PARTICLE DEFINITIONS AND THE INFORMATION LOSS PARADOX.

by

Alex Venditti

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Physics
University of Toronto

Copyright © 2013 by Alex Venditti
Abstract

Particle definitions and the information loss paradox.

Alex Venditti
Doctor of Philosophy
Graduate Department of Physics
University of Toronto
2013

An investigation of information loss in black hole spacetimes is performed. We demonstrate that the definition of particles as energy levels of the Harmonic oscillator will not have physical significance in general and is thus not a good instrument to study the radiation of black holes. This is due to the ambiguity of the choice of coordinates on the phase space of the quantum field. We demonstrate how to identify quantum states in the functional Schrödinger picture.

We demonstrate that information is truly lost in the case of a Vaidya black hole (a black hole formed from null dust) if we neglect back reaction. This is done by quantizing the constrained classical system of a Klein-Gordon field in a Vaidya background. The interaction picture of quantum mechanics can be applied to this system.

We find a physically well motivated vacuum state for a spherically symmetric spacetime with an extra conformal Killing vector. We also demonstrate how to calculate the response of a particle detector in the a LeMaitre-Tolman-Bondi spacetime with a self-similarity.

Finally, some of the claims and confusion surrounding Unruh radiation, Hawking radiation and the equivalence principle are investigated and shown to be false.
Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Charles Dyer for his constant support, patience and physical insight which made this thesis possible. Also, Prof. Pierre Savard, Prof. Harald Pfeiffer and Prof. Mike Luke for their suggestions and insightful questions. I would like to thank Prof. Rob Mann for not only reading my entire thesis and offering helpful suggestions but also for his mentor ship during my undergraduate years.

I would also like to thank Daniel O’Keeffe, James Mracek and T Nathan Oldridge for listening to my crazy ideas over the years and providing feedback. Also the staff at the Department of Physics, especially Krystyna Biel, for their patience and help with all administrative matters.

Last but not least I would like to thank my parents for their constant support throughout my PhD and throughout my life.
Contents

1 Introduction 1

2 Preliminaries 4
  2.1 Functional Schrödinger formalism 4
  2.2 The Heisenberg picture 8
  2.3 Particle detectors 11
    2.3.1 Particles in Minkowski spacetime 14
  2.4 Black hole radiation and the information loss paradox 15

3 Particle definitions in functional Schrödinger formalism 17
  3.1 Introduction 17
  3.2 Particles in Minkowski Space 19
    3.2.1 Response of Unruh-Dewitt detector 25
  3.3 Particles in de Sitter Spacetime 28
    3.3.1 Particle Interpretation 31
    3.3.2 Response of Unruh-Dewitt detector 34
  3.4 A `natural’ vacuum state in FRW 38
  3.5 Discussion 40

4 The response of particle detectors in Vaidya spacetimes 43
  4.1 Introduction 43
4.2 Quantization .................................................. 46
4.3 Response of particle detectors ............................. 53
4.4 Discussion ................................................... 59

5 Choosing a vacuum state in a spherical spacetime with a CKV 63
5.1 Introduction ............................................... 63
5.2 A self-similar LTB spacetime ............................. 65
5.3 Conformally coupled Klein-Gordon field ................. 71
5.4 Response of Particle Detectors ........................... 75
5.5 Discussion .................................................. 77
5.6 Proof of null response of (5.51) ......................... 78

6 Hawking radiation, Unruh radiation and the equivalence principle 80

7 Conclusion .................................................. 87

Bibliography .................................................. 90
Chapter 1

Introduction

It was first demonstrated by Parker et. al ([1], [2], [3]) that the definition of a vacuum state for a quantum field theory (QFT) was non-unique when working with a QFT on a non-Minkowski background. Consequently the mathematical definition of particles in QFT was also non-unique, leading to particle production in curved background geometries. The ambiguity in vacuum state arises from the fact that in curved backgrounds there are in general no symmetries and therefore no way to restrict the choice of vacuum state. Minkowski spacetime has a ten parameter group of symmetries that one can uniquely define the vacuum state to be invariant under.

Fulling et. al ([4], [5]) then demonstrated that even in a Minkowski spacetime it was possible to define alternative vacuum states. The new vacuum state known as the Rindler vacuum is not invariant under the usual group of symmetries on Minkowski space, though it is the natural vacuum state for uniformly accelerated observers. Unruh ([6]) later elucidated this result by showing that a particle detector would click when accelerated uniformly in the usual Minkowski space vacuum state.

The concept of particle detector introduced by Unruh [6] is that of a two state quan-
Chapter 1. Introduction

tum system with some energy difference between the two states. A particle is said to be
detected when an excitation occurs. The particle detector definition of particles agrees
with the usual formulation of particles in terms of energy levels of the harmonic oscillator
only in Minkowski spacetime. In more general spacetimes, the two definitions disagree.

Hawking used the techniques of Parker ([1], [2], [3], [7]) in [8] to show that black holes
emit radiation. The spectrum is uniquely determined by the properties of the black hole
which can usually be characterized by three numbers, spin ($J$), electric charge ($Q$) and
mass $M$. This immediately led to an apparent contradiction with the laws of quantum
mechanics [9], specifically the law of unitary time evolution of the quantum state. More
generally it led to the scenario where the laws of physics were such that one could not
predict the past from the current state of a system if a black hole were involved ([9]).
Roughly speaking, a large star of a given mass $M$ could collapse to form a black hole or
a mass $M$ of books could also form a black hole. Both black holes would be the same
mass and therefore give off the same spectrum of radiation. This potentially problematic
scenario is referred to as the information paradox.

There have been a variety of proposed resolutions to the information paradox over
the years. Hawking [8], Wald and Unruh ([10]) argued that information is intrinsically
lost when a black hole is present. Wilczek ([11]) and others have argued that once back
reaction of the radiation on the background geometry was taken into account then infor-
mation might be able to escape in the evaporation process. The possible resolution that
is of most interest here is the one proposed by Krauss, Vachaspati, et. al ([12]). The
claim is basically that thermal radiation is generated by event horizons. In a realistic
black hole collapse however it would take any observer outside the Schwarzschild radius
an infinite amount of proper time to see the event horizon form. Since at no time does
any observer see an event horizon, that observer would never see thermal radiation and
therefore information may be able to escape.

In chapter 3 we investigate the methodologies used by Vachaspati, et. al [12] to demonstrate that information can escape from black holes formed from gravitation collapse. In particular we find new ambiguities in the definition of particles used that have not been noticed before. We demonstrate how using the Unruh-Dewitt detector can give the actual (physical) particle content of a spacetime.

In chapter 4 a Vaidya metric is considered as a model for a black hole formed from gravitational collapse. Physically a Vaidya metric is a spherically symmetric shell of null radiation travelling radially inward or outward. The inward case is considered here so that initially there is flat spacetime with null radiation moving inwards to form a singularity. Some surprising results are found with direct application to the information paradox.

In chapter 5 an inhomogeneous cosmological model is studied with dust expanding in a spherically symmetric fashion. The metric has spherical symmetry plus an extra conformal Killing vector field that is orthogonal to the spherically symmetric surfaces. It is demonstrated how to calculate the response of a detector following a given trajectory through the spacetime for conformally coupled fields. The procedure in this chapter can be applied to any spherically symmetric spacetime with an orthogonal conformal Killing vector.

Finally in chapter 6 we investigate some of the relationships between Hawking and Unruh radiation and the equivalence principle that have been claimed in the literature [13], [14]. We discuss some of the incorrect conclusions reached by assuming a connection between Hawking and Unruh radiation and the equivalence principle.
Chapter 2

Preliminaries

2.1 Functional Schrödinger formalism

The Schrödinger equation [15] is given in general as

\[ \hat{H} |\Psi(t, \vec{x})\rangle = i \frac{\partial}{\partial t} |\Psi(t, \vec{x})\rangle \quad (2.1) \]

where \( \hat{H} \) is the Hamiltonian operator of the system and \( \vec{x} \) is a vector of real numbers representing some possible values of the complete set of observables \( \hat{\vec{x}} \) for the system.

The probability to find the observables with values in the infinitesimal range \( d\vec{y} \) centered around \( \vec{y} \) at time \( t \) is given by \( |\langle \vec{y} | \Psi(t, \vec{x}) \rangle|^2 d\vec{y} \) where \( |\vec{y}\rangle \) is an eigenstate of the operators \( \hat{\vec{x}} \) with eigenvalues \( \vec{y} \). \( \vec{x} \) could be a finite or infinite dimensional vector.

The specific system of interest to us here is that of a non-interacting real Klein-Gordon scalar field of mass \( m \) in some background spacetime. The covariant action functional for this system is [7]

\[ S = \frac{1}{2} \int d^4 x \sqrt{-g} \left( \partial_a \phi(x) \partial_b \phi(x) g^{ab} + m^2 \phi(x)^2 + \xi R \phi(x)^2 \right) \quad (2.2) \]

where \( g^{ab} \) is the spacetime metric, \( g = \det(g_{ab}) \), \( \phi(x) \) is a a scalar field and \( R \) is the Ricci scalar of the spacetime. Roman indices \( (a \text{ and } b) \) always run from 0 to 3. Finding
the maxima (and minima) of the above action functional by setting the variation of the action with respect to $\phi(x)$ as follows

$$\frac{\delta S}{\delta \phi(x)} = 0$$  \hspace{1cm} (2.3)

gives the Klein-Gordon equation below

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b \phi(x)) - m^2 \phi(x) - \xi R \phi(x) = 0$$  \hspace{1cm} (2.4)

To find the Hamiltonian formulation we note that the Lagrangian is given by

$$L = \frac{1}{2} \int d^3 x \sqrt{-g} \left( \partial_\alpha \phi(x) \partial_\beta \phi(x) g^{\alpha\beta} + m^2 \phi(x)^2 + \xi R \phi(x)^2 \right)$$  \hspace{1cm} (2.5)

where the integration is over some 3-dimensional spacelike hypersurface $\Sigma$. The conjugate momentum is given by the formula

$$\Pi(x) = \frac{\partial L}{\partial (\partial_0 \phi(x))} = \sqrt{-g} g^{\alpha 0} \partial_\alpha \phi(x)$$  \hspace{1cm} (2.6)

where $\partial_0 \phi(x)$ is the derivative of $\phi(x)$ with respect to the $x^0$ coordinate (the time coordinate). Performing the Legendre transformation we obtain the following Hamiltonian

$$H = \Pi(x) \partial_0 \phi(x) - L$$

$$= \frac{1}{2} \int d^3 x \sqrt{-g} \left( \left( \Pi(x) \right)^2 - g^{\alpha 0} \partial_\alpha \phi(x) \right)^2 \frac{1}{g^{00}}$$

$$- \partial_\alpha \phi(x) \partial_\beta \phi(x) g^{\alpha\beta} - m^2 \phi(x)^2 - \xi R \phi(x)^2$$

where the Greek indices $\alpha$ and $\beta$ always go over the spatial indices (1, 2, 3). For simplicity one can always choose coordinates such that $g_{\alpha 0} = 0$. In these coordinates the Hamiltonian takes the form

$$H = \frac{1}{2} \int d^3 x \left( \frac{\Pi(x)^2}{\sqrt{-g} g^{00} \det g_{\alpha \beta}} \right) - \sqrt{-g} \left( \partial_\alpha \phi(x) \partial_\beta \phi(x) g^{\alpha\beta} - m^2 \phi(x)^2 - \xi R \phi(x)^2 \right)$$  \hspace{1cm} (2.8)

Quantization of the above Hamiltonian operator consists of implementing the equal time commutation relations below

$$[\dot{\phi}(x^0, \vec{x}), \Pi(x^0, \vec{y})] = i \delta^3(\vec{x} - \vec{y})$$  \hspace{1cm} (2.9)
The ‘\(\phi\)-space’ implementation of these relations is given by the following

\[
\hat{\phi}(x) \rightarrow \phi(x) \ , \ \hat{\Pi}(x) \rightarrow -i \frac{\delta}{\delta\phi(x)} \quad (2.10)
\]

In the above implementation the Schrödinger equation (in coordinates where \(g^{00} = 0\)) is given as

\[
\frac{1}{2} \int_{\Sigma} d^3 x \left( -\frac{1}{\sqrt{-g^{00} \det g_{\alpha\beta}}} \delta^2 - \sqrt{-g} \left( \partial_\alpha \phi \partial_\beta \phi g^{\alpha\beta} - m^2 \phi^2 - \xi R\phi^2 \right) \right) \Psi(t, \phi) = i \frac{\partial \Psi(t, \phi)}{\partial t} \quad (2.11)
\]

where we have suppressed the dependence of the \(\phi\) field on \(x\) for compactness. The above is the Schrödinger equation for the wave functional \(\Psi(t, \phi)\) as opposed to the abstract state vector \(|\Psi(t, \phi)\rangle\). The relation to the abstract vector notation is given by the following relations

\[
\langle \phi | \hat{\phi} | \Psi(t, \phi) \rangle = \phi \Psi(t, \phi) \ , \ \langle \phi | \hat{\Pi} | \Psi(t, \phi) \rangle = -i \frac{\delta}{\delta\phi} \Psi(t, \phi) \quad (2.12)
\]

where \(|\phi\rangle\) is an eigenvector of the \(\hat{\phi}\) operator.

As an example we will write down the Schrödinger equation in Minkowski spacetime in coordinates where the metric takes the form

\[
ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (2.13)
\]

We will always use the \((+1, -1, -1, -1)\) signature unless otherwise stated. For a specific choice of coordinates on the phase space \((\phi(x), \Pi(x))\) the Schrödinger equation is

\[
\frac{1}{2} \int_{\Sigma} d^3 x \left( -\frac{\delta^2}{\delta\phi^2} + \left( \nabla_\phi^2 + m^2 \phi^2 + \xi R\phi^2 \right) \right) \Psi(t, \phi) = i \frac{\partial \Psi(t, \phi)}{\partial t} \quad (2.14)
\]

where \(\nabla = (\partial_x, \partial_y, \partial_z)\) and \(\Sigma\) is now a constant \(t\) hypersurface. To solve this equation we have to choose some convenient coordinates for the phase space. Since the constant
Chapter 2. Preliminaries

$t$ hypersurfaces are homogeneous, isotropic and flat we can expand in a basis of complex exponentials as follows

$$
\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\delta}{\delta \phi(t, \vec{y})} = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\delta}{\delta a_{\vec{k}}} e^{-i\vec{k} \cdot \vec{y}}
$$

(2.15)

where $a_{\vec{k}}$ are complex functions of time with $a^*_{\vec{k}} = a_{-\vec{k}}$ holding so that the $\phi$ fields are real ($*$ denotes complex conjugation). The coefficients $a_{\vec{k}}$ are the coordinates which give the configuration of the $\phi$ field. In this basis the Schrödinger equation becomes

$$
\frac{1}{2} \int d^3k \left(-\frac{\delta^2}{\delta a_{\vec{k}} \delta a_{-\vec{k}}} + \left(\vec{k}^2 + m^2 + \xi R\right) a_{\vec{k}} a_{-\vec{k}}\right) \Psi(t, \phi) = i \frac{\partial \Psi(t, \phi)}{\partial t}
$$

(2.16)

where the $\Psi(t, \phi)$ functional can now be thought of as a function of the $a_{\vec{k}}$ variables. The above equation can be solved for each pair of variables $(a_{\vec{k}}, a_{-\vec{k}})$. In order to interpret each solution we must develop a particle interpretation.

The Hamiltonian for a real Klein-Gordon field in flat spacetime can be written [15] as

$$
H = \int d^3k \omega(k) \left(\gamma^+_{\vec{k}} \gamma_{\vec{k}} + \frac{1}{2}\right)
$$

(2.17)

where

$$
\gamma_{\vec{k}} = \sqrt{\frac{\omega}{2}} a_{\vec{k}} + \frac{1}{\sqrt{2\omega}} \frac{\delta}{\delta a_{-\vec{k}}}
$$

(2.18)

$$
\gamma^+_{\vec{k}} = \sqrt{\frac{\omega}{2}} a_{-\vec{k}} - \frac{1}{\sqrt{2\omega}} \frac{\delta}{\delta a_{\vec{k}}}
$$

(2.19)

and

$$
\omega^2(k) = \vec{k}^2 + m^2 + \xi R
$$

(2.20)

The pair $(\gamma_{\vec{k}}, \gamma^+_{\vec{k}})$ are the annihilation and creation operators respectively for particles in mode $-\vec{k}$. We can now define states analogously to the Heisenberg picture by specifying their eigenvalues with respect to the number operator $N_{\vec{k}} = \gamma^+_{\vec{k}} \gamma_{\vec{k}}$. For example the solution to the Schrödinger equation given by

$$
\Psi(t, \phi) = \exp \left(\int d^3k \left(-\frac{\omega}{2} a_{\vec{k}} a_{-\vec{k}} + i \frac{\omega}{2} t\right)\right)
$$

(2.21)
is annihilated by $\gamma_k$, that is $\gamma_k \Psi = 0$, and is therefore a vacuum state with respect to all particles. The above wave functional is a product of many wave functionals, one for each mode $k$ as shown below

$$\Psi(t, \phi) = \prod_{k \in H} \Psi_k(t, a_k)$$

(2.22)

where $H$ denotes the ‘half-space’ of the momentum $k$ where $k$ and $-k$ are identified. So the vacuum state for a single mode in $H$ is simply

$$\Psi_k(t, a_k) = \exp \left( -\frac{\omega}{2} a_k a_{-k} + i \frac{\omega}{2} t \right)$$

(2.23)

Using the fact that $a_k^* = a_{-k}$ we write $a_k$ in terms of its real and imaginary parts, $b_k + ic_k$, for each distinct $k \in H$. The wave function takes the following form

$$\Psi_k(t, a_k) = \exp \left( -\frac{\omega}{2} (b_k^2 + c_k^2) + i \frac{\omega}{2} t \right)$$

(2.24)

so that we can write each mode in $a_k$ as a product of two modes (one in $b_k$ and one in $c_k$). It follows that a multi-particle state in the $b_k$ mode is given by the function

$$\Psi_k(t, b_k) = H_n(\sqrt{\omega} b_k) \exp \left( -\frac{\omega}{2} b_k^2 + i \omega \left( \frac{1}{2} + n \right) t \right)$$

(2.25)

where $H_n$ denote the Hermite polynomials. The integer $n$ denotes the number of particles in this state.

### 2.2 The Heisenberg picture

In the Heisenberg picture the operators are time dependent and the state is time independent. For the Klein-Gordon system of equation (2.2) the Heisenberg picture field operators $\hat{\phi}(x)$ and $\hat{\Pi}(x)$ satisfy the canonical commutation relations (2.9).
Implementation of the commutation relations can be done by expanding the field operators as

$$\hat{\phi}(x) = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$  \hspace{1cm} (2.26)

where $a_k$ satisfies

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}$$  \hspace{1cm} (2.27)

and $f_k$ and $f_k^*$ are a complete set of solutions to the Klein-Gordon equation (2.4) that satisfy the following normalization.

$$\langle f_k, f_{k'} \rangle = \int dx^3 \sqrt{-g} g^{0a} (f_k^* \partial_a f_k - f_k \partial_a f_k^*)$$

$$= i\delta_{kk'}$$  \hspace{1cm} (2.28)

where the integration is done over a three dimensional constant $x^0$ hypersurface. The above normalization is conserved so that it is independent of which constant $x^0$ hypersurface the integration is done over. It follows from the commutation relations that the modes also satisfy

$$\langle f_k, f_{k'}^* \rangle = 0$$  \hspace{1cm} (2.29)

$$\langle f_k^*, f_{k'}^* \rangle = -i\delta_{kk'}$$  \hspace{1cm} (2.30)

The above conditions can be shown to be equivalent to the canonical commutation relations (2.9) [7].

