}{M} \sum_t x_{jt})$ is the mean return of asset $j$;

- $\bar{y}(=\frac{1}{M} \sum_t y_t)$ is the mean return of assessed portfolio $Y$;

- $\lambda$ is the vector of portfolio weights ($\lambda \in \Lambda$);

- $\Lambda$ is the portfolio possibilities set ($\Lambda = \{\lambda \in \mathbb{R}^{n \times 1} : \lambda^T e = 1, \lambda \geq 0\}$);

- $\lambda_j$ is the weight of asset $j$ in a target portfolio $X\lambda$.

- $n$ is the number of available marketed assets;

- $M$ is the number of the states of nature (or the number of return observations).

Below is a brief review of the Lozano and Gutirrez (2007) SSD-DEA models.

3.5.1 Mean Return–Downside Absolute Semi-Deviation SSD-DEA Model

The first SSD-DEA model of Lozano and Gutirrez (2007) is called Mean Return–Downside Absolute Semi-Deviation SSD-DEA Model ($MR-DASD$). This model uses an input and an output that are the necessary condition of SSD efficiency as shown in Ogryczak and Ruszczynski (1999) and Ogryczak and Ruszczynski (2001). The model’s input and output are as follows,
Input

- Downside absolute semi-deviation\(^3\) (DASD).

Output

- Mean return (MR).

The SSD-DEA model \(MR-DASD\) is as follows,

**SSD-DEA Model \(MR-DASD\)**

\[
\begin{align*}
\text{max } & \delta_r + \delta_d \\
\text{s.t. } & \\
\sum_j \lambda_j &= 1 \\
\sum_j \bar{x}_j \lambda_j &= \bar{y} + \delta_r \\
\sigma_t &\geq \sum_j (\bar{x}_j - x_{jt}) \lambda_j \ t = 1, \ldots, M \\
\frac{1}{M} \sum_t \sigma_t &= d_J - \delta_d \\
\bar{y} + \delta_r &\geq 0 \\
\lambda_j &\geq 0, \ \sigma_t \geq 0, \ \delta_r \geq 0, \delta_d \geq 0
\end{align*}
\] (3.15)

where

- \(d_J(=\frac{1}{M} \sum_t \max (\bar{y} - y_t, 0))\) is the DASD of the return of the assessed portfolio \(Y\);

- \(\sigma_t\) is the deviation of the return of the portfolio \(X\lambda\) in period \(t\) below its mean return;

- \(\bar{\sigma}(=\frac{1}{M} \sum_t \sigma_t)\) is the average of \(\sigma_t\) or DASD of the portfolio \(X\lambda\);

- \(\delta_d\) is the reduction of DASD;

\(^3\)Standard deviation of returns below the mean return value
– $\delta_r$ is the increase in MR.

The SSD efficiency condition of the assessed portfolio $Y$ based on the optimal value of the objective function in the SSD-DEA model $MR$-$DASD$ is as follows,

**SSD Efficiency Condition $MR$-$DASD$**

If $\delta_d^* + \delta_r^* > 0$, then the assessed portfolio $Y$ is not DEA and SSD efficient which means that the optimal portfolio $X\lambda^*$ has a higher MR by an amount equal to $\delta_r^*$ and a lower DASD by an amount equal to $\delta_d^*$ than those of the assessed portfolio $Y$. However, $\delta_d^* + \delta_r^* = 0$ only implies DEA efficiency and not necessarily SSD efficiency of the assessed portfolio $Y$. The SSD inefficiency measure of the assessed portfolio $Y$ based on the the optimal value of objective function in the SSD-DEA model $MR$-$DASD$ is as follows,

**Inefficiency Measure $MR$-$DASD$**

The positive value of objective function, $\delta_r^* + \delta_d^*$, is the inefficiency measure in the $MR$-$DASD$ model.

### 3.5.2 Mean Return-Mean Downside Under–Achievement SSD-DEA Model

The second SSD-DEA model of Lozano and Gutirrez (2007) is called Mean Return–Mean Downside Under–Achievement SSD-DEA Model ($MR$-$MDUA$). This model uses outputs that are necessary conditions of SSD efficiency as shown in Ogryczak and Ruszczynski (2002). The model’s input and outputs are as follows,

**Input**

– constant equal to unity.
Chapter 3. Literature Review of Portfolio Selection Models based on SSD

Outputs

- Mean return (MR).

- Mean downside under-achievement (MDUA) which is the difference between mean return MR and downside absolute semi-deviation as safety measure DASD.

The SSD-DEA model \( MR-MDUA \) is as follows,

**SSD-DEA Model \( MR-MDUA \)**

\[
\begin{align*}
\text{max} & \quad \delta_r + \delta_s \\
\text{s.t.} & \\
\sum_j \lambda_j &= 1 \\
\sum_j \bar{x}_j \lambda_j &= \bar{y} + \delta_r \\
\sigma_t &\geq \sum_j (\bar{x}_j - x_{jt}) \cdot \lambda_j \quad t = 1, \ldots, M \\
\sum_j \bar{x}_j \lambda_j - \frac{1}{M} \sum_t \sigma_t &= \bar{y} - d_J + \delta_s \\
\bar{y} + \delta_r &\geq 0 \\
\lambda_j &\geq 0, \quad \sigma_t \geq 0, \quad \delta_r \geq 0, \quad \delta_s \geq 0
\end{align*}
\] (3.16)

where

- \( d_J (= \frac{1}{M} \sum_t \max (\bar{y} - y_t, 0)) \) is the DASD of the return of the assessed portfolio \( Y \);

- \( \sigma_t \) is the deviation of the return of the portfolio \( X\lambda \) in period \( t \) below its mean return;

- \( \bar{\sigma} (= \frac{1}{M} \sum_t \sigma_t) \) is the average of \( \sigma_t \) or DASD of the portfolio \( X\lambda \);

- \( \delta_s \) is the increase in mean downside under-achievement MDUA;
\( - \delta_r \) is the increase in MR.

The SSD efficiency condition of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD-DEA model \( MR-MDUA \) is as follows,

**SSD Efficiency Condition \( MR-MDUA \)**

If \( \delta_r^* + \delta_s^* > 0 \), then the assessed portfolio \( Y \) is not DEA and SSD efficient which means that the optimal portfolio \( X_\lambda^* \) has a higher MR by an amount equal to \( \delta_r^* \) and a higher MDUA by an amount equal to \( \delta_s^* \) than those of the assessed portfolio \( Y \). Again, however, \( \delta_r^* + \delta_s^* = 0 \) implies DEA efficiency and not necessarily SSD efficiency of the assessed portfolio \( Y \). The SSD inefficiency measure of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD-DEA model \( MR-MDUA \) is as follows,

**Inefficiency Measure \( MR-MDUA \)**

The positive value of the objective function, \( \delta_r^* + \delta_s^* \), is the inefficiency measure in the \( MR-MDUA \) model.

### 3.5.3 Mean Return–Weighted Absolute Deviation from Quantile SSD-DEA Model

The third SSD-DEA model of Lozano and Gutirrez (2007) is called Mean Return–Weighted Absolute Deviation from Quantile SSD-DEA Model (\( MR-WADQ \)). This model uses an input and an output that are necessary conditions of SSD efficiency as shown in Ogryczak and Ruszczynski (2002). The model’s input and output are as follows,

**Input**
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Weighted absolute deviation from quantile (WADQ).

Output

Mean return (MR).

The SSD-DEA model \textit{MR-WADQ} is as follows,

\textbf{SSD-DEA Model \textit{MR-WADQ}}

\[
\begin{align*}
\max & \quad \delta_r + \delta_q \\
\text{s.t.} & \quad \sum_j \lambda_j = 1 \\
& \quad \sum_j x_j \lambda_j = \hat{g} + \delta_r \\
& \quad \sigma_t^+ - \sigma_t^- = \sum_j x_j t \lambda_j - q_p, \ t = 1, \ldots, M \\
& \quad \frac{1}{M} \sum_t (\sigma_t^+ + \frac{1-p}{p} \cdot \sigma_t^-) = d_{q,J} - \delta_q \\
& \quad \hat{g} + \delta_r \geq 0 \\
& \quad \lambda_j \geq 0, \ \sigma_t^+ \geq 0, \ \sigma_t^- \geq 0, \ \delta_r \geq 0, \ \delta_q \geq 0
\end{align*}
\]

where

- \(q_{p,J}\) is the \(p\)th quantile of the return distribution of assessed portfolio \(J\), i.e. \(\frac{1}{M} \sum_t 1_{y_t < q_{p,J}} \leq p \leq \frac{1}{M} \sum_t 1_{y_t \leq q_{p,J}}\), in which \(1_{\text{cond}}\) is the indicator function that takes the value one if the condition is true and zero otherwise.

- \(d_{q,J}(= \frac{1}{M} \sum_t \max (y_t - q_{p,J}, \frac{1-p}{p} (q_{p,J} - y_t))\) is the WADQ for the \(p\)th quantile of the return distribution of assessed portfolio \(Y\) (deviation above the \(p\)th quantile are given a weight of one while deviations below the quantile are weighted with \(\frac{1-p}{p}\));

- \(\sigma_t^+\) is the deviation of the return of the benchmark portfolio \(X\lambda\) in period \(t\).
above the $p$th quantile;

- $\sigma_t^-$ is the deviation of the return of the benchmark portfolio $X\lambda$ in period $t$ below the $p$th quantile;

- $\delta_q$ is the reduction of WADQ;

- $\delta_r$ is the increase in MR.

The SSD efficiency condition of the assessed portfolio $Y$ based on the optimal value of the objective function in the SSD-DEA model $MR-WADQ$ is as follows,

**SSD Efficiency Condition $MR-WADQ$**

If $\delta_r^* + \delta_q^* > 0$, then the assessed portfolio $Y$ is not DEA and SSD efficient which means that the optimal portfolio $X\lambda^*$ has a higher MR by an amount equal to $\delta_r^*$ and lower WADQ by an amount equal to $\delta_q^*$ than those of the assessed portfolio $Y$. However, $\delta_r^* + \delta_q^* = 0$ implies DEA efficiency and not necessarily SSD efficiency of the assessed portfolio $Y$. The SSD inefficiency measure of the assessed portfolio $Y$ based on the the optimal value of the objective function in the SSD-DEA model $MR-WADQ$ is as follows,

**Inefficiency Measure $MR-WADQ$**

The positive value of the objective function, $\delta_r^* + \delta_q^*$, is the inefficiency measure in the $MR-WADQ$ model.

### 3.5.4 Mean Return–Tail Value-at-Risk SSD-DEA Model

The fourth SSD-DEA model of Lozano and Gutirrez (2007) is called Mean Return–Tail Value-at-Risk SSD-DEA Model ($MR-TVaR$). This model uses outputs that are necessary
conditions of SSD efficiency as shown in Ogryczak and Ruzsczynski (2001). The model’s an input and outputs are as follows,

**Input**

- constant equal to unity.

**Output**

- Mean return (MR).
- Tail Value at Risk (TVaR) which is the difference between mean return (MR) and weighted absolute deviation from quantile (WADQ).

