A Variant of Lehmer’s Conjecture in the CM Case

by

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Abstract

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Lehmer’s conjecture asserts that $\tau(p) \neq 0$, where $\tau$ is the Ramanujan $\tau$-function. This is equivalent to the assertion that $\tau(n) \neq 0$ for any $n$. A related problem is to find the distribution of primes $p$ for which $\tau(p) \equiv 0 \pmod{p}$. These are open problems. However, the variant of estimating the number of integers $n$ for which $n$ and $\tau(n)$ do not have a non-trivial common factor is more amenable to study. More generally, let $f$ be a normalized eigenform for the Hecke operators of weight $k \geq 2$ and having rational integer Fourier coefficients $\{a(n)\}$. It is interesting to study the quantity $(n, a(n))$. It was proved by S. Gun and V. K. Murty (2009) that for Hecke eigenforms $f$ of weight 2 with CM and integer coefficients $a(n)$

$$\{n \leq x \mid (n, a(n)) = 1\} = \frac{(1 + o(1))U_f x}{\sqrt{\log x \log \log \log x}}$$

for some constant $U_f$. We extend this result to higher weight forms.

We also show that

$$\{n \leq x \mid (n, a(n)) \text{ is a prime} \} \ll \frac{x \log \log \log \log x}{\sqrt{\log x \log \log \log x}}.$$  

(2)
Dedication

To my parents, who made me believe that anything is possible

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Chapter 1

Introduction

1.1 The $\tau$ function

Consider the cusp form of Ramanujan:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\piinz} = q \prod_{k=1}^{\infty} (1 - q^k)^{24},$$

where $q = e^{2\piiz}$. The coefficients $\tau(n)$ have received extensive arithmetic scrutiny following the groundbreaking investigations of Ramanujan himself.

Of the many problems that are open, there is Lehmer’s conjecture that asserts that for any prime $p$,

$$\tau(p) \neq 0.$$

Equivalently, for any $n \geq 1$,

$$\tau(n) \neq 0.$$

More generally, let

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\piinz}$$

be the Fourier expansion of a normalized eigenform of weight $k \geq 2$ and level $N$, and suppose $a(n) \in \mathbb{Z}$. We can ask a question about the vanishing of the coefficients $a(p)$ or $a(n)$. If the weight is 2, it is known that there are infinitely many primes $p$ for which $a(p) = 0$. If the weight is $\geq 4$, it is expected (but not known) that the set of primes $p$ for which $a(p) = 0$ is finite.

A problem closely related to the vanishing of $a(p)$ is to ask whether we can have

$$a(p) \equiv 0(\text{mod } p).$$

We might expect this to be rare, based on the heuristics reviewed below.

In the case of the Ramanujan $\tau$-function, it is known that

$$\tau(p) \equiv 0(\text{mod } p)$$

holds for primes $p = 2, 3, 5, 7, 2411, 7758337633$ and these are the only primes up to $10^{10}$ that satisfy this congruence [9], but it is not known if there are infinitely many such primes. Nor do we know any good upper
bounds of the number of such primes. In particular, is it true that
\[ \# \{ p \leq x : \tau(p) \equiv 0 \pmod{p} \} = o(\pi(x)) \]?

Or in general, is it true that
\[ \# \{ p \leq x : a(p) \equiv 0 \pmod{p} \} = o(\pi(x)) \]?

Since we have the Ramanujan-Petersson estimate, proved by Deligne [4], [5],
\[ |a(p)| \leq 2p^{(k-1)/2}, \]
we see that in the weight \( k = 2 \) case, for \( p > 3 \) the condition \( p|a(p) \) is equivalent to \( a(p) = 0 \). Heuristically, if the weight is \( > 2 \), then the number of primes \( p \) up to \( x \) for which \( p|a(p) \) may grow like \( \log \log x \), though we do not even know if these are of density zero. If we assume that the values of \( a(p) \) are equidistributed in the interval \([-2p^{k-1}/2, 2p^{k-1}/2]\), then we can roughly evaluate the number of such primes less than \( x \) in the following way. Given an \( a \in [-2p^{k-1}/2, 2p^{k-1}/2] \), we might expect that the probability that \( a(p) = a \) is
\[ \frac{1}{4p^{k-1} + 1}, \]
and therefore the number of \( p \leq x \) for which \( a(p) = a \) is asymptotically
\[ \frac{c\pi(x)}{p^{k-1}}. \]

Since the proportion of integers in the interval \([-2p^{k-1}/2, 2p^{k-1}/2]\) that are divisible by \( p \) is \( \frac{1}{p} \), we might expect that the probability that \( a(p) \equiv 0 \pmod{p} \) is proportional to \( \frac{1}{p} \).

Thus, we might expect
\[ \sum_{\substack{p \leq x \\atop a(p) \equiv 0 \pmod{p}}} 1 \sim c \sum_{p \leq x} \frac{p^{k-1} - 1}{p^{k-1}} \sim c \sum_{p \leq x} \frac{1}{p} \sim c \log \log x. \]

Besides the fact that this is a very rough heuristic, we note that the \( a(p) \) are not expected to be equidistributed. Indeed, there is a skewing due to the Sato-Tate measure. However, this will only affect the above heuristic by a constant.

Analogous to the vanishing of \( a(p) \) is the vanishing of \( a(n) \). Analogous to the question whether \( p|a(p) \) is whether \( (n, a(n)) \neq 1 \). In particular, we might ask whether
\[ \# \{ n \leq x : (n, a(n)) \neq 1 \} = o(x). \]

In fact, as explained in [10], this is not true, and the correct question to ask is the opposite, namely whether it is true that
\[ \# \{ n \leq x : (n, a(n)) = 1 \} = o(x). \]

This is the variant of Lehmer’s conjecture that we discuss. We remark that the motivation for the question arises from the possibility of using modular forms for new factoring algorithms.

Define \( L_1(x) = \log x \) and \( L_i(x) = \log L_{i-1}(x) \) for \( i \geq 2 \).
In a recent work [10] by V. K. Murty (1.1) was considered and the following theorem proved:
Theorem 1.1.1. (V. K. Murty): For a normalized Hecke eigenform $f$ with rational integer coefficients $a(n)$, one has

$$\# \{n \leq x \mid (n, a(n)) = 1\} \ll \frac{x}{L_3(x)}.$$ 

In [10] it was also anticipated that if $f$ has complex multiplication (CM), a stronger result should hold. Such a result was obtained in [6] for the case that $f$ has CM and is of weight 2:

Theorem 1.1.2. (S. Gun, V. K. Murty): Let $f$ be a normalized eigenform of weight 2 with rational integer Fourier coefficients $\{a(n)\}$. If $f$ is of CM-type, then there is a constant $U_f > 0$ so that

$$\# \{n \leq x \mid (n, a(n)) = 1\} = (1 + o(1)) \frac{U_f x}{\sqrt{L_1(x) L_3(x)}}.$$ 

The constant $U_f$ can be given explicitly.

In this thesis we study the analogue of Theorem 1.1.2 for CM forms of weight $> 2$. We obtain an asymptotic formula in this case as well. In order to do this, we need to surmount some technical obstacles. In [6], essential use is made of a result of Schaal [16] in order to obtain an estimate (of the form $1/p \log \log x$) for the sum

$$\sum_{\substack{q \leq x \mid \, \, a(q) \equiv 0 \pmod{p} \atop q \equiv 0 \pmod{p}}} \frac{1}{q}.$$ 

For weight $> 2$, the argument given in [6] breaks down and we need to find a replacement. In particular, we are not able to use Schaal’s estimate. Rather, we rely on a clever use of the Chebotarev density theorem. This allows us to establish the key Lemma 4.4.1. It will be of interest to see whether our technique can actually be used to strengthen Schaals’ theorem. We do not pursue this theme here, but hope to return to it in future work.

A new product that emerges in our estimates is:

$$\prod_{\substack{p \leq x \mid \, \, a(p) \equiv 0 \pmod{p} \atop a(p) \equiv 0 \pmod{p}}} \left(1 - \frac{1}{p}\right). \tag{1.2}$$

We need to assume that this product converges to a non-zero constant. Equivalently, we need to have that the sum

$$\sum_{\substack{a(p) \equiv 0 \pmod{p} \atop a(p) \equiv 0 \pmod{p}}} \frac{1}{p}$$

converges.

Denote by $Z'_f$ the set

$$Z'_f = \{p \text{ prime} \mid a(p) \neq 0, p|a(p)\},$$

and by $Z'_f(x)$ the number of primes in $Z'_f$ such that $p \leq x$:

$$Z'_f(x) = \{p \leq x \mid p \in Z'_f\}.$$ 

We can show that (1.2) converges to a constant in the asymptotic formula. This product did not emerge in the weight $k = 2$ case, because for $p > 3$ the condition $p|a(p)$ is equivalent
to $a(p) = 0$ and we know that

$$\sum_{a(p)=0} \frac{1}{p}$$

converges.

We also discuss cases when there can be at most finitely many primes in $\mathbb{Z}'_f$.

### 1.2 CM case for higher weight forms

Let

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$ 

Let $f$ be a normalized Hecke eigenform of weight $k \geq 2$ for $\Gamma_0(N)$ with complex multiplication and rational integer Fourier coefficients, and let $K$ be the imaginary quadratic field associated to $f$, i.e. there is a Hecke character $\Psi$ of $K$ with conductor $m$ such that

$$f(z) = \sum_{a \in \mathcal{O}_K, (a,m)=1} \Psi(a) e^{2\pi i N(a)z}.$$ 

Here the sum is over integral ideals $a$ of the ring of integers $\mathcal{O}_K$ of $K$ which are coprime to $m$, and $N(a)$ denotes the norm of $a$ in $K/\mathbb{Q}$. Thus,

$$a(n) = \sum_{\substack{N(a)=n, \\ (a,m)=1}} \Psi(a),$$

where $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ is the Fourier expansion of $f$ at infinity. In particular, if a prime $p$ is inert in $K$, then $a(p) = 0$.

Set

$$M_{f,1}(x) = \# \{ n \leq x \mid a(n) \neq 0, \ p|n \Rightarrow p \notin \mathbb{Z}'_f \}.$$ 

Then we show that there is a constant $u_f$ so that

$$M_{f,1}(x) = (1 + o(1)) \frac{u_f x}{(\log x)^{\frac{3}{2}}}.$$ 

Denote by

$$M_{f,1} = \{ n \mid a(n) \neq 0, \ p|n \Rightarrow p \notin \mathbb{Z}'_f \}.$$ 

Later on, whenever we are dealing with a set of natural numbers $E$, we will denote by $E(x)$ the cardinality of the set $\{ n \leq x, n \in E \}$.

We also show that there is a constant $\mu_f > 0$ so that

$$\prod_{p < x} \left(1 - \frac{1}{p}\right) \sim \frac{\mu_f}{(\log x)^{\frac{3}{2}}}.$$ 

where $\mu_f$ is given in Proposition 2.1.1. The main results of this thesis are the following theorems.
**Theorem 1.2.1.** Let $f$ be a normalized eigenform of weight $k \geq 2$ with rational integer Fourier coefficients \( \{a(n)\} \) and of CM-type. Then there is a constant $U_f > 0$ so that

\[
\#\{n \leq x \mid (n, a(n)) = 1\} = (1 + o(1)) \frac{U_f x}{\sqrt{L_1(x)L_3(x)}}.
\]

The constant $U_f$ is given explicitly in terms of $f$ during the course of the proof.

We also prove the following theorem

**Theorem 1.2.2.** Under the same hypothesis as above

\[
\#\{n \leq x \mid (n, a(n)) \text{ is prime}\} \ll \frac{xL_4(x)}{\sqrt{L_1(x)L_3(x)}}.
\]
Chapter 2

Vanishing of Fourier coefficients

2.1 Vanishing of $a(p)$

We denote by $d_K$ the discriminant of the field $K$. Recall that if $p$ is inert in $K$, then $a(p) = 0$, since in this case there exists no ideal of norm $p$, and so the corresponding coefficient in the definition of a CM-form is equal to zero. Also, $p$ ramifies in $K$ if and only if $p|d_K$.

We shall use repeatedly the following estimate

$$\#\{p \leq x \mid a(p) = 0 \text{ and } p \text{ splits in } K\} \ll \frac{x(\log \log x)}{(\log x)^2}.$$  

This follows by the method of Theorem 5.1 of [11].

The following result will be useful in establishing the main result.

**Proposition 2.1.1.** There is a constant $\mu_f > 0$ so that

$$\prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{\mu_f} = \frac{\mu_f}{\sqrt{\log z}} + O_f \left(\frac{1}{(\log z)^{3/2}}\right).$$

**Proof.** Using Rosen [15], Theorem 2, we have

$$\prod_{Np \leq z} \left(1 - \frac{1}{Np}\right)^{-1} = e^{\gamma \alpha_K} \log z + O_K(1).$$

Here the product is over primes $p$ of $K$ and $\alpha_K$ is the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$. Note that $\alpha_K = L(1, \chi_K)$, where $\chi_K$ is the quadratic character defining $K$ and $L(s, \chi_K)$ is the associated $L$-function, which is the extension to the whole complex plane of the Dirichlet $L$-series:

$$L(s, \chi_K) = \sum_{n=1}^{\infty} \frac{\chi_K(n)}{n^s}.$$  

It follows that

$$\prod_{Np \leq z} \left(1 - \frac{1}{Np}\right) = e^{-\gamma} L(1, \chi_K)^{-1} \frac{1}{\log z} + O_K \left(\frac{1}{(\log z)^2}\right),$$
and
\[
\prod_{Np \leq z} \left(1 - \frac{1}{Np}\right) \frac{1}{2} = \frac{e^{-\frac{1}{2} L(1, \chi_K)^{-\frac{1}{2}}}}{\sqrt{\log z}} + O_K \left(\frac{1}{(\log z)^{\frac{1}{2}}}\right).
\]

Now,
\[
\prod_{Np \leq z} \left(1 - \frac{1}{Np}\right) = \prod_{p \text{ splits in } K, p \leq z} \left(1 - \frac{1}{p}\right) \prod_{p \text{ inert in } K, p \leq \sqrt{z}} \left(1 - \frac{1}{p^2}\right) \prod_{p \text{ ramifies in } K, p \leq z} \left(1 - \frac{1}{p}\right).
\]

So,
\[
\prod_{p \leq z} \left(1 - \frac{1}{p}\right) = \prod_{p \leq \sqrt{z}, a(p) \neq 0} \left(1 - \frac{1}{p}\right) \prod_{p \leq \sqrt{z}, p \text{ inert in } K, a(p) \neq 0} \left(1 - \frac{1}{p}\right) \prod_{p \leq \sqrt{z}, p \text{ inert in } K, a(p) = 0} \left(1 - \frac{1}{p}\right) \prod_{p \leq \sqrt{z}, p \text{ splits in } K, a(p) = 0} \left(1 - \frac{1}{p}\right) = \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \prod_{p \leq \sqrt{z}, p \text{ inert in } K, a(p) = 0} \left(1 - \frac{1}{p}\right).
\]

which gives
\[
= \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \prod_{p \leq \sqrt{z}, p \text{ inert in } K, a(p) = 0} \left(1 - \frac{1}{p}\right) \prod_{p \mid d_K, a(p) = 0} \left(1 - \frac{1}{p}\right) = \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \prod_{p \mid d_K, a(p) = 0} \left(1 - \frac{1}{p}\right) = \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \mid d_K, a(p) = 0} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1}.
\]

where all the products are bounded by a constant, because \(\prod_{p \mid d_K, a(p) \neq 0} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}}, \prod_{p \mid d_K, a(p) = 0} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}}\) are finite.
products; \[ \prod_{p \text{ inert in } K} \left( 1 - \frac{1}{p^2} \right)^{-\frac{1}{2}} \] is convergent, and \[ \prod_{p \text{ splits in } K} \left( 1 - \frac{1}{p} \right)^{-1} \] is convergent, because \[ \sum_{p \text{ splits in } K} \frac{1}{p} \] converges (we show this later).

Thus, \[ \prod_{p \leq z, a(p) \neq 0} \left( 1 - \frac{1}{p} \right) = \frac{\mu_f}{\sqrt{\log z}} + O_f \left( \frac{1}{(\log z)^{3/2}} \right), \]

where \[ \mu_f = e^{-\gamma/2} L(1, \chi_K)^{-\frac{1}{2}} \prod_{p \text{ inert in } K} \left( 1 - \frac{1}{p^2} \right)^{-\frac{1}{2}} \prod_{p \text{ splits in } K} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq z, p|d_K, a(p) \neq 0} \left( 1 - \frac{1}{p} \right)^{\frac{1}{2}} \prod_{p \leq z, p|d_K, a(p) = 0} \left( 1 - \frac{1}{p} \right)^{-\frac{1}{2}}. \]

Here we split the primes for which \( a(p) = 0 \) into three sets: 1) \( p \) that split in \( K \) and \( a(p) = 0 \), 2) \( p \) that ramify in \( K \) and \( a(p) = 0 \) and 3) \( p \) that are inert in \( K \).

\[ \square \]

### 2.2 The number of non-zero Fourier coefficients

We begin by considering a slightly more general setting as in Serre ([19], §6), which parts of this section follow closely. Let \( \{a(n)\} \) be the Fourier coefficients of \( f \). Then \( a(n) \) is a multiplicative function, which means that \( a(nm) = a(n)a(m) \) whenever \( (n,m) = 1 \).

Denote by \( Z_f \) the set of all primes \( p \) that divide their coefficient \( a(p) \).

\[ Z_f = \{ p - \text{ prime} : p|a(p) \}. \]

Note that \[ Z_f = Z'_f \cup \{ p : a(p) = 0 \}. \]

Define the multiplicative function \[ a^0(n) := \begin{cases} 1, & \text{if } a(n) \neq 0, p|n \Rightarrow p \notin Z'_f \\ 0, & \text{otherwise.} \end{cases} \]

Note that for \( p - \text{ prime} \)

\[ a^0(p) := \begin{cases} 1, & \text{if } p \notin Z_f \\ 0, & \text{if } p \in Z_f \end{cases} \]

We want the asymptotic behaviour of \[ M_{f,d}(x) := \# \{ n \leq x \mid a(n) \neq 0, p|n \Rightarrow p \notin Z'_f, d|n \} = \sum_{dn \leq x} a^0(dn), \]

for any positive integer \( d \).
2.2.1 The case $d = 1$

Let us estimate

$$M_{f,1}(x) = \#\{n \leq x \mid a(n) \neq 0, \ p|n \Rightarrow p \notin \mathbb{Z}_f\} = \sum_{n \leq x} a^0(n).$$

**Definition 2.2.1.** A set of primes $P$ is called *frobenien* of density $\alpha$ if there exists a finite Galois extension $K/\mathbb{Q}$ and a subset $H$ of the group $G = \text{Gal}(K/)$ such that

- $H$ is stable under conjugation,
- $|H|/|G| = \alpha$,
- for $p$ sufficiently large, $p \in P \iff \sigma_p(K/\mathbb{Q}) \subseteq H$, where $\sigma_p(K/\mathbb{Q})$ denotes the class of a Frobenius automorphism associated to $p$.

**Definition 2.2.2.** A set of primes $P$ is called *regular* of density $\alpha$ if

$$\sum_{p \in P} \frac{1}{p^s} = \alpha \log \frac{1}{s-1} + \theta_P(s),$$

where $\theta_P(s)$ extends to a holomorphic function in the region $Re(s) \geq 1$.

**Lemma 2.2.3.** If $P$ is frobenien of density $\alpha$, then $P$ is regular of density $\alpha$.

*Proof.* (Serre [18]) Let $P$ be frobenien of density $\alpha$, with Galois group $G$ and the subset $H$ that satisfies the properties of the definition. Then

$$\sum_{p \in P} \frac{1}{p^s} = \frac{1}{|G|} \sum_{\chi} \chi(H) \log L(s, \chi) + g(s),$$

where $\chi$ runs through irreducible characters of $G$, $L(s, \chi)$ is the Artin L-function of the extension $K/\mathbb{Q}$ and character $\chi$, $g$ is a Dirichlet series that converges absolutely for $\Re(s) > 1/2$ (thus, it is holomorphic for $\Re(s) \geq 1$), and $\chi(H) = \sum_{h \in H} \chi(h)$.

From the elementary properties of the functions $L(s, \chi)$:

$$\log L(s, \chi) = \delta_{\chi} \log \frac{1}{s-1} + \theta_{\chi}(s),$$

where $\delta_{\chi} = \begin{cases} 0, & \text{if } \chi \neq 1, \\ 1, & \text{if } \chi = 1 \end{cases}$, and $\theta_{\chi}(s)$ is holomorphic for $\Re(s) \geq 1$. Thus,

$$\sum_{p \in P} \frac{1}{p^s} = \frac{|H|}{|G|} \log \frac{1}{s-1} + \theta_P(s) = \alpha \log \frac{1}{s-1} + \theta_P(s).$$

\[
\]

Consider the Dirichlet series

$$\varphi(s) = \sum_n \frac{a^0(n)}{n^s} = \prod_p \varphi_p(s)$$

where

$$\varphi_p(s) = \sum_{m=0}^{\infty} a^0(p^m)p^{-ms}. $$

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Thus, $\phi$ is analytic for $\Re$ where $n \in S$. This holds because $a^0(n) = a^0(p_1^{a_1})a^0(p_2^{a_2}) \cdots a^0(p_l^{a_l})$, where $n = p_1^{a_1}p_2^{a_2} \cdots p_l^{a_l}$ is the prime decomposition of $n$.

Let 

$$Z_f(x) = \# \{ p \leq x : p \in Z \}.$$

**Lemma 2.2.4.**

$$\log(\phi(s)) = \sum p \log(\phi_p(s)) = \sum p \frac{a^0(p)}{p^s} + \epsilon_1(s),$$

where $\epsilon_1(s)$ is also analytic in the neighbourhood of $s = 1$.

**Proof.**

$$\phi(s) = \prod_{n=1}^{\infty} a^0(n) \prod_p \phi_p(s).$$

Recall that

$$a^0(p) = \begin{cases} 1, & \text{if } p \notin Z_f, \\ 0, & \text{if } p \in Z_f. \end{cases}$$

Thus, $\phi_p(s)$ will start with $1 + \frac{1}{p^s}$ if and only if $p \notin Z_f$. So,

$$\phi(s) = \prod_p \phi_p(s) = \prod_{p \notin Z_f} \left(1 + \frac{1}{p^s} + \frac{a^0(p^2)}{p^{2s}} + \cdots \right) \cdot \prod_{p \in Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \cdots \right) =$$

$$= \prod_{p \notin Z_f} \left(1 + \frac{1}{p^s} \right) \left(1 + \frac{a^0(p^2)}{p^{2s}} + \cdots \right) \cdot \prod_{p \in Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \cdots \right) \cdot \prod_{p \notin Z_f} \left(1 + \frac{1}{p^s} \right) \cdot \epsilon_2(s),$$

where

$$\epsilon_2(s) = \prod_{p \notin Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \cdots \right) \cdot \prod_{p \in Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \cdots \right)$$

is analytic for $\Re(s) > 1/2$, $\epsilon_2(s) \neq 0$. So, $\log \epsilon_2(s)$ is analytic for $\Re(s) > \frac{1}{2}$. Thus,

$$\log \phi(s) = \sum_{p \in Z_f} \log \left(1 + \frac{1}{p^s} \right) + \log \epsilon_2(s) = \sum_{p \in Z_f} \frac{1}{p^s} + \epsilon_3(s) + \log \epsilon_2(s) = \sum_{p \notin Z_f} \frac{1}{p^s} + \epsilon_1(s),$$

where $\epsilon_1(s) = \log \epsilon_2(s) + \epsilon_3(s)$ is holomorphic for $\Re(s) > 1/2$.

Thus,

$$\log \phi(s) = \sum_{p \notin Z_f} \frac{1}{p^s} + \epsilon_1(s) = \sum_{p \notin Z_f} a^0(p) \frac{1}{p^s} + \epsilon_1(s)$$

We record another useful equality:

$$\log \phi(s) = \sum_{p \in Z_f} \frac{1}{p^s} + \epsilon_1(s) = \sum_{p \in Z_f} \frac{1}{p^s} - \sum_{p \notin S_1} \frac{1}{p^s} - \sum_{p \in S_2} \frac{1}{p^s} + \epsilon_1(s) = \sum_{p \in Z_f} \frac{1}{p^s} + \epsilon_4(s),$$

(2.1)
where $Z_f$ is written as a disjoint union of three sets $Z_f = S_1 \cup S_2 \cup S_3$, with
\[
S_1 = \{ p \text{ - prime} \mid p \text{ does not split in } K \}, \\
S_2 = \{ p \text{ - prime} \mid p \text{ splits in } K \text{ and } a(p) = 0 \}, \\
S_3 = \{ p \text{ - prime} \mid p \in Z_f, \ a(p) \neq 0 \}.
\]

Note that $S_1$ and $S_2$ together constitute the primes for which $a(p) = 0$.

The series $\sum_{p \in S_2} \frac{1}{p^s}$ converges at $s = 1$:
\[
\sum_{p \in S_2} \frac{1}{p^s} \bigg|_{s=1} = \sum_{p \in S_2} \frac{1}{p} = O \left( \int_2^x \frac{1}{t} \, d \left( \frac{t(\log t)^2}{(\log t)^2} \right) \right) = O(1).
\] (2.2)

Here we used the fact that $\# \{ p \leq t \mid p \in S_2 \} \ll \frac{(\log t)^2}{(\log t)^2}$.

Thus, the function $\sum_{p \in S_2} \frac{1}{p^s}$ is holomorphic as well.

\[
\sum_{p \in S_1} \frac{1}{p^s} = \sum_{p \in Z_f, a(p) \neq 0} \frac{1}{p^s} \text{ is holomorphic at } s = 1, \text{ because this sum ranges over at most finitely many primes (we show this in chapter 3).}
\]

So, the function $\epsilon_4(s) = \epsilon_1(s) - \sum_{p \in S_2} \frac{1}{p^s} - \sum_{p \in S_3} \frac{1}{p^s}$ is holomorphic as well.

**Lemma 2.2.5.** $S_1 = \{ p \text{ - prime} \mid p \text{ does not split in } K \}$ is frobenien of density $1/2$.

*Proof.* $K/\mathbb{Q}$ is a quadratic extension, so $G = \text{Gal}(K/\mathbb{Q})$ consists of 2 elements: $G = \pm 1$. We know that $1 \not\in H$. Take $H = -1$. Then $\sigma_p$ makes sense for $p$ that do not ramify. For $p$ sufficiently large $p \in S_1 \Leftrightarrow p$ remains prime in $K$. For those primes $\sigma_p = -1$.

Thus, the density is $\frac{|H|}{|G|} = \frac{1}{2}$. \qed

The orthogonality of characters gives us:
\[
\sum_{\chi} \chi(H) \chi(\sigma_p) = \begin{cases} 
0, & \text{if } \sigma_p \neq H, \\
\frac{|G|}{|H|}, & \text{if } \sigma_p = H,
\end{cases}
\]

Thus, $S_1$ is frobenien implies
\[
\sum_{p \in S_1} \frac{1}{p^s} = \sum_{\chi} \frac{1}{p^s} \left( \sum_{\chi} \chi(H) \chi(\sigma_p) \right) \frac{|H|}{|G|} = \frac{|H|}{|G|} \sum_{\chi} \sum_{p} \frac{1}{p^s} \chi(H) \chi(\sigma_p)
\]

If $\chi \neq 1$ this is analytic. Only $\chi = 1$ contributes
\[
\frac{|H|}{|G|} \sum_{p} \frac{1}{p^s} \chi(H) \chi(\sigma_p) = \frac{|H|}{|G|} \sum_{p} \frac{1}{p^s}.
\]

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Thus,
\[ \sum_{p \in S_1} \frac{1}{p^s} = \frac{|H|}{|G|} \sum \chi \sum \frac{1}{p^s} \overline{\chi(p)} = \frac{|H|}{|G|} \sum \frac{1}{p^s} + \epsilon_5(s) = \frac{1}{2} \sum \frac{1}{p^s} + \epsilon_5(s), \tag{2.3} \]
where \( \epsilon_5(s) \) converges at \( s = 1 \).

If \( \sum_{p \in \mathbb{Z}_f \atop a(p) \neq 0} \frac{1}{p^s} \) is convergent at \( s = 1 \), we have the following:

**Lemma 2.2.6.**
\[ \sum_{p \neq S_1} \frac{a(p)}{p^s} = \frac{1}{2} \log \left( \frac{1}{s-1} \right) + \epsilon_4(s), \]
where \( \epsilon_4(s) \) is analytic in the neighbourhood of \( s = 1 \).

**Proof.** If \( S_1 \) is regular of density \( \alpha \), then \( S_1' = \{ p \text{ - prime, } p \notin S_1 \} \) is regular of density \( 1 - \alpha \) ([18]). Since \( S_1 \) is Frobenius of density \( 1/2 \), we have \( S_1' \) is Frobenius of density \( (1 - 1/2) \), and so by Lemma 2.2.3, \( S_1' \) is regular of density \( 1/2 \). Thus,
\[ \sum_{p \notin S_1} \frac{1}{p^s} = \frac{1}{2} \log \left( \frac{1}{s-1} \right) + \theta_{S_1}(s), \tag{2.4} \]
where \( \theta_{S_1}(s) \) extends to a holomorphic function in the region \( \Re(s) \geq 1 \).

Put (2.1) and (2.4) together to get:
\[ \log \varphi(s) = \sum_{p \in S_1} \frac{1}{p^s} + \epsilon_4(s) = \frac{1}{2} \log \left( \frac{1}{s-1} \right) + \epsilon_7(s), \]
where \( \epsilon_7(s) = \epsilon_4(s) + \theta_{S_1}(s) \).

Also,
\[ \log \varphi(s) = \sum_{p \neq S_1} \frac{a_0(p)}{p^s} + \epsilon_1(s), \]
and so,
\[ \sum_{p} \frac{a_0(p)}{p^s} = \frac{1}{2} \log \left( \frac{1}{s-1} \right) + \epsilon_8(s), \]
where \( \epsilon_8(s) = \epsilon_7(s) - \epsilon_1(s) \). Thus, we get
\[ \varphi(s) = \frac{e^{\epsilon_7(s)}}{(s-1)^{1/2}} = \frac{\epsilon_9(s)}{(s-1)^{1/2}}, \]
where \( \epsilon_9(s) \) is analytic near \( s = 1 \) and \( \epsilon_9(s) \neq 0 \).

