A QUANTUM LIGHT SOURCE FOR LIGHT-MATTER INTERACTION

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Physics
University of Toronto

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Abstract

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2013

I present in this thesis the design, implementation and measurement results of a narrowband quantum light source based on cavity-enhanced Parametric Down-Conversion (PDC). Spontaneous Parametric Down-Conversion (SPDC) is the workhorse in the field of optical quantum information and quantum computation, yet it is not suitable for applications where deterministic nonlinearities are required due to its low spectral brightness. By placing the nonlinear crystal inside a cavity, the spectrum of down-conversion is actively modified, such that all the non-resonant modes of down-conversion experience destructive interference, while the resonant mode sees constructive interference, resulting in great enhancement in spectral brightness. I design and construct such a cavity-enhanced down-conversion source with record high spectral brightness, making it possible to use cold atoms as the interaction medium to achieve large nonlinearity between photons. The frequency of the photons is tunable and their coherence time is measured to be on the order of 10 nanoseconds, matching the lifetime of the excited state of typical alkali atoms. I characterize extensively the output of the source by measuring the second-order correlation function, quantifying two-photon indistinguishability, performing quantum state tomography of entangled states, and showing different statistics of the source.

The unprecedented long coherence time of the photon pairs has also made possible the encoding of quantum information in the time domain of the photons. I present a theoretical proposal of multi-dimensional quantum information with such long-coherence-
time photons and analyze its performance with realistic parameter settings. I implement this proposal with the quantum light source I have built, and show for the first time that a qutrit can be encoded in the time domain of the single photons. I demonstrate the coherence is preserved for the qutrit state, thus ruling out any classical probabilistic explanation of the experimental data. Such an encoding scheme provides an easy access to multi-dimensional systems and can be used as a versatile platform for many quantum information and quantum computation tasks.
To my family,

for their unwavering support.
Acknowledgements

I feel fortunate to have the opportunity to tackle some challenging problems during my study at the University of Toronto. The last few years has been focused and intense period of my life. Much of what I learned I will carry through the rest of my life. There are many people I want to thank who have made this an delightful and rewarding experience.

First and foremost, I want to thank my thesis supervisor, Aephraim Steinberg, without whom this work will not be possible. His breadth of knowledge and vision has made many discussions enjoyable and fruitful. He has a unique way of approaching and understanding problems intuitively, which for many times shows me the beauty of physics. He has been very supportive and allows us to pursue our own interests, which for me was to work on a project bridging Aephraim’s two existing labs. I started my work in the lab independently, and I am grateful to Rob Adamson for being a knowledgable mentor in the lab and very patient in answering my questions, no matter how dumb they are. Mirco Siercke has also been very helpful through the early times of the project. In 2007, I was fortunate to won a fellowship that supported me to visit Morgan Mitchell’s group at ICFO in Spain for three months. Morgan was a valuable source of knowledge and I was learning many experimental techniques from him. His group members also provided great help. In particular, I have spent many time in the lab with Florian Wolfgramm discussing problems, aligning optics and taking data. Florian also visited us in the summer of 2008 which was again a very productive period. I also owe many thanks to other members of Aephraim’s group. Amir Feizpour and Greg Dmochowski have helped numerous times in debugging and understanding the experiments. I also spent many hours in the lab with Yasaman Soudagar on the cluster quantum computation project, though the work is not presented in this thesis. Alex Hayat, our newly joint postdoc, is relentlessly resourceful in solving many theoretical and experimental issues. Chris Ellenor, Rockson Chang, Chao Zhuang, Krister Shalm, Lee Rozema, Dylan Mahler, Chris Pual, and the
rest of the group members have made the lab an interesting and productive place. I also
want to thank Alan Stummer for his expertise in electronics, and for his help in making
sure the circuits are doing what they were supposed to do.

I want to thank my parents for their unconditional support, and my brothers, Jian
and Kang, for their support in my decision of going to graduate school in Canada. I can
still remember lessons of science and life from them when I was a kid. I also want to
thank many friends who has made my life enjoyable beside academic research. Finally,
I want to thank my girlfriend, Wenwen Jiang, for keeping me grounded in the last year,
and for her supports that have made bad times bearable, and good times more enjoyable.
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List of Publications


*:Both authors contributed equally to this work.

4. Xingxing Xing, Amir Feizpour, Alex Hayat, and Aephraim M. Steinberg. Experimental demonstration of a 3-dimensional quantum system in the time domain, *in preparation.*
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<th>Description</th>
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<tr>
<td>AM</td>
<td>Amplitude Modulation</td>
</tr>
<tr>
<td>AWG</td>
<td>Arbitrary Waveform Generator</td>
</tr>
<tr>
<td>BBO</td>
<td>Barium Borate</td>
</tr>
<tr>
<td>BLIRA</td>
<td>Blue-Light-Induced Red Light Absorption</td>
</tr>
<tr>
<td>BS</td>
<td>Beam-Splitter</td>
</tr>
<tr>
<td>CDMA</td>
<td>Code Division Multi Access</td>
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<tr>
<td>ECDL</td>
<td>External Cavity Diode Laser</td>
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<tr>
<td>EOM</td>
<td>Electro-Optic Modulator</td>
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<tr>
<td>ERS</td>
<td>Error Rate per Symbol</td>
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<tr>
<td>FC</td>
<td>Fiber Coupler</td>
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<tr>
<td>FPC</td>
<td>Fiber Polarization Controller</td>
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<tr>
<td>FPD</td>
<td>Fast Photodiode</td>
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<tr>
<td>FPGA</td>
<td>Field Programmable Gate Array</td>
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<td>FWHM</td>
<td>Full Width Half Maximum</td>
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<td>HBT</td>
<td>Hanbury-Brown-Twiss</td>
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HOM  Hong-Ou-Mandel
HWP  Half-Wave Plate
KTP  Potassium Titanium Oxide Phosphate
LPF  Low-Pass Filter
MCA  Multi-channel Coincidence Analyzer
N00N  $|N, 0\rangle + |0, N\rangle$
OC  Output Coupler
OPO  Optical Parametric Oscillators
PBS  Polarizing Beam-Splitter
PDC  Parametric Down-Conversion
PDH  Pound-Drever-Hall
PFM  Phase-Flip Modulation
PID  Proportion-Integral-Derivative
PM  Phase Modulation
PPKTP  Periodically Poled Potassium Titanium Oxide Phosphate
PSD  Phase Sensitive detector
PSH  Phase Shifter
PZT  Piezoelectric Element
QND  Quantum Non-Demolition
QST  Quantum State Tomography

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<table>
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<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>QWP</td>
<td>Quarter-Wave Plate</td>
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<tr>
<td>SHG</td>
<td>Second Harmonic Generation</td>
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<td>SP</td>
<td>RF Power Splitter</td>
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<td>SPCM</td>
<td>Single Photon Counting Modules</td>
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<tr>
<td>SPDC</td>
<td>Spontaneous Parametric Down-Conversion</td>
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<tr>
<td>TDC</td>
<td>Time-to-Digital Converter</td>
</tr>
<tr>
<td>TEC</td>
<td>Thermoelectric Cooling</td>
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<tr>
<td>XPM</td>
<td>Cross-Phase Modulation</td>
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Chapter 1

Introduction

Quantum information processing is of fundamental and practical interests and has seen rapid development over the last few decades. For many tasks, the computational cost grows exponentially with the size of the system under consideration. Quantum computation can greatly increase the efficiency for many of such problems that are otherwise intractable. It was shown that universal simulation of physical processes is possible with quantum computers [49, 100], and that one can use quantum computers to factor large numbers [145] with exponential speedup and search databases [63] with quadratic speedup comparing to their classical counterparts.

Light is the ideal carrier for quantum information, yet it is generally difficult to introduce light-light interaction due to small nonlinearities in typical media. Many models of quantum computation based on optical systems are proposed to get around this problem to achieve quantum logic with light. One can use linear optics with post-selection to introduce effective nonlinearity, and it was shown to be efficient in principle to realize quantum computation [87]. If we have a large entangled state at disposal, Raussendorf and Briegel [134] showed that with one-qubit measurements and feedforward, one can also achieve the required non-linearity in quantum computation. Another model based on adiabatic evolution of Hamiltonian was shown to be universal for quantum computation [45, 4];
this transfers the difficulty in engineering nonlinearities to the exact implementation of the system Hamiltonian. Weak nonlinearity and Quantum Non-Demonition (QND) measurements were also shown to be an efficient resource for quantum computation, with improvements in scaling compared to models based on linear optics [118, 119].

To implement a quantum computer, we need a physical system that makes it easy to generate, manipulate and measure a quantum state. There are many different implementations of quantum information. For example, quantum information can be encoded and processed with nuclear magnetic resonance [56], superconductors [32], quantum dots [139, 39], NV centres [78, 43], rare-earth-ion doped crystals [137, 33, 171, 64], trapped ions [31], and many other systems.

There has been great development of sources of single and entangled photons such as atomic cascades, Spontaneous Parametric Down-Conversion (SPDC) [126], four-wave-mixing in atoms [154] and optical fibres [27, 99]. While these alone are not enough to realize practical quantum computation, they were frequently used for demonstrating many proof-of-principle experiments of quantum information processing protocols [20, 127, 128, 133, 96], as well as experiments of interest to fundamentals of quantum mechanics [7, 61, 84, 104, 167, 135]. It is natural to apply the techniques we learned from these experiments and extend to a more complex system towards practical quantum computation. One of the key elements here is again strong nonlinearities, or interactions between photons, which are extremely weak in typical medium such as glass or crystals. Atoms, on the other hand, provide strong near-resonance interaction with light, and they are proved to be a strong nonlinear medium for light [69]. There are many proposals [140, 124, 131, 161] of achieving effective non-linearity between photons with atoms being the interaction medium. To implement these proposals, we need a quantum light source that is compatible with atomic transitions. The goal of this thesis is to develop such a quantum light source which can serve as a light-matter interface. Together with ultracold atoms, we may be able to realize QND measurements on the photons [161], and
implement weak-nonlinearity-based quantum computation [118, 119].

The thesis is organized in the following way. In Chapter 2, I first introduce the proposal of weak-nonlinearity-based quantum computation which motivates this work. After that I introduce of some basic theoretical and experimental tools that will be used in the rest of the thesis.

Chapter 3 gives the details of the design of the quantum light source that is compatible with atomic transitions. Essentially the broadband spectrum of SPDC is modified with interference effect in a cavity. Design considerations, such as phase-matching conditions, compensation schemes, temperature tuning, as well as stabilization schemes, are given for the cavity. I demonstrate one of the spectrally brightest sources of down-conversion with such cavity enhancement. The output of the source is characterize in several ways, proving the quantum mechanical nature of the states from the source. I also show that different kinds of statistics, namely, Poisson, super-Poisson and sub-Poisson statistics, can all be measured from the same source with slightly different setups. Due to the cavity enhancement, the photons generated from our source is measured to have very long coherence time, making it possible to investigate some problems that are difficult due to the short-coherence-time nature of typical down conversions.

Specifically, the long coherence time of the photons make it possible to encode high dimensional quantum information in the time domain. Higher dimensional systems could be used to beat the channel capacity limit for qubits with superdense coding [9]. When applied to quantum key distribution in quantum cryptography, they could increase the coding density and increase the security margin [62]. Qudits (d-dimensional quantum systems) can also reduce the requirement in detection efficiency to achieve detection-loophole-free Bell Inequality [157], enable the test of Born rule [149, 146], and simplify certain computation tasks [116, 97]. Before this work, high dimensional quantum information was encoded with hyper-entanglement [8], time-bin entanglement [52, 153], orbital angular momentum [108, 115], spatial degree of freedom [97], and superconducting qudit
In Chapter 4, I exploit the long-coherence-time feature of the source and propose a theoretical framework of multidimensional quantum information in the time domain. I develop a theoretical framework for the encoding and decoding of multidimensional quantum information based on temporal modulation and spectral filtering of photon wavepackets. High detection efficiency and relatively low error rates are shown to be achievable with commercially available technologies. I also present experimental demonstrations of such a multi-dimensional quantum information encoding scheme, where I show methods and results of state preparation and projection using commercial electro-optic modulators and a narrowband filter cavity. I also characterize the quantum channel by measuring the coherence of the high-dimensional quantum states, and show that using the proposed encoding method, the coherence in some quantum superposition state is preserved.
Chapter 2

Concepts and techniques

2.1 Weak-nonlinearity-based quantum computation

2.1.1 The proposal

Photons do not interact with each other or the environment very strongly. This property is useful when using photons as the information carrier, but not suitable for information processing where large interactions (i.e., strong nonlinearities) are required. Photons can interact with each other in a nonlinear medium, though it is doubtful that any nonlinearity will be strong enough for single-photon-single-photon quantum logic gates [141, 142, 55]. Nemoto and Munro proposed a get-around scheme that relies on QND measurements to accomplish the photon-photon quantum logic gates [118, 119]. The QND measurement with two light fields can be done using 3rd nonlinearities of some Kerr medium [76]. The Hamiltonian of such an interaction is given by

\[ H_{\text{QND}} = \hbar \chi \hat{a}_s^{\dagger} \hat{a}_s \hat{a}_p^{\dagger} \hat{a}_p, \quad (2.1) \]

where the subscripts \( s \) and \( p \) stand for the signal and probe mode, and \( \chi \) describes the strength of the cross-Kerr nonlinear interaction of the medium.

When the signal is a Fock state and the probe is a coherent field, the input light fields
can be represented by the state $|\Psi_{in}\rangle = |n_s\rangle |\alpha_p\rangle$. After the cross-Kerr interaction, the output state becomes

$$|\Psi_{out}\rangle = e^{i\chi t} a_s^\dagger a_s a_p^\dagger a_p |n_s\rangle |\alpha_p\rangle = |n_s\rangle |\alpha_p e^{in_s\chi t}\rangle.$$  

(2.2)

The signal photon state remains unchanged while the probe field picks up a phase shift $e^{in_s\chi t}$ that is determined by the number of photons in the signal field. If such a phase shift is much larger than the phase uncertainty of the field, i.e., $n_s\chi t \gg |\alpha|$, the phase shift can be measured in a single-shot and we can infer the presence of the signal photon field without altering its state. In other words, one can achieve a photon number QND measurement on the signal field. The scheme is shown in Fig. 2.1.

Next we show how to achieve universal quantum computation with such QND measurements. Consider the signal state of the form $|\Psi_s\rangle = c_0 |0\rangle_s + c_1 |1\rangle_s$ interacting with a coherent field $|\alpha_p\rangle$. The state after the interaction can be written as $|\Psi_{s,p}\rangle = c_0 |0\rangle_s |\alpha_p\rangle + c_1 |1\rangle_s |\alpha_p e^{ixt}\rangle$. Let us introduce another single-photon field, the target, in the state $|\Psi_t\rangle = \frac{1}{\sqrt{2}}(|0\rangle_t + |1\rangle_t)$. Between the probe and the target, we use a second cross-Kerr medium that introduces a phase shift of equal magnitude but opposite sign to the first interaction, the final state becomes

$$|\Psi_{t,s,p}\rangle = c_0 |0\rangle_s |0\rangle_t |\alpha_p\rangle + c_1 |1\rangle_s |0\rangle_t |\alpha_p e^{ixt}\rangle + c_0 |0\rangle_s |1\rangle_t |\alpha_p e^{-ixt}\rangle + c_1 |1\rangle_s |1\rangle_t |\alpha_p\rangle.$$  

(2.3)

One can perform homodyne measurement on the coherent field with a local oscillator to measure its phase. When the phase of the local oscillator is chosen to be the same as the
phase of probe $|\alpha_p\rangle$, such a measurement corresponds to an $\hat{X}$-quadrature measurement where $\hat{X} = \hat{a} + \hat{a}^\dagger$, and the phase shift $e^{\pm i\chi t}$ can not be distinguished. More specifically, such phase measurements prepare the state to

$$
|\Psi_{t,s}\rangle = \langle \alpha_p | \hat{X} | \alpha_p \rangle [c_0 |0\rangle_s |0\rangle_t + c_1 |1\rangle_s |1\rangle_t] + \langle \alpha_p e^{i\chi t} | \hat{X} | \alpha_p e^{i\chi t} \rangle [c_1 |1\rangle_s |0\rangle_t + c_0 |0\rangle_s |1\rangle_t].
$$

(2.4)

With classical feedforward to a conditional $\hat{\sigma}_x$ gate and phase compensation of $\phi$ on the $s$ photon based on the $\hat{X}$-quadrature phase measurement, the final state of the signal and target becomes

$$
|\Psi_{\text{final}}\rangle = c_0 |0\rangle_s |0\rangle_t + c_1 |1\rangle_s |1\rangle_t.
$$

(2.5)

The procedure described above can deterministically entangle the signal photon and target photon and can be used to implement an entangling gate. With such two-qubit quantum logic gates and the single qubit rotations that can be accomplished using waveplates, one can implement a deterministic CNOT gate [118] and thus achieve universal quantum computation [121].

### 2.1.2 Ultracold atoms as a nonlinear medium

In classical nonlinear interactions, we can use the polarization $P(r,t)$ to describe the response of matter to an input electric field [22],

$$
\tilde{P}(t) = \chi^{(1)} \cdot \tilde{E}(t) + \chi^{(2)} \cdot \tilde{E}^2(t) + ...,
$$

(2.6)

where $\chi^{(i)}$ is the $i$-th order of the optical susceptibility. For simplicity, $\tilde{P}(t)$ and $\tilde{E}(t)$ are taken to be scalar, and instead of tensors, the susceptibilities $\chi^{(i)}$ are also treated as scalars. The electric field $E(t)$ can be decomposed into positive and negative frequency parts,

$$
\tilde{E}(t) = E(\omega)e^{-i\omega t} + E^*(\omega)e^{i\omega t}.
$$

(2.7)

The cross-Kerr nonlinearity describes the nonlinear effect between two field with frequency $\omega$ and $\omega'$. It is a third-order effect described by $\tilde{P}^{(3)}(t) = \chi^{(3)} \cdot \tilde{E}^3(t)$. The part
of the nonlinear polarization affecting the $\omega'$ beam is

$$P^{(3)}(\omega') = 6\chi^{(3)}|E(\omega)|^2E(\omega'). \quad (2.8)$$

Equivalently, we can describe the cross-Kerr effect with refractive indices

$$n = n_0 + 2n_2|E(\omega)|^2, \quad (2.9)$$

where $n_0$ and $n_2$ are the linear and nonlinear refractive index, respectively.

It is desirable to achieve a nonlinear refractive index $n_2$ as large as possible in order to implement a QND measurement [76]. There has been an enormous amount of work on optical nonlinearities. Optical nonlinearities at very low light levels have been observed in warm atomic vapours alone [38, 73], as well as in combination with tapered fibers [150] and hollow-core fibers [101, 156]. Ultracold atomic system is another candidate, and much work has been done with ultracold atomic clouds [67, 81, 28, 144] and Rydberg atoms [155, 54, 60, 42] to achieve strong nonlinear interactions. To date, ultracold atoms with electromagnetically induced transparency (EIT) [68, 103] exhibit by far the largest nonlinearity known in nature, with a value of 0.18 cm$^2$/W as shown in [69]. There have been rapid developments with EIT to accomplish many proof-of-principle demonstrations, such as optical quantum memory [106] where light is stored in and retrieved from the atoms [44, 26, 30], as well as mapping the quantum state of light to the atoms [109, 29]. Optimized pulse shape for storage with EIT has also been studied [59]. While it may be impossible to achieve a single photon-single photon phase shift as large as $\pi$ with EIT [141, 142, 55], a measurable cross-phase modulation [161, 71] between a single photon and classical field is enough for the purpose of QND measurement, and thus can be used to implement the weak-nonlinearity-based quantum computation [118, 119]. To perform a QND measurement at the single photon level is a challenging task for free-propagating photons, and there are also proposals to measure the cross-phase modulation at the single photon level via post-selection [24, 48], as a step towards the single-shot QND...
measurement. This thesis project serves as an important step to achieve measurable photon-photon interactions through ultracold atoms.

2.2 Optical cavity

Optical cavities are widely used in optics, and are a necessary component for lasers [113], interferometers [35, 36], and spectroscopic references [138]. They are resonators formed by interference of multiple reflections of an optical field in some confined spatial volume. Optical cavities are also the central elements for research in cavity quantum electrodynamics [112, 158]. When used appropriately, optical cavities can be used to greatly enhance the size of the non-linear effects and improve the spectral characteristics of the non-linear process. In this section, we will review some of the fundamentals of optical cavities, and give some calibration data for the cavities used in this thesis.

2.2.1 Cavity basics and cavity enhancement

One of the most common types of cavities are Fabry-Perot cavities (see Fig. 2.2), where the resonators consist of two identical highly reflective mirrors separated by a certain distance $L$. Standing wave pattern is formed when the distance is a multiple of the wavelength of the optical field, and in the frequency domain it shows up as a series of discrete modes. The separation of these modes, or the free spectral range (FSR), can be written as

$$\text{FSR} = \frac{c}{L_r} = \frac{1}{t_{rt}}$$  \hspace{1cm} (2.10)

where $c$ is the speed of light and $L_r$ is the round-trip optical path, and $t_{rt}$ is the round-trip time of the cavity. In the case of a Fabry-Perot, $L_r$ is twice of the distance between the cavity mirrors multiplied by the group index $n_g$, $L_r = 2n_gL$.

The linewidth of cavity, $\delta \nu$, is inversely proportional to the cavity ring-down time, $t_{rd}$,

$$\Delta \nu = \frac{1}{\tau_{rd}}.$$  \hspace{1cm} (2.11)
The finesse of the cavity can then be written as
\[ F = \frac{FSR}{\delta \nu} = \frac{t_{rd}}{t_{rt}}, \] (2.12)

From Eq. 2.12, we see that finesse \( F \) can be intuitively interpreted as the average number of round trips a light wavepacket takes inside the cavity before escape through the output coupler or other loss mechanisms. For a lossless cavity, finesse only depends on the reflectivity of the mirrors used to construct the cavity.

We will now derive the transmission of the cavity. As shown in Fig. 2.3, the phase shift \( \phi \) acquired by the electric field in each round-trip can be written as
\[ \phi = \frac{n \omega}{c} L_r. \] (2.13)

The electric field in each round trip is then
\[ E_1 = E_{in} \cdot t_1 e^{i \phi}, \] (2.14a)
\[ E_{i+1} = E_i \cdot r_1 r_2 e^{i \phi}, i \in [1, +\infty], \] (2.14b)
Figure 2.3: Multiple reflection and transmission in a cavity. The total reflection and transmission of the cavity are the results of interference of multiple reflection and transmission at each round-trip.

where the $r_i$ and $t_i$ reflection and transmission coefficients of mirror $i$ ($i = \{1, 2\}$), and they satisfy the following relation for a lossless mirror

$$|r_i|^2 + |t_i|^2 = 1; \quad (2.15a)$$

$$r_i^* t_i + r_i t_i^* = 0. \quad (2.15b)$$

Without loss of generality, we can choose $r_i$ to be real and $t_i$ to be pure imaginary.

Summing over all the terms of Eq. 2.14 gives the intra-cavity field, and the output field is just the transmission of the intra-cavity field at mirror 2. Thus the intra-cavity intensity can be written as

$$I_{cav} = |\sum_i E_i|^2 = \frac{I_0 \cdot |t_1|^2}{|1 - r_1 r_2 e^{i\theta}|^2}, \quad (2.16)$$

At resonance, the intensity, $I_{cav}$, reach its maximum,

$$I_{cav,max} = \frac{I_0 \cdot |t_1|^2}{(1 - r_1 r_2)^2}. \quad (2.17)$$

To understand the significance of Eq. 2.17, let us assume the two mirrors are identical such that $r_1 = r_2$ and $t_1 = t_2$. The maximum intensity in the cavity now becomes $I_{cav,max} = \frac{I_0}{|t_1|^2}$. For typical cavity mirrors with 99% reflectivity, $I_{cav,max} = 100I_0$. This shows the root of cavity enhancement - the constructive interference between multiple reflections during the cavity ring-down time can greatly increase the intracavity intensity.
2.2.2 Transmission of a cavity

Similar to the analysis we have shown for the cavity enhancement, we can also calculate the transmission of the cavity. As shown in Fig. 2.3, summing over all the terms of transmission amplitude in each round trip and calculating the intensity, the output of the cavity is given by

\[ I_{\text{out}} = I_{\text{cav}} \cdot |t_2|^2 = \frac{I_0 \cdot |t_1 t_2|^2}{|1 - r_1 r_2 e^{i\phi}|^2}. \]  \hspace{1cm} (2.18)

The intensity of the transmitted field depends on the round-trip phase \( \phi \) given by Eq. 2.13.