The $f_k$ and $f_k^*$ basis can be expanded in terms of any another complete set of solutions to the KG equation.

$$f_k = \sum_{k'} (\alpha_{kk'} g_{k'} + \beta_{kk'} g_{k'}^*)$$  \hspace{1cm} (2.31)

where $g_k$ and $g_k^*$ form another complete basis that satisfy the orthonormality relations (2.28), (2.29), (2.30). Using orthonormality it follows that

$$\alpha_{kp} = -i\langle f_k, g_p \rangle$$  \hspace{1cm} (2.32)

$$\beta_{kp} = i\langle f_k, g_p^* \rangle$$  \hspace{1cm} (2.33)
Using orthonormality one can express the $g_k$ and $g^*_k$ modes in terms of the $f_k$ and $f^*_k$'s.

$$g_k = \sum_j \left( \alpha^*_{jk} f_j - \beta_{jk} f^*_j \right) \quad (2.34)$$

Given the two complete orthonormal sets of modes it follows that the field operator can be expanded in terms of both sets of modes

$$\phi(x) = \sum_k \left( f_k a_k + f^*_k a^\dagger_k \right) = \sum_p \left( g_p b_p + g^*_p b^\dagger_p \right) \quad (2.35)$$

Using the relations (2.31), (2.34) and orthonormality we can express the $a_k$ operators in terms of the $b_k$, and $b^\dagger_k$'s and vice versa.

$$a_k = \sum_j \left( \alpha^*_{kj} b_j - \beta^*_{kj} b^\dagger_j \right) \quad (2.36)$$

$$b_k = \sum_j \left( \alpha_{jk} a_j + \beta^*_{jk} a^\dagger_j \right) \quad (2.37)$$

One can define two vacuum states $|0\rangle_a$ and $|0\rangle_b$ which are annihilated by the $a_k$ and $b_k$ operators respectively. If $\beta_{jk} \neq 0$ then the $a$ vacuum will be different than the $b$ vacuum.

$$a_k |0\rangle_b = - \sum_j \beta^*_{kj} b^\dagger_j |0\rangle_b \neq 0 \quad (2.38)$$

Consequently the number operator for the $a$ type particles in mode $j$ will have a non-zero expectation value in the $b$ vacuum.

$$\langle 0 | b^\dagger a j | 0 \rangle_b = \sum_k \beta_{jk}^2 \quad (2.39)$$

It is in this sense that particles are created. The spacetime can at one point be such that one set of modes seems most natural, say $f_k$ but evolve at a later time so that a different set of modes, $g_k$ are natural. If the initial state is the $f$ vacuum then it will at later time have $g$ type particles. The $g$ type particles would be ‘physical’ at this later time because they would be the most natural definition of modes given the form of the metric. Here, ‘most natural’ usually means that the modes are chosen to have the same symmetries as the spacetime.
Chapter 2. Preliminaries

2.3 Particle detectors

The Unruh-Dewitt detector is a conceptually independent definition of particles from the one given above. The detector is a two-level quantum mechanical system with each level having different energies. The ground state and excited state of the detector are given by the vectors $|E_0\rangle$ and $|E_1\rangle$ respectively. The detector is coupled, via a monopole interaction to the quantum field of interest, in our case a Klein-Gordon field. The monopole moment can be thought of as a type of charge that the detector has which determines the strength of the interaction with the Klein-Gordon field. The interaction part of the Hamiltonian is therefore given by

$$H_{int} = \epsilon \ m(\tau) \phi(x^c(\tau))$$

(2.40)

where $x^c(\tau)$ represents the path followed by the detector and $\tau$ is the parameter along the curve. $m(\tau)$ is the ‘monopole moment’ operator of the detector and $\phi(x^c(\tau))$ is the Klein-Gordon field. $\epsilon$ is a small coupling constant.

Working in the interaction picture we have that the monopole moment operator has the following time evolution

$$m(\tau) = e^{iH_0\tau} m(0) e^{-iH_0\tau}$$

(2.41)

where $H_0$ is the part of the Hamiltonian operator that corresponds to the detector’s evolution without interaction. The states $|E_1\rangle$ and $|E_0\rangle$ are the energy eigenstates of this part of the Hamiltonian $H_0 \ |E_{0(1)}\rangle = E_{0(1)} \ |E_{0(1)}\rangle$. We denote some ground state of the field by $|0\rangle$ and the combined ground state of the detector and the field by $|0\rangle |E_0\rangle$. The time evolution of some state satisfies

$$H_{int} |\Psi(\tau)\rangle = i \frac{\partial}{\partial \tau} |\Psi(\tau)\rangle$$

(2.42)

The solution to this equation is

$$|\Psi(\tau)\rangle = \mathcal{T} e^{i \int_{-\infty}^{\infty} H_{int}(\tau)d\tau} |\Psi(0)\rangle$$

(2.43)
where \( \mathcal{T} \) is the time-ordering symbol that when placed in front of a product of field operators denotes that the operators are in order of increasing coordinate time from left to right. The end points of the integration over \( \tau \) are \( \infty \) and \(-\infty\) in this case, however one can change these times to be any initial and final time. The probability amplitude to make a transition from the combined ground state at \( \tau = -\infty \) to some state where the detector is in the excited state and the field is in any state at time \( \tau = \infty \) is therefore given by

\[
\langle \phi | \langle E_1 | \mathcal{T} e^{i \int_{-\infty}^{\infty} H_{int}(\tau) d\tau} | 0 \rangle | E_0 \rangle
\]

Assuming the coupling constant \( \epsilon \) is small we have to first order in perturbation theory that the transition amplitude is

\[
i \epsilon \int_{-\infty}^{\infty} \langle \phi | \langle E_1 | m(\tau) \phi(x^e(\tau)) | 0 \rangle | E_0 \rangle d\tau
\]

Using the fact that \( m(\tau) \) commutes with \( \phi \) and equation (2.41) we have that the probability amplitude is

\[
i \epsilon \langle E_1 | m(0) | E_0 \rangle \int_{-\infty}^{\infty} e^{i(E_1 - E_0)\tau} \langle \phi | \phi(x^e(\tau)) | 0 \rangle d\tau
\]

This is the probability amplitude for a transition from the combined ground state to some state in which the detector is excited but the field is in some specific state \( \langle \phi \rangle \). The total probability for any transition of the detector from its ground state to its excited state (regardless of the state of the field) can be obtained by taking the modulus squared of (2.46) and summing over all field states using the relation \( \sum_{\phi} | \phi \rangle \langle \phi | = \hat{1} \), where \( \hat{1} \) is the identity operator. The probability amplitude is given by

\[
e^2 \langle E_1 | m(0) | E_0 \rangle \mathcal{F}(E_1 - E_0)
\]

where

\[
\mathcal{F}(\omega) = \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau} d\tau'' e^{-i\omega(\tau'' - \tau')} G^+(x(\tau'), x(\tau''))
\]
and \(G^+(x(\tau'), x(\tau'')) = \langle 0 \mid \phi(x^c(\tau'))\phi(x^c(\tau'')) \mid 0 \rangle \) is the Wightman function in the chosen state. The initial and final times have been changed from \(-\infty\) and \(\infty\) to \(\tau_0\) and \(\tau\) respectively. \(F(\omega)\) is known as the response function. It is the quantity one must calculate in order to obtain the transition rate of an actual detector and hence the particle content of the given quantum state in the background spacetime.

Another quantity of interest is what is called the ‘response rate’ of the detector. The response rate is the probability for the detector to make a transition per unit proper time along the detector world line. The following derivation of the response rate can be found in (Schlicht 2003) but is given here for completeness. The double integral in (2.48) is over a square in the \(\tau', \tau''\)-plane. The integral can be done by integrating over the lower triangle given by \(\tau'' < \tau'\) and integrating over the upper triangle given by \(\tau'' > \tau'\). These two integrations are expressed as follows

\[
F(\omega) = \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau'} d\tau'' e^{-i\omega(\tau'-\tau'')} W(\tau', \tau'') + \int_{\tau_0}^{\tau} d\tau'' \int_{\tau_0}^{\tau''} d\tau' e^{-i\omega(\tau'-\tau'')} W(\tau', \tau'') \tag{2.49}
\]

After making the re-labeling of the second term in (2.49) by switching \(\tau'\) and \(\tau''\) we see that the second term is the complex conjugate of the first term if we use the fact that \(W(\tau', \tau'') = W^*(\tau'', \tau')\) for real Klein-Gordon fields. Therefore (2.49) becomes

\[
F(\omega) = 2 \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau'} d\tau'' \Re \left( e^{-i\omega(\tau'-\tau'')} W(\tau', \tau'') \right) \tag{2.50}
\]

where \(\Re\) denotes the real part. Differentiating with respect \(\tau\) we obtain the response rate as

\[
\dot{F}(\omega) = 2 \int_{\tau_0}^{\tau} d\tau'' \Re \left( e^{-i\omega(\tau'-\tau'')} W(\tau, \tau'') \right) \tag{2.51}
\]

where \(\dot{} = d/d\tau\). Finally we make the substitution \(s = \tau - \tau''\) to obtain

\[
\dot{F}(\omega) = 2 \int_{\tau_0}^{\tau-\tau_0} ds \Re \left( e^{-i\omega s} W(\tau, \tau - s) \right) \tag{2.52}
\]

The response rate (2.52) is a different definition of particles from that given by the multiparticle state (2.25).
It can be shown that the zero particle state of the set of states given by (2.25) in Minkowski spacetime corresponds to the Poincare invariant vacuum and that the response rate (2.52) of inertial Unruh-Dewitt detectors will be zero. In spite of this we will show in chapter 3 that any choice of a complete set of states, such as the states (2.25), is equivalent to a choice of coordinates of the phase space of the field. Hence, any choice of states will not in general represent anything physical such as the response of a particle detector for some class of observers.

### 2.3.1 Particles in Minkowski spacetime

In Minkowski spacetime in the Heisenberg picture the KG field is usually expanded in the following Lorentz invariant manner

\[ f_k = \exp(-i\omega t + i\vec{k} \cdot \vec{x})/\sqrt{4(2\pi)^3}\omega \] with \( \omega = \sqrt{k^2 + m^2} \), where \( m \) is the mass. The field operator is

\[ \phi(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3}\omega} \left( a_k e^{-i\omega t + i\vec{k} \cdot \vec{x}} + a_k^\dagger e^{i\omega t - i\vec{k} \cdot \vec{x}} \right) \] (2.53)

Plugging this expression into the general Hamiltonian operator for the KG field (2.7) we get

\[ H = \int d^3k \omega \left( a_k^\dagger a_k + \frac{1}{2} \right) \] (2.54)

A complete set of states based on the above plane wave solutions is given by acting on the vacuum state defined by \( a_k |0\rangle = 0 \) with the \( a_k^\dagger \) operators

\[ \Pi_j a_{k(j)}^{\dagger n_j} |0\rangle \] (2.55)

It is known that the multi-particle states given above do represent physical particles for inertial observers as these states are used to represent physical particles in particle physics experiments [15]. The response of an Unruh-Dewitt detector confirms that the multi-particle states represent physical particles; as the response rate of a detector with an energy gap \( \omega(k) \) in the state \( a_k^{\dagger n} |0\rangle \) is proportional to \( n \), the number of particles of energy \( \omega \) in that state.
2.4 Black hole radiation and the information loss paradox

In [8] it was derived that any spherically symmetric massive body that collapses to form a black hole will emit a spectrum of thermal radiation with temperature given by $T = \kappa / 2\pi$ where $\kappa$ is the surface gravity of the black hole. That is, the expectation value of the number of particles of frequency $\omega$ in a black hole spacetime with mass $M$ is given by

$$\langle n_\omega \rangle = \frac{\Gamma_{\omega l}}{\exp(8\pi\omega M) - 1}$$

where $\Gamma_{\omega l}$ is a 'greybody' factor that gives the fraction of waves with frequency $\omega$ and spherical dependence $Y_{lm}(\theta, \phi)$ that escapes to future null infinity (i.e. does not get reflected by the geometry outside the black hole).

In [9], Hawking demonstrates that the probability distribution of particles is in fact a thermal one given by the probability distribution

$$P(n_\omega) = \frac{(1 - x)(x\Gamma_{\omega l})^n}{(1 - (1 - \Gamma_{\omega l})x)^{n+1}}$$

where $x = \exp(-\omega/T)$, $T$ is the temperature and $P(n_\omega)$ is the probability to find $n$ particles with frequency $\omega$.

Hawking uses the fact that the quantum state representing the outgoing radiation is a thermal one to show that at the end of the black hole evaporation process there is equal probability to emit a given configuration of particles so long as the sum of the masses of all the particles is equal to the black hole mass $M$ [9]. Thus two different spherically symmetric mass configurations of equal mass would evaporate away into identical quantum states and there would be no way to determine the original mass configuration; this is the information paradox.
The quantum state is found to be thermal on future null infinity (see figure 6.1), but not at any point in the interior of the spacetime. It may be problematic that one is drawing conclusions about the information that is present after the black hole evaporates from the behaviour of a quantum state on future null infinity as the points representing future null infinity do not exist anywhere in the physical spacetime. For instance, it may be the case that there exists deviations from thermality away from null infinity and that these deviations are precisely what allows the information about the matter that makes up the black hole to be transmitted.

The above mechanism for transmitting information is what is being investigated in chapters 4 and 5. We show the invalidity of some of the methods used to investigate this idea and state the correct ones in chapter 3. This mechanism is also what is investigated in [12], however there they formulate the idea equivalently as:

Since an outside observer never sees formation of a horizon in a finite time, radiation observed by him is never quite thermal.

Another approximation made in [8] and [9] is that the black hole is assumed to be ‘quasi-static’ for all time. By quasi-static we mean that the calculation of emitted radiation can be done at any time as if the black hole were static. To address whether this approximation hides some of the information about the initial quantum state one would need to look at the back-reaction of a quantum field on spacetime geometry.
Chapter 3

Particle definitions in functional Schrödinger formalism

3.1 Introduction

It is well known that when doing QFT in a curved background in the Heisenberg picture there are several different complete sets of solutions in which one can expand the field operators. Thus we could have

\[
\phi(x) = \int d^3k \left( u_\vec{k} a_{\vec{k}} + u_{\vec{k}}^* a_{\vec{k}}^\dagger \right) \quad (3.1)
\]

\[
= \int d^3k \left( \bar{u}_\vec{k} \bar{a}_{\vec{k}} + \bar{u}_\vec{k}^* \bar{a}_{\vec{k}}^\dagger \right) \quad (3.2)
\]

where \( u_\vec{k} \) and \( \bar{u}_\vec{k} \) are two different solutions to the Klein-Gordon equation and \( a_{\vec{k}}, \bar{a}_{\vec{k}} \) are two different annihilation operators. In virtually all cases, a complete set of modes \( (u_{\vec{k}}, u_{\vec{k}}^\dagger) \) is chosen so that they are ‘natural’ solutions in some coordinate system \( x^c \) chosen on the background spacetime. Here ‘natural’ means any coordinate dependent criteria used to single out solutions. For instance, the solutions might be the separable solutions to the Klein-Gordon equation in one or more coordinates chosen on the background spacetime (i.e. \( u_{\vec{k}} = A(x^0)B(x^1)C(x^2)D(x^3) \)).
Chapter 3. Particle definitions in functional Schrödinger formalism

If there exists a time-like (conformal) Killing vector field on the spacetime then it is natural to choose coordinates so that the (C.)K.V. is given by \( \partial_{x^0} \). One can then choose the modes that are ‘positive frequency’ with respect to \( \partial_{x^0} \).

\[ L_{x^0} u_k^\pm = i \omega u_k^\pm, \quad \omega > 0 \]  

where \( L_{x^0} \) is the Lie derivative with respect to the time \( x^0 \).

Hence, in the usual representation of the ambiguity of particle definitions, the choice of solutions is ‘induced’ by a choice of coordinates on the spacetime. Coordinate systems are usually tied to observers on the spacetime, so for each observer we will have a natural definition of particle states. In this sense the definitions of particle states are only as varied as the sets of observers in the spacetime. Coordinates may also be chosen so that a symmetry of the spacetime becomes manifest, but again we have that the definition of particles is tied to some property of the background spacetime so the definitions of particles are only as varied as the possible symmetries of the background spacetime. And if there are no symmetries at all there is no preferred definition to choose from.

This need not be the case as is apparent when one looks at things in the Schrödinger formalism. There is far more freedom in the choice of coordinates on phase space than on the spacetime. This is especially apparent when considering a field theory which has infinite degrees of freedom as opposed to the spacetime which only has four dimensions.

In section 3.2, it will be shown that the definition of particles as energy levels of the harmonic oscillator is equivalent to picking coordinates on phase space. It will also be shown that the calculation of the Wightman function and hence the response of an Unruh-Dewitt detector will not depend on the way coordinates are chosen on phase space. By the principle of general covariance this demonstrates that the true (physically accurate) definition of particles is given by the response of the Unruh-Dewitt detector and
that the expectation of the number operator for energy levels of the harmonic oscillator is in general physically meaningless. This is an explicit realization of a statement made in the literature[16] (see pg. 15 and pg. 60).

In section 3.3 an explicit example is given to demonstrate the points made in section 3.2. The particle expectation number of a conformal scalar field in the conformal vacuum seems to indicate that there is not a thermal distribution of particles and rather that the distribution becomes thermal as $t \to \infty$. Despite this indication, it is shown that a comoving observer will in fact see a thermal distribution of particles for all time.

Also, in section 3.3 we employ a new method to determine the wave functional corresponding to the conformal vacuum. This method is more reliable than the method found elsewhere [17] as it does not rely on how one renormalizes the stress energy tensor operator.

In section 3.4 it is shown that a canonical transformation can be done on phase space so that the Hamiltonian of a free Klein-Gordon field can be put in the form of a set of harmonic oscillators with constant mass and frequency. From this we obtain an obvious definition of particles based on energy levels of the harmonic oscillator that is different from those found elsewhere [12] [18] [19]. We will explain how the results of this section are at odds with some results in the literature[12] [18] [19].

3.2 Particles in Minkowski Space

The massless, real scalar field is usually quantized by expressing it as the following expansion

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} b_k(t) \exp(i\vec{k} \cdot \vec{x})$$  \hspace{1cm} (3.4)
where the $b_{\vec{k}}(t)$ are complex functions of time that must satisfy $b_{\vec{k}}^* = b_{-\vec{k}}$ in order for $\phi$ to be real. It is then substituted into the action

$$S = \int dt \int d^3k \sqrt{-g} \phi_{\vec{k}} \phi_{b^*} g^{ab}$$

with $g_{ab}$ taken to be the flat spacetime metric. In the above notation $|a = \partial_a$ was used.

One then treats the $b_{\vec{k}}$'s as the generalized coordinates that must be quantized via the canonical commutation relations. Quantizing the $b_{\vec{k}}$'s will lead to the “usual” results in Minkowski spacetime; the Poincare invariant vacuum will have a zero expectation value for the number operator of the $b_{\vec{k}}$ particles. In flat spacetime the zero expectation of the number operator corresponds to the zero response rate of Unruh-Dewitt detectors traveling along inertial trajectories.

Substituting (3.4) into (3.5) in Minkowski spacetime in the usual Cartesian coordinates we obtain

$$S = \frac{1}{2} \int dt \int d^3k \left( |b_{\vec{k}}(t)|^2 - |\vec{k}|^2 |b_{\vec{k}}(t)|^2 \right)$$

where the dot over the $b_{\vec{k}}(t)$ denotes differentiation with respect to $t$. This is the action for an infinite set of complex $b_{\vec{k}}(t)$'s. For our purposes we can simply deal with the real part of each $b_{\vec{k}}(t)$ one mode at a time. The action for this single real-variable is

$$S = \frac{1}{2} \int dt \left( \dot{x}(t)^2 - \omega^2 x(t)^2 \right)$$

where $\omega = |\vec{k}|$ and $x(t)$ is understood to be the real part of a single mode $b_{\vec{k}}(t)$. The Hamiltonian for the system is

$$H = \frac{1}{2} \left( \dot{x}^2 + \omega^2 x^2 \right)$$

where $p = \dot{x}$. At this point we remark that this is a harmonic oscillator with a fixed frequency and mass. Coordinates will be chosen on phase space so that the new Hamiltonian takes the form of a harmonic oscillator with time-dependent mass and frequency. The new Hamiltonian will have a natural interpretation for particle states (certainly as
natural as in de Sitter spacetime as will be shown below).

