COMPARISONS OF SPHERICAL SHELL AND PLANE-LAYER MANTLE CONVECTION MODELS

by

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Abstract

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Plane-layer geometry convection models remain useful for modelling planetary mantle dynamics however they yield significantly warmer mean temperatures than spherical shell models. For example, in a uniform property spherical shell with the same radius ratio, $f$, as the Earth’s mantle; a bottom heating Rayleigh number, $Ra$, of $10^7$ and a nondimensional internal heating rate, $H$, of 23 (arguably Earth-like values) are insufficient to heat the mean temperature, $\theta$, above the mean of the non-dimensional boundary value temperatures (0.5), the temperature in a plane-layer model with no internal heating. This study investigates the impact of this geometrical effect in convection models featuring uniform and stratified viscosity.

To address the effect of geometry, heat sinks are implemented to lower the mean temperature in 3D plane-layer isoviscous convection models. Over 100 models are analyzed, and their mean temperatures are used to derive a single equation for predicting $\theta$, as a function of $Ra$, $H$ and $f$ in spherical and plane-layer systems featuring free-slip surfaces.

The inclusion of first-order terrestrial characteristics is introduced to quantitatively assess the influence of system geometry on planetary scale simulations. Again, over 100 models are analyzed featuring a uniform upper mantle viscosity and a lower mantle viscosity that increases by a factor of 30 or 100. An effective Rayleigh number, $Ra_\eta$, is defined based on the average viscosity of the mantle. Equations for the relationship between $\theta$, $Ra_\eta$, and $H$ are derived for convection in a spherical shell with $f = 0.547$ and plane-layer geometries.

These equations can be used to determine the appropriate heating rate for a plane-layer convection model to emulate spherical shell convection mean temperatures for effective Rayleigh numbers comparable to the Earth’s value and greater. Comparing cases with the same $H$ and $Ra_\eta$, the increased lower mantle viscosity amplifies the mismatch in mean temperatures between spherical shell and plane-layer models. These findings emphasize the importance of adjusting heating rates in plane-layer geometry models and have important implications for studying convection with temperature-dependent parameters in plane-layer systems. The findings are particularly relevant to the study of convection in super-Earths where full spherical shell calculations remain intractable.
TO MY PARENTS, JANICE AND WILLIAM
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A.1 Phase diagram for $(Mg,Fe)_2SiO_4$ at range of pressures on y-axis and $T = 1873K$. $\alpha$ olivine changes to $\beta$ spinel and then $\gamma$ spinel at increased temperatures and/or pressures. Reprinted from Katsura and Ito (1989) with permission of the publisher John Wiley & Sons, Inc.
Chapter 1

Introduction

1.1 History

Various planetary bodies have different interior structure, and some of these differences can be manifested on the surface of the planet. During planetary formation, the planet undergoes differentiation, where denser elements ‘sink’ towards the center, creating layers within the planet with distinct characteristics. Most rocky planets are thought to be composed of layers: a core, a mantle and a crustal surface. The Earth is composed of a solid inner core, a hot liquid outer core, a hot convecting mantle divided into lower and upper layers and a crustal surface (see Figure 1.1).

The lithosphere is the outermost shell of a rocky planet, and it remains rigid over long time scales (~ $10^9$ yr) (Schubert et al., 2001). On the Earth, the lithosphere is composed of the crust and uppermost mantle. Below the lithosphere is the asthenosphere, a hotter and weaker part of the upper mantle. The boundary between the lithosphere and asthenosphere is defined by the response to stress. The mechanical lithosphere is rigid and deforms elastically (as observed in the flexing of the seafloor under the weight of seamounts, (e.g., Watts, 1972, 2001)), while the asthenosphere deforms viscously like a fluid.

Inside rocky planets the hot mantle convects, carrying heat towards the surface of the planet, and the nature of the convection is expressed in the appearance of the lithosphere. The lithosphere can be a single, solid rigid layer (a result of stagnant-lid convection) as on Mars, or it can be divided into mobile segments, as on Earth, in which case mantle convection manifests as plate tectonics (a class of mobile-lid convection). There are also transitional states between stagnant-lid and mobile surface convection, referred to as sluggish-lid and episodic mantle convection, where in the latter, the lithosphere is comprised of a single rigid surface for long periods before foundering, and being overturned and replaced with newer/younger mantle material. Evidence of long-lived, ongoing plate motion on the Earth appears in the magnetic field of the seafloor, seafloor sediment thickness, in the occurrence of earthquakes and even in the appearance of volcano locations.

The study of mantle convection is very important for understanding how the Earth functions, from how heat is lost from the interior to how the continental crust is replenished. While plate tectonics is a relatively new subject, only widely accepted in the 1960s, the interaction of the plates with the mantle is crucial for determining how the mantle transports heat. On Earth, the oceanic plates are the surface of the mantle and correspond to the thermal boundary layer formed at the top of the convecting mantle (across which the temperature varies continuously from the surface value to the mean mantle
temperature) (e.g., Turcotte and Oxburgh, 1967; Schubert et al., 2001). According to the boundary layer model developed for the mantle by Turcotte and Oxburgh (1967), the ascending convection is associated with ocean ridges and the descending convection is associated with ocean trenches. Ocean ridges are mountain ranges located along the seafloor, that circumscribe the Earth’s surface, while narrow ocean trenches comprise the deepest part of the ocean. These ridges and trenches define boundaries between plates. At trenches, one plate bends and the heavier, colder and denser material descends into the mantle, driven by gravitational forces, in a process called subduction (Davies, 1988a,b). At mid-ocean ridges, hot buoyant mantle material rises to create new lithosphere, thus replenishing the material lost through subduction. This cycle of destroying and creating new lithosphere is controlled by mantle convection.

Attempting to model mantle convection in rocky planets, to enhance our knowledge of planetary evolution, is the motivation for the work presented here. This thesis presents a study of mantle convection using computational models, and focuses on the thermal structure associated with different computational geometries used in convection models. Modelling the correct thermal structure can help determine when the onset of stagnant-lid convection versus mobile plates will occur in other planets.

1.1.1 Mantle Convection

Convection is a term describing the flow of heat carried by motion in a fluid. Density gradients in the mantle are largely derived from horizontal temperature gradients (and also chemical/compositional horizontal gradients). In the thermal boundary layers (across which the temperature varies continuously from the surface value to the mean mantle temperature) this buoyancy causes instabilities, allowing fluid to leave the boundary layer and rise or fall throughout the system interior. Subsolidus convection occurs from diffusion or dislocation creep in a solid material (see Section 1.2.2 for more details).
Figure 1.2: Convection cell in the mantle. Hot upwelling plume in red and cold downwelling (subducting) slab in blue.

The mantle is a visco-elastic solid, meaning it behaves both viscously and elastically in response to a stress. The viscous nature of the mantle is evident in the slow creep of the mantle manifesting itself as plate motion on the surface of the Earth. The elastic nature of mantle rock is evident in the transmission of shear waves within the mantle.

Heat is the main source of energy driving convection in the mantle. Heat in the mantle is derived from internal sources (radioactive decay of the elements uranium, thorium and potassium), heat released from the core and secular cooling of the planet as a whole (residual heat left over from planetary formation and a higher production of radioactive heating in the past), and is discussed in detail in Section 1.3.

Figure 1.2 shows a depiction of a mantle convection cell, with a hot upwelling plume (red) and cold subducting slab (blue). Figure 1.1 shows a depiction of mantle convection with an upwelling plume, passive upwelling at a mid-ocean ridge and subducting slabs (downwellings). It also shows large-low shear velocity provinces, ultra-low-velocity-zones and areas of post-perovskite. The Earth’s top and bottom thermal boundary layers are influenced by strong chemical heterogeneity.

1.2 Modelling the mantle as a convecting fluid

1.2.1 Laminar flow and momentum free fluids

Fluid flow can be described as laminar or turbulent. Laminar flow occurs when the fluid particles move in well defined paths with no mixing between the layers. Mixing occurs in turbulent flow (e.g., Kundu and Cohen, 2004). The change in flow type can be determined with a nondimensional parameter, the Reynolds number, \( Re = u d / \nu \), representing the ratio of inertial forces to viscous forces. If the Reynolds
number is high, the inertial forces dominate and the flow is turbulent. Whereas, if the Reynolds number is low, the viscous forces dominate.

With a low Reynolds number it is possible to determine if heat or momentum diffuse faster by comparing these two diffusion time scales. The ratio of these diffusion time scales is the Prandtl number, \( Pr = \frac{\nu}{\kappa} \), where \( \nu \) is the kinematic viscosity and \( \kappa \) is the thermal diffusivity. For large Prandtl number fluids, the momentum diffusivity (i.e., kinematic viscosity) dominates, and convection rather than conduction dominates the movement of energy. When \( Pr \) is small, heat diffuses quickly compared to the momentum.

In the Earth’s mantle the Prandtl number is around \( 10^{24} \) because the kinematic viscosity is large around \( 10^{18} \) m\(^2\)/s and the thermal diffusivity is small around \( 10^{-6} \) m\(^2\)/s, so momentum diffusivity dominates and the slow flow is dominated by viscous forces so the Reynolds number is low. In the derivation of the nondimensional equations of motion for a convecting fluid (see Chapter 2), the inertial term is multiplied by \( 1/Pr \). Therefore, the inertial term for mantle convection is negligible and the Earth’s mantle is a momentum free fluid. Because the Coriolis force is small compared to the viscous force, the effects of rotation can also be ignored. A momentum free fluid can be understood given an example with two cups, one containing honey, one containing water. The water is much less viscous compared to the honey. If both cups are stirred with a spoon and the spoon is removed (i.e., the driving force), the water will continue to rotate but the honey will stop quite quickly. This is due to the viscous forces resisting the motion in the honey filled cup. The Earth’s mantle has a very large viscosity which eliminates all motion in the absence of a driving force.

### 1.2.2 Creep flow

Creep flow is characterized by a small velocity or large viscosity. Creep occurs when a crystalline solid deforms slowly, like a fluid under the influence of stresses, such as at high temperature and pressure. The rocky mantle of a planet experiences both high temperatures and high pressures, increasing toward the centre of the planet.

In the mantle, deformation occurs by atom migration or dislocation creep (Schubert et al., 2001) as first proposed by Gordon (1967). There are empty spaces in the crystalline lattice and an equilibrium number of spaces determined by temperature dependence. In diffusion creep, atoms move due to the movement of adjacent vacancies. This diffusion of atoms is activated at a particular temperature. The activation energy is the sum of energy required for making a vacancy and energy preventing movement of an atom into an adjacent empty site (Schubert et al., 2001).

Dislocations are line imperfections in a crystalline lattice. Dislocation creep occurs by dislocation slip and dislocation climb. In dislocation slip, the dislocation line moves through a crystalline lattice and breaks the interatomic bonds while conserving mass. In dislocation climb, the line moves by adding atoms to the lattice, requiring movement of atoms from elsewhere in the lattice structure. Dislocation creep is also activated by temperature at low stress levels (Schubert et al., 2001).

Diffusion creep results in a linear relationship between strain-rate, \( \dot{\varepsilon} \), and deviatoric stress, \( \tau \), giving rise to a Newtonian viscosity. Dislocation creep follows a nonlinear relation between strain-rate and stress which results in a nonlinear viscous rheology (Schubert et al., 2001). A Newtonian viscosity fluid is modelled in this study.
1.2.3 Mantle structure

The structure of the Earth’s upper mantle can be derived from seismic waves. In the first few kilometers of depth, there is little change in the velocities of the seismic waves, velocities then increase down to a depth of around 200 km, at which depth the velocities begin to decrease. The upper region is referred to as the seismic lid (LID) and the lower velocity region as the low-velocity-zone (LVZ).

The seismic LID is widely interpreted as approximating the Earth’s surface thermal boundary layer, indicating the transition from lithosphere to asthenosphere. Below ocean basins, the LVZ can be interpreted as the convergence of the temperature geotherm and solidus of mantle rock and thus represents the starting depth of partial melting of mantle rocks (Schubert et al., 2001).

The seismic discontinuity at 660 km depth indicates an important constraint in modelling mantle convection. The seismicity of the mantle terminates at this depth, which led to the theory of layered mantle convection (see Section 1.4).

The lower mantle has a very gradual change in seismic properties, starting from the transition zone and descending to about 300 km above the core (Johnson, 1967). Above the core are ultra-low velocity zones (ULVZ) and large low shear velocity provinces (LLSVP) (e.g., Garnero and Helmberger, 1996), as seen in Figure 1.2. While a large part of the mantle between the transition zone and 300 km above the core is approximately homogenous, there are large and fine-scale heterogeneities in the lowermost mantle directly above the core (e.g., Haddon, 1982; Scherbaum et al., 1997; Weber, 1993; Thomas et al., 2004; Lay and Garnero, 2011).

1.2.4 Minerals in the mantle

Meteorites are fragments of asteroids that can be considered as remnants of the building blocks of terrestrial planets, including Earth, and from their composition it can be determined that the mantle is mostly composed of peridotite (olivine and pyroxene) and the core is mostly composed of iron (e.g., McDonough and Sun, 1995). The bulk composition of the mantle is mostly olivine, pyroxenes, garnet, stishovite (a hard, dense form of SiO$_2$), and magnesiowüstite (or ferropericlase) as well as olivine’s phase changes to spinel and perovskite. Olivine is a mixture of magnesium and iron silicates, $(Mg,Fe)_2SiO_4$ (e.g., Ringwood, 1975; Green and Falloon, 1998).

The seismic discontinuity at 410 km is associated with the phase change from olivine to spinel, while the discontinuity at 660 km represents the transformation of spinel to magnesium perovskite and magnesiowüstite. More details on the minerals in the mantle are described in Appendix A.

The lower mantle is mostly homogenous until about 200 km above the core-mantle boundary. This layer above the core-mantle boundary is referred to as the $D''$ layer, a designation that is a remnant of Bullen’s division of the Earth into layers A through G and subsequent further division of layer D into $D'$ and $D''$ (Bullen, 1936, 1940, 1950). This homogenous nature is due to the lack of many phase transformations at the high pressures in the lowermost mantle, so silicate perovskite and magnesiowüstite make up the bulk composition of the lower mantle.

The $D''$ layer may be the source of deep mantle plumes (e.g., Williams and Garnero, 1996), and may be compositionally different from the rest of the lower mantle (e.g., Kellogg et al., 1999; Schubert et al., 2001; Tackley, 2008). It contains regions of perovskite and post-perovskite (e.g., Murakami et al., 2004; Oganov and Ono, 2004; Peltier, 2007). This region also contains ultra-low velocity zones (ULVZ) just above the core-mantle boundary (Garnero and Helmberger, 1996) as well as the much bigger “large low-
shear-velocity provinces” (LLSVP) located beneath the Pacific and Africa (e.g., Garnero and McNamara, 2008; Thorne et al., 2013).

1.3 Sources of heat in the mantle

Heat in the mantle comes from various sources, isotopes decaying and producing heat, heat from the freezing of the inner core and secular cooling of the planet as a whole (i.e., heat from planetary accretion and a higher radiogenic heating rate in the past). The observed heat flux and age of the Earth can be reconciled if mantle heat loss is dominated by convection.

1.3.1 Advective versus diffusive heat transfer in the mantle

There are three regimes in the interior of a planet (Jeanloz and Morris, 1986). In the region between the thermal boundary layers, heat transport is dominated by vertical advection over other modes of heat transportation. Most of the mantle (and outer core) is in this regime where temperature increases with pressure/depth and heat is predominantly transported by advection (Ito and Katsura, 1989). There are also regions where heat transport occurs by a combination of advection and conduction, as occurs in the thermal boundary layers at the base and top of the mantle. The third regime is characterized by predominately conductive heat transfer, as found in the crust (Schubert et al., 2001).

1.3.2 The Earth’s heat budget

Heat loss from the Earth is something that can be directly observed. Pollack et al. (1993) calculated that the average heat flux from the continents is 65 mW/m² and the average heat flux from the oceans is 101 mW/m². Multiplying the heat flux by the area of the continents and oceans respectively gives the total heat flow out the Earth, $44.2 \times 10^{12}$ W or an average heat flux of 87 mW/m². More recent work by Jaupart et al. (2007) interpreted the same data set differently and arrived at an average total surface heat flux from Earth of 46 TW ± 3 TW. Davies and Davies (2010) presented a further revised estimate of the Earth’s surface heat flux, arriving at an average value of 47 TW (rounded from 46.7 TW with an error of ± 2 TW). While the exact amount of heat production from different sources on the Earth is not known, estimates of ranges of heat flow can be determined (e.g., Pollack et al., 1993; Jaupart et al., 2007; Lay et al., 2008; Davies and Davies, 2010).

Part of the Earth’s heat comes from the decay of radioactive elements, uranium (235, 238), thorium (232) and potassium (40). An upper limit on the amount of radiogenic heating can be derived by using $\psi = Q/M$ where $\psi$ is the amount of heat production per unit mass, $Q$ is heat flow out of Earth and $M$ is the mass of the radioactive matter in the mantle. Using Pollack’s (1993) original estimate for surface heat flux, and assuming the entire Earth is involved in radiogenic heating, the upper limit on $\psi$ is $7.42 \times 10^{-12}$ W/kg. Geochemical studies suggest the core cannot contain radioactive elements and so only the mantle is involved in heat production, bringing the upper limit to $\psi = 11.1 \times 10^{-12}$ W/kg. However, a large fraction of this heat production actually comes from radiogenic material in the crust, which reduces the upper limit to $\psi = 9.22 \times 10^{-12}$ W/kg. This radiogenic heat production decreases with time as the elements decay from radioactive isotopes to stable isotopes. Due to the half-lives of the radioactive elements, there was more heat production in the mantle in that past than currently. Figure 1.3 shows the decay of elements going backwards in time from present day. The figure assumes about
Figure 1.3: Plot as a function of time, going backwards from present day, of heat production in the mantle due to the radiogenic decay of uranium, thorium and potassium isotopes. Reprinted from Schubert et al. (2001) with the permission of Cambridge University Press.

80% of the present surface heat flux comes from current sources and the remaining 20% from cooling of the Earth, therefore, the upper limit on \( \psi \) is reduced to \( 7.38 \times 10^{-12} \text{ W/kg} \).

The estimate of how much of the present heat loss is coming from the freezing of the core has actually increased in recent years. Original estimates had 3-4 TW of heat coming from the core, but new estimates put this value at 5-15 TW (Lay et al., 2008).

In the course of the Earth’s history, the abundance of heat producing sources has reduced by a factor of four (Jaupart et al., 2007). A substantial amount of the heat currently being lost by the mantle is from heat produced and trapped in the past when the heating rate was higher. The loss of this heat is one of the components of the heat loss that contributes to secular cooling of the Earth.

1.3.3 Secular cooling

There is a discrepancy between the heat input into the Earth system and the heat output at the surface. The poorly constrained difference of about 8-28 TW (17% - 60%) of heat is not accounted for by the present day radiogenic decay or cooling of the core. This extra heat comes from secular cooling of the mantle (e.g., Pollack et al., 1993; Jaupart et al., 2007; Lay et al., 2008; Davies, 1999; Davies and Davies, 2010).

During the accretion of matter that forms a planet, gravitational potential energy is released. As the planet becomes differentiated (separation of core and mantle by density) more energy is released. Finally, the cooling and crystalization of magma oceans on the Earth or in the interior (Labrosse et al., 2007) will also have released energy, heating the Earth.

Presently, the thermal profile of the mantle has much uncertainty. Figure 1.4 shows the possible range of mantle geotherms based on melting curves which constrain the upper and lower bounds of the
1.4 Viscosity of the mantle

Radial variations in the viscosity of the mantle can be inferred by observing variations in seismic velocities and their corresponding density variations. A two-layer viscosity model for the Earth with a high viscosity lower mantle and a lower viscosity upper mantle can be constrained using tomography and geoid data determined from postglacial rebound rates. The upper mantle viscosity is dominated by temperature variations while the lower mantle viscosity has a stronger pressure dependence. The observational constraints on the viscosity in the asthenosphere is $10^{19} - 10^{21}$ Pa s (e.g., Haskell, 1935, 1936, 1937; Peltier, 1989; Mitrovica, 1996). Viscosity variations can also be expressed and observed as dynamic topography on the surface of a planet. A larger lower mantle viscosity corresponds to more dynamic topography on the surface (e.g., Richards and Hager, 1984; Ricard et al., 1989).

Mantle viscosity can be determined by looking at the mantle response to unloading of mass on the surface (postglacial rebound) and at lower mantle pressures by laboratory experiments with mantle minerals (e.g., Karato, 1998; Tsehn and Carter, 1987; Karato and Wu, 1993).

There are two interpretations of the seismic discontinuity at 660 km possible for mantle convection. The mantle can be considered to be convecting as a whole (e.g., Davies, 1977; Peltier and Jarvis, 1982), or in layers (e.g., Turcotte and Oxburgh, 1967, 1972; Richter, 1973; McKenzie et al., 1974). While it...
was originally supposed that the layered mantle explained the observed seismic discontinuity at 660 km, the current consensus is for whole mantle convection. Figure 1.5 shows the two types of convection.

1.4.1 Whole mantle convection

Mantle properties can also be determined based on geoid (gravity) data by looking at geoid anomalies associated with density anomalies in the mantle. Since buoyancy driven flow in the mantle is controlled by density changes, differences between the observed geoid and inferred density can limit the range of mantle viscosity values.

Haskell (1935) determined from the postglacial rebound from Scandinavia and Finland (Fennoscandian ice sheet) that the mean whole mantle viscosity is $\mu = 10^{21}$ Pa s. The asthenosphere viscosity is smaller than the average mantle viscosity by a few orders of magnitude. Schubert and Hey (1986) found it to be $10^{17} - 10^{18}$ Pa s, while later studies have found higher values of about $10^{20}$ Pa s (e.g., Kaufmann and Lambeck, 2000; Zhao et al., 2012). Further studies of mantle viscosity have agreed with earlier results and found a mean mantle viscosity of $\mu = 10^{21}$ Pa s (e.g., Peltier, 1989; King, 1995; Mitrovica, 1996; Peltier, 1996; Peltier and Jiang, 1996; Giunchi and Spada, 2000; Forte and Mitrovica, 2001).

A consensus was reached by some studies (e.g., Hager, 1984; Richards and Hager, 1984; Gurnis and Davies, 1985; Ricard and Vigny, 1989; Forte and Mitrovica, 2001) to determine the lower mantle viscosity increases by a factor of 30 over the upper mantle viscosity. However, while it is generally agreed upon that the lower mantle viscosity is larger than the upper mantle viscosity, the proposed amount
of increase varies between 10-100 times the upper mantle viscosity (e.g., Forte et al., 1991; King and Masters, 1992; King, 1995; Peltier, 1996; Peltier and Jiang, 1996; Kaufmann and Lambeck, 2000; Forte and Mitrovica, 2001; Mitrovica and Forte, 2004; Zhao et al., 2012).

Figure 1.6 shows a summary of viscosity profiles determined by different methods. The figure shows the variation in possible viscosity profiles inferred from the data, as well as a two-layer viscosity profile given by the average viscosity in each layer. This figure shows the contrast between upper and lower mantle viscosity to be about a factor of 30. It is this contrast that is used as guide for the simple viscosity profile utilized in this study.

### 1.4.2 Layered mantle

In the past, many geodynamists were of the view that the mantle convects in two-layers. With such a two-layer model, the heat would conduct out of the lower mantle and into the upper mantle which would convect to remove heat from the entire system. Thus two-layer mantle convection is less efficient at removing heat from the system compared with whole-mantle convection. This inefficiency would lead to higher mean temperatures inside the mantle. Figure 1.7 shows the two possible geotherms for whole mantle and two-layer convection models.

By definition, stratified mantle models do not exchange mass between the distinct layers and have the further complication that each layer convects almost independently to maintain their different composition (e.g., Richards and Hager, 1984). Any transport of material across these boundaries would mix the mantle. However, the observed increase in seismic velocity from the LVZ to about 400 km is consistent with a mantle with homogenous composition (Schubert et al., 2001; Tackley, 2008). In addition, seismic tomography data has shown slabs penetrating into the lower mantle, and thereby passing through 660 km depth (e.g., Deal and Nolet, 1999; Steinberger, 2000; Butler, 2009). Therefore, the current consensus
among scientists is for whole mantle convection.

1.5 Numerical modelling of mantle convection

Improvements in computational resources have allowed mantle convection simulations to evolve from two-dimensional (2-D) calculations (e.g., Jarvis and Peltier, 1982; Jarvis and Peltier, 1986; Travis et al., 1990a; King et al., 1992) to three-dimensional (3-D) plane-layer models (e.g., Houseman, 1988; Parmentier et al., 1994; Sotin and Labrosse, 1999) and finally to 3-D spherical models (e.g., Bercovici et al., 1989a, 1989b; Zhang and Yuen, 1995; Bunge and Richards, 1996; van Keken and Zhong, 1999; Tackley, 2000b; Zhong et al., 2000; Tackley et al., 2001; Shahnas et al., 2008; van Heck and Tackley, 2008). Three-dimensional spherical models still require large computational power to model mantle convection with Earth-like convective vigour (i.e., high Rayleigh number) (e.g. Schuberth et al., 2009; Wolstencroft et al., 2009; Tan et al., 2011; Crameri et al., 2012) or complex rheologies (e.g., van Heck and Tackley, 2008; Stein and Hansen, 2008). With an emerging interest in Super-Earths, it may be necessary to model even larger Rayleigh number systems with large domain sizes, suggesting the continuing importance of plane-layer models, which more easily accommodate computational requirements. However, plane-layer geometry models often overlook a consequence of system geometry, a change in thermal structure of the system, including high mean temperatures and weaker plumes. This thesis looks at how to quantify the thermal differences in the two geometries, to understand the limitations when applying previous plane-layer geometry results to current studies.

1.5.1 Models featuring plate tectonics

The interaction between mantle and plate dynamics is an active area of research. Numerous laboratory experiments and theoretical studies have shown that a high Rayleigh number homogenous fluid between
two boundaries forms a regular flow pattern with polygonal cells (e.g., Houseman, 1988; Davies, 1988a). These cells have horizontal length scales related to the thickness of the fluid system (Davies, 1988a). Comparing these results with the Earth, however, shows surprising contrasts. The horizontal scale of the Pacific plate, for example, is much larger than the depth of the mantle, hence the convection length scale found in Bénard-Rayleigh convection differs substantially from observed plate sizes. Moreover, the plates are generally irregular in size and shape and the underlying convection cells will reflect these characteristics (Lux et al., 1979; Davies, 1988a; Bunge and Richards, 1996; Lowman et al., 2001; Monnereau and Quéré, 2001). It has been found (Lux et al., 1979; Bunge and Richards, 1996) that the flow has horizontal scales comparable to the size of the plate, leading one to consider associating one convection cell with each plate (Lux et al., 1979; Bunge and Richards, 1996). However, forcing horizontal length scales to correspond to the lengths of plates will require different size solution domains for different plate lengths.

There are still many basic questions about the effect of certain parameters on mantle convection (such as Rayleigh number, internal heating rate and even solution domain size) that have not been fully explored. With the added complexity of including plate-like features in the convection models, it can be difficult to analyze the effect of a single parameter on the nature of convection. The studies described here are completed in the absence of plates. The models feature a free-slip upper surface. The inclusion of plates will increase the temperature of the overall system in both spherical shell and plane-layer convection models (e.g., Lowman et al., 2001). However, before addressing complexities that will heat both systems, this study focuses on parameters which can cool the system in order to compare different geometries. The results discussed here provide insight into the best parameter values to use in future, more complex, calculations.

1.5.2 Plane-layer models

Studies performed in laboratory tank experiments are performed in a plane-layer geometry. In addition, these experiments are subject to the presence of sidewalls and their effects, as the fluid has to be contained in a box. Numerous plane-layer numerical convection experiments have been performed with the benefit that plane-layer computational models can use periodic sidewalls (as in the studies described below) and avoid any effect from sidewalls.

Two-dimensional plane-layer systems typically feature vastly different convective planforms from their 3D counterparts. In a 2D plane-layer model when upwellings and downwellings develop they are representing long sheet-like upwellings and downwellings. Regardless of Rayleigh number, internal heating rate or aspect ratio, all 2D models produce sheet-like features. Three-dimensional calculations can feature a variety of convective planforms from sheet-like to bimodal (i.e., perpendicular rolls) to columnar upwelling and downwelling plumes.

Three-dimensional plane-layer convection models are commonly used to study mantle convection (e.g., Lowman et al., 2004, O’Farrell and Lowman, 2010; Deschamps et al., 2010; Stein et al., 2013; O’Farrell et al., 2013). However, these 3D models do not capture all the complexity necessary for modelling planets. The upper and lower surface areas in a plane-layer geometry are equal in size, but these areas are quite different in spherical shells with Earth-like core dimensions. Thus the lower mantle volume is much larger in a plane-layer geometry than in a spherical shell. This affects the amount of heating from the core, as well as internal heat production in the layer. In addition, it affects the amount of higher viscosity fluid present in the lower mantle. For example, with the additional volume in a
plane-layer model, subducting slabs have more space to enter the lower mantle and spread across the core-mantle boundary (CMB). All these factors will impact the convection planforms seen in plane-layer models. This study focuses on characterizing the effect of this geometric difference with the hope of adjusting plane-layer models appropriately in order to account for the difference.

1.5.3 Spherical shell models

The surface area in a spherical shell model is larger than the surface area of a plane-layer model with the same basal area. Consequently, a spherical shell system cools much more efficiently than a plane-layer system at the same convective vigour (measured quantitatively by the Rayleigh number). Previous studies in a spherical shell geometry have shown the effect of the ratio of inner radius, $r_i$, to outer radius, $r_o$, on the mean temperature and convection planform of the systems (e.g., Bercovici et al., 1989a, 1989b; Jarvis, 1993; Vangelov and Jarvis, 1994; Jarvis et al., 1995; Shahnas et al., 2008, Behounkova and Choblet, 2009; Deschamps et al., 2010; Choblet, 2012; Deschamps et al., 2012). They showed that decreasing the ratio $f = r_i/r_o$ decreased the mean temperature of a system with a fixed Rayleigh number and internal heating rate.

In particular, Jarvis et al. (1994) showed that for isoviscous systems with no internal heating and an arbitrary Rayleigh number, the mean temperatures of spherical shell systems are lower than the average of the boundary temperatures, which is the mean temperature in a plane-layer with no internal heating, regardless of Rayleigh number. Moreover, the mean temperature decreases as $f$ decreases. This, in part, motivates the study described in Chapter 3.

Conductive cooling of a sphere

The effect on heat transfer at the inner surface radius, versus the outer surface radius of a spherical shell is illustrated by considering the steady state solutions for conductive heat flow. A spherical shell with no internal heating and only isothermal basal heating reaches a steady state when there are no long term heating or cooling trends in the mean temperature of the system.

The conductive cooling equation for a simple solid or stationary fluid is

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)$$

(1.1)

where $\rho$ is density, $C_p$ is the specific heat capacity, $T$ is temperature, $t$ is time, and $k$ is the thermal conductivity.

When $k$, $\rho$, and $C_p$ are constant, equation (1.1) reduces to the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$  

(1.2)

where $\kappa$ is the thermal diffusivity.

For steady state solutions, the temperature in the system must therefore satisfy

$$\nabla^2 T = 0$$

(1.3)

for all time, $t$. Equation (1.3) is solved in a spherical shell geometry where $r$ is the radius. For a nondimensional solution in a spherical shell, assume the Laplacian operator is nondimensionalized and
the nondimensional boundary conditions are given by \( T = 1 \) at \( r_i \), the inner radius, and \( T = 0 \) at \( r_o \), the outer radius of the spherical shell, where \( r_o - r_i = 1.0 \) and \( r_i/r_o = f \).

Expanding equation (1.3) into spherical coordinates for the Laplace operator gives

\[
\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2}.
\]

(1.4)

In the simple case of a sphere with no angular variation, there is no \( \theta \) or \( \phi \) dependence, and the above equation (1.4) can be simplified to

\[
\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = 0.
\]

(1.5)

The solution to this equation (1.5) (derived in Appendix B) and therefore the equation for temperature as a function of radius in a purely conductive spherical shell is given by

\[
T(r) = \frac{r_i r_o}{r} - r_i
\]

(1.6)

\[
\Rightarrow T(r) = \frac{f}{r(1-f)^2} - \frac{f}{(1-f)}
\]

(1.7)

The plane-layer case occurs in the limit where \( f \to 1 \), which can be considered as the limit when \( r_i \to \infty \) with \( f = r_i/(r_i + 1) \). Taking the limit of equation (1.6), as \( f \to 1 \), with \( r \) some radius with \( r = r_i + z \) and \( 0 < z < 1 \) in the spherical shell, yields:

\[
\lim_{r_i \to \infty} \frac{r_i r_o}{r_i + z} - r_i = \lim_{r_i \to \infty} \frac{1 - z}{1 + z/r_i} = 1 - z
\]

(1.8)

(1.9)

Therefore, in the limit where \( f \to 1 \) the equation for temperature is just \( T(z) = 1 - z \) where \( z \) takes on values from 0 to 1.

Figure 1.8 shows profiles of the steady state temperatures in a conducting sphere of thickness \( 1 = r_o - r_i \) for different \( f \) values from \( f = 0.1 \) to \( f = 1 \) (the plane-layer case). This figure shows the effect of curvature on different sized spherical shells. As \( f \) is decreased, or as the spherical shell develops more curvature and has a smaller inner core relative to the shell thickness, the temperature profile cools across the shell depth and moves further from the straight line representing plane-layer convection. Clearly, the curvature of the sphere has a large effect on the mean temperature of the system.

1.6 Mixed mode convection and boundary layer theory

The mean temperature in a convection system is determined by numerous parameters, such as the thermal expansivity variability, viscosity structure, thermal conductivity variability, and in systems with both basal (bottom) and internal heating, the nondimensional internal heating rate, \( H \) and Rayleigh number, \( Ra \).

The nondimensional internal heating rate, \( H \), represents heating from internal heat sources from radiogenic decay and secular cooling. The Rayleigh number, \( Ra \), helps describe the vigour of convection
Figure 1.8: Profiles of mean temperature versus $r - r_i$ values for different spherical shell curvatures $f$ in purely conductive shells with no internal heating. $r - r_i$ is the height above $r_i$ in the conducting shell and $f = r_i/r_o$. 
1.6.1 Parameterizations of mean temperature and heat flow in a basally heated fluid

Previous studies (e.g., Jarvis et al., 1995; Parmentier et al., 1994; Sotin and Labrosse, 1999; Deschamps and Sotin, 2000; Shahmas et al., 2008; Deschamps et al., 2010) have attempted to parameterize equations for the mean temperature in convecting fluids. These studies used theoretical predictions based on boundary layer theory and data from numerical simulation to derive the parameterizations.

Jarvis et al. (1995) derived equations for the mean temperature of a convecting fluid in uniform property axi-symmetric spherical shells (a 2D slice of a sphere with no \( \theta \) dependence). They showed that decreasing the curvature value \( f \) (where \( f = r_i/r_o \), and \( r_i \) is the inner radius of the spherical shell and \( r_o \) is the outer radius of a spherical shell) decreased the mean temperature of the system. Figure 1.9 shows a plot of the laterally averaged mean temperature with depth (referred to as a geotherm from this point) for the different values of \( f \) considered.

The thermal boundary layer is the region at the top and bottom of the convecting system across which the temperature varies continuously from the surface/basal value to the approximately isothermal interior temperature. These regions can be seen in Figure 1.9 at the top and bottom where the temperature changes with a different gradient to the nearly isothermal temperature across the middle of the mantle.
1.6.2 Derivation of an equation for the mean temperature of a convecting fluid heated by internal sources and an isothermal bottom boundary

Considering a local boundary layer Rayleigh number, $Ra_\delta = (\alpha g T_{bdy} \delta^3) / (\nu \kappa)$, for a boundary layer of thickness $\delta$ and a temperature change across the boundary layer of $T_{bdy}$, and assuming $Ra_\delta$ converges to a constant value in the steady-state, the following relation is retrieved:

$$
\frac{1}{\delta} \sim \left( \frac{\alpha g T_{bdy}}{\nu \kappa} \right)^{1/3},
$$

where $\alpha$ is the thermal expansion coefficient, $g$ the gravitational acceleration, $\nu$ the kinematic viscosity in the thermal boundary layer and $\kappa$ the thermal diffusivity.

Now heat flux, $q$ across the top boundary scales as

$$
q \sim \frac{k T_{bdy}}{\delta}
$$

where $k$ is the thermal conductivity. Combining equation (1.11) and (1.10) means the heat flux through the top of the system is given by

$$
q_t \sim k \left( \frac{\alpha g}{\nu \kappa} \right)^{1/3} T_{bdy}^{4/3}
$$

Non-dimensionalizing the heat flux by using the conductive heat flux across the system in the absence of convection yields

$$
q'_t = \frac{q_t}{k \Delta T / d}
$$

where a prime indicates a nondimensional quantity, $d$ is the depth of the system and $\Delta T$ is the temperature difference across the convecting system. Combining (1.13) with (1.12) yields

$$
q'_t \sim \left( \frac{\alpha g d^3}{\nu \kappa} \right)^{1/3} T_{bdy}^{4/3} \frac{(\Delta T)^{1/3}}{\Delta T^{4/3}}
$$

and thus

$$
q'_t \sim Ra^{1/3} \theta^{4/3}
$$

where $\theta = T_{bdy} / \Delta T$ is the nondimensional temperature change across the thermal boundary (which is approximately the average temperature in the whole layer), and $Ra$ is the Rayleigh number based on the viscosity in the upper thermal boundary layer.