The SSD-DEA model $MR-TVaR$ is as follows,

**SSD-DEA Model $MR-TVaR$**

$$
\begin{align*}
\text{max } & \delta_r + \delta_q \\
\text{s.t. } & \\
\sum_j \lambda_j &= 1 \\
\sum_j \bar{x}_j \lambda_j &= \bar{y} + \delta_r \\
\sigma_t^+ - \sigma_t^- &= \sum_j x_{jt} \lambda_j - q_p \quad t = 1, \ldots, M \\
\sum_j \bar{x}_j \lambda_j - \frac{1}{M} \sum_t (\sigma_t^+ + \frac{1-p}{p} \cdot \sigma_t^-) &= \bar{y} - d_{qJ} + \delta_q \\
\bar{y} + \delta_r &\geq 0 \\
\lambda_j &\geq 0, \quad \sigma_t^+ \geq 0, \quad \sigma_t^- \geq 0, \quad \delta_r \geq 0, \quad \delta_q \geq 0
\end{align*}
$$

(3.18)

where

- $q_{p,J}$ is the $p$th quantile of the return distribution of assessed portfolio $J$, i.e.

$$
\frac{1}{T} \sum_t 1_{x_{jt} < q_p} \leq p \leq \frac{1}{M} \sum_t 1_{x_{jt} \leq q_p} ,
$$

in which $1_{\text{cond}}$ is the indicator function that takes value one if the condition is true and zero otherwise.
\[-d_{qJ}(= \frac{1}{M} \sum_t \max (y_t - q_{pJ}, \frac{1-p}{p}(q_{pJ} - y_t)))\] is the WADQ for the \textit{pth} quantile of the return distribution of assessed portfolio \textit{Y} (deviation above the \textit{pth} quantile are given a weight of one while deviations below the quantile are weighted with \(\frac{1-p}{p}\));

- \(\sigma_t^+\) is the deviation of the return of the benchmark portfolio \(X\lambda\) in period \(t\) above the \textit{pth} quantile;

- \(\sigma_t^-\) is the deviation of the return of the benchmark portfolio \(X\lambda\) in period \(t\) below the \textit{pth} quantile;

- \(\delta_q\) is the reduction of weighted absolute deviation from quantile (WADQ);

- \(\delta_r\) is the increase in MR.

The SSD efficiency condition of the assessed portfolio \textit{Y} based on the optimal value of the objective function in the SSD-DEA model \textit{MR-TVaR} is as follows,

**SSD Efficiency Condition \textit{MR-TVaR}**

If \(\delta_r^* + \delta_q^* > 0\), then the assessed portfolio \textit{Y} is not DEA and SSD efficient which means the optimal portfolio \(X\lambda^*\) has a higher MR by an amount equal to \(\delta_r^*\) and a higher tail value-at-risk (TVaR) by an amount equal to \(\delta_q^*\) than those of the assessed portfolio \textit{Y}. However, \(\delta_r^* + \delta_q^* = 0\) implies DEA efficiency and not necessarily SSD efficiency of the assessed portfolio \textit{Y}. The SSD inefficiency measure of the assessed portfolio \textit{Y} based on the the optimal value of the objective function in the SSD-DEA model \textit{MR-TVaR} is as follows,

**Inefficiency Measure \textit{MR-TVaR}**

The positive value of the objective function, \(\delta_r^* + \delta_q^*\), is the inefficiency measure in
Chapter 3. Literature Review of Portfolio Selection Models based on SSD

3.5.5 Mean Return–Below Target Semi-Deviation SSD-DEA Model

The fifth SSD-DEA model of Lozano and Gutirrez (2007) is called Mean Return–Below Target Semi-Deviation SSD-DEA Model ($MR-BTSD$). This model uses input and output that are necessary condition of SSD efficiency as shown in Bawa (1975) and Fishburn (1977). The model’s input and output are as follows,

**Input**
- Below target semi-deviation (BTSD).

**Output**
- Mean return (MR).

The SSD-DEA Model $MR-BTSD$ is as follows,

**SSD-DEA Model $MR-BTSD$**

$$
\begin{align*}
\max & \quad \delta_r + \delta_d \\
\text{s.t.} & \\
\sum_j \lambda_j & = 1 \\
\sum_j \bar{x}_j \lambda_j & = \bar{y} + \delta_r \\
\sigma_t & \geq \sum_j (\tau - x_{jt}) \cdot \lambda_j \\
\frac{1}{M} \sum_t \sigma_t & = d_{r,t} - \delta_d \\
\bar{y} + \delta_r & \geq 0 \\
\lambda_j & \geq 0, \quad \sigma_t \geq 0, \quad \delta_r \geq 0, \quad \delta_d \geq 0
\end{align*}
$$

(3.19)
where

- \( \tau \) is the target return;

- \( d_{r,t} = \frac{1}{M} \sum t \max (\tau - y_t, 0) \) is the BTSD of the assessed portfolio \( Y \);

- \( \sigma_t \) is the deviation of the return of the portfolio \( X\lambda \) in period \( t \) below the target return \( \tau \);

- \( \delta_d \) is the reduction of BTSD;

- \( \delta_r \) is the increase in MR.

The SSD efficiency condition of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD-DEA model \( MR-BTSD \) is as follows,

**SSD Efficiency Condition MR-BTSD**

If \( \delta_r^* + \delta_d^* > 0 \), then the assessed portfolio \( Y \) is not DEA and SSD efficient which means that the optimal portfolio \( X\lambda^* \) has a higher MR by an amount equal to \( \delta_r^* \) and a lower below target semi-deviation (BTSD) by an amount equal to \( \delta_d^* \) than those of the assessed portfolio \( Y \). However, \( \delta_r^* + \delta_d^* = 0 \) implies DEA efficiency and not necessarily SSD efficiency of the assessed portfolio \( Y \). The SSD inefficiency measure of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD-DEA model \( MR-BTSD \) is as follows,

**Inefficiency Measure MR-BTSD**

The positive value of the objective function, \( \delta_r^* + \delta_d^* \), is the inefficiency measure in the \( MR-BTSD \) model.
3.5.6 Above Target Mean Return–Below Target Semi-Deviation SSD-DEA Model

The sixth SSD-DEA model of Lozano and Gutirrez (2007) is called Above Target Mean Return–Below Target Semi-Deviation SSD-DEA Model \((ATMR-BTSD)\). This model uses input and output that are necessary condition of SSD efficiency as shown in Holthausen (1981). The model’s input and output are as follows,

**Input**

- Below target semi-deviation (BTSD).

**Output**

- Above target mean return (ATMR).

The SSD-DEA model \(ATMR-BTSD\) is as follows,

**SSD-DEA Model \(ATMR-BTSD\)**

\[
\begin{align*}
\text{max} & \quad \delta_r + \delta_d \\
\text{s.t.} & \quad \sum_j \lambda_j = 1 \\
& \quad \sigma^+_t - \sigma^-_t = \sum_j x_{jt} \lambda_j - \tau \quad t = 1, \ldots, M \\
& \quad \frac{1}{M} \sum_t \sigma^+_t = \bar{\sigma}_r + \delta_r \\
& \quad \frac{1}{M} \sum_t \sigma^-_t = \bar{\sigma}_d - \delta_d \\
& \quad \sum_j \bar{x}_j \lambda_j \geq 0 \\
& \quad \lambda_j \geq 0, \quad \sigma^+_t \geq 0, \quad \sigma^-_t \geq 0, \quad \delta_r \geq 0, \quad \delta_d \geq 0
\end{align*}
\]

where
\( \tau \) is the target return;

\(- d_{\tau J}(= \frac{1}{M} \sum_t \max (\tau - y_t, 0) \) is the BTSD of the assessed portfolio \( Y \);

\(- \bar{y}_r(= \frac{1}{M} \sum_t \max (y_t - \tau, 0) \) is the ATMR of the assessed portfolio \( Y \);

\(- \sigma_t^+ \) is the positive deviation of the return of the portfolio \( X\lambda \) in period \( t \) above the target return \( \tau \);

\(- \sigma_t^- \) is the positive deviation of the return of the portfolio \( X\lambda \) in period \( t \) below the target return \( \tau \);

\(- \delta_d \) is the reduction of below target semi-deviation (BTSD);

\(- \delta_r \) is the increase in above target mean return (ATMR);

The SSD efficiency condition of the assessed portfolio \( Y \) based on the optimal value of the objective function in the SSD-DEA model \( ATMR-BTSD \) is as follows,

**SSD Efficiency Condition \( ATMR-BTSD \)**

If \( \delta_r^* + \delta_d^* > 0 \), then the assessed portfolio \( Y \) is not DEA and SSD efficient which means the optimal portfolio \( X\lambda^* \) has a higher ATMR by an amount equal to \( \delta_r^* \) and a lower BTSD by an amount equal to \( \delta_d^* \) than those of the assessed portfolio \( Y \). However, \( \delta_r^* + \delta_d^* = 0 \) implies DEA efficiency and not necessarily SSD efficiency of the assessed portfolio \( Y \). The SSD inefficiency measure of the assessed portfolio \( Y \) based on the the optimal value of the objective function in the SSD-DEA model \( ATMR-BTSD \) is as follows,

**Inefficiency Measure \( ATMR-BTSD \)**

The positive value of the objective function, \( \delta_r^* + \delta_d^* \), is the inefficiency measure in the \( ATMR-BTSD \) model.
3.5.7 DEA-Based Models’ Limitation

Lozano and Gutirrez (2007) have proposed six DEA models. The inputs and outputs of these DEA models are selected from the necessary rules of SSD efficiency. Therefore, these models can only determine the SSD inefficiency of a given portfolio, and cannot determine its optimal dominating portfolio.

3.6 Summary

SSD is the optimal investment rule for the selection of efficient portfolios for risk averse investors and leads to the maximum value of all VNM utility functions\(^4\). Based on the expected utility approach (Pratt, 1964), a portfolio with a random return vector, \(X\lambda\), dominates a portfolio with random return vector, \(Y\), by SSD (i.e. \(X\lambda >_{SSD} Y\)) if and only if

\[
E[u(X\lambda)] > E[u(Y)] \quad \forall u \in U_2.
\]  

\( (3.21) \)

where \(U_2 \equiv \{ u : R \rightarrow R \text{ s.t. } u'(t) \geq 0, u''(t) \leq 0, \forall t \} \).

In this Chapter, we discussed different SSD dominance definitions in Eqs. 3.1, 3.6, 3.10, and 3.13, which are all proved to be equivalent to the dominance definition (3.21) (Levy, 2006). Different SSD models are proposed in the literature based on these different SSD dominance definitions. Based on our review and the one done recently by Lozano and Gutirrez (2007) and Lizyayev (2012), all of the aforementioned SSD models are compared in Table 3.1. Based on this comparison, the SSD model developed by Kopa-Chovanec

\(^4\)As discussed previously in this chapter, VNM utility functions are the class of concave increasing utility functions with positive first derivative and negative second derivative.
(2008) is selected as the best since it is the only model that can provide an optimal dominating portfolio. However, as shown in Table 3.1, none of the aforementioned SSD models in the literature can differentiate the SSD efficient portfolio based on an investor’s risk aversion degree. In the next Chapter we purpose a new SSD efficiency model that can find the SSD efficient portfolio based on an investor’s degree of risk aversion.

Table 3.1: Comparison of SSD models (based on our review and Lizyayev (2012)).

<table>
<thead>
<tr>
<th>SSD efficiency models</th>
<th>SSD inefficiency</th>
<th>SSD dominant portfolio</th>
<th>SSD efficient portfolio</th>
<th>Account for risk aversion degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revealed-Preference SSD models</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Majorization SSD models</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Distribution-Based SSD models</td>
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<td>✓</td>
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<td>✓</td>
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<tr>
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<td>✓</td>
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<tr>
<td>DEA-Based SSD models</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Criteria for Comparison of SSD Models:
- **SSD inefficiency**: If the model identifies SSD efficiency of assessed portfolio.
- **SSD dominant portfolio**: If the model provides a dominating portfolio for the SSD inefficient assessed portfolio.
- **SSD efficient portfolio**: If the model provides an SSD efficient portfolio for the SSD inefficient assessed portfolio.
- **Account for risk aversion degree**: If the model provides an SSD efficient portfolio for the inefficient assessed portfolio based on investor’s risk aversion degree.
Chapter 4

Methodology
In this chapter, we introduce our proposed SSD efficiency model, called SSD-DP model, which, unlike the aforementioned SSD efficiency models in the literature, does not require a benchmark portfolio and provides an SSD efficient portfolio as a function of the quantified risk aversion degree for a given investor. We will prove that the result of our proposed SSD-DP model is SSD efficient.

In this chapter we utilize the SSD theory presented in Chapter 3. This chapter is organized as follows. In Section 4.1, we briefly discuss different classes of risk measures and their consistency with SSD. In Section 4.2, we present our proposed model, SSD-DP, and show that it is SSD efficient for any risk aversion degree. In Section 4.3, we compare our proposed SSD-DP model to Kopá and Chovanec (2008), based on theory and an empirical application. Also in this chapter, we show how our model could be implemented for different risk averse investors. The Chapter concludes with summary in Section 4.4.

