TOPOLOGICAL DYNAMICS OF AUTOMORPHISM GROUPS OF \( \omega \)-HOMOGENEOUS STRUCTURES VIA NEAR ULTRAFILTERS

by

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Abstract

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In this thesis, we present a new viewpoint of the universal minimal flow in the language of near ultrafilters. We apply this viewpoint to generalize results of Kechris, Pestov and Todorčević about a connection between groups of automorphisms of structures and structural Ramsey theory from countable to uncountable structures. This allows us to provide new examples of explicit descriptions of universal minimal flows as well as of extremely amenable groups. We identify new classes of finite structures satisfying the Ramsey property and apply the result to the computation of the universal minimal flow of the group of automorphisms of \( \mathcal{P}(\omega_1)/\text{fin} \) as well as of certain closed subgroups of groups of homeomorphisms of Cantor cubes. We furthermore apply our theory to groups of isometries of metric spaces and the problem of unique amenability of topological groups.

The theory combines tools from set theory, model theory, Ramsey theory, topological dynamics and ergodic theory, and homogeneous structures.
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## Contents

1 Introduction 1

2 Preliminaries 13
   2.1 Topological dynamics and ergodic theory 13
      2.1.1 Topological groups 13
      2.1.2 Uniform spaces 15
      2.1.3 Flows 15
      2.1.4 Compact right-topological semigroups 16
      2.1.5 Minimal flows 18
      2.1.6 The greatest ambit 19
   2.2 Set theory 22
   2.3 First order structures and their automorphism groups 25
      2.3.1 Fraïssé theory 27
      2.3.2 Jónsson structures 28
   2.4 Ramsey theory 29
   2.5 Kechris-Pestov-Todorčević correspondence and precompact expansions 30

3 General theory 35
   3.1 The greatest ambit 35
   3.2 The universal minimal flow 37
   3.3 Extreme amenability 39
   3.4 Non-archimedean groups 39

4 Automorphism groups 43
   4.1 Universality of $S_n$ 44
   4.2 Kechris, Pestov and Todorčević for uncountable structures and precompact expansions 45
      4.2.1 Jónsson classes 48
      4.2.2 Constructing $\omega$-homogeneous structures 50
   4.3 Dual approach 54
      4.3.1 Applications 58
      4.3.2 Extremely amenable groups 61
      4.3.3 Unique ergodicity 63
   4.4 Ramsey property for finite Boolean algebras with ideals 64
      4.4.1 Cantor cubes 70
4.4.2 Quotients of power set algebras $\mathcal{P}(\kappa)/[\kappa]^{<\lambda}$ ........................................ 73

5 More applications .................................................. 75
5.1 Generalized Urysohn spaces .................................. 75
5.2 A problem of Ellis ........................................... 77
5.3 Disjoint left ideals ............................................. 78

6 Problems ............................................................. 83

Bibliography .......................................................... 86
Chapter 1

Introduction

In this thesis, we study problems arising in topological dynamics and ergodic theory. We develop an alternate approach to the treatment of dynamical notions and present its applications. The theory combines tools from set theory, model theory, Ramsey theory, and topological groups and applies these tools to the study of minimal flows.

Topological dynamics and ergodic theory both study dynamical systems. Topological dynamics originated from investigations of asymptotic behaviour of trajectories of systems of differential equations. In particular, limit sets, closures of trajectories, minimal systems, the existence of fixed points, periodic points and other signs of repetitiveness are studied. For more on the origins of topological dynamics see [15] and for a more recent treatise, see [14]. Ergodic theory studies dynamical systems equipped with an invariant measure, conditions under which trajectories take on large measure and various phenomena of recurrence. A major breakthrough in the history of ergodic theory was the Ergodic Theorem proved by Birkhoff in 1931 ([10]). Recent work on measure-preserving actions of countable discrete groups can be found in a book by Kechris ([35]). Topological dynamics and ergodic theory have an intimate connection. For instance, the question of when a dynamical system can be equipped with a (unique) invariant measure has been well-studied.

A dynamical system is a continuous action of a topological group $G$ on a compact Hausdorff space $X$. Namely, we have a continuous map $\pi : G \times X \to X$ such that for every $g, h \in G$ and $x \in X$, $\pi(gh, x) = \pi(g, \pi(h, x))$ and for the identity $e \in G$, $\pi(e, x) = x$. When the action $\pi$ is understood, we write $gx$ instead of $\pi(g, x)$ and we call $X$ a $G$-flow or just a flow. A continuous map $\phi$ between $G$-flows $X$ and $Y$ is a homomorphism if for every $x \in X$ and $g \in G$ we have that $\phi(gx) = g\phi(x)$. A homomorphism which is also a homeomorphism is called an isomorphism. We say that $Y$ is a quotient of $X$ if there is a continuous homomorphism from $X$ onto $Y$.

My main focus lies in the study of minimal flows. There are flows with no non-trivial subflows, which correspond to points that are fixed or close to being periodic. A point $x$ in a $G$-flow $X$ is almost periodic if for every open subset $O$ of $G$ finitely many translates of the set of returns of $x$ into $O$, $\text{Ret}(x, O) = \{g \in G : gx \in O\}$, cover $G$. Minimal subflows of $G$-flows correspond exactly to closures of orbits of almost periodic points. The notion of a minimal flow is one of the most fruitful in topological dynamics and has been extensively studied since its origins (e.g. [15], [34], Furstenberg’s theory of minimal distal flows [19] and disjoint flows [17]). A minimal flow of particular interest is the universal minimal flow - a minimal flow that has every other minimal flow as its quotient. The universal minimal
flow exists and it is unique up to isomorphism. Groups whose minimal flows admit a unique invariant probability measure are called **uniquely ergodic**. If the universal minimal flow of a topological group $G$ is a singleton, then every action of $G$ on a compact Hausdorff space has a fixed point. In this case, we say that $G$ has a **fixed point compacta property** or that it is **extremely amenable**. Extreme amenability and unique ergodicity have a strong connection with Ramsey theory and they are subjects of intense research ([57],[34],[36],[51]). In the study of universal minimal flows, the notion of the greatest ambit proved to be useful. An **ambit** is a $G$-flow $X$ with a distinguished point $x_0$ with a dense orbit $Gx_0 = \{gx_0 : g \in G\}$ in $X$. The greatest ambit $S(G)$ with the distinguished point $e$ is defined by the same universality property for all ambits as the universal minimal flow has for minimal flows: If $(X, x_0)$ is another $G$-ambit, then there is a homomorphism from $S(G)$ onto $X$ sending $e$ to $x_0$. The greatest ambit is the smallest compactification of $G$ to which every uniformly continuous function from $G$ to $\mathbb{R}$ extends in a continuous manner. Here we consider $G$ to be a uniform space with the right uniform structure generated by the covers $\{Vg : g \in G\}$ for $V$ an open neighbourhood of the identity in $G$. This compactification was first described by Samuel ([58]) in the more general setting of arbitrary uniform spaces and is therefore called the **Samuel compactification**. The multiplication on $G$ can be extended to $S(G)$ turning it into a compact right-topological semigroup. Every action of $G$ can be extended to an action of $S(G)$ and the closures of $G$-orbits are equal to orbits under the action of $S(G)$. The universal minimal flow $M(G)$ is then a minimal left ideal of $S(G)$ and therefore it also has a structure of a compact right-topological semigroup. It follows that $M(G)$ has idempotent elements, i.e. elements $m$ such that $mm = m$, which has large consequences in Ramsey theory (e.g. [18],[24]).

Although proving existence and uniqueness of the greatest ambit and the universal minimal flow is not difficult, we rarely have an explicit description of these spaces even when the universal minimal flow is metrizable. If $G$ is a discrete group then the greatest ambit of $G$ is the space of all ultrafilters $\beta G$ on $G$, and the universal minimal flow $M(G)$ is isomorphic to any minimal left ideal of $\beta G$. Neither $\beta G$, nor $M(G)$ are metrizable for any infinite discrete group $G$. Balcar and Franěk identified [4] the algebra $B(G)$ of clopen subsets of the universal minimal flow of a discrete group $G$ with a subalgebra of “large” subsets of $\mathcal{P}(G)$. Large here means left syndetic, where a subset $A$ of $G$ is called **left syndetic** if there exist $g_1, g_2, \ldots, g_n \in G$ such that $G = \bigcup_{i=1}^{n} g_i A$. The algebra $B(G)$ is isomorphic to any maximal (with respect to inclusion) subalgebra of $\mathcal{P}(G)$ consisting of left syndetic subsets of $G$ and invariant under left translations by elements of $G$. Balcar and Franěk also studied forcing properties of $B(G)$. Many metrizable universal minimal flows of groups have been computed thanks to having extremely amenable subgroups. For instance, Pestov showed that the group $\text{Homeo}_{+}(I)$ of orientation-preserving homeomorphisms of the unit interval is extremely amenable and, as a consequence, that the universal minimal flow of the group of orientation-preserving homeomorphisms $\text{Homeo}_{+}(S)$ of the unit circle $S$ is isomorphic to $S$ itself. A main breakthrough came when Kechris, Pestov and Todorcević developed in [34] a general theory connecting the Ramsey property for finite structures with extremely amenable groups of automorphisms of countable structures, and as a consequence, provided explicit descriptions of universal minimal flows of various automorphism groups. As an example of an application of their theory, Kechris, Pestov and Todorcević used Nešetřil and Rödl’s result ([48]) that the class of finite linearly ordered graphs is a Ramsey class to show that the group of automorphisms of the random linearly ordered graph is extremely amenable. Furthermore, they identified the universal minimal flow of the groups of automorphisms of the random graph $\mathcal{G}$ with the space of all linear orderings on $\mathcal{G}$. Their theory started a new vital branch of research pursued by Lionel Nguyen van Thé ([50], [51]), Miodrag Sokič([64]), Julien...
Melleray, Todor Tsankov ([42]), and others, and had applications in theoretical computer science ([11]).

On the abstract level, the greatest ambit and the universal minimal flow are usually described in terms of C*-algebras. However, we go back to the original description of the Samuel compactification and show that this language is more suitable for a variety of applications as well as structural results. The greatest ambit for a discrete group $G$ is the Cech-Stone compactification $\beta G$, the space of all ultrafilters on $G$ with a basis of clopen subsets consisting of sets $\hat{A} = \{ u \in \beta G : A \in u \}$. The algebra in Cech-Stone compactifications was extensively studied by Hindman and Strauss. If $G$ is a topological group, then $S(G)$ is a quotient of $\beta G$. Equivalence classes of this quotient map are then closed subsets of $\beta G$, which dually correspond to filters on $G$. These were explicitly described by Samuel. Recently the equivalence classes were identified with so-called near ultrafilters by Koçak and Arvasi ([37]). The algebraic structure on $\beta G$ factors down to $S(G)$, which allows us to generalize results of Hindman and Strauss about $\beta G$ to $S(G)$.

In Chapter 3, we show that we can also describe the universal minimal flow as a space of filters on $G$. Equivalently, the universal minimal flow can be described as a space of near ultrafilters on the Boolean algebra $B(G)$ of clopen subsets of the universal minimal flow of $G$ with the discrete topology. The notions of a near filter and near ultrafilter were first defined by Koçak and Strauss in [36].

**Definition 12.** Let $G$ be a topological group and $\mathcal{N}$ a base of neighbourhoods of the identity in $G$. A family $u$ of subsets of $G$ has the near finite intersection property if for every $F \subset u$ finite and every $V \in \mathcal{N}$ we have

$$\bigcap_{A \in F} VA \neq \emptyset.$$ 

We call $u$ a near ultrafilter, if it is a maximal collection with the near finite intersection property with respect to inclusion. A near ultrafilter on a subalgebra $\mathcal{A}$ of $\mathcal{P}(G)$ is then a maximal collection of sets in $\mathcal{A}$ with the near finite intersection property.

Let $G$ be a topological group and let $\mathcal{N}$ be a basis of neighbourhoods of the identity in $G$. Let $\hat{G}$ denote the space of near ultrafilters on $G$ with the topology generated by closed sets $\hat{A} = \{ u \in \hat{G} : A \in u \}$ for $A \subset G$. We can naturally identify $G$ with a subset of $\hat{G}$ by identifying points $g \in G$ with near ultrafilters $\{ A \subset G : Vg \cap A \neq \emptyset \ \forall \ V \in \mathcal{N} \}$. The action of $G$ on $\hat{G}$ defined by $gu = \{ gA : A \in u \}$ is continuous and in terms of topological dynamics it is the greatest ambit action.

**Theorem 23 ([37]).** Let $G$ be a topological group. Then $\hat{G}$ is the greatest ambit for $G$ with the action defined above.

Let $G$ be a topological group and let $G_d$ denote the group $G$ with the discrete topology. Let $B(G)$ denote a subalgebra of $\mathcal{P}(G)$ that is isomorphic to the algebra of clopen subsets of the universal minimal flow for $G_d$. A core of the alternate approach we are developing is that we can describe the universal minimal flow of $G$ as the space of near ultrafilters on $B(G)$.

**Theorem 26.** The universal minimal flow for a topological group $G$ is the space of near ultrafilters on $B(G)$.

In the case of groups of automorphisms of structures, this approach provides an internal characterization of the Boolean algebras of clopen subsets of the greatest ambit as well as of the universal minimal flow. More formally, we extend the result of Balcar and Franek from discrete groups to non-archimedean groups. A topological group is called non-archimedean if it admits a basis $\mathcal{N}$ of neighbourhoods of
the identity in $G$ consisting of open subgroups. We also show that every non-archimedean group can be realized as a group of automorphisms of an ($\omega$-homogeneous) structure, while every automorphism group is easily seen to be non-archimedean. Indeed, let $\mathcal{A}$ be a first-order structure considered with the discrete topology and denote by $\text{Aut}(\mathcal{A})$ the group of automorphisms of $\mathcal{A}$ with the topology of pointwise convergence. Then the topology on $\text{Aut}(\mathcal{A})$ is determined by a basis of neighbourhoods of the identity consisting of open subgroups

$$\text{Aut}(\mathcal{A})_A = \{ g \in \text{Aut}(\mathcal{A}) : \forall a \in A \ ga = a \}$$

for $A$ a finitely generated substructure of $\mathcal{A}$.

If $G$ is non-archimedean and $\mathcal{N}$ is a basis of neighbourhoods of the identity in $G$ consisting of open subgroups and $\mathcal{N}$ is closed under intersections then

$$L = \{ VA : A \subset G, V \in \mathcal{N} \}$$

is a Boolean algebra.

We characterize the greatest ambit and the universal minimal flow of $G$ in terms of $L$. Recall that if $\mathcal{A}$ is a Boolean algebra, then the Stone space of $\mathcal{A}$ is the space $\text{St}(\mathcal{A})$ of all ultrafilters on $\mathcal{A}$ with the topology generated by clopen sets $a^* = \{ u \in \text{St}(\mathcal{A}) : a \in u \}$ for $a \in \mathcal{A}$.

**Theorem 28.** The greatest ambit $S(G)$ is equal to the Stone space of the Boolean algebra $L$ as above with the action defined by $gx = \{ gA : A \in x \}$ for $g \in G$ and $x \in \text{St}(L)$.

**Theorem 29.** The universal minimal flow $M(G)$ is the Stone space of a maximal $B(G)$ subalgebra of $L$ consisting of left syndetic sets such that $B(G)$ is invariant under left translation by elements of $G$. All such subalgebras of $L$ are isomorphic.

We then translate some dynamical properties into the language of near ultrafilters. For instance, we obtain an internal characterization of extreme amenability.

**Lemma 11.** A topological group $G$ is extremely amenable if and only if for every pair $A, B$ of left syndetic subsets of $G$ and every open neighbourhood $V$ of the identity in $G$ we have $VA \cap VB \neq \emptyset$.

In the remaining chapters, we show how natural the viewpoint of near ultrafilters is by simplifying and generalizing known results, and proving new results.

In Chapter 4, we generalize results of Kechris, Pestov and Todorcevic to uncountable structures and precompact expansions introduced by Nguyen van Thé ([49]).

A **structure** always means a first order structure in a language $L$, where we allow $L$ to be uncountable. When $A$ is a substructure of $B$, we write $A \leq B$. Kechris, Pestov and Todorcevic characterized extremely amenable groups of automorphisms of locally-finite Fraïssé structures via the Ramsey property of their finite substructures. A structure $\mathcal{A}$ is called **locally-finite** if the class of finitely-generated substructures, $\text{Age}(\mathcal{A})$, consists of finite structures. For instance, all purely relational structures are locally-finite. A **Fraïssé structure** is a countable $\omega$-homogeneous structure, a structure in which every partial isomorphism between finitely-generated substructures can be extended to a full automorphism. Every group of automorphisms of a countable structure is isomorphic to a group of automorphisms of a Fraïssé structure. Examples of Fraïssé structures are the random graph, random hypergraph, random poset, random distributive lattice, algebra of clopen subsets of the Cantor space or a countable
Chapter 1. Introduction

5

set (without any structure). Fraïssé structures are in one-to-one correspondence with the classes of their finitely-generated substructures, called Fraïssé classes. A class \( \mathcal{K} \) of finite structures satisfies the Ramsey property if for every \( A \leq B \in \mathcal{K} \) and every \( k = 2, 3, \ldots \) there is a structure \( C \in \mathcal{K} \) such that for every colouring of copies of \( A \) in \( C \) by \( k \) many colours there is a copy \( B' \) of \( B \) in \( C \) which is monochromatic (i.e. all copies of \( A \) in \( B' \) take on the same colour). For instance, Ramsey’s original theorem from 1928 shows that the class of finite linear orderings satisfies the Ramsey property.

In [34], the authors computed universal minimal flows of groups of automorphisms of locally-finite Fraïssé structures whose age admits a non-trivial expansion to a Fraïssé class \( \mathcal{K} \) of linearly ordered structures satisfying the Ramsey property and the so-called ordering property. Nguyen Van Thé extended their result to precompact expansions satisfying the expansion property. For purposes of this thesis, we introduce the notion of precompact expansion for a pair of classes rather than for pair of structures as originally defined. Let \( \mathcal{K} \) be a class of structures in a language \( L \) and let \( \mathcal{K}_R \) be a class of expansions of structures in \( \mathcal{K} \) in a language \( L^* \) such that \( L^* \setminus L = \mathcal{R} \) is a countable set of relational symbols. We say that \( \mathcal{K}_R \) is a precompact expansion of \( \mathcal{K} \) if every structure in \( \mathcal{K} \) has only finitely many expansions in \( \mathcal{K}_R \). We say that \( \mathcal{K}_R \) has the expansion property relative to \( \mathcal{K} \) if for every \( A \in \mathcal{K} \) there exists \( B \in \mathcal{K} \) such that

\[
A^*, B^* \in \mathcal{K}_R \quad (A^*|L = A \& B^*|L = B) \rightarrow A^* \leq B^*.
\]

Let \( L \) be a language and let \( L^* \) be its relational expansion by a set \( \mathcal{R} = \{ R_i : i = 1 \} \) of countably many relations. Let \( \mathcal{K} \) be a class of finite structures in \( L \) and \( \mathcal{K}_R \) a class of finite structures in \( L^* \) such that \( \mathcal{K}_R \) is a precompact expansion of \( \mathcal{K} \). Let \( A \) be a structure with \( \text{Age}(A) = \mathcal{K} \). We say that an interpretation \( \mathcal{R}^A \) of \( \mathcal{R} \) on \( A \) is normal if \( \text{Age}(A, \mathcal{R}^A) \subseteq \mathcal{K}_R \). Let \( a_i \) denote the arity of the relation \( R_i \) for every \( i \in I \). Any interpretation \( R^A_i \) of \( R_i \) in \( A \) can be viewed as a point in \( 2^{A^{a_i}} \) and a interpretation of all \( R_i \)'s is then a point in

\[
\prod_{i \in I} \{0, 1\}^{A^{a_i}}.
\]

Viewing \( \prod_{i \in I} \{0, 1\}^{A^{a_i}} \) with the product topology of \( \{0, 1\} \) discrete, we turn \( \prod_{i \in I} \{0, 1\}^{A^{a_i}} \) into a \( G \)-flow via the following action:

\[
g \in G, \; x \in \prod_{i \in I} \{0, 1\}^{A^{a_i}} \mapsto g(x)(i)(y_1, y_2, \ldots, y_{a_i}) = 1 \leftrightarrow x(i)(g^{-1}y_1, g^{-1}(y_2), \ldots, g^{-1}(y_{a_i})) = 1.
\]

We denote the closed subspace of \( \prod_{i \in I} \{0, 1\}^{A^{a_i}} \) consisting of all normal interpretations of \( \mathcal{R} \) on \( A \) by \( X_R \) and consider the inherited action of \( G \) on \( X_R \).

The reason to restrict ourselves to precompact expansions is the following proposition.

**Proposition 4 ([49]).** The space \( X_R \) is compact if and only if \( \mathcal{K}_R \) is a precompact expansion of \( \mathcal{K} \).

The above proposition is originally stated in a different language and only for countable structures. We provide an alternative proof covering uncountable structures as well.

We first show that the theory of Kechris, Pestov and Todorcević directly carries over to uncountable structures.

**Theorem 32.** Let \( A \) be an \( \omega \)-homogeneous structure and let \( (A, \mathcal{R}^A) \) be an expansion of \( A \) with a set of countably many relations \( \mathcal{R} \) such that \( (A, \mathcal{R}^A) \) is \( \omega \)-homogeneous as well. Suppose that \( \text{Age}(A, \mathcal{R}^A) \)
Corollary 6. Let \( \kappa \) be a cardinal satisfying \( \kappa^{<\kappa} = \kappa \) and let \((A, <)\) be the universal homogeneous ordered graph \((K_n,\text{-free graph, hypergraph, } A,\text{-free hypergraph, poset})\) of cardinality \(\kappa\). Let \(G_\prec\) be a dense subgroup of \(\text{Aut}(A, <)\) and let \(G \) be a dense subgroup of \(\text{Aut}(A)\). Then \(G_\prec\) is extremely amenable and the universal minimal flow of \(G\) is \(\overline{G\prec}\).

We can, however, construct \(\omega\)-homogeneous structures with \(\omega\)-homogeneous linear expansions in every cardinality.

Theorem 35. Let \(A\) be a graph \((K_n,\text{-free graph, hypergraph, } A,\text{-free hypergraph, poset})\) of an infinite cardinality \(\kappa\) and let \(<\) be an arbitrary linear ordering on \(A\) (respectively, an ordering extending the partial order if \(A\) is a poset). Then there exists a linearly ordered graph \((K_n,\text{-free graph, hypergraph, } A,\text{-free hypergraph, poset})\) \((A', <')\) of cardinality \(\kappa\) in which \((A, <)\) is embedded such that both \(A'\) and \((A', <')\) are \(\omega\)-homogeneous (and if \(A\) is a poset, \(<'\) extends the partial order on \(A'\)).
Chapter 1. Introduction

Proof Theorem in 7.5 [34] and its generalization Theorem 33 rely on the existence of a suitable \( \omega \)-homogeneous expansion. While in the class of countable structures every \( \omega \)-homogeneous structure whose age admits a reasonable Fraïssé expansion admits an \( \omega \)-homogeneous expansion via the Fraïssé construction, we do not know whether this is so for uncountable structures (even in cases of structures for which we can compute the universal minimal flow of their automorphism groups).

For instance, Glasner and Gutman computed in [20] the universal minimal flow of the group of homeomorphisms of a space of great interest - the Čech-Stone remainder \( \omega^* \) of discrete \( \omega \). In dual terms, the authors showed that the universal minimal flow of the group of automorphisms of the Boolean algebra \( \mathcal{P}(\omega)/\text{fin} \) of all subsets of \( \omega \) modulo the ideal of finite sets is a space of linear orderings on \( \mathcal{P}(\omega)/\text{fin} \).

Up to now, we do not know whether \( \mathcal{P}(\omega)/\text{fin} \) admits an \( \omega \)-homogeneous linear ordering. Using the description of the Boolean algebra of clopen subsets of the universal minimal flow of non-archimedean groups in Theorem 29, we first showed in [8] that the existence of an \( \omega \)-homogeneous expansion is not necessary if the expansion of age is order-forgetful. An order expansion \( K^* \) of a class \( K \) of structures is \textbf{order forgetful} whenever

\[
(A, <), (B, \prec) \in K^* \text{ and } A \cong B \text{ imply } (A, <) \cong (B, \prec).
\] (1.1)

One of the main results of the thesis is that we can extend the result to arbitrary expansions generalizing both the result of Glasner and Gutman as well as Theorem 32 and 33.

**Theorem 39.** Let \( A \) be a locally-finite \( \omega \)-homogeneous structure and let \( K_K \) be a precompact Fraïssé expansion of \( \text{Age}(A) = K \). Then the following are equivalent.

1. \( K_K \) satisfies the expansion property with respect to \( \text{Age}(A) \).
2. \( X_K \) is a minimal flow for \( \text{Aut}(A) \).

**Theorem 40.** Let \( A \) be a locally-finite \( \omega \)-homogeneous structure and let \( K_K \) be a Fraïssé precompact expansion of \( \text{Age}(A) = K \). Suppose that \( K_K \) satisfies the Ramsey property and the expansion property relative to \( K \). Then \( X_K \) is the universal minimal flow of the group of automorphisms of \( A \).

As a consequence of Theorem 40, we obtain results about uncountable structures analogous to results of Kechris, Pestov and Todorčević for countable structures without the requirement that the structures admit an \( \omega \)-homogeneous relational expansion.

If \( A \) is one of the following structures and \( A \) is infinite, then the universal minimal flow of \( \text{Aut}(A) \) is the space of all linear orderings on \( A \).

- \( \omega \)-homogeneous graph;
- \( \omega \)-homogeneous \( K_n \)-free graph, \( n=2,3,\ldots \);
- \( \omega \)-homogeneous hypergraph;
- \( \omega \)-homogeneous \( H \)-free hypergraph, where \( H \) is a class of irreducible finite hypergraphs;
- an infinite set \( M \) (\( \text{Aut}(M) \cong S_\kappa \) for \( \kappa = |M| \)).

The class of all finite posets expanded with arbitrary linear orderings does not satisfy the Ramsey property as proved by Sokić in [64]. However, Sokić proved that expanding finite posets with linear orderings extending the partial orderings produces a Ramsey class. Applying Theorem 40 we can therefore generalize Sokić’s result about the countable random poset to uncountable \( \omega \)-homogeneous posets.
Chapter 1. Introduction

Theorem 44. Let $\mathcal{P}$ be an $\omega$-homogeneous poset and let $\mathcal{K}$ be the class of finite linearly ordered posets such that the linear orderings extend the partial order. Then the universal minimal flow of $\text{Aut}(\mathcal{P})$ is the space of normal orderings on $\mathcal{P}$ induced by $\mathcal{K}$.

We say that a linear ordering on a finite Boolean algebra is **natural** if it is an antilexicographical extension of an ordering on its atoms. Applying Theorem 40, we give an alternative proof of a result by Glasner and Gutman ([20]).

**Theorem 46 ([20]).** Let $X$ be an $h$-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \text{Homeo}(X)$ equipped with the compact-open topology. Then $M(G) = \Phi(X)$, the space of maximal chains of closed subsets of $X$.

Since homogeneous Boolean algebras are in Stone duality (see Section 2.2) with $h$-homogeneous zero-dimensional compact Hausdorff spaces, the following theorem is just a dual version of Theorem 46.

**Theorem 45.** Let $B$ be a homogeneous Boolean algebra and $\mathcal{K}$ the class of naturally ordered finite Boolean algebras. Then the universal minimal flow of $\text{Aut}(B)$ is the space $\text{NO}_\mathcal{K}(B)$ of all linear orderings on $B$ that are natural when restricted to a finite subalgebra.

As for Boolean algebras, we call a linear ordering on a finite vector space over a given finite field **natural** if it is an antilexicographical extension of a linear ordering on a basis relative to a fixed linear ordering of the field. The following theorem was proved in [34] in the case of $\aleph_0$-dimensional vector spaces.

**Theorem 50.** Let $V$ be an infinite-dimensional vector space over a finite field and let $\mathcal{K}$ be the class of all finite dimensional naturally-ordered spaces. Then the universal minimal flow of $\text{Aut}(V)$ is the space $\text{NO}_\mathcal{K}(V)$ of all linear orderings on $V$ that are natural when restricted to a finite subspace.

It is a well-known result that $S_\infty(\mathbb{Z})$, the group of all bijections on $\mathbb{Z}$, is universal for groups of automorphisms of countable structures. We show that the same is true for uncountable structures. If $\kappa$ is an infinite number, we denote by $S_\kappa$ the group of bijections on a set of cardinality $\kappa$.

**Theorem 30.** Let $G$ be an infinite topological group and let $\kappa$ be an infinite cardinal number. Then the following are equivalent:

(a) $G$ is a subgroup of $S_\kappa$,

(b) $G$ is non-archimedean with a basis $\mathcal{N}$ of neighbourhoods of the identity of cardinality $\leq \kappa$ consisting of open subgroups, such that the family of all left translates of elements from $\mathcal{N}$ also has cardinality $\leq \kappa$,

(c) $G$ is a dense subgroup of a group of automorphisms of an $\omega$-homogeneous relational structure on a set of cardinality $\kappa$,

(d) $G$ is a dense subgroup of a group of automorphisms of a structure on a set of cardinality $\kappa$.

In [34], the authors gave a characterization of extremely amenable subgroups of $S_\infty(\mathbb{Z})$. We prove that the same holds for $S_\kappa$. While some of the implications are just modifications of Proposition 4.3 in [34], it is not clear whether the implication $(b) \Rightarrow (a)$ can be obtained by their methods and heavily relies on the dual approach developed here.
Theorem 51. Let \( G \) be an infinite subgroup of \( S_\kappa \). The following are equivalent:

(a) \( G \) is extremely amenable,

(b) (i) for every finite \( A \subset \kappa \), \( \{ g \in G : ga = a \ \forall a \in A \} = G_A = G_{(A)} = \{ g \in G : gA = A \} \) and

(ii) for every colouring \( c : G/G_A \longrightarrow \{1, 2, \ldots, k\} \) and for every finite \( B \supset A \), there is \( g \in G \) and \( i \in \{1, 2, \ldots, n\} \) such that \( c(hG_A) = i \) whenever \( h[A] \subset g[B] \).

(c) (i') \( G \) preserves countably many relations and (ii) as above.

If \( G \) is the group of automorphisms of a locally-finite \( \omega \)-homogeneous structure \( A \), then (b)(ii) simply says that \( \text{Age}(A) \) satisfies the Ramsey property.

Applying Theorem 51 we show that the examples of extremely amenable groups provided in [34] can be of arbitrary cardinality.

Groups of automorphisms of the following structures are extremely amenable:

1. an \( \omega \)-homogeneous linear order ([57]);
2. a homogeneous linearly ordered Boolean algebra \( B \) such that \( \text{Age}(B) \) is the class of finite naturally ordered Boolean algebras;
3. an \( \omega \)-homogeneous infinite-dimensional vector space \( V \) over a finite field \( F \) such that \( \text{Age}(V) \) is the class of finite naturally ordered finite vector spaces over \( F \);
4. an \( \omega \)-homogeneous linearly ordered \( (K_n\text{-free}) \) graph that is universal for all finite linearly ordered \( (K_n\text{-free}) \) graphs;
5. an \( \omega \)-homogeneous linearly ordered hypergraph that is universal for all finite linearly ordered hypergraphs;
6. an \( \omega \)-homogeneous linearly ordered \( \mathcal{A}\text{-free} \) hypergraph for \( \mathcal{A} \) a class of finite irreducible subgraphs that is universal for all finite linearly ordered \( \mathcal{A}\text{-free} \) hypergraphs;
7. an \( \omega \)-homogeneous linearly ordered poset with the linear ordering the partial order that is universal for all posets with linear orderings extending the partial order.

In [2], Angel, Kechris and Lyons developed a theory connecting ergodic theory for groups of automorphisms of structures and a combinatorial property for classes of finite structures. This was a major breakthrough providing tools to find uniquely ergodic groups. Before their result, only compact and extremely amenable groups were trivially known to be uniquely ergodic. The only nontrivial result had been the result of Glasner and Weiss ([21]) that the group \( S_\infty(\mathbb{Z}) \) of all permutations of the integers is uniquely ergodic.

Using our description of the Boolean algebras of clopen subsets of universal minimal flows of groups of automorphisms in Theorem 29, we show that if \( G \) is an amenable group of automorphisms of an \( \omega \)-homogeneous structure \( A \) such that \( A \) admits an order-forgetful Fraïssé expansion, then \( G \) is uniquely ergodic. This provides a short alternative proof of the result in [2] that the group of automorphisms of a countable-dimensional vector space over a finite field is uniquely ergodic as well as of the result for \( S_\infty(\mathbb{Z}) \).

It is an old open problem, the so-called Katowice problem, to determine whether it is consistent that the Boolean algebras \( \mathcal{P}(\omega)/\text{fin} \) and \( \mathcal{P}(\omega_1)/\text{fin} \) are isomorphic. The nonexistence of such an isomorphism
Chapter 1. Introduction

is easily proved to be consistent, for instance, under CH (the continuum hypothesis). Under CH, $|\mathcal{P}(\omega)/\text{fin}| = 2^\omega = \omega_1 < 2^{\omega_1} = |\mathcal{P}(\omega_1)/\text{fin}|$. Since $\mathcal{P}(\omega)/\text{fin}$ is an $\omega$-homogeneous Boolean algebra, Theorem 45 of Glasner and Gutman applies, identifying the universal minimal flow of $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$ with the space of linear orderings on $\mathcal{P}(\omega)/\text{fin}$ that are natural on every finite subalgebra. We compute the universal minimal flow of $\mathcal{P}(\omega_1)/\text{fin}$ in the case that $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ are not isomorphic. We then have that $\mathcal{P}(\omega_1)/\text{fin}$ is not homogeneous. Expanding $\mathcal{P}(\omega_1)/\text{fin}$ with a predicate $I$ for the ideal of countable subsets of $\omega_1$, we obtain an $\omega$-homogeneous structure $(\mathcal{P}(\omega_1)/\text{fin}, I)$ with $\text{Age}(\mathcal{P}(\omega_1)/\text{fin}, I)$ being a class of structures $(B, I_B)$ where $B$ is a finite Boolean algebra and $I_B$ is an ideal on $B$ such that $I_B \neq B$.

In order to apply Theorem 40, we prove the Ramsey property for certain classes of finite Boolean algebras with (possibly uncountably many) ideals. Let $L = \{\lor, \land, 0, 1, \neg\}$ be the language of Boolean algebras and let $J$ be a linearly ordered set. We denote by $L_J$ the language $L$ expanded by $|J|$-many unary symbols $\langle P_j : j \in J \rangle$, i.e. $L_J = \{\lor, \land, 0, 1, \neg, P_j : j \in J\}$. To simplify the notation, if $A$ is a structure in the language $L_J$, we often write $P_i$ in place of $P_i^A$ for every $j \in J$. The structure $(\mathcal{P}(\omega_1)/\text{fin}, I)$ is then an example of a structure in the language $L_{\{\omega\}}$ with $I = P_\omega$ interpreted as the ideal of countable subsets of $\omega_1$. More generally, if $\lambda < \kappa$ are infinite cardinals and $J = [\lambda, \kappa]$, then we consider $\omega$-homogeneous structures

$$\mathcal{P}_\lambda^\kappa = (\mathcal{P}(\kappa)/[\kappa]^{< \lambda}, P_\mu : \mu \in J),$$

in the language $L_J$. $\mathcal{P}(\kappa)/[\kappa]^{< \lambda}$ is the quotient of the power set algebra on $\kappa$ by the ideal of sets of cardinality less than $\lambda$ and $P_\mu$ is the ideal on $\mathcal{P}(\kappa)/[\kappa]^{< \lambda}$ consisting of equivalence classes of subsets of $\kappa$ of cardinality $\leq \mu$ for every $\mu \in J$.

We will consider ages of $\mathcal{P}_\lambda^\kappa$ for arbitrary $\lambda < \kappa$ infinite and two other classes of finite Boolean algebras. Note that we allow $|\lambda, \kappa|$ to be uncountable.

**Definition 15.**

1. Let $B_J$ denote the class of isomorphism types of finite Boolean algebras in the language $L_J$, where for each $j \in J$, $P_j$ is interpreted as an ideal with $0 \in P_i$ for every $i \in J$ and $P_i \subset P_j$ for $i < j \in J$. We require that every $A \in B_J$ has at least one atom not in any $P_j$ for $j \in J$.

2. Let $B_u$ denote the class of isomorphism types of finite Boolean algebras in the language $L_1$ with $P_0$ interpreted as an ideal such that every $A \in B_u$ has exactly one atom not in $P_0$ (i.e. for every $A \in B_u$ all atoms but one are in $P_0$). We consider the two-element algebra to be an element of $B_u$ and we assume that $0 \in P_0$.

3. Let $B_J^0$ denote the class of isomorphism types of finite Boolean algebras in the language $L_J$ where for each $j \in J$, $P_j$ is interpreted as an ideal with $0 \in P_i$ for every $i \in J$ and $P_i \subset P_j$ for $i < j \in J$. Moreover, for every $A \in B_J^0$ all atoms but one are in one of the $P_j$’s. We again consider the two-element algebra to be an element of $B_J^0$.

If $J$ is countable, then all the classes above are Fraïssé classes and we can therefore consider their Fraïssé limits. In each case, we obtain the countable atomless Boolean algebra $C$ with a chain of ideals $P_J^0$ for $j \in J$. The ideal $P_0^C$ in the limit of $B_u$ will be prime and therefore the filter dual to $P_0^C$ will be an ultrafilter. In the limit of $B_J^0$ we get that $\bigcup_{j \in J} P_j^C$ is a prime ideal. Speaking in the dual language of Stone spaces, we obtain the Cantor set with a chain of closed subsets in the case of $B_J$, the Cantor set
with a distinguished point in the case of $B_u$, and the Cantor set with a chain of closed sets intersecting in a single point in the case of $B_j$.

We prove that the classes in Definition 15 satisfy the Ramsey property by showing that they admit an order-forgetful reasonable expansion to a Fra"issé class satisfying the Ramsey property. We modify the definition of natural orderings on finite Boolean algebras given in [34]. Let $A$ be a finite structure in a language $L_J$. We call an ordering $<\!$ on atoms of $A$ proper if for every two atoms $a, b \in A$ and $i < j$, if $a \in P_i$ and $b \in P_j \setminus P_i$ or $b \notin P_k$ for any $k \in J$, then $a < b$.

**Definition 16.** Let $B$ be one of $B_J, B_u, B_j$ and let $A \in B$. We say that a linear ordering $<\!$ on $A$ is natural if it is an antilexicographical extension of a proper ordering on atoms of $A$. We denote by $NA(B)$ the class of all naturally ordered algebras from $B$.