A canonical transformation is a change of the ‘position’ and conjugate ‘momentum’ coordinates such that Hamilton’s equations are preserved. Specifically, the equations

\[
\begin{align*}
\dot{R} &= \frac{\partial K}{\partial Q} \\
\dot{Q} &= -\frac{\partial K}{\partial R}
\end{align*}
\]  

(3.9)
(3.10)

hold in the new \((Q, R)\) coordinates. \(K\) is the Hamiltonian in the new coordinates. The coordinate transformation can be specified in terms of a generating function \(G\). The generating function can in general be a function of any pair of the old and new coordinates and can depend explicitly on time. A type 2 transformation is one in which the generating function depends on the old position coordinates \((x)\) and the new momentum coordinates \((R)\). For details on how the generating function is used to generate the coordinate transformations see [20].

We will consider a type 2 canonical transformation (see [20]) taking the coordinates \((x, p)\) to \((Q, R)\) that has the following form:

\[
G_2(x, R) = f(t)xR + f(t)x^2 \frac{1}{2} + h(t)\frac{R^2}{2}
\]

(3.11)

where the generating function is explicitly a function of time through \(f(t)\) and \(h(t)\). Above we have restricted ourselves to a general homogeneous and quadratic generating function. This restriction was made because we want the new Hamiltonian to also be quadratic in the momenta and position coordinates; hence we need the new coordinates to depend linearly on the old ones. This can only be achieved by considering a quadratic generating function. The relations giving the coordinate transformations are

\[
\begin{align*}
p &= \frac{\partial G_2}{\partial x} = f(t)(R + x) \\
Q &= \frac{\partial G_2}{\partial R} = f(t)x + h(t)R
\end{align*}
\]  

(3.12)
(3.13)
The new Hamiltonian is given by

\[
H' = H + \frac{\partial G_2}{\partial t}
\]  
(3.14)

\[
= \left[ \frac{1}{2} (f - h)^2 + \frac{\omega^2 h^2}{2 f^2} - \frac{\dot{f} h}{f} + \frac{\dot{f} h^2}{2 f^2} + \frac{\dot{h}}{2} \right] R^2
\]  
(3.15)

\[
+ \left[ \frac{1}{2} + \frac{\omega^2}{2 f^2} + \frac{\dot{f}}{2 f^2} \right] Q^2
\]

\[
+ \left[ (f - h) - \frac{\omega^2}{f^2} h + \frac{\dot{f}}{f} - \frac{\dot{f} h}{f^2} \right] QR
\]

where \( \dot{} \equiv d/dt \). We have expressed everything in the new \( Q, R \) coordinates by applying the relations (3.12) and (3.13).

As explained previously, \( H' \) should have the form of a harmonic oscillator with time-dependent mass and frequency. Hence we require that the coefficient of the \( QR \) term go to zero. The following relation is obtained between \( f \) and \( h \).

\[
h = \frac{f^3 + \dot{f} f}{f^2 + \omega^2 + \dot{f}}
\]  
(3.16)

\( H' \) is now given by

\[
H' = \frac{1}{2} \left( \frac{R^2}{M(t)} + M(t)\Omega^2(t)Q^2 \right)
\]  
(3.17)

where

\[
\frac{1}{M(t)} \equiv (f - h)^2 + \frac{\omega^2 h^2}{f^2} - \frac{2\dot{f} h}{f} + \frac{\dot{f} h^2}{f^2} + \dot{h}
\]  
(3.18)

\[
M(t)\Omega^2(t) \equiv 1 + \frac{\omega^2}{f^2} + \frac{\dot{f}}{f^2}
\]  
(3.19)

\( H' \) is the Hamiltonian of a harmonic oscillator with time-dependent frequency and mass.

To quantize this system we note that \([\hat{Q}, \hat{R}] = i\). This follows from direct substitution of the coordinate transformations 3.12 and 3.13 into the canonical commutation relation \([\hat{x}, \hat{p}] = i\).
The new Hamiltonian $H'$ has the form of a harmonic oscillator with time-dependent frequency and mass. By analogy with the constant mass harmonic oscillator [15] we can define a natural particle interpretation by specifying the annihilation and creation operators for this system. This is in fact what was done in [12], [18] and [19]. From now on we deal with quantum operators so we drop the hats. Let $\gamma(t)$ be the annihilation operator and $\gamma^\dagger(t)$ the creation operator with $[\gamma(t), \gamma^\dagger(t)] = 1$ holding. By analogy with the time-independent harmonic oscillator we have that

$$
\gamma(t) = \frac{R}{\sqrt{2 M(t) \Omega(t)}} - i \sqrt{\frac{M(t) \Omega(t)}{2}} Q
$$

(3.20)

The Hamiltonian can then be written as

$$
H' = \Omega(t) \left( \gamma^\dagger(t) \gamma(t) + \frac{1}{2} \right)
$$

(3.21)

We can interpret $\gamma^\dagger(t)$ as creating a particle of energy $\Omega(t)$ at time $t$. A quantum state can be specified by giving the initial condition at some time $t = t_0$ so that it be the lowest energy state: $\gamma(t_0) \Psi(t_0) = 0$. However a solution to the Schrödinger equation demands that at a later time this state would no longer be the lowest energy state: $\gamma(t_0) \Psi(t) \neq 0$, where $t \neq t_0$. This behaviour is usually interpreted as particle production [21]. What we intend to show is that this interpretation is dependent on the coordinates on phase space and hence not physical.

For purposes of comparing the above particle picture with the response of an Unruh-Dewitt detector, we wish to identify the wave function for the usual Minkowski vacuum state in the new $Q, R$ coordinates. The old Hamiltonian (3.8) can be expressed as

$$
H = \frac{1}{2} \omega \left( a^\dagger a + \frac{1}{2} \right)
$$

(3.22)

with $a = \frac{\hat{p}}{\sqrt{2 \omega}} - i \sqrt{\frac{\omega}{2}} \hat{x}$. The usual Minkowski vacuum state is the ground state of each of the harmonic oscillators of the KG field. For a single mode $a |\Psi(t)\rangle = 0$. We express the annihilation operator $a$ in terms of the $(Q, R)$ coordinates by using the relations...
Chapter 3. Particle definitions in functional Schrödinger formalism

(3.12) and (3.13). Since we wish to find the wave function we go to the ‘position’ space representation of the operators $Q$ and $R$.

$$\dot{Q} \rightarrow Q \quad \dot{R} \rightarrow -i \frac{\partial}{\partial Q}$$

The relation $a |\Psi(t)\rangle = 0$ becomes in the $(Q, R)$ coordinates, in terms of the wave function

$$\frac{\partial \Psi(t)}{\partial Q} + A(t)Q\Psi(t) = 0$$

where

$$A(t) \equiv \frac{\left( \frac{1}{\sqrt{2} \omega} - i \sqrt{\frac{\epsilon}{2}} \right)}{\left( \sqrt{\frac{\epsilon}{2} \frac{\hbar}{\omega}} - i \frac{\hbar}{\sqrt{2} \omega} \right)}$$

Integrating (3.24) we obtain the following wave function

$$\Psi(t) = \exp \left( \frac{-A(t)Q^2}{2} + B(t) \right)$$

where the function $B(t)$ is an arbitrary function of integration, independent of $Q$. It can be determined up to an additive constant by substitution of (3.26) into the Schrödinger equation

$$H'\Psi(t, Q) = i \frac{\partial \Psi(t, Q)}{\partial t}$$

It can be shown that the ansatz (3.26) is a solution to the above Schrödinger equation. It thus represents the usual vacuum state in the new coordinates $(Q, R)$.

We can now evaluate the expectation number of the particles, as defined by (3.20), in a given mode using the relation

$$\langle N_{\kappa} \rangle = \frac{\int da_{\kappa} \Psi^*(t)\gamma_{\kappa}^\dagger(t')\gamma_{\kappa}(t')\Psi(t)}{\int da_{\kappa} \Psi^*(t)\Psi(t)}$$

where we have taken the state $\Psi(t)$ and the particles $\gamma(t')$ at different times. This is done to account for the fact that the energy of particles as defined in (3.20) changes with time. One may want to take the particle energies to be constant (evaluated at a single
time) while the particle number is evaluated at the current time $t$. In this case we find the following expression for particle number

$$\langle N_{k} \rangle = \frac{\Re(A(t))}{4M(t')\Omega(t')} - \frac{\Re(A(t)))^{2}}{4M(t')\Omega(t')\ReA(t)} + \frac{M(t')\Omega(t')}{4} - \frac{1}{2}$$

(3.29)

where $\Re$ and $\Im$ are the real and imaginary part respectively. It can be checked that this expectation is in general non-zero by picking the function $f(t)$ arbitrarily (say $f(t) = e^{t}$).

The preceding calculations showed that by a change of variables an alternative definition of particles can be found based on the harmonic oscillator. The canonical vacuum on Minkowski spacetime is found to have a non-zero expectation value for these particles; this is counter to the physically correct zero expectation value of particles in this vacuum. We have thus demonstrated that the definition of particles as energy levels of the harmonic oscillator is akin to picking different coordinates on the phase space. Therefore, particles defined this way are in general unphysical. They do not in general correspond to what any observer would detect (not even an inertial observer with proper time $t$). Any single class of observers would have several different definitions of particles based on the energy levels of the Harmonic oscillator, were one to use this as a definition of particles.

### 3.2.1 Response of Unruh-Dewitt detector

There is a physically meaningful concept of a “particle”. It is based on the response rate of the Unruh-Dewitt detector. When the detector makes a transition to an excited state, a particle is said to be detected. The response of the detector is given by the integral of the two point function of the scalar field. The scalar field operator $\phi(x)$ is defined by its commutation relations with all other operators. The Dirac procedure for quantization is given by the following rule for calculating commutation relations:

$$[\hat{f}, \hat{g}] = i\{\hat{f}, \hat{g}\}$$

(3.30)
where the hat symbol over the Poisson bracket on the right hand side denotes the operator representation of the classical variable \( \{ f, g \} \). The commutation relations are unaffected by any canonical coordinate transformation on phase space because canonical coordinate transformations leave the Poisson bracket invariant; canonical transformations are defined to leave Hamilton’s equations unchanged. This guarantees that the \( \phi(x) \) field is invariant under change of coordinates on phase space and hence the response of an Unruh-Dewitt detector is invariant under the transformations given above.

What follows is a proof that the Poisson bracket is invariant under canonical transformations. Consider two sets of canonical coordinates given by \((q_i, p_j)\) and \(Q_m = Q_m(q_i, p_j)\), \(P_n = P_n(q_i, p_j)\), where the indices \(i, j, m\) and \(n\) take values from 1 to \(N\) where \(2N\) is the dimension of the phase space. The ‘direct conditions’ that \((q_i, p_j) \rightarrow (Q_m, P_n)\) defines a canonical transformation are given by:

\[
\begin{align*}
\left( \frac{\partial Q_m}{\partial p_n} \right)_{q,p} & = - \left( \frac{\partial q_n}{\partial P_m} \right)_{Q,P}, \\
\left( \frac{\partial Q_m}{\partial q_n} \right)_{q,p} & = \left( \frac{\partial p_n}{\partial P_m} \right)_{Q,P}, \\
\left( \frac{\partial P_m}{\partial p_n} \right)_{q,p} & = \left( \frac{\partial q_n}{\partial Q_m} \right)_{Q,P}, \\
\left( \frac{\partial P_m}{\partial q_n} \right)_{q,p} & = - \left( \frac{\partial p_n}{\partial Q_m} \right)_{Q,P}
\end{align*}
\] (3.31)

The coordinate representation of the Poisson bracket in the \((q, p)\) system is given by the following expression:

\[
\{ f, g \}_{q,p} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)
\] (3.32)

where \(f\) and \(g\) are arbitrary functions of the phase space coordinates. To show invariance of the Poisson bracket under canonical transformations we must show that:

\[
\{ f, g \}_{q,p} = \{ f, g \}_{Q,P}
\] (3.33)
Expanding the left hand side using (3.32) and using the chain rule we obtain the following:

\[
\{ f, g \}_{q,p} = \left( \frac{\partial Q_j}{\partial q_i} \frac{\partial f}{\partial Q_j} + \frac{\partial P_j}{\partial q_i} \frac{\partial f}{\partial P_j} \right) \left( \frac{\partial Q_k}{\partial p_i} \frac{\partial g}{\partial Q_k} + \frac{\partial P_k}{\partial p_i} \frac{\partial g}{\partial P_k} \right) - \left( \frac{\partial Q_j}{\partial p_i} \frac{\partial f}{\partial Q_j} + \frac{\partial P_j}{\partial p_i} \frac{\partial f}{\partial P_j} \right) \left( \frac{\partial Q_k}{\partial q_i} \frac{\partial g}{\partial Q_k} + \frac{\partial P_k}{\partial q_i} \frac{\partial g}{\partial P_k} \right)
\]

\[
= \{ Q_j, P_k \}_{q,p} \frac{\partial f}{\partial Q_j} \frac{\partial g}{\partial P_k} - \{ Q_j, P_k \}_{q,p} \frac{\partial f}{\partial P_j} \frac{\partial g}{\partial Q_k}
\]

where there is an implied summation over all indices. Using the equations (3.31) and the inverse function theorem we have that:

\[
\{ f, g \}_{q,p} = \delta_{jk} \left( \frac{\partial f}{\partial Q_j} \frac{\partial g}{\partial P_k} - \frac{\partial f}{\partial P_j} \frac{\partial g}{\partial Q_k} \right) \]

\[
= \{ f, g \}_{Q,P}
\]

Hence we have proven that the Poisson bracket is invariant under canonical transformations.

The above calculations demonstrate that quantum operators, and hence the correlation function, is invariant under canonical coordinate transformations. By the principle of general covariance we know that the response of a detector is a physically meaningful effect as opposed to the expectation of the number operator derived above. It is expected that the response of the detector following a given trajectory should have a coordinate independent meaning as it corresponds to the transition rate of a physical two-level system such as an electron orbiting a hydrogen atom.

The response of a detector for an inertial observer in the Minkowski vacuum ($\Psi$) is known to be null [22]. Contrast this with the generally non-zero particle number as specified by the expectation of the number operator above which is also in the Minkowski vacuum ($\Psi$). The point here is that when the response of a detector gives us a different particle number than the expectation of the number operator we should trust the detector.
3.3 Particles in de Sitter Spacetime

In this section we will quantize a conformally-coupled, massless, real scalar field in a de Sitter spacetime background and develop a mathematical notion of particles equivalent to that found in the literature [12] [18] [19]. It will be shown that the expectation of the number operator appears to indicate a non-thermal distribution of particles that becomes thermal as $t \rightarrow 0$ where $t$ is the conformal time. It is false to conclude that this would correspond to what a comoving observers sees or that the state is not a thermal one. It will be shown that the Unruh-Dewitt detector for a comoving, inertial observer will click as if in a bath of thermal radiation and hence does not agree with the expectation of the particle number operator.

The action for a massless, conformally coupled scalar Klein-Gordon field is

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left( \phi_\alpha \phi_\beta g^{\alpha \beta} - \frac{1}{6} R \phi^2 \right)$$

(3.36)

where $R$ is the Ricci scalar. The metric for the FLRW class of spacetimes that have flat spatial sections (of which a section of de Sitter is a special case [23]) in conformal time and comoving coordinates is

$$ds^2 = a^2(t) \left( dt^2 - dx^2 - dy^2 - dz^2 \right)$$

(3.37)

We note that the deSitter spacetime has the full ten parameter isometry group (like Minkowski spacetime) and is therefore not really a FLRW spacetime, which have only a six parameter isometry group. However, we chose deSitter spacetime for the purpose of comparison with the work in [19] and the choice does not affect our conclusions. Now we will expand the scalar field using the same expansion as above in section 3.2

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \alpha_k(t) \exp(ik \cdot \vec{x})$$

(3.38)

Substituting this into the action above with the background given by the metric (3.37) we obtain
Chapter 3. Particle definitions in functional Schrödinger formalism

\[ S = \frac{1}{2} \int dt d^3k \left[ a^2(t) \dot{\alpha}_{-\vec{k}} \dot{\alpha}_{\vec{k}} - \omega^2_k(t) \alpha_{\vec{k}} \alpha_{-\vec{k}} \right] \]  

(3.39)

with \( \omega^2_k(t) = a^2(t)k^2 + \frac{1}{6} a^4(t)R(t) \). Here the Ricci scalar is denoted to be \( R(t) \) because it will only be a function of time for the metric (3.37). Now we find the momenta conjugate to the coordinate \( a_{\vec{k}} \) which is given by the following formula

\[ \Pi_{\vec{k}} = \frac{\delta S}{\delta \dot{\alpha}_{\vec{k}}} = a^2(t) \dot{\alpha}_{-\vec{k}} \]  

(3.40)

The Hamiltonian is given via the Legendre transformation, with the result

\[ H = \int d^3k \Pi_{\vec{k}} \dot{\alpha}_{\vec{k}} - L \]  

(3.41)

\[ = \frac{1}{2} \int d^3k \left[ \frac{1}{a^2(t)} \Pi_{\vec{k}} \Pi_{-\vec{k}} + \omega^2_k(t) \alpha_{\vec{k}} \alpha_{-\vec{k}} \right] \]

The canonical commutation relations are given by

\[ [\dot{\alpha}_{\vec{k}}, \Pi_{\vec{q}}] = i \delta(\vec{k} - \vec{q}) \]  

(3.42)

with the following adjoint relations holding

\[ \Pi_{\vec{q}}^\dagger = \Pi_{-\vec{q}} \quad \dot{\alpha}^\dagger_{\vec{k}} = \dot{\alpha}_{-\vec{k}} \]  

(3.43)

We want a representation of the \( \Pi_{\vec{q}} \) and \( \dot{\alpha}_{\vec{q}} \) operators that acts on functions of the variables \( \alpha_{\vec{k}} \); we are dealing with the wave function representation. This representation is given by:

\[ \dot{\alpha}_{\vec{k}} \rightarrow \alpha_{\vec{k}} \quad \Pi_{\vec{k}} \rightarrow -i \frac{\partial}{\partial \alpha_{\vec{k}}} \]  

(3.44)

Now we will solve the Schrödinger equation for the wave functional of the scalar field. For this case the Schrödinger equation becomes

\[ \frac{1}{2} \sum_{\vec{k}} \left[ \frac{-1}{a^2(t) \epsilon^2} \frac{\partial^2 \Psi}{\partial \alpha_{\vec{k}} \alpha_{-\vec{k}}} + \omega^2_k(t) \alpha_{\vec{k}} \alpha_{-\vec{k}} \Psi \right] = i \frac{\partial \Psi}{\partial t} \]  

(3.45)
where we are working in a box or in the discrete momentum limit. \( \epsilon \) is a parameter with dimensions of length\(^{-3}\) that represents the spacing between consecutive values of the momenta and is taken to zero in the continuum limit. To solve the equation we use the following ansatz

\[
\Psi = \exp \left( -\epsilon \sum_q \left[ a_{q-q}(f_q(t) + ig_q) + \frac{\gamma_q(t) + i\delta_q(t)}{\epsilon} \right] \right)
\] (3.46)

In this case the functions \( f_q(t), g_q(t), \gamma_q(t) \) and \( \delta_q(t) \) are real and only depend on the magnitude of \( q \) and \( t \). Substituting this into the Schrödinger equation for this scenario we obtain the following equations

\[
\dot{\gamma}_k = \frac{-g_k}{a^2(t)} \tag{3.47}
\]
\[
\dot{\delta}_k = \frac{f_k}{a^2(t)} \tag{3.48}
\]
\[
\dot{f}_k = 4g_k f_k \tag{3.49}
\]
\[
\dot{g}_k = -\frac{2f_k^2 + 2g_k^2}{a^2(t)} + \frac{\omega_k^2(t)}{2} \tag{3.50}
\]

The above equations for \( f_k \) and \( g_k \) can be solved for the case of the portion of de Sitter spacetime with flat spatial ([23], page 125) sections given by \( a(t) = C/t \) and \( R = 12/C^2 \).

