Optimal Pairs Trading: Static and Dynamic Models

by

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Abstract

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Pairs trading has been a popular statistical arbitrage strategy among hedge funds. One important research field in pairs trading is to maximize the return under differential constraints and assumptions. In this thesis, we develop two models to optimize the performance of pairs trading. In the static model, we find the analytic solution of optimal thresholds for pairs trading to maximize the long run profit per unit time. Comparison is made between the optimal rules we developed and the common practice. To overcome limitations of the static model, we extend our research to dynamic pairs trading, where a continuous time Markov chain is used to model the change of parameters. Our objective is to maximize the expected return in the finite horizon under the Constant Relative Risk Aversion (CRRA) utility. Numerical examples are presented to illustrate the impact of price limits, risk aversion rate and regime switching on consumers’ investment decision.
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Chapter 1

Introduction

1.1 Background and Motivation

Since its birth in the 1980s, pairs trading has been one of the most popular statistical arbitrage strategies among major investment banks and hedge funds because the average annualized excess return of pairs trading is as high as 11%. However, the idea behind this is very simple. If prices of a pair always move together in the history, they are likely to continue with the same trend in the future. So when prices deviate, they are expected to converge in the future. The trader can simply take a short position in the overpriced stock and a long position in the underpriced one, and wait for the price difference to converge in the future. If convergence happens, he can clear positions in both stocks and make a profit.

A natural question for pairs trading is how to take and clear positions optimally (i.e. what is the optimal thresholds of taking and clearing positions). Of course, different investors have different definitions of "optimal" because their objective functions are different. As far as we know, no research has directly answered this simple but nontrivial question. Inspired by Bertram [2010], we decide to investigate into the optimal thresholds of pairs trading to maximize the profit per unit time in the long run. In this static model, we will derive the explicit formula for the expectation of the first passage time over a two-sided symmetric boundary for an OU process, and use this expectation to find the analytic solution of the optimal thresholds.

The limitation of this static model is the assumption that model parameters are time invariant. However, in the long run, it is unrealistic to assume constant parameters. Therefore, a dynamic model is needed to consider the change of parameters. Mudchanatongsuks, Primbs, and Wong [2008] and Fourin and Yan [2013] have laid a dynamic programming framework for pairs trading, and Wan [2006] has proposed a regime switching model for the change of parameters. In our dynamic model, we will also adopt the assumption that the regime switching follow a continuous time Markov chain, and we will incorporate the co-integrating factor into the return of the stocks. Unlike the static model, we will consider the finite horizon case where the investor wants to maximize his expected utility by maturity.

1.2 Common Practice

An important factor in pairs trading is the spread of the pair of stocks. Usually the spread is measured by the price ratio or the log of the ratio. For practitioners, the common practice of pairs trading is
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Figure 1.1: Common practice of pairs trading. Curve is the spread, and different colors mean different stages in a trading cycle.

summarized in three steps (see Figure 1.1): firstly, when the spread deviates from the mean by two standard deviations, short the overpriced stock and long the underpriced one (T1 in Figure 1.1); secondly, when the spread reverts, clear positions to make a profit (T2 in Figure 1.1); lastly, wait until the spread widens again to take another position. Methods to measure the spread include the price ratio, the squared Euclidean norm between the prices of a pair, and the log difference of the prices of a pair. Also, one does not have to follow the rule of two standard deviations in the first step, but can strategically change the triggering point according to the market condition and the actual movement of the pair. Similarly, it is not a fixed rule to clear positions exactly when the spread reverts back to the mean. One might want to wait longer to gain larger profit, but inevitably bears more risk. There is always a trade-off between the profit in one trade and the time it takes for a trade to complete. Therefore, setting the trading thresholds optimally is critical to balance the trade-off. As far as we know, no research has been conducted on the optimal thresholds of pairs trading in the literature. In our static model, we are going to present analytic results of the optimal thresholds to maximize the long run expected average profit, and draw practical insight from the results.

1.3 Mean Reversion in Pairs Trading

The most important feature of pairs trading is its mean reverting property. Various quantitative methods have been developed and applied to pairs trading in the literature. Three commonly used techniques are: distance method, co-integration and stochastic spread. The distance method is often used by practitioners. Nathi [2003] used 15 percentile of the distribution of distance as a trigger for trading and 5 percentile as the stop-loss barrier. Despite its model-free feature that prevents mis-estimation of parameter, the distance method provides little help in forecasting according to Do, Faff, and Hamza [2006]. Instead, Vidyamurthy [2004] developed a good framework for forecasting using the co-integration method, and analyzed the mean reversion of the residuals. Gatev and Rouwenhorst [2006] selected pairs and generated trading signals based on this method. The method of cointegration was further applied by Lin, McCRAE, and Gulati [2006] to develop a loss protection for pairs trading and Puspaningrum, Lin, and Gulati [2010] to develop algorithms to estimate trade duration and find optimal pre-set boundaries. The stochastic spread method, on the other hand, models the mean reverting process of pairs trading as an Ornstein-Uhlenbeck (OU) process. Elliott, Van D.H., and Malcolm [2005] provided an analytic
framework of pairs trading, which laid ground for prediction and decision-making based on the hidden OU process. Ekström, Lindberg, and Tysk [2011] explored optimal liquidation of pairs trading under the framework of OU process and analyzed the sensitivity of model parameters. Vladisavljevic [2004] and Boguslavsky and Boguslavskaya [2004] also based their research on the OU process. In our static model in this thesis, we will continue to use co-integration method and the OU process.

1.4 OU Process and First Passage Times

Since the static model is mainly based on the OU process and its first passage times, we will give a brief review on some relevant literature. The OU process (Uhlenbeck and Ornstein [1930]) is a special kind of stochastic process whose movement is highly related to the mean value of this process. The general form of the classic OU process is:

$$dX_t = -\theta(X_t - \mu)dt + \sigma dW_t$$

This process $X_t$ is determined by the reversion rate $\theta > 0$, the mean value $\mu$, and the scale of fluctuation $\sigma$. In this equation, $W_t$ is the standard Wiener process (or Brownian motion). When $X_t$ goes above its mean ($X_t > \mu$), we will have $E[dX_t] = E[-\theta(X_t - \mu)dt] < 0$ (because $E[dW_t] = 0$), which means that $X_t$ is expected to go down in the next moment (though it does not necessarily go down since the actual process is also determined by the value of $W_t$). Similarly, when $X_t$ is below the mean, it is more likely to go up in the next moment. We call this property mean reversion since whenever the process deviates from the mean, it will tend to move towards the mean in the future. Naturally, many mean reverting processes, such as the spread in pairs trading, are modeled by the family of OU processes.

The OU process has wide application in various fields, such as finance (Patie [2004]), biology (Ricciardi and Sacerdote [1979]), chemistry (Kim [1958], Buchete and Straub [2001]), physics (Tateno, Doi, Sato, and Ricciardi [1995], Lindenberg, Shuler, Freeman, and Lie [1975]), engineering (Roberts [1974], Khan, Datta, and Ahmad [2004]), software reliability and network security (Ma, Krings, and Millar [2009]).

Here we show some basic facts of the OU process. The first two moments of $X_t$ in the unconditional case is:

$$E[X_t] = \mu, \quad Var[X_s, X_t] = \frac{\sigma^2}{2\theta} e^{-\theta|s-t|}$$

and the conditional case:

$$E[X_t|X_0 = c] = \mu + (c - \mu)e^{-\theta t}, \quad Var[X_s, X_t|X_0 = c] = \frac{\sigma^2}{2\theta} (e^{-\theta|s-t|} - e^{-\theta(s+t)})$$

The first passage time (or first hitting time) is among the most important properties of the OU process. Suppose the process $X_t$ starts at a point $x$, then the first passage time is defined as the time it takes to reach another point $c$. Mathematically, the first passage time $\tau_{x,c}$ is defined as:

$$\tau_{x,c} = \inf \{ t > 0; \ X_t = c \mid X_0 = x \}$$

There is an abundant literature on the first passage time of the OU process. For the standard OU process where $\theta = 1$, $\mu = 0$ and $\sigma^2 = 2$, the probability density of $\tau_{0,c}$ is shown in Wang and Uhlenbeck.
where erf $x$ is the error function defined as \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \), and for $x > 0$, erf $x = 1 - \frac{2}{\pi} \int_0^x e^{-t^2} dt$.

To find the moments of $\tau_{x,c}$, the Laplace Transform has been shown by Siegert [1951] and Darling and Siegert [1953] as:

\[
E[e^{-\lambda \tau_{x,c}}] = \int_0^\infty f_{x,c}(t)e^{-\lambda t} dt = \begin{cases} D_{-\lambda}(x) \exp\left(\frac{x^2 - c^2}{4}\right), & \text{if } x < c, \\ D_{-\lambda}(c) \exp\left(\frac{x^2 - c^2}{4}\right), & \text{if } x > c, \end{cases}
\]

where $D_{-\lambda}(x)$ is the Weber function:

\[
D_{-\lambda}(x) = \left\{ \begin{array}{ll}
\int_0^\infty (1 + erf\left(\frac{t}{\sqrt{2}}\right)) \exp\left(-\frac{t^2}{2}\right) dt, & \text{for } \lambda < 1 \\
\frac{1}{\Gamma(\lambda)} \int_0^\infty (1 + erf\left(\frac{t}{\sqrt{2}}\right)) \exp\left(-\frac{t^2}{2}\right) dt, & \text{for } \lambda > 0
\end{array} \right.
\]

Note that for $0 < \lambda < 1$, the two equations agree.

One can calculate the moments of $\tau_{x,c}$ by integrating the probability density $f_{x,c}$, or differentiating the Laplace transform $E[e^{-\lambda \tau_{x,c}}]$ and setting $\lambda = 0$. Here we show the first two moments for $\tau_{x,0}$ and $\tau_{0,y}(x, y > 0)$ from Thomas [1975], Sato [1977] and Ricciardi and Sato [1988]:

\[
E[\tau_{x,0}] = \sqrt{\pi} \int_{-\infty}^0 (1 + erf\left(\frac{t}{\sqrt{2}}\right)) \exp\left(-\frac{t^2}{2}\right) dt = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!} \Gamma\left(\frac{k}{2}\right)
\]

\[
E[\tau_{0,y}] = \sqrt{\pi} \int_0^\infty (1 + erf\left(\frac{t}{\sqrt{2}}\right)) \exp\left(-\frac{t^2}{2}\right) dt = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k}{2}\right)
\]

\[
Var(\tau_{x,0}) = \sqrt{2\pi} \int_0^\infty \int_{-\infty}^t \int_0^{t-s} (1 + erf\left(\frac{r}{\sqrt{2}}\right)) \exp\left(\frac{r^2 + t^2 - s^2}{2}\right) dr ds dt - E(\tau_{0,x})^2
\]

\[
= E(\tau_{0,x})^2 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k}{2}\right) \Psi\left(\frac{k}{2}\right)
\]

\[
Var(\tau_{0,y}) = \sqrt{2\pi} \int_0^\infty \int_{-\infty}^t \int_0^{t-s} (1 + erf\left(\frac{r}{\sqrt{2}}\right)) \exp\left(\frac{r^2 + t^2 - s^2}{2}\right) dr ds dt - E(\tau_{y,0})^2
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k}{2}\right) \Psi\left(\frac{k}{2}\right) - E(\tau_{y,0})^2
\]

where erf $x$ is the error function defined as $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $\Gamma(x)$ is the gamma function defined as $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ and $\Psi(x)$ is the digamma function defined as $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

By the Markovian property and the symmetric property of the OU process, it is easy to find the first two moments of the first passage time from any point $x$ (not necessarily 0) to another point $c$ (not necessarily 0).
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Certainly, there are other types of first passage times. A more complex one is the first passage time over a two-sided boundary. In this case, the starting point $c$ is inside the range $[a, b]$, the first passage time is defined as the time from $c$ to either the upper bound $b$ or the lower bound $a$, whichever first. Mathematically, the definition of the first passage time over a two-sided boundary is:

$$
\tau_{a,c,b} = \inf \{ t > 0; \ X_t = a \ or \ X_t = b \ | \ X_0 = c \}
$$

Let the probability density function of $\tau_{a,c,b}$ be $f_{a,c,b}(t)$. Darling and Siegert [1953] showed that for $-a = b > 0$, the Laplace transform of $f_{-b,c,b}(t)$ is:

$$
E(e^{-\lambda \tau_{-b,c,b}}) = \frac{D_{-\lambda}(c) + D_{-\lambda}(-c)}{D_{-\lambda}(b) + D_{-\lambda}(-b)} \exp\left(\frac{c^2 - b^2}{4}\right)
$$

Though numerical methods have long existed, there is no known closed form of $f_{-b,c,b}(t)$ or the moments of $\tau_{-b,c,b}$. In Chapter 2 of this thesis, we will contribute to the literature on the closed form expectation of $\tau_{-b,c,b}$, both in the integral form and the polynomial form.

1.5 Stochastic Control and Pairs Trading

Stochastic control, or dynamic programming, has gained much popularity in finance. The advantage of stochastic control is that the resulting Hamilton-Jacobi-Bellman partial differential equation (HJB-PDE) directly leads us to the optimal solution. If lucky enough, analytic solutions can be found by solving the HJB-PDE. Merton [1969] first introduced this approach into portfolio optimization. In his seminal paper, he studied the optimal asset allocation and consumption when an individual can continuously adjust his investment in each asset and the rate of consumption in order to maximize his total utility by the maturity $T$. The optimal equation is a second order partial differential equation in terms of the current time $t$ and the wealth $W$. Under the Constant Relative Risk Aversion utility, he found that the optimal proportion of wealth invested in the risky asset is a constant, independent of $W$ and $t$, and the optimal consumption is a function of the wealth $W(t)$ and the time to maturity. The individual should consume all his wealth by the end of the maturity to maximize his total utility. In his model, he also discussed the infinite horizon case and the Constant Absolute Risk Aversion (CARA) utility. Since then, stochastic control has been widely used in finance, especially in the area of portfolio management. The limitation of Merton [1969] is that it failed to consider the transaction cost. In his paper, a rational investor should adjust his positions to maintain the proportion of wealth invested in the risky asset for infinitely many times in a finite time horizon. However, in the presence of proportional transaction cost, where a certain amount of transaction fee must be paid for each dollar transferring in or out of a risky asset, the investor has to make fewer transactions. Magill and Constantinides [1976] and Constantinides [1979] were the first to consider the proportional transaction cost in the model. In their work, the concept of region of inaction was introduced to describe the optimal policy for investment and consumption. The investor would not do any transaction unless the wealth invested in the risky asset fell out of a certain interval, and would put the minimal efforts on transaction just enough to bring the fraction of wealth inside the interval once it fell outside. Since then, Richard [1977] and Takar, Klass, and Assaf [1988] also considered the transaction cost in the stochastic control framework. Davis and
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Norman [1990] was the first to propose a comprehensive mathematical analysis on the problem. They considered the optimal proportion of wealth to allocate between one risky asset and one risk free asset when an investor was faced with proportional transaction cost. They solved the problem numerically to find the optimal policy. The optimal investment problem with proportional transaction cost in the literature has already been known as a singular stochastic control problem, where the rate of control is infinity at some points and 0 at others. To determine the region of inaction, one is always forced to solve the free boundary problem associated with the HJB equations. More study on the related problem can be found in Janeček and Shreve [2004], Leland [2000] and Shreve and Soner [1994]. Liu [2004] considered the multi-asset problem with transaction cost. Assuming that asset returns are independent, he successfully changed the multi-dimensional problem to one-dimensional problems characterized by ordinary differential equations. Muthuraman and Zha [2008] considered a more general case where the stock returns are correlated, and proposed a computational scheme for the multi-dimensional free boundary problem.

