Non-Contact Measurement of Dynamic Belt Span Tension in Automotive FEAD Systems

by

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A thesis submitted in conformity with the requirements for the degree of Master of Applied Science
Graduate Department of Mechanical and Industrial Engineering
University of Toronto

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Abstract

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The proposed tension measurement method focuses on measurement of tension in a single belt span of the multi-pulley Front End Accessory Drive (FEAD) system. The mean belt span tension is calculated from the measured belt span natural vibration frequency. The oscillation belt span tension is calculated from the measured belt span transverse displacement. The dynamic belt span tension is calculated using the mean and oscillation tensions, belt span support pulley rotations, and belt slip, where the slip equation is based on Euler’s equation. The proposed tension measurement method is validated using an experimental testing FEAD system which consists of a five pulley system and an automatic tensioner arm. Non-contact sensors are used to prevent disruption of the typical system behavior. Testing conditions simulate typical engine crankshaft rotation input. Results from experimental testing consistently produce results with percent error less than 10% for mean and maximum belt span dynamic tension.
Dedication

To Adam Oatman,

for being there on the successful days to share the excitement,
for being my support on the days that I struggled,
and for making me smile when I worried.
Acknowledgements

I would like to sincerely thank Prof. Jean Zu for having given me this opportunity to continue my studies, for her supervision and support, and for her continual confidence in me. As a mentor her personal guidance and lessons will prove invaluable, thank you for sharing with me so much of what you have learned from your career.

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Lastly, but not least, my family and friends. Thank you for your continued support and for your understanding of the time I must take away from home to pursue my studies.
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# Nomenclature

## Greek Letters

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<td>( \alpha_i )</td>
<td>Arc of slip</td>
<td>44</td>
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<td>( \beta )</td>
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<td>[rad/s²]</td>
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<td>( n^{th} ) belt loop frequency</td>
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<td>( \omega_n^i )</td>
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\[\omega^\text{Alt}_n\] Alternator 1 Frequencies, page 73 \[Hz\]

\[\omega^\text{CS}_n\] Crankshaft Frequencies, page 74 \[Hz\]

\[\omega^\text{Id}_1\] Idler 1 Frequencies, page 72 \[Hz\]

\[\omega^\text{Id}_2\] Idler 2 Frequencies, page 73 \[Hz\]

\[\omega^\text{sys}_n\] System frequencies, page 71 \[Hz\]

\[\omega_n\] Belt span transverse vibration frequency, page 17 \[\text{rad/s}\]

\[\omega^\text{ex}_n\] System excitation frequency, page 65 \[\text{rad/s}\]

\[\phi\] Belt span rotation angle in the z-plane, page 117 \[deg\]

\[\phi^\text{Galerkin}^\text{(x)}\] Galerkin basis function, page 21

\[\rho\] Linear belt density, page 13 \[\text{kg/m}\]

\[\tau\] Torque excitation amplitude, page 65 \[N\cdot m\]

\[\tau^\text{c}\] Constant torque input, page 65 \[N\cdot m\]

\[\theta^\text{i}\] Pulley rotation angle, page 34 \[\text{rad}\]

\[\varphi^\text{n}\] Pulley wrap angle, page 45 \[deg\]

\[\zeta^\text{n}\] Damping coefficient, page 41 \[\text{N.s/m}\]

**Vectors and Matrices**

\[M\] Belt span bending moment, page 119 \[N\cdot m\]

\[C^\text{b}\] Belt damping matrix, page 38

\[C^\text{r}\] Rotational damping matrix, page 38

\[C\] Damping matrix, page 38
G_d  Belt span discretized gyroscopic matrix, page 24

K_b  Belt stiffness Matrix, page 38

K_d  Belt span discretized stiffness matrix, page 24

K_r  Rotational stiffness matrix, page 38

K   Stiffness matrix, page 38

L   Cholosky of the inertia matrix, page 39

M   Inertia matrix, page 39

M_d  Belt span discretized mass matrix, page 24

P   Eigenvector matrix of the discrete system, page 40

q(t)  Coordinate transform of the discrete system, page 39

Q   Torque, page 37

R_t  Rotational matrix, page 38

T_c  Intermediate damping matrix, page 38

T_k  Intermediate stiffness matrix, page 38

v_i  Eigenvector of the discrete system, page 40

Λ   Eigenvalue matrix of the discrete system, page 40

q(t)  Decoupled damping matrix, page 41

Roman Letters

T   Kinetic energy, page 15              \[ J \]

\( \mathcal{U} \)  Strain energy, page 15   \[ J \]
$U_a$ Axial strain energy, page 118  
$\left[ J \right]$

$U_b$ Bending strain energy, page 119  
$\left[ J \right]$

$\mathcal{Y}$ Potential energy, page 15  
$\left[ J \right]$

$a$ Belt axial acceleration, page 16  
$\left[ \frac{m}{s^2} \right]$

$c$ Wave speed, page 16  
$\left[ \frac{m}{s} \right]$

$D$ Modal analysis constant, page 18

$d_b$ Belt depth, page 54  
$\left[ m \right]$

$d_i$ Pulley pitch diameter, page 54  
$\left[ m \right]$

$E$ Modal analysis constant, page 18

$EA_b$ Belt longitudinal stiffness, page 34  
$\left[ N/m/m \right]$

$I_i$ Pulley mass inertia, page 34  
$\left[ kg \cdot m^2 \right]$

$I_t$ Tensioner arm mass inertia, page 34  
$\left[ kg \cdot m^2 \right]$

$J$ Belt span moment of inertia about the belt cross sectional axis, page 119  
$\left[ m^4 \right]$

$k$ Axial Belt Stiffness, page 118  
$\left[ \frac{N}{m^2} \right]$

$k_{b_i}^s$ Belt span longitudinal stiffness coefficient, page 34  
$\left[ \frac{N}{m} \right]$

$k_{b_i}^p$ Pulley belt span stiffness coefficient, page 47  
$\left[ \frac{N}{m} \right]$

$k_{ri}$ Rotation stiffness coefficient, page 36  
$\left[ \frac{n \cdot s}{rad} \right]$

$l$ Belt span length, page 13  
$\left[ m \right]$

$l_b$ Belt length, page 54  
$\left[ m \right]$

$l_i$ Length of the $i^{th}$ belt span, page 34  
$\left[ m \right]$
<table>
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<tr>
<td>$l_i$</td>
<td>Quarter belt length change due to transverse oscillation</td>
<td>32</td>
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<tr>
<td>$l_{osc}$</td>
<td>Change in belt span length due to transverse vibration</td>
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<td>$m_b$</td>
<td>Belt mass</td>
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<td>$P$</td>
<td>Constant</td>
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<td>$p$</td>
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<td>$s$</td>
<td>Belt span eigenvalue</td>
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<td>$T$</td>
<td>Axial belt tension</td>
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<td>$t$</td>
<td>Time</td>
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<td>$T^m_i$</td>
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<td>$T^d_i$</td>
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<td>$T^o_i$</td>
<td>Oscillation tension</td>
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<td>$T^p_i$</td>
<td>Tension in span wrapped on the $i^{th}$ pulley</td>
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<td>$W$</td>
<td>Belt span eigenfunction</td>
<td>17</td>
</tr>
<tr>
<td>$w$</td>
<td>Belt span transverse displacement</td>
<td>14</td>
</tr>
</tbody>
</table>

Dimensions:
- $[m]$ for length and displacement
- $[kg]$ for mass
- $[N]$ for force and tension
- $[sec]$ for time
- $[m/s]$ for velocity

xvi
\( w_b \)  Belt width, page 54  \([\text{m}]\)

\( x \)  Translational coordinate, page 14  \([\text{m}]\)

\( x_i^p \)  Belt-Pulley slip, page 43  \([\text{m}]\)

\( x_i^a \)  Belt length change due to support pulley rotation, page 34  \([\text{m}]\)

\( X_i \)  X-Coordinate of the \( i^{th} \) pulley, page 54  \([\text{mm}]\)

\( y \)  Transverse coordinate, page 14  \([\text{m}]\)

\( Y_i \)  Y-Coordinate of the \( i^{th} \) pulley, page 54  \([\text{mm}]\)

\( y_i \)  Transverse belt displacement at the \( i^{th} \) displacement sensor, page 32  \([\text{m}]\)

\( z \)  Depth coordinate, page 14  \([\text{m}]\)

\( \mathbf{r}(t) \)  Coordinate transform of the discrete system, page 41
Chapter 1

Introduction

1.1 Motivation

Front End Accessory Drive (FEAD) systems are used in automobiles to transfer power from the engine to engine accessory components such as the air conditioner, alternator, water pump, etc. through the use of a serpentine belt. The introduction of new engine accessory components, such as the one-way clutch or automated starter-generator, have led to increased industry demand for a method to accurately measure the dynamic tension forces occurring in the FEAD system belt spans during system operation. This is especially true for the automated starter-generator as both the crankshaft and automated starter-generator will be used to drive the system causing large changes in the tension of each belt span in the system. Furthermore, if a real-time dynamic tension measurement device were available for automotive installation active tension control systems could be developed. Accurate belt tension measurement is essential as an over-tensioned belt will result in belt wear and higher than required bearing hubloads while an under-tensioned belt can result in belt squeal.
Chapter 1. Introduction

1.2 Literature Review

A brief review of the literature related to the study of FEAD system belt tension is presented. The review is divided into three sections. First, papers from the class of axially moving continua, i.e. the belt span, which are relevant to this work are reviewed. Second, the papers related to the whole of the discrete FEAD system where the belt spans are treated as axial springs and the pulleys as rotating lumped masses is discussed. Last, a review of work done on belt pulley slip and related friction equations is included.

1.2.1 Axially Travelling Continua

Published literature extensively covers the varying levels of complexity in the problem of an axially moving string or beam. Papers on the axially moving continua problem were initially published in the 1950’s and 1960’s. The papers focused the development of variations of the linear equation of motion. A resurgence of study on the subject occurred in the early 1990’s with a large body of work produced on the linear and nonlinear equations of motion. During this resurgence period many aspects of the moving continua problem were investigated, such as the inclusion of gyroscopic terms in the equation of motion or the study of more complex boundary conditions. This review will focus on the initial published papers and further review more complex papers which are applicable to this study.

In 1953 Sack [2] published the fundamental equation of motion for a string moving with constant axial velocity over smooth supports with the inclusion of forced oscillations and linear damping. The equation of motion was developed from the well-known wave equation the derivation of which can be found, among other sources, in the work of Rao [3]. Sack showed that the increased axial velocity of the continua corresponded to reduced natural vibration frequencies and stated “the longitudinal motion of the string
causes a reduction in the resonance frequencies the relative reduction amounting to $\frac{v^2}{c^2}$ [2], where $v$ is the belt axial velocity and $c$ is the wave speed. This publication was an extension of a treatment of the problem completed in 1897 by Skutsch [4] in which the superposition of two waves travelling in opposite direction in the continua was studied. Mahalingham [5], in a study of chain drives, recognized that the centrifugal tension of a chain wrapped on pulleys would increase the wave speed and reduce the instabilities of the axially moving continua. Mahalingham used a heavy chain, or “idealized uniform heavy string”, to study the effect of including centrifugal tension in the model and also included a study of forced vibration due to transverse and longitudinal excitation at the continua ends.

Miranker [6] reviewed derivation of the equation of motion for a tape moving between simple supports and then used Hamilton’s principle to derive the equation of motion with the assumption of constant axial tension. The equation is among the first to include the axial acceleration term. Miranker succeeded at providing a second solution for the axially travelling string equation by revisiting the concept of decomposing the solution into the summation of two waves traveling in opposite directions and then proceeded to consider forced vibrations of the system. As well, Miranker [6] showed that there is a periodic transfer of energy into and out of the axially moving systems varying at twice the free vibration frequency. In 1963 Swope and Aims [7] studied the travelling string equation with a boundary condition modeled for a string transversing an axially upright bobbin. The problem is solved using D’Alembert method and Method of Characteristics.

Mote [8] and Barakat [9] introduced the flexural rigidity of a beam into the equation of motion for a continua with constant axial velocity. Mote [8] considered periodic tension variations due to support eccentricities. Galerkin and numerical methods were used to solve the equation of motion for the fundamental frequencies and a band saw example
calculation was included. In 1973 Simpson [10] used Hamilton’s principle to derive the equation of motion for the translating beam with axial rigidity and then provided a study of the effect of axial velocity on frequency and mode shape for clamped-clamped boundary conditions.

Tabarrok et al. [11] developed coupled nonlinear equations for an axially accelerating cantilevered beam using rotational and translational equilibrium equations developed via Newton’s second law with only the restrictions of Euler-Bernoulli beam theory. Assumptions of small deflection gradients and an inextensible beam allowed the presentation of a simplified solution. The relation given, but later disregarded, between axial tension and beam slope is similar to the stationary equation offered by Tolonen [12].

Wickert et al. [13] presented derivation of the first order canonical form of the equation of general axially moving continua using the orthogonality relations of the eigenfunctions. A solution was presented using a modal method similar to that offered by Rao [3]. This solution method possesses greater accuracy at higher velocities compared to previous methods.

Also disregarding the assumption of constant axial tension, both Pakdemirli et al. [14] and Zwiers et al. [15] presented analysis of the accelerating string. Pakdemirli et al. [14] assumed the velocity term is sinusoidal and used Galerkins method to discretize the partial differential equation and obtain ordinary differential equations. Floquet theory was used to carry out stability analysis and the results were compared to an analytical solution. Even term approximations better captured the gyroscopic nature of the coupled string vibration and were better related to the analytical comparison.

Kong [16] showed that belt bending stiffness, EI, increased the tension, as well as the deflection, in the belt spans. The study found that these effect were more pronounced in
short belt spans, thick belts, or low tension belt spans.

Additional papers which have been reviewed and are relevant have been included in the works cited.

1.2.2 Lumped Mass Belt-Pulley Systems

In the FEAD system with fixed center pulleys there are two distinctive types of system vibration modes, rotational modes of the pulleys and transverse modes of the belt spans, Houser [17]. In the rotational modes the accessories rotate about their axes and the belt spans act as linear axial springs. In transverse modes, the belt spans vibrate transversely, similar to a taut string. Results indicate that vibration modes include both rotational and transverse vibration. The rotational and transverse vibration modes are uncoupled in the fixed center system and become coupled with the introduction of an automatic tensioner arm. However, in this study, it is assumed the system modes remain decoupled. Both rotational and transverse vibration induces dynamic tension. Transverse vibration of the belt spans was reviewed in the previous section. Here, studies of rotational vibration of the pulleys about their axis is presented.

Hawker [18] studied the natural frequencies of prototypical two and three pulley damped drive systems with and without an automatic tensioner arm. The effects of the belt stiffness and pulley inertia on belt tension were investigated with good agreement between theoretical and experiential testing. Hawker used the tensioner arm position equations developed by Ulsoy [19] to determine the linearized system equations.

Zu [20] developed the set of equations of motion for the FEAD system as a system of rotating pulleys connected by the belt spans treated as linear springs. The dynamic equations of motion were used to find the tension of a belt span as a function of the
pulley rotations. A transfer function is developed for the system from which the response of the system to crankshaft excitation is determined.

Barker, Oliver and Breig [21] included the change in belt length generated by stretching of the belt over the arc of the pulley. The equations are applicable in a system with or without an automatic tensioner arm. The tension force in the belt over the arc of the pulley was averaged from the tension in the two adjacent belt spans. The change in belt length was calculated using the belt stiffness coefficient and Hooke’s law. The system of linear equations developed were solved simultaneously. Balaji and Mockensturm [22] expanded on the work of Barker, Oliver and Breig by included a decoupler/isolator in the analysis and went on to study the tension fluctuation caused by the introduction of the automatic tensioner arm. Balaji and Mockensturm [22] also included damping in the system model.

Beikmann, Perkins and Ulsoy [23] evaluated a prototypical drive composed of a driven pulley, a driving pulley, and a dynamic tensioner. Modal tests on the experimental drive confirmed the theoretical predictions. Dynamic tensions in the spans are prescribed by torque variations in an adjacent driven accessory. The dynamic tensions parametrically excite transverse vibrations, leading to Mathieu type instabilities [19].

**Belt-Pulley Friction**

Differences in the tension values of the two spans adjacent to a pulley can cause the belt to slip, or creep, on the pulley. The slip will always from the span of lower tension to the span of higher tension. The slip can be measured using the difference in rotational speed of the pulley and axial speed of the belt as the belt will have a lower speed than the surface of the pulley when slip occurs [24], however these are difficult values to measure and other calculation methods have been developed. One of the first and most well-
established methods used to determine belt slip is Creep Theory, introduced by Reynolds [25] the theory was further formulated by Swift [26]. Creep Theory assumes that the belt stretches, or extends, while wrapped on the pulley and formulates creep is a function of the axial speed difference between the belt spans entering and exiting the pulley.

For application to inextensible belts Firbank [24] expanded upon the work of Euler [27] and introduced a new method of calculating belt slip, commonly referred to as Shear Theory. Firbank examined the shear strain in the belt at the pulley entry point and determined that there must exist a “zone of adhesion” over which the belt does not slip and the static friction coefficient, $\mu_s$, is applicable. Furthermore, he determined that at a certain point a “limiting value” is reached where the shear strains overcome the friction forces and the belt begins to slip on the pulley, in this zone the kinetic friction coefficient, $\mu_k$, is applicable. Thus, the arc of contact is divided into the arc of adhesion and the arc of slip, defined at the point of limiting friction. Examining the tension change over the arc of slip and the arc of adhesion Firbank introduces an equation relating the entering and exiting span tensions based on assumptions of low inertia, absence of tangential friction forces, belt adherence to the pulley surface, and constant $\mu_k$ and $\mu_s$ values. However, it is noted “the experimental evidence is that $\mu_k$ is not constant and depends on such factors as rubbing speed, presence of moisture, and dust, surface temperature, etc. Moreover, in the circumstances, the limiting value of the static friction coefficient $\mu_s$ is probably dependent on amount of vibration present in the drive” [24]. Ultimately, Firbank assumes these influences to be negligible in the interest of calculation ease. Firbank verified his theory by experiments and concluded that more power is transmitted by static friction than by kinetic friction.

Alciatore and Traver [28] investigated both Creep and Shear theory as well as application of shear theory in multi-pulley belt drives. For a multi-pulley system two compatibility
conditions are given, the first requiring that the total belt length remain constant, as the belt is inextensible, the second that the algebraic sum of the shear deformations must equal zero while running. As well, modifications for inclusion of bending effects and centrifugal tension and a review competed of Firbank’ work [24] extended it to included the slip equation for a driven pulley.

Meckstroth [29] designed a dynamic testing procedure to determine the friction characteristics between a pulley and belt in both wet and dry conditions. The experiments conducted re-confirmed the validity of Shear Theory. Typical K-ribbed belts ranging from 3-8 ribs were used in the experiments and friction coefficients were adjusted to include the extra friction force generated by the v-shape of the belt ribs. Meckstroth [29] found dynamic friction coefficients in the range of $\mu_k = 0.7 - 2.1$ for dry conditions and $\mu_k = 0.05 - 0.59$ for wet conditions. Tests for friction coefficients had good repeatability results.

Gerbert [30] presented a unified theory for belt creep and slip in V-belts and V-ribbed belts with focus on small pulleys with relatively thicker belts. Rubin [31] generated an exact solution for an extensible belt using nonlinear steady state equations of the string and including the effects of “Coulomb friction, radial and tangential accelerations, and the power loss due to friction between the belt and pulleys”. Gerbert and Sorge [32] further investigated Shear Theory and developed a detailed analysis of the v-belt friction forces. Kong and Parker [33] studied the two-dimensional forces of the belt in pulley groves for a two pulley system. The more detailed investigations into belt slip were briefly included here for completeness of the review, but are beyond the scope of this work. The friction modeling completed by Firbank [24] and Alciatore [28] will be used with simplifying assumptions.
1.3 Objectives

The primary objective of this work is to develop a dynamic tension measurement method for a belt span in a FEAD system. The final tension measured should accurately measure of the belt span mean tension while including the tension modulation which occurs due to belt and pulley vibration, namely the belt span dynamic tension. The measurement method should be applicable to the FEAD system regardless of crankshaft input profile, be it constant rotational input, accelerating rotational input, or an excitation frequency with varying torque amplitude. Also, the measurement method should be designed in such a way that the typical system behavior is not disturbed during measurement. For this reason, during experimental validation non-contact sensors will be used. The objectives are summarized as:

- Measurement method for dynamic tension in a FEAD system belt span,
- Measurement method is is applicable to all FEAD system operating conditions,
- and, the measurement method does not disrupt the system behavior.

1.4 Outline of Thesis

To find dynamic tension in the FEAD system the mean and oscillation belt span tensions are first found and then used to solve for the belt span dynamic tension. The mean tension of the belt span is found from the measured natural frequencies of the axially travelling string and beam equations. The axially travelling string equation is solved using modal analysis to find the frequency equation, or eigenvalue(s). The beam equations are solved using Galerkin method using three basis functions, namely, the mode shapes of the stationary beam with clamped-clamped-, clamped-pinned, and pinned-pinned boundary conditions to find the eigenvalues. This results in four belt span models, three with beam rigidity and varying boundary conditions and one without without beam rigidity. The
resulting four tension models are arranged such that mean tension is a function of belt span frequency and axial velocity. Nicely, the mean tension measurement includes the increase in tension due to centrifugal forces as the increased tension due to centrifugal tension increases the belt frequency.

As well, it is desired to capture the increase in tension due to belt span transverse vibration. This measure of tension is termed the span oscillation tension and will vary at twice the frequency of the span vibration. The oscillation tension is found by directly measuring the transverse displacement of the belt span and determining the resulting change in belt length, and thus increase in belt tension.

The dynamic tension of the belt span is found as a function of the span’s support pulley rotations and belt slip and also utilizes the mean and oscillation tensions. As the support pulley’s rotate the difference in respective arc lengths will increase or decrease the span tension due to deflections of the span length, which is treated as a linear spring. As well, during operation, the belt may slip from the “driven” support pulley and cause a decrease in span length with a corresponding decrease in belt tension. The belt slip is taken into account using the Shear Theory equation mentioned in the literature review. The equation takes into consideration the belt-pulley friction coefficients as well as the arc of slip. The dynamic tension incorporates the mean and oscillation tension as they are used to determine belt slip as well as to establish the span mean tension.

Validation of the proposed dynamic tension measurement method is completed using a FEAD testing system. The results obtained from the experimental testing system are compared with measured hubload values to determine tension measurement accuracy. The testing system consists of a five-pulley FEAD system and an optional tensioner arm connected with a v-ribbed belt as illustrated in Fig. 1.1, below:
Experimental measurement of span tension is focused on the belt span between the air conditioner pulley and alternator pulley. Non-contact laser displacement sensors are placed along the span length to determine the oscillation tension as well as the span natural frequency needed to calculate the mean tension. The four models of the belt span used to determine mean tension are compared with a contact validation hubload sensor measurement to determine the most accurate model. Non-contact laser rotation sensors are placed on the alternator and air conditioner pulleys to determine pulley rotation angles for dynamic tension measurement. As well, the air conditioner rotation measurement is used to determine axial belt speed. As it is most interesting to study the belt when axial belt tension is maximized, system excitation frequencies are selected which correspond to belt fundamental frequencies.
Chapter 2

Theoretical Modeling

2.1 Mean Tension from Fundamental Frequency

The belt span axial tension cannot be directly measured using any non-contact sensor currently available. It was therefore decided early in this project that another parameter in the system must be measured and used to calculate the belt span tension. The eigen-value(s) of the axially traveling belt span equation of motion relate the belt span natural vibration frequency to the belt span axial tension. This relationship allows the belt span natural frequency to be measured and used to calculate mean belt span axial tension. As the axial tension increases the belt span vibrates at higher natural frequencies. This is similar to a tuning a guitar string where the tuning pegs are turned to adjust the string tension and thus the tone, or frequency, the guitar string produces when plucked. The belt span of the FEAD system behaves in a similar manner to the guitar string with only the additional complication of axial speed, boundary conditions, and the rigidity of the belt to consider.

The inspiration for this measurement concept came from the well known Clavis gauge. The Clavis gauge is a reliable measurement method for the stationary belt span axial
tension which is commonly used in industry. Fawcett, Burdess, and Hewit in 1989 [34] first introduced the concept of using belt span frequency to indicate belt span tension. A method of measuring the stationary\(^1\) belt span axial tension using the measured span natural vibration frequency was later patented by Burdess in 1990 [35]. The patent is based on the transverse vibration natural frequency equation, Eq. (2.1), for a string of length, \( l \), and density, \( \rho \), pinned between simple supports with axial tension, \( T \). The equation is derived from the equation of motion for the stationary belt span which is available from many sources including *Mechanical Vibrations* by Rao [3].

\[
\omega_n = \frac{n \pi}{l} \sqrt{\frac{T}{\rho}} \quad (2.1)
\]

Variations of Eq. (2.1) take into consideration such variables as the belt flexural rigidity or different span end support, or boundary, conditions. In application, plucking the stationary belt span to excite transverse vibration and measuring the lowest natural vibration frequency, \( \omega_1 \), allows for easy calculation of the belt span axial tension. Rearrangement of Eq. (2.1) permits tension calculation for the first vibration mode\(^2\):

\[
T = \rho \left( \frac{\omega_1}{\pi} \right)^2 \quad (2.2)
\]

In this work a similar method to the Clavis gauge is developed to measure the mean tension of the axially travelling belt span. The belt span is modeled as a string or beam travelling between simple or clamped end supports as depicted in Fig. 2.1, next page.