As shown in Fig. 2.4, we characterize the transmission from a cavity to directly measure Eq. 2.18. The cavity transmits a series of narrow band in the frequency domain, which are the resonant modes of the cavity, and the non-resonant modes will be reflected. This is also the basis of the typical application of using the cavity as a spectral filter. For a typical cavity which we use in the later part of the thesis, the mirrors are separated by 1.0(1) mm, which gives a FSR of 150(15) GHz. The finesse is measured to be 329(5), from which we can infer the cavity linewidth \( \Delta \nu = 455(46) \) MHz. Note that even at resonance, the transmission of a cavity can be much less than 1, because of so-called cavity impedance mismatch. To illustrate this, we first write down the expression of the maximum transmission of the cavity at resonance

\[ I_{\text{out, max}} = I_{\text{cav}} \cdot |t_2|^2 = \frac{I_0 \cdot |t_1 t_2|^2}{(1 - r_1 r_2)^2}. \]  \hspace{1cm} (2.19)

We plotted Eq. 2.19 in Fig. 2.5, where the transmission is calculated for a linear cavity with the transmission amplitude of mirror 1 and 2 to be \( t_1 \) and \( t_2 \), respectively, according to Eq. 2.19. The transmission amplitude of mirror 1 is held constant, \( |t_1| = 0.01 \). A perfect destructive interference at the input coupler is reached when \( |t_2| = |t_1| \), where the cavity transmission reaches 100%. This is sometimes called critical coupling in other fields such as electrical or magnetic resonators (see, for example, [93]). When the condition \( |t_2| = |t_1| \) is not satisfied, the cavity transmission in the forward direction drops rapidly. It is common practice to choose two mirrors for a cavity from the same manufacture batch.

\[ \text{Chapter 2. Concepts and Techniques} \]
Figure 2.4: Cavity resonance. The light from a frequency stabilized laser is sent to the cavity. The length of the cavity is scanned by a PZT at the cavity to see multiple transmission peaks.
Figure 2.5: Cavity transmission due to impedance mismatch. The transmission is calculated for a linear cavity with the transmission amplitude to be $t_1$ and $t_2$, according to 2.19. The transmission amplitude of mirror 1 is held constant, $|t_1| = 0.01$.

in order to have similar reflection/transmission properties. Once we have the mirrors, we also wish to directly measure the transmission and reflection of the mirrors used to construct the cavity. For high reflection coatings, due to losses and noise in the system, it is can be difficult, though not impossible, to directly measure the reflection, e. g. , with careful alignment and control of beam size and detector placement. In many cases, we are more concerned about the behaviour of the composite system of the mirrors, i.e., the cavity. Next we will show how to measure the cavity impedance mismatch directly.

The cavity impedance mismatch can be probed by measuring the reflection from a cavity. Similar to the measurement of cavity transmission, the reflection can also be measured while scanning the cavity. For a ring cavity the reflection can be easily measured by placing a detector at the reflected mode which can be spatially separated from incident beam, see Fig. 2.2c and 2.2d for example. For a linear cavity such as shown in Fig. 2.2a, the measurement is slightly more complicated. If there is no birefringence in the cavity, we can use a Polarizing Beam-Splitter (PBS) and a Quarter-Wave Plate (QWP)
to accomplish the task. First adjust the input polarization to horizontal, and place the PBS and QWP before the cavity whose reflection we wish to measure. Set the QWP to $45^\circ$, such that upon two transmission through the QWP, the polarization will be flipped from H to V and vice versa. If the mode-matching is done correctly, the reflected beam should be in the same spatial mode as the incident beam, but now with opposite polarization, and will be reflected at the PBS. In such a measurement, the absolute amount of reflection does not matter, as long as we have a reference level corresponding to the off-resonant light. As shown in Fig. 2.6, the reflection of the cavities used later in our experiment are directly measured. We measured a reflection of 36% for the first cavity (details of this cavity can be found in Section 3.6), and reflection of 19% for the second cavity (Section 4.3.1). The measurements of reflection give upper bounds of impedance mismatch caused by unbalanced reflection coefficients of the mirrors. These measurements indicate that we have good cavity impedance matching given the high reflectivity mirrors involved. Note that the reflection can also be caused by mode-mismatch to the cavity, which will be discussed in the next section.

For practical cavities, there are also losses for each round-trip due to scattering at the mirror, as well as inside the cavity if there is any medium. To characterize the efficiency of a cavity, we define the escape efficiency $\eta$

$$\eta = \frac{t_{out}^2}{\sum_i t_i^2 + \sum_j \kappa_j}, \quad (2.20)$$

where $t_i$ is the transmission amplitude at each mirror including the output coupler $t_{out}$, and $\kappa_j$ describes scattering losses in the cavity, such as those due to a medium. In many cases where incoherent losses are detrimental such as the case in squeezing experiments, one would like as high of $\eta$ as possible. The output coupler transmission is typically chosen to be much larger than the losses, which will put a limit on the finesse of the cavity.
Figure 2.6: Measurement of cavity impedance mismatch. The reflection from the cavities used in our experiment are directly measured while the length of the cavities are scanned with a PZT. When on resonance, the light is maximally coupled into the cavity, and the reflections are measured to be 36% (a), and 19% (b). While far off resonance, the reflection is close to 100%.

2.2.3 Mode-matching and alignment of cavity

Within the paraxial approximation, the propagation of a coherent laser beam can be reduced to Gaussian optics where the radiation can be well approximated by a superposition of a set of Gaussian functions [138]. Formally, by solving the paraxial Helmholtz equation, the electric field amplitude can be written as

$$E(r, z) = E_0 \frac{w_0}{w(z)} \exp \left( -\frac{r^2}{w^2(z)} - ikz - i\frac{r^2}{2R(z)} + i\zeta(z) \right)$$  \hspace{1cm} (2.21)

where $r$ is the radial distance from the axis, $z$ is the axial distance from the beam waist, $w_0$ is the beam waist, $z_R$ is the Rayleigh range, $\zeta(z)$ is the longitudinal phase delay of the Gaussian beam, and the beam size $w(z)$ and radius of curvature $R(z)$ are

$$w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \hspace{1cm} (2.22a)$$

$$R(z) = z \left[ 1 + \left( \frac{z_R}{z} \right)^2 \right]. \hspace{1cm} (2.22b)$$
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Figure 2.7: Gaussian beam in a cavity. When the radius of curvature of the wavefront of the beam matches that of the cavity mirror, a mode-matching condition is satisfied and the cavity supports this mode of the beam.

One can use Eq. 2.21 and Eq. 2.22 to calculate the beam propagation in a cavity. The cavity can stably support different spatial distributions of the beam, which are superpositions of a set of orthogonal functions called the cavity modes. The first obvious cavity mode has a beam radius at the mirror that matches the radius of curvature of the mirror, as shown in Fig. 2.7, which corresponds to the fundamental mode of the cavity. This is an example of mode matching for the cavity. A useful and convenient technique to describe Gaussian beams is to define a complex beam parameter $q$

$$\frac{1}{q} = \frac{1}{R(z)} - \frac{i \lambda_0}{\pi n w^2},$$

(2.23)

and using the ABCD matrix [10] to describe the propagation

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}.$$ 

(2.24)

With this method, we can calculate the mode propagation for a cavity and design mode-matching optics accordingly, as shown in Fig. 2.8. A typical design process of an optical cavity includes the following procedures. First, a desired focus in a cavity is
Figure 2.8: Beam propagation and mode matching to a cavity. Shown is the beam waist as a function of propagation distance, where abrupt changes are made by the mode matching lenses and cavity mirrors. At a propagation distance of 0.48 m where the waist of the cavity is located, we reach a desired waist of 30 $\mu$m in size. The schematic is shown in the inset.

chosen, e.g., to give the best optical parametric gain of some non-linear medium. Then, the cavity mirrors and size is determined by the beam parameters as well as other design considerations such as FSR of the cavity. Next the reflectivity can be inferred for a certain cavity linewidth. Finally, a set of lenses and mirrors can be chosen as the result of the beam propagation calculation using the method shown above. For this specific cavity which we will be using for an Optical Parametric Oscillators (OPO) (see Section 3.4 for details), we reach a waist size of 30 $\mu$m in the cavity, matching the optimized beam focus for $\chi^{(2)}$ processes [21].

Optical cavities are sensitive devices that need dedicated efforts to construct and align. It is important to realize that the mode matching of a specific cavity mode, in most cases the fundamental, is not a trivial task. When first starting to align, it may
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Figure 2.9: Different cavity modes captured by a camera. (a)-(c), higher order cavity modes, which typically involve radial or axial discretization of the field distribution. (d) the fundamental mode of the cavity, a gaussian distribution for the optical field.

be confusing which mode is the desired fundamental mode. After some initial beam alignment and focus adjustment, one can use a CCD camera to help identify a certain mode. As shown in Fig. 2.9, the mode structure captured by the camera displays different features. Higher order cavity modes typically involve radial or axial discretization of the field distribution, whereas the fundamental mode of the cavity usually has a gaussian distribution. After successful identification of the cavity modes, one can then proceed to optimize mode-matching by tuning the coupling and focusing of the cavity. Note that sometimes the cavity itself needs to be aligned, in which case one needs to repeat the adjustment of the beam and the cavity alternatively, until a desired mode-matching is reached. We have aligned every cavity’s input coupling in this thesis to their fundamental mode. This ensures the best coupling efficiency and cavity stability.
2.3 Frequency stabilization

Frequency stabilization is critical in making sure the atomic compatibility of a narrow-band quantum light source. Over the years there has been tremendous work on frequency stabilization using lasers. The basic idea of frequency stabilization is simple: a laser is sent to some frequency reference (certain atomic transition or an optical cavity) for a measurement of frequency, and this measurement is then fed back to the laser to suppress frequency fluctuation and drift. The error signal, a signal that is directly proportional to the difference between the laser and reference frequency, is generated with such a measurement. There are many different ways of generating an error signal, which give rise to different locking techniques. From the atomic transition, one can use Doppler-free laser polarization spectroscopy [162] to lock the laser to an absolute frequency reference; Hansch-Couillaud locking [66] utilizes the polarization rotation in a Fabry-Perot cavity to either lock the laser to the cavity or vice versa; different spatial modes of a cavity can also interfere to give rise to an error signal - such techniques are also referred to as tilt locking [147]; frequency or phase modulation can also be used to generate an error signal, such as those used in the Pound-Drever-Hall locking technique [15] and frequency modulation spectroscopy [14].

2.3.1 Pound-Drever-Hall locking techniques

The Pound-Drever-Hall (PDH) technique is a robust and easy-to-implement way of locking a laser or a cavity. The main idea is to use phase (or frequency) modulation to generate frequency side-bands in addition to the carrier, and then probe a certain spectroscopic feature through the phase-shift of the beating signal formed by the carrier and side-bands.

Let us write the electrical field of the optical beam as $E(t) = Ae^{i\omega t}$, where $A$ and $\omega$ are the complex field amplitude and frequency, respectively. With a sinusoidal phase
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modulation, the field can be written in terms of Bessel functions

\[ E_{\text{mod}}(t) = A e^{i\omega t + i\beta \sin(\Omega t)} \]

\[ = A e^{i\omega t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{i n \Omega t}, \]

(2.25)

where \( J_n(\beta) \) is the n-th Bessel function of the first kind. For small modulation index \( \beta \), higher orders in Eq. 2.25 with \( n \geq 2 \) can be neglected, and the modulated field can then be approximated

\[ E_{\text{mod}}(t) \approx A e^{i\omega t} (J_0(\beta) + J_1(\beta) e^{i \Omega t} - J_1(\beta) e^{-i \Omega t}), \]

(2.26)

which is just a beating signal in time from the carrier with two side bands. After a certain dispersive medium, such as an atomic vapour cell or an optical cavity, the beating signal will pick up a frequency dependent phase-shift that can be easily detected. For the case of an cavity, whose transmission can be described by Eq. 2.18, we can calculate the shape of the error signal (the difference between a control target and the feedback from a system), as shown in Fig. 2.10. While at resonance, the error signal crosses zero, which can be used as the feedback signal. There are two more zero-crossing at offsets of the modulation frequency, which are caused by the side-bands. This technique works well when the modulation frequency is smaller than the frequency feature to be locked to.

The scheme is illustrated in in Fig. 2.11. Note that the same modulation-demodulation technique can be used to detect a phase shift, which will be discussed later in Section 2.4.1.

With the error signal generated by the Pound-Drever-Hall technique, one can then use Proportional-Integral-Derivative (PID) electronics to provide feedback to a PZT on the cavity (or the laser) for frequency stabilization. The performance of the stabilization significantly depends on the quality of the error signal and PID electronics. There is an optimum set of PID parameters given a certain error signal. The details of the PID electronic circuits is given in Appendix A.
Figure 2.10: Simulation of Pound-Drever-Hall locking technique. Shown is the error signal as a function of the laser frequency. While at resonance, the error signal crosses zero, which is the desirable feature as a feedback signal. There are two more zero-crossing caused by the side-bands.

Figure 2.11: Pound-Drever-Hall locking technique. A laser is phase modulated and sent to some frequency feature, e.g., atomic vapour cells or optical cavities. The transmission (or reflection) is then picked up by a photodiode and demodulated against the same RF signal used to modulated the laser to generate an error signal, which is used to stabilize the cavity (or the laser).
2.3.2 Spectroscopic features

Spectroscopy can be used to establish absolute frequency references using optical resonances of matter. Associated with the resonances are either dispersive or absorptive features which are related by the Kromers-Kronig relations [22]. Typically an optical cavity or molecular/atomic transition is used as the resonance.

Using the Pound-Drever-Hall technique, we can generate error signals from atomic transitions using Doppler-free saturated absorption spectroscopy (Fig. 2.12), and from optical cavity resonances (Fig. 2.13). The scope traces are recorded as the frequency of the laser is scanned. In both cases, we can clearly resolve the frequency feature and can choose a specific one to lock to using PID electronics. With high quality error signal generation, we can improve the performance of the locking and improve stability. In the lab, we typically achieve a stable locking for tens of minutes, which can be further improved if we employ laboratory temperature and climate control systems.

2.3.3 Laser linewidth narrowing

When designed appropriately, a locking technique can also provide linewidth narrowing in additional to frequency stabilization when locking a laser. In Fig. 2.11, the low-pass filter after the phase sensitive detector determines the bandwidth of the error signal. With a typical modulation frequency at 20 MHz, a low-pass filter at 1.5 MHz provides large enough bandwidth to cover slow drifting as well as fast fluctuation of the laser frequency. If we feed the high-frequency of the error signal to the current driver of the laser and low-frequency to the PZT, the laser linewidth will be reduced. Details about the implementation of this idea can also be found in Appendix A. We measure the laser linewidth after frequency stabilization. The linewidth of the laser is measured with a Fabry-Perot cavity to be 0.8 MHz, and after stabilization, the linewidth is 0.4 MHz.
Figure 2.12: Saturated absorption spectroscopy. (a) Setup of Doppler-free saturated absorption spectroscopy. (b) Error signal corresponding to Rb 85 transitions. (c) Error signal corresponding to Rb 87 transitions.
2.4 Calibration

In this section, we summarize the calibration of a few key elements in this thesis project. This includes the calibration of the phase measurement which uses the frequency modulation technique, the half- and quarter- waveplates used in tomography of the quantum states from the photon-pair source, as well as the electro-optical modulators used to encode high dimensional quantum information in the time domain of single photons.

2.4.1 Phase measurement

Similar to the Pound-Drever-Hall locking technique, the method of phase modulation-demodulation can be used to measure a phase-shift. In comparison to typical phase shift measurements where a spatial interferometer is used, measurement with modulation-demodulation technique does not use the spatial domain, but use the frequency domain. The phase modulation can then be thought of as the first Beam-Splitter (BS) as in

Figure 2.13: Error signal from a cavity with Pound-Drever-Hall technique. The transmission of the cavity and its demodulation are shown as a function of frequency.
the Mach-Zehnder interferometer, and the demodulation mixed the initial frequencies which can be treated as the second BS. A frequency dependent shift (dispersion) gives the phase difference. Using such frequency domain interferometer has the advantage of being insensitive to phase fluctuations caused by air turbulence etc., leading to better signal-to-noise ratio.

We characterize the performance of such phase measurements, and give the phase sensitivity for the electrical components used, as shown in Fig. 2.14a. In this characterization, we used an electrical driven phase shifter (Mini-circuits JSPHS-26) to produce a phase shift of interest. In an actual phase measurement, this can be the phase shift of a coherent beam propagating through an atomic medium, as described in Section 2.1.

With a sinusoidal drive signal, we acquire the output of the demodulation after a low-pass filter (Mini-circuits BLP-1.9), and fit the acquired data to a sinusoid, as shown in Fig. 2.14b. The fitting gives the signal-to-noise ratio of 25 dB for the acquired signal, and we can then infer the phase sensitivity. Given the phase shifter sensitivity of 0.6 rad/V, the low-pass filter bandwidth 1.9 MHz, and assuming the noise is white, we conclude a phase sensitivity of $1.7 \times 10^{-9}$ rad/$\sqrt{\text{Hz}}$. This noise level converts to a noise amplitude of $5.5 \times 10^{-6}$ rad when we use a 10 MHz low-pass filter, which is far below the quantum noise of a probe pulse with 1000 photons. Of course, in the practical implementation of the phase measurement, the detector noise and other technical noise in the laser beam and other parts of the system will contribute to the noise, and resulting in a less sensitive phase measurement system.

### 2.4.2 Wave plate calibration

As an important optical element, a wave plate has to be calibrated before use, especially for applications in quantum state tomography [2]. There could be many experimental imperfections that lead to errors, such as imperfect wave plate retardances, surface qualities, finite extinguishing ratio of polarizers used etc. A simple way to calibrate the
Figure 2.14: Calibration of phase measurement.
waveplates is to first align the axis of two polarizers to be $90^\circ$ with each other, and place the wave plate to be measured in between. Note it is important to have the incident beam at a normal angle to both polarizers and the wave plate. Next rotate the wave plate in small increments and record the photodiode readings after the second polarizer. Any wave plate (or birefringent material) can be modeled by the Jones matrix

$$J_{wp} = \begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix},$$

(2.27)

where $\phi_x$ and $\phi_y$ are the phase shifts of the light in $x-$ and $y-$polarized light, respectively. Eq. 2.27 describes a wave plate with its fast axis aligned to the $x$ axis. For arbitrary orientations, the Jones matrix is given by

$$J_{wp,\theta} = \begin{pmatrix} e^{i\phi_x} \cos^2 \theta + e^{i\phi_y} \sin^2 \theta & (e^{i\phi_x} - e^{i\phi_y}) \cos \theta \sin \theta \\ (e^{i\phi_x} - e^{i\phi_y}) \cos \theta \sin \theta & e^{i\phi_x} \sin^2 \theta + e^{i\phi_y} \cos^2 \theta \end{pmatrix},$$

(2.28)

where $\theta$ is the angle of the fast axis with respect to the $x$ axis. We are only concerned about the relative phase delay between $x$ and $y$ polarizations, so we define the retardance $\phi = \phi_y - \phi_x$, and the transmission through the calibration setup is given by

$$T(\phi, \theta) = |(e^{i\phi_x} - e^{i\phi_y}) \cos \theta \sin \theta|^2.$$

(2.29)

With measurements of transmission of the calibration setup, one can characterize the waveplates to know their axes and retardance.

We have used rotational mounts with a step motor driver (Newport ESP-300) to drive the waveplates. We find that there is a discrepancy between the displayed rotation angle and the measurement from a photodiode. There are also errors in wave plate retardance. These errors can be calibrated with the method described above. The result of the motor and waveplates calibration is listed in Table. 2.1. These errors need to be taken into account when using the waveplates for tasks like quantum state tomography.
Table 2.1: Results of waveplates calibration

<table>
<thead>
<tr>
<th>Type</th>
<th>Motor</th>
<th>Driver</th>
<th>Cycle</th>
<th>Fast axis</th>
<th>Retardance deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>QWP</td>
<td>A</td>
<td>Box1, Axis 2</td>
<td>360.08°</td>
<td>-85.43°</td>
<td>2.77°</td>
</tr>
<tr>
<td>QWP</td>
<td>B</td>
<td>Box2, Axis 2</td>
<td>362.32°</td>
<td>-78.96°</td>
<td>0.71°</td>
</tr>
<tr>
<td>QWP</td>
<td>C</td>
<td>Box2, Axis 1</td>
<td>378.88°</td>
<td>-26.07°</td>
<td>0.04°</td>
</tr>
<tr>
<td>HWP</td>
<td>D</td>
<td>Box1, Axis 3</td>
<td>359.65°</td>
<td>-14.06°</td>
<td>0.64°</td>
</tr>
<tr>
<td>HWP</td>
<td>E</td>
<td>Box1, Axis 1</td>
<td>360.06°</td>
<td>-11.83°</td>
<td>0.83°</td>
</tr>
<tr>
<td>HWP</td>
<td>F</td>
<td>Box2, Axis 3</td>
<td>360.02°</td>
<td>25.43°</td>
<td>0.13°</td>
</tr>
</tbody>
</table>

2.4.3 Electro-optic modulators

Electro-Optic Modulator (EOM) are widely used in optical communications and various optics experiments. They utilize the electro-optical effect of a non-linear crystal to modulate the phase, frequency, or amplitude of the beam. In this thesis, we use amplitude and phase EOMs to encode high-dimensional quantum information in the time domain of a single photon. It is important to characterize the performance of the EOMs in the experiment.

One important parameter of EOM, $V_\pi$, is defined as the voltage at which a $\pi$ phase shift is induced by the electric field. The setups for calibration of $V_\pi$ for phase and amplitude EOMs are shown in Fig. 2.15. A narrowband laser is used at the input to provide long-coherence light such that slight mismatch of path length in an interferometer is negligible. In the calibration, the coherence length of the light is on the order of $10^2$ meters, where the mismatch in path length is less than 1 meter. In the actual experiment, the EOM will be driven at high frequency on the order of 1 GHz. The frequency response of the EOM is almost flat over the range of 10 KHz to 1 GHz, thus we have used slow detector to monitor the intensity change at the output, and the EOM is driven with a signal generator at 15 KHz. Parameters such as $V_\pi$ and extinguishing ratio of the EOMs can be directly
measured using this method. The results of the calibration are shown in Fig. 2.16. From
the fitting, $V_\pi$ is measured to be 1.74 V at 15 KHz for the phase EOM, and 1.31 V at 15
KHz for the amplitude EOM. Moreover, the extinguishing ratio of the amplitude EOM
is measured to be 1.9%.

2.5 Photon detection

One of the essential tool in measuring the quantum properties of light is photon count-
ing. In this section we give a simplified discussion on theoretical and practical issues in
detecting photons. There are many ways of detecting very low-level light fields down to
single photon, such as semiconductor based detectors, photomultipliers, transition edge
detectors, and superconducting nanowire detectors. One of the most popular ways is
silicon-based single photon detectors working in Geiger mode. Such detectors respond
to one or many photons with a macroscopic pulse, and cannot resolve the number of
photons in the incident pulses. A typical curve of the wavelength dependency of pho-
ton detection efficiency for Single Photon Counting Modules (SPCM) by Perkin Elmer
is shown in Fig. 2.17. At 780 nm, the detection efficiency of the SPCM is about 58%,
making it suitable for many quantum optics experiments involving photon counting.

2.5.1 Photon counting

In the 1960s, Roy Glauber’s series of work [58] on photon counting and statistics, quantum
coherence theory, and the successful explanation of the Hanbury-Brown-Twiss (HBT)
effect [23] gave birth to modern quantum optics. The detection of optical fields is done
with a square-law detector. Classically, the signal is proportional to the square of the
electric field amplitude,

$$P_{\text{classical}}(t) \propto |E(t)|^2.$$  \hspace{1cm} (2.30)
Figure 2.15: Calibration setup for the phase EOM (a) and the amplitude EOM (b). For the calibration of the phase EOM, a Mach-Zehnder interferometer is constructed with fiber beam-splitters with the phase EOM in one of the arms. The phase of the EOMs are driven with a signal generator and the intensity of the light is monitored at the output of the interferometer (for the phase EOM) or the amplitude EOM.
Figure 2.16: Calibration data for phase EOM (a) and amplitude EOM (b). The photodiode voltages are measured with oscilloscopes while the phase of the EOMs are driven with 15 kHz sinusoidal signals.
Such formalism does not apply to a quantum field, since even a vacuum field will have non-zero signal at the detector according to Eq. 2.30. Instead, one can construct the positive and negative field operators

\[ \hat{E}^+(t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} d\omega \hat{a}(\omega) e^{-i\omega t}, \] (2.31a)
\[ \hat{E}^-(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} d\omega \hat{a}^\dagger(\omega) e^{i\omega t}, \] (2.31b)

where \( \hat{a}^\dagger(\omega) (\hat{a}(\omega)) \) is the creation (annihilation) operator for mode \( \omega \).

The detector signal is then proportional to the expectation value

\[ P(t) \propto \left\langle \hat{E}^-(t) \hat{E}^+(t) \right\rangle. \] (2.32)

With such a construction, the vacuum field will give no photocurrent as expected, since the result of the annihilation operator acting on the vacuum \( |0\rangle \) is zero.

### 2.5.2 Practical issues

In practice, the most common type of single photon detectors consist of a semiconductor p-n junction layer working in Geiger mode. An incident photon is absorbed by the p-n junction and creates an electron-hole pair which triggers an avalanche current. A
macroscopic photocurrent can then be detected with conventional electronics. After the detection, an quench must be applied to the detector to stop the avalanche so the detector is ready for the next photon.

There are many issues we need to consider before using a single photon detector, such as the SPCM manufactured by Perkin Elmer. Next, we will list some important characteristics that will lead to deviation from the ideal photon detection.

First, let us consider the dead time, after pulse and saturation of the detector. After the detection of a photon and photocurrent pulse being generated, the detector is working in avalanche mode such that even if a second photon is incident, no additional photocurrent pulse will be registered until the quenching is applied. There is a minimum time between two photon detection events, which is usually referred to as the dead time. Furthermore, an electron-hole pair may be trapped and released after quenching, which generates a photocurrent pulse even though no actual photon is incident, this is a so-called "after pulse" of the detection. The necessity of quenching also leads to the saturation of detection. To avoid missing photon detection events, the rate of incident photon should be far below saturation.

The second important characteristic of photon detection is the time jitter associated with the detector. The time jitter describes the uncertainty in time associated with the detection event. This will limit the time resolution of the detector. We list these deviations in Table 2.2 [122]. These practical parameters will be discussed in the context of photon and coincidence detection in this thesis.