For pairs trading, the correlation between a pair of stocks cannot be simply captured by the variance. In this case, co-integration is used on the returns of the stocks. Mudchanatongsuk et al. [2008] first proposed a stochastic control approach on pairs trading to capture the dynamic change of the spread, assuming the spread follows an OU process. Their objective was to maximize the power utility of wealth at maturity subject to the dynamics of the spread as an OU process. They assumed the money invested on the pair adds up to 0, and therefore the wealth at any time \( t \) is only determined by the wealth and the value of the spread, but not affected by the information of the actual prices of the pair. For trading with multiple pairs, Kim, Primbs, and Boyd [2008] considered the optimal allocation of wealth to each spread over a finite horizon in the dynamic programming framework. The spreads are assumed to follow the OU processes, and they showed that the optimality equations are a system of ordinary differential equation, which made the problem computationally tractable. In both papers, the information of the actual prices was not used, which makes their models less realistic. Inspired by Mudchanatongsuk et al. [2008], Tourin and Yan [2013] studied a more general model using stochastic control. They incorporated the co-integrating factor \( Z_t = \alpha + \log(S_{1t}) + \beta \log(S_{2t}) \) into the dynamics of one of the two stocks and showed \( Z_t \) also followed an OU process. In their research, investment at any time \( t \) in each of the stocks is determined by the stock prices as well as the wealth at that time. In order to get closed form solutions, they assumed the risk free rate of return in the money account is 0. Though this model provided a more general framework for pairs trading using stochastic control, there are still some limitations. Firstly, their analytic solutions are not valid when the risk free rate of return exists. Secondly, when the utility function is not CARA, but CRRA or other types of utility, then it will be hard to get the analytic solutions. Thirdly, when transaction cost exists, the investor cannot constantly adjust his position to meet the optimal policies in their model. In this case, the resulting problem will again be a free boundary problem. Qingshuo and Qing [2013] considered a fixed commission cost in each transaction, and also imposed stop-loss limits. In their model, the optimal policy can be found by solving a number of quasi-algebraic equations.

In traditional settings involving stochastic control, movement of the stock prices and mean spread are usually assumed to be uncertain but stationary, i.e. the model parameters are constant. Models under such assumptions often fail to optimally deal with the change of parameters. In the dynamic model, we propose a stochastic control approach on pairs trading to determine the optimal investment policy when parameters change in different regime states. The change of regime states are assumed to follow
a continuous time Markov chain. This assumption can also be found in Zhang [2001], Guo [2005] and Wan [2006]. Zhang [2001] considered an optimal selling rule by finding the optimal pre-selected target price and stop-loss limits. In their model, prices followed geometric Brownian motion coupled by a finite state Markov chain. They have derived analytic solutions for one-dimensional and two-dimensional cases. They have also proposed a way to calibrate the transition matrix of the continuous time Markov chain based by market data. In Guo [2005] the investor was trying to maximize his discounted expected payoff when there was a fixed amount $K$ to be paid back when he sold his stock. This was a optimal stopping problem. Again, the stock price was assumed to follow the geometric Brownian motion and the mean and variance were assumed to follow a continuous time Markov chain. They proved that the optimal stopping rule was of a threshold type for each state using the martingale theory. Wan [2006] proposed a general model for portfolio optimization. In this model, the investor was trying to maximize his discounted expected total wealth by deciding the fraction of wealth to allocate to multiple risky assets and one risk free asset. He also assumed model parameters to follow a finite state continuous time Markov chain. He derived the analytic solutions for both one-regime and two-regime case when only one risky asset and one risk free asset were considered. His one-regime case was the solution from Merton [1969]. The two-regime case was similar in the sense that the optimal investment should be a constant proportion of wealth to invest in each asset in each regime. Of course when regime states change, the optimal proportion should also change. More regime switching model can be found in Ishijima and Uchida [2011], Zhou and Yin [2003] and Bungton and Elliott [2002]. In our dynamic model, we will combine the stochastic control framework of pairs trading from Tourin and Yan [2013] and the regime switching model from Wan [2006] to study the optimal investment decisions in pairs trading.

1.6 Research Objectives

In this thesis, we are going to present two different models to optimize the performance of pairs trading. The first model is based on the common practice: take positions when the spread widens and clear positions when it reverts. Our objective is to find the optimal thresholds so that the investor can achieve the maximal profit per unit time in the long run. To achieve this objective, we need to know some basic properties (expectation, . . .) of two types of first passage time. The first passage time over the one-sided boundary has been extensively studied in the literature. Details can be found in Thomas [1975] and Ricciardi and Sato [1988]. There is also some literature on the first passage time over the two-sided symmetric boundary, which can be found in Darling and Siegert [1953]. However, as far as we know, the expectation for this type of first passage time is not explicitly known, though the numerical methods have long existed. Our first step to achieve our objective is to derive the explicit form of this expectation. With this explicit form, we will proceed to find the optimal thresholds of pairs trading to maximize the profit per unit time in the long run. There are mainly three cases to discuss depending on the threshold of clearing positions, and in each case, we need to solve a non-linear optimization problem with certain constraints. We need to compare the maximal value in each case to get the overall maximum in the whole domain. Also, to show the performance of our optimal trading rule, we need to use some real data to test the profitability for a given pair of stocks.

The limitation for the first model is the assumption that model parameters are constant. Also some information is wasted in the static model in the sense that actual stock prices are not considered, but only the spread of the prices affect our decision. To overcome this limitation and put more dynamic
flavor in our model, we turn to stochastic control in pairs trading. To model the change of parameters, we adopt the assumption that the regime switching of the parameters follow a continuous time Markov chain from Wan [2006], Zhang [2001] and Guo [2005]. Also, to make the model more realistic, we impose price limits on the stocks since this usually happens as a regulation. However, if there is no regulation, one can simply let the price limits in our model tend to infinity. To find the optimality condition, we will derive the HJB equations and solve it numerically. Lastly, we will show some numerical examples to see how investor’s decisions can be affected by change of model parameters, price limits and risk aversion rate.

1.6.1 Contributions

In summary, there are three main contributions in the static model, and each of them has its use in future research or on practical issues.

- **Explicit forms of the expectations for the first passage time over the two-sided symmetric boundary in the OU process.** For the expectation, we have derived both the integral form and the polynomial form. Our explicit form has the potential application in modeling interest rate and credit risk in finance since OU process is heavily involved in these fields. Application of OU process and first passage time in interest rate modeling can be found in Marsh and Rosenfeld [1983], and application in credit risk modeling can be found in Cariboni and Schoutens [2009] and Elliott, Jeanblanc, and Yor [2000].

- **Analytic solution of the optimal thresholds.** The solution of optimal thresholds can be found by solving an explicit equation. We have proved that the optimal thresholds of taking positions should be symmetric with the optimal thresholds of clearing positions around the mean of the spread. This result is counter-intuitive since people always clear positions when the spread reverts back to the mean in common practice. We have shown that the optimal trading rule cuts the transaction cost in a half compared to the common practice.

- **Profitability quantity.** This quantity is the byproduct when we derive the optimal thresholds. It has the potential to represent profitability of a pair of stocks in the long run. This quantity basically says that higher mean reversion rate and standard deviation of the spread would lead to higher profit per unit time. We have tested the performance of this quantity using five pairs.

As for the dynamic model, there are mainly two contributions:

- **Incorporation of continuous time Markov chain in the stochastic control framework of pairs trading.** The stochastic control approach has been introduced to pairs trading by Mudchanatongsuk et al. [2008] and improved by Tourin and Yan [2013]. Both of them assumed constant model parameters. Our model is more general and practical since we consider the optimal investment decisions when model parameters change.

- **Designing numerical schemes to solve a large system of the HJB equations.** The optimality condition for our model is $L$ HJB partial differential equations (PDE), where $L$ is the total number of regime states. In each PDE, there are three space dimensions (wealth and two stock prices) and a time dimension. For general cases, numerical methods are needed to solve the
systems of equations since analytic solutions are intractable. We designed a fully implicit numerical scheme based on the successive approximation approach by Chang and Krishna [1986] and Peyrl, Herzog, and Geering [2005].

1.6.2 Summary of Research Projects

The Static Model

To maximize the expected profit per unit time in the long run, the trader should choose the right entry and exit thresholds. If the thresholds are narrow, then the time it needs to complete a trade is small, but so is the profit in each trade. On the other hand, if thresholds are too wide, the profit in each trade is larger, but so is the total time needed to complete a trade. Bertram [2010] argued strongly for the role of time and derived analytic formula for the thresholds of a synthetic asset whose price is assumed to follow an OU process. It showed that the optimal thresholds were symmetric around the mean both in maximizing the return per unit time and the Sharpe ratio. In his paper, short selling of the synthetic asset is not allowed. Inspired by Bertram [2010], we consider that actual trading process of pairs trading instead of trading them as a synthetic asset. While there is always a waiting time between two trades in Bertram [2010], we prove that it is optimal for a pairs trading rule to keep running and never stop. In other words, when the trader clears positions of the synthetic asset in Bertram [2010], we show that the trader should also short sell the synthetic asset at the same time.

In summary, the static model contributes both theoretically and practically. From a theoretical point of review, we derive the analytic form of the expectation of the first passage time of OU process with two-sided boundary. From a practical point of view, we obtain the analytic formula of optimal thresholds for pairs trading, and the results are counter-intuitive. To compare with the common practice, we also show a step-by-step procedure on the daily data of Coca-Cola and Pepsi. Results show that the new optimal rule developed in the static model performs better than the common practice.

The Dynamic Model

As far as we know, there has been no study employing stochastic control on pairs trading that allows model parameters to change over time, or the structure of co-integration between the pair to break up. An example can be the pair of Google (GOOG) and Apple (AAPL). While GOOG increased steadily before and after the financial crisis, AAPL increased very slowly before financial crisis and much faster after the financial crisis. One might have experienced a huge loss if he assumed the relative performance of the pair is not changing at all. Another example is the exchange rate between the Chinese Yuan (CNY) and US Dollars (USD). In the past ten years, the exchange rate was very stable from the year 2003 to 2005, followed by a dramatic and steady appreciation from the year 2005 to 2008. The exchange rate came back to a steady state after 2008, until a series of Quantitative Easing policies by the US government starting in 2010, which led to another period of steady appreciation. There might be strong co-integration pattern in the steady periods, but the pattern broke up in the period of appreciation. In both examples, parameters in certain period of time may be relatively stable, but may differ significantly in other periods of time. Such phenomenon can be modeled by regime switching. Just like the market may switch between bear and bull market, the relative performance between a pair can also switch. In Wan [2006] and Guo [2005], the market regime switches according to a finite state continuous time Markov chain (CTMC). Similarly we adopt the same assumption, and let model parameters be determined by the
state space at each time $t$. We also introduce price limits on the pairs as part of regulation. For the case where there is no price limits, we can simply let the limits in our model go to 0 and infinity. Therefore, in our model, the trading exit time is not only determined by the maturity $T$ but also by the first passage time of the prices. Using stochastic control, we result in solving HJB equations with each PDE having three space dimensions and one time dimension. We designed a fully implicit finite difference scheme with successive approximation to solve the HJB equations. Convergence of the successive approximation is proven in Chang and Krishna [1986], and Peyrl et al. [2005]. Price limits are shown to have a huge impact on the investment decisions before maturity, and optimal investment on each of the stocks is a jump process with the change of regime states. We have also compared the performance between different risk aversion rate. As expected, with lower risk aversion rate, the expected return is higher, but volatility also gets higher.
Chapter 2

A Static Model of Pairs Trading

2.1 Model Description

In Avellaneda and Lee [2010], the co-integration is modeled as:

\[
\ln(P_t) - \ln(P_0) = \alpha(t - t_0) + \beta[\ln(Q_t) - \ln(Q_0)] + \epsilon_t, \quad t \geq 0,
\]  

(2.1)

where \(P_t\) and \(Q_t\) are the stock prices of a pair of assets at time \(t\). Notice that the drift rate \(\alpha\) is usually ignorable compared to fluctuation of the residual \(\epsilon_t\). The above model suggests that if we take a long position of 1 dollar in stock \(P\) at time \(t\), we should short \(Q\) for \(\beta\) dollars, and vice versa. In this model, we continue to use the relationship above and assume that the mean reverting process \(\epsilon_t\) follows an OU process. For simplicity, define \(X_t = \epsilon_t + \ln(P_0) - \beta \ln(Q_0)\) in Equation (2.1). Note that \(X_t\) is still an OU process since \(\ln(P_0) - \beta \ln(Q_0)\) is only a constant. A trading signal is generated when \(X_t\) reaches a preset threshold. We have the following two equations for the correlation of the pair and the dynamics of the residual \(X_t\):

\[
\ln(P_t) - \beta \ln(Q_t) = X_t,
\]  

(2.2)

\[
dX_t = \theta(\mu - X_t)dt + \sigma dW_t,
\]  

(2.3)

where \(\theta\) is the mean reversion rate, \(\mu\) is the mean of \(X_t\), \(W_t\) is the standard Wiener process, and \(\sigma\) is the standard deviation for the Wiener process in Equation (2.3).

Similar to Bertram [2010], we can transform Equation (2.3) into the dimensionless system by \(\tau = \theta t\) and \(Y_\tau = \frac{\sqrt{2\theta}}{\sigma}(X_t - \mu)\). Hence, we have:

\[
dY_\tau = -Y_\tau d\tau + \sqrt{2}dW_\tau
\]  

(2.4)

We call Equation (2.4) dimensionless system because \(Y_\tau\) is not dependent on the model parameters. Notice that the above transformation is linear, so that each value of \(X_t\) corresponds to a unique value of \(Y_\tau\).

We generate trading signals when \(Y_\tau\) reaches a preset threshold. For example, when \(Y_{\tau_1} = a\) \((a > 0)\), we short 1 dollar of stock \(P\) and long \(\beta\) dollars of stock \(Q\), and when \(Y_{\tau_2} = b\) \((b < a)\), we clear positions and make profit. The profit on \(P\) is \(r_1 = \frac{P_{\tau_2} - P_{\tau_1}}{P_{\tau_1}}\), or \(r_1 = \ln(P_{\tau_2}) - \ln(P_{\tau_1})\) in terms of the continuous compound rate of return. Similarly, \(r_2 = \beta[\ln(Q_{\tau_2}) - \ln(Q_{\tau_1})]\). From Equation (2.2), we can express the
return as $r = r_1 + r_2 = X_1 - X_2 = \tilde{a} - \tilde{b}$, where $\tilde{a} = a \frac{T}{\sigma} + \mu$ and $\tilde{b} = b \frac{T}{\sigma} + \mu$. Assume the transaction cost is $\tilde{c}$ and let $c = \tilde{c} \sqrt{\frac{\sigma}{T}}$ be the transaction cost in the dimensionless system, so the net profit for each transaction is $\tilde{a} - \tilde{b} - \tilde{c}$, or $a - b - c$ in the dimensionless system. Similarly, if we trade in at $Y_{\tau_1} = -a$, at which we go long 1 dollar of P and short $\sqrt{\beta}$ dollars of stock Q, then we trade out at $Y_{\tau_2} = -b$. As one can compute, the net profit in each trade in the dimensionless system is again $a - b - c$. Without loss of generality, we will assume positions are first taken at $Y_{\tau_1} = a$. It is intuitive that $b \in [-a, a]$. If $b > a$, then the trader will always lose since $a - b - c < 0$ for any $c \geq 0$. To rule out the case $b < -a$, a rigorous analysis will be given in Section 4.

Each trading cycle is composed of two parts: the first part is from taking positions to clearing positions, and the second part is simply waiting until the next trading opportunity. Notice that the rigorous analysis will be given in Section 4.

