Maintenance Models for A Two-unit Series System

by

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Abstract

We consider two maintenance models for a two-unit series system with economic dependence, composed of one deteriorating expensive unit and one cheaper unit with low replacement cost. In the first model, we apply the opportunistic age-based preventive replacement policy with two different age thresholds for the expensive unit, in order to save the system set-up cost. In the second model, we use condition based maintenance (CBM) with multivariate Bayesian control chart to calculate the posterior probability of the expensive unit in its warning state. Also, the OM control limit is introduced to provide opportunities for its full inspection when the cheaper unit fails during sampling intervals. In both two models, the semi-Markov decision process (SMDP) algorithms are developed to find two optimal control limits and minimum average costs in a long-run horizon.
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Chapter 1

Introduction

1.1 Introduction to Multi-unit Maintenance

During the last few decades, more researchers have focused on the system composed of several units in their maintenance research. Three major reasons may have contributed to this situation. First, systems are becoming more and more complicated and multifunctional. It requires new technologies and methodologies, more sophisticated maintenance models and control policies to solve these systems’ maintenance problems. Second, improvements of analytical and computational techniques, e.g., the availability of fast computers and various low-cost physical sensors, allow the complex systems to be inspected and monitored more frequently and effectively. Third, the industry as well as the researchers have realized that interactions between units in a system cannot be neglected and should be taken into account in any replacement and maintenance decision. These interactions between components not only complicate the modeling and maintenance optimization, but also offer the opportunity for group maintenance that may save the operating costs for the entire system. Therefore, maintenance models for multi-unit systems play an increasingly important role in maintenance decision research, and lead to considerable cost-savings in practical application.

Cho and Parlar (1991) give the definition of multi-unit maintenance models: “Multi-unit maintenance models are concerned with optimal maintenance policies for a system
consisting of several units of machines or many pieces of equipment, which may or may not depend on each other (i.e. economic/ stochastic dependency).” Optimal maintenance policies for such multi-unit systems would reduce to the simple combination of independent optimal maintenance policies for each separate unit, if the dependence among units does not exist. On the other hand, if units in the system are dependent upon each other, then an optimal decision on the replacement or repair for each single unit is not necessarily optimal for the whole system. A decision must be made to improve the whole system, rather than any subsystem. The interaction between units can be classified into three different kinds from Thomas (1986) analysis: economic dependence, structural dependence, and stochastic dependence.

- Economic dependence implies that performing replacement or maintenance on several subsystems jointly costs less money and/or time than on each subsystem separately. The cost structure of replacement and maintenance has interdependence between the units.

- Structural dependence means that some operating components in a multi-unit system have to be replaced, or at least dismantled, before failed components can be replaced or repaired. In other words, structural dependence between components indicates that subsystem cannot be maintained independently for physical connection. For example, a union formed by a bicycle chain and a cassette should always be maintained together rather than individually.

- Stochastic dependence occurs if the condition of one unit influences the lifetime distribution of other units in the system, or if there are causes outside the system which bring about simultaneous failures and hence correlate the lifetimes, which is
called common-cause failures. For example, it may be the case that the failure of one unit induces the failure of other units or causes a shock to other units.

Many multi-unit maintenance models consider only one of these dependencies, since combining more than one makes the model too complicated to solve or to analyze. Among these dependences, the structural dependence models are not so common, because most of them are only applicable to some specific systems with physical connections between units. Also, based on our literature review, there are a few articles published on this topic. For maintenance of systems with stochastic dependence, researchers have provided three main types of failure interaction between units in Murthy and Nguyen (1985) and Murthy and Nguyen (1985). Although this topic has attracted much attention since then, most articles still deal with two-component systems following same assumptions. Furthermore, maintenance models of systems with stochastic dependence require additional information, the failure interaction between units. The interaction is defined as the failure of one unit can induce failure of another unit with probability \( q \), without inducing an instantaneous failure, but affecting its failure rate. Such information about probability \( q \) is quite difficult to estimate and obtain in real applications. It generates that specific maintenance policies for systems with failure interactions are difficult to install in real industries. In general, economic dependence is the most universal in the continuous operating systems. Examples of such systems include ships, aircrafts, power plants, telecommunication systems, chemical processing facilities, and mass production lines. For this type of systems, the increased operating costs during unscheduled system shutdown might be much higher than maintenance costs.
Therefore, there is often a great potential for cost savings by implementing the preventive maintenance (PM) policy for a multi-unit system with economic dependence.

According to the above reasons, we will concentrate on the replacement and maintenance policies for multi-unit systems with economic dependence in this thesis. In the next section, we conduct a detailed literature review on the multi-unit systems models with economic dependence.

1.2 Literature Review on Multi-unit Maintenance Models with Economic Dependence

In the previous paragraphs, we classify maintenance models for multi-unit systems based on the different dependencies between units, and we concentrate on the maintenance policies for systems with economic dependence. In the next section, we further categorize multi-unit systems by their particular structures and connections between units in the system.

In order to calculate system reliability, reliability engineers often need to classify complex systems into five categories: series system, parallel system, series-parallel system, k-out-of-n system and standby system. A system is a series system if it operates when all its components are in operating conditions, and the system instantly fails if any component is out of operation. A system is in parallel configuration, if it is able to operate when at least one of its components is in operating conditions. A series-parallel system is a system where some of the components in series are replicated in parallel. A k-out-of-n system is a general version of the parallel system, and it is also named as a
majority parallel system. A system is a k-out-of-n system if it is able to operate when at least \( k \) of its components are in operating conditions, \( n \geq k \geq 1 \), with \( n-k \) elements are in active redundancy. In active redundancy all components operate simultaneously, even if they may not be required. If possible, it would be cheaper or more convenient to run only \( k \) components, and keep \( n-k \) on standby. Whenever a running component fails, replace it with one from standby components immediately, and there is no need to instantly repair the failed component. This system structure is called a standby system.

According to our review, reliability evaluations of these systems are fully developed, while there have been a few articles published on maintenance models of systems with the last four categories. First, because the maintenance policies for these systems always need to consider the number of repair shops, each repair shop’s capacity and spare parts inventory levels, it is difficult and complicated to generate a general optimal maintenance model for these types of systems. Second, the reliability of multi-unit systems with parallel, series-parallel, k-out-of-n or standby connection between units is much higher than that of multi-unit series systems. Hence, the multi-unit series system is the most common multi-unit system, where the optimal maintenance policies are needed to increase system reliability and decrease operating costs. In the following paragraphs, we start reviewing publications dealing with multi-unit series systems.

Among maintenance models for multi-unit series systems, we first review models for systems consisting of two units, since a two-unit series system is not only one of the most fundamental models in reliability and maintenance theory, but also is a frequently used model in practice.
Berg (1976) and Berg (1978) have first proposed a preventive replacement policy for a machine with two identical components which are subject to exponential failure. Considering the exponential failure assumption, there is an incentive for preventive replacement due to rewards that are decreasing with age. Under this opportunistic failure replacement policy, upon a component failure, the other component is simultaneously replaced by a new one if its age exceeds a pre-determined control limit $L$. A basic model is established first, and then non-negligible replacement time is considered into it. The author has applied a semi-Markov process to solve this problem and proved that the control limit exists. Phillips (1981) has considered a system consisting of two units, in which one unit is continuously inspected, while another is uninspected unless a special inspection is carried on. The author has labelled the continuously inspected one as ‘revealed’ unit and the uninspected one as ‘unrevealed’ unit. Under the assumption that inspection and repair intervals are negligible, the author reached a repair strategy, including that both units are repaired whenever the revealed unit fails regardless of the condition of the unrevealed one. Epstein and Wilamowsky (1982) have used an opportunistic replacement approach to the problem of a two-component system in which one fails exponentially and the other never fails but has a limited life. The author has developed a model for this problem and solved it through the dynamic programming. Liang (1985) has provided a piggybacking maintenance policy, in which PM of one component can only be performed in combination with corrective maintenance of another. The author has remarked that this type of policy can be far from optimal, in particular when the failure rate of the piggybacking is low. Thomas (1986) has proposed a probabilistic model indicating a robust maintenance circumstance, which is a scheduled
maintenance activity and a random maintenance activity with cost saving if they are combined. If the time elapsed for either maintenance activities exceeds a control limit, both two activities are conducted simultaneously. The long-run rates of single and joint replacement are evaluated when the random activity is uniformly distributed. For the case of two identical components and generally distributed lifetimes, van der Duyn Schouten and Vanneste (1990) have considered an efficient numerical procedure of a policy-iteration method to achieve the optimal control parameters $m$ and $M$. The author also show by using extensive numerical experiments that the best $(m,M)$-policy is nearly always less than 1% off from the overall optimal policy. Hence, this simple structure and near-optimality of $(m,M)$-policies, combined with the existence of an efficient computational procedure, make this class of $(m,M)$-policies of special interest for systems with two identical components.

The following paper is the first to consider the two-unit system maintenance policy in the CBM framework, and since then, many models on CBM for multi-unit systems have been studied. Barbera et al. (1999) have presented a CBM model considering exponential failures and fixed inspection intervals for a two-unit series system. The condition of each unit is monitored at equidistant time intervals. The condition signals for each unit are used to determine whether to repair an individual unit or to overhaul the whole system. Each unit becomes as good as new after a maintenance action has been performed. Each unit can fail only once within an inspection interval and when one or both units fail, the system fails. The cost of the long-run average cost of maintenance actions and failures is to be minimized. The author has used dynamic programming to achieve the optimal maintenance policy for this problem. Degbotse and Nachlas (2003) have considered an
age-based opportunistic maintenance (OM) policy for a two-unit series system, in which the theory of nested renewal processes is used to define model availability. The nesting of two renewal processes is shown to provide an effective mean for modeling time-based equipment availability under the opportunistic age replacement. Castanier et al. (2005) have also proposed a CBM policy for a two-unit deteriorating system, in which each unit is subject to gradual deterioration and is monitored by sequential non-periodic inspections. It can be maintained by good as new preventive or corrective replacements. Every inspection or replacement requires a set-up cost and a corresponding unit cost, but if actions on the two components are combined, the set-up cost is charged only once. A parametric maintenance decision framework is proposed to coordinate inspection/replacement of the two components in order to minimize the long-run maintenance cost of the system. A stochastic model is developed on the basis of the semi-regenerative properties of the maintained system state and the associated cost model is used to assess and optimize the performance of the maintenance model. Wang and Zhang (2007) have introduced a geometric process repair model into maintenance policy for a series repairable system consisting of two non-identical components and one repairer. The geometric process repair model was first proposed by Lin (1988). It is assumed that the ensuing survival times after repair constitute a decreasing geometric process and the ensuing repair times form an increasing geometric process for each component. A replacement policy \((M, N)\) is considered based on the numbers of failures of components 1 and 2. The explicit expression for the long-run expected cost per unit time is derived and the corresponding optimal replacement policy \((M^*, N^*)\) can be determined analytically or numerically. Jalali Naini et al. (2009) have also considered a
CBM policy for a series system composed of two components, which are subject to continuous deterioration and consequently stochastic failure. The components are inspected periodically and their deterioration degrees are monitored. Different maintenance actions, repair or replacement, with corresponding costs can be conducted into two components. The author has applied stochastic regenerative properties of the system to develop a stochastic model for analyzing this deterioration process. A novel approach is presented that simultaneously determines the optimal inspection time intervals and the appropriate maintenance actions for each component on the basis of the observed degrees of deterioration. This policy considers different criteria like reliability and long-run expected cost of the system. Zhuoqi et al. (2011) have also studied CBM and OM for a two-unit deteriorating series system with its condition being observed at equidistant intervals. It is assumed that each unit may have random failures and can fail only once during one inspection interval. When one unit fails, corrective maintenance action is performed and another unit’s condition will be inspected. If the condition of the inspected unit reaches a PM threshold, OM action will be performed simultaneously. Based on the model, a policy with CBM and OM is determined through minimizing discounted total maintenance cost using the value iteration algorithm. A numerical example is provided to illustrate how the policy proceeds and cost-savings are analyzed to validate the superiority of this maintenance policy. In this paper, only an approximate policy is proposed and its integrity to optimal policy is analyzed through simulation. Cheng et al. (2012) have investigated the optimal policy of continuously deteriorating system with economic dependence. This system is composed of two kinds of units, which are subjected to the deterioration failure described by the Gamma process and the Poisson
failure respectively. The author has presented a kind of improved OM policy, in which thresholds are determined by applying an iteration approach. Since considering the economic dependence, the opportunistic policy is used to coordinate the maintenance of the two units and minimize the long-run maintenance cost of the system by saving set-up costs.