The nondimensional bottom thermal boundary layer heat flux is given by

$$
q'_b = \frac{q_b}{k \Delta T / d} \sim Ra_b^{1/3} (1 - \theta)^{4/3}
$$

where $Ra_b$ is the Rayleigh number across the bottom thermal boundary layer (using the lower boundary viscosity value). Now letting $\eta_b = \Delta \eta \eta_t$ where $\Delta \eta$ is the viscosity difference across the depth of the fluid, $\eta_b$ is the viscosity in the bottom boundary and $\eta_t$ is the viscosity in the top boundary layer, $Ra_b = Ra / \Delta \eta$. Therefore, (1.16) becomes
Following Appendix C, the heat flux in a spherical shell of curvature \( f = r_i/r_o \) (where \( r_i \) and \( r_o \) are the inner and outer radius of the shell, respectively) is given by

\[
q'_t = q'_b f^2 + H \frac{(1 + f + f^2)}{3}.
\]  

(1.18)

Using the scaling relations for nondimensional heat flux derived above (equation (1.15) and (1.17)), equation (1.18) becomes

\[
\beta Ra^{1/3} \theta^{4/3} = \beta Ra^{1/3} (\Delta \eta)^{-1/3} (1 - \theta)^{4/3} f^2 + H \frac{(1 + f + f^2)}{3}
\]

(1.19)

where \( \beta \) is a proportionality constant to make (1.14) and therefore (1.15) and (1.16) an equality.

Equation (1.19) is equivalent to a twelfth order polynomial equation and so cannot be solved explicitly for \( \theta \). Thus, if \( H \neq 0 \), then \( \theta \) cannot be isolated. However, the case with no internal heating, where \( H = 0 \) yields

\[
\theta = (1 - \theta)(\Delta \eta)^{-1/4} f^{3/2}
\]

(1.20)

which solving for \( \theta \) gives

\[
\theta = \frac{f^{3/2}}{(\Delta \eta)^{1/4} + f^{3/2}}
\]

(1.21)

and in the isoviscous case when \( \Delta \eta = 1 \), equation (1.21) becomes

\[
\theta = \frac{1}{f^{-3/2} + 1}
\]

(1.22)

which is the equation Vangelov and Jarvis (1994) derived (by an alternate analysis) for mean temperature in an isoviscous spherical shell of curvature \( f \). For isoviscous convection, when \( f = 1 \) (plane-layer) equation (1.22) predicts a mean temperature of 0.5.

Shahnas et al. (2008) showed that the mean temperature based on actual observations from model calculations is better fit by an equation with \( f \) dependence in the numerator and proposed an equation of the form:

\[
\theta = \frac{f^{1/2}}{1 + f^{-3/2}}
\]

(1.23)

where the new numerator is able to “parametrize the effects of planform” (Shahnas et al., 2008) for different values of \( f \), and the equation (1.23) can be rewritten as

\[
\theta = \frac{f^2}{f^{3/2} + 1}.
\]

(1.24)

### 1.6.3 Adjusting for internal heating

Parmentier et al. (1994) found that the mean temperature in a purely internally heated plane-layer (\( \theta_i \)) is given by a power-law relationship
\[ \theta_i = \alpha \frac{H^3}{Ra_H^\gamma} \]  

(1.25)

where \( Ra_H \) is the Roberts-Rayleigh number (or thermal Rayleigh number) given by \( Ra_H = \frac{\rho g H d^5}{(k \kappa \nu)} \) and is related to the Bénard-Rayleigh number (\( Ra \)) by \( Ra_H = H Ra \).

Sotin and Labrosse (1999) found that the mean temperature in a mixed mode heated plane-layer (both internal and basal heat sources from an isothermal bottom boundary) can be approximated by a superposition of the mean temperatures from the basal heat source and internal heat source. Their equation

\[ \theta = \theta_{basal} + \theta_{internal} \]  

(1.26)

combined with equations (1.24) and (1.25) therefore imply

\[ \theta = \frac{f^2}{f^{3/2} + 1} + \alpha \frac{H^3}{Ra^\gamma}. \]  

(1.27)

In a plane-layer system the heat flux is conserved by \( q_t = q_b + H \) (see Appendix C). Therefore, in the case where the internal heating rate is so large that no more heat enters from the bottom of the system, (i.e., \( q_b = 0 \)), \( q'_t = H \) and equation (1.15) becomes

\[ H \sim Ra^{1/3} \theta^{4/3} \]  

(1.28)

which leads to

\[ \theta \sim \frac{H^{3/4}}{Ra^{1/4}} \]  

(1.29)

yielding predicted values for the exponents on \( H \) and \( Ra \).

Consider the heat flux out of a spherical shell of dimensional inner radius \( R_i \) and outer radius \( R_o \). Assuming the internal heating rate is so large that all the heat out of the top of the sphere is from internal sources (i.e., \( q_b = 0 \) and \( q_t = H \)), implies

\[ 4\pi R_o^2 q_t = H 4\pi (R_o^3 - R_i^3)/3 \]  

(1.30)

which simplifies to

\[ q_t = H (R_o - R_i) (1 + f + f^2)/3 \]  

(1.31)

(see Appendix C for derivation).

Now using equation (1.12) (and \( R_o - R_i = d \)) and nondimensionalizing as before yields

\[ \theta \sim \frac{(1+f+f^2) H^{3/4}}{Ra^{1/4}}. \]  

(1.32)

When \( f = 1 \), equation (1.32) is the same as the equation (1.29) which Sotin and Labrosse (1999) derived for the plane-layer case.
1.6.4 Accounting for modelling observations in the equations

Extending equation (1.26) to a spherical shell, the mean temperature in a spherical shell will be the sum of the basally heated component and internal heating component, or the combination of equations (1.22) and (1.32), yielding:

\[
\theta = \frac{1}{f^{-3/2} + 1} + C \left( \frac{(1+f+f^2)}{3} H \right)^{3/4} \frac{Ra^{1/4}}{Ra} \tag{1.33}
\]

where \(C\) is a constant that may depend on \(f\) and should reduce to the appropriate plane-layer value in the case where \(f = 1\).

Shahnas et al. (2008) found that the spherical data was best fit by multiplying the first term \(\theta_{basal}\) by \(f^{1/2}\). Their equation was

\[
\theta = \frac{f^{1/2}}{f^{-3/2} + 1} + C \left( \frac{(1+f+f^2)}{3} H \right)^{3/4} \frac{Ra^{1/4}}{Ra} \tag{1.34}
\]

where they found that \(C\) depends on the Rayleigh number of the calculation. Shahnas et al. (2008) assumed a linear dependence on \(f\) and a power law dependence on \(Ra\) and proposed the equation for \(C\) take the form

\[
C = C_{f=1} + \lambda Ra^\nu (1 - f) \tag{1.35}
\]

where \(C_{f=1}\) is the value of \(C\) corresponding to the \(f = 1\) plane-layer case. Shahnas et al. (2008) required that \(C_{f=1}\) does not depend on \(Ra\) as Sotin and Labrosse (1999) did not find any Rayleigh number dependence. Shahnas et al. (2008) also allowed the exponents on \(H\) to vary, to allow a more general fit of the equation. Therefore equations (1.34) and (1.35) can be combined to suggest the equation for mean temperature takes the form

\[
\theta = \frac{f^{1/2}}{f^{-3/2} + 1} + (C_{f=1} + \lambda Ra^\nu (1 - f)) \left( \frac{(1+f+f^2)}{3} H \right)^{3/4} \frac{Ra^{1/4}}{Ra} \tag{1.36}
\]

1.6.5 Parameterizations of mean temperature and heat flow in internally and bimodally heated fluids

Parmentier et al. (1994) determined an equation for mean temperature in a 3D fluid heated only from within (by internal sources). Their equation related the mean temperature \(T\) and the Roberts-Rayleigh number (Roberts, 1967), \(Ra_H = \alpha g \varepsilon d^5 / (k \kappa^2)\) and they found

\[
T = \frac{1.65 \, Ra_H^{0.234}}{Ra_H}. \tag{1.37}
\]

Later, Sotin and Labrosse (1999) examined three-dimensional plane-layer geometry models featuring Rayleigh numbers from \(10^5 - 10^7\) and nondimensional internal heating rates from 0 to 40. The final equation Sotin and Labrosse (1999) determined for mean temperature in a plane-layer is

\[
\theta = 0.5 + 1.02 \, H^{0.729} \frac{Ra^{0.232}}{Ra}. \tag{1.38}
\]
This equation was able to successfully predict the mean temperature in plane-layer convection models for the range of Rayleigh numbers examined by the authors.

Shahnas et al. (2008) attempted to fit mixed mode spherical shell results to their equation (1.36) and determined the following equation,

$$\theta = \frac{f^{1/2}}{1 + f^{-3/2}} + (1.318 + 0.251 R \alpha_B^{-1/3} (1 - f)) \left( \frac{1 + f + f^2}{R \alpha_B^{1/4}} \right)^{3/4} H^{-0.729}.$$  \hfill (1.39)

Equation (1.39) can be rearranged to solve for $H$ to determine the appropriate heating rate in a spherical geometry with a curvature factor appropriate for the Earth’s mantle ($f = 0.547$) to obtain a mean temperature of 0.5. This temperature is produced in a plane-layer geometry model with no internal heating. Setting $\theta = 0.5$, equation (1.39) yields an internal heating of $H = 26.26$ for Rayleigh number $10^7$.

The above equations are used as the starting point for determining parameterizations for the many numerical models in this work. The mean temperature of the system is an important parameter to study because it helps determine the gradient of the thermal boundary layer of the system. Temperature-dependent parameters also determine global system behaviour. This study will derive parameterizations for mean temperature in different geometries which can be used to obtain the thermal structure of spherical shell convection in plane-layer geometry models.

1.7 Mantle convection in other planets

The mantle convection models described above are motivated by an attempt to model mantle convection in the Earth. However, mantle convection models are fluid dynamical models which can also be applied to other planetary bodies (e.g., Tackley et al., 2001; O’Neill et al., 2007; Robin et al., 2007).

While it appears that the Earth is the only planet in our solar system to currently feature active plate tectonics, other planets show evidence of previous dynamic activity on their surface. Mars, Mercury and Venus show evidence of chasmata, scarps and volcanoes possibly arising from internal processes.

The thermal history of Mars and Venus likely includes the influence of subsolidus convection to remove heat from the interior of these planets. Mars is currently in a stagnant-lid convection regime wherein there is no active plate tectonics on the surface of the planet, and all heat must conduct through a thick immobile lithosphere. The surface of Venus is inferred to be relatively young compared with the age of the planet, and it has been suggested Venus could be in an episodic tectonic regime (Schubert et al., 2001). In the episodic regime, the lithosphere cools enough to ‘lock up’ and become a stagnant-lid, but after some time the interior of the planet becomes quite hot as heat cannot escape as efficiently through the stagnant-lid (i.e., by conduction) and instabilities grow in the upper thermal boundary layer so that the associated buoyancy overcomes the lithospheric yield stress to drag the thick, viscous, dense lithosphere into the mantle and subduct most of the surface.

Our solar system shows evidence for planets in different states of tectonic motion, however other solar systems might include lithospheres in the same mobile plate tectonic regime as Earth.

1.7.1 Super-Earths

An intriguing area of current research is the mantle dynamics of Super-Earths, terrestrial planets with masses up to an order of magnitude in excess of the Earth’s (e.g., Rivera et al., 2005; Valencia et al.,...
2006, 2009; O’Neill and Lenardic, 2007; van Heck and Tackley, 2011; Stein et al., 2011). A Super-Earth with a mass of ten terrestrial masses will have a radius of approximately twice that of the Earth and therefore both a deeper mantle and higher gravitational acceleration (e.g., Valencia et al., 2006, 2009).

The Rayleigh number for these Super-Earths could be orders of magnitudes higher than Earth’s (around $10^7$) which will require increased resolution to resolve the thermal gradients in the thermal structures. Earth-like Rayleigh number convection remains difficult to model (e.g., Wolstencroft et al., 2009) and increasing the Rayleigh number will provide more challenges. Some current studies (O’Neill and Lenardic, 2007; O’Neill et al., 2007; van Heck and Tackley, 2011; Stein et al., 2011) use two-dimensional plane-layer convection models to examine some of the first order effects different parameters have on mantle dynamics. The ability to use 2D and 3D plane-layer models to investigate Super-Earth dynamics will continue to be important for future studies and understanding their thermal structure and limitations will be necessary.

One important question to be answered is does plate tectonics occur on another planet, such as a Super-Earth. Studies have derived conflicting results on this matter (O’Neill and Lenardic, 2007; Stein et al., 2013) deriving mainly from different assumptions about the heating modes and viscosity of these planets. Ensuring the appropriate thermal structure is modelled will be critical for deducing the relevance of different results. This work, describing how to emulate spherical shell thermal structure in a plane-layer geometry, will be important in this aspect.

1.8 Outline

This study investigates the thermal properties of convection in different geometries and the effect of various parameters on mean temperature of the system. Chapter 2 describes the governing equations and numerical methods used in the convection models and their simplification and nondimensionalization. Chapter 3 investigates 3D plane-layer isoviscous convection experiments in different size solution domains. Some of the results and figures from Chapter 3 were published with modifications by O’Farrell and Lowman (2010). The 3D plane-layer calculations in Chapter 4 feature the depth-dependent viscosity described in Section 2.2.1. A systematic study of the effects of solution domain size, Rayleigh number, internal heating rate and viscosity stratification is performed in Chapters 3 and 4. The 3D results from Chapter 4 were published with modifications by O’Farrell et al. (2013). The implication and importance of these results are discussed in Chapter 5, along with a discussion of the effects of simplifications used throughout this study and suggestions for future work in the field.
Chapter 2

Methods

2.0.1 The Conservation Equations

Mantle convection can be studied numerically by solving a system of equations for convection in an infinite Prandtl number fluid. Specifically, it is modelled by solving equations for the conservation of mass, momentum and energy and an equation of state.

In dimensional form, the conservation of mass, or continuity equation, requires

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0 \quad (2.1)$$

where $\rho$ is the fluid density, which is a function of the position $x_i$, and $u_i$ are the component of the fluid’s velocity (the equations are written in summation notation where $i = 1, 2, 3$).

In a rotating frame of reference, the conservation of momentum, or Navier-Stokes equation, is given by

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + 2 \varepsilon_{ijk} \Omega_j u_k = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial \Phi}{\partial x_i} + \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} \quad (2.2)$$

where $t$ is time, $\Omega$ is the angular velocity of the rotating fluid, $P$ is pressure, $\tau_{ij}$ is the deviatoric stress tensor and $\Phi$ is the apparent gravitational potential in a rotating reference frame, given by $\Phi = U + 1/2 |\Omega \times r|^2$ where $U$ is the gravitational potential.

The conservation of energy, or heat equation, is given by

$$\rho C_p \left[ \frac{\partial T}{\partial t} - \frac{\alpha T}{\rho C_p} \frac{\partial P}{\partial t} + u_i \left( \frac{\partial T}{\partial x_i} - \left( \frac{\partial T}{\partial x_i} \right)_S \right) \right] = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \chi + \tau_{ij} \frac{\partial u_i}{\partial x_j} \quad (2.3)$$

where $C_p$ is the specific heat at constant pressure, $T$ is the temperature, $\alpha$ is the coefficient of thermal expansion (measuring how much density decreases with increasing temperature), $k$ the thermal conductivity, and $\chi$ the rate of internal heating generated per unit volume, $(\partial T/\partial x_i)_S$ is the change in temperature at constant entropy, $S$, which represents the adiabatic temperature gradient, and $\tau_{ij} \partial u_i / \partial x_j$ is the viscous generation of heat.

It is assumed that the mantle deforms by diffusion creep so that a linear dynamic viscosity, $\eta$, can be defined and
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\[ \tau_{ij} = \eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  

(2.4)

For modelling mantle convection with these equations further assumptions are required, for example \( \rho \) is considered continuous in the mantle and only depends on temperature. Accordingly, the partial differential equations are completed by a linearized equation of state,

\[ \rho(T) = \rho_0(1 - \alpha(T - T_0)). \]  

(2.5)

where \( \rho_0 \) is the reference density obtained when \( T = T_0 \), the surface temperature.

In this study, the Boussinesq approximation is adopted, which in its strongest form assumes that the density is constant, except in the buoyancy (i.e., driving force) term. In the buoyancy source term, the density is independent of pressure and depends only weakly on temperature. Furthermore, the thermodynamic parameters (\( \alpha, k, C_p, \eta \)) are assumed independent of temperature and pressure and can thus be treated as constant throughout the mantle. In its weaker form, the Boussinesq approximation allows for changes in one or more of the thermodynamic parameters while maintaining that the density only depends weakly on temperature.

The Boussinesq approximation is used in determining the equation of state, equation (2.5), which is the linear term of the Taylor expansion of density at \( (T - T_0) \). Specifically,

\[ \rho(T_0 + (T - T_0)) = \rho(T_0) + \frac{\partial \rho}{\partial T}|_{T_0}(T - T_0) + \frac{1}{2!} \frac{\partial^2 \rho}{\partial T^2}|_{T_0}(T - T_0)^2 + \ldots \]  

(2.6)

\[ \Rightarrow \rho(T) = \rho(T_0) - \rho(T_0)\alpha(T - T_0) + \ldots \]
\[ \Rightarrow \rho(T) = \rho(T_0)(1 - \alpha(T - T_0)) \]

The last equation in the derivation is the same as equation (2.5) where \( \rho(T_0) = \rho_0 \) and \( \alpha = -1/\rho_0[\partial \rho/\partial T]|_{T_0} \), which is the coefficient of thermal expansion.

Deep fluid layers will experience hydrostatic compression so that it cannot be assumed that density does not depend on pressure without some analysis. The height over which the density varies with pressure is referred to as the scale height, \( H_T = C_p/(g\alpha) \). For the mantle \( H_T \sim 10^7 \text{m} \). For density to be only weakly dependent on temperature requires \( \alpha \Delta T^* \ll 1 \) where \( \Delta T^* \) is the maximum value of \( (T - T_0) \). For the Earth \( \alpha \Delta T \sim 6 \times 10^{-2} \) (e.g., Parise et al., 1990; Hofmeister, 1999). This along with the requirement that \( d/H_T \) is small, must be satisfied by the convecting system in order to justify applying the Boussinesq approximation.

2.0.2 Nondimensionalization

The system of equations (2.1) - (2.5) can be nondimensionalized and used to study mantle convection in different systems. The distance can be nondimensionalized using the depth of the system, \( d \). Time can be nondimensionalized by the thermal diffusion time, \( d^2/\kappa \) and temperature by using the temperature difference across the system (\( \Delta T \)) when \( T_0 = 0 \) at the surface. The resulting nondimensionalization of variables is given by:
\[
\begin{align*}
  t &= t'd^2/\kappa \\
  u &= u'\kappa/d \\
  T &= \Delta TT' \\
  P &= P'\eta_0\kappa/d^2 \\
  \tau_{ij} &= \tau'_{ij}\eta_0\kappa/d^2
\end{align*}
\]  
\tag{2.7}

where all the primed variables are nondimensional.

The nondimensional variables are used to reformulate the equations governing convection in a fluid. In the mantle, the changes in density are small compared to the time-scale over which such changes occur. Applying the Boussinesq approximation, the conservation of mass equation (2.1) can be written in vector notation and nondimensionalized as follows:

\[
\begin{align*}
  \nabla \cdot (\rho_0 \mathbf{u}) &= 0 \quad (2.8) \\
  \nabla \cdot (\rho_0 \kappa d \mathbf{u}') &= 0 \quad (2.9) \\
  \rho_0 \kappa d \nabla \cdot \mathbf{u}' &= 0 \quad (2.10)
\end{align*}
\]

Finally resulting in

\[
\nabla \cdot \mathbf{u}' = 0 \quad (2.11)
\]

In the energy equation, (2.3), the adiabatic temperature gradient and the temporal derivative of pressure (\(\alpha T (\partial P/\partial t)\)) are of order \(d/H\) and are therefore negligible for the Boussinesq approximation. The remaining terms in the energy equation give:

\[
\begin{align*}
  \rho_o C_p \left[ \frac{\partial T'}{\partial t'} + u_i \frac{\partial T'}{\partial x_i} \right] &= k \frac{\partial^2 T'}{\partial x_i^2} + \chi \quad (2.12) \\
  \frac{\partial T'}{\partial t'} + u_i \frac{\partial T'}{\partial x_i} &= \kappa \frac{\partial^2 T'}{\partial x_i^2} + \frac{\chi}{\rho_o C_p} \quad (2.13) \\
  \frac{\kappa \Delta T}{d^2} \frac{\partial T'}{\partial t'} + \frac{\kappa}{d} u_i \frac{\Delta T}{d} \frac{\partial T'}{\partial x_i} &= \kappa \frac{\Delta T}{d^2} \frac{\partial^2 T'}{\partial x_i^2} + \frac{\chi}{\rho_o C_p} \quad (2.14) \\
  \frac{\partial T'}{\partial t'} + u_i \frac{\partial T'}{\partial x_i} &= \frac{\partial^2 T'}{\partial x_i^2} + \frac{d^2 \chi}{k \Delta T} \quad (2.15) \\
  \frac{\partial T'}{\partial t'} &= \nabla'^2 T' - \mathbf{u}' \cdot \nabla' T' + H \quad (2.16)
\end{align*}
\]

where \(H = d^2 \chi/(k \Delta T)\) is the nondimensionalized internal heating rate. The continuity equation can be used to simplify this further, by adding in a term equal to zero so that

\[
\frac{\partial T'}{\partial t'} = \nabla'^2 T' - \mathbf{u}' \cdot \nabla' T' + H \quad (2.17) \\
\frac{\partial T'}{\partial t'} = \nabla'^2 T' - \mathbf{u}' \cdot \nabla' T' - T' \nabla \cdot \mathbf{u}' + H \quad (2.18)
\]

The final, computationally easier to program, conservation of energy equation is given by:
\[
\frac{\partial T'}{\partial t'} = \nabla^2 T' - \nabla \cdot (\mathbf{u} T') + H.
\] (2.19)

The equation of motion can be written in vector notation as:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2(\Omega \times \mathbf{u}) = -\frac{1}{\rho} \nabla P + \nabla U + \frac{1}{2} \nabla |\Omega \times \mathbf{r}|^2 + \frac{1}{\rho} \nabla \cdot \tau.
\] (2.20)

Now use \( \nabla U = -g \hat{z} \) and decompose the pressure into hydrostatic (\( P_o \)) and nonhydrostatic (\( \tilde{P} \)) components, i.e., \( P = P_o + \tilde{P} \).

Multiplying (2.20) by \( \rho \) gives:

\[
\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2(\Omega \times \mathbf{u}) \right] = -\nabla P - \frac{\rho g}{\rho} \hat{z} + \frac{\rho}{2} \nabla |\Omega \times \mathbf{r}|^2 + \nabla \cdot \tau
\] (2.21)

Applying the Boussinesq approximation so that \( \rho = \rho_o \) (except in the buoyancy term, \( -\rho g \hat{z} \)) and using the hydrostatic equation \( \nabla P_o = -\rho_o g \hat{z} \), yields

\[
\rho_o \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2(\Omega \times \mathbf{u}) \right] = -\nabla P_o - \rho_o g \hat{z} + \frac{\rho_o g}{2} \nabla |\Omega \times \mathbf{r}|^2 + \nabla \cdot \tau
\] (2.22)

Equation (2.23) can be nondimensionalized using equations (2.7), \( \Omega = \Omega' \hat{z} \) and \( \tau' = 2 \eta' \dot{\varepsilon} \) where \( \eta' \) is the nondimensionalized viscosity and \( \dot{\varepsilon} \) is the strain rate (as given in equation (2.4)), so that

\[
\frac{\kappa^2}{d^3} \frac{\partial \mathbf{u}'}{\partial t'} + \frac{\kappa^2}{d^3} \mathbf{u}' \cdot \nabla \mathbf{u}' + \frac{2\Omega \kappa}{d} (\hat{z} \times \mathbf{u}') - \frac{\Omega^2 d^4}{2 \kappa^2} \nabla' |\hat{z} \times \mathbf{r}'|^2
\] =

\[
-\nabla' \tilde{P}' \eta_o K + \rho_o g \alpha \Delta T' \hat{z} + \nabla' \cdot \frac{\tau' \eta_o K}{d^3}
\] (2.24)

\[
\Rightarrow \quad \rho_o \left[ \frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla \mathbf{u}' + \frac{2\Omega d^2}{\kappa} (\hat{z} \times \mathbf{u}') - \frac{\Omega^2 d^4}{2 \kappa^2} \nabla' |\hat{z} \times \mathbf{r}'|^2 \right] \frac{\kappa^2}{d^3} =
\]

\[
-\nabla' \tilde{P}' \eta_o K + \rho_o g \alpha \Delta T' \hat{z} + \nabla' \cdot \frac{\tau' \eta_o K}{d^3}
\] (2.25)

\[
\Rightarrow \quad \left[ \frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla \mathbf{u}' + \frac{2\Omega d^2}{\kappa} (\hat{z} \times \mathbf{u}') - \frac{\Omega^2 d^4}{2 \kappa^2} \nabla' |\hat{z} \times \mathbf{r}'|^2 \right] \frac{\kappa \rho_o}{\eta_o} =
\]

\[
\frac{\rho_o g \alpha \Delta T d^3 T'}{\kappa \eta_o} + \left[ -\nabla' \tilde{P}' + \nabla' \cdot \tau' \right]
\] (2.26)
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$$\Rightarrow \frac{1}{Pr} \left[ \frac{\partial u'}{\partial t'} + u' \cdot \nabla' u' + \frac{2\Omega d^2}{\kappa} (\hat{z} \times u') - \frac{\Omega^2 d^4}{2\kappa^2} \nabla' |\hat{z} \times r'|^2 \right] =$$

$$RaT' \hat{z} + \left[ -\nabla' \tilde{P'} + \nabla' \cdot \tau' \right]$$

(2.27)

where $Pr = \nu / \kappa$ is the Prandtl number, and $\nu = \eta_o / \rho_o$ is the kinematic viscosity. $Ra = \rho_o g \alpha \Delta T d^3 / (\kappa \eta_o)$ is the Bénard-Rayleigh number (e.g., Chandrasekhar, 1961) and determines the vigour of convection in the system.

For a planetary mantle like the Earth’s, the Prandtl number is on the order of $10^{24}$. Also, in the Earth’s case $\Omega = 7 \times 10^{-5} s^{-1}$, and the multiplying pre-factor in the third term in equation (2.27), (i.e., $2\Omega d^2 / \kappa$), is of the order $10^{15}$ and is therefore much smaller than the Prandtl number. Consequently, the Coriolis force is small compared to the viscous force.

The centrifugal force can be absorbed into a modified pressure term so

$$P'_{\text{mod}} = \tilde{P'} - \frac{1}{Pr} \frac{\Omega^2 d^4}{2\kappa^2} |\hat{z} \times r'|^2$$

(2.28)

and equation (2.27) can then be written as

$$\frac{1}{Pr} \left[ \frac{\partial u'}{\partial t'} + u' \cdot \nabla' u' + \frac{2\Omega d^2}{\kappa} (\hat{z} \times u') \right] =$$

$$RaT' \hat{z} + \left[ -\nabla' \tilde{P'} + \frac{1}{Pr} \frac{\Omega^2 d^4}{2\kappa^2} \nabla' |\hat{z} \times r'|^2 + \nabla' \cdot \tau' \right].$$

(2.29)

Thus, the left-hand side of equation (2.29) can be eliminated to obtain

$$0 = RaT' \hat{z} + \left[ -\nabla' P'_{\text{mod}} + \nabla' \cdot \tau' \right].$$

(2.30)

Then the following relationships can be used to further simplify the equation of motion:

$$\frac{\partial P'_{\text{mod}}}{\partial x'_i} = \frac{\partial P'_{\text{mod}}}{\partial x'_j} \delta_{ij}$$

(2.31)

and

$$\sigma'_{ij} = -P'_{\text{mod}} \delta_{ij} + \tau'_{ij}$$

(2.32)

where $\sigma_{ij}$ is the stress tensor.

Thus

$$0 = RaT' \delta_{i3} - \frac{\partial P'_{\text{mod}}}{\partial x'_j} \delta_{ij} \delta_{ij} + \frac{\partial r'_{ij}}{\partial x'_j}$$

(2.33)

$$\Rightarrow 0 = RaT' \delta_{i3} + \frac{\partial \sigma'_{ij}}{\partial x'_j}.$$  

(2.34)

The equation of motion can then be written in vector notion as
\[ \nabla \cdot \sigma = -RaT \hat{z} \]  
(2.35)

where \( \sigma \) is a function of \( \tau' \) and thus spatial velocity gradients, as given in equation (2.4). Mantle convection in this work is studied by modelling an infinite Prandtl number, Boussinesq, fluid in a given geometry using the above nondimensionalized governing equations (2.11), (2.19) and (2.35).

### 2.1 Modelling plane-layer convection: MC3D

The parallelized code utilized in this study is MC3D the three-dimensional (3D) hybrid finite-difference and spectral method plane-layer convection model (described in Section 2.3) previously developed by C. W. Gable and B. Travis (e.g., Gable, 1989; Gable et al., 1991) and extensively benchmarked (e.g., Blankenbach et al., 1989; Busse et al., 1993; Koglin et al., 2005). A summary of the code’s method is given in Section 2.3. Computational changes to the code made during this study are described in Appendix D.

#### 2.1.1 Boundary conditions

In the calculations performed, the boundary condition on the sidewalls was specified as periodic (or a wrap-around condition). This is used to allow free lateral movement of the fluid in the system. The top and bottom of the solution domain are specified as free-slip boundaries (as the models are run in the absence of plates). The lower boundary, representing the core-mantle boundary, is held at a fixed nondimensional temperature of 1.0 and the upper boundary is held at a fixed nondimensional temperature of 0.0. These are appropriate thermal boundary conditions for modelling mantle convection when the time scales of convection in the neighbouring systems are considered. At the core-mantle-boundary the fluid in the core is convecting on a much shorter time scale than in the mantle. Thus on time scales relevant to mantle convection (millions of years), the core appears to be uniform in temperature when averaged over this long time scale. Similarly, at the surface, an ocean, an atmosphere or space (if the planet has no atmosphere) can be considered relatively uniform in temperature on the time scales relative to that of mantle convection. For example, an ocean overturns on time scales much smaller than the tens of millions of years it takes for a parcel of fluid to traverse the mantle depth.

Figure 2.1 shows a schematic of the 3D plane-layer calculations with the boundary conditions labelled.

#### 2.1.2 Calculations

For the isoviscous convection models presented in Chapter 3, 2D plane-layer models were initialized from a conductive state with a random perturbation and integrated forward in time until the mean temperature of the system showed no long term heating or cooling trends (i.e., statistically steady). Subsequently, the 2D model solutions were used to initialize the 3D models. The 2D models take less computation time than the 3D models, and by bringing the 2D model to a statistically steady state, the mean temperature of the system is closer to the value that will be obtained in a 3D convection model. An aspect ratio \( L_x \), 2D solution is imposed in a \( L_x \times L_y \times 1 \), 3D geometry (by extending the 2D solution in the y-direction by length \( L_y \)) and random perturbations in both the x- and y-directions are added at initialization. The 3D models were integrated forward until reaching a statistically steady temperature for domain geometries with ratios 1, 2, 3 and 5. The stratified viscosity convection models presented in
Chapter 2. Methods

Figure 2.1: Schematic of 3D plane-layer convection model boundary conditions.

Table 2.1: Table of resolutions for isoviscous plane-layer convection models in $L \times L \times 1$ solution domains geometry. For isoviscous convection models, $Ra_\eta = Ra$ since $\eta = 1$.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$Ra_\eta$</th>
<th>Solution domain size</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^5$</td>
<td>$10^5$</td>
<td>$L \times L \times 1$</td>
<td>$48L \times 48L \times 64$</td>
</tr>
<tr>
<td>$10^6$</td>
<td>$10^6$</td>
<td>$L \times L \times 1$</td>
<td>$72L \times 72L \times 96$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$10^7$</td>
<td>$L \times L \times 1$</td>
<td>$128L \times 128L \times 144$</td>
</tr>
<tr>
<td>$10^8$</td>
<td>$10^8$</td>
<td>$L \times L \times 1$</td>
<td>$256L \times 256L \times 288$</td>
</tr>
</tbody>
</table>

Chapter 4 were initialized similarly. Figure 2.2 shows two isosurfaces of the temperature field with the random perturbations in the 3D plane-layer geometry.

The resolution of the models was varied based on the Rayleigh number, $Ra$. The more vigorous the convection, the thinner the upper thermal boundary layer and the more resolution is necessary to resolve the large temperature gradient that occurs between the top of the system and the bottom of the upper thermal boundary layer. The resolutions for models with constant viscosity are listed in Table 2.1. For models with a stratified viscosity, the resolutions for each effective Rayleigh number (see Section 2.2.1) are listed in Tables 2.2 and 2.3.

For convection between isothermal boundaries, the boundary layer thickness scales as $Ra^{-1/3}$ (e.g., see Section 1.6 equations (1.11) and (1.15); Turcotte and Oxburgh, 1967; Roberts, 1979), so for each increase in Rayleigh number by a factor of 10, the boundary layer thins by a factor of slightly more than two and it is necessary to at least double the amount of vertical grid points (e.g., Jarvis and Peltier, 1982). It is this grid resolution that limits the thickness of a thermal boundary layer (and therefore how large a $Ra$) that can be accurately modelled. Ideally, a minimum of three vertical grid points is required for solving centered difference calculations to accurately resolve gradients in the boundary layer (e.g.,
Figure 2.2: Snapshot of temperature field of 3D plane-layer convection model initial conditions. Isosurfaces of constant temperature are shown for nondimensional values of 0.45 (blue) and 0.65 (red) which show the initial perturbations added to the 3D plane-layer convection models.

Table 2.2: Table of resolutions for plane-layer convection models in $L \times L \times 1$ solution domains geometry featuring depth-dependent viscosity with $X = 30$ in equation (2.36) (see section 2.2.1).

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$Ra_{\eta}$</th>
<th>Solution domain size</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 10^5$</td>
<td>14930</td>
<td>$L \times L \times 1$</td>
<td>$64L \times 64L \times 72$</td>
</tr>
<tr>
<td>$5 \times 10^5$</td>
<td>24880</td>
<td>$L \times L \times 1$</td>
<td>$64L \times 64L \times 72$</td>
</tr>
<tr>
<td>$6 \times 10^5$</td>
<td>29850</td>
<td>$L \times L \times 1$</td>
<td>$72L \times 72L \times 96$</td>
</tr>
<tr>
<td>$7 \times 10^5$</td>
<td>34830</td>
<td>$L \times L \times 1$</td>
<td>$72L \times 72L \times 96$</td>
</tr>
<tr>
<td>$8 \times 10^5$</td>
<td>39800</td>
<td>$L \times L \times 1$</td>
<td>$72L \times 72L \times 96$</td>
</tr>
<tr>
<td>$9 \times 10^5$</td>
<td>44780</td>
<td>$L \times L \times 1$</td>
<td>$72L \times 72L \times 96$</td>
</tr>
<tr>
<td>$10^6$</td>
<td>49760</td>
<td>$L \times L \times 1$</td>
<td>$72L \times 72L \times 96$</td>
</tr>
<tr>
<td>$2 \times 10^6$</td>
<td>99510</td>
<td>$L \times L \times 1$</td>
<td>$98L \times 98L \times 108$</td>
</tr>
<tr>
<td>$2.7 \times 10^6$</td>
<td>134340</td>
<td>$L \times L \times 1$</td>
<td>$98L \times 98L \times 108$</td>
</tr>
<tr>
<td>$3 \times 10^6$</td>
<td>149270</td>
<td>$L \times L \times 1$</td>
<td>$98L \times 98L \times 108$</td>
</tr>
<tr>
<td>$4 \times 10^6$</td>
<td>199020</td>
<td>$L \times L \times 1$</td>
<td>$108L \times 108L \times 128$</td>
</tr>
<tr>
<td>$5 \times 10^6$</td>
<td>248780</td>
<td>$L \times L \times 1$</td>
<td>$108L \times 108L \times 128$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>497560</td>
<td>$L \times L \times 1$</td>
<td>$128L \times 128L \times 144$</td>
</tr>
</tbody>
</table>
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Table 2.3: Table of resolutions for plane-layer convection models in $L \times L \times 1$ solution domains geometry featuring depth-dependent viscosity with $X = 100$ in equation (2.36) (see section 2.2.1). A (*) indicates a case for which two different planforms (sheets or bimodal) can be obtained, depending on initial condition.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$Ra_{\pi}$</th>
<th>Solution domain size</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 \times 10^5$</td>
<td>7555.2</td>
<td>$L \times L \times 1$</td>
<td>$64L \times 64L \times 72^*$</td>
</tr>
<tr>
<td>$10^6$</td>
<td>15110</td>
<td>$L \times L \times 1$</td>
<td>$64L \times 64L \times 72^*$</td>
</tr>
<tr>
<td>$3 \times 10^6$</td>
<td>45331</td>
<td>$L \times L \times 1$</td>
<td>$108L \times 108L \times 128$</td>
</tr>
<tr>
<td>$5 \times 10^6$</td>
<td>75552</td>
<td>$L \times L \times 1$</td>
<td>$108L \times 108L \times 128$</td>
</tr>
<tr>
<td>$6 \times 10^6$</td>
<td>90662</td>
<td>$L \times L \times 1$</td>
<td>$108L \times 108L \times 128$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>151100</td>
<td>$L \times L \times 1$</td>
<td>128$L \times 128L \times 144$</td>
</tr>
<tr>
<td>$3.2 \times 10^7$</td>
<td>483530</td>
<td>$L \times L \times 1$</td>
<td>128$L \times 128L \times 144$</td>
</tr>
</tbody>
</table>

Jarvis and Peltier, 1982; Jarvis and Peltier, 1989). The resolution must also increase in the horizontal, as horizontal gradients caused by upwellings and downwellings must also be accurately resolved.