To solve them use equation (3.49) to solve for \( g_k \) and \( \dot{g}_k \) in terms of \( f_k \) and its derivatives and then substitute the result into equation (3.50). This gives the following equation for \( f_q \)

\[
\frac{\ddot{f}_k}{f_k} - \frac{3}{2} \frac{\dot{f}_k}{f_k} - \frac{2}{t} = -8t^4 f_k^3 + \frac{4k^2 f_k}{C^4 \overline{f}_k} + \frac{8f_k}{t^2 \overline{f}_k} \tag{3.51}
\]

Now we make the substitution \( f_k = (t \chi_k)^{-2} \) and obtain the following equation for \( \chi_k \).

\[
\dddot{\chi}_k + \left( 2k^2 - \frac{2}{t^2} \right) \chi_k = \frac{4}{C^4 \chi_k^3} \tag{3.52}
\]

This equation has the general solution [24]

\[
\chi_k = \left( u^2(t) + \frac{4}{C^4} \frac{v^2(t)}{W^2} \right)^{1/2} \tag{3.53}
\]
where \( u(t) \) and \( v(t) \) are two independent solutions to the equation \( \ddot{\chi}_k + (2k^2 - 2/t^2) \chi_k = 0 \) and \( W \) is their Wronskian. The initial conditions on these solutions are \( u(t_0) = \chi_0, \) \( \dot{u}(t_0) = \dot{\chi}_0, \) \( v(t_0) = 0 \) and \( \dot{v}(t_0) \neq 0. \) The equation \( \ddot{\chi}_k + (2k^2 - 2/t^2) \chi_k = 0 \) can be solved in general in terms of sines, cosines and \( t. \) Therefore we can write down the closed form expression for any state represented by the ansatz (3.46) for \( \Psi. \)

An exact solution to the equations (3.49) and (3.50) with \( a(t) = C/t \) and \( R = 12/C^2 \) in which we will be interested is the following

\[
\begin{align*}
 f_q(t) &= \frac{qC^2}{2t^2} \\
 g_q(t) &= -\frac{C^2}{2t^3}
\end{align*}
\]

It will be shown below that the wave functional represented by the ansatz (3.46) with the above choice for the functions \( f_q \) and \( g_q \) is the conformal vacuum\[22\] of the de Sitter spacetime.

### 3.3.1 Particle Interpretation

Now we develop a particle interpretation of the theory. The Hamiltonian can be expressed as in equation (3.41)

\[
H = \frac{1}{2} \int d^3k \left[ \frac{1}{a^2(t)} \Pi_k \Pi_{-k} + \omega_k^2(t) \alpha_k \alpha_{-k} \right]
\]

which is of the form of a harmonic oscillator with time-dependent mass and frequency, implying it is possible to develop a particle interpretation. Again, since we have a harmonic oscillator Hamiltonian with time-dependent mass and frequency there is an obvious definition of the annihilation and creation operators given respectively by:

\[
\begin{align*}
 b_k^\dagger &= \sqrt{\frac{\epsilon}{2}} \left( \sqrt{a(t) \omega_k(t)} \hat{a}_k^\dagger + i \frac{\Pi_{-k}}{\sqrt{a(t) \omega_k(t)}} \right) \\
 b_k &= \sqrt{\frac{\epsilon}{2}} \left( \sqrt{a(t) \omega_k(t)} \hat{a}_k - i \frac{\Pi_k}{\sqrt{a(t) \omega_k(t)}} \right)
\end{align*}
\]
The above annihilation/creation operators satisfy the usual commutation relations 
\[ [b_q, b^\dagger_k] = \delta_{q,k}. \] The Hamiltonian can be written in terms of these operators as
\[
H = \int d^3k \frac{\omega_k(t)}{a(t)} \left( b^\dagger_k b_k + 1 \right)
\] (3.58)
By analogy with the harmonic oscillator Hamiltonian we conclude that \( b^\dagger_k \) creates particles
of energy \( \omega_k(t)/a(t) \) at time \( t \). As before we can calculate the expectation of the number
operator for particles of type \( b_k \). It is given by the formula
\[
\langle N_k \rangle = \frac{\int d\alpha_k \Psi^* b^\dagger_k b_k \Psi}{\int d\alpha_k \Psi^* \Psi}
\] (3.59)
\[
\langle N_k \rangle = \frac{g^2_k}{a(t) \omega_k(t) f_k} + \frac{f_k}{a(t) \omega_k(t)} + \frac{\omega_k a(t)}{4 f_k} - 1
\]
The above expression gives the number of particles as defined by the creation/annihilation
operators above. This definition of particles in terms of the creation and annihilation op-
erators with time dependent mass and frequency is equivalent to the definition of particles
found in the literature [12] [18] [19] as will be shown in the appendix. The expectation
number is more easily calculated by using the annihilation/creation operators as opposed
to dealing with the wave functions of the multi-particle states directly.

Figure 3.3.1 shows a plot of number vs. frequency in the conformal vacuum state
given by (3.46) and (3.54). The expectation rises to infinity at \( \nu = 0 \) and rises for all \( \nu \) as
\( t \to 0 \) where \( t \) is the conformal time (equivalently as the comoving time goes to \( \infty \)). The
expectation of particles in the Planck distribution is known to be given by the following
distribution.
\[
N(\nu) = \frac{1}{e^{\beta \nu} - 1}
\] (3.60)
Comparing 3.3.1 with (3.60) or evaluating the expression in (3.59) and comparing it with
(3.60) shows us that there is no way that the expectation of the particle number is that
Chapter 3. Particle definitions in functional Schrödinger formalism

of a thermal state.

The response of an Unruh-Dewitt detector in the conformal vacuum is calculated below and is found to give a response as if it was in a bath of thermal radiation at any time. So we see that the expectation of the number operator gives a result in direct contradiction with the response of an Unruh-Dewitt detector indicating that this is a false particle number count. The definition of particles used to calculate (3.59) is equivalent to that used in [12] [19] (see section 3.5). This definition of particles therefore gives us unphysical results.

Figure 3.1: Number of particles vs frequency for various conformal times (t)

Number vs. Frequency

- $t = -20$
- $t = -10$
- $t = -5$
3.3.2 Response of Unruh-Dewitt detector

It was shown in section (3.2) that the expectation of the particle number operator used in [12] and [19] does not correspond to what some observer actually detects with their particle detector in a general spacetime. One must explicitly calculate the response rate of a particle detector traveling along the observer’s trajectory in order to know what the observer actually sees. In this subsection we will have to identify the conformal vacuum wave functional in order to compare the expectation of particle numbers (as calculated above) with the response of a particle detector. To find the wave functional of the conformal vacuum we employ a new procedure where one compares the expressions for the two point function obtained in the functional Schrödinger formalism with that in the normal Heisenberg picture.

The transition rate of an Unruh-Dewitt detector at time $\tau$ is given by [25]

$$\dot{F}_\tau(\nu) = 2\lim_{\epsilon \to 0} \int_{-\infty}^{0} ds \, Re( \exp(i\nu s)G^+_\epsilon(x(\tau), x(\tau - s)))$$  \hspace{1cm} (3.61)

where $G^+_\epsilon(x(\tau), x(\tau - s))$ is the two point function evaluated at the two specified points $(x(\tau)$ and $x(\tau - s))$ on the detector world line parameterized by $\tau$. $\epsilon$ appears in the two point function $(G^+_\epsilon(x(\tau), x(\tau - s)))$ as:

$$G^+_\epsilon(x(\tau), x(\tau - s)) = \sum_k \phi_k \phi_k^* e^{-\epsilon|k|}$$  \hspace{1cm} (3.62)

where $\phi_k$ are the modes of the Klein-Gordon field. Hence, $\epsilon$ is a small parameter used to regulate divergences in the calculation. Physically, $\epsilon$ can be said to correspond to the size of the detector [25].

The formula for the two point function in the interaction picture is given by

$$G^+(x, x') = \langle \phi(x) \phi(x') \rangle$$  \hspace{1cm} (3.63)
Chapter 3. Particle definitions in functional Schrödinger formalism

where \( \phi(x) \) is given by

\[
\phi(x) = \sum_k \left( a_k f_k(x) + a_k^\dagger f_k^*(x) \right)
\]  

(3.64)

where \( f_k(x) \) are solutions to the Klein-Gordon equation and \( a_k \) and \( a_k^\dagger \) are operators satisfying \([a_k, a_k^\dagger] = 1 \). The angled brackets (\( \langle \rangle \)) indicate that it is an expectation value taken in some quantum state.

In the functional Schrödinger formalism the two point function can be derived as follows. In the interaction picture the scalar field operator at all times is given in terms of the operator at a single time as

\[
\hat{\phi}(t, \bar{x}) = U(t, t_0) \hat{\phi}(t_0, \bar{x}) U(t, t_0)
\]  

(3.65)

where \( U(t, t_0) = \mathcal{T} \exp(-i \int_{t_0}^t H(t')dt') \) and \( \mathcal{T} \) is the time ordering operator. For convenience we will evaluate the two point function at the initial time \( t = t_0 \) and \( t' \neq t \). The two point function can be expressed in terms of the scalar field operator at time \( t \) as

\[
\langle \Psi(t) | U(t', t) \hat{\phi}(t, \bar{x}') U(t', t) \hat{\phi}(t, \bar{x}) | \Psi(t) \rangle
\]  

(3.66)

where we have assumed that \( |0\rangle = |\Psi(t)\rangle \) is the state at the initial time \( t \). This equality is obtained by considering the solution to the Schrödinger equation \( |\Psi(t)\rangle = \mathcal{T} U(t', t) |0\rangle \) from which we get the desired result at \( t' = t \).

After expanding to linear order in \( \Delta t = t' - t \) in equation (3.66) we obtain

\[
\langle \Psi(t) | \hat{\phi}(\bar{x}') \hat{\phi}(\bar{x}) | \Psi(t) \rangle + i \Delta t \langle \Psi(t) | \hat{H}(t), \hat{\phi}(\bar{x}') \hat{\phi}(\bar{x}) | \Psi(t) \rangle
\]  

(3.67)

Inserting the resolutions of the identity in the basis of equation (3.38)

\[
\hat{1} = \prod_{\bar{x} \in \mathcal{H}} \int da_{\bar{x}} |a_{\bar{x}}\rangle \langle a_{\bar{x}}|
\]  

(3.68)

and performing the commutator we obtain \([\hat{H}(t), \hat{\phi}(\bar{x}')] = -i \hat{\pi}((\bar{x}')/a^2(t) \). To make sure we have the right normalization we divide by the norm of the vacuum state. This gives the following expression for the two point function up to linear order in \( \Delta t \):
Chapter 3. Particle definitions in functional Schrödinger formalism 36

\[
\frac{\int \mathcal{D}\phi(x)\Psi^*(t)\phi(\vec{x})\Psi(t) - \frac{i\Delta t}{a^2(t)} \Psi^*(t) \frac{\delta}{\delta \phi(\vec{x})}(\phi(\vec{x})\Psi(t))}{\int \mathcal{D}\phi(x)\Psi^*(t)\Psi(t)} \tag{3.69}
\]

where the integrals over the field configuration \(\phi(x)\) are done at time \(t\).

To determine the exact wave functional \(\Psi(t, \phi(x))\) that represents the conformal vacuum in the Schrödinger picture we will compare the linearization of the two point function (3.69) to the linearization of the explicit expression for the two point function in the conformal vacuum in deSitter spacetime, which can be found in [22]:

\[
\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left( \frac{tt'}{C^2} \exp(-ik(t-t') + i\vec{k} \cdot (\vec{x} - \vec{x}')) \right) \tag{3.70}
\]

where \(t\) is the conformal time and \(\vec{x}\) is the comoving coordinate. Specifically, we will have to compare equation (3.69) to (3.70) expanded to linear order in \(\Delta t\).

To perform the integration over all fields in equation (3.69) we go to the expansion (3.38) of the field \(\phi(x)\). In terms of this expansion the integration measure becomes

\[
\int \mathcal{D}\phi(x) \rightarrow \prod_{\vec{q} \in \mathcal{H}} \int da_{\vec{q}} \tag{3.71}
\]

where \(\mathcal{H}\) denotes the “half-space” of the momenta which can be defined as the set \(\mathcal{H} = \{\vec{q} : q_x > 0\}\). This is done so that we do not integrate over both \(a_{\vec{q}}\) and \(a_{-\vec{q}}\) as these variables are complex conjugates of each other for a real scalar field. The integration is performed for the case \(t = t'\), \(\vec{x} \neq \vec{x}'\). Integrating over all complex variables \(a_{\vec{q}}\) for \(\vec{q} \in \mathcal{H}\) we obtain the simplified expression for the linearized two point function (3.69) as

\[
\int \frac{d^3 k}{(2\pi)^3} \left( \frac{1}{4f_k(t)} - \frac{\Delta t}{2a^2(t)} \left( i + \frac{g_k(t)}{f_k(t)} \right) \right) \exp(\vec{k} \cdot (\vec{x} - \vec{x}')) \tag{3.72}
\]

Comparing the linearization of (3.70) we easily find the expression for \(f_k(t)\) given in (3.54). We can then use this \(f_k(t)\) to solve for the \(g_k(t)\) using equation (3.49). The
expressions obtained can then be verified to be solutions of (3.50).

Verifying that the two point functions are the same to zero’th and first orders in $\Delta t$ is sufficient to establish the exact state functional because the two point function satisfies the homogeneous wave equation in a distributional sense (with cutoff)

$$g^{ab}\nabla_a \nabla_b \langle \Psi | \phi(x)\phi(x') | \Psi \rangle = 0$$  \hspace{1cm} (3.73)

where $\nabla_a$ is the covariant derivative taken with respect to the unprimed $x^a$ coordinate. This can be checked by using the field equations for $\phi(x)$. Therefore we have two solutions to the homogeneous wave equation in $x$ that agree for all $\vec{x}$ for $t = t'$ and whose first time derivatives agree for all $\vec{x}$ at $t = t'$. By uniqueness of solutions to linear PDE’s this means that the two point function calculated above is the two point function in the conformal vacuum.

The above procedure for identifying the conformal vacuum wave functional is easier than that found in [17] where it is done by renormalizing the stress energy tensor to verify that the above wave functional is the conformal vacuum.

The response rate of an Unruh-Dewitt detector for a comoving detector in the conformal vacuum above has been calculated [22] and is found to be a thermal response in the comoving observer’s proper time.

$$\hat{F}(\nu) = \frac{\nu}{2\pi(c^2\pi\nu - 1)}$$  \hspace{1cm} (3.74)

It is easily seen by substituting the solutions representing the conformal vacuum (3.54) in the expectation of the number operator (3.59) that the expectation of the number operator is not that of a thermal spectrum. The expectation of the number operator and the response of an Unruh-Dewitt detector are therefore inconsistent.
3.4 A ‘natural’ vacuum state in FRW

The Hamiltonian for a Klein-Gordon field on a FRW metric with flat spatial sections (3.37) takes the form of a harmonic oscillator with time-dependent mass and frequency (3.41).

Consider the real part of a single mode of the field.

\[ H = \frac{1}{2} \left( \frac{p^2}{M(t)} + M(t) \Omega^2(t) x^2 \right) \] (3.75)

where \( x(t) \) is a real variable, \( M(t) \equiv a(t) \) and \( \Omega^2(t) \equiv \dot{a}^2 + \xi a^2(t) R(t) \), where \( \xi \) is the coupling to the Ricci scalar of the field. It will be demonstrated that this Hamiltonian can be put into the form of a harmonic oscillator with fixed mass and fixed frequency by performing a canonical coordinate transformation on \( x \) and \( p \).

The new Hamiltonian must take the form

\[ H' = \frac{1}{2m} P^2 + \frac{m \omega^2}{2} Q^2 \] (3.76)

where \( m \) and \( \omega \) are constants and \( Q, P \) are the new generalized coordinate and its conjugate momentum respectively. To this end we write down the most general type 2 generating function (see [20] for the definition of canonical transformations) that could possibly give rise to a new Hamiltonian of the form (3.76).

\[ G_2 = f(t) x P + g(t) \frac{x^2}{2} + h(t) \frac{P^2}{2} \] (3.77)

The coordinate transformations are

\[ p = \left( f(t) - \frac{h(t) g(t)}{f(t)} \right) P + \frac{g(t)}{f(t)} Q \] (3.78)

\[ x = \frac{Q - h(t) P}{f(t)} \] (3.79)
Using the coordinate transformations and the equation $H' = H + \frac{\partial G}{\partial t}$ we find that the functions $f(t)$, $g(t)$ and $h(t)$ must satisfy

\begin{align}
\dot{h}(t) &= \frac{1}{m} - \frac{f^2(t)}{M(t)} + m\omega^2 h^2(t) \\
\dot{f}(t) &= \left( m\omega^2 h(t) - \frac{g(t)}{M(t)} \right) f(t) \\
\dot{g}(t) &= m\omega^2 f^2(t) - \frac{g^2(t)}{M(t)} - M(t)\Omega^2(t)
\end{align}

(3.80)

(3.81)

(3.82)

where $\dot{} \equiv d/dt$. Turning (3.80), (3.81) and (3.82) into difference equations (discretizing the derivatives $dh/dt$ to $\Delta h/\Delta t$) it becomes apparent that so long as $M(t)$ and $\Omega^2(t)$ are continuous and real then the above equations have real solutions. In terms of the FRW spacetime, this means that the equations have solutions everywhere except at the big bang singularity.

The canonical transformations (3.78) do not make sense at points where $f(t) = 0$. However, we can integrate (3.81) to obtain

\[ f(t) = C \exp \left( \int dt \left( m\omega^2 h(t) - \frac{g(t)}{M(t)} \right) \right) \]

(3.83)

where $C$ is an integration constant. Therefore, as long as the initial condition on $f(t)$ is chosen so that $C \neq 0$ then $f(t) \neq 0$ everywhere in the FRW universe, except at the big bang singularity where $M(t) = a(t) = 0$.

Thus we have a definition of particles, based on the energy levels of the harmonic oscillator, that is more natural than that used in other studies[12], [19], [18]. It is more natural because we have the Hamiltonian of a harmonic oscillator with constant mass and frequency. This again demonstrates that the definition of particles based on the energy levels of the harmonic oscillator used elsewhere [12], [19], [18] are physically meaningless and do not in general represent the particle spectrum seen by some observer.
3.5 Discussion

It was demonstrated that the definition of particles, as defined by energy levels of the harmonic oscillator, correspond to the choice of coordinates on phase space. By the principle of general covariance this definition is then, in general, unphysical and one should use a theoretical definition of particles that is independent of the particular coordinates chosen on phase space. The Wightman function and consequently the response of an Unruh-Dewitt detector is such a definition that is coordinate independent. The particle detector concept is thus more reliable than the energy levels of the harmonic oscillator concept and when these two definitions disagree one should trust the response of the Unruh-Dewitt detector.

Motivated by the above statements it was demonstrated that the conclusions found in other works [12] [19] are unreliable due to the definition of particles used. Using those authors’ definition it has been demonstrated that one cannot identify a thermal or non-thermal state. The response of the Unruh-Dewitt detector is the correct way to evaluate the response of an actual particle detector and this response has been found to disagree with the expectation of the number operator even in the standard case of de Sitter spacetime.

It was also demonstrated that there is a ‘more natural’ definition of particles based on the energy levels of the harmonic oscillator than that used in the cited papers. This was done by showing that one can put the Hamiltonian of a free Klein-Gordon field on FRW with flat spatial sections in the form of a sum of harmonic oscillators with time-independent frequency and mass. This system has an obvious definition of particles akin to the usual definitions on flat spacetime that disagrees with both those found in the cited papers. This further demonstrates the point that definitions of particles based on the energy levels of the harmonic oscillator have no physical meaning away from flat
Further, we have demonstrated a new procedure for identifying the wave functionals in the functional Schrödinger formalism of various states in the Heisenberg picture. This procedure may be applied to find which states correspond to the “ground state” wave functional in various spacetimes where the scalar field takes the form of a harmonic oscillator with time-dependent mass and frequency. It is easier and more reliable than the method used by Hill [17] as that author compares the stress energy tensor operator calculated in the Heisenberg and Schrödinger picture. A renormalization must be done when calculating the stress energy tensor operator, so this makes the calculation less reliable.