Besides maintenance models for two-unit series systems with economic dependence in the above papers, the following literatures are also about studies on maintenance models of multi-unit systems with economic dependence. However, the number of units in the system is general $n \ (n \geq 2)$, or the system structure belongs to parallel, series-parallel, k-out-of-n or standby.

The following authors have considered maintenance policies for multi-unit systems with general $n$ units. Wildeman et al. (1997) have introduced grouping maintenance activities into multi-unit maintenance to save the system-dependent set-up cost. The author has proposed a rolling-horizon that makes its maintenance policy available on both long and short terms. Duarte et al. (2006) have studied an algorithm to achieve the optimum frequency to perform PM in multi-unit system, in which each unit exhibits a linearly increasing hazard rate and a constant repair rate. Laggoune et al. (2009) have studied a PM approach for a multi-component series system subjected to random failures, which can be applied for continuous operating systems, such as, petrochemical plants, the hydrogen compressor in an oil refinery, etc. A solution procedure is provided based on a Monte Carlo simulation. Zhou et al. (2009) have proposed an opportunistic PM scheduling algorithm for the multi-unit series system based on dynamic programming
with the integration of the imperfect maintenance effect. Zhou et al. (2010) have also proposed a dynamic opportunistic PM optimization policy for multi-unit series systems by integrating multi PM techniques, including periodic and sequential PMs. Tian and Liao (2011) have investigated CBM based on proportional hazards model for a multi-component system, where the economic dependency exists among different components subject to condition monitoring.

The following authors have considered maintenance models for the multi-unit system with parallel, series-parallel or k-out-of-n structures.

The parallel multi-unit system has been reported in the following literature: Yeh (1995) has proposed a maintenance model for a two-component parallel system, in which one component after repair is not 'as good as new' but another one after repair is 'as good as new'. Billinton and Pan (1998) have also presented equations which can be used to evaluate the failure frequency and the failure rate of a two-component parallel redundant system. The optimal maintenance interval for this system can be found by applying these equations.

The series-parallel multi-unit system has been reported in the literature: Nourelfath and Ait-Kadi (2007) have considered a minimal cost configuration for a multi-state series-parallel system, which is subject to reliability constraints and a specified maintenance policy. In this policy, the number of maintenance teams is less than the number of repairable components, and the priorities between the system components are ordered. Lin and Wang (2010) have presented a hybrid genetic algorithm to optimize the periodic PM model in a series-parallel system. This hybrid genetic algorithm can be used
to determine the structure of reliability block diagrams, maintenance priority of components, and their maintenance periods. Wang and Tsai (2012) have also presented an extended hybrid genetic algorithm to optimize a bi-objective imperfect PM model for a series-parallel system.

The k-out-of-n multi-unit system has been reported in the literature: Wang and Pham (1997) have introduced OM into maintenance policy of a k-out-of-n system, and proposed two decision variables \((r, T)\) with consideration of reliability requirements. Pham and Wang (2000) have expanded the previous maintenance model with consideration of imperfect PM and partial failure. Karin S Smidt-Destombes et al. (2004) have proposed the maintenance model for a k-out-of-n system with identical, repairable components, which is initiated when the number of failed components exceeds some critical levels. A multi-server repair shop, spare part stock level, and the repair capacity are included in the model. Karin S Smidt-Destombes et al. (2006) have further developed the above model to establish a CBM policy under the same requirements.

From the foregoing publications it can be seen that there are some significant research trends in maintenance models for multi-unit systems with economic dependence. First, many authors consider models with the combination of preventive and corrective maintenance, and this policy is called OM. The combination of preventive and failure maintenance was systematically studied for the first time by researchers from the RAND corporation in the early sixties. In multi-unit systems, any shutdown or interruption due to failure of one or several units serves as the ‘opportunity’ to conduct PM for other units even though it is not as planned. An advantage of this OM is that set-up costs can be
reduced. Set-up cost is a fixed cost that only depends on the system and component, and it is shared by all maintenance activities carried out simultaneously on the system. For example, the set-up cost can include the downtime cost due to production loss on the occasion that the system cannot be used during maintenance, or the preparation cost associated with erecting a scaffolding, settling maintenance staff on site or testing a machine. However, a disadvantage of OM is that it is often not known in advance whether the PM should be taken on operational units during the system shutdown times. In most cases, an additional OM threshold is necessary to be included to determine whether to execute maintenance actions or not. Among the previous review papers, maintenance models with OM are shown in the Table 1.1. Second, almost all authors focus on maintenance policies for multi-unit systems composed of identical units with the same type of deterioration processes. Under this assumption, the maintenance policies are easy to build and the optimal thresholds are more accessible by applying dynamic programming. In the previous publications, authors who have merely considered identical units in the system are shown in the Table 1.1. Third, in recent years, there are increasing number of publications considering the CBM policy, which is more advanced and optimal than the age-dependent policy or the periodic PM policy, for multi-unit systems. Recently, CBM is becoming a growing trend in the maintenance research area, which will be further illustrated in Chapter 2. Papers that introduced CBM into the multi-unit system maintenance are shown in the Table 1.1.
### Table 1.1 List of reference papers with different maintenance policies

<table>
<thead>
<tr>
<th>Maintenance Policy trends</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opportunistic Maintenance</td>
<td>Berg (1976); Berg (1978); Liang (1985); Thomas (1986); Wang and Pham (1997); Pham and Wang (2000); Degbotse and Nachlas (2003); Castanier et al. (2005); Laggoune et al. (2009); Zhou et al. (2009); Zhou et al. (2010); Zhuoqi et al. (2011); Cheng et al. (2012); LAI (2012)</td>
</tr>
<tr>
<td>Identical units following similar deterioration</td>
<td>Berg (1976); Berg (1978); Liang (1985); Thomas (1986); van der Duyn Schouten and Vanneste (1990); Wang and Pham (1997); Wildeman et al. (1997); Billinton and Pan (1998); Smidt-Destombes et al. (2004); Smidt-Destombes et al. (2006); Jalali Naini et al. (2009); Laggoune et al. (2009); Zhou et al. (2009); Tian and Liao (2011)</td>
</tr>
<tr>
<td>Condition Based Maintenance</td>
<td>Barbera et al. (1999); Castanier et al. (2005); Jalali Naini et al. (2009); Tian and Liao (2011); Zhuoqi et al. (2011)</td>
</tr>
</tbody>
</table>
1.3 Research Motivation

After reviewing the literature on the maintenance models of the multi-unit system with economic dependence, we find that almost all existing papers have focused on the optimal maintenance policies for the system composed of identical units with the same deterioration. Although this system structure is very important, there are also many series systems composed of unequally important units with different deteriorating processes and maintenance costs. A common example of such a system is the automatic vehicle transmission system with a transmission gearbox and an electric transmission control unit, where the transmission control unit is an electronic device being used to control the operation of the gearbox. In this two-unit system, the transmission gearbox is much more expensive, especially in a heavy hauler truck, and more important than the electric device from the maintenance/replacement point of view. More examples of unequally important systems are as follows. A computer system may be broadly treated as a series system consisting of hardware and less expensive software. An automatic control system could be generally looked as a series system consisting of power supply unit and relatively expensive operating component. Hence, in this thesis, we have decided to concentrate on establishing specific maintenance policies for the multi-unit system composed of unequally expensive units with economic dependence between units.

Although maintenance models for the multi-unit system composed of general \( n(n \geq 2) \) units or with special connections within units, such as, parallel, series-parallel and k-out-of-n, have received an increasing attention in the last decade, we have focused on the two-unit series system with economic dependence, which is composed of two
unequally expensive units, for the following reasons. First, when the number of units increases, the mathematical models of OM, especially with CBM being considered, can be so complex that analytical solution is almost impossible and must resort to simulation tools. In our review for systems with general $n$ units, Laggoune et al. (2009) have applied Monte Carlo simulations; Zhou et al. (2009) have applied dynamic programming to find the optimal answers. Hence, in most papers the number of units is fixed to two for simplifying calculations. Second, we have assumed that the cheaper unit in our two-unit system follows a constant failure rate, which means its lifetime follows an exponential distribution. It is one of the most important reliability distributions. In many real applications, most electronic components, with no wear-out properties or significant infant mortalities, exhibit constant failure rates. The exponential distribution is also the right distribution to simulate the useful life period of the ‘bathtub curve’ of the lifetime of a population of products. It is also in many respects the simplest reliability distribution for the analysis. Furthermore, if the failure rates of all failure modes of a component are constant and independent, the overall failure rate of the component is also constant. Hence, a complex system can be viewed as a two-unit system: one is the most important and expensive unit, and another one represents the rest units in the system. For example, the series-parallel system in Figure 1.1 can be treated as a two-unit series system composed of one significant and expensive unit 0 and the other group of units serving as one unit with a constant failure rate. Thus, the study of a two-component series system is sufficient to establish the key relationships and results from a complex multi-unit system in research.
Furthermore, there are generally two important maintenance ideas nowadays to deal with the two-unit series system with economic dependence. One is group maintenance and another is OM. Group maintenance allows the two-unit system to be either entirely replaced with new units or remained in operation, which is first proposed by Sivazlian and Mahoney (1978). Researchers concentrate more on OM than group maintenance policy, because the former policy is more effective on the reduction of maintenance costs by preventing possible over-maintenance in the group maintenance policy. Hence, we have decided to introduce the OM idea into our maintenance models.

1.4 Research Contributions

In this thesis, we focus on the study of the maintenance policies for a two-unit series system with economic dependence, which is composed of one deteriorating expensive unit and one cheaper unit with low replacement cost. The main contributions of this thesis are summarized as follows.

An optimal age-based preventive replacement policy that minimizes the long run expected average cost per unit time has been investigated. In order to save set-up costs, which are always incurred when the system is maintained, for the two-unit series system,
the opportunistic replacement threshold is considered to determine whether to simultaneously replace the expensive unit at the time when the other cheaper unit fails. The optimal preventive and opportunistic replacement thresholds are calculated in the SMDP framework. The computing speed is rather fast, which leads to computational benefits and easy implementation in practice.

A multivariate Bayesian control chart for CBM application is considered in our second maintenance model for a two-unit series system with unequally expensive units. The expensive unit is monitored at equidistant sampling intervals, and the posterior probability of its condition in warning state is calculated and plotted on the control chart. Also, an additional OM control limit on the control chart is introduced to provide opportunities for the full inspection when the cheaper unit fails between the sampling epochs and condition information is obtained for the expensive one, leading to minimize the long-run average maintenance cost of the system by saving set-up costs. The SMDP algorithm is also applied to find two optimal control limits on the multivariate Bayesian control chart. Our new models contribute to the multi-unit system maintenance modeling and control, and have the potential to be applied in various real world applications.

1.5 Overview of the Thesis

The thesis is organized as follows:

- In Chapter 1, we provide a detailed literature review of the maintenance policies for multi-unit systems with economic dependence. In the literature, we find that the maintenance for the two-unit series system composed of unequally important units needs further analysis.
• In Chapter 2, we briefly introduce two maintenance models. One is the basic PM model, age-based PM policy. Another one is the advanced model, CBM with multivariate Bayesian control chart.

• In Chapter 3, we introduce the opportunistic maintenance idea into the age-based PM policy to create an optimal age-based preventive replacement policy for a two-unit series system. We further apply SMDP algorithm to find the optimal results. A numerical example is conducted to illustrate this approach.

• In Chapter 4, we extend the multivariate Bayesian control chart for CBM application by including two control limits, the opportunistic maintenance limit and the preventive maintenance limit. These two optimal limits are found by the SMDP algorithm. A numerical example is also conducted to illustrate this method.

• Finally, in Chapter 5, conclusions from our research are drawn and several directions of future work are provided.
Chapter 2

Maintenance Models

The maintenance models for different types of multi-units systems were reviewed in the previous chapter. We decided to establish the maintenance policy for a two-unit series system composed of one expensive unit and one cheaper unit with economic dependence between them. In order to simplify the model and make it more applicable to real world applications, we assume that the cheaper unit has a constant hazard rate, which means its lifetime follows the exponential distribution. Dealing with the expensive unit in the two-unit system, we apply the age-based PM policy, which is one of the most common and popular maintenance policies, in our basic model in Chapter 3. We further design a multivariate Bayesian control chart for CBM application in our advanced model in Chapter 4. The brief introductions of these two important maintenance policies are in the following chapter.