The solution domain size was varied for the different plane-layer models. Dimensions of $Ld \times Ld \times d$ for $L = 1, 2, 3$ and 5 in the isoviscous cases and $L = 2, 3$ and 5 in the stratified viscosity cases are specified.

2.2 TERRA

TERRA is the spherical shell convection code employed in this study. It is also a parallelized code and uses the finite-element method for calculations. It was developed by J. Baumgardner (Baumgardner, 1985), and parallelized by H.-P. Bunge (Bunge and Baumgardner, 1995; Bunge et al., 1996, 1997). A small summary of the code is given in Section 2.3 and any computational changes made during this study are described in Appendix D.

In the spherical-shell calculations performed, the boundary conditions on the inner and outer boundaries of the shell are specified as free-slip. The inner boundary, or core-mantle-boundary, is again specified to have a nondimensional temperature of 1.0 and the outer boundary is fixed at a nondimensional temperature of 0.0. The radial resolution of all the TERRA models was fixed at 128 radial layers and the lateral resolution varied with Rayleigh number. The grid for the spherical code is a refined icosahedron for each radial layer. Figure 2.3 shows the mesh produced by icosahedron projections on a sphere, with successive refinements. When the grid resolution is increased a triangle is drawn inside of the triangular faces of the icosahedron, thus three lines are drawn connecting the midpoints of each side of the triangle. This divides the original triangle into four new equal sized triangles. If this is repeated for each face of the icosahedron the grid resolution can be uniformly increased across the entire sphere. As the resolution is increased, or with an increase in the number of faces starting from an icosahedron, it better approximates a sphere.

In this study, all spherical shell models have the same ratio of inner radius, $r_i$, to outer radius, $r_o$, namely $f = r_i/r_o = 0.547$ which is the same ratio as for the Earth’s mantle. All calculations began from the same initial state, with a constant temperature and a random perturbation superimposed. All calculations are integrated forward in time until no long term heating or cooling trend could be observed (i.e., until the systems reached statistically steady states). This was verified by checking the
Figure 2.3: Mesh produced by projection of icosahedron on a sphere. (b) through (f) show successive mesh refinements as resolution is increased. Reprinted from Baumgardner (1985) with kind permission from Springer Science and Business Media.
observed difference in heat flow into and out of the system. In a spherical shell the surface heat flux \( q_{\text{top}} \) must satisfy
\[
q_{\text{top}} = q_{\text{bot}} f^2 + H (1 + f + f^2)/3 \quad \text{(where } f = r_i/r_o, \text{ see Appendix C)}
\]
which reduces to \( q_{\text{top}} = q_{\text{bot}} + H \) in the plane-layer \( f = 1 \). Thus the difference, \( q_{\text{diff}} = q_{\text{top}} - q_{\text{bot}} \) will represent the observed internal heating rate in the system.

### 2.2.1 Effective Rayleigh Number with stratified viscosity

In models featuring a stratified viscosity, an isoviscous upper mantle is specified and an increase in viscosity by a factor of 30 or 100 is specified across the lower mantle. This viscosity contrast is motivated by a crude approximation of the terrestrial profile. The radial viscosity profile of the Earth’s mantle has generally been inferred from two methods, studies of postglacial rebound (e.g., Haskell, 1935; Peltier, 1976; Mitrovica and Peltier, 1995; Mitrovica, 1996) and the interpretation of long-wavelength geoid anomalies constrained by mantle convection (e.g., Richards and Hager, 1984; Forte et al., 1991; King and Masters, 1992; Zhang and Christensen, 1993; Montagner, 1994). The viscosity profiles determined from these methods suggest a range of mean lower mantle viscosities but over the past two decades a consensus has formed supporting a lower mantle viscosity that increases by a factor of 30 or more, relative to the mean viscosity of the upper mantle (e.g., Hager 1984; Forte and Peltier 1987; Ricard and Vigny, 1989; Forte and Mitrovica, 1996). Consequently, in all of the models discussed, a simple logarithmic increase in viscosity is adopted starting at 670 km depth in a 2900 km deep mantle (i.e., corresponding to the mean depth of the appearance of the spinel to perovskite/magnesiowüstite phase change in the Earth). The logarithmic function allows for a slightly more gradual increase in viscosity than a step function which helps reduce the chance of numerical instabilities from a large gradient. Figure 2.4 shows the different viscosity profiles used in this work as a function of height above the core-mantle boundary.

In the plane-layer models, the exact dependence of viscosity on nondimensional height, \( z \), above the core-mantle boundary is given by
\[
\eta(z) = \begin{cases} 
1 & \text{if } 0.769 \leq z \leq 1.00 \\
1 + \frac{X-1}{\log(1000)} \log[1 + \frac{999(0.769-z)}{0.769}] & \text{if } 0 \leq z < 0.769 
\end{cases} 
\]
where \( z=0.769 \) is the nondimensional height above the core-mantle boundary (at \( z=0 \)), corresponding to a depth of 670 km, and \( X \) is viscosity contrast (either 30 or 100). For the spherical shell models the viscosity’s radial dependence is given by
\[
\eta(r) = \begin{cases} 
1 & \text{if } 1.977 \leq r \leq 2.208 \\
1 + \frac{X-1}{\log(1000)} \log[1 + \frac{999(0.769-(r-1.208))}{0.769}] & \text{if } 1.208 \leq r < 1.977 
\end{cases} 
\]
(see Figure 2.4) where \( R_o/d = 2.208 \) and \( r = 1.977 \) is the nondimensional radius corresponding to a depth of 670 km.

The depth-dependence of the viscosity of the systems leads to different possible definitions of the Rayleigh number. The Rayleigh number appearing in equation (2.35), can be defined with the surface viscosity (non-dimensional viscosity of 1.0) and designated as the surface Rayleigh number. Averaging the viscosity over the entire volume of the mantle allows for the definition of an average viscosity Rayleigh number, \( Ra_\eta \). This Rayleigh number is dependent on the system geometry as well as the viscosity depth-dependence because the ratio of system volume above and below a given depth changes with \( f \). Thus obtaining:
Figure 2.4: Nondimensional viscosity variation as a function of height, z, for plane-layer geometry calculations, or radius, r, for the spherical shell model. The black curve corresponds to the profile obtained from equations (2.36) and (2.37) while the dashed curve shows the viscosity profile described in Section 4.4 and employed in the calculations featured in Figure 4.20. X is the total viscosity contrast across the system and takes values of either 30 or 100 in this study.
in a plane-layer geometry and

\[
\text{Ra}_\eta = \frac{g\alpha \rho_0 \Delta T d^3}{(3 \kappa)^{1/3} \int^{1/\nu^2} \eta(z)dz},
\]

in the spherical shell geometry with a nondimensional mantle depth of 1.0 and an \( f \) value corresponding to the Earth value (\( f = 0.547 \)). Due to the lower mantle volume difference associated with geometry changes, the average viscosity Rayleigh number is not equivalent for the plane-layer and spherical shell models with the same surface Rayleigh number even when the depth-dependent viscosity is the same in both systems. To compare systems with different geometries this study compares models with equivalent average viscosity Rayleigh numbers (equations (2.38) and (2.39)).

The effective Rayleigh number can also be defined based on averaging the Rayleigh number at each depth. In this case

\[
\overline{\text{Ra}} = \int^1_0 \text{Ra}(z)dz = \frac{g \alpha \Delta T d^3}{\kappa \int^1_{1/\nu^2} 1/\nu(z)dz}
\]

in a plane-layer geometry and

\[
\overline{\text{Ra}} = \int^{2.2075}_{1.2075} \text{Ra}(r)dr = \frac{g \alpha \Delta T d^3}{\kappa \int^{2.2075}_{1.2075} 1/\nu(r)dr}
\]

in a spherical shell geometry with a mantle depth of 1.0 and an \( f \) value of 0.547.

The use of the different definitions of Rayleigh number will be explored in the results. The definition using the average viscosity gives the best correlation between results in different geometries.

### 2.3 Numerical Methods

#### 2.3.1 Plane-layer geometry numerical methods in MC3D

In the plane-layer geometry convection code, MC3D, the code can be considered in two parts, a flow-solver which solves the nondimensional momentum equation (2.35) with the continuity equation (2.11) and an energy-solver which solves the nondimensional energy equation (2.19) at each time-step.

The flow-solver uses the mass equation (2.11) and the definition of stress (2.32) to solve the nondimensional momentum equation (2.35) using Fourier transforms in the \( x \) and \( y \) directions in order to derive a set of ordinary differential equations (ODE) in the \( z \) direction. Beginning with ten equations (one from the mass equation, three from the momentum equation and six from the definition of stress) and 10 unknowns (three components of velocity, 6 components of stress and pressure), the pressure and normal components of stress (i.e., \( \sigma_{ii} \)) can be eliminated, leaving six equations (first order ODEs) and six unknowns (e.g., Gable, 1989; Nettelfield, 2005). These equations can be simplified further by transforming the horizontal components to toroidal and poloidal components. The set of ODEs in \( z \) can then be solved for the Fourier coefficients representing the toroidal and poloidal velocity and stress and the vertical velocity and stress. For the convection models studied here, the boundary condition is free-slip, meaning there is zero shear stress at the boundary and thus nothing to drive toroidal flow.
and so no toroidal component of stress or velocity. Thus, for these studies, a set of only four equations and four unknowns must be solved. After solving the momentum equation in spectral space, an inverse Fourier transform is applied to obtain velocities in spatial (Cartesian) space.

This solution from the flow-solver is then used in the nondimensional energy equation (in the energy-solver) to determine the new temperature field, at the updated time-step, using finite volume approximations. The Courant-Friedrichs-Levy condition is used to determine the size of the time-step and keep it small, so the solution does not go numerically unstable (Courant et al., 1928). The CFL criteria states that \( \Delta t \leq \frac{1}{2} \Delta x / \max(|u|) \). A fluid particle cannot move further than a half grid space in one time-step, and the maximum velocity in the fluid is used to limit the maximum time-step allowable.

A Fourier transform is applied to the temperature field to use it in spectral space to solve the momentum equation, and the cycle repeats.

### 2.3.2 Spherical shell code, TERRA

In the spherical shell convection code, TERRA, the dimensional equations (2.1) - (2.3) are solved using a finite-element method (Baumgardner, 1985). The velocity field is calculated with a multigrid solver, which is used to calculate the temperature field using finite-differences and advance the time-step.

### 2.3.3 Computational costs

Both computational models use distributed memory parallel programming and result in largely different computational costs. The spherical shell code, TERRA, uses approximately three times more resources than the plane-layer numerical model, MC3D. For example, for a stratified viscosity convection model with a lower mantle viscosity increase of 30, a Rayleigh number \( Ra = 10^7 \) and no internal heating \( (H = 0) \), TERRA uses 128 processors with 2 GB of memory each for about 2800 hours to reach a statistically steady state. MC3D uses 8 to 32 processors with 2 GB of memory each for only about 1000 hours to reach a statistically steady state.

### 2.4 Fitting the data

In this study, mean temperatures, Rayleigh numbers and nondimensional internal heating rates from numerical models of convection are used to parameterize an equation for mean temperature in the system. The data is used in inversions to seek a standard least squares solution to the linear system of equations \( \mathbf{b} = \mathbf{A} \mathbf{x} \). By linearizing equations for mean temperature derived from boundary layer analysis (see Section 1.6) or previous studies (e.g., Parmentier et al., 1994; Sotin and Labrosse, 1999; Shahnas et al., 2008), a standard least squares solution can be determined.

Consider an equation of the form (see Section 1.6)

$$\theta = \theta_b(f) + \alpha(f) \frac{H^\beta}{Ra^\gamma}. \tag{2.42}$$

Rearranging this equation (2.42) and taking the log of both sides, the equation can then be linearized as

$$\log(\theta - \theta_b(f)) = \log(\alpha(f)) + \beta \log(H) - \gamma \log(Ra^\gamma). \tag{2.43}$$
The solution for $\alpha$, $\beta$ and $\gamma$ minimizes the sum of squared errors obtained with the data points used for each inversion. In all cases described in this thesis, the covariance matrix of the solution $x$ gives very small off-diagonal values, indicating the trade-off between inverted parameters ($\alpha$, $\beta$, and $\gamma$) is close to zero. Thus, any change in one parameter does not change the other two parameters by a large amount.

For the isoviscous convection study presented in Chapter 3, the percent error was calculated to determine how well the equation determined was able to predict mean temperatures in plane-layer systems. Tables 3.5 - 3.7 list two types of error. $\% \text{error}_t$ is the absolute percentage value of disagreement between observations and predictions, while $\% \text{error}_i$ is the percent disagreement between observed and predicted values of temperature difference from a temperature of $\theta_b(f = 1) = 0.5$.

The inversions calculated in this work are performed using MATLAB software, but have been confirmed by using other independent least squares analysis statistical software (Personal communication, Prof. George B. Arhonditsis and Dr. Johann Bayer).
Chapter 3

Results: Isoviscous Mantle Convection Study

Some of the results and figures from this chapter have been published in *Emulating the thermal structure of spherical shell convection in plane-layer geometry calculations* by K. A. O’Farrell and J. P. Lowman (Physics of the Earth and Planetary Interiors, Vol. 182, 2010). This chapter explores isoviscous convection in plane-layer models and examines the effect of solution domain size, internal heating rate and Rayleigh number on the thermal structure of the calculation. The mean temperatures of the models are used to derive an equation for predicting mean temperatures in plane-layer convection models. The equation is then modified to fit spherical shell data, and can be used to determine the correct heating rate for plane-layer models in order to emulate spherical shell convection.

3.1 The effect of system geometry on thermal structure

Previous work by Vangelov and Jarvis (1994), Sotin and Labrosse (1999) and Shahnas et al. (2008) derived equations for predicting mean temperature in isoviscous mantle convection models with different system geometries and mixed heating modes. Shahnas et al. (2008) derived an equation for predicting mean temperature in an isoviscous convecting spherical shell system given the Rayleigh number, $Ra$, nondimensional internal heating rate, $H$, and the ratio of the radii of the inner and outer spherical surfaces $f = r_i/r_o$. These authors found that a system with a Rayleigh number of $10^7$, a value of $f = 0.547$ and an internal heating rate of $H = 23$ (i.e., Earth-like values), has a mean temperature of approximately 0.478 which is less than the average of the temperatures at the boundaries, 0.5. A mean temperature of 0.5 can be obtained in a plane-layer convection system with the same Rayleigh number and no internal heating. In order to eliminate the difference in mean temperatures between the two geometries, it is necessary to consider internal cooling in plane-layer convection systems in order to reduce their mean temperatures to values closer to those obtained in the spherical shell systems. Introducing cooling is motivated not by any physical properties, but by the geometric differences in the two systems.
3.2 Model specifications

This study explores the effect of heating mode on three-dimensional plane-layer convection heated from an isothermal bottom boundary as well as uniform internal heating or cooling. Results are presented from plane-layer geometry calculations featuring an isothermal bottom boundary and uniform internal heating or cooling. These are compared with spherical geometry results presented by Shahnas et al. (2008) and a single predictive equation for mean temperature in both systems is obtained. This equation depends on $Ra$, $H$ and $f$. In the limit when $f = 1$ (i.e., plane-layer) the equation reduces to a predictive equation for mean temperature in a plane-layer geometry and allows for either heat sources or heat sinks (i.e., negative $H$ values).

The purpose of this study was to be able to quantify the differences in temperature in different geometries. The motivation is to find a way to emulate spherical shell thermal properties in plane-layer solution domains. While there have been many advancements in 3D spherical shell convection models, such as the ability to run models with increased resolution in shorter computational time, (as discussed previously in Section 1.5), plane-layer convection models still remain a useful tool for exploring mantle convection.

The study explores the effect of different parameters: the solution domain size, the internal heating rate, both constant and depth-dependent, and the Rayleigh number. Through these studies a clear path towards emulating spherical shell results emerges.

3.3 Calculations

Calculations are performed in 3D with different Rayleigh numbers, $Ra$, nondimensional internal heating rates, $H$, and aspect ratios specified. Rayleigh numbers range from $10^5$ to $10^8$, and the internal heating rate is varied from 0 to 15. The mean internal cooling rate is varied between -15 and -5. The 3D solution domain geometry is varied between $1 \times 1 \times 1$ and $5 \times 5 \times 1$.

The grid resolution is chosen so as to resolve the temperature gradients in the thermal boundary layers and therefore changes with different Rayleigh numbers and solution domain sizes (e.g., Sotin and Labrosse, 1999). For an $L \times L \times 1$ solution domain geometry we specify 64, 96, 144 and 288 evenly spaced vertical grid cells and 48L, 72L, 128L, and 256L cells in the horizontal directions for cases with $Ra$ of $10^5$, $10^6$, $10^7$, and $10^8$, respectively.

The results of 72 experiments are listed in Tables 3.5 - 3.7 at the end of this chapter. The temporal averages listed in the tables for mean temperature, $\theta_{obs}$, surface heat flux, $Q_{top}$, and basal heat flux, $Q_{bot}$, were calculated once the volumetrically averaged temperature had reached a statistically steady state (e.g., Sotin and Labrosse, 1999; Wolstencroft et al., 2009). The averages were calculated over different time periods determined by the magnitude of the Rayleigh number. Each calculation was integrated forward for at least 100 transit times (where a transit time is defined as the time required to traverse the layer depth at a velocity equal to the root mean square velocity, $V_{rms}$).

For a steady solution, $Q_{top}$, must equal the heat input. Consequently, for calculations with internal heating or cooling $Q_{top} = Q_{bot} + H$ (see Appendix C). As a quantitative measure of the steadiness of the calculations, in Tables 3.5 - 3.7 $Q_{top} - Q_{bot}$ is included for the time periods over which the mean temperature, $\theta$, was averaged in each calculation. The results are deemed approximately steady when the difference in heat flux is approximately equal to the internal heating rate prescribed in the calculation.
3.4 Solution Domain Size

The effect of varying solution domain size on mean temperature and the planform of convection is examined. The motivation is to minimize future computation effort for mantle convection calculations while ensuring the domain size yields temperatures and thermal structure similar to larger solution domain sizes.

Solutions obtained in $1 \times 1 \times 1$ solution domain geometries can differ from solutions in larger domains. This section will describe the results from multiple solution domain size calculations and determine what minimum dimensions are necessary to model larger plane-layer convection systems accurately for the boundary conditions specified.

Calculations with low Rayleigh number (i.e., $10^5$) and lower internal heating and cooling rates $H = \pm 5$, reach steady state solutions in small solution domain sizes of $1 \times 1 \times 1$ and $2 \times 2 \times 1$. At this Rayleigh number the $1 \times 1 \times 1$ solution domain models feature steady bimodal planforms (see Figure 3.1) even with the inclusion of some heating or cooling. However the $2 \times 2 \times 1$ solution domain calculations converge on steady convective roll planforms. With increased solution domain size ($3 \times 3 \times 1$ and $5 \times 5 \times 1$) the convection planform features upwelling and downwelling plumes which are time-dependent as is the mean temperature of the system (see Figure 3.2).

Figure 3.1: Plot of temperature isosurfaces for $Ra=10^5$, $H=-5$ and a $1 \times 1 \times 1$ solution domain. The green isosurface is at non-dimensional temperature 0.43 while the blue isosurface is at 0.23. The steady temperature is 0.333. The grid resolution for this calculation is $48 \times 48 \times 64$.

The increase in time dependence of the temperature with increasing solution domain size from $1 \times 1 \times 1$ to $2 \times 2 \times 1$ can be seen in Figure 3.3 for internal heating rate $H = 10$ and Rayleigh number $10^5$.

Figure 3.4 shows temporally averaged profiles of the laterally averaged mean temperature as a function of height for 24 calculations with varying solution domain sizes. Panels a, b and c correspond to calculations with $Ra$ of $10^5$, $10^6$ and $10^7$, respectively. All 24 calculations feature a moderate amount of
Figure 3.2: Plot of temperature isosurfaces for \( Ra = 10^5 \), \( H = -5 \) and a \( 5 \times 5 \times 1 \) solution domain. The green isosurface is at non-dimensional temperature 0.42 while the blue isosurface is at 0.16. The mean temperature is 0.292. The grid resolution for this calculation is \( 240 \times 240 \times 64 \).
Chapter 3. Results: Isoviscous Mantle Convection Study

Figure 3.3: Time series of average temperatures in the full domain for $H=10$, $Ra=10^5$ and varying domain geometry. Constraining the model size to $1 \times 1 \times 1$ reduced the average temperature by 10%.

internal heating or cooling $|H| = 10$. The figure shows that for solution domains of $2 \times 2 \times 1$ or larger, the temperature profiles exhibit little dependence on domain size. This is consistent with the findings reported in previous studies (e.g., Houseman, 1988; Sotin and Labrosse, 1999).

For all positive values of $H$ investigated in the $1 \times 1 \times 1$ solution domain with Rayleigh number $10^5$, the average temperature was approximately 10-15\% lower than it was in the large solution domains examined (see Figure 3.4). Increasing the internal heating or cooling from $\pm 5$ to $\pm 10$ to $\pm 15$ does not change the steady nature of the temperature in $1 \times 1 \times 1$ boxes, even though it does increase/decrease the overall temperature. However, in agreement with previous studies (e.g., Travis et al., 1990a, 1990b) it was found that with all larger solution domains, increased internal heating or cooling leads to time-dependent behaviour at all the Rayleigh numbers.

It was found that for a fixed Rayleigh number or internal heating rate the solution domain geometry did affect the temperature, where the $1 \times 1 \times 1$ model results yield a lower temperature than in the other size boxes. However, once the solution domain is $2 \times 2 \times 1$ or larger, the temperature profiles are very similar (see Figure 3.4).

When the Rayleigh number is $\geq 10^6$, for all $H$ the effect of the domain size on the temperature profiles is decreased (see Figure 3.4c). In solution domain geometries of $2 \times 2 \times 1$ and larger, the planform for higher Rayleigh number calculations is characterized by time-dependent convection featuring cylindrical upwellings and downwellings (e.g., Houseman, 1988; Travis et al., 1990b; Parmentier et al., 1994).

Based on these results, the remainder of the study uses $2 \times 2 \times 1$ solution domain geometry to examine the thermal properties in plane-layer convection.
Figure 3.4: Temporally averaged mean horizontal temperatures as a function of height for convection featuring internal heating or cooling of magnitude 10. In panel (a), $Ra = 10^5$; in panel (b), $Ra = 10^6$; and in panel (c), $Ra = 10^7$. Each panel includes temperature profiles from calculations with $L = 1, 2, 3$ and 5. The profiles for $L = 2, 3$ and 5 all overlap in (a), and all profiles overlap in (b) and (c).
3.5 Effect of Heating/Cooling Rate

Increasing the internal heating rate leads to higher mean temperatures, as noted in previous studies (e.g., McKenzie et al., 1974; Parmentier et al., 1994; Sotin and Labrosse, 1999; Shahmas et al., 2008; Moore, 2008). Similarly, with a fixed Rayleigh number, increasing the internal cooling rate leads to lower mean temperatures. For a fixed internal cooling rate the mean temperature of the system increases with increasing Rayleigh number.

Figure 3.5 shows the dependence of temperature on internal heating rates given a solution domain of $2 \times 2 \times 1$ and Rayleigh number $10^5$. It shows that increasing $H$ leads to an overall increase in temperature and also a larger subadiabatic temperature gradient. Increased temperatures can also be seen in the images of the 3D temperature fields in Figure 3.6, where hot mantle fluid is gathered at the top of the solution domain.

![Figure 3.5: Average temperature with depth for $Ra=10^5$ and varying internal heating rates. The solution domain geometry is $2 \times 2 \times 1$ and the grid resolution is $96 \times 96 \times 64$.](image)

3.5.1 Internal Cooling Rate

Shahmas et al. (2008) examined a range of Rayleigh numbers and internal heating rates in a spherical model to derive predictive equations for properties such as the average temperature of the convecting fluid. They attempted to fit their results to the following equation (discussed further in Section 1.6),

$$
\theta = \frac{f^{1/2}}{1 + f^{-3/2} + (\sigma + \lambda Ra_B^\nu (1 - f)) \frac{(1 + f^2)^{3/4} H^{3/4}}{Ra_B^{1/4}}} 
$$

(3.1)

and they used their data to solve for $\sigma$, $\lambda$ and $\nu$. In equation (3.1), $\theta$ is the mean temperature, $f$ is the curvature factor, $H$ is the nondimensional internal heating rate and $Ra_B$ is the Bénard-Rayleigh
Figure 3.6: Plot of temperature isosurfaces for \( Ra=10^6 \), \( H=15 \) and a \( 2 \times 2 \times 1 \) solution domain. The yellow isosurfaces are at non-dimensional temperature 0.66 while the orange isosurfaces are at 0.96. The mean temperature is 0.813. The grid resolution for these calculations was \( 144 \times 144 \times 96 \).
number. Shahnas et al. (2008) found the following predictive equation

$$\theta = \frac{f^{1/2}}{1 + f^{-3/2}} + (1.318 + 0.251Ra_B^{0.123}(1 - f)) \frac{(1 + f + f^2)^{3/4}H^{0.729}}{Ra_B^{1/4}}. \quad (3.2)$$

This predictive equation can be used to determine the mean temperature in a spherical shell with a curvature factor appropriate for the Earth’s mantle ($f = 0.547$) and specified $Ra$ and $H$ values. For convection in a plane-layer geometry with $Ra = 10^7$ and $H = 0$, the mean temperature of the system, 0.5, is that obtained for an internal heating rate of 26.26 in a spherical shell where $Ra = 10^7$ and $f = 0.547$. This value of $H$ is too high for the Earth’s internal heating. The conclusion drawn from these results is that plane-layer models need some sort of heat sink in order to accurately simulate the planet’s geotherm. Hence the next part of the study explores a suite of models employing heat sinks, specifically cases where $H = -5$, -10 and -15.

For small Rayleigh numbers, namely $Ra = 10^5$, the mean temperature of the system is again dependent on the size of the solution domain. For solution domains $1 \times 1 \times 1$, the mean temperature is about 10% higher than for larger domains. The planform of convection is also bimodal in this geometry. At larger Rayleigh numbers or in larger solution domains the mean temperature is time-dependent and the convection planform has various upwellings and downwellings consistent with planforms found in spherical models with similar Rayleigh numbers (e.g., Shahnas et al., 2008).

![Figure 3.7: Average temperature with depth for Ra=10^6 and varying internal heating rates. The solution domain geometry is 2 \times 2 \times 1 and the grid resolution is 144 \times 144 \times 96.](image)

Figures 3.7 and 3.8 show plots of horizontally average temperature as a function of height in the solution domain for fixed values of $Ra$, and various internal heating rates. These profile plots of horizontally averaged temperature show a systematic increase in average temperature with increasing internal heating rate. They also show a similar decrease in average temperature with increasing internal cooling rate. The profiles in both figures show anti-symmetric behaviour around the centre temperature profile for $H = 0$. This symmetric solution is required by the code, as a substitution of $H = -H$, $T = -T$, $f = f$.\[\text{\textcopyright 2008 Springer} \]
$z = -z$ and $\mathbf{u} = (u_x, u_y, -u_z)$ along with a switch of the boundary conditions is also a solution to the equations of motion. The upper boundary layers of models with heat sources grow equally in size to lower boundary layers of profiles from models with the same magnitude of heat sink.

Figure 3.8: Average temperature with depth for $Ra=10^7$ and varying internal heating rates. The solution domain geometry is $2 \times 2 \times 1$ and the grid resolution is $256 \times 256 \times 144$.

Figure 3.9 show isosurfaces of plane-layer convection with $Ra = 10^6$ and $H$ values of -15, 0 and 15. The plane-layer model with no internal heating $H = 0$ features sheet-like convection, whereas the models with heating and cooling feature time-dependent upwellings and downwellings.

Figure 3.10 shows isosurfaces of plane-layer convection with $Ra = 10^7$ and $H$ values of -15, 0 and 15. The calculation with internal heating has a high mean temperature expressed in the convection planform which features strong dominant downwellings. The calculation with no internal heating has similar strength upwellings and downwellings. The plane-layer model with internal cooling features dominant upwellings which is more similar to the convective planform observed in spherical shell convection with moderates heat sources (see Figure 3.22).

### 3.5.2 Effect of Rayleigh number on internal heating and cooling rate

Figure 3.11 shows horizontally averaged temperature as a function of height for models featuring a fixed heating ($H = 10$) or cooling ($H = -10$) rate and varying Rayleigh numbers. As Rayleigh number increases, the temperature profiles shift from having a positive temperature gradient in the region between the thermal boundary layers to a more isothermal region between the boundary layers.
Figure 3.9: Isosurfaces for calculations with $Ra=10^6$ and varying internal heating rates, $H = -15, 0$ and 15. The isosurfaces are plots for 0.1 above and below the mean temperature of the calculation. The solution domain geometry is $2 \times 2 \times 1$ and the grid resolution is $144 \times 144 \times 96$.

Figure 3.10: Isosurfaces for calculations with $Ra=10^7$ and varying internal heating rates, $H = -15, 0$ and 15. The isosurfaces are plots for 0.15 above and below the mean temperature of the calculation. The solution domain geometry is $2 \times 2 \times 1$ and the grid resolution is $256 \times 256 \times 144$. 
3.6 Deriving an equation for mean temperature in plane-layer geometry

Sotin and Labrosse (1999) showed that for a plane-layer geometry the temporally averaged mean temperature in an infinite Prandtl number convecting fluid heated by an isothermal base and internal sources fits an equation of the form:

$$\theta = 0.5 + \sigma \frac{H^\beta}{Ra^\gamma}$$  \hspace{1cm} (3.3)

from equation (1.26) with $\theta_b = 0.5$, the average of the mean thermal boundary layer temperatures in a plane-layer convection model with no internal heating.

The results from this study featuring convection calculations with both uniform internal heating and cooling (Tables 3.5-3.7), are used to solve equation (3.3) for the parameters $\sigma, \beta$ and $\gamma$. The results of the inversions, presented in Table 3.1, are compared with the findings published in Sotin and Labrosse (1999) and described below.

There is a strong correlation between $\sigma$ and $\gamma$. The value of $\sigma$ is strongly influenced by the range of Rayleigh numbers used in the inversion and thus the exponent on Rayleigh number, $\gamma$. If the inversion is restricted to using only $Ra = 10^6$ data, the values of $\sigma$, $\gamma$ and $\beta$ are found to be close to those found by Sotin and Labrosse (1999). When the entire data range is included, the values for $\sigma$, $\gamma$ and $\beta$ differ somewhat, which may be due to the wider sampling of Rayleigh number space in this study.

The effect of planform on predicting mean temperature has been discussed previously by Shahnas
Table 3.1: Summary of parameters producing the best fit of observations to equations in the form of equation 3.3 for various restrictions on $\beta$ and/or $\gamma$.

<table>
<thead>
<tr>
<th></th>
<th>This study (fixed $\beta, \gamma$)</th>
<th>SL99 Results (fixed $\beta, \gamma$)</th>
<th>This study (fixed $\beta$)</th>
<th>SL99 Results</th>
<th>This study (fixed $\beta$)</th>
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</thead>
<tbody>
<tr>
<td>$\sigma$</td>
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<td>1.236</td>
<td>0.500</td>
<td>1.02</td>
<td>0.523</td>
</tr>
<tr>
<td>$\beta$</td>
<td>3/4</td>
<td>3/4</td>
<td>0.770</td>
<td>0.729</td>
<td>3/4</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1/4</td>
<td>1/4</td>
<td>0.183</td>
<td>0.232</td>
<td>0.183</td>
</tr>
</tbody>
</table>

et al. (2008). These authors note that there can be a fundamental change in the fit of data to an equation like (3.3) when changing either $Ra$ or $H$ results in an accompanying change in planform. The findings presented here include steady solutions with roll-type planforms for some $Ra = 10^5$ calculations with $|H|=5$. In contrast, Sotin and Labrosse (1999) report that they do not find any steady solutions for internally heated cases. However, their only calculation with $Ra = 10^5$ featured $H=10$, a value at which this study finds time-dependent convection. (It should be noted that for these parameters the mean temperature found by Sotin and Labrosse is 0.885 while for this study it was found to be 0.883 indicating good agreement between the modelling results.)

Analysis based on boundary layer theory (see Section 1.6) predicts $\beta = 3/4$ and $\gamma = 1/4$ for the exponents in equation (3.3). Adopting the exponents based on boundary layer theory and energy balance, Sotin and Labrosse (1999) extended equation (3.3) to a generalized equation for the mean temperature in a convecting fluid in a spherical shell geometry. They propose that in a spherical shell with ratio of inner to outer radius, $f$, the mean temperature will be given by:

$$\theta = \frac{1}{1 + f^{-3/2}} + C \left( \frac{(1 + f + f^2)^{3/4}}{H^{3/4}Ra^{\beta}} \right),$$

(3.4)

where $C$ may be a function of $f$. The expression obtained from (3.4) when $H = 0$, was first derived by Vangelov and Jarvis (1994). (For an alternative derivation, see Section 1.6.)

Shahnas et al. (2008) explored isoviscous convection for a range of Rayleigh numbers and internal heating rates in a spherical shell model and concluded that $C$ has a non-negligible dependence on $Ra$. Moreover, the exponent on $Ra$ deviates more strongly from the boundary layer theory prediction of $1/4$ than the exponent on $H$. This is further supported by evidence from this study as seen in the exponents listed in Table 3.1.

Shahnas et al. (2008) extended the work done by Sotin and Labrosse (1999) to predict the mean temperature of a convecting fluid in a spherical shell and derived an equation of the form:

$$\theta = \frac{1}{1 + f^{-3/2}} + (\sigma + \lambda Ra^{\nu}_B(1 - f))^\beta \left( \frac{(1 + f + f^2)^{3/4}}{Ra^{1/4}_B} \right)$$

(3.5)

where $\lambda$ and $\nu$ are determined by inverting the observations from a large number of spherical shell geometry calculations. Shahnas et al. (2008) constrained the values for $\lambda$ and $\nu$ by specifying a value for $\sigma$ obtained from the study by Sotin and Labrosse (1999), $\sigma = 1.236$. The proposed equation (3.5) reduces to the result obtained by Sotin and Labrosse (1999) for $f=1$ with the theoretically obtained values for the parameters $\beta$ and $\gamma$.

Fitting the observations in Tables 3.5-3.7 for the calculations in a $2 \times 2 \times 1$ geometry, $\beta$ and $\gamma$ are constrained to the values predicted by theory and the inversion results in a $\sigma$ value of 1.221. This finding agrees reasonably well with results from Sotin and Labrosse (1999) and Moore (2008), who determined
values of 1.236 and 1.24 for $\sigma$ respectively. The resulting equation in this study is:

$$\theta = 0.5 \pm 1.221 \frac{|H|^{3/4}}{Ra_B^{1/4}},$$

where the negative sign leads the second term if $H$ is negative. This equation produces a reasonable fit of the observations for calculations featuring internal heating and cooling, with minimal disagreement between observed and predicted mean temperatures. Comparing the predicted values of mean temperature, $\theta$, with those found in the plane-layer geometry calculations (Tables 3.5-3.7) this equation has an average error of 3.15 % and a standard deviation of 3.40 %.

Given the values of $\sigma, \beta$ and $\gamma$ in equation (3.6), the spherical data from Shahnas et al. (2008) can be used to determine $\lambda$ and $\nu$ in equation (3.5) to obtain a single predictive equation for mean temperature. This equation can be used for the plane-layer calculations presented in this study featuring heating and cooling, as well as the spherical shell results presented by Shahnas et al. (2008) featuring uniform internal heating.