\(^1\)Stationary indicates that the belt span has no axial speed.
\(^2\)In the first vibration mode the frequency used is the lowest measured natural vibration frequency and \( n = 1 \).
Figure 2.1: Axially Travelling Belt Span Model

The equation of motion for the belt span is developed in which vibration is limited to the transverse direction, or $z$-plane. The transverse vibration frequency is chosen to be measured, rather than longitudinal or rotational vibration frequencies, as the transverse vibration mode is the dominant mode of a traveling belt or string [36]. The equation of motion is solved using both modal analysis and Galerkin method to find the natural frequency, or eigenvalue, equation(s).

From the natural frequency, or eigenvalue, equation(s) the mean tension equations is found relating axial belt tension to belt transverse vibration frequency. As the belt transverse vibration frequency is measured using data sampled over a period of time it is generally considered a constant. Thus, the mean tension equation gives the mean tension value which fluctuates only with the changes in axial velocity. Stability analysis is completed to determine the stable velocity regions over which the mean tension equation is valid for application.
2.1.1 Belt Span Equation of Motion

The equation of motion for the translating belt span is derived using Hamilton’s Principle. The system is considered conservative and the equation of motion is found using the well-known equation Hamilton’s equation:

$$\delta \int_{t_1}^{t_2} [\mathcal{T} - (\mathcal{U} + \mathcal{V})] \, dt = 0$$  \hspace{1cm} (2.3)

The velocity terms related to the span are summed to find total kinetic energy, $\mathcal{T}$. The axial strain and bending moment strain are summed to find total belt strain energy, $\mathcal{U}$. The potential energy, $\mathcal{V}$, of the span is considered negligible as gravity is ignored as the belt mass is low. The list of assumptions used in the development of the equation of motion are itemized below [8], [2].

- axial belt velocities are sub-critical\(^3\)
- constant uniform material properties of the belt
- gravitational acceleration is negligible
- negligible belt damping
- small transverse belt displacements
- the effect of air drag is negligible
- uniform tension along the length of the belt

The belt span equation of motion is reprinted below from the detailed derivation in Appendix A, page 121:

$$\rho \left[ \frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} + v^2 \frac{\partial^2 w}{\partial x^2} \right] + EJ_b \frac{\partial^4 w}{\partial x^4} - T \frac{\partial w^2}{\partial x^2} = 0$$  \hspace{1cm} (2.4)

\(^3\)Sub-critical axial belt velocities refer to velocities less than the wave speed, defined on page 16, Eq. (2.6).
Where the equation terms represent:

- $\frac{\partial v}{\partial t} = a$, Axial Acceleration
- $\frac{\partial^2 w}{\partial t^2}$, Local or Transverse Acceleration
- $v^2 \frac{\partial w}{\partial x}$, Centripetal Acceleration
- $2v \frac{\partial^2 w}{\partial t \partial x}$, Coriolis Acceleration
- $\frac{EJ_b}{\rho} \frac{\partial^4 w}{\partial x^4}$, Belt Flexural Rigidity
- $T \frac{\partial^2 w}{\partial x^2}$, Axial Tension\(^4\)

Assuming flexural rigidity, $EJ_b$, is negligible, Eq. (2.4) leads to the equation of motion for the axially traveling string. For this equation to be valid the belt length should be sufficiently long as to cause the belt to behave with low rigidity as a string would. If the belt length is very short the end supports provide rigidity to the belt span and cause it to act more as a beam than as a string. If the belt span length is long it will become more limp and behave less rigidly. As well, as the flexural rigidity of the belt increases this assumption becomes less valid. The equation of motion for the axially travelling string is displayed in Eq. (2.5), below, and was previously derived in published papers by Swope and Ames [7] as well as Archibald and Emslie [37] and others.

$$\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} + \left( v^2 - c^2 \right) \frac{\partial w^2}{\partial x^2} = 0 \quad (2.5)$$

The wave speed in the axial belt span, $c$, is defined as:

$$c = \sqrt{\frac{T}{\rho}} \quad (2.6)$$

\(^4\)This term is a restorative term. The greater the belt curvature, the greater the acceleration while a belt with no curvature, i.e. in the equilibrium position, experiences no acceleration.
If flexural rigidity term remains in Eq. (2.4) the equation of motion for the axially traveling beam is defined as:

$$\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial t \partial x} + \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} + \left(v^2 - c^2\right) \frac{\partial w^2}{\partial x^2} + \frac{EJ_b}{\rho} \frac{\partial^4 w}{\partial x^4} = 0$$  \hspace{1cm} (2.7)

Variations of Eq. (2.7) have been previously derived in the literature by Tobarrok [11] and by Mote [8] who also used Hamilton’s principle in the derivation.

### 2.1.2 Solution of the String Equation of Motion Using Modal Analysis

The eigenvalues of the travelling string equation of motion, Eq. (2.5), are found using modal analysis as per the solutions developed by Hagedorn and DasGupta [38] as well as Wickert and Mote [39]. Modal analysis is a common and reliable method of solving for the span frequency, or eigenvalue, equations as well as for the equation of motion analytical solution. The full derivation is included to demonstrate the application of the method and for completeness.

Modal analysis begins with separation of the spatial and time variables. An initial solution for Eq. (2.5) is assumed as the real portion of the form:

$$w(x, t) = \Re[W(x)e^{i\omega t}]$$  \hspace{1cm} (2.8)

The spatial separation of the equation is the eigenfunction, or mode shape, $W(x)$. The time separation is $e^{i\omega t}$ where the eigenvalue, or natural frequency, is $\omega$. Upon substitution of the assumed solution into the equation of motion for the traveling string, Eq. (2.5), the eigenvalue problem is found as:

$$\left[(v^2 - c^2) W(x)'' + (a + 2v\omega i) W(x)' - \omega^2 W(x)\right]e^{i\omega t} = 0$$
For a non-trivial solution to the problem, \( e^{i\omega t} \neq 0 \), and thus the spatial portion of the equation must equal zero:

\[
\left( v^2 - c^2 \right) W''(x) + (a + 2v\omega i) W'(x) - \omega^2 W(x) = 0
\]  
\[ (2.9) \]

Knowing that the eigenfunctions are complex a solution is assumed of the form:

\[
W(x) = Be^{i\beta x}
\]
\[ (2.10) \]

Where \( \beta \) is an arbitrary constant. Substituting the assumed solution for the eigenfunction into the eigenvalue problem, Eq. (2.9), results in the intermediate equation:

\[
\left[ \left( c^2 - v^2 \right) \beta^2 + \left( a i - 2v\omega \right) \beta - \omega^2 \right] B e^{i\beta x} = 0
\]
\[ (2.11) \]

Again, for a non-trivial solution, \( Be^{i\beta x} \neq 0 \), and the \( \beta \)-roots of Eq. (2.11) are solved for as:

\[
\beta_1 = \frac{-ai + 2v\omega + (-a^2 - 4av\omega i + 4c^2\omega^2)^{\frac{1}{2}}}{2 \left( c^2 - v^2 \right)}
\]
\[
\beta_2 = \frac{-ai - 2v\omega + (-a^2 - 4av\omega i + 4c^2\omega^2)^{\frac{1}{2}}}{2 \left( c^2 - v^2 \right)}
\]

Substitution of the two \( \beta \)-roots into the assumed solution of the eigenfunction, Eq. (2.10), gives the solution of the eigenfunction:

\[
W(x) = D e^{i \beta_1 x} + E e^{i \beta_2 x}
\]
\[ (2.12) \]

The arbitrary constants, D and E, are determined when solving the assumed solution of the belt span equation of motion using the selected boundary conditions. In this
instance the travelling belt span pulley supports are modeled as pinned-pinned boundary conditions and thus the transverse displacement at each end of the span length is zero:

\[ w(0, t) = 0 \]

\[ w(l, t) = 0 \]

Upon substitution of the above boundary conditions into Eq. (2.12), the boundary condition matrix and coefficient column matrix are obtained:

\[
\begin{bmatrix}
1 & 1 \\
1 & \frac{1}{2(c^2 - v^2)}
\end{bmatrix} \begin{bmatrix}
i^{\frac{\alpha_1 + 2\nu \omega - \left(\alpha_1^2 - 4\nu \omega + 4c^2 \omega^2\right)^{\frac{1}{2}}}{2(c^2 - v^2)}
i^{\frac{\alpha_1 - 2\nu \omega - \left(\alpha_1^2 - 4\nu \omega + 4c^2 \omega^2\right)^{\frac{1}{2}}}{2(c^2 - v^2)}
\end{bmatrix} \begin{bmatrix}
D \\
E
\end{bmatrix} = 0
\] (2.13)

For a non-trivial solution the coefficients \( D \) and \( E \) should not be zero. To realize this, the boundary condition matrix must not have an inverse. A condition for this to be true is to set the determinate of the boundary condition matrix equal to zero. This condition is used to solve for the characteristic equation from which the belt span natural frequencies are found\(^5\).

\[
det \begin{bmatrix}
1 & 1 \\
e^{i\beta_1 t} & e^{i\beta_2 t}
\end{bmatrix} = 0
\]

\[e^{i\beta_2 t} - e^{i\beta_1 t} = 0\]

\[e^{i(\beta_2 - \beta_1) t} = 1\]

\[\cos \left[ (\beta_2 - \beta_1) t \right] - i \sin \left[ (\beta_2 - \beta_1) t \right] = 1\]

Ignoring the imaginary parts allows the eigenvalue to be formed as a function of \( n \), or

\(^5\)\( \beta_1 \) and \( \beta_2 \) substituted back for simplicity.
the modes of vibration:

\[ \cos[(\beta_2 - \beta_1) l] = 1 \]

\[(\beta_2 - \beta_1) l = 2n\pi \]

Substituting back in the values of \(\beta_1\) and \(\beta_2\) the \(n\), or infinite, number of conjugate pairs of complex eigenvalues for the travelling belt are solved as:

\[
\omega_n = \frac{\sqrt{a^2 l^2 (c^2 - v^2) + (4\pi^2 c^2 n^2) (c^2 - v^2)^2}}{2c^2 l} \pm \frac{av}{2c^2} \quad (2.14)
\]

Substituting in the belt wave speed, \(c\), and solving for tension gives the mean tension equation as a function of belt transverse vibration frequency and axial speed:

\[
T_m^m (\omega_n, v, t) = \rho \sqrt{-l^2 \left( \pi^2 a^2 n^2 - l^2 \omega_n^4 - 4\pi^2 n^2 \omega_n^2 v^2 \right) + l^2 \omega_n^2 \rho + 2\pi^2 n^2 \rho v^2} \quad (2.15)
\]

The mean tension equation, Eq. (2.15), requires the measurement of the belt span natural transverse vibration frequency and axial speed as a function of time to determine the mean tension in the belt span.

**Solution Note:** The solution for the travelling belt is found as [3]:

\[
w(x, t) = \sum_{n=1}^{\infty} \left\{ [W_n(x)] e^{i\omega_n t} \right\} \quad (2.16)
\]

Where the constants in the eigenvectors are solved from initial conditions.

### 2.1.3 Solution of the Beam Equation of Motion Using Galerkin Method

For the axially translating beam Galerkin’s method is used to find the eigenvalues of the belt span equation. This is a common method of finding approximate solutions to the
axially moving continua problem. The belt span equation of motion, Eq. (2.7) on page 17, is simplified by removing the axial acceleration component to simplify the solution. The fourth order partial differential equation becomes:

$$\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial t \partial x} + \left(v^2 - c^2\right) \frac{\partial w^2}{\partial x^2} + \frac{EJ_b}{\rho} \frac{\partial^4 w}{\partial x^4} = 0$$

(2.17)

In application of Galerkin’s method, a solution is assumed of the form [38]:

$$w(x, t) = \sum_n p_n(t) \phi_n(x)$$

(2.18)

The solution consists of modal coordinates, \(p_n(t)\), and the basis function, \(\phi_n(x)\). This assumed solution produces a solution for the axially travelling beam that is a summation of the basis functions. The modal coordinates are time functions and determine how much of each basis function is included in the final solution. The belt’s stationary boundary conditions are selected to be a combination of clamped and pinned boundary conditions. These are the boundary conditions that most closely mimic the belt running over the support pulleys without introducing a high degree of complexity. The selected boundary conditions lead to the choice of three bases functions for comparison: the eigenfunction of the clamped-clamped end stationary beam, eigenfunction of the pinned-pinned end stationary beam, which is the sinusoidal function, also used by Kong and Parker [40], and for comparison, eigenfunction of the clamped-pinned stationary beam.

The first basis function, the mode shape of the clamped-clamped stationary beam, is derived in Appendix D, Eq. (D.23), and re-printed here for convenience:

$$\phi_n^1(x) = C_n \left[ \cosh s_n x - \cos s_n x + \left( \frac{\cosh s_n l - \cos s_n l}{\sinh s_n l - \sin s_n l} \right) \left( \sinh s_n x - \sin s_n x \right) \right]$$
Where:

\[ s_1l = 4.730041 \]
\[ s_2l = 7.853205 \]
\[ s_3l = 10.995608 \]
\[ s_4l = 14.137165 \]

The clamped-clamped eigenfunction is graphically depicted as:

![Figure 2.2: Basis Function One: Clamped-Clamped Stationary Beam Eigenfunction](image)

The second basis function is the mode shape of the pinned-pinned stationary beam Eq. (D.11), also derived in Appendix D and reprinted for convenience:

\[ \phi_n^2(x) = C_n \sin \left( \frac{n\pi x}{l} \right) \]

Where:

\[ s_n = \frac{n\pi}{l}, \quad n = 0, 1, 2, ... \]

The pinned-pinned eigenfunction is graphically depicted as:
The third basis function is the mode shape of the clamped-pinned stationary beam Eq. (D.24), also found in Appendix D and reprinted for convenience:

\[ \phi_n^3(x) = C_n \left[ \sin s_n x - \sinh s_n x + \left( \frac{\sin s_n l - \sinh s_n l}{\cos s_n l - \cosh s_n l} \right) (\cosh s_n x - \cos s_n x) \right] \]

Where:

\[
\begin{align*}
s_1 l &= 3.926602 \\
s_2 l &= 7.068583 \\
s_3 l &= 10.210176 \\
s_4 l &= 13.351768
\end{align*}
\]

The clamped-pinned eigenfunction function is graphically depicted as:
Substitution of the above eigenfunctions into the assumed solution, Eq. (2.18), and the equation of motion, Eq. (2.7), and using the orthogonality condition of each of the basis functions results in the discretized equation of motion:

\[
M_d \ddot{p}_n(t) + G_d \dot{p}_n(t) + K_d p(t) = 0
\]  

(2.19)

Where the belt span discretized Mass, Gyroscopic, and Stiffness matrices are evaluated by numerical quadrature and redefined using using integration by parts [38]:

\[
M_d = \int_0^l \phi_m(x) \phi_n(x) dx = \delta_{mn}
\]  

(2.20)

\[
G_d = 2v \int_0^l \dot{\phi}_m(x) \dot{\phi}_n(x) dx = v \int_0^l [\phi_m(x) \dot{\phi}_n(x) - \phi_m(x) \dot{\phi}_m(x)] dx
\]  

(2.21)

\[
K_d = (v^2 - c^2) \int_0^l \phi_m(x) \phi_n''(x) dx + \frac{E J_b}{\rho} \int_0^l \phi_m(x) \phi_n'''(x) dx
\]

\[
= - (v^2 - c^2) \int_0^l \phi_m'(x) \phi_n'(x) dx + \frac{E J_b}{\rho} \int_0^l \phi_m''(x) \phi_n''(x) dx
\]  

(2.22)
A solution for the modal coordinate which contains the eigenvalue, $s_n$, is assumed [38]:

$$p(t) = Pe^{st}$$  \hspace{2cm} (2.23)

Substitution into the discretized equation of motion generates an intermediate equation:

$$\left[ M_d s_n^2 + G_d s_n + K_d \right] Pe^{st} = 0$$  \hspace{2cm} (2.24)

For the non-trivial solution $Pe^{st} \neq 0$ and the characteristic equation is found from the determinate of the first portion of the above equation:

$$det \left[ M_d s_n^2 + G_d s_n + K_d \right] = 0$$  \hspace{2cm} (2.25)

The eigenvalues, $s_n$ are found by solving the above characteristic equation and are a function of the axial belt tension, $T$, and axial belt velocity, $v$. For a given system, axial tension can be solved as a function of the eigenvalue, or transverse vibration frequency, and axial velocity. The eigenvalues are too large to justify printing in this document, however, the general trends are reviewed in Section 2.4.

2.1.4 Belt Stability

The previously derived tension equations are valid for application only in the subcritical stable axial belt velocity region(s). It is important to understand what region the belt is operating in at any given axial speed for both the string and beam mean tension models.

String Stability

For the axially travelling string equation belt velocities in the range below the wave speed are considered subcritical and axial belt velocities greater than the wave speed are considered supercritical. In the super critical region the wave cannot travel in the
opposite direction of the belt because the belt is moving faster than the wave, this causes divergence instability. The critical velocity is the transition point between the super and subcritical regions and is defined by setting the velocity equal to the wave speed:

\[ v_{cr} = c \]  

(2.26)

For a constant axial belt velocity the eigenvalues are defined by the well known equation [3]:

\[ \omega_n = \frac{n\pi}{cl} \left( c^2 - v^2 \right) \]  

(2.27)

The eigenvalues decrease as the velocity ratio, which is the axial belt velocity over wave speed, increases. By setting arbitrary values, such as density or belt length, to unity the relationship for the first four modes is shown in Fig. 2.5 below:

![Figure 2.5: Eigenvalue Variation with Constant Axial Velocity](image)

While the eigenvalues increases for each mode, they ultimately decreases to zero regardless of the mode as the velocity ratio approaches unity. At this point the belt loses its stiffness and experiences divergence-type instability [41]. As the axial belt velocity increases the eigenvalues become negative and the systems enters the supercritical region. This
relationship between increased axial velocity and decreased vibration frequency for a transversely vibrating string is well documented in the literature \[38\]. As well, it is evident that the axial belt tension is a dominant force in determining the stability of the system. The wave speed is dependent upon the axial tension of the continua, as per Eq. (2.6) page 16, so as the tension increases so does the wave speed. Thus, systems with higher axial tension will have a larger stable velocity region.

**Constant Axial Acceleration** For the system with constant axial acceleration the axial velocity is a function of time and acceleration and so the system stability becomes a function of time and acceleration. Velocity takes on the form \( v = at \). As before, the critical velocity is found by setting the velocity equal to the wave speed. For a given constant acceleration value there exists a critical time at which the velocity is equal to the belt wave speed. Finding the relationship for the critical time based on constant acceleration is done by setting the eigenvalue for constant acceleration, Eq. (2.14) page 20, to zero and solving for the critical time as a function of belt acceleration and wave speed:

\[
t_{cr} = \sqrt{2 \pi n \frac{c}{a}}
\]  

(2.28)

For a given system, relationship of acceleration to critical time is shown in Fig. 2.6 below:
At low time magnitude the value of the constant acceleration, and thus the velocity, must be near zero for the eigenvalue to be zero. This represents the zero velocity case at the system start up. As time increases the constant acceleration value corresponding to system instability asymptotically reaches zero. This is a logical conclusion as the velocity increases linearly with time. The constant acceleration decreases with respect to time based on the wave speed according to the inverse relationship of Eq. (2.28).

**Sinusoidal Axial Acceleration** Sinusoidal acceleration is important to review as it is the crankshaft input profile that most closely mimics the crankshaft input profile from pistons firing in an automotive engine. For the sinusoidal acceleration case, the velocity is a sinusoidal term and the acceleration a cosine term:

\[ v = \sin(zt) \]
\[ a = z \cos(zt) \]

Where \( z \) is an arbitrary acceleration constant. The wave speed now also becomes sinusoidal.
as the tension is a function of the sinusoidal acceleration:

\[ c = \sqrt{\frac{T + ma}{\rho}} = \sqrt{\frac{T_0 + mz \cos(zt)}{\rho}} \]  

(2.29)

Setting all other values equal to unity, the relationship between the wave speed and the axial velocity is graphed in Fig. 2.7:

![Figure 2.7: Wave Speed and Axial Speed for Sinusoidal Acceleration](image)

The relationship in Fig. 2.7 shows that if the initial axial belt tension, and thus wave speed, is sufficiently large in magnitude at the system startup then the axial belt velocity will never exceed the wave speed. This leads to a stable system. However, care must be taken to consider that the maximum axial velocity does not the minimum wave speed to have a stable system.

**Beam Stability**

The translating beam equation of motion, Eq. (2.7) page 17, has two stable velocity regions. To find the stable velocity regions the equation is discretized and the eigenvalues are found as per Section 2.1.3. The eigenvalues are a function of the beam’s axial velocity
and so solving for the eigenvalues gives a prediction of beam stability at a given axial velocity. The belt is considered stable when the eigenvalues, Eq. (2.25) page 25, are purely imaginary, or, when no real parts exist \[38\]. Using unit values, the system stability trends are analyzed for the clamped-clamped belt span with using unit values.

![Beam Stability](image)

Figure 2.8: Beam Stability

Fig. 2.8 illustrates the trends found in the imaginary and real parts of the beam eigenvalues. There is a stable velocity region starting from \(v = 0 \text{ m/s}\) which ends when the eigenvalues become real, at approximately \(v = 3 \text{ m/s}\) for a system with the unit values used here. There is a second smaller stable velocity region from \(v = 14 - 16 \text{ m/s}\) after which the system is unstable for all higher velocity values. The general trend of two stable velocity regions separated by an unstable region occurs for all systems however the velocity values at which the stable regions start and end will depend on the specific system values, such as flexural rigidity, linear density, and end support conditions.

### 2.2 Oscillation Tension from Belt Vibration

The oscillation tension is a result of the change in length of the belt span due to transverse vibration. As the belt span reaches the maximum transverse vibration amplitude the belt
span length increases and causes an increase in belt span axial tension. The oscillation tension will have a frequency twice that of the belt span vibration as the span will experience stretch twice in every vibration cycle. This is tension behavior illustrated in Fig. 2.9 where the span transverse vibration causes the belt span tension to oscillate.

To calculate the oscillation tension the change in belt span length due to transverse vibration is measured in the running FEAD system. Three span displacement sensors are placed at equal distances along the belt span length. The belt span displacement sensor readings are recorded as \( y_1, y_2, \) and \( y_3 \) as illustrated in Fig. 2.10.

The four belt span sub-lengths in Fig.2.10, \( l_1, l_2, l_3 \) and \( l_4 \), are calculated using Pythagorean
Theorem with the known axial length values and the measured displacement values.

\[
\begin{align*}
    l_1 &= \sqrt{\left(\frac{l}{4}\right)^2 + y_1^2} \\
    l_2 &= \sqrt{\left(\frac{l}{4}\right)^2 + (y_2 - y_1)^2} \\
    l_3 &= \sqrt{\left(\frac{l}{4}\right)^2 + (y_2 - y_3)^2} \\
    l_4 &= \sqrt{\left(\frac{l}{4}\right)^2 + y_3^2}
\end{align*}
\] (2.30a-d)

The change in belt span length is the difference between the deflected length, or sum of the calculated belt span sub-lengths, and the belt length in the undeflected state:

\[l_{osc_i} = (l_1 + l_2 + l_3 + l_4) - l_i\] (2.31)

The resulting oscillation tension is calculated using the belt span stiffness coefficient:

\[T_{i}^{o} = k_{b_i}^{s} l_{osc_i}\] (2.32)

## 2.3 Dynamic Tension from Support Pulley Rotation

To determine the dynamic belt span tension the FEAD system is modeled as a lumped mass system where the pulleys are masses connected by belt spans treated as linear springs. A diagram illustrating this concept is shown in Fig. 2.11, below.
The crankshaft is labeled as pulley #1 and the remaining pulleys are numbered in opposite direction of the belt rotation direction. The belt span numbers are labeled as the same number of the pulley following the belt span in the direction of belt movement.

2.3.1 System Equation of Motion from Lagrange’s Equation

The system of equations of motion for the FEAD system is found using Lagrange’s equations of motion for multiple particle systems. Before the equations can be derived some assumptions concerning the FEAD system are defined [28], [32]:

- Constant coefficient of friction between pulleys and the belt.
- Negligible inertia and flexural rigidity of the belt spans.
- Pulley support stiffness is not considered, the pulley support is assumed to be perfectly rigid.
- The belt is made of linearly elastic materials.
• The increase in belt tension due to belt bending around the pulleys is negligible as the belt is thin and flat.

• The viscous drag of the belt moving through air is negligible.

Using Lagrange’s method, the kinetic energy of the system is the sum of the kinetic energy of all the parts of the system, Wells [42]. In the lumped mass model the kinetic energy of the pulleys is the greatest as the pulleys have the largest mass. Other kinetic energy in the system, such as kinetic energy in the form of belt rotation or translation, is much less than that of the pulleys, justifying the assumption of ignoring belt inertia. Thus, the general form of the equation capturing kinetic energy of the system is:

\[ T = \frac{1}{2} \sum_{i=1}^{n} I_i \dot{\theta}_i^2 + I_t \dot{\theta}_t^2 \] (2.33)

Potential energy of the lumped mass system the is also the sum of the potential energy of all the system’s parts, Wells [42]. The pulleys do not contribute to the potential energy of the system as they are assumed to have fixed axis of rotation. Each span is treated as a linear spring and the strain energy of the system is contained in the linear deflection of the belt spans. The strain energy summation equation for the fixed pulley system is defined as:

\[ U_{sf} = \frac{1}{2} \sum_{i=1}^{n} k_{bi}^s x_i^2 \] (2.34)

Where the belt stiffness coefficient value \( k_{bi}^s \) is based on the longitudinal stiffness of the belt, \( EA_b \), and the each belt span length, \( l_i \), using the relationship: \( k_{bi}^s = \frac{EA_b}{l_i} \). The change in the length of the \( i^{th} \) belt span is defined as \( x_i^s \). The change in length of the belt span is found using the rotation, or difference in arc length, of the span’s support pulleys:

\[ x_i^s = (r_{i+1} \theta_{i+1} - r_i \theta_i) \] (2.35)

For the tensioner pulley, typically the \( n^{th} \) pulley, the strain energy is defined as:
\[ U_s = \frac{1}{2} k_b^n (r_n \theta_n - r_1 \theta_1 + l_t \theta_t \sin (\beta_n - \theta_t))^2 \]
\[ + \frac{1}{2} k_b^{n-1} (r_{n-1} \theta_{n-1} - r_n \theta_n + l_t \theta_t \sin (\beta_{n-1} - \theta_t))^2 \]  

(2.36)

The angles used in the Eq. (2.36) are illustrated in Fig. 2.12, next page.