Appendix: Relationship of particle number calculations to results in the literature [19], [18], [12]

Calculating the expectation of the number operator for some mode \( a_{\vec{k}} \) as done above can be easily related to the procedure found in other works [19] [18][12]. In these works the expectation of the number operator is found by using directly the functions representing the n-particle states of some mode. These functions are denoted \( H_n(t, a_{\vec{k}}) \) and are the Hermite polynomials multiplied by a Gaussian factor like \( \exp(-h(t)a_{\vec{k}}a_{-\vec{k}}) \) where \( h(t) \) denotes a complex function of time. We assume that these functions are orthonormal, that is:

\[
\int d\vec{k} H^*_m(t, a_{\vec{k}})H_n(t, a_{\vec{k}}) = \delta_{nm} \tag{3.84}
\]

They are also eigenstates of the number operator

\[
N_{\vec{k}} H_n(t, a_{\vec{k}}) = nH_n(t, a_{\vec{k}}) \tag{3.85}
\]
They form a complete basis of functions so we can write the wave function for a single mode \((a_k)\) at some time in terms of these particle basis state at that time

\[
\Psi(t, a_k) = \sum_n c_n(t) H_n(t, a_k)
\]  \hspace{1cm} (3.86)

It can be easily shown using the orthonormality (3.84), the eigenstate property (3.85) and the completeness (3.86) that the following equality holds.

\[
\langle N_k \rangle = \int da_k \Psi^*(t, a_k) N_k \Psi(t, a_k) = \sum_n n |c_n(t)|^2
\]  \hspace{1cm} (3.87)

Therefore, the calculation of the expectation of the number operator above is equivalent to that found in the referenced literature [19] [18] [12] for a single mode. It is important to note that the multi-particle states are not necessarily solutions of the Schrödinger equation which is why the coefficients in equation (3.86) are functions of time.
Chapter 4

The response of particle detectors in Vaidya spacetimes

4.1 Introduction

The results in this chapter have been published in the following article: [26].

In Hawking’s original paper on black hole radiation [8], a general spherically symmetric collapse of matter is considered in which there is no interaction between the quantum field and the classical matter composing the black hole. The trade-off for considering such a general collapse scenario was that one could not calculate the detailed spectrum of radiation everywhere in the spacetime.

Hawking later demonstrated that if the radiation coming from a black hole was exactly thermal then one would lose all information about the matter that formed the black hole [9]. The lose of information was expressed mathematically, in this case, by the possible non-unitary evolution of quantum states; this is the famous information loss paradox.
There have been several attempts to solve the information loss paradox by demonstrating that the information is not lost at all. One class of these solutions is to demonstrate that when the back reaction of the emitted radiation is allowed to affect the mass of the black hole then information is not lost at all; for example see [11], [27]. There have also been claims that a full theory of quantum gravity is unitary and therefore information preserving [28].

Other attempts claim simply that if the radiation is calculated in detail for specific spacetimes then one would find the radiation is not thermal and hence information can be carried away in the deviations from thermality [29], [30], [12].

A third class of attempts to resolve the information paradox is to demonstrate that the radiation is exactly thermal and information is lost and to demonstrate that the laws of physics would still be consistent [10].

This paper will demonstrate that, in the absence of back reaction in Vaidya spacetimes, a particle detector that follows a time-like trajectory which is coupled to a spherically symmetric, massless Klein-Gordon field will not be able to distinguish between the different null dust configurations that collapse to form the black hole. Hence, the conclusions of this paper are in line with that of [10], [9] and [8] and opposed to the conclusions of [29], [30], [12] with regards to the thermality of the outgoing radiation.

In this paper the Vaidya metric will be used as a model for a black hole formed by null dust (see [31] or [32]).

\[ ds^2 = (1 - 2m(v)/r)dv^2 - 2dvdr - r^2d\theta^2 - r^2 \sin^2(\theta)d\varphi^2 \] (4.1)

where \( v \) is the ingoing null coordinate that is constant on radially ingoing null trajectories.
We will work with a mass function $m(v)$ that is of the form

$$m(v) = \begin{cases} 
0 & v < 0 \\
f(v) & 0 \leq v \leq T \\
m_0 & v > T 
\end{cases} \quad (4.2)$$

where $f(v)$ is some increasing function that goes from 0 to $m_0$ and $T$ is some timescale. Physically, the Vaidya class of spacetimes with the above mass function correspond to a spacetime with a spherically symmetric shell with finite thickness of ingoing light collapsing to a point to form a black hole. Outside of the shell the spacetime is Schwarzschild and inside the spacetime is Minkowski. From now on we will refer to the regions $v < 0$, $0 < v < T$ and $v > T$ as regions $\alpha$, $\beta$ and $\gamma$ respectively.

The spherically symmetric modes (s-waves) of a complex, massless scalar field will be quantized in this background. The Wightman function will be calculated using only the spherically symmetric modes and the response of a particle detector will be evaluated for a constant $r$ observer. The constant $r$ observer is the Killing observer after the shell of null radiation has passed him, i.e. in the region $v > T$. The $r$ coordinate as usual is the arial coordinate. The particle detector response can be thought of as due only to the spherical modes. It will be shown that the response rate of the particle detector at a point outside of the collapsing shell will not depend on the mass function $m(v)$ if the detector is taken to be in its ground state at a null time outside of the collapsing null dust. Therefore the information about the matter used to form the black hole cannot be present in correlations in the outgoing radiation.
4.2 Quantization

The action for a complex, massless scalar field in the Vaidya background is

\[ S = \int d^4x \sqrt{-g} \partial_a \phi \partial_b \phi^* g^{ab} \]  

(4.3)

with \( g_{ab} \) taken to be the above metric. The above action will be only linear in time derivatives of the field. Therefore, the system is a constrained one and this will have to be taken into account when performing the canonical quantization.

It will be shown that the Hamiltonian for the system can be written in the form

\[ H = H_0 + H_1(v) \]  

(4.4)

The \( H_0 \) part will be the Hamiltonian for a massless complex scalar field in flat spacetime. It will be convenient to work in a picture where the states evolve under the \( H_1(v) \) part of the Hamiltonian while the operators evolve under the \( H_0 \) part. Thus the scalar field operators will evolve as if they were in flat spacetime and the rest of the time evolution due to the curved geometry will be shunted onto the state.

Assuming no angular dependence of the scalar field (s-waves) we can expand the field \( \phi \) as follows

\[ \phi(v, r) = \sum_k a_k(v) f_k(r) \]  

(4.5)

Substituting the above expansion in the action and substituting in the Vaidya metric the action reduces to

\[ S = \int dv \sum_{k,l} (m(v)a_k^* C_{kl} a_l - a_k^* B_{kl} a_l - i a_k^* A_{kl} \dot{a}_l) \]  

(4.6)

where the \( \dot{a}_k \) denotes differentiation of \( a_k \) with respect to the ingoing null time \( v \). The Hermitian matrices \( A, B \) and \( C \) are defined as
The action can be re-written in a more convenient form by ignoring a boundary term. The re-written action is

\[ S = \int dv L(v) \]

\[ = \int dv \left( -\frac{i}{2} a^\dagger A \dot{a} + \frac{i}{2} a^\dagger A a - a^\dagger B a + m(v) a^\dagger C a \right) \]

where \( a \) is a column vector of the modes defined by \( a := (a_1, a_2, \ldots)^t \) and \( a^\dagger := (a_1^*, a_2^*, \ldots) \), where * is complex conjugation. The conjugate momenta is given by the usual formulæ

\[ p = \frac{\delta S}{\delta \dot{a}} = -\frac{i}{2} a^\dagger A \]

\[ p^\dagger = \frac{\delta S}{\delta \dot{a}^\dagger} = \frac{i}{2} A a \]

where \( p \) is the row vector \( p := (p_1, p_2, \ldots) \) and \( p^\dagger \) is the corresponding column vector. Hence there are two infinite sets of primary constraints between the conjugate momenta \( (p_k \text{ and } p_k^\dagger) \) and the \( a_k, a_k^\dagger \). In order to quantize this system we follow the Dirac procedure outlined in [33]. We denote the constraints for some conjugate momenta \( p_k \) from the first set of constraints (4.13) and \( p_k^\dagger \) from the second set of constraints (4.14) respectively by

\[ G_{1,k} = p_k + \sum_l \frac{i}{2} a_l^\dagger A_{lk} = 0 \]

\[ G_{2,k} = p_k^\dagger - \sum_l \frac{i}{2} A_{kl} a_l = 0 \]

The canonical Hamiltonian defined by the Legendre transformation is

\[ H = p \dot{a} + a^\dagger p^\dagger - L = a^\dagger B a - m(v) a^\dagger C a \]
where $L$ is the Lagrangian given by (4.10). Using the primary constraints in equations (4.13), (4.14) we can re-write the Hamiltonian as

$$H = 2 p A^{-1} B A^{-1} p^i - 2 m(v) p A^{-1} C A^{-1} p^i + \frac{1}{2} a^\dagger B a - \frac{1}{2} m(v) a^\dagger C a$$

(4.16)

We now show that the time derivative of each of the constraints $G_{1,k}$, $G_{2,k}$ are equal to some multiple of themselves. The time evolution is generated by the Hamiltonian in equation (4.16). Using the normal Poisson bracket relations $\{a_k, p_l\} = \delta_{lk}$ and $\{a^\dagger_k, p^\dagger_l\} = \delta_{lk}$ with all other Poisson brackets zero, the following can be shown:

$$\dot{G}_1 = \{G_1, H\} = i G_1 (A^{-1} B - m(v) A^{-1} C)$$

(4.17)

$$\dot{G}_2 = \{G_2, H\} = -i (B A^{-1} - m(v) C A^{-1}) G_2$$

(4.18)

where $G_1$ is a row vector and $G_2$ is a column vector. Hence we have a complete and consistent system of constraints and Hamiltonian.

In order to quantize this system we observe that

$$\{G_1, G_2\} = i A \neq 0$$

(4.19)

These constraints are called “second class”. To quantize the system the Dirac procedure will be used where the commutation relations of the observables are given by

$$[\hat{f}, \hat{g}] = i \{f, g\}_{DB}$$

(4.20)

where $\hat{f}$ and $\hat{g}$ are the operators corresponding to the classical variables $f$ and $g$ respectively. $\{f, g\}_{DB}$ is the Dirac bracket given by the following formula

$$\{f, g\}_{DB} = \{f, g\} - \sum_{a,b} \{f, G_a\} F_{ab} \{G_b, g\}$$

(4.21)

where the $G_a$ are all of the second class constraints and $F_{ab}$ is the inverse of the matrix $\{G_a, G_b\}$. 
Chapter 4. The response of particle detectors in Vaidya spacetimes

Following the same procedure as seen in [12] we use the principal axis transformation ([20]) to choose the basis $a_k$ so that $A$ is the identity matrix and $C$ is some diagonal matrix with real entries denoted by $\lambda_k$ for each mode $a_k$. The variables $a_k$ are complex so a new basis can be chosen so that the negative eigenvalues of the matrix $A$ become equal to 1 by re-scaling the modes $a_k$ by $i$. The zero eigenvalues of the matrix $A$ can be ignored as any mode with a corresponding zero eigenvalue for $A$ would have no time derivative term appearing in the action for that mode (4.10) and would thus be non-dynamical.

Now that the $A$ matrix is the identity we can deal with a single set of modes $a_k, a_k^\dagger, p_k, p_k^\dagger$. The Dirac bracket is readily calculated between each pair of these variables and the commutation relations are given by the rule in (19). The commutation relations are

\begin{align}
[\hat{a}_k, \hat{p}_l] &= \frac{i}{2} \delta_{kl} \quad (4.22) \\
[\hat{a}_k, \hat{p}_k^\dagger] &= 0 \quad (4.23) \\
[\hat{a}_k, \hat{a}_l^\dagger] &= -\delta_{kl} \quad (4.24) \\
[\hat{a}_k^\dagger, \hat{p}_l] &= 0 \quad (4.25) \\
[\hat{a}_k^\dagger, \hat{p}_k^\dagger] &= \frac{i}{2} \delta_{kl} \quad (4.26) \\
[\hat{p}_k, \hat{p}_k^\dagger] &= \frac{1}{4} \delta_{kl} \quad (4.27)
\end{align}

The Hamiltonian in terms of the new basis which diagonalizes $A$ and $C$ is given as

\begin{equation}
\hat{H} = \sum_{k,l} \left(2\hat{p}_l B_{lk} \hat{p}_k^\dagger + \frac{1}{2} \hat{a}_k^\dagger B_{kl} \hat{a}_l + 1 \right)
-2m(v)\lambda_k \hat{p}_k \hat{p}_k^\dagger - \frac{1}{2} m(v) \hat{a}_k^\dagger \hat{a}_k \lambda_k \right) \quad (4.28)
\end{equation}

Now we divide this Hamiltonian into the flat spacetime part and the curved spacetime part as in equation (4.4) with

\begin{align}
\hat{H}_0 &= \sum_{k,l} \left(2\hat{p}_l B_{lk} \hat{p}_k^\dagger + \frac{1}{2} \hat{a}_k^\dagger B_{kl} \hat{a}_l \right) \quad (4.29) \\
\hat{H}_1 &= \sum_{k,l} \left(-2m(v)\lambda_k \hat{p}_k \hat{p}_k^\dagger - \frac{1}{2} m(v) \hat{a}_k^\dagger \hat{a}_k \lambda_k \right) \quad (4.30)
\end{align}
To work in the interaction picture we demand that the operators become time dependent and evolve under $\hat H_0$. Using the commutation relations obtained above we solve the Heisenberg equations for the fundamental operators.

\begin{align}
i\hat p_k &= [\hat p_k, \hat H_0] = \sum_l \left( \frac{1}{2} \hat p_l - \frac{i}{4} \hat a_l^\dagger \right) B_{lk} \tag{4.31} \\
i\hat a_k &= [\hat a_k, \hat H_0] = \sum_l B_{kl} \left( i\hat p_l^\dagger - \frac{1}{2} \hat a_l \right) \tag{4.32} \\
i\hat p_k^\dagger &= [\hat p_k^\dagger, \hat H_0] = \sum_l B_{kl} \left( -\frac{1}{2} \hat p_l - \frac{i}{4} \hat a_l \right) \tag{4.33} \\
i\hat a_k^\dagger &= [\hat a_k^\dagger, \hat H_0] = \sum_l \left( i\hat p_l + \frac{1}{2} \hat a_l \right) B_{lk} \tag{4.34}
\end{align}

Taking linear combinations of the above equations they can be solved relatively easily. The general solution is

\begin{align}
\hat a &= -2ie^{iBv}\hat J^\dagger + 2i\hat D^\dagger \tag{4.35} \\
\hat p &= \hat J e^{-iBv} + \hat D \tag{4.36}
\end{align}

where $\hat J^\dagger$ and $\hat D^\dagger$ are column vectors of operator integration constants. To be clear $\hat J^\dagger = (\hat j_1\dagger, \hat j_2\dagger, ...)^t$ so that the $\dagger$ acting on the vector of operators also acts on each individual operator.

Imposing the commutation relations between the $\hat a$, $\hat a^\dagger$, $\hat p$ and $\hat p^\dagger$’s we find that the only non commuting pair of variables from the set $(\hat J_a, \hat J_a^\dagger, \hat D_a, \hat D_a^\dagger)$ are $\hat J_a$ and $\hat J_a^\dagger$. The following commutation relations are found

\[ [\hat J_a, \hat J_b^\dagger] = \frac{1}{4} \delta_{ab} \tag{4.37} \]

In particular the operators $\hat D_a$ and $\hat D_a^\dagger$ commute with everything. From now on we choose $\hat D = 0$. With this choice the flat spacetime Hamiltonian $\tilde H_0$ will not depend on time $(v)$, as it would if $\hat D \neq 0$, which is a physically reasonable requirement. Further, the choice $\hat D = 0$ is necessary to have the same relations between the operators $\hat p$ and $\hat a$
as in the classical relations (4.13) and (4.14) in a basis where the matrix $A$ is the identity.

In terms of the above solutions to the Heisenberg equations we can re-write $\hat{H}_0$ as

$$\hat{H}_0 = \sum_{k,l} 4 \hat{J}_l B_{lk} \hat{J}^\dagger_k$$

(4.38)

Since $B_{lk}$ is Hermitian it can be diagonalized by performing a unitary transformation on the operators $\hat{J}_k$. To this end we re-define the operators as $\hat{J}^\dagger = U^\dagger \hat{\chi}^\dagger /2$, where $U^\dagger$ is unitary. The flat spacetime Hamiltonian becomes

$$\hat{H}_0 = \sum_k \omega_k \left( \hat{\chi}^\dagger_k \hat{\chi}_k + 1 \right)$$

(4.39)

where the $\omega_k$ are the real eigenvalues of the matrix $B$ and $[\hat{\chi}_k, \hat{\chi}_l^\dagger] = \delta_{lk}$ holding. It will be shown in section (4.3) that $\omega_k > 0$. Therefore, the $\hat{\chi}_k$ operators are the annihilation operators of positive energy particles with their adjoint being the creation operators. The “curved” or time-dependent part of the Hamiltonian becomes

$$\hat{H}_1 = -m(v) \hat{\chi}^\dagger \exp(i\Omega v) \bar{\Lambda} \exp(-i\Omega v) \hat{\chi} - m(v) Tr(\bar{\Lambda})$$

(4.40)

where here $\bar{\Lambda}$ is the $C$ matrix in equation (4.9) in the new basis which diagonalizes the matrix $B$. $\Omega$ is a diagonal matrix with the real diagonal entries being the $\omega_k$ that are in the $\hat{H}_0$ part of the Hamiltonian. Specifically we have

$$\Omega := UBU^\dagger = \text{diag}([\omega_1, \omega_2, ...]$$

(4.41)

$$\bar{\Lambda} := (UCU^\dagger)^\dagger$$

(4.42)

The state of the Klein-Gordon field will be taken to be given by

$$\hat{\chi}_k |0\rangle = 0$$

(4.43)

Hence $|0\rangle$ is the vacuum for the operators $\chi_k$ and $\chi_k^\dagger$. It will be shown below, when we calculate the $\hat{\phi}$ field explicitly, that this state is the unique state such that the particle
detectors of constant $r$ observers will have a null response when $m(v) = 0$. For the mass function (4.2) the state has this interpretation in region $\alpha$. It is the most natural choice of vacuum state for spacetimes of the form (4.1) with (4.2) holding because in region $\alpha$ the observers are in a Minkowski spacetime and completely causally disconnected from the non vacuum region of spacetime (see figure 4.1).

![Conformal diagram for a Vaidya spacetime.](image)

Figure 4.1: Conformal diagram for a Vaidya spacetime. The region in between $v = 0$ and $v = T$ is the non vacuum region. Any massive observer with finite energy (that cannot approach the speed of light) starts on $I^-$ and can end either in the singularity or at $I^+$; hence the observer must travel through the non vacuum region.