2.1 Age-based PM Policy

From 1960\textsuperscript{th}, researchers started to think about the mathematical representation of the component reliability and maintenance models have been introduced to achieve minimum cost for operating the system. There are mainly two categories of maintenance models, corrective and preventive maintenance. Corrective maintenance (CM), also called reactive maintenance, is a traditional maintenance technique, in which repair actions caused by each system failure. PM is the maintenance that occurs when a system is still
operational. Among various PM models, there are two common and prevailing policies, age-based PM policy and periodic PM policy. The periodic PM policy, also called the block PM policy, is a constant-interval based preventive replacement method, in which failure replacements are performed immediately after failures happen and preventive replacements are performed at constant intervals. This policy is to find the optimal preventive replacement interval to minimize the total expected replacement cost per unit time.

Age-based PM is similar to the periodic PM, except that instead of conducting preventive replacements at fixed intervals, in which performing preventive replacements shortly after failure replacements are possible. In the age-based PM, it is the age of the item that determines when to perform preventive replacements. Failure replacements are still executed at system failure times. When either failure or preventive replacement occurs, the time clock is reset to zero, and the preventive replacement occurs only when the item has been used for the specific age. In the thesis, we assume that all maintenance actions in our models are perfect, which means the system is restored to ‘as good as new’ condition after the maintenance. We do not consider minimal or imperfect maintenance in our models. Under this assumption, the system condition is entirely renewed after both maintenance and replacement actions.

Age-based PM policy can be mathematically represented by the Renewal Reward Processes in stochastic models. From Jardine and Tsang (2013), the total expected replacement cost per unit time is:
\[ C(t_p) = \frac{C_p R(t_p) + C_f \left[ 1 - R(t_p) \right]}{\int_0^{t_p} R(t) \, dt} \] (2.1)

\( C_f \) is the total cost of a preventive replacement; \( C_p \) is the total cost of a failure replacement with restriction \( C_p < C_f \); \( R(t) \) is the system reliability function. \( t_p \), the predetermined age for system to conduct PM, can be obtained by minimizing \( C(t_p) \) through optimization methods.

2. 2 CBM with Multivariate Bayesian Control Chart

Although the age-based PM policy is an improvement compared to the earlier CM policy, the cost of PM is increasing heavily and becoming a financial burden of many companies. In order to save the PM cost, the actual condition of the system is taken into account to determine more precisely when to conduct PM. Along with the development of high speed calculation technology and the decreasing cost of equipment sensors, one more effective maintenance approach, CBM, is introduced to reduce the maintenance cost and unscheduled downtime, and to improve the system reliability.

The system condition information, collected through condition monitoring, is used to decide maintenance actions in the CBM process, which is based on the condition that the equipment goes through multiple degraded states before failure. The optimal maintenance actions can be scheduled for preventing the equipment catastrophic failure and minimizing the total operating and maintenance cost. CBM is the kind of maintenance policy to execute maintenance actions only when there is evidence that the breakdown is
approaching. There are three key steps in the CBM process: data acquisition, data processing and maintenance decision-making step. According to Jardine et al. (2006), data acquisition step is to collect the data related to the system health condition. Data processing step is to process and analyze the acquired data. In maintenance decision-making step, effective maintenance policies will be obtained on the condition of the analyzed information.

In recent years, academic researchers have increasingly recognized the economic benefits by applying statistical process control (SPC) techniques to analyze the acquired information from machine for its maintenance decision-making. Among SPC techniques, control charts are the most valuable tools. For example, Wu and Makis (2008) have considered an observable failure state in their contributions to the economic and economic-statistical design of multivariate chi-square chart for CBM application. Liu et al. (2012) have provided an economic and economic-statistical design of an $\bar{X}$ control chart for a two-identical unit series system with CBM. The system is described as a five-state continuous time Markov chain.

However, in many real applications, the system condition is only partially observable through sampling or inspection procedures. The unobservable deterioration process defined by the hidden system state is stochastically related to the observation process. On the other hand, when the system failure might occur and can be observed in most real situations, an observable failure state should be considered in the CBM model. For this situation, a multivariate Bayesian control chart for CBM application is introduced. This multivariate Bayesian control chart is proved to be an optimal SPC tool for both short and
long term production by Makis (2008); Makis (2009). In this model assumption, a system can be in one of two unobservable operational states, healthy state and warning state, or in an observable failure state. The observation process is represented by characteristics following the multivariate normal distribution. We collect the samples from the observation process at equidistant sampling epochs. Each sample is used to calculate the posterior probability that the system is in the warning state, and the posterior probabilities at different sampling epochs are continuously plotted on the multivariate Bayesian control chart. When the value of the posterior probability exceeds the predetermined control limit, a control chart alarm occurs. Then, we need to perform a full inspection of the system to determine the real state of the system. If the real state is the warning state, this chart alarm is called as a true alarm, otherwise, it is called a false alarm if it is in the healthy state. The system is replaced when it fails during sampling intervals or when there is a true alarm from the multivariate Bayesian control chart.

Figure 2.1 Bayesian control chart
Figure 2.1 shows an illustration of the Bayesian control chart, where condense dots denote the posterior probabilities of the system in warning state at sampling epochs $i\Delta$, and 4-point star denotes the chart alarm from the Bayesian control chart. Because this maintenance policy is optimal and easy to implement, it is easily applicable in real industrial machines with condition monitoring already in place.
Chapter 3

Optimal Age-based Preventive Replacement Policy for a Two-unit Series System

From our previous review in Chapter 1, many previous papers have considered that the multi-unit system consists of the identical units. In many real applications however, the system composed of unequally important units is still very important, which needs further maintenance studies. For example, a complex system includes an expensive mechanical device and an electrical control unit with a relatively low value. Hence, we have decided to focus on the maintenance model for this type of two-unit series system that includes one expensive deteriorating unit and the other cheaper unit with the constant failure rate. For the expensive unit, we apply the most common and popular maintenance technique, age-based preventive replacement policy, to prevent its failure from deterioration. A combined OM age threshold is further considered at the failure times of the cheap one in order to save the set-up cost. The SMDP algorithm is applied to find these two pre-specified age thresholds in order to achieve the minimum operating cost per unit time for the entire system.

3.1 Model Assumptions

In this model, we consider a series system, composed of two units, one deteriorating unit 0 with the increasing failure rate and the other unit 1 with the constant failure rate. When
either of them fails, the total system has to be ceased operating and the failure replacement of the failing unit has to be carried on immediately.

In our assumption, because unit 0 is relatively expensive compared with unit 1, an age-based preventive replacement policy is applied to improve its reliability and to decrease its operating cost per unit time. The time at which the preventive replacement occurs depends on the age of unit 0. When unit 0 operates for the specific time $t_p$, its preventive replacement needs to be applied. Since unit 1 is assumed to have the constant failure rate, which means that its failure might happen randomly, there is no need to do the preventive replacement before its failure. Because unit 1’s lifetime follows the exponential distribution that has the key feature of being memoryless, the whole system goes back to its renewed state only after the preventive or failure replacement of unit 0.

Furthermore, we assume that there is no structural or stochastic interaction between two units in the system, but we focus on the economic dependence, which is represented by the set-up cost. For example, the set-up cost may consist of the down-time cost due to the production loss, or of the preparation cost associated with opening the system, settling the maintenance staffs on site and pre-testing the system, whose magnitude does not depend on the number of units replaced. Since the set-up cost is saved at the opportunities when two units are replaced simultaneously, our policy is created by introducing the opportunistic replacement threshold $t_{OR}$ for unit 0 in order to decrease the operating cost for the entire system.
3. 2 Opportunistic Replacement Maintenance Policy

In the opportunistic replacement maintenance policy, when unit 1’s first failure happens after unit 0 continuous operation time $t_{OR}$, we are going to execute the failure replacement of unit 1 and preventive replacement of unit 0 simultaneously for saving the system set-up cost. Since the time taken for failure replacements of unit 1 is much smaller than the system mean time between failures, the period can be ignored and treated as zero. We assume that the failure and preventive replacement times for unit 0 are $T_{FR}$ and $T_P$, respectively. In this model, the optimal preventive replacement threshold $t_p$ and opportunistic preventive replacement threshold $t_{OR}$ should be found to minimize the system long-run expected cost per unit time $C(t_p, t_{OR})$ in the steady state.

![Figure 3.1 Age-based Preventive Replacement for the two-unit series system](image)

This optimal age-based preventive replacement policy for a two-unit series system is clearly shown in the Figure 3.1. When unit 0 fails, we do the failure replacement of unit 0.
When unit 1 fails, if unit 0 has been operating for \( t_{OR} \) at that time, we conduct the failure replacement on unit 1 and preventive replacement on unit 0 simultaneously to save the system set-up cost; if unit 0 has not been operating for \( t_{OR} \), we only do failure replacement of unit 1. When unit 0 has been continuously operating for \( t_p \), we conduct preventive replacement of unit 0.

For further calculation, we assume that the lifetime of unit 0 is a random variable \( T_0 \) following a general distribution \( F_0(t) \), with the failure rate \( r_0(t) \). According to the assumption that unit 0 is a deteriorating unit, \( r_0(t) \) is an increasing function of time \( t \).

Since Weibull analysis is the world's most popular method of analyzing and predicting failures and lifetimes of all types of devices, we assume that the lifetime of unit 0 follows the Weibull distribution. Hence, \( F_0(t) = 1 - \exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right) \); \( r_0(t) = \frac{\beta}{\eta} \left(\frac{t}{\eta}\right)^{\beta-1} \), \( (t \geq 0) \).

According to the assumption that unit 1 has a constant failure rate, which means its lifetime follows the exponential distribution \( f_1(t) = \lambda e^{-\lambda t} (t \geq 0) \); \( r_1(t) = \lambda (t \geq 0) \), the number of failures of unit 1 constitutes a Poisson process with a constant intensity rate \( \lambda \).

In order to derive the expected replacement cost for operating the system in one renewal cycle, the costs of different maintenance actions need to be set up. We assume that the failure replacements of unit 0 and unit 1 cost \( C_{F0} \) and \( C_{F1} \), respectively. The cost of preventive replacement of unit 0 equals to \( C_{P0} \), with the restriction \( C_{P0} < C_{F0} \), in which the preventive maintenance cost must be less than the failure replacement cost in practical applications. Moreover, the preventive or failure replacement of either unit has
to shut down the system, where an additional set-up cost $C_s$ will be generated to system operating cost. In next paragraphs, the minimum long-run expected cost per unit time $C(t_p, t_{OR})$ with two corresponding optimal age thresholds can be found under the SMDP framework.

3. 3 SMDP Framework

- SMDP State Definition

In this model, we utilize a discrete semi-Markov Decision Process (SMDP) formulation to compute the long-run expected average cost per unit time for the entire system. This framework helps to determine the optimal replacement policy, which includes two optimal time thresholds $t_{OR}$ and $t_p$ ($t_{OR} < t_p$) to determine the opportunistic replacement and preventive replacement of unit 0. In order to treat the infinite and continuous age state space, we divide the real age axis into a set of equally spaced age intervals. We assume that the maximum life span of unit 0 is a constant $M (M \in \mathbb{R})$, which usually equals to the machine's mean lifetime $\mu$ plus twice its lifetime standard deviation $\delta$, $\mu + 2\delta$. Then, a large number $L \in \mathbb{N}$ should be chosen to discretize the interval $[0, M]$, which leads to the equidistant age interval $\Delta = \frac{M}{L}$.

According to the above definition, we assume that the opportunistic replacement threshold equals to $t_{OR} = w\Delta = \frac{wM}{L}$ and the preventive replacement threshold equals to
\[ t_p = k\Delta = \frac{kM}{L} \], where \( 0 \leq w < k \leq L \) and \( w, k \in \mathbb{N} \), in which the opportunistic replacement threshold must be beneath the preventive replacement threshold based on our opportunistic replacement maintenance policy.