As a consequence of Theorem 40, we compute universal minimal flows of $(P(\kappa)/[\kappa]^{<\lambda}, P_\mu : \mu \in J)$, in particular $\mathcal{P}(\omega_1)/\text{fin}$.

**Theorem 66.** Let $G$ be the group of automorphisms of $\mathcal{P}(\omega_1)/\text{fin}$ and let $I$ be the ideal of countable subsets of $\omega_1$. If there is no isomorphism between $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$, then the universal minimal flow of $G$ is the space $\text{NO}(\mathcal{P}(\omega_1)/\text{fin}, I)$ of linear orderings on $(\mathcal{P}(\omega_1)/\text{fin}, I)$ that are natural when restricted to a finite subalgebra.

Moreover, we obtain results about subgroups of the group of homeomorphisms of the Cantor set. For instance:

**Theorem 63.** Let $G$ be a group of homeomorphisms of the Cantor set $E$ fixing a point $x$. Then the universal minimal flow of $G$ is the space of maximal chains of closed subsets of $E$ containing $\{x\}$.

In Chapter 5, we further apply results in Chapter 3 to simplify the proof of Pestov that groups of isometries of generalized Urysohn spaces are extremely amenable. A generalized Urysohn space is a metric space that contains an isometric copy of every finite metric space and every isometry between finite metric spaces extends to the isometry of the full space. Our proof combines our reformulation of extreme amenability in the language of near ultrafilters (Lemma 11) with Nešetřil’s result that the class of finite linearly ordered metric spaces is a Ramsey class. The proof proceeds in a spirit similar to that of the proof of Theorem 40 about groups of automorphisms.

**Theorem 67 ([54]).** Let $(U,d)$ be a generalized Urysohn space. Then the group of isometries of $(U,d)$ is extremely amenable.

We reformulate Pestov's conjecture to the problem of Ellis in dual terms.

Finally, we make progress on the problem of Megrelishvili, Pestov and Uspenskij on whether uniquely amenable groups need to be precompact. Recall that a topological group $G$ is called amenable if every $G$-flow admits an invariant probability measure. If this measure is unique, then we say that $G$ is uniquely amenable.

Recall that a group is called precompact if every neighbourhood of the identity in $G$ is left syndetic (equivalently right syndetic). It follows that if a topological group $G$ is not precompact, then there is a neighbourhood $V$ of the identity in $G$ such that there exist infinitely many pairwise disjoint left (equivalently right) translates of $V$. This inspired us to define the following cardinal. Let $V$ be a symmetric neighbourhood of the identity. We set

$$\kappa_V = \min\{|\Gamma| : \Gamma \subset G, \forall a, b \in \Gamma a \neq b \implies Va \cap Vb = \emptyset \& V^2\Gamma = G\}.$$
We show that a non-precompact group cannot be uniquely amenable if it moreover has a neighbourhood $V$ of the identity satisfying the following condition.

**Definition 17.** Let $G$ be a topological group and let $V$ be a symmetric neighbourhood of the identity in $G$. We say that $V$ satisfies the property $N$ if $\kappa_V$ is infinite and

$$\kappa_V = \kappa_V^\mu.$$

If $G$ is a non-precompact group with a neighbourhood $V$ satisfying the property $N$, we show that $S(G)$ contains two disjoint minimal left ideals. Then if $G$ is amenable, there are at least two different invariant probability measures, each supported on one of the minimal left ideals. We actually show that there are infinitely many subsets of $G$ with disjoint closures in $S(G)$ such that each of them contains a minimal left ideal in its closure in $S(G)$. Note that two subsets $A, B \subseteq G$ have disjoint closures in $S(G)$ if and only if for every neighbourhood $W$ of the identity in $G$ it holds that $WA \cap WB = \emptyset$. We call a subset $T$ of $G$ **thick**, if the closure of $T$ in $\beta(G_d)$ contains a minimal left ideal. The construction is a generalization of a result in [12] for discrete groups.

**Theorem 70.** Let $G$ be a topological group with a symmetric neighbourhood $V$ of the identity satisfying the property $N$. Then there are $\kappa_V$-many pairwise disjoint thick subsets $\{T_\mu : \mu < \kappa_V\}$ such that $\{VT_\mu : \mu \in \kappa_V\}$ are also mutually disjoint.

**Corollary 11.** If a topological group $G$ satisfies property $N$, then $S(G)$ has disjoint left ideals and therefore $G$ is not uniquely amenable.

We have verified the property $N$ for the classes of locally compact groups, groups of density less than $\aleph_\omega$ (in particular all Polish groups), groups containing an open subgroup of infinite index (in particular discrete groups), topological vector spaces, and groups of isometries of homogeneous metric spaces of density $\kappa$ such that for every sequence $\langle \lambda_n : n \in \omega \rangle$ of cardinals smaller than $\kappa$, $|\bigcup_{n \in \omega} \lambda_n| < \kappa$ (in particular Urysohn spaces). Corollary 11 for locally compact groups, topological vector spaces and groups of density less than $\aleph_\omega$ also follows from a result of Jan Pachl, who showed in [52] that they satisfy a stronger property of ambitability (which is a notion of interest for harmonic analysis).

In Chapter 2, we introduce notions and facts from topological dynamics, model theory, Ramsey theory, and the Kechris-Pestov-Todorčević correspondence needed in the rest of the thesis. The last chapter is devoted to problems arising from our study.
Chapter 2

Preliminaries

2.1 Topological dynamics and ergodic theory

In general, dynamics studies continuous actions of topological groups on topological spaces, while ergodic theory studies measure preserving action of topological groups on measure spaces.

In the thesis, we focus on abstract topological dynamics studying continuous actions of topological groups on compact Hausdorff spaces.

2.1.1 Topological groups

A Group is a tuple \((G, \cdot, -1, e)\), where \(\cdot\) is a binary operation called multiplication (if \(G\) is commutative, we often use + instead and call + addition), \(-1\) is a unary operation called inversion and \(e\) is the identity element such that:

1. \(\cdot : G \times G \to G\) is associative: \(g \cdot (h \cdot k) = (g \cdot h) \cdot k\) for every \(g, h, k \in G\),
2. \(-1 : G \to G\) satisfies \(g \cdot g^{-1} = e = g^{-1} \cdot g\) for all \(g \in G\),
3. and for every \(g \in G\) it holds \(g \cdot e = e \cdot g = g\).

We often omit \(\cdot\) and write \(gh\) in place of \(g \cdot h\). For every \(g \in G\), we denote by \(L_g : G \to G, h \mapsto gh\) and \(R_g : G \to G, h \mapsto hg\) the left and right translations by \(g\) respectively.

Example 1. The following are infinite groups.

1. \((\mathbb{Z}, +, -, 0)\).
2. \((\mathbb{Q}, +, -, 0)\).
3. \((\mathbb{Q}, \cdot, -1, 1)\).
4. \((\mathbb{R}, +, -, 0)\).
5. \((\mathbb{R}, \cdot, -1, 1)\).
6. \((\mathbb{C}, +, -, 0)\).
7. \((\mathbb{C}, \cdot, -1, 1)\).
Topological group is a group equipped with a $T_1$ topology $\tau$ such that the multiplication and inversion are continuous. The topology $\tau$ is determined by a base $\mathcal{B}$ of open neighbourhoods of the identity element $e$. The collection of left (equivalently right) translates of elements from $\mathcal{B}$ forms a basis for $\tau$. Moreover, if a collection of subsets of a group containing $e$ satisfies certain properties, then it forms a base of neighbourhoods of $e$ for a group topology on $G$.

**Theorem 1** (Base at the identity). Let $G$ be a topological group and let $\mathcal{B}$ be an open base of the identity $e$ of $G$. Then $\mathcal{B}$ satisfies the following.

1. For every $V \in \mathcal{B}$ there is $W \in \mathcal{B}$ such that $W^2 \subset V$;
2. For every $V \in \mathcal{B}$ there is $W \in \mathcal{B}$ such that $W^{-1} \subset V$;
3. For every $V \in \mathcal{B}$ and every $g \in V$, there is $W \in \mathcal{B}$ such that $Wg \subset V$;
4. For every $V \in \mathcal{B}$ and every $g \in V$, there is $W \in \mathcal{B}$ such that $gW \subset V$;
5. For every $V \in \mathcal{B}$ and every $g \in G$ there is $W \in \mathcal{B}$ such that $gWg^{-1} \subset V$;
6. $\bigcap \mathcal{B} = \{e\}$.

Conversely, if $G$ is a topological group and $\mathcal{B}$ a family of subsets satisfying conditions (1) – (7), then the family $\{Vg : g \in G, V \in \mathcal{B}\}$ is a base for a $T_1$- topology on $G$ that turns $G$ into a topological groups. The family $\{gV : g \in G, V \in \mathcal{B}\}$ is a base for the same topology.

**Theorem 2.** If $G$ is a topological group equipped with a $T_1$-topology, then the topology is actually $T_3$.

Due to continuity of multiplication, left and right translations are homeomorphisms.

**Example 2** (Groups of homeomorphisms). An important class of topological groups is the class of groups of homeomorphisms of compact Hausdorff spaces. Let $X$ be a compact Hausdorff space. Then $\text{Homeo}(X)$, the set of all homeomorphisms of $X$ together with composition, inversion and the identity function as operations on $\text{Homeo}(X)$ is a group. Two common topologies turning $\text{Homeo}(X)$ into a topological group are the topology of point-wise convergence and the compact-open topology. The topology of pointwise convergence is the subspace topology of $X^X$ with the product topology. A basis for the compact-open topology $\tau$ on $\text{Homeo}(X)$ consists of sets $[K, O] = \{f \in \text{Homeo}(X) : f[K] \subset O\}$ for $K \subset X$ compact and $O \subset X$ non-empty open. In general, the topology of pointwise-convergence is coarser than the compact-open topology. They coincide for $X$ discrete. We will always consider $\text{Homeo}(X)$ with the compact-open topology.

**Example 3** (Polish groups). If $X$ is a compact separable space, then $\text{Homeo}(X)$ is an example of a Polish group, where a Polish group is a separable completely metrizable topological group. The group of all bijections on $\mathbb{Z}$ denoted by $S_\infty(\mathbb{Z})$ with the topology of pointwise convergence is a Polish group and groups of automorphisms of countable first-order structures are closed subgroups of $S_\infty(\mathbb{Z})$. Groups of automorphisms are in detail described later.

**Example 4** (Locally compact groups). Locally compact groups have many special properties and a number of articles on abstract topological dynamics only consider actions by locally compact groups. A group is locally compact if it is locally compact as a topological space. It is equivalent to having a base of the identity element consisting of open sets with compact closures. An important feature of locally compact groups is that they can be equipped with a natural left-invariant measure, so called Haar measure.
2.1.2 Uniform spaces

**Definition 1** (Uniform space). Let $X$ be a set and let $U$ and $V$ be covers of $X$. We say that $U$ **star refines** $V$ if for every $A \in U$ there is a $B \in V$ such that whenever $C \in U$ and $A \cap C \neq \emptyset$ then $C \subset B$.

A pair $(X, U)$ is a **uniform space** if $X$ is a set and $U$ is a collection of covers of $X$ satisfying the following properties

(a) $\{X\} \in U$.

(b) If $U, V \in U$ then there is $W \in U$ that star refines both $U$ and $V$.

(c) If $U \in U$ and $V$ is a cover such that $U$ star refines $V$ then $V \in U$.

A family $U$ of covers of $X$ satisfying the above properties is called a **uniformity** on $X$ and every cover in $U$ is called a **uniform cover**.

A uniformity is called **separated** if for every $x, y \in X$ there is a uniform cover $U$ and $A, B \in U$ such that $A \cap B = \emptyset$ and $x \in A$ and $y \in B$.

Every topological group naturally admits multiple uniform structures. For our purposes, we will always consider the right uniformity. If $G$ is a topological group and $N$ a basis of neighbourhoods of the indentity in $G$, then the **right uniformity** on $G$ is generated by covers

$$\{Vg : g \in G\}$$

for $V \in N$.

2.1.3 Flows

Now, we are ready to define dynamical systems. In the sense of abstract topological dynamics, a **dynamical system** is a continuous action of a topological group $G$ on a compact Hausdorff space $X$, it means a continuous map

$$\pi : G \times X \to X$$

such that for every $g, h \in G$ and $x \in X$ we have $\pi(gh, x) = \pi(g, \pi(h, x))$ and $\pi(e, x) = x$ for $e$ the identity in $G$. When the action $\pi$ is understood, we write $gx$ instead of $\pi(g, x)$ and we call $X$ a **$G$-flow** or just a **flow**.

For every $g \in G$, the map $(g, \cdot) : X \to X, x \mapsto gx$ is a homeomorphism. It is easy to prove that

$$G \to \text{Homeo}(X)$$

g \mapsto (g, \cdot)

is a continuous homomorphism. Conversely, every continuous homomorphism $G \to \text{Homeo}(X)$ defines a continuous action of $G$ on $X$.

If $X$ is a $G$-flow and $Y \subset X$ is a closed invariant subspace, then we call $Y$ a **subflow** of $X$.

**Example 5.** Every action of the discrete group $\mathbb{Z}$ on a compact Hausdorff space is given by a single homeomorphism $f : X \to X$ corresponding to $1 \in \mathbb{Z}$ and every $n \in \mathbb{Z}$ then corresponds to the $n$-th iterate $f^n$ of $f$. 
An orbit of a point \( x \) in a \( G \)-flow \( X \) is the set \( Gx = \{gx : g \in G\} \). Closures of orbit in a \( G \)-flow \( X \) are subflows of \( X \) and they are of major interest in topological dynamics.

**Definition 2** (Effective and free action). An action of a topological group \( G \) on a compact Hausdorff space \( X \) is called

(1) **effective** if for every \( g \in G \) there is \( x \in X \) such that \( gx \neq x \). In other words, the homeomorphism of \( X \) given by \( g \) is not the identity.

(2) **free** if for every \( g \in G \) and \( x \in X \) we have \( gx \neq x \). It means that all homeomorphisms of \( X \) given by elements of \( G \) are fixed-point-free.

**Theorem 3** (Veech [74];[34]). Every locally compact topological group admits a free action.

Opposite to groups admitting free actions are **extremely amenable** groups that fix a point under any action.

A continuous map \( \phi \) between \( G \)-flows \( X \) and \( Y \) is a **homomorphism** if for every \( x \in X \) and \( g \in G \) it holds that \( \phi(gx) = g\phi(x) \). We say that \( Y \) is a **quotient** of \( X \) if there is a continuous homomorphisms from \( X \) onto \( Y \). The flows \( X \) and \( Y \) are **isomorphic** if there is a homeomorphisms between \( X \) and \( Y \) which is a homomorphism.

### 2.1.4 Compact right-topological semigroups

Two prominent flows we will focus on, the greatest ambit and the universal minimal flow, have a structure of a compact right-topological semigroup.

A **semigroup** is a structure \((S, \cdot)\) where \( S \) is the underlying set and \( \cdot \) is a binary operation \( \cdot : S \times S \to S \) which is associative. A structure \((S, \cdot, e)\) is a **monoid** if \((S, \cdot)\) is a semigroup and \( e \) is a both-sided identity, it means \( es = se = s \) for every \( s \in S \). Obviously, every group is a monoid and every monoid is a semigroup. Every semigroup can easily be turned into a monoid by adding an extra element for the identity.

**Example 6.** The following are infinite semigroups which are not groups.

(1) \((\mathbb{N}, +)\) and it can be turned into a monoid by specifying \( 0 \) as the identity element;

(2) \((\mathbb{N}, \cdot)\) and it can be turned into a monoid by specifying \( 1 \) as the identity element;

(3) \((\mathbb{R}_+, +)\) and it can be turned into a monoid by adding \( 0 \) as the identity element;

(4) \((\mathbb{R}_+, \cdot)\) and it can be turned into a monoid by specifying \( 1 \) as the identity element;

(5) \((X^X, \circ)\) - all mappings from a set \( X \) to itself with the operation of composition. The semigroup \((X^X, \circ)\) can been turned into a monoid by specifying the identity map \( \text{Id} \) as the identity element;

(6) \((C(X), \circ)\) - all continuous mappings from a topological space \( X \) to itself with the operation of composition. The semigroup \((C(X), \circ)\) can been turned into a monoid by specifying the identity map \( \text{Id} \) as the identity element.
A topological semigroup is a semigroup \((S, \cdot)\) where \(S\) is a topological space and \(\cdot\) is a continuous map \(S \times S \to S\) where \(S \times S\) is taken with the product topology. Every topological group is thus a topological semigroup.

\((S, \cdot)\) is a right-topological semigroup if \(S\) is a topological space and right translations \(R_s : S \times S, r \mapsto rs\) are continuous for every \(s \in S\).

A subset \(I\) of a semigroup \(S\) is a left ideal if \(SI = I\), a right ideal if \(IS = I\) and an ideal if it is both left and right ideal. If \(I, J\) are left (right, both-sided) ideals, then so is \(I \cap J\). A left ideal \(L\) is a minimal left ideal if every non-empty minimal left ideal \(L'\) contained in \(L\) is equal to \(L\). If \(L\) is a minimal left ideal of \(S\), then for every \(l \in L\) it holds \(Sl = Ll = L\). Similarly, we define a minimal right ideal and a minimal ideal. If \(S\) is a compact right-topological semigroup and \(L\) is a minimal left ideal of \(S\), then \(L\) is closed. Indeed, if \(l \in L\), then \(R_l : S \to S\) is continuous with the image \(L\). It follows that \(L\) is compact and therefore closed in \(S\). If \(R\) is a minimal right ideal, then \(R\) need not be closed, but its closure remains a right ideal. Minimal ideal need not be closed either, but its closure is an ideal. Minimal left (right, both-sided) ideals need not exist in arbitrary semigroup, however they do in compact right-topological semigroups.

**Theorem 4** ([29]). If \(S\) is a compact right-topological semigroup, then \(S\) has a minimal both-sided ideal \(K(S)\). \(K(S)\) is the union of all minimal left (respectively right) ideals in \(S\).

Moreover, all left ideals are topologically the same.

**Theorem 5** ([29]). In a compact right-topological semigroup \(S\), minimal left ideals are closed and homeomorphic. In fact, if \(L\) and \(L'\) are minimal left ideals and \(s \in L'\), then \(R_s|L\) is a homeomorphism from \(L\) onto \(L'\).

An element \(s\) of a semigroup \(S\) is called an idempotent if \(ss = s\). Existence of idempotents in compact right-topological semigroups is of profound importance with vast applications in Ramsey theory. For instance, simpler proofs of van der Wardnen’s, Hales-Jewett’s and Hindman’s theorems by Furstenberg ([18]), or Gowers’ theorem ([24]).

**Lemma 1** (Namakura; Ellis). If \((S, \cdot)\) is a compact right-topological semigroup, then there exists an idempotent in \(S\).

For a semigroup \(S\), we denote by \(I(S)\) the set of all idempotents in \(S\). Let us introduce a partial ordering of the idempotents in \(S\). Let \(i, j \in I(S)\). We set

\[ i \leq j \iff ij = i \& ji = i. \]

We say that an idempotent is minimal if it is minimal with respect to \(\leq\). Minimal idempotents are exactly those contained in \(K(S)\).

**Theorem 6.** ([29]) Let \(S\) is a compact right-topological semigroup and let \(L\) and \(R\) be a minimal left and right ideal respectively. Then there is a unique idempotent \(i\) in \(H_i = L \cap R\) and \(H_i = Ri = iL = iSi\). \(H_i\) is the largest subgroup of \(S\) that contains \(i\) as the identity.

Subgroups \(H_i\) in the above theorem are maximal subgroups of \(K(S)\) and they form a partition of \(K(S)\), since they partition minimal left (right) ideals.
Theorem 7 ([29]). Let $S$ be a compact right-topological semigroup and let $i \in I(S)$. Then the following are equivalent.

1. $Si$ is a minimal left ideal.
2. $iSi$ is a group.
3. $iSi = H_i$.
4. $iS$ is a minimal right ideal.
5. $i$ is a minimal idempotent.
6. $i \in K(S)$.
7. $K(S) = SiS$.

Theorem 8 ([29]). Let $S$ be a compact right-topological semigroup. Each of \{Si : i \in I(K(S))\}, \{iS : s \in I(K(S))\} and \{iSi : i \in I(K(S))\} are partitions of $K(S)$ into minimal left ideals, minimal right ideals of $S$ and maximal subgroups of $K(S)$ respectively.

All maximal subgroups of $K(S)$ are isomorphic and they are both algebraically and topologically isomorphic as long as they belong to the same minimal right ideal.

Theorem 9 ([29]). Let $S$ be a compact right-topological semigroup and $i, j \in K(S)$. Let $g$ be the inverse of $iji$ in the group $iSi$. Then $\phi : iSi \rightarrow jSj$ defined by $\phi(s) = jsgj$ is an isomorphism between $iSi$ and $jSj$. If $i, j$ belong to the same right ideal, then $\phi$ is a moreover a homeomorphism.

We will be extending group actions to actions by compact right-topological semigroups. If $S$ is a right-topological semigroup, $X$ a compact Hausdorff space and $\pi : S \times X \rightarrow X$, then $\pi$ is an action of $S$ on $X$ if for every $s, t \in S$ and $x \in X$ we have $\pi(st, x) = \pi(s, \pi(t, x))$.

2.1.5 Minimal flows

When classifying points of a $G$-flow $X$, the “simplest type” is a fixed point, i.e. a point $x \in X$ such that $gx = x$ for all $g \in G$. When every $G$-flow has a fixed point, then we say that $G$ has the fixed point compacta property or that it is extremely amenable. The question to determine whether extremely amenable groups exist was asked by Mitchell in [45]. The first example of such a group was constructed by Herer and Christensen ([28]) in the setting of pathological submeasures and until Gromov and Milman showed in [27] that the group of unitary operators on a Hilbert space equipped with the pointwise convergence topology is extremely amenable it was not clear how “natural” the phenomenon is. In [34], Kechris, Pestov and Todorčević developed a general theory connecting finite Ramsey theory and dynamics of groups of automorphisms of locally-finite countable $\omega$-homogeneous structures producing many more examples of extremely amenable groups (see more below).

The second simplest type of points would be periodic points. However, the closure of the orbit of a periodic point is the orbit itself and therefore the classification of orbit closures is trivial. Moreover, not every dynamical system has periodic points.

A natural generalization is to relax the condition on the orbit to be closed which yields a notion of points that are close to being periodic. A point $x$ in a $G$-flow $X$ is almost periodic if for every
non-empty open subset $O$ of $G$ there are finitely many left translates of the set of returns of $x$ into $O$, $\text{Ret}(x, O) = \{ g \in G : gx \in O \}$, that cover $G$.

Orbit closures of almost periodic points in a $G$-flow $X$ correspond to minimal subflows of $X$. A flow is minimal if it has no nontrivial closed invariant subsets. Equivalently, a flow is minimal if every orbit is dense.

By compactness, every flow has a minimal subflow, which in turn means that every flow has an almost periodic point.

A minimal flow of particular interest is the universal minimal flow $M(G)$. The minimal flow is defined by its universality property with respect to all minimal flows - $M(G)$ is a minimal flow which has every minimal flow as its quotient. It means that for every minimal flow $M$, there is a quotient mapping $q : M(G) \longrightarrow M$ such that the following diagram commutes.

$$
\begin{array}{ccc}
G \times M(G) & \longrightarrow & M(G) \\
\downarrow \text{Id} \times q & & \downarrow q \\
G \times M & \longrightarrow & M
\end{array}
$$

**Theorem 10.** The universal minimal flow exists for every topological group $G$ and it is unique up to isomorphism.

It is easy to see that a topological group is extremely amenable if its universal minimal flow is a singleton.

### 2.1.6 The greatest ambit

In the study of the universal minimal flow of a topological group $G$, the notion of the greatest ambit, $S(G)$, proved to be useful. An ambit is a $G$-flow $X$ with a distinguished point $x_0$ with a dense orbit $Gx_0 = \{ gx_0 : g \in G \}$ in $X$. Note that a minimal flow with any point as the distinguished point is an ambit. The greatest ambit is a compactification of $G$ with the identity $e$ of $G$ as the distinguished point. Multiplication on $G$ extends to an ambit action $G \times (S(G), e) \longrightarrow S(G)$ and it is universal with respect to all other $G$-ambits. If $(X, x_0)$ is an ambit, then there is a continuous surjection $q : S(G) \longrightarrow X$ with $q(e) = x_0$ such that the following diagram commutes.

$$
\begin{array}{ccc}
G \times (S(G), e) & \longrightarrow & S(G) \\
\downarrow \text{Id} \times q & & \downarrow q \\
G \times (X, x_0) & \longrightarrow & X
\end{array}
$$

Considering $G$ with the right uniform structure, the greatest ambit is the smallest compactification of $G$ to which every uniformly continuous function from $G$ to a compact space extends. Multiplication on $G$ can be extended to multiplication on $S(G)$: Considering the ambit action $G \times S(G) \longrightarrow S(G)$, for
every $s \in S(G)$, the map $(\cdot, s) : G \rightarrow S(G)$ is uniformly continuous and therefore can be extended to $(\cdot, s) : S(G) \rightarrow S(G)$. This defines an extension of the multiplication on $G$ to $S(G)$ turning $S(G)$ into a semigroup. Since every right translation is continuous, $S(G)$ is a right-topological semigroup, which provides us with powerful algebraic and topological tools to study the structure of flows.

If $\pi : G \times X \rightarrow X$ is an action, then for every $x \in X$ the map $(\cdot, x) : G \rightarrow X, g \mapsto gx$ is uniformly continuous. Therefore, $(\cdot, x)$ can be extended to $(\cdot, x) : S(G) \rightarrow X$ defining an extension of $\pi$ to a semigroup action $S(G) \times X \rightarrow X$. By compactness of $S(G)$, the orbit closure $\overline{\pi(x)}$ is then equal to the orbit $S(G)x$ for every $x \in X$.

For a discrete group $G$, the greatest ambit is $\beta(G)$, the Čech-Stone compactification of $G$. Its points are all ultrafilters on $G$ and a basis for clopen sets consists of $A = \{u \in \beta(G), A \in u\}$ for $A \subset G$. The greatest ambit action of $G$ on $\beta(G)$ is given by

$$gu = \{gA : A \in u\}$$

for $g \in G$ and $u \in \beta(G)$. The multiplication extends to all of $\beta(G)$ via

$$uv = u - \lim \{gv : g \in G\}.$$

Combinatorially, $A \in uv$ if and only if

$$\{g : g^{-1}A \in v\} \in u.$$

The following theorem explains the connection between the greatest ambit and the universal minimal flow.

**Theorem 11.** The universal minimal flow is isomorphic to any minimal left ideal of $S(G)$. Consequently, the universal minimal flow is itself a compact right-topological semigroup.

Since $M(G)$ is a left ideal of $S(G)$, for every $m \in M(G)$, the right translation $R_m : S(G) \rightarrow M(G), s \mapsto sm$ has the image in $M(G)$. By minimality, the image of $R_m$ is actually equal to $M(G)$. Moreover, $R_m$ is a $G$ homomorphism from $S(G)$ to $M(G)$. The following theorem says that all homomorphisms from $S(G)$ to $M(G)$ are of this form and describes automorphisms of $M(G)$.

Recall that for a semigroup $S$, $I(S)$ denotes the set of idempotents in $S$.

**Theorem 12.** Let $S(G)$ be the greatest ambit of a topological group $G$ and let $M(G)$ be its minimal left ideal.

1. If $i \in I(M(G))$, then $R_i : S(G) \rightarrow S(G)$ is a retraction onto $M(G)$.
2. Every $G$-homomorphism $f : S(G) \rightarrow M(G)$ has the form $f(x) = R_y(x) = xy$ for some $y \in M(G)$.
3. For every $i \in I(M(G))$, the group $iM(G)$ is isomorphic to the group of $G$-automorphisms of $M(G)$.

We now introduce notions of dynamically large subsets of discrete groups - syndetic, piecewise syndetic and thick sets - and we provide simple facts about them.

We call a subset $A$ of a discrete group $G$ left syndetic if $G$ can be covered by finitely many left translates of $A$, i.e. there are $g_1, g_2, \ldots, g_n \in G$ such that $\bigcup_{i=1}^n g_i A = G$. Left syndetic sets characterize minimal flows in the following way.
Lemma 2. Let $X$ be a $G$-flow. The following are equivalent.

1. $X$ is minimal.

2. For every non-empty open subset $O$ of $X$ and every $x \in X$, the set $\text{Ret}(x, O) = \{ g \in G : gx \in O \}$ is left syndetic.

We can describe left syndetic sets in terms of $\beta(G)$.

Lemma 3. A subset $A$ of $G$ is left syndetic if and only if the closure of $A$ in $\beta(G)$ intersects every minimal left ideal of $\beta(G)$.

A subset $A$ of $G$ is called **piece-wise syndetic** if there exist finitely many elements $g_1, g_2, \ldots, g_n$ such that

$$\left\{ h(\bigcup_{i=1}^{n} g_i A) : h \in G \right\}$$

has the finite intersection property.

Lemma 4. A subset $A$ of $G$ is piece-wise syndetic if and only if the closure of $A$ in $\beta(G)$ intersects $K(\beta(G))$.

A subset $A$ of $G$ is called **thick** if the family

$$\{ gA : g \in G \}$$

has the finite intersection property.

We can immediately see the following.

Lemma 5. A $\subset G$ is piece-wise syndetic if and only if there are $g_1, g_2, \ldots, g_n$ such that $\bigcup_{i=1}^{n} g_i A$ is thick.

We can also recognize thick sets by looking at their closures in $\beta(G)$.

Lemma 6. A $\subset G$ is thick if and only if the closure of $A$ in $\beta(G)$ contains a minimal left ideal of $\beta(G)$.

The following lemma describes a relationship between thick, left syndetic and piece-wise syndetic sets.

Lemma 7. Let $A \subset G$.

1. $A$ is thick if and only if $A$ intersects every left syndetic subset of $G$.

2. $A$ is left syndetic if and only if $A$ intersects every thick subset of $G$.

3. $A$ is piece-wise syndetic if there are $S, T \subset G$ such that $A = S \cap T$ with $S$ left syndetic and $T$ thick.

A useful combinatorial characterization of thick sets is the following.

Lemma 8. $A \subset G$ is thick if and only if for every finite subset $F$ of $G$ there exists $g \in G$ such that $Fg \subset A$. 
2.2 Set theory

In this section we introduce set-theoretic notions important for our investigation. The language of set theory contains one non logical binary symbol $\in$ for containment. We typically work with axioms of ZFC, Zermelo-Fraenkel set theory with the axiom of choice.

Ordinals and cardinals

An **ordinal** is a transitive set well-ordered by $\in$. A set $x$ is **transitive** if for every $y \in x$ it holds that $y \subset x$ and $x$ is **well-ordered** by $\in$ if $x$ is totally-ordered by $\in$ and every subset $y$ of $x$ has an $\in$-minimal element. All ordinals form a proper class which is well-ordered by $\in$.

An ordinal $\alpha$ is a **successor ordinal** if $\alpha = \beta + 1$ for some ordinal $\beta$. Otherwise, $\alpha$ is a **limit ordinal**.

A subset $A$ of an ordinal $\alpha$ is a **cofinal sequence** in $\alpha$ if for every $\beta \in \alpha$ there is $\gamma \in A$ such that $\beta \leq \gamma$.

If $x, y$ are sets, $x$ has cardinality less than or equal to the cardinality of $y$ if there is an injection from $x$ to $y$. An ordinal is a **cardinal** if every smaller ordinal has smaller cardinality. If $x$ is a set and $\kappa$ is a cardinal, we say that $x$ has **cardinality** $\kappa$ if there exists a bijection between $x$ and $\kappa$. As a consequence of the axiom of choice, this uniquely defines a cardinality for every set. For every cardinal $\kappa$, we denote by $\kappa^+$ the least cardinal greater than $\kappa$. For an ordinal $\alpha$ and a sequence of cardinals $\langle \lambda_\mu : \mu < \alpha \rangle$,

$$\sum_{\mu < \alpha} \lambda_\mu$$

denotes the cardinality of $\bigcup_{\mu < \alpha} (\{\mu\} \times \lambda_\mu)$,

the disjoint union of cardinals $\lambda_\mu$, and

$$\prod_{\mu < \alpha} \lambda_\mu$$

denotes the cardinality of $\prod_{\mu < \alpha} \lambda_\mu$,

the Cartesian product of the cardinals $\lambda_\mu$. For two cardinals $\lambda \leq \kappa$, we denote by $[\kappa]^{< \lambda}$ the family of all subsets of $\kappa$ of cardinality less than $\lambda$. For a cardinal $\kappa$, $2^\kappa$ is the cardinality of the power set of $\kappa$.

**Theorem 13** (Cantor). For a set $A$, the power set $\mathcal{P}(A)$ has cardinality larger than $A$.

**Cofinality** of a cardinal $\kappa$ is the least cardinality of a cofinal sequence in $\kappa$. A cardinal $\kappa$ is called **regular** if it is equal to its cofinality, otherwise $\kappa$ is **singular**.

Recursively, we define an increasing enumeration of infinite cardinals:

$$\aleph_0 = \omega$$

$$\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$$

$$\aleph_\gamma = \omega_\gamma = \bigcup\langle \aleph_\alpha : \alpha < \gamma \rangle$$ for $\gamma$ a limit ordinal

We usually use $\aleph_\alpha$ to denote the cardinal and $\omega_\alpha$ the corresponding ordinal.

**Continuum Hypothesis** (CH) is the statement that $2^{\aleph_0} = \aleph_1$. CH was proved to be consistent with the axioms of ZFC by Kurt Gödel, while the negation of CH was proved to be consistent with the axioms of ZFC by Paul Cohen. Therefore, CH is independent of the axioms of ZFC.
Generalized Continuum Hypothesis (GCH) is the statement that for every infinite cardinal $\kappa$, $2^\kappa = \kappa^+$, equivalently $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal $\alpha$. GCH is again consistent with ZFC by a result of Gödel, while it consistently fails in Cohen’s model. Moreover, Easton proved that GCH can fail at any instance, it means that arbitrarily large cardinal $\aleph_\alpha$ can satisfy $\aleph_\alpha^+ < 2^{\aleph_\alpha}$.

Boolean algebras, filters and ideals

**Definition 3** (Prime ideal). Let $X$ be a set and let $\mathcal{I}$ be a collection of subsets of $X$. We say that $\mathcal{I}$ is an **ideal** on $X$ if $\mathcal{I}$ is downwards closed and closed under finite unions:

1. If $A \in \mathcal{I}$ and $B \subset A$ then also $B \in \mathcal{I}$,
2. if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

We call $\mathcal{I}$ a **prime ideal** if it moreover satisfies a third condition:

3. If $A, B \subset X$ and $A \cap B \in \mathcal{I}$ then either $A \in \mathcal{I}$ or $B \in \mathcal{I}$.

We say that an ideal is **non-trivial**, if $X \not\in \mathcal{I}$.

In what follows, we only consider non-trivial ideals.

**Example 7.** (1) The collection of all finite subsets of a set $X$ is an ideal on $X$.

2. If $X$ is a measure space, then the sets of measure $0$ form an ideal.

3. If $X$ is a topological space, then the collection of all meager subsets is an ideal.

A dual notion to an ideal is the notion of a filter.

**Definition 4** (Ultrafilter). A collection $\mathcal{F}$ of subsets of a set $X$ is called a **filter** if it is closed under supersets and finite intersections:

1. If $A \in \mathcal{F}$ and $A \subset B$ then also $B \in \mathcal{F}$,
2. if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

We call $\mathcal{F}$ an **ultrafilter** if it is a maximal filter with respect to inclusion. Equivalently if it moreover satisfies a third condition:

3. For every $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

A filter $\mathcal{F}$ is called **non-trivial** if $\emptyset \notin \mathcal{F}$.

In this paper, we only consider non-trivial filters.

It is easy to see that if $\mathcal{I}$ is an ideal then $\mathcal{F} = \{ X \setminus A : A \in \mathcal{I} \}$ is a filter and vice versa. We say that $\mathcal{F}$ is the dual filter to the ideal $\mathcal{I}$ and $\mathcal{I}$ is the dual ideal to $\mathcal{F}$. The dual filter to a prime filter is an ultrafilter.

**Example 8.** Filters dual to the ideals in Example 7 are the filter of cofinite sets, the filter of sets of measure $1$ and the filter of subsets of the second category respectively.
Let us recall the definition of the Boolean algebra $P(\omega_1)/\text{fin}$ and similar algebras. If $X$ is a set and $P(X)$ denotes its power set, then $P(X)$ with the operations of $\cup, \cap$, complementation, $\emptyset$ and $X$ is a Boolean algebra. If $\lambda$ is at most the cardinality of $X$ then the set $\mathcal{I}$ of all subsets of $X$ of cardinality less than $\kappa \leq \lambda$ is an ideal on $P(X)$. The quotient $P(X)/\mathcal{I}$ of $P(X)$ by the ideal $\mathcal{I}$ consists of equivalence classes $[E] = \{F \subset X : \Delta(E, F) < \kappa\}$ for $E \subset X$ and $\Delta(E, F)$ the symmetric difference of $E$ and $F$. It is easy to see that the operations on $P(X)$ preserve the equivalence classes, and hence define a structure of a Boolean algebra on $P(X)/\mathcal{I}$. We usually write $P(X)/\text{fin}$ to denote the quotient algebra of the power set algebra on $X$ modulo the ideal of finite sets.

Below we give an abstract definition of a Boolean algebra, ideal and a quotient algebra.

**Definition 5** (Boolean algebra). $B = (B, \lor, \land, 0, 1, \neg)$ is a Boolean algebra if $\lor, \land$ are binary operations called join and meet respectively, $\neg$ is a unary operation called complementation and $0, 1$ are constants such that for all $a, b, c \in B$ the following axioms hold:

- $(a \lor b) \lor c = a \lor (b \lor c)$ and $(a \land b) \land c = a \land (b \land c)$ - associativity,
- $a \lor b = b \lor a$ and $a \land b = b \land a$ - commutativity,
- $a \lor 0 = a$ and $a \land 1 = a$ - identity,
- $a \lor (b \land c) = (a \lor b) \land (a \lor c)$ and $a \land (b \lor c) = (a \land b) \lor (a \land c)$ - distributivity,
- $a \lor \neg a = 1$ and $a \land \neg a = 0$ - complements.

**Example 9.** If $X$ is a topological space, the set of all Borel subsets of $X$ with the operations $\cup, \cap$, complementation, $X$ and $\emptyset$ is a Boolean subalgebra of $P(X)$.

There is a canonical partial ordering $<$ on a Boolean algebra $B$, given by $a < b$ if and only if $a \land b = a$ (if and only if $a \lor b = b$.)