### 2.6 Quantum correlations and measurements

Since its very birth in the 1920s, the question of what distinguishes quantum mechanics from its classical counterpart has been at the centre of discussion and debate. Despite Einstein’s successful explanation of the photoelectric effect by quantizing the light field,
Table 2.2: Practical detector parameters [122]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Efficiency @780 nm</td>
<td>64</td>
<td>%</td>
</tr>
<tr>
<td>After-pulse probability</td>
<td>0.5</td>
<td>%</td>
</tr>
<tr>
<td>Dead time</td>
<td>50</td>
<td>ns</td>
</tr>
<tr>
<td>Saturation</td>
<td>15</td>
<td>MHz</td>
</tr>
<tr>
<td>Detector jitter*</td>
<td>450(50)</td>
<td>ps</td>
</tr>
</tbody>
</table>

*: Experimentally measured value.

the effect itself can actually be accounted for with semiclassical theory, in which the light field is treated classically and the matter is quantized with discrete levels. The question of whether the light field necessarily needs to be described by quantum mechanics was not completely resolved even for years after the success of Dirac’s quantization of the electromagnetic field and the development of quantum electrodynamics. We use the measurements of quantum correlations to characterize the narrowband source. In this section, we will introduce the fundamentals of quantum correlations and methods of measuring them experimentally using coincidence and correlation measurements.

2.6.1 Coincidence detection

In the 1960s, Roy Glauber [58] successfully developed the theory on coherence and photon statistics. When there are two or more detectors, we are interested in the correlations in these detection events, or coincidences. The correlation function in terms of the normal ordered field operator for two detectors can be written as

\[
P(t_1, t_2) \propto \left\langle \hat{E}_1^-(t_1) \hat{E}_2^-(t_2) \hat{E}_2^+(t_2) \hat{E}_1^+(t_1) \right\rangle. 
\]

(2.33)

where \( \hat{E}_i^{\pm}(t_j) \) is the field operator at detector \( i \) and time \( t_j \), and \( P(t_1, t_2) \) corresponds to the probability density of detecting two photons at time \( t_1 \) and \( t_2 \). Experimentally,
for a coincidence gate window of $T$, the coincidence rate is given by

$$R_{1,2} \propto \int_0^T dt_1 \int_0^T dt_2 P(t_1, t_2). \quad (2.34)$$

Next we consider different correlation times and their effect on the coincidence rate. As shown by Steinberg et al. [151], when the detector response time and coincidence window $T$ is both much larger than the coherence time of the state of the light field, the calculations can be greatly simplified. Substituting Eq. 2.31 and 2.33 into Eq. 2.34, and taking the limit $T \to \infty$, we have

$$R_{1,2} \propto \lim_{T \to \infty} \int_0^T dt_1 \int_0^T dt_2 \left\langle \hat{E}_1^{-}(t_1)\hat{E}_2^{-}(t_2)\hat{E}_2^{(+)}(t_2)\hat{E}_1^{(+)}(t_1) \right\rangle$$

$$= \lim_{T \to \infty} \frac{1}{(2\pi)^2} \int \int \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \int_0^T dt_1 dt_2 e^{it_1(\omega_1 - \omega_1')} e^{it_2(\omega_2 - \omega_2')} \left\langle \hat{a}_1^\dagger(\omega_1)\hat{a}_2^\dagger(\omega_2)\hat{a}_2(\omega_2')\hat{a}_1(\omega_1') \right\rangle$$

$$= \int d\omega_1 d\omega_2 \left\langle \hat{a}_1^\dagger(\omega_1)\hat{a}_2^\dagger(\omega_2)\hat{a}_2(\omega_2)\hat{a}_1(\omega_1) \right\rangle, \quad (2.35)$$

In the last step of the derivation in Eq. 2.35, we have used the equality

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_0^T dt_1 e^{it_1(\omega - \omega')} = \delta(\omega - \omega'), \quad (2.36)$$

where $\delta$ is the Dirac delta function.

If, however, the characteristic time of the light field is much longer than the coincidence gating window $T$ and detector response time, instead of the time integral in Eq. 2.34 the coincidence rate is directly proportional to the probability density (Eq. 2.33), $\tilde{R}_{1,2} \propto P(t_1, t_2)$. We refer to such a measurement as “time-resolved coincidence detection”. In such a measurement, we typically use a long detection measurement time, during which the arrival time of an individual photon is recorded. Compared to the classical photocurrent which is continuous, the photon arrivals can be discrete. The detection of the photon arrival time collapses the state of the photon to a certain time interval. We append our time-tagging electronics design in Appendix B.

Coincidences can also be a result of detection of two independent photons within a certain time interval. For independent photons with no time correlations, the two
detection events can be described by two random variables and the accidental rate can be written as:

\[ R_{acc} = R_1 \cdot R_2 \cdot \tau_c \] (2.37)

where \( R_{1,2} \) are the average count rates, and \( \tau_c \) is the coincidence window. Since the accidental coincidences are completely random, they only contribute a background level that can be accounted for later.

It is worth noting that from a coincidence measurement we can also infer the losses of the photons in propagation. Ideally, for a perfect pair generating source, the coincidence rate should be equal to the singles rate, and the collection efficiency

\[ \eta_{c,\text{ideal}} = \frac{R_{1,2}}{R_1} = \frac{R_{1,2}}{R_2} = 1. \] (2.38)

In practice, the losses and detector efficiency will reduce the collection efficiency, and we have

\[ \eta_{c,1} = \frac{R_{1,2}}{R_1}, \] (2.39)

\[ \eta_{c,2} = \frac{R_{1,2}}{R_2}, \] (2.40)

for two different collection channels.

### 2.6.2 Correlation measurements

The measurement of correlation lies at the heart of quantum optics. In fact, the birth of quantum optics stemmed from the debates over Hanbury Brown and Twiss’s observation [23] of intensity correlation without collecting the phase information between two separated photomultiplier tubes. A simplified schematic of the HBT experiment is given in Fig. 2.18.

Glauber [58] successfully gave quantum mechanical explanation of the HBT effect. The correlation of photocurrent in the HBT effect can be explained by the second order
correlation function (forth-order in field) which is defined as

$$g^{(2)}(\tau) = \frac{\langle \hat{E}_1^-(t) \hat{E}_2^-(t + \tau) \hat{E}_2^+(t + \tau) \hat{E}_1^+(t) \rangle}{\langle \hat{E}_1^-(t) \hat{E}_1^+(t) \rangle \cdot \langle \hat{E}_2^-(t) \hat{E}_2^+(t) \rangle}$$ \hspace{1cm} (2.41)

where $\hat{E}_{1,2}^{(\pm)}(t)$ are the electric field operator defined by Eq. 2.31.

When the time delay between the two detection events is zero, Eq. 2.41 reduces to

$$g^{(2)}(0) = \frac{\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \rangle}{\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \cdot \langle \hat{a}_2^\dagger \hat{a}_2 \rangle}.$$ \hspace{1cm} (2.42)

In comparison, the classical correlation function is defined as

$$g_{\text{classical}}^{(2)}(\tau) = \frac{\langle I_1(t + \tau) I_2(t) \rangle}{\langle I_1(t) \rangle \cdot \langle I_2(t) \rangle}.$$

Classically, this correlation Eq. 2.43 will be no less than 1, $g_{\text{classical}}^{(2)}(\tau) \geq 1$. which can be easily derived using the Cauchy-Schwarz inequality. Quantum mechanically, however, $g^{(2)}(\tau)$ can be less than 1, and the classical Cauchy-Schwarz inequality is violated. For this reason, if some measurement reveals that $g^{(2)}(\tau) < 1$, we say the system is quantum, since it cannot be explained by any classical theory.

We consider $g^{(2)}(0)$ as defined in Eq. 2.42 for two cases: a field with a thermal distribution and a single-photon field. For simplicity, we restrict the field to a single mode problem, but it is not difficult to extend the analysis to multi-mode case. The
density matrix of a thermal state is given by [16]:

$$\rho_{\text{thermal}} = \sum_n \frac{n_0^n}{(1 + n_0)^{n+1}} |n\rangle \langle n|,$$

where $n_0$ is the average photon number. With some algebra, we have $g^{(2)}(0) = 2$ for thermally distributed light field.

For a single photon field, $|\Psi\rangle = |1\rangle$, and the numerator in Eq. 2.42 is zero due to the fact that the operator is second order in the annihilation operator, which means $g_0^{(2)} = 0$ for the single photon field. This has been demonstrated many times and serves as the main criteria for a single photon source [61, 17, 74]. Note, however, $g^{(2)}(0) = 0$ is not sufficient to imply a purely single-photon state. In fact, states like $|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$, which is pure, or states described by $\rho = |\alpha|^2 |0\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|$, where $|\alpha|^2 + |\beta|^2 = 1$, will have $g^{(2)}(0) = 0$. In this sense, even a complete mixture of the states $|0\rangle$ and $|1\rangle$ is non-classical.

### 2.6.3 Entanglement and non-classicality

One of the most astounding results of quantum mechanics is that it predicts stronger correlations than any classical theory would allow. In the previous section, we have shown that the single-photon state, as an example of a quantum state, can violate the classical Cauchy-Schwarz inequality. A perhaps more striking example of non-classicality - entanglement - is shown to have experimentally measurable deviation from classical theory by John Bell in 1963 [11]. Based on Bell’s work, Clauser et.al. [34] proposed an experimental measurable inequality based on optical photons to provide a decisive test between quantum mechanics and any local hidden-variable theorem. Freedman et.al. later implemented the proposal, and together with Aspect et.al.[53, 7] provided experimental support for quantum mechanics and by disproving local hidden variable theories. Based on these works, entanglement can now be defined as the excessive correlation between separated parts of a quantum system that exceeds the maximum amount of correlation.
allowed by classical theory.

There many classes of non-classical quantum states. For two qubits, the entangled states that have maximal violation of Bell’s inequality are called Bell states, and in the computational basis they are \( \{ \phi_\pm, \psi_\pm \} = \{ (|00\rangle \pm |11\rangle) \sqrt{2}, (|01\rangle \pm |10\rangle) \sqrt{2} \} \). Another family of quantum states in quantum information are so called \(|N, 0\rangle + |0, N\rangle \) (N00N) states [89], with promising applications in quantum metrology and optical lithography [159, 114]. These states have the form \( \Psi_{\text{N00N}} = \frac{1}{\sqrt{2}} (|N, 0\rangle_{1,2} + |0, N\rangle_{1,2}) \) for mode 1 and 2, hence the name. The most remarkable property of such states can be seen with an interferometer that uses mode 1 and 2 in its two arms. Assume a phase shift of \( \phi \) between the two arms of the interferometer. At one of the output ports of the interferometer, if one measures \( A_N = |N, 0\rangle \langle 0, N| + |0, N\rangle \langle N, 0| \), the result is \( \langle A_N \rangle = \cos(N\phi) \), which has acquired \( N \) times the phase shift of single photons or coherent beams in the same interferometer.

### 2.6.4 Quantum state tomography

To completely characterize a quantum system, one can perform quantum state tomography to reconstruct its density matrix. Note that the dimension of the density matrix, or the Hilbert space dimension, scales exponentially as the number of particles in the system. The standard method to measure a two-level quantum system was developed and demonstrated first by James et.al. [77]. Following the same treatment, we can extend the technique to higher-dimensional systems.

We will show in the following a generic way to reconstruct the density matrix for high dimensional systems. Consider a case where we encode the quantum information in a n-dimensional system, the \( n \times n \) density matrix can then be written as

\[
\hat{\rho} = \sum_{i=0}^{n^2-1} r_i \hat{\lambda}_i
\]  

(2.45)

where \( \hat{\lambda}_0 \) is the identity matrix and together with \( \hat{\lambda}_{i=1,2,...,n^2-1} \) form a complete set of
matrices (e.g. the generators of the SU(N) group), and \( r_i \) is the coefficient of the decomposition. The measurement projectors are defined by the specific phase profiles written on the EOM. To completely determine the \( n \times n \) density matrix, we need \( n^2 \) linearly independent measurements. The measurement result is then \( n_\nu = Tr\{\Psi_\nu \hat{\rho}\} = \langle \Psi_\nu | \hat{\rho} | \Psi_\nu \rangle \), where \( \Psi_\nu \) is the \( \nu \)-th element of \( n^2 \) linearly independent projectors. Substituting the expression 2.45 for \( \hat{\rho} \), we have

\[
 n_\nu = Tr\{\Psi_\nu \hat{\rho}\} = \langle \Psi_\nu | \sum_{\mu=0}^{n^2-1} r_\mu \hat{\lambda}_\mu | \Psi_\nu \rangle = \sum_{\mu=0}^{n^2-1} r_\mu B_{\nu,\mu} \tag{2.46}
\]

where \( B_{\nu,\mu} = \langle \Psi_\nu | \hat{\lambda}_\mu | \Psi_\nu \rangle \) and can be calculated beforehand. Given that \( B_{\nu,\mu} \) is non-singular. We can then find the coefficients of the density matrix decomposition:

\[
 r_\nu = \sum_{\mu=0}^{n^2-1} n_\mu (B_{\nu,\mu})^{-1} \tag{2.47}
\]

In experiments, as long as we have a set of linearly independent measurements \( n_\nu \), we can reconstruct the quantum state using the method given above. However, this method does not guarantee a physical density matrix due to inevitable noise in the measurements. To solve this problem, we can use the maximum likelihood method to find a density matrix that fits the data best, while constraining the optimization process to physical density matrices only \[77\].
Chapter 3

The quantum light source

Light is the ideal information carrier due to its large available bandwidth and excellent coherence properties, and thanks to the invention of the low-loss waveguides for light, optical fibres, light has been used extensively for communication purposes. In quantum information and quantum communication, photons, the quanta of light, can be used to encode information as well using different degrees of freedom, such as polarization, temporal mode, spatial mode, frequency, orbital angular momentum etc. For quantum information processing purposes, though, photons are not well suited since the inter-particle interaction is too weak, even in ‘strong’ nonlinear media. The best nonlinear medium known in nature are the cold atoms [69] with narrow resonances where the nonlinearity is greatly enhanced. It is the quest of this project to build a quantum light source that is compatible with such narrowband atomic transitions, and use it as a light-matter interface in quantum information processing.

3.1 Classical light source

Similar to most other quantum optics experiments, we need a well controlled classical laser system to build the quantum light source. The laser system used in the experiment is as shown in Fig. 3.1. The main component of the system is a commercial higher
power laser system (Toptica TA-SHG-110), which consists of an External Cavity Diode Laser (ECDL), a tapered amplifier and a Second Harmonic Generation (SHG) stage. The ECDL is frequency-stabilized with Doppler-free saturated absorption spectroscopy with a Rubidium cell, which guarantees the blue laser from the SHG is twice the transition frequency of a certain Rubidium line around 780 nm. After line-narrowing electronics, we have a laser line width of 0.4 MHz for the 780 nm laser. The tapered amplifier outputs 450 mW of red light that is coupled into the SHG cavity. The SHG cavity outputs a maximum of 120 mW of frequency stabilized 390 nm UV light. Part of the fundamental light before the tapered amplifier (~10 mW) is also fiber coupled serving as the locking laser for different cavities, to make sure those cavities are also stabilized to the same atomic transition.

### 3.2 Spontaneous parametric down-conversion

SPDC, the work horse of the field of optical quantum information, is a process where a high frequency pump photon splits into two lower frequency photons while conserving
energy and momentum. The process happens extremely rare, roughly on the order of $10^{-10}$ down-converted pairs per pump photon. In addition, the resultant down-converted photons are typically broadband, on the order of 100 GHz - 1 THz, much wider than the typical width of atomic transitions, which is on the order of 10 MHz. With the recent development of extremely high brightness sources [83, 47], one may use direct passive filtering to get the linewidth down by 6 to 7 orders of magnitude at the cost of attenuation in brightness of the same orders of magnitude. Indeed, it has been shown [65] that SPDC with direct passive filtering can generate single photons at atomic resonances, despite the extremely high loss. This is not an ideal option to have atomic resonant quantum light source, and we wish to use a process that can greatly enhance the spectral brightness. To do this, we will briefly review the process of SPDC.

### 3.2.1 Phase-matching conditions

SPDC is the process of splitting of a high energy pump photon into two lower energy photons, historically called signal and idler, during which the energy and momentum have to be conserved, as shown in Fig. 3.2. The process is parametric in the sense that the state of nonlinear material is not altered before and after the process. The condition of energy and momentum conservation is also referred to as the phase-matching condition,

\[
\omega_p = \omega_s + \omega_i, \quad (3.1a)
\]
\[
\vec{k}_p = \vec{k}_s + \vec{k}_i. \quad (3.1b)
\]

The Hamiltonian that describes the SPDC process is given by

\[
H = g^* \hat{a}_p^\dagger \hat{a}_s \hat{a}_i + g \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger \quad (3.2)
\]

where $g$ is the coupling constant describing the nonlinearity of the medium. We have assumed a two-mode output of the SPDC process. In many cases, the pump is strong
Figure 3.2: The process of SPDC. a) schematic of a high-frequency pump photon splitting into two lower energy photons. b) energy conservation. c) momentum conservation.

and can be treated classically, and the two-mode Hamiltonian can be written as

\[ H = g^\prime \hat{a}_s \hat{a}_i + g^\prime \hat{a}_s^\dagger \hat{a}_i^\dagger \]  \hspace{1cm} (3.3)

where we have absorbed the classical pump field amplitude into the c-number \( g^\prime \).

Such process happens in a non-centrosymmetric medium which has a non-vanishing \( \chi^{(2)} \). There are many nonlinear crystals for \( \chi^{(2)} \) processes that one can choose from. Typically, the nonlinear coefficients are quite small, ranging from \( 1 - 10^3 \) \( \text{pm/V} \). In most cases, one cannot simply choose the crystal with the largest nonlinear coefficients, since the phase-matching conditions usually poses tight limits on the range of wavelengths that can be phase-matched. In the next section, we will discuss a technique that allows us to extend the range of phase-matching.

### 3.2.2 Quasi-phase matching

The phase matching condition Eq. 3.1 poses a limit on the wavelength range that can be phase-matched for a given crystal. For example, by angle and temperature tuning, the phase-matching condition can be reached for the wavelength range from 440 nm to 3550 nm...
nm \[82\]. Also, it is often the case that the largest non-linear coefficient of the nonlinear medium cannot be used in order to satisfy the phase-matching condition. It is desirable to engineer the crystal in a way that the phase matching condition can be extended to a broader wavelength range, as well as utilizing the largest nonlinear coefficient available of the nonlinear crystal.

Quasi-phase-matching is a technique frequently used to extend the phase-matching range. As shown in Fig. 3.3a, the domain in the nonlinear crystal is periodically reversed using strong electric field. By applying periodic poling to the non-linear crystal, quasi-phase-matching induces an extra wavevector from the effective crystal momentum to the momentum conservation condition Eq. 3.1b. The momentum conservation now becomes

$$\vec{k}_p = \vec{k}_s + \vec{k}_i + \frac{2\pi}{\Lambda},$$

(3.4)

where $\Lambda$ is the poling period of the crystal.

To determine the poling period of Potassium Titanium Oxide Phosphate (KTP) crystal such that it can be phase matched for the desired down-conversion process, $\gamma_{\text{pump}}^{2\omega} \rightarrow \gamma_{\text{signal}}^{\omega} + \gamma_{\text{idler}}^{\omega}$, we re-write Eq. 3.7, and substitute with $k = \frac{2\pi}{\lambda}$, we have

$$\Lambda = \frac{\lambda}{2n_p(\lambda) - n_s(\lambda) - n_i(\lambda)},$$

(3.5)

where we have assumed degenerate down-conversion $\lambda_s = \lambda_i = \lambda_p/2 = \lambda$, and $n_p$, $n_s$ and $n_i$ are the refractive index for the pump, signal and idler, respectively.

The refractive index of KTP crystal is given by the Sellmeier equation\[82\]

$$n_x = \sqrt{3.29100 + \frac{0.04140}{\lambda^2 - 0.03978} + \frac{9.35522}{\lambda^2 - 31.45571}}$$

(3.6a)

$$n_y = \sqrt{3.45018 + \frac{0.04341}{\lambda^2 - 0.04597} + \frac{16.98825}{\lambda^2 - 39.43799}}$$

(3.6b)

$$n_z = \sqrt{4.59423 + \frac{0.06206}{\lambda^2 - 0.04763} + \frac{110.80672}{\lambda^2 - 86.12171}}$$

(3.6c)

where $x$, $y$ and $z$ are the axes of the refractive index ellipsoid. Eq. 3.6 is plotted in Fig. 3.4, where a horizontal dashed line is drawn to guide the eye for the refractive index of 780...
Figure 3.3: Quasi-phase-matching and periodic poling. (a) Periodic poling. (b) The growth of down-converted field amplitude as a function of crystal length for different phase-matching conditions: phase mismatched, 1st-order quasi-phase-matching, 3rd-order quasi-phase-matching, and perfect phase-matching.
Figure 3.4: Wavelength dependence of refractive index of KTP crystals. Dashed line represents the desired phase-matching configuration at 390 nm - 780 nm. The poling period can then be calculated using Eq. 3.5.
Figure 3.5: Calculation of period of poling of KTP crystal (1st order) using the Sellmeier equation. The horizontal line shows the 1st order poling period for phase matching at 780 nm.

nm light polarized along the z-direction. One appropriate configuration to be quasi-phase-matched is to have a z-polarized pump photon \( \gamma_{p,z}^{2\omega} \) to split into an y-polarized signal photon and a z-polarized idler photon, \( \gamma_{p,z}^{2\omega} \rightarrow \gamma_{s,y}^{\omega} + \gamma_{i,z}^{\omega} \). We calculate the poling period of the KTP crystal for this configuration, and the result is shown in Fig. 3.5. To phase-match at \( \omega = 780 \) nm (fundamental), the poling period is calculated to be 2.23 \( \mu m \), as indicated by the horizontal green line in the figure. Due to practical limitations in manufacturing, the calculated poling period may be too small to have a good quality poling. In that case, multiples of this poling period can be used, which corresponds to a higher-order quasi-phase-matching. For quasi-phase-matching of Periodically Poled Potassium Titanium Oxide Phosphate (PPKTP) crystal, one usually uses 3rd-order or 5th-order poling.

The phase matching bandwidth can also be determined for a certain poling period.
We first write down the phase mismatch as

$$\Delta k = \vec{k}_p - \vec{k}_s - \vec{k}_i - \frac{\pi}{\Lambda}. \quad (3.7)$$

By setting $\Delta k L = \pi$, where $L$ is the length of the crystal, the down-conversion will go completely out of phase. The offset of the frequency to the perfect quasi-phase-matching frequency $f_0$ ($\Delta k(f_0) = 0$) that satisfies $\Delta k(f_0 + \Omega) = \pi/L$ is called the phase matching bandwidth $\Omega$, which can be calculated

$$\Omega = \frac{\Delta k \cdot c}{n_p(\lambda/2)} \quad (3.8)$$

For our case, the bandwidth is calculated to be 128 GHz for degenerate down-conversion with a 20 mm long PPKTP crystal at 780 nm.

### 3.3 Optical parametric oscillators

As shown in Eq. 3.8, the phase-matching-bandwidth is inversely proportional to the length of the crystal. Due to the long PPKTP crystal used, the phase-matching bandwidth is much smaller than other experiments with popular Barium Borate (BBO) crystal (see, for example, [94, 95]). However, this is still 5 orders of magnitude higher than the desired atomic transition. Passive filtering with an narrow-band filter seems at first a plausible solution, but the amount of loss it involves makes such filtering impractical where the interested bandwidth of down-converted photons are orders of magnitude narrower than the phase matching bandwidth.

OPO provide an elegant solution [125, 92, 91]. By placing the nonlinear crystal inside a cavity, the spectrum of down-conversion is actively modified, such that only those resonant with the cavity can be generated, and all other modes of down-conversion experience destructive interference. Due to the multiple round-trips the photons take before exiting the cavity, the effective length of down-conversion crystal is increased by the finesse of the cavity, and the spectral brightness is thus greatly enhanced. By
designing the cavity linewidth to match the atomic transition linewidth, one can generate photon pairs that are resonant with atoms.

Since the OPO has active gain, with enough pump power the oscillator can output coherent light where individual photon statistics becomes classical. In order to keep the down-conversion at the regime we are interested in, that is, the photon statistics is still quantum, we need to operate the OPO far below threshold such that the stimulated process can be safely omitted.

3.3.1 Model

In the regime of SPDC, the weakness of non-linearity allows the use of perturbation theory to solve the problem. The detection is done with photon counting techniques as shown in Section 2.5, where higher order terms involving more than a pair of photon is typically ignored. In another regime, where the cavity is operated above its threshold, an OPO is formed to greatly enhance the non-linearity, and the perturbation theory does not apply. Instead, one can use classical theory to describe the coherent output of an OPO [40]. Just under threshold, the OPO will generate squeezed state of light as first shown by Yurke [169] and Collett [37]. While operating above but near threshold, a twin-beam will be generated with quantum correlation [72]. Lu and Ou [102] showed that when operating far below threshold, the stimulated process of the OPO can be neglected while the non-linearity is enhanced, and one should still observe photon statistics of the spontaneous process.

As shown by Lu and Ou [102], the output operator of a degenerate OPO on resonance is related the input as

\[
\hat{a}_{\text{out}}(\omega_0 + \omega) = G_1(\omega)\hat{a}_{\text{in}}(\omega_0 + \omega) + g_1(\omega)\hat{a}_{\text{in}}^\dagger(\omega_0 - \omega) + G_2(\omega)\hat{b}_{\text{in}}(\omega_0 + \omega) + g_2(\omega)\hat{b}_{\text{in}}^\dagger(\omega_0 - \omega),
\]

(3.9)
where

\[
\begin{align*}
G_1(\omega) &= \frac{\gamma_1 - \gamma_2 + 2i\omega}{\gamma_1 + \gamma_2 - 2i\omega}, \\
g_1(\omega) &= \frac{4\epsilon\gamma_1}{(\gamma_1 + \gamma_2 - 2i\omega)^2}, \\
G_2(\omega) &= \frac{2\sqrt{\gamma_1\gamma_2}}{\gamma_1 + \gamma_2 - 2i\omega}, \\
g_2(\omega) &= \frac{4\epsilon\sqrt{\gamma_1\gamma_2}}{(\gamma_1 + \gamma_2 - 2i\omega)^2},
\end{align*}
\]

(3.10)

where \(\epsilon\) is the single-pass parametric gain amplitude, \(\hat{b}_{in}\) represents the vacuum mode that mixes into the cavity, and \(\gamma_1, \gamma_2\) is the decay constant for mode \(\hat{a}_{in}\) and \(\hat{b}_{in}\), which is due to the leakage through the two cavity mirrors.