Figure 2.12: System Angles

So that the strain energy from belt deflections of the system with a tensioner is written as:

\[ U_s = U_{s_t} + U_{s_f} \]  

(2.37)

If the pulleys, including tensioner pulley, have rotational springs attached at the axis then the potential energy of the spring rotation is included using the equation:

\[ U_r = \frac{1}{2} \sum_{i=1}^{n} k_r \theta_i^2 + \frac{1}{2} k_t \theta_t^2 \]  

(2.38)
So that the total potential energy in the lumped mass FEAD system is defined as:

\[
\mathcal{V} = \mathcal{U}_s + \mathcal{V}_r \\
= \frac{1}{2} \sum_{i=1}^{n-2} \left( k_b^s (r_{i+1} \theta_{i+1} - r_i \theta_i)^2 + k_r^s (r_n \theta_n - r_1 \theta_1 + l_t \theta_t \sin (\beta_n - \theta_t))^2 + \right. \\
\left. + \frac{1}{2} k_{bn-1}^s (r_{n-1} \theta_{n-1} - r_n \theta_n + l_t \theta_t \sin (\beta_{n-1} - \theta_t))^2 + \frac{1}{2} k_t^2 \right)
\]

(2.39)

The Lagrangian is defined as the difference between the total kinetic and total potential energy of the system [42]. For the system with the tensioner arm on the \(n^{th}\) pulley is defined as:

\[
\mathcal{L} = \mathcal{T} - \mathcal{V} \\
= \frac{1}{2} \sum_{i=1}^{n} (I_i \dot{\theta}_i^2) + I_t \dot{\theta}_t^2 - \frac{1}{2} \sum_{i=1}^{n-2} \left( k_b^s (r_{i+1} \theta_{i+1} - r_i \theta_i)^2 \right) - \frac{1}{2} \sum_{i=1}^{n} \left( k_r^s (r_n \theta_n - r_1 \theta_1 + l_t \theta_t \sin (\beta_n - \theta_t))^2 \right) - \frac{1}{2} k_t^2 \\
- \frac{1}{2} k_{bn-1}^s (r_{n-1} \theta_{n-1} - r_n \theta_n + l_t \theta_t \sin (\beta_{n-1} - \theta_t))^2
\]

(2.40)

The system of equations for the discrete system is then found using the Lagrangian equation for a system viscous damping forces and external forces:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = F_{q_r}
\]

(2.41)

Where:

\[
F_{q_r} = \frac{\partial \mathcal{P}}{\partial \dot{\theta}_i}
\]

(2.42)

\(\mathcal{P}\) is the power function and is used to determine generalized forces [42]. For the discrete system in consideration damping forces occur due to of the belt spans and damping in the rotation of the pulleys. As well, torque forces on the pulleys must be considered. For this system, the power function is defined as:
Application of equation 2.41 results in a system of equations of motion which represent
the system as a whole and where the number of the equations is equal to the number of
degrees of freedom in the system. For the five pulley system illustrated in Fig. 2.12 the
system of equations without damping are defined as:

\[ Q_1 = I_1 \ddot{\theta}_1 + \left( (k_{b_4}^* + k_{b_5}^*) r_4^2 \theta_4 - k_{b_4}^* r_4 \dot{r}_1 \theta_2 \\
- k_{b_5}^* r_5 \dot{r}_1 \theta_5 - k_{b_5}^* r_4 l t \sin(\theta_{to} - \beta_4) \theta_5 + k_{r_1} \theta_1 \right) \] (2.44a)

\[ Q_2 = I_2 \ddot{\theta}_2 + \left( (k_{b_2}^* + k_{b_3}^*) r_2^2 \theta_2 - k_{b_2}^* r_2 \dot{r}_3 \theta_3 - k_{b_3}^* r_1 \dot{r}_2 \theta_1 + k_{r_2} \theta_2 \right) \] (2.44b)

\[ Q_3 = I_3 \ddot{\theta}_3 + \left( (k_{b_4}^* + k_{b_5}^*) r_3^2 \theta_3 - k_{b_4}^* r_3 \dot{r}_4 \theta_4 - k_{b_5}^* r_2 \dot{r}_3 \theta_2 + k_{r_3} \theta_3 \right) \] (2.44c)

\[ Q_4 = I_4 \ddot{\theta}_4 + \left( (k_{b_4}^* + k_{b_5}^*) r_4^2 \theta_4 - k_{b_4}^* r_4 \dot{r}_5 \theta_5 - k_{b_5}^* r_3 \dot{r}_3 \theta_3 \\
+ k_{b_4}^* r_4 l t \sin(\theta_{to} - \beta_5) + k_{r_4} \theta_4 \right) \] (2.44d)

\[ Q_5 = I_5 \ddot{\theta}_5 + \left( (k_{b_5}^* + k_{b_4}^*) r_5^2 \theta_5 - k_{b_5}^* r_5 \dot{r}_1 \theta_1 - k_{b_4}^* r_4 \dot{r}_5 \theta_4 \\
+ (k_{b_5}^* r_5 l t \sin(\theta_{to} - \beta_4) - k_{b_4}^* r_5 l t \sin(\theta_{to} - \beta_5)) \theta_5 + k_{r_5} \theta_5 \right) \] (2.44e)

\[ Q_6 = I_6 \ddot{\theta}_6 + \left[ - k_{b_5}^* r_1 l t \sin(\theta_{to} - \beta_4) \theta_1 + k_{b_4}^* r_4 l t \sin(\theta_{to} - \beta_5) \theta_4 \\
+ (k_{b_5}^* r_5 l t \sin(\theta_{to} - \beta_4) - k_{b_4}^* r_5 l t \sin(\theta_{to} - \beta_5)) \theta_5 \\
+ (k_{b_5}^* l_2^2 \sin(\theta_{to} - \beta_4)^2 - k_{b_4}^* l_2^2 \sin(\theta_{to} - \beta_5)^2) \theta_1 + k_{r_1} \theta_1 \right] \] (2.44f)

This system of equations, with damping, are arranged in matrix form:

\[ \mathbf{M} \ddot{\mathbf{\theta}}(\mathbf{t}) + \mathbf{C} \dot{\mathbf{\theta}}(\mathbf{t}) + \mathbf{K} \mathbf{\theta}(\mathbf{t}) = \mathbf{Q} \] (2.45)
The stiffness matrix can be simplified into two matrices as per Zu [20], the belt stiffness matrix, $K_b$, and the rotational stiffness matrix $K_r$:

$$K = K_b + K_r$$

The belt stiffness matrix can be further simplified into the product of the intermediate stiffness matrix, $T_k$, and the rotational matrix, $R_t$:

$$K_b = R_tT_k$$

Leading to:

$$K = R_tT_k + K_r \quad (2.46)$$

The belt damping matrix is similarly reduced using the same method as the stiffness matrix. The damping matrix, $C$, is broken down into the belt damping matrix, $C_b$, and the rotational damping matrix $C_r$.

$$C = C_b + C_r \quad (2.47)$$

Where:

$$C_b = R_tT_c \quad (2.48)$$

Leading to:

$$C = R_tT_c + C_r \quad (2.49)$$

Matrices are defined in Appendix B for both the undefined system and the 5-pulley system used in experiments.


2.3.2 System Modal Analysis

The eigenvalues and eigenvectors are solved using a Cholesky factorization of the inertia matrix, $M$. The Cholosky, $L$, of the mass inertia matrix, $M$, is used to normalize the system’s equations of motion as per the method presented by Inman [43]. The normalized eigenvectors are then found and used to decouple the system of equations. This Cholesky factorization of the inertia matrix produces an upper triangular matrix such that:

$$L^T L = M$$

Using the Cholesky factorization, the system is normalized using the first of two a coordinate system changes:

$$\theta(t) = (L^T)^{-1} q(t) \quad (2.50)$$

Pre-multiplying Eq. (2.45) by $L^{-1}$ and using the coordinate transformation of Eq. (2.50) results in the mass normalized equations of motion:

$$L^{-1} M(L^T)^{-1} \ddot{q}(t) + L^{-1} C(L^T)^{-1} \dot{q}(t) + L^{-1} K(L^T)^{-1} q(t) = L^{-1} Q$$

Noting that the identity matrix is formed by:

$$I = L^{-1} M(L^T)^{-1}$$

The Mass normalized equation of motion becomes:

$$I \ddot{q}(t) + \tilde{C} \dot{q}(t) + \tilde{K} q(t) = L^{-1} Q \quad (2.51)$$

The mass normalized stiffness matrix is defined as:

$$\tilde{K} = L^{-1} K(L^T)^{-1}$$
If damping is considered, the mass normalized damping matrix is similarity defined as:

\[ \tilde{C} = L^{-1}C(L^T)^{-1} \]

The matrix of eigenvectors, \( P \), and the matrix of eigenvalues, \( \Lambda \), are solved using the mass normalized stiffness matrix:

\[ (P, \Lambda) = \text{eig} \, \tilde{K} \]  

(2.52)

Where:

\[ P = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} & v_n \end{bmatrix} \]  

(2.53)

and \( v_n \) is the \( n^{th} \) eigenvector of the discrete system. The eigenvalues are defined in the \( \Lambda \) matrix as:

\[ \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ sym \lambda_{n-1} & 0 \end{bmatrix} \]  

(2.54)

Where:

\[ \lambda_n = \omega_n^2 \]  

(2.55)

The system of equations is now de-coupled using the matrix of eigenvectors, \( P \). The second coordinate change equation is shown below:

\[ q = Pr(t) \]  

(2.56)
Each term in Eq. (2.51) is pre-multiplied by $P^T$ and subject to the coordinate change, Eq. (2.56). This results in the decoupled system of equations:

$$P^TIP\ddot{r}(t) + P^T\ddot{C}Pr(t) + P^TKPr(t) = R$$

Due to the orthonormal properties of the eigenvectors $P^TP = I$ and the decoupled equations of motion can be written as:

$$I\ddot{r}(t) + B\dot{r}(t) + \Lambda r(t) = R$$

(2.57)

The matrix $\Lambda$ was previously defined in Eq. (2.54). The matrix $B$ is defined as:

$$B = P^T\ddot{C}P = \begin{bmatrix}
2\zeta_1\omega_1 & 0 & \cdots & 0 \\
0 & 2\zeta_2\omega_2 & \ddots & \\
& \ddots & \ddots & \\
0 & \cdots & 2\zeta_{n-1}\omega_{n-1} & 2\zeta_n\omega_n \\
\end{bmatrix}$$

(2.58)

If external forces are applied to the system:

$$R = P^TL^{-1}Q$$

(2.59)

Else:

$$R = 0$$

Once a solution is found for the decoupled equations of motion the solution is transformed back from the uncoupled coordinate system, $r(t)$, to original coordinate system, $\theta(t)$,
using the reverse coordinate change:

\[ \theta(t) = (L^T)^{-1} Pr(t) \]  

(2.60)

### 2.3.3 Dynamic Tension

Once the rotation of each pulley in the system is found rotation tension, or, tension change caused by support pulley rotation, is defined using the \( T_k \) and \( T_c \) matrices \[a\]:

\[ T^r = T_k \theta(t) + T_c \dot{\theta}(t) \]  

(2.61)

Eq. (2.61) includes only the changes in tension caused by support pulley rotation. The dynamic belt span tension for each belt span in the system is found using of the mean belt span tension, Eq. (2.15) page 20, tension belt span oscillation, Eq. (2.32) page 32 along with the previously defined tension from belt span rotation:

\[ T = T^m + T^o + T^r \]

(2.62)

Expansion of Eq. (2.62) leads to a series of equations for tension in each belt span of the system. Assuming the tensioner pulley is the \( n \)th pulley of the \( n \) pulley system, the equations for each span the system is defined as:

\[ T_i = T_i^m + T_i^o + k_{b_i} (r_i \theta_i - r_{i+1} \theta_{i+1}) + c_{b_i} (r_i \dot{\theta}_i - r_{i+1} \dot{\theta}_{i+1}) \]  

(2.63a)

\[ T_{n-1} = T_{n-1}^m + T_{n-1}^o + k_{b_{n-1}} \left[ r_{n-1} \theta_{n-1} - r_i \theta_i + l_t \sin (\theta_{to} - \beta_p^i) (\theta_t - \theta_{to}) \right] + c_{b_{n-1}} \left[ r_{n-1} \dot{\theta}_{n-1} - r_i \dot{\theta}_i + l_t \sin (\theta_{to} - \beta_p^i) \dot{\theta}_t \right] \]  

(2.63b)

\[ T_n = T_n^m + T_n^o + k_{b_n} \left[ r_n \theta_n - r_1 \theta_1 + l_t \sin (\theta_{to} - \beta_n) (\theta_t - \theta_{to}) \right] + c_{b_n} \left[ r_n \dot{\theta}_n - r_1 \dot{\theta}_1 + l_t \sin (\theta_{to} - \beta_n) \dot{\theta}_t \right] \]  

(2.63c)
However, these equations are limited as they assume all belt stretching occurs in the belt spans and does not take into consideration stretching of portions of the belt that are in wrapped on the pulleys. This increases the calculated stretching in the free belt spans and in turn overestimates the tension in the belt spans. The solution is to include the belt slip from the pulley “driven” by the belt span \([44]\). Ignoring the effect of damping and adding the slip let out by stretching on the pulley, \(x_{p}^{i}\), the final equation for rotation tension is defined as:

\[
T_{i} = T_{m}^{i} + T_{o}^{i} + k_{b_{i}}^{s} (r_{i} \theta_{i} - r_{i+1} \theta_{i+1} - x_{p}^{i}) \quad (2.64a)
\]

\[
T_{n-1} = T_{m}^{n-1} + T_{o}^{n-1} + k_{b_{n-1}}^{s} [r_{n-1} \theta_{n-1} - r_{n} \theta_{n} + l_t \sin (\theta_{t_{o}} - \beta_{n}^{p}) (\theta_{i} - \theta_{t_{o}}) - x_{p}^{n}] \quad (2.64b)
\]

\[
T_{n} = T_{m}^{n} + T_{o}^{n} + k_{b_{n}}^{s} [r_{n} \theta_{n} - r_{1} \theta_{1} + l_t \sin (\theta_{t_{o}} - \beta_{n}) (\theta_{i} - \theta_{t_{o}}) - x_{p}^{n}] \quad (2.64c)
\]

Where the pulley arc lengths and belt slip are displayed schematically for a belt span translating between two stationary pulleys:

![Figure 2.13: Pulley Arc Angles and Slip](image)

A common method for finding belt slip, \(x_{p}^{i}\), is to average the tensions in the spans adjacent to the pulley of interest and then calculate the corresponding change in length based on belt stiffness \([20], [22]\). This method involves interpolating the tensions in all belt spans.
of the system. Here, a method of measuring the belt tension in the belt span of interest without otherwise necessary measurement or calculation of all system pulley rotations. To achieve this aim an iterative method of calculating belt slip using belt-pulley frictions equations is proposed.

Friction Rotation Equations

The well known Euler equation for belt-pulley slip is modified using shear theory by Firbank [24] and Alciatore and Traver [28] giving the slip equation for a driving pulley as:

\[
\frac{T_1}{T_2} = e^{\mu_k \alpha} \left[ 1 + \frac{(\varphi - \alpha)}{2} \mu_s \right]
\]

(2.65)

and for the driven pulley:

\[
\frac{T_1}{T_2} = \frac{e^{\mu_k \alpha}}{\left[ 1 + \frac{(\varphi - \alpha)}{2} \mu_s \right]}
\]

(2.66)

Where the pulley wrap angles are defined in Fig. 2.14.
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Figure 2.14: Lumped Mass System Wrap Angles

Application of Eq. (2.65) and Eq. (2.66) require the following assumptions which are applicable to the FEAD system:

- The belt is flexible and inextensible, and is sufficiently thin to make shear and bending forces negligible.
- The belt does not slip on the pulley at the entering point.
- The friction coefficients, \( \mu_k \) and \( \mu_s \), are constant.
- Inertia forces may be ignored due to low mass and/or speeds.

The arc of slip, where the static friction forces are overcome and the belt slips on the pulley, is defined as \( \alpha_i \). In the angle of slip the surface speed of the pulley will be greater than that of the belt. \( \mu_k \) is the coefficient of kinetic friction, and \( \mu_s \) is the coefficient of static friction. As well, by definition \( T_1 > T_2 \). The tension change over the arc of slip, \( \alpha \),
is $T_2(e^{\mu_k \alpha} - 1)$. Where $T_2$ is the lower tension value. The tension change over the arc of adhesion, defined as $\varphi_i - \alpha_i$, is $\frac{(\varphi_i - \alpha_i)}{2}T_2e^{\mu_k \alpha}\mu_s [24]$.

Figure 2.15: Arc of Slip and Arc of Adhesion

To develop the equation for belt slip on the $(i + 1)^{th}$ driven pulley in the FEAD system the mean tension and vibration tension in the $i^{th}$ are measured and summed and defined as:

$$T_i = T_i^m + T_i^o$$

Using Eq. (2.66) the tension in the $(i + 1)^{th}$ span is then defined as:

$$T_{i+1} = T_i \left[1 + \frac{(\varphi_{i+1} - \alpha_{i+1})}{2}\mu_s\right]e^{\mu_k \alpha}$$

The tension in the span wrapped on the pulley is taken as an average of the adjacent pulley spans:

$$T_{i+1}^p = \frac{T_i + T_{i+1}}{2}$$

and the change in tension is defined as:

$$\Delta T_{i+1}^p = T_{i+1}^p - T_o$$ (2.67)
Where $T_o$ is the initial tension in the system. The change in belt length, or slip from pulley $(i+1)$ is then defined as:

\[
x_{i+1}^p = \frac{\Delta T_{i+1}^p}{k_{b_{i+1}}^p} = \frac{T_i^m + T_o}{2k_{b_{i+1}}^p} \left( 1 + \frac{1 + \left( \frac{\phi_{i+1} - \alpha_{i+1}}{2} \right)}{e^{\mu_k \alpha}} \right) - \frac{T_o}{k_{b_{i+1}}^p} \tag{2.68}
\]

Knowing the value of $x_{i+1}^p$ the true value of the tension in the $i^{th}$ span is then calculated using equation Eq. (2.64).

### 2.4 Summary of Tension Models

A total of three tension equations have been derived, the mean belt span tension from belt vibration frequency, oscillation belt span tension from belt vibrations, and dynamic belt span tension from mean belt span tension, oscillation belt span tension, belt slip, and belt span support pulley rotations.

The mean tension equation for the string model, Eq. (2.15) page 20, was derived from the belt span frequency equation and is reprinted below for convenience:

\[
T_i^m = \rho \sqrt{-l^2 \left( \pi^2 a^2 n^2 - l^2 \omega_n^4 - 4\pi^2 n^2 \omega_n^2 v^2 \right) + l^2 \omega_n^2 \rho + 2\pi^2 n^2 \rho v^2} \div 2\pi^2 n^2
\]

When considering the travelling beam, i.e. the string with the inclusion of the flexural rigidity term, the mean tension equation becomes very difficult to solve for. Instead, a plot is made relating frequency and tension at different axial speeds, or crankshaft RPM. The graphs are completed for the three basis functions chosen during Galerkin discritization as well as the travelling string for comparison.
It can be seen in Fig. 2.16 that for a given frequency value the boundary conditions do have a large effect on the corresponding axial tension in the belt span. The clamped-clamped boundary conditions increase the rigidity of the belt and, as more rigid objects vibrate with higher frequencies, the axial belt tension is correspondingly low. The tension
for a given frequency increases for the clamped-pinned boundary condition and increases further for the pinned-pinned boundary condition. This is an expected result as the pinned boundary condition models the belt span as being less rigid. The string model has even higher tension values corresponding to a given frequencies values as expected due to the low rigidity of the string model.

The oscillation tension which is generated from length change during belt transverse vibration, Eq. (2.32) page 32, is also reprinted for convenience:

\[ T_i^o = k_{b_i} l_{osc_i} \]

Last, dynamic belt span tension due to support pulley rotation and belt slip was defined in Eq. (2.64), page 43, and also reprinted:

\[
T_i = T_i^m + T_i^o + k_{b_i} (r_{i+1} \theta_{i+1} - r_i \theta_i - x_i^p) \\
T_{n-1} = T_{n-1}^m + T_{n-1}^o + k_{b_{n-1}} [r_{n-1} \theta_{n-1} - r_n \theta_n + l_t \sin (\theta_{to} - \beta_n) (\theta_t - \theta_{to}) - x_n^p] \\
T_n = T_n^m + T_n^o + k_{b_n} [r_n \theta_n - r_1 \theta_1 + l_t \sin (\theta_{to} - \beta_n) (\theta_t - \theta_{to}) - x_1^p]
\]

Where belt slip is defined in Eq. (2.68), previous page.
Chapter 3

Experiments

Validation of the tension equations presented in Section 2.4 is done by experiment. In this chapter the experiments carried out to validate the theory are fully explained. The experimental FEAD system is outlined and the sensors used to collect data are listed with selection justification. As well, data modification, including filters, resampling, and Fast Fourier Transforms, are explained with examples included.

3.1 Testing System

The data acquisition system was designed to allow for measurement of a FEAD span tension under a variety of axial velocity profiles. The system is a typical FEAD system consisting of a crankshaft, air conditioner, alternator, two idler pulleys, and an optional tensioner attached to the second idler pulley.

3.1.1 Characteristics

The apparatus set-up is defined as the FEAD system and the accompanying sensors and data acquisition equipment. As mentioned, the FEAD system consists of the crankshaft pulley, idler pulley, air conditioner pulley, alternator pulley, and an optional tensioner
arm with tensioner pulley. The system layout was chosen such that the third belt span is perpendicular to the ground. The crankshaft is labeled as pulley #1 and the remaining pulleys are numbered in opposite direction of the belt rotation direction. The span numbers are labeled with the same number as the pulley following it in the direction of belt movement. This configuration can be seen in the two systems below, Fig. 3.1, one with and one without the tensioner arm.

(a) Without Tensioner Arm

![Diagram of belt system with and without tensioner arm](image-url)
The system characteristics are defined in the Tables 3.1, 3.2, and 3.3, next page.
### Table 3.1: System Characteristics

<table>
<thead>
<tr>
<th>Pulley Name:</th>
<th>C/S</th>
<th>Idler 1</th>
<th>A/C</th>
<th>Alt</th>
<th>Idler 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulley #:</td>
<td>#1</td>
<td>#2</td>
<td>#3</td>
<td>#4</td>
<td>#5</td>
</tr>
<tr>
<td>X-Coordinate</td>
<td>$X_i$ [mm]</td>
<td>0</td>
<td>120</td>
<td>218</td>
<td>250</td>
</tr>
<tr>
<td>Y-Coordinate</td>
<td>$Y_i$ [mm]</td>
<td>0</td>
<td>-50</td>
<td>-12</td>
<td>212</td>
</tr>
<tr>
<td>Pitch Diameter</td>
<td>$d_i$ [mm]</td>
<td>163.76</td>
<td>70</td>
<td>122.76</td>
<td>69.15</td>
</tr>
<tr>
<td>Pitch Diameter Ratio</td>
<td>$d_1/d_i$</td>
<td>1</td>
<td>2.34</td>
<td>1.33</td>
<td>2.49</td>
</tr>
<tr>
<td>Wrap Angle</td>
<td>$\varphi_i$ [deg]</td>
<td>235</td>
<td>87</td>
<td>135</td>
<td>174</td>
</tr>
<tr>
<td>Span Lengths</td>
<td>$l_i$ [mm]</td>
<td>57</td>
<td>43</td>
<td>224</td>
<td>141</td>
</tr>
<tr>
<td>Inertia</td>
<td>$I_i$ [kg · m²]</td>
<td>$4 \times 10^{-3}$</td>
<td>$1.8 \times 10^{-4}$</td>
<td>$2.0 \times 10^{-4}$</td>
<td>$5.2 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
For the tensioner arm:

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Symbol</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulley Inertia</td>
<td>$I_t$</td>
<td>$[kg \cdot m^2]$</td>
<td>$3.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>Pulley Pitch Diameter</td>
<td>$d_i$</td>
<td>$[mm]$</td>
<td>71.90</td>
</tr>
<tr>
<td>Pulley Pitch Diameter Ratio</td>
<td>$d_1/d_i$</td>
<td>-</td>
<td>2.28</td>
</tr>
<tr>
<td>Pivot X-Coordinate</td>
<td>$X_t$</td>
<td>$[mm]$</td>
<td>233</td>
</tr>
<tr>
<td>Pivot Y-Coordinate</td>
<td>$Y_t$</td>
<td>$[mm]$</td>
<td>97</td>
</tr>
<tr>
<td>Arm Length</td>
<td>$l_t$</td>
<td>$[mm]$</td>
<td>70.02</td>
</tr>
<tr>
<td>Spring Rate</td>
<td>$k_t$</td>
<td>$[N \cdot mm/\text{rad}]$</td>
<td>10.07</td>
</tr>
<tr>
<td>Arm Inertia</td>
<td>$I_{arm}$</td>
<td>$[kg \cdot m^2]$</td>
<td>$1.2 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 3.2: Tensioner Arm Characteristics

The serpentine belt properties are defines as:

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Symbol</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal Stiffness</td>
<td>$EA_b$</td>
<td>$[N/m/m]$</td>
<td>$4.00 \times 10^5$</td>
</tr>
<tr>
<td>Width</td>
<td>$w_b$</td>
<td>$[mm]$</td>
<td>24.20</td>
</tr>
<tr>
<td>Depth</td>
<td>$d_b$</td>
<td>$[mm]$</td>
<td>4.15</td>
</tr>
<tr>
<td>Cross Sectional Area</td>
<td>$A_b$</td>
<td>$[mm^2]$</td>
<td>100.40</td>
</tr>
<tr>
<td>Estimated Effective Length</td>
<td>$l_b$</td>
<td>$[m]$</td>
<td>1.29</td>
</tr>
<tr>
<td>Mass</td>
<td>$m_b$</td>
<td>$[kg]$</td>
<td>1.28</td>
</tr>
<tr>
<td>Linear Density</td>
<td>$\rho_{nom}$</td>
<td>$[kg/m]$</td>
<td>$9.89 \times 10^{-2}$</td>
</tr>
<tr>
<td>Modulus of Elasticity</td>
<td>$E$</td>
<td>$[N/m^2]$</td>
<td>$4.00 \times 10^8$</td>
</tr>
<tr>
<td>Mass Moment of Inertia</td>
<td>$I_b$</td>
<td>$[kg \cdot m^2]$</td>
<td>$1.12 \times 10^{-3}$</td>
</tr>
<tr>
<td>Second Moment of Area</td>
<td>$J_b$</td>
<td>$[m^4]$</td>
<td>$1.44 \times 10^{-11}$</td>
</tr>
<tr>
<td>Flexural Rigidity</td>
<td>$EJ_b$</td>
<td>$[N \cdot m^2]$</td>
<td>$5.74 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 3.3: Belt Properties (7pk aramid cord belt)

1About the pivot point.
3.1.2 Setup

The system was mounted on a flat metal support panel with access from behind for attachment of the motor and sensor equipment. A variety of data sensors were used. The rotational speed of the crankshaft, air conditioner, and alternator pulleys were measured using rotation sensors. The transverse displacement of the third belt span was measured at the midpoint and quarter points using laser displacement sensors. A transducer was used to measure the torque of the crankshaft. As well, a hubload sensor was used on the first idler pulley.