4.1 Risk Measures and Their Consistency with SSD

Distortion risk measures and spectral measures are two major classes of risk measures for modeling risk averse preferences. Distortion risk measures, introduced by Denneberg (1990) and Wang (1996), are developed to adjust the portfolio probability distribution function in a way that gives higher probability to loss values and less to return values. Every distortion risk measure is associated with a nondecreasing distortion function $g$, with $g(0) = 0$, and $g(1) = 1$, and $g : [0, 1] \rightarrow [0, 1]$. Computation of a distortion risk
measure is based on the Choquet integral representation, introduced by Wang (1996). The distortion risk measure $\rho_g$, with distortion function $g$, for a portfolio with a random loss vector $Z (= -X)$, is defined as

$$
\rho_g(Z) = \int_{-\infty}^{0} [g(S_Z(z)) - 1)] dz + \int_{0}^{\infty} g(S_Z(z)) dz,
$$

where $S_Z(z) = 1 - F(Z \leq z)$, is the portfolio's loss decumulative distribution function. Distortion function $g$ transforms a probability distribution $S_Z(z)$ to a new distribution $g(S_Z(z))$. It can be shown (Wang, 1996) that Eq. (4.1) is equivalent to the expected value under the distorted probability distribution. Therefore, the distortion risk measure estimator of the empirical distribution of a portfolio with a discrete random loss vector $Z = -X$, will be

$$
\rho_g(Z) = \sum_{j=1}^{M} Z(j) \left[ g \left( \frac{M - j + 1}{M} \right) - g \left( \frac{M - j}{M} \right) \right] = \sum_{j=1}^{M} Z(j) \Delta g_j,
$$

where $M$, is the number of a portfolio’s loss observations, $Z(j)$, is the $j$-th element of the vector of sorted loss values of the portfolio from smallest to largest, and $\Delta g_j = g \left( \frac{M - j + 1}{M} \right) - g \left( \frac{M - j}{M} \right)$, represents the distorted probability density function value for the loss level $Z(j)$.

Alternatively, the spectral risk measure (see for e.g. Acerbi, 2002) is a weighted average of the quantiles of a portfolio with random return vector $X$, using a nonincreasing weight function referred to as a spectral (or risk aversion) function and denoted by $\emptyset(p)$. 
Spectral risk measure $H_\emptyset$, is defined as

$$H_\emptyset (X) = - \int_0^1 \emptyset(p) F_X^{(-1)}(p) \, dp,$$

(4.3)

where $F_X^{(-1)}(p)$, is the first quantile function of a portfolio with a random return vector $X$. It can be shown (Acerbi, 2002) that the spectral measure of a portfolio with a discrete random return vector $X$, has the consistent estimator

$$H_\emptyset (X) = - \sum_{j=1}^M X_{(j)} \emptyset_j;$$

(4.4)

where $M$ is the number of return observations, $X_{(j)}$ is the vector of sorted portfolio return values from smallest to largest, and $\emptyset_j$ is the spectral function value at return level $X_{(j)}$, which is computed as $\emptyset_j = \int_{j/M}^{(j+1)/M} \emptyset(p) \, dp$, for which $\sum_{j=1}^M \emptyset_j = 1$ should hold.

Any spectral measure has an equivalent distortion risk measure and vice versa (see for e.g. Sripoontchitta et al 2010). Spectral measures are related to distortion risk measures by equivalence of the spectral function to the first derivative of the distortion function as

$$\emptyset(p) = g'(p).$$

(4.5)

In order to be able to use either the spectral or distortion measures for modeling risk averse preferences in SSD models, these measures should be consistent with SSD. In financial decision making, the risk measure should be consistent with SSD, since risk modeling should be conducted through the expected utility theory from which SSD rules
are derived (Hardy and Wirch, 2001). For example, assume we have two portfolios with random return vectors $X_1$ and $X_2$. If a portfolio with the random return vector $X_1$ dominates a portfolio with the random return vector $X_2$ by SSD (i.e. $X_1 >_{SSD} X_2$), then the risk of Portfolio 1 should be less than that of Portfolio 2. A number of the spectral and distortion risk measures are consistent with SSD. Hardy and Wirch (2001) proved that a distortion risk measure is consistent with SSD if and only if its associated distortion function, $g$, is continuous, differentiable and strictly concave, i.e. $g''(p) < 0$. From the equivalence relation in Eq. (4.5), any spectral measure will be SSD consistent if and only if its corresponding distortion risk measure is consistent with SSD.

### 4.2 SSD-DP Model Development

We develop our proposed model in this section. First, we choose an appropriate risk measure for modeling risk aversion, then we develop our proposed model based on this risk measure, and finally, we show that our model leads to a portfolio that is SSD efficient for any chosen risk aversion degree.

#### 4.2.1 Investigating an Appropriate Risk Measure for Modeling Risk Averse Preferences

In order to provide an SSD efficient portfolio based on a quantified risk aversion degree, we want to develop an SSD efficiency model by a risk measure that is capable of modeling risk averse preferences and is consistent with SSD. As mentioned earlier, based on
the review done by Lizyayev (2012), the SSD efficiency model developed by Kopa and Chovanec (2008) is selected as the best. The Kopa and Chovanec (2008) model is based on the comparison of the Conditional Value at Risk (CVaR) of portfolios (see dominance definition Eq. (3.13)). However, the Kopa and Chovanec model gives the same SSD efficient portfolio for all risk averse investors. This model cannot be customized for a specific risk aversion degree, since CVaR is only consistent with SSD when compared at all confidence levels and not at a given confidence level, $\alpha$, where $\alpha$ could be an indicator of risk aversion degree. As is well known, the distortion function of the CVaR at a given confidence level, $\alpha$, for a portfolio with random loss vector $Z = -X$, is given by

$$g_{\alpha}^{CVaR}(p) = \min\left(\frac{p}{1 - \alpha}, 1\right) \quad p \in [0, 1].$$

Since the CVaR distortion function is not differentiable at $p = 1 - \alpha$ and ignores the amount of loss for all the percentiles below $1 - \alpha$, this risk measure is not consistent with SSD, based on the criteria proved by Hardy and Wirch (2001).

Alternatively, in order to find an SSD efficient portfolio according to an investor’s risk aversion degree, we propose to use a risk measure that is a function of risk aversion degree, and is consistent with SSD. The Dual Power Transform (DP), introduced by Yaari (1987), and the Quadratic risk measure (QRM), introduced by Sereda et al (2010), are risk measures capable of modeling risk averse preferences, and are both consistent risk measures with SSD. These risk measures can be either represented as distortion or
spectral risk measures. The distortion functions of DP and QRM are given by

\[ g_{\nu}^{DP}(p) = 1 - (1 - p)^{\frac{1}{\nu}} \quad p \in [0, 1] \quad \nu \in (0, 1], \]  

(4.7)

\[ g_{k}^{QRM}(p) = p + k (p - p^2) \quad p \in [0, 1] \quad k \in [0, 1], \]  

(4.8)

where \( \nu \) and \( k \) are the risk aversion degree parameters, and when \( \nu \) approaches 0, or \( k \) approaches 1, we have greater risk aversion. Note that at \( \nu = 1 \), or \( k = 0 \), the investor will be risk neutral. Since the distortion functions in Eqs. (4.7) and (4.8) are continuous, differentiable and strictly concave (i.e. \( g''(p) < 0 \)), both risk measures DP and QRM are consistent with SSD, based on the criteria proved by Hardy and Wirch (2001). Note that in the limit where the investor is considered risk neutral, i.e. \( \nu \to 1 \) and \( k \to 0 \), both \( g_{\nu}^{DP} \to p \) and \( g_{k}^{QRM} \to p \), as expected. Conversely, for the limit where the investor is extremely risk averse, i.e. \( \nu \to 0 \) and \( k \to 1 \), \( g_{\nu}^{DP} \to 1 \), as expected, but \( g_{k}^{QRM} \to 2p - p^2 \). Furthermore, Figures 4.1 and 4.2 show the distortion functions of DP and QRM at different risk aversion degrees, respectively. In Figure 4.2 we see that the QRM’s distortion function does not show an extreme case of risk aversion. Hence, the risk measure of QRM does not include a range of all possible degrees of risk aversion. On the other hand Figure 4.1 shows the extreme case of risk aversion for a most risk averse investor as the risk aversion degree approaches \( \nu = 0 \). Therefore, we base our model on DP as the appropriate risk measure for modeling risk aversion.

By combining Eqs. (4.2) and (4.7) we can calculate the DP distortion risk measure value for a given portfolio accounting for investor risk aversion. Note that for a risk
neutral investor $\rho_{\nu_{DP}}(Z) = \mathbb{E}[Z]$, and for an extremely risk averse investor $\rho_{\nu_{DP}}(Z) = Z_{(M)} = \max(Z)$.

Using the equivalence relation between distortion and spectral risk measures in Eq. (4.5), together with the DP’s distortion function of Eq. (4.7), we derive the spectral function for DP as

$$\psi_{\nu_{DP}}(p) = \frac{1}{\nu} (1 - p)^{\frac{1-\nu}{\nu}} \quad \nu \in (0, 1]. \quad (4.9)$$

Similar to the distortion function, in the limiting case of a risk neutral investor, the spectral function gives $H_{\psi_{\nu_{DP}}}(X) = \mathbb{E}[-X] = \mathbb{E}[Z]$, and for an extremely risk averse investor $H_{\psi_{\nu_{DP}}}(X) = -X_{(1)} = Z_{(M)}$.

Since the DP distortion risk measure is consistent with SSD, its corresponding spectral measure in Eq. (4.9) will also be SSD consistent. In order to compute the DP of a portfolio, we can either use its distortion function or spectral function. For example, consider a hypothetical asset or portfolio, $A$, with equally likely discrete returns represented...
Figure 4.2: QRM distortion function plot for risk averse investors with risk aversion degrees \( k = \{0, 0.3, 0.7, 1\} \).

by vector \( X = (-1, -0.5, 4.5)^T \). In Table 4.1 we show the DP calculation for Portfolio A based on the distortion risk measure of Eq. (4.2) and distortion function of Eq. (4.7) for varying risk aversion degrees. In Table 4.2, DP is also calculated for the portfolio, but using the spectral risk measure of Eq. (4.4) and spectral function of Eq. (4.9). As expected, the results are equivalent and the DP value increases as risk aversion increases. Furthermore, at \( \nu = 1 \) (i.e. the risk neutral investor case) \( \rho_g(Z) = -0.99 = \mathbb{E}[Z] = -\mathbb{E}[X] \), and at \( \nu \to 0 \), \( \rho_g(Z) \to 1 = \max(Z) \).