The total time for each trading cycle is $T = \tau_1 + \tau_2$. Suppose there are $N_r$ transactions completed in $[0, T]$, so the net profit is $NP_r = (a - b - c)N_r$. By the elementary renewal theorem, the expected profit per unit time is given by

$$
\mu = \lim_{\tau \to \infty} \frac{E[NP_r]}{\tau} = (a - b - c) \lim_{\tau \to \infty} \frac{E[N_r]}{\tau} = a - b - c,
$$

where $E[T] = E[\tau_1] + E[\tau_2]$. Also we know that the expected time of one cycle in the real system is $E[T] = \frac{E[T]}{\beta}$. In this model, our objective is to find optimal thresholds to maximize the expected return per unit time $\mu$.

Notice that the expected return per unit time in real system is $\tilde{\mu} = \tilde{a} - \tilde{b} - \tilde{c} = \frac{\sigma \sqrt{\beta}}{\sqrt{2}} \frac{a - b - c}{E[T]} = \sqrt{\frac{\beta}{2}} \sigma \mu$. The coefficient $\sigma \sqrt{\theta/2}$ is only determined by the prices of the pairs, and is a constant once the model parameters are known. Therefore, maximizing the real return is the same as maximizing the return in the dimensionless system. The constant $\sigma \sqrt{\theta/2}$ contains intuitive and important information: a larger mean reversion rate $\theta$ means a higher trading frequency, and a larger $\sigma$ means a bigger fluctuation of $X_t$, both leading to a higher profit in each trade.

Since both the time and scale are linearly transformed into the dimensionless system, we can first obtain the optimal thresholds in the dimensionless system and then transform back to the real system. For simplicity, we will only write in the notation of the dimensionless system afterwards.

### 2.2 First Passage Times

It is crucial to find the expectation of the first passage time over one-sided and two-sided boundaries in order to find the optimal thresholds. In this section, we will give a brief review on the first passage time over one-sided boundary, and derive the expectation of the first passage time over two-sided boundaries.

A major contribution of this model lies in finding a polynomial form of the expectation over two-sided
boundary.

### 2.2.1 First Passage Time over A One-sided Boundary

For one-sided boundary, [Thomas 1975, Sato 1977] and [Ricciardi and Sato 1988] expressed the expectation as an infinite sum of polynomials. To summarize, for \( x > 0 \) and \( y > 0 \), the expectation of \( T_{x,0} \), the first passage time from \( x \) to \( 0 \) is:

\[
E[T_{x,0}] = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \left( \frac{\sqrt{2} x}{k} \right)^k \frac{k}{k!} \Gamma \left( \frac{k}{2} \right),
\]

and the expectation of \( T_{0,y} \), the first passage time from \( 0 \) to \( y \) is:

\[
E[T_{0,y}] = \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{\sqrt{2} y}{k} \right)^k \frac{k}{k!} \Gamma \left( \frac{k}{2} \right).
\]

Hence, the expectation \( E[T_{a,b}] \) for the case \( a > 0 \) can be written as:

\[
E[T_{a,b}] = \begin{cases} 
E[T_{a,0}] - E[T_{b,0}], & \text{for } b > 0 \\
E[T_{a,0}] + E[T_{0,-b}], & \text{for } b \leq 0 
\end{cases}
\]

By symmetry of an OU process, we can also get the expectation for the case \( a < 0 \) by \( E[T_{a,b}] = E[T_{-a,0}] + E[T_{0,b}] \) for \( b > 0 \) and \( E[T_{a,b}] = E[T_{-a,0}] - E[T_{-b,0}] \) for \( b < 0 \). Similarly, there are explicit results for the variance of the first passage time over a one-sided boundary (shown in Section 1). One can find the variance of this type of first passage time between any two points by the symmetric property of an OU process.

### 2.2.2 First Passage Time over A Two-sided Boundary

For the first passage time over a two-sided symmetric boundary, [Darling and Siegert 1953] derived the Laplace transform of \( T_{-a,a,b} \), the first passage time from \( b \) to cross the boundary \((-a,a)\) as given by

\[
E[e^{-\lambda T_{-a,a,b}}] = \frac{D_{-\lambda}(b) + D_{-\lambda}(-b)}{D_{-\lambda}(a) + D_{-\lambda}(-a)} \exp \left( \frac{b^2 - a^2}{4} \right),
\]

where \( D_{-\lambda}(b) \) is the Weber function, which can be shown as:

\[
D_{-\lambda}(x) = \sqrt{\frac{2}{\pi}} \exp \left( \frac{x^2}{4} \right) \int_0^{\infty} t^{-\lambda} \exp \left( -\frac{t^2}{2} \right) \cos \left( \frac{\lambda \pi}{2} \right) t \cos(\lambda t) dt, \quad \text{for } \lambda < 1.
\]

Define \( m(\lambda, x) = D_{-\lambda}(x) + D_{-\lambda}(-x) \). We have the following from the Weber function (2.12):

\[
m(\lambda, x) \big|_{\lambda=0} = 2 \sqrt{\frac{2}{\pi}} \exp \left( \frac{x^2}{4} \right) \int_0^{\infty} \exp \left( -\frac{t^2}{2} \right) \cos(\lambda t) dt = 2 \exp \left( -\frac{x^2}{4} \right),
\]

\[
\frac{\partial m(\lambda, x)}{\partial \lambda} \bigg|_{\lambda=0} = -2 \sqrt{\frac{2}{\pi}} \exp \left( \frac{x^2}{4} \right) \int_0^{\infty} \ln(t) \exp \left( -\frac{t^2}{2} \right) \cos(\lambda t) dt.
\]
To get Equation (2.13), we need to use the fact that \( \int_0^\infty y^{2n} \exp\left(-\frac{y^2}{2}\right) dy = \sqrt{\pi} \frac{(2n)!}{n!} \left(\frac{\sqrt{2}}{2}\right)^{2n+1} \) for \( n = 1, 2, 3, \ldots \). Therefore, if we let \( y = xt \) in Equation (2.13) and use Taylor expansion on \( \cos(xt) \), we can get:

\[
\int_0^\infty \exp\left(-\frac{y^2}{2}\right) \cos(yt) dt = \frac{1}{x} \int_0^\infty \exp\left(-\frac{y^2}{2x^2}\right) \cos(y) dy
\]

\[
= \frac{1}{x} \int_0^\infty \exp\left(-\frac{y^2}{2x^2}\right) \sum_{n=0}^\infty (-1)^n \frac{y^{2n}}{(2n)!} dy
\]

\[
= \frac{1}{x} \sum_{n=0}^\infty (-1)^n \frac{1}{(2n)!} \int_0^\infty \exp\left(-\frac{y^2}{2x^2}\right) y^{2n} dy
\]

\[
= \sqrt{\frac{\pi}{2}} \sum_{n=0}^\infty (-1)^n \frac{1}{n!} \left(\frac{\sqrt{2x}}{2}\right)^{2n}
\]

\[
= \sqrt{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2}\right)
\]

Hence we have Equation (2.13). Taking the first derivative on both sides of Equation (2.11) and setting \( \lambda = 0 \), we have:

\[
E[-T_{-a,a,b}] = \frac{\partial m(\lambda,b)}{\partial \lambda} \bigg|_{\lambda=0} m(\lambda,a) \bigg|_{\lambda=0} - \frac{\partial m(\lambda,a)}{\partial \lambda} \bigg|_{\lambda=0} m(\lambda,b) \bigg|_{\lambda=0} \exp\left(-\frac{b^2-a^2}{4}\right)
\]

By using Equations (2.13) and (2.14), and multiplying both sides by \(-1\), we can get the expectation of \( T_{-a,a,b} \) as follows:

\[
E[T_{-a,a,b}] = \sqrt{\frac{\pi}{2}} |h(b) - h(a)|.
\]

where

\[
h(x) = \exp\left(\frac{x^2}{2}\right) \int_0^\infty \ln(t) \exp\left(-\frac{t^2}{2}\right) \cos(tx) dt.
\]

The following proposition will further simplify the expression and make it handy to find the optimal thresholds in the next section.

**Proposition 1.** The integral form of \( h(x) \) shown above can be expressed as an infinite sum of polynomials and a constant:

\[
h(x) = -\frac{1}{2} \sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{(\sqrt{2}x)^{2n}}{(2n)!} \Gamma(n) + C,
\]

where \( C = \int_0^\infty \ln(t) \exp\left(-\frac{t^2}{2}\right) dt \).

*Proof:* see Appendix A

From Equation (2.15) and Equation (2.17), we can simplify the expectation as

\[
E[T_{-a,a,b}] = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(\sqrt{2}a)^{2n} - (\sqrt{2}b)^{2n}}{(2n)!} \Gamma(n)
\]
Similarly we can find the second moment of $T_{a,a,b}$, which is used in computing the variance per unit time in Section 4. We have:

$$\frac{\partial^2 m(\lambda, x)}{\partial \lambda^2} \bigg|_{\lambda=0} = 2 \sqrt{\frac{2}{\pi}} \exp\left(\frac{x^2}{4}\right) \int_0^\infty \ln(t)^2 \exp\left(-\frac{t^2}{2}\right) \cos(xt) dt - \frac{\pi^2}{2} \exp\left(-\frac{x^2}{4}\right),$$

and the expectation of the second moment becomes:

$$E[T^2_{a,a,b}] = \exp\left(\frac{b^2 - a^2}{4}\right) [g_1(a,b) - g_2(a,b)]$$

where

$$g_1(a,b) = \left. \frac{\partial^2 m(\lambda,b)}{\partial \lambda^2} \right|_{\lambda=0} m(\lambda,a) \bigg|_{\lambda=0} - \left. \frac{\partial m(\lambda,a)}{\partial \lambda} \right|_{\lambda=0} \left. \frac{\partial m(\lambda,b)}{\partial \lambda} \right|_{\lambda=0},$$

and

$$g_2(a,b) = \left. \frac{\partial^2 m(\lambda,a)}{\partial \lambda^2} \right|_{\lambda=0} m(\lambda,b) \bigg|_{\lambda=0} + \left. \frac{\partial m(\lambda,a)}{\partial \lambda} \right|_{\lambda=0} \left. \frac{\partial m(\lambda,b)}{\partial \lambda} \right|_{\lambda=0} - 2 \left( \left. \frac{\partial m(\lambda,a)}{\partial \lambda} \right|_{\lambda=0} \right)^2 \left. m(\lambda,b) \right|_{\lambda=0}.$$

Unlike the first moment, the integral form cannot be simplified to the polynomial form, leaving the second moment difficult to use. The variance can be found by the first two moments, but only in a very complicated integral form.

### 2.3 Optimal Thresholds

With the polynomial form of the expectation in Section 3, we are now ready to find the optimal thresholds for pairs trading. The main goal is to maximize the expected return per unit time. As explained in Section 2, we take positions when $Y_\tau$ reaches the opening threshold $a$ (or $-a$), clear positions when it reaches the closing threshold $b$ (or $-b$), and wait for the next opportunity until $Y_\tau$ reaches an opening threshold again. In this section, we will discuss three cases with the values of $a$ and $b$. Throughout this section, we assume $a \geq 0$. Since the OU process is symmetric, the case for $a < 0$ will be exactly the same.

**Case 1:** $0 \leq b \leq a$

From Equation (2.7), our objective function is given by $f(a,b) = \frac{a-b-c}{E[\tau_1] + E[\tau_2]}$, where $E[\tau_1]$ and $E[\tau_2]$ are explicitly shown by Equation (2.10) and Equation (2.18). In order for $f(a,b)$ to be non-negative, we have to restrict $a - b - c \geq 0$. The optimization problem is:

$$\begin{align*}
\max_{a,b} \quad & f(a,b) = \frac{a-b-c}{E[\tau_1] + E[\tau_2]} = \frac{a-b-c}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right)} \\
\text{subject to} \quad & 0 \leq b \leq a - c
\end{align*} \quad (2.19)$$

To find the optimal solution in the domain $0 \leq b \leq a - c$, we need to use the fact that $\frac{\partial f(a,b)}{\partial b} < 0$
for any $a$ in the domain. To prove this, we have

$$
\sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n+1} - \left( \frac{\sqrt{2}b}{(2n+1)!} \right)^{2n+1} = \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right) - \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}b}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)
$$

$$
\geq \sqrt{2}(a - b - c) \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right) - \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}b}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)
$$

$$
= \sqrt{2}(a - b - c) \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}b}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)
$$

The first inequality is due to the fact that $c \geq 0$ and the second inequality is due to the fact that $a \geq b \geq 0$. With the above inequality, we can get

$$
\frac{\partial f(a, b)}{\partial b} = -\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n+1} \Gamma\left( \frac{2n+1}{2} \right) + (a - b - c) \sqrt{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}b}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right) \leq 0
$$

Equality only holds when $a = b$ and $c = 0$. So for any given $a$ and $c$, the optimal value of $b$ is $b^* = 0$. Therefore, the original maximization problem is now:

$$
f(a) = f(a, 0) = -\frac{a - c}{\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)}
$$

Setting $\frac{df(a)}{da} = 0$, we can find the optimal value $a^*$ by solving the equation:

$$
\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n+1} \Gamma\left( \frac{2n+1}{2} \right) = (a - c) \sqrt{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)
$$

(2.20)

The existence and uniqueness of the solution to Equation (2.20) can be easily shown. When $c = 0$, $a = 0$ is a solution. When $c > 0$, if we let $a \to c$, we will have

$$
\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n+1} \Gamma\left( \frac{2n+1}{2} \right) > (a - c) \sqrt{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)
$$

If we let $a \to \infty$, we will have

$$
\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n+1} \Gamma\left( \frac{2n+1}{2} \right) < (a - c) \sqrt{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n+1)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right),
$$

which proves the existence of the solution. To prove the uniqueness, we take derivative of both sides of Equation (2.20) with respect to $a$. We have

$$
c'(a) = \frac{(a - c) \sum_{n=1}^{\infty} \left( \frac{\sqrt{2}a}{(2n-1)!} \right)^{2n-1} \Gamma\left( \frac{2n}{2} \right)}{\sqrt{2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}a}{(2n)!} \right)^{2n} \Gamma\left( \frac{2n+1}{2} \right)} > 0
$$
Since \( c(a) \) is an increasing function of \( a \), there is a unique value of \( a \) that satisfies Equation (2.20) for any given \( c > 0 \).

To see that \( a^* \) is the maximizer rather than the minimizer, we let \( a \to c \) and \( a \to \infty \). For any \( c > 0 \), when \( a \to c \), it is easy to see that \( f(a) \to 0 \). Similarly, when \( a \to \infty \), we will have:

\[
f(a) = \frac{a - c}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1}}{(2n+1)!} \Gamma \left( \frac{2n+1}{2} \right)} = \frac{1 - \xi}{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1}}{(2n+1)!} \Gamma \left( \frac{2n+1}{2} \right)} \to 0
\]

because \( 0 \leq 1 - \xi < 1 \) and \( \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1}}{(2n+1)!} \Gamma \left( \frac{2n+1}{2} \right) \to \infty \).

Also we know that for any \( c \leq a < \infty \), \( f(a) \geq 0 \). Therefore we conclude that for \( c > 0 \), \( a^* \) maximizes \( f(a) \).

When \( c = 0 \), we have

\[
f(a) = \frac{a}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1}}{(2n+1)!} \Gamma \left( \frac{2n+1}{2} \right)} = \frac{\sqrt{2}}{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1}}{(2n+1)!} \Gamma \left( \frac{2n+1}{2} \right)}
\]

is a decreasing function of \( a \). When \( a \to 0 \), we have \( f(a) \to \sqrt{\frac{2}{\pi}} \). In this case, by solving Equation (2.20), we can still get \( a^* = 0 \).

**Remark:** The optimal solutions in this case are exactly the optimal thresholds for the conventional way of the pairs trading: take positions when the spread widens (\( Y_t = a^* \)) and clear positions when the spread reverts to the mean (\( Y_t = b^* = 0 \)). Note that when there is no transaction cost \( (c = 0) \), the gap between \( a \) and \( b \) should be infinitely close to 0, which means that the trader should constantly adjust his positions to make as many trades as possible in a given time. In this case, the trader values the trading frequency more than the profit per trade. This is also consistent with Bertram [2010].