The actual testing system set up with the attached sensors is shown in Fig. 3.2 and a schematic of the pulley system and sensor location in the system is shown in Fig. 3.3.
3.1.3 Sensor Selection

The data acquisition unit as well as sensors selected and accompanying data acquisition equipment are explained in this section. All data is sampled using discrete time sampling. Initially, all sensors were required to have a testing measurement frequency capacity of at least 5 kHz, however testing was completed with a higher sampling frequency of 10 kHz. This is well above the Nyquist frequency of approximately 300-400 Hz.

1. A ROTEC data acquisition unit was used to collect data from the sensors. Analog
channels (non speed channels) were sampled at 6000 Hz. Analog channels read 0-10V input. Speed channels are sampled at 60 times per revolution of the pulley. Speed channels read TTL signal 0-5 Volt.

2. Two types of speed encoders were used to measure rotational speed of the pulleys, an aluminium encoder and a sticker encoder. The aluminium speed encoders were used on the crankshaft and alternator pulleys. The aluminium speed encoders are made by Liten’s Automotive and are aluminium disks with 60 slots of equal width radially cut into them which are bolted to the pulley. An Optek infrared forked probed speed sensor was used with the aluminium encoder. The Optek probe reads a high voltage when the infrared beam passes between the forked ends and a low voltage when the encoder blocks the infrared beam. The voltage is transformed into speed based on the time interval between pulses using the ROTEC (Data acquisition equipment). A magnified images of the aluminium encoder and Optek probe are shown below:

![Aluminium Speed Encoder](Litens System Lab)  
(b) Optek Probe OPB971 BZ [45]

Figure 3.4: Crankshaft and Alternator Rotational Speed Measurement Equipment

The sticker encoder was used on the air conditioner pulley as an aluminium encoder could not be bolted to it. The sticker encoder in an adhesive sticker which has 60 white and 60 black radial divisions. A banner cable, which works similarly to the Optek probe, was used to measure rotational speed. The banner cable sensor
senses the difference in reflection between the white and black divisions. A voltage pulse is then sent to the ROTEC (Data acquisition equipment) where the speed recorded is based on the time interval between pulses. A magnified images of the sticker encoder and banner cable sensor are shown below:

![Sticker Speed Encoder](image1.png) ![Banner Cable](image2.png)

(a) Sticker Speed Encoder  (b) Banner Cable

Figure 3.5: Air Conditioner Rotational Speed Measurement Equipment (*Litens Systems Lab*)

3. A Wenglor laser was used to measure the displacements of the belt span. The top and bottom sensors were displacement sensors where $1 \text{ V} = 2 \text{ mm}$ displacement. The middle laser sensor was a displacement sensor where $-1 \text{ V} = 5 \text{ mm}$ displacement. The laser displacement sensors were set up such that the belt span was located in the middle of the sensor measurement range. The displacement measurement sensors were evenly placed along the length of the belt span as depicted in the figure below:
4. A string pot was used to measure the rotation of the tensioner arm about it’s axis. Series 150 in attachment pt_katalog, measures linear movement with nearly infinite resolution. The voltage output increases as string length increases.

Figure 3.7: Tensioner Arm Displacement Sensor [1]

5. The hubload sensor used on the first idler pulley was made at Liten’s Automotive Lab. It was made using a strain gauge and calibrated such that 1 V = 1000 N of force. Calibration was completed using hanging weights with known wrap angles as depicted in Fig. 3.8, next page.
3.1.4 Testing System Velocity Stability Regions

Before completing experimental testing to determine belt span tension values the axial velocity stability regions of the actual system are determined as a function of the belt span tension. This eliminates error which would occur if applying the equations to the FEAD system operating in the supercritical regions. The Galerkin method described in Section 2.1.4 is used to determine the system stability zones for the system modeled with flexural rigidity. For the system without flexural rigidity the stability analysis is completed for the system as per Section 2.1.4. Calculations for the experimental system used are completed using the system values from Tables 3.1, 3.2, and 3.3 on pages 53, 54, and 54.

Galerkin Beam Stability

The system stability with flexural rigidity is analyzed with an assumed axial tension of $T = 150 \, N$ for each of the basis functions used. A higher assumed tension will increase the wave speed and correspondingly increase the magnitude of the stability region of the graphs. As mentioned previously, the belt is considered stable when the eigenvalues are
purely imaginary, or, when no real parts exist.

Figure 3.9: Testing System Beam Stability

The clamped-pinned stability graph displays a stability region from \( v = 0 - 50\frac{m}{s} \), a large unstable region, and then a second stability region from approximately \( v = 170 - 185\frac{m}{s} \), after which the system is unstable. The clamped-clamped and pinned-pinned basis
functions have the same stability region from \( v = 0 - 50 \frac{m}{s} \) and then experience an unstable region. For the testing system a belt speed of \( v = 50 \frac{m}{s} \) corresponds to a crankshaft rotational speed of 5831 RPM. Thus, the system is operating in stable regions for all testing done at less than 5831 RPM.

**Modal Analysis String Stability**

For the belt span modeled with no flexural rigidity having an initial tension of \( T = 133 \text{N} \) traveling with constant axial velocity the following stability graph is applicable:

![Stability Graph](image)

**Figure 3.10: Actual System Eigenvalues Constant Velocity, \( T_o = 133 \text{N} \)**

It can easily be seen that the system is stable at this tension for any typical application of a FEAD system. The belt axial velocity in the above figure translates into a more intuitive crankshaft RPM input of up to 4,276 RPM.

\[
36.6714 \frac{m}{s} \left( \frac{1}{2 \pi r_c} \frac{\text{rot}}{m} \right) \left( 60 \frac{\text{sec}}{\text{min}} \right) = 4,276.8 \text{RPM} \quad (3.1)
\]

If the centrifugal tension is included in the total tension value, \( T = T_o + \rho v^2 \), the system will always be stable and the eigenvalue will asymptotically approach zero as the system...
speed increases:

![Graph showing actual system eigenvalues constant velocity, \( T_o = 133 N + \rho v^2 \)](image1)

Figure 3.11: Actual System Eigenvalues Constant Velocity, \( T_o = 133 N + \rho v^2 \)

A more appreciable figure for experimental application is the relationship between the initial tension in the system and the critical RPM speed of the crankshaft. For the third belt belt span, starting with an initial tension from \( T = 0 N \) to \( T = 500 N \) the critical RPM speed is plotted.

![Graph showing critical crankshaft RPM values for varying belt span tension](image2)

Figure 3.12: Critical Crankshaft RPM values for Varying Belt Span Tension

For sinusoidal crankshaft torque input, provided that the mean velocity is below the
critical velocity, the system is stable.

![Graph](image)

Figure 3.13: Actual Constant Acceleration vs. Critical Time

The above figure shows that the system can withstand high acceleration values while remaining in the subcritical axial belt velocity region.

In summary, provided the testing crankshaft RPM values remain below 4,276 RPM, the tension equations developed in Chapter 2 are valid for application to the experimental testing system’s third belt span.

### 3.2 Data Sets

A total of six data sets were collecting containing 6-11 trails in each set. Four data sets were collected without a tensioner arm and two were collected with an automatic tensioner arm. Two of the data sets consisted of a constant velocity, or steady state, velocity profile while the remaining four sets had sinusoidal torque input from the crankshaft generating
vibration within the system. The torque input from the crankshaft is given in the form:

\[ Q_{cs} = \tau_c + \tau \sin(\omega_{ext}) \]  

(3.2)

Where \( \tau_c \) is the crankshaft constant torque input and \( \tau \) is the crankshaft torque excitation amplitude which is excited at the frequency \( \omega_{ext} \). The six data sets are summarized:

<table>
<thead>
<tr>
<th>Set</th>
<th>( T_o ) (N)</th>
<th>( v ) (m/s)</th>
<th>( \tau ) (N \cdot m)</th>
<th>( \omega_{ex} ) (Hz)</th>
<th>Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>149</td>
<td>5 ~ 12</td>
<td>0</td>
<td>0</td>
<td>no</td>
</tr>
<tr>
<td>2</td>
<td>149</td>
<td>5</td>
<td>0 ~ 10</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>3</td>
<td>306</td>
<td>5</td>
<td>0 ~ 10</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>306</td>
<td>12</td>
<td>0 ~ 10</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>5</td>
<td>299</td>
<td>5 ~ 12</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>6</td>
<td>299</td>
<td>12</td>
<td>2 ~ 10</td>
<td>121</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table 3.4: Summary of Testing Conditions

Details of each trial are included in the following six testing condition tables.

<table>
<thead>
<tr>
<th>Trial #</th>
<th>( T_o ) (N)</th>
<th>( v ) (m/s)</th>
<th>( \tau ) (N \cdot m)</th>
<th>( \omega_{ex} ) (Hz)</th>
<th>Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>149</td>
<td>5.14</td>
<td>0</td>
<td>0</td>
<td>no</td>
</tr>
<tr>
<td>1-2</td>
<td>149</td>
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<td>0</td>
<td>no</td>
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<td>1-3</td>
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<td>0</td>
<td>no</td>
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<tr>
<td>1-4</td>
<td>149</td>
<td>7.72</td>
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<td>0</td>
<td>no</td>
</tr>
<tr>
<td>1-5</td>
<td>149</td>
<td>8.57</td>
<td>0</td>
<td>0</td>
<td>no</td>
</tr>
<tr>
<td>1-6</td>
<td>149</td>
<td>5.14-12.00</td>
<td>0</td>
<td>0</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 3.5: Set 1 Testing Conditions
### Table 3.6: Set 2 Testing Conditions

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$T_o$ (N)</th>
<th>$\nu$ (m/s)</th>
<th>$\tau$ (N·m)</th>
<th>$\omega_{ex}, \text{ (Hz)}$</th>
<th>Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>149</td>
<td>5.14</td>
<td>0</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-2</td>
<td>149</td>
<td>5.14</td>
<td>1</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-3</td>
<td>149</td>
<td>5.14</td>
<td>2</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-4</td>
<td>149</td>
<td>5.14</td>
<td>3</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-5</td>
<td>149</td>
<td>5.14</td>
<td>4</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-6</td>
<td>149</td>
<td>5.14</td>
<td>5</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-7</td>
<td>149</td>
<td>5.14</td>
<td>6</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-8</td>
<td>149</td>
<td>5.14</td>
<td>7</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-9</td>
<td>149</td>
<td>5.14</td>
<td>8</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-10</td>
<td>149</td>
<td>5.14</td>
<td>9</td>
<td>77</td>
<td>no</td>
</tr>
<tr>
<td>2-11</td>
<td>149</td>
<td>5.14</td>
<td>10</td>
<td>77</td>
<td>no</td>
</tr>
</tbody>
</table>
### Table 3.7: Set 3 Testing Conditions

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$T_o$ (N)</th>
<th>$v$ (m/s)</th>
<th>$\tau$ (N⋅m)</th>
<th>$\omega_{ex}$, (Hz)</th>
<th>Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-1</td>
<td>307</td>
<td>5.14</td>
<td>0</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-2</td>
<td>307</td>
<td>5.14</td>
<td>1</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-3</td>
<td>307</td>
<td>5.14</td>
<td>2</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-4</td>
<td>307</td>
<td>5.14</td>
<td>3</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-5</td>
<td>307</td>
<td>5.14</td>
<td>4</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-6</td>
<td>307</td>
<td>5.14</td>
<td>5</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-7</td>
<td>307</td>
<td>5.14</td>
<td>6</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-8</td>
<td>307</td>
<td>5.14</td>
<td>7</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-9</td>
<td>307</td>
<td>5.14</td>
<td>8</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-10</td>
<td>307</td>
<td>5.14</td>
<td>9</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>3-11</td>
<td>307</td>
<td>5.14</td>
<td>10</td>
<td>128</td>
<td>no</td>
</tr>
</tbody>
</table>
### Table 3.8: Set 4 Testing Conditions

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$T_o , (N)$</th>
<th>$v , (\frac{m}{s})$</th>
<th>$\tau , (N \cdot m)$</th>
<th>$\omega_{\text{ex}}, , (Hz)$</th>
<th>Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-1</td>
<td>307</td>
<td>12.00</td>
<td>0</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-2</td>
<td>307</td>
<td>12.00</td>
<td>1</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-3</td>
<td>307</td>
<td>12.00</td>
<td>2</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-4</td>
<td>307</td>
<td>12.00</td>
<td>3</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-5</td>
<td>307</td>
<td>12.00</td>
<td>4</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-6</td>
<td>307</td>
<td>12.00</td>
<td>5</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-7</td>
<td>307</td>
<td>12.00</td>
<td>6</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-8</td>
<td>307</td>
<td>12.00</td>
<td>7</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-9</td>
<td>307</td>
<td>12.00</td>
<td>8</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-10</td>
<td>307</td>
<td>12.00</td>
<td>9</td>
<td>128</td>
<td>no</td>
</tr>
<tr>
<td>4-11</td>
<td>307</td>
<td>12.00</td>
<td>10</td>
<td>128</td>
<td>no</td>
</tr>
</tbody>
</table>

### Table 3.9: Set 5 Testing Conditions

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$T_o , (N)$</th>
<th>$v , (\frac{m}{s})$</th>
<th>$\tau , (N \cdot m)$</th>
<th>$\omega_{\text{ex}}, , (Hz)$</th>
<th>Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-1</td>
<td>299</td>
<td>5.14</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>5-2</td>
<td>299</td>
<td>6.0021</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>5-3</td>
<td>299</td>
<td>6.8596</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>5-4</td>
<td>299</td>
<td>7.7170</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>5-5</td>
<td>299</td>
<td>8.5745</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>5-6</td>
<td>299</td>
<td>5.14-12.00</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
</tbody>
</table>
3.3 Data Modification

Before belt span tension can be calculated data modification of some measured data is required. Here the Fast Fourier Transform used to obtain belt span vibration frequency, the low-pass Butterworth filter used to filter the hubload sensor data, the resampling completed for the oscillation tension, and the envelope function used to better display data are discussed.

3.3.1 Fast Fourier Transform

The natural frequency of the belt span is found by transforming the data from the mid-span displacement sensor measurement into the frequency domain using a Fast Fourier Transform. The mid-span displacement sensor measurement is used as this measurement best captures the belt span’s first vibration mode. The resulting frequency plot will naturally contain several frequencies other than the belt span natural frequency. An example frequency plot containing many frequencies additional to the belt span natural frequency is illustrated in Fig. 3.14, next page.
Additional frequencies in the frequency plot could be the system excitation frequency, system natural vibration frequencies, the rotational frequencies of individual pulleys, or even the belt loop frequency. The natural frequency of the belt span is determined by calculating and removing these frequencies from the frequency plot.

The natural frequencies of the FEAD system are generated using modal analysis of the lumped mass FEAD system as per analysis in Section 2.3.2. Specifically, the eigenvalue equation, Eq. (2.52), on page 40 is used. The natural frequencies for the FEAD system without a tensioner arm are shown in Table 3.11a and in Table 3.11b for the FEAD system with a tensioner arm.
Table 3.11: System Frequencies, $\omega_{sys}^n$ (Hz)

<table>
<thead>
<tr>
<th>Pulley Frequency</th>
<th>No Tensioner Arm</th>
<th>With Tensioner Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{sys}^1$</td>
<td>0.00</td>
<td>$\omega_{sys}^1$</td>
</tr>
<tr>
<td></td>
<td>124.65</td>
<td>$\omega_{sys}^2$</td>
</tr>
<tr>
<td>$\omega_{sys}^2$</td>
<td>222.91</td>
<td>336.39</td>
</tr>
<tr>
<td>$\omega_{sys}^3$</td>
<td>373.55</td>
<td>373.57</td>
</tr>
<tr>
<td>$\omega_{sys}^4$</td>
<td>469.15</td>
<td>469.97</td>
</tr>
<tr>
<td>$\omega_{sys}^5$</td>
<td>1075.72</td>
<td>1075.72</td>
</tr>
</tbody>
</table>

Pulley frequencies are the rotation frequencies of the pulleys and are generated due to slight eccentricities in the pulley mass distribution. As the mass of the pulley is much larger than that of the belt, and the belt is supported by the pulley, the frequencies of the pulleys will appear in the frequency spectrum of the moving belt. These frequencies can be calculated using the equation below:

$$\omega_n^i = \frac{(Crankshaft\ RPM) \left( \frac{1}{60} \frac{min}{sec} \right) \left( \frac{d_{cs}}{d_i} \right)}{n}$$

(3.3)

Where $\omega_n^i$ is the $n^{th}$ frequency of the $i^{th}$ pulley. From experiments, these frequencies generally did not appear above the fourth frequency.

The belt loop frequency is the number of full rotations the belt makes per second. This frequency appears due to experimental error as the sensor is measuring the extra thickness of the belt label as it passes under the displacement sensor. The belt loop frequency is a function of the axial velocity, or crankshaft input RPM. It is calculated using Eq. (3.4) where $n$ can vary from 1 to $\infty$ depending on how often the sensor picks up the belt attachment label. From experiments, the belt loop frequency was generally not recorded past the fourth frequency. The equation for calculating belt loop frequency is defined
below:

$$\omega_b^n = (\text{Crankshaft RPM}) \left( \frac{1 \text{ min}}{60 \text{ sec}} \right) \left( 2\pi r_c \frac{m}{rot} \right) \left( \frac{1}{l_b} \right) n$$

(3.4)

Where $\omega_b^n$ is the $n^{th}$ belt loop frequency.

A complete list of the additional frequencies removed from the frequency plots are contained in Tables 3.12 - 3.17. The frequencies are calculated at the RPM speeds used in experiments, namely 600 – 1400 RPM. Frequencies for other speeds or systems can easily be calculated using the frequency equations listed in this section.

<table>
<thead>
<tr>
<th>Crankshaft RPM</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
<th>n=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>3.99 ± 2</td>
<td>7.98 ± 2</td>
<td>11.96 ± 2</td>
<td>15.95 ± 2</td>
<td>19.94 ± 2</td>
</tr>
<tr>
<td>700</td>
<td>4.65 ± 2</td>
<td>9.31 ± 2</td>
<td>13.96 ± 2</td>
<td>18.61 ± 2</td>
<td>23.26 ± 2</td>
</tr>
<tr>
<td>800</td>
<td>5.32 ± 2</td>
<td>10.63 ± 2</td>
<td>15.95 ± 2</td>
<td>21.27 ± 2</td>
<td>26.59 ± 2</td>
</tr>
<tr>
<td>900</td>
<td>5.98 ± 2</td>
<td>11.96 ± 2</td>
<td>17.95 ± 2</td>
<td>23.93 ± 2</td>
<td>29.91 ± 2</td>
</tr>
<tr>
<td>1000</td>
<td>6.65 ± 2</td>
<td>13.29 ± 2</td>
<td>19.94 ± 2</td>
<td>26.59 ± 2</td>
<td>33.23 ± 2</td>
</tr>
<tr>
<td>1400</td>
<td>9.31 ± 2</td>
<td>18.61 ± 2</td>
<td>27.92 ± 2</td>
<td>37.22 ± 2</td>
<td>46.53 ± 2</td>
</tr>
</tbody>
</table>

Table 3.12: Belt Loop Frequencies, $\omega_b^n$, (Hz)

<table>
<thead>
<tr>
<th>Crankshaft RPM</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>23.39 ± 2</td>
<td>46.79 ± 2</td>
<td>70.18 ± 2</td>
<td>93.58 ± 2</td>
</tr>
<tr>
<td>700</td>
<td>27.29 ± 2</td>
<td>54.59 ± 2</td>
<td>81.88 ± 2</td>
<td>109.17 ± 2</td>
</tr>
<tr>
<td>800</td>
<td>31.19 ± 2</td>
<td>62.38 ± 2</td>
<td>93.58 ± 2</td>
<td>124.77 ± 2</td>
</tr>
<tr>
<td>900</td>
<td>35.09 ± 2</td>
<td>70.18 ± 2</td>
<td>105.27 ± 2</td>
<td>140.36 ± 2</td>
</tr>
<tr>
<td>1000</td>
<td>38.99 ± 2</td>
<td>77.98 ± 2</td>
<td>116.97 ± 2</td>
<td>155.96 ± 2</td>
</tr>
<tr>
<td>1400</td>
<td>54.59 ± 2</td>
<td>109.17 ± 2</td>
<td>163.76 ± 2</td>
<td>218.34 ± 2</td>
</tr>
</tbody>
</table>

Table 3.13: Idler 1 Frequencies, $\omega_{id1}^n$, (Hz)
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#### Table 3.14: Air Conditioner Frequencies, $\omega_{n}^{AC}$, (Hz)

<table>
<thead>
<tr>
<th>Crankshaft RPM</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>13.34 ± 2</td>
<td>26.68 ± 2</td>
<td>40.02 ± 2</td>
<td>53.36 ± 2</td>
</tr>
<tr>
<td>700</td>
<td>15.94 ± 2</td>
<td>31.89 ± 2</td>
<td>47.83 ± 2</td>
<td>63.78 ± 2</td>
</tr>
<tr>
<td>800</td>
<td>17.79 ± 2</td>
<td>35.57 ± 2</td>
<td>53.36 ± 2</td>
<td>71.15 ± 2</td>
</tr>
<tr>
<td>900</td>
<td>20.01 ± 2</td>
<td>40.02 ± 2</td>
<td>60.03 ± 2</td>
<td>80.04 ± 2</td>
</tr>
<tr>
<td>1000</td>
<td>22.23 ± 2</td>
<td>44.47 ± 2</td>
<td>66.70 ± 2</td>
<td>88.93 ± 2</td>
</tr>
<tr>
<td>1400</td>
<td>31.13 ± 2</td>
<td>62.25 ± 2</td>
<td>93.38 ± 2</td>
<td>124.51 ± 2</td>
</tr>
</tbody>
</table>

#### Table 3.15: Alternator Frequencies, $\omega_{n}^{Alt}$, (Hz)

<table>
<thead>
<tr>
<th>Crankshaft RPM</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>24.90 ± 2</td>
<td>49.81 ± 2</td>
<td>74.71 ± 2</td>
<td>99.61 ± 2</td>
</tr>
<tr>
<td>700</td>
<td>29.05 ± 2</td>
<td>58.11 ± 2</td>
<td>87.16 ± 2</td>
<td>116.21 ± 2</td>
</tr>
<tr>
<td>800</td>
<td>33.20 ± 2</td>
<td>66.41 ± 2</td>
<td>99.16 ± 2</td>
<td>132.82 ± 2</td>
</tr>
<tr>
<td>900</td>
<td>37.35 ± 2</td>
<td>74.71 ± 2</td>
<td>112.06 ± 2</td>
<td>149.42 ± 2</td>
</tr>
<tr>
<td>1000</td>
<td>41.51 ± 2</td>
<td>83.01 ± 2</td>
<td>124.52 ± 2</td>
<td>166.02 ± 2</td>
</tr>
<tr>
<td>1400</td>
<td>58.11 ± 2</td>
<td>116.21 ± 2</td>
<td>174.32 ± 2</td>
<td>232.43 ± 2</td>
</tr>
</tbody>
</table>

#### Table 3.16: Idler 2 Frequencies, $\omega_{n}^{Id2}$, (Hz)

<table>
<thead>
<tr>
<th>Crankshaft RPM</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>22.78 ± 2</td>
<td>45.55 ± 2</td>
<td>68.33 ± 2</td>
<td>91.10 ± 2</td>
</tr>
<tr>
<td>700</td>
<td>26.572 ± 2</td>
<td>53.14 ± 2</td>
<td>79.72 ± 2</td>
<td>106.29 ± 2</td>
</tr>
<tr>
<td>800</td>
<td>30.37 ± 2</td>
<td>60.74 ± 2</td>
<td>91.10 ± 2</td>
<td>121.47 ± 2</td>
</tr>
<tr>
<td>900</td>
<td>34.16 ± 2</td>
<td>68.33 ± 2</td>
<td>102.49 ± 2</td>
<td>136.66 ± 2</td>
</tr>
<tr>
<td>1000</td>
<td>37.96 ± 2</td>
<td>75.92 ± 2</td>
<td>113.88 ± 2</td>
<td>151.84 ± 2</td>
</tr>
<tr>
<td>1400</td>
<td>53.14 ± 2</td>
<td>106.29 ± 2</td>
<td>159.43 ± 2</td>
<td>212.58 ± 2</td>
</tr>
</tbody>
</table>
### Crankshaft RPM

<table>
<thead>
<tr>
<th>Crankshaft RPM</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
<th>n=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>10.00 ± 2</td>
<td>20.00 ± 2</td>
<td>30.00 ± 2</td>
<td>40.00 ± 2</td>
<td>50.00 ± 2</td>
</tr>
<tr>
<td>700</td>
<td>11.67 ± 2</td>
<td>23.33 ± 2</td>
<td>35.00 ± 2</td>
<td>46.67 ± 2</td>
<td>58.33 ± 2</td>
</tr>
<tr>
<td>800</td>
<td>13.33 ± 2</td>
<td>26.67 ± 2</td>
<td>40.00 ± 2</td>
<td>53.33 ± 2</td>
<td>66.67 ± 2</td>
</tr>
<tr>
<td>900</td>
<td>15.00 ± 2</td>
<td>30.00 ± 2</td>
<td>45.00 ± 2</td>
<td>60.00 ± 2</td>
<td>75.00 ± 2</td>
</tr>
<tr>
<td>1000</td>
<td>16.67 ± 2</td>
<td>33.33 ± 2</td>
<td>50.00 ± 2</td>
<td>66.67 ± 2</td>
<td>83.33 ± 2</td>
</tr>
<tr>
<td>1400</td>
<td>23.33 ± 2</td>
<td>46.67 ± 2</td>
<td>70.00 ± 2</td>
<td>93.33 ± 2</td>
<td>116.67 ± 2</td>
</tr>
</tbody>
</table>

Table 3.17: Crankshaft Frequencies, $\omega_n^{CS}$, (Hz)

The additional frequencies are now be removed from the frequency plot in Fig. 3.14. The frequency plot is reprinted below next to the edited version.