### 3.3.2 Cavity enhancement

We can calculate the cavity enhancement from Eq. 3.9 as

\[
R_{opo} = \langle E_{out}^{-}(t)E_{out}^{+}(t)\rangle = \frac{|r|^2 F \gamma_2}{\pi \Delta t_{rt}}
\]

(3.11)

where \(r = \epsilon \Delta t_{rt}\) is the single pass gain parameter and \(\Delta t_{rt}\) is the round-trip time, \(F = \frac{2\pi}{\Delta t(\gamma_1 + \gamma_2)}\) is the finesse of the cavity.

Compare to the single-pass rate

\[
R_{sp} = |r|^2 \Omega_{BW}/2\pi
\]

(3.12)

The average enhancement factor per mode is

\[
M = \frac{R_{opo}/\delta \nu_{opo}}{R_{sp}/\Omega_{BW}} = F^2 \gamma_2/\pi.
\]

(3.13)

Thus the brightness will increase as the square of the finesse of the cavity. This can be understood as follows: the finesse of the cavity roughly corresponds to the average number of round trips the photons take in the cavity. When the fields amplitudes from all round-trips add up constructively, the overall field will increase linearly as the number of round-trips, and the rate of down-conversion should increase quadratically.
Figure 3.6: Spectrum of a cavity-enhanced PDC.
The structure of the cavity modes resembles that of a frequency comb, as shown in Fig. 3.6, and each comb tooth is one cavity mode with the separation being the free spectral range. Such a structure may enable the implementation of one-way computation model as proposed by Menicucci et. al. [111]. The envelope of the cavity modes is a sinc function determined by the phase-matching bandwidth of the crystal used. We assume the cavity is in double resonance with a perfect compensation of the birefringence such that the horizontal and vertical polarizations have the same resonances. Due to interference, all but the cavity modes in the phase matching band are suppressed. When the pump frequency is twice the frequency of the center of the phase-matching, the pairs can emit into any two modes that satisfy the energy conservation condition as shown in Eq. 3.1a. The simulation of the coincidence counts as a function of the delay between the arrival time of the photon pair is shown in Fig. 3.7. The finite detector response time of 500 ps is used in the simulation. We have chosen the cavity free spectral range to be 500 MHz, the finesse to be 70, and the phase matching bandwidth to be 150 GHz. This gives the cavity linewidth of 7.1 MHz. From the simulation, the coincidence has an
overall exponential decaying envelope, with a decay time constant of 22 ns, determined by the cavity linewidth. Under this envelope, there are many narrow peaks with a width of 6.7 ps, determined by the phase-matching bandwidth. Limited by the detector speed, the width of the peak will be broadened to the detector jitter of 500 ps. These peaks correspond to individual cavity round-trips, and the separations between the peaks are 2 ns, which is just the reciprocal of the free spectral range of the cavity.

### 3.4 Cavity design

#### 3.4.1 Compensation schemes

In practical implementations of PDC, there are many factors that will affect the quality of the output state, such as focus parameters of the pump, transversal and longitudinal walk-off due to birefringence or dispersion, collection angle, bandwidth and filter issues, to name a few. Early developments of Type I and Type II sources have already considered some of these issues [94, 95]. Since then, these imperfections are studied and corrected more extensively and systematically, and large improvements have been made in terms of the quality and brightness of the sources [5, 92, 83, 47, 163].

In the case of PDC with PPKTP, where the typical geometry is collinear Type II phase-matching, and the propagation direction is perpendicular to the optical axis, many practical issues are eliminated by the design such as the pump-down-conversion walk-off, and necessity of angle-tuning which is difficult to perform in a cavity. However, other factors such as longitudinal walk-off can be more pronounced and they need to be dealt with properly. As shown previously in Section 3.2.2, the phase matching of PPKTP is achieved by adding an extra momentum term proportional to the reciprocal poling period. In the real space, the signal and idler of down-conversion will still spatially walk off in the propagation direction due to their different polarizations. This is the source of the longitudinal walk-off. Note that the walk-off between the pump and down-converted
photons is less an issue here because of the narrow bandwidth typically used for the pump.

We can calculate the longitudinal walk-off and determine the characteristic length of crystal where the signal and idler become completely separated. The refractive index of PPKTP at 780 nm is \( n_H = 1.7578 \) and \( n_V = 1.8464 \) for H and V polarizations. Due to this birefringence, the two photons from the same down-conversion pair will experience different group velocities. Neglecting the group velocity dispersion for the narrow bandwidth of interest, the group delay per unit length of crystal between the two polarizations is calculated to be 0.295 ps/m. For a crystal length of 20 mm, the delay is calculated to be 5.9 fs. With the cavity, such delay is increased due to the multiple round-trips. For a typical finesse of 70, the delay is calculated to be 0.41 ps. The distinguishing information introduced by this delay is much smaller than the coherence time of the down-conversion. In other words, it maybe possible to such delay without the compensation in the cavity, as long as one can achieve double resonance and phase matching of PDC at the same time. In our experiment, we choose to implement the compensation to tune the double resonance and to eliminate any distinguishing information caused by the delay.

One possible compensation scheme is shown in Fig. 3.8. The amount of delay is proportional to the length of propagation inside the birefringent material. We write the (post-selected) state of the photon pair as

\[
|\Psi\rangle_{\text{pair}} \approx \int_0^Ldl \int d\omega_1 d\omega_2 f(\omega_1, \omega_2) \hat{a}_1^\dagger e^{i\omega_1 t + ik_1(\omega_1)l} \hat{a}_2^\dagger e^{i\omega_2 t + ik_2(\omega_2)l} \delta(\omega_p - \omega_1 - \omega_2) |0\rangle, \tag{3.14}
\]

where \( f(\omega_1, \omega_2) \) is the joint spectral function of the two-photon state, \( \delta \) is the Kronecker delta function that guarantees the energy conservation, \( L \) is the length of the crystal. The group delay can then be calculated from the frequency dependence of the wave number \( k_i(\omega)(i = \{1, 2\}) \) for two different polarizations. We can expand \( k_i(\omega) \) in a Taylor series and keep the lowest 3 orders, which means the group delay is linearly dependent on the propagation length in the crystal. By measuring the arrival time of the photons, we can then infer the location where the photon pair is generated. In this case, the time
correlation measurement projects the two-photon state to the product state

$$|\Psi\rangle_{\text{pair},t} \approx \int_0^L dl \left| H_t \right| \left| V_{t + \Delta t_g(l)} \right>,$$  \hspace{1cm} (3.15)

where $|H_t\rangle$ ($|V_t\rangle$) is the H-polarized (V-polarized) single-photon state at time $t$, and $\Delta t_g(l) = l/v_{gV} - l/v_{gH}$ is the time delay due to difference in group velocity for two polarizations. The compensation can be accomplished by using a KTP crystal that is half the length of the PPKTP, with its domain-axis rotated by 90° with respect to the PPKTP crystal. The KTP crystal should have the same crystal cut as the PPKTP crystal for a complete compensation. Without periodic poling, the KTP crystal is not phase matched thus will not provide nonlinear gain to the process. The compensation effectively creates a symmetry point at the middle of the crystal, as shown in Fig. 3.8. After the compensation, the arrival time of the two photons can no longer reveal the position inside the crystal where the pair is generated. Such fundamental indistinguishability gives rise to entanglement and the two-photon state after compensation can be written as the entangled state

$$|\Psi\rangle_{\text{pair},t} \approx \frac{1}{\sqrt{2}} \int_0^L dl \left| H_t \right| \left| V_{t + \Delta t_g(l)/2} \right> + \left| V_t \right| \left| H_{t + \Delta t_g(l)/2} \right>.$$  \hspace{1cm} (3.16)

Based on the compensation example given in the previous section, the extension of compensation inside a cavity is straightforward. For a linear cavity, the light propagates both forward and backward. With two KTP crystals at half length located at both sides of the PPKTP crystal, the round-trip group delay of the two polarizations is completely compensated, as shown in Fig. 3.9a. However, if the length of the compensation crystal deviates from half of the PPKTP crystal, the group delay accumulates from each round trip and the double resonance condition will not be satisfied. In practice, exact half length crystal is challenging to manufacture. A solution to this problem is to use one compensation crystal with the same length of the PPKTP crystal, and the two crystals can be cut and polished at the same time to guarantee the same length. The delay between each round-trip can be compensated, though there will be residual group delay to
Figure 3.8: Compensation of group delay caused by birefringence in PDC. (a) Scheme of group velocity compensation with a half-length crystal whose axis is rotated by 90°. The photon pairs with different polarizations (green and red dot) experiences a propagation length dependent delay due to birefringence. With the compensation crystal, the photon pairs generated at the beginning of the crystal have the same delay as the pairs generated at the end of the crystal. (b) The time delay between the two photons as a function of the position of pair generation inside the crystal. With such compensation, for any given time delay there are two possible locations in the crystal that the pair could have been generated.
Figure 3.9: Different schemes of compensation in a cavity. (a) Compensation with two half-length KTP crystal inside the cavity. (b) Compensation with one full-length KTP crystal inside the cavity, and a half-length crystal outside of the cavity.

provide distinguishing information of the position the pair is generated. Thus, in addition to the intra-cavity compensation, we can use another half-length crystal to compensate for the remaining group delay, and such distinguishing information of photon pairs is eliminated. The requirement of length precision is greatly reduced since the photons only pass through the crystal once. This compensation scheme is shown in Fig. 3.9b. We implement this compensation scheme in the experiment, with slight modifications to the cavity geometry, for reasons given in the next section. With proper compensation of the birefringence walk-off, and taking into account many round-trips the photons can take before exiting the cavity, the full two-photon state can be written as

\[ |\Psi\rangle_{cav,t} \approx \int_{0}^{t_g/2} dt' \sum_{m} g(m) \left( |H\rangle_t |V\rangle_{t+m\tau+\nu} + |V\rangle_t |H\rangle_{t+m\tau+\nu} \right), \]  

where the time delay \( t_g/2 \) is due to a half-length crystal \( L/2 \), \( \tau \) is the time the photons
take to complete a full round-trip, \( m \) is the difference in the number of round-trips the two photons take, and \( g(m) \) describes the probability distribution for \( m \). With a Lorentzian-lineshape cavity, \( g(m) \) is a double sided exponential decay with its peak at \( m = 0 \). Moreover, the width of \( g(m) \) is associated with the cavity parameter finesse \( F \), which corresponding to the average number of round trips the photon take before exiting the cavity.

### 3.4.2 Design considerations

After we choose the compensation scheme, we need to consider different cavity geometries and pick a design that is best for the purpose of cavity enhancement. There are many geometries available as discussed in Chapter 2. The example we gave above for the compensation is with a standing-wave cavity where the opposite propagation directions are spatially overlapping inside and outside of the cavity. Another type of cavity is traveling wave cavity, such as triangle cavity or bow-tie cavity. With traveling wave cavity, one could distinguish two beams in opposite directions. This is advantageous for cavity-enhanced down-conversion where we need independent access to classical locking beam and down-converted photons. In addition, traveling wave cavity may provide higher extraction efficiency with the same finesse compared to the standing wave cavity. This is because for the same number of round trips, the light in the traveling wave cavity goes through half of the number of crystal surfaces, and thus less loss of photons due to scattering. Both triangle and bow-tie cavities are traveling wave cavities, which one is more suitable for our below-threshold OPO? Considering the geometry and mode structure, bow-tie cavity provides less distortion from the Laguerre-Gaussian modes, since the angle of incidence to the mirrors can be made much smaller than that of triangle cavity [120, 13], which helps to improve coupling efficiency with less mode correction and matching optics. Thus we use bow-tie cavity as the geometry for our cavity-enhanced PDC.
The birefringence of PPKTP and KTP crystal is sensitive to temperature changes thus we can tune the phase-matching condition of the PPKTP crystal by temperature. To find the phase-matching temperature, we first perform the single-pass SHG with the PPKTP crystal alone. The crystal is pumped with 20 mW of 780 nm infrared light with the same focusing parameters of the fundamental cavity mode. The temperature dependence of the second harmonic at 390 nm is shown in Fig. 3.10. For this PPKTP crystal, the phase matching is found to be around 42°C. The PPKTP and compensating KTP crystals are placed in two ovens with temperature sensors and Thermoelectric Cooling (TEC) elements. A home-made temperature controller based on the Wavelength Electronics HTC-3000 chip is used to control temperature fluctuations to be within $5 \times 10^{-3}$°C. After finding the phase-matching temperature of the PPKTP crystal, the cavity is then assembled and aligned using techniques discussed in Chapter 2. To tune the double resonance of the cavity, the temperature of the KTP crystal is varied until the group delay between two different polarizations is completely compensated and the two polarization resonance peaks overlap.

The OPO cavity is double-resonance with the down-converted light, but operates with single pass of the 390 nm pump from the laser system (Toptica TA-SHG-110). There are several ways to mode match the pump to the cavity. The most straight-forward way of alignment is through the alignment of SHG with PPKTP crystal using the backwards injection of 780 nm to the cavity. When optimized for the coupling of the SHG from the cavity to the optical fiber for the pump, the mode of the pump is also well matched to the cavity. An equivalent mode matching technique is through the interference of the pump and the SHG from the PPKTP crystal. When the visibility of the interference is maximized, the pump mode is also well matched to the cavity.

To achieve maximum down-conversion efficiency, one can follow the optimal focusing parameters given by Boyd and Kleinman [21], $L/b = 2.84$, where $L$ is the crystal length and $b$ is the confocal parameter of the Gaussian beam. For a crystal length of 20 mm,
Figure 3.10: Temperature tuning of the PPKTP phase-matching condition. The PPKTP is injected with 20 mW of infrared light at 780 nm, and the second harmonic at 390 nm is measured at the photodiode when the temperature of the PPKTP crystal is tuned by a TEC elements. The x-axis is the voltage readouts of a thermistor as a temperature monitor. The width of the sinc corresponds to a temperature change of roughly 1 degree.
this corresponds to a beam waist of 29.6 $\mu m$ for the pump. One could also choose a larger focus in order to reduce the thermal lensing [152] and gray-tracking [19] effect in PPKTP crystal at the cost of a lower efficiency of down-conversion. Since the OPO cavity is typically pumped with very low power, on the order of 1 mW, the effects of thermal lensing and gray-tracking can be neglected and we have chosen a pump focus of 30 $\mu m$.

Since the photon pairs are spectrally overlapping with the classical beam used to lock the cavities, we need to take extra care to protect the single photon detectors from any exposure to classical laser beam. An optical chopper can be used to enable/disable a locking phase and a collection phase. The duty cycle of the chopper should be less than 50% to ensure no overlap of the locking phase and collection phase. The design of the chopper is shown in Fig. 3.11, where the beams are located in positions such that the ‘beam logic’ is alternated, that is, the collection port is never open when the locking beam is on.

In Fig. 3.12, we have shown the scope trace showing the periodic locking with an optical chopper. The signal is the photodiode voltage corresponding to the power of the locking beam. The bandwidth of the PID controller determines the range of fluctuation we can correct for with the electronics and is set to be 10 kHz which covers most of the acoustic vibrations from the optical mounts and is much lower than the resonance frequency ($\sim 600$ kHz) of the peizoelectric elements (PI-PL033.30). With the optimized parameters of PID circuits, the lock is established within a time that is fast compared to the locking period, typically equal or less than 5 mS (200 Hz). The chopper is also mounted on a vibration absorbing stage to give minimal disturbance to the locking of the cavity.

We have already discussed some practical issues associated with photon and correlation detection in Section 2.5.2. Specifically, the accidental rate will now be different from the formula given in Eq. 2.37. With the chopper and gating, there are two different sources for accidentals - one from the dark count of the detector, and the other the
Figure 3.11: Beam positions on the chopper. The locking beam and collection beam are alternated by the chopper to realize the ‘beam logic’, that is, the collection port is never open when the locking beam is on.

Figure 3.12: Periodic locking of the cavity.
unpaired down conversion photons.

\[ R_{\text{Acc}} = \eta_{\text{gate}} R_{1,\text{dark}} R_{2,\text{dark}} \tau_{\text{coin}} + \eta_{\text{chopper}} R_{1}' R_{2}' \tau_{\text{coin}} \] (3.18)

where \( R_{i,\text{dark}} \) is the dark counts at detector \( i \), \( R_{1}', R_{2}' \) are the singles rate with the background dark counts subtracted, \( \eta_{\text{gate}} \) is the gating duty cycle which we set to 50\%, and \( \eta_{\text{chopper}} \) is the chopper duty cycle which is 25\% theoretically and is 24\% taking into the finite beam size relative to the opening of the chopper. Note that the accidental counts have different scaling than the actual photon pairs as a function of pump power.

### 3.5 Output characterization

In this section, we will characterize output of the cavity-enhanced PDC. The time correlation of the photon pairs is measured showing interesting characteristics of such a source. With additional filtering, we have shown a source of narrow-band photons that is compatible with rubidium D2 transitions. We also measure the second-order correlation of the output from the source, and show that we can measure different statistics from the same source with different setups.

#### 3.5.1 Time-resolved coincidence counting

The time-resolved coincidence data as a function of the difference between the arrival time of the photon pairs is shown in Fig. 3.13. Each data point is coincidence counts accumulated for 10 minutes at a pump power of 0.2 mW. The size of the bin is set to 100 ps, which is chosen to be smaller than the detector jitter of 450(50) ps such that the time resolution is detector-limited. The overall envelope of the coincidence is fitted to an double-sided exponential with a time constant of 21 ns, which translates into a linewidth of 7.6 MHz. The separation between individual peaks within the envelope is measured to be 2.15(2) ns, which corresponds to the cavity round-trip time. The free spectral range
Figure 3.13: Coincidence from the cavity source, counted for 10 minutes at a pump power of 0.2 mW. The size of the bin is set to 100 ps.

of the cavity can then be inferred to be the inverse of the cavity round-trip time, 465(20) MHz. The free spectral range can also be calculated from the cavity length measurement. The length of the cavity is measured to be 60.5(5) cm. Taking into account the refractive indices the PPKTP and KTP crystal, the optical path of a round-trip is then 63.7(5) cm, and the free spectral range 471(4) MHz is in excellent agreement with the result from two-photon correlation measurement.

The coincidence rate is measured to be 4300 pairs/s at 0.2 mW of blue pump, which corresponding to a count rate of 86000 pairs/s/mW taking into account the chopper duty cycle. The threshold of the OPO corresponds to at least 1 photon per ringdown time, which means our cavity has a threshold of about 90 mW, and indeed we are operating
far below threshold with 0.2 mW of pump. Given the phase matching bandwidth of 130 GHz, there are about 660 cavity modes within that bandwidth. We estimate about 1/330 of the output photons is in the degenerate mode, and given the cavity linewidth of 7.6 MHz, we can infer the brightness of the source to be 34 pairs/(s × mW × MHz). We have also designed and built another similar source at 795 nm, with an even higher brightness, 70 pairs/(s × mW × MHz) [163]. Besides the implementation differences such as coupling efficiencies, the different brightness is due to higher absorption of pump at 390 nm in the PPKTP crystal. We compare the spectral brightness of sources using PDC from the literature [94, 95, 57, 51, 130, 90, 160, 50, 91, 47, 163, 65], as shown in Table 3.1. In Fig. 3.14, we plot the brightness in logarithm scale for different source developments presented in Table 3.1. Our source is about 2 orders of magnitude brighter than the previous brightest source of PDC [91]. Such improvement is partially explained by the following reasons: our cavity source uses a PPKTP crystal twice as long, which leads to a factor of 4 in total brightness; the cavity has a higher finesse with a factor of 1.3, which translates into a brightness increase of 1.7. We also use an active lock to stabilize the cavity and no interference filter is used afterwards, which account for the remaining factors of the improvement in spectral brightness.

### 3.5.2 Indistinguishable photon pairs

Indistinguishability of fundamental particles lies at the core of quantum mechanics, and is at the root of many important effects and phenomena. First demonstrated by Hong, Ou and Mandel in 1987 [75], the Hong-Ou-Mandel (HOM) interference provides a direct measurement of distinguishability of the elementary particles of light, photons. This interference effect is describes as follows. Consider two single photons, which are otherwise identical, are in two different modes, mode 1 and mode 2, see Fig 3.15. The two-photon state can be written as \(|1, 1\rangle_{1,2} = a_1^\dagger a_2^\dagger |0\rangle\) These two modes are then combined at a 50-50 beamsplitter. For simplicity, we consider a lossless beamsplitter:
Figure 3.14: Spectral brightness of PDC. The top two points represent the brightness of our work at 795 nm and 780 nm, respectively.
Table 3.1: Comparison of spectral brightness of PDC sources

<table>
<thead>
<tr>
<th>Reference</th>
<th>Geometry</th>
<th>Nonlinear crystal</th>
<th>Brightness (pairs/(s × mW × MHz))</th>
</tr>
</thead>
<tbody>
<tr>
<td>[94]</td>
<td>Single-pass</td>
<td>BBO</td>
<td>$3.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>[95]</td>
<td>Single-pass</td>
<td>BBO</td>
<td>$3.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>[57]</td>
<td>Single-pass</td>
<td>BBO</td>
<td>$2.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>[51]</td>
<td>Single-pass</td>
<td>PPKTP</td>
<td>$1.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>[90]</td>
<td>Single-pass</td>
<td>PPKTP</td>
<td>$8.6 \times 10^{-3}$</td>
</tr>
<tr>
<td>[130]</td>
<td>Single-pass</td>
<td>PPKTP</td>
<td>$4.2 \times 10^{-2}$</td>
</tr>
<tr>
<td>[160]</td>
<td>Cavity-enhanced</td>
<td>KbNO$_3$</td>
<td>0.12</td>
</tr>
<tr>
<td>[91]</td>
<td>Cavity-enhanced</td>
<td>PPKTP</td>
<td>0.7</td>
</tr>
<tr>
<td>[47]</td>
<td>Single-pass</td>
<td>PPKTP</td>
<td>0.3</td>
</tr>
<tr>
<td>this work [163, 166]</td>
<td>Cavity-enhanced</td>
<td>PPKTP</td>
<td>70 @ 795 nm, 34 @ 780 nm</td>
</tr>
</tbody>
</table>

\[
|t|^2 + |r|^2 = 1 \tag{3.19a}
\]

\[
t \cdot r^* + t^* \cdot r = 0 \tag{3.19b}
\]

The action of such beamsplitter can be then described by the following unitary transformation [46, 132]:

\[
\hat{a}_3^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_2^\dagger) \tag{3.20a}
\]

\[
\hat{a}_4^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_2^\dagger) \tag{3.20b}
\]

There are four possible outcomes for the two photons after the beamsplitter, namely transmission-transmission, reflection-reflection, transmission-reflection, and reflection-transmission. Using the operator formalism, the state after the beamsplitter is then

\[
\hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle = (\hat{a}_3^\dagger^2 - \hat{a}_3^\dagger \hat{a}_4^\dagger + \hat{a}_4^\dagger \hat{a}_3^\dagger - \hat{a}_4^\dagger^2) |0\rangle = (\hat{a}_3^\dagger^2 - \hat{a}_4^\dagger^2) |0\rangle. \tag{3.21}
\]
A key step in the derivation of Eq. 3.21 is the canceling out of terms $\hat{a}_3 \hat{a}_4^\dagger$ and $\hat{a}_4 \hat{a}_3^\dagger$, which means the probability amplitude of detecting one photon at each detector cancels out. Physically this describes the effect that the two photons will always exit the beamsplitter at the same port; in other words, the photons are bunched after the beamsplitter. If we put two detectors at mode 3 and 4 after the beamsplitter, there will be no coincidences, which can only happen if the photons are indistinguishable.

Now if we introduce distinguishing information to the two photons, such as the delay in arrival time at the beamsplitter, this HOM effect will then be washed out. With this delay, the terms $\hat{a}_3^\dagger(t_1) \hat{a}_4^\dagger(t_2)$ and $\hat{a}_4^\dagger(t_1) \hat{a}_3^\dagger(t_2)$ in Eq. 3.21 no longer cancel out completely, and the two-photon state becomes

$$\hat{a}_1^\dagger(t_1) \hat{a}_2^\dagger(t_2) |0\rangle = (\hat{a}_3^\dagger(t_1) \hat{a}_3^\dagger(t_2) - \hat{a}_3^\dagger(t_1) \hat{a}_4^\dagger(t_2) + \hat{a}_4^\dagger(t_1) \hat{a}_3^\dagger(t_2) - \hat{a}_4^\dagger(t_1) \hat{a}_4^\dagger(t_2)) |0\rangle. \quad (3.22)$$

The temporal overlap of the two incident photons $\hat{a}_1^\dagger |0\rangle$ and $\hat{a}_2^\dagger |0\rangle$ now determines the value of $-\hat{a}_3^\dagger(t_1) \hat{a}_4^\dagger(t_2) + \hat{a}_4^\dagger(t_1) \hat{a}_3^\dagger(t_2) |0\rangle$, which is the probability amplitude of detecting one photon each at detectors 3 and 4 - the coincidence detection. Varying this delay between photon 1 and 2, the visibility of the HOM interference will also change from 0 to, ideally, 1. This experimental signature is sometimes called the HOM dip. Note that the derivation above is done with the distinguishing information in the temporal mode. In fact, the same calculation applies to any mode that can be used to distinguish the photons. Thus, the HOM interference is an useful technique that can be used to measure the distinguishability of the photons.

The shape of the HOM interference for Type-II down-conversion is a triangle dip when the group delay between the two polarization is varied. This is because the probability of generating a pair is the same along the PPKTP crystal in SPDC, and the distinguishing information is introduced by different group delays which is linear in propagation length. Effectively this means the overlap of two down-converted photon wavepacket is also linear in propagation length, thus we expect a triangle shape in the correlation measurement. The delay can be introduced with half-length of crystal, as shown in the compensation
schemes in Section 3.4. A better technique to introduce the delay is to use a Michelson interferometer which allows continuous delay adjustment by varying the position of one of the mirrors (Fig. 3.16).

