DIFFERENTIAL ORBITAL ELEMENT-BASED SPACECRAFT FORMATION CONTROL STRATEGIES

by

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Spacecraft formation flight is an important technology for upcoming scientific and Earth observation missions. The topic of this work is the control of spacecraft formations specifically through the control of the differential, mean orbital elements of a spacecraft. Orbital perturbations can disturb formation geometry in an undesirable fashion so active control is required to maintain a precise relative trajectory.

A number of control strategies are proposed. The first is an impulsive thrust strategy that is valid for formations in both eccentric and circular orbits. A general $N$-thrust per orbit formulation is presented. For the two-thrust case, an analytical solution to the constraint equations is presented and the closed-loop stability of the formation is considered. Two-thrust performance is shown to achieve superior position control with similar $\Delta V$ over previously proposed control strategies.

The geomagnetic Lorentz force is a propellantless means of altering a spacecraft’s orbit. A spacecraft with a significant surface charge experiences the Lorentz force due to the spacecraft’s velocity relative to the Earth’s magnetic field. Application of the Lorentz force to the formation control problem is a major contribution of this work. It is identified that the relative spacecraft state is not completely controllable with the Lorentz force alone, necessitating control strategies that combine conventional thruster actuation with the Lorentz force. Emphasis is placed on minimizing the thruster actuation and maximizing the use of the Lorentz force. Strategies that employ both continuous and impulsive thruster actuation with the Lorentz force are considered. Results show that the majority of the required actuation can be achieved using the Lorentz force.

Investigation of optimal impulsive thrusting with continuous Lorentz force actuation motivates the development of novel optimal control theory for linear time-varying systems with both continuous and impulsive inputs. The necessary conditions to minimize a hybrid quadratic performance index are derived. A continuous and a discrete Riccati equation are required to solve the optimal control problem: the former yields the continuous solution between impulsive actions; the latter provides a new boundary condition at impulsive application times. Necessary and sufficient conditions are derived for optimal impulsive application times. Numerical simulations validate the solutions.
DEDICATION

To my parents.
Acknowledgements

Thank you to my mother and father for their unwavering love, support and understanding, without which this work would not have been completed.

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I am grateful for my Ultimate team mates, with whom I relieved so much stress, battling it out on the fields at Lamport, Downsview or wherever else we ended up on a given night.

Lastly, Angela: Thank you.

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March, 2014
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**Acronyms**

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<tr>
<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>EDT</td>
<td>Electrodynamc tether</td>
</tr>
<tr>
<td>GCI</td>
<td>Geocentric Inertial</td>
</tr>
<tr>
<td>GVE</td>
<td>Gauss’s variational equations</td>
</tr>
<tr>
<td>HCW</td>
<td>Hill Clohessy-Wiltshire</td>
</tr>
<tr>
<td>HEO</td>
<td>Highly elliptical orbit</td>
</tr>
<tr>
<td>IGRF</td>
<td>International Geomagnetic Reference Field</td>
</tr>
<tr>
<td>LAO</td>
<td>Lorentz-augmented orbit</td>
</tr>
<tr>
<td>LEO</td>
<td>Low Earth orbit</td>
</tr>
<tr>
<td>LQR</td>
<td>Linear quadratic regulator</td>
</tr>
<tr>
<td>LVLH</td>
<td>Local vertical/local horizontal</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear time-invariant</td>
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<tr>
<td>LTV</td>
<td>Linear time-varying</td>
</tr>
<tr>
<td>MMS</td>
<td>Magnetospheric Multiscale</td>
</tr>
<tr>
<td>PCO</td>
<td>Projected circular orbit</td>
</tr>
<tr>
<td>SAR</td>
<td>Synthetic aperture radar</td>
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</table>
## Nomenclature

### Variables

- $a$: semimajor axis
- $b$: semiminor axis
- $b$: Magnetic field vector
- $e$: Eccentricity
- $e_0$: Elementary charge
- $e$: Orbital element vector
- $f$: True anomaly
- $h$: Specific angular momentum magnitude
- $h$: Specific angular momentum vector
- $i$: Inclination
- $j_{e/i}$: Electron/ion current density
- $k$: Boltzmann constant
- $m$: Mass
- $m$: Magnetic dipole vector
- $n$: Mean motion
- $n_{e/i}$: Electron/ion density
- $p$: semilatus rectum
- $q$: Electrical charge
- $r$: Orbit radius magnitude
- $r$: Position vector
- $\dot{r}$: Velocity vector
- $t$: Time
- $u$: Continuous input/control vector
- $v$: Impulsive input/control vector
- $B$: Magnetic field magnitude
- $C$: Capacitance
- $C_{21}$: Rotation matrix from frame 1 to frame 2
- $E$: Eccentric anomaly
- $F$: Vectrix
- $H_0$: Magnetic dipole strength
- $\mathcal{H}_{c/d}$: Continuous/discrete Hamiltonian
- $I$: Electrical current
- $J$: Cost
- $J_n$: $n^{th}$ zonal harmonic coefficient
- $M$: Mean anomaly
- $N$: Number of thrusters per orbit
- $R_{sc}$: Spacecraft radius
- $R_{\oplus}$: Earth’s equatorial radius
- $S$: Plasma sheath thickness
- $T$: Orbital period
- $T_{e/i}$: Electron/ion thermal energy
- $V_{sc}$: Spacecraft surface potential
- $W$: Controllability Gramian
- $\alpha$: In-plane phase angle
- $\beta$: Out-of-plane phase angle
- $\epsilon_0$: Permittivity of free space
- $\eta$: Eigenvector
- $\zeta$: Differential element error

### Operators

- $\dot{\cdot}$: Derivative with respect to $t$
- $\dot{\cdot}$: Derivative with respect to $t_k$
- $\delta(t - \tau)$: Dirac Delta function
- $\|\cdot\|$: Euclidean norm
- $e^{(\cdot)}$: Matrix exponential
- $(\cdot)^\times$: Skew symmetric matrix operator

### Superscripts

- $\cdot$: Mean quantity
- $(\cdot)^*$: Optimal quantity
- $(\cdot)^+$: Post-impulse quantity
- $(\cdot)^-$: Pre-impulse quantity
- $(\cdot)^\circ$: Unit quantity

### Subscripts

- $c$: Chief
- $d$: Deputy
- $h$: Normal component of LVLH frame
- $i$: Inertial reference frame
- $l$: Local vertical/local horizontal reference frame
- $k$: Impulsive action index
- $ns$: Non-singular element
- $o$: Orbital reference frame
- $r$: Radial component of LVLH frame
- $r$: Reference quantity
- $L$: Lorentz force-related
- $\theta$: Along-track component of LVLH frame
- $0$: Initial quantity
- $\oplus$: Earth-related
Chapter 1

INTRODUCTION

The core topic of this thesis is the control of spacecraft formations. True to the lab of its production, this work is an amalgam of research in the areas of orbital mechanics and control theory. The former is essential in understanding the behaviour of the relative motion of one spacecraft with respect to another and to understanding how that motion is perturbed. The latter is necessary to mitigate the effect of those perturbations and maintain a specific relative trajectory.

1.1 Formation Flying History and Prospects

While each chapter in this thesis provides a review of the existing research relevant to its topic, we begin here with a historical overview of spacecraft formation flight itself.

The precursor to autonomous spacecraft formation flight, and what began research into the relative motion of one spacecraft with respect to another, was the terminal guidance and rendezvous problem for two spacecraft. Research into this problem was motivated by the requirement of the Apollo moon mission for the command module to separate from its launch vehicle, circle around and dock with the lunar module contained within the launch vehicle, before departing to the moon as one spacecraft. This work would also be important for future space missions such as the Apollo-Soyuz Test Project. The linearized relative spacecraft motion for spacecraft in circular orbits was described by Clohessy and Wiltshire. Curiously, unbeknownst to Clohessy and Wiltshire at the time of their work, American astronomer George W. Hill derived the same differential equations to describe the relative motion of the Moon with respect to the Earth nearly 80 years prior.

It is important to distinguish spacecraft formation flight, which is the near- or fully-autonomous flight of several spacecraft flying in a particular relative geometry, with the concept of a constellation
of spacecraft, which involves having several similar, but independently acting spacecraft either spaced out along a common orbit or in complementary orbits.

At the end of 1990s, the completion of the Global Position System (GPS) spurred new research in guidance and navigation, resulting in innovations, such as Carrier-Phase Differential GPS (CDGPS) techniques for relative spacecraft state determination, that enabled the possibility of autonomous spacecraft flying. A number of early formation-flying demonstration missions, such as Orion, Emerald and TechSat21 were planned. Ambitious scientific missions requiring formation flight were also planned, such as the New Millennium Interferometry mission, which sought to implement a three-spacecraft optical stellar interferometer in heliocentric orbit with a variable baseline from 100 m – 1 km. Spacecraft formation flight turned out to be harder to realize than anticipated, as none of the missions cited above actually flew, whether for financial reasons or technical setbacks; however, the research resulting from their interrupted development laid the groundwork for future missions.

Some of the first successful formation-flying missions include the GRACE gravity mission, the TanDEM-X mission, and the PRISMA formation flying demonstration mission. Launched in 2002, the Gravity Recovery and Climate Experiment (GRACE) mission is a two-spacecraft formation flying mission designed to map the Earth’s mass distribution to an unprecedented accuracy. The two spacecraft flew in a leader-follower formation, separated by 200 km and coarsely controlled to ±50 km. The TanDEM-X mission, developed by the German Aerospace Centre (DLR) and Astrium GmbH, began in 2010 as the first Earth observation mission to employ formation flight. Together with the TerraSAR-X spacecraft, launched in 2007, TanDEM-X formed the first reconfigurable synthetic aperture radar (SAR) interferometer, designed to generate an accurate digital elevation model. Flying in a unique ‘HELIX’ formation, spacecraft separations varied from 200 m to 2 km and with a position error ranging from an average 5 m to 25 m. The Swedish Space Corporation’s PRISMA formation flying mission’s purpose was to demonstrate the next generation of precise, autonomous formation flying algorithms and technologies. Autonomous formation-keeping, rendezvous from large separations and proximity operations such as inspection within 5 m – 100 m of a target were demonstrated.

Interest in formation flight continues today, with a number of missions still being planned. The upcoming CanX-4/5 mission is Canada’s first foray into spacecraft formation flight. Similar in concept to the canceled Orion and Emerald missions, the mission has ambitious goals of demonstrating sub-metre level formation control of nanosatellites (mass < 10 kg), with launch expected in 2014. The Magnetospheric Multiscale (MMS) mission is an ambitious four-spacecraft NASA mission launching in late 2014 or early 2015 into a highly elliptical orbit. The tetrahedral formation will study the Earth’s magnetic field. PROBA-3 is the European Space Agency’s entry into the field of formation flight.
Chapter 1. Introduction

The two-spacecraft mission, also launching into a highly elliptical orbit, doubles as both a demonstration of precise autonomous formation control and a scientific solar coronagraph and occulter mission.\textsuperscript{16}

1.2 About This Thesis

Essential to any of the missions discussed above is a minimized $\Delta V$ budget. Mission life is almost always dictated by how long the propellant on a spacecraft will last. The underlying design goal for the formation control laws in this thesis is to minimize the thruster $\Delta V$ required to reconfigure a spacecraft formation, or to maintain metre-level spacecraft formation-keeping.

Motivated by this goal, control law formulation is done exclusively with mean differential orbital elements. Differential orbitals are the differences between the orbital elements of one spacecraft and another. The 'mean' element refers to the inclusion of the secular effect of the $J_2$ zonal harmonic in the dynamics of the orbital elements. Mean orbital elements are an attractive state representation for long-term spacecraft formation-keeping, since control laws based on mean element dynamics ignore any periodic, non-accumulating orbital element fluctuations and only correct for secular change. The resulting control laws result in a reduced $\Delta V$ for formation keeping over Cartesian-based control laws.

This thesis considers two kinds of formation control actuation:

1. conventional, impulsive thrust-based formation control;
2. control of a spacecraft formation using the geomagnetic Lorentz force.

The formulation of control laws based on these two means of spacecraft actuation and the evaluation of their respective performances constitutes a large portion of the work in this thesis. The chapters corresponding to each control methodology describe the state-of-the-art of each in detail. For now, only a brief introduction to each is given.

Impulse thrust formation control uses periodic thrusts to realize near-instantaneous velocity changes to affect a spacecraft’s relative motion with respect to another, rather than having some continuously-acting thruster make constant corrections. Impulse control is an appealing control paradigm since it provides spacecraft with large amounts of vibration-free time during which observations or measurements could be taken. Continuous thrusting is often impractical for spacecraft with pointing requirements if the attitude of the spacecraft must constantly change to align itself with the correct acceleration vector.

The use of the geomagnetic Lorentz force as a means of actuation requires a spacecraft to generate, store and modulate a significant electrical charge. The relative velocity of the charged spacecraft with
respect to the Earth’s magnetic field results in the spacecraft experiencing the geomagnetic Lorentz force with a strength proportional to the magnitude of the charge. For modest charge magnitudes, a spacecraft can experience a sufficiently strong Lorentz force to mitigate orbital perturbations that are normally detrimental to spacecraft formation geometries. Thus, the Lorentz force represents a propellantless means of spacecraft actuation, or at least, a means of significantly reducing the required thruster propellant for a mission. Formation-keeping strategies that combine the Lorentz force with thruster actuation are referred to as “Lorentz-augmented” strategies.

In researching Lorentz-augmented formation-keeping strategies, an interesting control problem arose, and whose solution has become another significant portion of this research. The problem is that of determining the optimal control of a system with continuous and impulsive inputs, and the optimal timing of those inputs.

1.2.1 Objectives

The research objectives of this thesis are presented below.

1. Regarding impulsive-thrust formation control:
   1.1 develop a generalized impulsive formation-keeping strategy for nonzero orbit eccentricities;
   1.2 ascertain stability of the developed impulsive formation-keeping strategy;
   1.3 quantify performance in comparison to existing strategies.

2. Regarding Lorentz-augmented formation flight:
   2.1 ascertain controllability of the Lorentz-augmented relative spacecraft state;
   2.2 develop Lorentz-augmented formation-keeping strategies;
   2.3 evaluate formation-keeping performance for different orbit inclinations;
   2.4 estimate power consumption for a spacecraft employing a Lorentz-augmented formation-keeping strategy.

3. Regarding the control of systems that have both continuous and impulsive control inputs:
   3.1 derive necessary conditions for an optimal control law;
   3.2 derive necessary and sufficient conditions for determining optimal application time of an impulsive control;
   3.3 synthesize controllers for Lorentz-augmented formation-keeping.
1.2.2 Outline

The work in this thesis draws upon fundamental concepts from linear systems theory, the calculus of variations and astrodynamics. Chapter 2 reviews the concepts of a dynamical system, reachability and controllability, particular as it applies to linear time-varying systems. It also provides a short summary of the calculus of variations, which is relied upon heavily in Chapter 6.

Chapter 3 presents a detailed review of the astrodynamics concepts that provide the basis for both the formulation of the control laws and the simulation of the spacecraft dynamics. Keplerian motion, orbit description using orbital elements (both classical and non-singular with respect to eccentricity), and dominant perturbations such as the higher order spherical harmonics of the Earth’s gravity field, atmospheric drag, and the geomagnetic Lorentz force are covered. The notion of “mean” orbital elements, whose dynamics are the basis for this work’s approach to formation control, is discussed in detail. Lastly, the relative dynamics of one spacecraft with respect to another are discussed both from a Cartesian and an orbital element perspective. The $J_2$ zonal harmonic is identified as the primary perturbation of concern for spacecraft formations.

Chapter 4 presents a $N$-impulse formation-keeping strategy based on mean differential elements, designed to mitigate the effect of the $J_2$ perturbation. It generalizes the impulse strategy presented by Vadali et al.\textsuperscript{17} that is only suitable for formations in circular orbits. The constraint equations are formulated in both the classical and non-singular differential elements. Novel contributions in this chapter include the generalization of the control strategy for nonzero orbit eccentricities, a stability analysis of the control strategy for $N = 2$ thrusts and a discussion of when using the circular orbit strategy is no longer suitable. Simulation results are presented for formations in both low and high eccentricity orbits.

Chapter 5 introduces the geomagnetic Lorentz force as an alternative means of spacecraft formation control. The total charge of the spacecraft normalized by its mass (a quantity known as the charge-to-mass ratio, or simply, the specific charge) is the sole control variable for the Lorentz force. Due to a fundamental lack of controllability that is demonstrated in the chapter, Lorentz force actuation must be used in concert with some thruster actuation. Three novel methods for combining the two actuation methods are proposed: (1) a geometric decomposition method that takes an acceleration control vector from a pre-designed control strategy and decomposes it into a vector that is parallel with the direction of the Lorentz force and a vector that is perpendicular to it, the latter being realized by the spacecraft by thrusters; (2) a linear quadratic regulator is designed to combine the continuous thruster actuation with the Lorentz force and; (3) an ad hoc strategy for combining impulsive thrusting with Lorentz force
actuation. Each control strategy uses a differential orbital element formulation. The effectiveness of each strategy is demonstrated in simulation, with a detailed look at how performance of the different control strategies changes with orbit inclination.