4.2.2 Modeling SSD based on the DP Risk Measure

In this section we propose an SSD efficiency model, SSD-DP, which accounts for investor risk aversion. Consider a portfolio problem with \( n \) assets and \( M \) equally likely discrete return observations collected in matrix \( X = (x_1, x_2, \ldots, x_n) \), where \( X^m = (x^m_1, x^m_2, \ldots, x^m_N) \), for \( m = 1, \ldots, M \), is the \( m \)-th row of matrix \( X \), and represents the \( m \)-th return observation for all \( n \) assets. In order to find an SSD efficient portfolio, the SSD-DP model looks
Table 4.1: DP calculation for Portfolio $A$, based on the distortion risk measure of Eq. (4.2) and distortion function of Eq. (4.7) for varying risk aversion degrees.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$Z(j)$</th>
<th>$\Delta g_j^{\nu \rightarrow 0}$</th>
<th>$\Delta g_j^{\nu = 0.3}$</th>
<th>$\Delta g_j^{\nu = 0.7}$</th>
<th>$\Delta g_j^{\nu = 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4.5</td>
<td>0</td>
<td>0.03</td>
<td>0.21</td>
<td>0.33</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0</td>
<td>0.23</td>
<td>0.35</td>
<td>0.33</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0.74</td>
<td>0.44</td>
<td>0.33</td>
</tr>
</tbody>
</table>

$$\sum_{j=1}^{3} Z(j) \Delta g_j^{\nu} = 1 \quad 0.72 \quad -0.33 \quad -0.99$$

Table 4.2: DP calculation for Portfolio $A$, based on the spectral risk measure of Eq. (4.4) and spectral function of Eq. (4.9) for varying risk aversion degrees.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$X(j)$</th>
<th>$\Phi_j^{\nu \rightarrow 0}$</th>
<th>$\Phi_j^{\nu = 0.3}$</th>
<th>$\Phi_j^{\nu = 0.7}$</th>
<th>$\Phi_j^{\nu = 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0.74</td>
<td>0.44</td>
<td>0.33</td>
</tr>
<tr>
<td>2</td>
<td>-0.5</td>
<td>0</td>
<td>0.23</td>
<td>0.35</td>
<td>0.33</td>
</tr>
<tr>
<td>3</td>
<td>4.5</td>
<td>0</td>
<td>0.03</td>
<td>0.21</td>
<td>0.33</td>
</tr>
</tbody>
</table>

$$-\sum_{j=1}^{3} X(j) \Phi_j^{\nu} = 1 \quad 0.72 \quad -0.33 \quad -0.99$$

for a weighted portfolio with return vector $X\lambda$, created from the set of assets in matrix $X$, with weight vector $\lambda$, which holds the minimum DP value at a given risk aversion degree. Acerbi (2002) proved that if a risk measure is coherent, then its risk surface is convex; hence, the risk surface has a unique global minimum. A risk measure is coherent if and only if its spectral function, $\Phi(p)$, is a positive and decreasing function such that $\int_0^1 \Phi(p) dp = 1$. Since the spectral function of DP in Eq. (4.9) has the mentioned characteristics of the coherent risk measure, then it has a unique global minimum. As will be discussed later in this chapter, the SSD-DP model produces an optimal portfolio (i.e. a minimum DP value) that is SSD efficient. We define our SSD-DP model as follows,

$$\min_{\lambda} DP_\nu (X\lambda),$$

(4.10)
where $\text{DP}_\nu(X\lambda) = \rho_{s\text{DP}_\nu}(-X\lambda)$ (see Eq. (4.2)) or $\text{DP}_\nu(X\lambda) = H_{\text{DP}_\nu}(X\lambda)$ (see Eq. (4.4)).

Thus, in order to compute the value of DP for a portfolio $X\lambda$ with discrete return values we may either use the distortion risk measure of Eq. (4.2), or the spectral measure of Eq. (4.4). Since both measures lead to an equivalent result.

However, computation of both these measures is based on the sorted return values of portfolio $X\lambda$, which is unknown at the time of solving the optimization model (4.10).

To resolve this issue, we may use Acerbi’s (2002) piecewise linear representation of the spectral risk measure, which is not dependant on the known sorted values of a portfolio’s returns. Therefore, we can write the piecewise linear representation of Acerbi (2002), for DP at risk aversion degree $\nu$, as

$$\Gamma_{\text{DP}_\nu}(X\lambda, \vec{\varphi}) = \sum_{j=1}^{M} \Delta \Phi_{\nu}^j \left\{ \varphi_j - \frac{1}{j} \sum_{i=1}^{M} (\varphi_j - (X\lambda)_i)_+ \right\}, \quad (4.11)$$

where $\Phi_{\nu}^j$ is the spectral function value at a given risk aversion degree $\nu$, defined as

$$\Phi_{\nu}^j = \int_{\frac{j}{M}}^{\frac{j+1}{M}} \Phi_{\text{DP}_\nu}(p) \, dp,$$

and $(\varphi_j - (X\lambda)_i)_+ = \max(0, \varphi_j - (X\lambda)_i)$.

Also $\Delta \Phi_{\nu}^j$ is defined as

$$\Delta \Phi_{\nu}^j = \begin{cases} \Phi_{\nu}^{j+1} - \Phi_{\nu}^j & j = 1, \ldots, M - 1 \\ -\Phi_{\nu}^M & j = M \end{cases},$$

for which $\sum_{i=j}^{M} \Delta \Phi_{\nu}^i = -\Phi_{\nu}^j$, should hold. The spectral function $\Phi_{\text{DP}_\nu}(p)$, is based on the DP’s spectral function of Eq. (4.9). Defining $a_{ij} = (\varphi_j - (X\lambda)_i)_+$, we can rewrite the piecewise linear representation of Eq. (4.11) as

$$\Gamma_{\Phi_{\text{DP}_\nu}}(X\lambda, \vec{\varphi}) = \sum_{j=1}^{M} \Delta \Phi_{\nu}^j \left\{ \varphi_j - \frac{1}{j} \sum_{i=1}^{M} a_{ij} \right\}, \quad (4.12)$$
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The optimization model (4.10) now becomes a linear programming model as

$$\min_{\lambda, \varphi, \tilde{a}} \sum_{j=1}^{M} \Delta \varphi_j \cdot \left\{ \varphi_j - \frac{1}{j} \sum_{i=1}^{M} a_{ij} \right\},$$

s. t.

$$a_{ij} \geq (\varphi_j - (X\lambda)_i) \quad i, j = 1, \ldots, M,$$

$$a_{ij} \geq 0 \quad i, j = 1, \ldots, M,$$

(4.13)

The SSD-DP linear programming model (4.13) has $M^2 + M + n$ variables and $2M^2$ constraints.


As we mentioned before, the Kopa and Chovanec (2008) model is the only model in the literature that can provide an SSD efficient portfolio for all risk averse investors. This model uses CVaR as a risk measure and as discussed in Chapter 3 is defined in Eq. 3.14. The Kopa and Chovanec SSD model in Eq. 3.14 compares the value of CVaR for the two portfolios $X\lambda$ and $Y$, at all confidence levels, $\alpha_j$, $j = 1, 2, \ldots, M$. The slack variable $D_j$ represents the difference of the CVaR values for portfolios $X\lambda$ and $Y$ at each confidence level $\alpha_j$. The objective function of the Kopa and Chovanec model in Eq. 3.14 seeks to maximize the sum of the slack variables $D_j$. In other words, the optimization seeks to find a portfolio with return vector $X\lambda$ which holds smaller or equal CVaR values at all confidence levels, $\alpha_j$, than those of the benchmark portfolio $Y$, with
at least one strict inequality. The CVaR of a portfolio with discrete loss vector \((-X\lambda)\), at the confidence level \(\alpha_j\), is the minimum value of the piecewise linear representation introduced by Rockafellar and Uryasev (2002), already discussed in Eq. 3.12 in Chapter 3.

Instead of CVaR, the proposed SSD-DP model uses DP as the risk measure in order to find the SSD efficient portfolio for an investor with a specific risk aversion degree. We can prove that the DP value for the optimal portfolio in the SSD-DP model is equal to the sum of the weighted values of CVaR for the optimal portfolio at confidence levels \(\alpha_j, j = 1, 2, \ldots, M\).

**Theorem 1.** The DP value for the optimal portfolio in the SSD-DP model is equal to the sum of the weighted values of CVaR for the optimal portfolio at confidence levels \(\alpha_j, j = 1, 2, \ldots, M\).

**Proof.** The SSD-DP model looks for a portfolio with a return vector \(X\lambda\), which holds the minimum of Eq. (4.11), i.e.

\[
\min_{\lambda, \vec{\phi}} \Gamma_{\text{DP}}(X\lambda, \vec{\phi}) = \min_{\lambda, \vec{\phi}} \sum_{j=1}^{M} \Delta \theta_j^\alpha j \left\{ \phi_j - \frac{1}{j} \sum_{i=1}^{M} (\phi_j - (X\lambda)_i)^+ \right\}. \tag{4.14}
\]

By combining Eqs. (3.12) and (4.14) the DP value of the the optimal SSD-DP portfolio, with return vector \(X\lambda\), is determined by

\[
\min_{\lambda, \vec{\phi}} \Gamma_{\text{DP}}(X\lambda, \vec{\phi}) = \min_{\lambda} \sum_{j=1}^{M} -\Delta \theta_j^\alpha j \text{CVaR}_{\alpha_j}(-X\lambda). \tag{4.15}
\]
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From Eq. (4.16) we can conclude that the DP value for the optimal portfolio in the SSD-DP model is equal to the sum of the weighted values of CVaR at confidence levels $\alpha_j$, $j = 1, 2, \ldots, M$ and the proof is complete. \hfill \Box

From Theorem 1 we can prove that the optimal portfolio of the SSD-DP model for a given risk aversion degree $\nu$, is SSD efficient.

**Theorem 2.** A weighted portfolio with a discrete return vector $X\lambda^*$, created from the set of assets in matrix $X$, with weight vector $\lambda^*$, which holds a minimum DP value at a given risk aversion degree $\nu$, is SSD efficient.

**Proof.** Consider a portfolio with discrete return vector $X\lambda^*$, which holds the minimum DP value. In order to test the SSD efficiency of this portfolio we utilize the SSD dominance definition in Eq. (3.13), discussed in Chapter 3. A portfolio with a discrete return vector $X\lambda^*$, is not SSD efficient if there exists a portfolio with weight vector $\zeta$, such that

$$CVaR_{\alpha_j} (-X\zeta) \leq CVaR_{\alpha_j} (-X\lambda^*) \quad j = 1, \ldots, M,$$

with at least one strict inequality. Since the spectral function of DP in Eq. (4.9) is coherent, its spectral function, $\theta_j^\nu$ is a positive and decreasing function. Hence, the coefficient

$$\Delta \theta_j^\nu = \begin{cases} 
\theta_j^\nu - \theta_j^\nu & j = 1, \ldots, M - 1 \\
-\theta_M^\nu & j = M 
\end{cases}$$

in Eq. (4.16) will be negative. Therefore, from
Eq. (4.17), and the fact that $\Delta \phi_{DP}^j$ is negative, we conclude that

\[
\sum_{j=1}^{M} -\Delta \phi_{DP}^j j CVaR_{\alpha_j}(-X\zeta) < \sum_{j=1}^{M} -\Delta \phi_{DP}^j j CVaR_{\alpha_j}(-X\lambda^*).
\]  

(4.18)

Since a portfolio with a discrete return vector $X\lambda^*$ already holds the minimum DP value, we obtain a contradiction to Eq. (4.16) in Theorem 1. Therefore, we conclude that a portfolio with weight vector $\zeta$ does not exist and the proof is complete.

From Theorem 1 and Theorem 2, we note that while the Kopa and Chovanec (2008) model seeks to find the SSD efficient portfolio with smaller or equal CVaR values, at all confidence levels $\alpha_j$, than those of its benchmark portfolio, our proposed model tries to find the SSD efficient portfolio by minimizing the sum of the weighted values of CVaR at all confidence levels, $\alpha_j$, for a given risk aversion degree. Since the optimal portfolios of both the Kopa and Chovanec (2008) and the SSD-DP models are SSD efficient, neither of these optimal portfolios dominates the other, by SSD. However, the proposed SSD-DP model provides an SSD efficient portfolio based on quantified risk aversion degrees, whereas the Kopa and Chovanec (2008) model provides a single SSD efficient portfolio for all risk averse investors, regardless of their risk aversion degrees.

To further illustrate how the results of the Kopa and Chovanec (2008) and the SSD-DP models are different, consider five assets $A, B, C, D$ and $E$ with equally likely discrete return observations as shown in Table 4.3. For an assumed benchmark portfolio with weight vector $(0, 0, 0, 1, 0)^T$ the Kopa and Chovanec optimal portfolio weight vector is $(0, 0.71, 0.29, 0, 0)^T$ as shown in Table 4.4. Also shown in Table 4.4 are the optimal
Table 4.3: Assets $A$, $B$, $C$, $D$, and $E$ hypothetical discrete returns.

<table>
<thead>
<tr>
<th></th>
<th>$A$ Returns</th>
<th>$B$ Returns</th>
<th>$C$ Returns</th>
<th>$D$ Returns</th>
<th>$E$ Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.5</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$B$</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>-3</td>
</tr>
<tr>
<td>$C$</td>
<td>0.3</td>
<td>-0.2</td>
<td>7</td>
<td>4.5</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 4.4: The optimal portfolio weight vectors for the Kopa and Chovanec (2008) model and the the SSD-DP model at varying risk aversion degrees.