Using the values of $\sigma, \beta$ and $\gamma$ in equation (3.6), the equation determined for spherical geometry in this study is

$$\theta = \frac{f^{1/2}}{1 + f^{3/2}} + (1.221 + 0.284Ra_B^{0.114}(1 - f)) \frac{(1 + f + f^2)^{3/4}H^{3/4}}{Ra_B^{1/4}}$$

where the values of $\sigma, \lambda$, and $\nu$ all fall within error of the values found by Shahnas et al. (2008) which used Sotin and Labrosse’s (1999) study for the plane-layer results. Shahnas et al. (2008) found values of $\sigma = 1.236, \lambda = 0.256$ and $\nu = 0.120$. Comparing the predicted values of mean temperature, $\theta$, with the spherical shell data from the Shahnas et al. (2008) calculations, this equation has an average error of 0.901 % and a standard deviation of 1.14 %.

However, if $\sigma$ is fixed at 1.02, the result for $\sigma$ obtained in the Sotin and Labrosse (1999) paper, the resulting plane-layer equation is given by

$$\theta = 0.5 \pm 1.02 \frac{|H|^{0.674}}{Ra_B^{0.219}},$$

whose values of $\beta$ and $\gamma$ are within error (standard deviation of $\beta$ is 0.044, and $\gamma$ is 0.007) only 0.007 off the values from the Sotin and Labrosse (1999) results.

When the data used in the inversion is restricted to only the $Ra = 10^6$ data, the equation becomes

$$\theta = 0.5 \pm 1.02 \frac{|H|^{0.761}}{Ra_B^{0.233}}$$

which is also, within error (standard deviation of $\beta$ is 0.021, and $\gamma$ is 0.003), about 0.008 from the values of Sotin and Labrosse (1999). Thus the values of $\beta$ and $\gamma$ depend on the restriction on $\sigma$ as well as the data range used in the inversion.

The success of a predictive equation is dependent on the values chosen for the parameters $\sigma, \beta$ and $\gamma$. As previously mentioned, $\sigma$ and $\gamma$ are strongly correlated. Equation (3.6) uses the theoretically predicted values for the exponents and only varies $\sigma$. With this restriction the misfit of equation (3.6) and the observed temperatures is greatest at higher Rayleigh numbers (e.g., $10^7$), the range we are most interested in.

In order to improve the fit of the data to predictions, it is necessary to allow all three parameters $\sigma$,
\( \beta \) and \( \gamma \), to vary. In this case we find

\[
\theta = 0.5 \pm 0.500 \frac{|H|^{0.770}}{Ra_B^{0.183}},
\]  

(3.10)

where the negative sign before second term is used in cases where \( H \) is negative.

Using the constraints determined in equation (3.10), the spherical shell data are best fit by:

\[
\theta = \frac{f^{1/2}}{1 + f^{-3/2}} \pm (0.500 + 1.997 Ra_B^{-0.112}(1 - f)) \frac{(1 + f + f^2/3)^{3/4} H^{0.770}}{Ra_B^{0.183}}.
\]

(3.11)

For the spherical shell data from Shahnas et al. (2008), the average misfit of the observations by equation (3.11) is 1.7% and no more than 3.8% in any case when \( f = 0.547 \).

The disagreement between observation and prediction increases as \( |H| \) increases. Shahnas et al. (2008) also noted this, and that their data was best fit when the exponent \( \beta \) was below 3/4. The plane-layer geometry results presented here suggest that \( \beta \) is slightly above 3/4. To obtain a good fit for both plane-layer and spherical shell geometry data with one equation, one option is to return to specifying \( \beta = \frac{3}{4} \), the theory based estimate. Using this restriction on \( \beta \), the inversion of plane-layer geometry results yields the following equation for plane-layer geometry data:

\[
\theta = 0.5 \pm 0.523 \frac{|H|^{3/4}}{Ra_B^{0.183}}.
\]

(3.12)

It is worth noting that while \( \gamma \) has the freedom to vary, it does not change from the previous form of the equation (3.10). Using this equation, a new equation fit to the spherical data is obtained,

\[
\theta = \frac{f^{1/2}}{1 + f^{-3/2}} \pm (0.523 + 1.924 Ra_B^{-0.103}(1 - f)) \frac{(1 + f + f^2/3)^{3/4} |H|^{3/4}}{Ra_B^{0.183}}.
\]

(3.13)

Equation (3.13) reduced the average misfit of the spherical shell temperatures to 0.88% with a standard deviation of 1.1%. In comparison, the average misfit for the parameterization determined by Shahnas et al. (2008) is 0.82%.

Tables 3.5-3.7 list the disagreement between observed mean temperature \( \theta_{obs} \) and the value of \( \theta \) predicted by (3.12), \( \theta_{pre} \). Also listed (under % error) are the absolute value of the disagreement between the observed and predicted temperature as a percentage of the predicted temperature and the absolute value of the disagreement between the observed and predicted departure of the temperature from a temperature of 0.5. The latter quantity is a measure of the error on the term arising from internal heat sources or sinks and is listed under % errori.

Figure 3.12 is a plot of the temperatures predicted by equation (3.12) versus the observed mean temperatures from the \( 2 \times 2 \times 1 \) solution domain geometry calculations listed in Tables 3.5-3.7. The diagonal line on the plot indicates the position of points that would correspond to a perfect agreement between observation and prediction by equation (3.12). The average misfit of the observed temperatures is 1.26%, the maximum misfit is 3.3% and the standard deviation of the errors is 1.7%.

Figure 3.13 is a plot of the temperatures predicted by equation (3.13) versus the observed mean temperatures from the spherical shell data with different \( f \) values from the Shahnas et al. (2008) data. The diagonal line on the plot indicates the position of points that would correspond to a perfect agreement between observation and prediction by equation (3.13). The average misfit of the observed temperatures for \( f = 0.547 \) is 0.88%, the maximum misfit is 2.4% and the standard deviation of the
Figure 3.12: Observed mean temperature versus predicted temperature from equation (3.12) for the $2 \times 2 \times 1$ solution domain geometry calculations listed in Tables 3.5-3.7. The solid diagonal line indicates perfect agreement between observation and prediction.
The average misfit of the observed temperatures with various $f = 0.3, 0.4, 0.7, 0.8$ values is 2.47% with a maximum misfit of 3.9% (for the $f = 0.3$ value) and a standard deviation of the errors of 2.0%. The average misfit of the observed temperatures with $f = 0.4$ is 2.42% with a maximum misfit of 2.7% and a standard deviation of 0.22%. These errors show the difficulty in predicting mean temperatures for low $f$ values as expressed by Shahnas et al. (2008). However, the improved fit when $f = 0.4$ demonstrates that this equation (3.13) predicts very well for mid- to high-range $f$ values, relevant to Earth-like convection.

Figure 3.13: Observed mean temperatures for different $f$ value spherical shells from Shahnas et al. (2008) data versus predicted temperature from equation (3.13). The solid diagonal line indicates prefect agreement between observation and prediction.

Figure 3.14 plots observed mean temperatures from $2 \times 2 \times 1$ plane layer geometry calculations ($f=1$) versus $H$ in $\theta - H$ space. The continuous curves in the figure represent lines of constant $Ra$ for the different Rayleigh numbers used in this study. The values of $\theta$ are calculated from equation (3.13) for fixed values of Rayleigh numbers. Perfect agreement between observations and equation (3.13) is indicated if the data points land exactly on the curves. Strong agreement between observation and parameterization is obtained as $Ra$ increases.

$\theta$ decreases as $Ra$ increases for a fixed heating rate and similarly $\theta$ increases as $Ra$ increases for a fixed cooling rate.

Figure 3.15 plots predicted mean temperatures versus Rayleigh number using equation (3.13) for $H = 0$ and $H = 25$ in the different geometries. The straight curves represent convection with $H = 0$ in plane-layer (blue) and spherical shell geometries (red), where $f = 0.547$. When $H = 0$ or 25, the plane-layer system exhibits much higher mean temperatures at any Rayleigh number compared to the
Table 3.2: Summary of Shahnas et al. (2008) spherical shell models with varying $f$ observed mean temperatures ($\theta_{\text{obs}}$) and the predicted mean temperatures using equation (3.13) from this study.

<table>
<thead>
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<th>$Ra$</th>
<th>$H$</th>
<th>$f$</th>
<th>$\theta_{\text{obs}}$</th>
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<td>0.547</td>
<td>0.647</td>
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<td>2.10</td>
</tr>
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Figure 3.14: Contours of $Ra$ in $H - \theta$ space calculated from equation (3.12). Squares indicate the observed $(H, \theta)$ co-ordinates of the models used in this study for solution domains of $2 \times 2 \times 1$. 
spherical shell system with the same internal heating rate. For $Ra$ from $10^7$ to $10^8$, the discrepancy between the mean temperature of the plane-layer system with no internal heating and the spherical shell $H = 25$ increases. At large Rayleigh numbers, especially those relevant to convection in super-Earths, the plane-layer systems are much hotter than their spherical shell counterparts and the need for internal cooling is even more important.

Figure 3.15: Contours of $H$ in $Ra - \theta$ space calculated from equation (3.13). The blue curve represents $H = 0$ in plane-layer convection, the red curve represents $H = 0$ in spherical shell convection with $f = 0.547$. The pink and green curve represents convection with $H = 25$ in plane-layer and spherical shell systems, respectively.

Figure 3.16 plots contours of constant $H$ in $f - \theta$ space for different $Ra$ using equation (3.13). The plots show that for all $H$ and $f$ examined, as the Rayleigh number is increased in magnitude, the overall effect on the mean temperature becomes less dramatic. Therefore the space between contours of constant $H$ is reduced for increasing $Ra$. This should be expected as the exponent on $H$ in equation (3.13) is less than 1.0. Increasing the internal heating rate by a factor of 2.53 is required to double the contribution of internal heating to the overall temperature.

Figure 3.16 also shows that $\theta$ less influenced by $H$ with different $f$ values at higher Rayleigh numbers. This further indicates the importance of cooling at higher Rayleigh numbers. In Figure 3.16c, an $H$ value above 20 is needed to obtain $\theta = 0.5$ in a spherical shell with $f = 0.547$ and yet this is obtained in plane-
layer convection at the same Rayleigh number with no internal heating.

Figure 3.17 plots contours of constant $f$ values in $H - \theta$ space for Rayleigh numbers of $Ra = 10^5$, $Ra = 10^6$ and $Ra = 10^7$ respectively. In all three cases, an increase in the $f$ value results in a higher mean temperature for any given $H$ value. The gradient of the curves of constant $f$ value decreases with increasing Rayleigh number. Changes in internal heating result in larger changes in mean temperature at lower Rayleigh numbers than similar changes at higher Rayleigh numbers.

3.7 Emulating spherical shell results in plane-layer geometry

The equation (3.13) developed in this study has different exponent values from the results published in Shahnas et al. (2008), however the equation reproduces mean temperatures that are in agreement with the previous study. For example, the Shahnas et al. (2008) equation requires $H = 26.3$ to produce a spherical shell with $f = 0.547$, $Ra = 10^7$ and $\theta = 0.5$. The equation in this study (3.13) requires a value of 25.8 which is only a 2% disagreement with the Shahnas et al. (2008) result.

Equation (3.13) can be rearranged to solve for $H$, a heating or cooling rate, which will produce a specified mean temperature given $Ra$ and $f$. For a given set of parameters $f$, $Ra$ and $H$ describing a spherical shell convection system, the associated $\theta$ can be calculated. Then, by specifying $f = 1$, the rearranged equation (3.13) will determine the $H$ needed to yield the corresponding mean temperature $\theta$ for the same $Ra$ in a plane-layer calculation.

This can be tested by selecting some spherical cases and comparing them with plane-layer models featuring parameter values determined by the above method. Similarities and differences will be discussed between the spherical shell and plane-layer systems.

The first comparison attempts to reproduce the temperature profile observed in a spherical shell with $H = 0$ by modelling convection in a plane-layer ($f = 1$) with internal cooling. When $H = 0$, the mean temperature of the system is given by the first term in equation (3.13) and is independent of $Ra$. For a spherical shell with the Earth’s core to planet radius ratio ($f$) of 0.547, the predicted mean temperature of the convecting system is 0.213 (determined from equation (3.13)). With a Rayleigh number of $10^5$, equation (3.13) requires that a constant internal cooling rate of $H = -6.7$ to reproduce the mean temperature of 0.213 found in the spherical system. Figure 3.18 shows the isosurfaces of temperature for a $2 \times 2 \times 1$ geometry calculation with $H = -6.7$. The mean temperature of the plane-layer system converges on 0.212 which is very close to the predicted value. However, as seen in Figure 3.19, the temperature profile for the plane-layer system (red) is not a good approximation to the spherical shell profiles from Jarvis et al. (1995). The figure from Jarvis et al. (1995) shows different $f$ value spherical shells with $H = 0$. The closest curve to this above calculation would be the curve with $H = 0$ and $f = 0.5$.

Increasing the Rayleigh number to $10^6$, equation (3.13) requires a constant internal cooling rate of $H = -13.1$ to reproduce the mean temperature of 0.213 found in the spherical system. A plane-layer $2 \times 2 \times 1$ geometry model with $H = -13.1$ and $Ra = 10^6$, converges on a mean temperature of 0.208. The disagreement with the spherical shell is only 2.3%. Figure 3.20 shows a snapshot of the temperature isosurfaces for this calculation.

This example indicates that it is therefore possible to obtain the desired mean temperature for a spherical shell system with no internal heating in a plane-layer calculation. However, the mean horizontal temperature profile as a function of depth does differ between the two calculations, as discussed in
Figure 3.16: Contours of $H$ in $f-\theta$ space calculated from equation (3.13). In panel (a), $Ra = 10^5$; in panel (b), $Ra = 10^6$; and in panel (c), $Ra = 10^7$. In each panel, the contours plotted correspond to $H = 20, 10, 0, -10$ and -20 from top to bottom.
Figure 3.17: Contours of $f$ in $H - \theta$ space calculated from equation (3.13) for $Ra = 10^5$, $Ra = 10^6$, $Ra = 10^7$. The contours represent $f = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ from bottom to top in all three plots.
Figure 3.18: Plot of temperature isosurfaces 0.1 above and below the mean temperature of 0.212 for $H = -6.7$, $Ra = 10^5$ in a $2 \times 2 \times 1$ geometry. The grid resolution is $96 \times 96 \times 64$. 
Figure 3.19: Plot of temperature profiles for various $f$ values modified from Jarvis et al. (1995) and a plane-layer calculation with $H = -6.7$, $Ra = 10^5$ in a $2 \times 2 \times 1$ geometry.

previous studies (e.g., Jarvis et al., 1995). The temperature profile for the plane-layer system with $H = -13.1$ has an accumulation of hot fluid near the upper boundary and cold fluid at the lower boundary. This is commonly observed in plane layer models (e.g., Vangelov and Jarvis, 1994) but is less common in spherical shell geometries. This effect, however, significantly decreases with increasing Rayleigh number.

One goal of this study it to find a way to emulate spherical shells with internal heating in plane-layer geometries. Using a spherical shell model from the Shahnas et al. (2008) study with $f = 0.547$, $Ra = 10^7$ and $H = 23.533$, the mean temperature of the system is reported to be 0.475. Equation (3.13) predicts that to obtain this temperature in a plane layer geometry ($f=1$) requires an internal cooling rate of $H = -0.89$. Indeed, a plane-layer calculation with $Ra = 10^7$ and $H = -0.89$ has the same mean temperature as the spherical shell calculation to within a 1% error.

Figure 3.21 shows a temperature field snapshot from a plane-layer calculation with $Ra = 10^7$ and $H = -0.89$. The planform of convection is time-dependent, with transient upwellings and downwellings. This is similar to the planform observed in the spherical shell calculation by Shahnas et al. (2008), shown in Figure 3.22, which the plane-layer model emulates.

Figure 3.23 shows time averaged horizontally averaged temperature profiles from the plane-layer calculation with $Ra = 10^7$ and $H = -0.89$ and the corresponding spherical shell calculation from Shahnas et al. (2008). The figure shows that the spherical shell calculation has more hot fluid near the upper boundary layer. However, there is good overall agreement between the two temperature profiles.
Figure 3.20: Plot of temperature isosurfaces for $Ra=10^6$, $H=-13.1$, in a $2 \times 2 \times 1$ solution domain. The blue isosurfaces are at non-dimensional temperature 0.1 while the green isosurfaces are at 0.3. The mean temperature is 0.208. The grid resolution is $144 \times 144 \times 96$. 
Figure 3.21: Temperature isosurfaces from a snapshot of the calculation featuring a Rayleigh number of $10^7$ and $H = -0.89$ with average mean temperature $\theta = 0.475$. Isosurface values are 0.275 for the downwellings and 0.675 for the upwellings. Solution domain size is $2 \times 2 \times 1$ and the calculation resolution is $256 \times 256 \times 144$. 
Figure 3.22: Temperature isosurfaces from a snapshot of the calculation featuring a Rayleigh number of $10^7$, $f = 0.547$ and $H = 23.533$ with average mean temperature $\theta = 0.475$. Reprinted from Shahnas et al. (2008) with permission from Elsevier.

3.7.1 Varying $f$ values

Jarvis et al. (1995) observed that varying the value of $f$, in a non-internally heated fluid, changed the planform of convection in spherical models with $Ra = 10^5$. With small values of $f$ (0.1 or 0.3), isoviscous convection developed only one plume emanating from the core boundary; however, for larger values of $f$ (0.5 and 0.7), the planform had many upwellings and downwellings (see Figure 3.24). Jarvis et al. (1995) also observed a difference in mean horizontal temperature profiles as a function of depth with varying $f$ values (see Figure 3.25).

One goal of this study is to emulate the results for different $f$ valued spheres in a plane-layer geometry. Using equation (3.7), the mean temperature in a sphere can be predicted for $H = 0$, $Ra = 10^5$ and various values of $f$ (0.1, 0.3, 0.5, 0.7, 0.9). These temperatures can then be used in equation (3.7) (or (3.13)) with $f = 1$ to calculate the necessary cooling rate, $H$, in a plane-layer to simulate the mean temperature in the spherical shell with the particular $f$ value. Table 3.3 shows the observed and predicted mean temperatures for the different $f$ value spheres emulated in plane-layer models with a cooling rate determined with equation (3.7).

Emulating small $f$ spherical shell temperatures in a plane-layer geometry is attempted here. Using the values for $\theta$ when $H = 0$ for different $f$ values from Jarvis et al. (1995), or predicted by equation (3.13) for $Ra = 10^5$, equation (3.13) can be inverted to determine the $H$ value necessary to emulate the temperature in a particular $f$ valued sphere. Figure 3.25 shows horizontally averaged temperature with depth for five different $f$ values from the Jarvis et al. (1995) paper. Figure 3.26 shows the horizontally averaged temperature with depth for the plane-layer models with $H$ adjusted to mimic the mean temperatures in the same five models with varying $f$ from the Jarvis et al. (1995) results.

Figure 3.27 shows a plot of $\theta$ versus $f$, where the solid line represents the curve of predicted mean temperatures determined by equation (3.13) for spherical shells with $H = 0$ and a given $f$ value. The
Figure 3.23: Comparison of time averaged mean lateral temperatures as a function of height for a solution obtained in a plane-layer geometry with $H = -0.89$ (red solid curve) and a spherical shell geometry with $H = 23.533$ and $f = 0.547$ (black dashed curve). Both calculations feature Bénard-Rayleigh numbers of $10^7$. The spherical shell calculation data is from the study by Shahnas et al. (2008).
Figure 3.24: Contours of temperature (left) and radial velocity (right) from 3D spherical shell models with varying $f$ values. For the temperature plots the solid lines indicate positive temperature perturbations, or hot fluid rising, and the dashed lines indicate negative temperature perturbations or downwellings. (a) $f = 0.1$, (b) $f = 0.3$, (c) $f = 0.5$, (d) $f = 0.7$, (e) $f = 0.9$. Reprinted from Jarvis et al. (1995) by permission of the publisher Taylor & Francis Ltd.
Because of the ambiguity of the spectral information we will use our visual estimate of $I = 30$ for model E while recognizing that is certainly a crude estimate. (Curvature effects are minimal at $f = 0.9$ and our conclusions are not significantly influenced by our choice of $I$ for this model.) In such a thin shell model, the solution appears to be more affected by three-dimensionality than by the curvature or closed nature of the solution domain.

The Nusselt number, $\text{Nu}$, (or dimensionless heat flux) computed for this model is $\text{Nu} = 10.4$. This is very close to the Nusselt number, $\text{Nu} = 10.5$, obtained for a unit-aspect-ratio in two dimensional plane layer geometry (e.g., Blankenbach et al., 1989).

Although the solution domain is closed its lateral extent is so large relative to the shell thickness that the flow field easily adopts its preferred aspect ratio of approximately unity. At the lower values of $f$ considered here this is not so. The aspect ratios for the five models A to E (beginning with A) are 1.3, 2.0, 1.9, 1.4, and 1.0.

The mean radial temperature profiles for models A to E are shown superimposed in Figure 3. At $f = 0.9$, the profile is very similar to the familiar symmetric profile from Benard convection in plane layers. As $f$ decreases, the mean dimensionless temperature, $T/A$, decreases and a pronounced asymmetry develops between the large temperature drop across the inner thermal boundary layer and the small drop across the outer.

This asymmetry is required to conduct roughly the same heat through the inner and outer boundaries which have different surface areas. The Nusselt number also decreases from 10.4 at $f = 0.9$ to 9.2 at $f = 0.7$, 6.6 at $f = 0.5$, 4.4 at $f = 0.3$, and 2.6 at $f = 0.1$.

Figure 3.25: Profiles of horizontally averaged temperature as a function of distance from the inner radius, $R_1$ for different values of $f$ indicated on each curve. Reprinted from Jarvis et al. (1995) by permission of the publisher Taylor & Francis Ltd.
Chapter 3. Results: Isoviscous Mantle Convection Study

Figure 3.26: Horizontally averaged temperature profiles as a function of depth for $5 \times 5 \times 1$ plane-layer models with $H$ values determined to emulate spherical shells with $H = 0$ and different $f$ values. The grid resolution is $240 \times 240 \times 64$. 
plot shows very good agreement between observed and predicted temperatures for \( f = 0.5 \). For smaller values of \( f \), the mean temperatures in the plane-layer models (green squares) differ more from the predicted spherical shell temperatures (red line). In general the agreement is not as good as larger Rayleigh number calculations (see Section 3.7). This is not so surprising, as equation (3.13) had a better fit for larger Rayleigh number calculations.

\[
\text{Figure 3.27: The solid red line represents the curve of predicted } \theta \text{ versus } f \text{ value, determined by equation (3.13). The green points represent the observed mean temperature } \theta \text{ in the plane-layer models listed in Table 3.3 which are emulating the predicted temperatures in a given } f \text{ value spherical shell.}
\]

The profiles from the plane-layer geometry have a different shape than the profiles from the spherical shell geometry (compare Figures 3.25 and 3.26). This could be due to the constant heating/cooling rate used throughout the entire plane-layer.

The temperatures in small \( f \) value spheres correspond to the temperatures with large negative \( H \) values in the plane-layer model which results in more vigorous convection and complex planforms compared to small \( H \) values (i.e., many upwellings and downwellings rather than single plume or bimodal convection). Nevertheless, equation (3.13) had improved success at emulating spherical shells with internal heating as they possess many upwellings and downwellings.

### 3.8 High Rayleigh number convection

An important application of 3D plane-layer convection modelling is in modelling convection parameters relevant for research in the mantle dynamics of Super-Earths. Super-Earths are terrestrial planets with masses that range from 1-10 Earth masses (e.g., Rivera et al., 2005; Valencia et al., 2006, 2009).
Table 3.3: Summary of plane-layer models emulating different \( f \) value spheres with \( Ra = 10^5 \) and \( H = 0 \).

<table>
<thead>
<tr>
<th>( Ra )</th>
<th>( H )</th>
<th>( f )</th>
<th>Resolution</th>
<th>Solution Domain</th>
<th>( \theta_{\text{obs}} )</th>
<th>( \theta_{\text{pre}} )</th>
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<td>( 10^5 )</td>
<td>-13.7</td>
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<td>( 5 \times 5 \times 1 )</td>
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<td>0.185</td>
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<tr>
<td>( 10^5 )</td>
<td>-3.9</td>
<td>0.7</td>
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<td>( 5 \times 5 \times 1 )</td>
<td>0.343</td>
<td>0.309</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>-0.89</td>
<td>0.9</td>
<td>( 240 \times 240 \times 64 )</td>
<td>( 5 \times 5 \times 1 )</td>
<td>0.461</td>
<td>0.437</td>
</tr>
</tbody>
</table>

Improvements in spherical shell convection calculations are approaching the resolution necessary for modelling convection at large Rayleigh numbers, relevant for the Earth, however large scale modelling of many calculations at such Rayleigh numbers remains intractable. Thus, the ability to use plane-layer convection models to emulate these high Rayleigh number results will continue to be an important and useful application in the future (e.g., van Heck and Tackley, 2011; Lenardic and Crowley, 2012).

Scaling approximations determined by Valencia et al. (2006) state that \( R \sim M^{0.262}, \rho \sim M^{0.196}, g \sim M^{0.503}, q \sim M^{0.476}, \) and \( \eta \sim M^{-0.64} \) where \( R \) is the radius, \( M \) mass, \( \rho \) the average density of the planet, \( g \) the gravitational acceleration, \( q \) heat flux and \( \eta \) the viscosity of the Super-Earth mantle. Consider a Super-Earth with ten terrestrial masses. It will have a radius approximately twice that of Earth and therefore a deeper mantle and larger gravitational acceleration using the scaling developed by Valencia et al. (2006). Using these approximate scalings, a Rayleigh number of approximately \( 10^8 \) and nondimensional internal heating rate of \( H = 35 \) are both values within the range of the Super-Earth parameter regime. Equation (3.13) can be used to determine the mean temperature expected and heating rate required in plane-layer convection to model this system.

Using an Earth-like value of \( f = 0.55 \), equation (3.13) predicts a mean temperature of 0.438 for the spherical shell system of the Super-Earth with \( Ra = 10^8 \) and \( H = 35 \). Obtaining this temperature in a plane-layer model with \( f = 1 \) requires a cooling rate of \( H = -5.22 \). Figure 3.28 shows a snapshot of the temperature isosurfaces from a plane-layer calculation with \( Ra = 10^8 \) and \( H = -5.22 \). This model converges on a mean temperature of 0.443, a disagreement of less than 1.1% from the spherical shell model prediction. The convective planform features strong upwelling features, another feature expected in spherical shell convection models (See Figure 3.22). In contrast, a plane-layer model with \( Ra = 10^8 \) and \( H = 35 \) would yield a mean temperature of 0.757, according to equation (3.13), and would feature strong downwelling features as seen in Figure 3.10 (when \( H = 15 \) with an Earth-like Rayleigh number \( Ra = 10^7 \)).

### 3.9 \( V_{rms} \)

The system root-mean-square velocity is an alternative parameter to consider when comparing convection in different geometries. The root-mean-square values of temporally averaged nondimensional velocities of the systems (\( V_{rms} \)) are listed in Table 3.4. Figure 3.29 shows a plot of \( V_{rms} \) as a function of \( H \). The \( V_{rms} \) shows symmetry around the \( H = 0 \) case for all Rayleigh numbers examined (similar to Figures 3.7 and 3.8). Figure 3.29 also shows that the \( V_{rms} \) has a large decrease with even a small amount of internal heating or cooling, reflecting the transition to more diffuse upwellings and downwellings, respectively.

In addition to the symmetry around the \( H = 0 \) case, the \( V_{rms} \) values increase with increasing Rayleigh
Figure 3.28: Temperature isosurfaces snapshot from a calculation featuring a Bénard-Rayleigh number of $10^8$ and $H = -5.22$ with average mean temperature $\theta = 0.443$ in a $2 \times 2 \times 1$ plane-layer geometry calculation with resolution $512 \times 512 \times 288$. The isosurface values used to depict the upwellings are 0.64 and for downwellings are 0.24.
Figure 3.29: Temporally averaged root-mean-square of system nondimensional velocities for the $2 \times 2 \times 1$ solution domain geometry calculations listed in Table 3.4. Vertical bars indicate a standard deviation of the values used in the averaged time series.
number, reflecting the more diffuse upwellings and downwellings in the planforms of the more vigorous convection. As the Rayleigh number is increased, there is a different increase in $V_{\text{rms}}$ for any given fixed $H$ value. However, for a given $|H|$, the increase in $V_{\text{rms}}$ with increasing $Ra$ is almost identical. Therefore, the use of cooling (rather than heating) in the plane-layer models does not affect other parameters, such as the root-mean-square velocity of the system.

### 3.10 Surface heat flux

Deriving an accurate parametric equation for surface heat flux has its challenges as has been discussed previously by Shahnas et al. (2008). However, general trends are worth noting, particularly in the application of these results for isoviscous convection to the parameter range relevant to terrestrial mantle convection. Using an equation of the form $Q_{\text{top}} = \Gamma Ra^{1/3} \theta^{4/3}$ (a general form of equation 1.15 from Section 1.6) (e.g., Sotin and Labrosse, 1999; Shahnas et al., 2008) the heat fluxes from the $2 \times 2 \times 1$ calculations presented in Tables 3.5-3.7 can be inverted to find parameter value of $\Gamma = 0.695$ given the theoretical values for $\zeta = 1/3$ and $\psi = 4/3$. The equation for predicting mean surface heat flux is given by

$$Q_{\text{top}} = 0.695 Ra^{1/3} \theta^{4/3},$$

(3.14)

Figure 3.30 plots the observed versus predicted heat flux for the plane-layer and spherical shell data, and shows the large disagreement between observations and predictions using these theoretical values.

Allowing all the parameters to vary and inverting the plane-layer data from this study combined with the spherical shell heat flux data from Shahnas et al. (2008) gives parameter values of $\Gamma = 0.4818$, $\zeta = 0.2803$, and $\psi = 0.2959$. The refined equation for predicting mean surface heat flux is given by

$$Q_{\text{top}} = 0.4818 Ra^{0.2803} \theta^{0.2959}.$$  

(3.15)

Figure 3.31 shows a plot of the observed surface heat flux from the plane-layer models and the

<table>
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<th>Model</th>
<th>$Ra$</th>
<th>$H$</th>
<th>$\theta$</th>
<th>$V_{\text{rms}}$</th>
<th>Standard Deviation</th>
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<td>104.12</td>
<td>10.15</td>
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<td>9.82</td>
</tr>
<tr>
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<td>-5</td>
<td>0.307(s)</td>
<td>154.20</td>
<td>0.00</td>
</tr>
<tr>
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<td>10</td>
<td>0.883</td>
<td>109.01</td>
<td>10.72</td>
</tr>
<tr>
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<td>105.38</td>
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<td>-5</td>
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<td>480.45</td>
<td>42.49</td>
</tr>
<tr>
<td>$Ra_{1e6+10A2}$</td>
<td>$10^5$</td>
<td>10</td>
<td>0.635</td>
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<td>41.33</td>
</tr>
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<td>35.52</td>
</tr>
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<td>0.299</td>
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</tr>
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<td>0.652</td>
<td>1578.45</td>
<td>146.64</td>
</tr>
</tbody>
</table>
Figure 3.30: Plot of observed surface heat flux $Q_{obs}$ from this plane-layer study and the spherical shell models in Shahnas et al. (2008) study versus predicted surface heat flux given by equation (3.14). The plane-layer data is indicated by $f = 1$, $Ra$ values with similar labels to Figure 3.32 and the spherical shell data is indicated by the pink circles ($f = 0.547$) for a range of Rayleigh numbers.
predicted surface heat flux given by equation (3.15).

The equation for mean surface heat flux determined from these plane-layer observations differs from the equation determined by Shahnas et al. (2008) for their set of spherical shell data. Shahnas et al. (2008) found values of $\Gamma = 1.192$, $\zeta = 0.274$, and $\psi = 1.022$. This shows the difficulty in matching heat flux in spherical shell and plane-layer systems. Figure 3.32 shows the observed surface heat flux from Shahnas et al. (2008) spherical shell data with the predicted values using equation (3.15) derived using the plane-layer heat flux data from this study.

Figure 3.32 shows increased disagreement between observations and predicted values of heat flux when using spherical shell data with equation (3.15). Using the predicted values of $\theta$ from equation (3.13) in equation (3.15) does not change the misfit between observed and predicted surface heat flux.

This parameterization for $Q_{top}$ can have disagreement with observations by more than 10% but is sufficient to allow comparison of the general trends in the data with the surface heat flux behaviour shown by the spherical system as determined by Shahnas et al. (2008). Figure 3.33 plots predicted $Q_{top}$ as a function of $H$ for different values of Rayleigh number in both spherical shell and plane-layer geometry systems. The dashed lines in Figure 3.33 show predicted surface heat flux, $Q_{top}$, from the plane-layer geometry calculations and indicate that in systems featuring internal heating or cooling,
Figure 3.32: Plot of observed surface heat flux $Q_{\text{obs}}$ from the spherical shell models in Shahmas et al. (2008) study versus predicted surface heat flux given by equation (3.15).
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$Q_{\text{top}}$ increases or decreases, respectively, when compared with a case where $H = 0$. The total heat flux from the plane-layer calculations is not very sensitive to $H$ for any of the $Ra$ examined, except when $\theta$ is close to zero. When internal heating or cooling is added in a plane-layer geometry the heat flux through the basal surface area of the system decreases or increases, respectively, so that the net heat flux from the convecting layer does not change by a magnitude comparable to the addition $H$. The heat flux from spherical shell calculations are much more sensitive to the addition of internal heating (solid lines in Figure 3.33). When $f$ is small, the contribution of heat from the core to the heat flux at the surface diminishes. Consequently, when compared with a plane-layer system, a reduction in basal heat flux in the spherical shell system is not comparable to the effect of the addition of internal heating on the net surface heat flux.

Following the examples of calculations discussed previously, Figure 3.33 shows that although matching the temperatures in the plane layer calculation with $Ra = 10^6$ and $H = -13.1$ results in a surface heat flux that disagrees with the analogous spherical case (with $H = 0$ and $f = 0.547$) by a factor of a third, this is approximately 50% better than the disagreement between the spherical case and a plane layer case with $H = 0$. In contrast, $Q_{\text{top}}$ from a plane layer case with $Ra = 10^7$ and $H = 23.53$ better matches surface heat flux from the spherical case with the same parameters when compared with $Q_{\text{top}}$ from the plane-layer calculation with $Ra = 10^7$ and $H = -0.89$, (i.e., the adjusted cooling rate using equation 3.13). The disagreement in $Q_{\text{top}}$ is almost double in the comparison with the adjusted heating rate, despite the agreement of $\theta$. (O’Farrell and Lowman, 2010)

### 3.11 Extensions of the isoviscous convection study

#### 3.11.1 Alternate analysis of heat flux

Moore (2008) found an improved way to predict heat fluxes in plane-layer geometry. He derived equations using the Nusselt number, $Nu$, which represents the ratio of convective to conductive heat flux across a boundary (surface), or the nondimensional surface heat flux of a system. Moore (2008) proposed an equation based on the form $Nu_{\text{top}} - 1 = Nu_{\text{bot}} - 1 + H$ (similar to equation (C.9) derived in Appendix C). The Nusselt number should equal one at the critical Rayleigh number implying the following scaling, $Nu - 1 \sim (Ra - Ra_C)$. Then to ensure the heat flux scales with the internal heating as in the purely internally heated case, the $H$ term is scaled by 0.5 so that when $H$ is very large, half the heat is contributing to the surface flux since half the heat will flow out of the bottom boundary. Fitting data from 2D studies, Moore specifically proposed

$$Nu_{\text{top}} - 1 = 0.5H + 0.206(Ra - Ra_C)^{0.318} \quad (3.16)$$

where $Ra_C$ is the critical Rayleigh number for the onset of convection in the absence of internal heating, and the critical Rayleigh number for free-slip boundaries is $Ra_C = 27\pi^4/4 \sim 658$.

Figure 3.34 shows MC3D surface heat flux data from this plane-layer geometry study versus predicted surface heat flux given by the equation (3.16) from Moore (2008). Moore’s equation is able to predict the surface heat flux from the plane-layer calculations quite well, for plane-layer models featuring internal heating and cooling.

Following Moore’s study, the data from this study with plane-layer convection can be used to derive a fit to an equation of the form
Figure 3.33: Surface heat flux, $Q_{\text{top}}$, as a function of $H$ and $Ra$. Solid curves indicate predicted heat flux for spherical shell geometry calculations for $f = 0.547$ based on the findings of Shahnas et al. (2008). Dashed curves indicate predicted heat flux for the plane-layer geometry calculations of this study. Predicted heat flux determined by equation (3.31).
Figure 3.34: Observed MC3D data from this study versus predicted heat flux using the Moore (2008) equation (3.16). The dashed line shows perfect agreement between observations and predictions.
\[(Nu - 1) - 0.5H = \gamma(Ra - Ra_C)^\beta. \quad (3.17)\]

Using the data from the plane-layer geometry study (Tables 3.5-3.7), \(\gamma\) and \(\beta\) in equation (3.17) can be re-calculated.

The resulting equation is

\[Nu - 1 = 0.5H + 0.212(Ra - Ra_C)^{0.318}, \quad (3.18)\]

which is quite close to the equation derived by Moore (2008). The standard deviation for \(\gamma\) is 0.0420 and for \(\beta\) it is 0.0030.

Figure 3.35 shows MC3D surface heat flux data from this plane-layer geometry study versus predicted surface heat flux given by the equation (3.18).