The poor performance of the ad hoc continuous/impulsive control strategy from Chapter 5 led to the question of how to combine continuous and impulsive control input in an optimal fashion. The problem for spacecraft flight formation is momentarily put aside in Chapter 6 in order to develop the required control theory to respond to this question. The research in this chapter extends and, in some cases, fills in the gaps of existing hybrid optimal control theory. The term “hybrid” is a criminally overused term in the field of control — or perhaps just in the Spacecraft Dynamics and Control Lab — so to be clear, when it is used in this thesis, it refers to the combined use of continuous and impulsive inputs. This chapter presents a full derivation of the necessary conditions for the optimal control of a system with continuous and impulsive inputs leading to the developing of a hybrid linear quadratic regulator (LQR). The optimization of the impulsive application time is also considered. A variational approach is taken to develop necessary and sufficient conditions for a minimum with respect to application time. Novel contributions include explicitly showing how the Riccati equations behaves at impulsive application times, the derivation of the sufficient condition for a minimum with respect to impulse application time, and hybrid control synthesis for single- and multi-state systems.

Chapter 7 investigates the application of hybrid optimal control to the spacecraft formation flying problem. A hybrid LQR with prescribed impulsive application time is designed for the formation-keeping problem, while a hybrid LQR with optimal impulse application times is applied to a formation reconfiguration example. In both examples, the state penalty weights for the hybrid LQRs are chosen such that the Lorentz targets only the states that it can control, while the impulsive thrusts target the states that are uncontrollable. Some discussion is presented on the topic of controller synthesis for the optimally-timed hybrid LQR, as it not trivial. Performance of the hybrid LQR at different inclinations is discussed and comparisons are drawn between it and the previous formation-keeping strategies from Chapters 4 and 5.

Chapter 8 departs from the formation-keeping control problem in order to provide a closer look at the power requirements needed by a formation-flying spacecraft using a Lorentz-augmented formation-keeping strategy. A simple spacecraft charging model is presented. Spacecraft architecture is based on a Lorentz-augmented spacecraft concept originally proposed by Peck. In this work, it is assumed that charge regulation is achieved through some form of particle-beam emission. Due to the complexity of modeling spacecraft charging in low Earth orbit, it is not expected that the obtained power estimates are very accurate, but rather, serve to identify the order of magnitude of power required by a Lorentz-
augmented spacecraft as well as identify complicating factors for charge maintenance and regulation.

1.2.3 About the Simulation Results

Simulation results are an integral part of the work in this thesis. So, a discussion on their development and implementation is warranted. Early on in this research FORTRAN was chosen to be the language in which the simulations would be programmed. It is a language designed for scientific computation and remains a preferred language for such applications. There exists a large amount of freely available mathematical subroutines packages such as LAPACK (Linear Algebra PACKage)\textsuperscript{19} and Expokit\textsuperscript{20} as well as several astrodynmic models such as the International Geomagnetic Field\textsuperscript{21} and the International Reference Ionosphere model\textsuperscript{22} that can be leveraged.

All the spacecraft formation flying simulation results are based on the numerical integration of the inertial equations of motion, including the corresponding disturbance forces (Eq. (3.1), see Chapter 3). A custom-coded, adaptive-step Dormand-Prince\textsuperscript{23} differential equation solver is used to integrate the equations of motion. The same algorithm was used for solving the time-varying Riccati equations as well. Post-processing of the data as well as all figure generation was done in MATLAB\textsuperscript{®}. Spacecraft charging simulations was also performed using MATLAB\textsuperscript{®} and Simulink\textsuperscript{®}. 
Chapter 2

Mathematical Preliminaries

This chapter reviews some fundamental concepts regarding dynamical systems, their linear approximation, as well as concepts pertaining to the calculus of variations. These concepts are used throughout this thesis. Material in the chapter is taken from Refs. 24–26.

2.1 Dynamic Systems

The first mathematical description of a nonlinear, physical system is often given in the form of an $n$-component vector of ordinary, continuous, differential equations of the form

$$\dot{x}(t) = f(x(t), u(t), t),$$  \hspace{1cm} (2.1)

defined on the interval $t \in [t_0, t_f]$, where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathcal{U}$ is an input vector affecting the state. Eq. (2.1) is often called the state equation and associated with it is the output equation,

$$y(t) = g(x(t), u(t), t).$$ \hspace{1cm} (2.2)

Together, Eq. (2.1) and Eq. (2.2) are known as the system equations. For some given initial conditions,

$$x(t_0) = x_0,$$ \hspace{1cm} (2.3)

and a given control history $u(t)$, for $t \in (t_0, t_1]$, the function $\varphi$ is the transition function that gives the state at some time $t_1$

$$x(t) = \varphi(x(t_0), u(t), t_1).$$ \hspace{1cm} (2.4)
Assuming that there is a reference or target state \( x_r \) that we wish our state to arrive at in some finite time, then the control problem is that of finding a control \( u(t) \), \( t \in (t_0, t_1] \) such that

\[
x_r = \varphi(x(t_0), u(t), t_1)
\]

for \( t_1 < \infty \).

The notions of reachability and controllability are fundamental to the control problem. Supposing for a given system there is a set of admissible controls, \( \mathcal{U} \), which is a bounded set in \( \mathbb{R}^m \), then a state \( x_r \) is said to be reachable from an initial state \( x_0 \) if there exists a control \( u(t) \in \mathcal{U} \) that satisfies Eq. (2.5). If a state \( x_1 = 0 \) is reachable from \( x(t_0) \) at \( t_0 \), then \( x(t_0) \) is said to be controllable at \( t_0 \), i.e., if a piecewise continuous function \( u(t) \in \mathcal{U} \) exists such that

\[
0 = \varphi(x(t_0), u(t), t_f)
\]

for a time \( t_f \in [t_0, \infty) \), then \( x(t_0) \) at \( t_0 \) is controllable. More generally, a system is said to be controllable if every state \( x_0 \) at any time \( t \) in the interval on which the system is defined, is controllable.

### 2.1.1 Linear Systems

These concepts applied to linear time-varying (LTV) and linear time-invariant systems are used throughout this thesis. Consider an LTV system described by

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (2.7)
\]

\[
y(t) = C(t)x(t) + D(t)u(t).
\]

The transition function, or general solution, for such a system is

\[
x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau,
\]

where \( \Phi(t, t_0) \) is the state transition matrix. For LTV systems, the state transition matrix is obtained by solving the matrix differential equation

\[
\frac{d\Phi(t, t_0)}{dt} = A(t)\Phi(t, t_0),
\]

\[
(2.10)
\]
often by numerical integration, with \( \Phi(t_0, t_0) = 1_{n \times n} \). For an LTI system, the state transition matrix is simply the matrix exponential
\[
e^{A(t-t_0)} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (t-t_0)^k.
\]

### Controllability

The controllability of an LTV system is determined by considering the matrix function
\[
W(t_0, t) = \int_{t_0}^{t} \Phi(t, \tau)B(\tau)B^T(\tau)\Phi^T(t, \tau) d\tau.
\]

An LTV system is controllable if the matrix \( W(t_0, t) \), commonly known as the controllability Gramian, is invertible for all \( t_0 < t < \infty \). The inverse of the controllability Gramian, \( W^{-1}(t_0, t) \), can be shown to be related to the minimum energy needed to transfer a system from one state to another. If the inverse does not exist, then the state cannot be driven to a desired state, regardless of the energy input into the system.

In most cases, the critical question is simply whether or not a system is controllable. A non-invertible controllability Gramian has at least one eigenvalue equal to zero. The number of zero eigenvalues indicates how many states of the system are controllable.

### 2.2 Calculus of Variations

The calculus of variations provides the necessary tools for the derivation of control laws in Chapter 6. Its fundamental concepts are summarized here.

Consider a cost function \( J(x) \) that maps the vector \( x(t) \in \mathcal{X} \) to \( \mathbb{R} \). Additionally, let \( \mathcal{X} \) be the vector space of all continuously differentiable functions that map a scalar \( t \in \mathbb{R} \) to \( \mathbb{R}^n \). Thus, \( J \) is function that maps a function space to the set of real numbers; such a function is known as a functional.

Let the vector function \( x^*(t) \) be a local minimum of \( J \) such that
\[
J(x^*) \leq J(x)
\]
for all \( x(t) \) sufficiently close to \( x^*(t) \). Let us approximate the behaviour of \( J \) near \( x^*(t) \) by
\[
x^* + D(x - x^*),
\]
where
\[
D(x - x^*)
\]
where $\mathcal{D}$ is a linear operator mapping $x(t) \in \mathcal{X}$ to $\mathbb{R}$ and has the property

$$\lim_{x \to x^*} \frac{|f(x) - f(x') - \mathcal{D}(x - x')|}{||x - x'||} = 0.$$  \hspace{1cm} (2.15)

Such a linear operator $\mathcal{D}$ is the derivative of $J$ at $x^*$. If the minimum $x^*(t)$ is perturbed by $\epsilon \eta(t)$, where $\epsilon$ is a small, real number and $\eta(t) \in \mathcal{X}$ is an arbitrary vector function, then if $\mathcal{D}$ exists, the perturbed cost functional can be written as

$$J(x^* + \epsilon \eta) = J(x^*) + \mathcal{D}(\epsilon \eta) + o(\epsilon),$$  \hspace{1cm} (2.16)

where $o(\epsilon)$ has the property

$$\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0.$$

We call $\mathcal{D} = \delta J(x^*)$ the first variation of $J$. If $J(x^* + \epsilon \eta)$ is considered a function of $\epsilon$,

$$f(\epsilon) = J(x^* + \epsilon \eta),$$  \hspace{1cm} (2.17)

then since $J(x^*)$ is a local minimum, $f(\epsilon)$ must have a minimum at $\epsilon = 0$. It follows that

\[
\left. \frac{df(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{|f(x^* + \epsilon \eta) - J(x^*)|}{\epsilon} = \lim_{\epsilon \to 0} \frac{|\epsilon \delta J(x^*)\eta + o(\epsilon)|}{\epsilon} = |\delta J(x^*)\eta| = 0.
\]

We conclude that since $\eta(t)$ is arbitrary, a necessary condition for a local minimum is

$$\delta J(x^*) = 0.$$  \hspace{1cm} (2.18)

It should be noted that for $x \in \mathcal{X}$, where $\mathcal{X}$ is a subset of $\mathbb{R}^n$, a local minimum can also occur at points on the boundary of $\mathcal{X}$ and where $\delta J$ does not exist. While Eq. (2.18) is a necessary condition for a minimum, it is not sufficient for a minimum. On its own, it only indicates an extremum of $J$.

A second condition is needed to differentiate between minima and maxima. To obtain such a condition, we introduce a function, $\mathcal{P}(x, x)$, known as the inner product, which maps $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ and satisfies:

1. $\mathcal{P}(x_1 + x_2, x_3) = \mathcal{P}(x_1, x_3) + \mathcal{P}(x_2, x_3)$
2. $\mathcal{P}(\alpha x_1, x_2) = \alpha \mathcal{P}(x_1, x_2)$
3. \( P(x_1, x_2) = P(x_2, x_1) \)

For an \( x \) close to \( x^* \), if an inner product \( P \) exists, the functional \( J \) has a second derivative that is the quadratic form induced by \( P \), so that we approximate \( x \) by

\[
x = x^* + D(x - x^*) + P(x - x^*, x - x^*). \tag{2.19}
\]

Then, the functional that is the quadratic form induced by \( P \) is also called the second variation of \( J \) at \( x^* \) and we write

\[
P(x - x^*, x - x^*) = \delta^2 J(x^*)(x - x^*). \tag{2.20}
\]

If the minimum \( x'(t) \) is again perturbed by \( \epsilon \eta(t) \), where \( \epsilon \) is a small, real number and \( \eta(t) \in X \) is an arbitrary vector function, then we write

\[
J(x' + \epsilon \eta) = J(x') + \epsilon \delta J(x') \eta + \epsilon^2 \delta^2 J(x') \eta + o(\epsilon^2) \tag{2.21}
\]

where

\[
\lim_{\epsilon \to 0} \frac{o(\epsilon^2)}{\epsilon^2} = 0.
\]

Since \( x' \) is a minimum, it must hold that \( \delta J(x') = 0 \). Ignoring \( o(\epsilon^2) \), it follows then that

\[
J(x' + \epsilon \eta) - J(x') = \epsilon^2 \delta^2 J(x') \eta \geq 0 \tag{2.22}
\]

Since \( \epsilon^2 > 0 \), then a sufficient condition for \( x' \) to be a local minimum of \( J \) is that the second variation, \( \delta^2 J(x') \), must be positive definite, i.e.,

\[
\delta^2 J(x') \geq 0. \tag{2.23}
\]

### 2.2.1 The Minimum Principle

The variational principles can be readily applied to the optimal control problem to derive some form of Pontryagin’s Minimum Principle. Different flavours of the principle exist depending on, among other things, whether the the terminal state is specified, or in some final terminal set, \( S \); or whether the terminal state is free, or varying with the terminal time; or whether some of it is penalized, etc. Below, the Minimum Principle is derived for a general LTV system, with a fixed end time, \( t_f \), and a terminal cost.

Consider the LTV system described by Eq. (2.7) and Eq. (2.8) and a control interval of \( t \in [t_0, t_f] \).
We assume a scalar, continuous function $L(x(t), u(t))$ and some cost for the terminal state, $K(x(tf))$ and desire to minimize

$$J(x(t), u(t)) = K(x(tf)) + \int_{t_0}^{tf} L(x(t), u(t)) \, dt.$$  \hfill (2.24)

Adjoining the dynamics of the system with a vector of Lagrange multipliers, $\lambda(t)$, we obtain the following constrained cost functional

$$J(x(t), u(t)) = K(x(tf)) + \int_{t_0}^{tf} L(x(t), u(t)) + \lambda^T(t)((A(t)x(t) + B(t)u(t) - \dot{x}(t)) \, dt,$$  \hfill (2.25)

In the case of these dynamics constraints, we do not known whether an optimal solution necessarily exists, but we proceed under the assumption of a minimum and proceed to develop necessary conditions for one. We begin by perturbing slightly the optimal control trajectory such that

$$u(t) = u^*(t) + \epsilon \eta(t),$$  \hfill (2.26)

where $\epsilon$ and $\eta$ are a small number and an arbitrary vector, respectively. Consequently, small perturbations are induced on the optimal state trajectory, such that

$$x(t) = x^*(t) + \epsilon \psi(t),$$  \hfill (2.27)

and

$$\dot{x}(t) = \dot{x}^*(t) + \epsilon \dot{\psi}(t),$$  \hfill (2.28)

since small variations in the control cause small variations of the state. The functions $\psi(t)$ and $\eta(t)$ are continuous on the control interval and at the endpoints $\eta(t_0) = \eta(tf) = 0$ and $\psi(t_0) = 0$. The expressions $\delta x(t) = \epsilon \psi(t)$ and $\delta u(t) = \epsilon \eta(t)$ are called the variations of $x(t)$ and $u(t)$, respectively. Next, we define the Hamiltonian function as

$$H(x(t), u(t), \lambda(t)) = L(x(t), u(t)) + \lambda^T(t)((A(t)x(t) + B(t)u(t) - \dot{x}(t))$$  \hfill (2.29)

In what follows, the arguments of the Hamiltonian are omitted for brevity, but it should be understood that $H = H(x(t), u(t), \lambda(t))$. Furthermore, when the Hamiltonian is evaluated at optimal quantities, it is written as $H = H^* = H(x^*(t), u^*(t), \lambda^*(t))$. Following the same methodology as the previous section, we calculate the first variation of the cost
functional, $\delta J$, with a first order Taylor series expansion:

$$
\delta J(x'(t), u'(t)) + o(\epsilon) = J(x(t), u(t)) - J(x'(t), u'(t))
$$

$$
= \int_{t_0}^{t_f} \left[ H(x(t), u(t), \lambda(t)) - \lambda^T(t) \dot{x}(t) - \left( H(x'(t), u'(t), \lambda(t)) - \lambda^T(t) \dot{x}'(t) \right) \right] dt + H(x(t_f)) - H(x'(t_f)) + o(\epsilon)
$$

$$
= \epsilon \int_{t_0}^{t_f} \left( \frac{\partial H}{\partial x} \psi(t) + \frac{\partial H}{\partial u} \eta(t) - \lambda^T(t) \dot{\psi}(t) \right) dt + \epsilon \frac{\partial H}{\partial x} \psi(t_f) + o(\epsilon)
$$

$$
= \epsilon \int_{t_0}^{t_f} \left( \frac{\partial H}{\partial x} + \lambda^T(t) \right) \psi(t) + \frac{\partial H}{\partial u} \eta(t) \right) dt + \epsilon \left( \lambda^T(t_0) \dot{\psi}(t_0) - \lambda^T(t_f) \dot{\psi}(t_f) + \frac{\partial H}{\partial x} \dot{\psi}(t_f) \right) + o(\epsilon).
$$

We know that at an extremum the first variation, $\delta J = 0$. So,

$$
0 = \epsilon \int_0^T \left[ \left( \frac{\partial H}{\partial x} + \lambda^T(t) \right) \psi(t) + \frac{\partial H}{\partial u} \eta(t) \right] dt + \epsilon \left( \lambda^T(t_0) \dot{\psi}(t_0) - \lambda^T(t_f) \dot{\psi}(t_f) + \frac{\partial H}{\partial x} \dot{\psi}(t_f) \right) + o(\epsilon)
$$

(2.30)

must hold. Ignoring the residual higher-order term $o(\epsilon)$, the following conditions for an extremum are obtained:

$$
\dot{\lambda}(t) = -\frac{\partial H}{\partial x}^T
$$

(2.31)

$$
\frac{\partial H}{\partial u} = 0
$$

(2.32)

along with the boundary condition

$$
\lambda(t_f) = \frac{\partial H}{\partial x}^T.
$$

(2.33)

Let us consider for a moment how the minimized Hamiltonian varies with time:

$$
\frac{d H^*}{dt} = \frac{\partial H^*}{\partial t} + \frac{\partial H^*}{\partial x} \dot{x}(t) + \frac{\partial H^*}{\partial u} \dot{u}(t) + \frac{\partial H^*}{\partial \lambda} \dot{\lambda}(t)
$$

(2.34)

Taking the partial derivative of the Hamiltonian with respect to the costates yields the dynamics of the system

$$
\frac{\partial H}{\partial \lambda} = \dot{\lambda}(t)
$$

(2.35)

Substituting both the above equation and Eq. (2.31) into Eq. (2.34) yields

$$
\frac{d H^*}{dt} = \frac{\partial H^*}{\partial t} - \lambda^* t \dot{x}'(t) + \dot{x}'^T(t) \dot{\lambda}'(t) + \frac{\partial H^*}{\partial u} \dot{u}'(t) = 0
$$

The partial derivative of the Hamiltonian with respect to time vanishes because the Hamiltonian is not an explicit function of time. The partial derivative with respect to the control also vanishes, due to the
selection of the optimal control. We conclude then, that if the Hamiltonian is not an explicit function of time, then the optimal Hamiltonian is invariant with respect time over the control interval, \( t \in [t_0, t_f] \).