<table>
<thead>
<tr>
<th>Assets</th>
<th>$\nu = 0.0001$</th>
<th>$\nu = 0.3$</th>
<th>$\nu = 0.7$</th>
<th>$\nu = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>0.71</td>
<td>0.78</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C$</td>
<td>0.29</td>
<td>0.22</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$D$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

portfolio weights for SSD-DP for varying risk aversion degrees of $\nu = 0.0001$, 0.3, 0.7, and 1. As expected, for an extremely risk averse investor ($\nu = 0.0001$), the resulting SSD-DP consists of the single asset that has the least amount of loss, i.e. Asset $A$. If more than one assets have the least loss, at the degenerate case of $\nu \rightarrow 0$, the algorithm would not be able to differentiate among these assets, and one would need to look at the next least loss value. For the case of a risk neutral investor, $\nu = 1$, Asset $E$ has the highest expected return and therefore it is chosen as the optimal portfolio. Figure 4.3 plots the empirical cumulative distribution function (ECDF) for the optimal portfolio of Kopa and Chovanec with those determined by SSD-DP. As can be seen, in no cases does the Kopa and Chovanec optimal portfolio dominate the SSD-DP, by SSD, or vice versa. Finally, in Table 4.5 we present the DP values of the optimal portfolios for different risk aversion degrees. As can be seen, the DP values of the SSD-DP model are always lower than those of Kopa and Chovanec.
Figure 4.3: Each plot represents the empirical cumulative distribution function (ECDF) of optimal portfolios of the Kopa and Chovanec (2008) model versus the SSD-DP model at a specific risk aversion degree.

Table 4.5: The optimal portfolio DP values for the Kopa and Chovanec (2008) model and the SSD-DP model at varying risk aversion degrees.

<table>
<thead>
<tr>
<th></th>
<th>$\nu = 0.0001$</th>
<th>$\nu = 0.3$</th>
<th>$\nu = 0.7$</th>
<th>$\nu = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$DP_{\nu}^{RC}$</td>
<td>0</td>
<td>-0.31</td>
<td>-0.79</td>
<td>-1.01</td>
</tr>
<tr>
<td>$DP_{\nu}^{SSD-DP}$</td>
<td>-0.3</td>
<td>-0.35</td>
<td>-1.02</td>
<td>-2.66</td>
</tr>
</tbody>
</table>

In this section, we present an empirical study to compare the performance of the SSD-DP model to that of Kopa and Chovanec (2008), based on actual historical data. A full empirical analysis of the SSD-DP model will be presented in Chapter 6. The study is based on a set of 15 stocks (labeled S1 to S15) from the computer software sector. Weekly return rates, adjusted for dividends, for the period of January 2007 to December 2010 were collected from Yahoo! Finance (http://finance.yahoo.com/). The iShares Barclays Treasury Bond exchange traded fund (NYSE: SHY), which seeks to approximate the price and yield of the US short term treasury market, is used as the proxy for the risk free asset (labeled as S16). As shown in Table A.1 of Appendix A, most of the stock returns are not normally distributed. We conveniently assumed that short selling was not allowed.

The empirical analysis was performed by implementing a 24 week sliding window trading strategy, where for any given week, the past 24 weeks were used to determine the optimal portfolio weights, and a readjustment of the portfolio was done on a weekly basis. Thus, the year 2007 had 28 weeks of trading, whereas the subsequent years, 2008, 2009 and 2010, all had 52 weeks of trading.

As mentioned previously, the Kopa and Chovanec (2008) model requires a benchmark portfolio to determine an SSD efficient portfolio. We simulated 20 random benchmark portfolios to determine their corresponding SSD efficient portfolios according to the Kopa

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1Some of the descriptive statistics of the stocks are presented in Table A.1 of Appendix A.
and Chovanec model. For each year, we plotted the best and worst performing trading strategies that offer the highest and lowest investment value at the end of the year (see Figure 4.4). In Figure 4.5 we plot the performance of the trading strategy based on the SSD-DP model for different risk aversion degrees, namely, $\nu = 0.0001, 0.3, 0.7, 1$. As expected, as risk aversion was increased, the resulting trading strategy led to a more conservative performance. We emphasis that our model will not lead to a dominant portfolio over Kopa and Chovanec model, rather that neither of the two models will dominate each other. In fact, if the Kopa and Chovanec model is used with an appropriate benchmark portfolio, it would provide the exact same portfolio as ours for the given risk aversion degree. However, the Kopa and Chovanec model does not provide a mechanism to pick a given benchmark portfolio based on risk aversion degree (or any other criteria, for that matter), whereas, as stated before, our method results in a unique SSD efficient portfolio for a given risk aversion degree. In Figure 4.6 we present the DP values based on the returns of the SSD-DP model trading strategies. Specifically, the $x$-axis represents the risk aversion degree, $\nu$, used to find the optimal portfolios of the trading strategies, as depicted in Figure 4.5. Since the DP calculation is itself dependent on $\nu$, the figure provides four panels representing the four DP calculations for $\nu = 0.0001, 0.3, 0.7, 1$, i.e. $DP_\nu$. We emphasize that the DP calculation is a risk measure that skews the weights of loss returns based on the risk aversion degree. As discussed previously, as $\nu \to 0$, i.e. an extremely risk averse investor, DP assigns a maximum weight to the maximum loss. For $\nu = 1$, i.e. a risk neutral investor, DP assigns equal weighting to all return values, assuming they are all equally distributed. Therefore, as can be seen in Figure 4.6, when $\nu = 0.0001$, the DP value is the maximum loss of the trading strategies, and as expected,
Figure 4.4: Trading strategy performance of Kopa and Chovanec (2008).

Figure 4.5: Trading strategy performance of the SSD-DP model for various $\nu$. 
the less the risk aversion, the greater this value becomes. Conversely, for the case where \( \nu = 1 \), the DP calculation presents the expected loss of the trading strategies. Since the trading strategies generally led to positive returns, the DP calculations are negative.

![Figure. 4.6: Yearly DP values, DP\( \nu \), of the SSD-DP model for various \( \nu \).](image)

### 4.4 Summary

Based on the review done by Lizyayev (2012), the SSD efficiency model developed by Kopa and Chovanec (2008) is selected as the best. This model is based on the comparison of the Conditional Value at Risk (CVaR) of portfolios in the SSD dominance definition. However, the Kopa and Chovanec (2008) model gives the same SSD efficient portfolio for all risk averse investors. This model cannot be customized for a specific risk aversion degree, since its risk measure, CVaR, is only consistent with SSD when compared to
another portfolio’s CVaR at all confidence levels, not at a specific confidence level.

In this chapter, we introduced a new SSD efficiency model, SSD-DP, which unlike existing SSD efficiency models in the literature, provides an SSD efficient portfolio as a function of investors’ risk aversion degrees. In order to provide an SSD efficient portfolio based on a quantified risk aversion degree, we developed the SSD-DP model based on the Dual Power Transform (DP) that is capable of modeling risk averse preferences and is consistent with SSD. Using the piecewise linear representation of spectral measures (explored by Acerbi, 2002), the proposed SSD-DP model is based on the linear programming technique. The linear programming model finds the SSD efficient portfolio by minimizing DP for a given risk aversion degree. The resulting optimal portfolio was shown to be SSD efficient. In order to compare the results of the SSD-DP model with the Kopa and Chovanec (2008) model, we proved that the DP value for the optimal portfolio in SSD-DP model is equal to the sum of weighted values of CVaR at confidence levels \( \alpha_j = 1 - \frac{j}{M}, j = 1, 2, \ldots, M \). Therefore, while the Kopa and Chovanec model seeks to find an SSD efficient portfolio with smaller or equal CVaR values at all confidence levels \( \alpha_j, j = 1, 2, \ldots, M \), than those of its benchmark portfolio, our proposed model tries to find the SSD efficient portfolio by minimizing the sum of weighted values of CVaR at all confidence levels \( \alpha_j, j = 1, 2, \ldots, M \), as a function of an investor’s risk aversion degree. Since both models are proved to provide SSD efficient portfolios, there is no SSD dominance relation between these two. An empirical study was performed to compare the performance of the SSD-DP model to that of Kopa and Chovanec (2008), based on actual historical data. We also showed how the SSD-DP model can be utilized in trading
strategies to match an investor’s risk aversion degree to the performance of the trading strategy.
Chapter 5

Results
An empirical study was performed to analyze the SSD-DP portfolio selection performance for various risk aversion degrees. The study was based on 15 stocks from the Energy (E1 to E5), Health (H1 to H5), and Software System (S1 to S5) sectors. Five-day weekly return rates, adjusted for dividends, during January 2007 to December 2011 were collected from Yahoo! Finance (http://finance.yahoo.com/). The iShares Barclays Treasury Bond exchange traded fund (NYSE: SHY), which seeks to approximate the price and yield of the US short term treasury market, was used as the proxy for the risk free asset (Rf). As expected, an analysis of data showed that the stocks’ returns were not normally distributed, with significant skewness and kurtosis, which means that Mean-Variance analysis may not be appropriate for portfolio selection for risk averse investors (Xiong and Idzorek, 2011). We conveniently assumed that short selling was not allowed in this empirical study.

This chapter is comprised of four sections; first we analyze the portfolio selection and performance of the SSD-DP model based on the 15 individual stocks from the Energy, Health and Software System sectors and the risk free asset. Then, we compare the results to portfolio selection and performance of the SSD-DP model based on the sectors’ indices and the risk free asset. In the third section, we assess the performance skill of the SSD-DP model by generating random portfolios. Finally, we explore the impact of higher moments on optimal portfolios in the SSD-DP model based on simulated data in a controlled test.
5.1 Portfolio Selection and Performance Analysis based on Sectors’ Individual Stocks

Here, we present the portfolio selection and performance of the SSD-DP model based on the 15 stocks from the Energy (E1 to E5), Health (H1 to H5), and Software System (S1 to S5) sectors and the risk free asset. The empirical analysis was performed by implementing a 24 week sliding window trading strategy where for any given week, the past 24 weekly return rates of the individual stocks and the risk free asset were used to determine the optimal portfolio weights by the SSD-DP model for various risk aversion degrees, namely, $\nu = 0.0001, 0.1, 0.2, ..., 1$, and a readjustment of the portfolio was done on a weekly basis.

In order to show how the SSD-DP model works, we analyzed the optimal portfolios of the first window of data \(^1\). Using the similar graphs to those of Maringer (2008) in our analysis, Figure 5.1 shows scatter plots of the moments of the stocks’ returns for the first window of data where stocks are labeled based on their industrial sector. Points on the same horizontal level refer to the same stock. Some of the stocks were not included in any of the optimal portfolios at any risk aversion degree and are represented by red font. Blue font depicts stocks that are included in at least one of the optimal portfolios.

We note that some of the optimal portfolios consist of stocks that are dominated in the Mean-Variance space (for example, S3 is included in the portfolio but is clearly dominated in Mean-Variance analysis by a number of stocks that were not included in any optimal portfolio, such as E2 and E4). The main reason for this type of selection is the structure

\(^1\)Weights of the sectors’ stocks and the risk free asset in the optimal portfolios of the first window of data are presented in Table B.1 of Appendix B.
of the higher order moments (skewness (third moment) and kurtosis (fourth moment)) of the returns. The impact of higher moments of stocks’ returns on the SSD-DP model optimal portfolio selection is discussed in more detail in Subsection 5.4.1.

Figure 5.1: Stocks in the mean-volatility, mean-skewness and mean-kurtosis space. Blue font: stocks included in at least one of the optimal SSD-DP portfolios; Red font: stocks never included.