An atom of a Boolean algebra $B$ is a non-zero element $a \in B$ such that $0$ is the only element $<$-below $a$. A Boolean algebra is called atomless if it does not have any atoms. Every finite Boolean algebra has atoms. There is exactly one countable atomless Boolean algebra up to an isomorphism - the algebra of all clopen subsets of the Cantor set.

An ideal on a Boolean algebra is defined similarly as an ideal on a set. An ideal on a Boolean algebra $B$ is a subset of $B$ closed under finite meets and downwards closed under the canonical partial order.

If $B$ is a Boolean algebra and $\mathcal{I} \subset B$ an ideal on $B$, then $B/\mathcal{I}$ denotes the quotient algebra of $B$ by $\mathcal{I}$ consisting of equivalence classes

$$[a] = \{b \in B : (a \land \neg b) \lor (\neg a \land b) \in \mathcal{I}\}$$

for $a \in B$. The operations on $B/\mathcal{I}$ are inherited from $B$. Note that $(a \land \neg b) \lor (\neg a \land b)$ corresponds to the symmetric difference if $B$ is a power set algebra.

The category of Boolean algebras with homomorphisms is equivalent to the category of compact Hausdorff zero-dimensional spaces with continuous mappings via the Stone duality.

**Stone representation theorem**

In 1936, M.H. Stone [67] proved that every Boolean algebra $B$ is isomorphic to the Boolean algebra of all clopen subsets of a compact Hausdorff zero-dimensional space $\text{St}(B)$. The points of $\text{St}(B)$ are all
ultrafilters on \( \mathcal{B} \) and the sets \( \mathcal{A} = \{ u \in \text{St}(\mathcal{B}) : A \in u \} \) for \( A \in \mathcal{B} \) form a clopen base of the topology on \( \text{St}(\mathcal{B}) \). This gives us a one-to-one correspondence that also extends to homomorphisms: If \( f : \mathcal{B} \rightarrow \mathcal{C} \) is a homomorphism between two Boolean algebras, then \( \text{St}(f) : \text{St}(\mathcal{C}) \rightarrow \text{St}(\mathcal{B}) \) given by

\[
  u \mapsto \text{“the ultrafilter on } \mathcal{B} \text{ generated by } \{ f^{-1}(A) : A \in u \} \text{”}
\]

is a continuous map. If \( f \) is injective, then \( \text{St}(f) \) is surjective. In terms of category theory, \( \text{St} \) is a contravariant functor which is an equivalence between the category of Boolean algebras with homomorphisms and the category of compact Hausdorff zero-dimensional spaces with continuous mappings.

**Example 10.**

1. The Čech-Stone compactification \( \beta \omega \) of discrete \( \omega \) is the Stone space of \( \mathcal{P}(\omega) \).
2. The Čech-Stone remainder \( \omega^* = \beta \omega \setminus \omega \) is the Stone representation of the Boolean algebra \( \mathcal{P}(\omega)/\text{fin} \).
3. The Stone space of the countable atomless Boolean algebra is the Cantor space.

### Cantor cubes and free Boolean algebras

**Definition 6.** Let \( \mathcal{A} \) be a Boolean algebra and \( X \subset \mathcal{A} \). We say that \( \mathcal{A} \) is **free** over \( X \) if every mapping of \( X \) into a Boolean algebra extends to an homomorphism. We call \( X \) a set of **generators** of \( \mathcal{A} \).

It is a well-known fact, that if a free Boolean algebra \( \mathcal{A} \) has a set of generators of cardinality \( \kappa \) then every set of generators for \( \mathcal{A} \) has cardinality \( \kappa \). Moreover, for every infinite \( \kappa \) there is exactly one (up to isomorphisms) free Boolean algebra with \( \kappa \)-many generators. We can obtain the free Boolean algebras by the standard construction of a free object in the variety of Boolean algebras known from universal algebra. However, free Boolean algebras have an easy interpretation as algebras of clopen subsets of Cantor cubes \( \{0,1\}^\kappa \) for arbitrary infinite \( \kappa \). In other words, free Boolean algebras are in Stone duality with Cantor cubes.

For more information on free Boolean algebras, see [38].

### 2.3 First order structures and their automorphism groups

By a **structure** we mean a structure \( \mathcal{A} = (A, f_k^A, R_i^A, c_k^A) \) in some first order language \( L = (f_i, R_j, c_k) \). By \( \text{Age}(\mathcal{A}) \) we denote the class of finitely-generated substructure of \( \mathcal{A} \). For a substructure \( \mathcal{A} \) of a structure \( \mathcal{A} \), the binomial \( (\mathcal{A})^k \) denotes the collection of all copies of \( A \) in \( \mathcal{A} \).

Among the structures we consider, there are graphs, hypergraphs and posets. By a **graph** we mean an unordered graph. A **\( K_n \)-free graph** is a graph that does not contain the complete graph on \( n \)-vertices as an induced subgraph. A **hypergraph** \( H \) is a structure in a finite signature \( L = \{ R_i : i < k \} \) of relational symbols, where each \( R_i \) is closed under permutations, i.e. if \( R_i \) has arity \( l \) and \( \sigma \) is a permutation on \( \{0,1,\ldots,l-1\} \), then \( (h_0, h_1, \ldots, h_{l-1}) \in R_i^H \subset H^l \) implies that all \( h_0, h_2, \ldots, h_{l-1} \) are distinct and \( (h_{\sigma(0)}, h_{\sigma(1)}, \ldots, h_{\sigma(l-1)}) \in R_i^H \). In other words, we can think of \( R_i^H \) as a collection of subsets of \( H \) of size \( l \). A hypergraph \( H \) is called **irreducible** if it contains at least two elements and whenever \( x \neq y \) in \( H \) then there exists \( i < k \) and \( S \in R_i^H \) such that \( \{x, y\} \subset S \). Let \( \mathcal{A} \) be a class of irreducible hypergraphs in signature \( L \). A hypergraph \( \mathcal{H} \) is \( \mathcal{A} \)-**free** if no element of \( \mathcal{A} \) can be embedded into \( \mathcal{H} \). By a **poset**, we mean a partially ordered set.
Let \( A \) be a structure and let \( G \) be its group of automorphisms. Considering \( A \) as a discrete space, we can equip \( G \) with the topology of pointwise convergence turning it into a topological group. Since \( A \) is discrete, the topology of pointwise convergence coincides with the compact-open topology. A base of neighbourhoods of the identity consists of open (hence clopen) subgroups

\[
G_A = \{ g \in G : ga = g \quad \forall a \in A \}
\]

for \( A \) a finitely-generated substructure of \( A \).

A structure \( A \) is \textbf{locally-finite} if finitely-generated substructures of \( A \) are finite, i.e. \( \text{Age}(A) \) consists of finite structures. All relational structures are locally-finite.

**Example 11.** (1) Gromov and Milman showed in [27] that the group of unitary operators on a Hilbert space \( U(l^2) \) equipped with the pointwise convergence topology It means that \( U(l^2) \) is extremely amenable.

(2) Glasner and Weiss proved in [22] that the universal minimal flow of the group \( S_\infty(Z) \) of all bijections of \( Z \) is the space \( LO(Z) \) of all linear orderings on \( Z \) with the following action

\[
(g \in S_\infty(Z), < \in LO(Z), m, n \in Z) \mapsto (m(g <)n \leftrightarrow g^{-1}m < g^{-1}n).
\]

Here we consider \( LO(Z) \) as a closed subspace of the compact space \( 2^{\mathbb{Z}^2} \) with the product topology.

The flow in Example 11(2) is a special case of the so-called \textbf{logic action} that turned out to be very useful in applications.

Let \( A \) be a structure in a language \( L \) and let \( G \) be the automorphism group of \( A \). Let \( L^* \) be an expansion of \( L \) by relations, it means \( L^* \setminus L = \{ S_i : i \in I \} \) consists solely of relational symbol of finite arities \( a_i \) for \( i \in I \). A structure \( A^* \) in \( L^* \) is an \textbf{expansion} of \( A \) if \( A^*|L = A \) and \( A \) is a \textbf{reduct} of \( A^* \).

Let \( A \) be the universe of \( A \). Any interpretation \( S_i^A \) of \( S_i \) in \( A \) can be viewed as a point in \( \{0, 1\}^{a_i} = 2^{a_i} \), so an interpretation of all \( S_i \)'s is a point in

\[
\prod_{i \in I} 2^{a_i}.
\]

Viewing \( \prod_{i \in I} 2^{a_i} \) with the product topology with \( 2 = \{0, 1\} \) discrete, we can turn \( \prod_{i \in I} 2^{a_i} \) into a \( G \)-flow via the following action:

\[
\phi \in G, \ x \in \prod_{i \in I} 2^{a_i} \mapsto (\phi(x)(i)(y_1, y_2, \ldots, y_{a_i}) = 1 \leftrightarrow x(i)(\phi^{-1}y_1, \phi^{-1}(y_2), \ldots, \phi^{-1}(y_{a_i})) = 1).
\]

In this terminology, the flow \( LO(Z) \) in Example 11(2) is a subflow of \( 2^{\mathbb{Z}^2} \). By minimality of \( LO(Z) \), the orbit \( S_\infty(Z) \cdot < \) is dense in \( LO(Z) \) for every \( < \in LO(Z) \).

Next theorem shows that \( S_\infty(Z) \) is universal for groups of automorphisms of countable structures. We will see in Chapter 4 that the same holds for structures with an uncountable underlying set.

**Theorem 14** ([9]). Let \( G \) be an infinite topological group. Then the following are equivalent.

(1) \( G \) is a closed subgroup of \( S_\infty(Z) \).

(2) \( G \) is a group of automorphisms of a countable structure.
Chapter 2. Preliminaries

(3) $G$ is a group of automorphisms of a countable $\omega$-homogeneous structure.

(4) $G$ has a countable base of the identity of open subgroups of countable index.

Therefore when studying groups of automorphisms of structure, we can restrict our attention only to “nice” structures - $\omega$-homogeneous structures. A structure $\mathcal{A}$ is called $\omega$-homogeneous if every partial isomorphism between finitely-generated substructures of $\mathcal{A}$ can be extended to a full automorphism of $\mathcal{A}$. If $\mathcal{A}$ is an $\omega$-homogeneous Boolean algebra, we often call $\mathcal{A}$ just homogeneous. The following two sections provide recipes for constructing $\omega$-homogeneous structures.

2.3.1 Fraïssé theory

In [16], Fraïssé established a one-to-one correspondence between countable $\omega$-homogeneous structures and countable classes of finitely-generated structures closed under substructures, amalgamations and satisfying the joint embedding property.

A class of finitely-generated structures $\mathcal{F}$ of a given language is called a Fraïssé class, if it satisfies the following conditions:

(HP) Hereditary property: if $A$ is a finitely-generated substructure of $B$ and $B \in \mathcal{F}$, then also $A \in \mathcal{F}$.

(JEP) Joint embedding property: if $A, B \in \mathcal{F}$ then there exists a $C \in \mathcal{F}$ in which both $A$ and $B$ embed.

(AP) Amalgamation property: if $A, B, C \in \mathcal{F}$ and $i : A \rightarrow B$ and $j : A \rightarrow C$ are embeddings, then there exist $D \in \mathcal{F}$ and embeddings $k : B \rightarrow D$ and $l : C \rightarrow D$ such that $k \circ i = l \circ j$.

Remark 1. The original definition of a Fraïssé class contains an additional condition on countability. We relax this condition and call classes of structures satisfying (HP), (JEP) and (AP) Fraïssé classes even if they are uncountable.

For every countable Fraïssé class $\mathcal{F}$ there is a unique countable $\omega$-homogeneous structure, so called Fraïssé limit of $\mathcal{F}$, that contains a copy of each structure in $\mathcal{F}$ as a substructure. Conversely, if $\mathcal{A}$ is a countable $\omega$-homogeneous structure, then isomorphism types of structures in $\text{Age}(\mathcal{A})$ form a countable Fraïssé class.

Example 12. The following are locally-finite Fraïssé limits of Fraïssé classes of their finite substructures:

(a) A countably infinite set is a Fraïssé limit of the class of finite sets.

(b) Rado graph is a Fraïssé limit of the class of finite graphs.

(c) The countable atomless Boolean algebra is a Fraïssé limit of the class of finite Boolean algebras.

(d) A countably infinite-dimensional vector space over a finite field $F$ is a Fraïssé limit of the class of finite vectors spaces over $F$.

(e) The countable dense linear order without endpoints, $\langle \mathbb{Q}, < \rangle$ a Fraïssé limit of the class of finite linear orderings.

(f) The random poset is a Fraïssé limit of the class of finite posets.
2.3.2 Jónsson structures

In [32] and [31], Jónsson generalized Fraïssé’s construction ([16]) to uncountable structures of cardinality \( \kappa \), whenever \( \kappa^{<\kappa} = \kappa \). In the first article, [32], Jónsson was looking for conditions on a class \( \mathcal{K} \) of relational structures that would give rise to a universal structure for \( \mathcal{K} \). In the second article, [32], he used the amalgamation property by Fraïssé to answer in positive a question of R. Baer whether the universal structures that would give rise to a universal structure for \( \mathcal{K} \). In [32] and [31], Jónsson generalized Fraïssé’s construction ([16]) to uncountable structures of cardinality \( \kappa \).

Chapter 2. Preliminaries

28

Let us make precise what we mean by a universal and a homogeneous structure.

**Definition 7** (Universality; [32]). Let \( \mathcal{K} \) be a class of relational structures and let \( \alpha \) be an ordinal. We say that a structure \( A \in \mathcal{K} \) is \((\aleph_\alpha, \mathcal{K})\)-**universal** if \( A \) has cardinality \( \aleph_\alpha \) and every \( B \in \mathcal{K} \) of cardinality \( \leq \aleph_\alpha \) can be embedded into \( A \).

**Definition 8** (Homogeneity; [31]). Let \( \mathcal{K} \) be a class of relational structures and let \( \alpha \) be an ordinal. We say that a structure \( A \) is \((\aleph_\alpha, \mathcal{K})\)-**homogeneous** if \( A \in \mathcal{K} \), the cardinality of \( A \) is \( \aleph_\alpha \) and the following condition is satisfied. If \( B \) is a substructure of \( A \) in \( \mathcal{K} \) and \( B \) has cardinality smaller than \( \aleph_\alpha \), then every embedding of \( B \) into \( A \) can be extended to an automorphism of \( A \).

The following proposition gives us a tool to check homogeneity of structures. The proof goes by a back-and-forth argument.

**Proposition 1** (Extension property). Let \( \mathcal{K} \) be a class of structures and let \( A \) be a structure of cardinality \( \aleph_\alpha \) for some ordinal \( \alpha \). Then \( A \) is \((\aleph_\alpha, \mathcal{K})\)-homogeneous if and only if \( A \in \mathcal{K} \) and for every \( B, C \in \mathcal{K} \) of cardinalities less than \( \aleph_\alpha \) and embeddings \( i : B \to A \) and \( j : B \to C \), there exists and embedding \( k : C \to A \) such that \( i = k \circ j \).

The main results of [32] and [31] are the existence and uniqueness of a universal homogeneous structure for a Jónsson class of cardinality \( \aleph_\alpha \) whenever \( \aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha \).
Theorem 15 ([31]). Let $\alpha$ be a positive ordinal with the following two properties:

(i) If $\lambda < \omega_\alpha$ and if $n_\nu < \aleph_\alpha$ whenever $\nu < \lambda$, then $\sum_{\nu<\lambda} n_\nu < \aleph_\alpha$.
(ii) If $n < \aleph_\alpha$, then $2^n \leq \aleph_\alpha$.

If $\mathcal{K}$ is a Jónsson class for $\alpha$, then there exists a unique $(\aleph_\alpha, \mathcal{K})$-universal homogeneous structure.

Nowadays, we abbreviate the conditions (i) and (ii) in the previous theorem as

$$\aleph_\alpha < \aleph_\alpha = \aleph_\alpha$$

and say that $\aleph_\alpha$ to the weak power $\aleph_\alpha$ is equal to $\aleph_\alpha$.

An assumption on cardinality is essential; for instance, it is consistent both that there exists and that there does not exist a universal graph of cardinality $\aleph_1$ (see [61]).

Corollary 1 ([31]). If the General Continuum Hypothesis holds and $\mathcal{K}$ is a Jónsson class for every positive ordinal $\alpha$, then there is an $(\aleph_\alpha, \mathcal{K})$-universal homogeneous structure for every positive $\alpha$.

2.4 Ramsey theory

The original Ramsey’s Theorem was proved by F.P. Ramsey in 1928 to classify binary relations.

Theorem 16 (Ramsey’s Theorem). For every $k < n \in \mathbb{N}$ and $l \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every colouring of the $k$-element subsets of $N$ by $l$-many colours there is an $n$-element subset $X$ of $N$ such that all $k$-element subsets of $X$ have the same colour.

In the 1970’s and 1980’s, the Ramsey property for structures was first studied.

Definition 9 (Ramsey property). A class $\mathcal{K}$ of finite structures satisfies the Ramsey property if for every $A \leq B \in \mathcal{K}$ and $k \geq 2$ a natural number there exists $C \in \mathcal{K}$ such that

$$C \rightarrow (B)^A_k,$$

i.e. for every colouring of the copies of $A$ in $C$ by $k$ colours, there is a copy $B'$ of $B$ in $C$, such that all the copies of $A$ in $B'$ have the same colour. We then say that $B'$ is monochromatic and we call $\mathcal{K}$ a Ramsey class.

Example 13. (1) The class of finite sets and the class of finite linear orders satisfy the Ramsey theorem by Ramsey’s theorem.

(2) The class of finite linearly ordered graphs was proved to be a Ramsey class by Nešetřil and Rödl in [48].

(3) The class of finite linearly ordered hypergraphs was proved to be a Ramsey class also by Nešetřil and Rödl in [48] and independently by Abramson and Harrington in [1].

(4) The class of finite Boolean algebras is a Ramsey class by the Dual Ramsey Theorem of Graham and Rothschild [26].
(5) The class of finite vector spaces over a finite field was proved to be a Ramsey class by Graham, Leeb, Rothschild in [25].

(6) Sokić showed in [64] that the class of finite posets with linear orderings extending the partial order is a Ramsey class.

(7) The class of finite linearly ordered metric spaces was recently shown to be a Ramsey class by Nešetřil in [47].

The class of finite graphs itself is not a Ramsey class. However every finite graph has a finite Ramsey degree in the class of all finite graphs. Therefore the class of all finite graphs has an expansion to a Ramsey class. In general, this is often the case and aside from proving the Ramsey property, the difficulty also lies in looking for appropriate expansion.

**Definition 10 (Ramsey degree).** The arrow notation

\[ C \to (B)_{k,l}^A \]

means that for every colouring \( c \) of the copies of \( A \) in \( C \) by finitely many colours, there exists a copy \( B' \) of \( B \) in \( C \) such that the copies of \( A \) in \( B' \) take on at most \( l \)-many colours.

We say that \( A \in \mathcal{K} \) has a finite Ramsey degree in \( \mathcal{K} \) if there exists \( l \in \mathbb{N} \) such that for every \( k \in \mathbb{N} \) and every \( B \in \mathcal{K} \) there exists \( C \in \mathcal{K} \) such that

\[ C \to (B)_{k,l}^A. \]

The smallest such \( l \) is called the **Ramsey degree** of \( A \) in \( \mathcal{K} \).

### 2.5 Kečkiš-Pestov-Todorčević correspondence and precompact expansions

Kečkiš, Pestov and Todorčević generalized in [34] Pestov’s result that the group of automorphisms of \((\mathbb{Q},<)\) is extremely amenable to many other linearly ordered structures. They characterized extremely amenable groups of automorphisms of countable locally-finite \( \omega \)-homogeneous structures by the Ramsey property for finite substructures. A class \( \mathcal{K}_< \) of finite structures is called a **Fraïssé order class** if it is a Fraïssé class in a language containing a binary symbol \(<\) interpreted as a linear order in each structure in \( \mathcal{K}_< \). Let \( \mathcal{K} \) be a class of structures in a language \( L \) and let \( L^* = L \cup \{<\} \), where \(<\) is a binary symbol not contained in \( L \). Let \( \mathcal{K}_< \) be a class of \( L^* \)-structures with \(<\) interpreted as a linear order. We say that \( \mathcal{K}_< \) is an **order expansion** of \( \mathcal{K} \) if \( \mathcal{K}_<|L = \mathcal{K} \). If \( \mathcal{A} \) is a structure with \( \text{Age}(\mathcal{A}) = \mathcal{K} \) and \(<\) is a linear ordering on \( \mathcal{A} \), we say that \(<\) is a **normal ordering** with respect to \( \mathcal{K}_< \) if \( \text{Age}(\mathcal{A},<) \subseteq \mathcal{K}_< \). We denote the space of normal orderings on \( \mathcal{A} \) with respect to \( \mathcal{K}_< \) by \( \text{NO}_{\mathcal{K}_<}(\mathcal{A}) \).

**Theorem 17 ([34]).** Let \( G \) be a closed subgroup of \( S_\infty \). The following are equivalent.

1. \( G \) is extremely amenable.

2. \( G \) is a group of automorphisms of a Fraïssé limit of a Fraïssé order class satisfying the Ramsey property.
Remark 2. As we will see later, the condition on being a Fraïssé order class in Theorem 17 (2) can be replaced by requiring the Fraïssé class to consist of rigid elements, it means structures with trivial automorphism groups. The requirement of rigidity is however essential.

As a corollary of Theorem 17, we obtain the following examples of extremely amenable groups.

Corollary 2. Groups of automorphisms of the following structures are extremely amenable.

1. \((\mathbb{Q}, <)\) (Pestov, [57]).

2. \(\omega\)-homogeneous linearly ordered random graph ([34]).

3. \(\omega\)-homogeneous normally ordered countable atomless Boolean algebra with respect to natural orderings. ([22]; [34]).

4. \(\omega\)-homogeneous normally ordered \(\aleph_0\)-dimensional vector space over a finite field with respect to natural orderings. ([34]).

5. \(\omega\)-homogeneous poset with a linear ordering extending the partial order ([64]).

As we saw in Example 13, the class of finite Boolean algebras and the class of finite vector spaces over finite fields are Ramsey classes which are not order classes and groups of automorphisms of their Fraïssé limits are not extremely amenable. However, they both admit an order expansion to a Fraïssé order class with the Ramsey property via so-called natural orderings.

Definition 11 ([34]).

(BA) We call a linear ordering on a finite Boolean algebra natural, if it is an antilexicographical extension of an arbitrary linear ordering on its atoms.

(VS) We call a linear ordering on a finite vector space over a finite field \(G\) natural, if it is an antilexicographical extension of a linear ordering on a basis relative to a fixed linear ordering of the field \(G\) (see [68]).

The class of naturally-ordered finite Boolean algebras was proved to be a Fraïssé class in [34]. The class of naturally-ordered finite vector spaces over a finite field is a Fraïssé class by a result of Thomas ([68]). To deduce that both classes satisfy the Ramsey property, we need the following definition from [34]. We say that a linear expansion \(K_\prec\) of a class \(K\) of structures is order-forgetful whenever

\[(A, <), (B, <) \in K \text{ and } A \cong B \implies (A, <) \cong (B, <).\]  

Proposition 2 ([34]). Let \(K_\prec\) be an order-forgetful linear expansion of a class of finite structures \(K\) which is hereditary. Then the following are equivalent:

1. \(K\) satisfies the Ramsey property.

2. \(K_\prec\) satisfies the Ramsey property.

Classes of finite naturally-ordered Boolean algebras and finite naturally-ordered vector spaces over a finite field are obviously order-forgetful linear expansions of classes of finite Boolean algebras and finite vector spaces over a finite field respectively. Hence they are Ramsey classes since the latter two are by [26] and [25] respectively.
Using extreme amenability of the groups of automorphisms of \(\omega\)-homogeneous linearly ordered structures, the authors computed universal minimal flows of groups of automorphisms of \(\omega\)-homogeneous structures without linear orderings provided certain coherence properties are satisfied.

Let \(\mathcal{K}\) be a class of finite structure and let \(\mathcal{K}_<\) be its linear expansion. We say that \(\mathcal{K}_<\) satisfies the **ordering property** if for every \(A \in \mathcal{K}\), there is \(B \in \mathcal{K}\) such that whenever \(\prec\) is a linear ordering on \(A\), \(\prec'\) is a linear ordering on \(B\) and \((A, \prec), (B, \prec') \in \mathcal{K}\), then \((A, \prec)\) is a substructure of \((B, \prec')\). We say that \(\mathcal{K}_<\) is **reasonable** if for every \(A, B \in \mathcal{K}\), embedding \(\pi: A \rightarrow B\), and a linear ordering \(\prec\) on \(A\) with \((A, \prec) \in \mathcal{K}_<\), there is a linear ordering \(\prec'\) on \(B\) with \((B, \prec') \in \mathcal{K}_<\) such that \(\pi: (A, \prec) \rightarrow (B, \prec')\) remains an embedding.

An important feature of countable structures used in [34] is the uniqueness of countable \(\omega\)-homogeneous structures which insures that the reduct of the Fraïssé limit of a reasonable Fraïssé order class \(\mathcal{K}_<\) is equal to the Fraïssé limit of the class \(\mathcal{K}\) of reducts of elements in \(\mathcal{K}_<\). It however in general does not carry onto uncountable structures.

**Proposition 3.** [34] Let \(L \supset \{>\}\) be a language and let \(\mathcal{K}_<\) be a Fraïssé order class in \(L\). Let \(L_0 = L \setminus \{<\}\) and let \(\mathcal{K} = \mathcal{K}_<|L_0\). Denote by \(\mathbf{F}\) and \(\mathbf{F}_<\) the Fraïssé limits of \(\mathcal{K}\) and \(\mathcal{K}_<\) respectively. The following are equivalent.

1. \(\mathbf{F} = \mathbf{F}_<|L_0\).
2. \(\mathcal{K}_<\) is reasonable.

Together with Theorem 17 that characterizes extreme amenability, the following two theorems are the main theoretical results obtained in [34]. The first one shows that the ordering property is equivalent to minimality of a space of linear orderings and the second one shows that the Ramsey property ensures universality.

**Theorem 18** ([34], Theorem 7.4). Let \(L \supset \{<\}\) be a language, \(L_0 = L \setminus \{<\}\), \(\mathcal{K}_<\) a countable reasonable Fraïssé class in \(L\) with \(<\) interpreted as a linear order, and let \(\mathcal{K}\) be the reduct of \(\mathcal{K}_<\) to the language \(L_0\). Let \(\mathbf{F}_<\) denote the Fraïssé limit of \(\mathcal{K}_<\), \(\mathbf{F}\) the reduct of \(\mathbf{F}_<\) to \(L\) (so that \(\mathbf{F}_< = (\mathbf{F}, <)\)) and let \(G\) be the automorphism group of \(\mathbf{F}\). Let \(X^* = G < \mathbf{F}\). Then the following are equivalent.

1. \(\mathcal{K}_<\) satisfies the ordering property.
2. \(X^*\) is a minimal \(G\)-flow.

**Theorem 19** ([34], Theorem 7.5). Let \(L \supset \{<\}\) be a language, \(L_0 = L \setminus \{<\}\), \(\mathcal{K}_<\) a countable reasonable Fraïssé class in \(L\) with \(<\) interpreted as a linear order, and let \(\mathcal{K}\) be the reduct of \(\mathcal{K}_<\) to the language \(L_0\). Let \(\mathbf{F}_<\) denote the Fraïssé limit of \(\mathcal{K}_<\) and \(\mathbf{F}\) the reduct of \(\mathbf{F}_<\) to \(L\), and let \(G\) and \(G_0\) be the automorphism groups of \(\mathbf{F}_<, \mathbf{F}\) respectively. Let \(X^* = G < \mathbf{F}\).

If \(\mathcal{K}_<\) satisfies the Ramsey property and the ordering property, then \(G\) is extremely amenable and \(X^*\) is the universal minimal flow of \(G_0\).

Theorem 19 allowed the authors to provide explicit descriptions of universal minimal flows of automorphism groups of various countable structures.

**Corollary 3.** [34]
Chapter 2. Preliminaries

<table>
<thead>
<tr>
<th>Group of automorphisms of</th>
<th>universal minimal flow</th>
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<tbody>
<tr>
<td>countable set</td>
<td>linear orderings</td>
</tr>
<tr>
<td>random graph</td>
<td>linear orderings</td>
</tr>
<tr>
<td>random $K_n$-free graph</td>
<td>linear orderings</td>
</tr>
<tr>
<td>random hypergraph</td>
<td>linear ordering</td>
</tr>
<tr>
<td>countable atomless Boolean algebra</td>
<td>normal orderings induced by natural orderings</td>
</tr>
<tr>
<td>countably dimensional vector space over a finite field</td>
<td>normal orderings induced by natural orderings</td>
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</table>

Sokić applied the machinery developed in [34] to countable posets in his Ph.D. thesis [64]. By a result of Schmerl ([59]), there are exactly five countable $\omega$-homogeneous posets. Sokić analyzed their Fraïssé classes and fully characterized which of them admit an expansion to a reasonable Fraïssé order class with the ordering property and the Ramsey property.

One of Sokić’s results is the following theorem.

**Theorem 20 ([64]).** The universal minimal flow of the group of automorphisms of the random poset $P$ is the space of linear orderings on $P$ that extend the partial order.

Lionel Nguyen van Thé extended in [49] results about linear expansions in [34] to precompact expansions by countably many relations and computed the universal minimal flow of the group of automorphisms of a complete tournament.

For purposes of this thesis, we introduce the notion of precompact expansion for a pair of classes rather than for pair of structures as originally defined. Let $K$ be a class of structures in a language $L$ and let $K_R$ be a class of expansions of structures in $K$ in a language $L^*$ such that $L^* \setminus L = R = \{R_i : i \in I\}$ is a countable set of relational symbols. We say that $K_R$ is a **precompact expansion** of $K$ if every structure in $K$ has only finitely many expansions in $K_R$. Considering the logic action of the group $G$ of automorphisms of $A$ on $\prod_{i \in I} 2^{A_i}$, we denote by $X^*$ the closure of the orbit $G((R^*_A)_{i \in I})$.

The reason to focus on precompact expansions only is the following proposition first proved for countable structures.

**Proposition 4 ([49]).** The space $X^*$ is compact if and only if $K_R$ is a precompact expansion of $K$.

An analogue of the ordering property is the expansion property. Let $A$ be a structure and let $A^*$ be a precompact relational expansion of $A$. We say that Age($A^*$) has the **expansion property relative to** Age($A$) if for every $A \in$ Age($A$) there exists $B \in$ Age($A$) such that

$$A^*, B^* \in \text{Age}(A^*)(A^*|L = A \& B^*|L = B \rightarrow A^* \leq B^*).$$

Theorems analogous to Theorem 18 and Theorem 19 of Kechris, Pestov and Todorčević are the following.

**Theorem 21 ([49]).** Let $F$ be a Fraïssé and $F^*$ a Fraïssé precompact relational expansion of $F$. The following are equivalent:

1. The flow $X^*$ is minimal.
2. Age($F^*$) has the expansion property relative to Age($F$).
Theorem 22 ([49]). Let $\mathcal{F}$ be a Fraïssé and $\mathcal{F}^*$ a Fraïssé precompact relational expansion of $\mathcal{F}$. Assume that $\text{Age}(\mathcal{F}^*)$ consists of rigid elements. The following are equivalent:

(1) The flow $X^*$ is the universal minimal flow of $G$.

(2) $\text{Age}(\mathcal{F}^*)$ has the Ramsey property and the expansion property relative to $\text{Age}(\mathcal{F})$. 
Chapter 3

General theory

3.1 The greatest ambit

It is a well-known fact that the greatest ambit for a discrete group $G$ is the Čech-Stone compactification $\beta(G)$ of $G$, the space of all ultrafilters on $G$ with the topology generated by clopen sets

$$\tilde{A} = \{u \in \beta(G) : A \in u\}$$

for $A \subseteq G$. In terms of Stone duality, $\beta(G)$ is the Stone space of the power set algebra $\mathcal{P}(G)$ of $G$. The action of $G$ on $\beta G$ extends the multiplication in $G$:

$$G \times \beta(G) \rightarrow \beta(G)$$

$$u \mapsto \{gA : A \in u\}.$$ (3.1)

(3.2)

We can further extend the multiplication to all of $\beta(G)$ turning $\beta(G)$ into a right-topological semigroup. For $u, v \in \beta(G)$, define

$$u \cdot v = \{A \subseteq G : \{g \in G : g^{-1}A \in v\} \in u\}.$$ (3.3)

In other words,

$$uv = u - \lim\{gv : g \in G\}.$$ (3.4)

Algebra and topology in the Čech-Stone compactification of discrete semigroups have been extensively studied by Neil Hindman and Dona Strauss. For a comprehensive treatment see [29].

If $G$ is a non-discrete topological group and $\beta(G)$ is the Čech-Stone compactification of $G$, then the action defined in (3.1) is not continuous. The role of the greatest ambit is then played by the Samuel compactification $S(G)$ - the smallest compactification of $G$ such that every uniformly continuous function from $G$ to a compact Hausdorff space continuously extends to $S(G)$, where we consider $G$ with the right uniform structure. Recall that the right uniformity on $G$ is generated by covers $\{Va : a \in G\}$ for $V \in \mathcal{N}$, where $\mathcal{N}$ is a basis of open neighbourhoods of the identity in $G$.

Note that $S(G)$ must however be a quotient of $\beta G$ by the universal property of the greatest ambit: Let $G_d$ denote the group $G$ considered with the discrete topology. Since $(S(G), e)$ is the greatest ambit for $G$, it is also an ambit for $G_d$. Since $\beta(G_d)$ is the greatest ambit for $G_d$, there is a quotient mapping $\phi$ from $\beta(G_d)$ onto $S(G)$ such that
The quotient mapping $\phi$ defines an equivalence relation $\sim$ on $\beta(G_d)$. We give the description of $\sim$ provided by Samuel in [58] (in the more general setting of for arbitrary uniform spaces). Let $G$ be a topological group with the identity $e$ and $\mathcal{N}$ a basis of neighbourhoods of $e$ defining the topology on $G$. For an ultrafilter $u \in \beta(G_d)$, we define a filter $u^*$ to be generated by 

$$\{VA : A \in u, V \in \mathcal{N}\}.$$ 

Then for $u, v \in \beta(G)$ we set 

$$u \sim v \text{ if and only if } u^* = v^*.$$ 

The quotient space 

$$S(G) = \beta(G_d)/\sim$$

with the quotient topology is called the Samuel compactification of $G$ and it is the greatest ambit of $G$. The multiplication on $\beta(G_d)$ also factors to $S(G)$, making $S(G)$ a right-topological semigroup with the multiplication extending the multiplication on $G$. If $u$ is a filter on $G$, then the $\sim$-saturation of $u$ is exactly $u^*$.

In [36] Koçak and Strauss introduced the notion of a near ultrafilter and provided an alternative description of the Samuel compactification in the setting of uniform spaces. Explicit description for topological groups appeared soon after by Koçak and Arvasi in [37].

**Definition 12.** A family $u$ of subsets of $G$ has the **near finite intersection property** if for every $F \subset u$ finite and every $V \in \mathcal{N}$ we have 

$$\bigcap_{A \in F} VA \neq \emptyset.$$ 

We call $u$ a **near ultrafilter**, if it is a maximal collection with the near finite intersection property with respect to inclusion.

A near ultrafilter on a subalgebra $\mathcal{A}$ of $\mathcal{P}(G)$ is then a maximal collection of sets in $\mathcal{A}$ with the near finite intersection property.

Note that if $G$ is a discrete group, then the notion of a near ultrafilter coincides with that of an ultrafilter.

**Remark 3.** If $A, B \subset G$ then there is $V \in \mathcal{N}$ with $VA \cap VB \neq \emptyset$ if and only if there is $W \in \mathcal{N}$ such that $A \cap WB \neq \emptyset \neq WA \cap B$.

Let $G$ be a topological group and let $\hat{G}$ denote the space of all near ultrafilters on $G$. A base for closed sets of $\hat{G}$ is given by $\hat{A} = \{u \in \hat{G} : A \in u\}$ for $A \subset G$.

Similarly as the Čech-Stone compactification can be described as ultrafilters of zero sets, the Samuel compactification can be described as the space of all near ultrafilters.
Theorem 23 ([37]). Let $G$ be a topological group. Then $\hat{G}$ with the multiplication $gu = \{gA : A \in u\}$ for $g \in G$ and $u \in \hat{G}$ is the greatest ambit of $G$.

Remark 4. The correspondence between $\hat{G}$ and $\{u^* : u \in \beta(G_d)\}$ is the following
$$u^* \leftrightarrow \bigcup \{v \in \beta(G_d) : v \supset u^*\}.$$ It means that $u \sim v$ if and only if $u \cup v$ is a subset of one near ultrafilter on $G$.

The multiplication on $\hat{G}$ is analogous to multiplication in $\beta(G_d)$.

Theorem 24 ([37]). Let $u, v \in \hat{G}$. Then $A \in uv$ if and only if $A_V = \{g \in G : g^{-1}VA \in v\} \in u$
for every $V \in \mathcal{N}$.

3.2 The universal minimal flow

In [4], the authors described the universal minimal flow of a discrete group $G$ as the Stone space of a certain Boolean algebra $B(G)$ of left syndetic subsets of $G$.

Definition 13. A subset $S$ of $G$ is called left syndetic if there exist finitely many $g_1, g_2, \ldots, g_n \in G$ such that $\bigcup_{i=1}^n g_iS = G$.

For a $G$-flow $X$, a nonempty open subset $O$ of $X$ and a point $x \in X$, denote by $\text{Ret}(x, O)$ the set of elements of $G$ that bring $x$ into $O$. Formally,
$$\text{Ret}(x, O) = \{g \in G : gx \in O\}.$$

Lemma 9. The following are equivalent for a $G$-flow $X$
(i) $X$ is minimal,
(ii) for every non-empty open set $O \subset X$, $\bigcup_{g \in G} gO = X$,
(iii) for every $x \in X$ and a non-empty open set $O \subset X$, the set $\text{Ret}(x, O)$ is left syndetic.

An algebra $B$ of subsets of $G$ is called left syndetic if every $B \in B$ is left syndetic and for every $g \in G$ also $gB \in B$, i.e. $B$ is closed under left translations.

Theorem 25 ([4]). Let $G$ be a discrete group and let $B(G)$ be a maximal left syndetic subalgebra of the power set algebra of $G$ with respect to inclusion. Then the universal minimal flow $M(G)$ of $G$ is the Stone space of $B(G)$, $\text{St}(B(G))$, with the action $gu = \{gA : A \in u\}$ for $g \in G$ and $u \in \text{St}(B(G))$.