As shown earlier, the two-photon state after the cavity, with proper delay compensation can be written as $|\Psi\rangle = \int dt \int_{t_0}^{t} dt' \sum_m |H\rangle_t |V\rangle_{t+m\tau+t'} + |V\rangle_t |H\rangle_{t+m\tau+t'}$. If the two photons are identical in every degree of freedom except for their different spatial mode and the arrival time at the polarizing beamsplitter, such delay can then serve as the distinguishing information of the two photons and the HOM interference visibility depends on this delay. From the width of the interference pattern, one can infer the two-photon coherence time.

In Fig. 3.17, we show the interference measurement of coherence time/distinguishability. The cavity is pumped with 0.2 mW of 390 nm laser, and each data point is coincidence counts collected in 10 s. The interference is between photon pairs with different arrival times at the detectors - there are two possible locations of pair generation for the same
The incident light separates at the PBS, and the polarizations get flipped with transmitting twice through the QWP set at 22.5°.

Figure 3.16: Delay adjustment between the pairs of photons with different polarization. The data is then fit to a triangular dip, which is characteristic for type II down-conversion, with a visibility of 89(2)%. The coherence length of the two-photon state is measured to be 0.92 mm at Full Width Half Maximum (FWHM), corresponding to a bandwidth of 156 GHz, which translates to a coherence time of 6.4 ps. The measured bandwidth of the photon pair is slightly larger than the estimation of phase matching bandwidth, which we attribute to contributions from the first side lobes of the sinc phase-matching function. Compared to a free-space SPDC, such HOM interference will also occur at multiples of 60 cm of delays, which corresponding to different cavity round-trips (see Fig. 3.13).

Another interesting effect resulted from the HOM interference is the generation of
Figure 3.17: HOM interference of indistinguishable photon pairs. The coincidence counts are collected for 10 s at each data point. The group delay between two polarizations is varied by adjusting one arm of the Michelson interferometer, providing distinguishing information of arrival times of the photon pairs. The triangle shape of the dip reflects the linear dependence on propagation distance of the group delay between the photon pairs.
the simplest non-trivial state of the class $|N,0\rangle + e^{i\phi}|0,N\rangle$, commonly referred to as N00N states. Metrologically, N00N states can be used for applications such as super-resolving phase measurements [114, 3] and quantum lithography [18] to beat the classical limit. As shown in Eq. 3.21, when sitting at the bottom of the HOM dip, the photons are indistinguishable and therefore the state after the PBS is $(\hat{a}_3^+ - \hat{a}_4^+)|0\rangle = |2,0\rangle_{3,4} - |0,2\rangle_{3,4}$, which is a two-photon N00N state.

In the experiment, we first adjust the delay between the pairs so that the HOM dip is maximized. The state of the photon pair after the polarizing beam splitter should be the N00N state. To demonstrate the N00N state, we place a Half-Wave Plate (HWP) before the polarizing beam splitter, and count the coincidences as a function of the HWP angle. The result is shown in Fig. 3.18. The periodicity of an interference fringes with classical light is 180 degrees, as shown in Fig. 3.18(a). The periodicity of the coincidence is half of that in the classical interference. The variation of the two-photon fringes is due to different coupling efficiency in two arms combined with the mode overlap of H and V photons from the Michelson delay line.

### 3.5.3 Quantum state tomography

To completely characterize the state of the photon pair, we can also perform quantum state tomography as is discussed in Section 2.6.4. The experimental setup is quite similar to that of the HOM dip measurement. Though instead of polarizing beam splitter after the delay adjustment, we use a beam splitter to eliminate the polarization information of the two photons. After the beam splitter, the photon pair is then sent to two polarization analyzers each consisting of a QWP, a HWP and a polarizer. The schematic of the experiment is shown in Fig. 3.20.

Taking into account different scalings of signal coincidences (linear dependence) and the accidental coincidences (quadratic) in pump power, we reduce the pump power to about 0.1 mW to reduce the accidentals contribution. The counting time is set to 10 s
Figure 3.18: N00N state phase sensitivity. (a) Interference fringes with classical light at detector 1 and detector 2. (b) Interference fringes in the coincidence. The periodicity is half of that in the classical interference. See text for more details.
at each setting of basis controlled by the ESP-300 motorized wave plate driver. The raw data of the coincidence measurement for different basis is shown Table 3.2, where the counts are organized in a way such that the rows and columns corresponding to the basis the two photons are measured in.

To reconstruct the quantum state, we can use the quantum state tomography method outlined in Section 2.6.4. First the accidental contributions was subtracted from the raw counts to get the actual counts corresponding to the photon pairs. With the linear inversion method, we calculate the density matrix to be

$$
\rho_{\text{linear}} = \begin{pmatrix}
0.0097 & -0.0047 + 0.0020i & -0.0717 - 0.0182i & -0.0053 + 0.0535i \\
-0.0047 - 0.0020i & 0.5175 & -0.4831 - 0.1224i & 0.0853 + 0.0374i \\
-0.0717 + 0.0182i & -0.4831 + 0.1224i & 0.4728 & 0.0161 - 0.0201i \\
-0.0053 - 0.0535i & 0.0853 - 0.0374i & 0.0161 + 0.0201i & 0.0000
\end{pmatrix}
$$

(3.23)

The eigenvalues of the matrix are 1.0002, 0.1063, 0.0027, -0.1093. The first eigenvalue is larger than 1 while the last is negative, which means the density matrix we get from linear inversion is not positive semi-definite and therefore not physical. Instead of linear inversion, we can also use the maximum likelihood method [77] to reconstruct the density
Table 3.2: Raw counts of coincidence measurements

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>V</th>
<th>D</th>
<th>A</th>
<th>R</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>45</td>
<td>913</td>
<td>343</td>
<td>568</td>
<td>443</td>
<td>495</td>
</tr>
<tr>
<td>V</td>
<td>999</td>
<td>18</td>
<td>669</td>
<td>465</td>
<td>576</td>
<td>407</td>
</tr>
<tr>
<td>D</td>
<td>511</td>
<td>496</td>
<td>58</td>
<td>941</td>
<td>455</td>
<td>664</td>
</tr>
<tr>
<td>A</td>
<td>432</td>
<td>447</td>
<td>829</td>
<td>951</td>
<td>511</td>
<td>279</td>
</tr>
<tr>
<td>R</td>
<td>444</td>
<td>503</td>
<td>654</td>
<td>74</td>
<td>71</td>
<td>878</td>
</tr>
<tr>
<td>L</td>
<td>517</td>
<td>504</td>
<td>336</td>
<td>670</td>
<td>980</td>
<td>96</td>
</tr>
</tbody>
</table>

Figure 3.20: Schematic of quantum state tomography. After delay compensation adjustment, the photons are sent to a BS and polarization analyzers. The polarization correlation data is then recorded and analyzed using quantum state tomography.
matrix to guarantee a physical density matrix. Using the density matrix from linear inversion (Eq. 3.23) as the initial guess, the optimization of maximum likelihood gives

\[
\rho_{\text{mlh}} = \begin{pmatrix}
0.0021 & -0.0036 - 0.0329i & -0.0050 + 0.0288i & 0.0007 - 0.0015i \\
-0.0036 + 0.0329i & 0.5401 & -0.4583 - 0.1186i & 0.0226 + 0.0179i \\
-0.0050 - 0.0288i & -0.4583 + 0.1186i & 0.4533 & -0.0124 - 0.0098i \\
0.0007 + 0.0015i & 0.0226 - 0.0179i & -0.0124 + 0.0098i & 0.0045 \\
\end{pmatrix}
\]

(3.24)

with the eigenvalues 0.9751, 0.0249, 0.0000 and 0.0000, which means the density matrix is now a physical one. The pictorial representation of the density matrix \(\rho_{\text{mlh}}\) is given in Fig. 3.21.

From this density matrix, we calculate the purity to be 0.95. To understand the source of discrepancy in \(\rho_{\text{linear}}\) and \(\rho_{\text{mlh}}\), we calculate the error of density matrix elements.
in linear inversion due to statistical errors,

\[
\Delta \rho_{\text{linear}} = \begin{pmatrix}
0.0027 & -0.0083 + 0.0084i & -0.0073 - 0.0088i & 0.0141 + 0.0095i \\
-0.0083 - 0.0084i & 0.0202 & -0.0094 + 0.0113i & 0.0092 + 0.0075i \\
-0.0073 + 0.0088i & -0.0094 - 0.0113i & 0.0190 & 0.0078 - 0.0079i \\
0.0141 - 0.0095i & 0.0092 - 0.0075i & 0.0078 + 0.0079i & 0.0010
\end{pmatrix}
\]  

(3.25)

The discrepancies between the elements of \( \rho_{\text{linear}} \) and \( \rho_{\text{mlh}} \) are comparable to \( \Delta \rho_{\text{linear}} \), thus the statistical error is one of the major error sources in this measurement. In principle it is possible to increase to the counts by increasing the pump power or the total measurement time. In experiments we observe the purity reduced for higher pump power, which is attributed to some parasitic effects associated with high pump power that will be discussed in the next section. Our cascade locking system limits the total measurement time. With better electronics, it should be possible to increase the counting time without losing the laser frequency and cavity length stablization.

### 3.5.4 Pump power dependence

Blue-Light-Induced Red Light Absorption (BLIRA) is the absorption of red light due to the presence of a high intensity blue beam [107]. It is an incoherent absorption and is detrimental to the ideal squeezing process with \( \chi^{(2)} \) nonlinear crystals, especially in the case of PPKTP crystals where the interaction length is long due to quasi-phase-matching [6]. There are different models of BLIRA, contributing the effect to some trapping states either by the existence of free-carrier trapping states [107] or blue two-photon absorption and subsequent generation of trap states [6].

We measure the pump power dependence of red absorption directly, as shown in Fig. 3.22. For different polarizations the absorption is different. For H-polarization, the absorption is almost linear with the pump power, whereas the quality of the linear fitting for V-polarization is less satisfactory. This may be associated to the polarization of the
Figure 3.22: Power-dependent loss of the red light. The transmission of 780 nm laser light is measured for different power of 390 nm laser which is used as the pump for down-conversion for H- (a) and V-polarizations (b). The data is then fitted to a linear model (red line).

Blue light which induces the trapping sites for the carrier. Also the absolute value of the absorption will depend on the mode overlap of the red and blue, and for down-converted photons the absorption maybe more severe. The effect of BLIRA will pose a limit on how strongly we can pump the down-conversion. Follow a perhaps over-simplified model where the absorption is linearly dependent on the blue power, one can extrapolate the fitting in Fig. 3.22 and show to have 100% absorption, one need $10(1) \times 10^2$ mW for the H-polarization and $7(2) \times 10^2$ mW of blue light for V-polarization. In practice, we pump the crystal at levels a lot lower than these two powers, and we do not expect the BLIRA will have strong effect on the down-conversion photons.

The absorption of blue light has another effect other than BLIRA, that is, the temperature of the crystal may be slightly raised due to the absorption. The cavity double resonance is very sensitive to temperature changes. Actually using the temperature dependence of the double resonance, we can characterize the change of the temperature of
the crystal at 10 mW of pump to be 50 mK. Such change of temperature is not enough to shift the phase-matching conditions. Nonetheless, for every pump power one need to carefully re-adjust the double-resonance to compensate such temperature drift.

3.6 Spectral Filtering

In Section 3.5.2, we have inferred the two-photon bandwidth is 156 GHz. Given the free spectral range of the cavity FSR_{opo} = 470 MHz, there are more than 300 modes within the phase matching bandwidth. In order to pick out the degenerate mode, we need extra spectral filtering. To have good extinguishing ratio of the degenerate modes and other modes, the bandwidth of the filter should be less than the FSR_{opo}. Furthermore, the tunability is also desirable in order to generate pairs at different detunings to the atomic transition. This is a very narrow and precise frequency range, making common frequency filters such as commercial interference filter or fiber Bragg grating (see, for example, [123] and references therein) not applicable. A. Cerè et. al. demonstrate a narrowband filter based on atomic transition [25], where they have reached a transmission of 14%. Optical cavities, on the other hand, can provide narrowband and high transmission filtering capabilities at the same time. Here we use an optical cavity as the filter to pick out one of the 300 modes that is resonant with certain Rb transition.

3.6.1 Filter cavity design

One possible design of the filter cavity is to have the free spectral range matched with the phase matching bandwidth, FSR_{f} = 156 GHz, which means at most there will be only one modes that’s resonant with the filter cavity. We choose the linear cavity geometry for simplicity in construction and alignment. Since there are no dispersive medium in the cavity, from Eq. 2.10, the separation between the mirrors can be calculated as 
\[ L = \frac{c}{2 \times FSR_{f}} = 0.96\text{mm}. \]
Note that a smaller FSR_{f} can also be possible as long as the
Chapter 3. The quantum light source

Figure 3.23: Schematic of filter cavity. The broadband photons and classical locking beams are sent to the cavity from opposite directions, where a chopper is placed to alternate the beam going to/from the cavity. The chopper is driven with a synchronized signal as the chopper of the OPO cavity.

The least common multiple of FSR_{opo} and FSR_f is larger than the phase matching bandwidth, and the difference between any modes of the OPO cavity and any modes of the filter is much larger than the linewidth of the OPO cavity, except for one common resonance mode. We summarized the parameters of the filter cavity in Table 3.3.

Since the degenerate mode of the OPO cavity is very narrow and we wish sometimes to have the photon pairs frequency tunable, some active locking of the filter cavity is necessary. We can use the same technique as described in Section 2.3.1 to lock the filter cavity. Since the photons from down-conversion are the same frequency as the classical laser we use to lock the cavity, the same technique of time-interval locking with a chopper can be used.

Now we have two different choppers for the generation of photon pairs and the filtering. If the chopper drivers are run independently, even at the same frequency, the phase slip may cause the collection port at two cavities out of phase. To solve this problem, we
## Table 3.3: Parameters of the filter cavity

<table>
<thead>
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<th>Parameters</th>
<th>Values</th>
<th>Units</th>
</tr>
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<tr>
<td>Geometry</td>
<td>Two-mirror FP cavity</td>
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<tr>
<td>Radius of curvature</td>
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<td>mm</td>
</tr>
<tr>
<td>Length (center)</td>
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<td>mm</td>
</tr>
<tr>
<td>FSR</td>
<td>150(15)</td>
<td>GHz</td>
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<tr>
<td>$\Delta \nu$</td>
<td>455(46)</td>
<td>MHz</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>329(5)</td>
<td>-</td>
</tr>
<tr>
<td>Transmission</td>
<td>30(1)%</td>
<td>-</td>
</tr>
</tbody>
</table>

will need to phase lock the drivers. This is done with the PLL function of the chopper drivers (Thorlabs MC2000). Note the encoder position of the chopper is different from the beam location on the chopper, the phase shift should be set such that the collection beams at the OPO and filter cavities are synchronized. The traces of synchronized locking are shown in Fig. 3.24. The photodiode signals have been normalized to have the same peak-to-peak value.

It is also necessary to gate the single photon detectors even after the protection with choppers. This is because the chopper only protects the detector for the idler mode after the filter cavity, and when the filter cavity is locked, classical light can still leak back into the transfer fiber, be reflected from some surface and get into the signal mode. Even though every step in such process has very high loss, the leakage of classical light to the single photon detectors may still be much more than the signal we are interested in. This can be solved using an extra gating circuit synchronized by the chopper locking signal. The reader can refer to the gating specification of the single photon detector [122] for electrical requirements of the gating circuits.
Figure 3.24: Synchronization of locking of the OPO and filter cavities.
3.6.2 Narrowband photons after filtering

With the filter cavity in place for one of the down-conversion (spatial) modes, we expect to spectrally suppress most of the photons in the phase-matching bandwidth, except for those that are resonant with the filter cavity. Since the linewidth of the filter cavity is less than the free spectral range of the OPO cavity, and with the free spectral range of the filter cavity is matched with the phase-matching bandwidth, only one mode is expected to have high transmission through the cavity. In addition, by the phase-matching condition, the sum of the frequencies of the photons from the down-converted pair should add up to that of the pump. In the case of filtering, only the degenerate mode of the OPO cavity satisfies this condition. If locked appropriately, the degenerate mode of the OPO cavity will be resonant with the desired atomic transition. Thus by filtering one down-conversion modes, we expect to get narrowband photon pairs that is resonant with atoms.

The time correlation measurement of the photon pairs from the filtered down-conversion is shown in Fig. 3.25. For this measurement, the pump power of the OPO cavity is set to 4 mW, with a total counting time of 45 minutes. While the filter cavity greatly reduces the bandwidth of the down-conversion photon pairs, it also decreases the number of coincidence counts dramatically. We measured the singles counts to be 200/s and 480,000/s respectively, and the coincidence counts to be about 30/s. The transmission of the filter cavity is then estimated to be 0.04% for the down-conversion photons. From the total number of OPO cavity modes $\sim 330$, the cavity transmission of 30%, and collection efficiency of 50%, we also estimate the filtering efficiency through the cavity to be 0.06%. The additional loss in photons may be associate to the residual fluctuation after the locking. The coincidence is grouped into time bins with the size of 0.22 ns, which is chosen to be half of the jitter of the single photon detectors. For ideal single mode of cavity-enhanced PDC, we expect a double-sided exponential decay in the correlation measurement. Experimentally, we also observe residual fringe pattern in Fig. 3.25, with the visibility of 63(6)% near the centre after background subtraction using param-
Figure 3.25: Coincidence after spectral filtering, with 0.22 ns of time bin. To show the residual fringes in (a), the same data is plotted again in (b) without error bars.

Parameters from the fitting. Compared to the unfiltered correlation measurement as shown in Fig. 3.13, the visibility is significantly reduced, indicating the fringes result from a much less number of modes interfering. We contribute these fringes to leakage of nearest side bands in the frequency comb to the degenerate mode. In fact, with a simplified model of three-mode structure, we write the filtered output $|\Psi_f\rangle$ as

$$|\Psi_f\rangle = \int d\omega \left( f(\omega) + \eta_f (\omega + \Delta_{FSR}) + \eta_f (\omega - \Delta_{FSR}) \right) a^{\dagger}_\omega e^{i\omega t} |0\rangle$$

(3.26)

where $f(\omega)$ is the spectral function of the OPO cavity with appropriate normalization, $\eta_f$ is the relative transmission amplitude of the the side bands to the degenerate mode, and $\Delta_{FSR}$ is the free spectral range of the OPO cavity. From the fringe visibility $V = 4\eta_f^2$, we can infer the transmission to be $\eta_f^2 = 16(2)\%$. On the other hand, with Eq. 2.18 and parameters of the filter given in Table 3.3, we can calculate the transmission of the filter cavity of the side bands to be 18%. These values of visibility are in good agreement with each other. Thus we conclude that the major contribution to the detected coincidences counts comes from the degenerate mode of the OPO cavity. To see the envelope of the photon pulse more clearly, we used a bin size of 2 ns in Fig. 3.26. The shift of the centre
Figure 3.26: Coincidence after spectral filtering, with 2 ns of time bin.

agrees well with the extra 3 m of transfer fiber used, and approximately 60 cm of free space propagation, which translates to a delay of about 18 ns.

3.7 Measurement of quantum statistics

In this section, we will examine the photon statistics of the cavity-enhanced PDC source. As shown earlier, since we operate the cavity far below threshold, and within each ring-down time of the cavity the average number of photon pairs is much less than 1, therefore, the stimulation process can be neglected, and the spontaneous emission of photon pairs is dominant. If we look at statistics of one photon of the pair, it should display thermal distribution, or bunching. On the other hand, if we look at the statistics of the signal photon conditioned on the detection of an idler photon, it should display anti-bunching, as shown in Section. 2.6.2. In the following we show experiments that measures the
bunching and anti-bunching of photons from the same photon-pair source with different measurement setups.

### 3.7.1 Poisson statistics

Poisson statistics is used to describe a series of events in a certain process with a known average rate and are independent of each other. In quantum optics, the measurement of arrival time of photons in a coherent state is a good example of such process. Another example is the measurement of coincidence counts in a certain time interval that is much larger than the coherence time of the photon pairs.

The probability distribution function for a random variable $X$ with Poisson statistics can be written as

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (3.27)$$

where $\lambda$ is the expectation value of the random variable $X$.

As shown in Fig. 3.27, we have measured the the coincidence counts in a second from a cavity-enhanced PDC source. The pump power has been reduced significantly such that the coincidence rate is on the order of a few per second. The time interval of the measurements is much larger than the coherence time of the photon pairs, and thus the coincidence counts in a second can be treated as the random variable with Poisson statistics. This is shown in Fig. 3.27 with good agreements between experimental data and a model with Poisson distribution.

### 3.7.2 Photon bunching - 2nd-order correlation

It is first shown by Yurke and Potasek [168] that thermal statistics can be obtained from a pure quantum state of squeezed vacuum. With a two-mode squeezed-vacuum state, if one traces over one mode, say the idler, the signal mode will be projected to a state described by a thermal distribution.
Figure 3.27: Distribution of coincidence counts and Poisson statistics. The coincidence counts in a second is recorded and plotted with a histogram to shown the occurrence of each value of counts. The bar plot is the theoretical model based on Poisson statistics.
The state of the two-mode squeezed vacuum can be written as

$$\Psi_{spdc} = e^{iHt} |0\rangle = \text{sech}|\xi| \sum_{n=0}^{\infty} \left( \frac{\xi}{|\xi|} \tanh |\xi| \right)^n |n_s, n_i\rangle$$

(3.28)

where $\xi = g't$, $g'$ being the coupling strength described in Eq. 3.3, and $|n_s,i\rangle = \frac{1}{n!}(a^\dagger)^n |0\rangle$ is the Fock state for signal and idler mode. Note Eq. 3.28 describes the single frequency (or temporal) mode of the down-conversion. By tracing over one field, say, the idler, one can suppress the signal-idler correlation and the density matrix of the system now becomes

$$\rho_s = \sum_{0}^{\infty} P_n |n_s\rangle \langle n_s|$$

(3.29)

where $P_n$ is photon number distribution of a thermal state $P_n = \frac{n^n}{(1+n_0)^{n+1}}$, and $n_0 = \sinh^2 |\xi|$ is the average photon number of the thermal state.

For simplicity, let us take the limit of a small coupling strength $\xi$. Expand Eq. 3.28 with Taylor expansion, we have

$$\Psi_{spdc} \approx |0_s, 0_i\rangle + \xi |1_s, 1_i\rangle + \xi^2 |2_s, 2_i\rangle + O(\xi^3).$$

(3.30)

Taking the partial trace on the idler field, the second-order correlation function $g^{(2)}(0)$ for the signal field can then be calculated $g^{(2)}(0) = \frac{\xi^4}{\xi^2 \xi^2} = 2$, as expected for a thermal distribution. In many experiments, such demonstration of a thermal statistics for the continuously pumped SPDC is difficult. This is because the jitter in single photon detectors $\tau_{jitter}$ are typically on the order of 100 ps, which is much larger than the coherence time of the photons $\Delta t \sim 0.1 - 10$ ps. The larger jitter in the detector will effectively smear out the value of $g^{(2)}(0)$. For example, an excess value in $g^{(2)}(0)$ (compared to 1) of $3 \times 10^{-3}$ is measured in [16]. This can be modeled by incorporating a multi-mode description of Hamiltonian that is proportional to

$$\int d\omega f(\omega)a_{s\omega}^\dagger (\omega_p/2 + \omega)a_{i\omega}^\dagger (\omega_p/2 - \omega) + c.c.,$$

(3.31)

where $f(\omega)$ describes the spectrum of the down-conversion, $\omega_p$ is the pump frequency and the phase-matching condition is implicit in the signal and idler creation and annihilation
operators. The multi-mode down-conversion can be written as

\[ \Psi_{\text{spdc,mm}} \approx |0_s,0_i\rangle + \xi \int d\omega f(\omega)a_s^\dagger(\omega_p/2 + \omega)a_i^\dagger(\omega_p/2 - \omega) |0\rangle \\
+ \frac{\xi^2}{2} \int d\omega d\omega' f(\omega)f(\omega')a_s^\dagger(\omega_p/2 + \omega)a_s^\dagger(\omega_p/2 + \omega')a_i^\dagger(\omega_p/2 - \omega)a_i^\dagger(\omega_p/2 - \omega') |0\rangle \\
+ O(\xi^3). \quad (3.32) \]

Equivalently, we can analyze the down-conversion in the time domain. The multi-mode nature of Eq. 3.32 is then translate to different arrival times of the photon wavepackets. For our cavity-enhanced PDC source, the bandwidth of the photons after filtering can be significantly reduced thanks to the cavity. As shown earlier, the lifetime of the photon pair can be as large as 21 ns, much longer than the detector jitter \( \tau_j = 450(50) \) ps, making it possible to measure the \( g^{(2)}(0) \) close to 2. On the other hand, if the photon pairs after the cavity is not filtered, we expect the lifetime of the photons \( \tau_p \) to be much shorter, on the order of 1 ps. In this case, the \( g^{(2)}(\tau) \) will be broadened by the slow detector.

To understand this problem qualitatively and for simplicity, we assume the spectrum function \( f(\omega) \) is made up of \( n \) discrete modes with equal amplitude. The signal-signal second-order correlation function is then

\[ g_{ss}^{(2)}(0) = \frac{G_{ss}^{(2)}(0)}{G_{ss}^{(2)}(\infty)} = \frac{G_{ss}^{(2)}(0)}{(G_{s}^{(1)}(0))^2} \\
= \left| \sqrt{2}\xi^2 + (n-1)\frac{\xi^2}{2} \right|^2 \\
= \frac{n^2\xi^4}{n^2 \xi^4 - 1} \right)^2. \quad (3.33) \]

For the single-mode case, \( n = 1 \), and \( g_{ss}^{(2)}(0) = 2 \) as expected for the signal (idler) field. As the number of modes increases, \( g_{ss}^{(2)}(0) \to 1 \), which appears to be un-correlated photons. For example, for \( n = 300 \), this simple model gives a \( g_{ss}^{(2)}(0) \approx 1.002 \).