We conclude by stating the Minimum Principle with a terminal cost at a fixed end time. Let \( u^*(t) \) be a control in the set of admissible controls \( U \), such that it transfers the state \( x(t_0) = x_0 \) to some unspecified final state \( x^*(t_f) = x_f \). Let \( x^*(t) \) be the corresponding state trajectory for the control interval \( t \in [t_0, t_f] \).

For \( u(t) \) to be a control that minimizes the cost functional \( J(x, u) \), a function \( \lambda^*(t) \) must exist such that:

1. The states, \( x^*(t) \) and costates, \( \lambda^*(t) \) are solutions to the equations
   \[
   \dot{x}^*(t) = \frac{\partial H^*}{\partial \lambda},
   \]
   \[
   \dot{\lambda}^*(t) = -\frac{\partial H^*}{\partial x},
   \]
   respectively, and have the boundary conditions
   \[
   x(t_0) = x_0, \quad x(t_f) \text{ free;}
   \]
   \[
   \lambda(t_0) \text{ free,} \quad \lambda(t_f) = \frac{\partial K}{\partial x}^T.
   \]

2. The Hamiltonian, \( H(x(t), u(t), \lambda(t)) \), as a function of \( u(t) \), has a minimum as at \( u(t) = u^*(t) \) for \( t \in [t_0, t_f] \), i.e.,
   \[
   H(x^*(t), u^*(t), \lambda^*(t), t) \leq H(x(t), u(t), \lambda(t), t) \quad \text{for all} \ u(t) \text{ in} \ U.
   \]

3. The Hamiltonian satisfies the relation
   \[
   H(x^*(t), u^*(t), \lambda^*(t)) = H(x^*(t_f), u^*(t_f), \lambda^*(t_f)) = \int_{t_f}^{t} \frac{\partial H(x^*(\tau), u^*(\tau), \lambda^*(\tau))}{\partial t} d\tau
   \]
   \[
   = H(x^*(t_f), u^*(t_f), \lambda^*(t_f)).
   \]

A number of proofs of the Minimum Principle can be found in Ref. 26.
Chapter 3

Orbital and Spacecraft Formation Dynamics

A review of fundamental orbital mechanics is presented in this chapter. A discussion of orbital perturbations is also given, with a focus on gravitational perturbations due to the Earth’s oblateness, atmospheric drag and the influence of the Earth’s magnetic field on a charged spacecraft. Following this, the dynamics of spacecraft formation flying are introduced, both from a relative Cartesian and a differential orbital element point of view. Much of the material is taken from Refs. 27, 28.

3.1 Orbital Mechanics

For a spacecraft orbiting the Earth, the equation governing the motion of the spacecraft is

\[ \ddot{r} = -\frac{\mu_{\odot}}{r^3} r + f_d, \]  

where \( r \) is the position vector of spacecraft, \( \mu_{\odot} \) is the Earth’s gravitational constant, the Earth is assumed to be a point mass, and \( f_d \) is the disturbance force vector, which is the sum of any additional disturbance forces. While Cartesian coordinates in an inertial reference frame are well suited for orbit simulations, since they give a position and velocity for only one instant in time, they offer little insight into the overall path of the spacecraft. Fortunately, a well-known analytical solution to the spacecraft’s motion exists for the case of no disturbance forces, i.e., \( f_d = 0 \), and was first described by the German mathematician/astronomer Johannes Kepler. The motion can be described by either an ellipse, parabola
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or hyperbola (collectively known as conic sections), with one focus at the central body (e.g., the Earth). The latter two possibilities describe the path of a spacecraft leaving or entering the vicinity of the central body and are not considered further here. The ellipse describes the path of the spacecraft as it orbits the central body. The size and shape of an ellipse can be described by two quantities: semimajor axis, \( a \), and eccentricity, \( e \). The semimajor axis is a quantity of length and governs the size of the ellipse, whereas eccentricity varies from 0 to 1 and describes the degree to which the ellipse is a circle. When \( e = 1 \) then the motion is parabolic and for \( e > 1 \) the motion is hyperbolic. The geometry of the ellipse is illustrated in Fig. 3.1. The ellipse can be described by the set of polar coordinates, \( r, f \), where the radius, \( r \) is

\[
r = \frac{a(1 - e^2)}{1 + e \cos f}
\]  

(3.2)

and the angle \( f \) is the true anomaly. The quantity \( p = a(1 - e^2) \) is known as the semilatus rectum and appears frequently in orbital mechanics. The semiminor axis of the ellipse is \( b = a \sqrt{1 - e^2} \). The parameter \( \eta := \sqrt{1 - e^2} \) is defined.

The semimajor axis and eccentricity are two quantities obtained from fundamental constants of integration of Eq. (3.1). Another important constant of integration is the angular momentum per unit mass vector of a spacecraft

\[
\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}.
\]  

(3.3)

There are a number of different expressions for its magnitude:

\[
h = \sqrt{\mu} \sqrt{p},
\]  

(3.4)

\[
= r^2 \dot{f}.
\]  

(3.5)
The motion of the spacecraft in an orbit is periodic and the time needed to complete an orbit is given by:

$$T = 2\pi \sqrt{\frac{a^3}{\mu_{\oplus}}}.$$  \hfill (3.6)

The quantity $n = \sqrt{\mu_{\oplus}/a^3}$ is known as the mean motion. Note that, in general, the mean motion is not equal to the time derivative of the true anomaly; however, a spacecraft can relate its current true anomaly, which does not necessarily vary linearly with time, to the mean anomaly, $M$, which does vary linearly with time, with a rate given by the mean motion. The mean anomaly does not, in general, represent a physical quantity in the geometry of an orbit, except in the case of circular orbits, where it is equal to the true anomaly. The mean anomaly at a time $t$, measured from some initial time $t_0$ is given by:

$$M = n(t - t_0).$$  \hfill (3.7)

The true anomaly and mean anomaly are related via a third angle known as the eccentric anomaly, $E$. The true and eccentric anomalies are related via

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$  \hfill (3.8)

and the relation between the mean and eccentric anomalies is

$$M = E - e \sin E.$$  \hfill (3.9)

The orbit of the spacecraft has so far been described only in its orbital plane. What remains is to orient the orbit in an inertial frame fixed at the centre of the Earth. The Geocentric Inertial (GCI) frame is defined by the one-axis pointing in the direction of the vernal equinox, the three-axis aligned with the geographic north pole and the two-axis completes the right-hand rule. Three angles are needed to locate the orbit in the GCI frame. The inclination, $i$, describes the angle between the three-axis of the GCI frame and the orbit’s angular momentum vector. The right ascension of the ascending node, $\Omega$, is the angle between the one-axis of the GCI frame and the vector extending from the centre of the Earth to the point on the orbit where it crosses the equator, moving from south to north. The last angle, the argument of periapsis, $\omega$, is the angle from the right ascension vector to the periapsis vector – the point on the orbit with minimum radius – and is measured in the plane of the orbit. These angles are illustrated in Fig. 3.2. The transformation from the GCI frame, $\mathcal{F}_I$, to the orbital frame, $\mathcal{F}_O$, can be achieved with a 3-1-3 Euler rotation sequence where $\Omega$, $i$ and $\omega$ are the corresponding Euler angles.
The orbital element set \( e = [a \ e i \ \Omega \ \omega \ M]^T \) fully describes the current state of a spacecraft in orbit. This set is the classical set of orbital elements. Alternative orbital element sets also exist, such as the orbital element set \( e_{ns} = [a \ i \ q_1 \ q_2 \ \theta]^T \), where

\[
q_1 = e \cos \omega,
q_2 = e \sin \omega,
\theta = \omega + f,
\]

and serves to eliminate the singularities that arise when \( e = 0 \). The element \( \theta \) is known as the true latitude and the elements \( q_1 \) and \( q_2 \) are a polar representation of the eccentricity vector, \( e = [e \cos \omega \ e \sin \omega]^T \). For an element that changes linearly with the time, the mean latitude \( \lambda = M + \omega \) can be used in place of \( \theta \).

**3.1.1 Orbit Perturbations**

The effect of a perturbation on an orbit, whether in the form of a natural disturbance or a control force, is central to this thesis. So far, we have only considered Keplerian orbits, i.e., \( f_d = 0 \). When the disturbance forces are nonzero, the effect on the orbit geometry is not immediately obvious when considering the inertial Cartesian state of a spacecraft. A more practical approach is to understand how perturbing accelerations influence a spacecraft’s orbital elements. The French mathematician and astronomer Joseph-Louis Lagrange originally developed the method of variation of parameters to study the disturbed motion of two bodies and used it to derive his set of planetary equations.27 It
was German mathematician Carl Gauss, however, who extended Lagrange’s planetary equations and explicitly related accelerations expressed in a spacecraft orbital frame to changes in the classical orbital elements. It is Gauss’s variational equations (GVE) that are used in this thesis and are given as

\[
\begin{bmatrix}
\frac{da}{dt} \\
\frac{de}{dt} \\
\frac{di}{dt} \\
\frac{d\Omega}{dt} \\
\frac{d\omega}{dt} \\
\frac{dM}{dt}
\end{bmatrix} =
\begin{bmatrix}
\frac{2a^2 e \sin f}{h} & \frac{2a^2 p}{rh} & 0 \\
\frac{p \sin f}{h} & \frac{(p + r) \cos f + re}{h} & 0 \\
0 & 0 & \frac{r \cos \theta}{h} \\
0 & 0 & \frac{r \sin \theta}{h \sin \iota} \\
-\frac{p \cos f}{he} & \frac{(p + r) \sin f}{he} & -\frac{r \sin \theta}{h \tan \iota} \\
\frac{b(p \cos f - 2re)}{ahe} & \frac{b(p + r) \sin f}{ahe} & 0
\end{bmatrix}
\begin{bmatrix}
u_r \\
u_\theta \\
u_h
\end{bmatrix}.
\] (3.10)

Here \(u_r\) are components of a perturbing acceleration and \(\cdot\), \(\cdot\)\(_r\) and \(\cdot\)\(_h\) refer to the radial, along-track and normal directions of the local spacecraft orbital reference frame. The GVE are represented by the notation \(\dot{e} = B(e)u(t)\), with it being understood that the GVE vary with time, according to the true anomaly, although the time argument is not explicitly written. The temporal derivatives of the GVE are important in the optimization problems posed in later chapters. They are calculated using the chain rule and with the assumption that all orbital elements are time-invariant, with the exception of true anomaly:

\[
\frac{dB(e)}{dt} = \frac{\partial B(e)}{\partial f} \frac{df}{dt}.
\] (3.11)
where

\[
\frac{\partial \mathbf{B}(e)}{\partial f} = \begin{bmatrix}
\frac{2a^2 e \cos f}{h} & -\frac{2a^2 e \sin f}{h} & 0 \\
\frac{p}{h} \cos f & -\frac{p}{h} \sin f \left(1 + \frac{1 - e^2}{(1 + e \cos f)^2}\right) & 0 \\
0 & 0 & \frac{p}{h} \left(-\frac{\sin \theta}{1 + e \cos f} + \frac{e \sin f \cos \theta}{(1 + e \cos f)^2}\right) \\
0 & 0 & \frac{p}{h \sin i} \left(\frac{\cos \theta}{1 + e \cos f} + \frac{e \sin f \sin \theta}{(1 + e \cos f)^2}\right) \\
\frac{p}{he} \sin f & \frac{p}{he} \left(\frac{\cos f + \cos f + e}{1 + e \cos f}\right) & -\frac{p}{he} \left(\frac{\cos f + e}{1 + e \cos f}\right)
\end{bmatrix}
\]

(3.12)

and from Eq. (3.5), \(df/dt = h/r^2\).

The corresponding variational questions for the non-singular orbital elements \(q_1, q_2, \lambda\), are formed by taking the temporal derivatives of their respective relationships to the classical elements and then substituting in the appropriate classical GVE. The resulting variational equations are given by

\[
\begin{bmatrix}
\frac{dq_1}{dt} \\
\frac{dq_2}{dt} \\
\frac{d\lambda}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{p \sin \theta}{h} & \frac{(p + r) \cos \theta + re \cos \omega}{h} & \frac{re \sin \omega \sin \theta}{h} \\
\frac{-p \cos \theta}{h} & \frac{(p + r) \sin \theta + re \sin \omega}{h} & \frac{-re \cos \omega \sin \theta}{h} \\
\frac{pe \cos f}{h(1 + \eta)} - \frac{2r \eta}{h} & \frac{(p + r) e \sin f}{h(1 + \eta)} & \frac{-r \sin \theta}{h \tan i}
\end{bmatrix} \begin{bmatrix}
u_r \\
u_\theta \\
u_\phi
\end{bmatrix}.
\]

(3.13)

### 3.1.2 Disturbance Forces

#### Earth’s Gravity

For many orbital dynamics applications, it does not suffice to approximate the Earth’s gravitational influence as a point mass; the spatial distribution of its mass and its mass density must be accounted for. The gravitational field of a near-spherical Earth can be modeled by considering the effect of an infinitesimal piece of mass on the Earth, \(dm\), and integrating that result over the volume of the Earth to obtain a gravitational potential, \(V\). Doing so and expressing the potential in terms of spherical
harmonics, according to Ref. 29, yields

\[ V = \frac{\mu_\oplus}{r} \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{R_\oplus}{r} \right)^n P_{n,m} (\sin \phi) \left( C_{n,m} \cos(m\lambda) + S_{n,m} \cos(m\lambda) \right) \right], \quad (3.14) \]

where \( P_{n,m} \) are the associated Legendre polynomials and \( \lambda \) and \( \phi \) are the spacecraft’s longitude and latitude, respectively. The coefficients \( C_{n,m} \) and \( S_{n,m} \) must be empirically determined and describe the Earth’s mass distribution. They are particular to the gravity model being employed. The gravitational force can be determined by taking the gradient of the gravitational potential:

\[ \mathbf{f}_g = -\nabla V(\mathbf{r}). \quad (3.15) \]

The zonal harmonics correspond to \( J_n = -C_{n,0} \). The \( J_2 \) harmonic, caused by the oblateness of the Earth, is the dominant harmonic and can cause significant perturbations on the orbit of spacecraft in low Earth orbit. Brouwer\textsuperscript{30} identified that the zonal harmonics introduce short- and long-period oscillations in the orbital elements of a spacecraft as well as cause secular growth. If one is interested in the long-term evolution of an orbit, Brouwer showed that the effect of the short and long-term oscillations can be removed, leaving only the secular growth. The result is a set of mean orbital elements that approximates the true set of osculating orbital elements with mean values that change linearly with time. The mean rates of change of the classical mean orbital elements are

\[ \dot{a} = 0, \quad (3.16) \]
\[ \dot{e} = 0, \quad (3.17) \]
\[ \dot{i} = 0, \quad (3.18) \]
\[ \dot{\Omega} = \frac{3}{2} J_2 \frac{\hat{n}}{\hat{p}} \left( \frac{R_\oplus}{\hat{p}} \right)^2 \cos \hat{i}, \quad (3.19) \]
\[ \dot{\omega} = \frac{3}{4} J_2 \frac{\hat{n}}{\hat{p}} \left( \frac{R_\oplus}{\hat{p}} \right)^2 (5 \cos^2 \hat{i} - 1), \quad (3.20) \]
\[ \dot{M} = \dot{n} + \frac{3}{4} J_2 \frac{\hat{n}}{\hat{p}} \left( \frac{R_\oplus}{\hat{p}} \right)^2 (3 \cos^2 \hat{i} - 1). \quad (3.21) \]
The mean drift rates for the non-singular elements $\dot{q}_1$, $\dot{q}_2$ and $\dot{\lambda}$ are

$$\dot{q}_1 = \dot{\epsilon} \cos \omega - \dot{\epsilon} \sin \omega \dot{\omega}$$

$$= - \dot{q}_2 \dot{\omega} \tag{3.22}$$

$$\dot{q}_2 = \dot{\epsilon} \sin \omega + \dot{\epsilon} \cos \omega \dot{\omega}$$

$$= \ddot{q}_1 \dot{\omega} \tag{3.23}$$

$$\dot{\lambda} = \dot{M} + \dot{\omega} \tag{3.24}$$

To obtain a spacecraft’s inertial position and velocity from a set of mean orbital elements, one must first convert the mean elements to the correct osculating elements. In Ref. 28, Schaub and Junkins present a first-order mapping algorithm based on Brouwer theory between the classical osculating and mean orbital element sets.

**Atmospheric Drag**

Atmospheric drag, caused by the collision of atmospheric particles against a spacecraft, is a significant orbital perturbation for spacecraft in low Earth orbit. It is a function of a number of factors including the relative velocity of the spacecraft with respect to the atmosphere, the density of the atmosphere and both the ballistic coefficient of the spacecraft and its projected surface area in the direction of its velocity.