**Case 2:** \(-a \leq b \leq 0\)

Here, we do not exclude \( b = 0 \) for the feasibility of our optimal solution. For \( b \leq 0 \), the optimization problem is written as:

\[
Max_{a,b} f(a, b) = \frac{a - b - c}{E[T_1] + E[T_2]} = \frac{a - b - c}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2a})^{2n+1} - (\sqrt{2b})^{2n+1}}{(2n+1)!} \Gamma \left( \frac{2n+1}{2} \right)} \\
subject\ to \quad -a \leq b \leq \min \{0, a - c\} \tag{2.21}
\]

Here, we require \( a \geq \frac{c}{2} \) for feasibility. Notice that \( f(a, b) \) is bounded inside the domain. First of all, for any \( a \), \( f(a, b) \) is boundary since \( b \) is bounded and \( f(a, b) \) is continuous in \( b \). To prove \( f(a, b) \) is bounded in \( a \), we discuss two cases: \( c > 0 \) and \( c = 0 \). When \( c > 0 \), if we let \( a \to \frac{c}{2} \), then \( b \to -\frac{c}{2} \) in the domain. So we will have \( f(a, b) \to 0 \). If we let \( a \to \infty \), we will get \( f(a, b) \to 0 \) for any \( b \) in the domain. When \( c = 0 \), if we let \( a \to 0 \), we will have \( b \to 0 \), and \( f(a, b) \to \sqrt{\frac{2}{\pi}} \). Again, if we let \( a \to \infty \) when \( c = 0 \), we will get \( f(a, b) \to 0 \) for any \( b \) in the domain. So for both cases, \( f(a, b) \) is bounded in \( a \) and the minimal value \( f(a, b) \to 0 \) appears when \( a \) approaches its boundary. Since \( f(a, b) \) is continuous in both \( a \) and \( b \), the maximal value exists on the closed set of the domain.

Setting the gradient of \( f(a, b) \) to 0, we have:
\[ E[\tau_1] + E[\tau_2] = (a - b - c)(\frac{\partial E[\tau_1]}{\partial a} + \frac{\partial E[\tau_2]}{\partial a}) \]
\[ E[\tau_1] + E[\tau_2] = -(a - b - c)(\frac{\partial E[\tau_1]}{\partial b} + \frac{\partial E[\tau_2]}{\partial b}) \]

Therefore we have:
\[ \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) = \sum_{n=0}^{\infty} \frac{(\sqrt{2}b)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) \]  \hspace{1cm} (2.22)

Since \( g(x) = \sum_{n=0}^{\infty} \frac{(\sqrt{2}x)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) \) is an increasing function, for Equation (2.22) to hold, we must have \( a^2 = b^2 \). Since \(-a \leq b \leq \min\{0, a - c\} \), the optimal solution can only be \( b^* = -a^* \), where \( a^* \) can be found by solving the equation:
\[ \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+3}{2}\right) - \frac{c}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) \] \hspace{1cm} (2.23)

With the same argument in case 1, we can show the existence, uniqueness of the solution \( a^* \) in Equation (2.23).

However, we still have to check that \( b^* = -a^* \) is the global maximal by showing that \( f(a^*, b^*) \geq f(a, b) \) for any \( a, b \) on the boundary. For any \( b = -a \), we can prove that \( f(a, b) \leq f(a^*, b^*) \) by the same argument as in case 1. For any \( b \to a - c \) where \( c > 0 \), we have \( f(a, b) \to 0 < f(a^*, b^*) \). When \( c = 0 \), we have \( a^* = b^* = 0 \) and it is easy to check that \( f(a^*, b^*) \geq f(a, b) \) for any \( b \to a - c \). When \( b \to 0 \), we can show that \( \max_{a \geq \frac{1}{2}} f(a, b) \leq f(a^*, b^*) \) by Proposition 2.

Remark: The only difference between Equation (2.23) and Equation (2.20) is the term of \( c \). Equation (2.23) will be the same as Equation (2.20) if the transaction cost in Equation (2.20) is reduced to a half. Therefore we can expect case 2 to have a higher return than case 1 for a given value of \( c \). A formal statement and rigorous proof is given by Proposition 2 at the end of this section.

Case 3: \( b < -a \)

In this case, one may expect more profit in each trading cycle, but the expected time in each cycle is longer. Different from the two earlier cases, only the first passage time over the one-sided boundary is used. The optimization problem is:
\[ \max_{a, b} f(a, b) = \frac{a - b - c}{E[\tau_1] + E[\tau_2]} = \frac{a - b - c}{\sum_{n=0}^{\infty} \frac{(-\sqrt{2}b)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+3}{2}\right)} \]
\[ \text{subject to} \quad b < -a \] \hspace{1cm} (2.24)

where \( \tau_1 \) and \( \tau_2 \) are both first passage times over a one-sided boundary.

This time, the expected time for one trading cycle \( E[\tau_1] + E[\tau_2] = \sum_{n=0}^{\infty} \frac{(-\sqrt{2}b)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+3}{2}\right) \) does not depend on \( a \). For a given value of \( b \), the objective function \( f(a, b) = \sum_{n=0}^{\infty} \frac{(-\sqrt{2}b)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+3}{2}\right) \) is a linearly increasing function of \( a \). To maximize \( f(a, b) \), \( a \) should be as large as possible. In this case, since \( b < -a \), the largest \( a \) tends to the boundary \( a^* = -b \) for any fixed value of \( b \). If \( b > 0 \), the optimal solution will be infeasible since we restrict \( a \geq 0 \). When \( b \leq 0 \), the problem goes back to case 2 and we only need to
solve Equation \((2.23)\) to get the value of \(a^*\) and thus get \(b^*\) by \(b^* = -a^*\).

Out of the three cases, we have seen two different optimal rules which gives two different values of \(a^*\) and \(b^*\). We call the optimal rule in case 1 the “Conventional Optimal Rule” since it clears position exactly when the spread reverts to the mean at \(b^* = 0\), which is consistent with the common practice. In contrast, we call the rule in case 2 the “New Optimal Rule”, which basically allows no waiting time between the two trades. Since the “New Optimal Rule” cuts the transaction costs in a half compared to the “Conventional Optimal Rule”, it is intuitive that the “New Optimal Rule” performs better than the “Conventional Optimal Rule”. Formally, we state the proposition below:

**PROPOSITION 2.** When there is no transaction cost \((c = 0)\), the maximal return in case 1 is the same as the maximal return in case 2. When transaction cost exists \((c > 0)\), the maximal return in case 1 is strictly smaller than the maximal return in case 2.

Proof: see Appendix B.

Graphically, the comparison between the two rules in the theoretical level are shown in Figure 2.1. The “New Optimal Rule” (red curve) is always better than the “Conventional Optimal Rule” (blue curve). The advantage is more apparent when the transaction cost increases, despite the fact that the expected returns for both rules decrease as the transaction cost increases.

The comparison is only made in terms of the profit per unit time since it is our objective in this model. So “better” only means more profit per unit time. For traders who are more concerned about the risk, we show the variance per unit time for these two methods in Figure 2.2. Naturally the risk of “New Optimal Rule” is always higher than the “Conventional Optimal Rule” since the expected return is higher. To take risk into account, Sharpe ratio or the mean-variance optimization can be considered. In Figure 2.2 we can see that when the transaction cost increases to a very large value, the change of variances of both rules is small, but the change of the expected return is relatively large. Even when we consider the risk, “New Optimal Rule” can be more preferable when transaction cost is large enough. However, in this model, we will only focus on the expected return per unit time.

### 2.4 Numerical Examples

In this section, we will apply the two optimal rules derived in Section 4 and compare them with the common practice using actual daily data. Comparison is made in two aspects. Firstly, for the same pair of stocks, we compare the profitability for different trading rules. Secondly, we compare the profitability among different pairs under the same rule.

#### 2.4.1 Comparison of Different Trading Rules

One of the most commonly used pairs is Coca-Cola (KO) and Pepsi (PEP). We collected 756 daily prices of the pair KO-PEP from Yahoo-Finance from November 30, 2009 to November 29, 2012. As shown in Figure 2.3, their prices moved together.
Figure 2.1: Comparison between the “Conventional Optimal Rule” (case 1), and “New Optimal Rule” (case 2). Optimal thresholds of the two rules developed in this model are dependent on the transaction cost, thus they are shown to be curves instead of straight lines.

Figure 2.2: Comparison of the variance between the “Conventional Optimal Rule” (case 1), and “New Optimal Rule” (case 2).
Chapter 2. A Static Model of Pairs Trading

Figure 2.3: Actual adjusted daily prices of KO and PEP are shown in this graph. Time = 0 is the starting date on November 30, 2009.

Let the prices of PEP and KO be \( P_t \) and \( Q_t \), respectively. Applying linear regression, we get

\[
\ln(P_t) - \beta \ln(Q_t) = X_t,
\]

where \( \beta = 0.2187 \). The residual \( X_t \) is assumed to follow an OU process

\[
dX_t = \theta(\mu - X_t)dt + \sigma dW_t.
\]

In this model, we use the Maximum-Likelihood (ML) method to estimate the parameters based on Hu and Long [2007]. The log likelihood for the process \( X_t \) is given by:

\[
L(X|\mu, \theta, \sigma) = -\frac{n}{2} - \frac{1}{2} \sum_{i=1}^{n} \ln(1 - e^{-2\theta(t_i - t_{i-1})}) - \frac{\theta}{\sigma^2} \sum_{i=1}^{n} \frac{X_{t_i} - \mu - (X_{t_{i-1}} - \mu)e^{-\theta(t_i - t_{i-1})}}{1 - e^{-2\theta(t_i - t_{i-1})}}
\]

Maximizing \( L(X|\mu, \theta, \sigma) \), we get the estimation for the parameters: \( \mu = 3.4241, \theta = 0.0237 \) and \( \sigma = 0.0081 \). Assuming that the parameters are constant during the data collection period, we can apply our optimal pairs trading rules. We compare in Figure 2.4 our “New Optimal Rule” and “Conventional Optimal Rule” with two common practices, which take positions at one standard deviation (we call it “1-\( \sigma \) Rule”) or two standard deviations (“2-\( \sigma \) Rule”) and clear positions when the spread reverts back to the mean. Since thresholds of common practices do not change with the transaction cost, the total return should be straight lines. Similarly, since thresholds of the two optimal rules vary with the transaction cost, the total return should be curves.

As predicted in Section 4, the “New Optimal Rule” performs best. There is a trend of decreasing profit for all of the rules as the transaction cost \( c \) increases, but the “New Optimal Rule” performs increasingly better as \( c \) increases. The “Conventional Optimal Rule” does not distinguish itself from the “1-\( \sigma \) Rule” when the transaction cost is small, but tends to perform better as \( c \) increases. However, the result is not exactly as we expected. For example, there is a sudden drop in return at \( c = 0.006 \) dollars with both the “New Optimal Rule” and “Conventional Optimal Rule”, and their profits are even less than
Figure 2.4: Comparison between the four rules in this model using daily prices of KO and PEP. Generally “New Optimal Rule” performs best. “Conventional Optimal Rule” performs slightly better than the “1-σ Rule” when $c$ is small, and significantly better when $c$ is larger. “2-σ Rule” performs worst.

the “1-σ Rule”. Possible cause might be that model parameters might have been poorly estimated, and even that $X_t$ might not have been an OU process. To see the impact of model parameters, we conduct sensitivity analysis with each of the three parameters $\mu$, $\theta$, and $\sigma$, which is presented in Appendix D. We find the return to be very sensitive to the mean of the spread but not to the reversion rate $\theta$ nor to the standard deviation $\sigma$.

We also show the actual trading process for the “New Optimal Rule” in Table 2.1 and Figure 2.5. In each trade, we assume that the transaction cost $c = 0.02$ dollars for each dollars invested. We transform the transaction cost into the dimensionless system and obtain the optimal thresholds as $\tilde{a}^* = 0.991$ and $\tilde{b}^* = -0.991$. Then we transform back to get the real thresholds as $a^* = a^* \sqrt{2\theta} + \mu = 3.4611$ and $b^* = b^* \sqrt{2\theta} + \mu = 3.3871$. A trading is triggered whenever $X_t$ reaches $\tilde{a}^*$ or $\tilde{b}^*$. The last trade is not counted since positions cannot be cleared within our trading period.

There have been a total of five trades over the three years with an average earning per trade at 6% and the total earning over the whole period at 33.33%.

2.4.2 Comparison between Different Pairs

We consider five pairs: Coca-Cola and Pepsi (KO_PEP), Target and Wal-mart (TGT_WMT), Dell and Hewlet-Packard (DELL_HPQ), RWE AG and E.On Se (RWE_EO AN), and Chevron and Exxon Mobile (CVX_XOM). We computed the net returns of the five pairs under four trading rules given

\footnote{RWE and E. On Se are german utility companies}
Figure 2.5: Dynamics of the spread is shown in the blue curve. Dashed lines are the optimal trading thresholds $a$ and $b$. Block points are the day of opening and closing positions. They are not exactly on the dashed lines because trading is discrete. The graph shows that there are more trading opportunities in the first year.

Table 2.1: Details of transaction for each trade

<table>
<thead>
<tr>
<th>Trades</th>
<th>Status</th>
<th>Date</th>
<th>KO Prices($)</th>
<th>KO Action</th>
<th>KO Prices($)</th>
<th>KO Action</th>
<th>PEP Prices($)</th>
<th>PEP Action</th>
<th>Returns (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trade 1</td>
<td>Open</td>
<td>Dec/10/2009</td>
<td>29.290</td>
<td>Sell $0.22</td>
<td>61.84</td>
<td>Buy $1</td>
<td></td>
<td></td>
<td>8.67</td>
</tr>
<tr>
<td>Trade 2</td>
<td>Open</td>
<td>Mar/15/2010</td>
<td>26.825</td>
<td>Buy $0.22</td>
<td>66.15</td>
<td>Clear positions</td>
<td>Sell $1</td>
<td>8.85</td>
<td></td>
</tr>
<tr>
<td>Trade 3</td>
<td>Open</td>
<td>Apr/28/2011</td>
<td>33.705</td>
<td>Clear positions</td>
<td>69.72</td>
<td>Clear positions</td>
<td>Sell $1</td>
<td>9.02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Close</td>
<td>Jul/26/2011</td>
<td>34.595</td>
<td>Clear positions</td>
<td>64.07</td>
<td>Clear positions</td>
<td>Sell $1</td>
<td>9.02</td>
<td></td>
</tr>
<tr>
<td>Trade 4</td>
<td>Open</td>
<td>Jul/26/2011</td>
<td>34.595</td>
<td>Sell $0.22</td>
<td>64.07</td>
<td>Buy $1</td>
<td></td>
<td></td>
<td>7.72</td>
</tr>
<tr>
<td></td>
<td>Close</td>
<td>Jul/26/2012</td>
<td>39.425</td>
<td>Clear positions</td>
<td>71.22</td>
<td>Clear positions</td>
<td></td>
<td>5.72</td>
<td></td>
</tr>
</tbody>
</table>
the actual daily prices between November 30, 2009 and November 29, 2012. The profitability indicator $\sigma \sqrt{\theta/2}$ and the net returns for the five pairs under the four trading rules are summarized in Table 2.2.