To prove Theorem 25, the authors view $M(G)$ both as a quotient and as a subspace of $\beta(G)$. If $m \in M(G)$, then the right translation $R_m : \beta(G) \rightarrow M(G)$ is a quotient mapping. The dual homomorphism $\rho_m : \text{Clopen}(M(G)) \rightarrow \mathcal{P}(G)$ from the algebra of clopen subsets of $M(G)$ to the algebra of clopen subsets of $\beta(G)$ can be described as
$$\rho_m(A) = \text{Ret}(m, A).$$
Here we use that \( A \in \text{Clopen}(M(G)) \) can be identified with a subset of \( G \). Image of \( \rho_m \) is then isomorphic to \( B(G) \).

Now, let \((G, \tau)\) be a topological group and let \( \mathcal{N} \) be a basis of neighbourhoods of the identity in \( G \). Similarly as the greatest ambit \( S(G) \) can be described as the space of near ultrafilters on \( G \), it turns out that the universal minimal flow can be described as the space of near ultrafilters on \( B(G) \leq \mathcal{P}(G) \) isomorphic to the algebra of clopen subsets of the universal minimal flow for \( G \) with the discrete topology. This is a core ingredient of the new approach we are developing.

**Theorem 26.** The universal minimal flow for a topological group \( G \) is the space of near ultrafilters on \( B(G) \), a subalgebra of \( \mathcal{P}(G) \) isomorphic to the clopen algebra of the universal minimal flow of \( G_d \).

**Proof.** Since the multiplication on \( \beta(G_d) \) factors to the multiplication on \( S(G) \), we get the following commutative diagram for every \( m \in M(G_d) \subset \beta(G_d) \).

\[
\begin{array}{ccc}
\beta(G_d) & \xrightarrow{R_m} & M(G_d) \\
\sim & & \sim \\
S(G) & \xrightarrow{R_m^*} & M(G)
\end{array}
\]

By Remark 4, \( \sim \) is the equivalence relation on \( \beta(G_d) \) given by \( u \sim v \) if and only if \( u \cup v \) is a subset of one near ultrafilter on \( G \). If \( \mathcal{F} \) is a filter on \( G_d \), then the saturation of \( \mathcal{F} \) under \( \sim \) is equal to

\[
[\mathcal{F}]_\sim = \{VA : V \in \mathcal{N}, A \in \mathcal{F}\}.
\]

If \( \mathcal{C} \) is a centered family of subsets of \( G \), then \( \mathcal{C} \uparrow \) denotes the filter generated by \( \mathcal{C} \).

Let \( M(G_d) \) be a minimal left ideal of \( \beta(G_d) \). Then the quotient \( M(G) = M(G_d) / \sim \) is a minimal left ideal of \( S(G) \) and therefore \( M(G) \) is isomorphic to the universal minimal flow of \( G \).

Let \( \rho_m \) denote the Boolean algebra homomorphism dual to \( R_m \). Note that since \( R_m \) is surjective, \( \rho_m \) is injective. By Theorem 25, \( \text{Clopen}(M(G_d)) \) is isomorphic to a maximal left syndetic subalgebra \( B(G) \) of \( \mathcal{P}(G) \). We identify \( \text{Clopen}(M(G_d)) \) with \( B(G) \).

We can assume that \( m \) is an idempotent, which implies \( R_m|M(G_d) \) is the identity mapping. Then \( \rho_m : B(G) \rightarrow \mathcal{P}(G) \) is the identity as well.

First let \( u, v \in \text{St}(B(G)) = M(G_d) \) and suppose that \( u \sim v \). We need to show that \( u \) and \( v \) are contained in the same near ultrafilter, in other words that \( u \cup v \) has the finite near intersection property. Let \( \bar{u}, \bar{v} \) be ultrafilters on \( G_d \) such that \( R_m(\bar{u}) = u \) and \( R_m(\bar{v}) = v \) respectively. We have that \( u \sim v \) if and only if \( \bar{u} \sim \bar{v} \). But that happens exactly when \( \bar{u} \cup \bar{v} \) has the finite near intersection property.

Since \( u \subset \bar{u} \) and \( v \subset \bar{v} \), also \( u \cup v \) has the finite near intersection property.

To prove the other direction, note that

\[
\{[u]_\sim : u \in \text{St}(B(G))\}
\]

is a partition of \( \text{St}(B(G)) \) into closed sets and consequently

\[
\mathcal{Q} = \{\rho_m([u]_\sim) \uparrow : u \in \text{St}(B(G))\}
\]
defines a partition of $\beta(G_d)$ into closed sets. It follows that

$$[\rho_m(u)^\uparrow_\sim] \subset \rho_m([u]^\sim)^\uparrow$$

for every $u \in \text{St}(B(G))$.

Let $u, v \in \text{St}(B(G)) = M(G_d)$ and suppose that $u$ and $v$ are contained in the same near ultrafilter on $B(G)$. It means that

$$[\rho_m(u)^\uparrow_\sim] \cup [\rho_m(v)^\uparrow_\sim]$$

generates a filter on $G$. But that means that the closed subsets of $\beta(G_d)$ given by the filters $[\rho_m(u)^\uparrow_\sim]$ and $[\rho_m(v)^\uparrow_\sim]$ intersect and therefore must both be a subset of a single element of the partition $Q$. It implies that $u \sim v$.

\begin{proof}
\end{proof}

### 3.3 Extreme amenability

Recall that a topological group $G$ is **extremely amenable** if every $G$-flow has a fixed point. It means that every $G$-flow has a singleton as a subflow. In particular,

**Lemma 10.** A topological group $G$ extremely amenable if the universal minimal flow of $G$ is a single point if and only if every minimal $G$-flow is a single point.

As an immediate consequence of Theorem 26 we obtain the following theorem.

**Theorem 27.** A topological group $G$ is extremely amenable if and only if whenever $A, B \subset G$ belong to one syndetic algebra of subsets of $G$ and $V$ is an open neighbourhood of the identity element in $G$ then $VA \cap VB \neq \emptyset$.

The characterization of left syndetic sets in Lemma 3 provides the following characterization of extreme amenability.

**Lemma 11.** A topological group $G$ is extremely amenable if and only if for every $A, B \subset G$ left syndetic and every $V$ an open neighbourhood of the identity in $G$ we have $VA \cap VB \neq \emptyset$.

By Lemma 6, a subset $T \subset G$ is thick if it contains a minimal left ideal in its closure in $\beta(G_d)$. Therefore thick sets can help us recognize extreme amenability.

**Corollary 4.** A topological group $G$ is extremely amenable if and only if whenever $A$ belongs to a left syndetic algebra of subsets of $G$ and $V$ is an open neighbourhood of the neutral element in $G$ then $VG$ is thick.

### 3.4 Non-archimedean groups

A topological group is called **non-archimedean** if it admits a basis $\mathcal{N}$ of neighbourhoods of the identity in $G$ consisting of open subgroups. All groups of automorphisms of structures are non-archimedean groups and every non-archimedean group has a representation as a group of automorphisms.

We apply the theory developed in the previous sections to describe the greatest ambit and the universal minimal flow of a non-archimedean group $G$ as Stone spaces of certain subalgebras of $\mathcal{P}(G)$.
Greatest ambit

It is easy to see that if $G$ is non-archimedean and $\mathcal{N}$ is a basis of neighbourhoods of the identity in $G$ consisting of open subgroups and closed under intersections then

$$L = \{VA : A \subset G, V \in \mathcal{N}\}$$

is a Boolean algebra. Examining the Samuel compactification in this case, we get the following description of the greatest ambit.

**Theorem 28.** The greatest ambit $S(G)$ is equal to the Stone space of the Boolean algebra $L$ as above with the action defined by $gx = \{gA : A \subset x\}$ for $g \in G$ and $x \in \text{St}(L)$.

**Proof.** Recall that for $u \in \beta(G_d)$, $u^*$ denotes the filter generated by $\{VA : V \in \mathcal{N}, A \in u\}$ and that $S(G)$ can be identified with the quotient space $\{u^* : u \in \beta(G)\}$. We will show that $\phi : S(G) \rightarrow \text{St}(L)$, $u^* \mapsto u \cap L$ for $u \in \beta(G_d)$ is an isomorphism between $S(G)$ and the Stone space of $L$.

To show that $\phi$ is well-defined, let $u, v \in \beta(G_d)$ such that $u \cap L \neq v \cap L$. Then there is $A \in L$ such that $A \in u \cap L$ and $G \setminus A \in v \cap L$, which implies that $u^* \neq v^*$.

For injectivity, let $u, v \in \beta(G_d)$. Since $VV = V$ for every $V \in \mathcal{N}$, we have that $u \cap L$ is an ultrafilter on $L$ that generates $u^* \in S(G)$. It means that whenever $u \cap L = v \cap L$, then $u^* = v^*$.

Obviously, $\phi$ is surjective and preserves the actions of $G$ on $S(G)$ and $\text{St}(L)$. Therefore $\phi$ is an isomorphism.

The multiplication on $\text{St}(L)$ can be described explicitly as an analogue of the multiplication on $\beta(G_d)$.

**Lemma 12.** Let $u, v \in \text{St}(L)$, $A \subset G$ and $V \in \mathcal{N}$. Then $VA \in uv$ if and only if $\{g \in G : g^{-1}[VA] \in v\} \in u$.

**Proof.** Let $u', v' \in \beta(G_d)$ be representatives of the equivalence classes of $u, v$ respectively. We need to show that the definition of $uv$ in the claim corresponds to $u'v' \cap L$. By the definition of ultrafilter multiplication, $VA \in u'v' \cap L$ if and only if $U = \{g \in G : g^{-1}[VA] \in v'\} \in u'$. First, we show that we can replace $u'$ with $u$ : Notice that $VV = U$, since for every $s \in V$ and $g \in G$, we have that $sg \in U$ if and only if $(sg)^{-1}[VA] = g^{-1}s^{-1}[VA] = g^{-1}[VA] \in v$ if and only if $g \in U$. Therefore, $VA \in u'v' \cap L$ if and only if $U \in u' \cap L = u$. Second, we show that $v'$ can be replaced by $v$ : For every $g \in G$ and $W \in \mathcal{N}$, $g^{-1}[VA] \in v'$ implies $Wg^{-1}[VA] \in v$. Pick a $W \in \mathcal{N}$ such that $gWg^{-1} \subset V$. Then $Wg^{-1} \subset g^{-1}V$, so $Wg^{-1}VA \subset g^{-1}VV = g^{-1}VA$. But obviously, $g^{-1}VA \subset Wg^{-1}VA$, hence $Wg^{-1}VA = g^{-1}VA$ which shows that $g^{-1}VA \in v'$ if and only if $g^{-1}VA \in v' \cap L = v$, which concludes the proof.

Universal minimal flow

Let $G$ be again a non-archimedean group with a basis of neighbourhoods $\mathcal{N}$ of the neutral element $e$ consisting of open subgroups and let $L$ be as above. The universal minimal flow $M(G)$ for $G$ being a subspace of the greatest ambit $S(G)$, is itself a Stone space. Hence we can consider its Boolean algebra of all clopen subsets $B(G)$. For $m \in M(G)$ and $\emptyset \neq U \in B(G)$, denote again by $\text{Ret}(m, U)$ the set of elements of $G$ that bring $m$ into $U$, i.e. $\text{Ret}(m, U) = \{g \in G : gm \in U\}$. Since $\text{hRet}(m, U) = \text{Ret}(m,uhU)$ and $M(G)$ is compact, there are finitely many $g_1, g_2, \ldots, g_n \in G$ such that $\bigcup_{i=1}^n g_i \text{Ret}(m, U) = G$. Recall that subsets of $G$ whose finitely many left translates cover $G$ are called **left syndetic**.

Now, we are ready to imitate a proof for discrete semigroups from [4] to characterize $B(G)$.
Theorem 29. The universal minimal flow $M(G)$ is the Stone space of a maximal left syndetic subalgebra $B(G)$ of $L$. All maximal left syndetic subalgebras of $L$ are isomorphic.

Proof. Let $B(G)$ denote the algebra of clopen subsets of $M(G)$ and let $m \in M(G)$. Let $R_m : S(G) \rightarrow S(G)$ denote the right translation by $m$, it means $R_m(u) = um$.

Since $M(G)$ is a minimal left ideal of $S(G)$, $S(G)m = M(G)$, so $R_m$ actually maps $S(G)$ onto $M(G)$. $S(G)$ being a right-topological semigroup, $R_m$ is continuous. Since $R_m$ is onto, the dual homomorphism between the Boolean algebras of clopen sets $\rho_m : B(G) \rightarrow L$ is injective. If $X \in L$, we denote by $\hat{X}$ the clopen subset of $\text{St}(L)$ consisting of all ultrafilters containing $X$. By Lemma 12, we know that for every $VA \in L$,

$$\rho_m(\overline{VA} \cap M(G)) = \{ u \in S(G) : VA \in um \}$$

$$= \{ g \in G : g^{-1}[VA] \in m \}$$

$$= \text{Ret}(m, \overline{VA}).$$

So $B(G)$ is isomorphic to

$$\mathcal{A} = \{ \text{Ret}(m, \overline{VA}) : A \subset G, V \in \mathcal{N} \},$$

which is a subalgebra of $L$ consisting exclusively of left syndetic sets.

Now, we show that $\mathcal{A}$ is invariant under left translations by $G$: Let $h \in G$, then

$$h\text{Ret}(m, \overline{VA}) = h\{ g \in G : g^{-1}[VA] \in m \}$$

$$= \{ x \in G : x^{-1}[hVA] \in m \}$$

$$= \text{Ret}(m, \overline{hVA}).$$

It remains to show that $\mathcal{A}$ is a maximal left syndetic algebra. Let $B \supset \mathcal{A}$ be a left syndetic algebra. Then $\text{St}(B)$ with multiplication $gu = \{ gA : A \in u \}$ for $g \in G$ and $u \in \text{St}(B)$ is a minimal flow. The identity embedding of $i : \mathcal{A} \hookrightarrow B$ induces a surjective $G$-homomorphism $\text{St}(i) : \text{St}(B) \rightarrow \text{St}(\mathcal{A}) \cong M(G)$. By uniqueness of $M(G)$, it means that also $\text{St}(B) \cong M(G)$. Since every $G$-homomorphism from $M(G)$ to itself is an isomorphism, so is $\text{St}(i)$, which is only possible if $\mathcal{A} = B$.

For the second part of the theorem, let us assume that $\mathcal{A}$ is a maximal left syndetic subalgebra of $L$. To achieve the conclusion, it is enough to find an $m \in M(G)$ such that $\mathcal{A} = \{ \text{Ret}(m, O) : O \in B(G) \}$.

$\text{St}(\mathcal{A})$ with $G$-multiplication as above is a minimal $G$-flow. Hence there is a $G$-quotient mapping $\phi : M(G) \rightarrow \text{St}(\mathcal{A})$ inducing an injection $\mathcal{A} \rightarrow B(G), A \mapsto \phi^{-1}(A)$. Let $e$ be the identity in $G$. Consider $p \in \text{St}(\mathcal{A})$ given by $p = \{ A \in \mathcal{A} : e \in A \}$ and its preimage $m$ under $\phi$. Given $A \in \mathcal{A}$, denote by $\hat{A}$ the clopen set $\{ u \in \text{St}(\mathcal{A}) : A \in u \}$. Then $\text{Ret}(p, A) = \text{Ret}(m, \phi^{-1}[A])$, since $\phi(gm) = g\phi(m) = gp$ for all $g \in G$ and $\phi$ is surjective.
We have that
\[
\text{Ret}(p, \bar{A}) = \{g \in G : gp \in \bar{A}\}
= \{g \in G : g^{-1}[A] \in p\}
= \{g \in G : e \in g^{-1}[A]\}
= \{g \in G : ge \in \bar{A}\}
= A.
\]

Consequently,
\[
\mathcal{A} = \{\text{Ret}(p, A) : A \in \mathcal{A}\} \subset \{\text{Ret}(m, O) : O \in B(G)\}.
\]

By maximality of \(\mathcal{A}\), we get that \(\mathcal{A} = \{\text{Ret}(m, O) : O \in B(G)\}\).
Chapter 4

Automorphism groups

This chapter is devoted to groups of automorphisms of structures, which are a representation of non-archimedean groups. We present results of the author from [8] and [6] and their generalizations. We show that the group of all bijections on a set of cardinality $\kappa$ is universal for groups of automorphisms of ($\omega$-homogeneous) structures with the underlying set of cardinality at most $\kappa$ and therefore all non-archimedean groups of size at most $\kappa$. We first verify that results in Kechris, Pestov and Todorčević ([34]) introduced in Chapter 2 generalize to uncountable structures in the setting of precompact expansions developed by Lionel Nguyen van Thé ([49]). In Theorem 32 and Theorem 33 we compute universal minimal flows of groups of automorphisms of locally-finite $\omega$-homogeneous structures $A$ such that $\text{Age}(A)$ admits a precompact expansion to a reasonable Fraïssé class $K$ of rigid structures satisfying the Ramsey and expansion properties and such that there is an expansion $(A, R)$ of $A$ in the language of $K$ which is also $\omega$-homogeneous and $\text{Age}(A, R) = K$.

While this is not a limitation in the class of countable structures where for every $\omega$-homogeneous structure whose age admits a reasonable Fraïssé expansion we can obtain $\omega$-homogeneous expansion via Fraïssé construction, in the case of uncountable structures we only know how to do so for highly homogeneous and universal structures (Jóńsson structures) or we construct such expansions “by hand.”

Jóńsson's construction only provides structures of cardinalities $\kappa$ that satisfy $\kappa^{<\kappa} = \kappa$. Jóńsson's structures are $\kappa$-universal and $<\kappa$-homogeneous, however Theorem 33 only requires $\omega$-homogeneity. We construct $\omega$-homogeneous structures (and their $\omega$-homogeneous order expansion) in every cardinality to extend the pool of examples of extremely amenable groups and groups for which we have an explicit description of the universal minimal flow.

In general, there are often many non-isomorphic structures of the same theory and the same size. If we start with an $\omega$-homogeneous structure and manage to construct an $\omega$-homogeneous structure with $\omega$-homogeneous expansion satisfying assumptions of Theorem 33 we have no guarantee that we arrive at the same structure we started with as in the case of Fraïssé or Jóńsson structures. We do not know whether we can find such $\omega$-homogeneous expansions even for structures for which the universal minimal flow of its automorphism group is of the type as in Theorem 33. For instance, $\mathcal{P}(\omega)/\text{fin}$ is an uncountable homogeneous Boolean algebra and it was shown that the universal minimal flow of $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$ is a space of linear orderings on $\mathcal{P}(\omega)/\text{fin}$ by Glasner and Gutman in [20]. However, it is not know whether $\mathcal{P}(\omega)/\text{fin}$ expended by any of the linear orderings in the universal minimal flow is an $\omega$-homogeneous structure. We solve this obstruction in the proof of Theorem 40 showing that the requirement of an
\(\omega\)-homogeneous expansion is not necessary. Theorem 40 fully generalizes Theorem 7.5 in [34]. We apply the dual approach from Chapter 3 describing the Boolean algebra of clopen subsets of the universal minimal flow of a non-archimedean group \(G\) as a syndetic subalgebra of the power set algebra on \(G\).

Motivated by the old Katowice problem to determine whether Boolean algebras \(P(\omega)/\text{fin}\) and \(P(\omega_1)/\text{fin}\) can consistently be isomorphic, we compute the universal minimal flow of \(\text{Aut}(P(\omega_1)/\text{fin})\), while the universal minimal flow of \(\text{Aut}(P(\omega)/\text{fin})\) was originally computed by Glasner and Gutman ([20]). In order to apply our machinery to \(P(\omega_1)/\text{fin}\), we prove the Ramsey property for a class of finite Boolean algebras with an ideal and show that this class admits an expansion to a Fraïssé order-forgetful class. More generally, we prove analogous results for classes of finite Boolean algebras with chains of ideals under inclusion. This has further applications to dynamics of groups of automorphisms of quotient of power set algebras and the groups of homeomorphisms of Cantor cubes.

### 4.1 Universality of \(S_\kappa\)

Let \(\kappa\) be a cardinal number endowed with the discrete topology and denote by \(S_\kappa\) the group of all bijections on \(\kappa\). In what follows, we consider \(S_\kappa\) as a topological group with the topology of pointwise convergence. Recall that the topology is given by a basis of neighbourhoods of the neutral element consisting of open subgroups \(S_A = \{g \in S_\kappa : g(a) = a, a \in A\}\) where \(A\) is a finite subset of \(\kappa\). We can observe that a subset \(H\) of \(S_\kappa\) is closed if and only if it contains every \(g \in S_\kappa\) such that for any \(A \subset \kappa\) finite there exists an \(h \in H\) with \(g|A = h|A\).

Let \(G \leq S_\kappa\) be a subgroup and let \(A\) be a subset of \(\kappa\). We denote the point-wise stabilizer of \(A\) as:

\[
G_A = \{g \in G : ga = a, a \in A\}
\]

and the set-wise stabilizer as:

\[
G(A) = \{g \in G : gA = A\}.
\]

Let \(L\) be a first order language and let \(\mathcal{A}\) be an \(L\)-structure with the universe of cardinality \(\kappa\). Then the group of automorphisms of \(\mathcal{A}\), \(\text{Aut}(\mathcal{A})\), is a closed subgroup of \(S_\kappa\). It is clear that the topology on \(\text{Aut}(\mathcal{A})\) is given by \(\text{Aut}(\mathcal{A})_A\) for \(A\) a finitely-generated substructure of \(\mathcal{A}\).

Recall that if every partial isomorphism between two finitely generated substructures of \(\mathcal{A}\) can be extended to an automorphism of the whole structure \(\mathcal{A}\), then we say that \(\mathcal{A}\) is \(\omega\)-homogeneous. In the case of Boolean algebras, \(\omega\)-homogeneity coincides with the notion of homogeneity (see [46]). A Boolean algebra \(\mathcal{B}\) is homogeneous if for every \(b \in \mathcal{B}\) the relative Boolean algebra \(\mathcal{B}|b = \{c \in \mathcal{B} : c \leq b\}\) is isomorphic to \(\mathcal{B}\).

It was shown in [9] that closed subgroups \(S_\infty(\mathbb{Z}) = S_\omega\) correspond to groups of automorphisms of countable structures. We will show that an analogous result holds for \(S_\kappa\).

**Theorem 30.** Let \(G\) be an infinite topological group and let \(\kappa\) be a cardinal number. Then the following are equivalent:

(a) \(G\) is a subgroup of \(S_\kappa\),

(b) \(G\) is a non-archimedean group with a basis \(N\) of neighbourhoods of the identity of cardinality \(\nu \leq \kappa\) consisting of open subgroups such that the family of all left translates of elements from \(N\) has cardinality \(\mu \leq \kappa\),
(c) \( G \) is a dense subgroup of a group of automorphisms of an \( \omega \)-homogeneous relational structure on a set of cardinality \( \kappa \).

(d) \( G \) is a dense subgroup of a group of automorphisms of a structure on a set of cardinality \( \kappa \).

Proof. The equivalence of (a), (c) and (d) follows from the Theorem 4.1.1 in [30]. (b) trivially follows from (a), so we only need to establish that also (b) implies (a). We proceed as in Theorem 1.5.1 in [9]. Let \( \lambda = \max\{\nu, \mu\} \leq \kappa \). Let \( \{U_i : i \in \lambda\} \) be an enumeration of a basis for the topology on \( G \) given by all left translates of subgroups from \( \mathcal{N} \). For every \( g \in G \), define \( \phi(g) = \pi_g \in S_\kappa \) as follows

\[
\pi_g(i) = j \iff gU_i = U_j \quad \text{for} \quad i \in \lambda \quad \text{and} \quad \pi_g(i) = i \quad \text{otherwise}.
\]

Obviously, \( g \mapsto \pi_g \) is an injective homomorphism of \( G \) into \( S_\kappa \). To show that it is continuous, let \( A \subset \lambda \) be finite and let \( S_{\kappa,A} = \{\pi \in S_\kappa : \pi(i) = i, i \in A\} \) be a basic open subgroup of \( S_\kappa \). Then \( \phi^{-1}(S_{\kappa,A}) = \{g \in G : gU_i = U_i, i \in A\} \). Since for every \( i \in A \), \( U_i = h_i V_i \) for some \( h_i \in G \) and some subgroup \( V_i \in \mathcal{N} \),

\[
\phi^{-1}(S_{\kappa,A}) = \bigcap_{i \in A}\{g \in G : gh_i V_i = h_i V_i\} = \bigcap_{i \in A}\{g \in G : gh_i = h_i\} = \bigcap_{i \in A}\{g \in G : g \in h_i V_i h_i^{-1}\},
\]

which is an intersection of finitely many open sets, hence open. Similarly, let \( H \in \mathcal{N} \) be an open subgroup of \( G \), so \( H = U_i \) for some \( i \in \lambda \). Then \( \phi(H) = \{\pi_h : h \in H\} = \{\pi \in S_\kappa : \pi(i) = i\} \cap \phi(G) \), hence \( \phi \) is a homeomorphism onto its image.

Previous theorem implies that the class of groups of automorphisms of structures and the class of a groups of automorphisms of \( \omega \)-homogeneous structures are the same.

### 4.2 Kechris, Pestov and Todorčević for uncountable structures and precompact expansions

We generalize results of Kechris, Pestov and Todorčević introduced in Chapter 2 to uncountable structures and precompact expansions.

**Theorem 31.** Let \( L \) be a first-order language and let \( L^* \) be an expansion of \( L \) with countably many relational symbols \( \mathcal{R} \). Let \( A \) be an \( \omega \)-homogeneous \( L \)-structure and let \( (A, \mathcal{T}) \) be an expansion of \( A \) in the language \( L^* \). Then \( S \in \mathcal{G}_{\mathcal{T}} \) if and only if for every \( B \in \text{Age}(A) \), \( (B, S|B) \in \text{Age}(A, \mathcal{T}) \).

Proof. Let \( S \in \mathcal{G}_{\mathcal{T}} \). It means that for every \( B \in \text{Age}(A) \), there exists \( g \in G \) such that \( gT|B = S|B \). Thus \( g : g^{-1}(B) \rightarrow B \) is an isomorphism between \( g^{-1}(B), T|g^{-1}(B) \) and \( (B, S|B) \), which shows that \( (B, S|B) \in \text{Age}(A, \mathcal{T}) \).

Conversely, suppose that \( S \in \mathcal{G}_{\mathcal{T}} \) and \( (B, S|B) \in \text{Age}(A, \mathcal{T}) \) for every \( B \in \text{Age}(A) \). This means that for every \( B \in \text{Age}(A) \) there is an embedding \( i : (B, S|B) \rightarrow (A, \mathcal{T}) \). If \( C = i(B) \), then \( i^{-1} \) is an isomorphism between \( (C, \mathcal{T}|C) \) and \( (B, S|B) \) and in particular \( i^{-1} \) is an isomorphism between \( C \) and \( B \). By \( \omega \)-homogeneity of \( A \), \( i^{-1} \) can be extended to a \( g \in G \). Then \( S|B = gT|B \) and thus \( S \in \mathcal{G}_{\mathcal{T}} \). \( \Box \)

**Theorem 32.** Let \( A \) be an \( \omega \)-homogeneous structure and let \( (A, \mathcal{T}) \) be an expansion of \( A \) with a set of countably many relations \( \mathcal{R} \) such that \( (A, \mathcal{T}) \) is \( \omega \)-homogeneous as well. Suppose that \( \text{Age}(A, \mathcal{T}) \) is a precompact expansion of \( \text{Age}(A) \). Let \( G \) be a dense subgroup of \( \text{Aut}(A) \). Then the following are equivalent:

(a) \( \text{Age}(A, \mathcal{T}) \) satisfies the expansion property relative to \( \text{Age}(A) \),
Chapter 4. Automorphism groups

46

(a) \( \tilde{\Gamma} \) is a minimal \( G \)-flow.

Proof. \( (a) \Rightarrow (b) \) Let \( S \in \tilde{\Gamma} \). We would like to show that also \( \mathcal{T} \in \tilde{\mathcal{S}} \). By the previous theorem, this is equivalent to showing that \( \text{Age}(\mathcal{A}, \mathcal{T}) \subset \text{Age}(\mathcal{A}, S) \).

To that end, let \( (B, \mathcal{T}|B) \in \text{Age}(\mathcal{A}, \mathcal{T}) \) and find \( D \in \text{Age}(\mathcal{A}) \) given by the expansion property for \( B \). Let \( D_0 \) be an isomorphic copy of \( D \) in \( A \) and let \( i : (B, \mathcal{T}|B) \longmapsto (D_0, S|D_0) \) be an embedding ensured by the expansion property. Then \( i \) is an isomorphism between \( (B, \mathcal{T}|B) \) and \( (i(B), S|i(B)) \in \text{Age}(\mathcal{A}, S) \), showing \( (B, \mathcal{T}|B) \in \text{Age}(\mathcal{A}, S) \).

\( (b) \Rightarrow (a) \) Given \( B \in \text{Age}(\mathcal{A}) \) we need to find \( D \in \text{Age}(\mathcal{A}) \) such that whenever \( (B, \mathcal{T}') \in \text{Age}(\mathcal{A}, \mathcal{T}) \) and \( (D, S) \in \text{Age}(\mathcal{A}, \mathcal{T}) \), there is an embedding of \( (B, \mathcal{T}') \) into \( (D, S) \). Fix \( (B, \mathcal{T}') \in \text{Age}(\mathcal{A}, \mathcal{T}) \). For every \( C \in \text{Age}(\mathcal{A}) \) consider the set

\[
X_C = \{ S \in \tilde{\Gamma} : (B, \mathcal{T}'|B) \cong (C, S|C) \}.
\]

Then \( \tilde{\Gamma} = \bigcup_{C \in \text{Age}(\mathcal{A})} X_C \). Since each \( X_C \) is open, there are \( C_1, C_2, \ldots, C_n \in \text{Age}(\mathcal{A}) \) such that \( \tilde{\Gamma} = \bigcup_{i=1}^n X_{C_i} \), by compactness of \( \tilde{\Gamma} \). Let \( \mathcal{D}_i \) be the substructure of \( \mathcal{A} \) generated by \( \bigcup_{i=1}^n C_i \). We show that whenever \( (\mathcal{D}_i, S) \in \text{Age}(\mathcal{A}, \mathcal{T}) \), then \( (B, \mathcal{T}') \leq (\mathcal{D}_i, S) \). It means we need to find \( S' \in \tilde{\Gamma} \) extending \( S \). By minimality of \( \tilde{\Gamma} \), we know that there exists an embedding \( i : (\mathcal{D}_i, S) \longmapsto (\mathcal{A}, \mathcal{T}) \). By \( \omega \)-homogeneity of \( \mathcal{A} \), there is \( g \in G \) extending \( i \) to all of \( \mathcal{A} \). Then we get that \( g^{-1} \mathcal{T}|D_{\mathcal{T}'} = S \), so \( g^{-1} \mathcal{T} \in \mathcal{G} \) is the sought for extension of \( S \).

Now we repeat the procedure for every interpretation \( \mathcal{T}' \) on \( B \) with \( (B, \mathcal{T}') \in \text{Age}(\mathcal{A}, \mathcal{T}) \) and set \( D \) to be the substructure of \( \mathcal{A} \) generated by

\[
\bigcup_{(B, \mathcal{T}') \in \text{Age}(\mathcal{A}, \mathcal{T})} \mathcal{D}_{\mathcal{T}'}.
\]

Then \( D \) is a witness of the expansion property for \( B \) and we are done.

\[ \square \]

Theorem 33. Let \( \mathcal{A} \) be an \( \omega \)-homogeneous structure and let \( (\mathcal{A}, \mathcal{T}) \) be a relational expansion of \( \mathcal{A} \) with a set of countably many relations \( \mathcal{R} \) such that \( (\mathcal{A}, \mathcal{T}) \) is also \( \omega \)-homogeneous. Suppose that \( \mathcal{A} \) is locally-finite and that \( \text{Age}(\mathcal{A}, \mathcal{T}) \) is a precompact expansion of \( \text{Age}(\mathcal{A}) \) consisting of rigid elements. Let \( \mathcal{H} = \text{Aut}(\mathcal{A}) \) and \( \mathcal{H}_\mathcal{T} = \text{Aut}(\mathcal{A}, \mathcal{T}) \).

(a) Suppose that \( \text{Age}(\mathcal{A}, \mathcal{T}) \) satisfies the Ramsey property. Then \( \overline{\mathcal{T}} \) is the universal ambit among those \( \mathcal{H} \)-ambits whose base point is fixed by the action of \( \mathcal{H}_\mathcal{T} \).

(b) Suppose that \( \text{Age}(\mathcal{A}, \mathcal{T}) \) satisfies both the Ramsey and the expansion properties. Then \( \overline{\mathcal{H} \mathcal{T}} \) is the universal minimal \( \mathcal{H} \)-flow.

Proof. \( (a) \) Let \( (X, x_0) \) be an \( \mathcal{H} \)-flow such that \( \mathcal{H}_\mathcal{T} x_0 = \{ x_0 \} \). Let \( \Phi \) be the closure of the following set in the compact Hausdorff space \( \overline{\mathcal{H} \mathcal{T}} \times X \):

\[
\{(h \mathcal{T}, h x_0) : h \in \mathcal{H}\} \subset \overline{\mathcal{H} \mathcal{T}} \times X.
\]

We will show that \( \Phi \) is a graph of a function \( \phi : \overline{\mathcal{H} \mathcal{T}} \longrightarrow X \). Having proved this, it is easy to verify that \( \phi \) works: Since \( \Phi \) is closed, \( \phi \) is continuous. Also, \( \phi \) is an \( \mathcal{H} \)-homomorphism: If \( (S, x) \in \Phi \), then there is
a net \( \{ h_i : i \in I \} \) such that \( \{ h_i T : i \in I \} \) converges to \( S \) and \( \{ h_i x_0 : h_i \in I \} \) converges to \( x \). Let \( h \in H \). Then \( (h h_i T, h_i x_0) \in \Phi \) for every \( i \in I \), and \( \{ (h h_i, h_i x_0) : i \in I \} \) converges to \( (h S, h x) \in \Phi \). It follows that \( \phi(h S) = h x = h \phi(T) \). Finally, since \( H x_0 \) is dense in \( X \), \( \phi \) is surjective.

First, let us show that for every \( S \) in \( \mathcal{HT} \) there is an \( x \in X \) such that \( (S, x) \in \Phi \). Indeed, let \( \{ h_i : i \in I \} \) be a net such that \( \{ h_i T \} \) converges to \( S \). Then \( \{ h_i x_0 \} \) is a net in \( X \), so by compactness of \( X \) there is a subnet \( h_i^T x_0 \) converging to some \( x \in X \). Then \( (h_i^T, h_i^T x_0) \) converges to \( (S, x) \in \Phi \).

Second, we prove that if \( (S, x_1), (S, x_2) \in \Phi \), then \( x_1 = x_2 \). Let \( \{ h_i : i \in I \} \) and \( \{ g_j : j \in J \} \) be nets such that \( (h_i T, h_i x_0) \) is a net converging to \( (S, x_1) \) and \( (g_j T, g_j x_0) \) is a net converging to \( (S, x_2) \) and suppose that \( x_1 \neq x_2 \).

As \( X \) is a compact Hausdorff space, it is regular, so there are open neighbourhoods \( U_1, U_2 \) of \( x_1, x_2 \) respectively and \( V \) a neighbourhood of the diagonal \( \Delta = \{(x, x) : x \in X \} \) such that \( V \cap (U_1 \times U_2) = \emptyset \). Without loss of generality we may assume that \( h_i x_0 \in U_1 \) for every \( i \in I \) and \( g_j x_0 \in U_2 \) for every \( j \in J \).

For every \( y \in X \), there is a neighbourhood \( U_y \) of \( y \) such that \( U_y \times U_y \subset V \). Again by regularity, there are open \( \overline{V}_y \) for \( y \in X \) such that \( \overline{V}_y \subset U_y \). By compactness, we can find \( y_1, y_2, \ldots, y_n \in X \) such that \( X = \bigcup_{i=1}^n \overline{V}_{y_i} \).

Let \( \text{Homeo}(X) \) denote the group of homeomorphism of \( X \) with the compact open topology. Since the action of \( H \) on \( X \) is continuous, the map \( \phi : H \to \text{Homeo}(X) \) defined by \( \phi(h)(x) = h x \), is continuous. Since \( \overline{V}_{y_i} \subset U_{y_i} \) for every \( i \), the set

\[
O = \bigcap_{i=1}^n \{ f \in \text{Homeo}(X) : f(\overline{V}_{y_i}) \subset U_{y_i} \}
\]

is an open neighbourhood of the identity in \( \text{Homeo}(X) \). Let \( O_H = \phi^{-1}(O) \). Then \( O_H \) is an open neighbourhood of the identity element in \( H \) and whenever \( h \in O_H \), then \( h \overline{V}_{y_i} \subset U_{y_i} \) for \( i = 1, 2, \ldots, n \), so \((y, hy) \in V \) for every \( y \in X \). Since the topology on \( H \) is determined by finite substructures of \( \mathcal{A} \), there exists \( B \in \text{Age}(\mathcal{A}) \) such that \( G_B \subset O_H \). Since \( \{ (h_i T, h_i x_0) : i \in I \} \) converges to \( (S, x_1) \) and \( \{ (g_j T, g_j x_0) : j \in J \} \) converges to \( (S, x_2) \), there are \( i_0 \in I \) and \( j_0 \in J \) such that \( h_{i_0} x_0 \in U_1, g_{j_0} x_0 \in U_2 \) and

\[
h_{i_0} T | B = g_{j_0} T | B = S | B.
\]

That is to say, for every \( b_1, b_2 \in B \),

\[
h_{i_0}^{-1}(b_1) T h_{i_0}^{-1}(b_2) \leftrightarrow g_{j_0}^{-1}(b_1) T g_{j_0}^{-1}(b_2).
\]

If we denote \( h_{i_0}^{-1} B = C \) and \( g_{j_0}^{-1} B = D \), then \( (C, T | C) \) and \( (D, T | D) \) are isomorphic via \( \rho : h_{i_0}^{-1} B \leftrightarrow g_{j_0}^{-1} B \). Since \( \mathcal{A}, T \) is \( \omega \)-homogeneous, there exists \( r \in H_T \) extending \( \rho \) to all \( (\mathcal{A}, T) \). It means that \( r h_{i_0}^{-1}(b) = g_{j_0}^{-1}(b) \) for every \( b \in B \), in other words \( r h_{i_0}^{-1} B = g_{j_0}^{-1} B \). By the choice of \( B \), \( g_{j_0} r h_{i_0}^{-1} \in O_H \). So \( (h_{i_0} x_0, g_{j_0} r h_{i_0}^{-1}(h_{i_0} x_0)) \in V \). But \( r \in H_T \), so \( r x_0 = x_0 \) and therefore \( (h_{i_0} x_0, g_{j_0} x_0) \in V \). But also \( (h_{i_0} x_0, g_{j_0} x_0) \in U_1 \times U_2 \), which is a contradiction.