The force is a non-conservative one since the orbital energy of the spacecraft is continuously decreased. From Ref. 29, the force due to atmospheric drag is given by

$$f_{\text{drag}} = -\frac{1}{2} c_D A \rho v_{\text{rel}}^2 \frac{v_{\text{rel}}}{v_{\text{rel}}}, \tag{3.25}$$

where $c_D$ is the spacecraft’s ballistic coefficient, $A$, is its projected surface area in the direction of its velocity and

$$v_{\text{rel}} := \dot{r}(t) - \omega_{\oplus} \times r(t), \tag{3.26}$$

is the spacecraft’s velocity relative to the Earth’s atmosphere, which rotates with the Earth. The atmospheric density, $\rho$, is a function of the spacecraft’s altitude. Modeling atmospheric density accurately is, in general, difficult. There are a number of models available that trade complexity for accuracy to different degrees. For the work in this thesis, the Harris-Priester model, as described in Ref. 31, is used and is valid for altitudes from 120 km to 2000 km.
Since drag acts on all spacecraft in a formation in LEO, the difference in drag force is what is of particular importance when considering drag for formation flying applications. There is almost always be a difference in the projected area of individual spacecraft resulting in differences in the drag force experienced. As with gravitational perturbations, the goal of formation maintenance is to correct for the effect of this force differential.

The Geomagnetic Lorentz Force

Fundamentally, the Lorentz force is the force experienced by a charged particle moving through a magnetic field. In the context of this work, the spacecraft is the particle, and the magnetic field in question belongs to the Earth. A spacecraft with a net electrical charge experiences the geomagnetic Lorentz force resulting from the spacecraft’s motion relative to the Earth’s magnetic field. Unlike the two previous perturbations, which are generally considered to be disturbances with undesired effects on spacecraft formations, the Lorentz force can be a useful perturbation that can be exploited for formation control purposes, if the spacecraft possesses a mechanism to store and modulate an electrical charge. The per-unit mass Lorentz force vector is

$$\vec{f}_L(t, e) = \frac{q(t)}{m} \left( \vec{E}(t) + \vec{v}_{rel}(t) \times \vec{b}_m(t, e) \right),$$  \hspace{1cm} (3.27)

where \( q \) is the charge strength, \( m \) is the object’s mass, \( \vec{E} \) is the electrical field strength and \( \vec{v}_{rel} \) is the velocity of the spacecraft with respect to the magnetic field, which like the Earth’s atmosphere, rotates with the Earth and is given Eq. (3.26). The vector \( \vec{b}_m \) is the magnetic field vector at the location of the spacecraft. Typically in spacecraft applications the electrical field strength is negligible, since the phenomenon of the Debye sheath shields the spacecraft from local electrical fields. Furthermore, in LEO when observed in the Earth-fixed frame, the magnetic field of the Earth is essentially time-invariant, so the resulting electrical field is negligible.

The ratio of charge-to-mass, \( q/m \), is a useful measure of the amount of charge needed to perform maneuvers that is independent of spacecraft mass. It is referred to as the specific charge. The Lorentz force per specific charge is defined as

$$\tilde{\vec{f}}_L(t, e) := \vec{v}_{rel}(t) \times \vec{b}_m(t, e).$$  \hspace{1cm} (3.28)
3.2 Spacecraft Formation Flying Dynamics

This section reviews key concepts in the dynamics of spacecraft formation flying. When considering a spacecraft formation, it is typical to designate one spacecraft as the chief and consider the motion of the remaining spacecraft relative to the chief spacecraft. Any other spacecraft is referred to as a deputy.

In Keplerian orbits, if the orbital energies of the two spacecraft are matched, i.e., \( a_{\text{deputy}} - a_{\text{chief}} = 0 \), a spacecraft formation does not experience any drift. However, when perturbations such as the \( J_2 \) zonal harmonic or atmospheric drag are considered, this energy matching condition is not sufficient to ensure no formation drift. The perturbative net force that each spacecraft experiences is slightly different. This difference in force that each spacecraft experiences is the primary cause of the degradation of a desired formation geometry. For atmospheric drag, differences in the projected area of the spacecraft in the direction of its velocity result in differences in the drag experienced. Differences in spacecraft attitude or configuration could each cause such differences.

3.2.1 Local Vertical/Local Horizontal Frame

We begin by introducing the local vertical/local horizontal (LVLH) frame, \( \mathcal{F}_L \), sometimes referred as the Hill frame. Referring to Fig. 3.3, the frame has its origin at the chief spacecraft and moves with the chief along its orbit. The one-axis, known as the radial direction, is aligned with the outward pointing inertial position vector of the chief spacecraft. The three-axis is aligned with the orbit's angular momentum vector and is referred to as the out-of-plane or normal direction. The two-axis, known as the along-track direction, completes the right-hand rule. The components of the axes are denoted with the subscripts \((\cdot)_r\), \((\cdot)_h\) and \((\cdot)_\theta\), respectively.

![Figure 3.3: Local vertical/local horizontal frame.](image-url)
The relative spacecraft position can be obtained by rotating the relative spacecraft position from the GCI frame to the LVLH frame using a rotation matrix:

\[
x_l = [x_r, x_\theta, x_h]^T = C_{li}(r_d - r_c),
\]

(3.29)

where the rotation matrix \(C_{li}\) is defined by

\[
C_{li} = \begin{bmatrix}
\frac{r_c}{h_c} & \frac{r_c^\times h_c}{h_c} & \frac{h_c}{h_c}
\end{bmatrix}^T,
\]

(3.30)

where \(r_c\) and \(h_c\) are the position and angular momentum vectors of the chief, respectively, expressed in the GCI frame. The notation \((\cdot)\times\) denotes the skew symmetric matrix operator, which realizes the cross product operation:

\[
a^\times = \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}.
\]

(3.31)

The relative velocity in the LVLH frame is obtained by differentiating Eq. (3.29)

\[
x_l = C_{li}(\dot{r}_d - \dot{r}_c) + C_{li}(\dot{r}_d - \dot{r}_c).
\]

(3.32)

### 3.2.2 Relative State Representation Using Orbital Elements

The relative Cartesian representation of the spacecraft state is certainly the most intuitive representation. At a glance, one knows exactly where the deputy spacecraft is with respect the chief. Like its absolute counterpart, however, the relative Cartesian state gives no insight into the relative trajectory the deputy spacecraft follows. Furthermore, to realize any form of feedback control using the relative Cartesian state, a reference relative trajectory needs to be generated for an entire orbital period. This point is considered further in Section 3.2.4.

Taking the difference between the deputy and chief spacecraft orbital elements is an alternative means of representing the relative spacecraft state. The differences between the two sets of elements

\[
\delta e(t) = e_d(t) - e_c(t),
\]

\[
= [\delta a \; \delta e \; \delta i \; \delta \Omega \; \delta \omega \; \delta M]^T,
\]

(3.33)
are known as the differential orbital elements. Unlike the relative Cartesian state, the differential elements, in the Keplerian case, remain constant with time, which makes them advantageous to use for control applications.

For the non-Keplerian case that considers only the $J_2$ perturbation, mean differential elements elegantly capture the influence of the perturbation on the relative state. The full mean orbital element dynamics can be written in the form

$$
\dot{\bar{e}}(t) = A(\bar{e}) + \frac{\partial e(e)}{\partial \bar{e}} B(e) u(t),
$$

(3.34)

where the nonlinear function $A(\bar{e})$ corresponds to the secular drift rates due to $J_2$, given in Eqs. (3.16)-(3.21) and the vector $u(t) = [u_r(t) \ u_\theta(t) \ u_h(t)]^T$ is any additional acceleration being experienced by the spacecraft, expressed in the LVLH frame. The function $e(e) = \bar{e}$ is the mapping from osculating to mean elements. A first-order mapping algorithm can be found in Ref. 28. Any osculating element changes due to an applied acceleration are mapped to a corresponding mean element change using

$$
D(e) := \frac{\partial e(e)}{\partial \bar{e}}.
$$

(3.35)

This partial derivative can be found in Ref. 32. The dynamics of the mean differential elements are therefore

$$
\delta \dot{\bar{e}}(t) = \dot{\bar{e}}_\delta(t) - \dot{\bar{e}}_c(t),
$$

$$
= A(\bar{e}_c + \delta \bar{e}(t)) - A(\bar{e}_c) + D(e_c + \delta e(t))B(e_c + \delta e(t))u_c(t).
$$

(3.36)

The chief spacecraft is assumed to be uncontrolled, so $u_c(t) = 0$ and is removed. The terms are linearized about the chief mean orbital elements to yield

$$
\delta \dot{\bar{e}}(t) \approx \frac{\partial A(\bar{e}_c)}{\partial \bar{e}} \delta \bar{e}(t) + \left( D(e_c) + \frac{\partial D(e_c)}{\partial \bar{e}} \delta e(t) \right) \left( B(e_c) + \frac{\partial B(e_c)}{\partial \bar{e}} \delta e(t) \right) u(t).
$$

(3.37)

Since there is no bound on $\delta \bar{e}$, the terms $\frac{\partial B(e_c)}{\partial \bar{e}} \delta e$ and $\frac{\partial D(e_c)}{\partial \bar{e}} \delta e$ can be arbitrarily large. Regarding the term $\frac{\partial D(e_c)}{\partial \bar{e}} \delta e$, Schaub\textsuperscript{28} notes that the off-diagonal terms in $D(e_c)$ are already $O(J_2)$ or smaller, so it is reasonable to conclude that $\frac{\partial D(e_c)}{\partial \bar{e}} \delta e$ can also be safely ignored for sufficiently small $\delta e$.

Breger and How\textsuperscript{33} analyze the error associated in neglecting the term $\frac{\partial B(e_c)}{\partial \bar{e}} \delta e$ by numerically calculating the difference in the GVE when evaluated at the deputy and chief orbital elements, $\Delta B = B(e_c + \delta e) - B(e_c)$. To limit the effect of linearization error, they assign a mission-dependent cut-off value.
that the difference $\|\Delta B\|_2$ cannot exceed. For formations in LEO, a cut-off value of $0.01\|B(e_c + \delta e)\|_2$ results is a maximum relative position and velocity of 25 km and 40 m/s, respectively. For the same cut-off value in highly elliptical orbits, the reported maximum relative position and velocities are 50 km and 2 km/s, respectively. Breger and How’s results show that for formations sizes of several kilometres, like those considered in this work, the term $\frac{\partial B(e)}{\partial e}$ can be safely neglected in the linearized dynamics. Therefore, the linear time-varying dynamics of the differential elements are

$$\delta \dot{e}(t) = \frac{\partial A(\bar{e}_c)}{\partial \bar{e}} \delta \bar{e}(t) + D(e_c)B(e_c)u(t),$$

(3.38)

where $\frac{\partial A(\bar{e}_c)}{\partial \bar{e}}$ is the matrix of partial derivatives of the secular drift rates with respective to the chief orbital elements are given by:

$$\tilde{A} := \frac{\partial A(\bar{e}_c)}{\partial \bar{e}} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ \frac{\partial \dot{\Omega}}{\partial e} & \frac{\partial \dot{\Omega}}{\partial e} & \frac{\partial \dot{\Omega}}{\partial i} \\ \frac{\partial \dot{\omega}}{\partial e} & \frac{\partial \dot{\omega}}{\partial e} & \frac{\partial \dot{\omega}}{\partial i} \\ \frac{\partial \dot{M}}{\partial e} & \frac{\partial \dot{M}}{\partial e} & \frac{\partial \dot{M}}{\partial i} \end{bmatrix}.$$ 

(3.39)

The three nonzero orbital element drift rates are all functions of only $\bar{a}, \bar{e}$ and $\bar{i}$, which are constant. As a result, the matrix $\tilde{A}$ is time-invariant.

An identical derivation for the dynamics of the mean differential element error

$$\zeta(t) = \bar{e}_d(t) - \bar{e}(t)$$

$$\dot{\zeta}(t) = \tilde{A}(\bar{e}_r)\zeta(t) + D(e_r)B(e_r)u(t).$$

(3.40)

Both Eq. (3.38) and Eq. (3.40) are used in this thesis as plant models for controller design.
Non-Singular Elements

Non-singular elements can be used to eliminate numerical singularities encountered when \( e = 0 \). The differential elements \( \delta q_1, \delta q_2 \) and \( \delta \lambda \) are introduced:

\[
\delta q_1 = \cos \omega \delta e - e \sin \omega \delta \omega, \tag{3.41}
\]
\[
\delta q_2 = \sin \omega \delta e + e \cos \omega \delta \omega, \tag{3.42}
\]
\[
\delta \lambda = \delta M + \delta \omega. \tag{3.43}
\]

For \( e \neq 0 \), the classical differential elements \( \delta e \) and \( \delta \omega \) can be recovered using

\[
\delta e = \frac{q_1 \delta q_1 + q_2 \delta q_2}{e}, \tag{3.44}
\]
\[
\delta \omega = \frac{q_1 \delta q_2 - q_2 \delta q_1}{e^2}. \tag{3.45}
\]

Ref. 32 also gives the transformation \( D(e) \) in terms of the differential true latitude, \( \delta \theta = \delta f + \delta \omega \). It can be expressed in terms of the classical elements:

\[
\delta \theta = \frac{\sin f(2 + \cos f)}{\eta^2} \delta e + \delta \omega + \frac{(1 + e \cos f)^2}{\eta^3} \delta M. \tag{3.46}
\]

3.2.3 Differential Elements and Relative Spacecraft Position

From Ref. 34, the mapping between osculating differential orbital elements and relative position in the LVLH frame is

\[
x_r = \delta r \tag{3.47}
\]
\[
x_\theta = r(\delta \Omega \cos i + \delta \theta) \tag{3.48}
\]
\[
x_h = r(\delta i \sin \theta - \delta \Omega \sin i \cos \theta), \tag{3.49}
\]

where

\[
\delta r = r \left[ \left( 1 - \frac{e}{\eta} \right) \sin f(1 + e \cos f) \eta \right] \frac{\delta a}{a} - \frac{1}{\eta^2} \cos f(1 + e \cos f) \delta e + \frac{e}{\eta^3} \sin f(1 + e \cos f) \delta M_0 \right], \tag{3.50}
\]

and \( \delta M_0 \) is the initial differential mean anomaly.

Eq. (3.47) – Eq. (3.49) are useful for understanding how a change in a differential element affects a
spacecraft’s relative position.

**Partial Derivatives of Drift Rates**

The linear, time-varying, mean differential element system requires the partial derivatives of the secular drift rates with respect to the orbital elements. They are presented here. All quantities in this section are mean quantities, so the bar notation is dropped for this section. Regarding the derivatives of the drift rates with respect to $e$, $q_1$ and $q_2$, it is simpler to consider the chain rule product

$$\frac{\partial \dot{\alpha}}{\partial \eta} \cdot \frac{\partial \eta}{\partial \beta},$$

(3.51)

for $\alpha = \Omega, \omega, M$ and $\beta = e, q_1, q_2$ and $\eta = \sqrt{1 - e^2} = \sqrt{1 - q_1^2 - q_2^2}$. To begin with,

$$\frac{\partial \eta}{\partial e} = -e \eta,$$

(3.52)

$$\frac{\partial \eta}{\partial q_i} = -q_i \eta, \quad \text{for } i = 1, 2.$$  

(3.53)

The remaining partial derivatives are as follows:

$$\frac{\partial \dot{\Omega}}{\partial a} = \frac{21}{4} l_2 n \left( \frac{R_\oplus}{p} \right)^2 \left( \frac{\cos i}{a} \right),$$

(3.54)

$$\frac{\partial \dot{\Omega}}{\partial i} = \frac{3}{2} l_2 n \left( \frac{R_\oplus}{p} \right)^2 \sin i,$$

(3.55)

$$\frac{\partial \dot{\Omega}}{\partial \eta} = 6 l_2 n \left( \frac{R_\oplus}{p} \right)^2 \left( \frac{\cos i}{\eta} \right),$$

(3.56)

$$\frac{\partial \dot{\omega}}{\partial a} = -\frac{21}{8} l_2 n \left( \frac{R_\oplus}{p} \right)^2 \left( \frac{5 \cos^2 i - 1}{a} \right),$$

(3.57)

$$\frac{\partial \dot{\omega}}{\partial i} = -\frac{15}{4} l_2 n \left( \frac{R_\oplus}{p} \right)^2 \sin 2i,$$

(3.58)

$$\frac{\partial \dot{\omega}}{\partial \eta} = -3 l_2 n \left( \frac{R_\oplus}{p} \right)^2 \left( \frac{5 \cos^2 i - 1}{\eta} \right),$$

(3.59)

$$\frac{\partial \dot{M}}{\partial a} = -\frac{3 n}{2a} - \frac{21}{8} \eta l_2 n \left( \frac{R_\oplus}{p} \right)^2 \left( \frac{3 \cos^2 i - 1}{a} \right),$$

(3.60)

$$\frac{\partial \dot{M}}{\partial i} = -\frac{9}{4} \eta l_2 n \left( \frac{R_\oplus}{p} \right)^2 \sin 2i,$$

(3.61)
\[
\frac{\partial M}{\partial \eta} = -\frac{9}{4} \frac{J_2}{e^2} \left( \frac{R\oplus}{p} \right)^2 (3 \cos^2 i - 1),
\] (3.62)

\[
\frac{\partial \dot{\lambda}}{\partial a} = \frac{\partial M}{\partial a} + \frac{\partial \dot{\omega}}{\partial a},
\] (3.63)

\[
\frac{\partial \dot{\lambda}}{\partial i} = \frac{\partial M}{\partial i} + \frac{\partial \dot{\omega}}{\partial i},
\] (3.64)

\[
\frac{\partial \dot{\lambda}}{\partial \eta} = \frac{\partial M}{\partial \eta} + \frac{\partial \dot{\omega}}{\partial \eta}.
\] (3.65)

Lastly, the partial derivatives of the drift rates of \(q_1\) and \(q_2\) can be written as functions of \(\dot{\omega}\) and its partial derivatives with respect to \(a, i, q_1\) and \(q_2\).

\[
\frac{\partial q_1}{\partial a} = -q_2 \frac{\partial \dot{\omega}}{\partial a},
\] (3.66)

\[
\frac{\partial q_1}{\partial i} = -q_2 \frac{\partial \dot{\omega}}{\partial i},
\] (3.67)

\[
\frac{\partial q_1}{\partial q_1} = -q_2 \frac{\partial \dot{\omega}}{\partial q_1},
\] (3.68)

\[
\frac{\partial q_1}{\partial q_2} = -\dot{\omega} - q_2 \frac{\partial \dot{\omega}}{\partial q_2},
\] (3.69)

\[
\frac{\partial q_2}{\partial a} = q_1 \frac{\partial \dot{\omega}}{\partial a},
\] (3.70)

\[
\frac{\partial q_2}{\partial i} = q_1 \frac{\partial \dot{\omega}}{\partial i},
\] (3.71)

\[
\frac{\partial q_2}{\partial q_1} = \dot{\omega} + q_1 \frac{\partial \dot{\omega}}{\partial q_1},
\] (3.72)

\[
\frac{\partial q_2}{\partial q_2} = q_1 \frac{\partial \dot{\omega}}{\partial q_2}.
\] (3.73)

### 3.2.4 Reference Trajectories for Spacecraft Formations

Although all of the control work that is presented in this thesis uses mean differential orbital elements feedback, the reference differential elements are still be based on a desired Cartesian relative trajectory. This section briefly reviews relative motion of a spacecraft in the LVLH frame and relative formation geometry.