<table>
<thead>
<tr>
<th></th>
<th>$\sigma \sqrt{\theta/2}$</th>
<th>New</th>
<th>Conventional</th>
<th>1-(\sigma)</th>
<th>2-(\sigma)</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>RWE_EOAN</td>
<td>0.00142</td>
<td>39%</td>
<td>38%</td>
<td>39%</td>
<td>23%</td>
<td>35%</td>
</tr>
<tr>
<td>TGT_WMT</td>
<td>0.00126</td>
<td>66%</td>
<td>28%</td>
<td>29%</td>
<td>28%</td>
<td>38%</td>
</tr>
<tr>
<td>KO_PEP</td>
<td>0.00088</td>
<td>33%</td>
<td>26%</td>
<td>20%</td>
<td>13%</td>
<td>23%</td>
</tr>
<tr>
<td>DELL_HPQ</td>
<td>0.00063</td>
<td>54%</td>
<td>40%</td>
<td>40%</td>
<td>0%</td>
<td>34%</td>
</tr>
<tr>
<td>CVX_XOM</td>
<td>0.00071</td>
<td>32%</td>
<td>26%</td>
<td>26%</td>
<td>28%</td>
<td></td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td>45%</td>
<td>32%</td>
<td>31%</td>
<td>18%</td>
<td><strong>32%</strong></td>
</tr>
</tbody>
</table>

The last row of Table 2.2 is the average return of the trading rules. It is clear that the “New Optimal Rule” out-performs the rest. Moreover, if we invest only on the two most profitable pairs – RWE_EOAN and TGT_WMT – according to $\sigma \sqrt{\theta/2}$, the average return will increase significantly. That is, if the investor invests only on the two most promising stocks according to the profitability indicator ($\sigma \sqrt{\theta/2}$), the return would have been $52.5\% = (39\% + 66\%) / 2$, an increase of 17.04\% over the average shown in the last row of Table 2.2. The increases under other trading rules are also significant: 4.91\% for the “Conventional Optimal Rule”, 10.41\% for the “1-\(\sigma\) Rule”, and 41.76\% for the “2-\(\sigma\) Rule”. This demonstrates that the profitability indicator $\sigma \sqrt{\theta/2}$ can be used as a guideline to select a profitable pair for trading.
Chapter 3

A Dynamic Model of Pairs Trading

3.1 The Model

We consider a portfolio with two correlated stocks and a bank account. An optimal investment decision is to be determined to maximize expected utility at the maturity $T$, or at the exit time when prices fall out of the pre-set price limits. Investment decisions are made given the actual stock prices $S_1^t, S_2^t$ and wealth $W_t$ at time $t$. In this model, we use the Constant Relative Risk Aversion (CRRA) $U(w) = \frac{w^\theta}{\theta}$ ($U$ is a map from the wealth $w$ to utility) as our objective function, where $\theta(< 1)$ represents the degree of risk aversion. In this model, we study the power utility ($\theta < 1$ and $\theta \neq 0$), but all the methods in this study can also be applied to the log utility and exponential utility.

3.1.1 Assumptions

There are mainly two types of uncertainties in this model: movement of stock prices and change of regime states. Movement of stock prices directly affects the investment decisions, and regime states determine the model parameters of the stock prices. Before modeling the trading process, we need to state the widely accepted assumptions about these uncertainties.

Assumption 1:

Regime states $\xi(t) \in \{1, 2, 3, \ldots, L\}$ follow a continuous Markov chain independent of the stock prices. Moreover, the generating matrix is given by $Q = (q_{ij})_{LL}$. The stationary transition probability from state $i$ to $j$ at time $t$ is given by $P_{ij}(t) = \{\xi(t+s) = j | \xi(s) = i\}, \ t \geq 0$.

Assumption 2:

Movement of stock prices follows a geometric Brownian motion (GBM) and model parameters are determined by the regime state. Similar to Tourin and Yan [2013], we write the dynamics of two stocks $(S_1(t), S_2(t))$ and a risk free asset $M(t)$ as:

$$dS_1(t) = S_1(t)[\mu_1(\xi(t)) + \sigma(\xi(t))z(t)]dt + S_1(t)[\sigma_{11}(\xi(t))dB_1(t) + \sigma_{12}(\xi(t))dB_2(t)]$$  \hspace{1cm} (3.1)

$$dS_2(t) = S_2(t)\mu_2(\xi(t))dt + S_2(t)[\sigma_{21}(\xi(t))dB_1(t) + \sigma_{22}(\xi(t))dB_2(t)]$$  \hspace{1cm} (3.2)
Together with the maturity \( T \), suppose at time \( \tau \) stopped. Let \( \tau \) and \( \sigma \) be the stopping times for \( S_1(t) \) and \( S_2(t) \) respectively before considering the co-integrating factor, and \( \sigma_1 \), \( \sigma_2 \), and \( \sigma_3 \) are the associated standard deviation with \( B_1(t) \) and \( B_2(t) \) in each stock. \( \delta(\xi(t)) \) in \( S_1(t) \) measures the impact of \( \xi(t) \) on the first stock.

Our model is quite similar to the model proposed by [Tourin and Yan 2013]. Actually, when the co-integration exists and the regime state is stable, our model is exactly the same as theirs. However, our model is more general in the sense that we have considered that fact that the co-integration may break up. In this case, we set our parameter \( \delta(\xi(t)) = 0 \) to denote the case where the two stocks are not correlated. Therefore, the problem turns to the famous Merton's problem where analytic solutions are known.

**Remark:** Same as [Tourin and Yan 2013], \( z(t) \) can be shown to be an OU process. It only appears in the drift of the first stock, but it can be easily extended to both stocks.

### 3.1.2 Price Limits

Price limits usually appear as part of the regulation in day trading. They play an important role in keeping the loss below a pre-set value in case stocks of interest undergo fundamental changes. Take General Motors (GM) and Ford (F) for an example. The price ratio has been steady from 2006 to 2008 until GM went bankrupt in 2009. Traders who bet the spread to revert can suffer from huge loss from a long position in GM. In this case, price limits can stop further loss. It is up to the investor to decide whether or not to impose price limits on the trading. If he does choose to impose price limits, he can determine the limits based on his judgment of the market.

Suppose price limits for \( S_1(t) \) and \( S_2(t) \) are \([S_1(t), \bar{S}_1(t)]\) and \([S_2(t), \bar{S}_2(t)]\) at time \( t \) where \( \bar{S}_k(t) \) and \( \underline{S}_k(t) \) are functions of \( t \) for \( k = 1, 2 \). If stock prices hit the limits, then trading is automatically stopped. Let \( \tau_1 \) and \( \tau_2 \) be the stopping time for \( S_1(t) \) and \( S_2(t) \) defined as follows:

\[
\tau_1 = \inf \left\{ t > 0; S_1(t) \geq \bar{S}_1(t) \lor S_1(t) \leq \underline{S}_1(t) \mid S_1(0) \in [\underline{S}_1(0), \bar{S}_1(0)] \right\}
\]

\[
\tau_2 = \inf \left\{ t > 0; S_2(t) \geq \bar{S}_2(t) \lor S_2(t) \leq \underline{S}_2(t) \mid S_2(0) \in [\underline{S}_2(0), \bar{S}_2(0)] \right\}
\]

Together with the maturity \( T \), the total trading time is written as: \( \tau = \min \{\tau_1, \tau_2, T\} \).

### 3.1.3 Dynamic Programming Model

Suppose at time \( t \), our wealth is \( W(t) \). We need to decide the optimal fraction of wealth \( \pi_1(t) \) and \( \pi_2(t) \) to invest in the two stocks. Therefore, the fraction of wealth invested in the risk-free asset is \( 1 - \pi_1(t) - \pi_2(t) \). Dynamics of the wealth at time \( t \) can be shown as:

\[
dW(t) = \pi_1(t) \frac{dS_1(t)}{S_1(t)} + \pi_2(t) \frac{dS_2(t)}{S_2(t)} + (1 - \pi_1(t) - \pi_2(t)) \frac{dM(t)}{M(t)}
\]

\[
= \left[ (1 - \pi_1(t) - \pi_2(t))r_1(\xi(t)) + \pi_1(t) \mu_1(\xi(t)) + \pi_2(t) \mu_2(\xi(t)) \right] dt
\]

\[
+ \left[ \pi_1(t) \sigma_1^2(\xi(t)) + \pi_2(t) \sigma_2^2(\xi(t)) \right] dB(t)
\]
where \( \bar{\sigma}_i(\xi(t)) = (\sigma_{11}(\xi(t)), \sigma_{12}(\xi(t))) \) for \( i = \{1, 2\} \), \( B(t) = (B_1(t), B_2(t)) \) and \( \bar{\mu}_1(\xi(t)) = \mu_1(\xi(t)) + \sigma(\xi(t))z(t) \).

Let \( W(t) = w, S_1(t) = s_1, S_2(t) = s_2, \xi(t) = i \). Define \( \pi(s) \) to be the control at time \( s \), i.e. \( \pi(s) = (\pi_1(s), \pi_2(s)) \), and let \( \tilde{\pi} = \{ \pi(s); \ t \leq s \leq \tau \} \) to be the set of controls from time \( t \) to time \( \tau \). Define the expected utility at exit time \( \tau \) under control \( \tilde{\pi} \) by \( J_{\tilde{\pi}}(t, w, s_1, s_2, i) = E[U(W^{t, w, s_1, s_2, i, \tilde{\pi}(\tau)})]. \) Define the admissible control set \( A(t) \) the same as in \textbf{Tourin and Yan 2013}, i.e. each pair of control \((\pi_1, \pi_2)\) must be real-valued, progressively measurable and provide a unique solution in our problem. Also, for all \( s \in [t, T] \), \((\pi_1(s), \pi_2(s), S_1(s), S_2(s))\) must satisfy the integrability condition

\[
E \left[ \int_t^T [\pi_1(s)S_1(s)]^2 + [\pi_2(s)S_2(s)]^2 \right] ds < \infty
\]

Our objective is to find the optimal control \( \tilde{\pi^*} \) for the maximal expected utility \( J(t, w, s_1, s_2, i) \) defined as:

\[
J(t, w, s_1, s_2, i) = J_{\tilde{\pi}^*}(t, w, s_1, s_2, i) = \sup_{\tilde{\pi} \in A(t)} J_{\tilde{\pi}}(t, w, s_1, s_2, i)
\]

**Proposition 3.** Let \( J(t, w, s_1, s_2, i) \) be denoted as \( J^i \) for simplicity where the superscript \( i \) denote the current regime. We write the partial differentials of \( J^i \) as \( J_1^i, J_2^i, J_\theta^i, \ldots \) Also, we let the model parameters be \( \bar{\mu}_1(\xi(t)) = i = \mu_1^1, \mu_2(\xi(t)) = i = \mu_2^1, \ldots \) The control \( \pi = (\pi_1, \pi_2) \) at time \( t \) is the fraction of wealth to invest in both stocks, and the boundaries are \([S_k(t), \bar{S}_k(t)] = [\underline{S}_k, \bar{S}_k]\) for \( k = \{1, 2\} \). The objective function at each regime state \( i \) must satisfy the following partial differential equations:

\[
0 = J_1^i + \sup_{\pi} \sum_{j=1}^2 \mathcal{L}_{\pi}^j J^i = J_1^i + \sup_{\pi} \left( s_1 \mu_1^1 j_1^i + s_2 \mu_2^1 j_2^i + w'[1 - \pi_1 - \pi_2])r + \pi_1 \mu_1^1 + \pi_2 \mu_2^1 \right) J_1^i
\]

subject to:

\[
J(\tau, w, s_1, s_2, i) = \frac{[w(\tau)]^\theta}{\theta}
\]

where \( \mathcal{L}_{\pi}^j \) is the infinitesimal generator of \( J^i \) when investment in each stock at time \( t \) is \((\pi_1, \pi_2)\).

**Proof.** Let \( t < \tau \), and we choose a small enough \( h \) so that \( t + h < \tau \). Control \( \tilde{\pi} \) is chosen so that from \( t \) to \( t + h \) it is a random admissible control and from \( t + h \) to \( \tau \) it is the optimal control. Conditioned on the regime state at time \( t + h \), we have:

\[
J(t, w, s_1, s_2, i) = \sup_{\tilde{\pi} \in A(t)} J_{\tilde{\pi}}(t, w, s_1, s_2, i) = \sup_{\tilde{\pi} \in A(t)} \sum_{j=1}^L P_j(h) E[J(t + h, w^{\tilde{\pi}}(t + h), s_1(t + h), s_2(t + h), j)]
\]
Rearrange the equation above, we get:

\[
0 = \sup_{\tilde{\pi} \in A(t)} \left( \sum_{j \neq i} P_{ij}(h) E[J(t + h, w^\tilde{\pi}(t + h), s_1(t + h), s_2(t + h), j)] \\
+ (P_{ii}(h) - 1) E[J(t + h, w^\tilde{\pi}(t + h), s_1(t + h), s_2(t + h), i)] \\
+ E[J(t + h, w^\tilde{\pi}(t + h), s_1(t + h), s_2(t + h), i) - J(t, w, s_1, s_2, i)] \right)
\]

Divide both sides by \( h \) and let \( h \to 0 \), together with Equations (3.1), (3.2) and (3.4), we have:

\[
0 = \sup_{\tilde{\pi} \in A(t)} \left( \lim_{h \to 0} \sum_{j=1}^L q_{ij} J(t, w, s_1, s_2, j) + \lim_{h \to 0} \frac{E[J(t + h, w^\tilde{\pi}(t + h), s_1(t + h), s_2(t + h), i) - J(t, w, s_1, s_2, i)]}{h} \right) \\
= J^*_i + \sup_{\tilde{\pi}} \mathcal{R}^*_i J^i
\]

Here we used the fact that \( \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \) for \( j \neq i \), and \( \lim_{h \to 0} \frac{P_{ii}(h) - 1}{h} = q_{ii} \)

Equivalently we can write the terminal conditions as:

\[
J(T, w, s_1, s_2, i) = J(t, w, \Sigma_1, s_2, i) = J(t, w, \Sigma_1, s_2, i) = J(t, w, s_1, S_2, i) = J(t, w, s_1, S_2, i) = \frac{[W(\tau)]^\theta}{\theta}
\]

Of course, a more general model would distinguish the exit time \( \tau_1, \tau_2 \) and \( T \) by adding discounting factors so that Equation (3.5) can be changed to \( J(\pi, w, s_1, s_2, i) = e^{-\lambda \tau} [W(\tau)]^\theta \) where \( \lambda \) is the discounting rate. Since the main objective in this model is to consider the regime switching effect on pairs trading, we do not want to make it more complex by adding more factors.

**COROLLARY 1.** The optimal investment rule \( \pi_1^*, \pi_2^* \) for a given regime state \( i \) at a fixed time \( t \) is:

\[
\begin{pmatrix}
\pi_1^* \\
\pi_2^*
\end{pmatrix} = - \begin{pmatrix}
\frac{\sigma_1}{\sigma_1^2} & \frac{\sigma_1 \sigma_2}{\sigma_1^2 \sigma_2^2} \\
\frac{\sigma_1 \sigma_2}{\sigma_1^2 \sigma_2^2} & \frac{1}{\sigma_2^2}
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{J^*_i w \mu_1^i - r^i}{J^*_i w \mu_2^i - r^i} \\
\frac{J^*_i w \mu_2^i - r^i}{J^*_i w \mu_2^i - r^i}
\end{pmatrix}
\]

\( (3.7) \)

**Proof.** It is a widely accepted assumption that the value function \( J^i \) is concave with respect to the wealth \( w \), i.e. \( J^i_{ww} < 0 \). This provides the necessary condition for us to get the maximizers by taking the first derivatives with respect to \( \pi_1 \) and \( \pi_2 \):

\[
w J^i_{w \mu_1^i - r^i} + w^2 J^i_{ww} \left[ \tilde{\sigma}_1^2 \pi_1 + \tilde{\sigma}_1 \tilde{\sigma}_2 \pi_2 \right] + \tilde{\sigma}_1^2 s_1 w J^i_{s_1 w} + \tilde{\sigma}_1 \tilde{\sigma}_2 s_2 w J^i_{s_2 w} = 0
\]

\[
w J^i_{w \mu_2^i - r^i} + w^2 J^i_{ww} \left[ \tilde{\sigma}_2^2 \pi_1 + \tilde{\sigma}_1 \tilde{\sigma}_2 \pi_2 \right] + \tilde{\sigma}_2^2 s_2 w J^i_{s_2 w} + \tilde{\sigma}_1 \tilde{\sigma}_2 s_1 w J^i_{s_1 w} = 0
\]

Solving the above equations will get us the optimal control. 