![Figure 3.35: Observed MC3D data from this study versus predicted heat flux using the derived equation (3.18) from the plane-layer geometry data.](image)

For both of these equations (3.16) and (3.18), there is good agreement between the observed and predicted values of heat flux, as the data points fall close to the line of perfect agreement. The mean squared error of heat flux using Moore’s equation (3.16) is 2.59, while the mean squared error for the new equation (3.18) derived from data from this study is only 1.19.
3.11.2 Alternate form of parameterization for mean temperature in convecting systems

Deschamps et al. (2010) also developed an extension to the work by Shahnas et al. (2008) for predicting mean temperatures in a spherical shell. Deschamps et al. (2010) focused on refining the equation for different spherical shell sizes and in doing so lost the ability to regain the plane-layer equation of the form proposed by Sotin and Labrosse (1999) when $f = 1$. Their results did not restrict the equation to reduce to an $f = 1$ solution, because it is not a spherical topology. Deschamps et al. (2010) derived the following equation

$$\theta = \theta_m + (1.68 - 0.8f) \frac{(1 + f + f^2)/3}{Ra^{0.234}} H^{0.779}$$

(3.19)

where $\theta_m = \frac{f^2}{(1+f)}$ and this equation was successful at predicting mean temperatures in spherical shell models.

Figure 3.36 shows observed mean temperature for Deschamps et al. (2010) mixed heating spherical data with various $f$ values versus predicted mean temperatures for both the equation (3.19) and equation (3.13).

Figure 3.36: Observed Deschamps et al. (2010) data versus predicted mean temperature using the equation from Deschamps et al. (2010) (equation (3.19)) and the MC3D equation (3.13) developed in this study.

This figure shows good agreement in both equations with the data presented. The average error for
the temperature predicted with equation (3.19) is 1.8% with a standard deviation of 2.2%. The average error from equation (3.13) generated in this study is 2.7% with a standard deviation of 3.3%

Deschamps et al. (2010) used their own spherical shell data to generate their predictive equation (3.19), so it is expected that their equation is better able to predict their mean temperatures.

In order to better compare equation (3.19) with equation (3.13) generated in this study, it is important to compare the equations predictive temperatures with those observed in a different spherical study, specifically the Shahnas et al (2008) study used to generate the equation in this study.

Figure 3.37 shows observed mean temperatures from Shahnas et al. (2008) spherical data versus predicted mean temperatures for both the equation (3.19) and equation (3.13). The average error for equation (3.19) is 1.7% with standard deviation of 1.7%, while the average error from equation (3.13) is only 0.88% with a standard deviation of 1.1%.

Thus, the equation generated in this study is better able to predict the mean temperatures from Shahnas et al. (2008) for spherical shells with $f = 0.547$. The equation (3.19) from Deschamps et al. (2010) is also generated using various $f$ value spherical shell data. As has been previously discussed (e.g., Shahnas et al., 2008), it is difficult to find a single predictive equation for all $f$ valued spherical shells. Equation (3.13) was able to predict mean temperatures in spherical shells with an Earth-like $f$ value of 0.547 better than this new form of the equation derived by Deschamps et al. (2010). However, the equation from this study is not able to predict mean temperatures in smaller $f$ value spheres as well as
the equation from Deschamps et al. (2010). Finally, since one intention of this study is to use plane-layer models to emulate spherical shell models, it is necessary to test equation (3.19) for plane-layer data.

Figure 3.38 shows observed mean temperatures from this plane-layer study versus temperatures predicted with Deschamps et al. (2010) equation (3.19) and the equation (3.13) derived in this study. Using equation (3.19) to predict mean temperatures in a plane-layer ($f = 1$), the average error is 3.3% with a standard deviation of 2.7%, compared with equation (3.13) which has an average error of only 1.3% with a standard deviation of 1.7%.

Figure 3.38: Observed $2 \times 2 \times 1$ plane-layer data from this study versus predicted mean temperatures using Deschamps et al. (2010) equation (3.19) and equation (3.13) developed in this study.

Figure 3.39 shows curves of predicted mean temperature $\theta$ for Deschamps et al. (2010) equation (3.19) and the equation (3.13) derived in this study with the plane-layer observed temperatures for $Ra = 10^5$. It shows that both the equation from Deschamps et al. (2010) and the equation derived in this study are very similar for low Rayleigh numbers, with similar disagreement in both cases between observed and predicted temperatures for a given $H$ value.

Figure 3.40 shows curves of predicted mean temperature $\theta$ for Deschamps et al. (2010) equation (3.19) and the equation (3.13) derived in this study with the plane-layer observed temperatures for $Ra = 10^7$. Figure 3.40 shows that the equation determined in this study is best used for high Rayleigh number plane-layer calculations. Since one purpose of this study is to emulate spherical shell results in a plane-layer model, it is necessary for the predictive equation to reduce to an appropriate plane-layer case. Thus equation (3.13) from Chapter 3 remains the best equation for the purpose of predicting
Figure 3.39: Curves of predicted mean temperature $\theta$ in $H - \theta$ space using Deschamps et al. (2010) equation (3.19) and equation (3.13) developed in this study. The observed $2 \times 2 \times 1$ plane-layer data from this study for $Ra = 10^5$ is plotted as well.
Figure 3.40: Curves of predicted mean temperature $\theta$ in $H - \theta$ space using Deschamps et al. (2010) equation (3.19) and equation (3.13) developed in this study. The observed $2 \times 2 \times 1$ plane-layer data from this study for $Ra = 10^7$ is plotted as well.
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spherical shell mean temperatures and emulating the thermal structure of spherical shell results in a plane-layer calculation.

This Section 3.11 has described alternative parameterizations for both surface heat flux and mean temperature. The parameterizations for surface heat flux described by Moore (2008) was able to improve the predictive misfit for the plane-layer data.

The parameterization for mean temperature determined by Deschamps et al. (2010) performed as well as the equation (3.13) determined in this study for spherical shell data with \( f \sim 0.55 \). The Deschamps et al. (2010) parameterization was better able to predict mean temperatures in small \( f \) valued spheres, as this data was used in their inversion to determine their predictive equation. As one goal of this study was to emulate spherical shell temperatures in plane-layer models, it is important that the predictive equation used can be reduced to the \( f = 1 \) case. The equation (3.13) determined in this study had reduced misfit between observed and predicted temperatures in plane-layer models compared with the equation (3.19) from Deschamps et al. (2010).

3.12 Depth-dependent heating/cooling

In addition to matching mean temperatures in spherical shells, it is worth noting the similarity (or difference) between planforms of convection and temperature profiles in the different geometries. Vangelov and Jarvis (1994) published mean horizontal temperature as a function of depth for 2D axi-symmetric spherical shell, cylindrical annulus and planar convection models which demonstrate the difference in mean temperature between the various geometries (see Figure 3.41).

As discussed above, one of the goals of this study is to emulate spherical shell thermal properties in plane-layer convection models. Internal cooling was employed in plane-layer calculations to obtain the mean temperatures observed (and predicted) in spherical shells with modest internal heating rates or high Rayleigh numbers. While internal cooling was able to lower the mean temperature in plane-layer models, it also altered the shape of the temperature profiles. In an attempt to reconcile the shape of the temperature profiles in the spherical shell and plane-layer geometries, the effect of stratified cooling with depth is examined.

Different heating profiles can be specified in an attempt to modify the plane-layer temperature profiles and improve their agreement with the spherical shell temperatures profiles. A depth-dependent heating rate is considered here to simulate the reduced volume with depth in a spherical shell compared with a plane-layer (e.g., Nettelfield, 2005). Figure 3.42 shows a schematic of the differing volumes of layers with the same thickness, \( z_o - z_i \), in different geometries with the same surface area, \( A \). At a given depth, the volume of a spherical shell layer differs from the volume of a plane layer. To compensate for this geometric difference in the heat input, a depth-dependent function for \( H \) can be specified.

For example, the heating profile in a spherical shell will vary with \( r^2 \) and a possible heating profile in a plane-layer is \( H(z) = H_c(r_i + z)^2/r_o^2 \) with \( H_c \) corresponding to the heat sinks in the plane-layer. The source concentration decreases with depth to account for the difference in total layer volume at a given depth in the spherical and plane-layer models. The depth-dependent profile has a mean \( H \) given by \( H_c \frac{1}{3r^2} (r_o^3 - r_i^3) \) where the mean \( H \) is given by the predictive equation (3.13), for the given \( r_i \) and \( r_o \) of the spherical shell being emulated (see Appendix E for details).

It is expected that using the depth-dependent \( H \) profile described above with equation (3.13) should improve the success of emulating spherical shell convection with an Earth-like \( f \) value of 0.547. It was
Figure 3.41: Horizontally averaged temperature profiles from 2D axi-symmetric spherical shell, cylindrical and plane-layer convection models. In a, $f = 0.5$ for the axi-symmetric spherical shell and cylindrical model; in b, $f = 0.5$ for the axi-symmetric spherical shell but only $f = 0.25$ for the cylindrical model. Reprinted from Vangelov and Jarvis (1994) with permission of the publisher John Wiley & Sons, Inc.
Figure 3.42: Differing volumes of layers of thickness $z_o - z_i$ in spherical shell and plane-layer geometries with surface area $A$. 
noted above that a non-internally heated sphere with \( f = 0.547 \) has a mean temperature of 0.213. In order to emulate this temperature in a plane-layer geometry it is necessary to employ some sort of internal cooling. In section 3.7 (see Figure 3.20), an example was presented to emulate a spherical shell with \( Ra = 10^6 \) and \( H = 0 \), which necessitated specifying \( H = -13 \) in the plane-layer geometry. Using the depth-dependent \( H \) profile described above (\( H_c = -21 \)), a mean temperature of 0.207 is observed in a \( 2 \times 2 \times 1 \) plane-layer calculation.

Figure 3.43 shows a temperature field snapshot for this \( 2 \times 2 \times 1 \) solution domain plane-layer model with \( Ra = 10^6 \), a mean value of \( H = -13 \) and a mean temperature of 0.207. Figure 3.44 shows the temperature profiles for plane-layer systems with an average internal cooling rate of \( H = -13 \), one curve (red) is from a calculation with constant \( H \) and the other profile is from a calculation with the depth-dependent \( H \) (\( H_c = -21 \)). The horizontal temperature profile from the plane-layer model with depth-dependent \( H \) (see Figure 3.44) shows only a small improvement in attempting to emulate the spherical shell profile. Namely, the higher temperature fluid at the top of the system did not penetrate as deeply as in the uniform cooling case. The mean temperature in both cases is quite similar with a value of 0.211 for the constant cooling (\( H = -13 \)) and 0.207 for the depth-dependent cooling, however, the uniform cooling model is better able to match the spherical shell temperature than the depth-dependent cooling model.

3.12.1 Emulating spherical shell profiles with different depth-dependent \( H \)

In general, the depth-dependent cooling profile described above is not able to diminish the accumulation of hot fluid below the upper boundary layer, a feature not seen in spherical shells to the same degree. This led to exploration of a simple linear fit for \( H \) with depth. By varying \( H \) linearly with depth, it is possible to analyze the affect of varying the gradient of internal cooling and heating sources.

The gradient, \( m \), of a linear function \( H(z) = m(2z-1)/2 \) is varied from 50 to -50 to test the sensitivity of the temperature profile to linear variations in \( H \). Figure 3.45 shows the gradients of \( H(z) \) and the snapshots of horizontally averaged temperature profiles as a function of height for various gradient values in a plane-layer model with \( Ra = 10^6 \) and an average value of \( H = 0 \). It was only when the slope reached a value of 20 or higher that there was a significant change in the temperature profile from the isothermal profile obtained between the upper and lower thermal boundary layers in the \( m = 0 \) case. Regardless of the value of the negative slope, there was no change in the temperature profile. Only positive slopes, have values which alter the shape of the temperature profile without altering the mean temperature of the system.

Figure 3.46 shows isosurfaces of constant temperature for two plane-layer models with \( Ra = 10^6 \), one with a negative gradient of \( H \) and one with a positive gradient, both with values of 20. As the negative gradient \( H(z) \) values had no effect on the temperature profiles, they also had no effect on the planforms of convection. Figure 3.46a shows that even with a slope of -20, the planform remains sheet-like convection rolls. However, with a positive slope of 20, the planform of convection is no longer sheet-like but instead features time-dependent upwellings and downwellings similar to spherical shell models (as in Figure 5.1).

The value of \( H \) can also be shifted to emulate a spherical shell with no internal heating. A plane-layer model with \( Ra = 10^6 \) and an average internal cooling rate of \( H = -13 \) is necessary to obtain a temperature of 0.213 (the temperature in a spherical shell with \( H = 0 \) and \( f = 0.547 \) as determined above in Section 3.7). Figure 3.47 shows snapshots of the horizontally averaged temperature with height for different gradients of \( H(z) \) and a net cooling rate of \( H = -13 \) with \( Ra = 10^6 \).
Figure 3.43: Temperature isosurfaces for a $2 \times 2 \times 1$ plane-layer model with a mean internal cooling rate of $H = -13$ implemented with a depth-dependent $H$ profile (with $H_c = -21$). $Ra = 10^6$ and the mean temperature is 0.207. The blue isosurface is at a temperature of 0.1 and the green isosurface at a temperature of 0.3. The grid resolution is $144 \times 144 \times 96$. 

$Ra = 10^6$ $\theta = 0.207$
Figure 3.44: Average temperature with depth for $Ra = 10^6$ with constant (red) and $r^2$ depth dependent (green) cooling rates with an average value of $H = -13$. The solution domain geometry is $2 \times 2 \times 1$ and the grid resolution is $144 \times 144 \times 96$. 
Figure 3.45: (a). Profiles of $H(z)$ for different linear gradients of the heating source distribution in the mantle. (b). Snapshots of horizontally averaged temperature with height in the mantle for each of the corresponding gradients of $H(z)$.
Figure 3.46: Temperature isosurfaces for $2 \times 2 \times 1$ plane-layer models with $Ra = 10^6$. The left frames features a model with a gradient of -20 for the $H(z)$ profile and right frames feature a model with a gradient of 20. The mean temperature in both systems is 0.5 and the average $H$ value is 0. The dark green isosurface is 0.4 and the lighter green isosurface is at a nondimensional temperature of 0.6.

Figure 3.47: Snapshots of horizontally averaged temperature with height in the mantle for different gradients of $H(z)$ all shifted to a mean of -13.
The different gradients of $H$ with a mean cooling rate of $H = -13$ alter the shape of the temperature profile. The planform of convection remains similar, with all planforms having columnar upwellings and downwellings. These different gradients of $H$ were able to alter the temperature profiles of the system while keeping the same planform of convection in each case. However, the mean temperature value of the system did not stay constant with different gradients. Figure 3.48 shows plots of volume average temperature against time for the different gradients in Figure 3.47.

Figure 3.48: Volumed averaged mean temperature as a function of time for different gradients of $H(z)$ all shifted to a mean of -13.

In summary, the gradient required for $H(z)$ in order to implement change in planforms of convection and temperature profiles was much larger than expected when the mean $H$ value is 0. There is no physical or geometrical argument for such a large gradient in $H$. When the gradient is altered for a nonzero average $H$, the mean temperature of the system also changes and the equations derived in previous chapters cannot be used.

While attempting to adjust the internal heating or cooling rates at different depths might address accounting for the different volume of heat generating sources at different layers in a sphere, it does not account for the larger basal surface area in the plane-layer models, compared with the spherical shell. By having the highest cooling rate in the layer directly above the base of the mantle in the plane-layer calculations (featuring linear profiles with positive $m$), more heat is drawn from the lower boundary, which offsets the cooling and increases the temperature of the plane-layer system. While the use of a uniform cooling rate appears less refined, it allows for similar or improved results compared to using depth-dependent cooling rates. In particular, high Rayleigh number vigorous convection has a more uniform temperature between the boundary layers thus allowing the uniform cooling to produce a good match for emulating spherical shell temperatures in plane-layer convection systems.
Table 3.5: Summary of isoviscous convection experiments and results for $Ra = 10^5$

<table>
<thead>
<tr>
<th>Model</th>
<th>$H$</th>
<th>Resolution</th>
<th>Solution Domain</th>
<th>$\theta_{obs}$</th>
<th>$\theta_{pre}$</th>
<th>$% error^*$</th>
<th>$% error^*$</th>
<th>$Q_{top}$</th>
<th>$Q_{top}/Q_{bot}$</th>
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<td>48 x 48 x 64</td>
<td>1 x 1 x 1</td>
<td>0.094</td>
<td>-0.017</td>
<td>7.591</td>
<td>16.074</td>
<td>1.71</td>
<td>-15.00</td>
</tr>
<tr>
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<td>-15</td>
<td>96 x 96 x 64</td>
<td>2 x 2 x 1</td>
<td>0.094</td>
<td>-0.017</td>
<td>7.591</td>
<td>16.074</td>
<td>1.71</td>
<td>-15.00</td>
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<td>144 x 144 x 64</td>
<td>3 x 3 x 1</td>
<td>-0.001</td>
<td>-0.017</td>
<td>1.811</td>
<td>3.684</td>
<td>1.97</td>
<td>-14.96</td>
</tr>
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<td>H1-1A4</td>
<td>-15</td>
<td>240 x 240 x 64</td>
<td>5 x 5 x 1</td>
<td>-0.002</td>
<td>-0.017</td>
<td>1.854</td>
<td>3.771</td>
<td>1.95</td>
<td>-15.01</td>
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<td>360 x 360 x 64</td>
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<td>-0.017</td>
<td>1.854</td>
<td>3.771</td>
<td>1.95</td>
<td>-15.01</td>
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<td>-0.017</td>
<td>1.854</td>
<td>3.771</td>
<td>1.95</td>
<td>-15.01</td>
</tr>
</tbody>
</table>

$s$ indicates steady state solution temperatures

Table 3.6: Summary of isoviscous convection experiments and results for $Ra = 10^6$

<table>
<thead>
<tr>
<th>Model</th>
<th>$H$</th>
<th>Resolution</th>
<th>Solution Domain</th>
<th>$\theta_{obs}$</th>
<th>$\theta_{pre}$</th>
<th>$% error^*$</th>
<th>$% error^*$</th>
<th>$Q_{top}$</th>
<th>$Q_{top}/Q_{bot}$</th>
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<td>0.631</td>
<td>11.15</td>
<td>-15.05</td>
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<td>-15</td>
<td>144 x 144 x 96</td>
<td>2 x 2 x 1</td>
<td>0.183</td>
<td>0.183</td>
<td>0.546</td>
<td>0.315</td>
<td>10.65</td>
<td>-15.02</td>
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<tr>
<td>H1-1A3</td>
<td>-15</td>
<td>216 x 216 x 96</td>
<td>3 x 3 x 1</td>
<td>0.183</td>
<td>0.183</td>
<td>0.546</td>
<td>0.315</td>
<td>10.65</td>
<td>-14.99</td>
</tr>
<tr>
<td>H1-1A4</td>
<td>-15</td>
<td>360 x 360 x 96</td>
<td>5 x 5 x 1</td>
<td>0.183</td>
<td>0.183</td>
<td>0.546</td>
<td>0.315</td>
<td>10.65</td>
<td>-14.76</td>
</tr>
<tr>
<td>H1-1A5</td>
<td>-15</td>
<td>540 x 540 x 96</td>
<td>10 x 10 x 1</td>
<td>0.183</td>
<td>0.183</td>
<td>0.546</td>
<td>0.315</td>
<td>10.65</td>
<td>-14.99</td>
</tr>
<tr>
<td>H1-1A6</td>
<td>-15</td>
<td>720 x 720 x 96</td>
<td>15 x 15 x 1</td>
<td>0.183</td>
<td>0.183</td>
<td>0.546</td>
<td>0.315</td>
<td>10.65</td>
<td>-14.99</td>
</tr>
</tbody>
</table>

$s$ indicates steady state solution temperatures
Table 3.7: Summary of isoviscous convection experiments and results for $Ra \geq 10^7$

<table>
<thead>
<tr>
<th>Model</th>
<th>$H$</th>
<th>Resolution</th>
<th>Solution Domain</th>
<th>$\theta_{obs}$</th>
<th>$\theta_{pre}$</th>
<th>% error $^a$</th>
<th>% error $^b$</th>
<th>$Q_{top}$</th>
<th>$Q_{top} - Q_{bot}$</th>
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<tbody>
<tr>
<td>Ra/c7H+15A1</td>
<td>-15</td>
<td>128 x 128 x 144</td>
<td>1 x 1 x 1</td>
<td>0.366</td>
<td>0.292</td>
<td>1.158</td>
<td>3.942</td>
<td>28.88</td>
<td>-15.00</td>
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<tr>
<td>Ra/c7H+15A2</td>
<td>-15</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.299</td>
<td>0.292</td>
<td>0.946</td>
<td>3.221</td>
<td>28.66</td>
<td>-14.38</td>
</tr>
<tr>
<td>Ra/c7H+15A3</td>
<td>-15</td>
<td>384 x 384 x 144</td>
<td>3 x 3 x 1</td>
<td>0.357</td>
<td>0.292</td>
<td>9.138</td>
<td>31.106</td>
<td>31.34</td>
<td>-9.85</td>
</tr>
<tr>
<td>Ra/c7H+10A1</td>
<td>-10</td>
<td>128 x 128 x 144</td>
<td>1 x 1 x 1</td>
<td>0.344</td>
<td>0.347</td>
<td>0.521</td>
<td>2.222</td>
<td>31.7</td>
<td>-9.94</td>
</tr>
<tr>
<td>Ra/c7H+10A2</td>
<td>-10</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.343</td>
<td>0.347</td>
<td>0.567</td>
<td>2.418</td>
<td>30.97</td>
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<tr>
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<td>-10</td>
<td>384 x 384 x 144</td>
<td>3 x 3 x 1</td>
<td>0.354</td>
<td>0.347</td>
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<tr>
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<td>-10</td>
<td>640 x 640 x 144</td>
<td>5 x 5 x 1</td>
<td>0.344</td>
<td>0.347</td>
<td>0.475</td>
<td>2.026</td>
<td>31.02</td>
<td>-10.24</td>
</tr>
<tr>
<td>Ra/c7H-5A1</td>
<td>-5</td>
<td>128 x 128 x 144</td>
<td>1 x 1 x 1</td>
<td>0.394</td>
<td>0.409</td>
<td>2.623</td>
<td>17.033</td>
<td>36.19</td>
<td>-5.07</td>
</tr>
<tr>
<td>Ra/c7H-5A2</td>
<td>-5</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.401</td>
<td>0.409</td>
<td>1.421</td>
<td>9.231</td>
<td>35.07</td>
<td>-4.71</td>
</tr>
<tr>
<td>Ra/c7H-5A3</td>
<td>-5</td>
<td>384 x 384 x 144</td>
<td>3 x 3 x 1</td>
<td>0.414</td>
<td>0.409</td>
<td>0.914</td>
<td>5.934</td>
<td>35.39</td>
<td>-4.25</td>
</tr>
<tr>
<td>Ra/c7H-0.89A2</td>
<td>-0.89</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.475</td>
<td>0.475</td>
<td>0.000</td>
<td>0.000</td>
<td>40.79</td>
<td>-0.94</td>
</tr>
<tr>
<td>Ra/c7H+5A1</td>
<td>5</td>
<td>128 x 128 x 144</td>
<td>1 x 1 x 1</td>
<td>0.599</td>
<td>0.591</td>
<td>1.337</td>
<td>8.681</td>
<td>42.08</td>
<td>4.42</td>
</tr>
<tr>
<td>Ra/c7H+5A2</td>
<td>5</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.593</td>
<td>0.591</td>
<td>0.271</td>
<td>1.758</td>
<td>39.79</td>
<td>4.99</td>
</tr>
<tr>
<td>Ra/c7H+5A3</td>
<td>5</td>
<td>384 x 384 x 144</td>
<td>3 x 3 x 1</td>
<td>0.594</td>
<td>0.591</td>
<td>0.491</td>
<td>3.187</td>
<td>39.90</td>
<td>4.45</td>
</tr>
<tr>
<td>Ra/c7H+10A1</td>
<td>10</td>
<td>128 x 128 x 144</td>
<td>1 x 1 x 1</td>
<td>0.649</td>
<td>0.653</td>
<td>0.567</td>
<td>2.418</td>
<td>41.84</td>
<td>10.04</td>
</tr>
<tr>
<td>Ra/c7H+10A2</td>
<td>10</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.652</td>
<td>0.653</td>
<td>0.199</td>
<td>0.850</td>
<td>41.04</td>
<td>10.05</td>
</tr>
<tr>
<td>Ra/c7H+10A3</td>
<td>10</td>
<td>384 x 384 x 144</td>
<td>3 x 3 x 1</td>
<td>0.650</td>
<td>0.653</td>
<td>0.398</td>
<td>1.699</td>
<td>40.97</td>
<td>10.00</td>
</tr>
<tr>
<td>Ra/c7H+10A5</td>
<td>10</td>
<td>640 x 640 x 144</td>
<td>5 x 5 x 1</td>
<td>0.637</td>
<td>0.653</td>
<td>2.466</td>
<td>10.523</td>
<td>40.68</td>
<td>8.88</td>
</tr>
<tr>
<td>Ra/c7H+15A1</td>
<td>15</td>
<td>128 x 128 x 144</td>
<td>1 x 1 x 1</td>
<td>0.697</td>
<td>0.708</td>
<td>1.582</td>
<td>5.385</td>
<td>43.77</td>
<td>14.92</td>
</tr>
<tr>
<td>Ra/c7H+15A2</td>
<td>15</td>
<td>256 x 256 x 144</td>
<td>2 x 2 x 1</td>
<td>0.694</td>
<td>0.708</td>
<td>1.921</td>
<td>6.538</td>
<td>43.37</td>
<td>14.97</td>
</tr>
<tr>
<td>Ra/c7H+15A3</td>
<td>15</td>
<td>384 x 384 x 144</td>
<td>3 x 3 x 1</td>
<td>0.697</td>
<td>0.708</td>
<td>1.596</td>
<td>5.433</td>
<td>43.24</td>
<td>14.86</td>
</tr>
<tr>
<td>Ra/c8H-5.22A2</td>
<td>-5.22</td>
<td>512 x 512 x 288</td>
<td>2 x 2 x 1</td>
<td>0.443</td>
<td>0.438</td>
<td>1.050</td>
<td>7.355</td>
<td>76.77</td>
<td>-5.04</td>
</tr>
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</table>
Chapter 4

Results: Stratified Viscosity Mantle Convection Study


4.1 Effect of stratified viscosity

The isoviscous study presented in Chapter 3 and O’Farrell and Lowman (2010) neglected many important properties of planetary mantles, including some which are known to affect the mean temperature of the system. Previous studies have shown the effect of viscosity stratification on mantle mean temperature (e.g., Gurnis and Davies, 1986a,b; Hansen et al., 1993; Bunge et al., 1996; Bunge et al., 1997; Lenardic et al., 2006; Höink and Lenardic, 2008). This next section considers the influence of a stratified mantle viscosity on the thermal structure of mantle convection in spherical shell and plane-layer models. Models with varying viscosity profiles have different effective Rayleigh numbers compared to systems with constant viscosity throughout. Thus it is necessary to consider the effective Rayleigh number based on the average mantle viscosity when comparing systems. In addition, the same viscosity profile specified between \( r_i \) and \( r_o \) will generate different mean viscosity values in geometries with different \( f \) values (where \( f \) is the ratio of inner radius to outer radius) and \( r_o - r_i = 1.0 \). In this study a Rayleigh number definition for effective Rayleigh number is defined to account for these differences in mean system viscosity.

For the same effective Rayleigh number, isoviscous convection models feature higher mean temperatures than are found in models with a stratified viscosity that increases with depth. Stratified viscosity with an increased viscosity in the lower mantle leads to a thickening of the lower thermal boundary layer as the deep mantle is more viscous. Thickening of the thermal boundary layer, arising from increased stability, leads to a decrease in the mean heat flux across the lower boundary of the system. Thus the fluid between the boundary layers receives a reduced heat flux from below compared to the heat flux that would occur through a lower viscosity boundary layer. However, a lower viscosity upper thermal
boundary layer provides little resistance to the removal of the heat arriving from the base. This difference in resistance to heat transfer across the thermal boundary layers leads to a lower mean temperature in systems featuring a stratified viscosity.

A viscosity profile motivated by a simple approximation of the Earth’s radial viscosity variation is used in this study (see Figure 2.4 in Section 2.2.1). The stratified viscosity decreases the mean temperature in both spherical shell and plane-layer convection systems since both systems experience an increase in lower thermal boundary thickness and a corresponding heat flux reduction through the bottom boundary layer (e.g., Bunge et al., 1996).

This chapter explores the effect of Rayleigh number, nondimensional internal heating rate and different stratified viscosity profiles in both spherical shell and three-dimensional plane-layer convection models. Equations for predicting mean temperature in both geometries are determined using effective Rayleigh number and nondimensionalized internal heating rate. Using a fixed effective Rayleigh number, it is possible to combine the system of equations for different geometries to obtain similar thermal structure by varying the internal heating rate in the plane-layer geometry based on the predictions from the spherical shell equation. As in the case with isoviscous convection, this system of equations is able to predict and emulate the spherical shell temperatures in plane-layer geometry models.

This study begins by analyzing results from Table 4.1 (at the end of this chapter) for the spherical shell model TERRA and continues with the analysis of the plane-layer model (results listed in Tables 4.2 - 4.7). The tables list volumetric time averages for mean temperature, surface heat flux and nondimensionalized internal heating rate. Temperature averages are taken when the system reaches a statistically steady state (i.e., exhibiting neither heating nor cooling trends, which is verified by the value $q_{\text{diff}}$ in the Table of spherical geometry calculations and $q_{\text{top}} - q_{\text{bot}}$ in the plane-layer calculation tables (a perfectly steady-state would have $q_{\text{diff}} = H$, or $q_{\text{top}} - q_{\text{bot}} = H$). In the plane-layer cases, the difference in heat flux is almost exactly the internal heating prescribed. The value of $q_{\text{diff}}$ in the spherical table is given by $q_{\text{diff}} = 3(q_{\text{top}} - q_{\text{bot}}f^2)/(1 + f + f^2)$ (and derived in Appendix C), and shows good agreement with the prescribed internal heating rate $H$. The spherical shell calculations exhibit much more time-dependence and variation in temperature than the plane-layer models, which is reflected in the small differences between $q_{\text{diff}}$ and $H$ in the latter.

4.2 Spherical Shell Calculations

4.2.1 Effect of Rayleigh number and nondimensional internal heating rate on mean temperature and planforms of convection

This study begins by examining the effect of internal heating rate and Rayleigh number on the mean temperature and planforms of convection in spherical shell systems. All spherical shells considered here feature a ratio of inner radius to outer radius of $f = 0.547$ which is Earth-like (e.g., Gutenberg, 1913; Jeffreys, 1939). Table 4.1 includes results from all spherical shell calculations ($f = 0.547$) with different $H$ values. Figure 4.1 shows snapshots of the temperature fields from nine calculations with Rayleigh number increasing towards the bottom and internal heating rate increasing towards the right. A first apparent difference with the isoviscous case is that the mean temperature varies with Rayleigh number in the case with no internal heating ($H = 0$). This slight change in mean temperature with Rayleigh number is more apparent in Figure 4.2 which shows horizontally average temperature with height for
Chapter 4. Results: Stratified Viscosity Mantle Convection Study

Ra
6
=10
Ra
5
=6\times10
Ra
7
=10
1.0
0.0
0.5
Ra
H=0
H=25
\theta_{\text{obs}}=0.216
\theta_{\text{obs}}=0.356
\theta_{\text{obs}}=0.099
\theta_{\text{obs}}=0.098
\theta_{\text{obs}}=0.377
\theta_{\text{obs}}=0.585
\theta_{\text{obs}}=0.660
\theta_{\text{obs}}=0.076
H
H=10
\theta_{\text{obs}}=0.405

Figure 4.1: Pairs of diametrically opposed views of temperature field isosurface snapshots from spherical shell geometry calculations with \( f = 0.547 \). Rayleigh number increases downward and internal heating rate, \( H \), increases towards the right. The isosurfaces are constant temperatures of 0.1 above the mean temperature for \( H = 0 \), and 0.2 above the mean temperature for all \( H > 0 \). \( Ra = 6 \times 10^5 \) is equivalent to \( Ro_\eta = 3.57 \times 10^4 \); \( Ra = 10^6 \) is equivalent to \( Ro_\eta = 5.95 \times 10^4 \) and \( Ra = 10^7 \) is equivalent to \( Ro_\eta = 5.95 \times 10^5 \).

spherical shells with \( H = 0 \) and increasing \( Ra \). In general, Figure 4.1 shows that the mean temperature of the models decreases with increasing Rayleigh number for fixed internal heating rates (\( H > 0 \)). Distinct planform regimes are found for different combinations of the Rayleigh number and \( H \). For a fixed \( H \) value, different Rayleigh numbers develop different characteristic number of plumes. In general, for the parameter ranges explored, the number of plumes increases with increasing effective Rayleigh number for a given \( H \) value.

Figure 4.3 plots different convective planforms for each combination of Rayleigh number and internal heating rate examined in this study. In general for all cases examined, once the system reaches a statistically steady state (with no long term increasing or decreasing in mean temperature) there is a fixed number of plumes for low internal heating rates and Rayleigh numbers. For low Rayleigh numbers (\( Ro_\eta \leq 4.76 \times 10^4 \), or \( Ra \leq 8 \times 10^5 \)) with no internal heating (\( H = 0 \)) the models exhibit tetrahedral planforms with four strong upwellings which remain positionally steady. When the Rayleigh number is increased to \( Ro_\eta = 5.95 \times 10^4 \) (\( Ra = 10^6 \)) the planform transitions from the four plumes to five plumes. In this case, the planform is often two plumes merged into a single large upwelling on one side of the sphere (as in Figure 4.1, \( Ra = 10^6, H = 0 \)) but the large plume can temporarily break apart into its two
Figure 4.2: Temporally averaged profiles of laterally averaged mean temperature as function of height above the core for a fixed internal heating rate $H = 0$ and varying Rayleigh number $Ra = 4 \times 10^5, 5 \times 10^5, 6 \times 10^5, 8 \times 10^5, 10^6, 3 \times 10^6, 6 \times 10^6, 10^7$
smaller plumes. These cases are listed in transitional states (i.e., 4-5 plumes) and feature a time varying number of plumes. With increased Rayleigh number there is an increase in the number of upwellings in cases with moderate to high internal heating rates ($H \geq 5$). In general for all Rayleigh numbers and with $H \geq 20$ the planforms develop a time-dependent number of upwellings (see black and red circles in Figure 4.3).

Figure 4.4 shows temporally averaged mean lateral temperature as a function of height in the system (geotherm) for a fixed Rayleigh number $Ra_\eta = 3 \times 10^5$ and varying internal heating rates.

As $H$ increases, the mean temperature of the system increases and the temperature gradient between the thermal boundary layers increases. Figure 4.4 also shows a larger increase in mean temperature between no internal heating and $H = 5$ than between any other increase in internal heating rate by 5.

Increasing the Rayleigh number decreases the mean temperature of the system for a fixed internal heating rate (see Figure 4.5). The fluid interior becomes more isothermal and convective vigour increases in these systems. Figure 4.5 also shows a change in geotherm profile between $Ra = 10^6$ and $Ra = 3 \times 10^6$. This corresponds to the change in planform from the transitional planform of 4, sometimes 5, plumes for $Ra = 10^6$ to exactly 5 steady plumes for $Ra = 3 \times 10^6$. 

---

**Figure 4.3:** Convective planform for different Rayleigh numbers and internal heating rates. Solid green triangles represent tetrahedral planforms; unfilled blue diamonds are 4-5 plume transitioning planforms; solid cyan squares are 5 and 6 plume planforms; half filled red circles are planforms with generally 7 plumes (but as few as 5 at times) and solid black circles represent time-dependent variable number of plumes. The grey triangle represents a transitional planform that alternates between 2 and 3 plumes. $Ra_\eta \times 16.80 = Ra$ for all spherical shell geometry calculations.
Figure 4.4: Temporally averaged profiles of laterally averaged mean temperature as a function of height above the core for a fixed Rayleigh number $Ra = 5 \times 10^6$ or $Ra_T = 2.98 \times 10^5$, $H$ varies from 0, 5, 10, 15, 20, 25
Figure 4.5: Temporally averaged profiles of laterally averaged mean temperature as function of height above the core for a fixed internal heating rate $H = 10$ and varying Rayleigh number $Ra = 5 \times 10^5, 8 \times 10^5, 10^6, 3 \times 10^6, 5 \times 10^6, 6 \times 10^6, 10^7$.
Figure 4.6: Temperature fields from three isoviscous spherical shell calculations. Rayleigh number varies from top to bottom, $5 \times 10^5$, $10^6$, $10^7$, $f = 0.547$ and $H = 23.532$ in all three models. The orange isosurface is a nondimensional temperature of 0.77 and the blue isosurface is 0.14. Reprinted from Shahnas et al. (2008) with permission from Elsevier.