Much of the original work done regarding the relative motion of spacecraft in the LVLH frame was done in the context of terminal guidance and rendezvous. Clohessy and Wiltshire derived a set of linear differential equations for the relative motion in a circular reference orbit. Hill developed
the same linear equations for describing the Moon’s motion with respect to the Earth, and they are commonly known as the Hill Clohessy-Wiltshire (HCW) equations. Although they possess an elegant closed-form solution, they are not suitable for describing relative motion in eccentric orbits. Using true anomaly, $f$, instead of time as the independent variable, Tschauner and Hempel\textsuperscript{35} and, independently, de Vries,\textsuperscript{36} developed a set of differential equations suitable for relative motion in elliptic orbits, known as the Tschauner-Hempel (TH) equations. Sengupta and Vadali\textsuperscript{34} present the general solution to the TH equations for relative, periodic motion. The geometry of the relative, periodic trajectory is parametrized in terms of a position vector $\rho = [\rho_1 \rho_2 \rho_3]^T$, and two phase angles, $\alpha_0$ and $\beta_0$, illustrated in Fig. 3.4. The length quantities $\rho_1$ and $\rho_3$ describe the magnitude of the in-plane and out-of-plane motion, respectively, and $\rho_2$ is an offset in the along-track direction. The phase angles $\alpha_0$ and $\beta_0$ correspond to the initial phase angle in the in-plane and out-of-plane, respectively. From Sengupta and Vadali,\textsuperscript{34} the relative position is given by

$$x_r = \rho_1 \sin(f + \alpha_0),$$

$$x_\theta = 2\rho_1 \cos(f + \alpha_0) \left( \frac{1 + (e/2) \cos f}{1 + e \cos f} \right) + \frac{\rho_2}{1 + e \cos f},$$

$$x_h = \rho_3 \frac{\sin(f + \beta_0)}{1 + e \cos f}. \quad (3.74)$$

$$\quad (3.75)$$

$$\quad (3.76)$$

The parameters $\rho$, $\alpha_0$ and $\beta_0$ fully define the relative geometry of a periodic formation.
Judicious selection of \( \rho \) results in several well-known formations. The along-track formation, which is simply the deputy spacecraft either leading or following the chief spacecraft in the along-track direction, is defined by choosing \( \rho_1 = \rho_3 = 0 \) and selecting \( \rho_2 = d \), where \( d \) is the desired separation distance between to two spacecraft. For a circular chief orbit, along-track separation remains constant over an orbit, whereas for an eccentric orbit, along-track separation varies periodically with true anomaly in the same way as the absolute orbit radius, with \( \rho_2 \) being analogous to the semilatus rectum. For a circular orbit, active control on a fixed along-track separation amounts to mitigating \( J_2 \) and drag disturbances, whereas in eccentric orbits, maintaining a fixed distance also means fighting the natural relative dynamics and significantly increases formation-keeping control effort.

The projected circular orbit (PCO) formation is obtained when \( \rho_3 = 2\rho_1 = s \), where \( s \) is the desired relative orbit radius. In circular orbits, the result is a circular trajectory in the along-track/out-of-plane plane of the LVLH frame. The phase angles \( \alpha_0 \) and \( \beta_0 \) are used to define the deputy’s initial position on the circular trajectory. When \( \rho_2 = 0 \), the circle has its origin at the chief spacecraft. For nonzero eccentricities, the trajectory remains closed but ceases to be circular, instead taking on a triangular shape with rounded corners. Despite it not being circular for nonzero eccentricities, we continue to refer to the formation as a PCO. Once again, attempting to enforce a circular path in an eccentric reference orbit is considerably more expensive in terms of control effort than maintaining one of the natural non-circular closed formed paths.

The parameters \( \rho, \alpha_0 \) and \( \beta_0 \) map directly to a set of osculating differential elements. Once again from Sengupta and Vadali:\textsuperscript{34}

\[
\begin{align*}
\delta a &= 0, \\
\delta e &= -\frac{\rho_1}{a} \sin \alpha_0, \\
\delta i &= \frac{\rho_3}{p} \cos(\beta_0 - \omega), \\
\delta \Omega &= -\frac{\rho_3}{p} \sin(\beta_0 - \omega) \sin i, \\
\delta M_0 &= \frac{\rho_1}{a} \eta \cos \alpha_0, \\
\delta \omega &= \frac{\rho_2}{p} - \frac{\delta M_0}{\eta^3} - \delta \Omega \cos i.
\end{align*}
\] (3.77)

For the Keplerian case, the set of differential elements corresponding to a desired geometry described by \( \{\rho, \alpha_0, \beta_0\} \) is time-invariant. For eccentric, non-Keplerian orbits, however, the reference differential are no longer time invariant. Inspection of Eqs. (3.79) and (3.80) reveals why. Included in both expressions
Chapter 3. Orbital and Spacecraft Formation Dynamics

is the chief argument of perigee, which, in general, precesses under the influence of $J_2$. For many formation flying control applications, it is assumed that the chief spacecraft is uncontrolled. Solving for $\beta_0$ shows that it is a function of the chief argument of perigee:

$$\beta_0 = \arctan\left(\frac{\delta \Omega \sin i}{\delta i}\right) + \omega. \quad (3.83)$$

Therefore, even if the $J_2$ differential drift rates are mitigated through control, secular drift in the chief’s absolute argument of perigee still causes $\beta_0$ to change over time and result in an altered formation geometry. If the goal is to maintain a constant $\beta_0$ value, then the reference differential orbital elements must be recalculated periodically using Eqs. (3.77)-(3.82). Fig. 3.5 illustrates the effect of $\beta_0$ drift due to only the drift of the chief argument of perigee on the geometry of a projected circular orbit (PCO) in comparison to the complete effects of $J_2$, that is, including the effect of the differential drift rates. The effect of chief periapsis drift can actually be accounted for without reference state recalculations by using the non-singular differential orbital element set to represent the relative state of the deputy spacecraft. The reason why chief periapsis drift is accounted for can be seen in the linearized differential drift rates of $q_1$ and $q_2$. Consider Eq. (3.69) and Eq. (3.72): both equations contain the chief spacecraft’s argument of periapsis drift rate, accounting for its drift in the linearized dynamics of the differential
elements. Argument of periapsis drift of the chief spacecraft does not appear in the linearized dynamics of classical differential elements. The differences between using classical and non-singular element sets for formation keeping are be explored further in Chapter 4.

Relative Position Error

An important metric of formation-keeping performance is the relative position error of the deputy spacecraft, i.e., how closely the deputy spacecraft is tracking its reference relative position in the LVLH frame. There are many ways to obtain a reference trajectory in the LVLH frame, such as using Eq. (3.74)–Eq. (3.76), or the HCW equations (for circular chief orbits) or other closed-form equations such as those in Ref. 37. Alternatively, the relative position can be converted from the reference differential elements using the linear mapping between the elements and relative position and velocity, as in Refs 28, 32.

In this thesis, the relative position error is calculated in the following manner. Each simulation integrates the inertial Cartesian states of the chief and deputy spacecrafts in the GCI frame, so we begin with the chief inertial Cartesian state. As illustrated in Fig. 3.6, the current chief state is converted first to osculating orbital elements then to mean orbital elements using the first order mapping found in Schaub and Junkins.\textsuperscript{28} The mean reference differential elements that describe the desired formation are then added to the chief mean orbital elements to obtain the reference deputy orbital elements. The reference deputy orbital elements are then converted back to the reference inertial Cartesian of the deputy spacecraft. With the reference inertial state and the original chief inertial state, the relative reference Cartesian state in the LVLH frame is calculated. The position error is the difference between the actual relative deputy position and the reference position.

No Drift $J_2$ Condition

For the case of Keplerian orbits, Eq. (3.77) is the standard energy matching condition to ensure that there is no along-track secular growth between deputy and chief spacecrafts: if the two spacecraft have the same semimajor axes, their periods are the same and their relative along-track separation does not change. Unfortunately, once $J_2$ is considered, this no-drift condition no longer holds due to differences in the orbit element drift rates. To replace the Keplerian no-drift condition, Vadali and Alfriend\textsuperscript{17} give an alternate expression for $\delta a$ that introduces a semimajor axis mismatch between chief and deputy spacecraft that minimizes along-track drift due to $J_2$ secular rate differences. That expression is

\[
\delta a = a J_2 \left( \frac{R_\oplus}{p} \right)^2 \left( \frac{4 + 3 \eta}{2} \right) \left( 3 \cos^2 i - 1 \right) \frac{\delta \varepsilon}{\eta^2} \delta i - \sin 2i \delta \tilde{i} \right)
\]  

(3.84)
We derive this expression in Appendix A. For the simulation results in this thesis, this expression, not Eq. (3.77), is used to calculate the reference $\bar{\delta}a$.

### 3.2.5 Differential $J_2$ Drift Rate Magnitudes

As implied in Eq. (3.40), small differences in chief and deputy semimajor axes, eccentricities and inclinations all result in differences in the $J_2$ force experienced and, consequently, the mean drift rates. A moment is taken to explore the magnitudes of the partial derivatives of the drift rates and how they change from low Earth orbit (LEO) to highly elliptical orbit (HEO) – the two orbital regimes that are considered in this thesis.

All nine partial derivatives with respect to semimajor axis, eccentricity and inclination contain the term $G = J_2 n \left( \frac{R_{\oplus}}{p} \right)^2$. The $J_2$ coefficient and the Earth’s equatorial radius in metres, $R_{\oplus}$, are constant of order $O(10^{-3})$ and $O(10^6)$, respectively. In LEO, for $a < 8000$ km, the mean motion, $n$, is $O(10^{-3})$ and the semilatus rectum, $p$, is $O(10^6)$ and only slightly greater than $R_{\oplus}$, making the square term in $G$ approximately $O(10^6) - O(10^{-1})$. For formations in LEO, $G$ is approximately $O(10^{-6}) - O(10^{-7})$.

The three partial derivatives with respect to semimajor axis all contain an additional $a$ term in their denominators. In LEO, this term typically reduces the orders of $\frac{\partial \dot{\Omega}}{\partial a}$ and $\frac{\partial \dot{\omega}}{\partial a}$ to $O(10^{-12})$, but they can be as large as $O(10^{-11})$ at certain inclinations. In general for both LEO and HEO, these partial derivative terms contribute little to formation drift. The partial $\frac{\partial \dot{\Omega}}{\partial a}$ is larger due to the first order mean motion difference and is $O(10^{-9})$. The along-track drift due to differences in mean motion can be significant; however, the effect of different $J_2$ on mean anomaly is on the same order as for right ascension and
argument of perigee.

After the term $\frac{\partial M}{\partial a}$, the three partial derivatives with respect to inclinations are the most significant in terms of magnitude for formations in LEO. The partials $\frac{\partial \Omega}{\partial i}$, $\frac{\partial \omega}{\partial i}$ and $\frac{\partial M}{\partial i}$ are in general $O(10^{-7})$ and can be as large as $O(10^{-6})$ for particular inclinations. The magnitudes of these partial derivatives causes formations with large out-of-plane separations that correspond to differences in inclination (per Eq. (3.79)) to experience significant differential $J_2$ drift. For HEO, this effect diminishes as $G$ reduces in magnitude due to increasing semimajor axis.

Depending on the inclination, the partial derivatives with respect to eccentricity can be significant in both LEO and HEO. In LEO, the derivatives can be $O(10^{-8})$ for near-equatorial orbits, but are typically smaller, assuming an eccentricity $e < 0.05$. In HEO, with $e > 0.6$, the parameter $\eta = \sqrt{1-e^2}$ grows small. As a result, $\frac{\partial \Omega}{\partial e}$, $\frac{\partial \omega}{\partial e}$ and $\frac{\partial M}{\partial e}$ can be of order $O(10^{-7})$, depending on inclination. Per Eq. (3.78), HEO formations with a large in-plane separation and small in-plane phase angle are susceptible to this differential $J_2$ drift.

In summary, both large differential inclinations and differential eccentricities can cause significant formation drift. The inclination of the chief orbit plays a significant role in determining the magnitude of the drift. Formations with out-of-plane separations see the largest drift due to right ascension in polar orbits. At inclinations of 45°, drift due to differential inclination is largest in mean anomaly and argument of periapsis. For LEO, a large differential eccentricity causes drift in near-equatorial orbits. For HEO, a large differential eccentricity causes significant drift across most inclinations. Drift due to a differential semimajor axis can be significant due to the mismatch in orbital periods.

### 3.3 Chapter Summary

This chapter began with a brief review of the fundamental orbital mechanics that underpin the work in this thesis. This review included a summary of Keplerian two-body motion, a discussion of several orbital perturbations and how these perturbations influence a spacecraft’s orbital elements. Following the summary was a presentation of a number of spacecraft formation flying concepts that are used throughout this thesis. The representation of one spacecraft’s state relative to another was described both in the Cartesian LVLH reference frame and in terms of mean differential orbital elements. The linearized mean differential element dynamics were presented as the primary plant of interest; this thesis focuses on the control of this plant. The chapter concluded with a discussion of calculating reference differential element state and a corresponding Cartesian reference trajectory.
Chapter 4

Impulsive Formation Control

In the Keplerian case, a formation is passively stable so long as each spacecraft in the formation has the same orbital energy, i.e., the difference in the semimajor axes of the orbits is zero. Once perturbations are considered, however, this condition is no longer sufficient: the $J_2$ zonal harmonic introduces secular drift in spacecrafts’ orbital elements and, in general, causes the degradation of formation geometry. To maintain a desired relative state, some form of feedback control is typically necessary.

Impulsive formation control is an appealing control paradigm since it provides spacecraft with large amounts of vibration-free time during which observations or measurements could be taken. Both the Tandem-X\cite{38,39} interferometry mission and the autonomous formation flying demonstration mission PRISMA\cite{11} have successfully employed Schaub and Alfriend’s\cite{40} three-impulse mean orbit element formation-keeping strategy. Schaub and Alfriend’s strategy, while computationally simple, is not optimal, nor does it consider any perturbations acting on the spacecraft. Other differential element formation-control strategies include Beigelman and Gurfil’s\cite{41} optimal, analytical impulsive formation-control and fuel-balancing strategy and Breger and How’s\cite{33,42,43} numerically-optimized model-predictive controller for formation maintenance. Additionally, formation reconfiguration using two numerically-determined thrusts was investigated by Sengupta et al.\cite{44} Vadali et al.\cite{17} develop a set of constraint equations for a $N$-impulse strategy to mitigate the effect of $J_2$ for formations in circular orbit and also investigated spacecraft fuel balancing.