Consequently, the state space for the SMDP is defined by \( S = \{0,1,\ldots,k-1,PR,OR\} \). State 0 denotes that unit 0 is in its ‘as good as new’ condition. States from 1 to \( k-1 \) denote the age states of unit 0. In order to simplify our calculation and decrease the number of states, the SMDP is defined to directly transfer into state 0 after the failure of unit 0, and we exclude the failure state in our SMDP state space. State \( PR \) denotes the preventive replacement of unit 0. State \( OR \) denotes the combination of opportunistic replacement of unit 0 and failure replacement of unit 1. After the SMDP state definition, we further calculate SMDP transition probability between different states.

- **SMDP Transition Probability**

According to the model description, we assume that unit 0 can fail at any time during the equidistant age interval \( \Delta \), but unit 1 can fail only at the end of each age interval. Hence, the probability that unit 1 fails at the end of age interval \( \Delta \) equals to

\[ F_1(\Delta) = \int_0^\Delta f_1(t) \, dt = 1 - e^{-\lambda\Delta} \]  

in calculation.

Then, the SMDP transition probability \( p_{i,j} \) can be calculated as:
When $0 \leq i \leq w-2$:

\[
p_{i,j} = \begin{cases} 
\Pr(T_0 > (i+1)\Delta \mid T_0 > i\Delta) = \frac{R_0((i+1)\Delta)}{R_0(i\Delta)}, & \text{when } j = i+1; \\
\Pr(T_0 < (i+1)\Delta \mid T_0 > i\Delta) = 1 - \frac{R_0((i+1)\Delta)}{R_0(i\Delta)}, & \text{when } j = 0; \\
0, & \text{otherwise.}
\end{cases}
\]

In equation (3.1), when the current state $i$ is below the opportunistic replacement threshold $t_{OR}$, the SMDP process can transit either to the next age state $i+1$, or to the renewed state 0, if unit 0 fails between the time $i\Delta$ and $(i+1)\Delta$.

When $w-1 \leq i \leq k-2$:

\[
p_{i,j} = \begin{cases} 
\Pr(T_0 > (i+1)\Delta \mid T_0 > i\Delta) \cdot \Pr(Unit \text{ survives } \Delta) \\
\Pr(T_0 > (i+1)\Delta \mid T_0 > i\Delta) \cdot \Pr(Unit \text{ fails at the end of } \Delta) \\
\Pr(T_0 < (i+1)\Delta \mid T_0 > i\Delta) = 1 - \frac{R_0((i+1)\Delta)}{R_0(i\Delta)}, & \text{when } j = 0; \\
0, & \text{otherwise.}
\end{cases}
\]

In equation (3.2), when the current state $i$ is between the opportunistic replacement threshold $t_{OR}$ and the preventive replacement threshold $t_p$, the SMDP process can transit to one of three states: the next age state $i+1$ on the condition that both units 0 and 1 survive until the age $(i+1)\Delta$; the state $OR$ on the condition that unit 0 survives
until the age \((i+1)\Delta\), while unit 1 fails at the end of this age interval; the renewed state 0 on the condition that unit 0 fails between the time \(i\Delta\) and \((i+1)\Delta\).

When \(i = k - 1\):

\[
p_{i,j} = \begin{cases} 
\Pr(T_0 > (i+1)\Delta | T_0 > i\Delta) \cdot \Pr(\text{Unit 1 survives } \Delta) & \text{when } j = PR; \\
\Pr(T_0 > (i+1)\Delta | T_0 > i\Delta) \cdot \Pr(\text{Unit 1 fails at the end of } \Delta) & \text{when } j = OR; \\
\Pr(T_0 > (i+1)\Delta | T_0 > i\Delta) = 1 - \frac{R_0((i+1)\Delta)}{R_0(i\Delta)} & \text{when } j = 0; \\
0 & \text{otherwise.}
\end{cases}
\]

In equation (3.3), when the current state \(i\) belongs to the last age state \(k - 1\), the SMDP process can transit to one of three states: the state \(PR\) if both unit 0 and 1 survive until the time \((i+1)\Delta\); the state \(OR\) or the renewed state 0 with the same corresponding probabilities in equation (3.2).

Because the lifetime of unit 0 follows the Weibull distribution in the above assumption, we can further substitute \(\exp\left[\left(\frac{i\Delta}{\eta}\right)^\alpha - \left(\frac{(i+1)\Delta}{\eta}\right)^\alpha\right]\) for \(\frac{R_0((i+1)\Delta)}{R_0(i\Delta)}\) in equations (3.1), (3.2) and (3.3). After calculating all the transition probabilities between all states, the other two SMDP characteristics, the expected sojourn times and expected costs, in the linear equations for SMDP algorithm need to be further derived.
• SMDP Expected Sojourn Times

When $0 \leq i \leq k-1$:

$$
\tau_i = \Delta \frac{R_0((i+1)\Delta)}{R_0(i\Delta)} + \int_0^{\Delta} \left( t + T_{FR} \right) \frac{d}{dt} \left[ -\frac{R_0(i\Delta+t)}{R_0(i\Delta)} \right].
$$

(3.4)

In equation (3.4), when the current state $i$ is one of the age states from 0 to $k-1$, the expected sojourn time until the next decision epoch can be derived based on two conditions. When unit 0 survives until the time $(i+1)\Delta$, the sojourn time equals to the age interval $\Delta$. When unit 0 fails between the time $i\Delta$ and $(i+1)\Delta$, the sojourn time equals to the survival time of unit 0 in this age interval plus its failure replacement time.

Assumed that unit 0’s lifetime follows the Weibull distribution, we can further derive function (3.4) into:

$$
\tau_i = T_{PR} \left( 1 - \exp \left[ \left( \frac{i\Delta}{\eta} \right)^\beta - \left( \frac{(i+1)\Delta}{\eta} \right)^\beta \right] \right) + \frac{\int_0^{\Delta} \exp \left[ \frac{(i\Delta+t)}{\eta} \right] dt}{\exp \left[ \frac{i\Delta}{\eta} \right]}.
$$

(3.5)

According to the model assumption, we get:

$$
\tau_{PR} = \tau_{OR} = T_p.
$$

(3.6)

In equation (3.6), according to our assumption that the failure replacement time of unit 1 is ignored, the expected sojourn times in states $PR$ and $OR$ both equal to the preventive replacement time of unit 0.
• SMDP Expected Costs:

When $0 \leq i \leq w-2$:

$$C_i = (C_{F1} + C_S) \left(1 - e^{-i\Delta}\right) \frac{R_0((i+1)\Delta)}{R_0(i\Delta)} + (C_{F0} + C_S) \left[1 - \frac{R_0((i+1)\Delta)}{R_0(i\Delta)}\right];$$  \hfill (3.7)

In equation (3.7), when the current state $i$ is below the opportunistic replacement threshold $t_{OR}$, two maintenance costs will be incurred based on different conditions. One is the sum of unit 1’s failure replacement cost and the set-up cost on the condition that unit 0 survives until the time $(i+1)\Delta$, while unit 1 fails at the end of this age interval. The other is the sum of unit 0’s failure replacement cost and the set-up cost on the condition that unit 0 fails between the time $i\Delta$ and $(i+1)\Delta$.

When $w-1 \leq i \leq k-1$:

$$C_i = (C_{F0} + C_S) \left[1 - \frac{R_0((i+1)\Delta)}{R_0(i\Delta)}\right];$$ \hfill (3.8)

In equation (3.8), when the current state $i$ is between the opportunistic replacement threshold $t_{OR}$ and the preventive replacement threshold $t_p$, the sum of unit 0’s failure replacement cost and the set-up cost will be incurred on the condition that unit 0 fails after the time $i\Delta$ and before $(i+1)\Delta$.

According to the model description, we can get:

$$C_{PR} = C_{P0} + C_S;$$ \hfill (3.9)
\[ C_{OR} = C_{P_0} + C_{F_1} + C_S. \]  

(3.10)

For equations (3.7) and (3.8), we can further substitute \( \exp \left[ \left( \frac{i\Delta}{\eta} \right)^\beta - \left( \frac{(i+1)\Delta}{\eta} \right)^\beta \right] \) for \( \frac{R_0((i+1)\Delta)}{R_0(i\Delta)} \) based on the assumption that unit 0’s lifetime follows the Weibull distribution.

- The Policy-iteration Algorithm for the SMDP Model

With the above derivations, for the fixed values of preventive replacement threshold \( t_p \in [0,M] \) and opportunistic replacement threshold \( t_{OR} \in [0,t_p) \), the long-run expected average cost per unit time \( g(t_{OR},t_p) \) can be obtained by solving the following system of linear equations Tijms (1994):

\[
\begin{align*}
\nu_i &= c_i - g(t_{OR},t_p)\tau_i + \sum_{j \in S} p_{i,j} \nu_j, \text{ for } i \in S \\

\nu_s &= 0,
\end{align*}
\]

(3.11)

where \( s \) is an arbitrarily chosen state.

Hence, the optimal preventive replacement threshold and optimal opportunistic replacement threshold can be found:

\[
(t_{OR}^*, t_p^*) = \arg \inf_{t_{OR} \in [0,t_p)} g(t_{OR},t_p), \quad (t_p \in [0,M])
\]

(3.12)
which minimizes the long-run expected average cost per unit time \( g(t_{or}, t_p) \). It can be obtained through iteratively solving the system of equations (3.11) considering different combination of the values of \( t_{or} \) and \( t_p \). In the following section, we use a numerical example to illustrate this procedure.

### 3.4 Numerical Example

In this numerical example, it is assumed that the model parameters are already known and we pursue the optimization under this assumption. However, in many real applications, the model parameters of unit 0 need to be estimated first by applying the Weibull Analysis techniques, such as, Weibull plot and statistical software, in Jardine and Tsang (2013). The constant failure rate of unit 1 can be estimated through the use of accelerated high temperatures, or other stress methods, operating “run to failure” tests performed on a sample of devices randomly selected from its parent population. In our model, the parameters of Weibull distribution for unit 0 and the constant failure rate for unit 1 are given by:

\[
\beta = 2.5; \; \eta = 15; \; \lambda = 0.05.
\]

Since unit 0 follows the Weibull distribution, both the mean \( \mu \) and standard deviation \( \sigma \) can be determined analytically using the following expressions:

\[
\mu = \eta \Gamma\left(1 + \frac{1}{\beta}\right); \; \sigma^2 = \eta^2 \left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right],
\]

where \( \Gamma(z) \) is the gamma function.
Hence, the unit 0’s maximum life span \( M = \mu + 2\delta = 13.3090 + 2 \times 5.6950 = 24.6990 \).

The time required for maintenance actions are given by:

\[
T_{FR} = 8; T_p = 5.
\]

The following maintenance costs will be considered in the example:

\[
C_{F0} = 1200; C_{F1} = 50; C_{p0} = 500; C_S = 90.
\]

We first compute the optimal preventive replacement threshold and opportunistic replacement threshold that minimize the long-run expected average cost. We choose the partition parameter \( L = 32 \), and use the linear equations (3.11) to acquire the optimum results as shown in Table 3.1. We have coded the algorithm in MATLAB (R2008a) on the Intel Core i7-4770, 3.40 GHz with 16 GB RAM.

**Table 3.1: The optimal age-based maintenance policy for a two-unit series system**

<table>
<thead>
<tr>
<th>Optimal preventive replacement threshold ( t_p )</th>
<th>Optimal opportunistic replacement threshold ( t_{OR}^* )</th>
<th>Minimum average cost</th>
<th>Computation time (Sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.8058</td>
<td>5.4029</td>
<td>56.8755</td>
<td>0.0124</td>
</tr>
</tbody>
</table>

It is shown in Table 3.1 that the computation algorithm takes only 0.0124 seconds for each run to find the corresponding minimum average cost per unit time with two fixed age thresholds. This method is extremely fast for off-line computations and quite easy to be used in real applications. The optimal values of \( t_p \) and \( t_{OR} \) are 10.8058 and 5.4029, which provides the minimum average cost 56.8755. It means that under this maintenance
policy the expected average cost per unit time for the entire system over a long-run horizon equals to 56.8755. Figure 3.2 shows the different combination of values of the preventive replacement threshold \( t_p \) and opportunistic replacement threshold \( t_{OR} \) versus its corresponding long-run expected average cost.