In general, risk averse investors prefer portfolios with high expected return, low variance, positive skewness and low kurtosis (Xiong and Idzorek, 2011). In Figure 5.2 we plot the first four moments of the SSD-DP optimal portfolios for varying risk aversion degrees of the first window of data. The leftmost graph shows the expected returns versus the risk aversion degrees that are used for finding the optimal portfolios and the next three graphs show the expected return versus volatility, skewness and kurtosis of the optimal portfolios. Points on the same horizontal level refer to the same optimal portfolio. As expected, decreasing risk aversion, i.e. when $\nu$ approaches 1, leads to portfolios with higher
expected returns, higher volatility, and in general lower skewness and higher kurtosis. This is due to the fact that as investors become less risk averse, $DP_\nu$ penalizes negative returns less and gives more weight to the positive return values, therefore, optimal portfolios become more riskier and yet more profitable.

In order to analyze the constituents of the optimal portfolios, Figure 5.3 shows the optimal portfolios’ cumulated weights based on the stocks’ volatility groups and industrial sectors. In Figure 5.3(a), we have divided the stocks into five groups with the lowest to the highest volatility and plotted the stacked bar graphs showing the separate contribution of each volatility group and risk free asset to the cumulated weight of the optimal portfolios. We can see that as the investor becomes less risk averse, the risk free asset is replaced by more volatile stocks in the optimal portfolios. It is noteworthy that as risk aversion decreases, the medium volatility group is preferred over the high volatility group. This is due to the fact that the stocks with medium volatility provide highest expected return in this first window of data. In a similar analysis, we have grouped stocks based on their industrial sector and plotted the stacked bar graphs showing the amount invested within each sector and the risk free asset in Figure 5.3(b). As can be seen, as risk aversion decreases, the optimal portfolio consists of an increasing amount of investment in the Energy sector. The reason is that the Energy sector has stocks with the highest expected returns in the first window of data, as was shown in Figure 5.1.

In Figure 5.4 we plot the performance of the trading strategy based on the SSD-DP model for different risk aversion degrees, namely, $\nu = 0.0001, 0.3, 0.7, 1$. For a dollar value of investment at the beginning of each year, the $y$-axis shows the change in the
investment value during the year. As expected, as the risk aversion was increased, the resulting trading strategy led to a more conservative performance. In Figure 5.5 we present the $DP_\nu$ values based on the returns of the SSD-DP model trading strategies. Specifically, the $x$-axis represents the risk aversion degree, $\nu$, used to find the optimal portfolios of the trading strategies, as depicted in Figure 5.4. Since the DP calculation is itself dependent on $\nu$, the Figure provides four panels representing the four DP calculations for $\nu = 0.0001, 0.3, 0.7, 1$, i.e. $DP_\nu$. We emphasize that the DP calculation is a risk measure that skews the weights of loss returns based on the risk aversion degree.

As discussed previously, as $\nu \to 0$, i.e. an extremely risk averse investor, DP assigns a maximum weight to the maximum loss. For $\nu = 1$, i.e. a risk neutral investor, DP assigns equal weighting to all return values, assuming they are all equally distributed. Therefore, as can be seen in Figure 5.5, when $\nu = 0.0001$, the DP value is the maximum

Figure. 5.2: Moments of SSD-DP optimal portfolios in mean-volatility, mean-skewness and mean-kurtosis space.
Figure 5.3: Cumulative weights of stocks in the SSD-DP optimal portfolios based on stocks’ volatility groups and stocks’ industrial sectors of the first window of data.
loss of the trading strategies and, as expected, the less the risk aversion, the greater this value becomes. Conversely, for the case where $\nu = 1$, the DP calculation leads to the expected loss of the trading strategies. Since the trading strategies generally led to positive returns, the DP calculations are negative.

Figure 5.6 is similar to Figure 5.3(b) but depicts the cumulated weights of optimal portfolios of trading strategies for all windows of data. We can see that as the investor becomes less risk averse, the risk free asset in the trading strategies is replaced by more profitable and riskier sectors in the optimal portfolios. Also, as expected, the optimal portfolios for the most conservative, $\nu = .0001$ and least conservative, $\nu = 1$, trading strategies are less diversified than the rest of the trading strategies. This is due to the fact that the portfolio selection criteria for strategies with $\nu = .0001$ and $\nu = 1$ are limited to the stocks with minimum loss and maximum expected return, respectively.

In this section we empirically analyzed the SSD-DP portfolio selection results for various risk aversion degrees. We showed that as expected, decreasing risk aversion leads to optimal SSD-DP portfolios with higher expected return, higher volatility, lower skewness and higher kurtosis. Also optimal portfolios in the SSD-DP model may consist of stocks that are dominated in Mean-Variance analysis which can be explained by the structure of moments higher than mean and variance of stocks’ returns, that are ignored in the Mean-Variance analysis. We also saw that as risk aversion decreases, the resulting trading strategies based on the SSD-DP model led to less conservative performances. Finally, we showed that the optimal portfolios of trading strategies for an extremely risk averse investor and risk neutral investor are less diversified than the rest of trading
strategies and as risk aversion decreases the risk free asset is replaced by more profitable, yet riskier stocks in the optimal portfolios. In the following section, the results of the optimal SSD-DP portfolios based on sectors’ stocks are compared to those based on the sectors’ indices.

Figure. 5.4: Trading strategy performance of the SSD-DP model based on all sectors’ stocks for various $\nu$.

5.2 Portfolio Selection and Performance Analysis based on Sector Indices

In the previous section we used the individual stocks from the Energy, Health and Software System sectors to create optimal portfolios for varying risk aversion degrees. Here, instead of using sectors’ individual stocks, we use custom sector indices to create the SSD-DP optimal portfolios. This experiment will help us to see the difference between
Figure 5.5: Yearly DP values, $DP_\nu$, of the SSD-DP model based on sectors’ stocks for various $\nu$.

Figure 5.6: Cumulative weights of stocks across the different industries in the SSD-DP optimal portfolios of trading strategies for various risk aversion degrees $\nu$. 

\[ \text{Cum. weights} \]

\[ \text{Year} \]

\[ \nu = .0001, 0.3, 0.7, 1 \]
the performance of index versus individual stock picking in the SSD-DP optimal portfolios. Also, by only having three sector indices and the risk free asset to invest in, it is much easier to analyze the asset allocation of the optimal portfolios. For this reason, we created three sector indices of Energy, Health and Software Systems by using the stocks within each sector and weighting the stocks according to their market capitalization. In a similar analysis as before, we performed a 24 week sliding window trading strategy but this time, sector indices’ returns and the risk free asset were used to determine the optimal portfolios by the SSD-DP model for varying risk aversion degrees.

In Table 5.1 we show the expected return, expected volatility and $DP_{\nu}$ of optimal portfolios of individual stocks relative to those of sectors’ indices of the first window of data, for various risk aversion degrees $\nu = 0.0001, 0.3, 0.7, 1$. We can see that the optimal portfolios created from the individual stocks are more profitable, more volatile and lead to less $DP_{\nu}$ than those of sector indices. The reason is that the SSD-DP model based on the sectors’ individual stocks has the flexibility of using only those stocks in each sector which are more profitable and less risky, while the model based on the sector indices includes all stocks for the given sector.\(^2\)

Table 5.1: Expected return, volatility and $DP_{\nu}$ of optimal SSD-DP portfolios created from sectors’ stocks relative to those created from sectors’ indices of the first window of data.

<table>
<thead>
<tr>
<th></th>
<th>$\nu = 0.0001$</th>
<th>$\nu = 0.3$</th>
<th>$\nu = 0.7$</th>
<th>$\nu = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stocks</td>
<td>Indices</td>
<td>Stocks</td>
<td>Indices</td>
</tr>
<tr>
<td>Expected Return</td>
<td>0.0010</td>
<td>0.0010</td>
<td>0.0023</td>
<td>0.0010</td>
</tr>
<tr>
<td>Volatility</td>
<td>0.0015</td>
<td>0.0015</td>
<td>0.0017</td>
<td>0.0014</td>
</tr>
<tr>
<td>$DP_{\nu}$</td>
<td>0.0008</td>
<td>-0.0009</td>
<td>-0.0001</td>
<td>-0.0015</td>
</tr>
</tbody>
</table>

\(^2\)Weights of the sectors’ indices and the risk free asset in the optimal portfolios of the first window of data are presented in Table B.2 of Appendix B.
In Figure 5.7, similar to Figure 5.3(b), we plot the cumulated weights of the optimal portfolios of sector indices, of the first window of data. By comparing the results in Figure 5.7 to those in Figure 5.3(b), we can see that the amount of investment in the Health sector has significantly decreased. The reason is that the poor performance of some of the stocks in the Health sector of the first window of data (see H1 and H3 in Figure 5.1), affect the overall performance of Health index and makes it less attractive in the SSD-DP portfolio optimization.

![Figure 5.7: Cumulative weights of sectors’ indices in the SSD-DP optimal portfolios of first window of data.](image)

Figure 5.8 shows the performance of the trading strategies for all windows of data based on the sector indices. For a dollar value of investment at the beginning of the year, the $y$-axis shows the change in the investment value during each year. In Table 5.2, we compare the final investment value of trading strategies based on the sectors’ stocks to those based on the sectors’ indices, at the end of each year and in Table 5.3 we compare
the respective $DP_\nu$ values. We can see that trading strategies based on neither sectors’
stocks nor sectors’ indices offer higher final investment value in all years. However, the
results suggest that, in the majority of times, at a given risk aversion degree, $\nu$, a trading
strategy with a higher final value of investment in Table 5.2, has lower $DP_\nu$ value in
Table 5.3. To verify the existence of the observed pattern, we perform a statistical
significance analysis. To do so, at any given risk aversion degree, $\nu$, we calculate the
difference of the $DP_\nu$ value of the trading strategy with the lower final investment value
from the one with higher value. We see that the median of the difference in the majority
of times is negative. Therefore, we test the null and alternative hypotheses as follows,

$H_0$: At a given risk aversion degree, $\nu$, the median of the difference of the $DP_\nu$ values
of the strategy with the lower final investment value from the one with the higher value
is zero,

$H_1$: At a given risk aversion degree, $\nu$, the median of the difference of the $DP_\nu$ values
of strategy with the lower final investment value from the one with the higher value is
less than zero.

Since the difference of the $DP_\nu$ values is not normally distributed\(^3\), we use the
Wilcoxon signed rank test (see for example Gibbons and Chakraborti, 2011), to test
the null and alternative hypotheses. At a 5% significance level, the null hypothesis is
rejected at a p-value of 0.0001. This result suggests that, at a given risk aversion degree,
a trading strategy with a lower $DP_\nu$ value is more likely to lead to a higher investment
value at the end of the trading period, which is a very favorable result.

\(^3\)Performing the Jarque-Bera test at 5% significance level, the hypothesis for the normal distribution
of the difference of $DP_\nu$ values is rejected at 5% significance level with p-value of 0.001.
Table 5.2: Comparison of final wealth of trading strategies of SSD-DP portfolios created from the sectors’ individual stocks vs. those created from sectors’ indices in each year, for various risk aversion degrees.