Thus, we have the following decomposition:
\[ \varphi(s) = \frac{1}{(s-1)^{1/2}} \cdot (e_0 + \epsilon_1(s-1) + e_2(s-1)^2 + \ldots). \]

From (2.3) we get:
\[ \sum_{p \notin S_1} \frac{1}{p^s} = \frac{1}{2} \sum p^{-s} + \epsilon_{10}(s), \]
where $\epsilon_{10}(s) = -\epsilon_5(s)$. Thus,
\[
\log \varphi(s) = \frac{1}{2} \sum_p \frac{1}{p^s} + \epsilon_{11}(s),
\]
where $\epsilon_{11}(s) = \epsilon_{10}(s) + \epsilon_4(s)$.

So
\[
\varphi(s) = \exp \left\{ \frac{1}{2} \sum_p \frac{1}{p^s} \right\} \cdot e^{\epsilon_{11}(s)}.
\]

Also,
\[
(\zeta(s))^{\frac{1}{2}} = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-\frac{1}{2}} = \exp \left\{ -\frac{1}{2} \sum_p \log \left( 1 - \frac{1}{p^s} \right) \right\}
\]
\[
= \exp \left\{ -\frac{1}{2} \left( -\sum_p \frac{1}{p^s} + \sum_p \left( \frac{c_2}{p^{2s}} + \frac{c_3}{p^{3s}} + \ldots \right) \right) \right\} = \exp \left\{ \frac{1}{2} \sum_p \frac{1}{p^s} \right\} \cdot \epsilon_{12}(s),
\]

where $\epsilon_{12}(s)$ is analytic near $s = 1$ and $\epsilon_{12}(s) \neq 0$. Thus,
\[
\varphi(s) = (\zeta(s))^{\frac{1}{2}} \cdot h(s),
\]

where $h(s) = \frac{e^{\epsilon_{11}(s)}}{\epsilon_{12}(s)}$ is analytic near $s = 1$ and $h(s) \neq 0$. We will use (2.5) later.

**Some preliminary lemmas.**

Put
\[
b^0(x) = \sum_{n \leq x} a^0(n) \log \frac{x}{n}.
\]

The next three lemmas show that
\[
b^0(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \varphi(s)x^s \cdot s^{-2} ds.
\]

Consider the following integral:
\[
J(w) = \int_{2-i\infty}^{2+i\infty} e^{ws} \frac{s^{-2}}{s^w} ds,
\]

where $w$ is a real number. It converges, since $\left| \frac{e^{ws}}{s^2} \right| = \frac{e^{2w}}{4+\pi^2}$, where $s = 2 + it$.

**Lemma 2.2.7.**
\[
J(w) = \begin{cases} 
0, & \text{if } w \leq 0 \\
\frac{2\pi i w}{2\pi}, & \text{if } w \geq 0
\end{cases}
\]

**Proof.** (From [7]) Put
\[
J(w, T) = \int_{2-iT}^{2+iT} e^{ws} \frac{s^{-2}}{s^w} ds.
\]

Then
\[
J(w) = \lim_{T \to \infty} J(w, T).
\]

Consider the two cases:
1) $w \geq 0$. Let $T > 2$. By the Cauchy’s integral formula

$$J(w, T) - 2\pi i \text{Res} \left( \frac{e^{ws}}{s^2}, s = 0 \right) = \int_{\gamma} \frac{e^{ws}}{s^2} ds,$$

where $\gamma$ denotes the left semicircle connecting the points $2 + iT$ and $2 - iT$ and is traversed from $2 + iT$ to $2 - iT$.

$$\text{Res} \left( \frac{e^{ws}}{s^2}, s = 0 \right) = \lim_{s \to 0} \frac{d}{ds} \left( s^2 \cdot \frac{e^{ws}}{s^2} \right) = \lim_{s \to 0} w \cdot e^{ws} = w$$

Thus,

$$J(w, T) - 2\pi iw = \int_{\gamma} \frac{e^{ws}}{s^2} ds.$$

The length of the semicircle is $\pi T$. Also $|s| \geq T - 2$ and $\Re(s) \leq 2$, thus

$$\left| \frac{e^{ws}}{s^2} \right| \leq \frac{e^{2w}}{(T - 2)^2},$$

and so

$$|J(w, T) - 2\pi iw| = \int_{\gamma} \frac{e^{ws}}{s^2} ds \leq \frac{\pi T e^{2w}}{(T - 2)^2} \to 0, \text{ as } T \to \infty.$$

Thus,

$$J(w) = \lim_{T \to \infty} J(w, T) = 2\pi iw.$$

2) $w \leq 0$. Again we use Cauchy’s theorem, only we will integrate along the other semicircle, so that there are no poles inside the area bounded by the closed curve of integration. Then,

$$J(w, T) = \int_{\gamma} \frac{e^{ws}}{s^2} ds.$$

The length of the arc is $\pi T$; $|s| \geq T$ implies that $\left| \frac{e^{ws}}{s^2} \right| \leq \frac{e^{2w}}{T^2}$. Thus

$$|J(w, T)| \leq \frac{\pi e^{2w}}{T} \to 0, \text{ as } T \to \infty.$$

$$J(w) = 0 \text{ for } w \leq 0.$$

\[\square\]

Lemma 2.2.8.

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^s s^{-2} ds = \begin{cases} 0, & \text{if } 0 < y \leq 1 \\ \log y, & \text{if } y \geq 1 \end{cases}$$

Proof. This is an immediate consequence of the previous Lemma with $w = \log y$. \[\square\]

Lemma 2.2.9.

$$b^0(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \varphi(s) x^s s^{-2} ds$$

Proof. The statement follows from Lemma 2.2.8 with $y = \frac{x}{n}$;

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \varphi(s) x^s s^{-2} ds = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{n=1}^{\infty} a^0(n) \left( \frac{x}{n} \right)^s s^{-2} ds =$$
\[ \frac{1}{2\pi i} \sum_{n=1}^{\infty} a^0(n) \int_{2-i\infty}^{2+i\infty} \left( \frac{x}{n} \right)^s s^{-2} ds = \sum_{n=1}^{\infty} a^0(n) \log \frac{x}{n} = b^0(x). \]

In the next Lemma we shall state some properties of \( \zeta(s) \) which we will need in order to determine the behaviour of the function \( \varphi(s) \). Here \( c_1, c_2, \ldots \) denote positive constants, \( s = \sigma + ti \) is a complex variable.

**Lemma 2.2.10.** 1. \( \zeta(s) - (s - 1)^{-1} \) is holomorphic for \( \sigma > 0 \).

2. There exists \( c_1 \) such that \( \zeta(s) \neq 0 \) for \( \sigma \geq 1 - \frac{c_1}{(\log |t|)^9} \), \( |t| \geq 3 \), and for \( \sigma \geq 1 - \frac{c_1}{(\log 3)^9} \), \( |t| \leq 3 \).

3. There exists \( c_2 \) such that
   \[ |\zeta(s)| < c_2 \log |t| \]
   for \( \sigma \geq 1 - \frac{1}{\log |t|} \), \( |t| \geq 3 \), and \( c_3 \) such that
   \[ |\log \zeta(s)| < c_3 (\log |t|)^9 \]
   for \( \sigma \geq 1 - \frac{c_1}{(\log |t|)^9} \), \( |t| \geq 3 \).

4. There exist \( c_4, c_5 \) and \( c_6 \) such that
   \[ |\zeta(s)| < c_4 \quad \text{and} \quad |\log \zeta(s)| < c_5 \]
   for \( 1 - \frac{c_1}{(\log 3)^9} \leq \sigma \leq 1 - c_6 < 1 \), \( |t| \leq 3 \).

**Proof.** The properties given in parts 1, 2 and 3 of the lemma are contained in §42, §48 and §64 of Landau [7]. Part 4 is an immediate consequence of the rest of the lemma.

The next lemma follows immediately from Lemma 2.2.10 and the definition of \( \varphi(s) \). Recall that \( \varphi(s) = \left( \zeta(s) \right)^{\frac{1}{2}} h(s) \).
Lemma 2.2.11. For suitable positive constants $d_1$, $d_2$ and $d_3$, we have

1. The function $\varphi(s)$ is holomorphic for $\sigma \geq 1 - \frac{d_1}{(\log |t|)^9}$, $|t| \geq 3$ and for $\sigma \geq 1 - \frac{d_1}{(\log 3)^9}$, $|t| \leq 3$ except for a singularity at $s = 1$. (See figure 2.2)

2. $|\varphi(s)| < d_2(\log |t|)^k_1$ for $\sigma \geq 1 - \frac{d_1}{(\log |t|)^9}$, $|t| \geq 3$, where $k_1 > 0$, and $|\varphi(s)| < d_3$ for $\sigma = 1 - \frac{d_1}{(\log 3)^9}$, $|t| \leq 3$.

Lemma 2.2.12. If $|s - 1| \leq d_1(\log 3)^{-9}$, then

$$|x^s\varphi(s)s^{-2} - x^s h(1)(s - 1)^{-\frac{1}{2}}| = O(x^s).$$

Proof. By Lemma 2.2.10: part 1), $(s - 1)\zeta(s)$ is holomorphic for $\sigma > 0$, and

$$\lim_{s \to 1}(s - 1)\zeta(s) = 1.$$  \hfill (2.6)

$$\varphi(s) \cdot s^{-2} = (\zeta(s))^{\frac{1}{2}} h(s)s^{-2} = (s - 1)^{-\frac{1}{2}}((s - 1)\zeta(s))^{\frac{1}{2}} h(s)s^{-2},$$

where $((s - 1)\zeta(s))^{\frac{1}{2}} h(s)s^{-2}$ is holomorphic and bounded when $|s - 1| \leq d_1(\log 3)^{-9}$ and hence can be
expanded as a convergent power series in the form:

\[ ((s - 1)\zeta(s))^{\frac{1}{2}} h(s)s^{-2} = \sum_{k=0}^{\infty} w_k (s - 1)^k. \]

From (2.6) we have that \( w_0 = h(1). \)

Thus,

\[ \varphi(s)s^{-2} = h(1)(s - 1)^{-\frac{1}{2}} + (s - 1)^{\frac{1}{2}} \sum_{k=1}^{\infty} w_k (s - 1)^{k-1}, \]

and so

\[ |x^s \varphi(s)s^{-2} - x^s h(1)(s - 1)^{-\frac{1}{2}}| = O \left( (s - 1)^{\frac{1}{2}} \sum_{k=1}^{\infty} w_k (s - 1)^{k-1} \right) = O(x^s), \]

for \(|s - 1| \leq d_1 (\log 3)^{-9}. \) \(\square\)

Figure 2.2: The branches \( D \) and \( C \) lie in the desired region
Lemma 2.2.13. There is a constant $u_f$ such that

$$M_{f,1}(x) = (1 + o(1)) \frac{u_fx}{(\log x)^{1/2}}.$$ 

Proof.

$$\varphi(s) = \sum_n a^0(n) \frac{1}{n^s} = \sum_{n \in M_{f,1}} \frac{1}{n^s}$$

$\varphi(s)$ extends to a holomorphic function in the following region:

The branches $C$ and $D$ are defined by $\Re(s) = 1 - \frac{b}{\log A}$, with $T = 2 + |\Im(s)|$. And in this region $|\varphi(s)| = O(\log^A T)$ as $T \to \infty$.

Indeed, take $b = c_1$ and $A \geq 9(\log \log 3)(\log \log 2)^{-1}$ (so that the branches $C, D$ lie in the desired region) and apply Lemma 2.2.10.

Recall, that we defined

$$b^0(x) = \sum_{n \leq x} a^0(n) \frac{x}{n},$$

and established that

$$b^0(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \varphi(s)x^s \cdot s^{-2} ds.$$ 

Lemma 2.2.14.

$$b^0(x) = \frac{h(1)}{\sqrt{\pi}} x(\log x)^{-\frac{1}{2}} + O(x(\log x)^{-1}) \quad (2.7)$$

Proof. Cauchy’s theorem shows that this integral is equal to the analogous integral along the left edge of the region in question (i.e. along branches $D, E, F, E', C$ in figure 2.3). The branches $C$ and $D$ contribute negligible amounts of the order $\frac{x}{\log N}$ for any $N$. The integral along the circle $F$ tends to 0 as the radius of the circle tends to 0. Thus, the main term is provided by the two integrals along the horizontal segments $E$ and $E'$. To evaluate these we need the development of $\frac{\varphi(s)}{s^2}$ in the neighbourhood of $s = 1$:

$$\frac{\varphi(s)}{s^2} = \frac{1}{\sqrt{s-1}} (e_0 + e_1(s-1) + \ldots + e_k(s-1)^k + \ldots)$$

From Lemma 2.2.12 we have:

$$\int_{1-d'(\log 3)^{-9}}^{1} (x^s \varphi(s)s^{-2} - x^s h(1)(s-1)^{-\frac{1}{2}}) ds = O\left(\int_{1-d'(\log 3)^{-9}}^{1} x^s ds\right) = O\left(\frac{x}{\log x}\right).$$
Hence,

\[ \left\{ \int_{E'} + \int_{E} \right\} x^s \varphi(s) s^{-2} ds = \int_{1-d_1 (\log 3)^{-9}}^{1} h(1) x^{s^-} (s^- - 1)^{-\frac{1}{2}} ds^- - \int_{1-d_1 (\log 3)^{-9}}^{1} h(1) x^{s^+} (s^+ - 1)^{-\frac{1}{2}} ds^+, \]

where \( s^+ \) and \( s^- \) indicate the upper edge and the lower edge respectively on the cut. Since \( (s^+ - 1) = (1 - s^+) e^{\pi i} \) and \( (s^- - 1) = (1 - s^-) e^{-\pi i} \), it follows that

\[ \left\{ \int_{E'} + \int_{E} \right\} x^s \varphi(s) s^{-2} ds = \]
\[
= h(1) \int_{1-d_1(\log 3)^{-9}}^1 x^+(1 - s^+) - \frac{1}{2} \left( e^{\frac{\pi i}{2}} - e^{-\frac{\pi i}{2}} \right) ds^+ + O \left( \frac{x}{\log x} \right) = \\
= 2ih(1) \int_{1-d_1(\log 3)^{-9}}^1 x^+(1 - s) - \frac{1}{2} ds + O \left( \frac{x}{\log x} \right)
\int_{1-d_1(\log 3)^{-9}}^1 x^+(1 - s) - \frac{1}{2} ds = \Gamma \left( \frac{1}{2} \right) x(\log x) - \frac{1}{2} + O \left( \frac{x}{\log x} \right),
\]

Thus,

\[
\left\{ \int_{E'} + \int_{E} \right\} x^+ \varphi(s) s^{-2} ds = \frac{2\pi ih(1)}{\Gamma \left( \frac{1}{2} \right)} x(\log x) - \frac{1}{2} + O \left( \frac{x}{\log x} \right),
\]

and so

\[
b^0(x) = \frac{h(1)}{\Gamma \left( \frac{1}{2} \right)} x(\log x) - \frac{1}{2} + O \left( \frac{x}{\log x} \right).
\]

We apply this result with \( x + \delta x \), with \( \delta \sim \frac{1}{(\log x)^{K+1}} \) and find the needed estimate for \( \sum_{n \leq x} a^0(n) \). We have

**Lemma 2.2.15.** Suppose that

\[
b^0(x) = B \frac{x}{(\log x)^{\frac{1}{2}}} + O \left( \frac{x}{\log x} \right). \tag{2.8}
\]

Then

\[
M_{f,1}(x) = B \frac{x}{(\log x)^{\frac{1}{2}}} + O \left( x(\log x)^{-\frac{3}{2}} \right).
\]

**Proof.** Let \( \delta = \delta(x) = o(1) \) be a positive function of \( x \) to be chosen later, and suppose that \( x(1 + \delta) \) is an integer. Then, since

\[
\log(x(1 + \delta)) = \log x + O(\delta),
\]

\[
b^0(x(1 + \delta)) = B(x(1 + \delta)) x(\log(x(1 + \delta)) - \frac{1}{2}) + O \left( x(1 + \delta)(\log x(1 + \delta))^{-1} \right) = \\
= B(x(1 + \delta)) \left\{ (\log x)^{-\frac{1}{2}} + (-1/2)(\log x)^{-\frac{3}{2}} O(\delta) + O((\log x)^{-\frac{3}{2}}) \right\} + \\
+ O \left( x(1 + \delta)(\log x)^{-1} \right) = \\
= B x(\log x)^{-\frac{1}{2}} \left( 1 + \delta + O(\delta(\log x)^{-1}) \right) + O \left( x(\log x)^{-1} \right). \tag{2.9}
\]

By definition

\[
b^0(x(1 + \delta)) - b^0(x) = \sum_{n=1}^{x(1+\delta)} a^0(n) \log x(1 + \delta)/n - \sum_{n=1}^{x} a^0(n) \log x/n
\]

\[
= \log(1 + \delta) \sum_{n=1}^{x} a^0(n) + \sum_{n=x+1}^{x(1+\delta)} a^0(n) \log x(1 + \delta)/n \\
\geq \log(1 + \delta) M_{f,1}(x), \tag{2.10}
\]

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since the second sum is not negative. Similarly

\[ b^0(x(1 + \delta)) - b^0(x) = \sum_{n=1}^{x(1+\delta)} a^0(n) \log x(1 + \delta)/n - \sum_{n=1}^{x} a^0(n) \log x/n \]
\[ = \log(1 + \delta) \sum_{n=1}^{x(1+\delta)} a^0(n) + \sum_{n=x+1}^{x} a^0(n) \log x/n \]
\[ \leq \log(1 + \delta) M_f,1(x(1 + \delta)), \]

since the second sum is not positive. By (2.8), (2.9) and (2.10)

\[ M_f,1(x) \leq \left\{ b^0(x(1 + \delta)) - b^0(x) \right\} / \log(1 + \delta) \]
\[ = \left\{ b^0(x(1 + \delta)) - b^0(x) \right\} (1 + O(\delta)) \delta^{-1} \]
\[ = B x (\log x)^{-\frac{1}{2}} \left\{ 1 + O(\delta) + O((\log x)^{-1}) + O(\delta^{-1}(\log x)^{-1}) \right\}. \]

By (2.8), (2.9) and (2.11)

\[ M_f,1(x(1 + \delta)) \geq \left\{ b^0(x(1 + \delta)) - b^0(x) \right\} / \log(1 + \delta) \]
\[ = \left\{ b^0(x(1 + \delta)) - b^0(x) \right\} (1 + O(\delta)) \delta^{-1} \]
\[ = B x (\log x)^{-\frac{1}{2}} \left\{ 1 + O(\delta) + O((\log x)^{-1}) + O(\delta^{-1}(\log x)^{-1}) \right\}. \]

If we replace \( x \) by \( x/(1 + \delta) \) in (2.13), we obtain

\[ M_f,1(x) \geq B x (\log x)^{-\frac{1}{2}} \left\{ 1 + O(\delta) + O((\log x)^{-1}) + O(\delta^{-1}(\log x)^{-1}) \right\}. \]

We now choose \( \delta \) so that all the error terms of (2.12) and (2.14) are of a smaller order of magnitude than the first term. We can take

\[ \delta = x^{-1}[x(\log x)^{-\frac{1}{2}}]. \]

Then the error terms of (2.12) and (2.14) are

\[ O \left( x(\log x)^{-\frac{1}{2}}(\log x)^{-1} \right) = O \left( x(\log x)^{-\frac{3}{4}} \right). \]

Hence, from (2.12) and (2.14) if follows that

\[ M_f,1(x) = B \frac{x}{(\log x)^{\frac{1}{2}}} + O \left( x(\log x)^{-\frac{3}{4}} \right), \]

which is the result of the lemma. \( \square \)

We use (2.7) and Lemma 2.2.15 with \( B = u_f = \frac{h(1)}{\sqrt{\pi}} \) to obtain the result of Lemma 2.2.13. \( \square \)

2.2.2 Convolution with a secondary function.

Now consider another function \( n \mapsto b(n) \) with the following properties:

1. There is an integer \( d \) so that \( b(n) \neq 0 \) implies that all prime divisors of \( n \) are prime divisors of \( d \).

2. We have \( |b(n)| \leq 4^\nu(n) \), where \( \nu(n) \) is the number of distinct prime divisors of \( n \).

Let us set

\[ \xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}. \]
We see that
\[ \sum_{m \leq x} |b(m)| \leq \sum_{m \leq x} |b(m)| \left( \frac{x}{m} \right)^{\frac{1}{4}} \leq \sum_{m \in \mathbb{Z}} |b(m)| \left( \frac{x}{m} \right)^{\frac{1}{4}} = \]

since \( b(m) = 0 \) for the rest of \( m \)

\[ = \sum_{p | m \Rightarrow p \mid d} |b(m)| \left( \frac{x}{m} \right)^{\frac{1}{4}} \leq \sum_{p | m \Rightarrow p \mid d} 4^{\nu(m)} \left( \frac{x}{m} \right)^{\frac{1}{4}} = x^{\frac{1}{4}} \sum_{p | m \Rightarrow p \mid d} \frac{4^{\nu(m)}}{m^{\frac{1}{4}}} = \]

\[ = x^{\frac{1}{4}} \prod_{p \mid d} \left( 1 + \frac{4}{p^{\frac{1}{4}}} + \frac{4}{(p^2)^{\frac{1}{4}}} + \ldots \right) = x^{\frac{1}{4}} \prod_{p \mid d} \left( 1 + \frac{4}{p^{\frac{1}{4}} - 1} \right) \]

The number of factors in the product \( \prod_{p \mid d} \left( 1 + \frac{4}{p^{\frac{1}{4}} - 1} \right) \) is the number of distinct prime divisors of \( d \), i.e. it is \( \nu(d) \). For \( p \geq 5^4 \) we have \( 1 + \frac{4}{p^\frac{1}{4} - 1} \leq 2 \). Thus,

\[ \prod_{p \mid d} \left( 1 + \frac{4}{p^{\frac{1}{4}} - 1} \right) \ll 2^{\nu(d)} \]

and so

\[ \sum_{m \leq x} |b(m)| \ll x^{1/4} 2^{\nu(d)}. \] (2.15)

Moreover, using (2.15), we have

\[ \sum_{z < m < 2z} |b(m)| m \ll z^{-3/4} 2^{\nu(d)}. \] (2.16)

Indeed,

\[ \sum_{z < m < 2z} \frac{|b(m)|}{m} \ll \frac{1}{z} z^{-3/4} 2^{\nu(d)} \]

we use (2.15)

\[ \ll \frac{1}{z} (2z)^{1/4} 2^{\nu(d)} = \sqrt{2} z^{-3/4} 2^{\nu(d)}. \]

Let \( c = a^0 \ast b \) be the Dirichlet convolution (i.e., \( c(n) = \sum_{d|n} a^0(d)b \left( \frac{n}{d} \right) \)) and consider the function

\[ \psi(s) = \sum_n \frac{c(n)}{n^s} = \sum_n \frac{a^0(n)}{n^s} \cdot \sum_n \frac{b(n)}{n^s} = \varphi(s) \xi_d(s) \]

by the property of Dirichlet series.

Then, we have

\[ \sum_{n \leq x} c(n) = \sum_{n \leq x} \sum_{r | n} a^0(r)b \left( \frac{n}{r} \right) = \sum_{m \leq x} \sum_{m \leq x} \sum_{m \leq x} b(m)a^0(r) = \sum_{m \leq x} b(m) \sum_{r \leq x/m} a^0(r). \]

The contribution from the terms with \( \sqrt{x} \leq m \leq x \) is

\[ \sum_{\sqrt{x} \leq m \leq x} b(m) \sum_{r \leq x/m} a^0(r) \leq \sum_{\sqrt{x} \leq m \leq x} b(m) \cdot \frac{x}{m} \leq x \sum_{\sqrt{x} \leq m \leq x} \frac{|b(m)|}{m}. \]
Decomposing the sum into dyadic intervals $U < m \leq 2U$ and using (2.16) shows that the summation is $O(x^{-3/8}2^{\nu(d)})$:

$$\sum_{\sqrt{x} \leq m \leq x} \frac{|b(m)|}{m} \leq \sum_{l=1}^{k} \sum_{2^{l-1} \sqrt{x} \leq m \leq 2^l \sqrt{x}} \frac{|b(m)|}{m} \leq \sum_{l=1}^{k} 2^{\nu(d)} \cdot 2^{-3(l-1)/4} \cdot x^{-3/8} =$$

$$= 2^{\nu(d)} \cdot x^{-3/8} \sum_{l=0}^{k-1} \left( \frac{1}{2^{3/4}} \right)^{l} = O(x^{-3/8}2^{\nu(d)}),$$

hence the whole expression is $O(x^{5/8}2^{\nu(d)})$. Lemma 2.2.13 implies

$$\sum_{n \leq x} c(n) = \sum_{m \leq \sqrt{x}} b(m) \left\{ \left( u_f + O \left( \frac{1}{\log x} \right) \right) \frac{x}{m(\log x/m)^{1/2}} \right\} + O(x^{5/8}2^{\nu(d)}).$$

(2.17)

Note that

$$\left( \frac{\log x}{m} \right)^{-1/2} = (\log x - \log m)^{-1/2} = (\log x)^{-1/2} + O((\log m)(\log x)^{-3/2}).$$

Using this and (2.16), the right hand side of (2.17) is equal to

$$\left( u_f + O \left( \frac{1}{\log x} \right) \right) \frac{x}{(\log x)^{1/2}} \left( \xi_d(1) + O(x^{-3/8}(\log x)^{-1}2^{\nu(d)}) \right) + O(x^{5/8}2^{\nu(d)}).$$

Summarizing this discussion, we have proved the following

**Proposition 2.2.16.** We have

$$\sum_{n \leq x} c(n) = u_f \xi_d(1) \frac{x}{(\log x)^{1/2}} + O \left( \frac{x2^{\nu(d)}}{(\log x)^{1/2}} \right)$$

uniformly in $d$.

**2.2.3 The case of general $d$.**

Consider the Dirichlet series

$$\psi_d(s) = \sum_{n} \frac{a^0(dn)}{n^s}. \quad (2.18)$$

We may write it as

$$\psi_d(s) = \left( \sum_{n_1=1}^{\infty} \frac{a^0(dn_1)}{n_1^s} \right) \left( \sum_{n_2=1}^{\infty} \frac{a^0(n_2)}{n_2^s} \right) \left( \sum_{n_3=1}^{\infty} \frac{a^0(n_3)}{n_3^s} \right) \left( \sum_{n_4=1}^{\infty} \frac{a^0(n_4)}{n_4^s} \right) \cdots$$

$$= \left( \sum_{n_1=1}^{\infty} \frac{a^0(dn_1)}{n_1^s} \right) \left( \sum_{n_2=1}^{\infty} \frac{a^0(n_2)}{n_2^s} \right) \left( \sum_{n_3=1}^{\infty} \frac{a^0(n_3)}{n_3^s} \right) \left( \sum_{n_4=1}^{\infty} \frac{a^0(n_4)}{n_4^s} \right) \cdots \left( \sum_{n_{2d-1}=1}^{\infty} \frac{a^0(n_{2d-1})}{n_{2d-1}^s} \right)^{-1}.$$

Thus, we see that

$$\psi_d(s) = \varphi(s) \xi_d(s) = \psi(s),$$

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where as before
\[ \varphi(s) = \sum_{n_3=1}^{\infty} \frac{a^0(n_3)}{n_3^s} \]
and
\[ \xi_d(s) = \left( \sum_{n_1=1}^{\infty} \frac{a^0(dn_1)}{n_1^s} \right) \left( \sum_{n_2=1}^{\infty} \frac{a^0(n_2)}{n_2^s} \right)^{-1}. \]

We have a factorization
\[ \xi_d(s) = \prod_{p|d} \xi_{p,d}(s), \]
where
\[ \xi_{p,d}(s) = \left( \sum_{m=0}^{\infty} \frac{a^0(p^m + \text{ord}_p d)}{p^{ms}} \right) \left( \sum_{m=0}^{\infty} \frac{a^0(p^m)}{p^{ms}} \right)^{-1}, \]
and \text{ord}_p d is the power of \( p \) in the prime decomposition of \( d \).

It makes sense to consider \( \xi_{p,d}(s) \) for only those \( p \) that divide \( d \).
Note that if \( p \in Z_f \), then \( a^0(p^m) = 0 \) for all \( m \geq 1 \). Thus, for \( p|d, p \in Z_f \) we have
\[ \xi_{p,d}(s) = 0, \]
and so \( \xi_d(s) = 0 \) if at least one \( p|d \) satisfies \( p \in Z_f \).

The following Lemma is from [6].

**Lemma 2.2.17.** If \( p \notin Z_f \), then
\[ \xi_{p,d}(1) = a^0(p^{\text{ord}_p d}) + O \left( \frac{1}{p} \right). \] \hspace{1cm} (2.19)

**Proof.** This follows from the formula for \( \xi_{p,d}(s) \). \( \Box \)

We write
\[ \xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \]
and suppose that \( \xi_d(s) \) (that is, the coefficients \( \{b(n)\} \)) satisfies the conditions of Section 2.2.2. Recall that
\[ M_{f,d}(x) = \# \{ n \leq x | a(n) \neq 0, d|n, p|n \Rightarrow p \notin Z'_f \}. \]
We have
\[ M_{f,d}(x) = \sum_{dn \leq x} a^0(dn), \]
and by Proposition 2.2.16, we deduce the following [6].

**Proposition 2.2.18.** If \( \xi_d \) satisfies the hypotheses of Section 2.2.2, then we have
\[ M_{f,d}(x) = \frac{u_f \xi_d(1)x/d}{(\log x/d)^{1/2}} + O \left( \frac{x\nu(d)}{d(\log x/d)^{3/2}} \right) \]
uniformly in \( d \).
We begin with some preliminary results. Let us set $i_f(p)$ to be the least integer $i \geq 1$ for which $a_0(p^i) = 0$. If for a given $p$, there is no such $i$, then let us set $i_f(p) = 0$. In particular, if $p$ divides the level $N$, then $i_f(p) = 1$. Note that

$$i_f(p) = 1 \Rightarrow a(p) = 0 \text{ or } p | a(p)$$

The following is Lemma 4.6 from [6].

**Lemma 2.2.19.** For $p \nmid N$, $p \not\in Z_f$, we have

1. $i_f(p) \in \{0, 1, 2, 3, 5\}$.