(b) Since \( H_T \) is extremely amenable, every \( H \)-flow has a fixed point under the restricted action by \( H_T \). In particular, every minimal flow has a fixed point. Every point of a minimal flow has a dense orbit, so \( \mathcal{HT} \) is universal among all minimal \( H \)-flows by part (a). Since \( \text{Age}(\mathcal{A}, T) \) satisfies the expansion property, \( \mathcal{HT} \) is a minimal \( H \)-flow by the previous theorem. Altogether we get that \( \mathcal{HT} \) is the universal minimal flow of \( H \).
Corollary 5. Let \((A, T)\) and \(H\) be as above and let \(G\) be a dense subgroup of \(H\). Then the universal minimal flow of \(G\) is \(G^H\).

4.2.1 Jónsson classes

In the previous section, we presented a method which shows how to compute universal minimal flows of groups of automorphisms of structures provided that the structures admit a certain relational expansion. To obtain corresponding results for Jónsson classes, we require that the structures of those classes permit an expansion by countably many relations to a Jónsson class satisfying a mild condition of reasonability as in the case of Fraïssé classes and linear orders in [34].

Definition 14 (Reasonable class). Let \(L \supset \{R\}\) be a language with \(R\) a set of countably many relational symbols. Let \(K_R\) be a class of \(L\)-structures. Let \(L_0 = L \setminus \{R\}\) and \(K = K_R|L_0\). We say that \(K_R\) is reasonable, if whenever \(A, B \in K\), \(A\) is a substructure of \(B\) and \(S\) is an interpretation of \(R\) on \(A\) with \((A, S) \in K_R\), there exists an interpretation \(S'\) of \(R\) on \(B\) such that \((B, S') \in K_R\) and \((A, S)\) is a substructures of \((B, S')\).

The following proposition is Proposition 5.2 from [34] adjusted to Jónsson classes and precompact expansions.

Proposition 5. Suppose that \(\alpha\) is a positive ordinal such that \(\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha\). Let \(K_R\) be a Jónsson class for \(\alpha\) in a language \(L \supset \{R\}\) with \(R\) a set of countably many relations. Suppose that \(K_R\) is closed under substructures. Let \((A, T)\) be the \((\aleph_\alpha, K_R)\)-universal homogeneous structure. Set \(L_0 = L \setminus \{R\}\) and \(K = K_R|L_0\). Then the following are equivalent:

(a) \(K_R\) is reasonable,

(b) \(K\) is a Jónsson class and \(A = (A, T)|L_0\) is an \((\aleph_\alpha, K)\)-universal homogeneous structure

Proof. \((\Rightarrow)\) Obviously, \(K\) satisfies conditions I’,II,V. and IV. To verify the joint embedding property III., let \(B, C \in K\). Let \(S, S'\) interpretation of \(R\) on \(B, C\) respectively such that \((B, S), (C, S') \in K_R\). Since \(K_R\) satisfies III., there is \(D_R \in K_R\) such that \((B, S), (C, S') \subseteq D_R\). Then \(D = D_R|L_0\) is a witness of a joint embedding of \(B\) and \(C\) in \(K\).

It remains to show the amalgamation property IV’. Fix \(B, C, D \in K\) and embeddings \(i : B \longrightarrow C\) and \(j : B \longrightarrow D\). Let \(S\) be an ordering on \(B\) such that \((B, S) \in K_R\). Since \(K_R\) is reasonable, we can find linear orders \(S', S''\) on \(C, D\) respectively such that \((C, S'), (D, S'') \in K_R\) and \(i : (B, S) \longrightarrow (C, S'), j : (B, S) \longrightarrow (D, S'')\) are still embeddings. Amalgamation property for \(K_R\) provides us with \(E_R \in K_R\) and embeddings \(k : (C, S') \longrightarrow E_R, l : (d, S'') \longrightarrow E_R\) such that \(k \circ i = l \circ j\). Let \(E = E_R|L_0\) and embeddings \(k : C \longrightarrow E, l : D \longrightarrow E\) are witnesses of amalgamation of \(C\) and \(D\) over \(B\) in \(K\).

Finally, we check that \(A\) is \((\aleph_\alpha, K)\)-universal and homogeneous. Let \(B \in K\) of cardinality \(\leq \aleph_\alpha\) and let \(S\) be an interpretation of \(R\) on \(B\) such that \((B, S) \in K_R\). Since \((A, T)\) is \((\aleph_\alpha, K_R)\)-universal, there is an embedding \(i : (B, S) \longrightarrow (A, T)\) which is also an embedding from \(B\) to \(A\). It means that \(A\) is \((\aleph_\alpha, K)\)-universal.

To show that \(A\) is \((\aleph_\alpha, K)\)-homogeneous, it is enough to check the extension property in Proposition 1. For that, let \(B \leq C\) be structures in \(K\) with cardinality \(< \aleph_\alpha\) and let \(i : B \longrightarrow A\) and \(j : B \longrightarrow C\) be embeddings. Denote by \(S = i^{-1}(T|i(B))\). Then \((B, S) \in K_R\). Since \(K_R\) is reasonable, there exists \(S'\)
on $C$ with $(C, S') \in \mathcal{K}_R$ such that $j : (B, S) \longrightarrow (C, S')$ is also an embedding. Since $(A, T)$ is $(\aleph_\alpha, \mathcal{K}_R)$-homogeneous, it satisfies the extension property, i.e. there is an embedding $k : (C, S') \longrightarrow (A, T)$ such that $i = k \circ j$ and we are done.

$(\Leftarrow)$ Fix $B, C \in \mathcal{K}$ and an embedding $i : B \longrightarrow C$. Let $S$ be an interpretation of $R$ on $B$ such that $(B, S) \in \mathcal{K}_R$. Then there is an embedding $j : (B, S) \longrightarrow (A, T)$, which is of course also an embedding from $B$ to $A$. Since $A$ is homogeneous, it satisfies the extension property in Proposition 1., so there is an embedding $k : C \longrightarrow A$ extending $j$. Let $S' = j^{-1}(T \mid j(C))$. Then $(C, S') \in \mathcal{K}_R$ and $i : (B, S) \longrightarrow (C, S')$ is an embedding. \hfill \square

We are ready to apply Theorem 51 and Theorem 33 to Jónsson structures. We remind the reader that the structures need not be countable.

**Theorem 34.** Let $L \supset \{R\}$ be a relational signature and let $\alpha$ be a positive ordinal such that $\aleph_\alpha^\text{\textless} = \aleph_\alpha$. Let $\mathcal{K}_R$ be a reasonable Jónsson class in the signature $L$ with $R$ a countable set of relational symbols and let $\mathcal{K}_R$ be closed under substructures. Denote by $(\mathcal{A}, T)$ the $(\aleph_\alpha, \mathcal{K}_R)$-universal homogeneous structure.

(a) If $\text{Age}(\mathcal{A}, T)$ satisfies the Ramsey property and $G_R$ is a dense subgroup of $\text{Aut}(\mathcal{A}, T)$, then $G_R$ is extremely amenable.

(b) If $\text{Age}(\mathcal{A}, T)$ satisfies both the Ramsey property and the expansion property, then the universal minimal flow of any dense subgroup $G$ of $\text{Aut}(\mathcal{A})$ is $G < \mathbb{G}$.

Let us turn to concrete examples of Jónsson classes to which we can apply the above theorem.

**Proposition 6.** Let $\mathcal{K}$ be one of the following classes:

- graphs or graphs with arbitrary linear orderings,
- $K_n$-free graphs or $K_n$-free graphs with arbitrary linear orderings,
- hypergraphs or hypergraphs with arbitrary linear orderings,
- $\mathcal{A}$-free hypergraphs or $\mathcal{A}$-free hypergraphs with arbitrary linear orderings, where $\mathcal{A}$ is a class of finite irreducible hypergraphs,
- posets or posets with linear orderings extending the partial order.

Then $\mathcal{K}$ is a Jónsson class for every positive ordinal $\alpha$ and it is closed under substructures.

**Proof.** It is easy to see that in all cases, $\mathcal{K}$ satisfies conditions I.,II., V. and that it is closed under substructures which is a strengthening of VI. for every $\alpha$. We show that they also satisfy III. and IV.:

To satisfy the joint embedding property III. for structures $A, B \in \mathcal{K}$, we take $C$ to be the disjoint union of $A$ and $B$ and the embeddings to be the identity. If $A, B$ also possess a linear order, then let elements of $A$ precede elements of $B$ in $C$.

The amalgamation property IV’. is proved similarly as in the case of finite structures: Let $A, B, C \in \mathcal{K}$ and let $i : A \longrightarrow B, j : A \longrightarrow C$ be embeddings. Let the underlying set of $D$ be a quotient of the disjoint union of $B$ and $C$ via an equivalence relation $\sim$, where $b \sim c$ if and only if $b \in B, c \in C$ and there is an $a \in A$ such that $i(a) = b, j(a) = c$. Let $k : B \longrightarrow D$ and $l : C \longrightarrow D$ be the identity injections. Then obviously, $k \circ i = l \circ j$. Now we equip $D$ with a structure of the correct type to make sure that $k$ and $l$ are embeddings:
If $\mathcal{K}$ is a class of hypergraphs in a signature $L$ and $R_i \in L$ has arity $n$, then
\[(d_1, d_2, \ldots, d_n) \in R^D_i \subset D^n\]
if and only if either
\[(k^{-1}(d_1), k^{-1}(d_2), \ldots, k^{-1}(d_n)) \in R^B_i \subset B^n\]
or
\[(l^{-1}(d_1), l^{-1}(d_2), \ldots, l^{-1}(d_n)) \in R^C_i \subset C^n.\]

It means that $R^D_i = R^B_i \cup R^C_i$ when we identify $B, C$ with their corresponding images $k(B), l(C)$. Notice that $D$ will be a graph ($K_n$-free graph, $A$-free hypergraph) if $A, B, C$ are.

If $\mathcal{K}$ is the class of posets with $\mathcal{S}$ the symbol for the partial order, then $\mathcal{S}^D$ is the transitive closure of $(k \times k)(\mathcal{S}^B) \cup (l \times l)(\mathcal{S}^C)$.

If $\mathcal{K}$ is a class of linearly ordered structures with $<$ the symbol for the linear order, then $<^D$ on $D$ is an arbitrary linear ordering extending the transitive closure of $(k \times k)(<^B) \cup (l \times l)(<^C)$. In the next section, we will need to be more careful when amalgamating linear orders: we will set elements of $B$ to precede elements of $C$ whenever we can. Formally, for $b \in B$ denote by $(-, b) = \{a \in A : k(a) <^B b\}$ and for $c \in C$ denote by $(-, c) = \{a \in A : l(a) <^C c\}$. Let $<^D$ be the extension of both $k \times k(<^B)$ and $l \times l(<^C)$ such that whenever $b \in B, c \in C$ then $k(b) <^D l(c)$ if and only if $(-, b) \subseteq (-, c)$. It is easy to see that $<^D$ extends $<^D$ if $\mathcal{K}$ is the class of posets with a linear ordering extending the partial order. So in all cases, $\mathcal{K}$ is a Jónssson class for every $\alpha$.

We know that if $\mathcal{K}$ is one of the ordered classes in the proposition above, then finite structures in $\mathcal{K}$ satisfy the ordering property and the Ramsey property. Also, the ordered classes are obviously reasonable. Therefore, Theorem 34 applies to $\mathcal{K}$.

**Corollary 6.** Let $\kappa$ be a cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let $(\mathcal{A}, <)$ be the universal homogeneous ordered graph ($K_n$-free graph, hypergraph, $A$-free hypergraph, poset) of cardinality $\kappa$. Let $G_R$ be a dense subgroup of $\text{Aut}(\mathcal{A}, <)$ and let $G$ be a dense subgroup of $\text{Aut}(\mathcal{A})$. Then $G_R$ is extremely amenable and the universal minimal flow of $G$ is $\overline{G^<}$.

### 4.2.2 Constructing $\omega$-homogeneous structures

In this section, we overcome the restriction on the size of structures given by Jónssson’s construction for the price of losing universality and only keeping $\omega$-homogeneity. This is however sufficient to compute universal minimal flows of groups of automorphisms of such structures. Hence we obtain many more examples of explicitly computed universal minimal flows.

We present a construction of $\omega$-homogeneous (linearly ordered) graphs, $K_n$-free graphs, hypergraphs, $\mathcal{H}$-free hypergraphs (where $\mathcal{H}$ is a class of finite irreducible hypergraphs) and posets of arbitrary uncountable cardinality:

Let $\mathcal{A}$ be a graph ($K_n$-free graph, hypergraph, $\mathcal{H}$-free hypergraph, poset) of cardinality $\kappa$ and let $<$ be an arbitrary linear ordering on $\mathcal{A}$ (respectively an ordering extending the partial order if $\mathcal{A}$ is a poset). We will construct an $\omega$-homogeneous structure $\mathcal{A}_\kappa$, $<_\kappa$ of cardinality $\kappa$ in which $(\mathcal{A}, <)$ is embedded and such that $\mathcal{A}_\kappa$ itself is $\omega$-homogeneous.
By induction, we construct a chain of superstructures \(((A_\lambda, <_\lambda) : \lambda < \kappa)\) of \((A, <)\) such that \((A_\lambda, <_\lambda) \leq (A_\mu, <_\mu)\) whenever \(\lambda < \mu < \kappa\) and \((A_\kappa, <_\kappa) = \bigcup_{\lambda < \kappa} (A_\lambda, <_\lambda)\). In step \(\lambda\), we deal with a pair of isomorphic finite substructures \((F_\xi, <_\xi)\) and \((G_\lambda, <_\lambda)\) of \((A_\kappa, <_\kappa)\) and an isomorphism \(\phi_\lambda : F_\lambda \rightarrow G_\lambda\) and we construct \((A_{\lambda+1}, <_{\lambda+1})\) with an automorphism \(\psi_{\lambda+1} : A_{\lambda+1} \rightarrow A_{\lambda+1}\) extending \(\phi_\lambda\) that is order-preserving if \(\phi_\lambda\) is. Moreover, we make sure that if \(\xi < \lambda\) and \((F_\xi, <_\xi) = (F_\lambda, <_\lambda)\) and \((G_\xi, <_\xi) = (G_\lambda, <_\lambda)\), \(\phi_\xi = \phi_\lambda\) then \(\psi_\xi \in \psi_\lambda\).

To ensure that \(A_\kappa\) and \((A_\kappa, <_\kappa)\) are both \(\omega\)-homogeneous, we need to consider every triple \(((F_\lambda, <_\lambda), (G_\lambda, <_\lambda), \phi_\lambda : F_\lambda \rightarrow G_\lambda)\) cofinality of \(\kappa\)-many times for every \(\lambda < \kappa\). In order to do so, we fix a bookkeeping function a bijection \(f : \kappa \rightarrow \kappa \times \kappa\) - which we only need to satisfy that whenever \(f(\lambda) = (\mu, \xi)\), then \(\mu \leq \lambda\). We will describe \(f\) in detail later.

If \(\lambda\) is limit, then \((A_\lambda, <_\lambda) = \bigcup_{\mu < \lambda} (A_\mu, <_\mu)\). If \(\lambda = \mu + 1\), then we construct \((A_\lambda, <_\lambda)\) from \((A_\mu, <_\mu)\) as follows.

Let \((F_\mu, <_\mu), (G_\mu, <_\mu)\) be the pair of finite substructures of \((A_\mu, <_\mu)\) and \(\phi_\mu : F_\mu \rightarrow G_\mu\) the automorphism given by \(f\) in step \(\mu\) (as described later). Denote by \((B, <_B)\) the union of all \((A_\xi, <_\xi)\) such that \(\xi \leq \mu\) and \((F_\xi, <_\xi) = (F_\mu, <_\mu), (G_\xi, <_\xi) = (G_\mu, <_\mu)\) and \(\phi_\xi = \phi_\mu\) and let \(\psi_B\) be the union of the corresponding \(\psi_\xi\)’s. We need to set \(\psi_\lambda|B = \psi_\mu\). For every \(a \in A_\mu \setminus B\) we add two new points \(a^1\) and \(a^{-1}\) to be the image and the preimage of \(a\) under \(\psi_\lambda\) respectively. In the next step, we need to add for every \(a \in A_\mu \setminus B\) another two points \(a^2, a^{-2}\) to be the image of \(a^1\) and the preimage of \(a^{-1}\) under \(\psi_\lambda\) respectively. We continue in the same manner \(\omega\)-many times until we have taken care that every point has its image and preimage. Formally, let \(A^\#_\mu\) be a copy of \(A_\mu\) for \(z \in \mathbb{Z}\) and denote by \(a^z\) the element of \(A^\#_\mu\) corresponding to an \(a \in A_\mu\). We identify \(A^\#_{\mu + 1}\) with \(A_\mu\).

We examine two cases depending on whether the triple \(((F_\mu, <_\mu), (G_\mu, <_\mu), \phi_\mu)\) appears for the first time throughout the construction (i.e. \(B = \emptyset\)) or not. However, in both cases the construction follows a similar pattern.

**Case 1:** \(B = \emptyset\). Let \(F^z_\mu\) and \(G^z_\mu\) denote the corresponding copies of \(F_\mu\) and \(G_\mu\) in \(A^\#_\mu\) respectively. The underlying set of \(A_\lambda\) will be a quotient of the disjoint union of \(A^z_\mu\) for \(z \in \mathbb{Z}\) by an equivalence relation \(\sim\) gluing \(G^z_\mu\) with \(F^{z+1}_\mu\) via \(\phi_\mu:\)

\[
a^z_1 \in F^{z_1}_\mu, a^z_2 \in G^{z_2}_\mu \rightarrow (a^z_1 \sim a^z_2 \leftrightarrow (\phi_\mu(a_1) = a_2 \land z_1 = z_2 + 1)).
\]

If \(a^z \in A^z_\mu\) and \(a \notin F_\mu\) then we write \(a^z\) to mean its \(\sim\)-class \([a^z]_{\sim} = \{a^z\}\). If \(a \in F_\mu\), then we write \(a^z\) to mean its \(\sim\)-class \([a^z]_{\sim} = \{a^z, \phi_\mu(a)^{z-1}\}\).

Let us define \(\psi_\lambda : A_\lambda \rightarrow A_\lambda\) to be the mapping \(a^z \mapsto a^{z+1}\) whenever \(a \notin F_\mu\) and \(z \in \mathbb{Z}\) and \(a^z \mapsto \phi_\mu(a)^z\) whenever \(a \in F_\mu\).

If \(\phi_\mu\) is order preserving, let us transfer the linear order \(<_\mu\) on \(A_\mu\) to \(A^z_\mu\) for every \(z \in \mathbb{Z}\):

\[
a^z <^z_\mu b^z \text{ if and only if } \psi_\lambda^{-z}(a^z) <^z_\mu \psi_\lambda^{-z}(b^z)
\]

We define the structure on \(A_\lambda\) inductively using amalgamation as described in the previous section. If \(\phi_\mu\) is order preserving, then we use amalgamation for linearly ordered structures letting elements of \(A^{z_1}_\mu\) precede elements of \(A^{z_2}_\mu\) for \(z_1 < z_2\) whenever we can as described in the previous section. Otherwise, we use amalgamation for unordered structures and in the end obtain \(<_\lambda\) as an arbitrary linear extension of \(<_\mu\) (and extending the partial order on \(A_\lambda\) if \(A\) is a poset).

Let \(n \in \omega\) and let \(A^\#_{\lambda}^{-n}\) denote the subset of \(A_\lambda\) consisting of points in \(A^z_\mu\) for \(z \in \{-n + 1, -n + \ldots\}\).
2, \ldots, n}$, i.e.

$$A_{\lambda}^{n-} = \left( \bigcup_{z=-n+1}^{n} A_{\mu}^{z} \right) / \sim.$$ 

Similarly let $A_{\lambda}^{n+}$ denote the quotient

$$A_{\lambda}^{n+} = \bigcup_{z=-n}^{n} A_{\mu}^{z} / \sim$$

of the disjoint union of $A_{\mu}^{z}$ for $z \in \{-n, -n+1, \ldots, n\}$ by $\sim$. We have that $A_{\lambda}^{0} = A_{\mu}$. Suppose that the structure on $A_{\lambda}^{n+}$ has been defined. Then the structure on $A_{\lambda}^{(n+1)-}$ is given by amalgamating $A_{\lambda}^{n+}$ with $A_{\mu}^{n+1}$ along

$$F_{\mu}^{n+1} \longrightarrow A_{\lambda}^{(n+1)-}, a^{n+1} \mapsto (\phi_{\mu}(a))^{n} \text{ and } F_{\mu}^{n+1} \longrightarrow A_{\lambda}^{n+1}, a^{n+1} \mapsto a^{n+1}.$$ 

When the structure on $A_{\lambda}^{n-}$ has been defined, then the structure on $A_{\lambda}^{n+}$ is given by amalgamation of $A_{\lambda}^{n-}$ and $A_{\mu}^{n}$ along

$$F_{\mu}^{n-1} \longrightarrow A_{\lambda}^{n-}, a^{-n+1} \mapsto a^{-n+1} \text{ and } F_{\mu}^{n-1} \longrightarrow A_{\lambda}^{n}, a^{-n+1} \mapsto (\phi_{\mu}(a))^{-n}.$$ 

If $\phi_{\mu}$ is order preserving, then $A_{\lambda}^{n+}$ (respectively $A_{\lambda}^{n-}$) plays the role of $B$ and $A_{\mu}^{n+1}$ (respectively $A_{\lambda}^{n-}$) the role of $C$ in the amalgamation of linear orders described in the previous section.

We can see that $A_{\lambda}^{n+} \leq A_{\lambda}^{n+1} \leq A_{\mu}^{n+1}$ for every $n \in \omega$, so we can define the structure on $A_{\lambda}$ as the union of the chain $\langle A_{\lambda}^{n+} : n \in \omega \rangle$ (equivalently $\langle A_{\lambda}^{n-} : n \in \omega \rangle$):

$$A_{\lambda} = \bigcup_{n \in \omega} A_{\lambda}^{n+} = \bigcup_{n \in \omega} A_{\lambda}^{n-}.$$ 

Case 2: $B \neq \emptyset$. The underlying set of $A_{\lambda}$ will then be a quotient of the disjoint union of $A_{\mu}^{z}$ for $z \in \mathbb{Z}$ by an equivalence relation $\sim$ gluing corresponding copies of $B$ in $A_{\mu}^{z}$ via $\psi_{B}$:

$$a_{1}^{z_{1}} \in A_{\mu}^{z_{1}}, a_{2}^{z_{2}} \in A_{\mu}^{z_{2}} \mapsto (a_{1}^{z_{1}} \sim a_{2}^{z_{2}} \mapsto a_{1}, a_{2} \in B \land \psi_{B}^{-1}(a_{1}) = \psi_{B}^{-1}(a_{2})).$$ 

We identify $\psi_{B}^{0}$ with $\psi_{B}$.

If $a^{z} \in A_{\mu}^{z}$ and $a \notin B$ then we write $a^{z}$ to mean its $\sim$-class $[a^{z}]_{\sim} = \{a^{z}\}$. If $a \in B$, then we write $a$ to mean its $\sim$-class $[a]_{\sim} = \{a^{z} : z \in \mathbb{Z}\}$.

Let us define $\psi_{\lambda} : A_{\lambda} \longrightarrow A_{\lambda}$ to be the mapping $a^{z} \mapsto a^{z+1}$ whenever $a \notin B$ and $z \in \mathbb{Z}$ and $a \mapsto \psi_{B}(a)$ whenever $a \in B$.

As in Case 1, we can obtain the structure on $A_{\lambda}$ inductively by amalgamation along $B$ in $\omega$-many steps. However, we can obtain $A_{\lambda}$ via amalgamation of $A_{\mu}^{z}, z \in \mathbb{Z}$ along $B \longrightarrow A_{\mu}^{z}, b \mapsto \psi_{\lambda}(b)$ in one step:

- If $A$ is a hypergraph in a signature $L$ and $R_{i} \in L$ has arity $n$, then

  $$(a_{1}^{z_{1}}, a_{2}^{z_{2}}, \ldots, a_{n}^{z_{n}}) \in R_{i}^{A_{\lambda}},$$

  if and only if

  $$(z_{1} = z_{2} = \ldots = z_{n}) \land (\psi_{\lambda}^{-1}(a_{1}^{z_{1}}), \psi_{\lambda}^{-1}(a_{2}^{z_{2}}), \ldots, \psi_{\lambda}^{-1}(a_{n}^{z_{n}})) \in R_{i}^{A_{\mu}}.$$
Chapter 4. Automorphism groups

Notice that $A_\lambda$ will be a graph ($K_n$-free graph, $H$-free hypergraph) if $A_\mu$ is.

- If $A$ is a poset with a partial order $\prec$, then $\prec_\lambda$ is the transitive closure of the relation obtained as in the case of hypergraphs.

It remains to extend the linear order $\prec_\mu$ to a linear order $\prec_\lambda$ on $A_\lambda$. If $\phi_\mu$ is not order preserving, let $\prec_\lambda$ be an arbitrary linear ordering extending $\prec_\mu$ (and extending the partial order $\prec_\lambda$ on $A_\lambda$ in case that $A$ is a poset). If $\phi_\mu$ is order preserving, we will extend $\prec_\mu$ to $\prec_\lambda$ to make sure that $\psi_\lambda$ is order preserving.

We can also describe $\prec_\lambda$ explicitly, without an inductive construction. We again let elements in $A^{\mu}_\mu$ precede elements in $A^{\mu}_\mu$ whenever $z_1 < z_2$ and we are free to do so: Let $a^z \in A^{\mu}_\mu$ and denote by

$$(-, a^z) = \{ b \in B : b \prec_\mu \psi_\lambda^{-z}(a) \}.$$ 

Let $a^{z_1}_1 \in A^{\mu}_\mu$ and $a^{z_2}_2 \in A^{\mu}_\mu$ for some $z_1, z_2 \in \mathbb{Z}$ and $a^{z_1}_1 \neq a^{z_2}_2$.

- If $z_1 = z_2$, then $a^{z_1}_1 \prec_\mu a^{z_2}_2$ if and only if $\psi_\lambda^{-z_1}(a^{z_1}_1) \prec_\mu \psi_\lambda^{-z_2}(a^{z_2}_2)$.

- If $z_1 \neq z_2$, then $a^{z_1}_1 \prec_\mu a^{z_2}_2$ if and only if $(-, a^{z_1}_1) \subseteq (-, a^{z_2}_2)$ or $(-, a^{z_1}_1) = (-, a^{z_2}_2)$ and $z_1 < z_2$.

Obviously, $\prec_\lambda$ is antireflexive and antisymmetric. Let us check transitivity: Let $a^{z_1}_1, a^{z_2}_2, a^{z_3}_3 \in A_\lambda$ and $a^{z_1}_1 \prec_\mu a^{z_2}_2$ and $a^{z_2}_2 \prec_\mu a^{z_3}_3$. We have five possible cases:

1. $z_1 = z_2 = z_3$. Then $\psi_\lambda^{-z_1}(a^{z_1}_1) \prec_\mu \psi_\lambda^{-z_2}(a^{z_2}_2) \prec_\mu \psi_\lambda^{-z_3}(a^{z_3}_3)$ so $\psi_\lambda^{-z_2}(a^{z_2}_2) \prec_\mu \psi_\lambda^{-z_3}(a^{z_3}_3)$ by transitivity of $\prec_\mu$.

2. $z_1 = z_2 \neq z_3$. Then $\psi_\lambda^{-z_1}(a^{z_1}_1) \prec_\mu \psi_\lambda^{-z_2}(a^{z_2}_2)$, so $(-, a^{z_1}_1) \subseteq (-, a^{z_2}_2)$, and either $(-, a^{z_2}_2) \subseteq (-, a^{z_3}_3)$ or $(-, a^{z_2}_2) = (-, a^{z_3}_3)$ and $z_2 < z_3$.

3. $z_1 \neq z_2 = z_3$. Then $(-, a^{z_1}_1) \subseteq (-, a^{z_2}_2)$ or $(-, a^{z_1}_1) = (-, a^{z_2}_2)$ with $z_1 < z_2$ and $\psi_\lambda^{-z_2}(a^{z_2}_2) \prec_\mu \psi_\lambda^{-z_3}(a^{z_3}_3)$, so $(-, a^{z_2}_2) \subseteq (-, a^{z_3}_3)$.

4. $z_1 = z_3 \neq z_2$. Then $(-, a^{z_1}_1) \subseteq (-, a^{z_2}_2) \subseteq (-, a^{z_3}_3)$ and at least one inclusion is proper, since otherwise $z_1 < z_2 < z_3$. So we get $(-, a^{z_1}_1) \subseteq (-, a^{z_3}_3)$, which implies $\psi_\lambda^{-z_1}(a^{z_1}_1) \prec_\mu \psi_\lambda^{-z_3}(a^{z_3}_3)$.

5. $z_1 \neq z_2 \neq z_3, z_1 \neq z_3$. Then we have the following four cases:

(i) $(-, a^{z_1}_1) \subseteq (-, a^{z_2}_2) \subseteq (-, a^{z_3}_3)$.

(ii) $(-, a^{z_1}_1) \subseteq (-, a^{z_2}_2) = (-, a^{z_3}_3)$ and $z_2 < z_3$.

(iii) $(-, a^{z_1}_1) = (-, a^{z_2}_2) \subseteq (-, a^{z_3}_3)$ and $z_1 < z_2$

(iv) $(-, a^{z_1}_1) = (-, a^{z_2}_2) = (-, a^{z_3}_3)$ and $z_1 < z_2 < z_3$.

In all cases we get that $a^{z_1}_1 \prec_\lambda a^{z_3}_3$, hence $\prec_\lambda$ is transitive.

If $A$ is a poset with $\prec$ the symbol for the partial order and $\prec_\lambda$ is the extension of $\prec$ to $A_\lambda$ described above, let us verify that $\prec_\lambda$ extends $\prec_\lambda$: If $a^{z_1}_1, a^{z_2}_2 \in A^{\mu}_\mu$ for some $z \in \mathbb{Z}$, then $a^{z_1}_1 \prec_\mu a^{z_2}_2$ if and only if $a^{z_1}_1 \prec_\lambda a^{z_2}_2$ by definition. If $a^{z_1}_1 \in A^{\mu}_\mu, a^{z_2}_2 \in A^{\mu}_\mu$ for some $z_1 \neq z_2$ and $a^{z_1}_1 \prec_\lambda a^{z_2}_2$, then there is $a \in B$ such that $\psi_\lambda^{-z_1}(a^{z_1}_1) \prec_\mu a$ and $a \prec_\mu \psi_\lambda^{-z_2}(a^{z_2}_2)$. It means that $a \in (-, a^{z_1}_1)$ while $a \notin (-, a^{z_2}_2)$, showing that $a^{z_1}_1 \prec_\lambda a^{z_2}_2$. 
Let us go back to the definition of the book-keeping function \( f \): When \((\mathbb{A}_\mu, \prec_\mu)\) has been constructed, let \( \{(F^\xi_\mu, \prec^\xi_\mu), (G^\xi_\mu, \prec^\xi_\mu), \phi^\xi_\mu) \mid \xi < \kappa \} \) be an enumeration of all triples such that \((F^\xi_\mu, \prec^\xi_\mu)\) and \((G^\xi_\mu, \prec^\xi_\mu)\) are finite substructures of \((\mathbb{A}_\mu, \prec_\mu)\) and \(\phi^\xi_\mu\) is an isomorphism between \(F^\xi_\mu\) and \(G^\xi_\mu\) (this is possible since \(\kappa < \omega = \kappa\) for every infinite \(\kappa\)). If \( f(\mu) = (\nu, \xi) \), then the triple we consider in step \( \mu \) is \(((F^\xi_\mu, \prec^\xi_\mu), (G^\xi_\mu, \prec^\xi_\mu), \phi^\xi_\mu)\). The inequality \( \nu \leq \mu \) ensures that \(F^\xi_\mu\) and \(G^\xi_\mu\) are finite substructures of already defined \(\mathbb{A}_\nu \leq \mathbb{A}_\mu\). The inclusion \(\mathbb{A}_\nu \subset \mathbb{A}_\mu\) for \(\nu < \mu < \kappa\) provides that for every \(\mu < \kappa\), every triple \(((F_\mu, \prec_\mu), (\mathbb{G}_\mu), \phi)\) of two finite substructures of \((\mathbb{A}_\mu, \prec_\mu)\) and an isomorphism between \(F_\mu\) and \(G_\mu\) will be considered unboundedly many times throughout our construction.

The outcome of the construction can be formulated as the following theorem.

**Theorem 35.** Let \( \mathcal{A} \) be a graph \((K_n\)-free graph, hypergraph, \(\mathcal{H}\)-free hypergraph, poset) of an infinite cardinality \(\kappa\) and let \(<\) be an arbitrary linear ordering on \(\mathcal{A}\) (respectively an ordering extending the partial order if \(\mathcal{A}\) is a poset). Then there exists a linearly ordered graph \((K_n\)-free graph, hypergraph, \(\mathcal{H}\)-free hypergraph, poset) \((\mathcal{A}', <')\) of cardinality \(\kappa\) in which \((\mathcal{A}, <)\) is embedded and such that both \(\mathcal{A}'\) and \((\mathcal{A}', <')\) are \(\omega\)-homogeneous (and if \(\mathcal{A}\) is a poset, <' extends the partial order on \(\mathcal{A}'\)).

If we start with \( \mathcal{A} \) a graph \((K_n\)-free graph, hypergraph, \(\mathcal{H}\)-free hypergraph, poset) and a linear order \(<\) on \(\mathcal{A}\) such that \(\text{Age}(\mathcal{A}, <)\) is the class of all finite linearly ordered graphs \((K_n\)-free graphs, hypergraphs, \(\mathcal{H}\)-free hypergraphs) or posets with the linear order extending the partial order, the construction provides us with a structure for which we are able to compute the universal minimal flow of its group of automorphisms. This can easily be arranged for instance by requiring that \((\mathcal{A}, <)\) contains a copy of the countable \(\omega\)-homogeneous ordered graph \((K_n\)-free graph, hypergraph, \(\mathcal{H}\)-free hypergraph, poset).

**Theorem 36.** Let \( \mathcal{A} \) be an \(\omega\)-homogeneous graph \((K_n\)-free graph, hypergraph, \(\mathcal{H}\)-free hypergraph, poset) and let \(<\) be a linear ordering on \(\mathcal{A}\) (extending the partial order if \(\mathcal{A}\) is a poset) such that \((\mathcal{A}, <)\) is \(\omega\)-homogeneous as well. Suppose that \(\text{Age}(\mathcal{A}, <)\) is the class of all finite linearly ordered graphs \((K_n\)-free graphs, hypergraphs, \(\mathcal{H}\)-free hypergraphs) or posets with linear orderings extending the partial order.

If \( G \) is a dense subgroup of \( \text{Aut}(\mathcal{A}) \), then the universal minimal flow of \( G \) is the space of all linear orderings on \( \mathcal{A} \) (respectively the space of all linear orderings extending the partial order if \( \mathcal{A} \) is a poset).

### 4.3 Dual approach

In the previous section, we were able to compute universal minimal flows of groups of automorphisms of locally-finite \(\omega\)-homogeneous structures that allow an \(\omega\)-homogeneous expansion by countably many relations such that the age of the expanded structure is a Ramsey class and a precompact expansion of the original class. While this is not a restriction for countable structures, there is no known construction of \(\omega\)-homogeneous expansions for uncountable structures even if the classes of finite substructures satisfy the desired properties. For instance, if \( G \) is an arbitrary uncountable \(\omega\)-homogeneous graph, we do not in general know whether there exists a linear ordering \(<\) on \( G \) such that \((G, <)\) is \(\omega\)-homogeneous. We do not know whether existence of such an ordering follows from the theorem below which lets us identify the universal minimal flow of \( \text{Aut}(G) \) with the space of all linear orderings on \( G \). We will use the description of the greatest ambit and the universal minimal flow as Stone spaces of certain Boolean algebras provided in Theorem 29 to avoid the obstruction. This allows us to fully generalize the theory of Kechris, Pestov and Todorčević to uncountable structures.
Let us first introduce some notation. Let $L$ be a language and let $L^*$ be its relational expansion by a set $R$ of countably many relations. Let $K$ be a class of finite structures in $L$ and $K_R$ a class of finite structures in $L^*$ such that $K_R$ is a precompact expansion of $K$. Let $A$ be a structure with $\text{Age}(A) = K$. Similarly as for linear orderings, we say that an interpretation $T$ of $R$ on $A$ is normal if $\text{Age}(A,T) \subseteq K_R$. We denote the space of all normal interpretations of $R$ on $A$ by $X_R$. A basis of open sets in $X_R$ consists of sets of the form

$$(B,S)^* = \{ x \in X_R : x|B = (B,S) \}$$

for $B$ a finite substructure of $A$ and $S$ interpretations of $R$ on $B$ such that $(B,S)$ is isomorphic to a structure in $K_R$.

If $R = \{ < \}$ and $<$ in every structure in $K_R = K_<$ is interpreted as a linear ordering, we call the elements of $X_R = X_<$ normal linear orderings and often denote the space of all normal linear orderings by $\text{NO}_{K_<}(A)$.

Originally, the author proved in [8] results analogous to those in [34] for uncountable structures in a special case of order expansions that are order-forgetful. Recall that an order expansion $K$ of a class of structures is order-forgetful whenever

$$(A,\prec),(B,\prec) \in K \text{ and } A \cong B \text{ imply } (A,\prec) \cong (B,\prec). \quad (4.1)$$

**Theorem 37.** [8] Let $A$ be a locally-finite $\omega$-homogeneous structure such that $\text{Age}(A)$ satisfies the Ramsey property. Suppose that $K$ is a Fraïssé order class that is an order-forgetful expansion of $\text{Age}(A)$. Then the space $\text{NO}_K(A)$ of all linear orderings $\prec$ on $A$ such that $\text{Age}(A,\prec) = K_R$ is the universal minimal flow for every dense subgroup of $\text{Aut}(A)$.

The following proposition reasons why we restrict ourselves to precompact expansions. It provides an alternate proof of Proposition 4 by Nguyen van Thé about countable structures and covers uncountable structures as well.

**Proposition 7.** Let $A$ be a structure with $\text{Age}(A) = K$. $X_R$ is compact if and only if $K_R$ is a precompact expansion of $K$.

**Proof.** Suppose first that $X_R$ is compact and let $A \in K$. Let $\{(A,S_i) : i \in I\}$ be the collection of all expansions of $A$ belonging to $K_R$ for some index set $I$. Then $\{(A,S_i)^* : i \in I\}$ is an open cover of $X_R$ consisting of mutually disjoint sets and therefore having no proper subcover. By compactness of $X_R$, it follows that $I$ is finite.