![Original Frequency Plot](image1.png)

![Edited Frequency Plot](image2.png)

Figure 3.15: Original and Edited Frequency Plots, Trial 1-1

The three lowest frequencies with significant magnitude in Fig. 3.15a are $F_1$, $F_4$, and $F_5$. These frequencies are related to the first, second and fourth belt loop frequencies in Table 3.12, page 72. The next lowest frequency is $F_6$, this frequency is a crankshaft frequency found in Table 3.17, page 74. The remaining two frequencies are the belt span’s natural
frequency and after the original frequencies plot is edited this is shown in Fig. 3.15b. Many of the lower frequencies in Fig. 3.15b have been deleted as the MATLAB code is used to remove the additional frequencies from the frequency plot does not target high amplitude frequencies only. To prevent the actual belt span frequency from being deleted from the frequency plot the expected belt span frequency is calculated using the initial belt span tension, $T_o$, and frequencies in the expected frequency range are not deleted from the frequency plot.

Similar modifications were made for all data sets trials. Below are some sample original and edited frequency plots from the experimental data sets. In Fig. 3.16 the crankshaft excitation frequency of $76.9806 \text{ Hz}$ is removed to reveal the much lower amplitude natural belt frequency of $86.9751 \text{ Hz}$. Similar frequency plots were generated for all of Set 2 as in this set only the torque excitation amplitude changes.

![Figure 3.16: Original and Edited Frequency Plot, Trial 2-3](image)

In Fig. 3.17 many low frequency values with high amplitude are generated in the FEAD system which are removed to find the lower amplitude belt span natural frequency of
126.3428 Hz.

Figure 3.17: Original and Edited Frequency Plot, Trial 3-2

Similar edited was completed for all data.

3.3.2 Data Filters

The hubload tension validation data is filtered with a low pass Butterworth filter. Filtering is completed to allow the data to be used in a plot in a more visually pleasing manner in some instances. The low pass frequency was set to 1 Hz. Some examples of measured tension validation data collected from the hubload sensor during Set 3 trials and filtered using the low pass Butterworth filter are shown in Fig. 3.18. It is noteworthy that the filter is erroneous at low time values as the filter does not have enough data to generate accurate results.
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(a) Trial 3-1

(b) Trial 3-2

(c) Trial 3-3

(d) Trial 3-4
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(e) Trial 3-5  

(f) Trial 3-6  

(g) Trial 3-7  

(h) Trial 3-8
3.3.3 Data Resampling

When the oscillation tension was measured it was found that it exhibits large fluctuations or “spikes” at a frequency very close to 3.95 Hz for a crankshaft input speed of 600
**Chapter 3. Experiments**

*RPM*. This, interestingly, is the belt loop frequency, or the number of full cycles the belt makes per second, see Eq. (3.4) page 72 for a definition. The large fluctuations in the graph are, in fact, the sensor picking up the increased height of the belt at the tag or seam of the belt. This is obviously an error in the oscillation tension measurement. To eliminate these spikes the data was resampled at a frequency designed to eliminate the spike frequency. The data was then resampled again to return the sampling frequency to the original sampling frequency of 10 kHz.

For the third data set on the first trial the oscillation tension is plotted below:

![Figure 3.19: Oscillation Tension - Original Signal, Trial 5-1](image)

Below is a graph of both the original oscillation tension calculation with the oscillation tension calculated after the sensor data was resampled:
While the resampling method does in fact removed the frequency at which the tension modulates the change in amplitude, the desired measurement, remains preserved.

### 3.3.4 Envelope Function

An envelope function is used to display the measured tension validation data. This is done to allow for easier visual comparison of the measured validation tension with the calculated tension results. The signals envelope function is an outline of the data’s maximum or minimum values. The maximum and minimum envelope functions are then used to generate and area plot which visually displays the data range.
In Fig. 3.21 the blue area in the background represents the range of the measured validation tension.

### 3.4 Tension Validation

To validate the tension measurement methods a hubload sensor is placed in the first idler pulley in the experimental system, see Fig. 3.3, page 56. The rotational inertia of the first idler pulley is small in comparison to tension forces and therefore is ignored in calculations. This leads to the conclusion that the tension forces in the belt spans adjacent to the first idler pulley are equal. To determine the axial tension in the third belt span the rotary inertia of the air conditioner pulley, or second pulley, is included in further calculations. A force diagram is depicted in Fig. 3.22 followed by the derivation of the tension validation equations.
From the assumption of negligible first idler pulley rotational inertia it can be concluded that:

\[ T_1 = T_2 = T \]

Using trigonometry the tension validation equation for the first and second belt spans is easily found as:

\[ T_2(t) = \frac{H}{2 \sin \left( 90 - \frac{\varphi}{2} \right)} \quad (3.5) \]

To determine the belt tension forces in belt spans not adjacent to the first idler pulley non-negligible pulley inertias must be included. Focusing on the third belt span, the rotation force balance of the third pulley:

\[ I_3 \ddot{\theta}_3 = r_3 (T_3 - T_2) \]

Solving for \( T_3 \):

\[ T_3 = \frac{I_3 \ddot{\theta}_3}{r_3} + T_2 \quad (3.6) \]

Substituting Eq. (3.5) in Eq. (3.6) gives the tension validation equation for the third belt span as a function of pulley rotational acceleration and the first idler pulley hubload measurement:

\[ T_3(t) = \frac{I_3 \ddot{\theta}_3(t)}{r_3} + \frac{H(t)}{2 \sin \left( 90 - \frac{\varphi}{2} \right)} \quad (3.7) \]
Where the rotational acceleration is defined as:

\[
\ddot{\varphi}_3(t) = \frac{\dot{\varphi}_3(t_2) - \dot{\varphi}_3(t_1)}{\Delta t}
\] (3.8)

**Substitution of Testing System Values**

From Table 3.1, page 53, the wrap angle of the idler pulley is $\varphi_1 = 87^\circ$. Substitution into equation Eq. (3.5) leads to:

\[
t_2(t) = H(t) \ast 0.6893
\] (3.9)

Also from Table 3.1 the radii and inertia of the air conditioner are $r_3 = 0.0614$ $m$ and $I_3 = 0.0020$ $kg \cdot m^2$, respectively. Substitution into equation Eq. (3.7) leads to:

\[
t_3(t) = \ddot{\varphi}_3(t)\left(\frac{0.002 k g \cdot m^2}{0.0614 m}\right) + H(t) \ast 0.6893
\] (3.10)

The validation tension is calculated using Eq. (3.10).

### 3.5 Automated MATLAB Code

A complete set of automated MATLAB code has been included in Appendix E which is capable of analyzing data sensor and system data input to generate mean and dynamic tension outputs as a function of time. Required system data mass, damping, and stiffness matrices as defined in Section 2.3.1 and are also specified in the MATLAB code files. As well, the belt span longitudinal stiffness, $E_{A_B}$, Flexural Rigidity, $E_{J_B}$, and linear density, $\rho$, are required. Sensor data should be in column tabular format saved in “name” .txt files with the file name and location specified for easy scanning by the program. A sample data set from experiments is shown in Table 3.18.
<table>
<thead>
<tr>
<th>Time (s)</th>
<th>RPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>599.53</td>
</tr>
<tr>
<td>0.0002</td>
<td>601.60</td>
</tr>
<tr>
<td>0.0003</td>
<td>598.46</td>
</tr>
<tr>
<td>0.0004</td>
<td>602.39</td>
</tr>
<tr>
<td>0.0005</td>
<td>601.07</td>
</tr>
<tr>
<td>0.0006</td>
<td>600.98</td>
</tr>
<tr>
<td>0.0007</td>
<td>601.74</td>
</tr>
<tr>
<td>0.0008</td>
<td>600.37</td>
</tr>
<tr>
<td>0.0009</td>
<td>601.34</td>
</tr>
</tbody>
</table>

Table 3.18: Rotation Sensor Data Sample

Required data sets include the first support pulley rotation speed, the second support pulley rotation speed, and three evenly spaced displacement sensor readings from the span of interest. The data should be measured in base units of *seconds*, *RPM*, and *meters*.
Chapter 4

Results and Discussion

The results generated from experimental testing and generated using the proposed dynamic tension measurement are compared with the validation tension and presented here. The mean belt span tension and oscillation tension as developed in Section 2 are calculated and presented. After summation of the mean and oscillation tensions the dynamic belt span tension results are presented. Percent error results are compared for different testing sets.

4.1 Mean Tension

Here, the mean tension of a belt spans is calculated using the method outlined in Section 2.1, and, more specifically, frequency equations listed in Section 2.4 page 47. However, first the natural frequencies for the belt span in different operating conditions is presented and trends discussed.

4.1.1 Belt Span Transverse Vibration Frequencies

As discussed in the theoretical modeling Section, page 20, the mean tension of a belt span is calculated using the span’s natural vibration frequency. In the experiment section,
page 69, the span frequencies are found using a Fast Fourier Transform of the mid-span displacement sensor readings. For each of the six data sets the collected, the span’s first vibration frequency, as measured from the Fast Fourier Transform frequency plots, are plotted in Fig. 4.1 below. This figure shows the general vibration frequency trends with respect to crankshaft input speed and amplitude of crankshaft torque excitation.

Figure 4.1: Measured Span Transverse Vibration Frequencies

In Fig. 4.1a an increase in initial axial tension, $T_o$, in the belt span generates an increase in magnitude of the belt span’s first natural frequency. This proportional relationship between initial belt span tension and natural frequency also appears in Fig. 4.1b. The proportional relationship between axial belt span tension and belt span natural frequency is expected and confirms the relationship found in the mean tension equation, Eq. (2.15), page 20. Additionally, in Fig. 4.1a, for a given initial tension the frequency increases as the axial speed increases\(^1\). This is contradictory to the established tension-frequency relationship found in Eq. (2.15), where an increased axial belt velocity leads to a decreased belt span natural vibration frequency. The increase in belt belt span vibration frequency found in the Fig. 4.1 is most likely due to the increased centrifugal forces experienced by the belt span while travelling at higher axial speeds. The centrifugal forces increases

\(^1\)Axial belt velocity increases with increased crankshaft input speed.
the belt span axial tension and thus increases the natural frequency with a degree of influence that is greater than the decrease in natural frequency which results from the increased axial velocity. In Fig. 4.1b the introduction of torque excitation amplitude has little effect on the belt span’s natural frequency. This is expected as belt span natural frequency is not a function of the excitation frequency or excitation amplitude.

4.1.2 Mean Tension Results

The frequencies shown in Fig. 4.1 are used to calculate mean tension for for each of the experimental data sets. The mean tension values calculated using the string equation, the beam equation with pinned-pinned boundary conditions, the beam equation with pinned-clamped boundary conditions, and the beam equation with clamped-clamped boundary conditions are compared with the experientially measured tension to determine which method is the most accurate measure of mean belt tension.

A sample of the mean tension values calculated using each of the four methods mentioned above are plotted in Fig. 4.2. The sample was selected to include a variety of operating conditions, low and high axial belt speed, low and high torque excitation values, and low and high initial tension values. The measured validation tension is displayed both as it was measured and after passing it through a low pass Butterworth filter with a 1 Hz cutoff frequency. The filtered data displays the measured data trend in a manner which is easier to visualize.
Chapter 4. Results and Discussion

(a) Trial 1-1 - Low Axial Belt Speed, Low Initial Tension

(b) Trial 1-5 - High Axial Belt Speed, Low Initial Tension

(c) Trial 5-1 - Low Axial Belt Speed, High Initial Tension

(d) Trial 4-1 - High Axial Belt Speed, High Initial Tension
Chapter 4. Results and Discussion

(e) Trial 2-1 - Low Crankshaft Torque Excitaiton Amplitude, Low Initial Tension, Low Axial Belt Speed

(f) Trial 2-11 - High Crankshaft Torque Excitaiton Amplitude, Low Initial Tension, Low Axial Belt Speed

(g) Trial 3-1 - Low Crankshaft Torque Excitaiton Amplitude, Low Initial Tension, High Axial Belt Speed

(h) Trial 3-11 - High Crankshaft Torque Excitaiton Amplitude, Low Initial Tension, High Axial Belt Speed
(i) Trial 1-6 - Low Initial Tension, Axial Acceleration, Without Tensioner Arm

(j) Trial 5-6 - High Initial Tension, Axial Acceleration, With Tensioner Arm

Figure 4.2: Measured Mean Tension and Validation Tension Sample

In Fig. 4.2 it is apparent that, regardless of operating conditions, the beam equation with pinned-pinned boundary conditions most accurately measures the belt span mean tension. The string equation consistently gave tension values that were too high. The beam equations with clamped-clamped and clamped-pinned boundary conditions gave tension values that were too low. The string equation models the belt with too low a value of flexural rigidity, namely zero. This low rigidity model vibrates at lower frequencies for a given axial tension, thus when calculating tension from the more rigid experimental system the resulting tension is too high. The clamped boundary conditions of the beam equation increase the belt rigidity and the model assumes the vibration frequency would be higher for a given axial tension. The belt span being modeled best as a rigid beam with pinned-pinned boundary conditions most accurately captures the true nature of the belt span. This result is in agreement with the boundary conditions modeled by Kong and Parker [40].
For each of the data sets the percent error of the mean tension measurement method is calculated by comparing the calculated mean tension with the measured validation tension. The results for each of the data sets is graphed in Fig. 4.3, below.

![Graphs showing mean tension percent error for different sets under different conditions.](image)

**Figure 4.3: Measured Mean Tension Percent Error**

For each of the data sets the mean tension measurement error percent is less than 10% as shown in Fig. 4.3. For the testing completed with constant axial speed, data sets 1 and 5, the mean percent error is fairly constant regardless of axial belt speed. For the testing completed with crankshaft torque excitation, sets 2, 3, 4, and 6, the mean percent error increases as the crankshaft torque excitation amplitude increases. However, this error is negligible and is most likely due to measurement error.

### 4.2 Oscillation Tension

The oscillation tension results are calculated as per the method developed in the oscillation tension theory Section 2.2, page 30. The oscillation tension which occurs when the spans vibrate transversely is shown to be significant in the measurement of belt span tension when the excitation frequency is close to the span natural frequency. Also, the effect of oscillation tension increases when the crankshaft input has a large excitation frequency.
amplitude as the belt experiences greater displacement in the transverse direction. This can be seen in the graphs below, both taken from data set 2, the first graph is a system running with zero excitation amplitude and the second is the system running with a torque excitation amplitude of $10 \, N \cdot m$.

![Graphs](image)

(a) Trial 2-1 - Low Crankshaft Torque Excitation Amplitude  
(b) Trial 2-11 - High Crankshaft Torque Excitation Amplitude

Figure 4.4: Vibration Tension Comparison

The tension in Fig. 4.4a is much less than in the figure with higher torque excitation frequency, Fig. 4.4b. As well, large “spikes” are seen in the data, these were previously determined to be measurement error and were removed using a resampling technique as discussed in Section 3.3.3, page 79. Some further samples of the resulting measure for oscillation tension are shown in Fig 4.5.
(a) Trail 6-1 - Low Axial Speed, Low Crankshaft Torque Excitation Amplitude, With Tensioner Arm

(b) Trail 4-10 - High Axial Speed, High Crankshaft Torque Excitation Amplitude, Without Tensioner Arm

(c) Trail 1-1 - Low Axial Speed, Zero Crankshaft Torque Excitation Amplitude, Without Tensioner Arm

(d) Trail 1-5 - High Axial Speed, Zero Crankshaft Torque Excitation Amplitude, Without Tensioner Arm

Figure 4.5: Oscillation Tension Sample Calculations
With resampling oscillation tension still increases with increased torque excitation amplitude and when the system is excited at a belt natural frequency. Additionally, with increased axial speed more energy is introduced into the system causing increased belt span amplitude vibration and thus increased oscillation tension.

4.3 Dynamic Tension

Calculating dynamic tension is done by measuring the rotations of the belt span’s support pulleys, as outlined in Section 2.3 while utilizing the mean and oscillation tension calculated in Section 2.1 and Section 2.2 respectively. For the system without crankshaft excitation frequency the dynamic tension can be calculated using only the change in pulley length generated from pulley rotations, as in Eq. (2.64), page 43. However, the system with crankshaft excitation will generate rotational inertia forces large enough to cause the belt to slip on the pulley(s). In a system with torque excitation frequency, not taking into consideration the belt slip will generate large errors in dynamic tension measurement, as seen in Fig. 4.6.

![Graphs showing belt tension with and without slip](image)

(a) Modeled Without Slip  
(b) Modeled With Slip

Figure 4.6: Belt Slip - Trial 2-2

In Fig. 4.6a belt slip has not been taken into consideration and the tension measurement
becomes increasingly erroneous as the tension error, or slip error, accumulates. It should be noted that the tension could just as easily increase, it just depends on the system operating conditions.

To solve this problem belt slip is introduced to the dynamic tension equation. The derivation of the belt slip equation’s inclusion in the dynamic frequency equation is found in Section 2.3.3. The kinetic and static friction coefficients are chosen to be $\mu_k = 0.5$ and $\mu_s = 0.5$. The values of the friction coefficients were taken as the midpoint of the range of friction coefficient values found in the literature [16]. However, no references were found in the literature for a typical arc of slip, $\alpha$, and the corresponding angle of adhesion, $(\varphi - \alpha)$, as these values are dependent on system operating conditions. A solution was found by choosing the arc of slip values such that the mean belt span tension remains equal to the mean belt span tension found using the frequency method, previously discussed. This is a practical assumption and valid as the measured mean tension has low percent error.

The arc of slip is discussed in terms of its fraction of the total arc angle. This fraction is represented by the variable $p$ and is related to the angle of slip, $\alpha$, using Eq. (4.1).

$$p = \frac{\alpha}{2\pi r \varphi} \quad (4.1)$$

The angel of slip as a fraction of the total wrap angle is calculated for each of the data sets and displayed in Table 4.1.
Table 4.1: Arc of Slip Fraction of Wrap Angle, $p$

<table>
<thead>
<tr>
<th>Trial</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
<th>Set 5</th>
<th>Set 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.075</td>
<td>0.059</td>
<td>0.367</td>
<td>0.496</td>
<td>0.273</td>
<td>0.325</td>
</tr>
<tr>
<td>2</td>
<td>0.083</td>
<td>0.211</td>
<td>0.367</td>
<td>0.504</td>
<td>0.283</td>
<td>0.159</td>
</tr>
<tr>
<td>3</td>
<td>0.073</td>
<td>0.289</td>
<td>0.371</td>
<td>0.512</td>
<td>0.293</td>
<td>0.185</td>
</tr>
<tr>
<td>4</td>
<td>0.085</td>
<td>0.375</td>
<td>0.379</td>
<td>0.527</td>
<td>0.305</td>
<td>0.189</td>
</tr>
<tr>
<td>5</td>
<td>0.129</td>
<td>0.461</td>
<td>0.387</td>
<td>0.543</td>
<td>0.313</td>
<td>0.313</td>
</tr>
<tr>
<td>6</td>
<td>0.093</td>
<td>0.539</td>
<td>0.394</td>
<td>0.563</td>
<td>0.117</td>
<td>0.293</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0.609</td>
<td>0.402</td>
<td>0.504</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>0.672</td>
<td>0.410</td>
<td>0.512</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>0.734</td>
<td>0.418</td>
<td>0.516</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.781</td>
<td>0.426</td>
<td>0.523</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.820</td>
<td>0.434</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Where the above data trends are displayed in Fig. 4.7, below.

It is evident that the angle of slip increases as the torque excitation amplitude increases. This is because the higher torque excitation amplitudes generate larger rotational inertia.
forces and generate more belt-pulley slip. For the constant velocity trials, Fig. 4.7a, the arc of slip is relatively constant. This is expected as the system is running at a relatively steady state.

The dynamic tension of the belt span can now be calculated using the defined friction coefficients and the arc of slip values. The belt span dynamic tension is compared with the measured validation tension. As mentioned when calculating the arc of slip angles the mean tension of the dynamic span tension is unchanged from the mean tension calculated in the previous section. A sample of measured dynamic tension data is shown in Fig. 4.8. The measured validation tension is represented with an envelope function plotted in blue. The calculated dynamic belt span tension is plotting using a line graph for easy visual comparison of data.

(a) Trial 1-1 - Low Axial Speed, Zero Crankshaft Torque Excitation Amplitude, Without Tensioner Arm

(b) Trial 5-1 - Low Axial Speed, Zero Crankshaft Torque Excitation Amplitude, With Tensioner Arm
(c) Trial 2-1 - Low Axial Speed, Zero Crankshaft
Torque Excitation Amplitude, Low Excitation Frequency, Low Initial Tension, Without Tension Arm

(d) Trial 2-10 - Low Axial Speed, High Crankshaft
Torque Excitation Amplitude, Low Excitation Frequency, Low Initial Tension, Without Tension Arm

(e) Trial 3-2 - Low Axial Speed, Low Crankshaft
Torque Excitation Amplitude, High Excitation Frequency, High Initial Tension, Without Tensioner Arm

(f) Trial 3-10 - Low Axial Speed, High Crankshaft
Torque Excitation Amplitude, High Excitation Frequency, High Initial Tension, Without Tensioner Arm
Chapter 4. Results and Discussion

(g) Trial 4-1 - High Axial Speed, Low Crankshaft Torque Excitation Amplitude, High Initial Tension, Without Tension Arm

(h) Trial 4-10 - High Axial Speed, High Crankshaft Torque Excitation Amplitude, High Initial Tension, Without Tension Arm

(i) Trial 1-6 - Axial Acceleration Without Tensioner Arm

(j) Trial 5-6 - Axial Acceleration With Tensioner Arm

Figure 4.8: Measured Dynamic Tension and Validation Tension Samples

From the plots in Fig. 4.8 it can be seen that while the dynamic tension does follow the general trend of the measured validation tension the maximum and minimum tension values do contain errors. A linear damping coefficient of $c_b^* = 65 \frac{N s}{m}$ is introduced to the dynamic belt span tension equation, Eq. (2.64), page 43. This is a typical damping coefficient for lumped mass belt systems. The dynamic tension in the belt spans in now
generated as:

(a) Trial 1-1 - Low Axial Speed, Zero Crankshaft Torque Excitation Amplitude, Without Tensioner Arm

(b) Trial 5-1 - Low Axial Speed, Zero Crankshaft Torque Excitation Amplitude, With Tensioner Arm

(c) Trial 2-1 - Low Axial Speed, Zero Crankshaft Torque Excitation Amplitude, Low Initial Tension, Low Excitation Frequency, Without Tensioner Arm

(d) Trial 2-10 - Low Axial Speed, High Crankshaft Torque Excitation Amplitude, Low Initial Tension, Low Excitation Frequency, Without Tensioner Arm
Chapter 4. Results and Discussion

Figure 4.7: Damped Measured Dynamic Tension and Validation Tension Samples

The percent error for the damped and undamped measured tension is calculated and graphically compared. To determine the accuracy of the dynamic tension the maximum and minimum tension values are compared for each trial of each data set. The data sets are divided into two subcategories. Set 1 and Set 5 both have constant axial velocity and no torque excitation amplitude. Set 2, Set 3, Set 4, and Set 6 have constant axial speed...
with varying torque excitation amplitude.

Below are the comparison of the maximum undamped and damped tension for all data sets.