2. If $i_f(p) > 0$, then $a_f(p^i) = 0$ for every $i > 0$ with

$$i + 1 \equiv 0 \pmod{i_f(p) + 1}.$$

3. If $a(p^i) = 0$ for some $i > 0$, then $i + 1 \equiv 0 \pmod{i_f(p) + 1}$.

4. For $p$ sufficiently large (depending on $f$), we have $i_f(p) \in \{0, 1\}$.

Note that if $p \in Z_f$, then $a_0(p^i) = 0$ for $i \geq 1$.

**Proof.** Let us suppose that $i_f(p) > 0$. Thus, $a(p^i) = 0$ for some $i \geq 1$. Let us write $\alpha_p$ and $\beta_p$ for the roots of $X^2 - a(p)X + p$. Then, we have

$$a(p^i) = \frac{\alpha_p^{i+1} - \beta_p^{i+1}}{\alpha_p - \beta_p}. \tag{2.20}$$

Thus, $\alpha_p = \zeta \beta_p$, where $\zeta^{i+1} = 1$. Since $\zeta \in \mathbb{Q}(\alpha_p, \beta_p) = \mathbb{Q}(\alpha_p)$ and $[\mathbb{Q}(\alpha_p) : \mathbb{Q}] = 2$, we must have $\zeta^2 = 1$ or $\zeta^4 = 1$ or $\zeta^6 = 1$. This means that one of $\{\zeta + 1, \zeta^2 + \zeta + 1, \zeta^2 - \zeta + 1, \zeta^2 + 1\}$ is zero. This in turn means that one of $\{a(p), a(p^3), a(p^5), a(p^5)\}$ is zero.

$$\begin{cases}
\zeta^2 - 1 = 0 \\
\zeta^4 - 1 = 0 \\
\zeta^6 - 1 = 0
\end{cases} \Leftrightarrow \begin{cases}
\zeta^2 = 1 \\
\zeta^4 = 1 \\
\zeta^6 = 1
\end{cases}$$

1) $\zeta^2 = 1 \Rightarrow a(p) = \frac{\alpha_p^2 - \beta_p^2}{\alpha_p - \beta_p} = 0.$

2) $\zeta^4 = 1 \Rightarrow a(p^3) = \frac{\alpha_p^4 - \beta_p^4}{\alpha_p - \beta_p} = 0.$

3) $\zeta^3 = 1 \Rightarrow a(p^2) = \frac{\alpha_p^3 - \beta_p^3}{\alpha_p - \beta_p} = 0.$

4) $\zeta^6 = 1 \Rightarrow a(p^5) = \frac{\alpha_p^6 - \beta_p^6}{\alpha_p - \beta_p} = 0.$

This proves the first assertion.

The second follows from (2.20).

For the third assertion, we note that $\alpha_p = \zeta \beta_p$ where $\zeta^{i+1} = 1$. We also have $\zeta^{i+1} = 1$. Hence, $\zeta^j = 1$, where $i + 1 \equiv j (\text{mod } i_f(p) + 1)$. If $j > 0$, then $a(p^{j-1}) = 0$. But $0 \leq j - 1 < i_f(p)$, a contradiction, unless $j = 1$. But then $a(1) = 0$, which is also a contradiction. Hence, we must have $j = 0$, proving the third assertion.

The fourth assertion follows from [12], Lemma 2.5.
As before, let us set
\[ \varphi_p(s) = \sum_{m=0}^{\infty} \frac{a^0(p^m)}{p^{ms}}. \]
From the above lemma we deduce the following.

**Lemma 2.2.20.** We have for \( p \nmid N, p \not\in \mathbb{Z}_f, \)
\[
\varphi_p(s) = \begin{cases} 
\left(1 - \frac{1}{p^s}\right)^{-1} & \text{if } i_f(p) = 0, \\
\frac{1}{p^s}\left(\frac{1}{p^s-1} - \frac{1}{p^{i_f(p)+1}}\right) & \text{if } i_f(p) > 0.
\end{cases}
\]
Note that \( \varphi_p(s) = 1 \) for \( p \mid N \) or \( p \in \mathbb{Z}_f, \) because in this case \( a^0(p^i) = 0 \) for all \( i \geq 1. \)

Next, we evaluate \( \xi_a(1). \) We have the following.

**Proposition 2.2.21.** Writing
\[
\xi_a(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}
\]
we have that
1. \( b(n) = 0 \) if \( n \) is divisible by a prime that does not divide \( d, \) and
2. \( p \mid d, \) we have \( |b(p^m)| \leq 4 \) for all \( m. \)
In particular, the function \( n \mapsto b(n) \) satisfies the conditions of section 2.2.2. Moreover, we have for \( p \nmid N, p \not\in \mathbb{Z}_f, \)
\[
\xi_{p,d}(1) = \begin{cases} 
1 + p^{-1} - p^{v-2k_0+1} & \text{if } i_f(p) = 0, \\
\frac{1}{p^{i_f(p)+1}} & \text{if } i_f(p) > 1.
\end{cases}
\]
\( \xi_{p,d}(1) = 0, \) if \( p \in \mathbb{Z}_f. \)
Here \( v = \text{ord}_p d, \) where \( \text{ord}_p d := \max\{i\}, \) and \( k_0 \) is the smallest integer \( \geq \frac{v+1}{i_f(p)+1}. \)

**Proof.** By a calculation similar to that of Lemma 2.2.20, we see that
\[
\sum_{m=0}^{\infty} \frac{a^0(p^m+v)}{p^{ms}} = \begin{cases} 
\left(1 - \frac{1}{p^s}\right)^{-1} & \text{if } i_f(p) = 0, \\
\frac{1}{p^s}\left(\frac{1}{p^s-1} - \frac{1}{p^{i_f(p)+1}}\right) & \text{if } i_f(p) > 1.
\end{cases}
\]
Hence, writing \( i = i_f(p), \) we have
\[
\xi_{p,d}(s) = \frac{p^{(i+1)s} - 1 - p^{v+1-(k_0-1)(i+1)s} + p^{v-(k_0-1)(i+1)s}}{p^{i+1}s - p^s},
\]
which is equal to
\[
\left(1 - \frac{1}{p^{k_0(i+1)-v}s} + \frac{1}{p^{k_0(i+1)-v}s} - \frac{1}{p^{i+1}s}\right) \left(1 - \frac{1}{p^s}\right)^{-1}
\]
from which it follows that \( |b(p^m)| \leq 4. \) Moreover, as
\[
\xi_a(s) = \prod_{p \mid d} \xi_{p,d}(s),
\]
it follows also that \( b(n) = 0 \) unless every prime divisor of \( n \) also divides \( d \). The last assertion of the proposition follows from the above formulas.

**Remark 2.2.22.** Note that the dependence on \( d \) of \( \xi_{p,d} \) is only through \( \text{ord}_d d \). Thus, where the meaning is clear, for \( p \mid d \) and \( d \) squarefree, we shall write \( \xi_p \), since in the case of a squarefree \( d \) we have

\[
\text{ord}_d d = \begin{cases} 1, & \text{if } p \mid d \\ 0, & \text{if } p \nmid d \end{cases}.
\]

In the remainder of this section we will elaborate on the constant \( u_f \), and in particular, relate it to \( L \)-function values. From Lemma 2.2.20, we have

\[
\log \varphi(s) = \sum_p \log \varphi_p(s) = \sum_{p \not\equiv s} \log \varphi_p(s) + \sum_{p \equiv s} \log \varphi_p(s)
\]

\[
= - \sum_{i_f(p)=0} \log \left( 1 - \frac{1}{p^s} \right) - \sum_{i_f(p)=1} \log \varphi_p(s) + \sum_{i_f(p) > 1} \log \varphi_p(s)
\]

\[
= - \sum_{i_f(p) \equiv 0} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{i_f(p) > 1} \log \varphi_p(s).
\]

Note that \( \sum_{p \equiv s} \log \varphi_p(s) = 0 \) because \( \varphi_p(s) = 0 \) for \( p \in Z_f \); and \( i_f(p) = 1 \Leftrightarrow p \in Z_f \).

Also, note that by Lemma 2.2.19, (4), the third sum on the right ranges over a finite set of primes \( p \).

Denote by \( \chi_K \) the quadratic Dirichlet character that defines \( K \). This means that if \( d_K \) is the discriminant of \( K \), then \( \chi_K(n) = \left( \frac{n}{d_K} \right) \) – Jacobi symbol. If \( d_K = p_1^{\alpha_1} \cdots p_l^{\alpha_l} \) is the prime decomposition of \( d_K \), then

\[
\left( \frac{n}{d_K} \right) = \left( \frac{n}{p_1} \right)^{\alpha_1} \cdots \left( \frac{n}{p_l} \right)^{\alpha_l}, \text{ where } \left( \frac{n}{p_i} \right) \text{ is the Legendre symbol:}
\]

\[
\left( \frac{n}{p_i} \right) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{p_i}, \\ 1, & \text{if } n \not\equiv 0 \pmod{p_i} \text{ and } n \equiv x^2 \pmod{p} \text{ for some } x, \\ -1, & \text{if } n \not\equiv 0 \pmod{p_i} \text{ and there is no such } x. 
\end{cases}
\]

Let \( L(s, \chi_K) \) be the associated Dirichlet series. Let us denote by \( S, I, R \) the set of primes that split, stay inert or ramify in \( K \) respectively. Then, we have

\[
L(s, \chi_K) = \sum_{n=1}^{\infty} \frac{\chi_K(n)}{n^s} = \prod_p \left( 1 - \frac{\chi_K(p)}{p^s} \right)^{-1}
\]

Since

\[
\chi_K(p) = \left( \frac{d_K}{p} \right) = \begin{cases} -1, & \text{if } p \text{ is inert}, \\ 0, & \text{if } p \text{ ramifies}, \\ 1, & \text{if } p \text{ splits.}
\end{cases}
\]

we have

\[
- \sum_{p \in S} \log \left( 1 - \frac{1}{p^s} \right) = \frac{1}{2} \log \zeta(s) + \frac{1}{2} \log L(s, \chi_K) + \frac{1}{2} \sum_{p \in I} \log \left( 1 - \frac{1}{p^{2s}} \right) + \frac{1}{2} \sum_{p \in R} \log \left( 1 - \frac{1}{p^s} \right),
\]

and so
\[- \sum_{p \in S \setminus Z_f} \log \left( 1 - \frac{1}{p^s} \right) = \frac{1}{2} \log \zeta(s) + \frac{1}{2} \log L(s, \chi_K) + \frac{1}{2} \sum_{p \in I} \log \left( 1 - \frac{1}{p^{2s}} \right) + \frac{1}{2} \sum_{p \in R} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{p \in S \setminus Z_f} \log \left( 1 - \frac{1}{p^s} \right). \]

Moreover, if \( i_f(p) = 0 \), then \( a(p) \neq 0 \) and \( a(p^i) \neq 0 \) for any \( i \geq 2 \). Thus,

\[- \sum_{i_f(p) = 0, p \not\in Z_f} \log \left( 1 - \frac{1}{p^s} \right) = - \sum_{a(p) \neq 0, p \not\in Z_f} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{i_f(p) > 1, p \not\in Z_f} \log \left( 1 - \frac{1}{p^s} \right), \]

because \( p \not\in Z_f \Rightarrow a(p) \neq 0. \)

Thus,

\[
\log \varphi(s) = - \sum_{i_f(p) = 0, p \not\in Z_f} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{i_f(p) > 1, p \not\in Z_f} \log \varphi_p(s)
\]

\[
= - \sum_{p \in S \setminus Z_f} \log \left( 1 - \frac{1}{p^s} \right) - \sum_{p \in R \setminus Z_f} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{i_f(p) > 1, p \not\in Z_f} \log \varphi_p(s) + \sum_{i_f(p) > 1, p \not\in Z_f} \log \varphi_p(s)
\]

\[
= \frac{1}{2} \log \zeta(s) + \frac{1}{2} \log L(s, \chi_K) + \sum_{p \in I} \log \left( 1 - \frac{1}{p^{2s}} \right) + \frac{1}{2} \sum_{p \in R} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{p \in S \setminus Z_f} \log \left( 1 - \frac{1}{p^s} \right) + \sum_{i_f(p) > 1, p \not\in Z_f} \log \varphi_p(s)
\]

\[
= \frac{1}{2} \log \left( \frac{1}{s-1} \right) + \frac{1}{2} \log \left( \zeta(s)(s-1) \right) + \frac{1}{2} \log L(s, \chi_K) + \log C(s),
\]

where

\[
C(s) = \prod_{p \in I} \left( 1 - \frac{1}{p^{2s}} \right)^{\frac{i}{2}} \prod_{p \in R} \left( 1 - \frac{1}{p^s} \right)^{\frac{i}{2}} \prod_{p \in S \setminus Z_f} \left( 1 - \frac{1}{p^s} \right) \prod_{i_f(p) > 1, p \not\in Z_f} \varphi_p(s). \]
$C(s)$ is holomorphic because $\prod_{p \in S, \ p \in \mathbb{Z}_f} \left(1 - \frac{1}{p^s}\right)$ is holomorphic:

$$\prod_{p \in S, \ p \in \mathbb{Z}_f} \left(1 - \frac{1}{p^s}\right) = \prod_{p \in S, \ a(p) = 0} \left(1 - \frac{1}{p^s}\right) \cdot \prod_{p \in S, \ a(p) \neq 0} \left(1 - \frac{1}{p^s}\right) =$$

$$= \prod_{p \in S, \ a(p) = 0} \left(1 - \frac{1}{p^s}\right) \cdot \prod_{p \in \mathbb{Z}_f, \ a(p) \neq 0} \left(1 - \frac{1}{p^s}\right) \cdot \prod_{p \in R, \ a(p) \neq 0} \left(1 - \frac{1}{p^s}\right)^{-1},$$

because $p \in I \Rightarrow a(p) = 0$.

The product $\prod_{p \in S, \ a(p) = 0} \left(1 - \frac{1}{p^s}\right)$ converges at $s = 1$ since in (2.2) the sum $\prod_{p \in S, \ a(p) = 0} \frac{1}{p}$ converges. The product $\prod_{p \in \mathbb{Z}_f, \ a(p) \neq 0} \left(1 - \frac{1}{p^s}\right)$ ranges over at most finitely many primes. This gives us the convergence of $\prod_{p \in S, \ p \in \mathbb{Z}_f} \left(1 - \frac{1}{p^s}\right)$, as the product over the $p \in R$ is finite. Thus, we have convergence of $C(s)$.

Putting the above discussion together, we see that

$$\varphi(s) = \frac{\epsilon(s)}{(s-1)^{1/2}},$$

where

$$\epsilon(s) = L(s, \chi_K)^{\frac{1}{2}}C(s),$$

and so

$$u_f = \epsilon(1) = L(1, \chi_K)^{\frac{1}{2}}C(1).$$
Chapter 3

CM Hecke eigenforms

3.1 Examples of CM Hecke eigenforms

Let $K$ be an imaginary quadratic field. Let $\mathfrak{f}$ be an integral ideal of $K$ and denote by $I(\mathfrak{f})$ the group of fractional ideals of $K$ coprime to $\mathfrak{f}$. We consider Hecke characters $\Psi$ of $K$. Thus, $\Psi$ is a map

$$\Psi : I(\mathfrak{f}) \rightarrow \mathbb{Q}^\times.$$ 

We suppose that for a positive integer $r$, 

$$\Psi((\alpha)) = \alpha^r$$

for all $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\mathfrak{f}}$.

In order for such $\Psi$ to exist, we need to have

$$\zeta^r = 1$$

for any root of unity $\zeta \in K$ with $\zeta \equiv 1 \pmod{\mathfrak{f}}$. To such a $\Psi$, we associate the function $f = f_\Psi$ defined by

$$f_\Psi(z) = \sum_{\substack{a \in \mathcal{O}_K \\ (a, \mathfrak{f}) = 1}} \Psi(a)e^{2\pi i (Na)z}$$

for $z \in \mathbb{C}$, $\Im(z) > 0$.

Let us set $k = r + 1$ and $M = |d_K|N_f$. Let us also define the Dirichlet character $\epsilon$ modulo $M$ given by

$$\epsilon(a) = \left(\frac{d_K}{a}\right) \frac{\Psi((a))}{r}$$

for $a \in \mathbb{Z}$, $(a, M) = 1$. Then by a theorem of Hecke, $f_\Psi$ is a cusp form of weight $k$, level $M$ and character $\epsilon$.

This means that for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ with } c \equiv 0 \pmod{M},$$

we have

$$f_\Psi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}.$$ 

The $L$-function associated to $\Psi$ is given by

$$L(s, \Psi) = \prod_p \left(1 - \frac{\Psi(p)}{(Np)^s}\right)^{-1}$$

for $s > 1$. 

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where the product ranges over the primes $p$ of $K$ coprime to $f$. We can actually drop the latter condition by extending $\Psi$ by 0 at primes that divide $f$. Writing the above as an Euler product over rational primes gives the $L$-function of $f_{\Psi}$:

$$L(s, f_{\Psi}) = \prod_p \left( 1 - \frac{a(p)}{p^s} - \frac{\epsilon(p)p^r}{p^{2s}} \right)^{-1}.$$ 

In our work, we assume that $\epsilon = 1$ and all the $a(p)$ are rational integers.

**Example 1**: Let us consider an elliptic curve $E$ over $\mathbb{Q}$ with CM by $K$. Let $\Psi$ be the corresponding Hecke character. Then it satisfies the above hypotheses with $f_{\Psi}$ of weight 2, i.e. $f_{\Psi}$ is a normalized Hecke eigenform with CM and integer coefficients. We may also consider $\Psi_m$ for some $1 \leq m \in \mathbb{Z}$. The corresponding $f_{\Psi_m}$ is of weight $m+1$. Indeed, to see that the Fourier coefficients of $f_{\Psi_m}$ are rational integers, we observe that

$$\Psi^m(p) + \Psi^m(p) = (\Psi(p) + \Psi(p))^m - \sum_{j=1}^{m-1} \binom{m}{j} \Psi(p)^j \Psi(p)^{m-j}.$$ 

The sum on the right is

$$\sum_{j<m/2} \binom{m}{j} p^j \left( \Psi(p)^{m-2j} + \Psi(p)^{m-2j} \right).$$ 

Now we use induction on $m$ to deduce that the left hand side is a rational integer. Note that if $m$ is even, the above construction gives a form $f_{\Psi}$ with a nontrivial Nebentypus character. Very likely our methods will extend to this case.

**Example 2**: Consider the Dedekind $\eta$- function given by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = e^{\frac{1}{12} \pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

It is a modular form of weight $1/2$ and $\eta^r(z)$ is a modular form of weight $r/2$. Moreover, for $r = 2, 4, 6, 8, 10, 14, 26$ it is of CM type. This is a result of Ramanujan [14]. Also, it is a fact that

$$\eta(z) = \sum_{n \equiv 1 \pmod{6}} (-1)^{\frac{n-1}{2}} q^{\frac{n^2}{2}},$$

which shows that the powers of $\eta$ have integer coefficients. However, they may not be Hecke eigenforms. In [14] Ramanujan shows that the form $\eta^8(3z)$ is a form of weight 4 for $\Gamma_0(9)$ (in fact, a newform) with CM by $\mathbb{Q}(\sqrt{-3})$. Let us write

$$\eta^8(3z) = q \prod_{m=1}^{\infty} (1 - q^{3m})^8 = q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + \ldots = f_{K,\Psi}(z),$$

where $K = \mathbb{Q}(\sqrt{-3})$ and $\Psi$ is a Hecke character of $K$ with conductor $f = \sqrt{-3} \cdot \mathfrak{o}_K$, defined as follows:

If $\alpha$ is an ideal of $\mathfrak{o}_K$ coprime with $f$, then $\Psi(\alpha) = \alpha^3$, where $\alpha$ is the generator of $\alpha$ for which $\alpha \equiv 1 \pmod{f}$.

For this form we can show that there are no primes $p$ such that $p|\alpha(p)$ and $\alpha(p) \neq 0$. Indeed, if $p = (\alpha)$, $\alpha \equiv 1 \pmod{f}$,

$$\alpha = \frac{a + b\sqrt{-3}}{2}, \quad a, b \in \mathbb{Z}$$

then

$$4p = a^2 + 3b^2,$$
and
\[ a(p) = \alpha^3 + \pi^3 = \frac{1}{8}(\alpha^3 - 9ab^2) \equiv 0 \pmod{p} \]
implies \( p | a \), and so \( p | b \). Contradiction.

### 3.2 Characters of \( \mathbb{Q}(\sqrt{D}) \)

Consider the general case: \( K = \mathbb{Q}(\sqrt{D}) \). Let us suppose that \( \mathcal{O}_K = \mathbb{Z}[\sqrt{D}] \).
We do not assume that the class number \( h = h_K = 1 \). Choose representatives \( \mathfrak{c}_1, \ldots, \mathfrak{c}_h \) for the elements of the \( \mathfrak{f} \)-ideal class group.
Write \( p = p\mathfrak{p} \).
Say \( (\beta_p p) = (\alpha_p) \mathcal{C}, \alpha_p \equiv 1 \pmod{\mathfrak{f}}, \beta_p \equiv 1 \pmod{\mathfrak{f}} \), where \( \mathcal{C} \) is a fractional ideal. Then
\[ \text{denominator}(\mathcal{C})(\beta_p)p = \text{numerator}(\mathcal{C})(\alpha_p). \tag{3.1} \]

\[ \Psi((\beta_p))\Psi(p) = \Psi((\alpha_p))\Psi(\mathcal{C}) \]

\[ \beta_p^{k-1}\Psi(p) = \alpha_p^{k-1}\Psi(\mathcal{C}) \]
We have
\[ \Psi(p) = \Psi(\mathcal{C})\gamma_p^{k-1}, \]
where \( \gamma_p = \alpha_p/\beta_p \).
Then \( p | a(p) \) implies
\[ p | \Psi(p) + \Psi(\mathfrak{p}) = \Psi(\mathcal{C})\gamma_p^{k-1} + \Psi(\mathfrak{C})\gamma_p^{k-1} \]
Here \( \Psi(\mathcal{C}), \Psi(\mathfrak{C}), \gamma_p, \gamma_p \) need not be integers, but after multiplication of this equation by the common denominator \( D \) of \( \Psi(\mathcal{C}), \Psi(\mathfrak{C}), \gamma_p, \gamma_p \) we get the following equation, where \( A, B \) and \( \delta_p \) are integers:
\[ p | a(p) \cdot D = A\delta_p^{k-1} + B\delta_p^{k-1}. \tag{3.2} \]

Equation (3.1) implies that \( p | \text{numerator}(\mathcal{C}) \) or \( p | (\delta_p) \). This means that \( p | (\delta_p) \) except for finitely many primes (all the primes that divide the numerators of \( \mathcal{C}_i \)). This in turn, together with equation (3.2), implies that
\[ p | B\delta_p. \]
Since \( p | B \) is possible for finitely many primes \( p \) only, this means that \( p | \delta_p \). Since \( \mathfrak{p}|\delta_p \), we have
\[ p | \delta_p. \]
Thus, if \( p \) is sufficiently large, we can conclude that \( p | N((\delta_p)) = \delta_p \delta_p \).
We write \( \delta_p = a + b\sqrt{-D} \). Then
\[ p | \delta_p \Rightarrow p | a, \text{ and } p | b. \]

Since
\[ (p)(\beta_p)(\beta_p) = p\mathfrak{p}(\beta_p)(\beta_p) = \mathcal{C}(\alpha_p)(\pi_p) = \mathcal{C}(\gamma_p)(\pi_p)C = \mathcal{C}(a^2 + b^2D)C, \tag{3.3} \]
and since for \( p \) sufficiently large we have that \( p \) does not divide numerator nor denominator of \( \mathcal{C}\mathfrak{C} C \), we can conclude that \( p^2 \) divides the right hand side of the equation (3.3), which implies that \( p^2 | p \). This is a contradiction.
Thus we have shown that for \( p \) sufficiently large, namely \( p > \max \{ \text{all numerators and denominators of } \mathfrak{C}_i(\mathcal{C}_i) \}, \]
we have \( a(p) \not\equiv 0 \pmod{p} \).

We have proved the following proposition.

**Proposition 3.2.1.** Suppose \( \Psi \) is a Hecke character of the imaginary quadratic field \( K \) satisfying

- \( \Psi \) takes values in \( K \)
- For any prime \( p \) of \( K \) we have \( \Psi(p) + \Psi(\overline{p}) \in \mathbb{Z} \).

Then for \( p \) large enough

\[
\Psi(p) + \Psi(\overline{p}) \not\equiv 0 \pmod{p}.
\]
Chapter 4

Proof of Theorem 1.1.2

4.1 A sieve lemma

Recall that

$$
\mu_f = e^{-\gamma/2}L(1, \chi_K)^{-\frac{1}{2}} \prod_{p \text{ inert}} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} \prod_{p \text{ splits in } K \atop a(p) = 0} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \leq z \atop p \nmid d_K \atop a(p) = 0} \left(1 - \frac{1}{p}\right) \prod_{p \leq z \atop p \nmid d_K \atop a(p) \neq 0} \left(1 - \frac{1}{p}\right)^\frac{1}{2} \prod_{p \leq z \atop p \nmid d_K \atop a(p) = 0} \left(1 - \frac{1}{p}\right)^{-\frac{1}{2}},
$$

defined in 2.1.1.

**Lemma 4.1.1.** Let \( f \) be as in the previous section, a normalized Hecke eigenform of weight \( \geq 2 \) with complex multiplication. Let \( y_1 = (\log \log x)^{1+\epsilon} \) and set

$$
N_{y_1}(x) = \#\{n \leq x : q|n \Rightarrow q \geq y_1, a(n) \neq 0, q \notin Z_f\}.
$$

Then

$$
N_{y_1}(x) = \frac{U_f x}{(\log x \log \log x)^2} + O\left(\frac{x(\log \log \log x)^2}{(\log x)^2}\right),
$$

where

$$
U_f = u_f \mu_f \prod_{p \leq y_1 \atop \nu_f(p) = 1} \left(1 - \frac{1}{p^2}\right) \prod_{p \leq y_1 \atop \nu_f(p) \geq 2} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \leq 1 \atop \nu_f(p) > 1} \left(1 - \frac{\xi_{p,d(1)}}{p}\right).
$$

Note that the last two products are over the finite number of primes.

**Proof.** Set \( P_{y_1} = \prod_{p < y_1} p \). By the principle of inclusion-exclusion, we have

$$
N_{y_1}(x) = \sum_{d|P_{y_1}} \mu(d)M_{f,d}(x), \tag{4.4}
$$

where \( \mu \) is the Möbius function:

$$
\mu(n) = \begin{cases} 
1, & \text{if } n \text{ is a square-free positive integer with an even number of prime factors}, \\
-1, & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors}, \\
0, & \text{if } n \text{ is not square-free}.
\end{cases}
$$
\[
\log P_{y_1} = \sum_{p < y_1} \log p < \pi(y_1) \cdot \log y_1 = (1 + o(1)) \frac{y_1}{\log y_1} \cdot \log y_1 = (1 + o(1)) y_1,
\]
and so \( P_{y_1} \leq e^{(1+o(1))y_1} \). Thus for any \( d \mid P_{y_1} \), we have \( \log x \ll \log \frac{y_1}{x} \ll \log x \).

Now using Proposition 2.2.18, the right hand side of \((4.4)\) is
\[
= \frac{u_f x}{\sqrt{\log x}} \sum_{d \mid P_{y_1}} \mu(d) \frac{\xi_d(1) + O \left( \frac{2^{\nu(d)}}{\log x} \right)}{d}.
\]

The main term is
\[
= \frac{u_f x}{\sqrt{\log x}} \sum_{d \mid P_{y_1}} \mu(d) \frac{\xi_d(1)}{d} = \frac{u_f x}{\sqrt{\log x}} \sum_{d \mid P_{y_1}} \prod_{p \mid d} \frac{(-1) \xi_{p,d}(1)}{p}
\]
\[
= \frac{u_f x}{\sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{\xi_{p,d}(1)}{p} \right)
\]
\[
= \frac{u_f x}{\sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{1}{p} \right) \prod_{p < y_1} \left( 1 - \frac{1}{p^2} \right) \prod_{p < y_1} \left( 1 - \frac{\xi_{p,d}(1)}{p} \right)
\]
\[
= \frac{u_f x}{\sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{1}{p} \right) \prod_{p < y_1} \left( 1 - \frac{1}{p^2} \right) \prod_{p < y_1} \left( 1 - \frac{\xi_{p,d}(1)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-1}
\]

Note that by Proposition 2.2.21 for \( p \nmid N, \ p \not\in \mathbb{Z} \) we have: if \( i_f(p) = 0 \), then \( \xi_{p,d}(1) = 1 \). Also note that by Lemma 2.2.19, there are only finitely many primes \( p \) for which \( i_f(p) > 1 \), ensuring the convergence of
\[
\prod_{i_f(p) > 1} \left( 1 - \frac{\xi_{p,d}(1)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-1}.
\]

Now using Proposition 2.1.1, we see that the above sum is
\[
\frac{U_f x}{\sqrt{L_3(x) \log x}} \left( 1 + O_f \left( \frac{1}{\log \log \log x} \right) \right).
\]

The error term in \((4.5)\) is
\[
\ll \frac{x}{(\log x)^{3/2}} \sum_{d \mid P_{y_1}} \frac{|\mu(d)|}{d} 2^{\nu(d)}.
\]

The sum over \( d \) is
\[
\ll \prod_{i < y_1} \left( 1 + \frac{2}{7} \right) \ll \prod_{i < y_1} \left( 1 - \frac{1}{7} \right)^{-2} \ll L_3(x)^2.
\]

This proves the result.
We record here a variant of the above result.

**Lemma 4.1.2.** Suppose that \( p \leq y_1 \). We have

\[
\# \{ n \leq x \mid p | n, a_f(n) \neq 0, q | n \Rightarrow q \geq p \} \\
\ll \frac{x}{p(\log x)^2} \prod_{l < p} \left( 1 - \frac{1}{l} \right) + \frac{x}{(\log x)^2} e^{A \sqrt{\log p}} \frac{\log p}{p}. 
\]

### 4.2 Siegel zeros

Let \( L/\mathbb{Q} \) be a Galois extension of number fields with group \( G \) and \( n_L, d_L \) be the degree and the absolute value of the discriminant of \( L/\mathbb{Q} \) respectively. Suppose that Artin’s conjecture on the holomorphy of Artin \( L \)-functions is known for \( L/\mathbb{Q} \). Set

\[
\log M = 2 \left( \sum_{p | d_L} \log p + \log n_L \right). 
\]

Also, denote by \( d \) the maximum degree and by \( A \) the maximum Artin conductor of an irreducible character \( G \).