Due to the low count rate of photons after filtering, we will first measure the \( g^{(2)}(0) \) for unfiltered photons from the cavity. The experimental setup is shown in Fig. 3.28.
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Figure 3.28: Experiment layout to measure photon bunching.

After the generation and birefringence compensation, the photon pairs are separated by a PBS. Idler photons are then blocked and the signal photons are sent to a fiber BS and single-photon detectors. To measure the second-order correlation, the arrival time of the photons are recorded by the time-tagging electronics.

The measurement result is shown in Fig. 3.29, where the un-normalized $G^{(2)}(\tau)$ is plotted against the arrival time difference of two signal photons. The pump power of the cavity-enhanced PDC is measured to be 0.9 mW in this measurement. The measurement time window of the time-tagging electronics is set to 190 ns in order to capture the bunching as well as un-correlated photons. We have also included an extra time delay of about 40 ns to shift the zero-time delay to one side of the measurement window. The coincidence counts are binned into 2 ns windows. If the photon pairs are generated independently, $G^{(2)}(\tau)$ should be a flat line. We found that for our unfiltered cavity-enhanced PDC, $G^{(2)}(\tau)$ peaks at zero-time delay of the two signal photons, indicating bunching of the photons. We then fit the $G^{(2)}(\tau)$ to a double-sided exponential decay. From the fitting, we can then infer the value of $g^{(2)}(0)$ to be 1.15. This value is much larger than simple model result of $g^{(2)}(0)$ of 1.003, which we attributed to the leakage of
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Figure 3.29: Signal-signal correlation measurement at 0.9 mW of pump.

the idler photon into the signal mode, and the measured photon bunching is between the signal and idler photon. To verify this, we can measure the pump power dependence of the $g^{(2)}(0)$. For the PDC process, if we are measuring the first order effect, i.e., the pair generation, the rate scales linearly with the pump. For the second order effect, the rate scales quadratically with the pump. If the measured $g^{(2)}(0)$ is due to leakage, we then expect $g^{(2)}(0) - 1$ to be inversely proportional to the pump power.

Next we measure the second-order correlation for different pump powers, and find out the power dependence of $g^{(2)}(0)$. We tune the pump power to 1.2 mW and 1.5 mW, and the measurement results are shown in Fig. 3.30 and 3.31, respectively. At each pump power, the crystal temperature is adjusted to maintain the double resonance condition. The collection time is set to be the same for the measurement at 0.9 mW of pump power, and we follow the same procedure to measure the value of $g^{(2)}(0)$. The value of $g^{(2)}(0)$ for the two different pump powers, 1.2 mW and 1.5 mW, is calculated to
Figure 3.30: Signal-signal correlation measurement at 1.2 mW of pump.

Figure 3.31: Signal-signal correlation measurement at 1.5 mW of pump.
be 1.14 and 1.08, respectively. At high pump powers, we observe the quality of fitting decreases compared to that of low pump powers. Specifically, around zero delay time, the correlation function has an additional sharp feature. The time difference between such additional feature and the peak of the correlation function is around 40 ns. We attribute this to parasitic effects in the single photon detector at high count rates, where the exciton in the detector is trapped and recombined at a later time, and the additional peak of the correlation function then corresponds to the recombined photons getting back to the fiber and subsequently the other detector. Nonetheless, we observe the inverse proportionality of the value of $g^{(2)}(0)$ with the pump power, and thus conclude the major contribution of $g^{(2)}(0) - 1$ is from the leakage of the idler photon into the signal mode due to finite extinguishing ratio of the polarization beam splitters employed. We expect for the same reason, the leakage will contribute to the measurement of the filtered photons, where we expect a theoretical value of $g^{(2)}(0)$ to be 2. To avoid the leakage problem entirely, we need to use a different geometry where the down-conversion is spatially (non-collinear) or spectrally separate (non-degenerate), and the signal and idler can be separated with large extinguishing ratio to avoid any leakage. On the other hand, with careful characterization of the pump power dependence, it is possible using to infer the $g^{(2)}(0)$ of the signal field only. Our time-tagging electronics saturate at a count rate of 36 kilo-counts/second (see Appendix. B), which means we can not increase the pump power as we wish, and it is a challenge to identify the contribution from higher order terms in the down-conversion state. With the current setup and instruments, we cannot conclusively show the thermal statistics from 2 photon experiment.

### 3.7.3 Photon bunching - 3rd-order correlation

Is it possible to measure bunching statistics with the unfiltered photons from the down-conversion? Here, we explore some higher order correlation function than $g^{(2)}(0)$, such as the three-fold coincidence from three detectors: one as the trigger at the idler arm,
and two after a BS at the signal arm. The three-fold coincidence measurement using the field operators is given by

\[
G_{ssst}(t_i, t_{s1}, t_{s2}) = \left\langle \hat{E}_s^-(t_{s1}) \hat{E}_i^+(t_i) \hat{E}_i^-(t_i) \hat{E}_s^+(t_{s2}) \hat{E}_s^-(t_{s2}) \right\rangle, \tag{3.34}
\]

where \(\langle \rangle\) implies an ensemble average, and \(t_i, t_{s1}\) and \(t_{s2}\) are the time of the detection of the trigger photon and two signal photons. In the case we are considering, the system Hamiltonian is time independent, thus Eq. 3.34 is dependent only on the time difference, not absolute time, and we can write

\[
G_{ssst}(t_{s1} - t_i, t_{s2} - t_i) = G_{ssst}(t_i, t_{s1}, t_{s2}). \tag{3.35}
\]

To understand Eq. 3.35 qualitatively, we again assume a simplified mode structure \((n \text{ equal-amplitude modes})\) of the down-conversion and follow the same calculation procedure of \(g^{(2)}_{ss}(0)\) (Eq. 3.33), we find \(g^{(3)}_{ss}(0,0) = (1 + \frac{1}{n})^2\). For the single-mode case, \(g^{(3)}_{ss}(0,0) = 4\) and for \(n \gg 1\) \(g^{(3)}_{ss}(0,0) \to 1\). To understand the behavior at large \(n\) more precisely, we consider the triple coincidence Eq. 3.35 and evaluate the signal-signal autocorrelation \(R(\tau)\) and signal-idler cross-correlation \(C(\tau)\) \[164\]

\[
R(t_{k2} - t_{k1}) \equiv \left\langle \hat{E}_k^-(t_{k2}) \hat{E}_k^+(t_{k1}) \right\rangle, \quad k = \{s, i\}, \tag{3.36}
\]

\[
C(t_s - t_i) \equiv \left\langle \hat{E}_s^+(t_s) \hat{E}_i^+(t_i) \right\rangle. \tag{3.37}
\]

Substituting the down-conversion state (Eq. 3.32) and neglecting irrelevant phase factors, we have

\[
R(t_{k2} - t_{k1}) \propto |\xi|^2 \int d\omega |f(\omega)|^2 e^{i\omega(t_{k2}-t_{k1})}, \tag{3.38}
\]

\[
C(t_s - t_i) \propto |\xi| \int d\omega f(\omega) e^{-i\omega(t_s-t_i)}, \tag{3.39}
\]

where \(C(t_s - t_i)\) is the cross-covariance function of the signal and idler fields, and \(R(0) = |\xi|^2\) which physically corresponds to the number of pairs generated per pump pulse. Both \(R(t_{k2} - t_{k1})\) and \(C(t_s - t_i)\) drop rapidly as the time differences increase, roughly on the timescale of \(\tau_c\), where \(\tau_c\) is the coherence time of the photons.
We first study the unnormalized fourth-order signal-idler correlation (coincidence) function of the cavity-enhanced PDC with this formalism. With the quantum form of the Gaussian moment-factoring theorem \[143\], the coincidence between the signal and idler can be written as

\[
G_{si}^{(2)}(t_s - t_i) = R^2(0) + |C(t_s - t_i)|^2. \tag{3.40}
\]

The normalized second-order correlation function is then

\[
g_{si}^{(2)}(t_s - t_i) = 1 + \frac{|C(t_s - t_i)|^2}{R^2(0)} = 1 + \frac{1}{\xi^2} \left| \int d\omega f(\omega)e^{-i\omega(t_s - t_i)} \right|^2, \tag{3.41}
\]

For typical PDC, \(\xi \ll 1\) and the coincidence rate \(g_{si}^{(2)}(t_s - t_i) \gg 1\). This can be directly measured in our experiment with time-revolving coincidence detection.

Similarly, the triple coincidence function \[17, 12\] can be calculated as

\[
G_{ssi}^{(3)}(t_{s1} - t_i, t_{s2} - t_i) = \left[ R^2(0) + |R(t_{s2} - t_{s1})|^2 + |C(t_{s1} - t_i)|^2 + |C(t_{s2} - t_i)|^2 \right] R(0)
+ 2Re\{C(t_{s1} - t_i)C(t_{s2} - t_i)^*R(t_{s2} - t_{s1})\}, \tag{3.42}
\]

which is completely determined by the time dependence of \(R\) and \(C\). To simplify the analysis, let us consider the case where \(t_{s2} = t_i\) and study the time dependence of Eq. 3.42 on \(\tau = t_{s1} - t_i\). At time delays \(\tau \gg \tau_c\), Eq. 3.42 can be reduced to \(G^{(3)}(\infty, 0) = R^3(0) + R(0)|C(0)|^2\); at zero time delay \(\tau_1 = 0\), \(G^{(3)}(0, 0) = 2R^3(0) + 4R(0)|C(0)|^2\). We can then define a bunching parameter \(P\) for the triple coincidences as the ratio

\[
P(\tau) = \frac{G^{(3)}(\tau, 0)}{G^{(3)}(\infty, 0)}. \tag{3.43}
\]

For zero time delays \(\tau = 0\), \(P(0) = \frac{2+4K^2}{1+K^2}\), where \(K = \frac{|C(0)|}{R(0)}\), and \(2 \leq P(0) \leq 4\). For small values of \(K \ll 1\), we have the bunching parameter \(P(0) \to 2\), and \(P(0) \to 4\) for \(K \gg 1\). To evaluate the ratio \(K\) for our source, using Eq.3.36 and the multi-mode state Eq. 3.32, we can calculate \(C(0)\) and \(R(0)\), and thus

\[
K = \frac{|C(0)|}{R(0)} \approx \frac{1}{\xi} \int d\omega f(\omega), \tag{3.44}
\]
where $f(\omega)$ is the spectral amplitude function of the down-conversion, and $\xi$ is the coupling strength which is typically very small, $\xi \ll 1$, so we expect $K \gg 1$ and $P(0) \to 4$.

There are a few experimental limitations that need to be considered in practice. Firstly, the triple coincidence (Eq. 3.35) implicitly assumes the use of one photon number resolving (PNR) detector for the signal mode and one single-photon detector for the idler mode. The study of PNR detector is an active research area in its own right, and here instead we use a BS and two single-photon detectors at the signal mode to measure the three-fold coincidence. A typical setup of such measurement is shown in Fig. 3.32, where in the signal arm a beam splitter is used in order to measure higher-order terms in the signal mode. The annihilation operators for the two signal mode after the beam splitter, up to irrelevant phases, are given by

\begin{align}
\hat{a}_{s1} &= \frac{1}{\sqrt{2}} (\hat{a}_s + \hat{a}_{\text{vac}}), \\
\hat{a}_{s2} &= \frac{1}{\sqrt{2}} (\hat{a}_s - \hat{a}_{\text{vac}}),
\end{align}

where $\hat{a}_{\text{vac}}$ is the mixing of the vacuum mode at the beamsplitter. It is not difficult to show that $G^{(3)}_{s_1 s_2 i}(\tau_1, \tau_2) = \frac{1}{4} G^{(3)}_{ssi}(\tau_1, \tau_2)$, and the bunching parameter $P(\tau)$ (Eq. 3.43) remains unchanged.

Another experimental limitation comes from the finite time resolution in photon de-
tection. The single-photon detectors have finite time responses (with time scales of $\tau_{\text{jitter}}$) that are typically much slower than the coherence time of the photons $\tau_c$. Furthermore, the coincidence circuits usually have their own time resolution $\tau_{\text{coin}}$ which may be even longer than $\tau_{\text{jitter}}$. The time delays in Eq. 3.42 are only as precise as the time resolution of the slowest component in the coincidence detection. Let us rewrite the three-fold coincidence (Eq. 3.42) as

$$G^{(3)}_{\text{ssi}}(t_{s_1} - t_i, t_{s_2} - t_i) = \left[ R_0^2 + |R_{s_1 s_2}|^2 + |C_{i s_1}|^2 + |C_{i s_2}|^2 \right] R_0 + 2 R e \{ C_{i s_1} C_{i s_2}^* R_{s_1 s_2} \}, \quad (3.47)$$

where $R_0 = R(0)$, and $R_{nm}$ and $C_{nm}(n, m = \{i, s_1, s_2\})$ are the shortened forms of $R(t_m - t_n)$ and $C(t_m - t_n)$. Considering the finite time response of the system, the experimentally measurable triple coincidence is then

$$G_{\text{ssi}}^{(3)}(\tau_1, \tau_2) = B \ast G_{\text{ssi}}^{(3)}$$

$$= \frac{1}{T^3} \int \int \int dt_i dt_{s_1} dt_{s_2} b(t_i) b(t_{s_1} - t_i) b(t_{s_2} - t_i) G_{\text{ssi}}^{(3)}(t_{s_1} - t_i, t_{s_2} - t_i), \quad (3.48)$$

where $B = b(t_i) b(t_{s_1}) b(t_{s_2})$ is the response function of the three-fold photon-detection event at time $t_i, t_{s_1}$ and $t_{s_2}$, $T$ is the characteristic time of the detection, and the convolution describes the time broadening. For detector-resolution-limited cases, $b$ can usually be approximated by a Gaussian function whose width is determined by the detector resolution $T = \tau_{\text{jitter}}$, and when the coincidence circuit resolution is the limiting factor, $b$ becomes a rectangular function with a width of $T = \tau_{\text{coin}}$. In Fig. 3.33, we show the calculation of Eq. 3.48 for different time resolutions. In Fig. 3.33a and 3.33b, the time resolution is limited by the 10 ns coincidence detection. The peak at the centre ($\tau_1 = 0, \tau_2 = 0$) reaches a height which is twice as high as the ridges at $\tau_1 = 0, \tau_2 \to +\infty$. Such factor of 2, however, is not due to the statistics of stimulated emission as in thermal states. Instead, this factor of 2 is from two ways of getting a triple coincidence: 1) a photon pair at signal 1 and idler and an uncorrelated photon at signal 2, 2) a photon pair at signal 2 and idler and a third one at signal 1. The fast feature in time of Eq. 3.43 are washed out due to the slow coincidence window resolution. Fig. 3.33c and 3.33d show
the result when the time resolution is limited by the single photon detectors, where we assumed to be 400 ps here. The peak height is about 2.1 times the ridges on the side. The slight larger than 2 value is an indication of some quantum effect due to stimulated emission. For the single mode case, where the coherence time of the photons are much larger than the time resolution of the detection system, one can recover the full quantum effect with a peak height 4 times the ridges on the side. As shown in Fig. 3.33e and 3.33f, the photons are assumed to be filtered to a single mode with a linewidth of 5 MHz, and with the current detection conditions, a peak height of 4 is observed.

Based on our previous measurements, we have the coherence time $\tau_c = 6.4$ ps, the cavity round-trip time $\tau_{rt} = 2.1$ ns, the detector resolution $\tau_{jitter} = 450$ ps. Also assume a coincidence window $\tau_{coin} = 10$ ns, and $\tau_2 = 0$. With these conditions, we now proceed to simplify Eq. 3.48 to derive a model which is easier to verify experimentally. We set the time delay $\tau_2 = 0$, and derive the time dependence of $G_{ssi}^{(3)}(\tau, 0)$ as one varies $\tau_1$. Since the pair generation rate $R_0$ is time independent and is not broadened by $B$, $B \otimes |C_{is1}C_{is2}^*R_{is1is2}| \ll B \otimes |C_{is2}|^2 R_0$; and similar to the reasoning in Eq. 3.44, $|R_{is1is2}| \ll |C_{is2}| = C_0$. To compare the size of $R_0^2$ and $B \otimes |C_{is2}|^2$, note that since $\tau_{coin} > \tau_{rt}$, the broadening by $B$ reduces the peak value of $C_{is2}$ to a value on the order of $\tau_c/\tau_{rt} = 1/N$, where $N$ is the number of modes in the down-conversion. Again using Eq. 3.44, this amounts to comparing the size of $K/N$ and 1. For typical pump powers where the cavity is operated far below its threshold, $K/N \ll 1$, thus $R_0^2 \ll |C_{is2}|^2 \otimes H$. The three-fold coincidence taking into account the time resolution finally becomes

$$G_{ssi}^{(3)}(\tau, 0) \approx \left[ B \otimes |C_{is1}|^2 + B \otimes |C_{is2}|^2 \right] R_0.$$  \hspace{1cm} (3.49)

and

$$P'(\tau) = G_{ssi}^{(3)}(\tau, 0)/G_{ssi}^{(3)}(\infty, 0).$$  \hspace{1cm} (3.50)

Specifically, at zero time delay, $P'(0) \approx 2$. A full simulation of $P'(\tau)$ for our cavity source is shown in Fig. 3.34. Due to finite time resolutions, the peak value of the bunching
Figure 3.33: Calculation of $G_{\text{sisi}}^{(3)}$ for different time resolutions. (a)-(b): the time resolution is limited by the 10 ns coincidence detection. (c)-(d): the time resolution is limited by the single photon detectors. (e)-(f): the photons are filtered down to a single mode with a linewidth of 5 MHz. See text for more details.
parameter goes down from 4 to about 2. This bunching peak of 2 resulted from different time dependence of terms $B \otimes |C_{i_{s1}}|^2 R_0$ and $B \otimes |C_{i_{s2}}|^2 R_0$ in Eq. 3.49 which have double-side exponential decays whose widths are determined by the cavity ringdown time. These two terms mean different contributions of three-fold coincidence, and at zero delay, the three-fold coincidence will be twice as likely as that at large delays where $B \otimes |C_{i_{s1}}|^2 R_0 \rightarrow 0$. As discussed earlier, to prove a quantum effect due to stimulated emission, we need to have a factor larger than 2, and ideally this factor will reach 4.

The full experimental setup to measure the three-fold coincidence is shown in Fig. 3.35. After the pair generation and group-delay compensation, the photon pair is separated at a PBS, and the idler mode is used as the trigger which gives heralded single-photons in the signal mode. These single photons are then collected by a single-mode fiber and sent to a 50-50 fiber BS followed by two signal detectors. All the detectors are connected to a home-made Multi-channel Coincidence Analyzer (MCA) to collect the coincidence data.

The cavity source was pumped with a power of 1.5 mW of 390 nm laser. The time delays between any two of the three detector lines are calibrated with the time-tagging
electronics. The trigger-signal correlation are measured at the same time. The measurement result is shown in Fig. 3.36, which is similar to the signal-idler correlation measurements shown earlier. With the calibration of both trigger-signal1 and trigger-signal2 arms, we can fine-tune the electrical delays to a precision of 500 ps, which is the limit imposed by the single photon detectors.

After the calibration, we switched to the measurements using a home-made MCA, since our time-tagging electronics is only capable of measuring two channel data simultaneously. With the MCA, we have set the coincidence window to be $\tau_{\text{coin}} = 10$ ns, and collect data for different delays of signal-signal correlation from 0 ns to 100 ns. Ideally we would like to have the coincidence window to be as small as possible, and vary the delay in the increments of the coincidence window. This means, however, that the total data collection time will be very long. In fact, for a given curve of un-normalized $G^{(3)}_{\text{sssi}}(\tau)$, the measurement time goes as $1/\tau_{\text{coin}}^2$. The coincidence window of 10 ns is chosen that is much larger than the cavity round-trip time, and yet smaller than the cavity ring-down time to result in a reasonable total data collection time.

We measure the triple coincidence $G^{(3)}_{\text{sssi}}(\tau)$ (Eq. 3.42) for different delays. As shown
in Fig. 3.37, the triple coincidence is plotted against the delay $\tau$ between the trigger and signal 1. Each data point contains coincidence counts for 300 s, and the delay is varied between 0 ns and 100 ns in increments of 10 ns. The triple coincidence data is then fit to the theoretical model. From the fitting, we calculate the bunching parameter (Eq. 3.50) $P'(0) = 2.1(2)$. This results from the broadening due to the time resolution of our 10 ns coincidence window (see Eq. 3.49), which is much larger than the coherence time of the photon as well as the detector resolution. Even though the measured value of $P'(0)$ is larger than 2, it is not sufficient for us to conclude the thermal distribution of the down-converted photons due to large uncertainty. An straightforward improvement would be to collect the data for much longer to reduce the uncertainty. To reach an uncertainty 10 times smaller, one need to have 100 times more data with the same apparatus. This can be done in principle, yet experimentally tedious and challenging to stable the whole system for the extended time. Another solution is to use filtered photons whose coherence
Figure 3.37: Three-fold coincidence between trigger and signal modes. The delays between the trigger and two signal modes are adjusted to be zero in the beginning. An increment of 10ns in the delay are then introduced to one the signal mode. The photons are detected and counted by the MCA for 300 s at each delay.

time is much longer than the detector jitter or coincidence window. One can expect a full quantum effect with $P'(0) \rightarrow 4$. This can be challenging though due to the low count rate of the filtered photons. A better solution is to employ multi-channel time-tagging electronics instead of the current MCA device. This will allow simultaneously collection of data at all delays and the measurement time will be significantly reduced for the same amount of data. Also, typical resolution of the time-tagging electronics is much smaller than the detector jitter, and a statistically significant value showing the quantum statistics can be expected.
3.7.4 Photon anti-bunching

In the photoelectric effect, the quantization of the light field is introduced when the time scale of energy transfer of the light to matter is smaller than that determined by the energy flux of the light field. However, if we incorporate a probabilistic model of energy transfer, it is not conclusive that the concept of single photon is necessary for such cases. So what is necessary to prove the quantization of the light field? The answer lies in the sub-Poissonian statistics of the light field. In this section, we will introduce the measurement the anti-bunching effect of the photons. We first show that triggered on the detection of photons in one of the two output modes, the remainder displays approximately statistics of single photons. This is sometimes referred to as a heralded single-photon source.

In 1986, Grangier et al. [61] performed the famous experiment of a photon anti-correlation effect on a BS, conclusively demonstrated the quantum properties of the single photon. In that experiment, a heralded single-photon source from an atomic cascades was used, and the detection of a trigger photon “heralded” the presence of the other photon.

Our ability in generating pairs of photons has improved dramatically over the years. In particular, SPDC has proved to be a bright and easy-to-setup source for many quantum information and quantum optics applications. Heralded single photon source has also been successfully demonstrated with SPDC [17], but there are still some controversy [12] about the interpretation of the result due to the fact that the photon coherence time is much smaller than the detector jitter. With a cavity-enhanced down-conversion source, Höckel et al. [74] demonstrated anti-bunching of photons, yet the bunching effect is not clearly demonstrated.

In the following, we show the measurement of anti-bunching effect with our heralded single-photon source where one of the down-conversion modes, say, the idler, is used as the trigger, and the coincidence of two signal photons is conditioned on the detection of
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The conditional second-order correlation function can be written as

\[ g_c^{(2)}(\tau) = \frac{P(S_1, S_2|T)}{P(S_1|T)P(S_2|T)} = \frac{G_{ssi}^{(3)}(\tau)G_i^{(1)}(0)}{G_{si}^{(2)}(0)G_{si}^{(2)}(\tau)}, \]

where \( P(S_1, S_2|T) \) and \( P(S_{1,2}|T) \) are the detection rate of coincidence and singles count in the signal mode conditioned on the detection of a photon in the idler mode, \( G_{si}^{(2)} \) is the coincidence rate between the signal and idler, and \( G_i^{(1)}(0) \) is the rate of the trigger.

We experimentally demonstrate the anti-bunching effect with the same setup as in Fig. 3.35. The measurement result is shown in Fig. 3.38, where we have normalized the conditional second-order correlation to a large delay, much longer than the ring-down time of the cavity, such that the contributions to coincidence from correlated pairs are negligible compared to that from the accidentals. We then fit the data to a theoretical model to find the value of \( g_c^{(2)}(0) \). Since the coincidence window used \( \tau_{\text{coin}} = 10 \text{ ns} \), the dip in \( g_c^{(2)}(\tau) \) should be convoluted with this coincidence window and will be smeared out. Thus we have fitted the data to a model that takes into account this smearing effect. The \( g_c^{(2)}(0) \) was found to be 0.34, indicating an anti-bunching effect of the heralded single-photon source.

The value of \( g_c^{(2)}(0) \) can be further reduced by reducing the pump power, leading to a smaller value of \( \xi \) in Eq. 3.30. Reducing the pump power will also reduce the second-order correlation signal, which is quadratic in pump power. This will necessarily increase the data collection time quadratically. Currently our experiment is mainly limited by the method we use to measure \( G_c^{(2)}(\tau) \), namely we have to vary the delay by hand and collect the data at each delay for a certain amount of time. With a multi-channel time-tagging electronics, all the delay data can be collected simultaneously, and we expect to measure a much lower \( g_c^{(2)}(0) \) within a reasonable time at lower pump power.
Figure 3.38: Measurement of photon anti-bunching effect. The second order correlation function $g^{(2)}(\tau)$ is measured for different delays between two signal photons, triggered on the detection of an idler photon. We fit the data to an exponentially shaped function whose time constant can be interpreted as the coherence time of the photons. Taking into account the coincidence window of 10 ns, which effectively broadens the dip by convolution, this corresponding to a dip depth of 0.66, and the photons are anti-bunched.
Chapter 4

Time-domain multidimensional quantum information

Quantum information based on multidimensional systems has many advantages over that based on two-dimensional quantum systems (qubits), and is of fundamental and practical interests. It has been used to beat the classical channel capacity with super-dense coding[9], and the information capacity for quantum key distributing channels is shown to be improved compared to qubit-based systems [62]. Higher dimension of Hilbert space may prove easier to implement a detection-loophole-free test of Bell’s inequality [157]. Using higher dimensional systems one can test some of the most fundamental problems in quantum mechanics such as the test of Born rule [149, 146], and simplify certain computation tasks [116, 97] if one has access to high-dimensional quantum gates. In the past, high-dimensional quantum information has been implemented with many different systems. For example, high-dimensional quantum information can be encoded with hyper-entanglement [8], where multi-degree of freedom of the photon are entangled. With Franson interferometers one can also generate multidimensional time-bin entanglement [52, 153]. Different levels of harmonic oscillators can be used to implement multidimensional encodings [117]. Another type of useful high-dimensional systems involve the use
of orbital angular momentum of photons [108, 115].