![Graphs showing dynamic tension maximum value error percent for different conditions](image)

**Figure 4.10: Dynamic Tension Maximum Value Error Percent**

It can be seen in Fig. 4.10a and Fig. 4.10b that the introduction of damping greatly decreases the error of the constant velocity dynamic tension measurements. The damped percent error values for the constant axial speed trials are well within acceptable limits. For the trials with crankshaft torque excitation the introduction of damping produced...
mixed results. Set 2 and Set 3 both experienced increased error when damping was applied to higher torque excitation frequency trials. Set 4 and Set 6 experienced reduced percent error with the introduction of damping. This appears to indicate that damping proportional to excitation amplitude would be applicable to the dynamic tension measurement.

The minimum tension percent error is treated in a similar fashion. The percent error graphs comparing minimum calculated dynamic tension with the measured validation tension are shown below:

(a) Constant Axial Speed - Undamped

(b) Constant Axial Speed - Damped
The percent error results for minimum dynamic tension are higher than those found for maximum dynamic tension. This is acceptable as the maximum dynamic tension is generally of greater interest in FEAD systems as it contributes more to belt wear. Minimum tension values would be of interest if they were low enough to cause belt slip, etc. With the introduction of damping the reduction in percent error in the constant axial speed data sets is appreciable produces good results for Set 5. Set 1 values reduce to by approximately the same amount but were higher in the undamped case.

The data sets with crankshaft torque excitation once again saw a increase in percent error with the introduction of damping. The percent error without damping is rather consistent for all torque amplitude values. When damping is introduced a definite trend of increased percent error with increased crankshaft torque excitation amplitude is seen. The supports the suggestion of damping proportional to the amplitude of torque excitation.
4.4 Measurement Procedure

This section outlines the recommended procedure to measure dynamic belt span tension in running FEAD systems. The procedure follows the procedure used to write the code included in Appendix E.

1. Record required system values belt characteristics.

2. Set up displacement and rotation sensors on the span of interest as per Fig 4.12. Place one sensor in the center of the span to capture the first mode of vibration and one sensor one quarter of the way from each end of the span to capture the second mode.

3. Set the sampling frequency to be higher than the Nyquist frequency for the system. Use the initial belt tension to calculate the expected belt frequencies to determine the Nyquist frequency.

4. Run the system. Collect discrete data samples from the rotation and displacement sensors.

5. Perform Fast Fourier Transforms of the mid-span displacement data collected and remove additional frequencies. Record the belt span fundamental frequency from the edited frequency plot.
6. Linearly interpolate the data such that all data samples are recorded at equal time intervals. This is important for further calculations.

7. Calculate mean tension using the frequency Eq. (2.15), page 20.

8. Calculate oscillation tension from transverse belt deviation, Eq. (2.32), page 32.

9. Calculate the dynamic tension including the effects of belt slip Eq. (2.64), page 43.
Chapter 5

Conclusions

5.1 Summary and Conclusion

This study of dynamic belt span tension in FEAD systems generated interesting results which are summarized here. The proposed mean tension measurement method proved to be a highly accurate and reliable. The equation of motion with flexural rigidity and pinned-pinned boundary conditions was shown to most accurately model the moving belt span problem for application in tension measurement. The axial velocity and transverse vibration frequency parameters were easily measured in the belt span using a non-contact laser displacement sensor. The system frequencies, pulley frequencies, and belt loop frequency were removed from the Fast Fourier Transform of the mid-span displacement sensor to determine the belt span first natural frequency.

Oscillation tension was not negligible when the system is excited at the belt span natural frequency. However, when the system is excited at a frequency not near the belt natural frequency the oscillation tension may be considered negligible. The oscillation tension was measured using the span non-contact displacement sensors.
Dynamic belt span tension was accurately measured using the proposed method. Knowledge of belt-pulley friction coefficients is required, or an appropriate assumption can be made as was done here. The arc of slip was calculated by keeping the magnitude of mean dynamic tension equal to the mean tension calculated using the frequency method. The dynamic tension is most accurate for constant velocity systems with sufficient accuracy for some application in a system with crankshaft torque excitation. However, improved knowledge of belt damping may be required. The dynamic tension was found as a function of span support pulley rotation. As well, knowledge of mean and oscillation tension was required to calculate the belt slip of the driven pulley.

5.2 List of Contributions

The main contributions of this work are the measurement of mean and oscillation belt span tension and their use in the determination of dynamic belt span tension. The mean tension measurement uses the span frequency to accurately determine the mean tension. While this method has been used in industry for a stationary belt span this is the first use in a axially moving belt span. Calculating and removing system frequencies from the Fast Fourier Transform of the displacement sensor reading has allowed for the natural frequencies of the moving belt span to be determined and thus for mean tension to be determined. Oscillation tension is typically considered negligible, however, it was measured in this work using displacement sensors and shown that when the system excitation frequency is equal or near to the span natural vibration frequency the oscillation tension has a significant contribution to the total span tension. The dynamic tension was calculated using the support pulley arc lengths of rotation, and, while this is not new, the initial static tension was replaced with the measured mean tension to improve accuracy. As well, the use of the slip equation was applied to a single belt span rather than to the FEAD system of equations as it has previously been done. These combined
methods enable, for the first time, the dynamic belt tension in a single belt span of a FEAD system to be measured using non-contact sensors.

5.3 Recommendations for Future Study

While this study has been successful in the measurement of dynamic belt span tension there is room for improvement and expansion. First, the introduction of a better damping model would improve the dynamic belt span tension results, especially for systems with torque excitation. The equations for dynamic belt span tension in the spans adjacent to the FEAD system automatic tensioner arm are included in Eq. (2.64), page 43, however experimental measurement of dynamic tension in these belt spans was not completed. The belt spans adjacent to the automatic tensioner arm would have the added complication of the added moving support pulley frequency, this could also be removed from the belt span Fast Fourier Transform. Balaji and Mockensturm [22] completed FEAD system analysis with the inclusion of an isolator/decoupler and expansion of this dynamic tension model to include an isolator/decoupler would be very interesting.
Bibliography


Appendix A

Travelling Belt Span Equation of Motion Derivation

Here, the equation of motion for the axially travelling belt span with the inclusion of axial acceleration and bely rigidity is derived using Hamilton’s Principle:

\[ \delta \int_{t_1}^{t_2} [\mathcal{T} - (\mathcal{U} + \mathcal{V})] \, dt = 0 \]  \hspace{1cm} (A.1)

Where \( \mathcal{T}, \mathcal{U}, \) and \( \mathcal{V} \) are the kinetic, strain, and potential energy of the belt span respectively.

**Belt Span Kinetic Energy**  The kinetic energy in the traveling belt span is found in the transverse, longitudinal, and rotational motion\(^1\) of the belt. Fig. A.1 is a velocity diagram of a small section of the belt:

\(^1\)Rotation about the z-axis.
Assuming that the deflection angle, $\phi$, is very small the small angle assumptions can be made:

$$
\cos \phi \approx 1, \quad \frac{1}{\sqrt{1 + \tan^2 \phi}} \approx \frac{1}{\sqrt{1 + \frac{\partial^2 w}{\partial x^2}}}, \quad \sin \phi \approx \tan \phi,
$$

Thus, the velocity in the translational, or x-direction, is reduced to:

$$
v_x = v \cos \phi = v
$$

And the velocity in the transverse, or y-direction, is reduced to:

$$
v_y = \frac{\partial w}{\partial t} + v \sin \phi = \frac{\partial w}{\partial t} + v \tan \phi = \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial x}
$$

The magnitude of the velocity of the belt in the transverse plane is found using Pythagorean theorem as:

$$
v = \sqrt{v_y^2 + v_x^2}
$$

Finally, the kinetic energy of a belt, including transverse, longitudinal, and rotational kinetic energy, is described as [6]:

$$
\mathcal{T} = \int_0^l \frac{1}{2} \rho v^2 dx = \int_0^l \frac{1}{2} \rho \left[ \left( \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial x} \right)^2 + v^2 \right] dx \quad (A.2)
$$
Appendix A. Travelling Belt Span Equation of Motion Derivation

Belt Span Axial Strain Energy  The strain energy of the belt span system is found in the elastic deformation of the belt material. The diagram below depicts the forces acting on a small section of the belt:

![Figure A.2: Force Diagram](image)

The basic equation for elastic potential energy of a linearly axially stretched elastic belt, based on Hooke’s Law, is:

\[ U_a = k(dl)^2 = Tdl \]  \hspace{1cm} (A.3)

Where \( T \) is the span axial tension, \( l \) is the belt span length, and \( k \) is the belt span axial stiffness. The change in belt span length, with the assumed small transverse deflections, can be defined as [38]:

\[ dl = (ds - dx) = \left( \frac{dx}{\cos \phi} - dx \right) = \left[ \sqrt{1 + \left( \frac{\partial^2 w}{\partial x^2} \right)^2} - 1 \right] dx \]  \hspace{1cm} (A.4)

Using the well known series conversion for a polynomial equation:

\[ (1 + z)^p = \sum_{n=0}^{\infty} \frac{p!}{n!(p-n)!} z^n, \quad \text{Valid for } |z| < 1 \]

Equation A.4 expands into a series. Eliminating higher orders leads to:

\[ dl = \left( \sqrt{1 + \frac{\partial^2 w}{\partial x^2}} dx - dx \right) = \left[ \left( 1 + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \right) - 1 \right] dx = \frac{1}{2} \frac{\partial^2 w}{\partial x^2} dx, \quad \text{Valid for } \left| \frac{\partial^2 w}{\partial x^2} \right| \ll 1 \]  \hspace{1cm} (A.5)
Substitution of Eq. (A.5) into equation Eq. (A.3) gives the belt span axial strain energy equation:

\[ U_a = \frac{1}{2} \int_0^l T \frac{\partial^2 w}{\partial x^2} dx, \quad \text{Valid for} \quad \frac{\partial^2 w}{\partial x^2} \ll 1 \]  

(A.6)

**Belt Span Bending Strain Energy** If the belt span is sufficiently thick, strain energy is generated during belt bending.

This moment strain energy is quantified using the equation [10]:

\[ U_b = \frac{1}{2} \int_0^l \frac{M^2}{EJ} dx = \frac{1}{2} \int_0^l EJ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx \]  

(A.7)

**Equation of Motion** Substituting A.2, A.6, and A.7 into equation A.1 gives:

\[
\delta \int_{t_1}^{t_2} \int_0^l \left\{ \frac{1}{2} \rho \left[ \left( \frac{\partial w}{\partial t} \right)^2 + 2 \left( v \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} + v^2 \left( \frac{\partial w}{\partial x} \right)^2 + v^2 \right) - \left[ \frac{EJ}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{2} T \left( \frac{\partial w}{\partial x} \right)^2 \right] \right\} dx \, dt = 0
\]

Taking the variation:

\[
\int_{t_1}^{t_2} \int_0^l \left\{ \rho \left[ \left( \frac{\partial w}{\partial t} \delta \frac{\partial w}{\partial t} \right) + \left( v \frac{\partial w}{\partial t} \delta \frac{\partial w}{\partial x} \right) + \left( v \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial t} \right) + \left( v^2 \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial x} \right) \right] - \left( EJ \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial^2 w}{\partial x^2} \right) + \left( T \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial x} \right) \right\} dx \, dt = 0
\]
Integrating the above equation by parts term by term: Term #1:

\[
\int_{t_1}^{t_2} \int_{0}^{l} \rho \frac{\partial w}{\partial t} \frac{\partial w}{\partial t} \, dx \, dt = \int_{0}^{l} \rho \frac{\partial w}{\partial t} \delta w \, dx - \int_{t_1}^{t_2} \int_{0}^{l} \rho \frac{\partial^2 w}{\partial t^2} \delta w \, dx \, dt
\]

Term #2:

\[
\int_{t_1}^{t_2} \int_{0}^{l} \rho v \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} \, dx \, dt = \int_{0}^{l} \rho v \frac{\partial w}{\partial t} \delta w \, dx - \int_{t_1}^{t_2} \int_{0}^{l} \rho v \frac{\partial^2 w}{\partial t \partial x} \delta w \, dx \, dt
\]

Term #3:

\[
\int_{t_1}^{t_2} \int_{0}^{l} \rho v \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial t} \, dx \, dt = \int_{0}^{l} \rho v \frac{\partial w}{\partial x} \delta w \, dx - \int_{t_1}^{t_2} \int_{0}^{l} \rho \left( \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} + v \frac{\partial^2 w}{\partial x \partial t} \right) \delta w \, dx \, dt
\]

Term #4:

\[
\int_{t_1}^{t_2} \int_{0}^{l} \rho v^2 \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial x} \, dx \, dt = \int_{0}^{l} \rho v^2 \frac{\partial w}{\partial x} \delta w \, dx - \int_{t_1}^{t_2} \int_{0}^{l} \rho v \frac{\partial^2 w}{\partial x^2} \delta w \, dx \, dt
\]

Term #5:

\[
\int_{t_1}^{t_2} \int_{0}^{l} EJ \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial^2 w}{\partial x^2} \, dx \, dt = - \int_{t_1}^{t_2} EJ \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \, dt + \int_{t_1}^{t_2} \int_{0}^{l} EJ \frac{\partial^3 w}{\partial x^3} \delta \frac{\partial w}{\partial x} \, dx \, dt
\]

\[
\int_{t_1}^{t_2} \int_{0}^{l} EJ \frac{\partial^3 w}{\partial x^3} \delta \frac{\partial w}{\partial x} \, dx \, dt = \int_{t_1}^{t_2} EJ \frac{\partial^3 w}{\partial x^3} \delta w \, dt - \int_{t_1}^{t_2} \int_{0}^{l} EJ \frac{\partial^4 w}{\partial x^4} \delta w \, dx \, dt
\]

Term #6:

\[
\int_{t_1}^{t_2} \int_{0}^{l} T \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial x} \, dx \, dt = \int_{t_1}^{t_2} \int_{0}^{l} T \frac{\partial w}{\partial x} \delta w \, dx \, dt + \int_{t_1}^{t_2} \int_{0}^{l} T \frac{\partial^2 w}{\partial x^2} \delta w \, dx \, dt
\]
Collecting final terms:

\[
\int_{t_1}^{t_2} \int_0^l \left\{ \rho \left[ \frac{\partial^2 w}{\partial t^2} + v \frac{\partial^2 w}{\partial t \partial x} + \left( \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} + v \frac{\partial^2 w}{\partial x \partial t} \right) + v^2 \frac{\partial^2 w}{\partial x^2} \right] \\
+ EJ \frac{\partial^4 w}{\partial x^4} - T \frac{\partial w^2}{\partial x^2} \right\} \delta \omega dx \omega dt = 0
\]

If the integral is equal to zero then the integrand is also equal to zero, removing the integrand leads to the belt span equation of motion:

\[
\rho \left[ \frac{\partial^2 w}{\partial t^2} + v \frac{\partial^2 w}{\partial t \partial x} + \left( \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} + v \frac{\partial^2 w}{\partial x \partial t} \right) + v^2 \frac{\partial^2 w}{\partial x^2} \right] + EJ \frac{\partial^4 w}{\partial x^4} - T \frac{\partial w^2}{\partial x^2} = 0 \quad (A.8)
\]
Appendix B

FEAD Lumped Mass System

Matrices

The six degree of freedom system matrices used in the discrete system calculations are detailed here. For the system without the tensioner arm the tensioner arm length, \( l_t \), should be set to zero and the sixth column and row of the matrices deleted.

**General System With Tensioner**

This set of equations assumes that the tensioner pulley is the \( n^{th} \) pulley in the system. This places the tensioner is on the slackest span in the system, a common placement position.

\[
M = \begin{bmatrix}
I_1 & 0 & \cdots & 0 \\
& I_2 & \ddots & \\
& & \ddots & \ddots \\
& & & I_{n-1} \\
& & & & \text{sym} \\
& & & & I_n & 0 \\
& & & & & I_t
\end{bmatrix}
\]

(B.1)
### Appendix B. FEAD Lumped Mass System Matrices

The system matrices are given by:

\[ R_t = \begin{bmatrix} r_1 & 0 & \ldots & 0 & -r_1 \\ -r_2 & r_2 & \ddots & \vdots \\ 0 & -r_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & r_{n-1} \\ 0 & \cdots & 0 & l_t \sin(\theta_{to} - \beta_j^p) \\ 0 & \cdots & 0 & l_t \sin(\theta_{to} - \beta_j) \end{bmatrix} \]  \hspace{1cm} (B.2)

\[ T_k = \begin{bmatrix} k_{b_1}^s r_1 & -k_{b_1}^s r_2 & 0 & \cdots & 0 \\ 0 & k_{b_2}^s r_2 & -k_{b_2}^s r_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & k_{b_{n-1}}^s r_n & -k_{b_{n-1}}^s r_n \\ -k_{b_n}^s r_1 & 0 & \cdots & 0 & k_{b_n}^s r_n \\ & & & & k_{b_n}^s l_t \sin(\theta_{to} - \beta_j^p) \end{bmatrix} \]  \hspace{1cm} (B.3)

\[ T_c = \begin{bmatrix} c_{b_1}^s r_1 & -c_{b_1}^s r_2 & 0 & \cdots & 0 \\ 0 & c_{b_2}^s r_2 & -c_{b_2}^s r_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{b_{n-1}}^s r_n & -c_{b_{n-1}}^s r_n \\ -c_{b_n}^s r_1 & 0 & \cdots & 0 & c_{b_n}^s r_n \\ & & & & c_{b_n}^s l_t \sin(\theta_{to} - \beta_j^p) \end{bmatrix} \]  \hspace{1cm} (B.4)
\[ \begin{align*}
K_b & = \begin{pmatrix}
-\frac{k_{b2} l r 3}{2} & 0 & \ldots & 0 \\
\frac{k_{b2} + k_{b1} l r 2}{2} & -\frac{k_{b2} l r 3}{2} & \ldots & 0 \\
\frac{k_{b2} + k_{b1} l r 2}{2} & \frac{k_{b2} + k_{b1} l r 1}{2} & \ldots & 0 \\
\frac{k_{b2} + k_{b1} l r 1}{2} & \frac{k_{b2} + k_{b1} l r 1}{2} & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\end{pmatrix} \sin(\theta - \beta) \\
\end{align*} \]

\[ C_b = \begin{pmatrix}
\frac{c_{b2} l r 3}{2} & 0 & \ldots & 0 \\
\frac{c_{b2} + c_{b1} l r 2}{2} & -\frac{c_{b2} l r 3}{2} & \ldots & 0 \\
\frac{c_{b2} + c_{b1} l r 2}{2} & \frac{c_{b2} + c_{b1} l r 1}{2} & \ldots & 0 \\
\frac{c_{b2} + c_{b1} l r 1}{2} & \frac{c_{b2} + c_{b1} l r 1}{2} & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\end{pmatrix} \sin(\theta - \beta) \]
Appendix B. FEAD Lumped Mass System Matrices

\[
K_r = \begin{bmatrix}
  k_1^r & 0 & \cdots & 0 \\
  & k_2^r & & \\
  & & \ddots & \ddots & \ddots \\
  & & & \text{sym} & k_{n-1}^r \\
  & & & & k_{n-2}^r & 0 \\
  & & & & & k_n^r
\end{bmatrix}
\]  \tag{B.7}

\[
C_r = \begin{bmatrix}
  c_1^r & 0 & \cdots & 0 \\
  & c_2^r & & \\
  & & \ddots & \ddots & \ddots \\
  & & & \text{sym.} & c_{n-1}^r \\
  & & & & c_{n-2}^r & 0 \\
  & & & & & c_n^r
\end{bmatrix}
\]  \tag{B.8}

\[
\theta(t) = \begin{bmatrix}
  \theta_1 \\
  \theta_2 \\
  \vdots \\
  \theta_{n-1} \\
  \theta_n \\
  \theta_t
\end{bmatrix}
\]  \tag{B.9}

\[
Q = \begin{bmatrix}
  Q_1 \\
  Q_2 \\
  \vdots \\
  Q_{n-1} \\
  Q_n \\
  Q_t
\end{bmatrix}
\]  \tag{B.10}
Experimental Testing System

This set of equations relates to the testing system used in Fig. 2.11 page 33.

\[
M = \begin{bmatrix}
I_1 & 0 & 0 & 0 & 0 & 0 \\
I_2 & 0 & 0 & 0 & 0 & 0 \\
I_3 & 0 & 0 & 0 & 0 & 0 \\
I_4 & 0 & 0 & 0 & 0 & 0 \\
\text{sym.} & I_5 & 0 & 0 & 0 & 0 \\
I_t & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (B.11)

\[
R_t = \begin{bmatrix}
r_1 & 0 & 0 & 0 & -r_1 \\
-r_2 & r_2 & 0 & 0 & 0 \\
0 & -r_3 & r_3 & 0 & 0 \\
0 & 0 & -r_4 & r_4 & 0 \\
0 & 0 & 0 & -r_5 & r_5 \\
0 & 0 & 0 & l_t \sin(\theta_{to} - \beta_p^j) & l_t \sin(\theta_{to} - \beta_j)
\end{bmatrix}
\] (B.12)

\[
T_k = \begin{bmatrix}
k_{b_1}^s r_1 & -k_{b_1}^s r_2 & 0 & 0 & 0 & 0 \\
0 & k_{b_2}^s & -k_{b_2}^s r_3 & 0 & 0 & 0 \\
0 & 0 & k_{b_3}^s r_3 & -k_{b_3}^s r_4 & 0 & 0 \\
0 & 0 & 0 & k_{b_4}^s r_4 & -k_{b_4}^s r_5 & k_{b_4}^s l_t \sin(\theta_{to} - \beta_p^j) \\
-k_{b_5}^s r_1 & 0 & 0 & 0 & k_{b_5}^s r_5 & k_{b_5}^s l_t \sin(\theta_{to} - \beta_j)
\end{bmatrix}
\] (B.13)
\[ K_b = \begin{bmatrix}
-k_s b r_1^2 (\sin(\theta_o - \beta_j) - k_s b r_5^2 sin(\theta_o - \beta_j)^2 + k_s b r_5^2 \sin(\theta_o - \beta_j)_j) n_{\text{sym.}} & 0 & 0 & 0 & 0 \\
-k_s b r_2^2 (\sin(\theta_o - \beta_j) - k_s b r_5^2 sin(\theta_o - \beta_j)^2 + k_s b r_5^2 \sin(\theta_o - \beta_j)_j) n_{\text{sym.}} & 0 & 0 & 0 & 0 \\
-k_s b r_3^2 (\sin(\theta_o - \beta_j) - k_s b r_5^2 sin(\theta_o - \beta_j)^2 + k_s b r_5^2 \sin(\theta_o - \beta_j)_j) n_{\text{sym.}} & 0 & 0 & 0 & 0 \\
-k_s b r_4^2 (\sin(\theta_o - \beta_j) - k_s b r_5^2 sin(\theta_o - \beta_j)^2 + k_s b r_5^2 \sin(\theta_o - \beta_j)_j) n_{\text{sym.}} & 0 & 0 & 0 & 0 \\
-k_s b r_5^2 (\sin(\theta_o - \beta_j) - k_s b r_5^2 sin(\theta_o - \beta_j)^2 + k_s b r_5^2 \sin(\theta_o - \beta_j)_j) n_{\text{sym.}} & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ C_b = \begin{bmatrix}
(c_s b + k_s b)^2 & 0 & 0 & 0 & 0 \\
(c_s b + k_s b)^2 & 0 & 0 & 0 & 0 \\
(c_s b + k_s b)^2 & 0 & 0 & 0 & 0 \\
(c_s b + k_s b)^2 & 0 & 0 & 0 & 0 \\
(c_s b + k_s b)^2 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]
\[ T_c = \begin{pmatrix} c_{b_1} r_1 & -c_{b_1} r_2 & 0 & 0 & 0 & 0 \\ 0 & c_{b_2} & -c_{b_2} r_3 & 0 & 0 & 0 \\ 0 & 0 & k c_{b_3} r_3 & -c_{b_3} r_4 & 0 & 0 \\ 0 & 0 & 0 & c_{b_4} r_4 & -c_{b_4} r_5 & c_{b_4} l_t \sin(\theta_{to} - \beta_p) \\ -c_{b_5} r_1 & 0 & 0 & 0 & c_{b_5} r_5 & c_{b_5} l_t \sin(\theta_{to} - \beta_j) \end{pmatrix} \] (B.16)

\[ \theta(t) = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_t \end{pmatrix} \] (B.17)

\[ Q(t) = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_t \end{pmatrix} \] (B.18)
Appendix C

Belt Span Inertia and Moment Area Calculation

If the belt is assumed to be very slim, the mass moment of inertia about the belt span end is [46]:

\[
I_{be} = \int_0^l \rho l^2 dl = \frac{1}{3} \rho l^2 \tag{C.1}
\]

The mass moment of inertia about the belt span center is\(^1\):

\[
I_{bc} = \int_0^l \rho l^2 dl = \frac{1}{12} \rho l^2 \tag{C.2}
\]

The belt span second moment of area about the neutral axis:

\[
J_b = \frac{1}{12} \text{width} \cdot \text{height}^3 = \frac{1}{12} ab^3 \tag{C.3}
\]

\(^1\)The mass moment of inertia about the belt span center is used in calculations.
Specific Calculations for Belt Span Three

Calculations using values from table 3.3 for third belt span:

\[ I_{be} = \int_0^l \rho r^2 dr = \frac{1}{3} \rho l^3 = 0.004474 \text{ kg} \cdot \text{m}^2 \]

The mass moment of inertia about the span center:

\[ I_{bc3} = \frac{1}{12} \rho l^3 = \frac{1}{12} (0.0989)(0.22446)^2 = 0.001118 \text{ kg} \cdot \text{m}^2 \]

The belt span second moment of area about the neutral axis \(^2\):

\[ J_b = \frac{1}{12} \text{width} \times \text{height}^3 = \frac{1}{12} ab^3 = \frac{1}{12} (0.02415)(0.00415)^3 = 0.0000000014400 \text{ m}^4 \]

\(^2\)Applicable for any belt span length.
Appendix D

Stationary Beam with Axial Force

Eigenvalue Calculation

The general method is used as per Rao [3]. The equation of motion for a stationary beam with an axial force is the previously derived Eq. (2.7), page 17, with the acceleration and velocity set equal to zero:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho \frac{\partial^2 w}{\partial t^2} - T \frac{\partial^2 w}{\partial x^2} = 0 \quad (D.1)$$

A solution is obtained using the method of separation of variables. The assumed solution takes the form:

$$w(x, t) = W(x)(A \cos \omega t + B \sin \omega t)$$

Substitution of the assumed solution on into the equation of motion gives:

$$EI \frac{d^4 W(x)}{dx^4} - \rho \omega^2 W(x) - T \frac{d^2 W(x)}{dx^2} = 0 \quad (D.2)$$

Assuming a solution for the mode shape, $W(x)$, as:

$$W(x) = Ce^{sx} \quad (D.3)$$
Appendix D. Stationary Beam with Axial Force Eigenvalue Calculation

Leads to the axillary equation:

\[ EI s^4 - \rho \omega^2 - Ps^2 = 0 \]

Solving the axillary equation for the four roots of \( s \):

\[ s^2_1, s^2_2 = \frac{T^2}{2EI} \pm \left( \frac{T^2}{4EI^4} + \frac{\rho \omega^2}{EI} \right)^{\frac{1}{2}} \]  

(D.4)

This equation is later used to solved for the belt’s fundamental frequencies, \( \omega_n \), once the values of \( s \) are known from the frequency equation. Knowing the form of \( s_1, s_2, s_3, \) and \( s_4 \), and substituting them in a summation of Eq. (D.3) gives the solution of the mode shape equation, Eq. (D.1):

\[ W(x) = C_1 e^{is_1 x} + C_2 e^{-is_1 x} + C_3 e^{is_2 x} + C_4 e^{-is_2 x} \]

Which can also be expressed as:

\[ W(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x \]  

(D.5)

The natural frequencies and mode shapes are now found from the boundary conditions and Eq. (D.5).