There are different degrees of time-dependence and mean temperature variation in the stratified viscosity models, especially in comparison with the isoviscous case (discussed in Chapter 3 and Shahnas et al., 2008, Deschamps et al., 2010). With the increased lower mantle viscosity, the model planforms are more steady than in the isoviscous case with the same $Ra$ although the temperatures remain unsteady in time. Comparing cases with the same effective Rayleigh number $Ra_\eta$, the isoviscous models have many more plumes than the stratified viscosity models. The calculation with $Ra_\eta = 6 \times 10^5$, or $Ra = 10^7$ and $H = 25$ in Figure 4.1 shows only 6-7 plumes while the isoviscous case with a similar effective Rayleigh number has several time this many plumes (see top sphere in Figure 4.6). This observed transition from shorter wavelength to longer wavelength convection resulting from the introduction of a high viscosity lower mantle agrees with the findings of previous studies (e.g., Gurnis and Davies, 1986a; Zhang and Yuen, 1995; Bunge et al., 1996; Bunge et al., 1997; Lowman et al., 2001).

In the isoviscous case with no internal heating, the mean temperature of the system is only dependent
on the curvature factor, \( f \) (e.g., Jarvis et al., 1995; Shahnas et al., 2008; Deschamps et al., 2010; See also section 1.5.3 in Chapter 1). Jarvis et al. (1995) observed changes in planform for different \( f \) values in isoviscous convection which likely contributed to the difference in mean temperatures for a given \( f \) value. These authors, however, did not account for planform change in their derivation of a parametrization for mean temperature \( \theta_b \), in spherical shells heated only by an isothermal basal boundary. (See Boundary Layer Theory Section 1.6 for more details). With the stratified viscosity given in equation (2.37) and a fixed \( f \) value \( (f = 0.547) \), a weak dependence of mean temperature on Rayleigh number is observed (in this study) that may be explained by changes in planform. The results in Table 4.1 indicate a decrease in mean temperature for increasing Rayleigh number when \( H = 0 \) (or greater). In contrast, in an isoviscous model, when \( H = 0 \), the mean temperature for a spherical shell with \( f = 0.547 \) is approximately 0.22 for all of the effective Rayleigh numbers considered in this study.

With stratified viscosity and a fixed planform (four plumes), at a given heating rate \( (H = 10) \) the mean temperature of the system decreases with increasing Rayleigh number (see Figure 4.1). The planform change is accompanied by an increase in the number of upwelling plumes present. For a fixed internal heating rate, the planform change corresponds to a decrease in mean temperature of the system as more heat can be removed from the system by the increased number of upwellings which is associated with the increase in Rayleigh number.

### 4.2.2 Deriving a predictive equation for the mean temperature in a spherical shell

Shahnas et al. (2008) previously examined isoviscous convection in a spherical shell with variable \( f \) ranging from 0.3 to 1.0 and derived a predictive equation for the average temperature of a convecting fluid based on the radii ratio, \( f \), Rayleigh number, \( Ra \), and nondimensional internal heating rate, \( H \). Their equation took the form:

\[
\theta = \frac{f^{1/2}}{1 + f^{-3/2}} + (\sigma + \lambda Ra^\delta (1 - f))((1 + f + f^2)/3)^{3/4} \frac{H^\beta}{Ra^\gamma}, \tag{4.1}
\]

where \( \sigma, \lambda, \delta, \beta \) and \( \gamma \) are found by inverting the observations from a large number of spherical shell geometry calculations. Shahnas et al. (2008) found the maximum misfit between observed and predicted mean temperatures to be 2.06\% when \( f = 0.547 \). They also found that with \( f = 0.547 \) transitions between planforms at low \( H \) values occur abruptly and at larger \( H \) values planforms are less varied, leading to greater agreement between observations and predictions for larger \( H \). The derivation of equation (4.1) is given in the Boundary Layer Theory Section 1.6.

The results of this stratified viscosity study do not follow the same trends observed in the isoviscous case. Shahnas et al. (2008) and Deschamps et al. (2010) found that in an isoviscous spherical shell model with no internal heating, the mean temperature of the system was not dependent on the Rayleigh number but only on the curvature \( f \). With the addition of the lower mantle viscosity increase given by equation (2.37), the mean temperature of a system with no internal heating is now weakly dependent on Rayleigh number for fixed \( f = 0.547 \). Therefore, it is first necessary to determine the effect of \( Ra \) and \( H \) on mean temperature without the additional complexity of different core sizes (i.e., variable \( f \)). Here, the observations from 36 spherical shell calculations are fit to an equation of the form:
\[ \theta = \theta_b(f = 0.547) + \alpha(f = 0.547) \frac{H^{\beta'}}{(Ra_{\eta})^{\gamma'}} , \]  
\[ (4.2) \]

where \( \alpha(f = 0.547) \), \( \beta' \) and \( \gamma' \) are found by inverting the observations from TERRA (Table 4.1) when the mantle viscosity is given by equation (2.37).

In the absence of internal heat sources, the dependence of the mean temperature \( \theta_b \) on \( Ra_{\eta} \) may be explained by the fact that in the range of Rayleigh numbers investigated, the upper thermal boundary layer of the system is confined to an isoviscous region while the lower thermal boundary layer occupies a region with a radial viscosity gradient. Consequently, increasingly higher mean viscosity is obtained at the lower boundary as the boundary layer thins. Thus, as \( Ra_{\eta} \) increases, the development of instabilities at the lower boundary layer becomes inhibited relative to the development of instabilities at the top boundary. The result is that the efficiency of heat removal from the system increases with increasing Rayleigh number, faster than the efficiency of heat flow into the system. For high enough \( Ra_{\eta} \), the gradient in the viscosity across the bottom boundary will become inconsequential. This suggests that temperature decreasing with Rayleigh number will therefore likely be superseded by a different behaviour at sufficiently high Rayleigh number. (That is, the mean temperature will asymptotically approach some value greater than zero.)

A change in the power-law behaviour is also likely to occur. For example, for isoviscous Boussinesq convection, the exponents in the power-law dependence of the Nusselt number on the Rayleigh number changes so that there are transitions in the power-law behaviour in different Rayleigh number regimes. At low Rayleigh numbers, boundary layers are not present (e.g., Jarvis and Peltier, 1982; McKenzie et al., 1974), at higher \( Ra \) well developed boundary layers form (with isothermal fluid lying between the boundary layers), and eventually hard-turbulence appears (Ahlers et al., 2009; Chil`a and Schumacher, 2012). However, determination of the \( Ra_{\eta} \) at which this transition takes place is beyond the scope of this work.

In Figure 4.7a, observed mean temperature, \( \theta \), is plotted against Rayleigh number, \( Ra_{\eta} \) when \( H = 0 \). The contribution to the mean temperature arising solely from basal heating, \( \theta_b \), can be fit by a power law of the form \( \zeta(Ra_{\eta})^x \). Figure 4.7a shows mean temperature has a weak dependence on \( Ra_{\eta} \) with a slight decrease in mean temperature as \( Ra_{\eta} \) increases. Fitting the observations for \( H = 0 \) yields

\[ \theta_b(f = 0.547) = 0.266Ra_{\eta}^{-0.0936} . \]  
\[ (4.3) \]

Equation (4.3) can be used to estimate the mean temperature in a spherical shell with \( f = 0.547 \), no internal heating and the viscosity profile given by equation (2.37). Figure 4.7b shows the observed versus predicted temperatures for equation (4.3). The disagreement between the observed and predicted temperatures is sensitive to planform but the average error is 2.15\% and is no more than 5.45\% in any of the cases tested.

Employing equation (4.3), and solving for \( \alpha(f = 0.547) \) and the exponents on \( H \) and \( Ra_{\eta} \) in equation (4.2) by inverting all of the observations listed in Table 4.1 where \( H > 0 \) yields:

\[ \theta = 0.266Ra_{\eta}^{-0.0936} + 1.102 \frac{H^{0.665}}{Ra_{\eta}^{0.288}} . \]  
\[ (4.4) \]

Figure 4.8 plots observed mean temperatures versus predicted temperatures from the above equation. The average misfit is 1.49\% and the maximum disagreement is 3.50\% for cases that include internal
Figure 4.7: (a). Log-log plot of observed mean temperature $\theta_{\text{obs}}$ versus Rayleigh number $Ra_{\eta}$ for $H=0$ and $f = 0.547$. (b). Observed mean temperature versus predicted mean temperature from equation (4.3). The solid diagonal line represents perfect agreement between the observations and predictions. Solid green triangles represent tetrahedral planforms; unfilled blue diamonds are 4-5 plume transitioning planforms; and solid cyan squares are 5 and 6 plume planforms.
Figure 4.8: Observed mean temperature versus predicted mean temperature from equation (4.4). The solid diagonal line represents perfect agreement between the observations and predictions. Solid green triangles represent tetrahedral planforms; unfilled blue diamonds are 4-5 plume transitioning planforms; solid cyan squares are 5 and 6 plume planforms; half filled red circles are planforms with generally 7 plumes (but as few as 5 at times) and solid black circles represent time-dependent variable number of plumes. The grey triangle represents a transitional planform that alternates between 2 and 3 plumes.

heating.

The $H = 0$ case had only a very weak dependence on $Ra$ but some sensitivity to planform changes. Thus equation (4.3) tends to have larger errors when predicting mean temperature, since it cannot fully account for planform sensitivity. With the addition of moderate to substantial internal heating, the second term in equation (4.4) becomes the dominant factor and planform variability is diminished. Rayleigh number dependence of the system temperature is accounted for in the larger exponent on $Ra \eta$ in the second term of equation (4.4). Consequently, the misfit between observations and equation (4.4) is reduced for larger $H$. However, as noted by Moore (2008) parameterizations obtained with data characterized by $\theta < 1.0$ are probably not applicable when $\theta$ exceeds 1.0 (i.e., for very large $H$). In the latter case heat will be lost from the fluid to the lower boundary.
4.3 Plane-Layer Results

The study moves on to examine the influence of geometry on temperature in plane-layer convection. Following the isoviscous study, the hope is to obtain a set of equations relating spherical shell geometry and plane-layer geometry convection models so that for cases with the same Rayleigh number, $Ra_\eta$ and the same mean temperature $\theta$ there is a way to quantify the difference in nondimensional heating rate in the two systems. Tables 4.2 - 4.7 include results from all plane-layer calculations ($f = 1$) with different $H$ values.

Mean temperatures in plane-layer geometry convection show an even stronger dependence on planform than in the spherical shell systems. Depending on Rayleigh number and internal heating rate the convection planforms change from steady rolls (analogous to two-dimensional convection) to steady bimodal planform (orthogonal rolls) to time-dependent bimodal convection. Figure 4.9 shows snapshots of the temperature fields in $2 \times 2 \times 1$ solution domain cases with Rayleigh numbers of $Ra_\eta = 2.5 \times 10^4$, $Ra_\eta = 5 \times 10^4$ and $Ra_\eta = 2.5 \times 10^5$ and internal heating rates of $H = 0$ and 10. For low Rayleigh numbers, $Ra_\eta \leq 5 \times 10^4$, with no internal heating, the convection planform is characterized by rolls. If a moderate amount of internal heating is added ($H = 10$), the convection planforms become bimodal. With increased internal heating or increased Rayleigh number, the bimodal planforms become time-dependent.

Figure 4.10 indicates the convection planforms for the different Rayleigh numbers and internal heating rates observed in our $2 \times 2 \times 1$ solution domain calculations. For very low Rayleigh numbers, $Ra_\eta \leq 1.5 \times 10^4$, sheet-like upwellings and downwellings exist until a large internal heating rate is added ($H = 20$). For midrange Rayleigh numbers with low to no internal heating ($0 \leq H \leq 5$) we observe sheets transitioning to steady bimodal convection with increasing internal heating rates and/or increasing Rayleigh number. For large enough Rayleigh number $Ra_\eta \geq 1.49 \times 10^5$ we find bimodal convection for all $H$ values ($H \geq 0$). With no internal heating, a transition occurs from planforms featuring sheets to bimodal convection over a small range of Rayleigh numbers (approximately $Ra_\eta = 1.49 \times 10^5$ to $Ra_\eta = 1.54 \times 10^5$). In this range, the planform is dependent on the initial conditions of the system.

In Figure 4.11, the profiles of temporally averaged mean horizontal temperature show an increase in mean temperature with increasing internal heating rate. As in the spherical shell geometry models, increasing internal heating rate also increases the temperature with height between the thermal boundary layers.

As seen in Figure 4.12, fixing the internal heating rate at $H = 10$ and varying Rayleigh number results in a change in temperature with depth between thermal boundary layers. As Rayleigh number increase the mean temperature of the system decreases and the fluid interior becomes more isothermal (i.e., the convective vigour of the system increases). These are similar results to the profiles seen in spherical shell convection (see Figure 4.5).

4.3.1 Effect of solution domain size on plane-layer convection

The isoviscous convection study (see Chapter 3) found that for solution domain sizes of $2 \times 2 \times 1$ or larger, the temperature profiles exhibit little dependence on domain size. This is consistent with previous results reported by Houseman (1988) and Sotin and Labrosse (1999). Consequently, equation (3.3) is not dependent on solution domain for sizes greater than $2 \times 2 \times 1$. In a stratified viscosity system, solution domain size has a fundamental effect on mean temperature and the planform of convection.
Figure 4.9: Temperature field snapshots for plane-layer MC3D calculations. Rayleigh number increases downward and internal heating rate, $H$, increases to the right. For $H = 0$ the isosurfaces are shown for temperatures 0.1 above and below the mean temperature of the calculation. For $H = 10$, the isosurfaces are shown for temperatures 0.2 above and below the mean temperatures.
Figure 4.10: Convection planforms for various Rayleigh numbers and internal heating rates in a $2 \times 2 \times 1$ plane-layer model with the viscosity profile given by equation (2.36). The triangle represents the transition point for planform change from sheet-like planform to bimodal planform. $Ra_{\eta} = Ra/20.098$ for all plane-layer geometry calculations represented in this Figure.
Figure 4.11: Temperature as a function of height in the plane-layer system with a fixed Rayleigh number ($Ra_r = 2.5 \times 10^5$, or $Ra = 5 \times 10^6$) and varying internal heating rate, $H = 0, 5, 7.5, 10, 20$. 
Figure 4.12: Temperature as a function of height in the plane-layer system with a fixed internal heating rate, $H = 10$, and varying Rayleigh number, $Ra = 5 \times 10^6, 8 \times 10^5, 10^6, 3 \times 10^6, 5 \times 10^6, 6 \times 10^6, 10^7$. 
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Figure 4.13: Mean temperature as a function of Rayleigh number for plane-layer boxes of $2 \times 2 \times 1$ (pink unfilled squares), $3 \times 3 \times 1$ (green triangles) and $5 \times 5 \times 1$ (purple diamonds) when $H = 0$. The filled magenta square represents the transition point for planform change (from sheet-like convection to bimodal planform), where either planform can be obtained depending on the initial conditions of the system.

In calculations featuring the viscosity profile shown in Figure 2.4, Figure 4.16 shows the mean temperature with $H = 0$ for different Rayleigh numbers and solution domain sizes. There is a large drop in mean temperature in the $2 \times 2 \times 1$ system near $Ra_{\eta} = 1.5 \times 10^5$ corresponding to the planform change from sheets to bimodal convection. Figure 4.14 shows the temperature profiles for $2 \times 2 \times 1$ systems with increasing Rayleigh number and $H = 0$. There is a clear distinction between the two regimes, sheet-like convection and bimodal convection.

With increasing solution domain size, the difference in mean temperature between the convective sheet planforms and the bimodal planforms decreases. Figure 4.15 shows volume averaged temperatures over time for $2 \times 2 \times 1$ systems and $3 \times 3 \times 1$ systems with varying Rayleigh number. The different Rayleigh numbers converge to two different mean temperatures based on their planform of convection (sheet-like or bimodal). As the solution domain size is increased, the difference between the two temperatures decreases (i.e., the curves are closer together).

Moreover, the temperatures in the different solution domains begin to converge on the same mean temperature when $Ra_{\eta} \geq 1.5 \times 10^5$. This Rayleigh number corresponds to a change in planform from sheet-like convection to bimodal convection. After this drop, all the mean temperatures begin
Figure 4.14: Temperature profiles for plane-layer MC3D calculations with $2 \times 2 \times 1$ solution domains and no internal heating, $H = 0$ for various Rayleigh numbers. There is a clear drop in mean temperature between systems featuring sheet-like convection ($Ra$ of $5 \times 10^5$ to $2 \times 10^6$) and systems featuring bimodal convection ($Ra$ of $3 \times 10^6$ to $10^7$).

Figure 4.15: Volume average temperature for plane-layer MC3D calculations with $2 \times 2 \times 1$ solution domains and $3 \times 3 \times 1$ solution domains both with no internal heating, $H = 0$ for various Rayleigh numbers. There is a clear jump in mean temperature between systems featuring sheet-like convection and systems featuring bimodal convection. The curves on the right ($3 \times 3 \times 1$ solution domains) are closer in mean temperature than the curves on the left ($2 \times 2 \times 1$ solution domains).
to approach a fixed mean value. For a large solution domain size, $5 \times 5 \times 1$, the planforms transition with increasing Rayleigh number from sheets, to laterally stretched plumes, to columnar upwellings at $Ra_\eta = 1.5 \times 10^5$ (see Figure 4.17). In this geometry, the mean temperature shows a more gradual decrease with increasing Rayleigh number, compared with smaller geometries. However, a discontinuity in temperature change does still occur at $Ra_\eta = 1.5 \times 10^5$, corresponding to the disappearance of orthogonal rolls and laterally stretched plumes and the emergence of time-dependent columnar upwellings.

This research is partly motivated by wanting to reduce computational requirements while emulating convection with the same mean temperatures found in spherical shell geometries, thus this study proceeds using only $2 \times 2 \times 1$ solution domains. However, it is necessary to note that upwelling and downwelling morphology, as well as temperature, are sensitive to solution domain size at lower $Ra_\eta$. Moreover, solution domain size limits convective wavelength.
Figure 4.17: Temperature field snapshots for plane-layer MC3D calculations with $5 \times 5 \times 1$ solution domains and no internal heating. Isosurfaces are shown for temperatures 0.1 above and below the mean temperature of the calculations.
4.3.2 Deriving a parameterization for mean temperature in a plane-layer geometry

For the range of Rayleigh numbers and internal heating rates considered in this study the large drop in mean temperature corresponding to the change in planform of convection (as well as the dependence of the magnitude of the drop on the solution domain size) makes the determination of a single predictive equation for mean temperature difficult (if not impossible). However, it is possible to determine equations for the three Rayleigh number ranges corresponding to the three distinct regimes indicated in Figure 4.10. Specifically, there is a low Rayleigh number regime corresponding to only convective sheets; a midrange Rayleigh number regime corresponding to sheets that morph into bimodal planform with increasing $H$; and a large Rayleigh number regime corresponding to fully bimodal convection for all $H$ values. Since the intention of this study is to attempt to emulate large Rayleigh number spherical shell calculations using plane-layer geometry models, the main goal is to proceed to determine a predictive equation for mean temperature for cases with large Rayleigh numbers $Ra_\eta \geq 1.49 \times 10^5$ (or $Ra \geq 3 \times 10^6$).

For plane-layer convection, an equation of the form of equation (4.2) is also considered but for $\theta_b(f = 1)$ and $\alpha(f = 1)$. The results plotted in Figure 4.16 suggest that for the $H = 0$ case with $Ra_\eta \geq 1.49 \times 10^5$, $\theta_b$ can be approximated in equation (4.2) as 0.23, as there is only a very weak dependence of $\theta_b$ on Rayleigh number for the range being considered. This approximation allows the determination of an equation similar to that proposed by Sotin and Labrosse (1999) for isoviscous convection in a plane-layer model. A least squares fit of our numerical results is used to solve for the exponents in (4.2) and the following equation best fits the results:

$$\theta = 0.230 + 0.762 \frac{H^{0.701}}{Ra^{0.227}_\eta}. \tag{4.5}$$

Figure 4.18 shows observed mean temperatures versus predicted temperatures from the above equation (4.5). For $2 \times 2 \times 1$ solution domain geometry calculations, the diagonal line on the plot indicates perfect agreement between the observed and predicted temperatures obtained from equation (4.5). The average misfit is 1.96% and the maximum misfit is 5.15%.

The exponents in (4.2) can also be determined when $H > 0$ by performing a least square fit of the temperatures from the five models featuring planforms with sheet-like upwellings and downwellings by approximating $\theta_b$ as 0.350. The following equation fits these results:

$$\theta = 0.350 + 0.201 \frac{H^{0.960}}{Ra^{0.159}_\eta}. \tag{4.6}$$

Equation (4.6) has an average misfit of 0.13% and maximum misfit of only 0.29%. Finally, the numerical results for mid-range Rayleigh numbers corresponding to sheets that morph into bimodal planforms when $H > 0$ can be used to determine yet another set of exponents for equation (4.2) to obtain:

$$\theta = 0.350 + 2.00 \frac{H^{1.36}}{Ra^{0.519}_\eta}. \tag{4.7}$$

Equation (4.7) has an average misfit of 4.22% and a maximum misfit of 16.28%. Given that planform changes occur as $H$ increases in this regime the poor fit of the observations is not unexpected.

While the exponents in equations (4.6) and (4.7) differ largely from those in equation (4.5) and any equations obtained in isoviscous studies, equation (4.6) is still able to predict mean temperatures
Figure 4.18: Observed mean temperature versus predicted mean temperature from equation (4.5). The solid diagonal line represents perfect agreement between the observations and predictions. Convective planform is bimodal in all cases, filled pink diamonds represent steady planforms and unfilled blue diamonds represent time-dependent bimodal planforms.
successfully while for lower internal heating rates (i.e., $H < 20$) equation (4.7) has some success. However, the range of exponents found for different planform regimes indicate that a simple theory to describe system behaviour is unlikely to exist.

The exponents on $H$ and $Ra_\eta$ in equations (4.5) differ from those in equation (4.4). The exponent on $H$ is quite similar for both spherical and plane-layer results, but a larger difference in the exponent on $Ra_\eta$ is found. The two equations for different geometries differ largely in the coefficient $\alpha$. Based on isoviscous scalings that is not surprising since the exponent is not a function of $f$ but the coefficient, $\alpha$, is a function of $f$.

The findings from the plane-layer geometry models are fundamentally different from those of the spherical shell models because there are no discontinuities detected in the dependence of temperature on $Ra_\eta$ in the latter case. The discontinuities coincide with changes in planform which seem to occur sharply in the implicitly toroidal topology of boxes with periodic (wrap around) boundary conditions. For the range of $Ra_\eta$ and $H$ investigated, planforms with sheet-like features were not found in the spherical shell models. Previous authors (e.g., Jarvis et al., 1995; Shahnas et al., 2008) have reported the sensitivity of temperature to planform regime in isoviscous spherical shell models.

To model terrestrial or super-Earth mantle convection it is necessary to model large Rayleigh number convection. Consequently, equation (4.5) is focused on for the remainder of this study.

### 4.3.3 Temperature Profiles

Previous studies (e.g., Travis et al., 1990a, 1990b; Jarvis et al., 1995; Sinha and Butler, 2007; Deschamps et al., 2010) of isoviscous fluids that examined mixed heating mode convection in geometries with varying curvature $f$ have found that part of the temperature profile becomes subadiabatic when the internal heating rate is high and the Rayleigh number is relatively low. Temperatures within the system can reach or surpass the basal (bottom) boundary temperature which eliminates the bottom thermal boundary layer. This results in no basal heat flow into the fluid and effectively no formation of upwelling plumes. In an isoviscous fluid with a midrange Rayleigh number of $Ra = 10^6$ and an internal heating rate of $H = 25$ a plane-layer model features a mean temperature greater than one. However, with the introduction of the stratified viscosity profile given in equation (2.36), the mean temperature in a $2 \times 2 \times 1$ system with the same heating rate and effective Rayleigh number is only 0.55. Therefore while the high viscosity lower layer suppresses the thermal boundary layer instabilities (plume formation) in systems with low internal heating rate, it works to preserve these instabilities in systems with high internal heating rates.

Figure 4.19 show horizontally and temporally averaged temperature as a function of height and radius from plane-layer and spherical shell geometry models with $H = 0$ and $H = 20$. Figure 4.19a features models with Rayleigh numbers of $5 \times 10^5$ and in Figure 4.19b the Rayleigh number is $10^7$. Similar to isoviscous convection, the temperature profiles are stably stratified between the thermal boundary layers in both plane-layer and spherical shell models when the internal heating rate is high and Rayleigh number low (curves on right in panel a). At higher Rayleigh numbers the temperatures between the thermal boundary layers become more uniform in both geometries, even when $H=20$. In all cases the lower thermal boundary layer is present and active upwellings form.
Figure 4.19: (a). Temporally averaged profiles of laterally averaged mean temperature as function of height above the core for a fixed Rayleigh number $Ra = 5 \times 10^5$. Black curves are spherical shell models and red curves are plane-layer models with solid line representing $H = 0$ and dashed line representing cases with $H = 20$. (b). Same as a. but with $Ra = 10^7$. 
4.4 Viscosity profile versus viscosity contrast

The results above show successful parameterizations for mean temperature for the viscosity profile defined by equations (2.36) and (2.37), however tests of these parameterizations for alternate viscosity profiles featuring the same total viscosity contrast between the top and bottom of the system indicate that the parameterizations given in section 4.3.2 are dependent on the detail of the viscosity profile. In the absence of internal heating, the mean temperature, $\theta_b$, is not only a function of geometry and the total viscosity contrast across the convecting layer but in fact of the entire viscosity profile across the mantle depth.

Assuming a statistically steady system (such that the thermal boundary layer Rayleigh numbers have reached steady values) and satisfying the condition that the heat flow out of the system equals the heat flow in, the following equation for the mean temperature, $\theta_b$, is obtained (i.e., when $H=0$):

$$\theta_b = \frac{f^{3/2}}{(\Delta\eta)^{1/4} + f^{3/2}}. \tag{4.8}$$

In the isoviscous case this reduces to Vangelov and Jarvis’s (1994) result. (See Boundary Layer Section 1.6 for derivation of this equation).

For $f=1$, and a viscosity contrast of $\Delta\eta=30$ this equation implies $\theta=0.299$, differing substantially from the observed value of 0.230 employed in equation (4.5). Similarly, it fails for the models restricted to sheet-like planforms fit by equation (4.6) where $\theta_b=0.350$. As previously discussed, $\theta_b$ is quite sensitive to planform and the failure of the boundary layer theory result is not unexpected. Similar disagreement between boundary layer theory results and observations were reported previously for spherical shell isoviscous convection (Shahnas et al., 2008). In particular, the isoviscous result derived by Vangelov and Jarvis (1994) was abandoned by Shahnas et al. (2008) and Deschamps et al. (2010) for spherical isoviscous models because it does not work well in 3D systems that exhibit planform transitions.

Figure 4.20 plots mean temperature as a function of surface Rayleigh number, $Ra$, and effective Rayleigh number, $Ra_\eta$ for two sets of calculations featuring the same viscosity profiles in the regions of the thermal boundary layers but different viscosity structure between these layers (see Figure 2.4 in Chapter 2). All calculations feature no internal heating. The magenta squares correspond to the $2 \times 2 \times 1$ geometry results plotted in Figure 4.18 and feature the viscosity profile given by equation (2.36). The cyan squares correspond to $2 \times 2 \times 1$ geometry calculations employing the alternate viscosity profile given in Figure 2.4 (see Chapter 2). In general, the mean temperature is lower in systems with the red viscosity profile in Figure 2.4 as there is more vigorous heat advection across the region between the thermal boundary layers. In particular, downwellings encounter less resistance to penetrating the mid- to lower mantle and upwellings can travel between boundary layers faster in systems with the red viscosity profile (see Figure 2.4). Neither of the viscosity profiles used in the calculations results in temperatures that agree with equation (4.8).

Parameterizations based solely on the viscosity contrast across the convecting layer are unable to predict mean temperature successfully. However, parameterizations based on the effective Rayleigh number obtained with a particular viscosity profile, are not applicable to systems featuring different viscosity profiles. This result is not surprising as convective planform is dependent on the shape of the viscosity profile in these different systems. Furthermore, planform changes are a known complexity causing error in single parameterizations (e.g., Jarvis et al., 1995; Shahnas et al., 2008; Deschamps et al., 2010).
Figure 4.20: (a) Observed mean temperature versus surface Rayleigh number for plane-layer models featuring the viscosity profiles shown in Figure 1 with a viscosity contrast, $X$, of 30. The magenta squares represent observed mean temperatures for calculations featuring the (black) viscosity profile shown in Figure 2.4 and the cyan squares represent observed mean temperatures in calculations using the (red) viscosity profile shown in Figure 2.4. (b) Same as (a) but with effective Rayleigh number $\bar{Ra}$. 
4.5 Emulating spherical shell results in plane-layer geometry with a viscosity contrast of 30

For convection in a plane-layer $2 \times 2 \times 1$ solution domain, equation (4.5) can be rearranged to determine the necessary internal heating rate, $H$, for a given $Ra_\eta$ to obtain some desired mean temperature. For any spherical shell with $f = 0.547$, the mean temperature $\theta$ can be determined for a given $Ra_\eta$ and $H$ using equation (4.4). Thus, rearranging equation (4.5), allows determination of the necessary $H$ value to obtain the mean temperature of the spherical shell desired in a plane-layer geometry.

This claim can be tested, as in the isoviscous case (see Chapter 3), for cases with $Ra_\eta \geq 1.49 \times 10^5$ and different internal heating rates. Part of the goal in this study is to see how well temperatures can be matched in different geometries with large Rayleigh numbers. Spherical shell temperatures are emulated in plane-layer convection models using the heating rates predicted by inverting the previously determined plane-layer predictive equations.

Consider a Rayleigh number at the low end of the range of Rayleigh numbers for which equation (4.5) is valid. For a spherical shell with $f = 0.547$, when $H = 15$ and $Ra_\eta = 1.786 \times 10^5$ equation (4.4) predicts a mean temperature $\theta_{(4.4)} = 0.348$. Using equation (4.5) with $Ra_\eta = 1.786 \times 10^5$, an internal heating rate of $H = 3.55$ is needed in a plane-layer calculation to reproduce the mean temperature predicted in the spherical shell system. A $2 \times 2 \times 1$ plane-layer model with these parameters produces a mean temperature of $\theta = 0.337$. In this case, the disagreement between observed and predicted temperatures is 3.16%.

Figure 4.21 compares the profiles of the mean horizontal temperature for both the spherical shell calculation and plane-layer results. The means of the profiles differ slightly due to the different geometrical effects. Although both profiles are from models with a similar volume average temperature, in the spherical case the smaller radii sections contribute less to that average than the larger radii values of the profile. In the plane-layer geometry, the profile shows a small accumulation of hot fluid near the upper boundary which is absent in the spherical shell geometry profile at this Rayleigh number.

The spherical shell calculation has a stably stratified, slightly subadiabatic, temperature profile between the thermal boundaries, while the plane-layer system features some cold material in the upper mantle overlying hot material below. The lower thermal boundary layer (at the core-mantle boundary) is characterized by the same gradient in both the plane-layer and spherical shell results. This finding is consistent with the matching $\theta$ values of the models given that, relative to the fluid volume, the lower boundary of the plane-layer model is much larger than in the spherical shell case. Per unit area of the surface, the total heat flow across the lower boundary of the spherical shell model is therefore approximately $f^2$ times the total heat flow across the lower boundary of the plane layer case. The spherical shell case gains its additional heat through its internal sources. The similar basal fluxes and internal temperatures suggest plume vigour will be the same in both models. However, the upper thermal boundary has a modestly larger gradient in the plane-layer case than in the spherical shell calculations, resulting from more vigorous convection driven by a greater proportion of bottom heating.

Figure 4.21 also shows profiles of two other plane-layer geometry models. The dashed magenta curve shows the temperature profile for a plane-layer calculation with $Ra_\eta = 1.79 \times 10^5$ and $H = 15$ matching the heating rate of the spherical shell model. In this case, the observed profile and mean temperature differ greatly from the spherical shell model. Thus, only adjusting the effective Rayleigh number, by accounting for the different viscosity volumes in different geometries, is insufficient to properly match the
Figure 4.21: Time-averaged horizontally average temperature as a function of depth for a spherical shell \( f = 0.547 \) and three plane-layer \( (f = 1) \) calculations. The spherical shell calculation (solid black) uses \( Ra_\eta = 1.79 \times 10^5 \) and \( H = 15 \). The solid red line shows the temperature profile for a plane-layer calculation with \( Ra_\eta = 1.79 \times 10^5 \) and \( H = 3.6 \). The dashed magenta curve shows the temperature profile for a plane-layer calculation with \( Ra_\eta = 1.79 \times 10^5 \) and \( H = 15 \) matching the heating rate of the spherical shell model. The solid cyan curve shows the temperature profile for a plane-layer which matches the surface Rayleigh number and heating rate of the spherical shell with \( Ra = 3 \times 10^6 \) and \( H = 15 \).
two system geometry temperatures. The solid cyan curve shows the temperature profile for a plane-layer which matches the surface Rayleigh number and heating rate of the spherical shell with $Ra = 3 \times 10^6$ and $H = 15$. This last geotherm deviates the most from the spherical shell result, again showing the importance of matching not only the heating rates, but also the effective Rayleigh number in calculations done in different geometries.

To further support this method of comparing different geometries, a comparison of higher Rayleigh number cases in spherical shell and plane-layer systems is considered. For a spherical shell with $f = 0.547$, $Ra_\pi = 5.954 \times 10^5$ and $H = 20$, the TERRA calculation yields a mean temperature of $\theta = 0.312$ while equation (4.4) predicts a mean temperature of $\theta_{(4.4)} = 0.307$. Inverting equation (4.5) for $\theta = 0.307$, $Ra_\pi = 5.954 \times 10^5$ and $f = 1$, gives an internal heating rate of $H = 2.9$ needed in a plane-layer geometry to yield the same mean temperature as in the spherical shell. A $2 \times 2 \times 1$ geometry calculation with MC3D yields a mean temperature of $\theta = 0.300$ which is closer to the observed mean temperature in the spherical shell than was the case in the lower Rayleigh number calculations, reflecting the improvement in prediction at higher Rayleigh numbers. The plane-layer calculation shows a time-dependent bimodal planform while the spherical shell calculation has seven time-dependent upwellings: snapshots from both calculations are shown in Figure 4.22. Figure 4.22c shows the time averaged temperature profiles for the spherical shell calculation and the corresponding plane-layer calculation. A similar layer of hot fluid is found near the upper boundary in the plane-layer solution as in the lower Rayleigh number case (Figure 4.21).

Figure 4.23 shows contours for constant $\theta$ in $H - Ra$ space. As Rayleigh number increases, the difference between the internal heating rate $H$ required for matching temperatures in spherical shell (blue) and plane-layer (red) models increases.

### 4.6 Modelling a viscosity contrast of 100

The factor of 30 increase across the lower mantle depth was chosen based on previous numerical studies (e.g., Bunge et al., 1997) and studies modelling the geoid (e.g., Richards and Hager, 1984; Hager and Richards, 1989; Ricard et al., 1993) which determined this was a supported value for mantle viscosity increase. However, recent studies (e.g., Forte and Mitrovica, 1996; Mitrovica and Forte, 1998; Mitrovica and Forte, 2004) have found that an even larger lower mantle viscosity increase is possible, up to 2 or 3 orders of magnitude increase. Therefore, it is also important to consider the effect of a larger viscosity contrast across the layer and this section focuses on systems featuring $X = 100$ in equations (2.36) and (2.37) (see Chapter 2) based on Mitrovica and Forte’s (1998) study. Table 4.7 provides the results of the plane-layer calculations, which are all characterized by bimodal convective planforms and, based on the findings of section 4.3.2, can therefore be parameterized by a single equation. Table 4.8 provides the results of the spherical shell calculations. Increasing the lower mantle viscosity results in a larger discrepancy between the temperatures obtained in spherical shell and plane-layer models with the same effective Rayleigh number and internal heating rate. Thus, as the viscosity contrast is increased, in order to mimic spherical shell temperatures in plane-layer systems it becomes necessary to reduce the internal heating rate by an even greater amount.

For a viscosity increase of 100, the $\theta_b(f = 1)$ value from equation (4.5) is smaller than 0.23 (obtained with a viscosity contrast of 30). As in the case with a viscosity contrast of $X = 30$, for the $X = 100$ case the mean temperature also approaches a fixed temperature with increasing Rayleigh number. Specifically,
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\( Ra_{\eta} = 5.95 \times 10^5 \)

Figure 4.22: Snapshots of temperature field isosurfaces for a spherical shell geometry calculation with \( f = 0.547 \) and a plane-layer calculation, both with \( Ra_{\eta} = 5.95 \times 10^5 \). \( H = 20 \) in the spherical shell and \( H = 2.9 \) in the plane-layer calculation. The temperature field from the plane-layer calculation in panel (b) shows isosurfaces of constant temperature for values of 0.4 and 0.2. The spherical shell model snapshot in panel (a) shows an isosurface of constant temperature 0.4. Panel (c) shows horizontally and time averaged temperature as a function of depth for the spherical shell (solid black) and plane-layer (dashed red) models.
Figure 4.23: Contours of constant $\theta$ in $H - Ra_\eta$ space. The value of $H$ is determined by inverting equation (4.5) for different mean temperatures $\theta = 0.25, 0.3, 0.4, 0.5, 0.6$ (from bottom to top) in a plane-layer (red) system. While in a spherical shell (blue) the value of $H$ is determined by inverting equation (4.4) for mean temperatures $\theta = 0.1, 0.2, 0.3, 0.4, 0.5$ (from bottom to top).
the results indicate the mean temperature $\theta_b$ approaches 0.19 (see Table 4.7). For $H > 0$, the observed mean temperatures are used to determine a parameterization for $\theta$ of the same form as before (i.e., $\theta = \theta_b + \sigma \frac{H^\alpha}{Ra\eta}$) but for the larger viscosity contrast. If all parameters are allowed to vary in the inversion, the following equation

$$\theta = 0.190 + 1.212 \frac{H^{0.520}}{Ra^{0.255}}$$

best fits the results. This equation has an average error of 1.36% with a maximum error of 2.13% between observed and predicted mean temperatures.