Encoding information in the time domain of photons is usually difficult due to typically broadband nature of the photons. With our source of long-coherence-time entangled photons from cavity-enhanced PDC, it is now possible encode quantum information with temporal modulation of the phase and amplitude of the photons with commercially available EOMs. Such schemes may provide a convenient way for all optical quantum information encoding and decoding. Practically, the way the information is encoded in the time domain of photons is compatible with current fiber-optic networks. This is advantageous comparing to other encoding schemes such as those with orbital angular momentum or hyper-entanglement, where multiple spatial modes are used and are not compatible with the single-mode nature of optical fibers, and those with multi-dimensional time-bin systems where interferometric stability over the transmission line is required. In addition to quantum communication, the time-domain quantum information encoding schemes can also be extended to implement multi-dimensional time entanglement, which may prove to be useful for distributions of high-dimensional entanglement over long distances.

4.1 Time-domain encoding and decoding

4.1.1 State preparation and detection

We propose a scheme to encode the quantum information in the time domain of the photons, making use of the long coherence time of the cavity-enhanced PDC photons, where EOM and narrowband optical filters can be used to encode and decode high-dimensional quantum information. In such encoding schemes, the basis states of the multidimensional Hilbert space can ideally be an infinite set of orthonormal temporal profiles. In practice, the encoding dimension is limited by the modulation speed of the EOM as well as the photon bandwidth.

The scheme for state preparation and detection is shown in Fig. 4.1a. A narrowband
single photon is fed to the input, and the modulation with EOM prepares the photon in
a temporal profile \( f_k(t) \), which we call a “symbol”. The state of the photon can then be
described as

\[
|\Psi_k\rangle = \int d\omega F_k(\omega) a^\dagger_\omega |0\rangle
\]  

where \( F_k(\omega) \) is the Fourier transform of \( f_k(t) \). For an orthonormal set of profiles \( f_k(t) \),
the overlap \( p_{kj} \) between different profiles can be written as

\[
p_{kj} = |\langle \Psi_j | \Psi_k \rangle|^2 = \delta_{kj},
\]  

where \( \delta_{kj} \) is the Kronecker delta function. This requires the temporal profile function to
satisfy

\[
\left| \int dt f_j^*(t) f_k(t) \right|^2 = \delta_{kj},
\]  

and the set of \( f_k(t) \) forms a basis for the multidimensional Hilbert space.

To measure the photon in a given basis, we use an EOM in a storage loop followed by
a spectral filter (see Fig. 4.1a). In the storage loop, on each round-trip \( j \), the photon is
modulated with the EOM by the pattern \( f_j^*(t)(f_{j-1}^*(t))^{-1} \), such that after \( j \) round trips
it has acquired the mode structure \( f_j^*(t) \), and the photon state can then be written as

\[
U_j |\Psi_k\rangle = \int d\omega \left[ \int d\Omega F_j^*(\Omega) F_k(\omega + \Omega) \right] a^\dagger_\omega |0\rangle,
\]  

where the integral in the square bracket is the convolution of the two phase profiles in
the frequency domain.

A spectral filter can be modeled by the spectral function \( G(\omega) \) as

\[
G(\omega)\hat{a}_{in} = T(\omega)\hat{a}_T + R(\omega)\hat{a}_R,
\]  

where \( T(\omega) \) and \( R(\omega) \) is the spectral function for the transmission and reflection, and
\( a_{in} \), \( \hat{a}_T \) and \( \hat{a} \) are the annihilation operator for the incoming, transmitted and reflected
mode, respectively. At the detector side, the probability of transmission of the state 4.4
through the filter, $\bar{p}_{kj}$, is given by the spectral overlap of the filter transmission $T(\omega)$ and the spectrum of the state,

$$\bar{p}_{kj} = \int d\omega \left| \int d\Omega F_j^*(\Omega) F_k(\omega + \Omega) \right|^2 T(\omega) = \int dt \left| \int d\tau f_j^*(\tau) f_k(\tau) \mathcal{T}(t - \tau) \right|^2,$$

where $\mathcal{T}(t) = \int T(\omega)e^{i\omega t}d\omega$ is the time response function of the filter. In the limit of a sufficiently narrowband filter, the time scale of the filter response, $\tau_{\text{filter}}$, can be long compared to the time scale of the photon wavepacket. In this limit, the time convolution in Eq. 4.6 can be approximated by the product

$$\bar{p}_{kj} \approx \int d\tau f_j^*(\tau) f_k(\tau) \left| \mathcal{T}(t - \tau) \right|^2 = C|\langle \Psi_j | \Psi_k \rangle|^2,$$

which is proportional to the overlap of the $k$-th and $j$-th temporal patterns, with the proportionality constant $C = \int dt |\mathcal{T}(t)|^2$, which corresponding to the transmission probability of the narrowband filter. For $k \neq j$, the photon is reflected at the filter, and is sent back into the loop via a circulator. The cycle is repeated $d$ times for a Hilbert space of dimension $d$, and the photon incident in state $\Psi_k$ will be detected after $k$ round trips. This procedure thus corresponds to a projection in the d-dimensional computational basis.

Such projection measurement can be also extended to the characterization and distribution of multidimensional entanglement in the time domain, as shown in Fig. 4.1b. Entanglement cannot be created from shared randomness, local operations and classical communication, and it is sometimes desirable and necessary to distribute entanglement to remote parties, say Alice and Bob. Assume we have a source of time-entangled photon pairs, and each photon of the pair is sent to either Alice or Bob with a measurement device to perform a d-dimensional projective measurement. Comparing the results of the measurements from the two parties, we can then characterize the bipartite qudit entanglement, following the method shown in [136]. In our scheme, the photons can be sent over fiber-optic networks as the quantum channels, and it is robust against typical decoherence from the fibre. Compared to the Franson-type time-bin entanglement
our scheme relies on the long coherence time of the single photons, and it is not necessary to maintain interferometric stability over a long distance. Furthermore, practically speaking, our scheme may be more scalable since it does not require multiport interferometers. Thus, our scheme could prove to be one of the most adequate ways of qudit entanglement distribution over a long distance.

4.1.2 Two encoding schemes

In practice, filters have finite bandwidths, introducing deviations from Eq. 4.7. For photons encoded in computational basis state $k$, this means there are finite probability of the photons reaching the detector after $j(j \neq k)$ number of round-trips (“inter-symbol interference”). Different encoding schemes may have very different behaviour in inter-symbol interference. In the following, we propose and compare two possible encoding scenarios, and investigate the optimal parameter settings for each case in terms of Error Rate per Symbol (ERS) and channel capacity.

One n-dimensional phase profile scheme, common in classical Code Division Multi Access (CDMA) systems [41], is based on phase-flip modulation (PFM). In this scheme the single-photon pulse is divided into n time intervals with equal intensity integrals, and a phase of $0$ or $\pi$ radians is written on each of the n time sections. By assigning certain phase-profile patterns to these equal power-integral intervals, one can have a zero overlap integral (Eq. 4.7) for orthogonal phase profiles. A systematic method of finding a complete set of orthogonal states is implemented by using rows (or columns) of the Walsh matrix for encoding. An n-by-n Walsh matrix with $n>2$ exists if $n$ is a multiple of 4. The rows (or columns) of a Walsh matrix form a closed set such that the product of any two rows (or columns) of the matrix results in a vector that is also a row (or column) of the matrix,

$$W_{l,:} \times W_{m,:) = W_{k,:},$$

(4.8)

where $W_{l,:}$ is the $l$ – th row of the Walsh matrix. This is a many-to-one mapping that
Figure 4.1: Schematic of high-dimensional quantum communication. (a) Scheme of high-dimensional quantum communication based on single photons. (b) Scheme of entanglement distribution in the time domain.
many different pairs of rows of \( \{l, m\} \) result in the same vector \( W_{k,j} \). This property is very useful in practice when we consider the encoding and decoding with two profiles, in that instead of \( C_n^2 = n(n - 1)/2 \) numbers of possible resultant vectors, we only need to consider \( n \) vectors of the Walsh matrix.

An example of Walsh matrix for dimensional of 16 is shown in Fig. 4.3, where the Walsh matrix elements with values \( \pm 1 \) have been mapped into black (-1) and white (1) colours. Such a phase profile can be written as:

\[
\phi_i(t) = \sum_{i=1}^{n} \frac{\pi}{2} (W_{ij} + 1)(\Theta(t - t_j) - \Theta(t - t_{j+1})), \quad i \in [1, n]
\]  

(4.9)

where \( W_{ij} \) is the matrix element of the Walsh matrix, \( \Theta(t) \) is the step function, and \( t_j \) are the times that satisfy the following integral

\[
\int_{-\infty}^{+\infty} (\Theta(t_j) - \Theta(t_{j+1})) \cdot |f(t)|^2 dt = \frac{1}{n} \int_{-\infty}^{+\infty} |f(t)|^2 dt.
\]  

(4.10)

We have calculated the electric field amplitude of the photon in the frequency and time domain for such phase profiles, as shown in Fig. 4.2. Using the property Eq. 4.8, these are equivalently the resultant profiles after certain encoding and decoding. The bandwidth of the EOM is chosen to be 100 times of the bandwidth of the photons. For single photons with a bandwidth of 10 MHz, this corresponds to the EOM modulation bandwidth of 1 GHz, which is a conservative speed for commercial EOM. The encoding dimension is chosen to be \( d = 16 \), and 6 of all 16 possible resultant profiles are shown here. From the frequency domain picture, we show that except for the first profile, where no phase flip is present, all the other profiles have very low dc components (relative to the carrier frequency) in the spectrum. The first profile corresponds to decoding with the same profile as the encoding one. With a narrowband filter centred at the carrier frequency, such as an optical cavity locked to the centre frequency of the photons, one can then measure the photon in certain encoding phase profile with single-photon detectors. Together with storage loops consist of EOM and circulators, one can repeat
Figure 4.2: Decoding in the time and frequency domain (phase flip scheme). The encoding dimension is set to be $d = 16$, and there will be 16 different resultant profiles from any pair of encoding and decoding profiles. Shown here are 6 of all 16 possible profiles in this basis. Except for the first profile, all the other profiles have very low dc component (relative to the carrier frequency), which is essential to distinguish any two encoding profiles.

such measurement $d$ times corresponding to a projection in the $d$–dimensional Hilbert space.

4.1.3 Linear-phase-ramp encoding

An alternative modulation scheme is by linear-phase modulation, where a phase with linear time dependence is applied to the single-photon pulse, resulting in a frequency shift in the spectrum,

$$
|\Psi_k\rangle = \int d\omega F_k(\omega) e^{i\Delta_k t} a_\omega^\dagger |0\rangle = \int d\omega F_k(\omega + \Delta_k) a_\omega^\dagger |0\rangle ,
$$

(4.11)
Figure 4.3: Colour map of the Walsh matrix. The value of the matrix elements are mapped to black (value = -1) and white (value = 1). Each row (or column) of the matrix has different numbers of phase flips (values change from -1 to 1 or vice versa). The matrix is organized such that the number of flips is in ascending order for different rows (or columns).
where $\Delta_k$ is the rate of the change of the phase, and $F_k(\omega)$ is the Fourier transform of the spectral function of the photons. If the frequency shift due to the linear phase pattern is large enough so that the modulated spectrum has a small overlap with the non-modulated one, the two states can be considered nearly orthogonal, $\langle |\Psi_k\rangle \approx 0$ for $\Delta_k \gg \Delta_{\text{photon}}$, where $\Delta_{\text{photon}}$ is the spectral width of the photons.

For a d-dimensional qudit, the linear-phase-ramp rate $\Delta_k$ is chosen to be

$$\Delta_k = \frac{k\Delta_{\text{EOM}}}{d-1} \quad (4.12)$$

for integer numbers $k > 0$ and $d \geq 2$. Such choices of $\Delta_k$ means equal spacing of nearby profiles in the spectrum, resulting in minimum average overlaps between all possible pairs of profiles.

The encoding dimension is limited by the EOM speed and the bandwidth of single
photons used. To characterize the performance of the encoding scheme with a certain EOM speed, we define a dimensionless quantity $N$ as the ratio of the EOM and photon bandwidth,

$$N = \frac{\Delta_{EOM}}{\Delta_{\text{photon}}}$$

where $\Delta_{EOM}$ and $\Delta_{\text{photon}}$ are the EOM bandwidth and photon linewidth, respectively. Overlaps between different phase profiles (the basis states) give rise to errors of the encoding scheme. In order to have low error rates, the encoding dimension $d$ should satisfy $d \leq N$. We interpret $N$ as maximally allowed number of encoding dimensions, which will become clear in Section 4.2 after we do performance analysis for the encoding schemes.

In Fig. 4.4, we show the time- and frequency-domain representations of the photons for encoding dimension $d = 16$. Only the first 6 states is shown here. Similar to the conditions of the calculation for phase-flip scheme, the EOM bandwidth relative to the photon bandwidth, $N$, is chosen to be 100.

In the next section, we will analyze the performance of these encoding schemes, and calculate error rates based on realistic experimental conditions. In doing so, we wish to provide some guidance in choosing the encoding schemes for experimental implementations.

4.2  Performance analysis

4.2.1  Error rate analysis

In order to characterize the performance of this encoding scheme, we calculate the ERS, defined as the probability of incorrect projections normalized to the total detection rate, averaged over all possible inputs,

$$ERS = \frac{1}{d} \sum_{i=1}^{d} \frac{\sum_{j \neq i}^{d} \bar{p}_{ij}}{\sum_{j=1}^{d} \bar{p}_{ij}} = \frac{1}{d} \sum_{i=1}^{d} \left(1 - \frac{\bar{p}_{ii}}{\sum_{j=1}^{d} \bar{p}_{ij}}\right)$$

(4.14)
where $d$ is the encoding dimension and $\tilde{p}_{ij}$ is defined in Eq. 4.7.

For the phase-flip modulation scheme, the encoding dimension $d$ can be any number that is smaller than the Walsh Matrix dimension $n$. ERS depends on the size of the Walsh matrix. The larger the Walsh matrix, the larger number of phase flips on average is applied to the single photon, which leads to more high-frequency components in the photon spectrum. Therefore, the spectral overlap with the narrowband filter (chosen to match the un-modulated photon bandwidth) is reduced resulting in lower ERS. For infinitely fast EOM, we wish to choose the dimension of the Walsh matrix as large as possible to reduce ERS. For a given EOM speed, however, the Walsh matrix dimension is limited such that the fastest feature of the phase profile is slower than the EOM speed. In Fig. 4.5, we shown the ERS matrix for every input and output states in the computational basis for dimension $d = 32$ for two different values of $N$ as defined in Eq. 4.13. The diagonal of the matrix (anti-diagonal in the pictorial representation) corresponding to the decoding in the correct basis state, and have high detection probability. The anti-diagonals of the error matrix generally have lower error rates. Note that in practice, the code dimension $d$ can be chosen to be smaller than the full Hilbert space dimension $n$. To characterize the performance of the scheme, we wish to choose an optimal set of $d$ vectors within the $n$-dimensional vector space, and calculate its average ERS. In general, it is a computationally expensive optimization process to find the optimal set of size $d$ from $n$ possible codes. Instead, we observe that ERS is correlated with the number of flips (diagonal in Fig. 4.5) in the phase of the photon, and we have chosen $d$ phase profiles such that the products of two of such profiles (which corresponding to the encoding and decoding of the photon) have the largest average number of phase flips. This strategy works well when the encoding dimension $d$ is less than the bandwidth ratio $N$, as we can see from Fig. 4.5a. When $d \geq N$, the solution given by the strategy start to deviate from the actual optimum choice of phase profiles. In Fig. 4.5b this can be seen from the fact that the diagonal of the graph does not corresponds to the smallest error rate,
Figure 4.5: Error rate matrix for phase flip encoding scheme. The encoding dimension $d$ is chosen to be 32, and the bandwidth ratio $N$ is 100 (a) and 10 (b). The diagonal (positive slope in the figure) elements of the matrix correspond to decoding with the same profiles as the encoding, and can be interested as the measurement efficiency of those profiles.

and the optimization process described earlier will not give the optimal set for smallest ERS. However, this strategy may still be instructive in analyzing the performance of the encoding scheme.

We calculate the ERS as a function of encoding dimension $d$, using the strategy given above, for different Walsh matrix dimensions ($n=16, 32, 64$), and for two bandwidth ratios $N = 100$ and $N = 20$, as shown in Fig. 4.6. For $N = 100$, the EOM speed is fast compared to the phase profile time scales, and the ERS decreases when we increase the Walsh matrix dimension, as expected (Fig. 4.6a). In this case, we have the ERS less than 0.6 for encoding dimensions up to 20. In Fig. 4.6b, we study the opposite case, taking $N = 20$. Now the phase profiles have features faster than the EOM for Walsh matrix dimensions $n=32$ and 64, causing the optimization procedure described to deviate from the actual optimal set of vectors. In the limit of $n \gg N$, the ERS approaches the limit.
\((d - 1)/d\) when all the symbols are equally likely to be detected independent of input symbols. We show that for encoding dimensions up to 9, the encoding scheme with \(n=64\) performs worse than that with \(n=32\) due to the slow EOM. In practice, we should always choose the encoding dimension that minimizes the ERS. For this specific EOM speed \(N=20\), we have the ERS less than 0.75 for encoding dimensions up to 20. This is quite high an error rate that may be not practical in reality. Nonetheless, for small encoding dimensions, small error rate is still achievable.

For linear phase ramp scheme, our calculations show that this linear phase scheme results in ERS several orders lower than the phase flip scheme with fast EOM modulation. In practical EOM-based implementations, the speed of the EOM and the bandwidth of the photon can both affect the error rate in the detection. In Fig. 4.7, we have plotted the ERS as a function of encoding dimension, and for illustration purposes, we have relaxed the requirement of \(d \leq N\) and set \(N = 60\) for different EOM modulation bandwidth just to show the behavior of the scheme. For a constant \(N\), the ERS increases with increasing encoding dimension, because of increasing overlap of the spectra of the two encoding profiles. The rate of the ERS increase with \(d\) is larger for smaller \(N\), which is a result of smaller spectral spacing between the different profiles. When the encoding dimension \(d\) approaches or exceeds \(N\), the ERS saturates due to the fact that the spectra of the different profiles are already largely overlapping, and additional overlap does not affect the errors significantly. A faster EOM modulation speed leads to a smaller overlap, and thus a lower ERS. For example, with a 1 ns modulation speed for a 100 ns photon, the ERS can be made lower than 0.6% for encoding dimensions up to 10. The dominant factor of ERS is the spectral overlap of states with different phase profiles, which is determined by the spacing in frequency between two consecutive phase profiles. In Fig. 4.8, we have shown the ERS as a function of the normalized dimension \(d' = (d - 1)/N\), which physically corresponds the number of symbols one encodes per available frequency bin determined by the EOM speed. For a given \(d'\), the spacing in frequency is the same. The ERS for
Figure 4.6: **ERS** of the phase-flip encoding. (a) N=100. (b) N=20.
different EOM speed converges at $d' \sim 1$, as expected. For small $d'$, the ERS is slightly higher for higher dimensions. This is because for the same $d'$, the fixed frequency spacing leads to a fixed intersymbol interference rate for a given pair of symbols, but the larger number of symbols means the total ERS is higher.

With the linear-phase-ramp scheme, one can write arbitrary superpositions as well. An interesting class of superpositions for high dimensions is described by the theory of mutually unbiased bases (MUBs). MUBs have many applications such as quantum state reconstruction [165, 1] where they allow one to maximize information extraction per measurement and minimize redundancy. To illustrate our capability of encoding superpositions with the linear-phase-ramp scheme, we use a construction for MUBs of
Figure 4.8
dimension 4 given in [85],

\[
\begin{align*}
M_0 &= \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}, \\
M_1 &= \{\frac{1}{2}(1, 1, 1, 1), \frac{1}{2}(1, -1, -1), \frac{1}{2}(1, 1, i, -i), \frac{1}{2}(1, -1, 1, 1)\}, \\
M_2 &= \{\frac{1}{2}(1, -1, -i, -i), \frac{1}{2}(1, -1, i, i), \frac{1}{2}(1, 1, -i, i)\}, \\
M_3 &= \{\frac{1}{2}(1, -i, -i, -1), \frac{1}{2}(1, -i, i, 1), (1, i, -1)\}, \\
M_4 &= \{\frac{1}{2}(1, -i, -i, -1), \frac{1}{2}(1, -i, i, 1), \frac{1}{2}(1, i, -1)\}. \\
\end{align*}
\]  

(4.15)

Note $M_0$ is just the computational basis. We calculate the ERS matrix for one of the bases, $M_2$ for example, and the result is shown in Fig. 4.9. We have chosen $N = 100$ and since the encoding dimension $d = 4$ is small compared to $N$, we use a larger filter bandwidth $\Delta_f = 10\Delta_{\text{photon}}$ to reduce the detection loss caused by the filter. The diagonals of the ERS matrix correspond to the detection efficiencies when encoding and decoding with the same states. The state-dependent detection efficiencies are less than 1 due to the loss associated with the encoding scheme. The ERS matrix elements excluding the diagonals show the inter-symbol interference, with the largest value of around 0.03%.

We also consider the effects of dark counts and losses associated with the quantum channels, which are important in actual implementation of such encoding schemes. We show the ERS as a function of the loss in Fig. 4.10. The ERS are calculated for $N=100$ and dimensions $d = 2, 6, 10, 14, \text{and} 18$. The dark-count rate per symbol (DRS) is calculated to be $10^{-5} \times d$ for single photons with 100 ns coherent time, and a dark count rate of 100 /s is used for the single-photon detectors. The factor of $d$ in DRS is due to the fact that a $d-$dimensional state needs $d$ numbers of repetition in the EOM loop for a $d-$dimensional projection. With these parameters, we show that the inter-symbol interference dominates the ERS, and the effect of dark counts is negligible for losses as high as 70%. When we increase the loss to approach 1, the ERS approaches the asymptotic value of $1 - \frac{1}{d}$ as expected, where $d$ is the encoding dimension.
Figure 4.9: ERS matrix in the superposition basis $M_2$. The diagonals of the matrix correspond to the detection efficiency when encoding and decoding with the same states, while the rest show the inter-symbol interference. The detection efficiency is less than 1 due to the loss associated with the encoding scheme, and is state dependent. The largest inter-symbol interference is shown to be around 0.03%.
Figure 4.10: ERS as a function of loss in the linear-phase-ramp scheme. ERS increases as the loss of the quantum channel increases, due finite dark counts at the detector side. At large losses where most of the photons are lost, ERS approaches $1 - \frac{1}{d}$, $d$ being the encoding dimension.
4.2.2 Channel capacity

The ERS calculations in the previous section have shown the relation between the encoding dimension, the EOM speed and the photon bandwidth for a given filter bandwidth. A larger filter bandwidth gives higher transmission for the correct decoding profile, and at the same time, the inter-symbol interference is also increased. Therefore there is a trade-off in filter bandwidth between the transmission and ERS. To optimize over the filter bandwidth, one can calculate the channel capacity to achieve the maximum amount of quantum information that can be transmitted through the quantum channel.

A simplified model for a quantum channel of qudits is given in Fig. 4.11, which describes the information transmission over a lossy channel. Since the additive noise in typical fibre-optic communications is low (background light is negligible), we neglect the contribution from such noise. The effect of the loss of the channel may be complicated since it can be basis-dependent and state-dependent.

Mutual information measures the transmitted information over a certain quantum channels. We use the mutual information $I(X : Y)$ [121] between the input variable $X$ and output variable $Y$ to characterize its effects on information capacity. The channel capacity is defined as the maximum amount of mutual information between the input
and output

\[ C = \max_X I(X : Y). \]  \hspace{1cm} (4.16)

where the optimization is taken over the distribution of the input variable \( X \).

The mutual information is given by the entropy of the variable \( X \) and \( Y \) as

\[ I(X : Y) = H(X) - H(X|Y) \]
\[ = H(Y) - H(Y|X) \]
\[ = H(X) + H(Y) - H(X,Y), \]  \hspace{1cm} (4.17)

where \( H(X) \) and \( H(Y) \) are the marginal entropies, \( H(X|Y) \) and \( H(Y|X) \) are the conditional entropies, and \( H(X,Y) \) is the joint entropy for variables \( X \) and \( Y \). A pictorial representation of mutual information is given in Fig. 4.12.

Comparing the two encoding schemes, the linear-phase-ramp encoding scheme has several advantages relative to the phase-flip scheme for multidimensional encoding, including higher efficiency of detection and lower error rates. As shown in Fig. 4.6, the ERS of phase-flip scheme is much higher than that of the linear phase ramp (Fig. 4.7), when other parameters are held constant. For this reason, in our calculation of the mutual information, we will only consider the linear-phase-ramp scheme. For ideal modulation and demodulation, the detection efficiency should be unity and the ERS should be zero, and there is no information loss over the channel. Practically, however, the informa-
tion can be lost due to less-than-unity detection efficiency and finite ERS. The filter bandwidth affects both the detection efficiency and ERS. To characterize the quantum channel, one can calculate the mutual information between the input and the output for such a channel, and find an optimal filter bandwidth for a certain encoding scheme.