Suppose now that $K_R$ is a precompact expansion of $K$. For a relation $r \in R$, let $a_r$ denote the arity of $r$. To prove that $X_R$ is compact, it is enough to show that $X_R$ is a closed subspace of $\prod_{r \in R} \{0,1\}^{a_r}$. For $(A,S) \in K_R$, let $(\widehat{A},\widehat{S})$ denote the clopen set $\{ x \in \prod_{r \in R} \{0,1\}^{a_r} : x|A = S \}$. Note that we can write

$$X_R = \bigcap_{A \in K} \bigcup_{(A,S) \in K_R} (A,S).$$

Since each $A \in K$ has only finitely many expansions $(A,S) \in K_R$, the union $\bigcup_{(A,S) \in K_R} (A,S)$ is closed. We get that $X_R$ is an intersection of closed subsets of $\prod_{r \in R} \{0,1\}^{a_r}$ and therefore closed and consequently compact. \qed
The following two theorem characterize when \( X_R \) is a minimal flow.

**Theorem 38.** Let \( \mathcal{A} \) be a locally-finite \( \omega \)-homogeneous structure and let \( K_R \) be a Fraïssé precompact expansion of \( \text{Age}(\mathcal{A}) \). Then \( X_R \) is a minimal flow if and only if the isomorphism types of \( \text{Age}(\mathcal{A}, x) \) and \( \text{Age}(\mathcal{A}, y) \) form the same class for every \( x, y \in X_R \).

**Proof.** Let \( x, y \in X_R \) and \( A \in \text{Age}(\mathcal{A}, x) \). If \( X_R \) is minimal, then there is \( g \in G \) such that \( gA|y = x|A \). It follows that \( A \) is isomorphic to a structure in \( \text{Age}(\mathcal{A}, y) \).

Conversely, if \( A \) is isomorphic to \( B \in \text{Age}(\mathcal{A}, y) \) via a partial automorphism \( g : A \rightarrow B \), there is an extension \( \tilde{g} \in G \) of \( g \) by \( \omega \)-homogeneity of \( \mathcal{A} \). It follows that \( x \in G_{\tilde{g}}y \) for every \( x, y \in X_R \) witnessing the minimality of \( X_R \). \( \square \)

**Theorem 39.** Let \( \mathcal{A} \) be a locally-finite \( \omega \)-homogeneous structure and let \( K_R \) be a Fraïssé precompact expansion of \( \text{Age}(\mathcal{A}) = \mathcal{K} \). Then the following are equivalent.

1. \( K_R \) satisfies the expansion property relative to \( \text{Age}(\mathcal{R}) \).
2. \( X_R \) is a minimal flow and there is \( x \in X_R \) such that \( \text{Age}(\mathcal{A}, x) \subset K_R \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( x \in X_R \) and let \( (B, S)^* \subset X_R \) be a non-empty open basic set for some \( B \) a finite substructure of \( \mathcal{A} \) and \( (B, S) \) isomorphic to a structure in \( K_R \). Then

\[
\text{Ret}(x, (B, S)^*) = \{ g \in \text{Aut}(\mathcal{A}) : gx \in (B, S)^* \} = G_B M
\]

for some \( M \subset G \).

Due to Lemma 9, in order to prove that \( X_R \) is minimal, we need to show that \( G_B M \) is left syndetic. Let \( D \in \mathcal{K} \) be given by the expansion property for \( B \) and let \( \{ g_i : i \in I \} \) be the set of all embeddings of \( B \) into \( D \) and let \( \{ f_i : i \in I \} \) be automorphisms of \( \mathcal{A} \) extending \( g_i \) for each \( i \in I \) provided by \( \omega \)-homogeneity of \( \mathcal{A} \). Since \( D \) is finite, the set \( I \) is finite.

**Claim 1.** \( \bigcup_{i \in I} f_i G_B M = G \).

**Proof of claim.** Let \( g \in G \) and let \( D' = g^{-1} D \). By the expansion property, there is \( (B', S') \leq (D', x|D') \) such that \( (B', S') \) is isomorphic to \( (B, S) \) via an isomorphism \( h \). We have that \( g \) maps \( B' \) into \( D \) and therefore there is \( i \in I \) such that \( f_i^{-1} g|B' = h \). It follows that \( f_i^{-1} g \in G_B M \). \( \square \)

We finish up by showing that the expansion property implies that \( \text{Age}(\mathcal{A}, x) = K_R \) for every \( x \in X_R \). Indeed, let \( x \in X_R \) and \( B \in K \). Let \( D \in K \) witness the expansion property for \( B \). Let \( (B, S) \in K_R \). Then there is an embedding of \( (B, S) \) into \( (D, x|D) \) witnessing that \( (B, S) \in \text{Age}(\mathcal{A}, x) \).

(2) \( \Rightarrow \) (1) Assume that \( X_R \) is a minimal flow and let \( x \in X_R \) be such that \( \text{Age}(\mathcal{A}, x) = K_R \). By Theorem 38, \( \text{Age}(\mathcal{A}, x) = \text{Age}(\mathcal{A}, y) \) for every \( x, y \in X_R \), so \( x \) can be an arbitrary element of \( X_R \). For every \( B \in K \) we will find \( D \in K \) that witnesses the expansion property for \( B \). By minimality of \( X_R \), for every \( S \) with \( (B, S) \in K_R \) the set \( \text{Ret}(x, (B, S)^*) \) is left syndetic. Let \( g_1^S, g_2^S, \ldots, g_{n_S}^S \) be such that

\[
\bigcup_{i=1}^{n_S} g_i^S \text{Ret}(x, (B, S)^*) = G.
\]
Let $F \subset G$ be the union of $\{g_i^S, g_2^S, \ldots, g_n^S\}$ for all $S$ such that $(B, S) \in \mathcal{K}_R$. Since $\mathcal{K}_R$ is a precompact expansion of $\mathcal{K}$, $F$ is finite. We claim that $D \leq A$ generated by

$$\bigcup_{f \in F} fB$$

witnesses the expansion property for $B$. Let $S, T$ be such that $(B, S), (D, T) \in \mathcal{K}_R$ and let $(D, T) \hookrightarrow (A, x)$ be an embedding with image $(E, x|E)$. Choose an automorphism $g : A \to A$ that extends an isomorphism from $E$ to $D$. There is $i \in \{1, \ldots, n_S\}$ such that

$$(g_i^S)^{-1}g = k \in \text{Ret}(x, (B, S)^*)$$.

It means that $k^{-1} : (B, S) \to (k^{-1}B, x|k^{-1}B)$ is order preserving and

$$k^{-1}B = ((g_i^S)^{-1}g)^{-1}B = g^{-1}g_i^S B \leq (E, x|E)$$.

So $k^{-1}$ is an embedding of $(B, S)$ into $(E, x|E) \cong (D, T)$.

\[\square\]

**Theorem 40.** Let $A$ be a locally-finite $\omega$-homogeneous structure and let $\mathcal{K}_R$ be a Fraïssé precompact expansion of $\text{Age}(A) = \mathcal{K}$ consisting of rigid structures. Suppose that $\mathcal{K}_R$ satisfies the Ramsey property and the expansion property relative to $\mathcal{K}$. Then $X_R$ is the universal minimal flow of every dense subgroup $G$ of the group of automorphisms of $A$.

**Proof.** Let $x \in X_R$. According to Theorem 29, we need to prove that the algebra $B$ of sets $\text{Ret}(x, (B, S)^*)$ for $B$ a finite substructure of $A$ and $S$ such that $(B, S)$ is isomorphic to a structure in $\mathcal{K}_R$ is a maximal left syndetic subalgebra of the algebra $L = \{G_B M : M \subset G, B \leq A \text{ finite}\}$. By Theorem 39, $X_R$ is minimal. Henceforth, $B$ consists of left syndetic sets and $B$ is closed under left translations.

To show maximality of $B$, assume that there is $M \subset G$ and $B \leq A$ finite such that $G_B M \not\subset B$ and the algebra generated by $G_B M$ and $B$ is still a left syndetic algebra. Since $G_B M \not\subset B$, there is $(B, S)^* \subset X_R$ such that

$$\text{Ret}(x, (B, S)^*) \cap G_B M = G_B N \neq \emptyset \neq G_B P = \text{Ret}(x, (B, S)^*) \setminus G_B N$$

are left syndetic. Let $g_1, g_2, \ldots, g_n$ witness their left syndeticity, it means

$$\bigcup_{i=1}^n g_i G_B N = G = \bigcup_{i=1}^n g_i G_B P$$.

Set $C$ to be the finite substructure of $A$ generated by $\bigcup_{i=1}^n g_i B$.

Since $\mathcal{K}_R$ consists of rigid elements, we can view $\text{Ret}(x, (B, S)^*)$ as all copies of $(B, S)$ in $(A, x)$ via the one-to-one correspondence.

$$g \in \text{Ret}(x, (B, S)^*) \leftrightarrow (g^{-1}(B), x|g^{-1}(B))$$.

Under this identification, $\text{Ret}(x, (B, S)^*) = G_B N \cup G_B P$ corresponds to all copies of $(B, S)$ in $(A, x)$, denoted by $\left(\frac{A, x}{(B, S)}\right)$. 
Define a colouring
\[ c : \left( (A, x), (B, S) \right) \rightarrow \{ GBN, GBP \} \]
by \( c(B', S') = GBN \) if and only if there is \( g \in GBN \) such that \( g^{-1}B = B' \) and \( c(B', S') = GBP \) if there is \( g \in GBP \) with \( g^{-1}B = B' \). Since \( K_R \) is a Ramsey class, there is a copy \((C', T')\) of \((C, x|C)\) in \((A, x)\) that is monochromatic, say in colour \( GBN \). Let \( h : A \rightarrow A \) be an automorphism extending the isomorphism from \((C', T')\) to \((C, x|C)\). We claim that \( h \not\in \bigcup_{i=1}^{n} g_i GBP \). Suppose on the contrary that there is \( i \in \{1, 2, \ldots, n\} \) and \( k \in GBP \) such that \( h = g_i k \). We then have \( k = g_i^{-1}h \) and so \( k^{-1}(B, S) = (g_i^{-1}h)^{-1}(B, S) = h^{-1}g_i(B, S) \leq (C', T') \) which contradicts that all copies of \((B, S)\) in \((C', T')\) have colour \( GBN \). The proof for colour \( GBP \) is analogous.

We have shown that the condition on the existence of an \( \omega \)-homogeneous expansion in the theory of Kechris, Pestov and Todorčević is not necessary. However, we do not know whether it is automatically satisfied when all the other conditions are.

**Question 1.** Let \( A \) be a structure satisfying the assumptions of Theorem 40. Is there \( S \in X_R \) such that \((A, S)\) is \( \omega \)-homogeneous?

### 4.3.1 Applications

We apply Theorem 40 to concrete structures and compute universal minimal flows of their automorphism groups.

#### Group of bijections

Since the class of finite linear orders is an order-forgetful expansion of the class of finite sets satisfying the Ramsey property, we can provide an alternate proof for the result of Glasner and Weiss ([21]) about \( S_\omega \) and of Pestov ([55]) about \( S_\kappa \) for arbitrary infinite \( \kappa \).

**Theorem 41.** The universal minimal flow of \( S_\kappa \) is \( LO(\kappa) \), the space of all linear orderings on \( \kappa \).

#### Graphs

The class of finite graphs is a Fraïssé class and the class of finite linearly ordered graphs is its reasonable Fraïssé expansion satisfying the ordering and the Ramsey properties ([48]). The ordering property can be proved as a special instance of the Ramsey property. The same holds about the pair of classes of finite \( K_n \)-free graphs and their expansion with arbitrary linear orderings for any natural number \( n \geq 2 \).

We can thus generalize a theorem of Kechris, Pestov, Todorčević about countable graphs to uncountable ones.

**Theorem 42.** Let \( G \) be an \( \omega \)-homogeneous graph or an \( \omega \)-homogeneous \( K_n \)-free graph for \( n = 2, 3, \ldots \). Then the universal minimal flow of \( Aut(G) \) is \( LO(G) \), the space of all linear orderings on \( G \).

#### Hypergraphs

The class of finite hypergraphs is a Fraïssé class and the class of finite linearly ordered hypergraphs is its reasonable Fraïssé expansion satisfying the ordering and the Ramsey properties ([48] and [1]). The
ordering property can be proved as a special instance of the Ramsey property. The same holds about the pair of classes of finite $A$-free hypergraphs and their expansions with arbitrary linear orderings, where $A$ is a class of finite irreducible hypergraphs.

We can thus generalize a theorem of Kechris, Pestov, Todorčević about countable hypergraphs to uncountable ones.

**Theorem 43.** Let $H$ be an $\omega$-homogeneous hypergraph or an $\omega$-homogeneous $A$-free hypergraph for $A$ a class of irreducible finite hypergraphs. Then the universal minimal flow of $\text{Aut}(H)$ is $\text{LO}(H)$, the space of all linear orderings on $H$.

**Posets**

The class of finite posets admits a reasonable expansion by linear orderings extending the partial order to a Fraïssé class satisfying the Ramsey and the ordering properties (see [64]). Theorem 40 thus gives us the following generalization of Sokić’s result for countable posets.

**Theorem 44.** Let $\mathcal{P}$ be an $\omega$-homogeneous poset and let $\mathcal{K}$ be the class of finite linearly ordered posets such that the linear orderings extend the partial order. Then the universal minimal flow of $\text{Aut}(\mathcal{P})$ is $\text{NO}_K(\mathcal{P})$, the space of normal ordering on $\mathcal{P}$ induced by $K$.

**Boolean algebras**

As we noted earlier, $\omega$-homogeneous Boolean algebras are usually called just homogeneous. The age of a homogeneous Boolean algebra contains a copy of every finite Boolean algebra. Recall that a linear ordering $<$ on a finite Boolean algebra is **natural** if it is an antilexicographical order induced by a linear ordering of its atoms. Since both the class of finite Boolean algebras and the class of naturally ordered finite Boolean algebras are Fraïssé classes with the Ramsey property and they satisfy the assumptions of the theorem, we get the following result, which generalizes Theorem 8.2(ii) in [34] to uncountable homogeneous Boolean algebras.

**Theorem 45.** Let $\mathcal{B}$ be a homogeneous Boolean algebra and let $\mathcal{K}$ be the class of naturally ordered finite Boolean algebras. Then the universal minimal flow of $\text{Aut}(\mathcal{B})$ is the space $\text{NO}_K(\mathcal{B})$ of all linear orderings on $\mathcal{B}$ that are natural when restricted to a finite subalgebra.

Homogeneous Boolean algebras are in Stone duality (see Section 2.2) with $h$-homogeneous zero-dimensional compact Hausdorff spaces, so this is just a dual version of a result by Glasner and Gutman. Recall that a topological space $X$ is called **$h$-homogeneous**, if all non-empty clopen subsets of $X$ are homeomorphic.

**Theorem 46 ([20]).** Let $X$ be an $h$-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \text{Homeo}(X)$ be the group of homeomorphisms of $X$ equipped with the compact-open topology. Then $M(G) = \Phi(X)$, the space of maximal chains of closed subsets of $X$.

The space $\Phi(X)$ of maximal chains of closed subsets of a compact space was introduced by Uspenskij in [72]: Let $X$ be a compact space and denote by $\exp X$ the space of closed subsets of $X$ equipped with the Vietoris topology. Then the space $\Phi(X)$ of all maximal chains of closed subsets of $X$ is a closed subspace of $\exp \exp X$. The natural action of $\text{Homeo}(X)$ on $X$ induces an action on $\exp X$ and $\Phi(X)$,
which is the action considered in the theorem above. There is of course an explicit isomorphism between these two universal minimal flows described in Theorem 8.3 in [34] for countable structures. However, the same proof works for uncountable structures as well. We reproduce the isomorphism in Theorem 60 for uncountable structures with the proof.

Following a paper by van Douwen [73], if $\kappa$ is a cardinal number, we denote by $P(\kappa)/[\kappa]^{<\kappa}$ the quotient algebra of the Boolean algebra of all subsets of $\kappa$ by the ideal of sets of cardinality less than $\kappa$. It is easy to see that $P(\kappa)/[\kappa]^{<\kappa}$ is homogeneous for every cardinal $\kappa$. For $X \subset \kappa$, let $[X] \in P(\kappa)/[\kappa]^{<\kappa}$ denote its equivalence class.

Now we introduce two subgroups of $P(\kappa)/[\kappa]^{<\kappa}$: Denote by $T_\kappa$ the set of all bijections between subsets $A, B \subset \kappa$ with $\text{card}(\kappa \setminus A), \text{card}(\kappa \setminus B) < \kappa$. With the operation of composition, $T_\kappa$ is a monoid, but not a group. We can however assign to each $f \in T_\kappa$ an automorphism $f^*$ of $P(\kappa)/[\kappa]^{<\kappa}$, $f^*([X]) = [f[X]]$, mapping $T_\kappa$ onto a subgroup $T_\kappa^* = \{ f^* : f \in T_\kappa \}$ of $\text{Aut}(P(\kappa)/[\kappa]^{<\kappa})$. Since the automorphisms in $T_\kappa^*$ are induced by a pointwise bijection between subsets of $\kappa$, we call them trivial. Inside of $T_\kappa^*$ we have a normal subgroup of those automorphisms induced by a true permutation of $\kappa$, let us denote it by $S_\kappa^*$. Shelah [60] (see also [62]) proved that consistently every automorphism of $P(\omega)/\text{fin}$ is trivial. This has been extended to $P(\kappa)/\text{fin}$ for all cardinals $\kappa$ in [75]. Of course, consistently the two groups are different (e.g. under CH).

The next theorem shows that $T_\omega^*$ and $S_\omega^*$ do not coincide, hence $T_\omega^*$ (and thus consistently also $\text{Aut}(P(\omega)/\text{fin})$) is not simple.

**Theorem 47** ([73]). There is a homomorphism $h^*$ from $T_\omega^*$ onto $\mathbb{Z}$ with kernel $S_\omega^*$. (In particular, $T_\omega^* \neq S_\omega^*$.)

Van Douwen also identified all normal subgroups of $T_\kappa^*$.

**Theorem 48** ([73]). A subgroup $G$ of $T_\kappa^*$ is normal if and only if $\text{card}(G) = 1$ or $G \in \{ (h^*)^{-1}k\mathbb{Z} : k \in \mathbb{N} \}$.

Since $S_\kappa^*$ is dense in $\text{Aut}(P(\omega)/\text{fin})$, it follows that so are all non-trivial normal subgroups of $T_\kappa^*$. We can therefore apply Theorem 40 to obtain the following corollary.

**Corollary 7.** Let $K$ denote the class of all finite naturally-ordered Boolean algebras. The universal minimal flow for all normal subgroups of $T_\kappa^*$ and of $\text{Aut}(P(\omega)/\text{fin})$ is $\text{NO}_K(P(\omega)/\text{fin})$.

Note that even though the algebraic structure of $S_\kappa^*$ is inherited from $S_\kappa$, their topologies are radically different, therefore so are their universal minimal flows the space of normal orderings on $P(\kappa)/[\kappa]^{<\kappa}$ with respect to the class of naturally ordered finite Boolean algebras and $\text{LO}(\kappa)$ respectively (even in cardinality).

In the uncountable case, the situation is slightly different.

**Theorem 49** ([73]). If $\kappa > \omega$, then $T_\kappa^* = S_\kappa^*$.

Nevertheless, $T_\kappa^*$ is dense in $\text{Aut}(P(\kappa)/[\kappa]^{<\kappa})$, so Theorem 40 applies.

**Corollary 8.** Let $\kappa$ be a cardinal number. Then the universal minimal flow of $\text{Aut}(P(\kappa)/[\kappa]^{<\kappa})$ and $S_\kappa^* = T_\kappa^*$ is $\text{NO}_K(P(\kappa)/[\kappa]^{<\kappa})$, where $K$ is the class of naturally ordered finite Boolean algebras.
Chapter 4. Automorphism groups

Vector spaces over finite fields

Every infinite-dimensional vector space over a finite field \( F \) is \( \omega \)-homogeneous. Its age contains a copy of every finite vector space over \( F \) and forms a Fraïssé class satisfying the Ramsey property. So does the class of all finite naturally-ordered spaces over \( F \). Moreover, these classes satisfy the condition (4.1) on order-forgetfulness of Theorem 37, as noted in [34], p.144. Thus we can generalize the result in [34] about countable-dimensional vector spaces to those with uncountable dimension.

**Theorem 50.** Let \( V \) be an infinite-dimensional vector space over a finite field and let \( K \) be the class of all finite dimensional naturally-ordered spaces. Then the universal minimal flow of \( \text{Aut}(V) \) is the space \( \text{NO}_K(V) \) of all linear orderings on \( V \) that are natural when restricted to a finite subspace.

### 4.3.2 Extremely amenable groups

Next theorem characterizes extremely amenable subgroups of \( S_\kappa \). The implications \( (a) \Rightarrow (c) \) and \( (c) \Rightarrow (b) \) have identical proofs as for \( S_\infty(\mathbb{Z}) \) given in [34].

Recall that a topological group \( G \) is called **extremely amenable** if every action of \( G \) on a compact Hausdorff space has a fixed point. It is easily seen, that \( G \) is extremely amenable if and only if \( M(G) \) is a singleton.

**Theorem 51.** Let \( G \) be an infinite subgroup of \( S_\kappa \). The following are equivalent:

1. \( G \) is extremely amenable,
2. (i) for every finite \( A \subseteq \kappa \), \( \{ g \in G : ga = a \ \forall a \in A \} = G_A = G(\mathcal{A}) = \{ g \in G : gA = A \} \) and
   (ii) for every colouring \( c : G/G_A \to \{1, 2, \ldots, k\} \) and for every finite \( B \supseteq A \), there is \( g \in G \) and \( i \in \{1, 2, \ldots, n\} \) such that \( c(hG_A) = i \) whenever \( h[A] \subseteq g[B] \).
3. (i') \( G \) preserves countably many relations \( \mathcal{R} \) on \( \kappa \) with \( \text{Age}(\kappa, \mathcal{R}) \) consisting of rigid structures and
   (ii) as above.

**Remark 5.** Let \( \mathcal{A} \) be an \( \omega \)-homogeneous relational structure such that \( G \) is dense in its automorphism group. Since finitely generated substructures of \( \mathcal{A} \) are finite, (ii) of (b) simply says that \( \text{Age}(\mathcal{A}) \) satisfies the Ramsey property.

**Proof.** (a) \( \Rightarrow \) (c) Since \( G \) is extremely amenable, it has a fixed point under its natural action on \( X_\mathcal{R} \) for any \( \mathcal{R} \) a countable set of relational symbols. In particular, the canonical action of \( G \) on \( \text{LO}(\kappa) \) has a fixed point \( < \) and \( \text{Age}(\kappa, <) \) consists of rigid structures. Hence (i') holds. To prove (ii), suppose that \( c : G/G_A \to \{1, 2, \ldots, k\} \) is a colouring of left cosets of \( G_A \) by \( k \) many colours. Consider \( c \) as a point in the compact space \( X = \{1, 2, \ldots, k\}^{G/G_A} \) and an action of \( G \) on \( X \) given by \( gx(hG_A) = x(g^{-1}hG_A) \). Let \( Y \) be the closure of the orbit of \( c \) in \( X \). Since \( G \) is extremely amenable, the induced action of \( G \) on \( Y \) has a fixed point \( d \). As \( G \) acts transitively on \( G/G_A \), \( d \) must be a constant function, say with range \( \{i\} \supseteq \{1, 2, \ldots, k\} \) Let \( B \supseteq A \) be finite and let \( H = \{ hG_A \in G : hA \subseteq B \} \subseteq G/G_A \). Since \( d \in \overline{Gc} \), there is a \( g \in G \) such that \( g^{-1}H = d[H] \). Then \( c(ghG_A) = g^{-1}c(hG_A) = d(hG_A) \) for every \( h \in H \).

(c) \( \Rightarrow \) (b) Let \( \mathcal{R} \) be the set of relations given by (i'). Since for every finite set \( A \subseteq \kappa \), \( (\mathcal{A}, \mathcal{R}|A) \) is rigid, we have that whenever \( g \in G \) fixes \( A \) setwise (i.e. \( gA = A \)) it also fixes \( A \) pointwise (i.e. \( ga = a \) for every \( a \in A \)). Therefore we get \( G_A = G(\mathcal{A}) \).
(b) ⇒ (a) The proof resembles to that of Theorem 40. Let \( \mathcal{A} \) be an \( \omega \)-homogeneous relational structure with the underlying set \( \kappa \) such that \( G \) is dense in its group of automorphisms. By part (ii), \( \text{Age}(\mathcal{A}) \) satisfies the Ramsey property. By part (i), elements in \( \text{Age}(\mathcal{A}) \) are rigid, it means that they have trivial groups of automorphisms. Let \( A \) be a finite substructure of \( \mathcal{A} \). Since \( A \) is rigid, we can identify \( G/G_A = \langle A \rangle \) via \( gG_A \leftrightarrow g^{-1}A \).

We will show that the only subset of \( G \) of the form \( G_A K \) for \( K \subset A \) appearing in a left syndetic subalgebra of \( \{G_A M : A \subset A \text{ finite}, M \subset A\} \) is \( G \) itself. Suppose not. It means that there is \( G_A K \) left syndetic such that \( \emptyset \neq G \setminus G_A K = G_A N \) is left syndetic as well. Let \( g_1, g_2, \ldots, g_n \) witness left syndeticity of both \( G_A K \) and \( G_A N \).

It means \( \bigcup_{i=1}^n g_i G_A K = G = \bigcup_{i=1}^n g_i G_A N \).

Let \( B \) be the substructure of \( \mathcal{A} \) generated by \( \bigcup_{i=1}^n g_i A \).

Define a colouring \( c: \left( A \atop \mathcal{A} \right) \rightarrow \{0, 1\} \) by \( c(A') = 0 \) if and only if there is \( g \in G_A K \) such that \( g^{-1}A = A' \) and \( c(A') = 1 \) otherwise (it means there exists \( g \in G_A N \) such that \( g^{-1}(A) = A' \)).

Since \( \text{Age}(\mathcal{A}) \) satisfies the Ramsey property, there is a copy \( B' \) of \( B \) in \( \mathcal{A} \) that is monochromatic, say in colour 0. Let \( h: A \longrightarrow \mathcal{A} \) be an automorphism extending the isomorphism from \( B' \longrightarrow B \). We claim that \( h \notin \bigcup_{i=1}^n g_i G_A N \). Suppose the contrary and let \( i \in \{1, 2, \ldots, n\} \) and \( g \in G_A N \) be such that \( h = g_i g \). Then \( h^{-1}g_i A \subset B' \), but \( h^{-1}g_i A = (g_i g)^{-1}g_i A = g^{-1}g_i^{-1}g_i A = g^{-1}A \). Due to the definition of \( c \) that means that \( h^{-1}g_i A \) has colour 1 which contradicts \( B' \) being monochromatic in colour 0. The proof for colours 0 and 1 switched is analogous.

As an immediate consequence of the Theorem 51, we obtain examples of extremely amenable groups as groups of automorphisms of structures generalizing analogous results in [34] from countable to uncountable structures.

**Theorem 52.** Groups of automorphisms of the following structures are extremely amenable.

1. an \( \omega \)-homogeneous linear order ([57]);
2. a homogeneous linearly ordered Boolean algebra \( B \) such that \( \text{Age}(B) \) is the class of finite naturally ordered Boolean algebras;
3. an \( \omega \)-homogeneous infinite-dimensional vector space \( V \) over a finite field \( F \) such that \( \text{Age}(V) \) is the class of finite naturally ordered finite vector spaces over \( F \);
4. an \( \omega \)-homogeneous linearly ordered \((K_n\text{-free})\) graph that is universal for all finite linearly ordered \((K_n\text{-free})\) graphs;
5. an \( \omega \)-homogeneous linearly ordered hypergraph that is universal for all finite linearly ordered hypergraphs;
(6) an \(\omega\)-homogeneous linearly ordered \(A\)-free hypergraph for \(A\) a class of finite irreducible subgraphs that is universal for all finite linearly ordered \(A\)-free hypergraphs;

(7) an \(\omega\)-homogeneous linearly ordered poset with the linear ordering extending the partial order that is universal for all finite posets with linear orderings extending the partial order.

4.3.3 Unique ergodicity

Recall that a topological group \(G\) is called \textit{amenable} if every \(G\)-flow admits an invariant probability measure. If \(G\) is amenable and every minimal \(G\)-flow admits exactly one invariant probability measure, then \(G\) is called \textit{uniquely ergodic}. This is equivalent to saying that the universal minimal of \(G\) admits a unique invariant probability measure. In [2], Angel, Kechris and Lyons connected unique ergodicity of automorphism groups with a quantitative version of the ordering property using probabilistic methods. The authors proved unique ergodicity for the groups of automorphisms of the random graph, the random \(K_n\)-free graphs, the random \(n\)-uniform hypergraphs, the rooted \(\aleph_0\)-regular tree, countably infinite-dimensional vector spaces over finite fields or the isometry group of the countable Urysohn space with rational distances. This was a major breakthrough since until then only compact groups and extremely amenable groups were known to be uniquely ergodic. The only non-trivial result had been unique ergodicity of \(S_\infty(Z)\) proved by Glasner and Weiss ([21]).

While general results in [2] rely on involved machinery, the proof for some structures can be rather simple when looking at the Boolean algebra of clopen subsets of the universal minimal flow. The hope is to apply this approach to other structures as well and obtain analogous results via structural rather than probabilistic means.

**Theorem 53.** Let \(G\) be an amenable group of automorphisms of an \(\omega\)-homogeneous structure \(A\). Suppose that \(\text{Age}(A)\) has a reasonable expansion-forgetful precompact expansion to a Fraïssé class \(K_R\) satisfying the Ramsey and the expansion properties. Then \(G\) is uniquely ergodic.

**Proof.** Let \(X_R\) be the space of normal expansions of \(A\) induced by \(K_R\). By Theorem 40, \(X_R\) is the universal minimal flow for \(G\). Let \(\mu\) be an invariant measure on \(X_R\). Then \(\mu\) is uniquely determined by its values on the algebra of clopen subsets of \(X_R\). As in Theorem 40, if \(x \in X_R\) we can identify the clopen algebra of \(X_R\) with the sets of returns of \(x\) into clopen subsets of \(X_R\) : \(\{\text{Ret}(x,(B,S)^*) : B \subseteq A\text{ finite }, (B,S) \in K_R\}\).

Let \(B \subseteq A\) finite and let \(\{g_1, g_2, \ldots, g_n\}\) be all automorphisms of \(B\). By \(\omega\)-homogeneity of \(A\) there are \(\{f_1, f_2, \ldots, f_n\} \subseteq \text{Aut}(G)\) such that \(f_i|B = g_i\) for every \(i \in \{1, 2, \ldots, n\}\). Then we have that

\[
\bigcup_{i=1}^{n} f_i\text{Ret}(x,(B,S)^*) = G,
\]

and for every \(i \neq j\)

\[
f_i\text{Ret}(x,(B,S)^*) \cap f_j\text{Ret}(x,(B,S)^*) = \emptyset.
\]

Since \(\mu\) is invariant, it follows that \(1 = \mu(G) = n\mu(\text{Ret}(x,(B,S)^*))\), so \(\mu(\text{Ret}(x,(B,S)^*)) = \frac{1}{n}\).

We know that a countable set (without any structure) and \(\aleph_0\)-dimensional vector spaces over finite fields satisfy conditions in Theorem 53. Therefore Theorem 53 provides an alternative proof of unique ergodicity of their automorphism groups.
Theorem 54 (Glasner and Weiss, [21]). $S_\infty(\mathbb{Z})$ is uniquely ergodic.

Theorem 55 ([2]). The groups of automorphisms of an $\aleph_0$-dimensional vector space over a finite field is uniquely ergodic.

All groups considered in [2] are Polish groups and unique ergodicity was proved for all amenable groups considered as long as their universal minimal flow was metrizable. The authors asked whether this is always the case.

Question 2 ([2]). Let $\Gamma$ be an amenable Polish group with a metrizable universal minimal flow. Is $\Gamma$ uniquely ergodic?

4.4 Ramsey property for finite Boolean algebras with ideals

This research was motivated by an old open problem to determine whether the Boolean algebras $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ can be isomorphic. The question was asked at a seminar in Katowice in the 1970’s and is therefore called the Katowice problem. It is easily seen to consistently have a negative solution, e.g. under the Continuum Hypothesis. The first reaction of most is that such an isomorphism can never exist and it has been studied by many great mathematicians. Balcar and Frankiewicz ([5]) showed that this problem is really specific to $\omega$ and $\omega_1$, in particular $\mathcal{P}(\kappa)/\text{fin}$ cannot be isomorphic to $\mathcal{P}(\lambda)/\text{fin}$ for any pair of cardinals $\kappa, \lambda$ other than $\omega, \omega_1$. A natural approach is to assume existence of an isomorphism between $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ and to study consistency of its consequences. See for instance works of Juris Steprāns ([66]) or David Chodounský ([13]). Recently, Klaas Pieter Hart showed that another consequence is the existence of an automorphism of $\mathcal{P}(\omega)/\text{fin}$ which is not induced by an almost permutation. Therefore, the isomorphism cannot exist in any model where all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are induced by an almost permutation.

Inspired by the result of Hart, we compute the universal minimal flow of $\text{Aut}(\mathcal{P}(\omega_1)/\text{fin})$. If $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ are isomorphic, then so are $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$ and $\text{Aut}(\mathcal{P}(\omega_1)/\text{fin})$ and hence so are their universal minimal flows which were computed by Glasner and Gutman in [20]. If $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ are not isomorphic, then $\mathcal{P}(\omega_1)/\text{fin}$ is not an $\omega$-homogeneous structure, so we cannot directly apply Theorem 40. However, expanding $\mathcal{P}(\omega_1)/\text{fin}$ with a unary predicate for the ideal of countable sets, we obtain an $\omega$-homogeneous structure. We show that the age of the expanded structure admits an order-forgetful expansion to a Fraïssé class satisfying the Ramsey property. More generally, we consider Fraïssé classes consisting of finite Boolean algebras with an increasing chain of ideals and show that they admit an order forgetful expansion to a Fraïssé class satisfying the Ramsey property. Since the Fraïssé limit of the class of finite Boolean algebras is the countable atomless Boolean algebra whose Stone space is the Cantor set, this allows us to compute universal minimal flows of groups of homeomorphisms of the Cantor set fixing some closed sets. We can further generalize some of the results to Cantor cubes of arbitrary weight.

Let $L = \{\lor, \land, 0, 1, \neg\}$ be the language of Boolean algebras. Let $J$ be a linearly ordered set. We denote by $L_J$ the language $L$ expanded by $|J|$-many unary symbols $\langle P_j : j \in J \rangle$, i.e. $L_J = \{\lor, \land, 0, 1, \neg, P_j : j \in J\}$. To simplify the notation, if $A$ is a structure in the language $L_J$, we often write $P_j$ in place of $P^A_j$ for every $j \in J$.

We will consider three classes of finite Boolean algebras in the language $L_J$. 
Definition 15.

(1) Let $B_J$ denote the class of isomorphism types of finite Boolean algebras in the language $L_J$, where each $P_j$ for $j \in J$ is interpreted as an ideal with $0 \in P_j$ and $P_i \subset P_j$ for $i < j \in J$. Moreover, to ensure the amalgamation property, we require that every $A \in B_J$ has at least one atom not in any $P_j$ for $j \in J$.

(2) Let $B_u$ denote the class of isomorphism types of finite Boolean algebras in the language $L_1 = L_{\{0\}}$ with $P_0$ interpreted as an ideal such that every $A \in B_u$ has exactly one atom not in $P_0$, i.e., for every $A \in B_u$, all atoms but one are in $P_0$. We require that the two element algebra is an element of $B_u$ and $0 \in P_0$.

(3) Let $B^u_J$ denote the class of isomorphism types of finite Boolean algebras in the language $L_J$ where each $P_j$ for $j \in J$ is interpreted as an ideal with $0 \in P_i$ for every $i \in J$ and $P_i \subset P_j$ for $i < j \in J$. Moreover, for every $A \in B^u_J$, all atoms but one are in one of $P_j$. We again consider the two element algebra to be an element of $B^u_J$.

All the classes in Definition 15 are Fraïssé classes. Moreover, if $J$ is countable, then so is $B_J$ and $B^u_J$, while $B_u$ is always countable. Therefore, if $J$ is countable, we can considerFraïssé limits of the classes in Definition 15. In each case, we obtain the countable atomless Boolean algebra $C$ with a chain of ideals $P^u_j$ for $j \in J$. The ideal $P^u_0$ in the limit of $B_u$ will be prime and therefore the filter dual to $P^u_0$ will be an ultrafilter. In the limit of $B^u_J$ we get that $\bigcup_{j \in J} P^u_j$ is a prime ideal. Speaking in the dual language of Stone spaces, we obtain the Cantor set with a decreasing chain of closed subsets in the case of $B_J$, the Cantor set with a distinguished point in the case of $B_u$, and the Cantor set with a decreasing chain of closed sets intersecting in a single point in the case of $B^u_J$.

Let $\lambda < \kappa$ be infinite cardinals and let $\theta$ be the order type of $[\lambda, \kappa)$. Consider structures $(\mathcal{P}(\kappa)/[\kappa]^{<\lambda}, P_\mu : \mu \in [\lambda, \kappa])$ in the language $L_\theta$, where $\mathcal{P}(\kappa)/[\kappa]^{<\lambda}$ is the quotient Boolean algebra of the power set algebra on $\kappa$ by the ideal of sets of cardinality less than $\lambda$ and $P_\mu$ is the ideal of equivalence classes of subset of $\kappa$ of cardinality $< \mu$. Then isomorphism types of structures in $\text{Age}(\mathcal{P}(\kappa)/[\kappa]^{<\lambda}, P_\mu : \mu \in [\lambda, \kappa])$ form exactly the class $B_\theta$.

We will show that we can equip structures in each class in Definition 15 with linear orderings to obtain a Fraïssé class of order-forgetful structures. We alter the definition of natural orderings in [34] respecting the enumeration of the ideals.

We first specify which orderings of atoms we allow. Let $A$ be a finite Boolean algebra in the language $L_J$. We call an ordering $<$ of atoms of $A$ proper if for every two atoms $a, b \in A$ and $i < j$ if $a \in P_i$ and $b \notin P_j \setminus P_i$ or $b \notin P_k$ for any $k \in J$, then $a < b$.

Definition 16. Let $B$ be one of $B_J, B_u, B^u_J$ and let $A \in B$. We say that a linear ordering $<$ on $A$ is natural if it is an antilexicographical extension of a proper ordering of atoms of $A$. We denote by $\text{NA}(B)$ the class of all naturally ordered algebras from $B$.

The class $\text{NA}(B)$ is obviously order-forgetful for any choice of $B$, therefore it trivially satisfies the ordering property. Indeed, if $A \in B_J$ and $<, <'$ are two natural orderings on $A$ then $(A, <) \cong (A, <')$. In other words, we can take $A$ in place of $B$ in the definition of the ordering property.