Let \( C \) be the set of elements in \( G \) that map to the Cartan subgroup and also have trace zero. Then \( C \) is stable under conjugation and thus \( C \) is a union of conjugacy classes. Denote by \( \pi(x, C) \) the number of primes \( p \leq x \) with \( \text{Frob}_p \in C \). Then, \cite{11}, Theorem 4.1 asserts that for

\[
\log x \gg d^4(\log M), 
\]

there is an absolute and effective constant \( c > 0 \) so that

\[
\pi(x, C) = \left\lfloor \frac{|C|}{|G|} \log x - \left\lfloor \frac{|C|}{|G|} \log x \right\rfloor \right\rfloor + O \left( \left\lfloor \frac{|C|}{|G|} \log x \right\rfloor^2 x \exp \left\{ -c \frac{\log x}{d^{3/2} \sqrt{d^4(\log A)^2 + \log x}} \right\} \right), 
\]

where

\[
\text{Li } x = \int_2^x \frac{dt}{\log t}. 
\]

The term involving \( \beta \) is present only if the Dedekind zeta function \( \zeta_L(s) \) of \( L \) has a real zero \( \beta \) (the Siegel zero), in the strip

\[
1 - \frac{1}{4 \log d_L} \leq \Re(s) < 1. 
\]

For any prime \( p \), let \( \mathbb{Z}_p \) denote the ring of \( p \)-adic integers. By Eichler-Schimura-Deligne and since the Fourier coefficients of \( f \) are in \( \mathbb{Z} \), there is a continuous representation

\[
\rho_{p,f} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{Z}_p). 
\]

This representation is unramified outside the primes dividing \( N_p \). This means that for any prime \( q \) that does not divide \( N_p \) and for any prime \( q \) of \( \overline{\mathbb{Q}} \) over \( q \), \( \rho_{p,f}(\text{Frob}_q) \) makes sense. We note that while \( \rho_{p,f}(\text{Frob}_q) \) does depend on the choice of \( q \) over \( q \), its characteristic polynomial depends only on the conjugacy class of \( \rho_{p,f}(\text{Frob}_q) \) (hence only on \( q \)) and is given by

\[
T^2 - a(q)T + q. 
\]
We consider the reduction of the above representation modulo $p$

$$\overline{\rho}_{p,f} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_p).$$

The fixed field of the kernel of this representation determines a number field $L$ that is a Galois extension of $\mathbb{Q}$ with group the image of $\overline{\rho}_{p,f}$. Now, let $G = \text{Gal}(L/\mathbb{Q})$ (viewed as a subgroup of $GL_2(\mathbb{Z}/p)$) and let $C$ be the subset of elements of $G$ of trace zero. It is known that the subgroup $H = \text{Gal}(L/K)$ is Abelian and maps to a Cartan subgroup of $GL_2(\mathbb{Z}/p)$. The image of $G$ maps to the normalizer of this group. As $G$ has an Abelian normal subgroup of index 2, it is well-known that all irreducible characters of $G$ are monomial, and so Artin’s holomorphy conjecture holds for it.

Thus we can appeal to the above version of the Chebotarev density theorem. The extension $L/K$ is unramified outside of primes dividing $pN$, where $N$ is the level of $f$. We have $d = 2$, and

$$\log M \ll \log pN$$

as well as

$$\log A \ll \log pN.$$

For $p$ sufficiently large, it is known that $G$ maps onto the normalizer of a Cartan subgroup, and hence

$$p^2 \ll |G| \ll p^2.$$ 

Moreover, the size of $|C|$ satisfies

$$p \ll |C| \ll p.$$ 

Thus, if we set $\delta(p) = |C|/|G|$, we have for $p$ sufficiently large

$$\frac{1}{p} \ll \delta(p) \ll \frac{1}{p}.$$ 

Thus, we have the following result.

Let $\pi^*(x, p) := \# \{q \leq x | a(q) \equiv 0 \pmod{p}, a(q) \neq 0\}$. Then $\pi^*(x, p) \leq \pi(x, p)$.

**Lemma 4.2.1.** (Lemma 3.2 from [10], or Theorem 6.1 from [6]): Let $f$ be a CM-form of level $N$ as before. If $\log x \gg (\log pN)^2$, then we have

$$\pi^*(x, p) = \delta(p) Li x - \delta(p) Li x^{\beta} + O(x \cdot e^{c\sqrt{\log x}}),$$

where $\frac{1}{p} \ll \delta(p) \ll \frac{1}{p}$ and the implied constant is absolute and effective.

From the discussion above, we know that the stated bounds on $\delta(p)$ hold for $p$ sufficiently large. To deduce that they hold for all $p$, it suffices to show that $\delta(p) > 0$ holds for all $p$. This inequality follows from the fact the image of complex conjugation is an element of trace zero in the Galois group.

If the Dedekind zeta function $\zeta_K(s)$ of $K$ has a Siegel zero $\beta$ in the interval $1 - \frac{1}{4 \log d_K} \leq \Re(s) \leq 1$, then by a result of Heilbronn in ([2]) we know that there is a quadratic field $M$ contained in $L$ such that $\zeta_M(\beta) = 0$. From [21], for such $M$

$$\beta \leq 1 - \frac{1}{\sqrt{d_M}}.$$ 

Let $[L : M] = n$. Since $d_L \geq d_M^n$, we have

$$\beta \leq 1 - \frac{1}{d_L^{1/2n}}.$$ 

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Now, by inequality of Hensel [19], p. 129,
\[ \log d_L \leq 2n \log pn_L \]
and so
\[ \frac{1}{2n} \log d_L \leq \log pn_L. \]
Hence
\[ \beta < 1 - \frac{1}{pn_L}. \] (4.7)

4.3 Intermediate lemmas

Let \( 0 < \varepsilon < 1/2 \), and set \( y = L_2^{1-\varepsilon}(x) \).

Lemma 4.3.1. Let \( p < y \) be a fixed prime. Then we have
\[ \sum_{q \leq x}^* \frac{1}{q} = \delta(p)L_2(x) + O(L_3(x)), \]
where \( \sum_{q \leq x}^* \) means that the summation is over all primes \( q \leq x \) for which \( a(q) \neq 0 \), and \( \delta(p) \) is from Lemma 4.2.1.

Proof. By partial summation the sum is
\[ \sum_{q \leq x}^* \frac{1}{q} = \int_1^x \frac{1}{t} d\pi^*(t,p) = \frac{\pi^*(x,p)}{x} + \int_1^x \frac{\pi^*(t,p)}{t^2} dt. \]
The integral \( \int_1^x \frac{\pi^*(t,p)}{t^2} dt \) can be written as a sum
\[ \int_1^{(\log x)\gamma} \pi^*(t,p) \frac{dt}{t^2} + \int_{(\log x)\gamma}^x \pi^*(t,p) \frac{dt}{t^2}, \]
where \( \gamma \) is chosen in such a way that for \( (\log x)^\gamma \leq t \leq x \), we have \( \log t \gg (\log pN)^2 \). The first integral is
\[ \leq \int_1^{(\log x)\gamma} \frac{\pi(t)}{t^2} dt \ll L_3(x), \]
where \( \pi(t) = \# \{ p \leq t \mid p \text{ prime} \} \)
and the second integral is
\[ \int_{(\log x)\gamma}^x \frac{1}{t^2} \left( \delta(p)\mathrm{Li}(t) - \delta(p)\mathrm{Li}(t^\beta) + O(te^{-c\sqrt{\log t}}) \right) dt, \]
by Lemma 4.2.1.

The first term is equal to
\[ \delta(p) \int_{(\log x)\gamma}^x \frac{dt}{t \log t} + O(L_3(x)) = \delta(p)L_2(x) + O(L_3(x)). \]
Next, consider the term with the Siegel zero. Since by (4.7), \( \beta < 1 - \frac{1}{p\ln L} \), therefore the second term is

\[
\delta(p) \int_{(\log x)^{\gamma}}^{x} \frac{1}{t^2} \text{Li}(t) \, dt = \delta(p) \int_{(\log x)^{\gamma}}^{x} \frac{1}{t^2} \int_{2}^{t^\beta} \frac{du}{\log u} \, dt \\
= \delta(p) \int_{2}^{x^\beta} \frac{1}{\log u} \int_{\max\{ (\log x)^{\gamma}, u^\frac{1}{\beta} \}}^{x} dt \, du.
\]

We split the range of integration of \( u \) into two integrals:

\[
(I) \quad 2 \leq u \leq (\log x)^{\gamma\beta}, \\
(II) \quad (\log x)^{\gamma\beta} \leq u \leq x^\beta.
\]

The first range gives rise to the integral

\[
\delta(p) \int_{2}^{(\log x)^{\gamma\beta}} \frac{1}{\log u} \left\{ \frac{1}{(\log x)^{\gamma}} - \frac{1}{x} \right\} du \ll \delta(p)(\log x)^{(\beta - 1)} \ll 1.
\]

The second range gives rise to the integral

\[
\delta(p) \int_{(\log x)^{\gamma\beta}}^{x^\beta} \frac{1}{\log u} \left\{ \frac{1}{u^\beta} - \frac{1}{x} \right\} du.
\]

Set \( v = u^\frac{1}{\beta} \). Then \( v^\beta = u \) and \( \beta \log v = \log u \). Moreover, \( du = \beta v^{\beta - 1} \, dv \). Hence the integral is

\[
\delta(p) \int_{(\log x)^{\gamma\beta}}^{x^\beta} \frac{1}{\beta \log v} \left\{ \frac{1}{v} - \frac{1}{x} \right\} dv \ll \frac{\delta(p)}{(\log x)^{(1 - \beta)}} \int_{(\log x)^{\gamma}}^{x} \frac{dv}{v \log v} \\
\ll \frac{\delta(p)L_2(x)}{(\log x)^{\frac{1}{\beta} - \gamma}} \ll 1.
\]

Finally, using the elementary estimate \( e^{c\sqrt{u}} \gg u^2 \), we deduce that the O-term is

\[
\ll \int_{L_2(x)}^{\log x} \frac{du}{u^2} \ll 1.
\]

The term \( \frac{\pi^*(x, p)}{x} \) is of smaller order. This proves the lemma.

Define \( \nu(p, n) = \# \{ q^m | n \mid a(q^m) \equiv 0 \pmod{p} \} \), i.e. \( \nu(p, n) \) counts the number of primes \( q \) that divide \( n \) and \( a(q^m) \equiv 0 \pmod{p} \), where \( m \) is defined by \( q^m | n \). Note that for these primes \( q \) we have

\[
a(q^m) \equiv 0 \pmod{p} \Rightarrow a(n) \equiv 0 \pmod{p},
\]

and so \( \sum_{p|n} \nu(p, n) = 0 \) means that \( (n, a(n)) = 1 \).
Lemma 4.3.2. Assume that $p < y$. Then we have

$$
\sum_{n \leq x}^* \nu(p, n) = (1 + o(1)) \left( \frac{u_f \delta(p) x L_2(x)}{\sqrt{\log x}} \right),
$$

where $\sum_{n \leq x}^*$ means that the summation is over all natural numbers $n \leq x$ such that $a(n) \neq 0$.

Proof. Interchanging summation, we see that

$$
\sum_{n \leq x}^* \nu(p, n) = \sum_{q^m \leq x}^* \sum_{n \leq x}^* 1_{a(q^m) \equiv 0( \text{ mod } p)} 1_{q^m | n}
\begin{align*}
&= \sum_{q \leq x}^* \sum_{n \leq x}^* 1_{a(q) \equiv 0( \text{ mod } p)} 1_{q^m | n} + \sum_{q^m \leq x, m \geq 2}^* \sum_{n \leq x}^* 1_{a(q^m) \equiv 0( \text{ mod } p)} 1_{q^m | n}.
\end{align*}
$$

The contribution of terms $q^m$ with $m \geq 2$ is

$$
\begin{align*}
\sum_{q^m \leq x, m \geq 2}^* \sum_{n \leq x}^* 1_{a(q^m) \equiv 0( \text{ mod } p)} 1_{q^m | n} &\ll \sum_{q^m \leq x}^* \sum_{n \leq x}^* 1_{a(q^m) \equiv 0( \text{ mod } p)} 1_{q^m | n} \\
&= \int_{x}^{x^e} \frac{x}{\sqrt{\log(x/q^m)}} + \sum_{m \geq 2}^* \frac{x}{q^m}
\end{align*}
$$

by Proposition 2.2.18

$$
\ll \frac{x}{\sqrt{\log x}} \sum_{q^m \leq x}^* \frac{1}{q^m} + x \int_{x}^{x^e} \frac{dt}{t^2}
\ll \frac{x}{\sqrt{\log x}} + \frac{x}{x^e} \ll \frac{x}{\sqrt{\log x}}.
$$

Also, we have

$$
\sum_{q \leq x}^* \sum_{n \leq x}^* 1_{a(q) \equiv 0( \text{ mod } p)} 1_{q | n} = \sum_{q \leq x}^* \sum_{n \leq x}^* 1_{a(q) \equiv 0( \text{ mod } p)} 1_{q | n} + \sum_{x^{1/\log \log x} \leq q \leq x}^* \sum_{n \leq x}^* 1_{a(q) \equiv 0( \text{ mod } p)} 1_{q | n}.
$$

We show that the second double sum on the right of (4.8) contributes a negligible amount. For this we split
it into two parts.

\[ \sum_{x^{1/2 \log \log x} \leq q \leq x} \sum_{a(q) \equiv 0 \pmod{p}} n \mid n = \sum_{x^{1/2 \log \log x} \leq q \leq x} \sum_{a(q) \equiv 0 \pmod{p}} \sum_{n \leq x} 1 + \sum_{x^{1/2 \log \log x} \leq q \leq x} \sum_{a(q) \equiv 0 \pmod{p}} \sum_{n \leq x} 1 \]

Consider first the sum:

\[ \sum_{x \leq q \leq x} \sum_{n \leq x q \mid n} 1. \quad (4.9) \]

This is majorized by

\[ \sum_{n \leq x} \sum_{x^{1/2 \log \log x} \leq q \leq x} 1. \]

Note that the inner sum is bounded because we can have at most \( l = \left\lfloor \frac{1}{\epsilon} \right\rfloor \) distinct primes \( q \) such that \( q \mid n \) and \( q \geq x^{1/2 \log \log x} \). Indeed, assume \( q_1, \ldots, q_l \) are distinct primes that divide \( n \). Then

\[ x^{le} \leq q_1 \cdots q_l \leq x \]

\[ le \leq 1 \]

\[ l \leq \frac{1}{\epsilon} \]

Since the inner sum is bounded by \( \frac{1}{\epsilon} \), and by Proposition 2.2.18 we have

\[ \sum_{n \leq x} \sum_{x^{1/2 \log \log x} \leq q \leq x} 1 \leq \frac{1}{\epsilon} \cdot \sum_{n \leq x} 1 = \frac{1}{\epsilon} \cdot Mf,1(x) \ll \frac{x}{\sqrt{\log x}} \]

and so we see that (4.9) is

\[ \ll \frac{x}{\sqrt{\log x}} \quad (4.10) \]

Now, consider the quantity

\[ \sum_{x^{1/2 \log \log x} \leq q \leq x} \sum_{a(q) \equiv 0 \pmod{p}} \sum_{n \leq x} 1. \quad (4.11) \]

By Proposition 2.2.18, the inner sum is

\[ \sum_{n \leq x} \sum_{a(q) \equiv 0 \pmod{p}} 1 \leq \sum_{n \leq x} 1 = Mf,q(x) \ll \frac{x}{q \sqrt{\log x}} \]

Since

\[ \sum_{x^{1/2 \log \log x} \leq q \leq x} \frac{1}{q} = \int_{x^{1/2 \log \log x}}^{x} \frac{1}{u} d(\pi(u)) = \pi(u) \bigg|_{x^{1/2 \log \log x}}^{-u} + \int_{x^{1/2 \log \log x}}^{x} \frac{\pi(u)}{u^2} du = \]

\[ = \frac{\pi(u)}{u} \bigg|_{x^{1/2 \log \log x}}^{x} + \int_{x^{1/2 \log \log x}}^{x} \frac{1}{u \log u} du + O \left( \int_{x^{1/2 \log \log x}}^{x} \frac{1}{u \log u^2} du \right) = \]

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\[= \frac{\pi(u) x^\epsilon}{u x^{1/\log \log x}} + \log \log u \left( x^\epsilon x^{1/\log \log x} + O\left( \frac{1}{\log u} x^{1/\log \log x} \right) \right) = \]
\[= \log \log x^\epsilon - \log \log x^{1/\log \log x} + O(1) = \log(\epsilon \log x) - \log \left( \frac{1}{\log \log x} \log x \right) + O(1) = \]
\[= \log \epsilon + \log \log x - \log \log x + \log \log \log x + O(1) = \log \log \log x + O(1), \]

it follows that (4.11) is
\[\ll \frac{x L_3(x)}{\sqrt{\log x}}. \quad (4.12)\]

Putting (4.10) and (4.12) together, we deduce that
\[\sum_{q \leq x} \sum^*_{n \leq x} 1 = \sum_{q \leq x^{1/\log \log x}} \sum^*_{n \leq x} 1 + O\left( \frac{x L_3(x)}{\sqrt{\log x}} \right). \]
Now by Proposition 2.2.18, Lemma 2.2.17 (and the fact that in the sum \(a^0(q) = 1\)),

\[
\sum_{q \leq x^{1/\log x}} \sum_{n \leq x} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right) \leq \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right)
\]

By Proposition 2.2.18

\[
\sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right) = \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right)
\]

By Lemma 2.2.17

\[
\sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right) = (1 + o(1)) \frac{u_f x}{\sqrt{\log x}} \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right)
\]

Now applying Lemma 4.3.1, we see that this is

\[
= (1 + o(1)) \frac{u_f x}{\sqrt{\log x}} \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \left( \sum_{q \leq x^{1/\log x}} \frac{1}{a(q) \equiv 0 \pmod{p}} \right)
\]

This proves the lemma. \(\square\)
Lemma 4.3.3. Assume that \( p < y \). Then

\[
\sum_{n \leq x} \nu(p, n)^2 = (1 + o(1)) \frac{u_f S^2(p) x L^2(x)}{\sqrt{\log x}}.
\]

Proof. The sum in question is equal to

\[
\sum_{q_{m_1}^m \leq x} \sum_{a(q_{m_1}^m) \equiv 0 \pmod{p}} \sum_{q_{m_2}^n \leq x} \sum_{a(q_{m_2}^n) \equiv 0 \pmod{p}} \sum_{n \leq x} 1.
\]

By a small modification to the argument given in the proof of Lemma 4.3.2, we find that the contribution of terms with \( q_{m_1} = q_{m_2} \) is

\[
\ll x L_2(x) \sqrt{\log x}.
\]

Next, we consider the contribution \( S \) (say) of terms with \( q_{m_1}^m q_{m_2}^n > x^\varepsilon \). For estimating this, we may suppose that \( q_{m_1}^m > q_{m_2}^n \). Since \( q_2 > 2 \), we may suppose that \( x/2 \geq q_{m_1}^m \geq x^{\varepsilon/2} = z \) (say). Denote by \( S_1 \) the contribution of terms for which \( z \leq q_{m_1}^m \leq x^{\varepsilon} \) and by \( S_2 \) the contribution of all the remaining terms in \( S \).

Then by Proposition 2.2.18, we have

\[
S_1 \ll x \sum_{z \leq q_{m_1}^m \leq x^{\varepsilon}} 1 \sum_{q_{m_2}^n \leq q_{m_1}^m} q_{m_2}^n \sqrt{\frac{\log x}{q_{m_1}^m q_{m_2}^n}} \log \log(q_{m_1}^m)
\]

\[
\ll x L_2(x) \int_z^{x^{\varepsilon}} \frac{dt}{t \log t} \sqrt{\frac{x}{t}} \ll x L_2(x) \sqrt{\log x}.
\]

Next, we observe that

\[
S_2 \ll \sum_{\sqrt{x} < q_{m_1}^m \leq x^{\varepsilon}} 1 \sum_{n \leq x} \nu(p, n)
\]

and by Lemma 4.3.2, this is

\[
\ll x L_2(x) \sum_{\sqrt{x} < q_{m_1}^m \leq x^{\varepsilon}} 1 \frac{1}{\sqrt{\log x/q_{m_1}^m}} \ll x L_2(x) \sqrt{\log x}.
\]

It remains to estimate

\[
\sum_{q_{m_1}^m q_{m_2}^n \leq x^\varepsilon} \sum_{a(q_{m_1}^m) \equiv 0 \pmod{p}} \sum_{q_{m_1}^m || n, q_{m_2}^n || n} 1
\]

\[
= I + J, \text{ say}
\]

where in \( I \) we have the terms with \( m_1 > 1 \) or \( m_2 > 1 \) and in \( J \) we have the terms with \( m_1 = m_2 = 1 \). In
order to estimate $I$, suppose without loss of generality that $m_1 \geq 2$. Then by Proposition 2.2.18, we have

\[
I \ll x \sum_{q_1 \geq m_1} \frac{1}{q_1^{m_1}} \sum_{q_2 \leq x} \frac{1}{q_2^{m_2}} \sum_{q_1 \mid q_2} \frac{1}{q_1 \log q_1^{1 + q_2^{2}}}
\]

\[
\ll \frac{x}{\log x} \sum_{q_1 \geq m_1} \frac{1}{q_1^{m_1}} \left( \sum_{q_2 \leq x} \frac{1}{q_2} + \sum_{m_2 \geq 2} \frac{1}{q_2^{m_2}} \right)
\]

\[
\ll \frac{xL_2(x)}{\log x}.
\]

Next, we consider

\[
J = \sum_{q_1 \leq x} \sum_{n \leq x} 1.
\]

By Proposition 2.2.18 and Proposition 2.2.21, we have

\[
J = (1 + o(1)) \frac{ufx}{\log x} \sum_{q_1 \leq x} \frac{1}{q_1 q_2} + O \left( \frac{xL_2(x)}{\log x} \right)
\]

\[
= (1 + o(1)) \frac{ufx}{\log x} \left( \sum_{q \leq x} \frac{1}{q} \right)^2 + O \left( \frac{xL_2(x)}{\log x} \right)
\]

\[
= (1 + o(1)) \frac{ufx}{\log x} \left( \frac{\delta(p)L_2(x) + O(L_3(x))}{\log x} \right)^2 + O \left( \frac{xL_2(x)L_3(x)}{\log x} \right)
\]

\[
= (1 + o(1)) \frac{ufx \delta^2(p)xL_2(x)}{\log x} + O \left( \frac{\delta(p)xL_2(x)L_3(x)}{\log x} \right).
\]

This proves the lemma. \(\square\)

**Lemma 4.3.4.** Suppose $p < y$. Then

\[
\sum_{n \leq x} (\nu(p, n) - \delta(p)L_2(x))^2 \ll \frac{\delta(p)x}{\log x} L_2(x)L_3(x).
\]

**Proof.** This follows from Lemma 4.3.2 and Lemma 4.3.3. \(\square\)

**Lemma 4.3.5.** Assume $p < y$. Then

\[
\# \{n \leq x | \nu(p, n) = 0\} \ll \frac{xL_3(x)}{\delta(p)\sqrt{\log xL_2(x)}}.
\]
Proof. By Lemma 4.3.4 this is
\[
\ll \frac{1}{\delta^2(p)L_2(x)} \left\{ \delta(p) \frac{x}{\sqrt{\log x}} L_2(x) L_3(x) \right\} = \frac{xL_3(x)}{\delta(p)\sqrt{\log x} L_2(x)}.
\]

\[\square\]

4.4 Proof of Theorem 1.1.2

Break up the set \( \{ n \leq x \mid (n, a(n)) = 1 \} \) into the sets \( \{ n \leq x, p \mid n, (n, a(n)) = 1, q \mid n \Rightarrow q \geq p \} \) for \( p \)-prime, i.e. for each prime \( p \) put together all \( n \)'s such that \( p \) is their smallest prime divisor.

Denote by \( G_p(x) := \# \{ n \leq x, p \mid n, (n, a(n)) = 1, q \mid n \Rightarrow q \geq p, q \not\in \mathbb{Z}_f \} \).

Note that for \( p \in \mathbb{Z}_f \) we have \( G_p(x) = 0 \), because \( p \mid a(p) \Rightarrow p \mid a(p^m) \), so we get \( p \mid n, p \mid a(n) \), which means that \( (n, a(n)) \neq 1 \). In fact, a stronger statement holds: \( (n, a(n)) = 1 \) implies that if \( p \mid n \), then \( p \not\in \mathbb{Z}_f \).

Then
\[
\# \{ n \leq x \mid (n, a(n)) = 1 \} = \sum_{p \leq x} G_p(x) = A_1(x) + A_2(x) + A_3(x),
\]

where
\[
A_1(x) = \sum_{p \leq (\log \log x)^{1/2 - \varepsilon}} G_p(x), \quad A_2(x) = \sum_{(\log \log x)^{1/2 - \varepsilon} < p < (\log \log x)^{1+\varepsilon}} G_p(x),
\]
\[
A_3(x) = \sum_{p \geq (\log \log x)^{1+\varepsilon}} G_p(x).
\]

Estimating \( A_1(x) \), \( A_2(x) \) and \( A_3(x) \) requires the intermediate lemmas from Section 4.3. Recall that in Section 4.3 we assumed \( 0 < \varepsilon < 1/2 \) and set \( y = L_2^{1-\varepsilon}(x) \).

Now, using Lemma 4.3.5, we have
\[
A_1(x) \leq \sum_{p \leq L_2^{1/2 - \varepsilon}(x)} \# \{ n \leq x \mid p \mid n, (n, a(n)) = 1 \}
\]
\[
= \sum_{p \leq L_2^{1/2 - \varepsilon}(x)} \# \{ n \leq x \mid p \mid n, \sum_{p_1 \mid n} \nu(p_1, n) = 0 \}
\]
\[
= \sum_{p \leq L_2^{1/2 - \varepsilon}(x)} \# \{ n \leq x \mid p \mid n, \nu(p_1, n) = 0 \text{ for all } p_1 \mid n \}
\]

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\[ \sum_{p \leq L_2^{1-\epsilon}(x)} \# \{n \leq x \mid \nu(p, n) = 0 \} \]

by Lemma 4.3.5

\[ \ll \frac{xL_3(x)}{(\log x)^{\frac{1}{2}} L_2(x)} \sum_{p \leq L_2^{1-\epsilon}(x)} \frac{1}{\delta(p)} \]

\[ \ll \sum_{1 \leq p \leq L_2^{1-\epsilon}(x)} p, \text{ as } \delta(p) \gg \frac{1}{p} \]

\[ \ll \frac{x}{(\log x)^{\frac{1}{2}} L_2'(x)} = o \left( \frac{x}{(L_3(x) \log x)^{\frac{1}{2}}} \right). \]

Moreover, by Lemma 4.1.2 we have

\[ A_2(x) \leq \sum_{L_2^{1-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \# \{n \leq x \mid p|n, a(n) \neq 0, q|n \Rightarrow q \geq p \} \]

by Lemma 4.1.2

\[ \ll \frac{x}{\sqrt{\log x}} \sum_{L_2^{1-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \frac{1}{p} \prod_{l \leq p \text{ prime}} \left( 1 - \frac{1}{l} \right) \]

\[ \ll \frac{x}{\sqrt{\log x}} \cdot \frac{1}{\log(L_2^{1-\epsilon}(x))} \cdot \sum_{L_2^{1-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \frac{1}{p} \]

\[ \ll \frac{x}{\sqrt{\log x}} \cdot \frac{1}{L_3(x)} \cdot L_4(x) = o \left( \frac{x}{(L_3(x) \log x)^{\frac{1}{2}}} \right). \]

Recall the notation introduced in Lemma 4.1.1:

\[ y_1 = L_2(x)^{1+\epsilon} \text{ and } N_{y_1}(x) = \# \{n \leq x \mid q|n \Rightarrow q \geq y_1, a(n) \neq 0, q \nmid a(q) \}. \]

Then

\[ N_{y_1}(x) - \sum_{y_1 \leq q_1, q_2 \leq x} \sum_{a(q_1^n) \equiv 0 \pmod{q_2}} \sum_{q_2 | n, q_1^n \nmid q_1 a(q)} 1 \leq A_3(x) \leq N_{y_1}(x), \]

where \( \sum_{n \leq x} \) means that the summation is over all natural numbers \( n \leq x \) such that \( a(n) \neq 0 \) and \( q|n \) implies that \( q > y_1 \).

Indeed, the inequality \( A_3(x) \leq N_{y_1}(x) \) is obvious, since \( N_{y_1}(x) \) differs from \( A_3(x) \) by having no requirement
that \((a(n), n) = 1\). Now, denote

\[
M_{y_1}(x) = \#\{ (q_1, q_2, n) \mid q_1, q_2 \in [y_1, x], n \leq x, q_1^n \| n, q_2|n, a(n) \neq 0, a(q_1^n) \equiv 0 \pmod{q_2} \} = \sum^* \sum^{**} 1.
\]

If we denote by \(A_3, N_{y_1}\), and \(M_{y_1}\) the sets that correspond to the numbers \(A_3(x), N_{y_1}(x)\) and \(M_{y_1}(x)\), then

\[
M_{y_1} \supseteq \{ n \leq x \mid q|n, \Rightarrow q \geq y_1, a(n) \neq 0, (a(n), n) \neq 1 \} = N_{y_1} \setminus A_3,
\]

\[
N_{y_1} \setminus A_3 \subseteq M_{y_1},
\]

and so (going back to numbers of the elements in the sets) we have

\[
N_{y_1}(x) - M_{y_1}(x) \leq N_{y_1}(x) - (N_{y_1}(x) - A_3(x)) = A_3(x).
\]

By Lemma 4.1.1, to prove the theorem, it suffices to show that

\[
\sum^* \sum^{**} 1 = o \left( \frac{x}{(L_3(x) \log x)^{\frac{3}{2}}} \right),
\]

(4.13)

Note that \(q|n \Rightarrow q \not\mid a(q)\) implies that \(q_1 \neq q_2\).