**Pinned-Pinned Boundary Conditions**

For Pinned-Pinned boundary conditions the displacement and bending moment are zero at both ends of the belt:

\[ W(0) = 0 \]  

(D.6a)

\[ W(l) = 0 \]  

(D.6b)
Appendix D. Stationary Beam with Axial Force Eigenvalue Calculation

\[ \frac{\partial^2 W}{\partial x^2}(0) = 0 \quad (D.6c) \]
\[ \frac{\partial^2 W}{\partial x^2}(l) = 0 \quad (D.6d) \]

Where, from Eq. (D.5)

\[ W(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x \]
\[ \frac{\partial W}{\partial x}(x) = s_1 C_1 \sinh s_1 x + s_1 C_2 \cosh s_1 x - s_2 C_3 \sin s_2 x + s_2 C_4 \cos s_2 x \]
\[ \frac{\partial^2 W}{\partial x^2}(x) = s_1^2 C_1 \cosh s_1 x + s_1^2 C_2 \sinh s_1 x - s_2^2 C_3 \cos s_2 x - s_2^2 C_4 \sin s_2 x \]

Using boundary condition 1, Eq. (D.6a), gives:

\[ W(0) = C_1 + C_3 = 0 \]

\[ C_3 = -C_1 \quad (D.7) \]

Using boundary condition 3, Eq. (D.6c), gives:

\[ \frac{\partial^2 W}{\partial x^2}(0) = s_1^2 C_1 - s_2^2 C_3 = 0 \]

\[ C_1 = C_3 \quad (D.8) \]

From Eq. (D.7) and Eq. (D.8) it can be deduced that \( C_1 = C_3 = 0 \).

Using boundary condition 2, Eq. (D.6b) gives:

\[ W(l) = C_2 \sinh s_1 l + C_4 \sin s_2 l = 0 \]
Using boundary condition 4, Eq. (D.6d) gives:

\[ \frac{\partial^2 W}{\partial x^2} (l) = s_1^2 C_2 \sinh s_1 l - s_2^2 C_4 \sin s_2 l = 0 \]

For a non-trivial solution of \( C_2 \) and \( C_4 \) the determinant of the coefficients must be zero:

\[
\begin{vmatrix}
\sinh s_1 l & \sin s_2 l \\
s_1^2 \sinh s_1 l & -s_2^2 \sin s_2 l
\end{vmatrix} = 0
\]

\[-s_2^2 \sin s_2 l \sinh s_1 l - s_1^2 \sinh s_1 l \sin s_2 l = 0\]

\[-s^2 \sin(sl) \sinh(sl) - s^2 \sinh(sl) \sin(sl) = 0\]

\[-2s^2 \sinh(sl) \sin(sl) = 0\]

\[s^2 \sinh(sl) \sin(sl) = 0 \quad (D.9)\]

Since \( \sinh(sl) \) is only zero at \( sl = 0 \), Eq. (D.9) is governed by the \( \sin(sl) \) portion of the equation. The roots of the equation are then:

\[ s = \frac{n\pi}{l}, \quad n = 0, 1, 2, ... \]

The natural frequencies are then solved for by subbing the known values of \( s \), from above, into Eq. (D.4), and solving for \( \omega_n \):

\[ \omega_n = \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho}} \left( n^4 + \frac{n^2 l^2}{\pi^2 EI} \right)^{\frac{1}{2}} \quad (D.10) \]

Finally, the mode shapes for the pinned-pinned stationary beam are found using a com-
Appendix D. Stationary Beam with Axial Force Eigenvalue Calculation

Combination of the boundary condition equations as:

\[ W_n(x) = C_n \left[ \sin s_n x - \left( \frac{\sin s_n l}{\sinh s_n l} \right) \sinh s_n x \right] \]

\[ W_n(x) = C_n \left[ \sin \left( \frac{n\pi x}{l} \right) - \left( \frac{\sin n\pi}{\sinh n\pi} \right) \sinh \left( \frac{n\pi x}{l} \right) \right] \]

\[ W_n(x) = C_n \left[ \sin \left( \frac{n\pi x}{l} \right) \right] \quad (D.11) \]

Clamped-Clamped Boundary Conditions

For the clamped-clamped stationary beam, the boundary conditions are defined as the displacement and slope are zero at both ends of the belt:

\[ W(0) = 0 \quad (D.12a) \]
\[ W(l) = 0 \quad (D.12b) \]
\[ \frac{\partial W}{\partial x}(0) = 0 \quad (D.12c) \]
\[ \frac{\partial W}{\partial x}(l) = 0 \quad (D.12d) \]

Where, from Eq. (D.5):

\[ W(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x \]

\[ \frac{\partial W}{\partial x}(x) = s_1 C_1 \sinh s_1 x + s_1 C_2 \cosh s_1 x - s_2 C_3 \sin s_2 x + s_2 C_4 \cos s_2 x \]

\[ \frac{\partial^2 W}{\partial x^2}(x) = s_1^2 C_1 \cosh s_1 x + s_1^2 C_2 \sinh s_1 x - s_2^2 C_3 \cos s_2 x - s_2^2 C_4 \sin s_2 x \]
Appendix D. Stationary Beam with Axial Force Eigenvalue Calculation

Using boundary condition 1, or Eq. (D.12a), gives:

\[ W(0) = C_1 + C_3 = 0 \]

\[ C_3 = -C_1 \tag{D.13} \]

Using boundary condition 3, or Eq. (D.12c), gives:

\[ \frac{\partial W}{\partial x}(x) = s_1 C_1 \sinh s_1 x + s_1 C_2 \cosh s_1 x - s_2 C_3 \sin s_2 x + s_2 C_4 \cos s_2 x \]

\[ \frac{\partial W}{\partial x}(0) = s_1 C_2 + s_2 C_4 \]

\[ C_4 = -C_2 \tag{D.14} \]

So that Eq. (D.5) becomes:

\[ W(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x - C_1 \cos s_2 x - C_2 \sin s_2 x \]

\[ W(x) = C_1 (\cosh s_1 x - \cos s_2 x) + C_2 (\sinh s_1 x - \sin s_2 x) \tag{D.15} \]

Using boundary condition 2, or Eq. (D.12b), gives the result:

\[ W(l) = C_1 (\cosh s_1 l - \cos s_2 l) + C_2 (\sinh s_1 l - \sin s_2 l) = 0 \tag{D.16} \]

Using boundary condition 4, or Eq. (D.12d), gives:

\[ \frac{\partial W}{\partial x}(l) = C_1 (\sinh s_1 l + \sin s_2 l) + C_2 (\cosh s_1 l - \cos s_2 l) = 0 \tag{D.17} \]
For a non-trivial solution of Eq. (D.16) and Eq. (D.17) the determinant of the coefficients is set to zero:

\[
\begin{vmatrix}
(cosh s_1 l - cos s_2 l) & (sinh s_1 l - sin s_2 l) \\
(sinh s_1 l + sin s_2 l) & (cosh s_1 l - cos s_2 l)
\end{vmatrix} = 0
\]

This leads to the frequency equation:

\[
cosh s_1 l \cos s_2 l = 1 \quad (D.18)
\]

Using trigonometric identities the frequency equation can be transformed into

\[
\tan^2 \frac{sl}{2} - \tanh^2 \frac{sl}{2} = 0
\]

\[
\left( \tan \frac{sl}{2} + tanh \frac{sl}{2} \right) \left( \tan \frac{sl}{2} - tanh \frac{sl}{2} \right) = 0 \quad (D.19)
\]

The solution of Eq. (D.19) is:

\[
s_n l = 0, 4.7300, 7.8532, 10.9956, 14.1372, 17.2788, ... \quad (D.20)
\]

\[
s_n l \approx 0, \frac{(2n + 1) \pi}{2}, \quad n = 1, 2, 3, ...
\]

\[
s_n \approx 0, \frac{(2n + 1) \pi}{2l}, \quad n = 1, 2, 3, ...
\]

The values more closely match the true values, Eq. (D.20), for larger values of \( n \) in the series of Eq. (D.21).

The frequencies of the belt are calculated using Eq. (D.4) and the values for \( s \) from Eq. (D.21):

\[
\omega_0 = 0
\]
\[ \omega_n \approx \sqrt{\left( \frac{EI}{\rho} \right) \left[ \frac{(2n+1)\pi}{2l} \right]^2 \left\{ \left[ \frac{(2n+1)\pi}{2l} \right]^2 - \frac{P}{EI} \right\}} \]  

(D.22)

The mode shapes for the clamped-clamped belt are found by substituting Eq. (D.16) into Eq. (D.15):

\[ W_n(x) = C_n \left[ \cosh s_n x - \cos s_n x - \left( \frac{\cosh s_n l - \cos s_n l}{\sinh s_n l - \sin s_n l} \right) (\sinh s_n x - \sin s_n x) \right] \]  

(D.23)

Where the values for \( s_n l \) are known from Eq. (D.21). Note that these eigenvalues are the same as those for a beam without axial tension [3].

**Clamped-Pinned Boundary Conditions**

The Clamped-Pinned boundary condition are derived in a similar manner [3]:

\[ W_n(x) = C_n \left[ \sin s_n x - \sinh s_n x + \left( \frac{\sin s_n l - \sinh s_n l}{\cosh s_n l - \cosh s_n l} \right) (\cosh s_n x - \cos s_n x) \right] \]  

(D.24)

Where:

\[ s_1 l = 3.926602 \]
\[ s_2 l = 7.068583 \]
\[ s_3 l = 10.210176 \]
\[ s_4 l = 13.351768 \]
Appendix E

MATLAB Code

Main Program

The main program defines program characteristics, defines folder paths to save plots, calls subprograms, and calculates result percent error.

```matlab
function [ output_args ] = MAIN( " ")
%---------------------------------------------------------------------
% 1. Define plot characteristics, etc.
%---------------------------------------------------------------------
beep off
format long g
set(0,'DefaultAxesColorOrder',[0 0 1;1 0 0;0 1 0;0 1 1;1 0 1;0 0 0],...
    'DefaultAxesLineStyleOrder','-|--|-.')
%---------------------------------------------------------------------
% 2. Define file folder paths to save plots:
%---------------------------------------------------------------------
fnamesaveFFT = 'path';
fnamesaveHub = 'path';
fnamesaveOsc = 'path';
```
fnamesaveMeanTension = 'path';
fnamesaveDynamicTension = 'path';

% 3. Retrieve the FEAD System Characteristics, measured in base units:

% % M = Pulley inertia matrix
% K = Stiffness matrix
% R = Vector of pulley radii
% L = Vector of belt span lengths
% Lb = Total belt length
% dof = Degree of freedom of the FEAD system
% p = Belt linear density
% EA = Belt
% EJ = Belt flexural rigidity
% Span = Number of the belt span to determine tension in, eg. Span = 3
% l = Length of the belt span to determine tension in

% 4. Retrieve the System Testing Data, measured in base units:
[ TopDisp, MidDisp, BotDisp, UHub, FHub, time, DrivenRPM, DriverRPM,...
  RPM, ExFreq, To, m ] = InputData();
% TopDisp = Top displacement sensor data
% MidDisp = Middle displacement sensor data
% BotDisp = Bottom displacement sensor data
% UHub = Unfiltered hubload sensor data
% FHub = Low-Pass (1 Hz) Butterworth filtered hubload sensor data
% DrivenRPM = Driven span support pulley rotation speed data
% DriverRPM = Driver span support pulley rotation speed data
% time = Time vector applicable to all sensor data
% RPM = Crankshaft rotational speed.
% ExFreq = Crankshaft excitation frequency
% To = Initial belt tension while at rest
% m = Number of data points in the data sets
%
% 5. Generate System Frequencies:

[Fsys, F1, F2, F3, F4, F5, Fbl, Fo] = AdditionalFrequencies( RPM, M, K,...
    R, dof, Lb, To, l, p);

% Fsys = System frequencies
% F1 - F5 = Frequencies of the system pulley
% Fbl = belt loop frequency
% Fo = Expected belt span natural frequency based on initial belt tension
%
% 6. Generate the FFT data set for the middle span displacement sensor:

[fftAmp, fftFreq] = FastFourierTransform( time, MidDisp);

% fftAmp = Frequency plot amplitude vector
% fftFreq = Frequency plot frequency vector
%
% 7. Remove additional frequencies from the FFT and find the belt span
% natural frequency:

[BeltFreq, fftAmpEdit] = FrequencyRemoval( fftAmp, fftFreq, Fsys, F1,...
    F2, F3, F4, F5, Fbl, exfreq, dof, Fo);

% BeltFreq = Belt span first natural frequency
% fftAmpEdit = Frequency plot amplitudes with additional frequencies
% removed
%
% 8. If desired, generate the edited frequency plot:

FFTPlot( fftFreq, fftAmpEdit, fnamesaveFFT);

% 9. Calculate and plot the validation tension from the hubload data:
[ TVF, TVU ] = TensionValidation( m, HubUf, HubF, time,...
    DriverRPM, fnamesaveHub)
%
TVU = Unfiltered Validation Tension
% TVF = Filtered Validation Tension
%-------------------------------------------------------------------------------------------------------------
% Calculate and plot the mean tension:
%-------------------------------------------------------------------------------------------------------------
[ Tmean ] = MeanTension( m, RPM, TVF, TVU, time, AC, l, R, p,...
    TopDisp, MidDisp, BotDisp, EA, Fbl, fnamesaveMeanTension,...
    fnamesaveOsc, BeltFreq)
TmError = abs((mean(Tmean) - mean(TVU))/mean(TVU))*100;
%
%-------------------------------------------------------------------------------------------------------------
% Dynamic Tension
%-------------------------------------------------------------------------------------------------------------
[ T Tmax Tmin Hubmax Hubmin P ] = DynamicTension( To, m,...
    fnamesaveDynamicTension, time, Tpp, DrivenRPM, DriverRPM, TVU, R,...
    Span, EA, l)
MaxErrorPer = abs((Hubmax-Tmax)/Hubmax)*100
MinErrorPer = abs((Hubmin-Tmin)/Hubmin)*100
RangeErrorPer = abs(((Hubmax-Hubmin)-(Tmax-Tmin))/(Hubmax-Hubmin))*100
MeanErrorPer = abs((mean(TVF)-mean(T))/mean(TVF))*100
clear all
end

System Characteristics

This program defines the system constants. These include pulley radii, belt length, system degree of freedom, belt stiffness, pulley mass and inertia, etc. System matrices are also defined.


Appendix E. MATLAB Code
% Declare variables:

syms R1 R2 R3 R4 R5
syms L1 L2 L3 L4 L5 Lt B4 B5
syms W1 W2 W3 W4 W5
syms LW1 LW2 LW3 LW4 LW5
syms Kb Kt Oto
syms Kb1 Kb2 Kb3 Kb4 Kb5
syms Kr1 Kr2 Kr3 Kr4 Kr5 Krt
syms Cb1 Cb2 Cb3 Cb4 Cb5
syms Cr1 Cr2 Cr3 Cr4 Cr5 Crt
syms I1 I2 I3 I4 I5 U6 It
syms Oto dof

% Declare singulare constants:

Lb = 1.290; % [m] Estimated Effective Belt Length
dof=6;

Span = 3; % Span in which tension is to be measured
Oto=194*pi/180; % [rad] Initial tensioner arm rotation angle

% Declare pulley radii, span lengths, wrap angles, Inertia, and stiffness values:

% Radii:
R1=163.76/2000; % [m]
R2=70/2000; % [m]
R3=122.76/2000; % [m]
R4=69.152/2000; % [m]
R5=71.90/2000; % [m]
R=[R1 R2 R3 R4 R5];
% Length:
L1=56.91/1000; %[m]
L2=42.76/1000; %[m]
L3=224.46/1000; %[m]
L4=140.55/1000; %[m]
L5=141.10/1000; %[m]
Lt=70.0177/1000; %[m]
L=[L1 L2 L3 L4 L5 Lt];
% Wrap angles:
W1=235*pi()/180; %[rad]
W2=87*pi()/180; %[rad]
W3=135*pi()/180; %[rad]
W4=174*pi()/180; %[rad]
W5=104*pi()/180; %[rad]
W=[W1 W2 W3 W4 W5];
% Belt stiffness:
EA=400000;
Kb1=EA/LW1; %[N/m]
Kb2=EA/LW2; %[N/m]
Kb3=EA/LW3; %[N/m]
Kb4=EA/LW4; %[N/m]
Kb5=EA/LW5; %[N/m]
% rotational stiffness:
Kr1=0; %[N-m/rad]
Kr2=0; %[N-m/rad]
Kr3=0; %[N-m/rad]
Kr4=0; %[N-m/rad]
Kr5=0; %[N-m/rad]
Krt=0.01007*180/3.14159; %[N-m/rad]
% Pulley inertails:
I1=0.004; %[Kg-m^2]
I2=0.000175; %[Kg-m^2]
I3=0.0002; %[Kg-m^2]
I4=0.000521; %[Kg-m^2]
Appendix E. MATLAB Code

\[ I_5 = 0.000313; \quad \text{[Kg} \cdot \text{m}^2]\]
\[ I_t = 0.00122; \quad \text{[Kg} \cdot \text{m}^2]\]

\[ B_5 = (153+180) \times 3.14159/180; \quad \text{[rad]}\]
\[ B_4 = (110+180) \times 3.14159/180; \quad \text{[rad]}\]

\[ \text{--- Matrices:} \]

\[
R_t = \begin{bmatrix}
R_1 & 0 & 0 & 0 & -R_1 \\
-R_2 & R_2 & 0 & 0 & 0 \\
0 & -R_3 & R_3 & 0 & 0 \\
0 & 0 & -R_4 & R_4 & 0 \\
0 & 0 & 0 & -R_5 & R_5 \\
0 & 0 & 0 & Lt \cdot \sin(O_{to} - B_5) & Lt \cdot \sin(O_{to} - B_4)
\end{bmatrix};
\]

\[
T_k = \begin{bmatrix}
K_{b1} \cdot R_1 & -K_{b1} \cdot R_2 & 0 & 0 & 0 & 0 \\
0 & K_{b2} \cdot R_2 & -K_{b2} \cdot R_3 & 0 & 0 & 0 \\
0 & 0 & K_{b3} \cdot R_3 & -K_{b3} \cdot R_4 & 0 & 0 \\
0 & 0 & 0 & K_{b4} \cdot R_4 & -K_{b4} \cdot R_5 & K_{b4} \cdot Lt \cdot \sin(O_{to} - B_5) \\
-K_{b5} \cdot R_1 & 0 & 0 & 0 & K_{b5} \cdot R_5 & K_{b5} \cdot Lt \cdot \sin(O_{to} - B_4)
\end{bmatrix};
\]

\[ \text{--- Rotation stiffness matrix} \]

\[
K_r = \begin{bmatrix}
K_r_1 & 0 & 0 & 0 & 0 & 0 \\
0 & K_r_2 & 0 & 0 & 0 & 0 \\
0 & 0 & K_r_3 & 0 & 0 & 0 \\
0 & 0 & 0 & K_r_4 & 0 & 0 \\
0 & 0 & 0 & 0 & K_r_5 & 0 \\
0 & 0 & 0 & 0 & 0 & K_r_t
\end{bmatrix};
\]

\[ \text{--- Kb = belt stiffness matrix} \]

\[ K_b = R_t \cdot T_k; \]
\[ K = K_b + K_r; \]

\[ \text{--- Inertia matrix:} \]

\[
M = \begin{bmatrix}
I_1 & 0 & 0 & 0 & 0 & 0 \\
0 & I_2 & 0 & 0 & 0 & 0 \\
0 & 0 & I_3 & 0 & 0 & 0
\end{bmatrix};
\]
Additional Frequencies

This program calculates additional frequencies which occur in the system, these include pulley rotational frequencies, system frequencies, and the belt loop frequency.

```matlab
function [ Fsys, F1, F2, F3, F4, F5, Fbl, Fo] = AdditionalFrequencies( RPM, M, K, R, dof, Lb, To,l,p )

% 1. Calculate the System Frequencies: (See Section
L=chol(M);
Kt=inv(L)*K*inv(L');
[V,lambda]=eig(Kt);
for j =1:dof
    wn(j,1)=vpa(sqrt(lambda(j,j)))/(2*pi); %[Hz]
end
Fsys=wn;
Fsys=round(real(Fsys)*1000)/1000;
Fsys=double(Fsys);

% 2. Calculate the Pulley and Belt Loop Frequencies:
for j=1:dof
    F1(j)=(RPM/60)*(R(1)/R(1))*j;
    F2(j)=(RPM/60)*(R(1)/R(2))*j;
    F3(j)=(RPM/60)*(R(1)/R(3))*j;
end
```
Appendix E. MATLAB Code

\[ F_4(j) = \frac{RPM}{60} \times \frac{R(1)}{R(4)} \times j; \]
\[ F_5(j) = \frac{RPM}{60} \times \frac{R(1)}{R(5)} \times j; \]
\[ F_{bl}(j) = \frac{RPM}{60} \times \frac{2\pi R(1)}{L_b} \times \left( \frac{j}{j/L_b} \right); \]

end

% 2. Calculate the expected belt frequency based on initial tension:

syms k1 k2 v w k n a c l
expr = (c^2 - v^2) * k^2 + (a*i - 2*v*w) * k - w^2; % Travelling string EOM
K = solve(expr, k); % See modal analysis section for explanation.
k1 = K(1,1);
k2 = K(2,1);
V = k1 + k2 - (2*n*pi)/(l);
wn = solve(V, w);
n = 1; % Define as first natural frequency
c = sqrt(To/p); % Wave speed
a = 0; % Assume no axial belt acceleration
v = RPM * (1/60) * 2*pi * (R(1)/2000); % Calculate axial belt velocity. The speed
% can also be calculated using the support pulley rotation speed.
wn = eval(wn);
wn = wn * (1/(2*pi)); % Convert from rad/s to Hz.
Fo = wn(1,1); % Define the expected belt frequency
end

Fast Fourier Transform

This program calculates the Fast Fourier Transform of the middle span displacement
sensor measurements.

function [ fftAmp, fftFreq ] = FastFourierTransform( time, MidDisp )
%
% Move data vector of discrete sampled points mean to zero:
%--------------------------------------------------------------------------
x=MidDisp-mean(MidDisp);

% Define the data sampling period and frequency using the time vector:
%--------------------------------------------------------------------------
dt=time(2)-time(1);
fs=1/dt;

% Use next highest power of 2 greater than or equal to length(x) to calculate FFT.
%--------------------------------------------------------------------------
nfft= 2^(nextpow2(length(x)));

% Take fft, padding with zeros so that length(fftx) is equal to nfft
%--------------------------------------------------------------------------
fftx = fft(x,nfft);

% Calculate the number of unique points
%--------------------------------------------------------------------------
NumUniquePts = ceil((nfft+1)/2);

% FFT is symmetric, throw away second half
%--------------------------------------------------------------------------
fftx = fftx(1:NumUniquePts);

% scale the fft so that it is not a function of the length of x
%--------------------------------------------------------------------------
fftAmp=abs(fftx)/length(x); % amplitude not power

% Since half the FFT was "dropped", multiply fftAmp by 2 to keep the same
% energy. However, the DC component and Nyquist component, if exists, are
% unique and should not be multiplied by 2.
% if rem(nfft, 2) % odd nfft excludes Nyquist point
    fftAmp(2:end) = fftAmp(2:end)*2;
else
    fftAmp(2:end-1) = fftAmp(2:end-1)*2;
end

% This is an evenly spaced frequency vector with NumUniquePts points.
fftFreq = (0:NumUniquePts-1)*fs/nfft;