\( \square \)
3.2 Numerical Solutions

Though analytic solutions are the most desirable, they cannot be obtained for every HJB equation in stochastic control. In our case, the regime switching and first passage times make the HJB equations too complex to solve analytically, and we resort to numerical solutions. In this section, we will approximate each nonlinear PDE by iteratively solving linear PDEs until it converges. Due to the lack of analytic solutions, we have to resort to the comparison between our numerical solutions and the analytic solutions in [Tourin and Yan 2013] to verify our solutions.

3.2.1 Successive Approximation Approach

We adopt the successive approximation approach by Chang and Krishna [1986] and Peyrl et al. [2005]. The basic idea is to convert nonlinear PDE into linear PDE and solve them iteratively. Below we present convergence results from Chang and Krishna [1986] and Peyrl et al. [2005].

**Lemma 1.** Define $\overline{J}(t, w, s_1, s_2, i)$ as the solution to the following linear partial differential equation:

$$\overline{J}_t(t, w, s_1, s_2, i) + \sum_{i=1}^{n} L_i \overline{J}(t, w, s_1, s_2, i) = 0$$

subject to the boundary condition:

$$\overline{J}(\tau, w, s_1, s_2, i) = 0$$

then $\overline{J}(t, w, s_1, s_2, i) = J(t, w, s_1, s_2, i)$, which is the expected utility under control $\pi$ defined in Section 3.1.3.

**Lemma 2.** Define the iteration procedure as the following: suppose the control at each iteration $k$ $\pi^k = (\pi^k_1, \pi^k_2)$ ($k$ is a positive integer) is known. Let $\overline{J}^k(t, w, s_1, s_2, i)$ be the solution to Equation (3.8) at iteration $k$ with corresponding boundary conditions. We solve the equation for $\overline{J}^k(t, w, s_1, s_2, i)$ and update the control by:

$$\pi^{k+1} = \text{arg sup}_\pi \left\{ L_i \overline{J}^k(t, w, s_1, s_2, i) \right\}$$

We get the function $\overline{J}^{k+1}(t, w, s_1, s_2, i)$ by solving Equation (3.8) with control $\pi^{k+1}$ and the corresponding boundary conditions. Then for each iteration, we have:

$$\overline{J}^{k+1}(t, w, s_1, s_2, i) \geq \overline{J}^k(t, w, s_1, s_2, i)$$

**Lemma 3.** For each fixed point, we have:

$$\lim_{k \to +\infty} \pi^k = \pi^* \quad \text{and} \quad \lim_{k \to +\infty} \overline{J}^k(t, w, s_1, s_2, i) = \overline{J}(t, w, s_1, s_2, i)$$

where $\pi^*$ is the optimal control, and $\overline{J}(t, w, s_1, s_2, i)$ is the optimal function value for the given point.

**Remark:** The three lemmas together guarantee the convergence of the successive approximation approach. The iteration procedure defined in Lemma 2 separated the optimal feedback control with the PDE to avoid directly dealing with the nonlinear PDE.
3.2.2 Fully Implicit Finite Difference Scheme

The successive approximation approach has transformed the nonlinear PDEs into linear PDEs. Now we design a fully implicit finite difference scheme to solve the linear PDE system.

To solve Equation (3.5), we only need to consider how to solve \( J^i + \sum_{j=1}^N J^j = 0 \) in each iteration for given control \( \pi \). In order to reduce the dimensionality, we let \( J^i = \frac{\mu^i}{\sigma^i} H_i \) where \( H_i \) is only a function of \( t, w, s_1, s_2 \). Also, we replace \( t \) by \( T - t \) so that the terminal condition becomes the initial condition for computational purpose (it should be replaced back finally). Therefore, we need to solve:

\[
H_i = [s_1 \mu_1^i + \sigma_1^2 (\pi_1 \sigma_1^2 + \pi_2 \sigma_2^2) s_1 \theta] H_i^1 + [s_2 \mu_2^i + \sigma_2^2 (\pi_1 \sigma_1^2 + \pi_2 \sigma_2^2) s_2 \theta] H_i^2 + \theta [(1 - \pi_1 - \pi_2) r^i + \pi_1 \mu_1^i + \pi_2 \mu_2^i + \frac{\theta - 1}{2} |\pi_1 |^{\sigma_1^2} + \pi_2 |\sigma_2^2|] H^i \\
+ \frac{1}{2} |\pi_1 |^{\sigma_1^2} H_{s_1 s_1} + \frac{1}{2} |\pi_2 |^{\sigma_2^2} H_{s_2 s_2} + \sigma_1 \sigma_2 s_1 s_2 H_{s_1 s_2} + \sum_{j=1}^L q_{i j} H^j
\]

(3.9)

subject to the boundary conditions:

\[ H^i(0, s_1, s_2) = H^i(t, \bar{S}_1, s_2) = H^i(t, s_1, \bar{S}_2) = H^i(t, s_1, \bar{S}_2) = 1 \]

With the solution of \( H^i \), we get the control for the next iteration by:

\[
\pi_1^1 = \frac{s_1 H_i^1}{(1 - \theta) H^i} + \frac{(\mu_1^i - r^i) |\sigma_1^2| - (\mu_2^i - r^i) |\sigma_1^2|}{(1 - \theta) |\sigma_1^2|^{\sigma_1^2} - |\sigma_1^2|^{2} |\sigma_2^2|^{2}}
\]

(3.10)

\[
\pi_2^1 = \frac{s_2 H_i^2}{(1 - \theta) H^i} + \frac{(\mu_2^i - r^i) |\sigma_1^2| - (\mu_1^i - r^i) |\sigma_1^2|}{(1 - \theta) |\sigma_1^2|^{\sigma_1^2} - |\sigma_1^2|^{2} |\sigma_2^2|^{2}}
\]

(3.11)

The boundary condition in this problem is from Equation (3.6). The optimal solutions \( (\pi_1^1, \pi_2^1) \) are from a direct result from Equation (3.7) with substitution \( J^i = \frac{\mu^i}{\sigma^i} H_i \).

To solve Equation (3.9) at time \( t \), we discretize \( s_1 \) in the interval \([\underline{S}_1, \overline{S}_1]\) into \( N_1 + 1 \) points with the stepsize \( \Delta s_1 \) and let \( s_1^n \) be the \((n+1)th\) point in the interval. Here we let the boundary point be \( s_1^0 = \underline{S}_1 \) and \( s_1^{N_1} = \overline{S}_1 \). Discretize \( s_2 \) in the same way. The time interval is discretized into \( N + 1 \) points with stepsize \( \Delta t \). Let \((H_{i_1, i_2}^1)^n = H(t, s_{1_1}, s_{2_2}, i)\) be the discretized value function where \( n \) is a superscript in \((H_{i_1, i_2}^1)^n\).

Using the central difference method with the second order derivatives and cross derivatives and either forward, backward or central difference with the first derivatives on the right side of Equation (3.9) at time \((t_{n+1})\). Also we adopt forward difference for the first derivative in time \( H_i^t = \frac{(H_{i_1, i_2}^1)^{n+1} - (H_{i_1, i_2}^1)^n}{\Delta t} \). As a result, we get:
Chapter 3. A Dynamic Model of Pairs Trading

3.1 Case, linear interpolation or extrapolation is used to find \((t_{i+1, j+1})\) where \(S\) and each sub-matrix \(T\) is a large sparse matrix.

Developed by Van der Vorst [1992], which is known to have fast convergence rate and high accuracy in computing large sparse matrix.

\[
A_{i,i+1, j,j+1}^n + B_{i,i+1, j,j+1}^n + C_{i,i+1, j,j+1}^n + D_{i,i+1, j,j+1}^n + E_{i,i+1, j,j+1}^n + F_{i,i+1, j,j+1}^n + G_{i,i+1, j,j+1}^n + H_{i,i+1, j,j+1}^n + I_{i,i+1, j,j+1}^n + J_{i,i+1, j,j+1}^n + K_{i,i+1, j,j+1}^n + L_{i,i+1, j,j+1}^n + M_{i,i+1, j,j+1}^n + N_{i,i+1, j,j+1}^n + O_{i,i+1, j,j+1}^n + P_{i,i+1, j,j+1}^n + Q_{i,i+1, j,j+1}^n + R_{i,i+1, j,j+1}^n + S_{i,i+1, j,j+1}^n + T_{i,i+1, j,j+1}^n + U_{i,i+1, j,j+1}^n + V_{i,i+1, j,j+1}^n + W_{i,i+1, j,j+1}^n + X_{i,i+1, j,j+1}^n + Y_{i,i+1, j,j+1}^n + Z_{i,i+1, j,j+1}^n
\]

(3.12)

Coefficients \(A_{i,i+1, j,j+1}, B_{i,i+1, j,j+1}, \ldots\) in this equation are shown in Appendix E. Note that \((H_{i,i+1, j,j+1}^n)\) may not explicitly exist at time \(t\) since the boundary at time \(t_{n+1}\) might be different from that at time \(t_n\). In this case, linear interpolation or extrapolation is used to find \((H_{i,i+1, j,j+1}^n)\) based on the known grid points at time \(t_n\). To show Equation (3.12) in a matrix form, we define \(H^n = [(H^1)^n, (H^2)^n, \ldots, (H^L)^n]\), where we let \((H^i)^n = [(H_{i1}^n), (H_{i2}^n), \ldots, (H_{iN_1}^n), (H_{i1}^n), (H_{i2}^n), \ldots, (H_{iN_2}^n), \ldots, (H_{i1}^n), (H_{i2}^n), \ldots, (H_{iN_1}^n), (H_{i1}^n), (H_{i2}^n), \ldots, (H_{iN_2}^n)]\).

Therefore, the matrix form can be written as:

\[
PH^{n+1} = H^n + S
\]

(3.13)

where \(S\) is a vector of the boundary conditions shown in Appendix A and \(P\) is a large sparse square matrix of size \((L \times N_1 \times N_2)^2\):

\[
P = \begin{pmatrix}
T^1 & -\Delta t_{q_1, j, 2} I & -\Delta t_{q_1, j, 3} I & \cdots & -\Delta t_{q_1, j, L} I \\
-\Delta t_{q_2, 1} I & T^1 & -\Delta t_{q_2, j, 3} I & \cdots & -\Delta t_{q_2, j, L} I \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\Delta t_{q_{L, 1}} I & -\Delta t_{q_{L, 2}} I & \cdots & -\Delta t_{q_{L, j-1}} I & T^L 
\end{pmatrix}
\]

Each \(T^i\) in \(P\) is a block tridiagonal matrix:

\[
T^i = \begin{pmatrix}
TC_1^i & TR_1^i & \cdots & 0 \\
TL_1^i & TC_2^i & TR_2^i & \cdots \\
TL_2^i & TC_3^i & TR_3^i & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & TC_{N_1}^i & TR_{N_1}^i 
\end{pmatrix}
\]

and each sub-matrix \(TL_k^i, TC_k^i, TR_k^i\) is a tridiagonal matrix shown in Appendix E.

Since \(P\) is a non-symmetric large sparse matrix, we use bi-conjugate gradient stabilized method developed by Van der Vorst [1992], which is known to have fast convergence rate and high accuracy in computing large sparse matrix.
3.2.3 Numerical Algorithm

We summarize the numerical algorithm for solving the PDE system as the following:

Step 1: Start with an arbitrary control \( \pi \) and a given tolerance level \( \epsilon \), and define a positive number \( \text{Scale} \) to preclude unrealistic levels of accuracy later as in Forsyth and Labahn [2007] (Typically \( \text{Scale} = 1 \) for values expressed in dollars).

Step 2: Solve Equation (3.9) with the fully implicit finite difference scheme described in Section 3.2.2 (i.e. find the value of \( H \) at time \( n + 1 \) by solving Equation (3.13)) to get the discretized value function \( H = \{H^n ; n = 0, 1, \ldots N\} \). Update the control by Equations (3.10) and (3.11).

Step 3: Repeat Step 2 until the relative error \( \epsilon = \frac{|H^{k+1} - H^k|}{\max\{|H^k|, \text{Scale}\}} < \epsilon \), where \( H^k \) is the value function \( H \) at iteration \( k \).

Remark: The number of iterations needed to converge depends on \( \epsilon \). In our numerical experiments with \( L = \{1, 2, 3, 4\} \), convergence occurs in less than 30 iterations for \( \epsilon = 10^{-5} \) and the total number of iterations is found to increase with \( L \).

3.3 Numerical Example

Given regime states and stock prices, one can estimate parameters using Maximum-Likelihood shown in Appendix F. However, it is usually hard to accurately decide the regime states at each time. In addition, the focus of this model is on decision making for given parameters, instead of parameter estimation. Therefore, same as Guo [2005], we resort to simulation for the case of multiple regime states. In this section, just for the purpose of illustration, we study the case with two regime states with the parameters:

\[
\begin{align*}
& r(1) = r(2) = 0, \delta(1) = -1, \alpha(1) = 0.3, \beta(1) = -0.5, \sigma_{11}(1) = 1.2, \sigma_{12}(1) = 0.4, \mu_2(1) = 0.58, \sigma_{21}(1) = 0.6, \sigma_{22}(1) = 2, \\
& \mu_1(2) = 0.2, \delta(2) = -0.8, \alpha(2) = 0.1, \beta(2) = -0.8, \sigma_{11}(2) = 1.3, \sigma_{12}(2) = 0.2, \mu_2(2) = 0.25, \sigma_{21}(2) = 1, \sigma_{22}(2) = 1.5, \quad T = 1.
\end{align*}
\]

3.3.1 Numerical Results

Numerical experiments find out that investment decisions are essentially different for \( \theta \in (0, 1) \) and \( \theta < 0 \). In this section, we will try to provide intuitive explanation of optimal policies rather than trying to prove them mathematically.

Case 1: \( \theta \in (0, 1) \)

In this case, investors are less risk averse compared to Case 2. The CRRA utility \( U(w) = \frac{w^\theta}{\theta} \) is a positive valued function bounded below but not above. In an extreme case when \( \theta \to 1 \), investors become risk neutral and the utility function becomes the wealth \( U(w) = w \). We use \( \theta = 0.5 \) as an example. The optimal investment decisions are shown in Figure (3.1) and (3.2).

When \( s_1 \) and \( s_2 \) are both sufficiently small (near their lower bounds), \( \pi_1 \) decreases as \( s_1 \) increases but increases as \( s_2 \) increases. The intuition is that as stock 1 gets more expensive, it is more likely to be over-priced so that we should reduce our investment in stock 1 (or sell more of stock 1 if it is already
Figure 3.1: Comparison of optimal investment in stock 1 for regime state 1 when $\theta = 0.5$.

(a) $\pi_1$ at $t < T$

(b) $\pi_1$ at $t = T$

Figure 3.2: Comparison of optimal investment in stock 2 for regime state 1 when $\theta = 0.5$.

(a) $\pi_2$ at $t < T$

(b) $\pi_2$ at $t = T$
in a short position). On the other hand, as the price of stock 2 increases, it is slightly more likely that stock 1 is under-priced and we should invest more in stock 1. This is consistent with the closed form results in Tourin and Yan [2013].