In the picture we are working in, the state of the Klein-Gordon field evolves as

$$|\Psi(v)\rangle = \mathcal{T} \exp(-i \int_{v_0}^{v} \hat{H}_1(v')dv') |\Psi(v_0)\rangle$$

(4.44)

where $\mathcal{T}$ is the time ordering operator. If we pick the initial state to be the vacuum with respect to the above creation and annihilation operators (i.e. $\hat{\chi}_k |\Psi(v_0)\rangle = 0$) then we can see from the form of $\hat{H}_1$ that the state evolves in time only by the phase factor $\exp(i \int_{v_0}^{v} m(v')Tr(\bar{A}))$ which means that the vacuum state vector is mathematically the
same at all times. Notice however that a general state from the Fock space formed by
the creation operators will not evolve only by a phase factor and hence there will be
a change in the state due to the Vaidya background. This can be seen by expanding
equation (4.44) to first order to obtain
\[ |\Psi(v)\rangle \approx |\Psi(v_0)\rangle - i \int_{v_0}^{v} \tilde{H}_1(v')dv' |\Psi(v_0)\rangle \] (4.45)
So now if we have a one-particle state \( |\Psi(v_0)\rangle = \hat{\chi}^\dagger_k |0\rangle \) at time \( v = v_0 \) then to first order
we will have a linear combination of one particle states given by
\[ |\Psi(v)\rangle \approx \left(1 + i \int_{v_0}^{v} dv' m(v') Tr(\hat{\Lambda})\right) |\Psi(v_0)\rangle + \]
\[ \sum_{p,j,l} i \int_{v_0}^{v} dv' m(v') \exp(i\Omega v') \lambda_{p} \hat{\Lambda} \exp(-i\Omega v') j_{k} \hat{\chi}^\dagger_{l} |0\rangle \] (4.46)

### 4.3 Response of particle detectors

The response rate of a particle detector at a given frequency \( \nu \) as it follows some path
through spacetime is given by
\[ \dot{F}_\nu(\nu) = 2 \int_{0}^{\tau - \tau_0} ds \ \Re\{ \exp(-i\nu s)G^+(x(\tau), x(\tau - s))\} \] (4.47)
where \( x(\tau) \) is the path through spacetime (see [25], [22]) and \( \Re \) denotes the real part. \( \tau \)
is the proper time of the detector that parameterizes the path \( x(\tau) \). Also, the particle
detector is in the ground state at \( \tau_0 \). First, we must evaluate the Wightman function
\( G^+(x(\tau), x(\tau - s)) = \langle 0 | \hat{\phi}(x(\tau)) \hat{\phi}(x(\tau - s)) |0 \rangle \).

The scalar field operators are given by
\[ \hat{\phi}(v, r) = \sum_{k} \hat{a}_k(v) f_k(r) \] (4.48)
We have solved for the operator coefficients \( \hat{a}_k \) in equation (4.35). The \( \hat{\phi}(v, r) \) field
operators are then given by
\[ \hat{\phi}(v, r) = \sum_{l,a} -2i f_l(r) \exp(iBv)_{l,a} \hat{J}^\dagger_{a} \] (4.49)
where \( \exp(iBv) \) is a matrix. If we now make the substitution \( \hat{J}_a^\dagger = 1/2U_{ba}^*\hat{\chi}_b^\dagger \), which diagonalizes the Hamiltonian \( \hat{H}_0 \), we end up with the field operator as

\[
\hat{\phi}(v, r) = \sum_{b, c, d} -i f_i(r) U_{cb}^* \exp(i\Omega v) \hat{\chi}_b^\dagger
\]

(4.50)

Defining \( g_k(r) = f_i U_{cb}^* \) and substituting this into the above expression, we finally obtain.

\[
\hat{\phi}(v, r) = \sum_k -i g_k(r) \exp(i\omega_k v) \hat{\chi}_k^\dagger
\]

(4.51)

In order to solve explicitly for the \( g_k(r) \) modes we use the fact that the complex field \( \hat{\phi}(v, r) \) is a solution of the wave equation determined by the \( H_0 \) operator. It is not difficult to see that this will be the wave equation in the Vaidya background with \( m(v) = 0 \), i.e. the wave equation in Minkowski spacetime. By substituting the expansion (4.51) into the wave equation for the Vaidya background with \( m(v) = 0 \) we obtain that the equation

\[
r^2\partial_r^2 g_k(r) + 2i\omega_k r^2 \partial_r g_k(r) + 2r \partial_r g_k(r) + 2i\omega_k rg_k(r) = 0
\]

(4.52)

must hold for each mode \( g_k(r) \). The general solution to this equation is

\[
g_k(r) = C(k) + \frac{D(k)}{r} \exp(-2i\omega_k r)
\]

(4.53)

where \( C(k) \) and \( D(k) \) are arbitrary complex constants. We wish the modes we expand in to be physical and hence regular at \( r = 0 \) in Minkowski spacetime. The only way to achieve this is to choose \( C(k) = -D(k) \) so that the divergence there cancels.

The expression for the matrix \( B \) is given by (4.8) in terms of the original \( f_k(r) \) modes. The diagonalized matrix \( \Omega \) is given in terms of \( B \) as

\[
\Omega_{lk} = \sum_{a, b} U_{la} B_{ab} U_{kb}^*
\]

(4.54)

Using the definition of the matrix \( B \) and the definition of the modes \( g_k(r) \) we obtain:

\[
\Omega_{lk} = 4\pi \int_0^\infty dr r^2 \partial_r g_l^*(r) \partial_r g_k(r)
\]

(4.55)
As a check, the modes (4.53) with \( C(k) = -D(k) \), when substituted into the right hand side of (4.55) should be diagonal in \( l \) and \( k \). Substitution of the modes

\[
g_k(r) = D(k) \left( -\frac{1}{r} + \frac{\exp(-2i\omega_k r)}{r} \right)
\]  

(4.56)

into this integral can be shown to reduce to the single term

\[
\Omega_{lk} = 16\pi D^*(l)D(k) \int_{0}^{\infty} dr \omega_l \omega_k \exp(2i(\omega_l - \omega_k)r)
\]  

(4.57)

where \( \omega_l \) and \( \omega_k \) are real numbers. To show this matrix is diagonal we will put our system in a box of finite size so that we can index our momentum modes with integers. The radius of our system will be \( R \) and this will be the upper bound of the integral in (4.57). The discrete limit is obtained by making the replacements

\[
\omega_l, \omega_k \to \frac{\pi l}{R}, \frac{\pi k}{R}
\]  

(4.58)

where on the right hand side \( l \) and \( k \) are integers. The integral (4.57) becomes

\[
16\pi D^*(l)D(k) \int_{0}^{R} dr \frac{\pi^2}{R^2} lk \left[ \cos\left(\frac{2\pi}{R}(l - k)r\right) + i \sin\left(\frac{2\pi}{R}(l - k)r\right) \right]
\]  

(4.59)

This is easily shown to be equal to

\[
\frac{16\pi^3}{R} l^2 \delta_{lk} |D(l)|^2
\]  

(4.60)

Recall that the elements of \( \Omega_{lk} \) are \( \omega_k \) on the diagonal. Using this in the discrete limit we have that the above expression must be equal to \( \pi l/R \), in the discrete limit. This fixes the the magnitude of the complex constant to be

\[
|D(l)| = \frac{1}{4\pi \sqrt{l}}
\]  

(4.61)

The expression (4.60) is manifestly positive. Therefore, only the modes \( \omega_k > 0 \) need to be considered when we calculate the Wightman function.
The quantized complex Klein-Gordon field in (4.51) is not Hermitian and therefore not a valid observable. We will use the Hermitian observable that is formed from equation (4.51) which is given by the following expansion

\[
\Re \left( \hat{\phi}(v, r) \right) = \sum_k \frac{-i}{2} (g_k(r) \exp(i\omega_k v) \hat{\chi}_k^\dagger - g_k^*(r) \exp(-i\omega_k v) \hat{\chi}_k) \tag{4.62}
\]

with the \( g_k \) given by equation (4.56) and (4.61). The above Hermitian field obeys the correct equations of motion and the correct commutation relations and is therefore the real, quantum, spherically-symmetric Klein-Gordon field operator.

In order to calculate the Wightman function using the interaction picture we will first calculate the Feynman correlation function using the formula given in [15].

\[
\langle \varnothing | T \hat{\phi}_F(v_n, r_n) \ldots \hat{\phi}_F(v_1, r_1) | \varnothing \rangle = \frac{\langle 0 | T \hat{\phi}(v_n, r_n) \ldots \hat{\phi}(v_1, r_1) \exp(-i \int_{-\infty}^{\infty} dv \hat{H}_1(v)) | 0 \rangle}{\langle 0 | T \exp(-i \int_{-\infty}^{\infty} dv \hat{H}_1(v)) | 0 \rangle} \tag{4.63}
\]

The state \( | \varnothing \rangle \) signifies the vacuum state of the full Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_1(v) \) and \( | 0 \rangle \) is the vacuum state of the “free” Hamiltonian \( \hat{H}_0 \) and \( \hat{\phi}_F(v, r) \) denote the scalar field with the full Hamiltonian. Note that the contribution from the integrals at \( \infty \) cancel since \( \hat{H}_1(v) = 0 \) for \( v < 0 \) (in region \( \alpha \)). Also, the \( Tr(\Lambda) \) term in \( \hat{H}_1(v) \) cancels. For the case of two \( \hat{\phi} \) factors the expression becomes for \( v_2 > v_1 \)

\[
\langle \varnothing | \hat{\phi}_F(v_2, r_2) \hat{\phi}_F(v_1, r_1) | \varnothing \rangle = \langle 0 | \hat{\phi}(v_2, r_2) T \exp(-i \int_{v_1}^{v_2} dv \eta(v)) \hat{\phi}(v_1, r_1) | 0 \rangle \tag{4.64}
\]

where

\[
\eta(v) \equiv -m(v)\hat{\chi}^\dagger \exp(i\Omega v)\hat{\Lambda} \exp(-i\Omega v)\hat{\chi} \tag{4.65}
\]

where we have used the definition of the vacuum state (4.43) to evaluate the denominator in (4.63).
At this point we can specify the meaning of the vacuum state given in (4.43). Using the general formula for the Wightman function (4.63) we can obtain the Wightman function in Minkowski spacetime for the spherically symmetric modes of the real scalar field operator given in (4.62). In the continuum limit it is expressed by the integral

\[ G^+(t, r; t', r') = \frac{1}{16\pi rr'} \int_0^\infty \frac{d\omega}{\omega} \exp(-i\omega(t-t')) \sin(\omega r) \sin(\omega r') \]  

where we have used that \( v = t + r \) in Minkowski spacetime and we have used the relations (4.58) and the relation

\[ \frac{1}{R} \sum_k \to \int d\omega \]  

(4.67)

to go to the continuum limit. By plugging (4.66) into equation (4.47) it can be verified that constant \( r \) observers in Minkowski spacetime in the state (4.43) will have a null response for their particle detectors (see appendix for details).

We note that the \( v = \text{const} \) surfaces are not Cauchy surfaces and thus there is a concern that the field evolution will not be unique if we specify the evolution in terms of these surfaces. For Vaidya metrics of the form (4.1), (4.2) we have that for constant \( v \) as \( r \to \infty \) the metric becomes Minkowski spacetime. The limit \( r \to \infty \) can effectively be reproduced by considering \( m(v) = 0 \). Therefore an alternate characterization of the vacuum (4.43) is that it is the vacuum state such that constant \( r \) particle detectors do not click on past null infinity (see figure 4.2). Past null infinity is a Cauchy surface and thus specifying the state of the field there is sufficient to determine the evolution of the field uniquely throughout the spacetime.

Another way to see that the evolution of the Klein-Gordon field is unique is to note that imposing the condition that \( \hat{\phi} \) be finite at \( r = 0 \) as well as picking the modes to be positive frequency with respect to the inertial time \( t \) in Minkowski spacetime fixes all arbitrary constants present in the mode solutions to the Klein-Gordon equation.
The proper time of the constant $r$ observers will be used to parameterize the paths in formula (4.47). In order to evaluate the response rate of a particle detector for a constant $r$ observer it is convenient to re-write the equation (4.47) in terms of an integral over $v$ instead of the proper time of the time-like particle detector. To this end, make the substitution $p = \tau - s$ so that we now integrate over the variable $p$ instead of $s$. $\tau$ is a constant representing the proper time at which the response of the detector is evaluated. The response rate is given by:

$$\dot{F}_\tau(v) = 2 \int_{\tau_0}^{\tau} dp \Re\{\exp(-i\nu(\tau - p)))\} G^+(x(\tau),x(p))$$  \hspace{1cm} (4.68)$$

Since $\tau$ and $p$ are both proper times we can use the line element (4.1) to solve for them in terms of the null time $v$.

$$d\tau = \sqrt{1 - \frac{2m(v)}{r}} dv$$  \hspace{1cm} (4.69)$$

The proper times as functions of the null time are denoted as $\tau = \tau(v)$ and $p = \tau(v')$. In the integral over the proper time it is $p$ that is integrated over, so in terms of null time
we will be integrating over \( v' \). The final formula is then given by

\[
\dot{F}_v(v) = 2 \int_{v_0}^{v} dv' \, \Re(\xi(v', r)G^+(v, r; v', r))
\] (4.70)

where

\[
\xi(v', r) \equiv \sqrt{1 - \frac{2m(v')}{r}} \exp(-i\nu(\tau(v) - \tau(v')))
\]

where \( G^+(v, r; v', r) \) is given by equation (4.64) and the detector is in the ground state at \( v_0 \). The above formula represents the transition rate for an observer at radius \( r \) at time \( v \).

For mass functions of the form (4.2) it is clear from the form of the Wightman function given in (4.64) that, for \( v_1 \) and \( v_2 \) in region \( \gamma \), it will be independent of the mass function in region \( \beta \). Therefore, the Wightman function is completely independent of the configuration of null radiation that collapsed to form a black hole when evaluated in this region. Hence, from equation (4.70) we have that the transition rate of a detector that is in its ground state at some time in region \( \gamma \) is completely independent of the configuration of null radiation.

It is clear from equation (4.64) that the Wightman function will depend on the mass function if evaluated at null times such that \( v_1 \) is in region \( \alpha \) and \( v_2 \) is in region \( \gamma \) or vice versa. From equation (4.47) it is seen that this indicates that the transition rate of a detector at a time \( v \) in region \( \gamma \), that is in its ground state at some time \( v_0 \) in region \( \alpha \), will depend on the details of the mass function. This is expected since the detector is actually in the non vacuum region of the spacetime in this scenario (see figure 4.1).

### 4.4 Discussion

A particle detector that responds only to the spherically symmetric modes of the quantum Klein-Gordon field has been considered in the Vaidya class of spacetimes with a
mass function given by equation (4.2). There are two important cases to be considered for a particle detector that follows a time-like path through spacetime.

The first is when the particle detector is taken to be in its ground state at a time \( \nu \) in region \( \alpha \). The response rate of this detector at a time outside of the collapsing null dust (region \( \gamma \)) is seen to depend on the details of mass function and hence the details of the collapse. This is expected since a time-like particle detector must travel through the non vacuum region of the spacetime and hence “feel” the gravitational field caused by the collapsing matter (see figure 4.1).

The second case is when the particle detector is taken to be in its ground state at some time after the collapse (region \( \gamma \)). The response rate of the detector at any time in region \( \gamma \) is seen to only depend on the value of the mass function in region \( \gamma \). The response rate is independent of the form of the mass function in region \( \beta \) and hence is independent of the details of the collapse. The implication is that any radiation emitted from the black hole is totally independent of the configuration of the null dust that forms the black hole. Therefore any deviations from a thermal spectrum in the outgoing radiation could not carry information about the matter that collapsed to form the black hole. This is an unexpected result and is in contradiction to more recent claims that radiation being emitted from a black hole would deviate from thermality and hence might contain details of the configuration of stress-energy that formed the black hole [12]. Of course the investigation in [12] has to do with a black hole formed from massive matter and not null dust. However, on the point that general black holes emit non-thermal radiation that transmits information about the collapsing matter, this paper has conclusions opposite to [12]. The arguments here are in line with claims made in [8] and [9].

We emphasize that we have not calculated the explicit, numerical response of the
particle detector. We have demonstrated that whatever the response is, it must only depend on the final value of the mass function (4.2) in the Vaidya spacetime and be independent of the form of the function (4.2) which physically means that the response is independent of the null dust configuration. Therefore we cannot verify the response to be thermal as in [34], [35].

It is interesting to consider if the results obtained here carry over to the case of a black hole formed by time-like (rather than null) dust (see figure 4.2). Specifically the second scenario described above where we consider a detector that is in its ground state when it is outside of the collapsing matter. In the case of a massive dust collapse, a detector following a time-like path could remain outside of the collapsing matter for all time. If the response of these detector were in line with the Vaidya case, i.e. their responses do not depend on the configuration of the collapsing matter, then this would again have the implication that the outgoing radiation is totally independent of the collapsing matter.

Appendix

We demonstrate that the response of a constant $r$ particle detector in Minkowski space is null in the state with Wightman function given by (4.66). We perform the integral in (4.66) by inserting the usual exponential cut-off ($\exp(-\epsilon \omega)$), the result is

$$
\int_0^\infty \frac{d\omega}{\omega} \exp\left(-i\omega(t - t') - \epsilon \omega\right) \sin(\omega r) \sin(\omega r') = \ln \left( \frac{-4r^2 + (v - v' + i\epsilon)^2}{(v - v' - i\epsilon)^2} \right)
$$

To evaluate the response of the detector we plug (4.71) into (4.66) and plug the resulting equation into (4.70). The resulting expression for the response rate of constant $r$ observers
The response of particle detectors in Vaidya spacetimes is

\[
\frac{1}{16\pi r^2} \int_{-\infty}^{v} dv' \Re \left( \exp (-i\omega(v - v')) \ln (-2r + v - v' - i\epsilon) \right.
\]
\[
+ \exp (-i\omega(v - v')) \ln (-2r - (v - v') - i\epsilon)
\]
\[
-2 \exp (-i\omega(v - v')) \ln (v - v' - i\epsilon) \right)
\]  

(4.72)

The above integral can be evaluated numerically and gives zero for various values of \( \omega \) and \( r \) and for small values of \( \epsilon \). It has thus been demonstrated that the vacuum state used in (4.43) is one in which there is a null response for constant \( r \) detectors.
Chapter 5

Choosing a vacuum state in a spherical spacetime with a conformal Killing vector

5.1 Introduction

In order to calculate the response of particle detectors following a given trajectory in a dynamic spacetime, one often picks spacetimes that have a large amount of symmetry. Symmetries of the spacetime make PDE’s such as the Klein-Gordon equation separable, allowing one to give a complete set of solutions to the PDE in terms of solutions to several ODE’s. The correlation function can be obtained from the complete set of solutions and hence the response of an Unruh-Dewitt detector [6] can be calculated.

The response of detectors in anisotropic Bianchi spacetimes can be calculated due to the existence of three translational Killing vector fields [22]. In the eternal Schwarschild spacetime, existence of spherical symmetry and a global time translation Killing vector field allow these calculations to be done everywhere [22], [7].
In [8] black holes without the global time translation Killing vector field were considered. It was demonstrated that all black holes formed from collapsing matter radiate with the characteristic black body spectrum. However, the trade off for making an argument for black hole solutions formed from a general spherically symmetric matter is that the spectrum of radiation can only be calculated on future null infinity.

The problem with requiring more symmetry is that it often makes the physical system less realistic. For example, the global time translation Killing vector in the Schwarzschild spacetime specifies a black hole that was not formed at a finite time in the past. Hence, by assuming this symmetry one would not be able to obtain a physically realistic response of particle detectors away from future null infinity for black holes formed from collapsing matter. In the cosmological realm, homogeneity is assumed to be true only at the largest scales and it is likely that at smaller scales the geometry of the universe is more like that of a Swiss Cheese model [36]. The swiss cheese model contains inhomogeneous regions such as a Lemaitre-Tolman-Bondi (LTB) spacetime.

We give a physically well motivated prescription for choosing a vacuum state that can be applied to any conformally coupled quantum field in a general spherically symmetric spacetime with an extra conformal Killing vector field. The reason for considering this type of spacetime is that it is more physical than considering a spherically symmetric solution with a time-like Killing vector (Schwarzschild) which is completely non-dynamical. The late time response of comoving detectors is given for the self-similar LTB spacetime given in [36].


Chapter 5. Choosing a vacuum state in a spherical spacetime with a CKV

5.2 A self-similar LTB spacetime

This section is an overview of the results given in [36] that are needed for this paper (for a full discussion one should consult [36]).

A model for an inhomogeneous, spherically symmetric universe filled with pressureless dust is given by the following LTB metric.

\[ ds^2 = dt^2 - \exp(2\lambda)dr^2 - r^2S^2d\Omega^2 \]

where \( \lambda = \lambda(t,r) \) and \( S = S(t,r) \). The stress energy tensor is

\[ T_{ab} = \rho u_a u_b \]

where \( \rho \) is the density and \( u_a \) is the 4-velocity of the dust.