Consider a single equation for mean temperature in plane-layer systems which accounts for the difference in temperature due to a different viscosity increase in the $\theta_b$ term. As described in Sections 4.3.2 and 4.4 above, $\theta_b$ depends on the effective Rayleigh number and planform of convection of the system. Since $\theta_b$ is expected to change with a large viscosity increase ($X = 100$), which it does, it is possible to expect this $\theta_b$ term to account for all the change in temperature due to the large viscosity increase. Specifically, allowing the second term representing the contribution to mean temperature from internal heating (i.e., $\theta_i$) to contribute in a similar way as before. Specifying the same $\theta_i$ term as in equation (4.5), the new equation is

$$\theta = 0.190 + 0.762 \frac{H^{0.701}}{Ra^{0.227}}$$

and the average error between observed and predicted mean temperatures is 21.3%, which is much larger than equation (4.9). However, the isoviscous study (see Section 3.6) discussed the sensitivity of $\sigma$ to the values of $\gamma$ and $\beta$. Therefore, an improved equation would allow $\sigma$ to vary but continue by specifying the same $\beta$ and $\gamma$ values as in equation (4.5). The following equation

$$\theta = 0.190 + 0.565 \frac{H^{0.701}}{Ra^{0.227}}$$

best fits the observed temperatures, resulting in an average error of 1.29% and a maximum error of 4.91%. The success of equation (4.11) shows promise for the ability to determine a single equation for mean temperature in plane-layer systems featuring different viscosity contrasts (but the same viscosity profile). In this case, $\alpha(f = 1)$ in equation (4.2) would be dependent on the viscosity profile as well as the geometry curvature. Since this equation (4.11) has a reduced mean error, it is used in the remainder of this large viscosity contrast study.

In order to examine how this increased viscosity in the lower mantle affects the disagreement of the plane-layer and spherical shell geometry temperatures, it is useful to calculate the heating rates required in plane-layer models in order to match the temperatures observed in the spheres. It is important to consider cases by matching the effective Rayleigh number. Three cases are compared, isoviscous models and models featuring the viscosity profile corresponding to the solid curve in Figure 2.4 when $X=30$ and 100.

In particular, all three cases have $Ra\eta = 1.8 \times 10^5$ and $H = 15$ because observations of the mean temperature in the required three spherical shell calculations exist. In the isoviscous case, the parameterization given in Chapter 3 by equation (3.13) requires $H = 3.91$ in a plane-layer geometry in order to obtain the temperature of $\theta = 0.658$ found in a spherical shell with $f=0.547$. With $X=30$, the viscosity profile in Figure 2.4 requires $H = 3.46$ in a plane-layer model, in order to match the mean temperature
\( \theta = 0.346 \) found in the spherical shell with \( H = 15 \) and \( Ra_\pi = 1.8 \times 10^5 \). With \( X = 100 \), equation (4.11) requires \( H = 2.27 \) in the plane-layer model in order to mimic a corresponding spherical shell temperature of \( \theta = 0.254 \), observed in a spherical shell convection calculation performed with TERRA. Thus, for a given \( Ra_\pi \) and internal heating rate, increasing the lower mantle viscosity results in requiring a greater reduction in the internal heating in the plane-layer geometry in order to match the mean temperature in the spherical shell case.

As the lower mantle viscosity is increased, the spherical geometry can be thought of as transporting basal heat more efficiently than the plane-layer geometry system. This is owing to the fact that in the plane-layer models, more of the total volume of the convecting fluid is at a high viscosity and heat produced by internal sources is not removed from this region as efficiently by advection. Consequently, when the lower mantle viscosity is increased, more heat is trapped in the sluggishly moving lower mantle of a plane-layer, in comparison to a sphere.

For a viscosity increase of 100, it is also possible to determine the effect of \( Ra \) and \( H \) on mean temperature in a spherical shell. The observations from Table 4.8 are fit to an equation of the form (4.2). The mean temperature in a spherical shell with a viscosity contrast of 100 and no internal heating is again weakly dependent on Rayleigh number for fixed \( f = 0.547 \). Following the analysis in Section 4.2.2, the term \( \theta_b \) can be fit by a power law of the form \( \zeta(Ra_\pi)^x \). Fitting the observations for \( H = 0 \) yields

\[
\theta_b(f = 0.547) = 0.309 Ra_\pi^{-0.125}
\]  
(4.12)

which has an average error of 1.17% and maximum misfit of 1.96%.

Employing equation (4.12), and solving for \( \alpha(f = 0.547) \) and the exponents on \( H \) and \( Ra_\pi \) in equation (4.2) by inverting all of the observations listed in Table 4.8 where \( H > 0 \) yields:

\[
\theta = 0.309 Ra_\pi^{-0.125} + 0.648 H^{0.798} \frac{H^{0.798}}{Ra_\pi^{0.287}}
\]  
(4.13)

which has an average misfit of 2.48% and a maximum disagreement of 5.01%.

Following the same method as above for the plane-layer systems, it is possible to parameterize an equation for mean temperature in spherical shells with a viscosity contrast of 100 which preserves the same exponents on \( Ra_\pi \) and \( H \) as in equation (4.4), but allows \( \theta_b \) and \( \alpha(f = 0.547) \) to vary. The following equation

\[
\theta = 0.309 Ra_\pi^{-0.125} + 0.720 H^{0.665} \frac{H^{0.665}}{Ra_\pi^{0.268}}
\]  
(4.14)

best fits the observed temperatures, resulting in an average error of 3.48% and a maximum error of 6.77%.

Figure 4.24 shows constant contours of predicted \( \theta \) for \( 2 \times 2 \times 1 \) plane-layer systems in \( H - Ra \) space determined by equation (4.5) for a viscosity contrast of \( X = 30 \) (solid red) and equation (4.11) for a viscosity contrast of \( X = 100 \) (solid blue). It also shows constant contours of predicted \( \theta \) for spherical shells determined by equation (4.4) for a viscosity contrast of \( X = 30 \) (dashed red) and equation (4.14) for a viscosity contrast of \( X = 100 \) (dashed blue). It also shows observed mean temperatures from spherical shell models with viscosity contrasts of \( X = 30 \) (red) and \( X = 100 \) (blue). As the effective Rayleigh number increases, the blue curves (representing the higher viscosity contrast) indicate that the
Figure 4.24: Contours of constant $H$ in $\theta - Ra_{\eta}$ space. Curves of $\theta$ for constant $H$ are determined for plane-layer systems by equations (4.5) and (4.11) for plane-layer systems and equations (4.4) and (4.14) for spherical shell systems with viscosity contrasts of $X = 30$ (red) and $X = 100$ (blue) respectively. The solid curves represent plane-layer systems and the dashed curves represent spherical shell systems. Observations from spherical shell models with $X = 30$ (red) and $X = 100$ (blue) are plotted as points.
increased lower mantle viscosity results in lower mean temperatures. In particular, these curves indicate that with increased lower mantle viscosity ($X = 100$) the spherical shell temperatures are much lower than the smaller viscosity increase spherical shells, and will have temperatures below the $H = 0$ curve in the plane-layer system at high Rayleigh number. This emphasizes the increased importance of adjusting the internal heating rate in plane-layer models in order to obtain the lower mean temperatures observed in spherical shells.

### 4.7 Two-dimensional study of effect of stratified viscosity

The inclusion of stratified viscosity results in a much more complex system than observed in the isoviscous convection study in Chapter 3. In particular, a single predictive equation is not possible to obtain for predicting mean temperature in different geometries. The plane-layer systems also feature a dependence on solution domain size not previously observed in the isoviscous models. While some effects of solution domain size were examined (see Section 4.3.1), a study of a full suite of models with variable solution domain size is computationally expensive in 3D. However, such a study can be accomplished in 2D.

Current studies of large Rayleigh number convection (e.g., Sinha and Butler, 2009), such as in Super-Earths, (e.g., O’Neill and Lenardic, 2007; Armann and Tackley, 2012) are often executed in 2D plane-layer models which are computationally faster than both their 3D plane-layer and especially 3D spherical shell counterparts (one of the motivations for this thesis research). Therefore a study of the effect of mean temperature in 2D plane-layer systems with stratified viscosity could be of use to future studies such as those mentioned, when trying to emulate, or comment on, spherical shell results in plane-layer convection models.

This study examines the effect of solution domain size, referred to as aspect ratio in 2D, and Rayleigh number on mean temperatures in 2D plane-layer convection models featuring the viscosity profile given by equation (2.36) with $X = 30$ (black curve in Figure 2.4). The study features $L \times 1$ boxes that begin convecting from a conductive state with an initial perturbation. All models are run until a statistically steady temperature (no general increasing or decreasing in temperature) is achieved. Table 4.9 shows the Rayleigh number, aspect ratio and mean temperatures of the models used in this 2D study.

Figure 4.25 shows a plot of Rayleigh number as a function of aspect ratio with the temperature at each point plotted from the bar graph on the right. For low Rayleigh numbers and small aspect ratios, there is a clear pattern of decreasing temperature with increasing aspect ratio. However, this pattern falls apart for large aspect ratio systems. In fact, the systematic decrease in temperature with increasing aspect ratio ceases at smaller aspect ratios for larger Rayleigh numbers.

#### 4.7.1 Effect of aspect ratio on 2D plane-layer convection

The effect of aspect ratio, $A$ is examined first. Figure 4.26 shows mean temperature as a function of Rayleigh number. There is a clearer relationship between mean temperature and Rayleigh number for lower aspect ratios ($A = 2, 3, 4$) but this correlation is not similar to the possible relationship between temperature and Rayleigh number at larger aspect ratio $A = 6, 7$. Figure 4.27 shows a similar plot as Figure 4.26 but also includes higher Rayleigh numbers and non integer values of aspect ratio.

Figure 4.28 shows mean temperature versus aspect ratio for fixed Rayleigh numbers, $Ra = 5 \times 10^6$ and $Ra = 10^6$ and $H = 0$. For the lower Rayleigh number, $Ra = 5 \times 10^5$, there appears to be two
Figure 4.25: Plot of Rayleigh number as a function of aspect ratio with the temperature at each point plotted from the bar graph on the right.

Figure 4.26: Plot of mean temperature of 2D plane-layer systems as a function of Rayleigh number and aspect ratio, for low Rayleigh number and integer values of aspect ratio.
Figure 4.27: Plot of mean temperature of 2D plane-layer systems as a function of Rayleigh number and aspect ratio, featuring non integer values of aspect ratio.

distinct lines, one for systems up to aspect ratio 6 and one for systems above aspect ratio 7. That is to say, a strong correlation between temperature and aspect ratio exists.

It appears the results for $Ra = 10^6$ will also fit this same trend, however the correlation between temperature and larger aspect ratios $Asp \geq 7$ is not as straightforward as for the lower Rayleigh number case. This change in relationship between temperature and aspect ratio could be related to the pattern of convection in systems with different Rayleigh numbers.

4.7.2 Effect of pattern of convection on mean temperatures in 2D plane-layer convection

Figures 4.29 and 4.30 show the temperature fields of different aspect ratio 2D models with $Ra = 10^5$ and $H = 0$. These figures show the change in pattern of convection (i.e., number of convection cells) with increasing aspect ratio. Figures 4.27 and 4.28 demonstrate how the mean temperature changes for different aspect ratio systems. In particular, these figures show that large aspect ratio models do not follow the same trends as smaller aspect ratio models. Figure 4.29 and 4.30 offer an explanation as to why the larger aspect ratio models do not follow the same trend. Changes in pattern of convection result in changes to the mean temperature of the system. Thus, systems experiencing different patterns of convection are likely to have temperature that do not follow the trend of other models.

Figure 4.31 plots mean temperature versus aspect ratio for systems with two and four convection cells in each system (same data as those plotted in Figure 4.26).

Figure 4.31 shows strong correlation between mean temperature and aspect ratio for small aspect ratios. However, the inclusion of systems featuring planforms with four convection cells shows less
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Figure 4.28: Plot of mean temperature of 2D plane-layer systems as a function of aspect ratio for fixed Rayleigh numbers $Ra = 5 \times 10^5$ and $Ra = 10^6$.

Figure 4.29: Plots of temperature field of a $4 \times 1$ 2D plane-layer system with $Ra = 5 \times 10^5$ and $H = 0$. This systems exhibits two convection cells.

Figure 4.30: Plot of temperature field of a $8 \times 1$ 2D plane-layer system with $Ra = 5 \times 10^5$ and $H = 0$. This systems exhibits four convection cells.
correlation between mean temperature and aspect ratio. Thus the planform of convection will effect the mean temperature and correlation with aspect ratio of plane-layer systems.

4.7.3 Developing an equation for mean temperature when \( H = 0 \) in 2D plane-layer convection

Based on the results in Figure 4.28, it should be possible to derive an equation for mean temperature in 2D plane-layer systems with no internal heating with small to mid-range aspect ratio sizes (\( A \leq 6 \)). The results shown in Figure 4.28 show a clear correlation between mean temperature and aspect ratio and can be used to attempt a fit of the data to a to an equation of the form

\[
\theta_h = \alpha A_{sp}^\gamma. \tag{4.15}
\]

The temperatures from 2D plane-layer models with aspect ratios of \( A_{sp} = 2, 2.5, 3, 3.5, 4, 4.5, 5, 6 \) and 7 with stratified viscosity (\( X = 30 \) in equation (2.36) and Figure 2.4) exhibit two types of results, time dependent temperatures and steady temperatures. Fitting the results to the power law relationship in equation 4.15 for mean temperature and aspect ratio 1 \( \leq A_{sp} \leq 6 \) when \( H = 0 \) (for systems featuring steady temperature and convective rolls), the equation that best fits the data is given by

\[
\theta = 0.3809 \ast A_{sp}^{-0.1112} \tag{4.16}
\]

The mean error between observed and predicted values of mean temperature was 0.15% with a max...
error of 0.29%.

Figure 4.32 shows the observed versus predicted mean temperatures from equation (4.16) for 2D plane-layer systems with stratified viscosity and $H = 0$.

The excellent agreement between observed and predicted mean temperatures in smaller aspect ratio systems is due to the convective nature of these systems. The results used in determining equation (4.16) all featured steady temperatures and convective pattern with only two convection cells. Thus the effect of convective pattern must also be considered, not just the aspect ratio of the system.

Due to the difficulty in obtaining systems with patterns featuring only two convection cells in both large aspect ratio systems and large Rayleigh number systems, it is not possible to develop a single predictive equation for mean temperature when $H = 0$ that varies with aspect ratio. As mentioned previously (see Section 1.6), changes in patterns correspond to changes in temperature and it is very difficult to constrain these changes with a single equation for mean temperature.

These difficulties confirm that our system of equations developed above in 3D for a single solution domain size and $f$ value sphere are very successful at allowing plane-layer models to emulate spherical shell temperatures given the parameter ranges specified.
Table 4.1: TERRA calculations table of values. Heating parameters are defined in the text. $\theta_{(4.4)}$ is the mean temperature obtained from equation (4.4) for the given $Ra_{\pi}$ and $H$ and $q_{diff} = 3(q_{top} - q_{bot} f^2)/(1+f+f^2)$ where $q_{top}$ and $q_{bot}$ are the nondimensional surface and basal heat fluxes, respectively. All models feature 128 cells of radial resolution.

<table>
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<th>$Ra_{\pi}$</th>
<th>$H$</th>
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<th>$q_{top}$</th>
<th>$q_{diff}$</th>
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Table 4.2: Summary of experiments and results for plane-layer convection in a $2 \times 2 \times 1$ solution domain. Heating parameters are defined in the text. $\theta_{\text{predicted}}$ is the mean temperature obtained from equations (4.5), (4.6) and (4.7) for given $Ra\eta$ and $H$. Equation (4.5) is valid for all $Ra\eta \geq 1.49 \times 10^5$, equation (4.6) is valid for $Ra\eta \leq 1.5 \times 10^4$, and equation (4.7) is valid with $1.5 \times 10^4 < Ra\eta < 1.49 \times 10^5$. (s) indicates the solution obtains a steady state. A * indicates a value at which two different planforms (sheets or bimodal) can be obtained, depending on initial condition. This is referred to as the transition point in the text.

<table>
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<th>$Ra\eta$</th>
<th>$H$</th>
<th>Resolution</th>
<th>$\theta$</th>
<th>$\theta_{\text{predicted}}$</th>
<th>$q_{\text{top}}$</th>
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</tr>
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<td>0.236(s)</td>
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Table 4.3: (cont.) Summary of experiments and results for plane-layer convection in a $2 \times 2 \times 1$ solution domain. Heating parameters are defined in the text. $\theta_{\text{predicted}}$ is the mean temperature obtained from equations (4.5), (4.6) and (4.7) for given $Ra_\eta$ and $H$. Equation (4.5) is valid for all $Ra_\eta \geq 1.49 \times 10^5$, equation (4.6) is valid for $Ra_\eta \leq 1.5 \times 10^4$, and equation (4.7) is valid with $1.5 \times 10^4 < Ra_\eta < 1.49 \times 10^5$.

(s) indicates a steady state solution.

<table>
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<tr>
<th>$Ra$</th>
<th>$Ra_\eta$</th>
<th>$H$</th>
<th>Resolution</th>
<th>$\theta$</th>
<th>$\theta_{\text{predicted}}$</th>
<th>$q_{\text{top}}$</th>
<th>$q_{\text{top}} - q_{\text{bot}}$</th>
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Table 4.4: MC3D table of values for $3 \times 3 \times 1$ solution domains. Heating parameters are defined in the text. $\theta_{predicted}$ is the mean temperature obtained from equations (4.5), (4.6), (4.7) for given $Ra_\eta$ and $H$. Equation (4.5) is valid for all $Ra_\eta \geq 1.49 \times 10^5$, equation (4.6) is valid for $Ra_\eta \leq 1.5 \times 10^4$, and equation (4.7) is valid with $1.5 \times 10^4 < Ra_\eta < 1.49 \times 10^5$. (s) indicates a steady state solution.

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<th>$Ra$</th>
<th>$Ra_\eta$</th>
<th>$H$</th>
<th>Resolution</th>
<th>$\theta$</th>
<th>$\theta_{predicted}$</th>
<th>$q_{top}$</th>
<th>$q_{top} - q_{bot}$</th>
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<td>0.337(s)</td>
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<td>0.336(s)</td>
<td>0.350</td>
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</table>

Table 4.5: MC3D table of values for $5 \times 5 \times 1$ solution domains. Heating parameters are defined in the text. $\theta_{predicted}$ is the mean temperature obtained from equations (4.5), (4.6), (4.7) for given $Ra_\eta$ and $H$. Equation (4.5) is valid for all $Ra_\eta \geq 1.49 \times 10^5$, equation (4.6) is valid for $Ra_\eta \leq 1.5 \times 10^4$, and equation (4.7) is valid with $1.5 \times 10^4 < Ra_\eta < 1.49 \times 10^5$. (s) indicates a steady state solution.

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<th>Resolution</th>
<th>$\theta$</th>
<th>$\theta_{predicted}$</th>
<th>$q_{top}$</th>
<th>$q_{top} - q_{bot}$</th>
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<td>1.00E+07</td>
<td>497560</td>
<td>0</td>
<td>384 x 384 x 144</td>
<td>0.236</td>
<td>0.230</td>
<td>18.11</td>
<td>0.04</td>
</tr>
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</table>
Table 4.6: MC3D results for $2 \times 2 \times 1$ solution domains with the viscosity contrast indicated by the dashed curve in Figure 2.4 with $X = 30$. Heating parameters are defined in the text. (s) indicates a steady state solution. A * indicates cases for which two different planforms (sheets or bimodal) can be obtained, depending on initial condition.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$Ra_\eta$</th>
<th>$H$</th>
<th>Resolution</th>
<th>$\theta$</th>
<th>$\theta_{predicted}$</th>
<th>$q_{top}$</th>
<th>$q_{top} - q_{bot}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00E+05</td>
<td>54985 0</td>
<td>128 x 128 x 72</td>
<td>0.266(s)*</td>
<td>7.56</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.00E+05</td>
<td>54985 0</td>
<td>144 x 144 x 96</td>
<td>0.311(s)*</td>
<td>7.27</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00E+06</td>
<td>109970 0</td>
<td>128 x 128 x 72</td>
<td>0.258(s)*</td>
<td>9.46</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00E+06</td>
<td>109970 0</td>
<td>144 x 144 x 96</td>
<td>0.310(s)*</td>
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<td>0.00</td>
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<td></td>
</tr>
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<td>54985 0</td>
<td>216 x 216 x 128</td>
<td>0.254</td>
<td>15.32</td>
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<td></td>
</tr>
<tr>
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<td>216 x 216 x 128</td>
<td>0.220</td>
<td>15.58</td>
<td>-0.01</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>109970 0</td>
<td>256 x 256 x 144</td>
<td>0.208</td>
<td>17.85</td>
<td>-0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00E+06</td>
<td>109970 0</td>
<td>128 x 128 x 72</td>
<td>0.666(s)</td>
<td>17.93</td>
<td>15.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00E+06</td>
<td>109970 0</td>
<td>144 x 144 x 96</td>
<td>0.592</td>
<td>19.28</td>
<td>14.99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.00E+06</td>
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<td>216 x 216 x 128</td>
<td>0.462</td>
<td>25.68</td>
<td>15.00</td>
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<td></td>
</tr>
<tr>
<td>6.00E+06</td>
<td>659820 0</td>
<td>216 x 216 x 128</td>
<td>0.451</td>
<td>26.61</td>
<td>15.01</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>109970 0</td>
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<td>0.425</td>
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</tbody>
</table>

Table 4.7: MC3D results for $2 \times 2 \times 1$ solution domains with viscosity contrast of 100 and profile given by equation (2.36) with $X = 100$. Heating parameters are defined in the text. $\theta_{predicted}$ is the mean temperature obtained from equation (4.11) for the given $Ra_\eta$ and $H$. (s) indicates a steady state solution. A * indicates a case for which two different planforms (sheets or bimodal) can be obtained, depending on initial condition.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$Ra_\eta$</th>
<th>$H$</th>
<th>Resolution</th>
<th>$\theta$</th>
<th>$\theta_{predicted}$</th>
<th>$q_{top}$</th>
<th>$q_{top} - q_{bot}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00E+05</td>
<td>7555.2 0</td>
<td>128 x 128 x 72</td>
<td>0.264(s)*</td>
<td>0.190</td>
<td>5.63</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>5.00E+05</td>
<td>7555.2 0</td>
<td>144 x 144 x 96</td>
<td>0.320(s)*</td>
<td>0.190</td>
<td>5.91</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1.00E+06</td>
<td>15110 0</td>
<td>128 x 128 x 72</td>
<td>0.238(s)*</td>
<td>0.190</td>
<td>7.03</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1.00E+06</td>
<td>15110 0</td>
<td>144 x 144 x 96</td>
<td>0.310(s)*</td>
<td>0.190</td>
<td>7.52</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>5.00E+06</td>
<td>7555.2 0</td>
<td>216 x 216 x 128</td>
<td>0.200(s)</td>
<td>0.190</td>
<td>11.28</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>6.00E+06</td>
<td>90662 0</td>
<td>216 x 216 x 128</td>
<td>0.198(s)</td>
<td>0.190</td>
<td>11.89</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1.00E+07</td>
<td>151100 0</td>
<td>256 x 256 x 144</td>
<td>0.191(s)</td>
<td>0.190</td>
<td>19.67</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>3.20E+07</td>
<td>483530 0</td>
<td>256 x 256 x 144</td>
<td>0.191(s)</td>
<td>0.190</td>
<td>19.67</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>5.00E+06</td>
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<td>0.569(s)</td>
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</tr>
<tr>
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<td>0.509(s)</td>
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</tr>
<tr>
<td>1.00E+07</td>
<td>151100 0</td>
<td>256 x 256 x 144</td>
<td>0.412</td>
<td>0.411</td>
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</tr>
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<td>128 x 128 x 72</td>
<td>0.698(s)</td>
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<tr>
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<td>7555.2 15</td>
<td>216 x 216 x 128</td>
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</tr>
<tr>
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<td>90662 15</td>
<td>216 x 216 x 128</td>
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<td>0.470</td>
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</tr>
<tr>
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<td>0.439</td>
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<td>15.00</td>
<td></td>
</tr>
<tr>
<td>3.20E+07</td>
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<td>256 x 256 x 144</td>
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<td>0.381</td>
<td>31.28</td>
<td>15.00</td>
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</tr>
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</table>
Table 4.8: TERRA calculations with a viscosity contrast of 100 and profile given by equation (2.37) with $X = 100$. Heating parameters are defined in the text. $\theta_{(4.14)}$ is the mean temperature obtained from equation (4.14) for the given $Ra_\eta$ and $H$ and $q_{\text{diff}} = 3(q_{\text{top}} - q_{\text{bot}})/Q^2/(1 + f + f^2)$ where $q_{\text{top}}$ and $q_{\text{bot}}$ are the nondimensional surface and basal heat fluxes, respectively. All models feature 128 cells of radial resolution.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$Ra_\eta$</th>
<th>$H$</th>
<th>$\theta$</th>
<th>$\theta_{(4.14)}$</th>
<th>$q_{\text{top}}$</th>
<th>$q_{\text{diff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00E+06</td>
<td>18210</td>
<td>0</td>
<td>0.093</td>
<td>0.091</td>
<td>2.53</td>
<td>0.00</td>
</tr>
<tr>
<td>3.00E+06</td>
<td>54630</td>
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<td>0.079</td>
<td>0.080</td>
<td>3.64</td>
<td>0.00</td>
</tr>
<tr>
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<td>0</td>
<td>0.072</td>
<td>0.073</td>
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<tr>
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<td>0.059</td>
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<tr>
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<td>54630</td>
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<td>0.254</td>
<td>0.259</td>
<td>9.23</td>
<td>9.94</td>
</tr>
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<td>145680</td>
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<td>0.195</td>
<td>0.209</td>
<td>10.35</td>
<td>10.01</td>
</tr>
<tr>
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<td>182100</td>
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<td>0.193</td>
<td>0.199</td>
<td>10.53</td>
<td>10.10</td>
</tr>
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<td>0.152</td>
<td>0.155</td>
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<td>10.13</td>
</tr>
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<td>0.426</td>
<td>0.407</td>
<td>11.13</td>
<td>14.98</td>
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<td>3.00E+06</td>
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<td>15</td>
<td>0.319</td>
<td>0.315</td>
<td>12.10</td>
<td>15.00</td>
</tr>
<tr>
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<td>109260</td>
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<td>0.265</td>
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<td>0.254</td>
<td>0.239</td>
<td>13.39</td>
<td>15.31</td>
</tr>
</tbody>
</table>
Table 4.9: MC3D results for 2D $L \times 1$ solution domains with viscosity contrast of 30 and profile given by equation (2.36) with $X = 30$. $H = 0$ in all cases.

<table>
<thead>
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<th>Aspect Ratio</th>
<th>$\theta$</th>
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<td>0.345</td>
</tr>
<tr>
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<td>3</td>
<td>0.338</td>
</tr>
<tr>
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<td>3.5</td>
<td>0.322</td>
</tr>
<tr>
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<td>0.326</td>
</tr>
<tr>
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<td>6</td>
<td>0.313</td>
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<td>0.333</td>
</tr>
<tr>
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<td>0.326</td>
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<td>0.317</td>
</tr>
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</tr>
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<td>0.330</td>
</tr>
<tr>
<td>$10^6$</td>
<td>4</td>
<td>0.326</td>
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<tr>
<td>$10^6$</td>
<td>4.5</td>
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</tr>
<tr>
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<td>5</td>
<td>0.318</td>
</tr>
<tr>
<td>$10^6$</td>
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<td>0.318</td>
</tr>
<tr>
<td>$10^6$</td>
<td>7</td>
<td>0.329</td>
</tr>
<tr>
<td>$10^6$</td>
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<td>0.337</td>
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<tr>
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<td>0.319</td>
</tr>
<tr>
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</tr>
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Chapter 5

Discussion and Conclusions

5.1 Summary of results

This study has shown the necessary solution domain size and heating rates required in plane-layer convection in order to emulate spherical shell thermal properties in plane-layer models. For isoviscous calculations, a rectangular $2 \times 2 \times 1$ geometry is large enough to obtain the mean temperatures and planforms observed in larger geometry calculations (e.g., Figure 3.4). In order to emulate the temperatures of spherical shell convection using a plane-layer model, it can be necessary to include heat sinks which cool the overall temperature of the fluid. This study derived a predictive equation (3.13) for mean temperature in a spherical shell or plane-layer based on the $Ra$, $H$ and $f$ value of the system, which allows for heat sources or sinks (negative $H$ values) in the plane-layer models.

5.1.1 Isoviscous study summary

The isoviscous convection study in Chapter 3 used the approach employed by Sotin and Labrosse (1999) for determining the free parameters of a power-law equation that gives the mean temperature of a plane-layer convecting system featuring internal and basal heat sources. Although the data follows the power-law behaviour described by others (e.g., Parmentier et al., 1994; Sotin and Labrosse, 1999) the parameters that best fit the data do modestly deviate from the previously published results (Sotin and Labrosse, 1999). The best fit to observations of the mean temperatures of the systems modelled is affected by the range of the parameter space. For example, including more high Rayleigh number results in the inversion increases the disagreement with the findings of Sotin and Labrosse (1999), probably because convection at lower Rayleigh numbers takes on a different planform to higher Rayleigh number convection (see Figure 3.9 versus 3.10). Shahnas et al. (2008) previously noted that convection with different planforms is difficult to fit with a single power-law. Employing the power-law exponents that best fit the plane-layer geometry calculations, the study presented here is extended following the method published by Shahnas et al. (2008) and converge on a version of an equation for the mean temperature in a convecting infinite Prandtl number fluid in a spherical shell for a given Rayleigh number, internal heating rate and inner shell to outer shell radius, $f$. Given this equation (3.13), it is possible to predict the mean temperature, $\theta$, in a spherical shell for specified $Ra$, $H$ and $f$ and then invert the equation to determine $H$ for the same $Ra$ and $\theta$ when $f = 1$. Accordingly, a plane-layer calculation can be performed which will reproduce a good approximation of the spherical shell thermal structure. The
fit between spherical shell and plane-layer mean temperatures and temperature profiles is improved as Rayleigh number increases (compare Figure 3.19 with Figure 3.23).

In addition to deriving parameterizations for mean temperature in isoviscous systems, it is possible to derive parameterizations for mean surface heat flux (see Sections 3.10 and 3.11.1). However, these parameterizations are not as successful at predicting values as the equations for mean temperature. Moore (2008) determined a different parameterization from Shahnas et al. (2008) for heat flux, but his equation only reduced the misfit for plane-layer models. The form of the equation proposed by Shahnas et al. (2008) worked for spherical shell data, but in both cases the errors were a few percent which was higher than the errors obtained when matching mean temperatures.

Section 3.11.2 described an alternative parameterization for the equation for mean temperature in a spherical shell system. However, as explained, this equation worked best for different \( f \) valued spheres and not for plane-layer systems. As one goal of the study was to match mean temperatures in plane-layer and spherical shell geometries, the parameterization needed to perform optimally in both geometries. Therefore, the equation (3.13) followed the form proposed by Shahnas et al. (2008) was developed to minimize the error when matching temperatures in different geometries.

### 5.1.2 Stratified viscosity study summary

The stratified viscosity study in Chapter 4 used a similar approach to that of the plane-layer convection study in Chapter 3 to determine the free parameters of a power-law equation for mean temperature, this time for both spherical shell and plane-layer experiments. There were a number of differences between this study with increased lower mantle viscosity and the isoviscous study, and for this reason only a single \( f \) value sphere is studied, namely \( f = 0.547 \). For the spherical shell results with no internal heating, the mean temperature demonstrated a slight dependence on \( Ra \) which was not seen in the isoviscous spherical shell results presented by Shahnas et al. (2008) and Deschamps et al. (2010), nor the isoviscous plane-layer results in Chapter 3 or by Sotin and Labrosse (1999). Thus \( \theta_b \) (the contribution to mean temperature from basal heat sources) was given as a simple power function of \( Ra \) for the spherical shell data. The spherical data with internal heating \( H > 0 \) was then inverted to solve an equation of the form \( \theta = \theta_b + \alpha H^\beta / Ra^\gamma \), as was done in the isoviscous convection study in Chapter 3. This equation for spherical shell convection data was able to predict mean temperatures in a spherical shell for a range of Rayleigh numbers. The plane-layer results with stratified viscosity showed a dominant dependence on planform based on the range of Rayleigh numbers used. That is, for a given range of Rayleigh numbers a particular planform was obtained. As previously mentioned (e.g., Chapter 3, Jarvis et al., 1995; Shahnas et al., 2008), the change in planform can correspond to a large change in temperature which cannot be parameterized with a single equation. For the stratified viscosity results presented here, three different parameter regimes were determined; one with only sheet-like convection, one featuring sheet-like convection that morphs into bimodal convection with increased \( H \) and one regime featuring only bimodal convection for all \( H \) values. Three different equations of the form presented in Chapter 3 were determined, one for each planform regime. Since one of the goals of this study was to emulate high Rayleigh number spherical shell convection in a plane-layer model, the study continued using only an equation determined for large \( Ra_\eta \geq 1.49 \times 10^5 \). Ultimately, using the combination of two equations, one for spherical shell and one for plane-layer convection, an adjusted heating rate in plane-layer models can be determined to emulate a particular spherical shell heating mode given \( Ra_\eta \) and \( H \) (see Figure 4.22). In addition to determining a parameterization for mean temperature for a viscosity contrast of 30 across
the lower mantle, Section 4.6 describes the effect of increasing the lower mantle viscosity by a factor of 100. A new equation can be determined from the plane-layer data and used successfully to predict the mean temperature in a spherical shell with the same viscosity contrast. The stratified viscosity study began with a single profile for viscosity across the mantle and increased the lower mantle viscosity from 30 to 100. However, upon observing the results for a different viscosity profile (see Section 4.4), it was found that the equations described above are specific to the profile and viscosity contrast used in the numerical model. Thus a single equation for different viscosity contrasts and profiles was not achieved. Nevertheless, the results showed that as the lower mantle viscosity is increased the amount of heating required in a plane-layer model should be decreased in order to match the spherical shell temperature. This emphasizes the importance of adjusting the internal heating rate in plane-layer models when trying to emulate spherical shell results, especially in cases with increased lower mantle viscosity.

5.1.3 Derivation of equations

The equations developed in this study were derived from extensions of boundary layer theory (described in Section 1.6). However, as discussed, the mean temperatures from spherical shell model observations do not match the predicted values derived from theory. In particular, there is disagreement in the term representing the contribution to temperature from purely basal heating. The theory yields \(1/(f^{-3/2} + 1)\) for the basal heating term. Shahmas et al. (2008) found that multiplying this term by \(f^{1/2}\) best fit the spherical shell observations of mean temperature. When \(f = 1\), both these equations result in a temperature of 0.5, the mean temperature in a plane-layer system with no internal heating.

Deschamps et al. (2010) proposed that the term representing basal heating \(\theta_{basal}\) is

\[
\theta = \frac{f^2}{1 + f^2}.
\]  

(5.1)

Deschamps et al. (2010) also assumed an \(f\) dependence in the numerator and fit their spherical data to an equation of the form \(f^a/(1 + f^b)\) without employing any limits on the values of \(a\) or \(b\) implied by boundary layer theory. No multiplication of the terms in Deschamps et al.’s (2010) equation (5.1) by any \(f^a\) will alter the value of \(\theta\) when \(f = 1\).

The equation for mean temperature in a system with variable viscosity is given by

\[
\theta = \frac{f^{3/2}}{\Delta \eta^{1/4} + f^{3/2}}
\]  

(5.2)

which is equation (1.21) in Section 1.6. When \(f = 0.5\) in an isoviscous system (\(\Delta \eta = 1\)), this equation (1.21) predicts a mean temperature of 0.2. When the viscosity contrast across the layer depth is an increase of a factor of 30 (\(\Delta \eta = 30\)), this equation (1.21) predicts a mean temperature of 0.299 when \(f = 1\) and a temperature of 0.131 when \(f = 0.5\). If equation (1.21) is multiplied by \(f^{1/2}\) as suggested by Shahmas et al. (2008), this yields

\[
\theta = \frac{f^2}{\Delta \eta^{1/4} + f^{3/2}}
\]  

(5.3)

and there is still no effect on \(\theta\) in \(f = 1\) case. That is, \(\theta = 0.299\) with \(\Delta \eta = 30\). However, this value does not match the observed temperatures in the plane-layer systems in Chapter 4 featuring this type of viscosity stratification. Thus, an equation of this form featuring the viscosity contrast is not able to be developed in the study. As mentioned in Chapter 4, both the viscosity contrast and viscosity profile
contribute to the mean temperature of the system and so a single equation is specific to the type of viscosity stratification employed in the model.