![Figure 3.2 Expected average cost per unit time as a function of the preventive replacement threshold and opportunistic replacement threshold.](image)

Then, we compare this optimal age-based preventive replacement policy with the traditional age-based replacement policy that does not include the opportunistic replacement threshold for the same two-unit series system. According to the renewal theory, the total expected replacement cost per unit time, \( C(t_p) \), equals to the total expected replacement cost per cycle divided by the expected cycle length.
We can get the specific expression with the same model notations:

\[
C(\tau_p) = \frac{(C_{p0} + C_S)R_0(\tau_p) + (C_{F0} + C_S)(1 - R_0(\tau_p))}{(t_p + T_p)R_0(\tau_p) + \int_0^{\tau_p} (u + T_{FR})d(-R_0(u))} + \lambda(C_{F1} + C_S). \tag{3.14}
\]

Hence, the optimal predetermined replacement age limit can be found:

\[
\tau_p^* = \arg \inf_{\tau_p \in [0,M]} C(\tau_p). \tag{3.15}
\]

Solving equations (3.14) and (3.15), we get \(\tau_p = 10.6200\). The minimum long-run expected average cost for the entire system under the traditional age-based replacement policy is calculated as 60.5516. It is higher than the minimum average cost 56.8755 found by our optimal replacement model when the opportunistic replacement threshold is considered. Although this saving of 6.07% does not seem enormous, we find the cost saving increases with a larger system set-up cost.

We now investigate the effectiveness of system set-up cost on the difference between minimum expected average costs per unit time considering two maintenance policies. In Figure 3.3, by increasing system set-up costs, we compare two groups of minimum expected average costs under two different age-based replacement policies, using with and without the opportunistic replacement threshold. We further show the increment of cost saving rates between two policies along with the increasing system set-up costs in Figure 3.4. From this analysis, we conclude that the improved age-based preventive replacement policy for a two-unit series system, which includes the opportunistic
replacement threshold, can save more replacement cost when the system set-up cost becomes prohibitively high.

Figure 3.3 Minimum expected average costs based on two different maintenance policies vs. set-up costs

Figure 3.4 Rate of saving of the minimum average cost between two different maintenance policies vs. set-up costs
Chapter 4

Optimal Opportunistic Maintenance Policy for a Two-Unit System applying Bayesian Control Chart

In Chapter 3, we have utilized the age-based preventive replacement method to maintain the expensive unit in a two-unit series system. Nowadays, more effective maintenance approaches such as the condition based maintenance (CBM) are widely adopted to further reduce the maintenance cost and to prevent unexpected failures. Dealing with maintenance policies of two-unit systems, only CBM based on Proportional Hazards Model was considered before by Tian and Liao (2011). In this chapter, we propose a new maintenance policy for a two-unit series system with economic dependence. The maintenance model with a CBM based on multivariate Bayesian Control Chart considering the OM idea was proposed. For this two-unit system, one unit, e.g. the transmission gearbox, is more expensive than the other one, so that it is subject to condition monitoring and the obtained information, oil data or vibration data, is collected for the PM decision-making. The other unit, e.g. the electric transmission control unit, is assumed to be a ‘non-repairable’ device with low replacement cost. When this unit fails, it is replaced immediately, which simultaneously creates a PM opportunity for the expensive unit to save system set-up cost. The objective is to develop a policy minimizing the long-run expected average cost per unit time for the entire system. The Bayesian control scheme is introduced and the optimal maintenance policy is found under
an SMDP framework. The computational procedure is illustrated by a numerical example.

4.1 Notations

- Posterior probability at inspection epoch \( n \Delta \): \( \Pi_n \)

- PM limit: \( \bar{\Pi} \)

- OM limit: \( W \)

- Set-up cost of the system: \( C_{SET} \)

- PM cost of unit 0: \( C_{PM} \)

- Failure replacement cost of unit 0: \( C_F^0 \)

- Failure replacement cost of unit 1: \( C_F^1 \)

- Inspection time of unit 0: \( T_I \)

- PM time of unit 0: \( T_{PM} \)

- Failure maintenance time of unit 0: \( T_F \)

- Lost production cost per unit time of the idle system: \( C_{PL} \)

- Observable failure time of unit 0: \( \xi_0 \)
• Observable failure time of unit 1: \( \xi_i \)

• Sampling interval for CBM of unit 0: \( \Delta \)

4.2 Model Assumptions

The system is composed of two units: one expensive deteriorating unit with an increasing failure rate and one cheaper unit with the constant failure rate. The following assumptions are considered:

1. We denote the deteriorating unit and non-deteriorating unit as unit 0 and unit 1, respectively. As soon as one fails, the whole system has to be suspended to replace the failed unit.

2. Since unit 0 is relatively expensive, we establish a CBM with multivariate Bayesian control chart to monitor the posterior probability that it is in a warning state. We assume that the unit 0 can be categorized into one of three states: a healthy or "good as new" state (state 0), an unhealthy or warning state (state 1), and a failure state (state 2). Unit 0 starts operating in state 0 as a ‘good as new’ condition. Over time the unit 0 deteriorates with age and usage, then its failure rate increases as a result. Eventually the unit 0 can no longer be viewed as in ‘good as new’ condition, even though it can still operate, but jump to the warning state 1. For it is working in both state 0 and 1, unit 0’s state condition cannot be directly observed, but can be estimated by using the data obtained from unit condition monitoring. Unit 0’s failure state can be figured out immediately. The multivariate Bayesian control chart introduced in Chapter 2 is used to determine when the unit 0 is likely in the warning state by plotting the posterior probability.
3. Unit 1 has a constant failure rate and no need to be replaced before it fails. Unit 1 can be categorized into one of two states: a healthy state (state 0), and a failure state (state 1). In this model, we ignore the replacement time consuming when unit 1 fails, because it is relatively small compared with the CBM sampling interval and maintenance time of unit 0.

4. There is no stochastic or structural dependence between unit 0 and unit 1. We focus on the economic dependence between them, i.e., a system set-up cost is incurred whenever the system fails or PM of unit 0 is performed. The system set-up cost may consist of the preparation cost under certain conditions, e.g., inspecting a machine, putting up the scaffolding, and sending maintenance staff on site, etc. The system set-up cost can be saved when maintenance actions are executed simultaneously.

### 4.3 Model Formulation

This model studies a combination of CBM and OM policy for a two-unit series system. Unit 0 is subject to condition monitoring at equidistant, discrete time epochs, while unit 1 does not need PM. Whenever unit 1 fails, failure replacement of this unit is performed and simultaneously the posterior probability for unit 0 is calculated. If it is above the OM limit, full inspection of unit 0 is performed. If its condition is in the warning state, OM of unit 0 is performed simultaneously. Also, when the posterior probability of unit 0 exceeds a PM control limit, full inspection of this unit is performed possibly followed by PM.

The deterioration process of unit 0 \( \{X^0_t, t \in R_+\} \) is a continuous time homogeneous Markov process, with state space \( S_0 = \{0,1,2\} \) as discussed in the model assumption.
When unit 0 is in the state 0, the observation \( Y_n | X_{n\Delta} = 0 \sim \Sigma_0 \). When unit 0 is in the state 1, the observation \( Y_n | X_{n\Delta} = 1 \sim \Sigma_1 \), where \( \mu_0, \mu_1 \in \mathbb{R}^d, \Sigma_0, \Sigma_1 \in \mathbb{R}^{d \times d} \), are the known model parameters, which can be obtained by Maximum Likelihood Estimation method proposed in Jiang et al. (2012). When unit 0 fails, a failure signal is generated and failure replacement of unit 0 is carried out immediately. Unit 0 can make a transition from state 0 to state 1 with transition probability \( p_{01}^0 \) or from state 0 to state 2 with transition probability \( p_{02}^0 \), with \( p_{01}^0 + p_{02}^0 = 1 \). The instantaneous transition rate of unit 0, \( \lambda_{ij} \ (i, j \in S_0) \), is defined by

\[
\lambda_{ij} = \lim_{v \to 0^+} \frac{P(X_{t+v} = j | X_t = i)}{v}, \quad i \neq j \in S_0
\]

\[
\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}
\]

Hence, the unit 0 is assumed to start in a healthy state, and the transition rate Q-matrix for unit 0 is:

\[
\Lambda = \left( \lambda_{ij} \right)_{i,j \in S_0} = \begin{pmatrix}
-(\lambda_{01} + \lambda_{02}) & \lambda_{01} & \lambda_{02} \\
0 & -\lambda_{12} & \lambda_{12} \\
0 & 0 & 0
\end{pmatrix}
\]

To model the monotonic unit deterioration, we assume that the state process of unit 0 is non-decreasing with probability 1, i.e. \( \lambda_{ij} = 0 \) for all \( j < i \). It is also assumed that unit 0 is more inclined to fail in unhealthy state 1 than in healthy state 0, i.e. \( \lambda_{02} < \lambda_{12} \). The failure state 2 is an absorbing state. The transition probability matrix \( P(t) = \left( P_{ij}(t) \right)_{i,j \in S} \)
can be obtained by solving Kolmogorov backward differential equations (Ross (2010)), which follows:

$$
P^0(t) = \begin{pmatrix}
P^0_{ij}(t) & \text{for } i,j \in S_0 \\
0 & \text{for } i,j \notin S_0
\end{pmatrix}
= \begin{pmatrix}
\frac{e^{-\lambda_{ij}t} - e^{-\lambda_{ij}t'}}{\lambda_{ij} + \lambda_{ij} - \lambda_{ij}} & 0 \\
0 & 1 - e^{-\lambda_{ij}t'} & 0 & 1 - e^{-\lambda_{ij}t'}
\end{pmatrix},
$$

(4.3)

where $P^0_j(t) = P(X_{s+t} = j | X_s = i)$.

The condition of unit 0 is monitored at equidistant sampling intervals $\Delta, 2\Delta, \ldots$ and the data $Y_1, Y_2, \ldots \in \mathbb{R}^d$ are collected at these epochs, which represent partial information about the system state. In this paper, we assume that the observations have $d$-dimensional normal distribution $N_d(\mu_i, \Sigma_i)$ and are conditionally independent given the current system state $i$, i.e.

$$f_{Y_j|X_{s+t}}(y \mid i) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma_i}} \exp \left( -\frac{1}{2} (y - \mu_i)' \Sigma_i^{-1} (y - \mu_i) \right)$$

(4.4)

Where $\mu_o, \mu_t \in \mathbb{R}^d, \Sigma_o, \Sigma_t \in \mathbb{R}^{d \times d}$ are known observation process parameters.

The state process of unit 1 $\{X_{s+t}^1, t \in \mathbb{R}_+\}$ is also a continuous-time homogeneous Markov chain with states $S_1 = \{0,1\}$, and the constant failure rate of unit 1 is $\lambda^*$. When unit 1
fails, the failure replacement of unit 1 is carried out immediately. Hence, the transition probability matrix is:

\[
P^1(t) = \begin{pmatrix} e^{-\lambda t} & 0 \\ 1 - e^{-\lambda t} & 1 \end{pmatrix}.
\]  

(4.5)

### 4.4 Multivariate Bayesian Control Chart

The Bayesian control chart is used to monitor the posterior probability that the unit 0 is in the warning state. This method has recently received a lot of attention (Yin and Makis (2011), Kim et al. (2011), Makis (2009) and Nenes and Panagiotidou (2011)) from our discussion in Chapter 2, and it has been proven to be the optimal tool for decision making in quality control by Makis (2008). According to the theory of partially observable Markov decision process, the posterior probability that the system in the warning state is sufficient for optimal decision-making. Thus, at each sampling epoch the new observation data are collected and the posterior probability is updated using Bayes’ theorem. The posterior probability that unit 0 is in the warning state is denoted by:

\[
\Pi_n = Pr \left( X_{n\Delta}^0 = 1 \mid Y_1, \ldots, Y_n, \xi > n\Delta, \Pi_{(n-1)} \right).
\]  

(4.6)

In the Bayesian control chart, the posterior probability can be expressed as:

\[
\Pi_n = \frac{Pr \left( X_{n\Delta}^0 = 1, Y_1, \ldots, Y_n, \xi > n\Delta, \Pi_{(n-1)} \right)}{Pr \left( Y_1, \ldots, Y_n, \xi > n\Delta, \Pi_{(n-1)} \right)}.
\]  

(4.7)
Hence, applying Bayes’ Theorem, the posterior probability of unit 0, \( \Pi_n \) can be calculated as:

\[
\Pi_n = \frac{f(Y_n | 1) \left( P_{01}^0 (\Delta)(1 - \Pi_{(n-1)}) + P_{11}^0 (\Delta) \Pi_{(n-1)} \right)}{f(Y_n | 0) P_{00}^0 (\Delta)(1 - \Pi_{(n-1)}) + f(Y_n | 1) \left( P_{01}^0 (\Delta)(1 - \Pi_{(n-1)}) + P_{11}^0 (\Delta) \Pi_{(n-1)} \right)}, (4.8)
\]

where \( P_{ij}^0 (\Delta) \) are given in equation (4.3).