Our next observation is that, for a fixed $s_2$, when $s_1$ is sufficiently near its upper bound and time is before the maturity $T$ as shown in Figure (3.1(a)), $\pi_1$ increases as $s_1$ gets higher. This seemingly counter-intuitive behavior is a result of the exit time $\tau = \min \{\tau_1, \tau_2, T\}$. As $s_1$ gets near $\overline{S}_1$, $\tau$ is more likely to be determined by $\tau_1$. Therefore if $s_1$ increases to $\overline{S}_1$ in the near future, $\tau$ will be $\tau_1$ and the trading stops. In such a case, the investor will gain more if he increases his investment in stock 1. This investment behavior only disappears at time $T$ shown in Figure (3.1(b)). The reason is that at maturity, we will have $\tau = T$ no matter what the values of $s_1$ and $s_2$ are. In other words, the first passage times $\tau_1$ and $\tau_2$ do not affect the investment decisions at maturity.

Another observation from Figure (3.1(a)) is that when $s_1$ is large, increasing $s_2$ from its lower bound leads to the increase of $\pi_1$ first, and then decrease of $\pi_1$ as $s_2$ gets larger. The first part is intuitive in that as $s_2$ gets larger, stock 1 is relatively more under-priced and we should increase $\pi_1$. But when $s_2$ is large enough, the influence of $\tau_2$ get large. While it is still possible for $s_1$ to increase to $\overline{S}_1$, the risk is higher as $s_2$ gets closer to $\overline{S}_2$. Therefore it is reasonable for the investor to put less money in stock 1.

Same as the behavior of $\pi_1$, it is natural for $\pi_2$ to increase with the increase of $s_1$ and decrease with the increase of $s_2$, as shown in Figure (3.2(b)). However, the $\pi_2$ does not display a simple increasing or decreasing trend when time $t < T$. For the same reason as $\pi_1$, the counter-intuitive investment behavior of $\pi_2$ can be understood by the interplay between potential profit and risk from the exit time $\tau$.

Though we only showed the results for regime 1, the optimal investment decisions for regime 2 behave exactly the same as regime 1 except the scales are different.

**Case 2: $\theta < 0$**

The investors in this case are more risk averse. The CRRA utility $U(w) = \frac{w^\theta}{\theta}$ is now a negative valued function bounded above but not below. In this case, the utility function is different from Case 1 (though they are both CRRA utility). As an illustration, we set $\theta = -20$ and show the optimal investment decisions in Figure 3.3 and Figure 3.4.

This time the optimal behavior is more consistent with our intuition for $\pi_1$: the rise of $s_1$ leads to the decrease of $\pi_1$ and the rise of $s_2$ leads to the increase of $\pi_1$. However, the behavior of $\pi_1$ still differs between the time before maturity and the time at maturity in the sense that the rate of decrease of $\pi_1$
is different as $s_1$ increases, especially when $(s_1, s_2)$ is near $(\bar{S}_1, S_2)$. Again, it might be the result of the exit time $\tau$ and the risk aversion rate $\theta$, but an intuitive explanation is hard to find.

The behavior of $\pi_2$ before maturity $T$ is consistent with the intuition for most values of $s_1$ and $s_2$ except when $s_1$ is in the middle and $s_2$ is large. Again, it is hard to be explained intuitively since it is the result of $\tau$ and $\theta$. In this case we have to rely on the numerical solutions.

### 3.3.2 Simulation

We start with initial prices $s_1(0) = s_2(0) = 10$ and initial wealth $w(0) = 1$. We generate 1000 data for both stocks with equally spaced time points $t_k = \frac{kT}{1000} \in [0, T]$. We set the boundaries to be $[S_k(t), S_k(t)] = [1, 30]$ for both $k = 1, 2$. Regime states are set as the following:

<table>
<thead>
<tr>
<th>Regime States</th>
<th>Regime 1</th>
<th>Regime 2</th>
<th>Regime 1</th>
<th>Regime 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>1 $\rightarrow$ 82</td>
<td>83 $\rightarrow$ 377</td>
<td>378 $\rightarrow$ 744</td>
<td>745 $\rightarrow$ 1000</td>
</tr>
</tbody>
</table>

The transition rates between different states are estimated in the same way as Wan [2006] and Zhang [2001]. Since there are 449 data in regime 1 and regime 1 appears twice in the whole trading period, the transition rate for regime 1 is $q_{11} = -q_{12} = -\frac{1}{\text{449} \times 2} = -4.4543$. We do the same calculation for regime 2. Therefore the generating matrix is:

$$Q = \begin{pmatrix} -4.4543 & 4.4543 \\ 3.2698 & -3.2698 \end{pmatrix}$$ (3.14)

We show some examples to demonstrate the effect of regime switching, price limits and different levels of risk aversion $\theta$. In each example, we compare between $\theta = 0.5$ and $\theta = -20$.

**Example 1:** $\tau = T$  
Stock prices are shown in Figure 3.5. Since the prices fluctuate between $[1, 30]$ and never hit the boundaries, the trading process ends at $\tau = T$.

Results are shown in Figure 3.6 for the case $\theta = 0.5$. The wealth and utility both have a large increase. The dynamics of the wealth and utility look very similar, though scales are different. Since $\theta = 0.5 > 0$, the utility function $U(w) = \frac{w^2}{\theta}$ is close to the wealth $w$, which explains for their similar
Figure 3.5: Price movement for both stocks in example 1 from 0 to T. Red curve is the price of stock 1 and blue curve is the price of stock 2.

movement. Figure 3.6(c) and 3.6(d) show the jump of optimal investments between different regime states. It is very natural that such jumps appear in the optimal investment following the change of regime states. Such behavior can also be seen in other papers involving regime switching such as [Wan 2006]. Also note that by the end of the trading day investment in both stocks tends to 0 (may not be exactly at 0, depending on the parameters), which is also consistent with the numerical example as in Tourin and Yan [2013].

For the case \( \theta = -20 \) shown in Figure 3.7, the wealth dynamics behaves different from the dynamics of utility in terms of the rate of change. As we discussed before, there is an upper-bound for the utility function \( U(w) = \frac{w^\theta}{\theta} \) when \( \theta < 0 \). As \( w \) gets larger, the increase of utility is getting slower. In this case, the return of the wealth is lower than the previous case. This result is not surprising since the investor’s risk aversion rate is much higher than \( \theta = 0.5 \).

**Example 2:** \( \tau < T \) Stock prices are shown in Figure 3.8. The trading stops at \( \tau = \tau_2 = t_{574} \) when \( s_2 \) reaches \( S_2 \). When \( \theta = 0.5 \), the terminal wealth has a 150% return while for \( \theta = -20 \), return of the wealth is only 2.5%. Unlike the first example, the optimal investment in both stocks (Figure 3.9 and 3.10) are not close to 0 at the end of trading, since the trading exit when price limits are reached before the maturity \( T \).

Lastly, we compare the performance of \( \theta = 0.5 \) and \( \theta = -20 \) from the simulation of 10^6 sample paths. We generate regime states using the same \( Q \) in Equation (3.14), and then simulate the sample paths of prices based on the generated regime states. For each sample path we calculate the return by \( R_W = \frac{W(\tau) - W(1)}{W(1)} \). The convergence of the mean return and variance from the sample paths can be easily shown by numerical experiments. It is found out that the mean return is 1.12% for \( \theta = 0.5 \) and 0.57% for \( \theta = -20 \), and the variance is 2.2217 for \( \theta = 0.5 \) and 7.3698 \times 10^{-4} \) for \( \theta = -20 \). As the investor gets more risk averse, the expected return decreases, together with the volatility. Recall that \( w(0) = 1 \) at the beginning, histograms of the terminal wealth \( w(\tau) \) are shown in 3.11(a) and 3.11(b). Recall that for \( \theta = 0.5 \), the utility function is bounded below but not above. This leads to the fat tail on the right side in 3.11(a). Similarly, one may expect a fat tail on the left side of 3.11(b) when \( \theta = -20 \), since the
Figure 3.6: Trading results with $\theta = 0.5$ in example 1. In sub-figure (c) and (d), blue curves are for regime 1 and red curves are for regime 2.

Figure 3.7: Trading results with $\theta = -20$ in example 1. In sub-figure (c) and (d), blue curves are for regime 1 and red curves are for regime 2.
Figure 3.8: Price movement for both stocks in example 2 from 0 to $\tau = 0.574$. Red curve is the price of stock 1 and blue curve is the price of stock 2.

Figure 3.9: Trading results with $\theta = 0.5$ in example 2. In sub-figure (c) and (d), blue curves are for regime 1 and red curves are for regime 2.
utility function is bounded above but not below. However, this is not as obvious as in 3.11(a) since the variance is very small.

Figure 3.11: Distribution of the terminal wealth for different risk aversion rate
Chapter 4

Conclusion and Future Works

4.1 Summary

This thesis is mainly focused on optimizing the performance of pairs trading. Our research is motivated by the common practice of pairs trading: take positions when the spread widens and clear the positions when it reverts back to the mean. A natural question is when exactly should one take and clear positions in order to maximize his return. Certainly, the objectives can vary for different investors. In our static model, we answer this question with the objective to maximize the profit per unit time in the long run. Contrary to the current practices, we find that the optimal threshold of clearing positions is not the mean of the spread, but should be symmetric to the threshold of taking positions around the mean. Our static model is inspired by the work of Bertram [2010] and can be seen as an extension to his work. The limitation of our static model is that model parameters might not be constant in the long run. To consider the dynamics of stock prices and model parameters, we put pairs trading in the stochastic control framework. Our dynamic model is inspired by the work of Mudchanatongsuk et al. [2008], Fourin and Yan [2013], and Wan [2006]. In this model, our objective is to maximize the expected return in the finite horizon and model parameters are assumed to follow a continuous time Markov chain. We use numerical examples to show that price limits, risk aversion rate and the regime switching have a large impact on the investment decisions.

4.2 Main Contribution

In our static model, there are two main contributions. Firstly, we have derived an explicit formula of the expectation of the first passage time over the two-sided boundary for an OU process. The expectation is expressed both in the integral and polynomial form. The polynomial form is used throughout the static model. Secondly, we have derived the analytic solution for the optimal thresholds of pairs trading. We find that, in order to maximize the profit per unit time in the long run, the investor should clear positions and take the opposite positions at the same time. This result is counter-intuitive for the common practice.

In our dynamic model, we have developed a regime-switching framework that considers the change of model parameters in a continuous time Markov chain. Based on the successive approximation approach, we have also designed a numerical scheme to solve the system of partial differential equations. Our
numerical results show the impact of price limits, risk aversion rate and the regime switching on the investment decisions.

4.3 Limitations and Future Works

While the rules we develop in the static model are directly applicable, the performance relies on our assumption that the model parameters are constant. Though they may not change in a short period of time, they are likely to change in the long run. Parameters may change due to the change of overall market states, internal management issues within one company in the pair, and so on. Therefore, future research is needed to study the optimal thresholds when the model parameters change with time, or when the OU process breaks up. One way to limit loss when the OU process breaks up is to impose stop-loss barriers on the trading process. In this case, the optimal thresholds will be functions of model parameters, transaction cost, and the level of stop-loss barriers.

Though the dynamic model has considered the situation when model parameters change with time, the application of this model is still limited. On one hand, the change of model parameters might not always be modeled as a continuous time Markov chain. Therefore, this assumption, though widely accepted in the literature, might fail to reflect the reality. On the other hand, even if regime switches in a continuous time Markov chain, it is hard to determine the exact regime state at a given time point. Though we have the ML estimation for given data in a fixed regime, calibration of parameters when a regime state is not perfectly known still remains unsolved. Future work is required to estimate the probability of regime states given observations, where Bayesian updating is usually used. Also, knowing the probability of regime states, we still need to determine the optimal investment decisions. Finding optimal decisions when regime information is only partially observable is still challenging, but also quite interesting to explore.
Appendices

A Proof of Proposition 1

Proof. We use the symbolic calculation from Mathematica to get the following results:

\[
\exp\left(\frac{x^2}{2}\right) \int_0^\infty \ln(t) \exp\left(-\frac{t^2}{2}\right) \cos(xt) dt = -\frac{1}{2} \sqrt{\frac{\pi}{2}} \left[ \gamma + \ln(2) + \text{Hypergeometric}_1^{(1,0,0)}(0, \frac{1}{2}, \frac{x^2}{2}) \right]
\]

where \( \gamma = \int_0^\infty \exp(-t) \ln(t) dt \) is the Euler constant. Since the definite integral \( \int_0^\infty \exp(-x^2) \ln(x) dx = -\frac{\sqrt{\pi}}{4} [\gamma + 2 \ln(2)] \), it is easy to verify that \( C = \int_0^\infty \ln(t) \exp\left(-\frac{t^2}{2}\right) dt = -\frac{1}{2} \sqrt{\frac{\pi}{2}} [\gamma + 2 \ln(2)] \). The polynomial form of the Kummer confluent hyper-geometric function is \( \text{Hypergeometric}_1 F_1(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \), where \( (x)_n = x(x+1)(x+2)\ldots(x+n-1) \). By the chain rule we have \( \frac{d(x)_n}{dx} = (n-1)! \) at \( x = 0 \). Therefore it is easy to verify that:

\[
\text{Hypergeometric}_1 F_1^{(1,0,0)}(0, \frac{1}{2}, \frac{x^2}{2}) = \sum_{n=1}^{\infty} \frac{2^n(n-1)!}{(2n)!} x^{2n}
\]

Therefore we get:

\[
\exp\left(\frac{x^2}{2}\right) \int_0^\infty \ln(t) \exp\left(-\frac{t^2}{2}\right) \cos(xt) dt = -\frac{1}{2} \sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{(\sqrt{2}x)^{2n}}{(2n)!} \Gamma(n) + \int_0^\infty \ln(t) \exp\left(-\frac{t^2}{2}\right) dt
\]

\[\square\]
B Proof of Proposition 2

Let \( f^i(a_i, b_i) \) be the maximal expected return per unit time in case \( i \) with the optimal thresholds \( a_i \) and \( b_i \), for \( i = 1, 2 \). We only have to prove:

\[
f^1(a_1, b_1) = f^2(a_2, b_2), \quad \text{for } c = 0
\]

and

\[
f^1(a_1, b_1) < f^2(a_2, b_2), \quad \text{for } c > 0
\]

**Proof.** In both case 1 and case 2, the expected times of one trading cycle are both

\[
\sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right)
\]

In case 1, since \( b_1 = 0 \), the objective function is given by:

\[
F^1(a, c) = \frac{a - c}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right)}
\]

and the optimal function for case 2 is:

\[
F^2(a, c) = \frac{a - \frac{c}{2}}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right)}
\]

Obviously, when \( c = 0 \), we have \( F^1(a, c) = F^2(a, c) \) for any value of \( a \). Therefore [1] is proved.

To prove [2] we make use of the fact that the optimal values of \( a_1 \) and \( a_2 \) satisfy the equation [2.20] and [2.23] respectively. Therefore in the optimal solution we have:

\[
F^1(a_1, c) = \frac{a_1 - c}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a_1)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right)} = \frac{1}{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a_1)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right)}
\]

and similarly

\[
F^2(a_2, c) = \frac{a_2 - \frac{c}{2}}{\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a_2)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right)}
\]

Since \( \frac{1}{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right)} \) is a decreasing function of \( x \), to prove \( F^2(a_2, c) > F^1(a_1, c) \), we only need to prove \( a_2 < a_1 \). Let \( a(x) \) satisfy the following equation:

\[
\frac{1}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n+1}}{(2n+1)!} \Gamma\left(\frac{2n+1}{2}\right) = \frac{1}{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right)}
\]

It is easy to see that \( a(x) > x \) for any \( x > 0 \). Obviously, \( a_2 = a(\frac{c}{2}) \) and \( a_1 = a(c) \). Since \( c > 0 \), all we need is to prove \( a(x) \) is a strictly increasing function. Taking the derivative of \( x \) on both sides of the equation above, we get:

\[
a'(x) = \frac{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}a)^{2n}}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right)}{(a - x) \sum_{n=1}^{\infty} \frac{(\sqrt{2}a)^{2n-1}}{(2n-1)!} \Gamma\left(\frac{2n+1}{2}\right)} > 0
\]

Hence [2] is proved.

\[ \square \]
Figure 1: The red curve is from our explicit formula of the expectation in (2.18), and the blue curve is from the simulation. We fix the upper at $a = 0.8$ and lower boundaries at $-a = -0.8$. For any starting point $b \in (-0.8, 0.8)$ inside the boundaries, we can get the expected time for the OU process to reach the boundaries either by simulation or using our explicit formula.