The equations for spherical shell mean temperatures determined from boundary layer theory and energy conservation did not agree with observations. They did, however, provide some guidance regarding the form for the parameterization of mean temperature. This misfit has been previously reported in Shahnas et al. (2008) and Deschamps et al. (2010) papers which did not agree with Vangelov and Jarvis (1994) derivation which was unable to be confirmed by observations at the time. The boundary layer theory does not work well when a variety of different planforms is possible, as previously stated in many of these above isoviscous studies (e.g., Shahnas et al., 2008; Deschamps et al., 2010; Vangelov and Jarvis, 1994). In particular, it also does not work very well in studies featuring depth-dependent viscosity as presented in Chapter 4. However, the equations used in this study are successful at predicting mean temperatures in plane-layer and spherical shell models.

5.2 Applications

Full three-dimensional spherical shell convection models remain a computational challenge (Wolstencroft et al., 2009), particularly for problems requiring the resolution of high gradients or heterogeneous composition (e.g., temperature and stress-dependent rheologies or modelling continental and oceanic plates).

In addition, the increasing interest in exosolar planetary interior dynamics (e.g., Valencia et al., 2006, 2009; O’Neill and Lenardic, 2007, van Heck and Tackley, 2011) and associated mantle convection at Rayleigh numbers that may exceed the Earth’s by an order of magnitude or more means that plane-layer convection modelling continues to be one of the best options for reaching the relevant parameter range using some sort of high resolution 3D calculation.

Consequently, at the very least, plane layer convection studies will remain useful for parameter sensitivity studies (e.g., Moore, 2008; Choblet and Parmentier, 2009) and other studies requiring large numbers of calculations (i.e., data points).

The findings presented here also have important implications for rectangular laboratory tank models of mantle convection processes (e.g., Davaille et al., 2002; Jellinek et al., 2002; Gonnermann et al., 2004; Robin et al., 2007). An often cited criticism of laboratory tank modelling is that the models do not incorporate internal heat sources. This study suggests that this is quite appropriate for modelling convection in a box with Earth-like parameters since to mimic the Earth’s mean temperature in a plane-layer geometry requires only a very small amount of internal cooling if contributions to the heat budget from secular cooling are to be ignored, or a small amount of internal heating otherwise. In particular, the isoviscous convection study found that small amounts of internal cooling are necessary to model spherical shell temperatures, while the stratified viscosity convection study found that even larger amounts of internal cooling are necessary to emulate the corresponding spherical shell temperatures.

Using present day heat flux values for the Earth (see Section 1.3), $H = 20$ and $Ra_\eta = 10^7$ in a spherical shell will require $H = -2.3$ in an isoviscous plane layer model and $H = -5.3$ in a plane-layer model with a viscosity contrast of 30 from the upper to lower mantle. Therefore, no internal heating in a plane-layer model is needed to emulate the temperature from spherical shell convection using Earth-like values for $H$ and $Ra$. 
5.2.1 Very high $Ra$ convection

Both the isoviscous and stratified viscosity studies feature high Rayleigh number comparisons between the two geometries (see Figures 3.28 and 4.22). In order to model more Earth-like or Super-Earth type planets, it is necessary to model high Rayleigh number convection. However, when increasing the Rayleigh number of the system, it is also necessary to increase the resolution required to properly resolve the thinning boundary layers.

An important application of this work is to allow for high Rayleigh number calculations in plane-layer geometries to simulate the convection obtained in spherical shell geometries. Modelling of plate tectonics is a particularly difficult computation problem. Consequently, modelling plates in 2D plane-layer convection rather than in computationally expensive spherical models is still a useful alternative for high $Ra$ studies.

The study of Super-Earths, planets with 1-10 times the mass of the Earth is a growing field. However, such massive planets will have parameter values which require increased computational resources. The ability to model Super-Earths in plane-layer models is also important for studying the onset of plate tectonics and mantle dynamics in general.

5.3 Temperature-dependent parameters

5.3.1 The importance of cooling in isoviscous models

The isoviscous study in Chapter 3 describes the results of trying to emulate isoviscous spherical shell thermal properties in plane-layer convection models. Figure 5.1 show snapshots of temperature fields for spherical shell and plane-layer convection. While the planform of convection in the two systems is similar, thin columnar upwellings and downwellings, there are many more upwellings in the spherical shell than in the plane-layer model.

It was found that for the same heating rate in each geometry, the ratio of $\theta$ in the plane-layer case to $\theta$ in the spherical shell case increases with $Ra$. (That is, temperatures decrease faster with Rayleigh number in a spherical shell). Consequently, for $H > 0$ and $Ra$ greater than the $Ra$ where $\theta$ in the spherical shell equals 0.5, the discrepancy in temperature between plane-layer models with no internal heating and spherical shell models increases as the Rayleigh number increases (see Figure 5.2a). Thus at very high Rayleigh numbers, such as those relevant to Super-Earths, increased internal cooling would be necessary to obtain spherical shell-like temperatures in the plane-layer models.

These isoviscous results have important implications for the use of temperature-dependent parameters. If the plane-layer model being used features temperature dependent parameters but does not account for the much larger mean temperature found in the specific geometry, the results will not be appropriate to compare with spherical shell models. These plane-layer models with internal heating are significantly hotter than their spherical shell counterparts and some sort of internal cooling is needed to reduce the temperature of the system.

5.3.2 The importance of adjusting $H$ in stratified viscosity models

Figure 5.2b,c shows contours of constant $H$ as a function of $Ra\eta$ and $\theta$ for stratified viscosity models. For different $H$, the solid lines (red) indicate the temperatures from plane-layer convection calculations as a function of Rayleigh number and the dotted lines indicate the mean temperatures for the spherical shell
Chapter 5. Discussion and Conclusions

Figure 5.1: Snapshot of isosurfaces of constant temperature for spherical shell and plane-layer systems. (a) $Ra = 10^7$, $H = 23.53$, $\theta = 0.475$ with blue isosurfaces at nondimensional temperature of 0.14 and orange at 0.77 (from Shahnas et al., 2008). (b) $Ra = 10^7$, $H = -0.89$, $\theta = 0.464$ with blue isosurfaces at nondimensional temperature of 0.13 and yellow at 0.76.

calculations with $f = 0.547$. The calculated curves in panel (a) are based on equations for isoviscous convection from Chapter 3 and the calculated curves in panel (b) and (c) are based on equations for stratified viscosity convection from Chapter 4.

Considering an effective Rayleigh number of $5 \times 10^7$, we observe that for fixed $H$ the ratio of $\theta$ in the plane-layer systems to $\theta$ in the spherical shell models is smaller for isoviscous models than those with increased lower mantle viscosity (see section 3.5). Thus, temperature changes by a greater percentage when changing geometry in the stratified viscosity case. For example, in the models featuring a stratified viscosity the mean temperatures of spherical shell models with fixed high internal heating rates (e.g., $H = 30$) become colder than a plane-layer model with no internal heating at a lower effective Rayleigh number when compared with the isoviscous convection case. This effect is amplified with an increased lower mantle viscosity ($X = 100$). The contours indicate that, as in an isoviscous plane-layer convection calculation, internal cooling rather than internal heating is necessary to emulate the temperatures found in a spherical shell models with large Rayleigh numbers. Moreover, extra internal cooling is necessary to equalize temperatures in the stratified viscosity models when comparing cases with the same effective Rayleigh numbers, $Ra_\eta$.

The results summarized in Figure 5.2 show that these studies have important implications for plane-layer geometry studies which use temperature-dependent parameters. For example, the effect of a plane-layer geometry causes an increased temperature-difference across the upper thermal boundary layer of the convecting system which can promote the transition to stagnant-lid convection (e.g., Solomatov, 1995; Moresi and Solomatov, 1995, Richards et al., 2001, Stein and Hansen, 2008) when a temperature-dependent rheology is present.

Consider a simple approximation of the temperature-dependent viscosity based on an approximation of Arrhenius law such as $\eta = \exp(-ln(10^8)(T))$ (Turcotte and Schubert, 2002), and calculate the expected viscosity for different geotherms. Figure 5.3 shows the snapshots of the temperature fields for
Figure 5.2: Contours of constant $H$ generated by the predictive equations for isoviscous (a) and stratified viscosity with viscosity contrast of 30 (b) and 100 (c) for both plane-layer (red solid) and spherical shell (blue dashed) geometries. $H$ values of 0, 5, 10, 20, 30 are labelled on corresponding curves from bottom to top for each geometry. For the stratified viscosity case with $X = 30$, equations (4.4) and (4.5) are used to generate the curves, and for the viscosity contrast of $X = 100$, equations (4.11) and (4.14) are used to generate the curves.
the spherical shell model and three different plane-layer models, one with the adjusted heating rate, one with the same effective Rayleigh number as the spherical shell and non adjusted heating rate and one with the non adjusted Rayleigh number and internal heating rate. Figure 5.4 shows example geotherms for a spherical shell and different plane layer models which attempt to mimic the spherical results. The profiles show that using the same parameters in a plane-layer calculation as in the spherical shell will result in the plane-layer geotherm being too hot by approximately 65%.

Panel (b) of Figure 5.4 shows the temperature-dependent viscosity profiles found with the geotherms in (a). In the case with identical heating parameters this leads to a much smaller mean viscosity and a much larger drop in viscosity across the upper thermal boundary. In the example shown, the difference in the temperature across the upper thermal boundary layer could amount to a difference of several orders of magnitude in viscosity.

This comparison is oversimplified as the temperature profiles will be affected by the presence of temperature-dependent rheology. However, it demonstrates the first order effect geometry has on the mean temperature of a system which can cause large variations in the viscosity jump across the lithosphere. This is particularly important to consider in models featuring plate-like features, especially for determining when the onset of mobility occurs.

The decreased viscosity drop across the upper thermal boundary layer in spherical shell convection will affect the parameters at which the onset of stagnant-lid convection begins. Specifically, surface mobility will be maintained in the sphere for internal heating rates that may result in stagnant-lid convection in the plane-layer model. For rheologies featuring temperature and stress-dependence (including a specified yield stress) surface mobility can similarly be lost in a box for parameters that would allow mobility in a sphere. Furthermore, for mantles characterised by the same Rayleigh number (based on the mantle depth not planet radius), spherical shells with smaller cores are more likely to have mobile surfaces because they have smaller thermal contrast across their upper thermal boundary layers. The results presented here suggest an adjustment for plane-layer calculations to obtain appropriate temperatures so that extrapolation of the findings to a spherical convection scenario can be made.

### 5.4 Missing complexities in these models

The isoviscous and stratified viscosity convection studies neglect a number of important properties of planetary mantles, including those known to effect the temperatures of the convecting system, such as: temperature- (e.g., Tackley, 1993; Ratcliff et al., 1997; Solomatov and Moresi, 1997) and depth-dependent rheology (Gurnis and Davies, 1986a; Cserepes, 1993; Bunge et al., 1996), temperature- (Ghias and Jarvis, 2008; Tosi et al., 2013) and depth-dependent (Hansen et al., 1991; 1993; Tosi et al., 2013) thermal expansion coefficient, phase changes (e.g., Christensen and Yuen, 1985; Peltier and Solheim, 1992; Tackley et al., 1993; Solheim and Peltier, 1993, 1994a, 1994b; Yuen et al., 1994; Christensen, 1995; Hernlund et al., 2005; Tosi et al., 2010; Matyska et al., 2011), heterogeneous chemistry (e.g., Hansen and Yuen, 1994; Montague et al., 1998; Montague and Kellogg, 2000; McNamara and Zhong, 2004; McNamara and Zhong, 2005; Lassak et al., 2007) and the presence of plates (Bunge and Richards, 1996; Zhong et al., 2000; Tackley, 2000a, 2000b; Lowman et al., 2001; Monnereau and Quéré, 2001; Lowman et al., 2003; Stein et al., 2004; Gait and Lowman, 2007; Stein and Hansen, 2008; van Heck and Tackley, 2008; Lowman et al., 2011; Rolf and Tackley, 2011). However, it can be argued that the inclusion of a given physical property in a convection experiment will act to augment cooling or
Figure 5.3: Plots of snapshots of temperature fields for the (a) spherical shell calculation with $Ra_\eta = 1.79 \times 10^5$ ($Ra = 3 \times 10^6$) and $H = 15$, isosurface is constant temperature of 0.5. Temperature fields for (b) the plane-layer calculation with $Ra_\eta = 1.79 \times 10^5$ and $H = 3.6$ the adjusted heating rate, isosurfaces are 0.1 above and below mean temperature. (c) shows temperature field for the plane-layer model with $Ra_\eta = 1.79 \times 10^5$ and $H = 15$ which matches heating rate of spherical shell, isosurfaces are 0.2 above and below the mean temperature. (d) shows temperature field for the plane-layer model which matches the surface Rayleigh number and heating rate of the spherical shell with $Ra = 3 \times 10^6$ and $H = 15$, isosurfaces of constant temperature are 0.2 above and below the mean temperature.
Figure 5.4: Panel (a) shows time-averaged horizontally averaged temperature as a function of depth for a spherical shell \((f = 0.547)\) and three plane-layer \((f = 1)\) calculations. The spherical shell calculation (solid black) uses \(Ra_\eta = 1.79 \times 10^5\) and \(H = 15\). The solid red line shows the temperature profile for a plane-layer calculation with \(Ra_\eta = 1.79 \times 10^5\) and \(H = 3.6\). The dashed magenta curve shows the temperature profile for a plane-layer calculation with \(Ra_\eta = 1.79 \times 10^5\) and \(H = 15\) matching the heating rate of the spherical shell model. The solid cyan curve shows the temperature profile for a plane-layer which matches the surface Rayleigh number and heating rate of the spherical shell with \(Ra = 3 \times 10^6\) and \(H = 15\). Panel (b) shows the corresponding temperature-dependent viscosity profiles as a function of depth for each of the curves in (a) using a viscosity law \(\eta = exp(-ln(10^8)(T))\) with a viscosity increase of the order \(10^8\) across the mantle depth.
heating of a system regardless of the geometry of the system. Thus, an internal cooling rate, or at least a reduced internal heating rate, is a required feature in plane-layer mantle convection calculations because the model internal heat source is the most fundamental of convection control parameters (e.g., its magnitude does not depend on temperature).

The recognition that convection in nonuniform property fluids can converge on vastly different mean temperatures from the temperatures found in uniform property fluids was the motivation for the comparison of thermal structure dependence on geometry given a stratified viscosity. An increase in viscosity in the lower mantle has long been recognized to cool the mean temperature of a convecting layer (e.g., Gurnis and Davies, 1986a,b). It was shown that the effect is substantial in both a spherical and plane-layer system so that differences in temperature between the two remain at least as important as in isoviscous convection. Consequently, the main conclusion of the stratified viscosity study should apply regardless of the presence of the above mentioned complexities. Namely, it’s expected that in convecting systems with identical heating conditions and fluid characteristics, a plane layer geometry system will be substantially hotter than a spherical shell system with a core to planet radii ratio similar to Earth’s.

5.5 Future work

In future work, results from these experiments examining heating modes will allow for selection of the solution domain, heating mode and viscosity distribution that best reproduce Earth-like conditions. Future work could also include plates and examine the effect of different heating modes on plate movement and velocity.

The results described in this thesis have important implications and applications for future work on planetary mantles. The results describe a method to allow the study of spherical shell convection in plane-layer geometry models. This allows for a reduction in computational demands by only modelling 2D or small 3D plane-layer models rather than an entire sphere. This simplification will ease future parameter studies which require numerous calculations with various parameter values to gain an understanding of the underlying physics.

While this work began with a simplified isoviscous convection model, it progressed to include various depth-dependent viscosity profiles and parameterized equations for mean temperature were developed in all cases. However, these studies do not complete the parameter space story.

Section 3.7.1 described trying to match temperatures in different f valued spheres in plane-layer geometry by using a reduced constant internal cooling rate. Section 3.12 described an attempt to model isoviscous spherical shell convection in a plane-layer model using a depth-dependent internal heating rate $H$. It described the effect of many alternative depth dependent $H$ functions, however none were successful in altering the temperature profiles in plane-layer convection while keeping the mean temperature constant. Thus the rest of the study proceeded using only constant $H$ values.

Section 3.10 described an alternative parameterization of the data for mean surface heat flux. The method used (similar to Sotin and Labrosse (1999) and Shahmas et al., (2008)) had some success at predicting surface heat flux in plane-layer and spherical shell systems. When comparing plane-layer systems with adjusted heating rates, their temperatures very closely matched the mean temperatures from spherical shells. The equations for heat flux showed limited agreement between heat flux in the two different geometries. Additionally, when the internal heating rate in the sphere was increased, the disagreement between the heat flux from a sphere and plane-layer model was increased.
Moore (2008) developed an improved predictive equation for surface heat flux from a plane-layer convection system. However, this equation did not scale to spherical shell systems at all and thus did not improve the ability to match heat fluxes in different geometries. Consequently, the focus remained on the successful ability to predict and match mean temperatures in different geometries. The isoviscous convection study used previously calculated spherical shell results with different core sizes \( f \) values, and a full parameterized equation for different \( f \) values was obtained. While there was thorough undertaking in analyzing many possibilities for parameterizing the isoviscous data, the study presented here did not address the many more possibilities that exist in depth dependent viscosity calculations for analysis.

The depth-dependent viscosity study began with an analysis of a single profile and viscosity contrast in the models. With the addition of this one profile, the entire analysis of plane-layer and spherical shell convection had to be redone. The stratified viscosity presented a much more complex picture of convection and finding trends or the right regime to focus on took some time. In the isoviscous convection study, it was found that the terms representing internal heating and basal heating in the predictive equation both depended on the value of \( f \) in the sphere. A similar dependence is likely observed in spherical cases featuring stratified viscosity, however given the difficulty in emulating the temperatures in small \( f \) value spheres, this was not a focus of the stratified viscosity study. Future studies could examine the effect of different core sizes to develop an even more generalized equation for mean temperature in spherical shells with stratified viscosity.

Calculations featuring stratified viscosity in a spherical shell produced mean temperatures which were slightly dependent on Rayleigh number \( Ra \) in the absence of internal heating (see equation 4.3 in Section 4.2.2). However, the study only considered a single value for \( f \) (0.547) and a single viscosity contrast/profile. The study went on to consider a similar viscosity profile with a larger viscosity contrast (increase from 30 to 100 across the lower mantle). The results obtained indicate that this first term in the predictive equation, representing heating from basal sources \( \theta_b \), will also depend on \( Ra \) and likely \( f \) as in the isoviscous equation (see equation 3.13 in Section 3.6) as well as depending on the viscosity profile used.

Section 4.4 described the results from using a different viscosity profile but with the same viscosity contrast of 30 across the lower mantle. These results indicate that a full predictive equation for various viscosity contrasts will be specific to the viscosity profile used. Accounting for the viscosity contrast in a predictive equation using either \( \eta \) in the parameterization, or using an effective Rayleigh number based on average viscosity is not sufficient to fully parameterize the complexity generated from different viscosity profiles. Future work could explore the use of various viscosity profiles with the same viscosity contrast to determine a way to account for this variation in the parameterizations.

In addition to these extensions of the equations derived in this study, future work could examine some of the other parameters which will strongly affect the mean temperature of the system. In particular, temperature-dependent viscosity is an important parameter effecting mean temperature which could be examined in future studies. Section 5.3 describes the importance of temperature-dependent viscosity in determining the parameters necessary for the onset of stagnant-lid convection.

The study presented here determines various equations for mean temperature and heat flux in both plane-layer and spherical shell geometries with some different viscosity contrasts and profiles. While there are lots of avenues left to explore, the results presented help increase understanding of the underlying physics governing mantle convection. The study shows a way to adjust heating rates in plane-layer models in order to emulate spherical shell temperatures.
Appendix A

Minerals in the mantle

There are two types of pyroxenes in the mantle, orthopyroxene or \((\text{Mg, Fe})\text{SiO}_3\) and clinopyroxene which includes calcium and aluminum. Mantle derived samples usually contain peridotites which are mixtures of olivine and pyroxene and eclogites which are mixtures of pyroxene and garnet. Basalt becomes eclogite at depths of more than 80 km (e.g., Ringwood, 1975; Green and Falloon, 1998).

In 1952, Francis Birch found the upper mantle to be composed of olivine, pyroxene and garnet, which matched seismic data, and the lower mantle to be composed of dense oxides like periclase \((\text{MgO})\) and stishovite \((\text{SiO}_2)\). Birch found the transition zone to include pressure-dependent phase changes (Birch, 1952).

Laboratory experiments found the phase change between olivine and spinel to occur at pressures representative of the 410 km seismic discontinuity (Akimoto and Fujisawa, 1968; Ringwood and Major, 1970). Olivine transforms into spinel at around 13 -14 GPa, or a depth of 405 km, and at a temperature of approximately 1,700 K (Akaogi et al., 1989; Katsura and Ito, 1989). Figure A.1 shows the phase diagram for olivine’s transformation to spinel at different pressures and concentrations.

The pyroxene system does not experience such drastic phase changes with increased pressure (i.e., depth) or temperature and thus does not account for seismic discontinuities observed at 410 km and 660 km. The olivine-spinel phase change likely accounts for the discontinuity at 410 km. However, the discontinuity at 660 km is even larger and over a smaller range of depths (about 5 km). At 660 km seismic velocities increase as the density of the mantle is also found to increase by a large amount (e.g., Birch, 1952; Ringwood, 1975, 1982; Ringwood and Irifune, 1988).

Liu (1974) discovered a silicate structure with a high density of perovskite \((\text{MgSiO}_3)\) at pressures expected in the lower mantle. The 660 km seismic discontinuity represents transformation of spinel to magnesium perovskite and magnesiowüstite. The pyroxene reactions are likely responsible for the larger velocities gradients which extend below the 660 km by over 100 km (Irifune and Ringwood, 1987; Ito, 1989).

The phase change from olivine to spinel is exothermic, as indicated by the positive Clapeyron slope estimates of \(dp/dT = 2.5 \text{ MPa/ K} \) (Katsura and Ito, 1989) to \(dp/dT = 1.5 \text{ MPa/ K} \) (Akaogi et al., 1989) at the depth of the phase change. This reaction releases heat which enhances both hot upwellings and cold downwellings. As the hot upwelling rises, the spinel changes to olivine (an endothermic reaction) and the fluid increases in temperature, rising faster. As the cold downwelling sinks, the olivine changes to spinel (an exothermic reaction), cooling the fluid which descends faster. The phase change from
Figure A.1: Phase diagram for \((Mg,Fe)\textsubscript{2}SiO\textsubscript{4}\) at range of pressures on y-axis and \(T = 1873\)K. \(\alpha\) olivine changes to \(\beta\) spinel and then \(\gamma\) spinel at increased temperatures and/or pressures. Reprinted from Katsura and Ito (1989) with permission of the publisher John Wiley & Sons, Inc.

Spinels to perovskite, however, is endothermic, indicated by the associated negative Clapeyron slope, \(dp/dT \simeq -2.5\) MPa/K (Ito and Takahashi, 1989; Bina and Helffrich, 1994). This phase change absorbs heat from surroundings, which reduces buoyancy when crossing the boundary (e.g., Schubert et al., 2001).
Appendix B

Conductive cooling of a sphere

For steady state solutions, the temperature in a homogenous conductively cooling sphere must satisfy

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = 0.$$  \hspace{1cm} (B.1)

This can be further simplified to

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = 0$$  \hspace{1cm} (B.2)

$$\Rightarrow \frac{\partial T}{\partial r} = \frac{C_1}{r^2}$$  \hspace{1cm} (B.3)

where $C_1$ is a constant of integration.

A solution to this differential equation (B.3) is

$$T(r) = -\frac{C_1}{r} + C_2$$  \hspace{1cm} (B.4)

where $C_2$ is a second constant of integration. The constants must satisfy the boundary conditions where $T = 1$ at $r_i$ and $T = 0$ at $r_o$, and are found by solving the system of equations:

$$1 = -\frac{C_1}{r_i} + C_2$$  \hspace{1cm} (B.5)

$$0 = -\frac{C_1}{r_o} + C_2$$  \hspace{1cm} (B.6)

Taking equation (B.5) subtract equation (B.6) gives

$$1 = -C_1 \left( \frac{1}{r_i} - \frac{1}{r_o} \right)$$  \hspace{1cm} (B.7)

$$\Rightarrow 1 = -C_1 \left( \frac{r_o - r_i}{r_o r_i} \right)$$  \hspace{1cm} (B.8)

and thus $C_1 = (r_i r_o)/(r_i - r_o)$. Substituting this value into equation (B.6) gives $C_2 = r_i/(r_i - r_o)$.

Using the fact that $f = r_i/r_o$ and the assumption that the shell depth is 1, so $r_o - r_i = 1$,
\[
C_1 = -r_1 r_o \quad \text{(B.9)}
\]
and
\[
C_2 = -r_i. \quad \text{(B.10)}
\]

Furthermore, \( r_o \) can be rewritten as \( r_o = 1/(1 - f) \) and \( r_i \) can be rewritten as \( r_i = f/(1 - f) \) so the constants can be written

\[
\Rightarrow C_1 = \frac{-f}{(1 - f)^2} \quad \text{(B.11)}
\]
\[
\Rightarrow C_2 = \frac{-f}{1 - f} \quad \text{(B.12)}
\]

Therefore equation for temperature as a function of radius in a purely conductive spherical shell is given by

\[
T(r) = \frac{-C_1 r}{r} + C_2
\]
\[
\Rightarrow T(r) = \frac{r_1 r_o - r_i}{r} \quad \text{(B.13)}
\]
\[
\Rightarrow T(r) = \frac{f}{r(1 - f)^2} - \frac{f}{1 - f} \quad \text{(B.14)}
\]

Taking the limit as \( f \to 1 \) (the plane-layer limit), is equivalent to the limit when \( r_i \to \infty \) with \( f = r_i/(r_i + 1) \). Equation (B.15) becomes

\[
\lim_{r_i \to \infty} \frac{r_i r_o}{r_i + z} - r_i = \frac{r_i(r_i + 1)}{r_i + z} - \frac{r_i (r_i + z)}{r_i + z} \quad \text{(B.16)}
\]
\[
= \lim_{r_i \to \infty} \frac{r_i (r_i + 1) - r_i (r_i + z)}{r_i + z} = \frac{r_i (1 - z)}{r_i + z} \quad \text{(B.17)}
\]
\[
= \lim_{r_i \to \infty} \frac{r_i^2 + r_i - r_i^2 - r_i z}{r_i + z} = \frac{1 - z}{1 + z/r_i} = 1 - z \quad \text{(B.18)}
\]
Appendix C

Energy balance

By the conservation of energy, the heat flow out of the system must equal the heat flow into the system plus any internal heat generation in the system. Consider a spherical shell of inner radius, \( r_{\text{bot}} \) and outer radius, \( r_{\text{top}} \). The dimensional heat flow out of a spherical shell is denoted \( Q_{\text{top}} \), while the dimensional heat flow into the system at the basal boundary is denoted \( Q_{\text{bot}} \). The equation for heat flow from a spherical shell is given, in dimensional quantities, as:

\[
Q_{\text{top}} = Q_{\text{bot}} + \varepsilon \rho \frac{4}{3} \pi (r_{\text{top}}^3 - r_{\text{bot}}^3) \quad (C.1)
\]

where \( \varepsilon \) is heat per unit mass and \( \rho \) is the density. Equation (C.1) can be written in terms of heat flux as

\[
q_{\text{top}} 4\pi r_{\text{top}}^2 = q_{\text{bot}} 4\pi r_{\text{bot}}^2 + \varepsilon \rho \frac{4}{3} \pi (r_{\text{top}}^3 - r_{\text{bot}}^3) . \quad (C.2)
\]

Expanding equation (C.2) yields

\[
q_{\text{top}} 4\pi r_{\text{top}}^2 = q_{\text{bot}} 4\pi r_{\text{bot}}^2 + \varepsilon \rho \frac{4}{3} \pi (r_{\text{top}}^3 + r_{\text{bot}})(r_{\text{top}}^2 - r_{\text{bot}}r_{\text{bot}}^2) \quad (C.3)
\]

and dividing through by \( r_{\text{top}}^2 \) gives

\[
q_{\text{top}} 4\pi = q_{\text{bot}} 4\pi \frac{r_{\text{bot}}^2}{r_{\text{top}}^2} + \varepsilon \rho \frac{4}{3} \pi (r_{\text{top}} + r_{\text{bot}})(1 - \frac{r_{\text{bot}}}{r_{\text{top}}} + \frac{r_{\text{bot}}^2}{r_{\text{top}}^2}) \quad (C.4)
\]

Using the relation \( f = \frac{r_{\text{bot}}}{r_{\text{top}}} \), equation (C.4) can be simplified to

\[
q_{\text{top}} = q_{\text{bot}} f^2 + \varepsilon \rho \frac{1}{3}(r_{\text{top}} - r_{\text{bot}})(1 + f + f^2) \quad (C.5)
\]

Equation (C.5) is nondimensionalized using

\[
q'_{\text{top}} = \frac{q}{k \Delta T / d} \quad (C.6)
\]

where \( \Delta T \) is the temperature difference between the upper and lower boundaries of the spherical shell; \( q \) is the dimensional heat flux; \( k \) is the thermal conductivity and \( d \) is the depth of the mantle in the spherical shell. This yields
\[ q'_{\text{top}} \frac{k \Delta T}{d} = \frac{k \Delta T}{d} q'_{\text{bot}} f^2 + \varepsilon \rho d \frac{1}{3} (1 + f + f^2) \]  
(C.7)

which becomes

\[ q'_{\text{top}} = q'_{\text{bot}} f^2 + H \frac{(1 + f + f^2)}{3} \]  
(C.8)

where \( H = \varepsilon \rho d^2 / k \Delta T \) is the nondimensional internal heating rate.

In the plane-layer case, \( r_{\text{bot}} \rightarrow r_{\text{top}} \) so that \( f = 1 \) and equation (C.8) reduces to the plane-layer equation

\[ q'_{\text{top}} = q'_{\text{bot}} + H \]  
(C.9)

where \( H \) is the nondimensional internal heat generation.
Appendix D

Computational changes to numerical codes

D.1 Changes to the plane-layer geometry model, MC3D

MC3D was written as a Fortran 77 code for numerical mantle convection modelling. I assisted my supervisor Dr. Julian Lowman in migrating the code from Fortran 77 to Fortran 90, with notable help and guidance from the code’s original developer, Dr. Carl Gable. One of the main changes involved removing pointers used for memory allocation, and replacing them with modules featuring dynamic memory allocation statements and common blocks. These modules and common blocks can be placed to confine memory allocation to a small set of subroutines.

Later, I migrated the code from using the PGI compiler on a cluster at the UTSC campus (UTSCGrid) to using the Intel compiler on Sharncet and SciNet. I received helpful debugging tips and guidance from the helppages at both Sharncet and SciNet (e.g., Loken et al., 2010). The PGI compiler uses a default value of zero for all undefined variables, but the Intel compiler does not do this. This necessitated finding any instances in the code where a variable may have been undeclared, but assumed to be zero. In addition, some of the older subroutines written in Fortran IV or 77 had to be changed.

When the Intel compiler was updated at SciNet, this required another update of the code. In all instances, a test suite is run to ensure the code continues to produce the same results, and continues to agree with the benchmark cases.

In addition, I added a subroutine to compute the temporal average of the horizontally averaged temperatures with depth to be able to produce the geotherms in this thesis.

D.2 Changes to the spherical shell geometry model, TERRA

TERRA was provided to me as a Fortran 90 code. Only minor changes were required to succeed in compiling the code with the Intel compiler on SciNet, most of which are related to how much more strict this compiler is in regards to variable declaration and initialization.

The code originally used a fixed maximum value for the radial grid resolution of 128 radial layers. Therefore, the code filled arrays of fixed size 128, in particular in the definition of radial viscosity. Since I wanted to run a large Rayleigh number ($10^8$) spherical calculation, I needed to increase the
radial resolution to 256 radial layers. I changed the code to allow the specification of \( nr \) in the input parameters, and used \( nr \) as a parameter in the code for the number of radial layers. This allowed me to then run spherical shell models with 256 radial layers. These changes were discussed with Dr. Hans-Peter Bunge, who provided us with the code. (Note: The calculation I have been running with \( Ra = 10^8 \), \( H = 0 \) and a viscosity contrast of 30 has not yet reached a statistically steady state.)

In order to run stratified viscosity models with TERRA, it is necessary to prescribe the viscosity at each radial layer in the viscosity subroutine. This subroutine used an array filled with values of viscosity at each radial layer, so it was necessary to compute the viscosity at each layer from equation (2.37) and input this into an array. TERRA also required gradual increases in the lower mantle viscosity to avoid numerical instabilities. Thus, although not presented, the study described required that I prescribe arrays with a viscosity increase of a factor of 3, 10, 30 and 100 to use in the calculations in Chapter 4.
Appendix E

Depth-dependent $H$ derivation

Figure 3.42 shows a schematic of the differing volumes of layers with the same thickness, $z_o - z_i$, in different geometries with the same surface area, $A$. At a given depth, the volume of a spherical shell layer differs from the volume of a plane layer. To compensate for this geometric difference in the heat input, a depth-dependent function for $H$ can be specified.

Consider the internal heating rate in a plane-layer and spherical shell to be given by $H_{\text{box}}$ and $H_{\text{sphere}}$, respectively. If the total amount of nondimensional internal heating $H_{\text{total}}$ is equal in both geometries, the integral of $H_{\text{box}}$ over the layer from $z_i$ to $z_o$ will be equal to $H_{\text{sphere}}$ times the volume of the spherical shell between $r_i + z_i$ and $r_i + z_o$.

The total amount of nondimensional internal heating, $H_{\text{total}}$, in a plane-layer between $z_i$ and $z_o$ is $A$ times the integral of $H_{\text{box}}$ over the layer, which is

$$A \int_{z_i}^{z_o} H_{\text{box}}(z)dz \quad (E.1)$$

and the total amount of nondimensional internal heating $H_{\text{total}}$ in a spherical shell between $r_i + z_i$ and $r_i + z_o$ is $H_{\text{sphere}}$ times the volume of the spherical shell from $r_i + z_i$ to $r_i + z_o$, that is

$$H_{\text{sphere}} \frac{4}{3} \pi ((r_i + z_o)^3 - (r_i + z_i)^3). \quad (E.2)$$

Setting the surface of each section to be equal, so $A = 4 \pi r_o^2$. Equating the total amount of internal heating in the layer of the box and spherical shell section gives

$$4 \pi r_o^2 \int_{z_i}^{z_o} H_{\text{box}}(z)dz = H_{\text{sphere}} \frac{4}{3} \pi ((r_i + z_o)^3 - (r_i + z_i)^3) \quad (E.3)$$

which simplifies to

$$\int_{z_i}^{z_o} H_{\text{box}}(z)dz = H_{\text{sphere}} \frac{1}{3r_o^2} ((r_i + z_o)^3 - (r_i + z_i)^3) \quad (E.4)$$

Taking the derivative of both sides of equation (E.4) with respect to $z_o$, and applying the fundamental theorem of calculus ($d/dx \int_{a}^{x} f(z)dz = f(x)$) gives

$$H_{\text{box}}(z_o) = H_{\text{sphere}} \frac{1}{3r_o^2} (3(r_i + z_o)^2) \quad (E.5)$$
or
\[ H_{box}(z_o) = H_{sphere} \frac{(r_i + z_o)^2}{r_o^2}. \]  

(E.6)

It is necessary to prescribe an appropriate upper boundary (surface) cooling rate \((H_{top})\), so that the mean internal cooling rate \((H_{mean})\) is the value determined by inverting equation (3.13).

\(H_{mean}\) is the total amount of internal heating/cooling, \(H_{box}\), in the whole plane-layer, divided by the volume of the plane-layer model. Therefore,

\[
H_{mean} = \frac{A}{A} \int_0^1 H_{box}(z)\,dz 
\]

(E.7)

\[ \Rightarrow H_{mean} = \int_0^1 H_{box}(z)\,dz \]

(E.8)

Now, equation (E.4) gives an expression for \(\int_0^1 H_{box}(z)\,dz\) (with \(z_i = 0\) and \(z_o = 1\)) which can be used in equation (E.8) to get

\[
H_{mean} = H_{sphere} \frac{1}{3r_o^2} ((r_o)^3 - (r_i)^3).
\]

(E.9)

where \(r_i + 1 = r_o\).

So in a sphere with \(f = 0.547\), \(r_o = 2.2\) and \(r_i = 1.2\) (with \(r_o - r_i = 1\)), which in equation (E.9) gives \(H_{mean} = 0.614H_{sphere}\). So \(H_{top}\) (the prescribed surface heating or cooling rate) is given by \(H_{mean}/0.614\) in spherical shells with \(f = 0.547\).
Bibliography