To simplify equation (4.8) under the assumption \( \Sigma_0 \neq \Sigma_1 \), we introduce the following functions:

\[
\frac{f_{y_i|x_n}(y|0)}{f_{y_i|x_n}(y|1)} = \frac{1}{\sqrt{(2\pi)^d |\Sigma_0|}} \exp \left( -\frac{1}{2}(y - \mu_0)' \Sigma_0^{-1} (y - \mu_0) \right) = h \exp \left( \frac{1}{2} ((Y_n - B)' A(Y_n - B) + C) \right), (4.9)
\]

Where in (4.9), \( h \), \( A \), \( B \) and \( C \) are given by

\[
\begin{align*}
    h &= |\Sigma_0|^{-\frac{1}{2}} |\Sigma_1|^{-\frac{1}{2}}, \\
    A &= \Sigma_1^{-1} - \Sigma_0^{-1}, \\
    B &= A^{-1} \left( \Sigma_1^{-1} \mu_1 - \Sigma_0^{-1} \mu_0 \right), \\
    C &= \left( \mu_1 \Sigma_1^{-1} \mu_1 - \mu_0 \Sigma_0^{-1} \mu_0 \right) - B' \left( \Sigma_1^{-1} \mu_1 - \Sigma_0^{-1} \mu_0 \right).
\end{align*}
\]

(4.10)

We further define:

\[
V_n = (Y_n - B)' A(Y_n - B),
\]

so that (4.8) simplifies to
\[ \Pi_n = \frac{P^0_{01}(\Delta)(1 - \Pi_{(n-1)}) + P^0_{11}(\Delta)\Pi_{(n-1)}}{h \exp\left(\frac{1}{2}(V_n + C)\right)P^0_{00}(\Delta)(1 - \Pi_{(n-1)}) + \left(P^0_{01}(\Delta)(1 - \Pi_{(n-1)}) + P^0_{11}(\Delta)\Pi_{(n-1)}\right)}. \] (4.12)

### 4.5 Maintenance Policy

Our maintenance policy is defined by introducing the OM limit into the multivariate Bayesian control chart. Under this policy, there will be one of three scenarios happened during one sampling interval, which is clearly showed in the flowchart in Figure 4.1.

- The first scenario: When unit 1 fails first at time \( t \) before the failure of unit 0 and between its two consecutive inspection epochs, we check the posterior probability of unit 0 at the previous inspection epoch \( \left[ \frac{t}{\Delta} \right] * \Delta \), which is \( \Pi_{[t/\Delta] \Delta} \). If \( \Pi_{[t/\Delta] \Delta} \) exceeds the OM limit \( W \in \left[ 0, \bar{\Pi} \right] \) on the Bayesian control chart, full inspection of unit 0 is initiated. Following that, when unit 0 is found to be in the good state, it is left operational and only unit 1’s failure replacement is carried out, with the corresponding cost \( C_{SET} + C_{F} + T_I \times C_{PL} \). When unit 0 is found to be in the warning state, the preventive replacement of unit 0 and failure replacement of unit 1 are simultaneously triggered at a cost of \( C_{SET} + C_{F} + C_{PM} + (T_I + T_{PM}) \times C_{PL} \). Additionally, if posterior probability \( \Pi_{[t/\Delta] \Delta} \) is under the OM limit \( W \) on the Bayesian control chart, only failure replacement of unit 1 is performed with the cost \( C_{SET} + C_{F} \).
• The second scenario: When unit 0 posterior probability $\Pi_n$ exceeds an optimal PM limit $\Pi \in [0, 1]$ at the next sampling epoch before unit 1’s failure, the inspection of unit 0 is initiated, and the subsequent maintenance decisions depend on its inspection result. If it is found to be in the warning state, the unit 0 has to be preventively maintained with the corresponding cost $C_{SET} + C_{PM} + (T_I + T_{PM})C_{PL}$.

Otherwise, if it is in good state, it is left operational and the false alarm costs are considered as $C_{SET} + T_I C_{PL}$.

• The third scenario: When unit 0 fails before the Bayesian chart signals, only the failure maintenance of unit 0 is performed and the corresponding cost is $C_{SET} + C^0_F + T^0_F C_{PL}$.

![Figure 4.1 The Bayesian Control Scheme](image)
The objective is to find the optimal values of the PM limit $\bar{\Pi}^* \in [0, 1]$ and the OM limit $W^* \in [0, \bar{\Pi}^*)$ on the Bayesian control chart in order to minimize the long-run expected average cost per unit time. From the renewal theory, the long-run expected average cost per unit time $g(\bar{\Pi}, W^*)$ is calculated as the expected system cost incurred in one cycle (CC) divided by the expected cycle length (CL). A cycle is completed when both units in the system return back to their healthy states. Since the lifetime of unit 1 follows the exponential distribution with the memoryless feature, the renewal cycle is completed only after the preventive or failure maintenance on the unit 0. The cost minimization problem is equivalent to finding the optimal PM limit and optimal OM limit, and it can be formulated as:

$$\frac{E_{\bar{\Pi}^*, W^*}(CC)}{E_{\bar{\Pi}^*, W^*}(CL)} = \inf_{\bar{\Pi} \in [0, 1], W \in [0, \bar{\Pi}]} \left\{ \frac{E_{\bar{\Pi}, W}(CC)}{E_{\bar{\Pi}, W}(CL)} \right\}. \quad (4.13)$$

In the next section, we develop an efficient algorithm in the SMDP framework to determine the optimal values of the PM limit $\bar{\Pi}^*$ and OM limit $W^*$.

4.6 SMDP Algorithm

4.6.1 State Space Definition

In order to compute the long-run expected average cost in the SMDP framework, we choose a large $L \in \mathbb{N}$ to discretize the interval $[0, 1]$. We have found that when $L \geq 30$, the discretization leads to a sufficient degree of precision and fast computation. Then, the
state space for the SMDP is defined by \( S = \{0, 1, \ldots, L, L+1, L+2\} \). The SMDP is defined to be in state 0 if the unit 0 is in the new condition. The PM limit \( \overline{\Pi} = \frac{j}{L} \) and OM limit \( W = \frac{i}{L} \), where \( 1 \leq i \leq j \) and \( i, j \in \mathbb{N} \). State \( L+1 \) denotes the PM state, and \( L+2 \) denotes the failure state of unit 0.

Suppose that both units 0 and 1 are operational before reaching the sampling epoch \( n\Delta \).

If the posterior probability of unit 0, \( \Pi_n \), is below the OM limit and lies in the interval \( \left[ \frac{l-1}{L}, \frac{l}{L} \right) \), the SMDP is defined to be in state \( l \), where \( l < i \). If the posterior probability of unit 0, \( \Pi_n \), is between OM limit and the control limit, lying in the interval \( \left[ \frac{l-1}{L}, \frac{l}{L} \right) \), the SMDP is defined to be in state \( l \), for \( i \leq l < j \). If the posterior probability of unit 0 is above the control limit and lies in the interval \( \left[ \frac{l-1}{L}, \frac{l}{L} \right) \), the SMDP is defined to be in state \( l \), \( l \geq j \), and full inspection of unit 0 is conducted following with two possible outcomes. Suppose unit 1 fails before the sampling epoch \( n\Delta \), while unit 0 does not fail. Meanwhile, when the posterior probability of unit 0, \( \Pi_n \), is below the OM limit at the previous sampling epoch \( (n-1)\Delta \), the action chosen at the unit 1's failure time is just to replace unit 1 and let unit 0 continually operates until its next sampling epoch. On the contrary, when the posterior probability of unit 0, \( \Pi_n \), is between OM limit and PM limit at \( (n-1)\Delta \), the action chosen at the unit 1's failure time is to replace unit 1 and fully inspect unit 0 simultaneously. Based on different unit 0
conditions, the combination of unit 0’s PM and unit 1’s failure replacement, or only failure replacement of unit 1, will be conducted.

4.6.2 Derivation of Linear Equations

With the above definition of the SMDP state space, for the long-run average cost criterion, the SMDP is determined by the following quantities:

- \( p_{ij} \) = the probability that at the next decision epoch the system will be in state \( j \in S \) given the current state is \( i \in S \).

- \( \tau_i \) = the expected sojourn time unit the next decision epoch given the current state is \( i \in S \).

- \( c_i \) = the expected cost incurred until the next decision epoch given the current state is \( i \in S \).

With the above definitions, for the fixed PM limit \( \bar{\Pi} \in [0,1] \) and OM limit \( W \in [0, \bar{\Pi}] \), the long-run expected average cost per unit time \( g(\bar{\Pi}, W) \) can be obtained by solving the following system of linear equations (Tijms (1994)):

\[
\begin{align*}
\nu_k &= c_k - g(\bar{\Pi}, W) \tau_k + \sum_{j \in S} p_{k,j} \nu_j, \text{ for } k \in S \\
\nu_s &= 0,
\end{align*}
\]  

(4.14)

where \( s \) is an arbitrarily chosen state. The quantities \( \nu_k \) denote the ‘relative values’ at the condition that the initial state is \( k \) and the particular control limit policy,
parameterized by \( \overline{\Pi} \) and \( W \) in this case, is applied. Hence, the optimal control limit \( \overline{\Pi}^* \in [0, 1] \), the optimal OM limit \( W^* \in [0, \overline{\Pi}) \), and the corresponding optimal average cost \( g(\overline{\Pi}, W^*) = \inf_{\overline{\Pi} \in [0, 1], W \in [0, \overline{\Pi})} g(\overline{\Pi}, W) \) can be computed by iteratively solving the above system of linear equations considering different values of \( \overline{\Pi} \) and \( W \). The remainder of the mathematical analysis is the derivation of the expressions for transition probabilities \( p_{lm} \), sojourn times \( \tau_i \) and costs \( c_i \), with \( l, m \in S \).

### 4.6.3 SMDP Transition Probability

The SMDP transition probability \( p_{l,m} \), where \( \theta_1 = \frac{m-1}{L} \) and \( \theta_2 = \frac{m}{L} \) (1 \( \leq \) \( l \leq \) \( m \)), can be calculated as follows:

when \( l < W \), 1 \( \leq \) \( l \leq \) :

\[
p_{l,m} = P\left( \theta_1 \leq \overline{\Pi}_n < \theta_2, \xi_0 > n\Delta \mid \xi_0 > (n-1)\Delta, Y_1, \ldots, Y_{n-1}, \overline{\Pi}_{(n-1)} \right)
\]

\[
= P\left( \theta_1 \leq \overline{\Pi}_n < \theta_2 \mid \xi_0 > n\Delta, Y_1, \ldots, Y_{n-1}, \overline{\Pi}_{(n-1)} \right)
\]

\[
* \frac{P\left( \xi_0 > n\Delta \mid \xi_0 > (n-1)\Delta, Y_1, \ldots, Y_{n-1}, \overline{\Pi}_{(n-1)} \right)}{R_0 \left( \Delta \mid \overline{\Pi}_{(n-1)} \right)}
\]

(4.15)

In equation (4.15), when the current state \( l \) is beneath the OM limit \( W \), the SMDP state jumps to the state \( m \) (1 \( \leq \) \( l \leq \)) no matter whether unit 1 fails between sampling
epochs, \((n-1)\Delta\) and \(n\Delta\), or not. When unit 1 fails, we only conduct its failure replacement without considering the replacement time as above assumptions mentioned.

\[
p_{l,L+2} = P\left((n-1)\Delta < \xi_0 < n\Delta \mid \Pi_{(n-1)}\right) = 1 - R_0\left(\Delta \mid \Pi_{(n-1)}\right),
\]

(4.16)

In equation (4.16), the SMDP process will directly shift to the failure state when unit 0 fails between sampling epochs \((n-1)\Delta\) and \(n\Delta\).