Frequency Removal

This program removes the additional frequencies from the Fast Fourier Transform. The program also returns belt span natural vibration frequency.

function [ BeltFreq(1), fftAmpEdit ] = FrequencyRemoval( fftAmp,...
        fftFreq, Fsys, F1, F2, F3, F4, F5, Fbl, ExFreq, dof, Fo )
%UNTITLED8 Summary of this function goes here
% Detailed explanation goes here
% Define a new frequency amplitude vector which will be modified:
fftAmpEdit=fftAmp';

% Insure that the belt span natural frequency is not removed from the
% frequency plot:
for n=1:dof
    if (F1(n) > (Fo-4) && F1(n) < (Fo+4))
Appendix E. MATLAB Code

```matlab
F1(n) =0;
end
if (F2(n) > (Fo-4) && F2(n) < (Fo+2))
    F2(n) =0;
end
if (F3(n) > (Fo-4) && F3(n) < (Fo+4))
    F3(n) =0;
end
if (F4(n) > (Fo-4) && F4(n) < (Fo+4))
    F4(n) =0;
end
if (F5(n) > (Fo-4) && F5(n) < (Fo+4))
    F5(n) =0;
end
if (Fbl(n) > (Fo-4) && Fbl(n) < (Fo+4))
    Fbl(n) =0;
end
if (Fsys(n) > (Fo-4) && Fsys(n) < (Fo+4))
    Fsys(n) =0;
end
end
%-------------------------------------------------------
% Eliminate additional frequencies from the frequency plot:
%-------------------------------------------------------
for j=1:size(fftAmp,2)
    for n=1:dof
        if (fftFreq(j) < 10 && fftFreq(j) ~= 0 ) % Remove Small Frequencies
            fftAmpEdit(j)=0;
        end
        if (fftFreq(j) > (ExFreq*n-2) && fftFreq(j) < (ExFreq*n+2)) % Excitation Freq
            fftAmpEdit(j)=0;
        end
        if (fftFreq(j) > (F1(n)-2) && fftFreq(j) < (F1(n)+2)) % CS Freq
```
fftAmpEdit(j)=0;
end
if (fftFreq(j) > (F2(n)-2) && fftFreq(j) < (F2(n)+2)) % Idler 1 Freq
fftAmpEdit(j)=0;
end
if (fftFreq(j) > (F3(n)-2) && fftFreq(j) < (F3(n)+2)) % AC Freq
fftAmpEdit(j)=0;
end
if (fftFreq(j) > (F4(n)-2) && fftFreq(j) < (F4(n)+2)) % Alt Freq
fftAmpEdit(j)=0;
end
if (fftFreq(j) > (F5(n)-2) && fftFreq(j) < (F5(n)+2)) % Idler 2 Freq
fftAmpEdit(j)=0;
end
if (fftFreq(j) > (Fbl(n)-2) && fftFreq(j) < (Fbl(n)+2)) % Belt Loop Freq
fftAmpEdit(j)=0;
end
if (fftFreq(j) > (Fsys(n)-2) && fftFreq(j) < (Fsys(n)+2)) % Sys Freq
fftAmpEdit(j)=0;
end
end
end
%-----------------------------------------------------------------------------
% Determine the belt span natural frequency as the frequency with the
% highest amplitude in the frequency plot:
%-----------------------------------------------------------------------------
sorted=sort(fftAmpEdit,'descend');
count=0;
for j=1:size(sorted,1)
    if sorted(j)==0
        count=count+1;
    end
end
Appendix E. MATLAB Code

BeltFreq=zeros((size(sorted,1)-count),1);

%%Sort the belt frequencies by order of amplitude:
for j=1:(size(sorted,1)-count)
    BeltFreq(j,1)=fftFreq(find(fftAmp==sorted(j)));
end
end

Fast Fourier Transform Plot

This program plots the edited Fast Fourier Transform data with a marker for the largest amplitude frequency.

function [ output_args ] = FFTPlot( fftFreq, fftAmp, fnamesaveFFT)
    %------------------
    % Generate new amplitude vector for string:
    %------------------
    f=fftFreq;
    amp=fftAmp;
    %------------------
    % Sort the amplitude vector:
    %------------------
    amps=sort(fftAmp,'descend');
    %------------------
    % Generate a marker location for the belt span natural frequency:
    %------------------
    ind1=find(fftAmp==amps(1));
    %------------------
    % Generate the plot, title and labels:
    %------------------
    scrsz = get(0,'ScreenSize');
    figure('Position',[scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
plot(fftFreq,amp);
grid on
box on
axis([0 200 0 amps(1)+0.01])
axis square
title(' ');
xlabel('Frequency [Hz]');
ylabel('|Y(x)|');

% Add markers for the highest peaks:
mainPeriodStr1=num2str(fftFreq(ind1));
plot(fftFreq(ind1),amp(ind1),'r.', 'MarkerSize',25);
text(fftFreq(ind1)+2,amp(ind1),['F1 = ',mainPeriodStr1]);

% Save the diagram:
saveas(gcf,fullfile(fnamesaveFFT, sprintf('MiddleSpanFFT')),'eps');
saveas(gcf,fullfile(fnamesaveFFT, sprintf('MiddleSpanFFT')));
hold off
end

Tension Validation

This program calculates the tension in the belt span of interest using the installed hubload sensor.

function [ TVF, TVU ] = TensionValidation( m, HubUf, HubF, time, ...
        DriverRPM, fnamesaveHub)
% Appendix E. MATLAB Code

% Define hubload vectors:

% HU=HubUF(1:m-1)*0.6893;
% HF=HubF(1:m-1)*0.6893;

% Find AC Acceleration:

for j=1:m-1
    ACaccel(j)=((2*pi)/60)*(DriverRPM(j+1)-DriverRPM(j))/(t(j+1)-t(j));
end
t=t(1:m-1);

% Calculate Tension and Find Max and Min Tension for Graph Axis:

for j=1:m-1
    TVU(j)=ACaccel(j)*(0.0002/0.0614) + HU(j);
    TVF(j)=ACaccel(j)*(0.0002/0.0614) + HF(j);
end

% Plot:

scrsz = get(0,'ScreenSize');
figure('Position',[scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
plot(time, TVF,'r', time, HU(1:m-1), 'b')
legend('Inclusion of A/C Inertia','Uniform, T_2=T_3')
grid off
axis on
axis tight
axis square
box on
```matlab
axis square
title('')
xlabel('Time [s]');
ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesaveHub, sprintf('FilteredHubload')), 'jpg');
saveas(gcf, fullfile(fnamesaveHub, sprintf('FilteredHubload')));

scrsz = get(0, 'ScreenSize');
figure('Position', [scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca, 'LooseInset', get(gca, 'TightInset'))
hold on
plot(time, TVU, 'r', time, HF(1:m-1), 'b')
legend('Inclusion of A/C Inertia', 'Uniform, T_2=T_3')
axis tight
axis square
axis on
grid off
box on
title('')
xlabel('Time [s]');
ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesaveHub, sprintf('UnfilteredHubload')), 'jpg');
saveas(gcf, fullfile(fnamesaveHub, sprintf('UnfilteredHubload', Tor)));

scrsz = get(0, 'ScreenSize');
figure('Position', [scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca, 'LooseInset', get(gca, 'TightInset'))
hold on
plot(time, TVU, 'r', time, TVF)
legend('Uniltered', 'Filtered')
axis tight
axis square
axis on
```
grid off
box on
title('')
xlabel('Time [s]');
ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesaveHub, sprintf('BothHubload')), 'jpg');
saveas(gcf, fullfile(fnamesaveHub, sprintf('BothHubload')));
end

Mean Tension Calculation

This program calculates the mean tension in the belt span using the belt natural vibration frequency.

function [ Tpp ] = MeanTension( m, RPM, TVF, TVU, time, AC, l, R, p,...
     TopDisp, MidDisp, BotDisp, EA ,Fbl, fnamesaveMeanTension,...
     fnamesaveOsc, BeltFreq)
%------------------------------------------------------------------
% Define the beam eigenvalues as determined using Galerkin's Method:
%------------------------------------------------------------------
syms c v
SIcc = imag(((2000000*(455719711471443409*c^4)/4000000000000000000000+ (587249811714558191*c^2*v^2)/200000000000000 + (5082426523715582773484613*c^2)/197800000000000000000000 - (831846491167465391*v^4)/400000000000000 + (3683594141117116545800187*v^2)/197800000000000000000000 + 2267276022409835155461760724053/1564993600000000000000)ˆ(1/2))/201601 - (519939*c^2)/898 + (54749187*v^2)/403202 - 190000800479/386140)ˆ(1/2));
SIpp = imag((50*((67971071224849*c^4)/15625000000000000000000 + (43335177719640684783*c^2*v^2)/625000000000000000000000 + (33235763135411469997*c^2)/671875000000000000000000 - (3891869954290176626645679*v^4)/6250000000000000000000000000000 + (2950275671929072176556587*v^2)/268750000000000000000000 + 16251265888382061586218841/11556250000000000000000000000000)ˆ(1/2)- (54963047*c^2)/100000 + (970375461111*v^2)/5000000000 - 210164370813509/989000000)ˆ(1/2));
SIcp = imag(((500000000000*((65937724491326688854374453*c^4)/625000000000000000000000 - (1518700371080890824605267*v^4)/976562500000000000000 + (387780980022049359866302397539017*v^2)/6250000000000000000000000000000 + 8255013918148514617236860967707571382103/122265125000000000000000000000000)ˆ(1/2))/50303730949 - (27162097596758*c^2)/50303730949 + (9232624377380*v^2)/50303730949 - 77768594721686961648/248751949542805)ˆ(1/2));
%------------------------------------------------------------------
% Round the belt frequency:
%------------------------------------------------------------------
BeltFreq=round(BeltFreq*10)/10;
%------------------------------------------------------------------
% For each of the beam eigenvalues calculate the mean tension using
% the belt frequency
%------------------------------------------------------------------
\%---------------------------------------------------------------------

\texttt{SI=subs(SIpp,c,sqrt(T/p));}
\texttt{v=RPM*(1/60)*2*pi*(163.76/2000);
exprI1=eval(SI);
wnI1=eval(exprI1)/(2*pi);\% convert from rad/s to Hz for the plot}
\texttt{T1=1:m; \% define the tension length}
\texttt{wnpI1=zeros(size(T1));
for n=1:size(T1,2)
    T=T1(n);
    wnpI1(n)=eval(wnI1);
end}
\texttt{wnp=round(wnp);
ind=find(wnp==BeltFreq);
for k=1:10}
    \texttt{if isempty(ind) == 1
        ind=find(wnp==BeltFreq+k/10);
    end}
\texttt{end}
\texttt{Tpp=T1(ind(size(ind,2)));
for j=1:m-1
    Tpp(j)=T1(ind(size(ind,2)));
end}
\texttt{clear wnp T1}

\texttt{SI=subs(SIcc,c,sqrt(T/p));}
\texttt{v=RPM*(1/60)*2*pi*(163.76/2000);
exprI1=eval(SI);
wnI1=eval(exprI1)/(2*pi);\% convert from rad/s to Hz for the plot}
\texttt{T1=1:m; \% define the tension length}
\texttt{wnpI1=zeros(size(T1));
for n=1:size(T1,2)
    T=T1(n);
    wnpI1(n)=eval(wnI1);
wnp=round(wnp);
ind=find(wnp==BeltFreq);
for k=1:10
    if isempty(ind) == 1
        ind=find(wnp==BeltFreq+k/10);
    end
end
Tcc=T1(ind(size(ind,2))); 
for j=1:m-1
    Tcc(j)=T1(ind(size(ind,2))); 
end
clear wnp T1

SI=subs(SIcp,c,sqrt(T/p));

v=RPM*(1/60)*2*pi*(163.76/2000);
exprI1=eval(SI);
wnI1=eval(exprI1)/(2*pi); % convert from rad/s to Hz for the plot
T1=1:m; % define the tension length
wnpI1=zeros(size(T1));
for n=1:size(T1,2)
    T=T1(n);
    wnpI1(n)=eval(wnI1);
end
wnp=round(wnp);
ind=find(wnp==BeltFreq);
for k=1:10
    if isempty(ind) == 1
        ind=find(wnp==BeltFreq+k/10);
    end
end
Tcp=T1(ind(size(ind,2))); 
for j=1:m-1
    Tcp(j)=T1(ind(size(ind,2))); 
end
Tcp(j)=T1(ind(size(ind,2))); 
end  
clear wnp T1 c v  

% % Calculate the mean tension using the string frequency equation:
% % From paper calculations  
% expr=(c^2-v^2)*k^2+(a+1-2v*w)*k-w^2;  
K=solve(expr,k);  
k1=K(1,1);  
k2=K(2,1);  
% Using the boundary conditions of a pinned cable:  
V=-k1+k2-(2*n*pi)/(l);  
% The natural Frequencies are then:  
wn=solve(V,w);  
% Coosing the positive frequency:  
wn1=wn(1,1);  
% Solving the frequency equation for tension:  
wn1=subs(wn1,c,sqrt(T/p));  
T=solve(wn1-omegan,T);  
n=1; % first frequency will have the highest magnitude  
for j=1:m-1  
v=(AC(j)*(R(Span))*(2*pi/60));  
vp=(AC(j+1)*(R(Span))*(2*pi/60));  
a={(vp-v)/10000};  
Tstring(j)=real(eval(T(1,1)));  
end  
Tstring=Tstring(1:m-1);  

% % Calculate the oscillation tension:  
% [ fTmod ] = OscillationTension(TopDisp, MidDisp, BotDisp, time,...
Fbl, l, EA, fnamesaveOsc, m )

% Adjust vector lengths:

time=time(1:m-1);
Tstring=Tstring(1:m-1);
fTmod=fTmod(1:m-1);
Tcp=Tcp(1:m-1);
Tpp=Tpp(1:m-1);
Tcc=Tcc(1:m-1);

% Add oscillation tension to the mean tension:
Tstring = Tstring + fTmod;
Tcp = Tcp + fTmod;
Tpp = Tpp + fTmod;
Tcc = Tcc + fTmod;

% Calculate plot max and min values:
Tmax=[max(TVU), max(TVF), max(Tstring), max(Tpp), max(Tcp)];
Tmax=max(Tmax)+10;
Tmax2=[ max(TVF), max(Tpp)];
Tmax2=max(Tmax2)+10;
Tmin=[min(TVU),min(TVF),min(Tstring),min(Tpp),min(Tcp)];
Tmin=min(Tmin)-10;
Tmin2=[ min(TVF) ,min(Tpp)];
Tmin2=min(Tmin2)-10;

% Plot:

scrsz = get(0,'ScreenSize');
figure('Position',[scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
plot(time, TVF, time, TVU, time, Tstring, time, Tcc, time, Tcp, time, Tpp)
legend('Unfiltered Measured','Filtered Measured','Calculated String','Calculated Clamped
\-Clamped','Calculated Clamped \-Pinned','Calculated Pinned
\-Pinned')
axis tight
axis square
axis([0 max(time) Tmin Tmax])
grid off
box on
title('')
xlabel('Time [s]');
ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesaveMeanTension, sprintf('AllMeanTension',Tor)), 'jpg');
saveas(gcf, fullfile(fnamesaveMeanTension, sprintf('AllMeanTension',Tor)));

scrsz = get(0,'ScreenSize');
figure('Position',[scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
plot(time, TVU, time, Tpp)
legend('Filtered Measured','Calculated Pinned-Pinned')
axis tight
axis square
axis([0 max(time) Tmin2 Tmax2])
grid off
box on
title('')
xlabel('Time [s]');
ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesaveMeanTension, sprintf('MeanTension',Tor)), 'jpg');
saveas(gcf, fullfile(fnamesaveMeanTension, sprintf('MeanTension',Tor)));
clear Tmin Tmax vp T c v k a w n omegan wn wn1 V K k1 k2
Oscillation Tension Calculation

This program calculates the oscillation tension in the belt span from the span displacement sensor data.

```matlab
function [ fTmod ] = OscillationTension(TopDisp, MidDisp, BotDisp, time,...
    Fbl, l, EA, fnamesaveOsc, m )

% % Each nominal subspan is 1/4 of the total span length:
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% l=l*1000 % multiply by 1000 m/mm as all data sets are measured in mm.
% x=l/4; % 1/4 of the total span length, ie. one section
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% For each of the data sets the data must be "adjusted" for the distance
% % sensor is away from the stationary belt.
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tdis = mean(Top);
mdis = mean(Mid);
bdis = mean(Bot);
% % Adjust the values from the scanned data files using the values of
% % tdis, mdis, and bdis. Assign these values to the variables y1, y2, and
% % y3:
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
y1=TopDisp-tdis; %Top
y2=MidDisp-mdis; %Middle
y3=BotDisp-bdis; %Bottom
```
% Calculate deviation of each belt sub belt length:

for i=1:m
    l1(i)=sqrt(x^2+y1(i)^2);
    l2(i)=sqrt((y2(i)-y1(i))^2+x^2);
    l3(i)=sqrt((y2(i)-y3(i))^2+x^2);
    l4(i)=sqrt(x^2+y3(i)^2);
    lmod(i)=(l1(i)+l2(i)+l3(i)+l4(i))-l;
end
lmod=lmod';

% Calculate oscillation tension:

for i=1:m
    Tmod(i)=((EA/(l/1000))*lmod(i))/1000;
end

% Remove the belt loop frequency from the data by resampling, then
% resample back to the original frequency:

fs=10000; % Original sampling frequency. (Hz)
point=round(10000/Fbl); % Set the point based on unwanted frequency
if rem(point,2)==0 % Point must be odd
    point=point+1;
end
fsResamp = round(point*ceil(Fbl)); % "ceil(Fbl)" is the frequency to be
% removed, must be a positive integer
vResamp = resample(Tmod, fsResamp, fs); % Resample
tResamp = (0:numel(vResamp)-1) / fsResamp; % Reset the time vector
vAvgResamp = sgolayfilt(vResamp,1,point); % Savitzky-Golay smoothing filter
vResamp2 = resample(vAvgResamp, fs+1, fsResamp); % Resample
tResamp2 = (0:numel(vResamp2)-1) / fs; % Reset the time vector
fTmod = sgolayfilt(vResamp2,1,3); % Savitzky-Golay smoothing filter.

%Choosing a small point disrupts the signal the least

%---------------------------------------------------------------
% Plot:
%---------------------------------------------------------------

Tmin=min(Tmod)+1;
Tmax=max(Tmod)+1;
tmax=max(t);

scrsz = get(0,'ScreenSize');
figure('Position',[scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
plot(time,Tmod,tResamp2,fTmod)
legend('Original','Resampled')
grid off
box on
axis([0 tmax Tmin Tmax])
axis square
xlabel('Time (s)')
ylabel('Tension (N)')
hold off
saveas(gcf, fullfile(fnamesaveOsc, sprintf('BothTensionModulation',Tor)), 'jpg');

scrsz = get(0,'ScreenSize');
figure('Position',[scrsz(4)*.1 scrsz(4)*.1 scrsz(3)/2.2 scrsz(3)/2.2])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
plot(time,Tmod)
grid off
box on
axis([0 tmax Tmin Tmax])
Dynamic Tension Calculation

This program calculates the dynamic tension from the mean tension, oscillation tension, belt slip, and pulley rotation.
function [ T Tmax Tmin Hubmax Hubmin P] = DynamicTension( To, m, ...
     fnamesave, time, Tpp, DrivenRPM, DriverRPM, TVU, R, Span, EA, l)
%
% Define constants:
%
% Pulley Radii:
   r4=R(Span+1)/2; %[m]
   r3=R(Span)/2; %[m]
   phi4= 174*(pi/180); % Driven wrap angle in radians
   phi3= 135*(pi/180); % Driver wrap angle in radians
   kb3=EA/l; %[N/m]
   kp4=EA/(phi4*r4); %[N/m]
   kp3=EA/(phi3*r3); %[N/m]
%
% Transform the RPM to rad/sec:
%
   Rot4=DrivenRPM*((2*pi)/60); %[rad/s]
   Rot3=DriverRPM*((2*pi)/60); %[rad/s]
%
% Calculate speed in velocity:
%
   v3=Rot3*r3; %[m/2]
   v4=Rot4*r4; %[m/2]
%
% Calculate the support pulley rotation angles:
%
   Ang3=v3*0.0001;
   Ang4=v4*0.0001;
%
% Define friction coefficients:
%
   mewk=0.5; % Meckstroth
   mews=0.5;
% Define P, the fraction of the wrap angle that slips:
P=1;

% Calculate dynamic tension while iteratively calculating P:
alpha=phi4*P;
a=(exp(mewk*alpha)/(1+((phi4-alpha)/2)*mews))
T(1)= Tpp(1);
for j=2:m-1
    Ts4=real(T(j-1)/a);
    dTp4=((T(j-1)+Ts4)/2)-To;
    slip4=dTp4/kp4;
    T(j)=((Ang3(j)-Ang4(j)-slip4)*kb3)+T(j-1)-Tpp(j-1)+Tpp(j);
end
clear slip4 dTp4 Ts

aHIGH=P;
aLOW=0;
while round(mean(T)) ˜= round(mean(Tpp))
    if round(mean(T)) > round(mean(Tpp))
        aHIGH=P;
        P = (P+aLOW)/2
        alpha=phi4*P;
a=(exp(mewk*alpha)/(1+((phi4-alpha)/2)*mews));
        T(1)=((Arc3(1)-Arc4(1))*kb3)+Tpp(1);
        for j=2:m-1
            Ts4=real(T(j-1)/a);
            dTp4=((T(j-1)+Ts4)/2)-To;
            slip4=dTp4/kp4;
            T(j)=((Ang3(j)-Ang4(j)-slip4)*kb3)+T(j-1)-Tpp(j-1)+Tpp(j);
            clear slip4 dTp4 Ts
        end
    end
end
\begin{verbatim}
end
if round(mean(T)) < round(mean(Tpp))
  aLOW=P;
P = (P+aHIGH)/2
alpha=phi4*P;
a=(exp(mewk*alpha)/(1+((phi4-alpha)/2)*mews));
T(1)=((Arc3(1)-Arc4(1))*kb3)+Tpp(1);
for j=2:m-1
  Ts4=real(T(j-1)/a);
  dTp4=((T(j-1)+Ts4)/2)-To;
  slip4=dTp4/kp4;
  T(j)=((Ang3(j)-Ang4(j)-slip4)*kb3)+T(j-1)-Tpp(j-1)+Tpp(j);
  clear slip4 dTp4 Ts
end
end
end

%---------------------------------------------------------------------
% Define and calculate with damping if desired:
%---------------------------------------------------------------------

cb3=65
Td(1)=((Arc3(1)-Arc4(1))*kb3)+Tpp(1);
for j=2:m-1
  Ts4=real(Td(j-1)/a);
  dTp4=((Td(j-1)+Ts4)/2)-To;
  slip4=dTp4/kp4;
  Td(j)=((Ang3(j)-Ang4(j)-slip4)*kb3)+Td(j-1)-Tpp(j-1)+Tpp(j)...
                -cb3*(v3(j)-v4(j));
  clear slip4 dTp4 Ts
end

%---------------------------------------------------------------------
% Define the validation tension envelope area plot:
%---------------------------------------------------------------------

k=400;
\end{verbatim}
z=1;
up(1)=HIU(1);
down(1)=HIU(1);
tup(1)=t(1);
tdown(1)=t(1);
for j=2:round((m-1)/k)
    up(j)=max(HIU(z:(z+k-1)));  
down(j)=min(HIU(z:(z+k-1)));  
    ind1=find(up(j)==HIU(z:(z+k-1)));  
    ind2=find(down(j)==HIU(z:(z+k-1)));  
    tup(j)=t(ind1(end)+z);  
    tdown(j)=t(ind2(end)+z);  
    z=z+k;
    n=j;
end
up(n)=TVU(m-1);
down(n)=TVU(m-1);
tup(n)=t(m-1);
tdown(n)=t(m-1);

% Calculate plot max and min values:
Tmax=[max(T), max(TVU)];
Tmax=max(Tmax)+10;
Tmin=[min(T),min(TVU)];
Tmin=min(Tmin)-10;

% Plot:
scrsz = get(0,'ScreenSize');
figure('Position', [50 50 scrsz(3)*.8 scrsz(4)*.8])
set(gca,'LooseInset',get(gca,'TightInset'))
hold on
Appendix E. MATLAB Code

area(tup, up, 'FaceColor', 'b')
area(tdown, down, 'FaceColor', 'w')
plot(time, T, 'r')
hold off
box on
axis([0 max(t) Tmin Tmax])
legend('Measured Tension Range', 'area2', 'Dynamic Tension', ...
   'location', 'northwest')
xlabel('Time [s]'); ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesave, sprintf('DynamicTension') ), 'png');
saveas(gcf, fullfile(fnamesave, sprintf('DynamicTension') ));

scrsz = get(0, 'ScreenSize');
figure('Position', [50 50 scrsz(3)*.8 scrsz(4)*.8])
set(gca, 'LooseInset', get(gca, 'TightInset'))
hold on
area(tup, up, 'FaceColor', 'b')
area(tdown, down, 'FaceColor', 'w')
plot(time, Td, 'r')
hold off
box on
axis([0 max(t) Tmin Tmax])
legend('Measured Tension Range', 'area2', 'Measured Tension', ...
   'Damped Dynamic Tension', 'location', 'northwest')
xlabel('Time [s]'); ylabel('Tension [N]');
saveas(gcf, fullfile(fnamesave, sprintf('DampedDynamicTension') ), 'png');
saveas(gcf, fullfile(fnamesave, sprintf('DampedDynamicTension') ));

Tmax = max(T); 
Tmin = min(T); 
Hubmax = max(TVU); 
Hubmin = min(TVU); 
end