To solve the Einstein equations a self-similarity is imposed by making \( \lambda = \lambda(t/r) \) and \( S = S(t/r) \). The self-similarity implies the existence of a vector field \( (\xi_a) \) satisfying the equation

\[ \xi_a\|b + \xi_b\|a = 2g_{ab} \]

where \( \|b \) denotes covariant differentiation. \( \xi_a \) is called a homothetic Killing vector field. Self-similarity and the Einstein field equations imply that \( S \) and \( \lambda \) satisfy the following equations

\[ S'^2 = 2E + 2/S \]

\[ \exp(\lambda) = (S - sS')/\sqrt{1+2E} \]

where \( s = t/r \) and \( S' = \frac{dS}{ds} \). \( E \) is an integration constant representing the energy of the shell in the infinite past. The case \( E = 0 \) will be used. The solution to equation (5.4) in the \( E = 0 \) case is

\[ s + a = \pm \frac{\sqrt{2}}{3} S^{3/2} \]
where $a$ is an integration constant representing the degree of inhomogeneity in the metric.

In the $t, r$ coordinates the line element appears to be singular at $r = 0$ [36]. Since the behaviour of the Klein-Gordon field will be needed near $r = 0$ a coordinate transformation will be made to the coordinate $\omega$ given by $r = 2\omega^3/2t_0^2$, where $t_0$ is some arbitrary time scale. In the $\omega$ coordinate the line element is

$$ds^2 = dt^2 - \left(\frac{t + \tilde{\gamma}\omega^3}{t_0}\right)^{4/3} \left[\left(\frac{t + 3\tilde{\gamma}\omega^3}{t + \tilde{\gamma}\omega^3}\right)^2 d\omega^2 + \omega^2 d\Omega^2\right]$$  

where $\tilde{\gamma} = 6a/(27t_0^2)$.

It is generally true that for a spacetime with a conformal vector field (CKV) there exists a conformally transformed spacetime in which that C.K.V. becomes a Killing vector field. To prove this consider a spacetime metric $g_{ab}$ with a conformal Killing vector field satisfying

$$L_k g_{ab} = \phi(x^c) g_{ab}$$  

where $L_k$ is the Lie derivative along the trajectory of the vector field $k^a$. Consider a parameter $\tau$ along the trajectory of the vector field $k^a$ so that $k^a$ can be represented by the derivative $\frac{d}{d\tau}$. The conformally transformed spacetime given by

$$\bar{g}_{ab} = \exp \left(- \int d\tau \phi(\tau)\right) g_{ab}$$

is a solution of the equation

$$L_k \bar{g}_{ab} = 0$$

confirming that $k^a$ is a Killing vector field in the spacetime with metric $\bar{g}_{ab}$.

A coordinate can be chosen along the trajectories of the Killing vector field $k^a$ (say $\tau$) so that the metric is independent of this coordinate. In a spherically symmetric, 4 dimensional spacetime this guarantees that the Klein-Gordon equation is separable and
hence one can solve for a complete set of solutions.

The spacetime described above has a conformal Killing vector field given by

\[ \xi^a \frac{\partial}{\partial x^a} = t \frac{\partial}{\partial t} + \frac{\omega}{3} \frac{\partial}{\partial \omega} \]  

(5.11)

The coordinate \( y \) will be chosen so that \( \frac{d}{dy} \) is the conformal Killing vector field. The relationship to the \( t \) and \( \omega \) coordinates can be obtained by equating (5.11) and the relation

\[ \frac{d}{dy} = \frac{\partial \omega}{\partial y} \frac{\partial}{\partial \omega} + \frac{\partial t}{\partial y} \frac{\partial}{\partial t} \]  

(5.12)

We obtain

\[ \omega = A \exp(y/3) \quad t = B \exp(y) \]  

(5.13)

where \( A \) and \( B \) are integration constants that are independent of \( y \). We choose \( A = t_0 x \) and \( B = t_0 \), where \( x \) will be another new coordinate. The coordinate \( x \) is constant along curves generated by the self-similarity transformation.

The line element in the \( y, x \) coordinates is given by

\[ ds^2 = t_0^2 \exp(2y) \left[ dy^2 - \frac{b^2}{a^{2/3}} (dx + \frac{x}{3} dy)^2 - x^2 a^{4/3} d\Omega^2 \right] \]  

(5.14)

where \( a \equiv 1 + \gamma x^3 \) and \( b \equiv 1 + 3\gamma x^3 \) with \( \gamma = 6a/27 \). The \( \exp(2y) \) factor is the only dependence of the metric on \( y \). Therefore, \( d/dy \) is a conformal Killing vector in the above metric \( g_{ab} \) and a Killing vector in the metric \( \bar{g}_{ab} \) given by

\[ \bar{g}_{ab} = \exp(-2y) g_{ab} \]  

(5.15)

The conformal Killing vector \( d/dy \) becomes null \((g(d/dy, d/dy) = 0)\) when the coefficient of the \( dy^2 \) term in (5.14) vanishes. The surface where the C.K.V. \( d/dy \) becomes null defines a conformal Killing horizon [37]. So there is a conformal Killing horizon at the solutions of the following equation

\[ 1 - \frac{b^2}{a^{2/3}} \frac{x^2}{9} = 0 \]  

(5.16)
In order to count the number of positive roots in $x$ of the above equation we first factor the left hand side using the difference of squares.

$$
\left(1 - \frac{b}{a^{1/3}} \frac{x}{3}\right) \left(1 + \frac{b}{a^{1/3}} \frac{x}{3}\right) = 0 \quad (5.17)
$$

The factor on the right is always positive for $\gamma > 0$ and therefore has no roots for $x > 0$. Making the substitution $z = x^3$ into the factor on the left and simplifying gives the following equation, which has the same positive roots as (5.16):

$$
27\gamma^3 z^4 + 27\gamma^2 z^3 + 9\gamma z^2 + (1 - 27\gamma)z - 27 = 0 \quad (5.18)
$$

There is only one sign change in the coefficients of the polynomial on the left hand side of (5.18). Therefore there can be at most one positive root. Using $z = x^3$ we see that there is only one positive real root for $x > 0$ and hence only one conformal Killing horizon in the expanding, inhomogeneous universe ($t > 0$).

We now introduce the mathematical tools to study the evolution of a bundle of null geodesic curves in the above spacetime. The purpose is to find other types of horizons. The scalar expansion of a congruence of geodesic curves is defined to be

$$
\theta = u^a_{||a} \quad (5.19)
$$

where $u^a$ is a tangent vector to the congruence of curves given by $d/d\lambda$ where $\lambda$ is an affine parameter along the curves and $||a$ is the covariant derivative with respect to the coordinate $x^a$ [16]. This quantity defines how the cross-sectional area of a small bundle of curves changes. For example, a bundle of curves representing radially outgoing light rays in Minkowski spacetime would have positive expansion because the curves move away from each other as they move to increasing radius, while the expansion for radially ingoing light rays in Minkowski spacetime would be negative.
A trapped surface of a spherically symmetric spacetime is defined as the surface where the expansion is negative for both ingoing and outgoing sets of null geodesics [32]. The apparent horizon is the boundary of all the trapped surfaces, meaning that the expansion is zero for one set of radial null geodesics and negative for the other. For example, the event horizon of the Schwarzschild spacetime is an apparent horizon [32].

It can be shown that the conformal Killing horizon (CKH) is also an apparent horizon of the spacetime (5.7) by calculating the expansion of the congruence of the two independent, future-directed, radially moving, null geodesics. The ingoing and outgoing tangent vectors, denoted by $k^a_-$ and $k^a_+$ respectfully, are calculated by solving the equations

\begin{align}
  k^b k^a_{|b} &= 0 \quad (5.20) \\
  k^a k_a &= 0 \quad (5.21)
\end{align}

For convenience we will work in the conformally transformed spacetime $\bar{g}_{ab}$ given by (5.15). We can work in the conformally transformed spacetime because null geodesics remain null geodesics under conformal transformations. The tangent vectors in affine parameterization in the original spacetime can be obtained by a conformal transformation even though working in the barred spacetime $(\bar{g}_{ab})$ changes the geodesics from affine parameterization to non-affine parameterization. Taking the null geodesics to be defined by an affine parameter $\lambda$ in $\bar{g}_{ab}$ and working in the coordinates of (5.7) we obtain the following expression for (5.21)

\begin{equation}
  \left( \dot{y} - \frac{b}{a^{1/3}} (\dot{x} + \frac{x}{3} \dot{y}) \right) \left( \dot{y} + \frac{b}{a^{1/3}} (\dot{x} + \frac{x}{3} \dot{y}) \right) = 0 \quad (5.22)
\end{equation}

with

\begin{equation}
  k^a = (\dot{y}, \dot{x}, 0, 0) \quad (5.23)
\end{equation}

where $\dot{y} \equiv d\dot{y}/d\lambda$. Setting each of the factors to zero gives us an equation representing one of the two independent null geodesics.

\begin{equation}
  \dot{y} \pm \frac{b}{a^{1/3}} (\dot{x} + \frac{x}{3} \dot{y}) = 0 \quad (5.24)
\end{equation}
Chapter 5. Choosing a vacuum state in a spherical spacetime with a CKV

The geodesic equations for the null vector (5.20) reduce to the following expression

\[ \dot{y} - \frac{b^2 x}{3a^{2/3}}(\dot{x} + \frac{x}{3} \dot{y}) = C \]  

(5.25)

where \( C \) is an integration constant. Solving equations (5.24) and (5.25) we obtain an expression for \( k^\alpha_\pm \) in the conformally transformed spacetime

\[ k^\alpha_\pm = C \left( (1 + xb/3a^{1/3})^{-1}, \mp \frac{a^{1/3}}{b}, 0, 0 \right) \]  

(5.26)

It can be verified that in the original spacetime the tangent vectors given by the derivative with respect to an affine parameter are given by

\[ k^\alpha = Ce^{-2y} \left( (1 + xb/3a^{1/3})^{-1}, \mp \frac{a^{1/3}}{b}, 0, 0 \right) \]  

(5.27)

It must also be determined where in the spacetime that the null vectors \( k^\alpha \) are future pointing: the reason is because if we find that the expansion associated with \( k^\alpha \) is negative for example but it is past pointing then as time runs forward the expansion is actually positive. To accomplish this the \( k^\alpha \) vectors are expressed in the \((t, \omega)\) coordinates of (5.7). This is done because both the vectors \( d/dy \) and \( d/dx \) coordinates are time-like in certain regions while \( d/d\omega \) is space-like everywhere and \( d/dt \) is time-like everywhere.

The \( d/dt \) component of the null vectors are

\[ k^0_\pm = Ct \frac{e^{2y}}{1 \pm xb/3a^{1/3}} \]  

(5.28)

So while the component \( k^0_+ \) is always positive for \( x > 0 \), the component \( k^0_- \) becomes negative when \( 1 - xb/3a^{1/3} < 0 \). Hence the vector \( k^a_+ \) is always future pointing but \( k^a_- \) becomes past pointing when \( 1 - xb/3a^{1/3} = 0 \). Note that this is also the value of \( x \) where the CKH is found.

The expansion of each set of null geodesics is found to be given by the expressions

\[ \theta_\pm = \frac{2e^{2y}(2x \mp 3a^{1/3})}{(3a^{1/3} + xb)x a^{2/3}} \]  

(5.29)
It can be verified that for $\gamma > 8/27$ and $x > 0$, $\theta_+ < 0$ and $\theta_- > 0$ everywhere. However $k^a$ becomes past directed at the CKH so the true expansion associated with these null geodesic congruences becomes negative. Hence, the CKH is also an apparent horizon for $\gamma > 8/27$. This will be of importance later when we use the existence of the CKH to give a low frequency cut-off for the two point function.

### 5.3 Conformally coupled Klein-Gordon field

A conformally coupled Klein-Gordon (KG) field in 4 dimensional spacetime satisfies the equation

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b \psi) + \frac{1}{6} R \psi = 0$$

(5.30)

where $R$ is the Ricci scalar given by

$$R = -\frac{4}{3} \frac{27 \gamma^2 x^6 + 18 \gamma x^3 + 1}{ab}$$

(5.31)

The conformally coupled KG field has the property that if $\psi$ satisfies (5.30) in the metric $g_{ab}$ then the field $\tilde{\psi} = \phi^{-1} \psi$ will satisfy the KG equation in the conformally transformed metric $\tilde{g}_{ab} = \phi^2 g_{ab}$. We can solve for the conformal fields in the metric $\tilde{g}_{ab}$ given by (5.15) because the KG equation is separable there and then obtain the fields in the metric $g_{ab}$ by a conformal transformation.

In the metric where $d/dy$ is a Killing vector, the following ansatz is used.

$$\tilde{\psi}_{\nu \ell m} = \exp(-i\nu y) f_{\nu \ell}(x) Y_{\ell m}(\theta, \varphi)$$

(5.32)

The vacuum of the ansatz (5.32) is invariant under translations along the Killing vector $d/dy$ (because the modes are multiplied by a phase under shifts in $y$). Substituting (5.32) in the KG equation the following differential equation for $f_{\nu \ell}$ follows.
Chapter 5. Choosing a vacuum state in a spherical spacetime with a CKV

\[
\begin{align*}
\left( \frac{x^2}{9} - \frac{a^{2/3}}{b} \right) abx^2 \frac{d^2 f_{vl}}{dx^2} + \left( \frac{2i\nu}{3} abx^3 + \frac{d}{dx} \left( \left( \frac{x^2}{9} - \frac{a^{2/3}}{b^2} \right) abx^2 \right) \right) \frac{df_{vl}}{dx} \\
+ \left( -\nu^2 abx^2 + \frac{i\nu}{3} \frac{d}{dx} (abx^3) + l(l+1) \frac{b}{a^{1/3}} - \frac{2}{3} x^2 \left( 27\gamma^2 x^6 + 18\gamma x^3 + 1 \right) \right) f_{vl} = 0
\end{align*}
\]

(5.33)

The singularities at \( x = 0 \) are regular; the powers of \( x \) appearing the equation 5.33 have a finite negative power. Therefore (5.33) can be solved perturbatively by expressing the coefficients as a series expanded around \( x = 0 \). The two independent solutions take the form

\[
f_{vl} = x^l \sum_{n=0}^{\infty} a_n x^n \quad \text{(5.34)}
\]

\[
f_{vl} = x^{-l-1} \sum_{n=0}^{\infty} b_n x^n \quad \text{(5.35)}
\]

where \( l > 0 \). The solution (5.34) is regular at \( x = 0 \) while (5.35) is singular there. The line element (5.14) at \( x = 0 \) reveals that the metric is regular there. The \( d\Omega^2 \) term in (5.14) is multiplied by \( x^2 \), so it does not contribute at \( x = 0 \). The same behaviour occurs in flat spacetime in spherical coordinates at \( r = 0 \) even though it is a regular point, therefore we expect the spacetime to be completely well behaved at \( x = 0 \). Since the metric is well behaved at \( x = 0 \), it is expected that the modes will be also. Hence, only equation (5.34) will be used for the \( f_{vl} \) part of the modes.

Requiring a vacuum that is invariant along the Killing trajectory reduced the arbitrariness down to two complex constants (\( b_0 \) and \( a_0 \) in (5.34) and (5.35)). Imposing regularity at \( x = 0 \) implied \( b_0 = 0 \), reducing the choice of vacuum down to one complex constant (\( a_0 \)). Finally, the canonical commutation relations of the KG field operator imply the modes must satisfy the following normalization.

\[
(\psi_{\nu'}\psi_{\nu m'}, \psi_{\nu m}) = \int dx^3 \sqrt{-g} g^{0a} (\psi_{\nu'}^{*}\psi_{\nu m'}) \partial_a \psi_{\nu m} - \psi_{\nu m} \partial_a \psi_{\nu'}^{*} \psi_{\nu m'})
\]

\[
= i \delta(\nu - \nu') \delta_{\nu'} \delta_{mm'}
\]

(5.36)
where the integration is done over the $x^1$, $x^2$ and $x^3$ direction. The normalization (5.36) fixes one of the two remaining real constants leaving only a phase factor unfixed. The phase factor disappears when calculating correlation functions. Hence imposing the conditions of regularity at $x = 0$, invariance under $y$ translations and the normalization (5.36) uniquely specifies the vacuum state.

Using the ansatz (5.32) the normalization condition (5.36) reduces to the following condition on $a_0$.

$$|a_0|^2 = i \left( 2\pi ab x^2 \left[ \frac{2i\omega x}{3} |h_{\omega l}|^2 + \left( \frac{x^2}{g} - \frac{a^{2/3}}{b^2} \right) \left( h_{\omega l}^* h'_{\omega l} - h_{\omega l} h'_{\omega l}^* \right) \right] \right)^{-1} \quad (5.37)$$

where

$$h_{\omega l} = x^l \left( 1 + \sum_1^\infty a_n x^n \right) \quad (5.38)$$

The integration (5.36) is done over a constant $x$ surface so the condition (5.37) is evaluated at some constant $x = u$.

The correlation function in the vacuum state defined above (the vacuum state which is regular at $r = 0$ and has a conformal symmetry along $d/dy$) can be calculated by using the usual sum over all modes formula. The general formula can be found in [22].

$$G(x, x') = \sum_{l,m} \int_0^\infty d\omega \exp(-i\omega(y - y') - (y + y')) f_{\omega l}(x) f_{\omega l}^*(x') Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \quad (5.39)$$

Only the correlation function evaluated along the points $\theta = \theta'$ and $\varphi = \varphi'$ will be of interest. The following formula can be used to simplify the sum over modes:

$$\sum_{m=-l}^{m=l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta, \varphi) = (2l + 1)/4\pi \quad (5.40)$$

giving

$$G(x, x') = \sum_{l=0}^\infty \int_0^\infty d\omega \exp(-i\omega(y - y') - (y + y')) f_{\omega l}(x) f_{\omega l}^*(x') \frac{2l + 1}{4\pi} \quad (5.41)$$
Evaluating $|a_0|^2$ for at the lowest order in $x$, the following expression is obtained

$$|a_0|^2 = \frac{3}{4\pi(1 + gu^3)(1 + 3gu^3)u^3\omega} \tag{5.42}$$

where $u$ is the constant $x$ surface that the modes are normalized on. The Taylor expansion for $f_{\omega l}$ is given by

$$f_{\omega l} = a_0 x^l (1 + O(x^2)) \tag{5.43}$$

It can be seen that the correlation function will be divergent at $\omega = 0$ due to the normalization on the modes. To solve this an IR cutoff must be imposed. A value for the cutoff is given by the constant $x$ surface that is the conformal Killing horizon. This surface is also an apparent horizon, so it would be natural to impose that no wave mode have a wavelength that is larger than the apparent horizon. The single solution to (5.16) is denoted by $x = \beta$. The function $f_{\omega l}(x)$ has approximately the wavelength $2\pi/\omega$. Therefore only modes with $\omega > 2\pi/\beta$ will be included in the sum to calculate the correlation function.

$$G(x, x') = \sum_{l=0}^{\infty} \int_{2\pi/\beta}^{\infty} d\omega \exp(-i\omega(y - y') - (y + y')) f_{\omega l}(x)f_{\omega l}^*(x') \frac{2l + 1}{4\pi} \tag{5.44}$$

Performing the integration and summation gives

$$G(x, x') = N(u, \gamma) \frac{Ei\left(1, \frac{2\pi}{\beta} (\epsilon + i(y - y'))\right) \left(\frac{xx'}{u^2}\right)^{1/4} P\left(\frac{1}{2}, \frac{1}{2}, 1 + \frac{xx' + u^2}{u^2}\right)}{(1 - \frac{xx'}{u^2})^{3/2} \exp(y + y')} \tag{5.45}$$

where $N(u, \gamma)$ is only dependent on the normalization surface $x = u$ and the inhomogeneity parameter $\gamma$. $Ei(1, x)$ is the exponential integral special function and $P(a, b, x)$ is the associated Legendre polynomial of the first kind. They are given by the following.

$$Ei(1, x) = \int_1^{\infty} dy \exp(-yx)y^{-1} \tag{5.46}$$

$$P(1/2, 1/2, x) = \frac{(x + 1)^{b/2}F_1(-a, a + 1, 1 - b, 1/2 - 1/2x)}{(x - 1)^{b/2}\Gamma(1 - b)} \tag{5.47}$$

The correlation function is being considered for small values of $x$ so it will be expanded around $x = 0$ in a series. Keeping only the leading term the following is obtained

$$G(x, x') = N(u, \gamma) \frac{Ei\left(1, \frac{2\pi}{\beta} (\epsilon + i(y - y'))\right)}{\pi^{1/2} \exp(y + y')} \tag{5.48}$$
Chapter 5. Choosing a vacuum state in a spherical spacetime with a CKV75

The above function is the correlation function for an observer following the trajectory \( x = 0 \). It is also the leading order approximation to the correlation function for an observer at small \( x \).