In order to prove (4.13), let us write

\[
\sum^* \sum^{**} 1 = \sum^* \sum^{**} 1 =: B_1 + B_2.
\]

Let us consider \(B_1\) first.

\[
B_1 = \sum^* \sum^{**} 1 = \sum^* \sum^{**} 1 =: B_{11} + B_{12}
\]

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$$B_{11} = \sum_{y_1 \leq q_1, q_2 \leq x \atop a(q_1^m) \equiv 0 \pmod{q_2}} \frac{1}{q_1^m q_2} \prod_{p < q_1} \left( 1 - \frac{1}{p} \right) \ll \sum_{y_1 \leq q_1, q_2 \leq x \atop q_1 \neq q_2, m \geq 2} \frac{x}{q_1^m q_2}.$$
\[
\leq \frac{x}{L_3(x)} \sum_{y_1 \leq q_1 \leq x} \frac{1}{q_1^{m}} \cdot \frac{q_1^{m}}{(\log x)\sqrt{xy_1^2}} \cdot \log x \leq \frac{\sqrt{x}}{L_3(x)} \cdot \frac{1}{y_1^2} \cdot \sqrt{x} = \frac{x}{(\log x) L_3(x) y_1^2} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).
\]

Thus,
\[
B_{111} \leq B_{1111} + B_{1112} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).
\]

\[
B_{112} = \frac{x}{L_3(x)} \sum_{y_1 \leq q_1 \leq x} \frac{1}{q_1^{m}} q_2 \leq \frac{x}{L_3(x)} \sum_{y_1 \leq q_1 \leq x} \frac{1}{q_1^{m}} \sum_{q_2 > \sqrt{x}} \frac{1}{q_2} \ll
\]

\[
\leq \frac{x}{L_3(x)} \sum_{y_1 \leq q_1 \leq x} \frac{1}{q_1^{m}} \cdot \frac{L_1(x)}{\sqrt{y_1^2}} \ll \frac{\sqrt{L_1(x)}}{y_1 L_3(x)} \ll o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).
\]

Here we estimated the number of distinct \( q_2 > \sqrt{x} \) such that \( q_2 | a(q_1^m) \) as follows:
\[
(\sqrt{x})^l < q_2 \cdot q_2 \ldots < q_2 | \leq \sum_{m \geq 2} \frac{m(k-1)}{2} \leq (m+1)q_1 \leq x \implies L_1(x) \ll L_3(x),
\]

since \( y_1^m \leq q_1^m \leq x \) implies \( m \leq \frac{\log x}{\log y_1} \ll L_1(x) \ll L_3(x) \).

Thus,
\[
B_{11} \ll B_{111} + B_{112} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).
\]

For the remaining terms sum \( B_{12} \) we use Proposition 2.2.18:
\[
B_{12} \leq \sum_{y_1 \leq q_1 \leq x} \sum_{q_2 \leq (\log x)\sqrt{y_1^2}} \frac{1}{q_1^m q_2} \cdot |a(q_1^m q_2)| \cdot \sum_{m \geq 2} \frac{1}{q_2} \ll
\]

\[
\leq \sum_{y_1 \leq q_1 \leq x} \sum_{q_2 \leq (\log x)\sqrt{y_1^2}} \frac{1}{q_1^m q_2} \ll
\]

\[
\ll \frac{x}{(\log x)^2} \sum_{y_1 \leq q_2 \leq x} \frac{1}{q_2} \sum_{m \geq 2} \frac{1}{q_1^m} \ll
\]

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\[
\frac{x}{y_1 (\log x)^2} \sum_{y_1 \leq y_2 \leq x} \frac{1}{q_2} \ll
\]
\[
\ll \frac{x L_2(x)}{y_1 (\log x)^2} = \frac{x}{(\log x)^2 L_2(x)}.
\]

\[
B_2 = \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 =
\]
\[
= \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 + \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 =
\]
\[
=: D_1 + D_2.
\]

Then by Proposition 2.2.18, and because \(q_1 q_2 \leq 2q_1 \frac{k+1}{k-1} \ll x^{\frac{k+1}{k-1}} = x \frac{k+1}{k-1} \), we have

\[
D_1 \leq \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 \ll \frac{u_f \psi_{q_1 q_2}(1)}{q_1 q_2 (\log x)^2} \ll
\]
\[
= \frac{1}{q_1 q_2} \ll \frac{u_f \psi_{q_1 q_2}(1) \sqrt{k + 2}}{q_1 q_2 (\log x)^2} \ll
\]
\[
\ll \frac{x}{(\log x)^2} \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 =
\]
\[
= \frac{x}{(\log x)^2} \left\{ \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 + \sum_{y_1 \leq q_1 \leq \frac{x}{2q_2} \left( \log \log \frac{\log q_1}{q_1} \right) \leq \frac{x}{2q_2} \sum_{y_2 = 0 (\mod q_2)} 1 \right\} =: \frac{x}{(\log x)^2} (Q_1 + Q_2).
\]

We changed the order of summation in \(Q_1\) as follows. \(q_1 \geq \left( \frac{q_2}{2} \right)^{\frac{1}{k+1}} \geq \left( \frac{y_1}{2} \right)^{\frac{1}{k+1}} \). Also, \(q_1 \leq q_2 \log q_2 \) implies \(q_2 \geq \sqrt{\frac{q_1}{\log q_1}} \). Indeed, if we assume \(q_2 < \sqrt{\frac{q_1}{\log q_1}}\), then

\[
q_1 \leq q_2 \log q_2 < \frac{q_1}{\log q_1} \cdot \log \sqrt{\frac{q_1}{\log q_1}} = \frac{1}{2} q_1 \log q_1 \log \log q_1 < q_1.
\]
Contradiction. Thus, \( q_2 \geq \sqrt{\frac{d_1}{\log q_1}} \).

The sum \( Q_1 \) is then majorized by

\[
\ll \sum_{(\frac{d_1}{q_1})^{1/2} \leq q_1 \leq x} \frac{1}{q_1} \sum_{\frac{q_1}{a(q_1) \equiv 0 (mod q_2)}} 1.
\]

We note that the inner sum over \( q_2 \) is bounded. In fact, with \( 0 < |a(q_1)| \leq 2q_1^{1/2} \), there exists at most \( 2k + 2 \) primes \( q_2 \geq \sqrt{\frac{q_1}{\log q_1}} \) which divide \( a(q_1) \). Indeed, assume that there are \( s \) such primes \( q_2_1, q_2_2, \ldots, q_2_s \)


Thus, the right hand side is

\[
\ll \sum_{\frac{q_1}{a(q_1) \equiv 0 (mod q_2)}} 1 \ll \sqrt{\log q_1} \sum_{\frac{q_1}{a(q_1) \equiv 0 (mod q_2)}} 1 \ll \frac{1}{(L_2(x))^{\frac{1}{2}}},
\]

because

\[
\sum_{(\frac{d_1}{q_1})^{1/2} \leq q_1 \leq x} \frac{\sqrt{\log q_1}}{q_1^{3/2}} \text{ is convergent.}
\]

And so,

\[
Q_1 \ll \frac{1}{(L_2(x))^{\frac{1}{2}}} = o\left( \frac{1}{\left(\log \log \log x\right)^{\frac{1}{2}}} \right).
\]

Let us estimate \( Q_2 \). For Lemma 4.2.1 to be applicable we need \( x \cdot \exp(-c\sqrt{\log x}) = o\left( \frac{1}{\log \log \log x} \right) \), or,

\[
p = o\left( \frac{e^{\sqrt{\log x}}}{\log x} \right). \tag{4.14}
\]

Also, the condition \( \log x \gg (\log pN)^2 \) is equivalent to \( \log x \gg (\log p)^2 \), which is the same as \( x \gg e^{(\log p)^2} \) or

\[
p \ll e^{\sqrt{\log x}}. \tag{4.15}
\]

Since \( \log x \ll e^{A\sqrt{\log x}} \) for any \( A > 0 \), there exists \( \epsilon_1 > 0 \) such that \( e^{\epsilon_1 \sqrt{\log x}} = o\left( \frac{e^{\sqrt{\log x}}}{\log x} \right) \) and \( e^{\epsilon_1 \sqrt{\log x}} = 52 \)
Thus, if we take $0 < \varepsilon_1 < \min \{c, 1\}$ and
\[ p \leq e^{\varepsilon_1 \sqrt{\log x}}, \] (4.16)
then both conditions (4.14) and (4.15) will be satisfied.

We need to estimate the following sum:
\[ \sum_{p^2 \log p \leq q \leq x \atop a(q) \equiv 0 \pmod{p}} \frac{1}{q}, \]
where $\sum$ means that the summation is over those $q$ that $a(q) \neq 0$.

**Lemma 4.4.1.**
\[ \sum_{p^2 \log p \leq q \leq x \atop a(q) \equiv 0 \pmod{p}} \frac{1}{q} \ll \frac{1}{p} \log \log x + \int_{p^2 \log p}^{x} \frac{\pi^*(t, p)}{t^2} \frac{1}{t} \, dt. \] (4.17)

**Proof.** This proof is similar to the proof of Lemma 4.1 in [10].
\[ \sum_{p^2 \log p \leq q \leq x \atop a(q) \equiv 0 \pmod{p}} \frac{1}{q} \ll \int_{p^2 \log p}^{x} \frac{1}{t} \, d(\pi^*(t, p)) \ll \frac{\pi^*(t, p)}{t} \bigg|_{p^2 \log p}^{x} + \int_{p^2 \log p}^{x} \pi^*(t, p) \frac{1}{t^2} \, dt \]
The first term is small:
\[ \frac{\pi^*(t, p)}{t} \bigg|_{p^2 \log p}^{x} \ll \frac{\pi^*(x, p)}{x} = O \left( \frac{1}{p} \log \log x \right). \]

Estimate for the integral goes as follows. To use formula (4.6) we need condition (4.16) which is equivalent to $t \geq e^{\varepsilon_1 \sqrt{\log p}}$. We split the range of integration into two segments:
\[ \int_{p^2 \log p}^{x} \pi^*(t, p) \frac{1}{t^2} \, dt = \int_{p^2 \log p}^{e^\gamma (\log p)^2} \pi^*(t, p) \frac{1}{t^2} \, dt + \int_{e^\gamma (\log p)^2}^{x} \pi^*(t, p) \frac{1}{t^2} \, dt \]
where $\gamma := \frac{1}{\varepsilon_1^2}$.

Estimate for the second integral:
\[ \int_{e^\gamma (\log p)^2}^{x} \pi^*(t, p) \frac{1}{t^2} \, dt \ll \frac{1}{p} \int_{e^\gamma (\log p)^2}^{x} \frac{Li(t)}{t} \frac{1}{t^2} \, dt + \frac{1}{p} \int_{e^\gamma (\log p)^2}^{x} \frac{Li(t)^2}{t^2} \, dt + \frac{1}{p} \int_{e^\gamma (\log p)^2}^{x} \frac{1}{t} \, dt \ll \frac{1}{p} \log \log x \]
1) \[ \frac{1}{p} \int_{e^\gamma (\log p)^2}^{x} \frac{Li(t)}{t^2} \, dt \ll \frac{1}{p} \int_{e^\gamma (\log p)^2}^{x} \frac{1}{t \log t} \, dt \leq \frac{1}{p} \log \log x \]
2) Next, consider the term with the Siegel zero. We use a corollary from [10] of the result of Stark that says that if there is an exceptional zero $\beta$, then it satisfies
\[ \beta \leq 1 - \frac{1}{p^{\varepsilon_1}} \]

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for some constant $c_2$ depending on $N$. Using this bound, we see that the term containing the exceptional zero is

$$
\frac{1}{p} \int_{e^{\gamma \log p}^2}^x \frac{1}{t^{2-\beta} \log t} dt \leq \frac{1}{p} \int_{e^{\gamma \log p}^2}^x \frac{1}{t \log t} \cdot \exp \left\{ -\frac{\log t}{p^{c_1}} \right\} dt.
$$

This is

$$
\leq \frac{1}{p} \log \log x \cdot \exp \left\{ -\frac{\gamma (\log p)^2}{p^{c_1}} \right\} \ll \frac{1}{p} \log \log x.
$$

3) Finally, we deal with the integral coming from the O term:

$$
\int_{e^{\gamma \log p}^2}^x \frac{1}{t^2} e^{-\gamma \sqrt{\log t}} dt = \int_{e^{\gamma \log p}^2}^x \frac{1}{t} e^{-\gamma \sqrt{\log t}} dt = \int_{\gamma \log p}^{\log x} e^{-\gamma \sqrt{u}} du \ll \int_{\gamma \log p}^{\log x} e^{-\gamma \sqrt{u}} \left( 1 - \frac{1}{c \sqrt{u}} \right) du = -\frac{1}{c \sqrt{u}} e^{-\gamma \sqrt{u}} \bigg|_{\log x}^{\gamma \log p} \leq \frac{\sqrt{\gamma}}{c} \cdot \frac{\log p}{e^{\gamma \sqrt{\log p}}} = \frac{\sqrt{\gamma}}{c} \cdot \frac{\log p}{p^{c_1/2}} \ll \frac{1}{p},
$$

since earlier we choose $\epsilon_1 = \frac{1}{\sqrt{\gamma}} < c$, where $c$ is from (4.6).

Note, that here we used the following:

$$
\left( \sqrt{u} e^{-c \sqrt{u}} \right)' = \frac{e^{-c \sqrt{u}}}{2 \sqrt{u}} - \frac{c}{2} e^{-c \sqrt{u}}
$$

implies that

$$
\int \left( e^{-c \sqrt{u}} - \frac{e^{-c \sqrt{u}}}{c \sqrt{u}} \right) du = -\frac{2}{c} \sqrt{u} e^{-c \sqrt{u}} + C;
$$

also $e^{-c \sqrt{u}} \ll e^{-c \sqrt{u}} - \frac{1}{c} \frac{e^{-c \sqrt{u}}}{\sqrt{u}}$.

This implies

$$
\int_{e^{\gamma \log p}^2}^x \frac{1}{t} e^{-\gamma \sqrt{\log t}} dt = \int_{\gamma \log p}^{\log x} e^{-\gamma \sqrt{u}} du \ll \int_{\gamma \log p}^{\log x} \left( e^{-c \sqrt{u}} - \frac{1}{2c} \frac{e^{-c \sqrt{u}}}{\sqrt{u}} \right) du = \frac{1}{c \sqrt{u}} e^{-c \sqrt{u}} \bigg|_{\log x}^{\gamma \log p} \leq \frac{\sqrt{\gamma}}{c} \cdot \frac{\log p}{p^{c_1/2}} \ll \frac{1}{p} \log \log x,
$$

since $\gamma = \frac{1}{\epsilon_1}$ and we chose $\epsilon_1$ so that $\epsilon_1 \leq c$, which makes $c \sqrt{\gamma} > 1$.

Thus,

$$
\int_{e^{\gamma \log p}^2}^x \frac{\pi^*(t, p)}{t^2} dt \ll \frac{1}{p} \log \log x,
$$

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and so,
\[
\sum_{\substack{p^2 \log p \leq q \leq x \\ a(q) \equiv 0 \pmod{p}}} \frac{1}{q} \ll \frac{1}{p} \log \log x + \int_{p^2 \log p}^{e^{\gamma (\log p)^2}} \pi^*(t, p) \frac{1}{t^2} dt.
\]

We split our sum into two parts:
\[
Q_2 = \sum_{y_1 \leq q_2 \leq e^{\epsilon_1 \sqrt{\log x}}} \sum_{y_1 \leq q_2 \leq \epsilon_1 \sqrt{\log x}} \frac{1}{q_1 q_2} = \sum_{y_1 \leq q_2 \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{q_1} \leq \sum_{y_1 \leq q_2 \leq \epsilon_1 \sqrt{\log x}} \frac{1}{q_2} \ll \sum_{y_1 \leq q_2 \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \log \log x \frac{x}{\sqrt{q_2}} + \int_{q_2^2 \log q_2}^{e^{\epsilon_1 \sqrt{\log q_2}} \log Q} \pi^*(t, q_2) \frac{1}{t^2} dt) =
\]

For the first part we use the Lemma 4.4.1.
\[
Q_{21} = \sum_{y_1 \leq q_2 \leq \epsilon_1 \sqrt{\log x}} \frac{1}{q_1} \ll \sum_{y_1 \leq q_2 \leq \epsilon_1 \sqrt{\log x}} \frac{1}{q_2} \log \log x \frac{x}{\sqrt{q_2}}
\]

Consider the expression
\[
Q_{22} = \sum_{y_1 \leq q_2 \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \int_{q_2^2 \log q_2}^{e^{\epsilon_1 \sqrt{\log q_2}} \log Q} \pi^*(t, q_2) \frac{1}{t^2} dt.
\]

Switch the order of integration and summation:
\[
\sum_{y_1 \leq q_2 \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \int_{q_2^2 \log q_2}^{e^{\epsilon_1 \sqrt{\log q_2}} \log Q} \pi^*(t, q_2) \frac{1}{t^2} dt \ll \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} dt \sum_{e^{\epsilon_1 \sqrt{\log x}} \leq q_2 \leq \epsilon_1 \sqrt{\log x}} \pi^*(t, q_2) \frac{1}{q_2} dt \leq \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} dt \sum_{y_1 \leq q_2 \leq \epsilon_1 \sqrt{\log x}} \frac{1}{q_2} \ll \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} dt \frac{t}{\log t} \sqrt{\log t} \frac{1}{e^{\epsilon_1 \sqrt{\log t}}} dt =
\]

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Let $a \in e^{e^{\epsilon_1 \sqrt{\log t}}}$. Thus, $p \leq e^{e^{\epsilon_1 \sqrt{\log t}}}$. Then, since $p \leq y_1^2 \log y_1$, and $t \leq e^{1/(\log y_1)}$, we have:

$$q \leq e^{e^{\epsilon_1 \sqrt{\log t}}} \cdot \frac{1}{e^{e^{\epsilon_1 \sqrt{\log t}}}} \leq o \left( \frac{1}{\sqrt{L_3(x)}} \right).$$

The bounds for summation and integration were changed as follows:

$$q_2 \cdot \log q_2 \leq t \leq e^{e^{\epsilon_1 \sqrt{\log t}}} \cdot \log(y_1^2 \log y_1).$$

For $t$ we had the following bounds: $t \geq q_2 \cdot \log q_2$ and $t \leq e^{e^{\epsilon_1 \sqrt{\log t}}}$. The first inequality is satisfied if $q_2 \leq \sqrt{\frac{t}{\log t}}$, and the second inequality is equivalent to $q_2 \geq e^{e^{\epsilon_1 \sqrt{\log t}}}$. Here we estimated the number of the prime divisors $q_2 \geq e^{e^{\epsilon_1 \sqrt{\log t}}}$ of $a(q_1)$ as follows:

$$a(q_1) \leq 2q_1^{k-1} \leq 2t^{k-1}.$$ 

Let $a(q_1) = p_1 \cdots p_m$ be the decomposition of $a(q_1)$ into the product of primes (not necessarily distinct). Then, since $p_i \geq e^{e^{\epsilon_1 \sqrt{\log t}}}$, we have:

$$(e^{e^{\epsilon_1 \sqrt{\log t}}})^m \leq a(q_1) \ll t^{k-1} \approx e^{k-1} \log t.$$ 

Thus,

$$Q_{212} \ll \frac{1}{(\log \log x)^4},$$

and so

$$Q_{21} \ll \frac{1}{(\log \log x)^4}.$$ 

For the second sum $Q_{22}$, namely the sum over the primes in the interval $e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t} < q_2 \leq 2x^{k-1}$, we switch the order of summation:

$$Q_{22} = \sum_{\substack{e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t} < q_2 \leq 2x^{k-1} \\ q_2 \log q_2 \leq q_1 \leq e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} \leq \sum_{\substack{e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t} < q_2 \leq 2x^{k-1} \\ q_1 \leq e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} =$$

$$= \sum_{q_1 \leq e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t}} \frac{1}{q_1} \sum_{\substack{q_2 \leq 2x^{k-1} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_2} \ll \sum_{q_1 \leq e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t}} \frac{1}{q_1} \cdot \frac{\log(2x^{k-1})}{e^{e^{\epsilon_1 \sqrt{\log t}} \cdot \log t}} \ll \sqrt{\log x} \cdot \log \log x \ll \frac{1}{(\log \log x)^2}. $$

Put these two estimates together to obtain:
Then use this to obtain:

\[ Q_1 + Q_2 = o \left( \frac{1}{\sqrt{L_3(x)}} \right). \]

And so

\[ \frac{Q_1 + Q_2}{D_1} = o \left( \frac{1}{\sqrt{L_3(x)}} \right). \]

In order to estimate \( D_2 \), we write

\[
D_2 = \sum_{x^{1/3} \leq q_1 \leq x, q_1 \equiv 0 \pmod{q_2}} \sum_{n \leq x | n, q_2 | n} 1 \leq \sum_{x^{1/3} \leq q_1 \leq x, q_1 \equiv 0 \pmod{q_2}} \sum_{n \leq x | n, q_2 | n} 1 = \]

\[
\sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 + \sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 + \]

\[
\sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 =: J_1 + J_2 + J_3. \]

Here

\[
J_3 = \sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 \leq \sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 \leq \sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 \leq \]

\[
\sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} 1 \leq \sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} \frac{x}{q_1 q_2} = \sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} \frac{1}{q_1} \leq \]

\[
\sum_{x^{1/3} \leq q_1 \leq x | n, q_2 | n} \frac{1}{q_1} \leq \left\{ q_2 \mid q_2 \geq e^{\sqrt{\log x}}, q_2 \mid a(q_1), 0 \neq a(q_1) \leq 2e^{1/k_2} \right\} \leq \]

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We show that
\[ \frac{x^{\log x}}{e^{\sqrt{\log x}}} \sum_{q_1 \leq x} 1 \ll \frac{x^{\sqrt{\log x} L_2(x)}}{e^{\sqrt{\log x}}} = o \left( \frac{x}{(\log x)^{3/2} (\log \log x)^{1/2}} \right) \]

Here we used the following estimate for the number

\[ \# \left\{ q_2 \mid q_2 \geq e^{\sqrt{\log x}}, \quad q_2|a(q_1), \quad 0 \neq a(q_1) \leq 2x^{\frac{1}{k+1}} \right\} \leq \frac{k+1}{2} \sqrt{\log x}. \quad (4.18) \]

Indeed, for each fixed \( q_1 \) we are estimating the number of distinct primes \( q_2 \) such that \( q_2 \geq e^{\sqrt{\log x}} \) and \( q_2|a(q_1) \). Thus, if we denote the biggest possible such number by \( s \), we get

\[ e^{s \sqrt{\log x}} \leq q_2 q_2 \cdots q_2 \leq |a(q_1)| \leq 2x^{\frac{k-1}{k+1}} \leq 2x^{\frac{k+1}{2}} \]

\[ s \sqrt{\log x} \leq \frac{k+1}{2} \log x \]

and so

\[ s \leq \frac{k+1}{2} \sqrt{\log x} \]

which leads to the estimate (4.18).

In order to estimate \( J_1 \) and \( J_2 \), we write

\[ J_1 = \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \sum_{n \leq x} 1 + \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \sum_{n \leq x} 1 \]

\[ x^{\sqrt{\log x}} \sum_{n, q_2 | n} 1 \]

\[ = J_{11} + J_{12}, \]

and

\[ J_2 = \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \sum_{n \leq x} 1 + \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \sum_{n \leq x} 1 \]

\[ x^{\sqrt{\log x}} \sum_{n, q_2 | n} 1 \]

\[ = J_{21} + J_{22}. \]

We show that \( J_{11} \) and \( J_{21} \) are \( o \left( \frac{x}{(L_3(x)L_1(x))^{1/2}} \right) \). Similarly, one can show that \( J_{12} \) and \( J_{22} \) are \( o \left( \frac{x}{(L_3(x)L_1(x))^{1/2}} \right) \). We can write

\[ J_{11} \ll x \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \frac{1}{q_2} \sum_{n \leq x^{1/2}} \frac{1}{q_1} \left( \log \frac{x}{q_1 q_2} \right)^{1/2} \]

\[ \ll x \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \frac{1}{q_2} \int_{x^{1/2}}^x \frac{d\pi^*(t, q_2)}{t \left( \log \frac{x}{q_2} \right)^{1/2}} \]

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\[
\ll x \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \frac{1}{q_2} \left\{ \frac{\pi^*(t, q_2)}{t (\log \frac{x}{q_2})^{1/2}} \right\}_{t = x/(2q_2)}^{t = x/(2q_2)} + \int_{x/(2q_2)}^{x/(2q_2)} \frac{\pi^*(t, q_2) dt}{t^2 (\log \frac{x}{q_2})^{1/2}} \]

Then by using Lemma 4.2.1, we have

\[
J_{11} \ll x \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \frac{1}{q_2} \left\{ \frac{1}{\log t (\log \frac{x}{q_2})^{1/2}} \right\}_{t = x/(2q_2)}^{t = x/(2q_2)} + \int_{x/(2q_2)}^{x/(2q_2)} \frac{dt}{t (\log \frac{x}{q_2})^{1/2}} \]

\[
= \frac{x}{\log x} \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \frac{1}{q_2} \ll \frac{x}{y_1 \log x}. \]

Since for each pair of primes \( q_1, q_2 \) with \( y_1 \leq q_2 \leq e^{\sqrt{\log x}} \), \( x/(2q_2) \leq q_1 \leq x/q_2 \), there are at most two \( n \leq x \) with \( q_1 q_2 | n \) (either \( \frac{x}{2} < q_1 q_2 < x \) and \( n = q_1 q_2 \), or \( \frac{x}{2} = q_1 q_2 \) and \( n_1 = q_1 q_2, n_2 = 2q_1 q_2 \)), we have

\[
J_{21} \ll \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \sum_{\frac{x}{2q_2} \leq q_1 \leq \frac{x}{q_2}} \frac{1}{q_2} \ll \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \pi^*(x/q_2, q_2) \]

by Lemma 4.2.1

\[
= \frac{x}{\log x} \sum_{y_1 \leq q_2 \leq e^{\sqrt{\log x}}} \frac{1}{q_2} \ll \frac{x}{y_1 \log x}. \]

Hence

\[
D_2 \ll J_1 + J_2 + J_3 \ll \frac{x}{(\log x)^{1/2} L_3(x)}. \]

Now we put all the estimates together to conclude that

\[
B_1 + B_2 \ll B_1 + D_1 + D_2 \ll \frac{x}{(\log x)^{1/2} (\log \log x)^{3/2}} \]

which gives us the estimate (4.13).
Chapter 5

Proof of Theorem 1.2.2

5.1 Good primes

Let us now estimate

\[ \# \{ n \leq x \mid (n, a(n)) \text{ is prime} \} . \]

We call the primes \( p \) for which \( p \nmid a(p) \) good primes.

Lemma 5.1.1. If \( p \) is a good prime, then \( p \nmid a(p^m) \) for any \( m \).

Proof. Let \( p \) be a good prime. For \( m \geq 2 \)

\[ a(p^m) = a(p)a(p^{m-1}) - p^{k-1}a(p^{m-2}). \]

Thus,

\[ p | a(p^m) \Rightarrow p | a(p)a(p^{m-1}) \Rightarrow p | a(p^{m-1}) \text{ since } p \text{ is a good prime.} \]

We have

\[ p | a(p^m) \Rightarrow p | a(p^{m-1}) \Rightarrow p | a(p^{m-2}) \Rightarrow \cdots \Rightarrow p | a(p). \]

This is a contradiction. Hence \( p | a(p^m) \) for any \( n \).

In fact, the converse statement is also true: \( p | a(p) \Rightarrow p | a(p^m) \) for all \( m \in \mathbb{N} \), since \( p | a(p^m) \) if and only if \( p | a(p) \) or \( p | a(p^{m-1}) \).

But we see that for \( m \geq 3 \) we have \( p | a(p) \Rightarrow p^m | a(p^m) \) from equation (5.1). Thus, we have proved the following lemma.

Lemma 5.1.2. For weight \( k \geq 3 \) we have \( p | a(p) \) if and only if \( p^m | a(p^m) \) for any \( m \).

Remark 5.1.3. Note that this argument did not require the hypothesis that the modular form \( f \) is of CM-type.

We write

\[ \sum_{n \leq x \atop (n, a(n)) \text{ is a prime}} 1 = \sum_{n \leq x \atop n \text{ is squarefree}} 1 + \sum_{n \leq x \atop (n, a(n)) \text{ is a prime}} 1 =: \Sigma' + \Sigma''. \]

We split the first sum in two parts, depending on whether the prime that is the \( (n, a(n)) \) is a good or a bad prime.
\[ \Sigma' = \sum_{n \leq x} \sum_{l \text{ prime } n \text{ is squarefree } (n,a(n)) = l} 1 = \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } n \text{ is squarefree } (n,a(n)) = l} 1
\]

\[ = \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } n \text{ is squarefree } (n,a(n)) = l} 1 + \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } n \text{ is squarefree } (n,a(n)) = l} 1. \]

Write \( n = ml \). Since \( n \) is squarefree, we have \( l \nmid m \). Thus, \( a(ml) = a(m)a(l) \).

1. Consider the \( l \mid a(l) \) case: \( (lm, a(l)a(m)) = l \) implies that \( (m, a(l)/a(m)) = 1 \), in particular \( (m,a(m)) = 1, (m, a(l)) = 1 \). Thus

\[ \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } l \mid a(l) \text{ and } (n,a(n)) = l} 1 \]

\[ \leq \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } l \mid a(l) \text{ and } (m,a(m)) = 1, (m,a(l)) = 1} 1 =: \Sigma_1. \]

2. Consider the \( l \nmid a(l) \) case: \( (lm, a(l)a(m)) = l \) implies \( l \mid a(m) \).

\[ \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } l \nmid a(l) \text{ and } (n,a(n)) = l} 1 \]

\[ \leq \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } l \nmid a(l) \text{ and } (m,a(m)) = 1, (m,a(l)) = 1} 1 =: \Sigma_2. \]

5.2 Contribution of non-squarefree numbers

Let us estimate the number of non-squarefree numbers \( \Sigma'' \).

\[ \Sigma'' = \sum_{n \leq x} \sum_{n \text{ is non-squarefree } (n,a(n)) = l} 1 = \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } l \mid a(l) \text{ and } (n,a(n)) = l} 1 + \sum_{l \leq x} \sum_{n \leq x} \sum_{l \text{ prime } l \mid a(l) \text{ and } (n,a(n)) = l} 1 \]

\[ \leq \sum_{l \leq x} \sum_{m \leq x} \sum_{l \mid m \text{ and } (m,a(m)) = 1} 1 + \sum_{l \leq x} \sum_{p \mid m} \sum_{l \mid a(m) \text{ for some prime } p \nmid l} 1 =: \Sigma_3 + \Sigma_4. \]

\( \Sigma_3 \) counts the non-squarefree numbers \( n \) for which \( l \) comes from the non-squarefree part: \( l^2 \mid n \). Since \( l^2 \mid n \), \( n \leq x \), we have \( l \leq \sqrt{x} \).