In [34], the authors proved that naturally ordered finite Boolean algebras (without predicates) form a Fraïssé class. Working along their proof, we show that the same is true for the class $\text{NA}(B)$ (except for the condition of countability if $J$ is uncountable and $B = B_J$ or $B = B^u_J$).
Lemma 13. Let $B$ be one $B_f, B_u, B^*_f$ and let $L_B$ be the language of $B$. Then $NA(B)$ is a Fraïssé class.

Proof. We first show that $NA(B)$ is hereditary. Let $B \in B$ and let $A$ be its subalgebra. Suppose that $<$ is a natural ordering of $B$ given by an ordering of its atoms $b_1 < b_2 < \ldots < b_n$. Let $a_1 < a_2 < \ldots < a_k$ be the atoms of $A$. Write $a_i = b_{i1} \lor b_{i2} \lor \ldots \lor b_{ij_i}$, where $b_{i1}, b_{i2}, \ldots, b_{ij_i}$ are atoms of $B$ in $<$-increasing order. Then $a_i \in P_h$ for $P_h \in L_B$ if and only if $b_{ij_i} \in P_h$. This holds because the ideals are ordered by inclusion and $<$ respects this order, while listing elements not in any $P_h$ as the last. Moreover, if $B$ is either $B_u$ or $B^*_f$, then $A$ has exactly one atom not in any $P_h$ since $B$ does. Therefore $A \in B$.

It remains to prove (AP), since (JEP) follows by amalgamation along the two element algebra. Let $(A, <_A), (B, <_B), (C, <_C) \in NA(B)$ with atoms $a_1 <_A a_2 <_A \ldots <_A a_k, b_1 <_B b_2 <_B \ldots <_B b_n, c_1 <_C c_2 <_C \ldots <_C c_m$ respectively. Suppose that there are embeddings $f : (A, <_A) \to (B, <_B)$ and $g : (A, <_A) \to (C, <_C)$. We will find $(D, <_D)$ and embeddings $r : (B, <_B) \to (D, <_D)$ and $s : (C, <_C) \to (D, <_D)$ such that $r \circ f = s \circ g$.

Write $f(a_i) = b_{i1} \lor b_{i2} \lor \ldots \lor b_{ij_i}$, with $b_{i1}, b_{i2}, \ldots, b_{ij_i}$ some of the atoms of $B$ in the increasing order and $g(a_i) = c_{i1} \lor c_{i2} \lor \ldots \lor c_{ij_i}$ for $c_{i1}, c_{i2}, \ldots, c_{ij_i}$ some of the atoms of $C$ in the increasing order. Notice that $a_i \in P_h$ if and only if $b_{ij_i}, c_{ij_i} \in P_h$ by the same argument as for (HP).

We let the atoms of $D$ to be $\overline{b_{ij}}_{1 \leq i \leq k, 1 \leq j \leq \ell_i}$ and $\overline{c_{ij}}_{1 \leq i \leq k, 1 \leq j \leq \ell_i}$ all disjoint except for $\overline{b_{ij}} = \overline{c_{ij}}$ for $1 \leq i \leq k$. We need to determine how $<_D$ will behave on these atoms and where the atoms of $B,C$ will be sent by $r,s$ respectively.

First consider $\overline{b_{ij}}_{1 \leq i \leq \ell_i} \cup\overline{c_{ij}}_{1 \leq i \leq \ell_i}$ and set $\overline{b_{i1}} <_D \overline{b_{i2}} <_D \ldots <_D \overline{b_{ij_i}}, \overline{c_{i1}} <_D \overline{c_{i2}} <_D \ldots <_D \overline{c_{ij_i}} = \overline{b_{ij_i}}$

and define $<_D$ to be arbitrary on the rest to be proper.

For any linear ordering $<$, we denote by $(-\infty, a] = \{x : x \leq a\}$ and $[a, b) = \{x : a < x < b\}$. Then we let $r(b_{i1}) = \bigvee(-\infty, \overline{b_{i1}}], r(b_{i2}) = \bigvee[\overline{b_{i1}}, \overline{b_{ij_i}}], \ldots, r(b_{ij_i}) = \bigvee[\overline{b_{ij_i-1}}, \overline{b_{ij_i}}]$

$s(c_{i1}) = \bigvee(-\infty, \overline{c_{i1}}], s(c_{i2}) = \bigvee[\overline{c_{i1}}, \overline{c_{ij_i}}], \ldots, s(c_{ij_i}) = \bigvee[\overline{c_{ij_i-1}}, \overline{c_{ij_i}}].$

Then $b_{it} \in P_h$ if and only if $r(b_{it}) \in P_h$ for every $t = 1, \ldots, J_i$ and $h \in J$, since $<_D \{b_{ij}\}_{1 \leq j \leq \ell_i} \cup\{c_{ij}\}_{1 \leq j \leq \ell_i}$ is proper. Similarly, $c_{it} \in P_h$ if and only if $s(c_{it}) \in P_h$ for every $t = 1, \ldots, I_i$ and $h \in J$.

Now we extend $<_D$ to $\overline{b_{ij}}_{1 \leq i \leq \ell_i} \cup\overline{c_{ij}}_{1 \leq i \leq \ell_i}$ and require that the maps $b_{ij} \mapsto \overline{b_{ij}}$ and $c_{ij} \mapsto \overline{c_{ij}}$ are order preserving and extend otherwise arbitrarily to be proper. We extend $r$ and $s$ in the following way:

$r(b_{21}) = \bigvee(-\infty, \overline{b_{21}}], \ldots, r(b_{2j_2}) = \bigvee[\overline{b_{2j_2-1}}, \overline{b_{2j_2}}]$

$s(c_{21}) = \bigvee(-\infty, \overline{c_{21}}], \ldots, s(c_{2l_2}) = \bigvee[\overline{c_{2l_2-1}}, \overline{c_{2l_2}}].$

We proceed in the same manner to define $<_D$ on all atoms of $D$ and $r$ and $s$ on all atoms of $B$ and $C$ respectively.

The last remaining ingredient is to show that naturally ordered expansions of the classes in Definition 15 satisfy the Ramsey property. We first introduce some notation.

Let $(J, <_J)$ be a linear ordering and let $K$ be a class of structures in the language $L_J$ such that in every structure $X$ in $K$ each $P_i$ is interpreted as an ideal on $X$, $P_i \subseteq P_j$ if and only if $i <_J j$ and we
allow $X = P_i$ for some $i \in J$. We denote by $\text{NA}(K)$ the class of all structures in $K$ expanded by natural orderings that are proper on atoms.

Let $(X, <_X) \in \text{NA}(K)$ be such that $P_i^X = X$ for some $i \in J$ and let $i_0$ be the minimal such $i$. Let $(Y, <_Y) \in \text{NA}(K)$ be such that $\min\{i \in J : P_i^Y \neq \{0\}\} > i_0$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ denote the atoms of $X$ and $Y$ respectively. We define

$$(A, <_A) = (X, <_X) \ast (Y, <_Y)$$

to be the structure in $\text{NA}(K)$ with atoms $x_1, \ldots, x_n, y_1, \ldots, y_m$. The natural order $<_A$ on $A$ is given by $x_i <_A y_j$ for every $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ and $x_i <_A x_j$ if and only if $x_i <_X x_j$ and $y_i <_A y_j$ if and only if $y_i <_Y y_j$. The interpretations of $P_j$’s in $A$ are given by their interpretations in $X$ and $Y$: $x_i \in P_j^A$ if and only if $x_i \in P_j^X$ and $y_i \in P_j^A$ if and only if $y_i \in P_j^Y$.

Let $A \in \text{NA}(K)$ and let $j_0$ be the $J$-minimal element such that $P_{j_0}^A \neq \{0\}$. Denote by $A_0$ the structure generated by the atoms of $A$ that lie in $P_{j_0}^A$ and denote by $A_1$ the structure generated by the remaining atoms of $A$. Suppose that $X$ is a substructure of $A_0$ generated by $n$ atoms $x_1 < x_2 < \ldots < x_n$ and $Y$ is a substructure of $A_1$ generated by $m$ atoms $y_1 < y_2 < \ldots < y_m$ and $n > m$. We define

$$X \circ Y$$

to be the substructure of $A$ with atoms $b_1, b_2, \ldots, b_n$ where $b_i = x_i$ if $i = 1, 2, \ldots, n - m$ and $b_i = x_i \lor y_{-(n-m)}$ if $i > n - m$.

If $X \in \text{NA}(K)$ such that $P_i^X = \{0\}$ for every $i \in J$ and $j$ is an element of $J$, we define

$$X^j$$

to be the structure in $\text{NA}(K)$ with the same atoms as $X$ and all of its elements in the predicate $P_j$. Notice that $X^j$ is not a structure in $\text{NA}(B)$ for any choice of $B = B_1, B_2, B_3$.

If $X \in \text{NA}(K)$, then $X_r$ denotes the structure in $\text{NA}(K)$ with the same atoms and natural order as in $X$ but with $P_i^{X_r} = \{0\}$ for every $i \in J$. We can think of $X_r$ as of the reduct of $X$ to the language of naturally ordered Boolean algebras (without predicates).

**Theorem 56.** $\text{NA}(B_J)$ satisfies the Ramsey property.

**Proof.** Let $A \leq B$ be two structures in $\text{NA}(B_J)$ and let $k \in \{2, 3, \ldots\}$. We claim that there is $C \in \text{NA}(B_J)$ such that

$$C \longrightarrow (B)^A_k.$$

Denote by $M_B = \{j \in J : P_j^B \neq \{0\}\}$ and let $j_0$ be the $J$-minimal element of $M_B$. Notice that if $P_j^A \neq \{0\}$ then $j \in M_B$. We proceed by induction on $n = |M_B|$.

If $n = 0$, then the existence of desired $C$ is ensured by the Dual Ramsey Theorem.

Suppose that we can find $C$ for all pairs $A \leq B$ with $M_B$ of size $n$. We will prove the statement for $M_B$ of size $n + 1$. We find $C$ in two pieces $C_0, C_1 \in \text{NA}(K)$ such that $C_0^{P_i} = \{0\}$ for all $i \in J$ and $C = C_0^{j_0} + C_1$. The structures $A_r$ and $B_r$ have interpretations of all $P_i$’s equal to $\{0\}$, so we can find $C_0$ such that

$$C_0 \longrightarrow (B_r)^A_k$$

by the Dual Ramsey Theorem.
To find $C_1$, we denote by $A_1$ the structure generated by the atoms of $A$ not in $P_{j_0}$ with the natural ordering and interpretations of $P_j$'s inherited from $A$. Similarly, we define $B_1$ to be the structure generated by the atoms of $B$ not in $P_{j_0}$ with the natural ordering and interpretations of $P_j$'s inherited from $B$. Then $A_1$ and $B_1$ are elements in $\text{NA}(B_j)$ and $\{j \in J : P_j^{B_j} \neq \emptyset\}$ has at most $n$ elements. So we can use the induction hypothesis to obtain $C_1$ such that

$$C_1 \rightarrow (B_1)^{A_{(C_0)}(B_0)}_{(B_0)}.$$ 

Claim 2. Let $C = C_0^{j_0} * C_1$. Then

$$C \rightarrow (B)^{A_1}_k.$$ 

Proof. Let $c : \binom{C}{A} \rightarrow \{0, 1, \ldots, k - 1\}$ be an arbitrary colouring. Then for every $D \in \binom{C_1}{A_1}$ we define a colouring

$$c_D : \binom{C_0}{A_0} \rightarrow \{0, 1, \ldots, k - 1\} \text{ by } c_D(E) = c(E^{j_0} \circ D).$$

Using the Dual Ramsey Theorem, we can find a monochromatic $B_D \in \binom{C_0}{B_r}$ in colour $k_D$. Now consider the colouring

$$c' : \binom{C_1}{A_1} \rightarrow \binom{C_0}{B_r} \times \{0, 1, \ldots, k - 1\} \text{ given by } c'(D) = (B_D, k_D).$$

By the induction hypothesis, there is a monochromatic $B'_1 \in \binom{C_1}{B_1}$ in colour $(B'_r, k_{B_r})$ for some $B'_r \in \binom{C_0}{B_r}$ and $k_{B_r}$ equal to one of $0, 1, \ldots, k - 1$. Let $B' = (B'_r)^{j_0} \circ B'_1$. We verify that $\binom{C_0}{B_r}$ is monochromatic in the colour $k_{B_r}$. Let $A' \in \binom{C_A}{A}$ and denote by $A'_0$ the part of $A'$ that is in $P_{j_0}$ and by $A'_1$ the remaining elements of $A'$. Since every atom in $B'$ has an atom of $(B_r)^{j_0}$ below, also every atoms of $A'$ has an atom of $(B'_r)^{j_0}$ below. Below atoms in $A'_0$ there are no atoms from $B'_1$ while every atom in $A'_1$ has an atom both from $(B'_r)^{j_0}$ and $B'_1$ below. Therefore we can think of $A'$ as $E^{j_0} \circ D$ for some $E \in \binom{B'_r}{A}$ and $D \in \binom{B'_1}{A_1}$. It means that $c'(D) = (B'_r, k_{B_r})$ and therefore $c(A') = c_D(E) = k_{B_r}$ and we are done. 

\(\Box\)

Theorem 57. $\text{NA}(B_0)$ satisfies the Ramsey property.

Proof. Let $A, B \in \text{NA}(B_0)$. It means that $A, B$ are structures in the language $L_1$ and all but one atom of both $A$ and $B$ belong to $P_0$. Let $C_0$ be the Boolean algebra given by the Dual Ramsey Theorem for $A_r, B_r$ and $k$ colours. Let $C_1$ be the Boolean algebra with exactly one atom. We claim that $C = C_0^{j_0} * C_1$ satisfies

$$C \rightarrow (B)^{A_1}_k.$$ 

Let $c : \binom{C}{A} \rightarrow \{0, 1, \ldots, k - 1\}$ be an arbitrary colouring. Then $c$ induces a colouring $c' : \binom{C_0}{A} \rightarrow \{0, 1, \ldots, k - 1\}$ by $c'(E) = c(E^{j_0} \circ C_1)$. Let $B_0$ be a $c'$-monochromatic copy of $B$, in $C_0$. Then $B' = B_0^{j_0} \circ C_1$ is a $c$-monochromatic copy of $B$ in $C$.

\(\Box\)

Theorem 58. $\text{NA}(B_0^j)$ satisfies the Ramsey property.

Proof. Let $A, B \in \text{NA}(B_0^j)$ and $k \in \{0, 1, \ldots, k - 1\}$. It means that $A, B$ are structures in the language $L_j$ and all but one atom of both $A$ and $B$ belong to some of the ideals $\langle P_j, j \in J \rangle$. We show that there
exist \( C \in \text{NA}(\mathcal{B}_j^y) \) such that
\[ C \rightarrow (B)^A_k. \]

We proceed in the same manner by induction as in the proof of Theorem 56 ensuring that we can always pick \( C \) with exactly one atom not in any \( P_j \) for \( j \in J \). Let \( M_B = \{ j \in J : B \cap P_j \neq \{0\} \} \) and let \( j_0 \) be the \( J \)-minimal element of \( M_B \). We proceed by induction on \( n = |M_B| \).

If \( n = 0 \), then both \( A \) and \( B \) are the two element algebra, so we can take \( C \) to be a two element algebra as well to satisfy the Ramsey property. More generally, anytime \( A \) is the two element algebra, we can pick \( C = B \), since \( |(C)^A_k| = 1 \).

So suppose from now on that \( A \) is not the two element algebra. If \( n = 1 \) and \( M = \{ j \} \) for some \( j \in J \), then \( A \) and \( B \) have all but one atom in \( P_j \setminus P_{j-1} \). Then the existence of \( C \) is ensured by Theorem 57.

Suppose that we can find \( C \) for all pairs \( A \leq B \) with \( n \) and we shall prove the statement for \( M_B \) of size \( n + 1 \). As in the proof of Theorem 56, we obtain \( C \) as \( C_{j_0}^0 \ast C_1 \) where \( C_0 \) is provided by an application of the Dual Ramsey Theorem to satisfy
\[ C_0 \rightarrow (B)^{A_{j_0}}_k. \]

To find \( C_1 \), we define \( A_1 \) (respectively \( B_1 \)) to be the structure in \( \text{NA}(\mathcal{B}_j^y) \) that is generated by the atoms of \( A \) (respectively \( B \)) that are not in \( P_{j_0} \) with the natural ordering and interpretations of \( P_i \)'s inherited from \( A \) (respectively \( B \)). Then \( M_{B_1} \) has at most \( n \)-many elements and we can apply the induction hypothesis to get \( C_1 \in \text{NA}(\mathcal{B}_j^y) \) such that
\[ C_1 \rightarrow (B_1)^{A_1}_{|\langle C_0^1 \rangle^A_k} \]
holds. Set \( C = C_{j_0}^0 \ast C_1 \). Then \( C \in \text{NA}(\mathcal{B}_j^y) \). The verification that \( C \) satisfies
\[ C \rightarrow (B)^A_k \]
is identical to the proof of Theorem 56.

We have shown that \( \mathcal{B}_J, \mathcal{B}_u, \mathcal{B}_j^y \) are Fraïssé classes that admit order expansions by natural orderings to Fraïssé classes satisfying the ordering property and the Ramsey property. Therefore we can apply Theorem 40 to their Fraïssé limits and other \( \omega \)-homogeneous structures that have one of \( \mathcal{B}_J, \mathcal{B}_u, \mathcal{B}_j^y \) as their age.

We proved the Ramsey property for classes of finite Boolean algebras with chains of ideals and for their expansions with natural orderings. It is unclear what happens in the case that we do not require the ideals to be ordered by inclusion.

**Question 3.** Let \( \mathcal{B} \) be a class of isomorphism types of finite Boolean algebras in the language \( L_{\{0, 1\}} \), where \( P_0, P_1 \) are interpreted as non-trivial ideals with \( 0 \in P_i \) for \( i \in \{0, 1\} \). Is \( \mathcal{B} \) a Ramsey class? Does \( \mathcal{B} \) admit a precompact expansion to a Fraïssé class satisfying the Ramsey property and the expansion property relative to \( \mathcal{B} \).

Considering the Fraïssé limit of the class \( \mathcal{B} \) in Question 3, we obtain the countable atomless Boolean algebra \( C_\mathcal{B} \) with two ideals \( P_i^\mathcal{B} \) for \( i \in \{0, 1\} \) such that one is not included in the other. If \( \mathcal{F}_i \) is the dual
filter to $P^C_i$ for $i \in \{0, 1\}$, then the closed subsets of the Cantor set corresponding to $F_i$ intersect but one is not included in the other.

### 4.4.1 Cantor cubes

Assuming $J$ to be a countable linearly ordered set, we compute universal minimal flows of Fraïssé limits of $B_J, B_u, B^u_J$ and their naturally ordered analogues. Since the Fraïssé limit $C$ of the class of finite Boolean algebras is the countable atomless Boolean algebra, Fraïssé limits of $B_J, B_u, B^u_J$ are countable atomless Boolean algebras with each $P_j$ interpreted as an ideal. If $B$ is one of $B_J, B_u, B^u_J$, then the Fraïssé limit of $\text{NA}(B)$ is the same as the Fraïssé limit of $B$ with a distinguished normal ordering which is natural when restricted to every finite substructure.

In [34], it was shown that the universal minimal flow of the group of automorphisms of the countable atomless Boolean algebra $C$ is the space of normal orderings on $C$ induced by the natural orderings of its finite subalgebras. Applying Theorem 19, we obtain similar results for the classes $B_J, B_u, B^u_J$ and their expansions with natural orders.

We expend some of these results to arbitrary free Boolean algebras, dually to Cantor cubes using Theorem 40.

**Theorem 59.** Let $J$ be a countable linear order and let $B$ be one of the classes $B_J, B_u, B^u_J$. Let $C_B$ and $C^<_B$ be the Fraïssé limits of $B$ and $\text{NA}(B)$ respectively. Then

(a) the universal minimal flow of the group of automorphisms of $C_B$ is $\text{NO}_{\text{NA}(B)}(C_B),$

(b) the group of automorphisms of $C^<_B$ is extremely amenable.

The Stone space $E$ of the countable atomless Boolean algebra $C$ is the Cantor set. The universal minimal flow of the group of homeomorphisms of the Cantor set was first computed by Glasner and Weiss in [22]. They identified the universal minimal flow with the space of maximal chains of closed subsets introduced by Uspenskij in [72]: Let $X$ be a compact space and denote by $\exp X$ the space of closed subsets of $X$ equipped with the Vietoris topology. Then the space $\Phi(X)$ of all maximal chains of closed subsets of $X$ is a closed subspace of $\exp \exp X$. Since the universal minimal flow is unique, the space of maximal chains $\Phi(X)$ and the space of natural orderings on $C$ must be isomorphic. An explicit isomorphism was given in [34]. We cite the theorem for arbitrary homogeneous Boolean algebras with its proof as given in [8]. Homogeneous Boolean algebras are in Stone duality (see Section 2.2) with $h$-homogeneous zero-dimensional compact Hausdorff spaces. Recall that a topological space $X$ is called $h$-homogeneous, if all non-empty clopen subsets of $X$ are homeomorphic.

**Theorem 60 ([34]).** Let $G$ be the group of homeomorphisms of an $h$-homogeneous zero-dimensional compact Hausdorff space $X$ and let $B$ be the Boolean algebra of clopen subsets of $X$. Let $K$ denote the class of finite naturally ordered Boolean algebras. There exists an (explicit) $G$-isomorphism $\phi : \Phi(X) \rightarrow \text{NO}_K(B)$.

**Proof.** Given a maximal chain $M$ of closed subsets of $X$, for every clopen subset $C$ of $X$, let

$$M_C = \bigcap \{M \in M : M \cap C \neq \emptyset\}.$$
By the maximality of $\mathcal{M}$, $M_C \cap C$ is a single point for every clopen set $C$. If $C \subset D$ are two clopen subsets of $X$, then $M_C \supset M_D$, though they can also be equal. However, if $C$ and $D$ are disjoint, then $M_C$ and $M_D$ are different. If $A$ is a finite subalgebra of $\mathcal{B}$ and $a, b$ are two atoms of $A$ then we set $a <_M b$ if and only if $M_a \supset M_b$. This gives a total ordering of atoms of $A$ which in turn induces an antilexicographical ordering on $A$. These orderings cohere and produce a total order $<_\mathcal{M}$ of $\mathcal{B}$. Then $\phi(M) = <_\mathcal{M}$ is the sought for isomorphism from $\Phi(X)$ to $\text{NO}_K(B)$. 

If $\mathcal{I}$ is an ideal on $C$ and $\mathcal{F}$ is the dual filter to $\mathcal{I}$ (i.e. $\mathcal{F} = \{\neg a : a \in \mathcal{I}\}$), then $E_\mathcal{F} = \{e \in E : \mathcal{F} \subset e\}$ is a closed subset of $\mathcal{E}$. Therefore if $\{P^C_j : j \in J\}$ are the ideals in the Fraïssé limit $C_B$ for $\mathcal{B} = B_J$, then each filter $\mathcal{F}_j$ dual to $P^C_j$ corresponds to a closed subset $E_j = E_{\mathcal{F}_j}$ of $\mathcal{E}$. If $i < j \in J$, then $P^C_i \subset P^C_j$, so also $\mathcal{F}_i \subset \mathcal{F}_j$ and consequently $E_i \supset E_j$ (larger filter determines a smaller closed subset). So we have that the increasing chain $\{P^C_j : j \in J\}$ of ideals on $C_B$ corresponds to a decreasing chain $\{E_j : j \in J\}$ of closed subsets of the Cantor set $\mathcal{E}$.

**Theorem 61.** Let $\{E_j : j \in J\}$ be a decreasing chain of closed subsets of $\mathcal{E}$ as above. Let $G$ be the group of homeomorphisms of $\mathcal{E}$ such that $gE_i = E_i$ for every $i \in J$. Then the universal minimal flow of $G$ is the space of maximal chains of closed subsets of $\mathcal{E}$ containing $\{E_j : j \in J\}$.

**Proof.** Let $\phi : \Phi(\mathcal{E}) \to \text{NO}_{B_J}(C_{B_J})$ be the isomorphism given in Theorem 60. Recall that for a clopen subset $C$ of $\mathcal{E}$, we define

$$M_C = \bigcap \{M \in \mathcal{M} : M \cap C \neq \emptyset\}.$$ 

Note that $M_C \in \mathcal{M}$.

We first show that if $\mathcal{M}$ is a maximal chain of closed subsets of $\mathcal{E}$ that extends $\{E_j : j \in J\}$, then $\phi(M) = <_\mathcal{M}$ is a normal ordering on $C_{B_J}$ induced by $\text{NA}(B_J)$. Let $A$ be a finite substructure of $C_{B_J}$. Let $a, b$ be atoms of $A$ and suppose that $a \in P_i$ and $b \in P_j \setminus P_i$ for some $i < j$. Thinking of $a$ and $b$ as clopen subsets of $\mathcal{E}$, we have that $a \cap E_i = \emptyset$ while $b \cap E_i \neq \emptyset$. It follows that $M_a \supset M_i$ while $M_b \subset E_i \neq \emptyset$ and hence $M_a \supset M_b$ showing that $a <_M b$. If $a \in P_i$ and $b \notin P_j$ for any $j \in J$, then the same argument shows that $a <_M b$. It means that $<_\mathcal{M}$ is proper on atoms of any finite substructure of $C_{B_J}$ and therefore $<_\mathcal{M} \in \text{NO}_{B_J}(C_{B_J})$.

Secondly, we show that if $\mathcal{M}$ is a maximal chain that does not extend $\{E_j : j \in J\}$, then $<_\mathcal{M}$ is not natural on some finite substructure $A$ of $C_{B_J}$. If $S$ is a closed subset of $\mathcal{E}$, let $\mathcal{F}_S$ denote the corresponding filter on $\mathcal{C}_{B_J}$.

Suppose that $\mathcal{M}$ does not extend $\{E_j : j \in J\}$. So there exist $M \in \mathcal{M}$ and $i \in J$ such that $E_i \not\subset M \not\subset E_i$. Let $x \in M \setminus E_i, y \in E_i \setminus M$. Since $x \notin E_i$, there is $a \in \mathcal{F}_x$ that is not compatible with $\mathcal{F}_{E_i}$, i.e. $\neg a \in \mathcal{F}_{E_i}$, so $a \in P_i$. Similarly, since $y \notin M$, there is $b \in \mathcal{F}_y$ not compatible with $\mathcal{F}_M$, in other words $\neg b \in \mathcal{F}_M$. However, $b$ is compatible with $\mathcal{F}_{E_i}$ and therefore $b \notin P_i$. Let $a' = a \wedge \neg b \in x$. Then $a' \in \mathcal{F}_x$ and therefore $a'$ is compatible with $\mathcal{F}_M$. As $a \in P_i$ and $a' \leq a$, also $a' \in P_i$. Trivially $a' \wedge b = 0$, so we can consider a substructure $A$ of $C_{B_J}$ with atoms $a', b, \neg(a' \vee b)$. It follows that

$$M \cap a' \neq \emptyset = M \cap b$$

and consequently $M_{a'} \subset M_b$. It means that $b <_\mathcal{M} a'$ which contradicts $<_\mathcal{M}$ being proper on the atoms of $A$, since $a' \in P_i$ while $b \notin P_i$. 

If $\mathcal{B} = B_a$, then the ideal $P^C_0$ in the limit $C_B$ is a prime ideal and therefore the dual filter $\mathcal{F}_0$ is an
We are done.

Let \( \mathcal{F}_0 \) be the prime ideal dual to \( \mathcal{P} \). Theorem 62.

Let \( \mathcal{F}_0 \) be the prime ideal dual to \( \mathcal{P} \) and let \( \mathcal{F} \) be any ultrafilter on the countable atomless Boolean algebra \( \mathcal{C} \), then there exists \( h \in \text{Aut}(\mathcal{C}) \) such that \( h \mathcal{F}_1 = \mathcal{F}_2 \). For \( \mathcal{F}_1 \) and ultrafilter on \( \mathcal{C} \), let \( (\mathcal{C}, \mathcal{F}_1) \) denote the expansion of \( \mathcal{C} \) with one unary predicate interpreted as the ultrafilter \( \mathcal{F}_1 \).

**Proposition 8.** Let \( \mathcal{F} \) be any ultrafilter on the countable atomless Boolean algebra \( \mathcal{C} \). Then \( (\mathcal{C}, \mathcal{F}) \) is \( \omega \)-homogeneous.

**Proof.** Let \( (\mathcal{C}, P_0) \) be the Fraïssé limit of \( \mathcal{B}_n \) and let \( \mathcal{F}_0 \) be the ultrafilter dual to \( P_0 \). Since \( (\mathcal{C}, P_0) \) is \( \omega \)-homogeneous, so is \( (\mathcal{C}, \mathcal{F}_0) \). Let \( h \) be an automorphism in \( \text{Aut}(\mathcal{C}) \) such that \( h(\mathcal{F}) = \mathcal{F}_0 \). Let \( (A, \mathcal{F}_1|A) \) be a finite substructure of \( (\mathcal{C}, \mathcal{F}) \) and let \( \phi : A \rightarrow \mathcal{C} \) be a partial automorphism preserving \( \mathcal{F} \). Then \( h \phi \mathcal{F}_1 : hA \rightarrow \mathcal{C} \) is a partial automorphism preserving \( \mathcal{F}_0 \). By \( \omega \)-homogeneity of \( (\mathcal{C}, \mathcal{F}_0) \), find an extension \( \bar{\phi} \) of \( h \phi \mathcal{F}_1 \) to all \( \mathcal{C} \) preserving \( \mathcal{F}_0 \). Then \( h^{-1} \bar{\phi} \) is an automorphism in \( \text{Aut}(\mathcal{C}, \mathcal{F}) \) extending \( \phi \). \( \square \)

We can actually extend Proposition 8 to arbitrary free algebra.

**Proposition 9.** Let \( \mathcal{A}_\kappa \) be the free Boolean algebra on \( \kappa \)-many generators. Let \( \mathcal{F} \) be an ultrafilter on \( \mathcal{A}_\kappa \). Then \( (\mathcal{A}_\kappa, \mathcal{F}) \) is \( \omega \)-homogeneous.

**Proof.** Let \( (A, \mathcal{F}|A) \) be a finite substructure of \( (\mathcal{A}_\kappa, \mathcal{F}) \). Let \( S \) be the set of free generators of \( \mathcal{A}_\kappa \) and let \( S' \subset S \) be a finite subset of \( S \) generating \( A \). Let \( \mathcal{C} \) be a countable atomless Boolean subalgebra of \( \mathcal{A}_\kappa \) generated by a subset of \( S'' \) containing \( S' \). Then \( \mathcal{F}|\mathcal{C} \) is an ultrafilter on \( \mathcal{C} \) and by Proposition 8, \( (\mathcal{C}, \mathcal{F}|\mathcal{C}) \) is \( \omega \)-homogeneous. Let \( \phi : (A, \mathcal{F}|A) \rightarrow (\mathcal{A}_\kappa, \mathcal{F}) \) be a partial automorphism and let \( \bar{\phi} \) be its extension to \( (\mathcal{C}, \mathcal{F}|\mathcal{C}) \). Then \( \bar{\phi} \) is given by its values on \( S'' \). Define \( \hat{\phi} \) to be an automorphism on \( \mathcal{A}_\kappa \) which is \( \bar{\phi} \) on \( S'' \) and arbitrary on \( S \setminus S'' \) preserving \( \mathcal{F} \). Then \( \hat{\phi} \) is an extension of \( \phi \) preserving \( (A, \mathcal{F}|A) \) and we are done. \( \square \)

Therefore we can apply Theorem 40 to extend Theorem 59 to arbitrary free algebras with an ultrafilter.

**Theorem 62.** Let \( \mathcal{A}_\kappa \) be the free Boolean algebra on \( \kappa \) generators. Let \( \mathcal{F} \) be an ultrafilter on \( \mathcal{A}_\kappa \) and let \( \mathcal{P} \) be the prime ideal dual to \( \mathcal{F} \). Then the universal minimal flow of \( \text{Aut}(\mathcal{A}_\kappa, \mathcal{P}) \) is the space of normal orderings on \( (\mathcal{A}_\kappa, \mathcal{P}) \) that are natural when restricted to a finite substructure.

Dually, we obtain the following statement.

**Theorem 63.** Let \( 2^\kappa \) be the Cantor cube of weight \( \kappa \) for \( \kappa \) an infinite cardinal. Let \( x \in 2^\kappa \) and let \( G \) be the group of homeomorphisms of \( 2^\kappa \) such that \( gx = x \) for every \( g \in G \). Then the universal minimal flow of \( G \) is the space of maximal chains of closed subsets of \( 2^\kappa \) containing \( \{x\} \).
Proof. The proof goes along the same lines as the proof of Theorem 61.

Let \( x \in 2^\kappa \) and let \( F_x \) be the corresponding ultrafilter on \( A_\kappa \). Denote by \( P_x \) the ideal dual to \( F_x \) and let \( B \) be the class of finite substructures of \((2^\kappa, P_\kappa)\).

Let again \( \phi : \Phi(2^\kappa) \longrightarrow \mathcal{N}O_\kappa(2^\kappa) \) be the isomorphism as in Theorem 60. Suppose first that \( M \) is a maximal chain of closed subsets of \( 2^\kappa \) extending \( \{x\} \). Let \( A \) be a non-trivial finite substructure of \((A_\kappa, P_\kappa)\) and let \( a, b \) be atoms of \( A \) such that \( a \in P_\kappa \) and \( b \notin P_\kappa \). Then \( b \in F_x \) and so \( x \in b \). It follows that

\[
M_b = \bigcap \{ M \in M : b \cap M \neq \emptyset \} = \{ x \}.
\]

Since \( a \notin F_x \), \( M_a \) must be a superset of \( M_b \) showing that \( a \not< M b \), so \( a \not< M \) is proper on the atoms of \( A \).

Conversely, let \( M \in \Phi(X) \) not extend \( \{ x \} \). Then by maximality of \( M \), \( \bigcap M = \{ y \} \in M \) for some \( y \neq x \). Let \( a \in F_x \) such that \( \neg a \in F_y \), the ultrafilter on \( 2^\kappa \) corresponding to the point \( y \). Since \( M_a = \{ y \} \), we must have \( M_b \supset M_a \), which implies \( a \not< M \neg a \). However \( a \notin F_x \) and \( \neg a \in P_x \) showing that \( a \not< M \) is not proper on atoms of the algebra \( \{ 0, 1, a, \neg a \} \).

Let \( B = B_j \) and let \( \{ P_{j}^{CB} : j \in J \} \) be the ideals interpreting \( P_j \)'s for \( j \in J \) in \( C_B \). For each \( j \in J \), let \( F_j \) be the filter dual to \( P_{j}^{CB} \) and let \( E_j = \{ e \in E : E_j \subset e \} \) be the closed subset of \( E \) corresponding to \( F_j \). Then \( \{ E_j : j \in J \} \) is a decreasing chain of closed sets with the intersection a single point. Indeed,

\[
\bigcap_{j \in J} E_j = \{ e \in E : \bigcup_{j \in J} F_j \subset e \},
\]

so it suffices to show that \( F = \bigcup_{j \in J} F_j \) is an ultrafilter. Let \( a \in C_B \). Then the structure with the underlying set \( \{ 0, 1, a, \neg a \} \) is in \( B \). Therefore by the definition of \( B_j \), exactly one of \( a \) and \( \neg a \) is in \( P_{j_0}^{CB} \) for some \( j_0 \in J \), say \( a \). Then \( \neg a \in F_{j_0} \), which shows that \( F \) is an ultrafilter.

**Theorem 64.** Let \( \{ E_j : j \in J \} \) be a decreasing chain of closed subsets of \( E \) intersecting in a singleton as above. Then the group of homeomorphisms \( h \) of the Cantor cube such that \( hE_j = E_j \) for every \( j \in J \) is the space of chains of maximal closed subsets of \( E \) containing \( \{ E_j : j \in J \} \).

**Proof.** The proof is identical to the proof of Theorem 61.

We would naturally like to extend Theorem 61 and Theorem 64 to Cantor cubes. Therefore we ask the following question.

**Question 4.** Let \( A_\kappa \) be the free Boolean algebra on \( \kappa \)-many generators and let \( \{ P_i : i \in I \} \) be a chain of ideals on \( A_\kappa \) ordered under inclusion for some linearly ordered set \( I \). What conditions on \( \{ P_i : i \in I \} \) insure that \((A_\kappa, P_i : i \in I)\) is an \( \omega \)-homogeneous structures?

### 4.4.2 Quotients of power set algebras \( \mathcal{P}(\kappa)/[\kappa]^{< \lambda} \)

Throughout this section, let \( \lambda \leq \kappa \) be two infinite cardinals. We denote by \( \mathcal{P}(\kappa)/[\kappa]^{< \lambda} \) the quotient of the power set algebra on \( \kappa \) by the ideal of sets of cardinality less than \( \lambda \).

If \( \kappa = \lambda \), then \( \mathcal{P}(\kappa)/[\kappa]^{< \lambda} \) is homogeneous. Suppose that \( \lambda < \kappa \). For every \( \mu \in [\lambda, \kappa) \), we let \( P_\mu \) be the ideal on \( \mathcal{P}(\kappa)/[\kappa]^{< \lambda} \) consisting of equivalence classes of subsets of \( \kappa \) of cardinality \( \leq \mu \). Then the extended structure

\[
\mathcal{P}_\kappa = (\mathcal{P}(\kappa)/[\kappa]^{< \lambda}, P_\mu : \mu \in [\lambda, \kappa))
\]
becomes a homogeneous structure. To see that, let $A, B$ two be finite substructures of $\mathcal{P}_\lambda^\kappa$ and $\phi : A \rightarrow B$ an isomorphism. Let $A_0, A_1, \ldots, A_n$ be mutually disjoint subsets of $\kappa$ such that $\bigcup_{i=0}^n A_i = \kappa$ and $[A_0], [A_1], \ldots, [A_n]$ are atoms of $A$. Similarly, pick a disjoint partition $B_0, B_1, \ldots, B_n$ of $\kappa$ consisting of representatives of atoms of $B$. Without loss of generality, we can assume that $\phi([A_i]) = [B_i]$ for every $i = 0, 1, \ldots, n$. Then for every $i$, the sets $A_i$ and $B_i$ have the same cardinality. So we can pick a bijection $b_i : A_i \rightarrow B_i$ for every $i = 1, 2, \ldots, n$. Since $A_i$’s are mutually disjoint and so are $B_i$’s, also $b_i$’s are mutually disjoint. Let $b : \kappa \rightarrow \kappa$ be a bijection extending every $b_i$ and consider the induced automorphism $b_\omega : \mathcal{P}_\lambda^\kappa \rightarrow \mathcal{P}_\lambda^\kappa$. Then $b_\omega | A = \phi$ and $b_\omega$ is a total automorphism showing that $\mathcal{P}_\lambda^\kappa$ is $\omega$-homogeneous.