Figure 2: is fixed. The red curve is from our explicit formula of the expectation in (2.18), and the blue curve is from the simulation. We fix our starting point at $b = 0.1$ and let the upper boundary $a \in (0.1, 0.6)$. The lower boundary is defined accordingly.

C Comparison of Simulation and Analytic Solutions

We have obtained the explicit formula for the expectation of first passage time over two-sided boundaries in section 3. Now we want to verify this results by comparing it with simulations. For given $a$ and $b$, we simulate $10^6$ paths and for each path we get a $\tau_2$ defined in section 2. Then we calculate $E[\tau_2]$ by taking the average of all the simulated values of $\tau_2$. On the other hand we calculate the expectation directly from the explicit formula. Comparison is shown in figure 1 and figure 2.

The blue lines are the simulation results and thus vibration is expected. The red lines are the results from the explicit formula and therefore it is very smooth. The simulation and the explicit formula are very close, which verify our explicit formula.
D Sensitivity Analysis

Fix $\theta$ and $\sigma$ and we impose a small shock on $\mu$, we find the sensitivity analysis for $\mu$ in Table 1 and Figure 3.

<table>
<thead>
<tr>
<th>Mean</th>
<th>$\mu$</th>
<th>3.355633</th>
<th>3.372753</th>
<th>3.389874</th>
<th>3.406994</th>
<th>3.424111</th>
<th>3.441235</th>
<th>3.458356</th>
<th>3.475477</th>
<th>3.492597</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of change</td>
<td>-2%</td>
<td>-1.50%</td>
<td>-1%</td>
<td>-0.50%</td>
<td>0%</td>
<td>0.50%</td>
<td>1%</td>
<td>1.50%</td>
<td>2%</td>
<td></td>
</tr>
<tr>
<td>Total Return</td>
<td>0.11554</td>
<td>0.119631</td>
<td>0.13604</td>
<td>0.243107</td>
<td><strong>0.33301</strong></td>
<td>0.316707</td>
<td>0.207322</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Percentage of change</td>
<td>-65%</td>
<td>-64%</td>
<td>-99%</td>
<td>-27%</td>
<td>0%</td>
<td>-5%</td>
<td>-38%</td>
<td>-100%</td>
<td>-100%</td>
<td></td>
</tr>
</tbody>
</table>

For a small change of $\mu$, the change for the return is very large. Therefore, it is crucial to have a very accurate estimation of $\mu$. In our example of KO and PEP, we have a relatively accurate estimation of $\mu$ since the change in each direction results in a huge loss.
Appendices

Figure 4: Sensitivity Analysis for $\theta$. The black spot is the estimated value of $\theta$

Then we fix $\mu$ and $\sigma$ and conduct sensitivity analysis for $\theta$ shown in Table 2 and Figure 4.

<table>
<thead>
<tr>
<th>Reversion rate $\theta$</th>
<th>0.014201</th>
<th>0.016568</th>
<th>0.018935</th>
<th>0.021302</th>
<th>0.02367</th>
<th>0.026036</th>
<th>0.028403</th>
<th>0.03077</th>
<th>0.033137</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of change</td>
<td>-40%</td>
<td>-30%</td>
<td>-20%</td>
<td>-10%</td>
<td>0%</td>
<td>10%</td>
<td>20%</td>
<td>30%</td>
<td>40%</td>
</tr>
<tr>
<td>Total Return</td>
<td>0.228298</td>
<td>0.37313</td>
<td>0.358932</td>
<td>0.344309</td>
<td>0.33301</td>
<td>0.330835</td>
<td>0.322827</td>
<td>0.311096</td>
<td>0.281548</td>
</tr>
<tr>
<td>Percentage of change</td>
<td>-31%</td>
<td>12%</td>
<td>8%</td>
<td>3%</td>
<td>0%</td>
<td>-1%</td>
<td>-3%</td>
<td>-7%</td>
<td>-15%</td>
</tr>
</tbody>
</table>

The change of return is relatively inactive to the change of the reversion rate $\theta$. In fact, small changes of $\theta$ does not affect the return at all. It only affects the return when the change of $\theta$ is large enough. From Figure 4 we can see that a smaller value of $\theta$ actually gives a higher return.
Lastly we show the sensitivity analysis for $\sigma$ in Table 3 and Figure 5.

Table 3: Sensitivity analysis for $\sigma$

<table>
<thead>
<tr>
<th>Standard deviation $\sigma$</th>
<th>0.004877</th>
<th>0.005689</th>
<th>0.006502</th>
<th>0.007315</th>
<th><strong>0.00813</strong></th>
<th>0.008941</th>
<th>0.009753</th>
<th>0.010566</th>
<th>0.011379</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of change</td>
<td>-40%</td>
<td>-30%</td>
<td>-20%</td>
<td>-10%</td>
<td>0%</td>
<td>10%</td>
<td>20%</td>
<td>30%</td>
<td>40%</td>
</tr>
<tr>
<td>Total Return</td>
<td>0.264128</td>
<td>0.264128</td>
<td>0.275932</td>
<td>0.313926</td>
<td><strong>0.33301</strong></td>
<td>0.358932</td>
<td>0.373601</td>
<td>0.228298</td>
<td>0.228298</td>
</tr>
<tr>
<td>Percentage of change</td>
<td>-21%</td>
<td>-21%</td>
<td>-17%</td>
<td>-6%</td>
<td>0%</td>
<td>8%</td>
<td>12%</td>
<td>-31%</td>
<td>-31%</td>
</tr>
</tbody>
</table>

Similar as $\theta$, the change of $\sigma$ is not influential to the change of the total return. A larger value of $\sigma$ may result in a higher value in our example.
E Numerical Discretization

At each fixed point \((t; s_1, s_2, i; \pi_1, \pi_2)\), Equation \((3.12)\) can be written as:

\[
H^i = a H^i_{s_1} + b H^i_{s_2} + c H^i_{s_1 s_1} + d H^i_{s_2 s_2} + e H^i_{s_1 s_2} + f H^i + \sum_{j \neq i} H^j
\]

where

\[
a = s_1 \mu_1^i + s_1 \theta \sigma_1^i (\pi_1 \sigma_1^i + \pi_2 \sigma_2^i),
\]

\[
b = s_2 \mu_1^i + s_2 \theta \sigma_2^i (\pi_1 \sigma_1^i + \pi_2 \sigma_2^i),
\]

\[
c = \frac{1}{2} |s_1 \sigma_1^i|^2,
\]

\[
d = \frac{1}{2} |s_2 \sigma_2^i|^2,
\]

\[
e = s_1 s_2 \sigma_1^i \sigma_2^i
\]

\[
f = \theta ([1 - \pi_1 - \pi_2] r^i + \pi_1 \mu_1^i + \pi_2 \mu_2^i) + \frac{1}{2} [\pi_1 \sigma_1^i + \pi_2 \sigma_2^i]^2 \theta (\theta - 1) + q_{ii}
\]

If we use the forward difference for first derivatives and central difference for second derivatives and cross derivatives, coefficients in Equation \((3.12)\) can be written as:

\[
A^i_{1i+2} = -e \frac{\Delta t}{\Delta s_1 \Delta s_2},
\]

\[
B^i_{i+1i} = -a \frac{\Delta t}{\Delta s_1} - c \frac{\Delta t}{\Delta s_2},
\]

\[
D^i_{i+1i} = -b \frac{\Delta t}{\Delta s_2} - d \frac{\Delta t}{\Delta s_2}
\]

\[
C^i_{i+1i} = -A^i_{i+1i},
\]

\[
E^i_{i+1i} = 1 + a \frac{\Delta t}{\Delta s_1} + b \frac{\Delta t}{\Delta s_2} + 2c \frac{\Delta t}{\Delta s_1} + 2d \frac{\Delta t}{\Delta s_2} - f \Delta t
\]

\[
P^i_{i+1i} = -d \frac{\Delta t}{\Delta s_2},
\]

\[
G^i_{i+1i} = C^i_{i+1i},
\]

\[
M^i_{i+1i} = -c \frac{\Delta t}{\Delta s_1},
\]

\[
N^i_{i+1i} = A^i_{i+1i}
\]

Coefficients from backward difference and central difference for the first derivative are very similar to the forward difference. One only has to apply the same procedure on the discretization except for

\[
H^i_{sk} = \frac{(H^i_{s_1, i+1})^{n+1} - (H^i_{s_1, i+1})^{n+1}}{\Delta s_k} \text{(backward)} \quad \text{and} \quad H^i_{sk} = \frac{(H^i_{s_1, i+1})^{n+1} - (H^i_{s_1, i+1})^{n+1}}{2\Delta s_k} \text{(central)}.
\]

The tridiagonal matrices \(TL_k^i, TC_k^i, TR_k^i\) are known as:

\[
TL_k^i = \begin{pmatrix}
M^i_{k1} & G^i_{k1} & \cdots & 0 \\
N^i_{k2} & M^i_{k2} & G^i_{k2} & \cdots \\
N^i_{k3} & M^i_{k3} & G^i_{k3} & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
0 & \cdots & \cdots & N^i_{kN_2} & M^i_{kN_2}
\end{pmatrix}
\]
The boundary condition $S = [S^1, S^2, \ldots, S^L]'$ where each $S^k = [BS^k_1, BS^k_2, \ldots, BS^k_{N_2}]$ and each $BS^k_m$ is a vector of length $N_2$ for the discretization of $s_2$ when regime is $k$ and $s_1 = s_{i_1}$. When $m = 1$:

$$BS^k_1(1) = G^{i}_{1_{i_1}i_2} + M^{i}_{1_{i_1}i_2} + N^{i}_{1_{i_1}i_2} + C^{i}_{1_{i_1}i_2} + F^{i}_{1_{i_1}i_2}$$

$$BS^k_1(n) = G^{i}_{1_{i_1}i_2} + M^{i}_{1_{i_1}i_2} + N^{i}_{1_{i_1}i_2} \text{ for } 1 < n < N_2$$

$$BS^k_1(N_2) = G^{i}_{1_{i_1}i_2} + M^{i}_{1_{i_1}i_2} + N^{i}_{1_{i_1}i_2} + A^{i}_{1_{i_1}i_2} + D^{i}_{1_{i_1}i_2}$$

When $1 < m < N_1$:

$$BS^k_m(k) = C^{i}_{1_{i_1}i_2} + F^{i}_{1_{i_1}i_2} + N^{i}_{1_{i_1}i_2} \text{ for } 1 \leq n < N_2$$

$$BS^k_m(N_2) = G^{i}_{1_{i_1}i_2} + A^{i}_{1_{i_1}i_2} + D^{i}_{1_{i_1}i_2}$$

When $m = N_1$:

$$BS^k_{N_1}(1) = A^{i}_{1_{i_1}i_2} + B^{i}_{1_{i_1}i_2} + C^{i}_{1_{i_1}i_2} + F^{i}_{1_{i_1}i_2} + N^{i}_{1_{i_1}i_2}$$

$$BS^k_{N_1}(n) = A^{i}_{1_{i_1}i_2} + B^{i}_{1_{i_1}i_2} + C^{i}_{1_{i_1}i_2} \text{ for } 1 < n < N_2$$

$$BS^k_{N_1}(N_2) = A^{i}_{1_{i_1}i_2} + B^{i}_{1_{i_1}i_2} + C^{i}_{1_{i_1}i_2} + D^{i}_{1_{i_1}i_2} + G^{i}_{1_{i_1}i_2}$$
F Parameter Estimation

In each regime with given data, we can estimate parameters for the model using Maximum-Likelihood (ML). Suppose observations are made at \( t_0, t_1, \ldots, t_N \) with stock prices known as \( s_1(t_n) \) and \( s_2(t_n) \). For simplicity, let \( y(t_n) = (s_1(t_n), s_2(t_n)) \) and \( \Theta = (\mu_1, \alpha, \beta, \sigma_1, \mu_2, \sigma_2) \). Here we omit return of the risk-free asset \( r \) as it is not correlated with the stock prices and can be estimated separately. Since \( y(t) \) is Markovian, we can express the likelihood function as:

\[
f(y(t_1), y(t_2), \ldots, y(t_N); \Theta) = \prod_{n=0}^{N_1} f_\Theta(y(t_{n+1})|y(t_n))
\]

where

\[
f_\Theta(y(t_{n+1})|y(t_n)) = \frac{1}{2\pi \sqrt{\det(\Sigma)}} \times \exp \left\{ -\frac{1}{2} (y(t_{n+1}) - E[y(t_{n+1})|y(t_n)])^T \Sigma^{-1} (y(t_{n+1}) - E[y(t_{n+1})|y(t_n)]) \right\}
\]

\[
E[y(t_{n+1})|y(t_n)] = \begin{pmatrix} E[s_1(t_{n+1})|s_1(t_n)] \\ E[s_2(t_{n+1})|s_2(t_n)] \end{pmatrix}
\]

\[
\Sigma = \begin{pmatrix} Var[s_1(t_{n+1})|s_1(t_n)] & Cov[s_1(t_{n+1})|s_1(t_n), s_2(t_{n+1})|s_2(t_n)] \\ Cov[s_1(t_{n+1})|s_1(t_n), s_2(t_{n+1})|s_2(t_n)] & Var[s_2(t_{n+1})|s_2(t_n)] \end{pmatrix}
\]

Let \( \alpha = \mu_1 - \frac{1}{2} |\sigma_1|^2 + \delta \alpha, b = \delta, c = \delta \beta \) and \( \mu = \mu_2 - \frac{1}{2} |\sigma_2|^2 \). Suppose we know the information at time \( t_n: s_1(t_n) = s_1, s_2(t_n) = s_2 \). Let \( \Delta t_n = t_{n+1} - t_n \), then we have:

\[
E[s_1(t_{n+1})|s_1(t_n)] = s_1 e^{b \Delta t_n} - \frac{b(a + cs_2) + c \mu}{b^2} (1 - e^{b \Delta t_n}) - \frac{c \mu \Delta t_n}{b}
\]

\[
E[s_2(t_{n+1})|s_2(t_n)] = s_2 + \mu \Delta t_n
\]

\[
Var[s_1(t_{n+1})|s_1(t_n)] = c^2 |\sigma_2|^2 \left( \frac{\Delta t_n}{b^2} + \frac{3 e^{b \Delta t_n}}{2b^2} - \frac{2 e^{2b \Delta t_n}}{b^4} + \frac{e^{2b \Delta t_n}}{2b^5} \right)
\]

\[
\quad + |\sigma_2|^2 \left( \frac{e^{2b \Delta t_n} - 1}{2b} \right) + 2c \sigma_1 \sigma_2 \left( \frac{1}{2b^2} - \frac{e^{b \Delta t_n}}{b^2} + \frac{e^{2b \Delta t_n}}{2b^2} \right)
\]

\[
Var[s_2(t_{n+1})|s_2(t_n)] = |\sigma_2|^2 \Delta t_n
\]

\[
Cov[s_1(t_{n+1})|s_1(t_n), s_2(t_{n+1})|s_2(t_n)] = c |\sigma_2|^2 \left( \frac{e^{b \Delta t_n} - 1}{b^2} - \frac{\Delta t_n}{b} \right) + \sigma_1 \sigma_2 \frac{e^{b \Delta t_n} - 1}{b}
\]

Now perform a standard optimization technique to maximize the function \( f(y(t_1), y(t_2), \ldots, y(t_N); \Theta) \).
or equivalently, its log-likelihood to get the model parameters for a given regime.
Bibliography