We calculate the mutual information $I(X : Y)$ between the input $X$ and output $Y$, as shown in Fig. 4.13. To get channel capacity one has to take optimization over all possible input states, which is difficult in general [148], especially when the losses are state- and basis-dependent. Here we have limited our choice of parameters in the states of the form

$$|\Psi\rangle = \sqrt{1 - a^2}|\omega_k\rangle + ae^{i\phi}|\omega_j\rangle,$$

(4.18)

where $a$ and $\phi$ are the amplitude and phase parameters, respectively. In order to project the photon state to $|\Psi\rangle$, in general we need to introduce amplitude modulation, since such superposition in the frequency domain means an intensity beating pattern in the time domain, which necessarily leads to loss if we use only passive optical elements with no gain medium, such as EOM. Such loss is basis dependent, and when the superposition has equal amplitudes of different frequencies, the loss is maximized to 50%. However, this is not a limitation of the scheme, since such losses can be calibrated ahead of time for the demodulation basis used.

We can determine the optimum filter bandwidth using the calculated mutual information curve. As shown in Fig. 4.13, for encoding dimension $d = 16$, the optimal filter bandwidth $\Delta_f$ is found to be approximately 1.5$\Delta_{\text{photon}}$; for this value, the mutual information is 3.3 bits/symbol. With a even larger filter bandwidth, the mutual information decreases due to an increased ERS. Since losses are necessary for superpositions, the mutual information is reduced compared to that in the computational basis. The losses is maximum for equal-amplitude superpositions, and for this case the mutual informaton drops considerably, but is still well above 1 bit of information per detected symbol.
Figure 4.13: Mutual information for superpositions, as a function of filter bandwidth used. The encoding dimension is chosen to be $d = 16$, which correspond to 4 bits of information per detected symbol with a uniformly distributed input. For small filter bandwidth, the mutual information increases with the filter bandwidth due to higher detection efficiency. It peaks at about $1.5 \Delta_{\text{photon}}$ with a maximum 3.3 bit/symbol of information. Further increase of the filter bandwidth results in lower mutual information due to increased error rate. Since losses are necessary for superpositions, the mutual information is reduced compared to that in the computational basis.
4.3 Experiments

As shown in the previous chapter, the lifetime of the photons generated from the cavity is measured to be 21 ns. We use commercially available phase modulators (EOSPACE PM-OK5-10) at 10 GHz to encode quantum information to the photon. Due to the availability of low loss circulators and fast optical switch, other than implementing the ideal projection measurements in a basis with the storage loop, we will use a single-pass to implement one decoding vector at a time for the proof-of-principle demonstration. The schematic of the experiment is shown in 4.14. From the pair source of the OPO cavity, the signal mode is sent to the filter cavity, and the arrival time of the photons are measured and recorded by the Time-to-Digital Converter (TDC) electronics. The spectral filtering of the signal projects the idler to a single cavity mode that is narrowband, due to energy conservation in phase-matching conditions. The idler is then sent through the quantum channel for state preparation and projection with EOMs and a second cavity, and for clarity we call this cavity the detection cavity. A pre-calculated set of waveforms is loaded to an two-channel arbitrary waveform generator (Tecktronics AFG-3102) where the two channels are phase-locked with each other. To encode and decode information in the single photons, the EOMs are driven with the arbitrary waveform generator with desired phase profiles. The output of the EOMs are then sent to the detection cavity which is actively locked with a classical laser that is frequency stabilized to a reference atomic transition. Finally the photon is detected with SPCM and TDC for arrival time measurement.

4.3.1 Detection cavity

The detection cavity consists of two high-reflective mirrors in the confocal geometry, where the separation of the cavity mirrors is set to be the same as their radius curvature $R_1 = R_2 = 100$ mm, $L = 100(2)$ mm. The free spectral range of this cavity is thus
Figure 4.14: Experimental setup for high dimensional quantum information. The narrowband single photon from the OPO cavity is sent through the quantum channel for state preparation and projection with EOM and the detection cavity, after which the photon is sent to a SPCM and subsequent arrival time measurement.

FSR\textsubscript{det} = 1.50(3) GHz. Thanks to the good impedance matching of the cavity, it has a high transmission of 82%. This is made possible by using two high quality mirrors (99.0±0.2%) from the same coating run (Layertec). As shown earlier, the optimal detection cavity bandwidth is roughly 1.5\(\Delta\text{photon}\), \(\Delta\text{photon}\) being the single photon bandwidth. In practice, it takes some luck to purchase a coating that is precise enough to the specification that the linewidth of our cavity can be controlled to the precision of a few MHz. We measured the linewidth of the detection cavity to be 6.97(44) MHz, which agrees well with the linewidth estimation from the FSR\textsubscript{det} and the reflectivity of the mirrors. The parameters of this cavity is summerized in Table 4.1.

For such a narrowband source of single photons, the frequency stability is crucial to the experiment. We lock the 780 nm infrared laser of the Toptica TA-SHG-110 laser system to a \(\text{Rb} - 85\) transition line (\(F = 3 \rightarrow F' = 4\)), which offers an absolute frequency reference for our experiment. The infrared laser is then frequency doubled with a SHG
stage, yielding a blue laser as the pump of the down-conversion cavity. The degenerate mode of the down-conversion is then at the same frequency as the Rb reference. The other two optical cavities used in this experiment are also actively locked to the same 780 nm lasers using the Pound-Drever-Hall (PDH) locking technique. The modulation frequency of the PDH is at 20 MHz, which is efficient to generator sizable error signals for the cavities. To avoid the difficulty of separating the single photon pairs from the classical laser beams used to lock the cavities, we use mechanical choppers for the detection and spectral filtering cavities to alternate between injection of locking beam and collection of photon pairs, and similar to the case of the down-conversion cavity, the duty cycles of the choppers are also matched at 24%. The choppers are interlocked with PLL controllers at a frequency of 200 Hz. The synchronized locking trace for all three cavities is shown in Fig. 4.15. Single photons are collected and measured at the locking intervals of the locked periods of the trace.

Table 4.1: Parameters of the detection cavity

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry</td>
<td>Confocal</td>
<td>-</td>
</tr>
<tr>
<td>Radius of curvature</td>
<td>100</td>
<td>mm</td>
</tr>
<tr>
<td>Length(center)</td>
<td>100(2)</td>
<td>mm</td>
</tr>
<tr>
<td>FSR</td>
<td>1.50(3)</td>
<td>GHz</td>
</tr>
<tr>
<td>( \Delta \nu )</td>
<td>6.97(44)</td>
<td>MHz</td>
</tr>
<tr>
<td>( F )</td>
<td>215(14)</td>
<td>-</td>
</tr>
<tr>
<td>Transmission</td>
<td>82(2)%</td>
<td>-</td>
</tr>
</tbody>
</table>
Figure 4.15: Synchronization of locking of the down-conversion, filter and detection cavities. Shown are the traces of photodiode signal of the locking lasers. Single photons are collected and measured at the locking intervals of the locked periods of the trace.
4.3.2 Quantum state reconstruction

To demonstrate the state preparation and projection in the time domain, we perform Quantum State Tomography (QST) on a high-dimensional quantum system. For simplicity and without loss of generality, we use a quantum system with the Hilbert space size $d = 3$, however, the method to encode quantum information is not limited to this size. QST requires measurements in a complete (sometimes overcomplete) set of bases, and since our encoding scheme has basis- and state-dependent loss, we use weak coherent pulse to characterize the system. As shown in Fig. 4.16, the photon is either prepared in state $|\Psi_1\rangle = |1\rangle$ in the computational basis, or the superposition state $|\Psi_2\rangle = \frac{1}{2} (e^{-i\phi}|0\rangle + \sqrt{2}|1\rangle + |2\rangle)$. $\Psi_2$ is prepared with sinusoidal phase modulation with a phase EOM. In general the modulation can be written as

$$|S\rangle = J_{-1}(\beta)|0\rangle + J_0(\beta,\theta)|1\rangle + J_1(\beta)|2\rangle$$

(4.19)

where $J_i(\beta)$ is the $i$-th order Bessel function of the first kind, and $\beta$ is the modulation depth. $\Psi_2$ corresponds to a modulation depth $\beta = 1.162$. With such phase modulation, the size of the full Hilbert space for the system is the number of non-zero modulation side bands plus the carrier, though most of the power is concentrated in the carrier and first few sidebands. For example, the carrier together with the first sidebands contain about 95.3% of the total power at the modulation depth $\beta = 1.162$. In the experiment we restrict ourselves to the subspace of $d = 3$, and are only concerned with the carrier and first side bands of the phase modulation.

As shown in Fig. 4.16, we measure the cavity transmission with a photodiode for different basis. The cavity is being scanned during the experiment, and the transmission corresponds to the spectral function of the photons. Different input states are prepared, including the computational basis state $|\Psi_1\rangle$ and the superposition state $|\Psi_2\rangle$. From these traces we measure photodiode voltages and extract the probabilities for different projections, as shown in Table 4.2. The losses associate with different projections are
Table 4.2: Probabilities of different projections

| Measurements | $|\Psi_1\rangle$ | $|\Psi_2\rangle$ |
|--------------|-----------------|-----------------|
| $|0\rangle \langle 0|$ | 0.0374 | 0.2465 |
| $|1\rangle \langle 1|$ | 0.9006 | 0.4606 |
| $|2\rangle \langle 2|$ | 0.0620 | 0.2929 |
| $|0+1\rangle \langle 0+1|$ | 0.3297 | 0.0796 |
| $|0+2\rangle \langle 0+2|$ | 0.0365 | 0.0330 |
| $|1+2\rangle \langle 1+2|$ | 0.2899 | 0.3690 |
| $|0+i1\rangle \langle 0+i1|$ | 0.4153 | 0.1701 |
| $|0+i2\rangle \langle 0+i2|$ | 0.0456 | 0.6080 |
| $|1+i2\rangle \langle 1+i2|$ | 0.2579 | 0.0949 |

taken into account in the calculation of these probabilities.

Following the quantum state tomography protocol given in Chapter 2, we perform the state reconstruction for the two sets of measurements. The maximum likelihood method of the quantum state estimation gives the result for the state prepared in $\Psi_1$ as

$$
\rho_{H, \text{mlh}} = \begin{pmatrix}
0.0329 & -0.0776 - 0.0340i & -0.0010 + 0.0105i \\
-0.0776 + 0.0340i & 0.8354 & -0.1364 - 0.1593i \\
-0.0010 - 0.0105i & -0.1364 + 0.1593i & 0.0587
\end{pmatrix}
$$

(4.20)

and $\Psi_2$ as

$$
\rho_{\text{sup, mlh}} = \begin{pmatrix}
0.2654 & -0.2584 - 0.1679i & -0.1598 + 0.2291i \\
-0.2584 + 0.1679i & 0.4426 & -0.0047 - 0.2904i \\
-0.1598 - 0.2291i & -0.0047 + 0.2904i & 0.3101
\end{pmatrix}
$$

(4.21)

These density matrix is also plot in Fig. 4.17 and 4.18. From these density matrices, we calculate the purity of reconstructed states for $|\Psi_1\rangle$ and $|\Psi_2\rangle$ to be 0.91 and 0.84, fidelity 0.92 and 0.86, respectively. The less-than-one purity and fidelity of the reconstructed
Figure 4.16: Quantum information decoding in the time domain. The cavity transmission is recorded with a photodiode while the cavity is scanning, which corresponding to the measurement of the spectrum of the input (classical) light field. (a) Data trace when the input is one of the computational basis state ($|\Psi_1\rangle$). (b) Data trace when input is a superposition ($|\Psi_2\rangle$).
Figure 4.17: Reconstruction of density matrix $\rho$ when the input is prepared in state $|\Psi_1\rangle$, with maximum likelihood estimation. The density matrix is represented in the computational basis $\{ |1\rangle, |1\rangle, |2\rangle \}$.

Figure 4.18: Reconstruction of density matrix $\rho$ when the input is prepared in state $|\Psi_2\rangle$, with maximum likelihood estimation. The density matrix is represented in the computational basis $\{ |1\rangle, |1\rangle, |2\rangle \}$. 
states are due to the following sources: finite EOM distinguishing ratio (20 dB), finite signal generator bandwidth (with 30 MHz of modulation bandwidth, each period has only 33 discrete steps, which leads to discretization error), and the locking modulation at 20 MHz also contribute to the systematic error. The superposition state $|\Psi_2\rangle$ has smaller values of purity and fidelity compared to the computational basis state $|\Psi_1\rangle$ due to additional losses involved in superpositions encoding.

### 4.3.3 Characterization of a quantum channel

From the measurement results with weak coherent pulses, it is possible to extend quantum state tomography for the case of single photons. The detection cavity has to be actively stabilized to allow a larger transmission and the losses calibration from the weak coherent pulse can be used. However, the result of tomography depends on a reliable calibration of the measurement devices [129, 105] and in the case of QST with single photons, the calibration of the measurement devices requires photons to be prepared in known states. In this section, instead of quantum state tomography, we use the single photons to characterize the quantum channel consists of the fiber and EOM. The operation of the EOM on the single photon can be written as

$$|\Psi_i\rangle = A(t)e^{i\phi_i(t)} |0\rangle \quad (4.22)$$

In general, the amplitude modulation by EOM inevitably introduces loss, since deterministic noiseless amplifier at the single-photon level does not exist (see, for example, [170] and references therein). It is desirable to use an encoding scheme with pure phase profiles $\phi_i(t)$ only to avoid excessive losses. As in ref. [70], we use discrete frequency bins as the computational basis for our high dimensional quantum system. The simplest higher dimensional quantum system than a qubit is a 3-state system, or a qutrit. For the computational basis, the three eigenstate of the system can be encoded in the frequency
basis as

\[ |0\rangle = |\omega_p - \Delta_e\rangle, \quad (4.23a) \]
\[ |1\rangle = |\omega_p\rangle, \quad (4.23b) \]
\[ |2\rangle = |\omega_p + \Delta_e\rangle, \quad (4.23c) \]

where \( \omega_p \) is the center frequency of the narrowband photon, and \( \Delta_e \) the frequency shift by EOM modulation. A frequency shift of the photon corresponds to a linear-phase ramp in time, and the phase induced by the EOM can be thus written \( exp(\pm i\Delta t) \) for positive and negative frequency shift, respectively. Note that the phase is define modulo \( 2\pi \), so in practice we can drive the EOM with a sawtooth wave with the same ramping slope as the linear-phase ramp. The superposition basis can be achieved using linear combinations of Eq. 4.23, which corresponds to a different set of waveforms for the EOMs. With sinusoidal modulation, we can have the superposition states

\[ |S_\theta\rangle \approx J_{-1}(\beta, \theta) |\omega_p - \Delta_e\rangle + J_0(\beta, \theta) |\omega_p\rangle + J_1(\beta, \theta) |\omega_p + \Delta_e\rangle, \quad (4.24) \]

where \( J_i(\beta, \theta) \) is the \( i \)-th order Bessel function of the first kind, \( \beta = 1.2024 \) is half of carrier-suppressed modulation index, and \( \theta \) is the phase of the sinusoidal modulation. We consider only the carrier and first side bands of this modulation, and higher-order terms are omitted so that we consider only the 3-dimensional subspace of the full Hilbert space. Compared to the modulation described in Eq. 4.19, we introduce an additional phase \( \theta \) to the modulation function. Specifically, we chose the following two vectors for the superposition basis:

\[ |S_+\rangle = J_{-1}(\beta, \theta_+) |\omega_p - \Delta_e\rangle + J_0(\beta, \theta_+) |\omega_p\rangle + J_1(\beta, \theta_+) |\omega_p + \Delta_e\rangle, \quad (4.25a) \]
\[ |S_-\rangle = J_{-1}(\beta, \theta_-) |\omega_p - \Delta_e\rangle + J_0(\beta, \theta_-) |\omega_p\rangle + J_1(\beta, \theta_-) |\omega_p + \Delta_e\rangle, \quad (4.25b) \]

where \( \theta_+ - \theta_- = \pi \), and \( \langle S_+ | S_+ \rangle = 1 \) and \( \langle S_+ | S_- \rangle = 0 \). For state projection, these phase profiles are written to another EOM followed by a detection cavity. We summarize the phase profiles used in Table 4.3.
Table 4.3: Phase profiles written on EOM for state preparation and projection. The five profiles shown are represented as phasors of amplitude 1, which form a subset of all possible profiles that can be achieve by amplitude and phase modulation.

<table>
<thead>
<tr>
<th>State Preparation</th>
<th>Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>0\rangle)</td>
</tr>
<tr>
<td>(</td>
<td>1\rangle)</td>
</tr>
<tr>
<td>(</td>
<td>2\rangle)</td>
</tr>
<tr>
<td>(</td>
<td>S^+\rangle)</td>
</tr>
<tr>
<td>(</td>
<td>S^-\rangle)</td>
</tr>
</tbody>
</table>

Figure 4.19: The projection of a qutrit to different basis. The demodulation vector indices \(\{0, 1, 2\}\) correspond to the computational basis \(\{|0\rangle, |1\rangle, |2\rangle\}\) and indices \(\{4, 5\}\) correspond to the superposition basis \(\{|S^+\rangle, |S^-\rangle\}\).
Table 4.4: Coincidence counts in 15 minutes with background subtracted. The photon was prepared (columns) and measured (rows) in one of the vectors \{\ket{0}, \ket{1}, \ket{2}, \ket{S_+}, \ket{S_-}\}.

<table>
<thead>
<tr>
<th>Counts</th>
<th>\ket{0}</th>
<th>\ket{1}</th>
<th>\ket{2}</th>
<th>\ket{S+}</th>
<th>\ket{S-}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\bra{0}</td>
<td>638.3</td>
<td>5.5</td>
<td>79.9</td>
<td>220.8</td>
<td>391.9</td>
</tr>
<tr>
<td>\bra{1}</td>
<td>78.8</td>
<td>1152.1</td>
<td>62.1</td>
<td>486.8</td>
<td>568.6</td>
</tr>
<tr>
<td>\bra{2}</td>
<td>46.4</td>
<td>39.4</td>
<td>659.1</td>
<td>380.3</td>
<td>204.0</td>
</tr>
<tr>
<td>\bra{S+}</td>
<td>390.5</td>
<td>493.1</td>
<td>198.2</td>
<td>1087.6</td>
<td>48.7</td>
</tr>
<tr>
<td>\bra{S-}</td>
<td>245.2</td>
<td>446.2</td>
<td>280.0</td>
<td>56.7</td>
<td>1059.2</td>
</tr>
</tbody>
</table>

With the five vectors \{\ket{0}, \ket{1}, \ket{2}, \ket{S_+}, \ket{S_-}\} we first performed a truth table measurement, that is, to measure the pair-wise overlap of the five vectors. The data was shown in Table 4.4. Each measurement takes 15 minutes of counting, and the number shown are background subtracted. For any given basis, the detection probability was larger than 88%. The major contribution of the finite cross-talk for orthogonal vectors is from deviations from the ideal modulation waveform for the \textit{EOM} due to finite bandwidth of the waveform generator. Other contributions of the error includes finite single-photon purity of the source, polarization alignment to the axis of \textit{EOM} and fluctuation in the cavity lock. For the superposition basis \{\ket{S_+}, \ket{S_-}\}, the overlap is less than 5(2)%, which clearly shows the coherence between the computational basis. On the other hand, classical probabilistic theory predicts the overlap to be \(|J_0(\beta)|^2 = 44.9\%\), and our result cannot be explained by the classical probabilistic theory.

For further investigate the coherent behavior of the superposition vectors, we varied the phase in Eq. 4.24 in state preparation, and then project to the \ket{S_+} state. The result is shown in Fig. 4.21. We have also calculated the measurement probability under the same condition. There is no free parameter in the simulation, which shows the result we have is in good agreement with the theory. Such coherence cannot be explained by a
Figure 4.20: Graphical representation of truth table 4.4. The left axis in the x-y plane is the measured vectors and the right axis the prepared ones, and the z-axis is the number of photon counts in 15 minutes.
Figure 4.21: Measurement and simulation of the coherence in the superposition state. Shown with the axis to the left (blue) is coincidence counts as a function of relative phase between superpositions (See Eq. 4.24). The simulated result is overlaid with the measurement with single photons, with its axis to the right (green). There is no free parameter in the simulation of the measurement probability. One classical prediction based on probability theory is plotted in red for comparison.

classical model with probabilistic preparation of the different states.
Chapter 5

Conclusion

5.1 Major results

We have presented our theoretical and experimental results on a quantum light source based on cavity-enhanced PDC. This is an effort towards a quantum light-matter interface which can be used to implement the weak-nonlinearity-based quantum computation. Compared to previous developments based on down-conversion, we have shown a two-order improvement in spectral brightness, making it possible to use the source as a light-matter interface. With proper group delay compensation, we have measured the HOM interference of the photon pairs and shown a high degree of two-photon coherence, demonstrating a source of pairs of indistinguishable photons. We have also generated a two-photon N00N state from the source and shown the two-fold improvement in phase sensitivity. Using quantum state tomography, we have measured the density matrix of the photon pair and achieve high fidelity to the desired entangled state. We have also demonstrated both classical and quantum statistics from the same source with modifications in the measurement apparatus.

Exploiting the long coherence time of the photons from such quantum light source, we have shown a novel scheme of implementing high-dimensional quantum information in
the time domain. We have provided two examples of such encoding scheme and analyzed their performance in quantum communication settings. We have calculated the optimal parameters which can be applied to realistic physical implementations. We have experimentally demonstrated the encoding of a qutrit in the time domain of single photons from our narrowband quantum light source. The measurement of coherence of the qutrit rules out any classical probabilistic explanation and thus proves the quantumness of the state. With access to better electronics, it is possible to extend the current implementation to dimensions as high as 10, enabling a broad range of possible applications of qudit systems.

5.2 Future work

With a quantum light source of photons that is compatible with atomic transitions, the obvious next step is to bring the single photons and cold atoms together and characterize the interaction. There are many proposals of using cold atoms as the interaction medium to achieve large nonlinearities between photons (Cross-Phase Modulation (XPM)). While the measurement of XPM of a single photon and another light field at a single shot may be difficult, it is feasible to measure the average single-photon phase shift on a classical beam with enough averages. Such measurements serves as an important step towards the single-shot measurement of XPM of a single photon while not destroying the photon (QND measurement of the photon number), as required by the weak-nonlinearity-based quantum computation.

Besides the obvious goal of the source for XPM experiments, it is the hope that the development of such sources will see other applications in quantum information and quantum computation, as well as other problems of interest in fundamental quantum mechanics. For the narrowband source, based on the frequency-comb-like spectrum of the cavity-enhanced PDC, there has been proposals of implementing the one-way model using
the frequency bins [111]. Another interesting subject to pursue with the source is quantum metrology. We have shown increased birefringent sensitivity with the two-photon $\text{N00N}$ state. It is straightforward to extend to the time-domain $\text{N00N}$ metrology where the two superimposing modes are in different time intervals. This may see interesting applications of phase estimation where the optical power is limited. With our proposal and experimental demonstration of the high-dimensional quantum information in the time domain, there are several interesting experiments as well. Specifically, our system is well suited for characterization of high-dimensional quantum systems with symmetrically information complete positive operator-valued measurements [110, 79, 80]. It can also be used to demonstrate non-classicality using Kochen-Specker theorem [88, 86, 98].
Appendix A

PID controller

PID controllers are common feedback controllers that consists of proportional, integral and derivative gain stages. The block diagram of using a PID controller to control a process, e.g., frequency stabilization of a laser, is given in Fig. A.1, where the controller outputs

\[ K_p e(t) + K_i \int dt e(t) + K_d \frac{d}{dt} e(t) \]  

(A.1)

where \( K_p, K_i \) and \( K_d \) are the proportional, integral and derivative gains, and \( e(t) \) is the error function describing the difference of system response and the target.

In practice, the derivative gain stage slows the transient response of the controller and make it more sensitive to noise which reduces the system stability. Therefore, in many cases the derivative gain stage is omitted in actual implementation. We used the

![Block diagram of PID controller](image)

Figure A.1: Block diagram of PID controller.
optimized P, I parameters from Ziegler-Nichols method [172], and set $K_p = 0.45K_u$, $K_i = 1.2K_p/T_u$, where $K_u$ is the proportional gain at which the controller starts to oscillate, and the $T_u$ is the oscillation period. The circuit schematic for the PID controller is given below.
Appendix B

Time-to-digital converter and coincidence counting

The method to measure the arrival times of photons is described as follows. The single-photon detectors generate a short pulse with a rising edge of \(~300\) ps and a duration of \(35\) ns. This pulse is then discriminated with a fast comparator to generate a trigger to an integrator consisting of a resistor and capacity with a time constant of \(10\) ns. An analog-to-digital converter captures the integrator voltage every \(10\) ns. The exact arrival time of the photon can then be calculated from the integrator voltage.

To measure the coincidence, two of such time-to-digital converters are used and the logic is analyzed with a Field Programmable Gate Array (FPGA). The resolution in arrival time is measured to be \(21\) ps, much smaller than the jitter \((450\) ps\) of the single-photon counting module. We used an embedded controller (Rabbit RCM3200) to handle the communication between the FPGA (Altera Cyclone II) and the host computer. The data rate is currently limited by the speed of the embedded controller, and using assembly language we have reached a limit of \(36\) kps for the time tagging of the coincidence. For many time-tagging applications in this project, this is sufficient. The limit can be improved using faster embedded controllers or a better design of the architecture. The
schematic of the electrical circuits for our time-resolved coincidence counting circuits are given below.
Appendix B. Time-to-digital converter and coincidence counting

Photodiode "A" Timing

[Diagram of photodiode timing circuit]

Coincidence Timer

Steinberg Lab

University of Toronto Quantum Optics Group

Drawn by A. Stummer Oct'07

Description

Edited, not saved

Sheets 1/3

Rev. A1
Appendix B. Time-to-digital converter and coincidence counting
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