One might expect that the modes that define the vacuum state should oscillate infinitely rapidly as the apparent horizon is approached. For example, the modes defining the Unruh vacuum state oscillate infinitely rapidly near the future horizon of the eternal Schwarzschild spacetime. This is not the case for the other two standard vacuums on the Schwarzschild spacetime. However the Unruh vacuum is thought to be the physically correct vacuum for an object collapsing to form a black hole. In other words, it is the correct vacuum for a star that collapses to form a future horizon.

Solving (5.33) numerically for regular initial conditions at \( x = 0 \), it is found that the modes \( f_m(l)(x) \) seem to oscillate infinitely rapidly at the conformal Killing horizon. Hence the modes (5.32) that specify the vacuum state have the physically desired property that they oscillate infinitely rapidly at the conformal Killing horizon.

5.4 Response of Particle Detectors

The response rate of an Unruh-Dewitt detector will be calculated for observers co-moving with the dust of the above LTB metric. The expression for the response rate is given by

\[
\hat{F}_\tau(\omega) = \Re \left( \int_0^{\tau - \tau_0} ds \exp(-i\omega s) G(x(\tau), x(\tau - s)) \right)
\]

(5.49)

\( s \) is the amount of proper time passed since the initial time \( \tau_0 \) at which the detector is in its ground state. \( G(x, x') \) is the correlation function and \( \tau \) is the time at which the response rate of the detector is evaluated.
Chapter 5. Choosing a vacuum state in a spherical spacetime with a CKV

The $r$ and $t$ coordinates of the metric (5.1) will be used to parameterize a co-moving observer. $r$ is constant for a radially in falling dust particle and $t$ is the proper time of this dust particle. The coordinate transformation $(x, y) \rightarrow (t, r)$ is given by

$$ (x, y) = \left( \left( \frac{9r}{2t} \right)^{1/3}, \ln(t) \right) \quad (5.50) $$

The trajectory of a comoving dust particle can therefore be parameterized by the $y$ coordinate. Making the substitution $t = \exp(y)$ the integral becomes

$$ \hat{F}_y(\omega) = \Re \left( \frac{1}{\exp(y)} \int_{y_0}^y dy' \exp \left( -i\omega \left( e^{y'} - e^{y} \right) \right) Ei \left( 1, \frac{2\pi}{\beta} (\epsilon + i(y - y')) \right) \right) \quad (5.51) $$

where $y = \ln(\tau)$ is the time at which the detector response is evaluated and $y_0 = \ln(\tau_0)$ is the time at which the detector is taken to be in its ground state. The correlation function for the observer at $x = 0$ given by (5.48) was used.

It can be demonstrated that the response rate given by equation (5.51) goes to zero as $y \to 0$ for a fixed initial time $y_0$. This is true simply due to the fact that the $\exp(-y)$ factor dominates over integral part of the equation. A detailed proof of this result is given in section (5.6).

We have therefore demonstrated that the response of a comoving particle detector near $x = 0$ has a null response at late times in the vacuum we have described previously. The null response at late times confirms that the vacuum state we have chosen is physically reasonable because the spacetime (5.7) approaches Minkowski spacetime at late times. Hence, any geodesic particle detector should register zero particles.

A similar argument can demonstrate that any comoving detector (constant $r, \theta, \phi$ trajectory) will not click. This can be seen by using the relation $x = (9r/2t)^{1/3}$. As $t \to \infty$, $x \to 0$. The integral (5.49) can be divided into two parts, one for late times or $0 < x < x_0$, and one for early times $x > x_0 > 0$. The late time part of the response rate
is given by (5.51) and hence becomes zero at late times as before. The early time part of the response rate will also be exponentially damped by the \( \exp(-y) \) factor and hence tend to zero as \( y \to 0 \). Therefore, the response rate for a comoving (constant \( r \)) particle detector will also tend to zero at late times.

### 5.5 Discussion

The selection of a vacuum state is a non-trivial process for a general background metric. In the eternal Schwarzschild spacetime, there are three common choices for the vacuum state. The Hawking-Hartle vacuum [38] is a state that is regular on the past and future horizons. The Boulware [39] vacuum is symmetric with respect to the time translation Killing vector field \( \partial/\partial t \). Finally, the Unruh vacuum state [6] is symmetric with respect to the Killing vector on the past horizon and symmetric with respect to the Killing vector on past null infinity.

We have given a physically motivated prescription for choosing a vacuum state of a conformally coupled quantum field on the spacetime given in [36]. The response of a particle detector was given in this state so that the response along any trajectory could be given by integrating numerically. The late time response of comoving particle detectors was found to be zero in this vacuum.

The above prescription for the vacuum state can be used for any conformally coupled field on a spherically symmetric background with an extra conformal Killing vector field. In particular, there are solutions to the Einstein Field Equations representing spherically symmetric matter collapsing to form a black hole [40]. The solution in [40] also has a conformal Killing vector field. One could therefore study the detailed spectrum of Hawk-
A solution of Einstein’s equations representing matter collapsing to some central point \( (r = 0) \) should be regular along the line of points specified by \( r = 0 \) before some specified time when the singularity forms. The metric is regular at these points; hence we expect the vacuum state to also be regular there which gives us one of the conditions we have used to specify the above vacuum state. Further, we expect a conformally coupled field to have the same conformal symmetry as the background spacetime, which gives us the second condition on our vacuum state. These two conditions uniquely specify the vacuum state and for the above reasons we believe that it is the physically correct vacuum for spherically symmetric collapse solutions with a conformal Killing vector.

We note that the condition of regularity at \( r = 0 \) cannot be used on the eternal Schwarzschild spacetime because there is always a singularity in the metric at \( r = 0 \) and therefore there is no reason to expect that other scalars (such as a Klein-Gordon field) would not also blow up there. By ‘eternal Schwarzschild spacetime’ we mean the spherically symmetric spacetime that everywhere has an a time translational Killing vector. This spacetime cannot represent matter collapsing to form a black hole. The conditions we have used to define a vacuum are more physically realistic in the sense that they apply to a spacetime representing a black hole that was formed at some finite time in the past.

\[ H(y, y') = \exp \left( -i \omega \left( e^y - e^{y'} \right) \right) Ei(1, \frac{2\pi}{\beta} (\epsilon + i(y - y'))) \]  

\[ (5.52) \]
So the response rate (5.51) becomes

$$\exp(-y) \Re \left( \int_{y_0}^{y} dy' H(y, y') \right) \tag{5.53}$$

The following set of inequalities hold:

$$0 \leq \Re \left( \int_{y_0}^{y} dy' H(y, y') \right) \leq \left| \int_{y_0}^{y} dy' H(y, y') \right| \leq \int_{y_0}^{y} dy' \left| H(y, y') \right| \tag{5.54}$$

where we have assumed that the response rate is non-negative. The definition of the exponential integral function $Ei(1, z)$ for complex $z$ with $\Re(z) > 0$ is given by:

$$Ei(1, z) = \int_{1}^{\infty} \frac{\exp(-az)}{a} da \tag{5.55}$$

Using this definition we further have the following set of (in)equalities holding:

$$\int_{y_0}^{y} dy' \left| H(y, y') \right| \leq \int_{y_0}^{y} dy' \int_{1}^{\infty} da \left| \frac{\exp\left(-a \left( \frac{2\pi \epsilon}{\beta} (\epsilon + i(y - y')) \right) \right)}{a} \right| \tag{5.56}$$

$$= \int_{y_0}^{y} dy' \int_{1}^{\infty} da \frac{\exp\left(-\frac{2\pi \epsilon\alpha}{\beta} \right)}{a}$$

$$= (y - y_0) Ei(1, 2\pi \epsilon / \beta)$$

Combining all these results we have that the response rate obeys the following inequality

$$0 \leq \exp(-y) \Re \left( \int_{y_0}^{y} dy' H(y, y') \right) \leq \exp(-y)(y - y_0) Ei(1, 2\pi \epsilon / \beta) \tag{5.57}$$

which clearly demonstrates that the response rate goes to zero at late times ($y \to \infty$).
Chapter 6

Hawking radiation, Unruh radiation and the equivalence principle

Hawking radiation is commonly described as the radiation detected by a Killing observer in the Schwarzschild spacetime. It is the radiation seen by a constant radial observer outside of a black hole. Unruh radiation is commonly described as the radiation detected by a uniformly accelerating observer in Minkowski spacetime.

There are many forms of the equivalence principle. Some forms of the equivalence principle have weaker implications than others. For instance, a weak statement may be

Mass (measured with a balance) and weight (measured with a scale) are locally in identical ratio for all bodies (Newton’s Principia)

Einstein’s version of the weak form is

There is no way of distinguishing between the effects on an observer of a uniform gravitational field and of constant acceleration.

An example of a stronger statement of the principle is
It equates a gravitational field with a uniformly accelerating reference frame locally [14].

The crux of the second statement is that things are only equivalent ‘locally’; meaning ignoring effects that depend on the second and higher derivatives of the metric at a point, equivalently assuming a uniform gravitational field over the extent of the experiment.

Using the stronger statement it is often claimed in the literature that Hawking and Unruh radiation are linked by the equivalence principle ([14], [13]). In the Schwarzschild spacetime Killing observers have constant acceleration. By the strong statement of the equivalence principle all local experiments performed by a Killing observer in the Schwarzschild spacetime must give the same results as if the experiment was done in Minkowski spacetime by an observer with the same acceleration. It is then incorrectly claimed that the response of a particle detector in the Schwarzschild and Minkowski spacetimes must be the same because it is a local experiment and hence Hawking and Unruh radiation are equivalent.

The particle detector does only rely on information along some time-like trajectory. This can be seen from the integral representing the response.

$$ F(\omega) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \exp(-i\omega(\tau - \tau')) G^+(x(\tau), x(\tau')) $$  \hspace{1cm} (6.1)

where $\omega$ is the energy of the detected particle, $G^+$ is the correlation function and $x(\tau)$ is the trajectory the detector follows. So the detector does only depend on the information along a time-like trajectory and by using Fermi normal coordinates the metric can be written as Minkowski spacetime plus second order coordinate differences along the entire trajectory.

$$ g_{ab} = \eta_{ab} - \frac{1}{3} R_{abcd} y^{c} y^{d} + O(s^3) $$  \hspace{1cm} (6.2)

where $\eta_{ab} = diag(1, -1, -1, -1)$, $R_{abcd}$ is the Riemann tensor, $y^{c}$ are coordinates that are
zero along the trajectory \( \gamma \) around which the above expansion is evaluated.

The problem is that the particle detector is coupled to a quantum field (typically a Klein-Gordon field) and the state of the quantum field is not a local entity. Indeed a quantum state for a field is chosen by specifying the value of the modes and the normal derivatives of the modes evaluated on a Cauchy surface. For instance in the collapse spacetime given by the conformal diagram in figure 4.2 a quantum state for a massless field can be chosen by specifying the modes and their normal derivatives on past null infinity (\( J^- \)). \( J^- \) is a Cauchy surface for massless fields.

The quantum state in which there is Hawking radiation is the one where the modes are chosen on \( J^- \) to be a certain form so that there is no radiation incoming onto the black hole. The quantum state in which Unruh radiation occurs is the usual Minkowski vacuum state which is specified by choosing the modes and their normal derivatives on a space-like hypersurface given by a constant value of \( t \) in the usual Cartesian coordinates. In both cases we have to give information over a non-local neighbourhood (the null and space-like hypersurfaces).

A notable instance of incorrectly using the equivalence principle to link Unruh and Hawking radiation is in [14] where the authors claim to have found a ‘detailed violation of the equivalence principle’. The authors compare the response of a uniformly accelerated detector with proper acceleration \( a \) in Minkowski spacetime in the usual Poincare invariant vacuum to that of a detector following a Killing trajectory in the Schwarzschild spacetime in the Unruh vacuum state. The response of the detector in Minkowski spacetime is that of a detector in a thermal state with temperature

\[
kT = \frac{a}{2\pi} \tag{6.3}
\]

where \( k \) is the Boltzmann constant and \( a \) is the constant proper acceleration of the
Chapter 6. Hawking radiation, Unruh radiation and the equivalence principle

detector. The response of the detector in the Schwarzschild spacetime following a Killing trajectory given by \( r = R \) is thermal with a temperature of

\[
kT = \frac{\kappa}{2\pi} = \frac{1}{8\pi M \sqrt{1 - 2M/R}} \tag{6.4}
\]

where \( \kappa \) is the surface gravity at the horizon and the Schwarzschild metric takes the form

\[
ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - 2M/r} - r^2 d\Omega^2 \tag{6.5}
\]

The constant proper acceleration of the Killing trajectory \( r = R \) in Schwarzschild spacetime is given by

\[
a = \frac{M}{R^2 \sqrt{1 - 2M/R}} \tag{6.6}
\]

Since the temperature experienced by a detector following the trajectory \( r = R \) in Schwarzschild is given by (6.4) and not by (6.3) with \( a \) given by (6.6) then an observer can distinguish a uniformly accelerating frame in Minkowski spacetime from a frame following a Killing trajectory with equal acceleration in Schwarzschild spacetime by measuring the difference in temperature of the radiation. The authors incorrectly conclude that this is a violation of the equivalence principle. As explained above, it is not a violation because there is non-local information present in the quantum state.

It is noted in [14] that the acceleration of a succession of Killing trajectories, given as a function of \( R \) in the Schwarzschild spacetime approaches

\[
a = \frac{1}{4M \sqrt{1 - 2M/R}} \tag{6.7}
\]

as \( R \to 2M \) where \( R \) is the radial coordinate of the Killing observer. This acceleration is equal to that of the surface gravity at the horizon given in (6.4). Hence in the Unruh vacuum the Killing observer sees a bath of thermal radiation with temperature approaching (6.3) as the limit \( R \to 2M \) is taken. So near the horizon a Killing observer would not be able to tell they are not a uniformly accelerating observer in Minkowski spacetime simply
by measuring the temperature of the radiation. This led the authors of [14] to conclude that the equivalence principle is ‘restored’ at the horizon.

The explanation for a Killing detector near the horizon seeing thermal radiation with the Unruh temperature (6.3) is simple. From figure (6.1) one can see that the future and past horizons taken together form a Cauchy surface for massless fields. That is, specifying the fields and their normal derivatives on these surfaces specifies the fields in the entire spacetime. One can therefore think of the Unruh vacuum as being specified on the future and past horizon so as to give the same response rate near these surfaces as a uniformly accelerated detector in Minkowski spacetime. So the equivalence principle is not restored near the horizon because it was never violated and the responses of Killing detectors near the horizon are that of a thermal bath at the Unruh temperature because they were chosen to be so.
Another example of the above misconceptions is in the entropic gravity program [13]. In [13] the author attempts to describe gravity as an entropic force. An entropic force is succinctly described

An entropic force is an effective macroscopic force that originates in a system with many degrees of freedom by the statistical tendency to increase its entropy.

Therefore the force of gravity on a mass is due to the statistical tendency of the underlying degrees of freedom to increase their entropy. This is demonstrated by invoking the holographic principle which states that the entropy of any three dimensional volume can be encoded on some two dimensional surface surrounding the volume.

The entropic force on a system is proportional to the temperature of that system. For example (see [13] for details), a polymer of length $x$ that is emersed in a bath of temperature $T$ experiences a force

$$F = T \frac{\partial S(E,x)}{\partial x} \quad (6.8)$$

where $S$ is the entropy of the system and $E$ is the total energy of the heat bath. Hence there must be a non-zero temperature for there to be an entropic force.

The specific scenario of interest in [13] is that of an arbitrary mass density given by $\rho(\vec{r})$. A holographic screen is considered which lies along an equipotential surface of the Newtonian potential of this mass distribution. The author then states then the temperature associated with this screen is given by

$$kT = \frac{1}{2\pi} \frac{h \nabla \Phi}{kc} \quad (6.9)$$

where $\Phi$ is a potential function that gives the acceleration at the screen

$$a = -\nabla \Phi \quad (6.10)$$
and the derivation of $\Phi$ is taken in the direction of the outward pointing normal to the holographic screen.

It is in equation (6.9) that the errors described above make their appearance. Equation (6.9) describes the temperature of some holographic screen as the one that a uniformly accelerating observer would see in Minkowski spacetime. So here it is implied that a Killing observer hovering above some spherically symmetric mass distribution would see a bath of thermal radiation with temperature equal to the acceleration of that observer. If this mass distribution were a black hole this would be in contradiction to the Hawking temperature of that black hole. As correctly noted in [14], the temperature experienced by a Killing observer outside of a black hole would not be that given by (6.3).
Chapter 7

Conclusion

An investigation of possible solutions to the information paradox for certain background spacetimes has been initiated. Also some investigation into the nature of the Hawking and Unruh effects was done.

In chapter 3 the correct methodology was established in order to investigate the particle content of a given spacetime; black hole spacetimes being one example. It was demonstrated that the Unruh-Dewitt detector is the physically correct (coordinate invariant) method to determine the particle content as seen by some observer. It was also shown that the method of determining the particle content by counting the occupation numbers of the energy levels of the harmonic oscillator gives incorrect particle content even in the most simple case. The two methods were shown to differ in deSitter spacetime. It was shown that there is an extra ambiguity when dealing with calculations of field systems which is due to the choice of coordinates on phase space. This is the root of the discrepancy between the response of the Unruh-Dewitt detector and the energy levels of the harmonic oscillator.

In chapter 4 a Klein-Gordon field was quantized on a set of Vaidya background space-
times representing a spherically symmetric, finitely thick shell of null dust converging to a central point and forming a black hole. It was demonstrated that the form of the Hamiltonian for the Klein-Gordon system means that the response of detectors (and more generally the correlation functions) are independent of the specific distribution of the null dust and depend only on the final mass of the black hole. Hence, at least in the semi-classical realm it would be impossible to determine any information that went into forming the black hole.

In chapter 5 it was demonstrated that the conformally coupled Klein-Gordon equation is separable on spherically symmetric spacetimes with a conformal Killing vector that is orthogonal to the spherical surfaces. The response of a particle detector was calculated in a LTB spacetime that represents an inhomogeneous cosmology, more specifically it is dust that expands radially outwards after an initial big bang singularity. The response of a particle detector was shown to tend to zero for late times for comoving observers.

Chapter 6 was a slight deviation from the others. The meaning of the equivalence principle was discussed along with the Unruh and Hawking effects. It was demonstrated that some claims in the literature ([13], [14]) were incorrect due to serious misconceptions about the relation between the Unruh and Hawking effects and the equivalence principle. Specifically, a uniformly accelerating observer in Minkowski spacetime and a Killing observer in a black hole spacetime of equal acceleration will in general see different temperatures for the baths of thermal radiation they experience.

Overall, it was shown that at least semi-classically information is lost for Vaidya black holes. Vaidya spacetimes have a peculiar structure however. Specifically the fact that there is null (as opposed to massive) dust prevents observers with bounded acceleration from avoiding the null dust that makes up the black hole. It would be more physically
realistic and an altogether different scenario if the black hole were formed from time-like matter. The program initiated in chapter 5 could be extended to a spherically symmetric collapse spacetime with CKV. There do exist collapse spacetimes with time-like matter and a CKV ([40]). However, there are other problems with this spacetime, such as a singularity at the center of spherical shells that prevent the vacuum prescription of chapter 5 from being used.
Bibliography