Impulsive formation control using the relative Cartesian spacecraft state has also appeared in the literature. Gurfil\cite{45} presented both optimal Cartesian and classical element-based impulse strategies for formations in elliptic eccentric orbits. A computationally intensive but optimal, multi-spacecraft for-
4.1 Feedback Control Using Classical Orbital Elements

An impulsive feedback control strategy is formulated using the classical, mean differential orbital element set \( \delta \bar{e} = [\delta \bar{a} \ \delta \bar{e} \ \delta \bar{i} \ \delta \bar{\Omega} \ \delta \bar{\omega} \ \delta \bar{M}]^T \). The linear mean differential element dynamics are described in Eq. (3.39). As noted in Ref. 28, the mean-to-osculating transformation \( D(\bar{e}_c) \) has off-diagonal terms of order \( J_2 \) or smaller and for controller design purposes can be reasonably approximated by a \( 6 \times 6 \) identity matrix and the Gauss’s variational equations (GVE), represented by \( B(\bar{e}_c) \), are evaluated using chief mean orbital elements. Assuming that over a time interval \( t_0 \) to \( t_f \) there are \( N \) impulsive control thrusts that are applied at \( t_k, k = 1, ..., N \), the mean differential element dynamics are

\[
\delta \dot{\bar{e}}(t) = \tilde{A}(\bar{e}_c) \delta \bar{e}(t) + \sum_{k=1}^{N} B(\bar{e}_c(t_k)) u(t_k) \delta(t - t_k), \tag{4.1}
\]

where \( \delta(t) \) is the Dirac Delta function. Beginning at some initial time \( t_0 \), the mean differential elements at some time \( t_f \) can be calculated by integrating Eq. (4.1). From Chapter 2, Eq. (2.9), we obtain:

\[
\delta \bar{e}(t_f) = \int_{t_0}^{t_f} \left( \tilde{A}(\bar{e}_c(t)) \delta \bar{e}(t) + \sum_{k=1}^{N} B(\bar{e}_c(t_k)) u(t_k) \delta(t - t_k) \right) dt = \Phi(t_f, t_0) \delta \bar{e}(t_0) + \sum_{k=1}^{N} \Phi(t_f, t_k) B(\bar{e}_c(t_k)) v_k, \tag{4.2}
\]
where \( \Phi(t_2, t_1) = e^{\hat{A}\Delta t_21}, \Delta t_21 = t_2 - t_1, v_k = [v_r, v_\theta, v_h]^T \) is the three component thrust vector, and \( e(\cdot) \) is the matrix exponential. If \( \delta e(t_f) \) is the desired state at time \( t_f \), then the expression \( \delta \bar{e}(t_f) - \Phi(t_f, t_0)\delta \bar{e}(t_0) \) can be interpreted as the state error that needs to be corrected to reach the desired state.

For the system of classical differential orbital elements, the matrix \( \tilde{A} \) is nilpotent due to its structure (see Eq. (3.39)), such that \( \tilde{A}^k = 0 \) for \( k > 1 \). The nilpotency of \( \tilde{A} \) reduces the calculation of the matrix exponential to

\[
\Phi(t_f, t_k) = e^{\tilde{A}\Delta t_{fk}} = 1 + \tilde{A}\Delta t_{fk},
\]

(4.3)

without a loss in accuracy.

The general \( N \)-impulsive problem is to determine the thrust components and thrust application times necessary for the deputy spacecraft to have the desired mean differential elements at a specific, fixed end time, \( t_f \). The thrust components and application times can be determined numerically by solving the following optimization problem, where Eq. (4.2) is rearranged to form the equality constraint:

\[
\text{minimize} \quad J(v_k, t_k) = \sum_{k=1}^{N} v_k^Tv_k
\]

with respect to \( v_k, t_k, \quad k = 1, ..., N \)

subject to \( \quad 0 = \delta \bar{e}(t_f) - \Phi(t_f, t_0)\delta \bar{e}(t_0) - \sum_{k=1}^{N} \Phi(t_f, t_k)B(\bar{e}_c(t_k))v_k. \)

The constraint equations for the individual orbital elements are

\[
\delta a(t_f) = \delta a(t_0) + 2a^2 \sum_{k=1}^{N} \left[ e \sin f_k \quad \frac{p}{r_k} \quad 0 \right] v_k,
\]

(4.4)

\[
\delta e(t_f) = \delta e(t_0) + \sum_{k=1}^{N} \left[ \frac{p \sin f_k}{h} \quad \left( p + r \right) \cos f_k + \frac{r_k e}{h} \quad 0 \right] v_k,
\]

(4.5)

\[
\delta i(t_f) = \delta i(t_0) + \sum_{k=1}^{N} \left[ 0 \quad 0 \quad \frac{r_k \cos \theta_k}{h} \right] v_k,
\]

(4.6)

\[
\delta \Omega(t_f) = \delta \Omega(t_0) + \left( \frac{\partial \Omega}{\partial a} \delta a(t_0) + \frac{\partial \Omega}{\partial e} \delta e(t_0) + \frac{\partial \Omega}{\partial i} \delta i(t_0) \right) \Delta t_{f0}
\]
\begin{align}
+ \sum_{k=1}^{N} \left\{ \begin{bmatrix} 0 & 0 & \frac{r_k \sin \theta_k}{h \sin i} \end{bmatrix} v_k + \left( \frac{\partial \Omega}{\partial a} \Delta \delta a_{f_k} + \frac{\partial \Omega}{\partial e} \Delta \delta e_{f_k} + \frac{\partial \Omega}{\partial i} \Delta \delta i_{f_k} \right) \Delta t_{f_k} \right\}, \quad (4.7)
\end{align}

\begin{align}
\delta \omega(t_f) &= \delta \omega(t_0) + \left( \frac{\partial \dot{\omega}}{\partial a} \Delta \delta a_{f_0} + \frac{\partial \dot{\omega}}{\partial e} \Delta \delta e_{f_0} + \frac{\partial \dot{\omega}}{\partial i} \Delta \delta i_{f_0} \right) \Delta t_{f_0}
+ \sum_{k=1}^{N} \left\{ \begin{bmatrix} -\frac{p \cos f_k}{he} & (p + r_k) \sin f_k \frac{1}{he} & -\frac{r_k \sin \theta_k}{h \tan i} \end{bmatrix} v_k
+ \left( \frac{\partial \dot{\omega}}{\partial a} \Delta \delta a_{f_k} + \frac{\partial \dot{\omega}}{\partial e} \Delta \delta e_{f_k} + \frac{\partial \dot{\omega}}{\partial i} \Delta \delta i_{f_k} \right) \Delta t_{f_k} \right\}, \quad (4.8)
\end{align}

\begin{align}
\delta M(t_f) &= \delta M(t_0) + \left( \frac{\partial M}{\partial a} \Delta \delta a_{f_0} + \frac{\partial M}{\partial e} \Delta \delta e_{f_0} + \frac{\partial M}{\partial i} \Delta \delta i_{f_0} \right) \Delta t_{f_0}
+ \sum_{k=1}^{N} \left\{ \begin{bmatrix} \frac{b(p \cos f_k - 2r_k e)}{a e} & \frac{b(p + r_k) \sin f_k}{a e} & 0 \end{bmatrix} v_k
+ \left( \frac{\partial M}{\partial a} \Delta \delta a_{f_k} + \frac{\partial M}{\partial e} \Delta \delta e_{f_k} + \frac{\partial M}{\partial i} \Delta \delta i_{f_k} \right) \Delta t_{f_k} \right\}. \quad (4.9)
\end{align}

In Eqs. (4.4)-(4.9), the notation \( \Delta \delta(\cdot)_{f_k} \) denotes the change in the element due to an individual impulse, e.g., for \( k = 1 \),
\begin{equation}
\Delta \delta i_{f_1} = \frac{r_1}{k} \cos \theta_1 v_0. \quad (4.10)
\end{equation}

### 4.1.1 Two-Impulse Case

Limiting the problem to two impulses per orbit allows for the problem to be solved analytically, rather than resorting to numerical methods. Assuming that the firing times \( t_1 \) and \( t_2 \) are prescribed, Eq. (4.1) can be rearranged to solve for the two thrust vectors:
\begin{align}
\begin{bmatrix} v_1 \\
v_2 \end{bmatrix} &= \left[ \Phi(t_f, t_1)B(\bar{e}_1(t_1)) \vert \Phi(t_f, t_2)B(\bar{e}_1(t_2)) \right]^{-1} \left( \delta \bar{e}(t_f) - \Phi(t_f, t_0)\delta \bar{e}(t_0) \right).
\end{align}
\( (4.11) \)

Define
\begin{equation}
X(t_1, t_2) := \left[ \Phi(t_f, t_1)B(\bar{e}_1(t_1)) \vert \Phi(t_f, t_2)B(\bar{e}_1(t_2)) \right]. \quad (4.12)
\end{equation}

Thus, so long as \( X(t_1, t_2) \) is invertible, a solution to the thrusts exists. The most obvious case of when \( X \) is not invertible is when \( t_1 = t_2 \). In that case, the matrix is no longer full rank. Additional cases when \( X \) is not invertible are explored in Section 4.3.

An alternative solution, first presented by Vaddi\(^4^9\) for osculating elements, that does not require
the prescribing of thrust times also exists. If the effects of changes caused by impulses to $\delta a$, $\delta e$ and $\delta i$ on the differential drift rate of the ascending node are ignored (making this assumption is reasonable since these changes are typically small), then both Eq. (4.6) and Eq. (4.7) are functions of only the thrust times and the out-of-plane thrust components, $v_{h_1}$ and $v_{h_2}$.

A single thrust vector and time can be calculated to correct for errors in differential inclination and right ascension in the following fashion. Consider the equations:

\[
\delta i(t_f) - \delta i(t_0) = \frac{r_1 \cos \theta_1}{h} v_{hr},
\]

\[
\delta \Omega(t_f) - \delta \Omega(t_0) = \left( \frac{\partial \Omega}{\partial a} \delta a(t_0) + \frac{\partial \Omega}{\partial e} \delta e(t_0) + \frac{\partial \Omega}{\partial i} \delta i(t_0) \right) \Delta t_f = \frac{r_1 \sin \theta_1}{h \sin i} v_h.
\]

Taking the quotient of Eq. (4.13) and Eq. (4.14) results in $v_h$ being eliminated and $\theta_1$ can be solved for to obtain:

\[
\theta_1 = \arctan \left( \frac{\frac{\delta \Omega(t_f) - \delta \Omega(t_0)}{\delta i(t_f) - \delta i(t_0)}}{\frac{\partial \Omega}{\partial a} \delta a(t_0) + \frac{\partial \Omega}{\partial e} \delta e(t_0) + \frac{\partial \Omega}{\partial i} \delta i(t_0) \Delta t_f} \right). \tag{4.15}
\]

Similarly, the out-of-plane thrust component can be determined by squaring Eq. (4.13) and Eq. (4.14) and adding the equations together. This operation eliminates the $\sin \theta$ and $\cos \theta$ terms, and the thrust component is given by

\[
v_h = \pm \frac{h}{r_1} \left( \left( \delta i(t_f) - \delta i(t_0) \right) + \left( \delta \Omega(t_f) - \delta \Omega(t_0) - \left( \frac{\partial \Omega}{\partial a} \delta a(t_0) + \frac{\partial \Omega}{\partial e} \delta e(t_0) + \frac{\partial \Omega}{\partial i} \delta i(t_0) \right) \Delta t_f \right)^2 \sin^2 i \right)^{\frac{1}{2}}. \tag{4.16}
\]

To transform this one thrust solution into a two thrust solution, the firing time obtained from Eq (4.15) is adjusted such that $0 < \theta_1 < \pi$ and then the second firing time is then chosen to half an orbital period later, i.e., $\theta_2 = \theta_1 + \pi$. The out-of-plane thrust magnitude is then split equally between the two thrusts.

If the original $\theta_1$ is negative, the negative square root is taken for $v_h$, otherwise the positive root is taken. The first thrust magnitude then is $v_{h_1} = \frac{v_h}{2}$. The second out-of-plane thrust component is $v_{h_2} = -v_{h_1}$. The remaining four constraint equations, Eqs. (4.4), (4.5), (4.8) and (4.9) can be solved to uniquely determine the remaining four in-plane impulse components.

**Optimal Two-Impulse Firing Times**

The natural question that arises when using Eq. (4.11) to determine the impulsive thrusts is at what times should the thrusts be applied. Since the impulse magnitudes are a function of only the two firing
times, we have the optimization problem:

\[
\begin{align*}
\text{minimize} & \quad J(v_1, v_2) = v_1^T v_1 + v_2^T v_2 \\
& = (\delta \bar{e}(t_f) - \Phi(t_f, t_0)\delta \bar{e}(t_0))^T X^{-T}(t_1, t_2)X^{-1}(t_1, t_2)\left(\delta \bar{e}(t_f)\Phi(t_f, t_0)\delta \bar{e}(t_0)\right)
\end{align*}
\]

with respect to \( t_1, t_2 \)  \( (4.17) \)

subject to \( t_0 < t_k < t_f, \quad k = 1, 2. \)

Vaddi\(^{49} \) showed that when in-plane thrusts are small compared to the out-of-plane thrusts, using Eqs. (4.15), (4.16) are optimal when \( J_2 \) is not considered. The small in-plane thrust assumption holds for formations in near-circular orbits but is generally not the case for high eccentricity orbits.

### 4.2 Feedback Control Using Non-Singular Elements

The development of an analogous impulsive control strategy for non-singular differential elements, \( \delta e_{ns} = [\delta a \ \delta i \ \delta q_1 \ \delta \Omega \ \delta \lambda]^T \), is presented. All orbital element quantities are mean quantities unless otherwise indicated.

The development of the constraint equations for \( \delta q_1 \) and \( \delta q_2 \) is more involved than the other elements, so we begin with them. Differentiating Eq. (3.41) and Eq. (3.42) yields

\[
\begin{align*}
\delta q_1 &= \cos \omega \delta \dot{e} - q_2 \delta \dot{\omega} - \delta q_2 \dot{\omega}, \quad (4.18) \\
\delta q_2 &= \sin \omega \delta \dot{e} + q_1 \delta \dot{\omega} + \delta q_1 \dot{\omega}. \quad (4.19)
\end{align*}
\]

The absolute orbital elements are all chief spacecraft elements. Since the differential drift rate of mean eccentricity is zero, \( \delta \dot{e} \) is any change in \( \delta e \) due to thrusts. The term \( \delta \dot{\omega} \) captures both the nonzero effect of the differential drift of mean argument of periapsis and the influence of any control thrusts. The expanded dynamics are

\[
\begin{bmatrix}
\delta q_1 \\
\delta q_2
\end{bmatrix} = \begin{bmatrix}
-q_2 \frac{\partial \dot{\omega}}{\partial q_1} & -q_2 \frac{\partial \dot{\omega}}{\partial q_2} - \dot{\omega} \\
q_1 \frac{\partial \dot{\omega}}{\partial q_1} + \dot{\omega} & q_1 \frac{\partial \dot{\omega}}{\partial q_2}
\end{bmatrix} \begin{bmatrix}
\delta q_1 \\
\delta q_2
\end{bmatrix} + \begin{bmatrix}
-q_2 \frac{\partial \dot{\omega}}{\partial a} & -q_2 \frac{\partial \dot{\omega}}{\partial i} \\
q_1 \frac{\partial \dot{\omega}}{\partial a} & q_1 \frac{\partial \dot{\omega}}{\partial i}
\end{bmatrix} \begin{bmatrix}
\delta a \\
\delta i
\end{bmatrix}
\]
The state transition matrix for the two state system is

\[
\begin{bmatrix}
\frac{p \sin \theta}{h} & (p + r) \cos \theta_k + re \cos \omega \\
-\frac{p \cos \theta}{h} & (p + r) \sin \theta_k + re \sin \omega
\end{bmatrix}
\begin{bmatrix}
\frac{re \sin \omega \sin \theta}{h} & \tan i \\
\frac{re \cos \omega \sin \theta}{h} & -\tan i
\end{bmatrix}
\begin{bmatrix}
u_r(t) \\
u_\theta(t)
\end{bmatrix}.
\] (4.20)

The dominant term in the state matrix is the mean argument of perigee drift rate, \(\dot{\omega}\), allowing the state matrix to be approximated by

\[
\begin{bmatrix}
-q_2 \frac{\partial \dot{\omega}}{\partial q_1} & -q_2 \frac{\partial \dot{\omega}}{\partial q_2} - \dot{\omega} \\
q_1 \frac{\partial \dot{\omega}}{\partial q_1} + \dot{\omega} & q_1 \frac{\partial \dot{\omega}}{\partial q_2}
\end{bmatrix}
\approx
\begin{bmatrix}
0 & -\dot{\omega} \\
\dot{\omega} & 0
\end{bmatrix}.
\] (4.21)

Assuming \(N\) impulsive thrusts and using the general solution to an LTV system, the state at \(t = t_f\) is given by

\[
\begin{bmatrix}
\delta q_1(t_f) \\
\delta q_2(t_f)
\end{bmatrix}
= \begin{bmatrix}
\cos(\Delta t \dot{\omega}) & -\sin(\Delta t \dot{\omega}) \\
\sin(\Delta t \dot{\omega}) & \cos(\Delta t \dot{\omega})
\end{bmatrix}
\begin{bmatrix}
\delta q_1(t_0) \\
\delta q_2(t_0)
\end{bmatrix}
+ \sum_{i=1}^{N}
\begin{bmatrix}
\cos(\Delta t_i \dot{\omega}) & -\sin(\Delta t_i \dot{\omega}) \\
\sin(\Delta t_i \dot{\omega}) & \cos(\Delta t_i \dot{\omega})
\end{bmatrix}
\begin{bmatrix}
\frac{p \sin \theta_k}{h} & (p + r_i) \cos \theta_k + r_i e \cos \omega \\
-\frac{p \cos \theta_k}{h} & (p + r_i) \sin \theta_k + r_i e \sin \omega
\end{bmatrix}
\begin{bmatrix}
\frac{re \sin \omega \sin \theta_k}{h} & \tan i \\
\frac{re \cos \omega \sin \theta_k}{h} & -\tan i
\end{bmatrix}
\begin{bmatrix}
v_{r_i} \\
v_{\theta_i}
\end{bmatrix}.
\] (4.22)