Note that if \( l \mid a(l) \), then \( l^2 \mid n \) implies \( l^2 \mid a(n) \), and so \( l^2 \mid (m, a(n)) \). Thus, in the estimate of \( \Sigma_3 \) we can sum over \( l \nmid a(l) \).

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We have
\[
\Sigma_3 = \sum_{l \leq x} \sum_{m \leq \sqrt{x} \atop l \mid m \atop (ml, a(ml)) = 1} 1 = \sum_{l \leq \sqrt{x} \atop l \mid a(l)} \sum_{m \leq \sqrt{x} \atop (ml, a(ml)) = 1} 1 = \sum_{l \leq \sqrt{x} \atop l \mid a(l)} \sum_{m_0 \leq \sqrt{x} \atop l \mid a(l^2 m_0) \atop (l^2 m_0, a(l^2 m_0)) = 1} 1
\]

Let \( m = l^k m_1 \), where \( l \nmid m_1 \) (i.e. \( n = l^{k+1} m_1 \)). Since \( l \mid m \), we have \( k \geq 1 \). Then
\[
a(n) = a(l^{k+1} m_1) = a(l^{k+1}) a(m_1).
\]

As \( l \nmid a(l) \), by Lemma 5.1.1, \( l \nmid a(l^{k+1}) \). Hence
\[
l \mid a(n) \iff l \mid a(m_1).
\]

Since \( l \nmid m_1 \) and \( l \nmid a(l^k) \), we have
\[
(l^k m_1, a(l^k) a(m_1)) = l \Rightarrow \begin{cases} (m_1, a(m_1)) = 1 \\ l \mid a(m_1) \end{cases}.
\]

(5.2)

For each \( m_1 \) that satisfies condition (5.2), there are \( \ll \log \frac{x}{m_1} \) such \( m_1 \)'s, that are counted in the sum \( \Sigma_3 \). Indeed, for each \( m_1 \) we have
\[
m_1 < m_1 \cdot l < m_1 \cdot l^2 < \ldots < m_1 \cdot l^i \leq \frac{x}{l}
\]
\[
m_1 \cdot l^{i+1} \leq x
\]
\[
l^{i+1} \leq \frac{x}{m_1}
\]
\[
i \ll \log \frac{x}{m_1}.
\]

Thus, our sum \( \Sigma_3 \) can be estimated as follows
\[
\Sigma_3 \ll \sum_{l \leq \sqrt{x} \atop l \text{ prime}} \sum_{m_1 \leq \sqrt{x} \atop l \mid a(m_1) \atop (m_1, a(m_1)) = 1} \log \frac{x}{m_1} \leq \sum_{l \leq \sqrt{x} \atop l \text{ prime}} \sum_{m_1 \leq \sqrt{x} \atop (m_1, a(m_1)) = 1} \log \frac{x}{m_1}.
\]

To estimate this sum, we will break it up into two pieces:
\[
\sum_{l \leq \sqrt{x} \atop l \text{ prime}} \sum_{m_1 \leq \sqrt{x} \atop (m_1, a(m_1)) = 1} \log \frac{x}{m_1} = \sum_{l \leq x^{1/2} \atop l \text{ prime}} \sum_{m_1 \leq \sqrt{x} \atop (m_1, a(m_1)) = 1} \log \frac{x}{m_1} + \sum_{x^{1/2} < l \leq \sqrt{x} \atop l \text{ prime}} \sum_{m_1 \leq \sqrt{x} \atop (m_1, a(m_1)) = 1} \log \frac{x}{m_1} =: \Sigma_{31} + \Sigma_{32}.
\]

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Let us estimate the inner sum of $\Sigma_{31}$:

$$\sum_{m_1 \leq \frac{x}{l}, (m_1, a(m_1)) = 1} \log \frac{x}{m_1} \ll \int_{c_1}^{x/l^2} \log \frac{x}{t} d\mu(t) =$$

where $\mu(t) = \#\{m_1 \leq t \mid (m_1, a(m_1)) = 1\}$ and $c_1 = e^{x^e}$, since we want to make sure that the $L_3(t)$ that appears in the estimate for $\mu(t)$ is well defined,

$$= \log \frac{x}{t} \cdot \mu(t) \bigg|_{c_1}^{x/l^2} - \int_{c_1}^{x/l^2} \mu(t) \left( \log \frac{x}{t} \right)' dt =$$

$$= \log \frac{x}{t} \cdot \mu(t) \bigg|_{c_1}^{x/l^2} - \int_{c_1}^{x/l^2} \mu(t) \frac{t}{x} \cdot \left( -\frac{x}{t^2} \right) dt =$$

$$= \log \frac{x}{t} \cdot (1 + o(1)) \frac{U_f t}{\sqrt{L_1(t)L_3(t)}} \bigg|_{c_1}^{x/l^2} + \int_{c_1}^{x/l^2} \frac{U_f (1 + o(1))}{\sqrt{L_1(t)L_3(t)}} dt \ll$$

$$\ll \log l^2 \frac{x/l^2}{\sqrt{L_1(x/l^2)L_3(x/l^2)}} - \frac{c_1}{e^x} \log \frac{x}{c_1} + \int_{c_1}^{x/l^2} \frac{1}{\sqrt{L_1(t)L_3(t)}} dt \ll$$

$$\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \log l \frac{l^2}{l^2} + \frac{x/l^2}{\sqrt{L_1(x/L^2)L_3(x/L^2)}}.$$

Since for $l \leq x^{\frac{1}{2} - \alpha}$ the first term dominates the second one, this is seen to be

$$\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \frac{\log l}{l^2}.$$

Here we used the following:

$$\int_{c_1}^{x/l^2} \frac{1}{\sqrt{L_1(t)L_3(t)}} dt = \int_{c_1}^{x/l^2} \frac{1}{\sqrt{L_1(t)L_3(t)}} dt + \int_{c_1}^{x/l^2} \frac{1}{\sqrt{x/l^2}} \frac{1}{\sqrt{L_1(t)L_3(t)}} dt \ll$$

$$\ll \sqrt{x/l^2} + \frac{1}{\sqrt{L_1(x/L^2)L_3(x/L^2)}} (x/L^2 - \sqrt{x/l^2}) \ll \frac{x/l^2}{\sqrt{L_1(x/L^2)L_3(x/L^2)}}.$$

Then,

$$\Sigma_{31} \ll \sum_{l \leq x^{\frac{1}{4} - \alpha}} \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \frac{\log l}{l^2} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \sum_{l \leq x^{\frac{1}{4} - \alpha}} \frac{\log l}{l^2} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.$$

Let us estimate $\Sigma_{32}$:
The inner sum can be estimated:

$$\sum_{m_1 \leq l^2, (m_1, a(m_1)) = 1} \log \frac{x}{m_1} \ll \int_1^{x/l^2} \log \frac{x}{t} \, dt = t \cdot \log \frac{x}{t} - \int_1^{x/l^2} \frac{1}{t} \cdot \left( -\frac{x}{t} \right) \, dt =$$

$$= \frac{x}{l^2} \log l^2 - \log x + \int_1^{x/l^2} \frac{1}{t} \, dt \ll \frac{x}{l^2} \log l \ll \frac{x}{l^2} \log l.$$

Thus,

$$\Sigma_{32} = \sum_{\frac{x}{l^{1/2-\alpha}} \leq l \leq \sqrt{x}} \sum_{\frac{x}{l^{1/2-\alpha}} \leq l \leq \sqrt{x}} \log \frac{x}{m_1} \ll \sum_{\frac{x}{l^{1/2-\alpha}} \leq l \leq \sqrt{x}} \sum_{\frac{x}{l^{1/2-\alpha}} \leq l \leq \sqrt{x}} x \cdot \log l \ll \frac{x}{l^{1/2-\alpha}} \cdot \sum_{\frac{x}{l^{1/2-\alpha}} \leq l \leq \sqrt{x}} \frac{\log l}{l} \ll x^{1/2+\alpha} \cdot (\sqrt{x})^\beta = x^{1/2+\alpha+\beta/2} = o \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right).$$

Here we used the following estimate:

$$\log \frac{t}{l} \ll \frac{1}{l^{1-\beta}}$$

for any $0 < \beta < 1$

$$\sum_{l=x^{1/2-\alpha}}^{\sqrt{x}} \frac{\log l}{l} \ll \sum_{l=x^{1/2-\alpha}}^{\sqrt{x}} \frac{1}{l^{1-\beta}} \ll \int_1^{\sqrt{x}} \frac{1}{t^{1-\beta}} \, dt \sim \frac{1}{\beta} x^{\beta}$$

choose $\alpha$ and $\beta$ so that $\frac{1}{2} + \alpha + \frac{\beta}{2} < 1$.

Thus,

$$\Sigma_3 = \Sigma_{31} + \Sigma_{32} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.$$

Now let us estimate $\Sigma_4$, which is a bound for those non-squarefree numbers $n$ for which $l = (n, a(n))$ comes from the squarefree part of $n$:

$$\Sigma_4 = \sum_{l \leq x} \sum_{l^{1/2-\alpha} \leq m \leq \frac{x}{l}} 1 =$$

$$= \sum_{l \leq e^{1+\sqrt{\log x}}} \sum_{p^2 \mid m \text{ for some prime } p} \frac{1}{p} = \sum_{l \leq e^{1+\sqrt{\log x}}} \sum_{p^2 \mid n} \frac{1}{p} \left( \sum_{e^{1+\sqrt{\log x}} \leq l \leq x} \frac{1}{l} \right) =: \Sigma_{41} + \Sigma_{42};$$

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where \(0 < \epsilon < 1\), and \(\epsilon_1\) is chosen so that \(\epsilon_1 < \min\{c, 1\}\) with \(c\) from Lemma 4.2.1.

We have

\[
\Sigma_{42} = \sum_{e^{\epsilon_1 \sqrt{1 - \epsilon \log x}} \leq l \leq x} \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{m_0 \leq \frac{lp}{s}} \sum_{(m_0, a(m_0)) = 1} \sum_{l|a(p^s)a(m_0)} 1 \leq \\
\leq \sum_{x^{\epsilon_1 \sqrt{1 - \epsilon \log x}} \leq l \leq x} \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{m_0 \leq \frac{lp}{s}} \sum_{(m_0, a(m_0)) = 1} \left( \sum_{l|a(p^s)} 1 + \sum_{m_0 \leq \frac{lp}{s}} \sum_{(m_0, a(m_0)) = 1} \sum_{l|a(p^s)} 1 \right) =: \Sigma_{421} + \Sigma_{422}.
\]

\[
\Sigma_{421} = \sum_{e^{\epsilon_1 \sqrt{1 - \epsilon \log x}} \leq l \leq x} \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{m_0 \leq \frac{lp}{s}} \sum_{(m_0, a(m_0)) = 1} \sum_{l|a(p^s)} 1 \leq \\
\leq x \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{m_0 \leq \frac{lp}{s}} \sum_{(m_0, a(m_0)) = 1} \sum_{l|a(p^s)} 1.
\]

The number of \(l\)'s in the second sum is estimated as follows:

\[
\left( e^{\epsilon_1 \sqrt{1 - \epsilon \log x}} \right)^r \leq l_1 l_2 \cdots l_r = |a(p^s)| \leq (s + 1) (p^s)^{\frac{k-1}{2}} \ll (p^s)^{\frac{k-1}{2} + \epsilon} \ll x \frac{k-1}{2} + \epsilon,
\]

so

\[
r \ll \frac{\log x}{2} = \sqrt{\log x}.
\]

Hence,

\[
\Sigma_{421} \ll x \sum_{p^s \leq \frac{x}{l}} \frac{1}{p^s} \cdot \frac{1}{e^{\epsilon_1 \sqrt{1 - \epsilon \log x}}} \cdot \sqrt{\log x} \ll \frac{x}{\log x} = o \left( \frac{x}{L_1(x)L_3(x)} \right).
\]

\[
\Sigma_{422} = \sum_{e^{\epsilon_1 \sqrt{1 - \epsilon \log x}} \leq l \leq x} \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{m_0 \leq \frac{lp}{s}} \sum_{(m_0, a(m_0)) = 1} \sum_{l|a(p^s)} 1 \leq \\
\leq \sum_{e^{\epsilon_1 \sqrt{1 - \epsilon \log x}} \leq l \leq x} \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{q \leq \frac{p}{l}} \sum_{i \geq 1} \sum_{m_1 \leq \frac{lpq}{s}} \sum_{(m_1, a(m_1)) = 1} \sum_{l|a(q^i)} 1 \leq \\
\leq x \sum_{p^s \leq \frac{x}{l}} \sum_{s \geq 2} \sum_{q \leq \frac{p}{l}} \sum_{i \geq 1} \sum_{l|a(q^i)} 1 
\]

The number of \(l\)'s in the inner sum is estimated as in the estimate of the previous sum \(\Sigma_{421}\), it is \(\ll \frac{\log x}{\sqrt{\log x}} = \sqrt{\log x}\). Thus,
\[
\Sigma_{422} \ll x \sum_{p^s \leq \sqrt{\epsilon x} \atop s \geq 2} \frac{1}{p^s} \sum_{\eta' \leq \epsilon x \atop \eta_i \geq 1} \frac{1}{\eta^i} \cdot \frac{1}{e^{\epsilon_1 \sqrt{1-\epsilon \sqrt{\log x}}} \cdot \sqrt{\log x}} \ll
\]
\[
\frac{x \sqrt{\log x} L(x)}{e^{\epsilon_1 \sqrt{1-\epsilon \sqrt{\log x}}} \ll \frac{x}{\log x} = o \left( \frac{x}{L_1(x) L_2(x)} \right). \]

\[
\Sigma_{41} = \sum_{l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 \leq
\]
\[
\leq \sum_{l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \left( \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 + \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 \right) =: \Sigma_{411} + \Sigma_{412}.
\]

\[
\Sigma_{411} = \sum_{l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \left( \sum_{p^s \leq \frac{x}{2}} \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 + \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 \right) =: \Sigma_{4111} + \Sigma_{4112}
\]

\[
\Sigma_{4111} = \sum_{l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 \leq
\]
\[
\leq \sum_{l \leq y_1 \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 + \sum_{y_1 \leq l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 =: \Sigma_{41111} + \Sigma_{41112}
\]

\[
\Sigma_{4112} = \sum_{y_1 \leq l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \sum_{m_0 \leq \frac{x}{l \prod_p a(p)^s}} 1 \ll
\]
\[
\ll \sum_{y_1 \leq l \leq e^{1+\sqrt{\epsilon \sqrt{\log x}}} \atop s \geq 2} \sum_{p^s \leq \frac{x}{2}} \frac{x}{l p^s \sqrt{L_1 \left( \frac{x}{lp^s} \right) L_2 \left( \frac{x}{lp^s} \right)}}.
\]

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Since \( \frac{x}{lp^s} \geq \left( \frac{x}{lp^s} \right)^\epsilon \geq x^\epsilon \) implies \( x \sqrt{L_1 \left( \frac{x}{lp^s} \right) L_3 \left( \frac{x}{lp^s} \right)} \leq \frac{x}{lp^s \sqrt{L_1(x)L_3(x)}} \), we have

\[
\Sigma_{41112} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{l \leq e^{x^{-1} + y_{\epsilon = \infty, x_{\epsilon = 1} \leq 2}}} \frac{1}{l} \sum_{p^s \leq x \atop l \mid a(p^s)} \frac{1}{p^s}.
\]

We switch the order of summation here:

\[
\Sigma_{41112} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p^s \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{l \mid a(p^s)} \frac{1}{l} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p^s \leq x \atop s \geq 2} \frac{1}{p^s} \log \log(p^s) \ll \frac{x}{\sqrt{L_1(x)L_3(x)}},
\]

since

\[
\sum_{p^s \leq x \atop s \geq 2} \frac{1}{p^s} \log \log(p^s) \ll \sum_{p \leq \sqrt{x}} \sum_{s \geq 2} \frac{1}{p^s} \log \log p \ll \sum_{p \leq \sqrt{x}} \left( \frac{1}{p^2} \log \log p \left( 1 + \frac{1}{p^2} + \frac{1}{p^4} + \ldots \right) \right) \ll 1.
\]

\[
\Sigma_{41111} = \sum_{l \leq y_1} \sum_{p^s \leq \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right)^{1-\epsilon}} \sum_{m_0 \leq \frac{x}{p^s} \atop (m_0, a(m_0)) = 1} 1 \leq \sum_{l \leq y_1} \sum_{p^s \leq \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right)^{1-\epsilon}} \sum_{m_0 \leq \frac{x}{p^s} \atop (m_0, a(m_0)) = 1} 1 \ll \sum_{l \leq y_1} \sum_{p^s \leq \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right)^{1-\epsilon}} \frac{x}{lp^s \sqrt{L_1 \left( \frac{x}{lp^s} \right) L_3 \left( \frac{x}{lp^s} \right)}} \ll \sum_{l \leq y_1} \sum_{p^s \leq \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right)^{1-\epsilon}} \frac{x}{lp^s \sqrt{L_1 \left( \frac{x}{lp^s} \right) L_3 \left( \frac{x}{lp^s} \right)}} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{l \leq y_1} \frac{1}{l} \sum_{p^s \leq \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right)^{1-\epsilon}} \frac{1}{p^s}.
\]

Since the sum \( \sum_{p^s \leq \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right)^{1-\epsilon}, s \geq 2} \frac{1}{p^s} \) is convergent, we have

\[
\Sigma_{41111} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \log \log(y_1) \ll \frac{xL_4(x)}{\sqrt{L_1(x)L_3(x)}}
\]
This completes the estimate for $\Sigma_{4111}$.

$$\Sigma_{4112} = \sum_{l \leq e^{1+\sqrt{\log x}} \left( \frac{2}{3} \right)} \sum_{s \geq 2} \sum_{l \leq e^{1+\sqrt{\log x}} \left( \frac{2}{3} \right)} \frac{1}{l l_{a(p)}} \sum_{m_0 \leq \frac{l}{p}} \sum_{s \geq 2} \sum_{l \leq e^{1+\sqrt{\log x}} \left( \frac{2}{3} \right)} \frac{1}{l l_{a(m_0)}} = 1 \ll$$

$$\ll \sum_{l \leq e^{1+\sqrt{\log x}} \left( \frac{2}{3} \right)} \sum_{s \geq 2} \frac{x}{l p^s} \ll x \sum_{l \leq e^{1+\sqrt{\log x}} \left( \frac{2}{3} \right)} \frac{1}{l} \sum_{s \geq 2} \frac{1}{l}.$$ 

Since $\frac{1}{l} \geq \frac{1}{e^{1+\sqrt{\log x}} \log x} \geq \frac{1}{x^{1-\epsilon}}$ implies $(\frac{x}{l})^{1-\epsilon} \geq (\frac{x}{x^{1-\epsilon}})^{1-\epsilon} = x^{\epsilon(1-\epsilon)}.$

Let us estimate the inner sum $\sum_{s \geq 2} \frac{1}{p^s}$. For this we break it up into two sums with $0 < \alpha < \epsilon(1-\epsilon)$.

$$\sum_{s \geq 2} \frac{1}{p^s} = \sum_{x^{(1-\epsilon)} < p^s \leq x} \frac{1}{p^s} + \sum_{x^{(1-\epsilon)} < p^s \leq 2^\alpha} \frac{1}{p^s} =: \Sigma_{41121} + \Sigma_{41122}.$$ 

The first sum is bounded by

$$\sum_{x^{(1-\epsilon)} < p^s \leq 2^\alpha} \frac{1}{p^s} \leq \sum_{x^{(1-\epsilon)} < p^s \leq x} \frac{1}{p^s} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right) \ll \sum_{p \leq x^{(1-\epsilon)} \log x} \frac{1}{p^2} \ll \frac{1}{x^\alpha \log x} \ll \frac{1}{x^\alpha \log x}.$$ 

The second sum is bounded by

$$\sum_{x^{(1-\epsilon)} < p^s \leq 2^\alpha} \frac{1}{p^s} \leq \frac{1}{x^{(1-\epsilon)} \log x} \sum_{s \geq 2} \sum_{p \text{ prime} \leq x^{(1-\epsilon)} \log x} \frac{1}{p \cdot \log x} \sum_{p \leq x^{(1-\epsilon)} \log x} \frac{1}{p^2} \ll \frac{1}{x^{(1-\epsilon)} \log x} \ll \frac{1}{x^{(1-\epsilon)} \log x}.$$ 

where $\sum_{s \geq 2} \frac{1}{p^s} \leq \log x$, since $2^\alpha \leq p^s \leq x$.

Thus, since $\epsilon(1-\epsilon) - \alpha > 0$, both of these sums, $\Sigma_{41121}$ and $\Sigma_{41122}$, are bounded by $\frac{1}{x^\epsilon}$ for some $\epsilon > 0$. So we have

$$\Sigma_{4112} \ll x \sum_{l \leq e^{1+\sqrt{\log x}} \left( \frac{2}{3} \right)} \frac{1}{l} \left( \Sigma_{41121} + \Sigma_{41122} \right) \ll x \cdot \frac{1}{x^\epsilon} \cdot \log x \ll x^{1-\epsilon}.$$ 

This completes the estimate for $\Sigma_{411}$. The sum $\Sigma_{412}$ is the same as the sum $\Sigma_{411}$, only the condition $l_{a(p)}$ is replaced with $l_{a(m_0)}$. When estimating the sum $\Sigma_{411}$ we used this condition $l_{a(p)}$ in only one place, namely, in the estimate of the sum $\Sigma_{41112}$. Thus, when estimating sum $\Sigma_{412}$ everything will stay exactly the same, except that in the estimate of the corresponding piece $\Sigma_{41212}$ we have to use the condition $l_{a(m_0)}$ instead of $l_{a(p)}$.
\[ \Sigma_{41212} = \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{m_0 \leq \frac{x}{p^s}} \frac{1}{l_a(m_0)} = \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{q^t \leq \frac{x}{p^s}} \sum_{m_1 \leq \frac{x}{p^s q^t}} \frac{1}{l_a(q^t)} (m_1, a(m_1)) = 1 \]

\[ = \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{q^t \leq \frac{x}{p^s}} \sum_{m_1 \leq \frac{x}{p^s q^t}} \frac{1}{l_a(q^t)} =: \Sigma_{412121} + \Sigma_{412122} \]

\[ \Sigma_{412121} = \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{q^t \leq \frac{x}{p^s}} \sum_{m_1 \leq \frac{x}{p^s q^t}} \frac{1}{l_a(q^t)} =: \Sigma_{4121211} + \Sigma_{4121212} \]

\[ \Sigma_{412121} \ll \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{q^t \leq \frac{x}{p^s}} \frac{1}{lp^s q^t} = \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \frac{x}{l_a(q^t)} \frac{1}{lp^s q^t} =: \]

\[ \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{q^t \leq \frac{x}{p^s}} \frac{1}{lp^s q^t} = \]

\[ = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \left( \sum_{q^t \leq \frac{x}{p^s}} \frac{1}{lp^s q^t} + \sum_{q^t \leq \frac{x}{p^s}} \frac{1}{lp^s q^t} \right) =: \]

\[ = \Sigma_{4121211} + \Sigma_{4121212} \]

\[ \Sigma_{412122} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < t \leq e^{x^{1/\sqrt{\log x}}}} \sum_{p^s \leq \left( \frac{y}{x} \right)^{1-\varepsilon}} \sum_{q^t \leq \frac{x}{p^s}} \frac{1}{lp^s q^t} \ll \]

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Let us estimate the inner part of this sum by switching the order of summation and integration:

\[
\begin{align*}
&\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x} \frac{1}{p^s} \sum_{s \geq 2} \frac{1}{q^t} \log \log(q^t) \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.
\\
&\Sigma_{4121211} \leq \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < t \leq e^{1 + \sqrt{\log x}}} \frac{1}{t} \sum_{p' \leq x} \frac{1}{p^s} \sum_{s \geq 2} \frac{1}{q^t} \log \log(q^t) \ll \frac{x}{y_1 \sqrt{L_1(x)L_3(x)}} \leq o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right)
\\
&\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < t \leq e^{1 + \sqrt{\log x}}} \frac{1}{t} \sum_{p' \leq x} \frac{1}{p^s} \left(\frac{1}{t} \log L_2(x) + \int_{t^2 \log t}^{\frac{1}{t} \log t} \frac{1}{t^2} \pi^*(t, l) \frac{1}{t^2} dt + \sum_{q \leq t^2 \log t} \frac{1}{q}\right) =: \Sigma_{41212111} + \Sigma_{41212112} + \Sigma_{41212113}
\\
&\Sigma_{41212111} = \frac{xL_2(x)}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < t \leq e^{1 + \sqrt{\log x}}} \frac{1}{t} \sum_{p' \leq x} \frac{1}{p^s} \ll \frac{xL_2(x)}{y_1 \sqrt{L_1(x)L_3(x)}} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right)
\\
&\Sigma_{41212112} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x} \frac{1}{p^s} \sum_{y_1 < t \leq e^{1 + \sqrt{\log x}}} \frac{1}{t^2} \log t \frac{1}{t^2} \pi^*(t, l) \frac{1}{t^2} dt =
\\
&= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x} \frac{1}{p^s} \log y_1 \sum_{y_1 < t \leq e^{1 + \sqrt{\log x}}} \frac{1}{t \log t} \cdot \sqrt{\log t} \cdot \frac{1}{e^{1 + \sqrt{\log t}}} dt =
\\
&= \int_{y_1^2 \log y_1}^{x} \frac{1}{t \log t} e^{1 + \sqrt{\log t}} dt = -\frac{2}{\epsilon_1} \left.\frac{1}{e^{1 + \sqrt{\log t}}}\right|_{y_1^2 \log y_1}^{x} = o(1).
\end{align*}
\]

Let us estimate the inner part of this sum by switching the order of summation and integration:
Thus, 

\[ \Sigma_{412112} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{y_1 < l \leq e^{s+\sqrt{s}\log x}} \frac{1}{l} \int_{t^2 \log l}^{\frac{1}{t^2} (\log t)^2} \pi^*(t, l) \frac{1}{t^2} \, dt = \]

\[ = o \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right). \]

\[ \Sigma_{4121113} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{y_1 < l \leq e^{s+\sqrt{s}\log x}} \frac{1}{l} \sum_{q \leq t^2 \log l} \frac{1}{q} \leq \]

\[ \leq \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{q \leq x \atop q \geq \log^{2/3} (q^2)} \frac{1}{l} \sum_{l \mid a(q)} \frac{1}{q} \ll \]

\[ \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{q \leq x} \frac{1}{q} \frac{1}{q^2} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}. \]

This finishes the estimate for the sum \( \Sigma_{412111} \). 

\[ \Sigma_{412112} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < l \leq e^{s+\sqrt{s}\log x}} \frac{1}{l} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{q' \leq x \atop q' \leq q^2 \atop l \mid a(q')} \frac{1}{q'} \leq \]

\[ \leq \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{q' \leq x \atop l \geq q^2} \frac{1}{l} \sum_{l \mid a(q')} \frac{1}{q'} \ll \]

\[ \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{p' \leq x \atop s \geq 2} \frac{1}{p^s} \sum_{q' \leq x \atop l \geq q^2 \atop l \mid a(q')} \log \log (q') \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}. \]

This finishes the estimate for the sum \( \Sigma_{412112} \), and so also the estimate for \( \Sigma_{412121} \).
\[
\Sigma_{412122} \ll \sum_{y_1 < l \leq e^{c_1 \sqrt{\sqrt{\log x}}} \sqrt{\log x}} \sum_{p^s \leq (\frac{x}{l})^{1-\epsilon}} \sum_{l \mid a(q') \atop i \geq 2} \frac{x}{l p^s q^i} L_1 \left( \frac{x}{l p^s q^i} \right) L_3 \left( \frac{x}{l p^s q^i} \right) 
\]

\[
\leq \sum_{y_1 < l \leq e^{c_1 \sqrt{\sqrt{\log x}}} \sqrt{\log x}} \sum_{p^s \leq (\frac{x}{l})^{1-\epsilon}} \sum_{l \mid a(q) \atop i \geq 2} \frac{x}{l p^s q^i} L_1 \left( \frac{x}{l p^s q^i} \right) L_3 \left( \frac{x}{l p^s q^i} \right) + \sum_{y_1 < l \leq e^{c_1 \sqrt{\sqrt{\log x}}} \sqrt{\log x}} \sum_{p^s \leq (\frac{x}{l})^{1-\epsilon}} \sum_{l \mid a(q') \atop i \geq 2} \frac{x}{l p^s q^i} L_1 \left( \frac{x}{l p^s q^i} \right) L_3 \left( \frac{x}{l p^s q^i} \right) =: \Sigma_{4121221} + \Sigma_{4121222}.
\]

For the estimate of \(\Sigma_{4121221}\) we will need the following lemma:

**Lemma 5.2.1.** For \(u \geq x^\alpha\) and \(l \leq e^{c_1 \sqrt{\sqrt{\log x}}} \sqrt{\log x}\) we have

\[
\sum_{u^{1-\epsilon} \leq q \leq \frac{u}{l}} \frac{u}{q \sqrt{L_1 \left( \frac{u}{q} \right) L_3 \left( \frac{u}{q} \right)}} \ll \frac{1}{l} \cdot \frac{u}{\sqrt{L_1 (u) L_3 (u)}}. \tag{5.3}
\]

**Proof.** This proof is similar to the one of Lemma 4.4.1, which in turn is similar to the proof of Lemma 4.1 in [10]. We will use Lemma 4.2.1 here.

\[
\sum_{u^{1-\epsilon} \leq q \leq \frac{u}{l}} \frac{u}{q \sqrt{L_1 \left( \frac{u}{q} \right) L_3 \left( \frac{u}{q} \right)}} = \int_{u^{1-\epsilon}}^{u} \frac{u}{t \sqrt{L_1 \left( \frac{t}{u} \right) L_3 \left( \frac{t}{u} \right)}} d(\pi^*(t,l)) =
\]

\[
= \frac{u}{t \sqrt{L_1 \left( \frac{t}{u} \right) L_3 \left( \frac{t}{u} \right)}} \pi^*(t,l) \bigg|_{u^{1-\epsilon}}^{u} - \int_{u^{1-\epsilon}}^{u} \pi^*(t,l) \left( \frac{u}{t \sqrt{L_1 \left( \frac{t}{u} \right) L_3 \left( \frac{t}{u} \right)}} \right) \frac{dt}{l}
\]