When \(W\leq \cdot \leq \cdot \leq \cdot \):

\[
p_{l,m} = P\left(\theta_1 \leq \Pi_0 \leq \theta_2, \xi_0 > n\Delta, \xi_1 > n\Delta \mid (n-1)\Delta, \xi_0 > (n-1)\Delta, Y_1, ..., Y_{n-1}, \Pi_{(n-1)}\right)
= P\left(\theta_1 \leq \Pi_0 \leq \theta_2 \mid \xi_0 > n\Delta, \xi_1 > n\Delta, Y_1, ..., Y_{n-1}, \Pi_{(n-1)}\right) * R_0(\Delta \mid \Pi_{(n-1)}) * R_1(\Delta \mid \Pi_{(n-1)}),
\]

(4.17)

In equation (4.17), when the current state \(l\) is between the OM and PM limits, the SMDP process can jump to the state \(m\) \((l \leq \cdot \leq \cdot \leq \cdot\) under the condition that both units 0 and 1 do not fail until \(n\Delta\). When unit 1 fails between \((n-1)\Delta\) and \(n\Delta\), we will fully inspect the condition of unit 0, following one of two possible conditions. If its condition is in the warning state 1, the SMDP will directly shift to the PM state. If its condition is in the healthy state 0, the SMDP will go back to the state 0. These two transition probabilities can be calculated in equations (4.18) and (4.19).

\[
p_{l,L+1} = P\left(X_0^{\xi_1} = 1, \xi_1 < \xi_0, (n-1)\Delta < \xi_1 < n\Delta \mid \Pi_{(n-1)}\right),
\]

\[
= \int_0^{\xi_0} P\left(X_0^{\xi_1} = 1, \xi_1 < \xi_0, (n-1)\Delta < \xi_1 < n\Delta \mid \Pi_{(n-1)}, \xi_0 = (n-1)\Delta + u\right) \lambda^* e^{-\lambda^* u} du
\]

\[
= \int_0^{\Delta} \left[P_{01}^0(u)(1 - \Pi_{(n-1)}) + P_{11}^0(u)\Pi_{(n-1)}\right] \lambda^* e^{-\lambda^* u} du
\]

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\[
\begin{align*}
&= \left( \frac{\lambda_0 \lambda^* (1 - \Pi \alpha^-)}{(\lambda_0 + \lambda_{02} - \lambda_{12})(\lambda_{12} + \lambda^*)} + \frac{\lambda^* \Pi_{(n-1)}}{\lambda_{12} + \lambda^*} \right) * \left( 1 - e^{-(\lambda^* + \lambda_{12}) \Delta} \right) \\
&= - \frac{\lambda_{01} \lambda^* (1 - \Pi_{(n-1)})}{(\lambda_{01} + \lambda_{02} - \lambda_{12})(\lambda_{01} + \lambda_{02} + \lambda^*)} \left( 1 - e^{-(\lambda_{01} + \lambda_{02} + \lambda^*) \Delta} \right),
\end{align*}
\)  
(4.18)

\[
p_{l,0} = P(X^0_{\xi_1} = 0, \xi_1 < \xi_0, (n-1)\Delta < \xi_1 < n\Delta | \Pi_{(n-1)}) \\
= \int_0^\Delta P(X^0_{\xi_1} = 0, \xi_1 < \xi_0, (n-1)\Delta + u < \xi_1 < n\Delta | \Pi_{(n-1)}) \lambda^* e^{-\lambda^* u} du \\
= \int_0^\Delta P(X^0_{(n-1)\Delta + u} = 0 | \Pi_{(n-1)}) \lambda^* e^{-\lambda^* u} du \\
= \int_0^\Delta P_0^0(u)(1 - \Pi_{(n-1)}) \lambda^* e^{-\lambda^* u} du \\
= \frac{\lambda^* (1 - \Pi_{(n-1)})}{\lambda_{01} + \lambda_{02} + \lambda^*} \left( 1 - e^{-(\lambda_{01} + \lambda_{02} + \lambda^*) \Delta} \right),
\]

\[
p_{l,1,2} = P((n-1)\Delta < \xi_0 < \xi_1 < n\Delta | \Pi_{(n-1)}) + P((n-1)\Delta < \xi_0 < n\Delta < \xi_1 | \Pi_{(n-1)}).
\]  
(4.20)

In equation (4.20), the SMDP will transfer to the failure state when unit 0 fails first, between sampling epochs \((n-1)\Delta\) and \(n\Delta\), than unit 1’s possible failure.

When \(l > \Pi\):

\[
p_{l,L+1} = \Pi_{(n-1)}, \quad p_{l,0} = 1 - \Pi_{(n-1)},
\]  
(4.21)

In equation (4.21), when the current state \(l\) is above the PM limit, it will directly impose the full inspection of unit 0. Finally, after conducting the failure or preventive maintenance of unit 0, the SMDP will go back to state 0 with probability 1,

\[
p_{L+1,0} = p_{L+2,0} = p_{0,1} = 1.
\]  
(4.22)
Hence, for first term in equations (4.15), (4.17) can be calculated as follows:

\[
P(\theta_i \leq \Pi_n < \theta_2 \mid \xi_0 > n\Delta, \xi_j > n\Delta, Y_1, \ldots, Y_{n-1}, \Pi_{(n-1)})
= P(\theta_i \leq \Pi_n < \theta_2 \mid \xi_0 > n\Delta, Y_1, \ldots, Y_{n-1}, \Pi_{(n-1)})
= P(\theta_i < \Pi_n < \theta_2 \mid X_{n\Delta}^0 = 0)P(X_{n\Delta}^0 = 0 \mid \xi_0 > n\Delta, \Pi_{(n-1)}) + P(\theta_i < \Pi_n < \theta_2 \mid X_{n\Delta}^0 = 1)P(X_{n\Delta}^0 = 1 \mid \xi_0 > n\Delta, \Pi_{(n-1)})
\]

\[
P \left( \frac{1 - \theta_2}{\theta_2 k_{\Pi_{(n-1)}}^0} \right) - C < V_n \leq 2 \ln \left( \frac{1 - \theta_i}{\theta_i k_{\Pi_{(n-1)}}^0} \right) - C \mid X_{n\Delta}^0 = 0, X_{n\Delta}^1 = 0 \right) \left( \frac{k_{\Pi_{(n-1)}}^0}{k_{\Pi_{(n-1)}}^0 + k_{\Pi_{(n-1)}}^1} \right),
+ P \left( \frac{1 - \theta_2}{\theta_2 k_{\Pi_{(n-1)}}^0} \right) - C < V_n \leq 2 \ln \left( \frac{1 - \theta_i}{\theta_i k_{\Pi_{(n-1)}}^0} \right) - C \mid X_{n\Delta}^0 = 1, X_{n\Delta}^1 = 0 \right) \left( \frac{k_{\Pi_{(n-1)}}^0}{k_{\Pi_{(n-1)}}^0 + k_{\Pi_{(n-1)}}^1} \right),
\]

Where,

\[
h = \left| \Sigma_0 \right|^{\frac{1}{2}} \left| \Sigma_i \right|^{\frac{1}{2}},
\]

\[
k_{\Pi_{(n-1)}}^0 = P_{00}^0 \left( \Delta \right) \left( 1 - \Pi_{(n-1)} \right),
\]

\[
k_{\Pi_{(n-1)}}^1 = P_{01}^0 \left( \Delta \right) \left( 1 - \Pi_{(n-1)} \right) + P_{11}^0 \left( \Delta \right) \Pi_{(n-1)};
\]

Provost and Rudiuk (1996) shows that any indefinite quadratic form in multivariate normal vectors \( Q = G' AG \), where \( G \sim N_d \left( \mu, \Sigma \right), d \in N \), can be represented as the difference of two linear combination of independent chi-square variables. Applying this property, equation (4.23) can be simplified as:

\[
P(\theta_i \leq \Pi_n < \theta_2 \mid \xi_0 > n\Delta, \xi_j > n\Delta, Y_1, \ldots, Y_{n-1}, \Pi_{(n-1)})
= F(Q_0) * \left( \frac{k_{\Pi_{(n-1)}}^0}{k_{\Pi_{(n-1)}}^0 + k_{\Pi_{(n-1)}}^1} \right) + F(Q_1) * \left( \frac{k_{\Pi_{(n-1)}}^1}{k_{\Pi_{(n-1)}}^0 + k_{\Pi_{(n-1)}}^1} \right)
\]

(4.25)
Where \( F(Q_0) \) and \( F(Q_1) \) are the cumulative density equations of a quadratic form in normal vectors.

For conditional reliability of unit 0 and unit 1 in equations (4.15), (4.16), (4.17):

\[
R_0\left( t \mid \Pi_{(n-i)} \right) = P(\xi_0 > (n-1)\Delta + t \mid \xi_0 > (n-1)\Delta, Y_i, \ldots, Y_{n-1}, \Pi_{(n-i)}) \\
= P(X_{(n-i)\Delta+t} \neq 2 \mid \xi_0 > (n-1)\Delta, Y_i, \ldots, Y_{n-1}, \Pi_{(n-i)}) \\
= (1 - \Pi_{(n-i)})(1 - P^0_2(t)) + \Pi_{(n-i)}(1 - P^0_1(t)),
\]

\[
R_1\left( t \mid \Pi_{(n-i)} \right) = R_1(t) = e^{-\lambda^*t}. \tag{4.27}
\]

For equation (4.20):

\[
P((n-1)\Delta < \xi_0 < \xi_1 < n\Delta \mid \Pi_{(n-i)}) = \int_0^{\Delta} \left(1 - R_0\left( t \mid \Pi_{(n-i)} \right) \right) \lambda^* e^{-\lambda^*t} \, dt, \tag{4.28}
\]

\[
P((n-1)\Delta < \xi_0 < n\Delta < \xi_1 \mid \Pi_{(n-i)}) = \left(1 - R_0\left( \Delta \mid \Pi_{(n-i)} \right) \right) \exp(-\lambda^*\Delta). \tag{4.29}
\]

### 4.6.4 SMDP Expected Costs

When \( l = 0 \),

\[
C_l = C_{SET}. \tag{4.30}
\]

In equation (4.30), whenever the SMDP process goes back to the renewed state 0, the system set-up cost will be incurred.

When \( l < \infty \),
\[
C_i = \left( C_F^i + C_{SET}^i \right) \left[ P \left( (n-1) \Delta < \xi_1 < \xi_0 < n\Delta | \Pi_{(n-1)} \right) + P \left( (n-1) \Delta < \xi_1 < n\Delta < \xi_0 | \Pi_{(n-1)} \right) \right]. \tag{4.31}
\]

In equation (4.31), when the current state \( l \) is beneath the OM limit \( W \), if unit 1 fails first between sampling epochs \((n - 1) \Delta \) and \( n\Delta \), both the unit 1’s failure replacement cost and system set-up cost are incurred. In this situation, the SMDP process does not go back to the renewed state 0.

When \( W < l \leq \xi \),

\[
C_i = \left( T_i C_{pl} + C_F^i \right) \left[ P \left( (n-1) \Delta < \xi_1 < \xi_0 < n\Delta | \Pi_{(n-1)} \right) + P \left( (n-1) \Delta < \xi_1 < n\Delta < \xi_0 | \Pi_{(n-1)} \right) \right]. \tag{4.32}
\]

In equation (4.32), when the current state \( l \) is between the OM and PM limits, if unit 1 fails first between \((n - 1) \Delta \) and \( n\Delta \), the unit 1’s failure replacement cost with the additional inspection cost of unit 0 are incurred, for we need further inspect the unit 0's condition in this situation.