<table>
<thead>
<tr>
<th></th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>1.0707</td>
<td>1.0707</td>
<td>1.0497</td>
<td>1.0034</td>
<td>1.0240</td>
</tr>
<tr>
<td>Indices</td>
<td>1.0707</td>
<td>1.0497</td>
<td>1.0497</td>
<td>1.0034</td>
<td>1.0240</td>
</tr>
<tr>
<td>ν = 0.0001</td>
<td>1.0610</td>
<td>1.0491</td>
<td>1.0441</td>
<td>1.0282</td>
<td>1.0289</td>
</tr>
<tr>
<td>ν = 0.1</td>
<td>1.0657</td>
<td>1.0454</td>
<td>1.0350</td>
<td>1.0244</td>
<td>1.0227</td>
</tr>
<tr>
<td>ν = 0.2</td>
<td>1.0744</td>
<td>1.0421</td>
<td>1.0126</td>
<td>1.0229</td>
<td>0.9336</td>
</tr>
<tr>
<td>ν = 0.3</td>
<td>1.1610</td>
<td>1.0233</td>
<td>0.9977</td>
<td>1.0228</td>
<td>1.2089</td>
</tr>
<tr>
<td>ν = 0.4</td>
<td>1.0927</td>
<td>0.8960</td>
<td>0.8582</td>
<td>1.0204</td>
<td>1.4295</td>
</tr>
<tr>
<td>ν = 0.5</td>
<td>1.0506</td>
<td>0.7497</td>
<td>0.8217</td>
<td>1.0216</td>
<td>2.0030</td>
</tr>
<tr>
<td>ν = 0.6</td>
<td>1.1261</td>
<td>0.7379</td>
<td>0.8358</td>
<td>0.9628</td>
<td>2.3810</td>
</tr>
<tr>
<td>ν = 0.7</td>
<td>1.0506</td>
<td>0.7005</td>
<td>0.9704</td>
<td>0.8616</td>
<td>3.0799</td>
</tr>
<tr>
<td>ν = 0.8</td>
<td>0.9664</td>
<td>0.6863</td>
<td>0.9283</td>
<td>0.6928</td>
<td>3.4367</td>
</tr>
<tr>
<td>ν = 0.9</td>
<td>0.8661</td>
<td>0.7391</td>
<td>0.8544</td>
<td>0.7323</td>
<td>4.1759</td>
</tr>
</tbody>
</table>
Table 5.3: Comparison of $DP_\nu$ of trading strategies of SSD-DP portfolios created from the sectors’ individual stocks vs. those created from sectors’ indices in each year, for various risk aversion degrees.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu = 0.0001$</td>
<td>0.0031</td>
<td>0.0031</td>
<td>0.0074</td>
<td>0.0074</td>
<td>0.0095</td>
<td>0.0095</td>
<td>0.0029</td>
<td>0.0029</td>
<td>0.0031</td>
<td>0.0031</td>
</tr>
<tr>
<td>$\nu = 0.1$</td>
<td>0.0030</td>
<td><strong>0.0024</strong></td>
<td>0.0069</td>
<td>0.0059</td>
<td>0.0036</td>
<td>0.0038</td>
<td>0.0019</td>
<td>0.0017</td>
<td>0.0029</td>
<td><strong>0.0018</strong></td>
</tr>
<tr>
<td>$\nu = 0.2$</td>
<td>0.0012</td>
<td><strong>0.0014</strong></td>
<td>0.0047</td>
<td>0.0037</td>
<td>0.0030</td>
<td><strong>0.0022</strong></td>
<td>0.0014</td>
<td>0.0010</td>
<td>0.0064</td>
<td>0.0013</td>
</tr>
<tr>
<td>$\nu = 0.3$</td>
<td>0.0002</td>
<td>0.0009</td>
<td>0.0044</td>
<td>0.0027</td>
<td>0.0104</td>
<td>0.0019</td>
<td>0.0032</td>
<td>0.0006</td>
<td>0.0105</td>
<td>0.0014</td>
</tr>
<tr>
<td>$\nu = 0.4$</td>
<td>0.0056</td>
<td>0.0017</td>
<td>0.0083</td>
<td>0.0020</td>
<td>0.0120</td>
<td>0.0058</td>
<td>0.0048</td>
<td>0.0003</td>
<td>0.0110</td>
<td>0.0055</td>
</tr>
<tr>
<td>$\nu = 0.5$</td>
<td>0.0181</td>
<td><strong>0.0110</strong></td>
<td>0.0132</td>
<td><strong>0.0015</strong></td>
<td>0.0187</td>
<td>0.0103</td>
<td>0.0111</td>
<td>-0.0001</td>
<td>0.0107</td>
<td>0.0052</td>
</tr>
<tr>
<td>$\nu = 0.6$</td>
<td>0.0144</td>
<td><strong>0.0215</strong></td>
<td>0.0135</td>
<td><strong>0.0015</strong></td>
<td>0.0110</td>
<td>0.0150</td>
<td>0.0072</td>
<td>0.0027</td>
<td>0.0129</td>
<td>0.0042</td>
</tr>
<tr>
<td>$\nu = 0.7$</td>
<td>0.0095</td>
<td>0.0196</td>
<td>0.0130</td>
<td>0.0036</td>
<td>0.0030</td>
<td>0.0076</td>
<td>0.0064</td>
<td>0.0042</td>
<td>0.0124</td>
<td>0.0033</td>
</tr>
<tr>
<td>$\nu = 0.8$</td>
<td>0.0069</td>
<td><strong>0.0185</strong></td>
<td>0.0083</td>
<td>0.0061</td>
<td>-0.0075</td>
<td>-0.0004</td>
<td>-0.0044</td>
<td>-0.0017</td>
<td>0.0109</td>
<td>0.0016</td>
</tr>
<tr>
<td>$\nu = 0.9$</td>
<td>0.0031</td>
<td><strong>0.0164</strong></td>
<td>0.0074</td>
<td><strong>0.0088</strong></td>
<td>0.0095</td>
<td>-0.0095</td>
<td>0.0029</td>
<td>-0.0078</td>
<td>0.0031</td>
<td>0.0022</td>
</tr>
<tr>
<td>$\nu = 1$</td>
<td>0.0030</td>
<td>0.0107</td>
<td>0.0004</td>
<td>0.0051</td>
<td>-0.0466</td>
<td>-0.0148</td>
<td>-0.0101</td>
<td>-0.0133</td>
<td>0.0114</td>
<td>-0.0012</td>
</tr>
</tbody>
</table>

In Figure 5.9, similar to Figure 5.7, we plot the cumulated weights of the optimal portfolios of the sector indices, of all windows of data. In order to analyze the allocation of the sector indices in the optimal portfolios, the first four moments of the returns for the sectors indices and the risk free asset are calculated and shown in Figure 5.10. The sector indices experience a sharp drop in the expected return during the first and last months of 2008, the first months of 2009 and the last months of 2011. Theses low expected returns lead to the heavy allocation of the risk free asset in the optimal portfolios following these periods for all risk aversion degrees, as can be seen in Figure 5.9. Also, we can see that the Health sector clearly outperforms the other sectors as well as the risk free asset during mid to late 2009, which results in a heavy allocation to the Health sector for the trading strategies with lower risk aversion degree, i.e. $\nu \geq 0.7$. 
In this section we showed that more granularity of the sectors’ stocks relative to sector indices can lead to the optimal portfolios by the SSD-DP model with higher expected return and lower risk by $DP_\nu$, at a given risk aversion degree, $\nu$. Moreover, by comparing the final investment value of the trading strategies based on the sectors’ stocks relative to those based on the sectors’ indices, we statistically showed that, at a given risk aversion degree, $\nu$, strategies with lower $DP_\nu$ value are more likely to provide higher investment value at the end of investment period. Finally, we mentioned some instances of asset allocation in the SSD-DP model based on the sectors’ indices according to different market conditions. In the next section we assess the investment skill of our trading strategies for different levels of risk aversion degrees.

![Figure. 5.8: Trading strategy performance of the SSD-DP model based on sectors’ indices for various $\nu$.](image-url)
Figure 5.9: Cumulative weights of sectors’ indices across the different industries in the SSD-DP optimal portfolios of trading strategies for various risk aversion degrees $\nu$.

Figure 5.10: Expected return, volatility and skewness of sectors’ indices in each window of the trading strategies.
5.3 Measuring the Investment Skill of SSD-DP Trading Strategies Against a Benchmark

Measuring the performance skill of trading strategies against a benchmark is a common practice in financial industry. In order to measure the performance skill of the SSD-DP trading strategies against a benchmark, we need to define an appropriate market index. Since the empirical study discussed above is based on only five stocks from each of the three industries Energy, Health and Software Systems sectors, it is hard to define an appropriate market index that can mimic the performance of the Energy, Health and Software Systems sectors, with only 15 stocks, at the same time. For this reason, we perform an analysis based on 15 representative stocks from the Software Systems sectors. We selected the XSW exchange traded fund (ETF) as the benchmark index for the Software sector. By using the 15 stocks within Software Systems sector and weighting the stocks according to their market capitalization, a custom market index of for the Software Systems sector is also created. Figures 5.11 and 5.12 show the performance of the SSD-DP trading strategies against the XSW market index and custom index, respectively. For a dollar value of investment at the beginning of each year, the $y$-axis shows the change in the investment value during the year.

The performance of the XSW market index and the custom index are very similar, which means that the XSW is an appropriate benchmark for our 15 selected stocks of Software Systems sector. From Figures 5.11 and 5.12, we can see that in the majority of times the SSD-DP trading strategies with $\nu > 0.7$, offer higher final investment values.
Figure 5.11: Performance of the SSD-DP trading strategies based on 15 stocks of Software Systems sector for various $\nu$ Vs. XSW market index performance.

Figure 5.12: Performance of the SSD-DP trading strategies based on 15 stocks of Software Systems sector for various $\nu$ Vs. Software Systems sector custom index performance.
than those of the XSW market index and the custom index. Also, we can see that during the recession of 2008, the final investment values for all of the SSD-DP trading strategies, with \( \nu = 0.001, 0.3, 0.7, 1 \), are greater than those of the XSW market index and the custom index, which is a very promising result.

Although, the SSD-DP model is shown to outperform the benchmarks, measuring investment skill relative to a benchmark is shown to not be appropriate (see for e.g. Burns, 2004). One reason is that with the vast number of benchmarks to choose from, it is hard to define a specific index or combination of indices to measure the relative performance of trading strategies. Moreover, the constraints (such as number of stocks, maximum weight for each stock as so on) that a specific benchmark obeys might be different from the constraints of our empirical study. For this reason, in the next section, we measure the investment skill of SSD-DP trading strategies via random portfolio analysis, which is proved to be a more appropriate method (see for e.g. Burns, 2004).

### 5.4 Measuring Investment Skill of SSD-DP Trading Strategies via Random Portfolio Analysis

Burns (2004) has showed that measuring performance skill via random portfolios leads to a more appropriate analysis than the measurement against a benchmark. Therefore, in this section, we utilize random portfolio analysis to assess the performance skill of the trading strategies based on the SSD-DP model. This analysis is performed on the same set of stock data set used previously (see Section 5.1). In a similar analysis as before, we
performed a 24 week sliding window trading strategy with the same constraint of no short selling, but this time, in each window we assigned random weights to a random number of assets to construct a random portfolio. One hundred random trading strategies were then generated and the $DP_\nu$ values based on the returns of the trading strategies were calculated for various risk aversion degrees, namely, $\nu = 0.0001, 0.1, 0.2, ..., 1$. The performance skill of the trading strategies based on the SSD-DP model, derived in Section 5.1, are then compared to those of the random trading strategies. Thus, at any given risk aversion degree, $\nu$, the $DP_\nu$ value of the corresponding SSD-DP trading strategy is compared to $DP_\nu$ values of the one hundred random trading strategies. The number of the random trading strategies with equal or smaller $DP_\nu$ values than that of the SSD-DP trading strategy, at a given risk aversion degree $\nu$, are then divided by one hundred to calculate the p-value of the test, thus resulting in a p-value at each risk aversion degree. The process is repeated twenty times to obtain twenty p-values at each given risk aversion degree, $\nu = 0.0001, 0.1, 0.2, ..., 1$ (The p-value analysis is similar to the study done by Burns (2004)).