Let us estimate the first summand:

\[
\frac{u}{t \sqrt{L_1 \left( \frac{t}{u} \right) L_3 \left( \frac{t}{u} \right)}} \pi^*(t,l) = \frac{u}{t \sqrt{L_1 \left( \frac{t}{u} \right) L_3 \left( \frac{t}{u} \right)}} \left( \frac{1}{l} \text{Li}(t) - \frac{1}{l} \text{Li}(t^\beta) + O \left( \frac{t}{e^{c_1 \log t}} \right) \right),
\]

where \(\text{Li}(t) = \int_{2}^{t} \frac{ds}{\log s} \ll \frac{t}{\log t}\).
The first term is
\[
\frac{u \mathrm{Li}(t)}{t L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \ll \frac{1}{\log t}
\]
the second term is
\[
\frac{u \mathrm{Li}(t^\beta)}{t L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \ll \frac{t^\beta}{\log t} \ll \frac{1}{\log t}
\]
and the third term is
\[
\ll \frac{t}{\log t} \approx o \left( \frac{t}{\log t} \right).
\]
Thus,
\[
\frac{u}{t L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \frac{\pi^*(t, l)}{u^{1-\epsilon}} \ll \frac{1}{\log u} \ll \frac{u}{l \log L_2(u)} \approx o \left( \frac{u}{L_2(u)} \right).
\]

Estimate for the main term goes as follows:

First, we compute the derivative
\[
\left( \frac{u}{t L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \right)' =
\]
\[
= u(-1) \frac{1}{t^2 L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \cdot \left( t L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right) \right)' =
\]
\[
= - \frac{u}{t^2 L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \cdot \left( \frac{1}{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} +
\]
\[
+ \frac{t}{2} \frac{1}{\sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)}} \cdot \left( \frac{t}{u} \left( \frac{u}{t^2} \right) L_3 \left( \frac{u}{t} \right) + \frac{1}{L_2 \left( \frac{u}{t} \right)} \cdot \frac{1}{L_1 \left( \frac{u}{t} \right)} \frac{t}{u} \left( \frac{u}{t^2} \right) L_1 \left( \frac{u}{t} \right) \right) \right) =
\]
\[
= - \frac{u}{t^2 \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)}} + \frac{u}{2 t^2 \left( L_1 \left( \frac{u}{t} \right) \right)^{1/2} \sqrt{L_3 \left( \frac{u}{t} \right)}} + \frac{u}{t^3 L_1 \left( \frac{u}{t} \right) L_2 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)}.
\]

Thus, the main term is:

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where $c$ is any number that satisfies $0 < c_1 < 1$. Let us estimate these three integrals:

\[
- \int_{u^{1-\epsilon}}^{\infty} \pi^*(t, l) \left( \frac{u}{t \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)}} \right) dt = \int_{u^{1-\epsilon}}^{\infty} \pi^*(t, l) \left( \frac{u}{t^2 L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} - \frac{u}{2t^2 \left( L_1 \left( \frac{u}{t} \right) \right)^2 \sqrt{L_3 \left( \frac{u}{t} \right)}} - \frac{u}{t^3 L_1 \left( \frac{u}{t} \right) L_2 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \right) dt \leq \int_{u^{1-\epsilon}}^{\infty} \pi^*(t, l) \frac{u}{t^2 L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} dt =
\]

since $u \geq x^\alpha$ implies $t \leq e^{c_1 \sqrt{t \log 2}} \leq e^{c_1 \sqrt{t \log u}}$ which is equivalent to $u^{1-\epsilon} \geq e^{c_1 (\log t)^2}$, and so we can use Lemma 4.2.1

\[
= \int_{u^{1-\epsilon}}^{\infty} \left( \frac{1}{t} \mathbb{L}(t) - \frac{1}{t} \mathbb{L}(t^3) + O \left( \frac{1}{t \log t} \right) \right) \frac{u}{t^2 L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} dt =: I_1 + I_2 + I_3.
\]

Let us estimate these three integrals:

\[I_1 = \frac{u}{T} \int_{u^{1-\epsilon}}^{\infty} \frac{\mathbb{L}(t)}{t^2 \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)}} dt \ll \frac{u}{T} \int_{u^{1-\epsilon}}^{\infty} \frac{1}{t \log t \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)}} dt =
\]

substitution: $v \log u = \log t$, $t \in [u^{1-\epsilon}, \frac{u}{\epsilon v}]$, $v \in [1 - \epsilon, 1 - \epsilon \frac{\sqrt{n}}{\log u}]$

\[
= \frac{u}{T} \int_{1-\epsilon}^{1-\epsilon \frac{\sqrt{n}}{\log u}} \frac{\log u}{v \log \sqrt{1-v} \sqrt{L_2((\log u)(1-v))}} dv =
\]

\[
= \frac{u}{T} \int_{1-\epsilon}^{1-\epsilon \frac{\sqrt{n}}{\log u}} \frac{1}{v \sqrt{\log \sqrt{1-v} \sqrt{L_2((\log u)(1-v))}}} dv +
\]

\[
+ \frac{u}{T} \int_{1-\epsilon}^{1-\epsilon \frac{\sqrt{n}}{\log u}} \frac{1}{v \sqrt{\log \sqrt{1-v} \sqrt{L_2((\log u)(1-v))}}} dv =: I_{11} + I_{12},
\]

where $c_1$ is any number that satisfies $0 < c_1 < 1$.

Note that for $v \leq 1 - \frac{c_1 \sqrt{n}}{\log u}$ we have

\[
1 - v \geq \frac{c_1 \sqrt{n}}{\log u}
\]

\[
L_2((\log u)(1-v)) \geq L_2 \left( e^{L_2(u)} \right) = c_1 L_3(u)
\]

and so \[
\frac{1}{\sqrt{L_2((\log u)(1-v))}} \leq \frac{1}{\sqrt{c_1 L_3(u)}}.
\]

Thus,
We will also need the following variant of the above lemma:

This finishes the proof of lemma.

In the estimate of the integral we used the following substitution:

\[ I \ll 3 = 2 = \log \frac{1}{u} \ll u \ll \leq \leq \leq u \ll \frac{1}{L(u) L_3(u)} \]

\[ I_1 \ll \frac{u}{l \log u} \int_{1 - e^{(L_2(u))/\log u}}^{1} \frac{1}{v \sqrt{1 - v}} \, dv \leq \]

\[ \leq \frac{u}{l \log u} \int_{1 - e^{(L_2(u))/\log u}}^{1} \frac{1}{v \sqrt{1 - v}} \, dv \ll \frac{u}{l \log u} \cdot 2 \sqrt{e^{(L_2(u))/\log u}} \ll \]

\[ \ll \frac{u}{l \log u} = o \left( \frac{u}{l \log u} \right) \]

In the estimate of the integral we used the following substitution:

\[ \int_{1 - e^{(L_2(u))/\log u}}^{1} \frac{1}{v \sqrt{1 - v}} \, dv = \]

substitution: \( \sqrt{1 - v} = z \quad v \in \left[ 1 - e^{(L_2(u))/\log u}, 1 \right] \)

\[ dz = -\frac{1}{2\sqrt{1 - v}} \quad z \in \left[ \sqrt{e^{(L_2(u))/\log u}}, 0 \right] \]

\[ = 2 \int_{0}^{1} \frac{1}{1 - z^2} \, dz = (\log(1 + z) - \log(1 - z)) \bigg|_0^{\sqrt{e^{(L_2(u))/\log u}}} = \]

\[ = \log \left( 1 + \sqrt{e^{(L_2(u))/\log u}} \right) - \log \left( 1 - \sqrt{e^{(L_2(u))/\log u}} \right) \ll 2 \sqrt{e^{(L_2(u))/\log u}}, \text{ as } u \to \infty. \]

\[ I_3 \ll \int_{u^{-1}}^{u} \frac{u}{e^{c\sqrt{\log t}}} \, dt = \int_{u^{-1}}^{u} \frac{u}{t e^{c \sqrt{\log t}}} \, \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \, dt \ll \]

since \( \epsilon_1 \) was chosen so that \( \epsilon_1 < c \), which implies \( e^{c \sqrt{\log t}} \geq e^{\epsilon_1 \sqrt{\log t}} \log t \)

\[ \ll \int_{u^{-1}}^{u} \frac{u}{t \log t} e^{\epsilon_1 \sqrt{\log t}} \, \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \, dt \leq \]

\[ \leq \frac{u}{e^{\epsilon_1 \sqrt{\log u}}} \int_{u^{-1}}^{u} \frac{u}{t \log t} e^{\epsilon_1 \sqrt{\log t}} \, \sqrt{L_1 \left( \frac{u}{t} \right) L_3 \left( \frac{u}{t} \right)} \, dt \ll \]

\[ \ll \frac{u}{e^{\epsilon_1 \sqrt{\log u}}} \sqrt{L_1(u) L_3(u)} \leq \frac{u}{l \log L_1(u) L_3(u)}, \text{ since } l \leq e^{\epsilon_1 \sqrt{\log u}} \leq e^{\epsilon_1 \sqrt{\log u}}. \]

This finishes the proof of lemma.

We will also need the following variant of the above lemma:

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Lemma 5.2.2.

\[
\sum_{x^{1-\epsilon} \leq q \leq x} \frac{x}{q \sqrt{L_1 \left( \frac{x}{q} \right) L_3 \left( \frac{x}{q} \right)}} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.
\]  

(5.4)

Proof. The proof is done in the same way as the proof of the previous of Lemma. We will use Lemma 4.2.1 here.

\[
\sum_{x^{1-\epsilon} \leq q \leq x} \frac{x}{q \sqrt{L_1 \left( \frac{x}{q} \right) L_3 \left( \frac{x}{q} \right)}} \ll
\]

\[
\ll \int_{x^{1-\epsilon}}^{x} \frac{x}{t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} d(\pi(t)) =
\]

\[
= \frac{x}{t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} \pi(t) \bigg|_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} - \int_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} \pi(t) \left( \frac{x}{t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} \right)' dt
\]

Let us estimate the first summand:

\[
\frac{x}{t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} \pi(t) \bigg|_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} \ll \frac{x}{\log t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} \bigg|_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} \ll \frac{x}{\log x}.
\]

Estimate for the main term goes as follows. We use the derivative computed in the previous proof.

\[
- \int_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} \pi(t) \left( \frac{x}{t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} \right)' dt \leq
\]

\[
\leq \int_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} \pi(t) \frac{x}{t^2 \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} dt =
\]

\[
\ll \int_{x^{1-\epsilon}}^{\frac{x}{x^{1-\epsilon}}} \frac{x}{t \log t \sqrt{L_1 \left( \frac{x}{t} \right) L_3 \left( \frac{x}{t} \right)}} dt =
\]

substitution: \(v \log x = \log t\) \(\log x \, dv = \frac{1}{t} \, dt\) \(t \in \left[ x^{1-\epsilon}, \frac{x}{x^{1-\epsilon}} \right]\) \(v \in [1 - \epsilon, 1 - \epsilon / \log x]\)

\[
= x \int_{1-\epsilon}^{1-\epsilon / \log x} \frac{1}{v \sqrt{\log x \sqrt{1 - v} \sqrt{L_2(\log x(1 - v))}}} \, dv =
\]

\[
\ll \frac{x}{\sqrt{L_1(x)L_3(x)}},
\]

where the last integral is the same as in the proof of Lemma 5.2.1. This finishes the proof of lemma. \(\square\)
Use Lemma 5.2.1 for the sum $\Sigma_{4121221}$:

$$
\Sigma_{4121221} \ll \sum_{y_1 < l \leq e^{\sqrt{x}} \sqrt{1+\sqrt{x}}} \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{x}{p^r L_1 \left(\frac{x}{lp^s}\right)} L_3 \left(\frac{x}{lp^s}\right) \\
\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < l \leq e^{\sqrt{x}} \sqrt{1+\sqrt{x}}} \frac{1}{lp^s} \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{p^s} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \frac{1}{y_1 \log y_1} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).
$$

$$
\Sigma_{4121222} = \sum_{y_1 < l \leq e^{\sqrt{x}} \sqrt{1+\sqrt{x}}} \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \left(\frac{x}{p^r q_i}\right)^{1-\varepsilon} q_i \leq \sum_{l \mid a(q_i)} \sum_{i \geq 2} \frac{x}{lp^s q_i} \leq \frac{x}{lp^s q_i} \leq
$$

$$
\leq x \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{p^s} \sum_{(x)^{2(1-\varepsilon)} \leq q_i \leq x} \frac{1}{q_i} \sum_{l \mid a(q_i)} \sum_{i \geq 2} \frac{1}{l} \ll
$$

$$
\ll x \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{p^s} \sum_{(x)^{2(1-\varepsilon)} \leq q_i \leq x} \frac{1}{q_i} \log \log(q_i) =
$$

$$
= x \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{p^s} \left(\sum_{(x)^{2(1-\varepsilon)} \leq q_i \leq x} \frac{1}{q_i} \log \log(q_i) + \sum_{q_i \leq x} \frac{1}{q_i} \log \log(q_i) \right) =
$$

$$
=: \Sigma_{4121221} + \Sigma_{4121222}.
$$

$$
\Sigma_{4121221} = x \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{p^s} \sum_{(x)^{2(1-\varepsilon)} \leq q_i \leq x} \frac{1}{q_i} \log \log(q_i) \ll
$$

$$
\ll x \sum_{\frac{1}{q_i} \geq \log x} \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{q_i^{1-\alpha}} \ll x \sum_{\frac{1}{q_i} \geq \log x} \sum_{p^r \leq \left(\frac{x}{y_1}\right)^{1-\varepsilon}} \sum_{s \geq 2} \frac{1}{q_i^{1-\alpha}} \ll \frac{x}{(\log x)^{1-\alpha}}.
$$
Choose \( \alpha < \frac{1}{2} \), which gives us.

\[
\Sigma_{41212221} = o \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right).
\]

\[
\Sigma_{4121222} = x \sum_{p^s \leq (x)^{1-\epsilon}} \frac{1}{p^s} \sum_{(x)^{2(1-\epsilon)} \leq q^i \leq x \atop q \geq \log x} \frac{1}{q^i} \log \log(q^i) \ll
\]

\[
\ll x \sum_{(x)^{1-\epsilon}} \frac{1}{x^{x^2(1-\epsilon)}} \log \log((\log x)\log x) \cdot \# \{ q \mid q \leq \log x \} \cdot \# \{ i \mid q^i \leq x \} \ll
\]

\[
\ll x \cdot \frac{1}{x^{x^2(1-\epsilon)}} L_2(x) \cdot \frac{\log x}{L_2(x)} \cdot \log x \ll x^{1-\beta} = o \left( \frac{x}{\sqrt{L_1(x)L_3(x)}} \right).
\]

This finishes the estimate of \( \Sigma_{4121222} \), which together with the estimate for \( \Sigma_{4121221} \) gives us the estimate for \( \Sigma_{412122} \).

This, in turn, finishes the estimate for \( \Sigma_{41212} \), and so also the estimate for \( \Sigma_4 \):

\[
\Sigma_4 \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.
\]

Thus, the contribution of the non-squarefree integers is

\[
\Sigma'' \leq \Sigma_3 + \Sigma_4 \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.
\]

### 5.3 Contribution of squarefree numbers

Let us estimate the number \( \Sigma' \) of squarefree numbers \( n \) such that \( (n, a(n)) \) is prime.

We have

\[
\Sigma' = \Sigma_1 + \Sigma_2 = \sum_{l \leq x \atop l \text{ prime}} \sum_{m \leq \frac{x}{l}} 1 + \sum_{l \leq x \atop l \mid a(l)} \sum_{m \leq \frac{x}{l}} 1 =
\]

\[
= \sum_{l \leq x \atop l \text{ prime}} \sum_{m \leq \frac{x}{l}} 1 + \sum_{l \leq x \atop l \mid a(l)} \sum_{m \leq \frac{x}{l}} 1 =: F + F_4.
\]

#### 5.3.1 Estimation of \( F_4 \)

\[
F_4 = \sum_{l \leq x \atop l \mid a(l)} \sum_{m \leq \frac{x}{l}} 1 = \sum_{l \leq x^2 \atop l \mid a(l)} \sum_{m \leq \frac{x}{l}} 1 + \sum_{x^{1-\epsilon} \leq l \leq x \atop l \mid a(l)} \sum_{m \leq \frac{x}{l}} 1 =: F_{41} + F_{42}.
\]
where we used the fact that the sum
\[ \sum_{(l,a(l))} \frac{1}{l} \leq \sum_{l \leq x^{1-\varepsilon}} \frac{1}{l} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}}, \]
is bounded.

We use Lemma 5.2.2 for the estimate of \( F_{42} \):
\[ F_{42} \ll \sum_{x^{1-\varepsilon} < l \leq x} \frac{x}{l \sqrt{L_1(x) L_3(x)}} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}}. \]

### 5.3.2 Splitting of the sum \( F \)

\[
F = \sum_{l \leq y_1} \sum_{l - \text{prime}} \sum_{m \leq \frac{x}{l}} \sum_{y_1 < l \leq x} \sum_{(m,a(m))=1 \atop (m,a(l))=1} 1 = \\
= \sum_{l \leq y_1} \sum_{l - \text{prime}} \sum_{m \leq \frac{x}{l}} \sum_{d|a(m) \atop l|a(m)} \mu(d) + \sum_{y_1 < l \leq x} \sum_{l - \text{prime}} \sum_{m \leq \frac{x}{l}} \sum_{d|a(m) \atop l|a(m)} 1 \leq \\
\leq \sum_{l \leq y_1} \sum_{d|a(l)} \mu(d) \left( \sum_{m \leq \frac{x}{l}} \sum_{d|a(m) \atop (m,a(m))=1 \atop (m,a(l))=1} 1 \right) + \sum_{y_1 < l \leq x} \sum_{d|a(m) \atop (m,a(m))=1 \atop l|a(m)} 1 = \\
= \sum_{l \leq y_1} \sum_{d|a(l)} \mu(d) f \left( \frac{x}{l^2 d}, d \right) + \sum_{y_1 < l \leq x} \sum_{d|a(m) \atop (m,a(m))=1 \atop l|a(m)} 1 = \\
\sum_{l \leq y_1} \sum_{d|a(l)} \mu(d) f \left( \frac{x}{l^2 d}, d \right) + \sum_{l \leq y_1} \sum_{d|a(d) \atop d > \left( \frac{x}{l^2 d} \right)^{\frac{1}{x-2y}}} \mu(d) f \left( \frac{x}{l^2 d}, d \right) + \sum_{y_1 < l \leq x} \sum_{d|a(m) \atop (m,a(m))=1 \atop l|a(m)} 1 = \\
=: F_1 + F_2 + F_3.
\]
5.3.3 Estimation of the sum $F_2$

Let us estimate $F_2$:

$$F_2 = \sum_{l \leq y_1} \sum_{d \mid a(l)} \mu(d) f \left( \frac{x}{l}, d \right) \leq \sum_{l \leq y_1} \sum_{d \mid a(l), d > (\frac{x}{l})^{2(k+2)}} \frac{x}{ld} \leq \sum_{l \leq y_1} \frac{x}{l^{2(k+2)}} \cdot \# \{d - \text{squarefree}, d \mid a(l) \} \ll x^{1 - \frac{1}{2(k+2)^2}} \cdot l \ll x^{1 - \frac{1}{2(k+2)^2}} \cdot y_1^2 \ll x^{1 - \epsilon}.$$  

The estimate for the number of squarefree divisors of $|a(l)|$ was done as follows. First, we estimate $\omega(|a(l)|)$ – the number of distinct prime divisors of $|a(l)|$ using a trivial estimate:

$$\omega(|a(l)|) \leq \log |a(l)| \ll \log l.$$  

Then, since the number of squarefree divisors of $|a(l)|$ is equal to $2^{\omega(|a(l)|)}$, we have:

$$\# \{d - \text{squarefree}, d \mid a(l) \} \ll 2^{\log l} = l.$$  

5.3.4 Estimation of the sum $F_3$

Let us estimate $F_3$.

$$F_3 = \sum_{y_1 \leq l \leq x} \sum_{m \leq \frac{x}{l}, (m, a(m)) = 1, (m, a(l)) = 1, l \mid a(m)} 1 = \sum_{y_1 \leq l \leq e^{\epsilon \sqrt{\log x}}} \sum_{m \leq \frac{x}{l}, (m, a(m)) = 1, (m, a(l)) = 1, l \mid a(m)} 1 + \sum_{e^{\epsilon \sqrt{\log x}} < l \leq x} \sum_{m \leq \frac{x}{l}, (m, a(m)) = 1, (m, a(l)) = 1, l \mid a(m)} 1 : = F_{31} + F_{32}.$$  

To estimate the sum $F_{32}$ we will switch the order of summation and use the fact that $l$'s are big:

$$F_{32} = \sum_{e^{\epsilon \sqrt{\log x}} < l \leq x} \sum_{m \leq \frac{x}{l}, (m, a(m)) = 1, (m, a(l)) = 1, l \mid a(m)} 1 \leq \sum_{m \leq \frac{x}{e^{\epsilon \sqrt{\log x}}}, (m, a(m)) = 1} \sum_{e^{\epsilon \sqrt{\log x}} < l \leq x} \sum_{l \mid a(m)} 1 \ll \sum_{m \leq \frac{x}{e^{\epsilon \sqrt{\log x}}}, (m, a(m)) = 1} \sqrt{\log x} \ll \frac{x}{e^{\epsilon \sqrt{\log x}} \sqrt{\log x}} = o \left( \frac{x}{\sqrt{L_1(x) L_3(x)}} \right).$$

Let us estimate $F_{31}$. Since we are estimating the number of squarefree integers $n$, we have:
\[ F_{31} = \sum_{y_1 \leq \ell \leq e^{\log^{3/2} x}} \sum_{m \leq \frac{1}{\ell}, (m, a(m)) = 1} \sum_{p \leq \frac{1}{\ell}, (m, a(m)) = 1} \sum_{l \mid a(p)} 1 = \]

\[ = \sum_{y_1 \leq \ell \leq e^{\log^{3/2} x}} \left( \sum_{p \leq \frac{1}{\ell}, \frac{1}{\ell} \leq m \leq \frac{1}{\ell}} \sum_{l \mid a(p), (m, a(m)) = 1} 1 + \sum_{l \mid a(p), (m, a(m)) = 1} \right) \]

\[ =: F_{311} + F_{312}. \]

We will use Lemma 5.2.1 for the estimate of \( F_{312} \):

\[ F_{312} = \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \sum_{m \leq \frac{1}{l}, (m, a(m)) = 1} \sum_{p \leq \frac{1}{l}, (m, a(m)) = 1} 1 \ll \]

\[ \ll \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \sum_{m \leq \frac{1}{l}, (m, a(m)) = 1} \frac{x}{l} \sum_{l \mid a(p)} \frac{x}{L_1 \left( \frac{x}{l} \right) L_3 \left( \frac{x}{l} \right)} \]

since \( \frac{x}{l} \geq \frac{x}{e^{1/10} \log x} \ll x^\alpha \), we can use Lemma 5.2.1 here

\[ \ll \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \sum_{l \mid a(p)} \frac{x}{L_1 \left( \frac{x}{l} \right) L_3 \left( \frac{x}{l} \right)} \ll \frac{x}{\sqrt{L_1 \left( x \right) L_3 \left( x \right)}}. \]

\[ F_{311} = \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \sum_{p \leq \frac{1}{l}, \frac{1}{l} \leq m \leq \frac{1}{l}} \sum_{l \mid a(p), (m, a(m)) = 1} 1 \ll \]

\[ \ll \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \sum_{l \mid a(p)} \frac{x}{L_1 \left( \frac{x}{l} \right) L_3 \left( \frac{x}{l} \right)} \ll \]

\[ \ll \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \sum_{l \mid a(p)} \frac{x}{L_1 \left( \frac{x}{l} \right) L_3 \left( \frac{x}{l} \right)} \ll \]

since \( l \leq x^{1-\epsilon} \),

\[ \ll \frac{x}{\sqrt{L_1 \left( x \right) L_3 \left( x \right)}} \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \frac{1}{l} \sum_{p \leq \frac{1}{l}, \frac{1}{l} \leq m \leq \frac{1}{l}} \frac{1}{p} \]

\[ \ll \frac{x}{\sqrt{L_1 \left( x \right) L_3 \left( x \right)}} \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \frac{1}{l} \sum_{l \mid a(p)} \frac{1}{p} \]

\[ = x \frac{1}{\sqrt{L_1 \left( x \right) L_3 \left( x \right)}} \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \frac{1}{l} \sum_{l \mid a(p)} \frac{1}{p} \]

\[ \ll x \frac{1}{\sqrt{L_1 \left( x \right) L_3 \left( x \right)}} \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \frac{1}{l} \sum_{l \mid a(p)} \frac{1}{p} \]

\[ = x \frac{1}{\sqrt{L_1 \left( x \right) L_3 \left( x \right)}} \sum_{y_1 \leq l \leq e^{\log^{3/2} x}} \frac{1}{l} \sum_{l \mid a(p)} \frac{1}{p} \]
Let us estimate the inner sum:

$$\sum_{p \leq \left(\frac{x}{y_1}\right)^{1 - \epsilon}} \frac{1}{p} \leq \sum_{p \leq x} \frac{1}{p} \sum_{l \leq x} \frac{1}{p} + \sum_{l \leq x} \frac{1}{p} \ll$$

by Lemma 8.1

$$\ll \frac{1}{l} L_2(x) + \int_{l^2 \log l}^{e^{\frac{1}{8} (\log l)^2}} \frac{1}{l} \pi^*(t, l) \frac{1}{t^2} \, dt + \sum_{p \leq l^{\frac{1}{2} \log l}} \frac{1}{p}.$$ 

Thus,

$$F_{311} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 \leq l \leq e^{1/\log x}} \frac{1}{l} \left(\frac{1}{l} L_2(x) + \int_{l^2 \log l}^{e^{\frac{1}{8} (\log l)^2}} \frac{1}{l} \pi^*(t, l) \frac{1}{t^2} \, dt + \sum_{p \leq l^{\frac{1}{2} \log l}} \frac{1}{p}\right) =: F_{3111} + F_{3112} + F_{3113}.$$

$$F_{3111} \ll \frac{x L_2(x)}{\sqrt{L_1(x) L_3(x)}} \cdot \frac{1}{y_1 \log y_1} = o \left(\frac{x}{\sqrt{L_1(x) L_3(x)}}\right).$$

$$F_{3112} = \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 \leq l \leq e^{1/\log x}} \frac{1}{l} \int_{l^2 \log l}^{e^{\frac{1}{8} (\log l)^2}} \pi^*(t, l) \frac{1}{t^2} \, dt \leq$$

switch order of summation and integration

$$\leq \frac{x}{\sqrt{L_1(x) L_3(x)}} \int_{2y_1^2 \log y_1}^{x} \frac{1}{t^2} \left(\sum_{e^{1/\log x} \leq l \leq \sqrt{\frac{y_1}{\log x}}} \frac{1}{l} \pi^*(t, l)\right) \, dt \leq$$

since $$\#\{l \geq e^{1/\log x} \mid l \mid a(q)\} \ll \log x$$

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \int_{2y_1^2 \log y_1}^{x} \frac{1}{t^2} \cdot \sqrt{\frac{t \log t}{e^{1/\log x}}} \, dt =$$

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\[
\frac{x}{\sqrt{L_1(x) L_3(x)}} \int_{y_1^2 \log y_1}^{x} \frac{1}{t} \cdot \frac{1}{\sqrt{\log t}} \cdot \frac{1}{e^{\epsilon_1 \sqrt{\log t}}} dt = \\
\text{substitution } u = e^{\epsilon_1 \sqrt{\log t}}
\]

\[
\frac{x}{\sqrt{L_1(x) L_3(x)}} \left( -\frac{2}{\epsilon_1} e^{\epsilon_1 \sqrt{\log t}} \right) \bigg|_{y_1^2 \log y_1}^{x} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \cdot \frac{1}{e^{\epsilon_1 \sqrt{\log (y_1^2 \log y_1)}}}
\]

\[
= o \left( \frac{x}{\sqrt{L_1(x) L_3(x)}} \right).
\]

\[
F_{3113} = \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 \leq t \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{t} \sum_{p \leq t^{2 \log t}} \frac{1}{p} \ll
\]

\[
\leq \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{p \leq t \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{t} \sum_{l \geq \sqrt{\log p}} \frac{1}{l} \ll
\]

note that for each \( p \) the number of distinct \( l \)'s is estimated as follows:

\[
\# \{ l \geq \sqrt{\log p} \mid l | a(p) \} \ll 1,
\]

since

\[
\sqrt{\log p} \leq l_1 l_2 \cdots l_i \leq |a(p)| \leq p^{\frac{k-1}{2} + \epsilon},
\]

and so

\[
i \leq \frac{\log p}{\log \left( \sqrt{\frac{p}{\log p}} \right)} \ll 1.
\]

\[
\leq \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{p \leq x} \frac{1}{p} \cdot \frac{1}{\sqrt{\log p}} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{p \leq x} \frac{1}{p^{\frac{1}{2} - \alpha}} \ll
\]

\[
\ll \frac{x}{\sqrt{L_1(x) L_3(x)}}.
\]

Thus,

\[
F_3 \ll \frac{x}{\sqrt{L_1(x) L_3(x)}}.
\]

5.3.5 Estimation of \( F_1 \)

We have now shown that

\[
\Sigma' = F_1 + O \left( \frac{x L_4(x)}{\sqrt{L_1(x) L_3(x)}} \right)
\]

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To estimate $F_1$ we need to estimate the function $f \left( \frac{x}{p}, d \right)$ for $d \leq \left( \frac{x}{L_L} \right)^{\frac{1}{d+2}}$.

We will work with the function

$$f(x, u) = \sum_{m \leq x \atop (m, a(m)) = 1} 1 = \# \{ m \leq x \mid u \mid m, (m, a(m)) = 1 \},$$

and estimate it for $u \leq x^{\frac{1}{d+2}}$.