For the second terms of functions (4.31) and (4.32):

\[
P \left( (n-1) \Delta < \xi_1 < \xi_0 < n\Delta | \Pi_{(n-1)} \right) = \int_0^\Delta e^{-\lambda \xi_1} \frac{d}{dt} R_0 \left( t | \Pi_{(n-1)} \right) - R_0 \left( \Delta | \Pi_{(n-1)} \right). \tag{4.33}
\]

\[
P \left( (n-1) \Delta < \xi_1 < n\Delta < \xi_0 | \Pi_{(n-1)} \right) = R_0 \left( \Delta | \Pi_{(n-1)} \right) \left( 1 - e^{-\lambda \Delta} \right). \tag{4.34}
\]
When \( l > \bar{\Pi} \),

\[
C_l = T_l C_{PL},
\]

\[
C_{L+1} = C_{PM} + T_{PM} C_{PL},
\]

\[
C_{L+2} = T_F^o C_{P2} + C_F^0.
\]

### 4.6.5 SMDP Expected Sojourn Times

When \( l \leq \bar{\Pi} \):

\[
\tau_l = \Delta R (\Delta | \Pi_{(n-1)}) + \int_0^\Delta u \left( -\frac{dR(u | \Pi_{(n-1)})}{du} \right)
\]

\[
= \int_0^\Delta \left( 1 - \frac{\lambda_{01} e^{-\lambda_{12} t} - e^{-\left(\lambda_{01} + \lambda_{02}\right) t}}{\lambda_{01} - \lambda_{12} + \lambda_{02}} - e^{-\left(\lambda_{01} + \lambda_{02}\right) t} \right) dt.
\]

In equation (4.38), when the current state \( l \) is beneath the OM limit \( W \), the expected sojourn time can be calculated based on two situations. One situation is that unit 0 survives until the sampling epoch \( n\Delta \), where the sojourn time equals to the sampling interval \( \Delta \). The other situation is that unit 0 fails between the sampling epochs \((n-1)\Delta\) and, where the sojourn time equals to the unit 0’s survival time after the sampling epoch \((n-1)\Delta\).
When \( W < \lambda \leq \rho \):

\[
\tau_l = E\left( \text{Sojourn Time} \mid \Pi_{(n-1)} \right)
= \int_{0}^{\Delta} \frac{d}{du} \left[ -R_0\left( u \mid \Pi_{(n-1)} \right) \right] \lambda^* e^{-\lambda^* w} dw \text{ for } 0 < \xi_0 < \xi_1 < \Delta
+ \int_{0}^\Delta \frac{d}{dw} \left[ -R_0\left( u \mid \Pi_{(n-1)} \right) \right] \lambda^* e^{-\lambda^* w} dw \text{ for } 0 < \xi_1 < \xi_0 < \Delta
+ \int_{\Delta}^\infty \frac{d}{du} \left[ -R_0\left( u \mid \Pi_{(n-1)} \right) \right] \lambda^* e^{-\lambda^* w} dw \text{ for } 0 < \xi_0 < \Delta < \xi_1
+ \int_{\Delta}^\infty \frac{d}{dw} \left[ -R_0\left( u \mid \Pi_{(n-1)} \right) \right] \lambda^* e^{-\lambda^* w} dw \text{ for } \Delta < \xi_0, \Delta < \xi_1
+ \int_{\Delta}^\infty \frac{d}{du} \left[ -R_0\left( u \mid \Pi_{(n-1)} \right) \right] \lambda^* e^{-\lambda^* w} dw - wR_0\left( w \mid \Pi_{(n-1)} \right) \lambda^* e^{-\lambda^* w} dw
+ \left[ \int_{0}^{\Delta} R_0\left( u \mid \Pi_{(n-1)} \right) du - \Delta R_0\left( \Delta \mid \Pi_{(n-1)} \right) \right] e^{-\lambda^* \Delta}
+ \left[ \int_{0}^{\Delta} R_0\left( u \mid \Pi_{(n-1)} \right) du - \Delta R_0\left( \Delta \mid \Pi_{(n-1)} \right) \right] e^{-\lambda^* \Delta}
+ \left[ 1 - e^{-\lambda^* (\Delta \lambda^* + 1)} \right] R_0\left( \Delta \mid \Pi_{(n-1)} \right) + \Delta R_0\left( \Delta \mid \Pi_{(n-1)} \right) e^{-\lambda^* \Delta}.
\]

In equation (4.39), when the current state \( l \) is between the OM and PM limits, the expected sojourn time can be calculated according to different situations. If unit 0 fails first, the sojourn time is unit 0’s survival time after the sampling epoch \((n-1)\Delta\). If unit 1 fails first, the sojourn time is unit 1 survival time after \((n-1)\Delta\). If both units 0 and 1 survive until, the sojourn time is the length of the sampling interval.

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For $l > \overline{\Pi}$:

$$\tau_l = T_l, \tau_{L+1} = T_{PM}, \tau_{L+2} = T_F.$$  \hspace{1cm} (4.40)

To summarize, by substituting the above equations in equation (4.14) for the fixed PM limit $\overline{\Pi} \in [0, 1]$ and OM limit $W \in [0, \overline{\Pi})$, the long-run expected average cost per unit time $g(\overline{\Pi}, W)$ can be calculated. Furthermore, the two optimal limits, $\overline{\Pi}^*$ and $W^*$, with the corresponding minimum average cost per unit time can be computed.

### 4.7 Numerical Example

In the numerical example, we apply the simulated data to illustrate the above maintenance policy and the SMDP approach to find the optimal results. We assume that the model parameters are already given and proceed with the optimal computation under this assumption. However, in many real applications, the unit 0’s parameters need to be statistically estimated first before establishing the Bayesian control chart for the maintenance application. The parameters can be estimated using the expectation–maximization algorithm of Kim et al. (2012). The unit 1’s failure rate can be estimated by the “run to failure” test mentioned in Chapter 3.

According to the model assumption, unit 0’s deterioration follows a continuous-time homogeneous Markov chain with state space $S_0 = \{0, 1, 2\}$. States 0 and 1 are unobservable, representing the healthy and warning state respectively, and state 2 is the observable failure state. Unit 0’s sojourn time in the healthy state is exponentially
distributed with parameter $v_0 = \lambda_{01} + \lambda_{02}$, and the sojourn time in the warning state is exponentially distributed with parameter $v_1 = \lambda_{12}$. For unit 0, we define the parameters for the hidden process as: $\lambda_{01} = 0.15$, $\lambda_{02} = 0.02$ and $\lambda_{12} = 0.2$. The parameters for the observation process, the observation process $(Y_n : n \in N)$, are given by:

$$
\mu_0 = \begin{pmatrix} 0.21 \\ -0.01 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1.5 & 0.61 \\ 0.61 & 1.9 \end{pmatrix}; \mu_1 = \begin{pmatrix} 0.75 \\ 0.54 \end{pmatrix}, \Sigma_1 = \begin{pmatrix} 1.81 & 1.97 \\ 1.97 & 2.22 \end{pmatrix}.
$$

For unit 1, the constant failure rate is given by: $\lambda^* = 0.02$.

The following costs are considered in the numerical example:

$$
C_{PM} = 200, C_F^0 = 600, C_F^1 = 50, C_{SET} = 90, C_{PL} = 20.
$$

The system inspection and maintenance actions, the corresponding times are given by:

$$
T_I = 1, T_{PM} = 3, T_F = 10.
$$

We further assume that the sampling interval of this observation process for unit 0 equals to 5, $\Delta = 5$. For these parameters, we first compute the optimal values of PM limit $\overline{\Pi}^*$ and OM limit $W^*$, which minimizes the long-run expected average cost per unit time for this two-unit series system. In the SMDP algorithm, we choose the partition parameter $L = 32$, and use the linear equations (4.14) to find the optimal PM limit, OM limit and its corresponding minimum average cost per unit time, as shown in Table 4.1.
Table 4.1: The optimal Bayesian control chart for a two-unit series system

<table>
<thead>
<tr>
<th>Optimal PM Limit $\bar{\Pi}^*$</th>
<th>Optimal OM limit $W^*$</th>
<th>Minimum average cost Per unit time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8125</td>
<td>0.1875</td>
<td>44.6638</td>
</tr>
</tbody>
</table>

The algorithm takes 5.1163 seconds for each run on the code compiled by MATLAB (R2008a) on the Intel Core i7-4770, 3.40 GHz with 16 GB RAM. We find this computational time is much higher than the time by considering the optimal age-based preventive replacement policy for the two-unit series system in Chapter 3. We believe the calculation of SMDP transition probabilities in this case is much more complicated and difficult than the age-based case, but it is still quite fast for the off-line computation.

Figure 4.2 Expected average cost as a function of PM limit and OM limit.
The Figure 4.2 shows the corresponding long-run expected average operating cost per unit time for the entire system by applying different combination of values of the PM limit $\overline{\Pi}$ and OM limit $W$.

Additionally, we compare these optimal results with the results obtained from the multivariate Bayesian control chart for CBM application, which does not include the OM limit, where the preventive or failure maintenances are taken separately for each unit. For unit 0, when the posterior probability $\Pi_n$ exceeds an optimal control limit $\overline{\Pi} \in [0,1]$ on the Bayesian control chart, the full inspection is initiated with three different actions. If it is found to be in the good state, the unit 0 is left operational with the corresponding false alarm cost $C_{SET} + T_1 * C_{PL}$. If it is found to be in the warning state, the unit 0 must be stopped to conduct its preventive maintenance with the cost $C_{SET} + C_{PM} + (T_I + T_{PM}) * C_{PL}$. If the unit 0 fails before the chart signals, the failure replacement is triggered with the corresponding cost $C_{SET} + C_F^0 + (T_I + T_F^0) * C_{PL}$. For unit 1, when it fails, the failure replacement is carried out with cost $C_{SET} + C_F^1$.

For the above maintenance policies, we also use SMDP algorithm to calculate the optimal PM limit based on the Bayesian control chart technique and obtained its corresponding expected average cost per unit time. Figure 4.3 shows the corresponding long-run expected average operating cost per unit time for the entire system by using different values of the PM limit $\overline{\Pi}$.
Figure 4.3 Expected average cost as a function of PM limit

The long-run expected average cost per unit time without OM limit is calculated as 45.6764, which is bigger than the result of 44.6638 obtained from the model with OM limit, in the same two-unit series system. Although this increase is quite small, the system cycle length is usually very long in practical applications, such as; the system cycle length is typically well over 25,000 operational hours in the mining industry. Hence, the total saving is still considerable in actual system applications.

We find two other reasons to explain why the cost saving does not seem dramatic in our numerical examples. First, the Bayesian control chart without OM limit has proven to be an optimal CBM tool under the assumption of a 3-state hidden Markov model, and it has relatively little room to improve under the similar assumptions. Second, the system set-up cost $C_{SET}$ is relatively small compared with unit 0’s failure replacement and PM costs, $C_F^0$ and $C_{PM}$, respectively. Since OM limit is introduced to save the set-up cost for the
two-unit series system, the difference between operating cost per unit time from the Bayesian control charts with and without OM limit is not very impressive.

To summarize, this optimal opportunistic maintenance policy for a two-unit series system applying Bayesian control chart can be applied in some real world situations.
Chapter 5
Conclusions and Future Work

5.1 Conclusions

After the careful literature review on maintenance models for multi-unit systems with economic dependence, we have found that few papers have considered the maintenance policy for a two-unit system with unequally important units. Hence, this thesis has investigated two optimal maintenance policies for a two-unit series system with economic dependence, which is composed of one deteriorating expensive unit and one cheaper unit with low replacement cost.

An optimal age-based preventive replacement policy with two age thresholds, the preventive and opportunistic replacement thresholds, has been considered to save system set-up cost. An effective SMDP algorithm has been applied to find two optimal thresholds with the corresponding long-run expected average cost per unit time. This policy can be computed quickly, and can save system maintenance cost, especially under the condition when the system set-up cost is high.

The multivariate Bayesian control chart for CBM application has been developed for the multi-unit series system for the first time. The expensive unit is monitored at equidistant sampling intervals, and the posterior probability that this unit is in the warning state is calculated and plotted on the control chart. Also, an additional OM control limit on the control chart is introduced to provide opportunities for the full inspection when the
cheaper unit fails between the sampling epochs and the health condition of the expensive unit is obtained, leading to the reduction of the long-run average maintenance cost of the system by saving set-up costs. The SMDP algorithm is also applied to find two optimal control limits on the multivariate Bayesian control chart.

Our new models contribute to the fields of multi-unit system maintenance modeling and control, and have the potential to be applied in various real world situations.

5. 2 Suggestions for Future Work

For these two maintenance models for a two-unit series system, the lifetime of the cheaper unit is assumed to follow the exponential distribution with the memoryless property. In future research, more realistic distributions, such as Weibull or Gamma distributions could be considered for this unit. Furthermore, the non-negligible replacement time for the cheaper unit could be considered in future work.

Another interesting future research would be to test the effectiveness of our optimal opportunistic maintenance policy which uses Bayesian control chart for the two-unit series system in real industrial applications. Given the condition monitoring data, such as oil data and vibration data, for the expensive unit and the age or operating time data for the cheaper unit, an optimal maintenance policy considering real-world restrictions can be established.
References