Figure 5.13 shows the distribution of p-values across the various risk aversion degrees, $\nu = 0.0001, 0.1, 0.2, ..., 1$. In particular the distribution of p-values bounded between 0 and 0.5 are preferred, which means that less than 50% of the time, the $DP_\nu$ value of the SSD-DP trading strategy is less than those of the random strategies. From Figure 5.13 we can see that the distribution of p-values at all of the tested risk aversion degrees are bounded between between 0 and 0.5, which confirms the higher performance skill of the SSD-DP model relative to the random trading strategies. In the next section, we assess the impact of higher moments of stocks’ returns on the asset allocation in the SSD-DP
Chapter 5. Results

5.4.1 Higher Moments impact on SSD-DP Model Optimal Portfolios

In general, risk averse investors prefer portfolios with positive skewness and low kurtosis (Xiong and Idzorek, 2011). In order to assess the impact of the higher moments of stocks’ return distributions on optimal portfolios in the SSD-DP model, we implement a controlled test using the same example as that of Xiong and Idzorek (2011) with a similar analysis. We consider four assets A, B, C and D with equally likely discrete return observations with fixed expected returns, volatilities and correlation matrix, as are shown in Tables 5.4 and 5.5. From Table 5.4, we see that the sharp ratio (i.e. ratio of expected return to volatility) for all of the four stocks is almost equal. Asset A can be assumed as a bond index with low correlation to other assets. To analyze the impact of higher moments of the assets’ returns on the SSD-DP model’s optimal portfolios, four scenarios were generated with various skewness and kurtosis values for Asset C relative to the other assets (see Table 5.6). Monte Carlo simulation was used to simulate assets returns for the four assets using the given parameters. Using the simulated assets’ returns, the optimal SSD-DP portfolios were generated for various risk aversion degrees. Table 5.7 shows the optimal weights for Asset C for the four scenarios, highlighted in Table 5.6. In Scenario 2, where Asset C has high kurtosis, we see lower optimal weight across all risk aversion degrees compared with the other scenarios, and in Scenario 4, where asset C modeled with positive skewness, we see a higher optimal weight for the different risk
Figure 5.13: Distribution of p-values of random portfolio analysis, at all of the tested risk aversion degrees.

aversion degrees. The outcome of this experiment shows that the SSD-DP model has a preference for positive skewness and low kurtosis.

Table 5.4: Expected return and volatility of simulated assets.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Expected return (%)</th>
<th>Volatility (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>D</td>
<td>13</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 5.5: Correlation matrix of simulated assets.

<table>
<thead>
<tr>
<th>Asset</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0.34</td>
<td>0.32</td>
<td>0.32</td>
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<tr>
<td>B</td>
<td>0.34</td>
<td>1</td>
<td>0.82</td>
<td>0.71</td>
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<tr>
<td>C</td>
<td>0.32</td>
<td>0.82</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>0.32</td>
<td>0.82</td>
<td>0.71</td>
<td>1</td>
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</table>
Table 5.6: Skewness and kurtosis of simulated assets.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Asset</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Skewness</td>
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<td>0</td>
<td>0</td>
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<tr>
<td></td>
<td>Kurtosis</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
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<tr>
<td>Scenario 2: High kurtosis</td>
<td>Skewness</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
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<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Scenario 3: Negative skewness</td>
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<td>0</td>
<td>-1.5</td>
<td>0</td>
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<tr>
<td></td>
<td>Kurtosis</td>
<td>3.5</td>
<td>3.5</td>
<td>6</td>
<td>3.5</td>
</tr>
<tr>
<td>Scenario 4: Positive skewness</td>
<td>Skewness</td>
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<td>0</td>
<td>1.5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
<td>3.5</td>
<td>3.5</td>
<td>6</td>
<td>3.5</td>
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</tbody>
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Table 5.7: Optimal weights for Asset C for the four scenarios.

<table>
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<tr>
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<th>$\nu = 0.0001$</th>
<th>$\nu = 0.3$</th>
<th>$\nu = 0.50$</th>
<th>$\nu = 0.7$</th>
<th>$\nu = 1$</th>
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<tr>
<td>Scenario 1</td>
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<td>0.91</td>
<td>1.00</td>
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<td>0.86</td>
<td>1.00</td>
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<td>0.94</td>
<td>1.00</td>
</tr>
<tr>
<td>Scenario 4</td>
<td>0.00</td>
<td>0.14</td>
<td>0.47</td>
<td>0.92</td>
<td>1.00</td>
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</table>
5.5 Summary

In this chapter we have conducted various empirical analyses of our proposed SSD-DP portfolio selection model. The SSD-DP model is based on second order Stochastic Dominance (SSD) and is designed to find the SSD efficient portfolio as a function of investors’ risk aversion degree. We empirically showed that the SSD-DP model is a very promising tool for constructing trading strategies that choose SSD efficient portfolios based on investors’ risk aversion degrees. This model is capable of using the information of moments of assets’ return higher than mean and variance and is shown to provide trading strategies with high investment skill.
Chapter 6

Concluding Remarks
6.1 Discussion of Results

In this thesis, we introduced a new SSD efficiency model, SSD-DP, which unlike existing SSD efficiency models in the literature, provides an SSD efficient portfolio as a function of investors’ risk aversion degrees. In order to provide an SSD efficient portfolio based on a quantified risk aversion degree, we developed the SSD-DP model based on the Dual Power Transform (DP) that is capable of modeling risk averse preferences and is consistent with SSD. The proposed SSD-DP model is based on the linear programming technique. The linear programming model finds the SSD efficient portfolio by minimizing DP for a given risk aversion degree. The resulting optimal portfolio was shown to be SSD efficient. Based on the review done by Lizyayev (2012), the SSD efficiency model developed by Kopa and Chovanec (2008) is selected as the best. In order to compare the results of the SSD-DP model with the Kopa and Chovanec (2008) model, we proved that the DP value for the optimal portfolio in the SSD-DP model is equal to the sum of weighted values of CVaR at confidence levels $\alpha_j = 1 - \frac{j}{M}$, $j = 1, 2, \ldots, M$. Therefore, while the Kopa and Chovanec model seeks to find an SSD efficient portfolio with smaller or equal CVaR values at all confidence levels $\alpha_j, j = 1, 2, \ldots, M$, than those of its benchmark portfolio, our proposed model tries to find the SSD efficient portfolio by minimizing the sum of weighted values of CVaR at all confidence levels $\alpha_j, j = 1, 2, \ldots, M$, as a function of an investor’s risk aversion degree. Since both models are proved to provide SSD efficient portfolios, there is no SSD dominance relation between these two. However, the Kopa and Chovanec (2008) model can only provide an SSD efficient portfolio for a given benchmark and does not provide a mechanism to pick a given benchmark portfolio based on an investor’s risk
We conducted various empirical studies to analyze the results of our proposed SSD-DP model. In our first empirical study, for a given risk aversion degree, \( \nu \), we performed a sliding window trading strategy where in each window we found the optimal SSD-DP portfolio based on the past return rates of 15 stocks and a risk free asset. Then, a readjustment of the portfolios was done on a weekly basis. We saw that decreasing risk aversion led to optimal portfolios that had higher expected return, higher volatility, lower skewness and higher kurtosis. Also, we showed that the trading strategies resulted in less conservative performance for lower degrees of risk aversion.

In order to see the difference between the performance of trading strategies based on the sectors indices and those based on the sectors’ stocks, we performed the same analysis as before but this time on a set of sectors indices. We showed that more granularity of the sectors’ stocks relative to the sectors’ indices led to portfolios with higher expected return and lower \( DP_\nu \) values, at all risk aversion degrees. The reason was that the SSD-DP model based on the stocks has the flexibility of using only the stocks in each sector which are more profitable and less risky, while the model based on the sector indices includes all stocks for the given sector. By comparing the final investment value of the trading strategies based on the stocks to those based on the indices, we statistically showed that, at a given risk aversion degree, strategies with lower \( DP_\nu \) values are more likely to provide higher investment values at the end of the investment period, which is a very favorable result. The analysis based on the three sectors’ indices and the risk free asset, gave us the opportunity to analyze the asset allocation in the SSD-DP model according to the
different market conditions. We showed that the SSD-DP model allocated different sector indices to the optimal portfolios based on different market conditions across the varying risk aversion degrees.

In order to assess the performance skill of the trading strategies based on the SSD-DP model, we compared the SSD-DP model to a random portfolio analysis. We saw that the SSD-DP trading strategies are more likely (out of the 220 tests all the p-values\(^1\) to hold lower \(DP_\nu\) values than those of the random strategies, which is a very promising result.

In our last empirical analysis, we performed a controlled test to assess the impact of skewness and kurtosis on the SSD-DP model. As expected, we showed that the SSD-DP model has an overall preference to allocate assets with higher skewness and lower kurtosis to the optimal portfolios for all of the different risk aversion degrees.

In summary, the SSD-DP model is a very promising tool for constructing trading strategies that choose SSD efficient portfolios based on the investors’ risk aversion degree. By keeping the \(DP_\nu\) values of the trading strategies at low levels, the SSD-DP model provides a high chance of obtaining high investment value at the end of the investment periods.

6.2 Recommendations for Future Work

Several areas for future research arise from this work. Opportunities include:

\(^{1}\)P-values are calculated as the number of the random trading strategies with equal or smaller \(DP_\nu\) values than that of the SSD-DP trading strategy, at a given risk aversion degree \(\nu\), that are divided by one hundred. Therefore, Small p-values represent high investment skill of SSD-DP trading strategies.
• Introduce risk aversion degree for arbitrary order of Stochastic Dominance for portfolio selection

In this thesis we have developed an SSD model that is an applicable portfolio selection model for all risk averse investors. Our proposed model can be extended to orders of SD higher than SSD that can account for investor’s risk aversion degree. For instance this model can be extended based on third order stochastic dominance (TSD) which is for the class of risk averse investors with decreasing level of risk aversion as their wealth level increases;

• Develop a model that can provide investors risk aversion degree on the scale from 0 (most risk averse) to 1 (risk neutral)

Our proposed SSD-DP model is a function of investors’ risk aversion degree which is a parameter between zero (for an extremely risk averse investor) and one (for a risk neutral investor). In this thesis, we assume that the risk aversion parameter is provided by an investor as a value between zero and one. Therefore, one can look into a method that can calculate the inventor’s risk aversion degree and provide that as an input for our proposed model;

• Apply the SSD-DP model in different algorithmic trading models such as high frequency trading to match an investor’s risk aversion degree to the performance of the trading strategy

The proposed SSD-DP model is shown to be a very promising tool for trading strategies. Since our proposed model is based on linear programming
technique, is intuitive and easy to apply, and has the potential to be used in algorithmic trading models such as high frequency trading strategies;

- Perform sensitivity analysis by changing the parameters of trading strategy in empirical studies

The empirical analysis was performed by implementing a 24 week sliding window trading strategy. One can perform sensitivity analysis by performing empirical analysis for different units of return rates such as daily or monthly and also different number of sliding windows to analyze their impact on the performance of trading strategies;

- Apply the SSD-DP model in technical analysis for stock price and trend prediction

The proposed SSD-DP model can be used in technical analysis for predicting the trend and price of stocks using historical data.
Bibliography


BIBLIOGRAPHY


Appendix A
Table A.1 shows some of the descriptive statistics of 15 stocks, S1 to S15, for the sample period. The Jarque-Bera test at 95% confidence level confirms that majority of stocks have non-normal and asymmetric return distributions.

Table A.1: Descriptive statistics of all 15 stocks from computer software sector for the sample period January 2007 to December 2010.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean Return (%)</th>
<th>Std. Dev.</th>
<th>skewness</th>
<th>Kurtosis</th>
<th>Jarque Bera Pvalue (95%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>-0.0028</td>
<td>0.0367</td>
<td>-0.5118</td>
<td>3.9089</td>
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<tr>
<td>S2</td>
<td>0.0014</td>
<td>0.0369</td>
<td>-0.3049</td>
<td>3.7979</td>
<td>0.0176</td>
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<tr>
<td>S3</td>
<td>0.0018</td>
<td>0.0420</td>
<td>-0.8980</td>
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<td>S4</td>
<td>0.0016</td>
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<td>-0.3741</td>
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<td>S5</td>
<td>0.0052</td>
<td>0.0377</td>
<td>-0.3926</td>
<td>4.9425</td>
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<tr>
<td>S6</td>
<td>0.0050</td>
<td>0.0392</td>
<td>0.1874</td>
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<td>S7</td>
<td>0.0034</td>
<td>0.0557</td>
<td>0.0963</td>
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<td>0.0074</td>
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Appendix B
Table B.1: Weights of the sectors’ stocks and the risk free asset in the optimal SSD-DP portfolios of the first window of data, for various risk aversion degrees.

<table>
<thead>
<tr>
<th>ν</th>
<th>E1</th>
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Table B.2: Weights of sectors’ indices and the risk free asset in the SSD-DP optimal portfolios of the first window of data, for various risk aversion degrees.

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