The state transition matrix for the two state system is

\[
e^{\Delta t \dot{\omega}} = \begin{bmatrix}
\cos(\Delta t \dot{\omega}) & -\sin(\Delta t \dot{\omega}) \\
\sin(\Delta t \dot{\omega}) & \cos(\Delta t \dot{\omega})
\end{bmatrix}
\] (4.23)

and

\[
\int_{t_0}^{t_f} e^{\Delta t(\dot{\omega})} \begin{bmatrix}
-q_2 \frac{\partial \dot{\omega}}{\partial q_1} & -q_2 \frac{\partial \dot{\omega}}{\partial q_2} \\
q_1 \frac{\partial \dot{\omega}}{\partial q_1} & q_1 \frac{\partial \dot{\omega}}{\partial q_2}
\end{bmatrix} \delta a
+ \begin{bmatrix}
-q_2 \frac{\partial \dot{\omega}}{\partial q_1} & -q_2 \frac{\partial \dot{\omega}}{\partial q_2} \\
q_1 \frac{\partial \dot{\omega}}{\partial q_1} & q_1 \frac{\partial \dot{\omega}}{\partial q_2}
\end{bmatrix} \delta i
\] d\tau

\[
\approx
\begin{bmatrix}
\frac{q_1(t_f) - q_1(t_0)}{\dot{\omega}} & \frac{q_2(t_f) - q_2(t_0)}{\dot{\omega}} \\
\frac{q_1(t_f) - q_1(t_0)}{\dot{\omega}} & \frac{q_2(t_f) - q_2(t_0)}{\dot{\omega}}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \dot{\omega}}{\partial a} + \frac{\partial \dot{\omega}}{\partial i} \\
\frac{\partial \dot{\omega}}{\partial a} + \frac{\partial \dot{\omega}}{\partial i}
\end{bmatrix}.
\] (4.24)

The remaining constraint equations are determined in the same manner as in Section 4.1. Differential secular drift rates of the right ascension of the ascending node and mean latitude are approximated to first order. The constraint equations for \(\delta a\) and \(\delta i\) are identical and are not repeated. We also have:

\[
\delta \Omega(t_f) = \delta \Omega(t_0) + \left(\frac{\partial \Omega}{\partial a} \delta a(t_0) + \frac{\partial \Omega}{\partial i} \delta i(t_0) + \frac{\partial \Omega}{\partial q_1} \delta q_1(t_0) + \frac{\partial \Omega}{\partial q_2} \delta q_2(t_0)\right) \Delta t_f + \sum_{k=1}^{N} \begin{bmatrix}
0 & 0 \\
\frac{r_k \sin \theta_k}{h \sin i}
\end{bmatrix} v_k
\]
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\[ + \left( \frac{\partial \Omega}{\partial a} \Delta \delta a_{f_k} + \frac{\partial \Omega}{\partial i} \Delta \delta i_{f_k} + \frac{\partial \Omega}{\partial q_1} \Delta \delta q_{1f_k} + \frac{\partial \Omega}{\partial q_2} \Delta \delta q_{2f_k} \right) \Delta t_{f_k} \right), \]  

(4.25)

\[ \delta \lambda(t_f) = \delta \lambda(t_0) + \left( \frac{\partial \lambda}{\partial a} \Delta \delta a(t_0) + \frac{\partial \lambda}{\partial i} \Delta \delta i(t_0) + \frac{\partial \lambda}{\partial q_1} \Delta \delta q_{1}(t_0) + \frac{\partial \lambda}{\partial q_2} \Delta \delta q_{2}(t_0) \right) \Delta t_{f_0} \]

\[ + \sum_{k=1}^{N} \left\{ \frac{-p e \cos f}{h(1 + \eta)} - \frac{2r \eta}{h} \frac{(p + r) e \sin f}{h} - \frac{r \sin \theta}{h \tan \theta} \right\} v_k \]

\[ + \left( \frac{\partial \lambda}{\partial a} \Delta \delta a_{f_k} + \frac{\partial \lambda}{\partial i} \Delta \delta i_{f_k} + \frac{\partial \lambda}{\partial q_1} \Delta \delta q_{1f_k} + \frac{\partial \lambda}{\partial q_2} \Delta \delta q_{2f_k} \right) \Delta t_{f_k} \right). \]

(4.26)

Solution methods for the constraint equations are the same as in the prequel. For \( N > 2 \), the minimization problem posed in Section 4.1 applies. For \( N = 2 \), either Eq. (4.11) can be used to solve for two prescribed firing times, where in this case the state matrix is by taking partial derivatives of the non-singular mean element drift rates:

\[ \bar{A} = \left. \frac{\partial A_{ns}(\bar{e})}{\partial \bar{e}} \right|_{\bar{e} = \bar{e}_0} = \begin{bmatrix} \bar{A}_{0} \\ \bar{A}_{1} \\ \bar{A}_{2} \\ \bar{A}_{3} \end{bmatrix} \begin{bmatrix} 0_{2x4} \\ 0_{2x2} \end{bmatrix} \]

(4.27)

Alternatively, one can make use of the fact that the out-of-plane thrusts are approximately decoupled from the in-plane thrusts and can be solved for using Eq. (4.16) and Eq. (4.15) but substituting in the non-singular element drift rates. The four in-plane thrusts are determined by solving Eqs. (4.4), (4.22) and (4.26) for \( v_{r1,2} \) and \( v_{\theta1,2} \).

4.3 Stability of the Two-Impulse Control Strategy

The feedback law presented in Eq. (4.11) provides two corrective impulses for a pair of prescribed firing times. A natural question that arises is whether the chosen firing times result in a stable, closed-loop system. In this section, this question of input-output stability of the closed-loop formation using the feedback law from Eq. (4.11) is addressed.

We begin with the differential element state tracking error, \( \zeta(t) = \delta \bar{e}(t) - \delta \bar{e}_r \). Using Eq. (3.40), the
error dynamics for a two-impulse system, where impulses are small, are

$$\dot{\zeta}(t) = \dot{A}(\bar{e}_t)\zeta(t) + B(\bar{e}_t)u_1\delta(t - t_1) + B(\bar{e}_t)u_2\delta(t - t_2).$$  \hfill (4.28)

The assumption is a reasonable one, since the purpose of the control scheme is to maintain a current formation, not to perform any significant reconfigurations. Integrating Eq. (4.28), the evolution over time of the system is given by

$$\zeta(t) = e^{\dot{A}(t-t_0)}\zeta(t_0) + \int_{t_0}^{t} \left( e^{\dot{A}(\tau-t_1)}B(\bar{e}_t(t_1))u_1\delta(\tau-t_1) + e^{\dot{A}(\tau-t_2)}B(\bar{e}_t(t_2))u_2\delta(\tau-t_2) \right) d\tau \right)$$

$$= e^{\dot{A}(t-t_0)}\zeta(t_0) + e^{\dot{A}(t-t_1)}B(\bar{e}_t(t_1))v_1 + e^{\dot{A}(t-t_2)}B(\bar{e}_t(t_2))v_2$$  \hfill (4.29)

where $\delta$ is the Dirac Delta function. An interval of one orbital period of the chief spacecraft, $T$, is considered. Discretizing Eq. (4.29) with an interval of one orbital period, yields the following discrete-time system:

$$\zeta[k+1] = e^{\dot{A}T}\zeta[k] + \begin{bmatrix} e^{\dot{A}(T-t_1)}B(\bar{e}_t(t_1)) & e^{\dot{A}(T-t_2)}B(\bar{e}_t(t_2)) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= A_k\zeta[k] + B_kv[k].$$  \hfill (4.30)

Returning to the two-impulse control law in Eq. (4.11), $\delta(\bar{e}(t_f)) - \delta(\bar{e}(t_0))$ is equivalent to $-\zeta[k]$, so the expression for the control vector $v[k]$ can be obtained using:

$$v[k] = -X^{-1}(\bar{e}_c, t_1, t_2) \left( \zeta[k] + \dot{A}(\bar{e}_c)T\delta(\bar{e}(t_0)) \right).$$  \hfill (4.31)

Substituting Eq. (4.31) into Eq. (4.30) gives an analytical description of the closed-loop system:

$$\zeta[k+1] = \left( A_k - B_kX^{-1}(\bar{e}_c, t_1, t_2) \right)\zeta[k] - B_kX^{-1}(\bar{e}_c, t_1, t_2)\dot{A}(\bar{e}_c)\delta(\bar{e}(t_0))T$$  \hfill (4.32)

For bounded-input bounded-output stability,

$$|\lambda_i| < 1, \quad i = 1...6,$$

where $\lambda_i$ are the eigenvalues of the closed-loop state matrix $A_k - B_kX^{-1}$. 
4.4 Simulation Results

This section proceeds to evaluate the proposed control strategy for the purpose of formation keeping. Both formations in near-circular and highly elliptical orbits are considered. Formation-keeping performance is compared to Schaub and Alfriend’s\textsuperscript{50} three impulsive control strategy, which does not consider $J_2$ in its formulation. Control performance using classical and non-singular differential element sets is examined. The impact of using more than two thrusts is also considered.

For all simulation in this chapter and subsequent chapters, the reported $\Delta V$ quantities are calculated by taking the sum of the two-norm of each thrust vector applied in the control period.

The analytic two-impulse scheme, using Eqs. (4.15)-(4.16), and an optimal four-impulse scheme are considered in this section. For the control problem when $N > 2$, the solution is calculated by using the MATLAB\textsuperscript{R} nonlinear optimization function fmincon to solve the optimization problem posed in Section (4.1).

![Test formation trajectories for low and high eccentricity.](a) PCO formation trajectory. (b) Magnetospheric Multiscale formation trajectory.

Figure 4.1: Test formation trajectories for low and high eccentricity.

4.4.1 Low Eccentricity Projected Circular Orbit

The low eccentricity formation considered is a 1 km projected circular orbit (PCO) formation defined by the initial conditions in Table 4.1 and illustrated in Fig 4.1(a). Relative position errors for both classical differential element and non-singular element control strategies are presented in Figure 4.2. Note that as per Section 3.2.4, in order to maintain the desired $\varrho$, $a_0$, and $\beta_0$, the reference differential elements for the singular case are recalculated once per orbit to reflect drift in chief argument of perigee. This
recalculation is the cause of the sudden discontinuity in position error observed in out-of-plane error in Figure 4.2(a). Performance between classic and non-singular element strategies is very similar in terms of relative position error. The non-singular element scheme has better tracking in the out-of-plane direction, however, the classical element strategy is superior in both radial and along-track directions. The non-singular strategy does require less $\Delta V$ per orbit: 10.6 mm/s as opposed to 12.4 mm/s required by the classical element strategy.

In comparison to Schaub and Alfriend’s three-impulse strategy, which does not include the effect of $J_2$ on the differential elements, the two-impulse classical element strategy achieves the identical position error for the in-plane components. For the out-of-plane position error, the advantage of using mean elements is evident. The inclusion of the secular drift rates in the proposed control formulation results in lower out-of-plane position error when using either classical and non-singular element control strategies. The differences in out-of-plane error are presented in Figure 4.3. The $\Delta V$ required for the three-impulse strategy is 12.2 mm/s per orbit. Since Schaub’s three impulse strategy also uses the classical differential orbital element set, per orbit recalculation of the the reference elements is required to maintain a fixed $\varrho$, $\alpha_0$, and $\beta_0$.

For this low eccentricity formation, a numerical study reveals that increasing the number of thrusts per orbit provides no additional benefit. The optimal two-impulse solution has a total $\Delta V$ of 8.365 mm/s. Increasing $N$ does not yield any improved results and in fact the numerical optimizer continues to converge to a nominally two-impulse strategy, with all but two impulses being determined to be zero or near-zero. As a result, position error is not significantly different for these cases. Table 4.2 compares optimal firing times and thrust vectors for $N = 2, 4$ and 10.

### Table 4.1: Projected near-circular orbit initial conditions.

<table>
<thead>
<tr>
<th>Chief elements</th>
<th>Relative orbit parameters</th>
<th>Deputy differential elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>quantity</td>
<td>unit</td>
<td>value</td>
</tr>
<tr>
<td>$\bar{a}$</td>
<td>km</td>
<td>7189.795</td>
</tr>
<tr>
<td>$\vec{e}$</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>$\vec{i}$</td>
<td>deg</td>
<td>90.0</td>
</tr>
<tr>
<td>$\bar{\Omega}$</td>
<td>deg</td>
<td>0.0</td>
</tr>
<tr>
<td>$\bar{\omega}$</td>
<td>deg</td>
<td>0.003</td>
</tr>
<tr>
<td>$\bar{M}$</td>
<td>deg</td>
<td>359.997</td>
</tr>
</tbody>
</table>
4.4.2 Classical versus Non-singular Element Control

It has been emphasized that the formation-keeping results using the classical differential orbital element strategy were obtained by recalculating the reference differential orbital elements once per orbit to account for the drift of the chief spacecraft’s argument of perigee. The recalculation was based on maintaining a fixed $\varphi$, $\alpha_0$, and $\beta_0$. As discussed in Section 3.2.4, no such recalculation is necessary when...
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Figure 4.3: Relative out-of-plane position error comparison - PCO formation.

<table>
<thead>
<tr>
<th>$\theta$ [rad]</th>
<th>$v_r$ [mm/s]</th>
<th>$v_\theta$ [mm/s]</th>
<th>$v_h$ [mm/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0111</td>
<td>-0.0019</td>
<td>0.0027</td>
<td>0.0001</td>
</tr>
<tr>
<td>1.5714</td>
<td>-0.0041</td>
<td>-0.0026</td>
<td>8.3642</td>
</tr>
<tr>
<td>3.1463</td>
<td>0.0000</td>
<td>-0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>4.6210</td>
<td>0.0010</td>
<td>0.0024</td>
<td>4.4001</td>
</tr>
</tbody>
</table>

Table 4.2: Optimal thrust solutions for PCO maintenance, for $N = 2, 4, 10$.

using the non-singular differential element strategy. Figure 4.4 illustrates the difference in formation-keeping performance between using the classical element strategy with and without reference state recalculation.

Figure 4.4: Relative trajectory of a deputy in the along-track/out-of-plane plane using the two-impulse classical element strategy.

(a) No reference differential element update.  (b) Reference differential updated once per orbit according to a desired $\phi, a_0, \text{ and } \beta_0$. 
In both cases, the control strategy successfully tracks the desired reference trajectory. But as can be seen in Figure 4.4(a), without updating the reference differential elements, the reference relative trajectory changes over time, altering the formation geometry. Compensating for chief argument of perigee drift incurs a small $\Delta V$ cost. For the example in Figure 4.4, which used the initial conditions in Table 4.1, the total $\Delta V$ cost for formation keeping without reference state recalculation is 8.4 mm/s, compared to 12.2 mm/s used with reference state recalculation.

### 4.4.3 Formation Stability

The stability of the prescribed firing time two-thrust solution for the PCO formation is now considered. The stability criteria presented in Section 4.3 holds true if the maximum eigenvalue in modulus, $\lambda_m = \max \{ |\lambda_i| : \forall i \in [1, 6] \}$, is less than 1. An eigenvalue map of $\lambda_m$ can be constructed for all possible firing time pairs, here specified in terms of true anomaly, $f = (f_1, f_2) \in [0, 2\pi)$ and $f_2 \neq f_1$, to illustrate possible regions of instability where the maximum eigenvalue in modulus lies outside the unit disc.

The eigenvalue map for the low eccentricity PCO formation is shown in Figure 4.5(a). The lower left half of the plot is not populated with data, since only $f_2 > f_1$ are considered. It is seen that the majority of firing time combinations are stable. The region of instability occurs along the line $f_2 = f_1 + \pi$. At this point, the matrix $X$ becomes near-singular with a determinant of $O(10^{-17})$. Its inaccurate inversion results in an exceedingly high $\Delta V$ requirement. This result can be seen in the estimated $\Delta V$ map in Figure 4.5(b). As firing time pairs approach $f_2 = f_1 + \pi$, estimated $\Delta V$ grows. High $\Delta V$ also occurs, in general, as the $f_1$ approaches $f_2$.

![Figure 4.5: PCO stability analysis and estimated fuel cost.](image)
This instability region is exactly the region in which the alternative two-thrust solution (Eqs. (4.15), (4.16)) operates. By decoupling the in-plane and out-of-plane thrusts, the singularity that occurs in \( X \) at \( f_2 = f_1 + \pi \) is avoided.

### 4.4.4 High Eccentricity Magnetospheric Multiscale Formation

The formation proposed for NASA’s planned Magnetospheric Multiscale (MMS) mission is used as a realistic example of a formation in a highly elliptical orbit. The formation will operate in an orbit with a semimajor axis of \( a = 42,905 \text{ km} \) and eccentricities varying between \( e = 0.81818 \) and \( e = 0.9084. \) Although in these orbits the specific magnitude of the \( J_2 \) gravity perturbation is on the order of \( 1 \times 10^{-6} \text{ N/kg} \), differences in the mean orbital element drift rates of the chief and deputy spacecraft can cause formation drift on the order of \( 1 - 100 \text{ m/orbit} \), depending on the formation. Although four spacecraft will eventually comprise the entire MMS formation, for simplicity, only two spacecraft are considered here. The two-spacecraft formation, illustrated in Figure 4.1(b), is defined by the initial conditions in Table 4.3 and can be found in Ref. 15. The formation is a closed, periodic formation with an average chief-deputy separation of 32 km. Uncontrolled formation drift in the out-of-plane direction causes 500 m in position error after five orbits.