If $J = \theta$ is the order type of $[\lambda, \kappa)$, then $B_0$ is the class of all finite substructures of $\mathcal{P}_\lambda^\kappa$ up to an isomorphism.

Having shown that $\text{NA}(B_0)$ has all the desired properties, we are ready to apply Theorem 40 to compute universal minimal flows of the groups of automorphisms of $\mathcal{P}_\lambda^\kappa$.

**Theorem 65.** Let $\lambda \leq \kappa$ be two infinite cardinals and let $\theta$ be the order type of $[\lambda, \kappa)$. Denote by $K$ the class $\text{NA}(B_0)$. Then the universal minimal flow of the group of automorphisms of $\mathcal{P}_\lambda^\kappa$ is the space $\text{NO}_K(\mathcal{P}_\lambda^\kappa)$.

If there is no isomorphism between $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$, then the groups of automorphisms of $\mathcal{P}(\omega_1)/\text{fin}$ and $\mathcal{P}(\omega_1) = (\mathcal{P}(\omega_1)/\text{fin}, \mathcal{P}(\omega_1))$ are topologically and algebraically isomorphic. So we obtain the following theorem.

**Theorem 66.** Let $G$ be the group of automorphisms of $\mathcal{P}(\omega_1)/\text{fin}$ and let $K = \text{NA}(B(\omega_1))$. If there is no isomorphism between $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$, then the universal minimal flow of $G$ is the space $\text{NO}_K(\mathcal{P}(\omega_1))$.

If $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ are isomorphic, then the universal minimal flows of their groups of automorphisms are isomorphic. The universal minimal flow of the group of homeomorphisms of $\omega^*$ (the Stone space of $\mathcal{P}(\omega)/\text{fin}$) was first computed by Glasner and Gutman in [20] (see Theorem 45).

Following a paper by van Douwen [73], we introduce two dense subgroups of the group of automorphisms of $\mathcal{P}_\lambda^\kappa$:

$S_{\kappa, \lambda}^* \text{ and } T_{\kappa, \lambda}^*$.

Denote by $T_{\kappa, \lambda}$ the set of all bijections between subsets $A, B \subset \kappa$ with $|\kappa \setminus A|, |\kappa \setminus B| < \lambda$. With the operation of composition, $T_{\kappa, \lambda}$ is a monoid, but not a group. We can however assign to each $f \in T_{\kappa, \lambda}$ an automorphism $f^* \text{ of } \mathcal{P}_\lambda^\kappa$, $f^*([X]) = [f[X]]$, mapping $T_{\kappa, \lambda}$ onto a subgroup $T_{\kappa, \lambda}^* = \{ f^* : f \in T_{\kappa, \lambda} \}$ of the group of automorphisms of $\mathcal{P}_\lambda^\kappa$. Since the automorphisms in $T_{\kappa, \lambda}^*$ are induced by a pointwise bijection between subsets of $\kappa$, we call them trivial. Inside of $T_{\kappa, \lambda}^*$ we have a normal subgroup of those automorphisms induced by a true permutation of $\kappa$, let us denote it by $S_{\kappa, \lambda}^*$.

By density of $S_{\kappa, \lambda}^*$ and $T_{\kappa, \lambda}^*$ in the group of automorphisms of $\mathcal{P}_\lambda^\kappa$, we can apply Theorem 40 to prove the following.

**Corollary 9.** Let $\lambda \leq \kappa$ be infinite cardinals and denote by $\theta$ the order type of $[\lambda, \kappa)$. Let $K = \text{NA}(B_0)$. Then the universal minimal flow of the groups $S_{\kappa, \lambda}^*$ and $T_{\kappa, \lambda}^*$ is the space $\text{NO}_K(\mathcal{P}_\lambda^\kappa)$. 
Chapter 5

More applications

5.1 Generalized Urysohn spaces

The Urysohn universal metric space $(U, d)$ is a complete separable metric space that contains an isometric copy of every separable metric space and every isometry between finite subspaces can be extended to an isometry of the whole space. It was first constructed by P. Urysohn in 1925 (see [70]) who also showed that it is unique up to an isometry. Sierpiński [63] constructed a metric space of cardinality $\kappa$ universal for metric spaces of cardinality $\leq \kappa$ for every infinite cardinal $\kappa$ under GCH. After that, Urysohn space had been forgotten until Katetov’s presented a new construction of the Urysohn universal metric space in [33] providing analogous spaces of uncountable weights $\kappa$ satisfying $\kappa^{< \kappa} = \kappa$ that are universal for spaces of density $\leq \kappa$ and homogeneous with respect to subspaces of weight $< \kappa$, and showing their uniqueness. A new era of vibrant research about the Urysohn universal metric space was originated by a result of Uspenskij ([71]), who showed that the group of isometries of $(U, d)$ is universal for all Polish groups (i.e. every completely metrizable separable topological group can be algebraically and topologically embedded into it). Since then much research on the Urysohn space and its group of isometries has been done (e.g. [43], [44], [39], [51], [56], [40]). Dropping the requirement on separability and completeness, we obtain a definition of a generalized Urysohn space - a metric space that contains an isometric copy of every finite metric space and every finite isometry between two subspaces extends to an isometry of the whole space.

Pestov proved in [54] that groups of isometries of generalized Urysohn spaces with the compact-open topology are extremely amenable using concentration of measure phenomena. We give a simple proof of this result based on the Ramsey property for finite linearly ordered metric spaces (see [47]).

Let $(M, d)$ be an arbitrary metric space. Denote by $\text{Iso}(M)$ the group of all isometries of $M$. The topology on $\text{Iso}(M)$ is given by a basis of open neighbourhoods of the identity of the following form:

$$V(\varepsilon, X) = \{ \phi \in \text{Iso}(U) : d(x, \phi(x)) < \varepsilon \ \forall x \in X \},$$

where $\varepsilon > 0$ is a real number and $X$ is a finite subspace of $M$.

Denote by $(X)_\varepsilon$ the set of all copies $X'$ in $M$ such that there exists a bijection $\phi : X \rightarrow X'$ satisfying $d(x, \phi(x)) < \varepsilon$.

In answer to a question of Kechris, Pestov and Todorcević in [34], Nešetřil verified in [47] that the
class of finite metric spaces with arbitrary linear orderings satisfies the Ramsey property. It means that for every pair of finite linearly ordered metric spaces $X, Y$ such that $X$ is a linearly ordered subspace of $Y$, there exists a finite linearly ordered metric space $Z$ such that for every colouring of copies of $X$ in $Z$, there exists a copy of $Y$ in $Z$ in which all copies of $X$ take on the same colour.

**Theorem 67** ([54]). Let $(U, d)$ be a generalized Urysohn space, then $\text{Iso}(U, d)$ is extremely amenable.

**Proof.** Let $B$ be a left syndetic algebra of subsets of $G = \text{Iso}(U, d)$, $A, B \in B$ and let $V = V(\varepsilon, X)$ be a basic open neighbourhood of the identity for some $\varepsilon > 0$ and $X$ a finite subspace of $U$. We need to show that $VA \cap VB \neq \emptyset$. Without loss of generality, we can assume that there is $x \in X$ such that for every $y \neq z$ in $X$, $d(x, y) \neq d(x, z)$. If there is no $x \in X$ satisfying this requirement, we can pass from $X$ to $X' = X \cup \{x\}$ with $x \in U \setminus X$ by universality and $\omega$-homogeneity of $U$. Then $V(\varepsilon, X') \subset V(\varepsilon, X)$, so if $V(\varepsilon, X')A \cap V(\varepsilon, X')B \neq \emptyset$, then certainly $V(\varepsilon, X)A \cap V(\varepsilon, X)B \neq \emptyset$. Notice that, this assumption ensures that $\text{Iso}(X)$ is trivial and for $\varepsilon$ small enough (for instance set $\varepsilon < \min\{d(x, y), x \neq y, x, y \in X\}$)

$$V((\varepsilon), X) = \{g \in \text{Iso}(U) : gX \in (X)_\varepsilon\} = V(\varepsilon, X).$$

Let $g_1, g_2, \ldots, g_n$ witness left syndeticity of both $A$ and $B$:

$$g_iA = G = \bigcup_{i=1}^{n} g_iB.$$

Let $Y = \bigcup_{i=1}^{n} g_iX$ and $Y_i = \bigcup_{j=1}^{n} g_jX$ We will show that we can replace $g_i$ by $f_i$ for $i = 1, 2, \ldots, n$ so that $f_i$’s witness that $VA$ and $VB$ are left syndetic and $f_iX \cap f_jX = \emptyset$ whenever $i \neq j$ : Since $\text{Iso}(X)$ is trivial, we can identify $VA$ with those copies of $X'$ of $X$ in $U$ such that there exists $g \in A$ with $gX' \in (X)_\varepsilon$. Similarly, $g_iVA$ can be identified with those copies of $X'$ of $X$ in $U$ such that $gX' \subset (g_iX)_\varepsilon$ for some $g \in A$. Suppose that $i \leq n$ is the first index such that $Y_i \cap g_{i+1}X = S \neq \emptyset$. We will find a copy of $X$ within $\varepsilon$-distance from $g_{i+1}X$ that is disjoint from $Y_i$. Let

$$\varepsilon' = \min\{\varepsilon, \min\{d(x, y) : x \in Y_i \setminus S, y \in g_{i+1}X\}\}.$$  

By universality for finite metric spaces and $\omega$-homogeneity of $U$, there is a copy $X'$ of $X$ in the $\varepsilon'$-neighbourhood of $g_iX$ which is disjoint from $g_iX$ and by the choice of $\varepsilon'$ also disjoint from $Y_i$. Let $f$ be the isometry between $X$ and $X'$ (by rigidity of $X$ there is only one). Set $f_{i+1}$ to be an extension of $f$ to $U$. In the same manner we can continue for the remaining $g_{i+2}, \ldots, g_n$ with $g_{i+1}$ replaced by $f_{i+1}$.

Let us check that

$$\bigcup_{i=1}^{n} f_iVA = G.$$  

Let $g \in G$. Then there is $1 \leq i \leq n$ and $a \in A$ such that $g = g_ia$. If we show that $g \in f_iVA$, we will be done. We have that $g = g_ia \in f_iVA$ if and only if $g_i \in f_iV$, which holds since $g_i \in (f_iX)_\varepsilon$. The proof that $f_i$’s witness left syndeticity of $VB$ is analogous.

Let $<$ be an arbitrary linear ordering on $X$. Since $f_iX \cap f_jX = \emptyset$ whenever $i \neq j$, there is a linear ordering $<'$ on $Y' = \bigcup_{i=1}^{n} f_iX$ extending $\bigcup_{i=1}^{n} (f_i \times f_i)(<)$. Using the Ramsey property for finite linearly ordered metric spaces, find a finite metric subspace $Z$ of $U$ and $<$ a linear ordering on $Z$ such that

$$(Z, <) \rightarrow (Y', <')_2^{(X, <)}.$$
Identify now $VA$ with those copies $X'$ of $X$ in $U$ such that there is $h \in VA$ with $hX' = X$ (i.e. there exists a copy $X''$ of $X$ and $g \in A$ with $gX'' = X$ and $\min\{d(x', x'') : x' \in X', x'' \in X''\} < \varepsilon$). So $f_iVA$ corresponds to all those copies $X'$ of $X$ such that there is $h \in VA$ with $hX' = f_iX$. Similarly for $VB$ and $f_iVB$.

Colour copies of $(X, <)$ in $(Z, <)$ by 0 if they belong to $VA$, by 1 if they belong to $VB$, and by 2 otherwise. By the Ramsey property, there is a copy $(Y'', <''')$ of $(Y', <')$ in $(Z, <)$ that is monochromatic in a colour $r \in \{0, 1, 2\}$. Let $h : (Y'', <''') \rightarrow (Y', <')$ be the unique isometry and let $h'$ be its arbitrary extension to $U$. If $r = 0$, then $h' \notin \bigcup_{i=1}^{n} f_iVB$, if $r = 1$ then $h' \notin \bigcup_{i=1}^{n} f_iVA$ and if $r = 2$ then both $h' \notin \bigcup_{i=1}^{n} f_iVA$ and $h' \notin \bigcup_{i=1}^{n} f_iVB$. However, we chose $f_i$, $i = 1, 2, \ldots, n$ to witness syndeticity of both $VA$ and $VB$ which is a contradiction. 

\section{A problem of Ellis}

In order to study minimal subflows, the Ellis enveloping semigroup of a flow was defined. If $\pi : G \times X \rightarrow X$ is a $G$-flow, then the space $X^X$ of functions from $X$ to $X$ with the operation of composition and the product topology is a compact right-topological semigroup. For every $g \in G$, $(g, \cdot) : X \rightarrow X$ is a homeomorphism. We can therefore consider

$$E(X) = \overline{G}^{X^X},$$

the closure of $\{(g, \cdot) : g \in G\}$ in $X^X$. If the action $\pi$ is effective, then we can identify $G$ with a subgroup of $E(X)$, however in general with a coarser topology. $E(X)$ is called the Ellis enveloping semigroup of $X$. Being a closed subsemigroup of $X^X$, $E(X)$ also has a structure of a right-topological semigroup. Moreover, the action $\pi$ can be extended to a continuous action of $E(X)$. The semigroup $E(X)$ is naturally an ambit with the distinguished point being the identity element $e$ of $G$, therefore there is a surjective homomorphism of ambits $\phi_X : (S(G), e) \rightarrow (E(X), e)$. If $X = M(G)$, then minimal left ideals of $E(M(G))$ are isomorphic to $M(G)$ and $E(M(G))$ shares many properties with $S(G)$. Hence Ellis naturally asked:

**Ellis’ problem 1 ([57]).** Is $\phi_{M(G)} : (S(G), e) \rightarrow (E(M(G)), e)$ an isomorphism?

The problem trivially has a negative answer for nontrivial extremely amenable groups since then $E(M(G))$ is a singleton and $S(G)$ contains $G$. In the case of the discrete group of integers $\mathbb{Z}$, a negative answer was provided by Glasner in [23]. Pestov explored the problem for many topological groups and conjectured that Ellis’ problem has a positive answer only for precompact groups.

**Pestov’s conjecture 1.** Ellis’ problem has a positive answer exactly for precompact groups.

A topological group is called precompact if every open neighbourhood of the identity element is left (equivalently right) syndetic. If a group is precompact, then $S(G) = M(G) = E(M(G))$ and therefore Ellis problem has a positive answer for $G$.

We will work with the following reformulation of Ellis’ problem.

**Theorem 68 ([53]).** For a topological group $G$, the greatest ambit $S(G)$ is canonically isomorphic to the enveloping semigroup of the universal minimal flow $E(M(G))$ if and only if the points of $S(G)$ are separated by $G$-homomorphisms to $M(G)$. 


If $X$ and $Y$ are $G$-flows, we say that homomorphisms from $X$ to $Y$ separate points of $X$ if for every $x \neq y$ in $X$ there is a homomorphism $h : X \to Y$ such that $h(x) \neq h(y)$. So it is useful to know how $G$-homomorphisms from $S(G)$ to $M(G)$ look like and what the structure of $M(G)$ as a minimal left ideal of $S(G)$ is.

**Proposition 10.** For every $G$-homomorphism $\phi$ of $S(G)$ to itself, there is $u \in S(G)$ such that $\phi(v) = vu$ for every $v \in S(G)$.

Since $M(G)$ is a left ideal of $S(G)$, it follows that every $G$-homomorphism from $S(G)$ is a right translation by some element from $M(G)$. Thus homomorphisms from $S(G)$ to $M(G)$ separate points if and only if for every $x \neq y$ in $S(G)$ there is $m \in M(G)$ such that $xm \neq ym$.

Since $M(G)$ is a compact right-topological semigroup, it contains idempotents by Lemma 1. Recall that $m$ is an idempotent if $mm = m$. We denote the set of idempotents in $M(G)$ by $I(M(G))$.

$M(G)$ being a minimal left ideal of $S(G)$, it can be partitioned into maximal subgroups of $S(G)$ with respect to the property of having an idempotent in $I(M(G))$ as the identity element. For each $i \in I(M(G))$, the corresponding group $H_i$ is the intersection of $M(G)$ and the minimal right ideal $iS(G)$ of $S(G)$ and $H_i = iS(G)i$ (see Theorem 6).

Exploring the structure of $M(G)$ as a right-topological semigroup, we realize that we need not consider all elements of $M(G)$ to check which points of $S(G)$ are separated by homomorphisms to $M(G)$.

**Observation 1.** Let $r \in M(G)$ and let $i \in I(M(G))$ such that $r \in H_i$. Then $R_r : S \to M$ and $R_i : S \to M$ separate the same points of $S(G)$.

**Proof.** Consider $r \in M(G)$ and let $i(r)$ be the idempotent in $M(G)$ such that $r \in H_i$. In particular, we have that $ri = r = ir$. Then for $x, y \in S(G)$ it holds that $xr = yr$ if and only if $xir = yir$ if and only if $xi = yi$, since the right translation by $r$ is an automorphism of $M(G)$ and $xi, yi \in M(G)$. Therefore, the right multiplication by $r \in H_i$ separates exactly the same points as the right multiplication by $i$. \qed

As a consequence, we can translate Ellis’ problem for discrete groups into the dual language of Boolean algebras.

**Proposition 11.** Let $M(G)$ be a minimal left ideal in $\beta(G)$. Let $i \in I(M(G))$ and let $\rho_i$ denote the dual morphism to $R_i : \beta G \to M(G)$ from the algebra of clopen subsets of $M(G)$ into $P(G)$ with the image $B_i(G)$. Then $\{R_i : i \in I(M(G))\}$ separate points if and only if the subalgebra of $P(G)$ generated by

$$\bigcup_{i \in I(M(G))} B_i(G)$$

is equal to $P(G)$.

### 5.3 Disjoint left ideals

An amenable topological group is called uniquely amenable if every $G$-flow admits a unique invariant measure. If $S(G)$ has two disjoint minimal left ideals $M_1, M_2$, then $G$ cannot be uniquely amenable since $S(G)$ admits invariant measures supported on $M_1, M_2$ respectively. Groups that are easily uniquely amenable are precompact groups. Megrelishvili, Pestov and Uspenskij asked whether there are any others.
Question 5 ([41]). Is there a non-precompact group that is uniquely amenable?

Since the greatest ambit having disjoint minimal left ideals is a necessary condition for unique amenability, Question 5 will have a negative answer if we answer positively the following question.

Question 6. If $G$ is a non-precompact group, does the greatest ambit $S(G)$ contain disjoint left ideals?

In Chapter 2, we identified subsets of $G$ whose closures in $\beta(G)$ contain a minimal ideal in $\beta(G)$ as thick (see Lemma 6). Therefore finding two disjoint left ideals of $\beta(G)$ is equivalent to finding two disjoint thick subsets. In the case of a topological group, the greatest ambit ambit $S(G)$ has two disjoint minimal left ideals if and only if there are disjoint subsets $T_1, T_2$ of $G$ which are thick with respect to the discrete topology and a neighbourhood $V$ of the identity such that $VT_1 \cap VT_2 = \emptyset$.

Recall that a subset $T$ of $G$ is called thick, if the family $\{gA : g \in G\}$ has the finite intersection property.

The characterization of thick sets we will use is stated the following lemma introduced in the Preliminaries.

Lemma 8. Let $T$ be a subset of a topological group $G$. $T$ is thick if and only if for every finite subset $F$ of $G$ there is an $x \in G$ such that $Fx \subset T$.

The following theorem provides an affirmative answer to Question 6 for discrete groups.

Theorem 69 ([12]). Let $G$ be an infinite discrete group of cardinality $\kappa$. Then every thick subset of $T$ has $\kappa$ many mutually disjoint thick subsets.

Corollary 10. If $G$ is an infinite discrete group, then $\beta G$ contains disjoint minimal left ideals and therefore $G$ is not uniquely amenable.

Let $G$ be a topological group with a neighbourhood basis $\mathcal{N}$ of the identity in $G$. We use the description of the greatest ambit $S(G)$ via near ultrafilters by Koçak and Arvasi (Theorem 23). If $T$ is a thick subset of a $G$, then $\{VT : V \in \mathcal{N}\}$ contains a minimal left ideal of $S(G)$. Therefore, in order to generalize Corollary 10 to topological groups, we need to prove that if $G$ is not precompact then there are disjoint thick subsets $T_1, T_2 \subset G$ and a neighbourhood $V$ of the identity such that $VT_1 \cap VT_2 = \emptyset$. We succeed for non-precompact groups that possess a neighbourhood $V$ of the identity such that the number of left translates of $V$ necessary to cover $G$ is the same as for $V^6$, which we verify for certain classes of groups.

Non-precompact discrete groups are exactly infinite groups. For topological groups, infinity is replaced by the existence of a neighbourhood with infinitely many mutually disjoint translates.

Let $G$ be a topological group and let $V$ be a symmetric neighbourhood of the identity in $G$ (i.e. $V^{-1} = \{v^{-1} : v \in V\} = V$). We define $\kappa_V$ to be the smallest cardinality of a maximal disjoint system of left translates of $V$. Note that the number will remain the same if we replace left translates with right translates by symmetricity of $V$.

If $G$ is not precompact, then there is a neighbourhood $V$ such that $\kappa_V$ is infinite.

Definition 17. (a) Let $G$ be a topological group and let $V$ be a symmetric neighbourhood of the identity in $G$. We say that $V$ satisfies the property $N$ if $\kappa_V$ is infinite and

$$\kappa_V = \kappa_V^6.$$
(b) We say that $V$ satisfies the property $Q$ if for every finite set $F \subset G$ and a set $A \subset G$ of cardinality less than $\kappa_V$

$$FV^2A \neq G.$$ 

Note that if $\Gamma$ is a maximal subset of $G$ with respect to the property that for every $a, b \in \Gamma$ it holds that $Va \cap Vb = \emptyset$, then $V^2\Gamma = G$.

The following result shows that the property $N$ implies the property $Q$. The theorem generalizes a result by Solecki (see [65]) about Polish groups.

**Proposition 12.** If $V$ is a symmetric neighbourhood of the identity in a topological group $G$ and $\kappa_V = \kappa_V^2$ is infinite, then $V$ satisfies the property $Q$.

**Proof.** Let $V$ be a symmetric neighbourhood of the identity such that $\kappa = \kappa_V = \kappa_V^2$ is infinite. Let $A$ be a subset of $G$ of cardinality $< \kappa$ and suppose that there is $F \subset G$ finite such that $G = FV^2A$. We will show that then there exists $B \subset G$ of cardinality $< \kappa$ such that $G = V^{-2}V^2B = V^4B$, showing that $\kappa_V < \kappa$ which is a contradiction.

Suppose on the contrary that for every $B \subset G$ of size $< \kappa$ it holds that $G \neq V^4B$. We will prove that for every finite set $F \subset G$ the following statements holds.

(*) If $B \subset G$ has cardinality $< \kappa$ there exists a $g \in G$ such that $Fg \cap V^2B = \emptyset$, which is a contradiction with the existence of a finite $F \subset G$ with $FV^2A = G$.

We proceed by induction on the cardinality of $F$. If $F$ is the emptyset, then there is nothing to prove. Suppose that the statement (*) holds for all finite sets of size $n$. Let $F = \{f_1, f_2, \ldots, f_{n+1}\}$ and fix $B \subset G$, $|B| < \kappa$. Since $G \neq V^4B$, there exists $h \in G$ such that $V^2h \cap V^2B = \emptyset$. By induction, let $k \in G$ be such that $\{f_1, f_2, \ldots, f_n\}k \cap (V^2B \cup V^2Bh^{-1}B) = \emptyset$. If $f_n+1k \notin V^2B$, then we are done.

So let us assume that $f_{n+1}k \in V^2B$ and let $b \in B$ be such that $f_{n+1}k \in V^2b$. Then we have that $f_{n+1}k^{-1}b^{-1} \in V^2$. Set $g = kb^{-1}h$. Then $f_{n+1}h \in V^2g$ which is disjoint with $V^2B$. And if $1 \leq i \leq n$, then $f_i g = f_i k b^{-1} h \notin V^2B$, since $f_i k \notin V^2Bb^{-1}b$, which finishes the proof.

We are ready to generalize Theorem 69 to topological groups satisfying the property $N$.

**Theorem 70.** Let $G$ be a topological group with a symmetric neighbourhood $V$ of the identity satisfying the property $N$. Then there are $\kappa_V$-many pairwise disjoint thick subsets $\{T_\mu : \mu < \kappa_V\}$ of $G$ such that $\{VT_\mu : \mu \in \kappa_V\}$ are also mutually disjoint.

**Proof.** Let $\Gamma$ be a subset of $G$ of cardinality $\kappa_V$ such that for every $a, b \in \Gamma$ we have $Va \cap Vb = \emptyset$. Let $C = \{F_\sigma : \sigma \in \kappa_V\}$ be an enumeration of finite subsets of $\Gamma$. Let $C = \bigcup_{\mu < \kappa_V} C_\mu$ be a partition of $C$ such that for every $F \subset \Gamma$ finite there exist $F_\mu \in C_\mu$ such that $F \supset F_\mu$ for every $\mu \in \kappa_V$.

We will inductively construct $g_\sigma$ such that $V^3F_\sigma g_\sigma \cap V^3F_{\sigma'}g_{\sigma'} = \emptyset$ for every $\sigma, \sigma' \in \kappa_V$.

To see that this is possible, suppose that $g_\sigma$ has been constructed for all $\sigma < \lambda \in \kappa_V$ and we will find $g_\lambda$.

Consider $H = V^3A$ where $A = \bigcup_{\sigma < \lambda} F_\sigma g_\sigma$. We are looking for $g_\lambda$ such that $V^3F_\lambda g_\lambda \cap H = \emptyset$. That happens if and only if

$$g_\lambda \in G \setminus (F_\lambda^{-1}(V^3)^{-1}H) = G \setminus (F_\lambda^{-1}(V^3)^{-1}V^3A) = G \setminus (F_\lambda^{-1}V^6A).$$
Since the cardinality of $A$ is smaller than $\kappa_V$, the property $Q$ ensures that $F^{-1}_V A \neq G$ and therefore we can find $g_\lambda$ as needed.

Define $T_\mu = V^2 \cdot (\bigcup \{ F_\sigma g_\sigma : F_\sigma \in C_\mu \})$. Then $VT_\mu \cap VT_\nu = \emptyset$ whenever $\mu \neq \nu$, so it remains to show that $T_\mu$ are thick.

Let $F \subset G$ be finite. Since $V^2 \Gamma = G$, there is an $F' \subset \Gamma$ such that $F \subset V^2 F'$. Let $F_\sigma \in C_\mu$ be such that $F' \subset F_\sigma$. Then $Fg_\sigma \subset V^2 F_\sigma g_\sigma \subset T_\mu$, so $T_\mu$ is thick.

Since the closure of each $VT_\mu$ in $S(G)$ in Theorem 70 contains a minimal left ideal of $S(G)$, we can generalize Corollary 10 to topological groups with a neighbourhood of the identity satisfying the property $N$, thus answering Question 5 in the negative for a variety of topological groups.

**Corollary 11.** If a topological group $G$ satisfies property $N$, then $S(G)$ has disjoint left ideals and therefore $G$ is not uniquely amenable.

A notion closely related to $\kappa_V$ is that of a covering number. Let $V$ be an open neighbourhood of the identity in a topological group $G$. The covering number of $V$ is defined as $\lambda_V = \min \{|\Gamma| : V\Gamma = G\}$.

We can immediately see that $\kappa_V = \lambda_V^2$.

A cardinal invariant considered in the literature ([3], [69]) related to $\kappa_V$ and $\lambda_V$ is the index of narrowness. Let $G$ be a topological group and let $\kappa$ be a cardinal number. We say that $G$ is $\kappa$-narrow if $G$ can be covered by less than $\kappa$-many left (equivalently right) translates of any neighbourhood of the identity element. It is easy to see that the index of narrowness is equal to the supremum of $\kappa_V$ (respectively $\lambda_V$) for $V$ running through neighbourhoods of the identity.

**Lemma 14.** The following groups contain neighbourhoods of the identity satisfying the property $N$ if they are not precompact.

1. Groups containing an open subgroup with infinitely many cosets (e.g. infinite discrete groups, groups of automorphisms of structures containing infinitely many copies of some finitely-generated substructure).
2. Locally compact groups.
3. Groups in which the index of narrowness is attained (e.g. Polish groups, more generally groups with density $< \kappa_\omega$);
4. Groups of isometries of homogeneous metric spaces of infinite density $\kappa$ satisfying that for any $\lambda_n < \kappa$ for $n \in \omega$ it holds that $|\bigcup_{n \in \omega} \lambda_n| < \kappa$ (in particular generalized Urysohn spaces of density $\kappa$).

**Proof.**

1. Let $G$ be a topological group with an open subgroup $V$ with infinitely many cosets. Then $\kappa_V$ is infinite and $\kappa_V = \kappa_{V^n}$ for every finite power $n$.

2. Let $G$ be a locally compact group and let $W$ be a symmetric neighbourhood of the identity in $G$ with the compact closure such that $\kappa_W$ is infinite. Let $V$ be an open symmetric neighbourhood of the identity such that $V^{12} \subset W$. Then the closure $V^{12}$ of $V^{12}$ in $G$ is compact. Therefore there are finitely many translates of $V^2$ that cover $V^{12}$ showing that $\kappa_V = \lambda_V^2 = \lambda_{V^{12}} = \kappa_{V^n}$. 


(3) Let $\kappa$ be the index of narrowness and suppose that there is an open symmetric neighbourhood $V$ of the identity such that $\kappa_V = \kappa$. Then for every neighbourhood $W \subset V$ we have $\kappa_W = \kappa_V$, namely we can pick a symmetric neighbourhood $W$ such that $W^6 \subset V$, which in turn satisfies $\kappa_W = \kappa_{W^6} = \kappa$.

(4) Let $(X, d)$ be a homogeneous metric space with density $\kappa$ satisfying the required property. Let $x \in X$ and $\varepsilon > 0$ and let $V = V(\varepsilon, x) = \{ \phi \in \text{Iso}(X) : d(x, \phi(x)) < \varepsilon \}$. First, notice that $V$ is a symmetric neighbourhood of the identity. Second, notice that for $\phi \in \text{Iso}(X)$, $\phi V = \{ \psi \in \text{Iso}(U) : d(\psi(x), \phi(x)) \} < \varepsilon$. It means that $\phi V \cap \psi V = \emptyset$ exactly when $B(\phi(x), \varepsilon) \cap B(\psi(x), \varepsilon) = \emptyset$, where $B(x, \varepsilon)$ denotes the open $\varepsilon$-ball around $x$. It follows by homogeneity of $X$ that $\kappa_V$ is equal to the minimal cardinality of a maximal disjoint system of $\varepsilon$-balls in $X$. Suppose that for every $\varepsilon$, $\kappa_V(\varepsilon, x) < \kappa$. It means that for every $n$, there is a maximal disjoint system of $\frac{1}{n}$-balls in $X$, which in turn implies that for every $n$, there is a $\frac{1}{n}$-net $A_n$ in $X$ of cardinality $< \kappa$. Then $A = \bigcup_{n \in \mathbb{N}} A_n$ is dense in $X$. However, $|A| < \kappa$ by the assumption on $\kappa$, which is a contradiction. Therefore, there is $n_0$ such that the minimal cardinality of a maximal system of disjoint $\frac{1}{n_0}$-balls is $\kappa$. Consequently, $\kappa_V(\frac{1}{n_0}, x) = \kappa$ for any $x \in X$. Then for any symmetric neighbourhood $V$ satisfying $V^6 \subset V(\frac{1}{n_0}, x)$ we have $\kappa_V = \kappa_{V^6} = \kappa$, since $\kappa_V \leq$ the density of $X$.

Remark 6. We learnt that the existence of two disjoint left ideals for locally compact groups and groups in which the index of narrowness is attained follow from a result of Jan Pachl that these groups satisfy a stronger condition of ambitability arising in harmonic analysis (see [52]).

Question 7. Does every topological group contain a neighbourhood $V$ of the identity satisfying $\kappa_V = \kappa_{V^6}$ and $k_V$ is infinite?
Chapter 6

Problems

This Chapter collects problems that motived this thesis and problems that arouse while working on the thesis. The questions are quoted with numbers as they appear in the text.

In Chapter 4, we proved the following generalization of Theorem 7.5 in [34] from countable to uncountable structures.

**Theorem 40.** Let $\mathcal{A}$ be a locally-finite $\omega$-homogeneous structure and $\mathcal{K}_\mathcal{R}$ a Fraïssé precompact expansion of $\text{Age}(\mathcal{A}) = \mathcal{K}$ satisfying the expansion property and the Ramsey property. Then $X_\mathcal{R}$ is the universal minimal flow of the group of automorphisms of $\mathcal{A}$.

The obstruction we had to overcome was that while in the case of countable structures there is an interpretation $S \in X_\mathcal{R}$ of $\mathcal{R}$ on $\mathcal{A}$ such that $(\mathcal{A}, \mathcal{R}_\mathcal{A})$ is an $\omega$-homogeneous structures by the Fraïssé construction, we do not know whether an analogous situation appears for uncountable structures.

**Question 1.** Let $\mathcal{A}$ be a structure satisfying the assumptions of Theorem 40. Is there $S \in X_\mathcal{R}$ such that $(\mathcal{A}, X_\mathcal{R})$ is $\omega$-homogeneous?

In [2], the authors gave a characterization of unique ergodicity for groups of automorphisms of countable structures and stated a strong conjecture which we hope to attach with the near ultrafilter approach.

**Question 2 ([2]).** Let $\Gamma$ be an amenable Polish group with metrizable universal minimal flow. Is $\Gamma$ uniquely ergodic?

Let $L = \{\lor, \land, 0, 1, \neg\}$ be the language of Boolean algebras and let $J$ be an index set. We denote by $L_J$ the language $L \cup \{P_j : j \in J\}$, where each $J$ is a unary predicate. In Section 4.4.1, we proved the Ramsey property for classes of finite Boolean algebras of chains of non-trivial ideals under inclusion. It means classes of finite structures in the language $L_J$ that have a structure of a Boolean algebra with every $P_j$ for $j \in J$ is interpreted as a non-trivial ideal containing $0$, $J$ a linear order and $P_i \subset P_j$ whenever $i < j$. A natural question arises whether the class of finite Boolean algebras with non-trivial ideals is a Ramsey class.

**Question 3.** Let $\mathcal{K}$ be the class of isomorphism types of finite Boolean algebras in the language $L_{\{0,1\}}$, where $P_0, P_1$ are interpreted as non-trivial ideals with $0 \in P_i$ for $i \in \{0,1\}$. Is $\mathcal{K}$ a Ramsey class? Does $\mathcal{K}$ admit a precompact expansion to a Fraïssé class satisfying the Ramsey property and the expansion property relative to $\mathcal{K}$.

83
The motivation to study classes of finite Boolean algebras with ideals was to compute the universal minimal flow of the automorphism group of the Boolean algebra $\mathcal{P}(\omega_1)/\text{fin}$. We hope to relate this result to a surprising open problem in set theory.

**Question 8** (Katowice problem). *Is it consistent with axioms of set theory that the algebras $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{P}(\omega_1)/\text{fin}$ are isomorphic?*

If $\mathcal{K}$ is a countable Fraïssé class of finite Boolean algebras with ideals in a language $L_J$, then the Fraïssé limit of $\mathcal{K}$ is the countable atomless Boolean algebra $C$ with each $P_j$ interpreted as an ideal. As a result of the Fraïssé construction, $(C, P_j : j \in J)$ is $\omega$-homogeneous. Analogous structures to $C$ in higher cardinalities are free Boolean algebras on uncountably many generators. For every infinite cardinal $\kappa$ there is exactly one (up to isomorphism) free Boolean algebra $A_\kappa$ on $\kappa$-many generators. We showed that if $P$ is any prime ideal on $A_\kappa$, then $(A_\kappa, P)$ is $\omega$-homogeneous. The question remains how to construct an ideal $P$ or a chain of ideals $(P_j : j \in J)$ such that $(A_\kappa, P)$ or $(A_\kappa, P_j : j \in J)$ are $\omega$-homogeneous.

**Question 4.** Let $A_\kappa$ be the free Boolean algebra over $\kappa$-many free generators and let $\{P_i : i \in J\}$ be a chain of ideals on $A_\kappa$ ordered under inclusion for some linearly ordered set $J$. What conditions on $\{P_i : i \in J\}$ insure that $(A_\kappa, P_i : i \in J)$ is an $\omega$-homogeneous structure?

In Chapter 5, we investigated a problem of Ellis whether the greatest ambit and the Ellis enveloping semigroup of the universal minimal flow are isomorphic as ambits.

**Ellis problem 1.** *Is $\phi_M(G) : (S(G), e) \to (E(M(G)), e)$ an isomorphism?*

Any extremely amenable group is a counterexample to Ellis’ problem as well as the discrete group of integers ([23]). On the other hand, the problem easily has an affirmative answer for precompact groups and Pestov conjectured that this is the only case.

**Pestov’s conjecture 1.** *Ellis’ problem has a positive answer exactly for precompact groups.*

We applied near ultrafilters to make a progress on a question by Megrelishvili, Pestov and Uspenskij.

**Question 5 ([41]).** *Is there a non-precompact group that is uniquely amenable?*

Recall that an amenable topological group $G$ is called uniquely amenable, if every $G$-flow admits exactly one invariant measure.

If the greatest ambit $S(G)$ has two disjoint left ideals, then $G$ cannot be uniquely amenable. Therefore answering the following question in the affirmative would provide a complete solution to Question 5.

**Question 6.** *If $G$ is a non-precompact group, does the greatest ambit $S(G)$ contain disjoint left ideals?*

This question was fully answered for discrete groups in [12], where the authors showed that if $G$ is a discrete group of infinite cardinality $\kappa$, then $\beta G = S(G)$ contains $\kappa$-many disjoint minimal left ideals. We generalize their result to topological groups that admit an open neighbourhood $V$ of the identity whose covering number is infinite and equal to the covering number of $V^6$. However, we do not know whether all non-precompact groups satisfy this condition.

**Question 7 ([7]).** *Does every topological group contain a neighbourhood $V$ of the identity satisfying $\kappa_V = \kappa_{V^6}$?*
Recall that $\kappa_V$ is the minimal cardinality of the maximal pairwise disjoint set of left translates of $V$.

Due to recent progresses in the search of a counterexample to Question 7, we also ask the following question.

**Question 9.** Is there a non-trivial uniquely extremely amenable group?

It has been communicated to me by Jan Pachl that the existence of uniquely extremely amenable groups has been considered though never appeared in the literature.
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