It was estimated (Theorem 1.2.1) that

$$\sum_{m \leq x \atop (m, a(m)) = 1} 1 = (1 + o(1)) \frac{U_p x}{\sqrt{L_1(x)} L_3(x)}.$$  \hspace{1cm} (5.5)

We prove the following estimate:

**Lemma 5.3.1.** For $u \leq x^{\frac{1}{d+2}}$,

$$f(x, u) = (1 + o(1)) \frac{U_p x}{u \sqrt{L_1(x)} L_3(x)} + O \left( \frac{x (\nu(u))^2}{u \sqrt{L_1(x)} L_3(x)} \right) + O \left( \frac{x \nu(u) L_2(u)}{u \sqrt{L_1(x)} L_3(x)} \right).$$

**Proof.** To prove this, we use the same technique as in the proof of (5.5). We break up the set of all the $m$’s that satisfy $m \leq x, u \mid m$ and $(m, a(m)) = 1$ into the union of sets $\{ m \leq x \mid p \mid m, u \mid m, (m, a(m)) = 1, q \mid m \Rightarrow q \geq p \}$, i.e. we group them according to the smallest prime divisor of $m$ (slightly abusing notation we denote the sets and the numbers of elements in these sets by the same letter). Denote

$$G_p(x, u) = \# \{ m \leq x \mid p \mid m, u \mid m, (m, a(m)) = 1, q \mid m \Rightarrow q \geq p \}.$$

Note that some of these sets will be empty. If $m \in G_p(x, u)$, then $p \mid m, u \mid m$, and all the prime divisors of $m$ are $\geq p$. Thus, all the prime divisors of $u$ have to satisfy this condition, since $u \mid m$. So, if there exists $q_1$ such that $q_1 \mid u, q_1 < p$, then $G_p(x, u) = 0$.

Denote by $p_u$ the smallest prime divisor of $u$. Then $G_p(x, u) = 0$ if $p > p_u$, since in that case $p$ cannot be the smallest divisor of $n$.

As before, we denote by $\nu(p, u) = \# \{ q^m \mid n, a(q^m) \equiv 0 \pmod{p} \}$. Note that $\nu(p, u) = 0$ for all $p \mid n$ means that $(n, a(n)) = 1$.

$$\sum_{m \leq x \atop m \equiv 0 \pmod{u}} 1 = \sum_{p \text{ prime} \atop p > p_u} G_p(x, u) = \sum_{p \text{ prime} \atop p \leq p_u} G_p(x, u).$$

We split the sum $\sum_{p \text{ prime}} G_p(x, u)$ into three parts:

$$\sum_{p \text{ prime}} G_p(x, u) = A_1(x, u) + A_2(x, u) + A_3(x, u),$$

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where
\[ A_1(x, u) = \sum_{p \leq (\log \log x)^{1-\varepsilon}} G_p(x, u) \]
\[ A_2(x, u) = \sum_{(\log \log x)^{1-\varepsilon} \leq p \leq (\log \log x)^{1+\varepsilon}} G_p(x, u) \]
\[ A_3(x, u) = \sum_{p \geq (\log \log x)^{1+\varepsilon}} G_p(x, u) \]

Note that if \( p_u \leq (\log \log x)^{1+\varepsilon} \), then \( A_3(x, u) = 0 \), and if \( p_u \leq (\log \log x)^{1-\varepsilon} \), then \( A_2(x, u) = A_3(x, u) = 0 \).

The estimates of \( A_1(x, u) \), \( A_2(x, u) \) and \( A_3(x, u) \) depend on variants of the Lemma 4.1.1 and Lemma 4.1.2:

**Lemma 5.3.2.** (Analogue of Lemma 4.1.1 for the case \( u|n \))

\[
N_{y_1}(x, u) = \frac{U_f x}{u \sqrt{L_1(x)L_3(x)}} + O \left( \frac{x(L_3(x))^2}{u(\log x)^{1+\varepsilon}} \right).
\]

**Proof.** As before, set \( P_{y_1} = \prod_{p < y_1} p \).

Denote by \( M_{f,d}(x, u) \) the number
\[
M_{f,d}(x, u) = \# \{ n \leq x \mid a(n) \neq 0, d|n, u|n \}.
\]

Then
\[
N_{y_1}(x, u) = \sum_{d|P_{y_1}} \mu(d)M_{f,d}(x, u) \text{ by the principle of inclusion-exclusion.}
\]

Note that \( (d, u) = 1 \) because all prime divisors of \( u \) are \( \geq y_1 \). Thus,
\[
M_{f,d}(x, u) = M_{f,du}(x) \text{ in the notation of [6].}
\]

By Proposition 2.2.18 we have
\[
M_{f,du}(x) = \frac{u_f \xi_{du}(1) \frac{x}{du}}{(\log \frac{x}{du})^{1+\varepsilon}} + O \left( \frac{x 2^{\nu(du)}}{du (\log \frac{x}{du})^{1+\varepsilon}} \right).
\]

Here \( \xi_{du} \) satisfies the conditions 2.2.2 by Proposition 2.2.21.

Thus,
\[
N_{y_1}(x, u) = \sum_{d|P_{y_1}} \mu(d) \left( \frac{u_f \xi_{du}(1) \frac{x}{du}}{(\log \frac{x}{du})^{1+\varepsilon}} + O \left( \frac{x 2^{\nu(du)}}{du (\log \frac{x}{du})^{1+\varepsilon}} \right) \right) = \frac{u_f x}{u \sqrt{\log x}} \sum_{d|P_{y_1}} \mu(d) \left( \xi_{du}(1) + O \left( \frac{2^{\nu(du)}}{\log x} \right) \right),
\]

the last equality follows from the fact that \( u \leq x^{\frac{1}{\log \log x}} \) and \( d \leq y_1 = (L_2(x))^{1+\varepsilon} \).
The main term is

\[ \frac{u_f x}{u \sqrt{\log x}} \sum_{d | y_1} \frac{\mu(d) \xi_{du}(1)}{d} = \]

\[ = \frac{u_f x}{u \sqrt{\log x}} \sum_{d | y_1} \prod_{p | d} \left( \frac{-1}{p} \xi_{p, du}(1) \right) = \]

\[ = \frac{u_f x}{u \sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{\xi_{p, du}(1)}{p} \right) = \]

If \( i_f(p) = 0 \), then by Proposition 2.2.21 \( \xi_{p, du}(1) = 1 \); if \( i_f(p) = 1 \), and \( u \) is squarefree, then \( du \) is also squarefree, and by Proposition 2.2.21 \( \xi_{p, du}(1) = \frac{1}{p} \). Thus

\[ \frac{u_f x}{u \sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{\xi_{p, du}(1)}{p} \right) = \]

where the last product is finite,

\[ \frac{u_f x}{u \sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{1}{p} \right) \prod_{p < y_1} \left( 1 - \frac{1}{p^2} \right) \prod_{p < y_1} \left( 1 - \frac{\xi_{p, du}(1)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-1} = \]

\[ = \frac{u_f x}{u \sqrt{\log x}} \prod_{p < y_1} \left( 1 - \frac{1}{p} \right) \cdot C_f, \text{ say,} \]

use Proposition 2.1.1

\[ = C_f \left( \frac{u_f x}{u \sqrt{\log x} \sqrt{\log y_1}} \right) + O_f \left( \frac{1}{(\log y_1)^{3/2} u \sqrt{\log x}} \right) = \]

\[ = \frac{u_f C_f x}{u \sqrt{L_1(x) L_3(x)}} \left( \frac{1}{\sqrt{1 + \epsilon}} + O_f \left( \frac{1}{\log \log x} \right) \right). \]

\[ \square \]

Lemma 5.3.3. (Analogue of Lemma 4.1.2 for the case \( u | n \)) Suppose that \( p \leq y_1 \), \( u \leq x^{\frac{1}{n_1 + 1/3}} \), and let \( p_u \) denote the smallest prime divisor of \( u \). We have

1) \( \# \{ n \leq x \mid p | n, u | n, a_f(n) \neq 0, q | n \Rightarrow q \geq p \} \ll \)

\[ \ll \frac{x}{(u \log x)^2} \prod_{l \leq p} \left( 1 - \frac{1}{l} \right) + \frac{x}{u (\log x)^2} \frac{2^{1+\nu(n)} (\log p)^2}{p}. \]

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Estimate for $A_1(x, u)$:

$$A_1(x, u) = \sum_{p \leq (\log \log x)^{1/2-\epsilon}} G_p(x, u) =$$

$$= \sum_{p \leq (\log \log x)^{1/2-\epsilon}, p \nmid u} G_p(x, u) + \sum_{p \leq (\log \log x)^{1/2-\epsilon}, p \mid u} G_p(x, u)$$

$$= P_1(x, u) + P_2(x, u), \text{ say.}$$

Note that $G_p(x, u) = 0$ if $p > p_u$, where $p_u$ denotes the smallest prime divisor of $u$. Thus, the second sum $P_2(x, u)$ will consist of at most one nonzero term $G_{p_u}(x, u)$, since $G_p(x, u) = 0$ for $p > p_u$. We have

$$P_2(x, u) = \sum_{p \leq (\log \log x)^{1/2-\epsilon}, p \mid u} G_p(x, u) =$$

$$= \begin{cases} G_{p_u}(x, u), & \text{if } p_u \leq (\log \log x)^{1/2-\epsilon} \\ 0, & \text{otherwise.} \end{cases}$$

So, we need to estimate

$$G_{p_u}(x, u) = \# \{m \leq x \mid p_u | m, u | m, (m, a(m)) = 1, q | m \Rightarrow q \geq p_u \} =$$

$$= \sum_{m_0 \leq x, (m_0, a(m_0u)) = 1} 1 \leq \sum_{m_0 \leq x, (m_0, a(m_0u)) = 1} \frac{x}{u \sqrt{L_1(\frac{x}{u})} L_3(\frac{x}{u})} \ll \frac{x}{u \sqrt{L_1(x)L_3(x)}}, \text{ since } u \leq x^{1/(2k+2)}.$$
\[ P_1(x, u) \leq \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \{ n \leq x \mid p|n, (n, a(n)) = 1, u|n \} = \]

\[ = \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \# \{ n \leq x \mid pu|n, \nu(p_1, n) = 0 \text{ for all } p_1|n \} = \]

\[ = \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \# \left\{ m \leq \frac{x}{u} \mid p|m, \nu(p_1, mu) = 0 \text{ for all } p_1|mu \right\} = \]

\[ = \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \# \left\{ m \leq \frac{x}{u} \mid p|m, (u, m) = 1, \nu(p_1, mu) = 0 \text{ for all } p_1|mu \right\} \]

Since we are working in the squarefree case, we have \((m, u) = 1\), and so:

\[ \nu(p, mu) = \# \{ q^k||mu \mid a(q^k) \equiv 0 \pmod{p} \} = \]

\[ = \# \{ q^k||m \mid a(q^k) \equiv 0 \pmod{p} \} + \# \{ q^k||u \mid a(q^k) \equiv 0 \pmod{p} \} = \]

\[ = \nu(p, m) + \nu(p, u) \]

\[ \nu(p, mu) = 0 \text{ means that } \nu(p, m) = 0 \text{ and } \nu(p, u) = 0. \]

Thus,

\[ P_1(x, u) \ll \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \# \left\{ m \leq \frac{x}{u} \mid \nu(p, mu) = 0 \right\} = \]

\[ = \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \# \left\{ m \leq \frac{x}{u} \mid \nu(p, m) = 0, \nu(p, u) = 0 \right\} \leq \]

\[ \leq \sum_{p \leq (\log \log x)^{1/2-\epsilon}} \# \left\{ m \leq \frac{x}{u} \mid \nu(p, m) = 0 \right\} = \text{ using the result from [6]} \]

\[ = o \left( \frac{x}{u \sqrt{L_1(x) L_3(x)} \left( \frac{5}{2} \right)} \right) = o \left( \frac{x}{u \sqrt{L_1(x) L_3(x)}} \right), \]

since this estimate is obtained under the condition \(u \leq x^{\frac{1}{\log \log x}}\). Thus,

\[ A_1(x, u) \ll P_1(x, u) + P_2(x, u) = o \left( \frac{x}{u \sqrt{L_1(x) L_3(x)}} \right). \]

The estimate for \(A_2(x, u)\) is done in the same way as the estimate for \(A_2(x)\) before, only this time \(u\)
Since we are working in the squarefree case, we can write
\[ A_2(x, u) \leq \sum_{L_2^{\frac{1}{2}+\varepsilon}(x) < p < L_2^{1+\varepsilon}(x)} \sum_{\nu \mid n, u | n, a(n) \neq 0, q | n} \frac{1}{\nu} \prod_{l \text{ prime}} \left( 1 - \frac{1}{l} \right) \]
\[ \leq \frac{x}{u \sqrt{\log x}} \sum_{L_2^{\frac{1}{2}+\varepsilon}(x) < p < L_2^{1+\varepsilon}(x)} \frac{1}{p \log p} \]
\[ \leq \frac{x}{u \sqrt{\log x}} \cdot \frac{1}{\log(L_2^{1-\varepsilon}(x))} \cdot \sum_{L_2^{\frac{1}{2}+\varepsilon}(x) < p < L_2^{1+\varepsilon}(x)} \frac{1}{p} \]
\[ \leq \frac{x}{u \sqrt{\log x}} \cdot \frac{1}{L_3(x)} \cdot L_3(x) = o\left( \frac{x}{u(L_3(x) \log x)^{\frac{3}{2}}} \right) . \]

Let us estimate \( A_3(x, u) \). Let \( y_1 = L_2(x)^{1+\varepsilon} \) and \( N_{y_1}(x, u) = \# \{ n \leq x | q | n \Rightarrow q \geq y_1, a(n) \neq 0, u | n \} \). Note that all prime divisors of \( u \) have to be \( \geq y_1 \), otherwise \( N_{y_1}(x, u) = 0 \), because the set of numbers counted by \( N_{y_1}(x, u) \) is empty. Then
\[ N_{y_1}(x, u) - \sum_{y_1 \leq q_1, q_2 \leq x} \sum_{\nu \mid n, q_1 \equiv 0 \pmod{q_2}, q_1 | n, q_2 | n, u | n} \frac{1}{\nu} 1 \leq A_3(x, u) \leq N_{y_1}(x, u) . \]

Since we are working in the squarefree case, we can write
\[ N_{y_1}(x, u) - \sum_{y_1 \leq q_1, q_2 \leq x} \sum_{\nu \mid n, q_1 \equiv 0 \pmod{q_2}, q_1 | n, q_2 | n, u | n} \frac{1}{\nu} 1 \leq A_3(x, u) \leq N_{y_1}(x, u). \]

By Lemma 5.3.2, to prove Lemma 5.3.1, it suffices to show that
\[ \sum_{y_1 \leq q_1, q_2 \leq x} \sum_{\nu \mid n, q_1 \equiv 0 \pmod{q_2}, q_1 | n, q_2 | n, u | n} \frac{1}{\nu} 1 = o\left( \frac{x}{u \sqrt{L_1(x)}} \right) + O\left( \frac{x^3}{u \sqrt{L_1(x) L_3(x)}} \right) + O\left( \frac{x^3}{u \sqrt{L_1(x) L_3(x)}} \right) . \]
\[ (5.6) \]

In order to prove (5.6) we use the same technique as for the estimate of the sum \( B_2 \) in the proof of Theorem 1.1.2. Our sum differs from the sum \( B_2 \) by having an additional condition \( u | n \).
\[
\sum_{y_1 \leq q_1, q_2 \leq x \atop a(q_1) \equiv 0 \pmod{q_2}} \sum_{n \leq x \atop q \mid q \alpha(q)} 1 \leq \sum_{y_1 \leq q_1, q_2 \leq x \atop q_1 \neq q_2} \sum_{n \leq x \atop q_1 \mid n, q_2 \mid n, u \mid n} 1 = \\
= \sum_{y_1 \leq q_1, q_2 \leq x \atop q_1 \mid n, q_2 \mid n, u \mid n} \sum_{n \leq x \atop a(q_1) \equiv 0 \pmod{q_2}} \sum_{y_1 \leq q_1, q_2 \leq x \atop q_1 \mid u, q_2 \mid u, \; a(q_1) \equiv 0 \pmod{q_2}} 1 =: E_1(x, u) + E_2(x, u).
\]

where \( D_1(x, u) \) and \( D_2(x, u) \) are defined by \( y_1 \leq q_1 \leq x^{\frac{k}{k+1}} \) and \( x^{\frac{k}{k+1}} < q_1 \leq x \).

Since \( u \leq x^{\frac{k}{k+1}} \Rightarrow q_1 q_2 u \ll x^\alpha \) for some \( 0 < \alpha < 1 \), in the estimate for \( E_1(x, u) \) we have

\[
\frac{1}{\log \left( \frac{x}{q_1 q_2 u} \right)} \ll \frac{1}{\log x},
\]

and the \( \frac{1}{u} \) from the expression \( \frac{x}{q_1 q_2 u} \) will factor out from the sums.

For \( D_1(x, u) \) we use Proposition 2.2.18:

\[
D_1(x, u) \ll \sum_{y_1 \leq q_1 \leq x^{\frac{k}{k+1}} \atop y_1 \leq q_2 \leq 2q_1 \atop a(q_1) \equiv 0 \pmod{q_2}} M_{f, q_1 q_2 u}(x) \ll \sum_{y_1 \leq q_1 \leq x^{\frac{k}{k+1}} \atop y_1 \leq q_2 \leq 2q_1 \atop a(q_1) \equiv 0 \pmod{q_2}} u f x \xi_{q_1 q_2 u}(1) \frac{1}{q_1 q_2 u \log \left( \frac{x}{q_1 q_2 u} \right)^{\frac{k}{k+1}}}.
\]

We need to have \( q_1 q_2 u \leq x^\alpha < x \).

\[
q_1 q_2 \leq 2 x^{\frac{k+1}{k+2}}
\]

\[
q_1 q_2 u \leq 2 x^{\frac{k+1}{k+2}} u \leq x^\alpha < x
\]

\[
u \leq x^{-\frac{k+1}{k+2}}.
\]

Take \( \alpha = \frac{k+2}{k+1} \). Then with \( u \leq x^{\frac{k+2}{k+1}} \) we have \( q_1 q_2 u \leq x^\alpha \). Thus, \( \frac{x}{q_1 q_2 u} \gg x^{1-\alpha} = x^{\frac{1}{k+2}} \), and so

\[
\frac{1}{\log \left( \frac{x}{q_1 q_2 u} \right)^{1/2}} \ll \frac{\sqrt{2(k+2)}}{(\log x)^{3/2}} \ll \frac{1}{(\log x)^{1/2}}.
\]
Thus,

$$D_1(x, u) \ll \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{q_1, q_2 \leq x \\ q_1 | u, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} \frac{uf \xi_{q_1, q_2} u(1) \sqrt{2(k+2)}}{q_1 q_2 u (\log x)^{\frac{1}{2}}} \ll \frac{x}{u (\log x)^{\frac{1}{2}}} \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{q_1, q_2 \leq x \\ q_1 | u, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} \frac{1}{q_1 q_2} = o \left( \frac{x}{u \sqrt{L_1(x)L_2(x)}} \right),$$

because the sum

$$\sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{q_1, q_2 \leq x \\ q_1 | u, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} \frac{1}{q_1 q_2}$$

does not depend on u and was estimated for the number $D_1(x, u)$ in the proof of Theorem 1.1.2.

$D_2(x, u)$ is estimated in the same way as before, because everywhere \( \frac{x}{u q_1 q_2} \) will replace \( \frac{x}{q_1 q_2} \), and so \( \frac{1}{u} \) factors out from the sums.

The sum $E_2(x, u)$ contains those cases when $q_1 | u$ or $q_2 | u$. We split it into three parts:

$$E_2(x, u) = \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{n \leq x, u | n \\ q_1 | n, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} 1 + \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{n \leq x, u | n \\ q_1 | n, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} 1 + \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{n \leq x, u | n \\ q_1 | n, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} 1 =$$

recall that $u \leq x^{\frac{1}{k+2}}$

$$= \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{n \leq x, u | n \\ q_1 | n, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} 1 + \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{n \leq x, u | n \\ q_1 | n, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} 1 + \sum_{y_1 \leq x \leq \frac{x}{k-1}} \sum_{\substack{n \leq x, u | n \\ q_1 | n, q_2 | n, \ a(q_1) \equiv 0 \ (\text{mod } q_2)}} 1 =$$

$$= E_{21}(x, u) + E_{22}(x, u) + E_{23}(x, u).$$

Let us estimate these three quantities and show that their contribution to the sum $F_1$ is small.
\[ E_{21}(x, u) = \sum_{y_1 \leq q_1 \leq x} \sum_{y_1 \leq q_2 \leq x} \sum_{y_1 \leq q_2 \leq x} 1 \ll \sum_{y_1 \leq q_2 \leq x} \sum_{q_2 \mid n} \left( \sum_{n \leq x} 1 \right) \cdot \prod_{p \leq y_1} \left( 1 - \frac{1}{p} \right) \ll \]

\[ \ll \frac{1}{L_3(x)} \sum_{q_2 \mid n} \frac{x}{q_2 u} \ll \]

since \( q_2 \mid a(q_1) \Rightarrow q_2 \leq |a(q_1)| \leq 2q_1^\frac{1}{2} \leq 2u^\frac{1}{2} \leq 2x^\frac{1}{2} \leq 2x^\frac{k-1}{4} \leq 2x^\frac{1}{4} \),

and so \( q_2 u \leq 2x^\frac{k}{4} x^\frac{1}{2} \ll x^{1-\epsilon} \)

\[ \ll \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_1 | u, y_1 \leq q_2 \leq x} \frac{1}{q_2} \ll \]

since \( \sum_{y_1 \leq q_2 \leq x} \frac{1}{q_2} \leq \sum_{q_2 \mid a(q_1)} \frac{1}{q_2} \ll L_2(|a(q_1)|) \ll L_2(u) \)

\[ \ll \frac{x}{L_3(x) \sqrt{\log x}} \cdot \frac{\nu(u) L_2(u)}{u} . \]

\[ E_{22}(x, u) = \sum_{y_1 \leq q_1 \leq x} \sum_{y_1 \leq q_2 \leq x} \sum_{y_1 \leq q_2 \leq x} 1 \ll \sum_{y_1 \leq q_2 \leq x} \sum_{q_2 \mid n} \left( \sum_{n \leq x} 1 \right) \cdot \prod_{p \leq y_1} \left( 1 - \frac{1}{p} \right) \ll \]

\[ \ll \frac{1}{L_3(x)} \sum_{q_2 \mid n} \frac{x}{q_2 u} \ll \]

\[ = \frac{1}{L_3(x)} \sum_{q_2 \mid n} \left( \sum_{y_1 \leq q_1 \leq x} \frac{x}{q_1 u} \right) \sqrt{\log \frac{x}{q_1 u}} + \sum_{y_1 \leq q_2 \leq x} \frac{x}{q_2 u} \sqrt{\log \frac{x}{q_2 u}} \right) =: E_{221}(x, u) + E_{222}(x, u) . \]
We use Lemma 5.2.2 to estimate $E_{222}(x, u)$:

$$E_{222}(x, u) = \frac{1}{L_3(x)} \sum_{q_2 \leq y_1} \sum_{q_2 \mid a(q_1)} \frac{x}{q_1 u} \ll \sqrt{\log \frac{x}{q_1 u}}$$

by Lemma 5.2.2

$$\ll \frac{1}{L_3(x)} \sum_{q_2 \leq y_1} \frac{x}{u L_3(x) \sqrt{\log x}} \ll \frac{x}{L_3(x) \sqrt{\log x}} \nu(u)$$

$$E_{221}(x, u) = \frac{1}{L_3(x)} \sum_{q_2 \leq y_1} \sum_{q_2 \mid a(q_1)} \frac{x}{q_1 u} \ll \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \leq y_1} \sum_{q_2 \mid a(q_1)} \frac{1}{q_1} \ll \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \leq y_1} \sum_{q_2 \mid a(q_1)} \frac{1}{q_1} \ll E_{2211}(x, u) + E_{2212}(x, u) + E_{2213}(x, u)$$

$$E_{2211}(x, u) = \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \mid a(q_1)} \frac{1}{q_2} \ll E_{2212}(x, u) + E_{2213}(x, u)$$

$$E_{2212}(x, u) = \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \leq y_1} \int_{q_2 \log q_2}^{\frac{1}{e^{\gamma_1} (\log q_2)^2}} \pi^*(t, q_2) \frac{1}{t^2} dt \ll \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \leq y_1} \int_{q_2 \log q_2}^{\frac{1}{e^{\gamma_1} (\log q_2)^2}} \frac{1}{t \log t} dt \leq \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \leq y_1} L_2 \left( \frac{1}{e^{\gamma_1} (\log q_2)^2} \right) \ll \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \leq y_1} L_2(q_2) \ll \frac{x}{L_3(x) \sqrt{\log x}} \frac{\nu(u) L_2(u)}{u}$$

$$E_{2213}(x, u) = \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \mid a(q_1)} \frac{1}{q_1} \ll \frac{x}{u L_3(x) \sqrt{\log x}} \sum_{q_2 \mid a(q_1)} L_2(q_2) \ll \frac{x}{L_3(x) \sqrt{\log x}} \frac{\nu(u) L_2(u)}{u}$$

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Thus,

\[ E_{221}(x, u) \ll E_{2211}(x, u) + E_{2212}(x, u) + E_{2213}(x, u) \ll \frac{x}{L_3(x) \sqrt{\log x}} \cdot \frac{\nu(u)L_2(u)}{u}, \]

which gives us

\[ E_{22}(x, u) \ll \frac{x}{L_3(x) \sqrt{\log x}} \cdot \frac{\nu(u)L_2(u)}{u}. \]

\[ E_{23}(x, u) = \sum_{y_1 \leq q_1 \leq x} \sum_{y_1 \leq q_1 \leq x} 1 = \sum_{y_1 \leq q_1 \leq x} \sum_{y_1 \leq q_1 \leq x} 1 \ll \sum_{y_1 \leq q_1 \leq x} \sum_{y_1 \leq q_1 \leq x} \left( \sum_{n \leq x} 1 \right) \cdot \prod_{p \leq y_1} \left( 1 - \frac{1}{p} \right) \ll \frac{1}{L_3(x)} \sum_{y_1 \leq q_1 \leq x} \frac{1}{\sqrt{\log x}} \ll \frac{x}{\sqrt{\log x} L_3(x)} \cdot \frac{(\nu(u))^2}{u}. \]

Replace \( x \) with \( x^L \) in \( E_{21}, E_{22} \) and \( E_{23} \).

The estimate

\[ \sum_{l \leq y_1} \sum_{u | a(l)} \left( E_{21} \left( \frac{x}{l}, u \right) + E_{22} \left( \frac{x}{l}, u \right) + E_{23} \left( \frac{x}{l}, u \right) \right) \ll \]

\[ \ll \sum_{l \leq y_1} \sum_{u | a(l)} \left( \frac{x}{l \sqrt{\log x} L_3 \left( \frac{x}{l} \right)} \cdot \frac{(\nu(u))^2}{u} \right) \ll \frac{x}{\sqrt{\log x} L_3 \left( \frac{x}{l} \right)} \sum_{y_1 \leq u | a(l)} 1 \ll \frac{x}{\sqrt{\log x} L_3 \left( \frac{x}{l} \right)} \sum_{u \leq \left( \frac{x}{l} \right)} \frac{(\nu(u))^2}{u} \ll \]

\[ \ll \frac{x}{\sqrt{\log x} L_3 \left( \frac{x}{l} \right)} \sum_{l \leq y_1} \sum_{u | a(l)} \frac{(L_2(u))^2}{u} \ll \frac{x(L_2(y_1))^4}{\sqrt{\log x} L_3 \left( \frac{x}{l} \right)} \ll \frac{x(L_2(y_1))^4}{\sqrt{\log x} L_3 \left( \frac{x}{l} \right)} = o \left( \frac{x}{\sqrt{L_1(x) L_3(x)}} \right). \]

finishes the proof of Lemma 5.3.1. \( \square \)
Then, we have

\[ F_1 = \sum_{l \leq y_1} \sum_{d \mid a(l)} \mu(d) f \left( \frac{x}{l} \right) \leq \left( \frac{1 + o(1)}{ld} \right) L_1 \left( \frac{x}{l} \right) L_2 \left( \frac{x}{l} \right) + O \left( \frac{xL_1(x)}{l^2 L_3 \left( \frac{x}{l} \right)} \right) \]

by Lemma 5.3.1 and the fact that \( l \ll x^{\epsilon} \)

\[ = \sum_{l \leq y_1} \sum_{d \mid a(l)} \left( \frac{xU_f}{ld} \right) L_1 \left( \frac{x}{l} \right) + O \left( \frac{xL_1(x)}{l \cdot L_3 \left( \frac{x}{l} \right)} \right) \]

The error term is

\[ \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{l \leq y_1} \frac{1}{l} \sum_{d \mid a(l)} \frac{L_2^2(d)}{d} \ll \frac{xL_4(x)}{\sqrt{L_1(x)L_3(x)}}. \]

The main term of \( F_1 \) is

\[ \frac{xU_f}{\sqrt{L_1(x)L_3(x)}} \sum_{l \leq y_1} \frac{1}{l} \sum_{d \mid a(l)} \frac{\mu(d)}{d}. \]

Note that for \( x \) sufficiently large \( 2y_1^{\frac{2}{1+\epsilon}} \ll x^{\frac{1}{2+\epsilon}} \). Hence for all \( l \leq y_1 \) we have

\[ |a(l)| \leq 2^{l^{1+\epsilon}} \leq \left( \frac{x}{l} \right)^{\frac{1}{2+\epsilon}}. \]

Thus in the sum over \( d \) we can drop the bound, and so we see that it is equal to

\[ \sum_{d \mid a(l)} \frac{\mu(d)}{d} = \frac{\phi(|a(l)|)}{|a(l)|}. \]

The mail term is therefore

\[ \frac{xU_f}{\sqrt{L_1(x)L_3(x)}} \sum_{l \leq y_1} \frac{1}{l} \cdot \frac{\phi(|a(l)|)}{|a(l)|} \ll \frac{xL_4(x)}{\sqrt{L_1(x)L_3(x)}}. \]

Thus \( \Sigma' \ll \frac{xL_4(x)}{\sqrt{L_1(x)L_3(x)}} \). This combined with the estimate \( \Sigma'' \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \) would give us the desired estimate

\[ \# \{ n \leq x \mid (n, a(n)) \text{ is a prime} \} \leq \Sigma' + \Sigma'' \ll \frac{xL_4(x)}{\sqrt{L_1(x)L_3(x)}}. \]
Bibliography