<table>
<thead>
<tr>
<th>Chief elements</th>
<th>Deputy differential elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{a} )</td>
<td>km</td>
</tr>
<tr>
<td>( \delta \bar{a} )</td>
<td>m</td>
</tr>
<tr>
<td>( \bar{\varepsilon} )</td>
<td>0.81818</td>
</tr>
<tr>
<td>( \delta \bar{\varepsilon} )</td>
<td>( 10^{-3} )</td>
</tr>
<tr>
<td>( \bar{i} )</td>
<td>deg</td>
</tr>
<tr>
<td>( \delta \bar{i} )</td>
<td>deg, ( 10^{-4} )</td>
</tr>
<tr>
<td>( \bar{\Omega} )</td>
<td>deg</td>
</tr>
<tr>
<td>( \delta \bar{\Omega} )</td>
<td>deg, ( 10^{-3} )</td>
</tr>
<tr>
<td>( \bar{\omega} )</td>
<td>deg</td>
</tr>
<tr>
<td>( \delta \bar{\omega} )</td>
<td>deg, ( 10^{-3} )</td>
</tr>
<tr>
<td>( \bar{M} )</td>
<td>deg</td>
</tr>
<tr>
<td>( \delta \bar{M} )</td>
<td>deg, ( 10^{-2} )</td>
</tr>
</tbody>
</table>

Table 4.3: MMS formation initial orbital elements

Figure 4.6 illustrates the differential orbital element errors for three control strategies: the analytical two-impulse solution and an optimal four-impulse solution are compared once again to Schaub and Alfriend’s three-impulse solution. Note that when \( N = 4 \), as was seen with the PCO formation, the numerical optimizer converges to a minimum where two of the four thrusts are zero. For all three control strategies, singular elements are used.

The difference in performance between the two proposed control strategies and Schaub’s three-impulse strategy is substantial. Due to the inclusion of the effect of \( J_2 \), the two and four-impulse schemes have improved tracking of the differential ascending node, argument of perigee, and significantly better tracking of mean anomaly. This improved differential element tracking results in reduced relative
position errors, as shown in Figure 4.7. The improvement is most pronounced in the radial and along-track directions; however, all three directions have a superior tracking error. Relative position error statistics for the three cases are summarized in Table 4.4. Schaub and Alfriend’s three-impulse strategy does, however, have minimal effect on differential semimajor axis and eccentricity, whereas both the proposed two- and four-impulse strategies cause periodic jumps in those differential elements, which may be undesirable.

Figure 4.6: Comparison of differential element errors - MMS formation.

Figure 4.7: Relative position component errors - MMS formation.
For the PCO formation, the decoupled analytical control strategy was yielding a near-optimal solution, compared to the numerically determined strategy. This optimality of the analytical two-thrust solution is not seen for the case of the MMS formation. Total $\Delta V$ required for the two-impulse analytical strategy is 12.1 mm/s and for Schaub’s three-impulse strategy, it is 12.7 mm/s per orbit; the optimal four-impulse strategy uses 6.7 mm/s per orbit. The four-impulse scheme also has the best position tracking error performance. The two-impulse strategy has slightly larger position error than the four-impulse strategy; however, its $\Delta V$ requirement is similar to that of Schaub’s three-impulse scheme.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>RMS Error $x_r$ [m]</th>
<th>RMS Error $x_\theta$ [m]</th>
<th>RMS Error $x_h$ [m]</th>
<th>Max Absolute Error $x_r$ [m]</th>
<th>Max Absolute Error $x_\theta$ [m]</th>
<th>Max Absolute Error $x_h$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical Two-impulse</td>
<td>50</td>
<td>56</td>
<td>27</td>
<td>66</td>
<td>173</td>
<td>50</td>
</tr>
<tr>
<td>Optimal Four-impulse</td>
<td>39</td>
<td>87</td>
<td>13</td>
<td>52</td>
<td>107</td>
<td>21</td>
</tr>
<tr>
<td>Schaub’s Three-impulse</td>
<td>187</td>
<td>202</td>
<td>36</td>
<td>507</td>
<td>1095</td>
<td>61</td>
</tr>
</tbody>
</table>

Table 4.4: Position error statistics for MMS formation.

**Effect of Thrust Number**

Simulations show that having $N > 2$ control thrusts can result in some modest $\Delta V$ savings for this formation case, as seen in Figure 4.8(a), where $\Delta V$ costs are compared for different $N$. The numerical optimizer continues to converge to solutions where some of the thrust magnitudes are zero but it no longer converges to solutions with only two nonzero thrust for any $N$. Figure 4.8(b) plots the thrust magnitude profile over one orbit for different $N$, showing that the majority of the large magnitude thrusts are applied as the spacecraft is approaching apogee and in the period following apogee. While some benefit is gained in having $N > 2$, there is negligible $\Delta V$ improvement when $N \geq 10$.

**Formation Stability**

The stability analysis finds that there are a number of unstable firing time pairs for the high-eccentricity MMS formation. As was the case for the low-eccentricity PCO formation, the firing times pairs along the line $f_2 = f_1 + \pi$ are unstable, due to the matrix $X$ being ill-conditioned. In addition to these pairs, when firing time pairs are close to each other and near perigee, the formation also becomes unstable. These regions of instability are illustrated in Figure 4.9(a). The high eccentricity appears to play a significant role in the stability of the formation. The effect of eccentricity on formation stability can be seen by comparing stability plots for formations defined by the same differential elements but in lower eccentricities. The corresponding plots for MMS-like formations in orbits with eccentricity of $e = 0.718181$ and $e = 0.618181$ are shown in Figure 4.9(b) and 4.9(c), respectively. The region along
Figure 4.8: Controller performance for various impulse numbers, $N$.

$$f_2 = f_1 + \pi$$ remains for all three eccentricities; however, the regions close to perigee diminish in size and ultimately vanish as eccentricity decreases.

Figure 4.9: Stability comparison at different eccentricities.

The $\Delta V$ required for formation maintenance using Eq. (4.11) varies significantly for the MMS-like formation. A log-plot of the $\Delta V$ for different firing times is shown in Figure 4.10(a). The regions of instability correspond to areas with large $\Delta V$ requirements, making them doubly undesirable.

The stability analysis assumes that maintenance thrusts are sufficiently small to be considered impulsive. Figure 4.10(a) shows that for some firing time pairs, this assumption is clearly violated: for some firing time pairs, the required $\Delta V$ approaches 1 m/s. For a single 1 N hydrazine thruster, of a kind comparable to the ones used aboard the PRISMA formation flying mission, a 1 m/s change in velocity requires approximately two and a half minutes of thrust, assuming a 150 kg spacecraft. Such a burn can no longer be considered impulsive, therefore the stability analysis for that firing time pair is no longer accurate.
Chapter 4. Impulsive Formation Control

4.5 Need for Including Eccentricity

This work has extended Vadali et al.’s formation-keeping strategy for formations in circular orbits, presented in Ref. 17, to be valid for all eccentricities, $0 < e < 1$. In this section, we briefly explore when making a circular orbit assumption no longer holds and at what eccentricities the eccentric equations are necessary.

A PCO is considered for this study, that is to say, $\rho=[500 0 1000]^T$ m, $\alpha_0 = 0^\circ$ and $\beta_0 = 0^\circ$. As the eccentricity of the chief orbit increases, as discussed in Chapter 3, the formation does not project a circle in the along-track/out-of-plane plane of the LVLH frame, but rather, it makes more of a triangular path with rounded vertices. In order to accommodate larger eccentricities, a chief orbit with a semimajor axis of $a = 20800$ km and an inclination of $i = 30^\circ$ is considered. For this orbit and formation, uncontrolled drift due to $I_2$ results in a maximum position error ranging from 47 m to 57 m after fifteen orbits, depending on eccentricity.

Formation-keeping performance of the proposed eccentric-orbit control strategy and Vadali’s cir-

Figure 4.10: Two-thrust $\Delta V$ requirement for prescribed $f$. 

This result was verified by simulating a formation controlled with a stable firing-time pair but one with a large $\Delta V$ cost and including the dynamics of a 1 N hydrazine thruster. There are many such firing time pairs for the MMS-like formation in an orbit with $e = 0.618181$, as can been seen in Figs. 4.9(c) and 4.10(b). The simulated formation was stable when the full thrust magnitude was treated as impulsive and applied instantaneously. When the actuator dynamics were included, the formation geometry was not maintained.
circular strategy is compared in Table 4.5 in terms of RMS position error norm and required $\Delta V$. For consistency with Vadali et al.’s equations, the non-singular element formulation is used for the eccentric equations. Orbit eccentricities from $e = 0$ to $e = 0.4$ are considered. For circular orbits, the performance of the two strategies is virtually identical. Unsurprisingly, with the introduction of nonzero eccentricity, the performances of the two strategies begin to diverge. The eccentric equations consistently achieve small relative position errors and require less $\Delta V$ to do so. The larger the eccentricity, the larger the difference in performance. For the example considered here, the circular equations begin to fail to control the formation at $e \approx 0.38$. For $e \geq 0.38$, the relative position error grows unbounded when using the circular equations. We conclude that even for small eccentricities, some additional reduction in required $\Delta V$ can be realized by using the full eccentric-orbit strategy over the circular orbit strategy, although using the circular equations for formations in low eccentricity orbits does not appear to impact formation stability.

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>RMS Pos. Error [m]</th>
<th>$\Delta V$ [mm/s/orbit]</th>
</tr>
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<td>0.56</td>
<td>0.56</td>
</tr>
<tr>
<td>0.1</td>
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<td>0.46</td>
</tr>
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<td>0.3</td>
<td>1.51</td>
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<td>1.93</td>
<td>1.10</td>
</tr>
<tr>
<td>0.38</td>
<td>3.72†</td>
<td>1.27</td>
</tr>
<tr>
<td>0.4</td>
<td>15.03†</td>
<td>1.40</td>
</tr>
</tbody>
</table>

Table 4.5: RMS position error norm and $\Delta V$ requirement for circular and eccentric impulsive formation-keep strategies. (†) denotes systems that are unstable and whose error norms grow unbounded over time.

### 4.6 Chapter Conclusions

Novel constraint equations are presented for an $N$-impulse control strategy to mitigate the $J_2$ perturbation. Equations for both the classical and non-singular orbital element sets have been presented. These equations are a generalization of the control strategy for formations in circular orbits proposed in Ref. 17. For small eccentricities, the circular constraint equations perform similarly to the eccentric equations, although the latter consistently achieves smaller position error and require less $\Delta V$ per orbit. The inclusion of the linearized effect of the $J_2$ perturbation in the control law results in improved position tracking when compared to Schaub and Alfriend’s differential element-based three-impulse control law found in Ref. 50. The required $\Delta V$ for the proposed control law is equal to or less than what
is required by Schaub and Alfriend’s strategy. This improvement is evident for formations in both LEO and HEO. Several solution methods to the constraint equations were considered: for $N = 2$, in-plane and out-of-plane thrusts could be decoupled and solved separately, yielding a particular pair of firing times, separated by $\pi$ rad of true latitude. Otherwise, two firing times could be prescribed and the six constraint equations could be solved for the six thrust components. For $N > 2$, numerical optimization was required to determine thrust components and firing times. Stability of the prescribed $N = 2$ thrust solution can be determined. For formations in low eccentricity, nearly all firing times pairs were stable. The system is unstable only when $\theta_2 = \theta_1 + \pi$, where the inversion of the constraint equations is singular. For the MMS formation in HEO, additional unstable firing times were found when both firing times were close to perigee. In addition to being unstable, these firing time pairs also required a large amount of $\Delta V$, making them unattractive for formation keeping. While this instability appears to vanish at lower eccentricities, the high $\Delta V$ cost remains and applying the solution with realistic thruster dynamics results in an unstable formation.
Chapter 5

LORENTZ-AUGMENTED FORMATION CONTROL

Electromagnetic interaction with the Earth’s magnetic field has long been a means for spacecraft to realize some form of attitude control. More recently, methods exploiting similar interaction have been proposed for altering the orbit of spacecraft without the need for chemical propellant. Specifically, a spacecraft with a net electrical charge experiences the Lorentz force as it orbits. If the charge can be modulated, the Lorentz force can provide a useful acceleration that is normally provided by thrusters. It is effectively a propellantless alternative to thruster-based spacecraft formation control. The challenge is that the Lorentz force is constrained to act along the vector obtained from the cross product of the charge’s relative velocity and the local magnetic field vector.

This chapter explores control strategies that attempt to make use of the Lorentz force for the purpose of spacecraft formation maintenance in the presence of disturbances. While a desired formation geometry cannot be maintained with the Lorentz force alone — a result that is shown in this chapter — combining the Lorentz force with some limited thruster control can yield significant fuel savings. The goal for the hybrid Lorentz force/thruster control strategies within this chapter is to realize the majority of the required control effort with the Lorentz force, with the remainder provided by thrusters.

The technology required to realize a Lorentz-augmented spacecraft is currently at a low state of readiness. Specific charge quantities feasible with current technology are small and range from $1 \times 10^{-5}$ C/kg to $1 \times 10^{-1}$ C/kg. Although no Lorentz-augmented spacecraft has been constructed yet, spacecraft architecture has been explored in the literature: Peck$^{18}$ proposed charge control through the use of electron and ion beams with the charge being retained on a capacitive sphere. Spacecraft charging
dynamics were reported to have a time constant on the order of seconds. Lorentz-augmented spacecraft architecture and power consumption was further explored by Streetman and Peck. One key current technological limitation identified is that the proposed architecture is thought to be restricted to only a negative charge polarity.

The influence of the Lorentz force on a spacecraft’s orbital elements has been previously investigated by considering the force’s impact on orbital energy and angular momentum and by treating the Lorentz force as a perturbation and studying its influence via Gauss’s variational equations (GVE). Numerous applications for Lorentz-augmented spacecraft have been identified. Streetman et al. investigated new methods for maintaining a desired ground-track of spacecraft and mitigating the effects of the \( J_2 \) gravity perturbation with the Lorentz force. The use of the Lorentz force to dissipate the orbital energy of a spacecraft for Jovian insertion was studied by Atchison et al. Specific charge requirements for these applications, however, often exceeds what is currently technologically feasible.

Spacecraft formation flying was identified by Peck as an application that requires specific charge magnitudes in the range of what is considered achievable by adapting current technology. In-plane formation reconfiguration was demonstrated using a relative Cartesian tracking law by Peck et al. Pollock et al. presented analytical relative motion equations for Lorentz-augmented spacecraft valid for circular, inclined orbits. Spacecraft phasing and rendezvous as well as fly-around maneuvers using the Lorentz force were also explored. The stability of solutions to the relative dynamics of the spacecraft formations subject to the Lorentz force was investigated by Yamakawa et al. Pollock investigated a hybrid formation control scheme using the Lorentz force with the Coulomb force between the spacecraft in formation; however, this formation scheme is ineffective in low Earth orbit (LEO), since the Coulomb force is negligible between charged spacecraft due to the Debye length being much smaller than the scale of the spacecraft. The hybrid Lorentz-thruster scheme proposed in this chapter is not be limited by the Debye length since the Debye length only impacts the effectiveness of the Coulomb force, thus allowing the proposed control scheme to be applied to formations in LEO, as well as formations with baselines that are much larger than the Debye lengths in both low and high orbits.

A charged spacecraft moving in LEO attracts particles of the opposite charge from the local plasma environment, forming a particle sheath, sometimes called a Debye shield, with a thickness on the order of the Debye length around the spacecraft. One may expect the sheath then to effectively neutralize the charge of the spacecraft and negate any Lorentz force effects. This is not the case. The Debye shielding does not block magnetic fields from interacting with charge. This is corroborated by the observation of natural Lorentz force phenomena in the dust rings of Jupiter.
In this chapter, the \( j_2 \) zonal harmonic is again the primary perturbation the spacecraft formation experiences. A mean orbital element approach is taken for modeling the relative spacecraft state and designing a suitable control law. By using differential orbital elements, the approach to designing the control law is not limited to a specific orbit inclination or eccentricity. Through the appropriate choice of orbital element sets, singularities that occur when \( e = 0 \) or \( i = 0^\circ \) can be avoided.

### 5.1 The Geomagnetic Field

Simulation results in this thesis use the 11th Generation International Geomagnetic Reference Field (IGRF)\(^{21}\) to model the Earth’s magnetic field. This model represents the field with a geomagnetic scalar potential calculated by an infinite series of spherical harmonics:

\[
V(r, \theta, \phi, t) = R_\oplus \sum_{n=1}^{N} \sum_{m=0}^{n} \left( \frac{R_\oplus}{r} \right)^{n+1} \left[ g_n^m(t) \cos(m\phi) + h_n^m(t) \sin(m\phi) \right] P_n^m(\cos \theta),
\]  

(5.1)

where \( g_n^m \) and \( h_n^m \) are the Gauss coefficients found in Ref. 21, \( r \) is the radial distance from the centre of the Earth, \( \theta \) and \( \phi \) are the geocentric latitude and East longitude of the spacecraft, and \( P_n^m(\cos \theta) \) is the normalized associated Legendre polynomial of degree \( n \) and order \( m \). The magnetic field vector at a specific point can be obtained by taking the gradient of the potential at that point

\[
\vec{b}_m = -\nabla V.
\]  

(5.2)

For the design of control strategies, the simpler tilted dipole model is used, which expands the IGRF model to only the first degree. According to the IGRF-11 model, the first three Gaussian coefficients for the year 2010 are

\[
g_0^1 = -29496.5 \text{ nT}, \quad g_1^1 = -1585.9 \text{ nT}, \quad h_1^1 = 4946.1 \text{ nT},
\]

and the dipole strength is \( H_0 = \sqrt{g_0^2 + g_1^2 + h_1^2} \). If the effects of the Earth’s precession and nutation are ignored, the magnetic dipole unit vector in the geocentric inertial frame is given by\(^{62}\)

\[
\hat{\mathbf{m}} = F_m^T \begin{bmatrix} \sin \theta \cos \lambda_m \\ \sin \theta \sin \lambda_m \\ \cos \theta \end{bmatrix},
\]  

(5.3)
where the coelevation, $\theta_m'$, and the East longitude of the dipole, $\lambda_m$, are functions of the Gauss coefficients and time:

$$\theta_m' = \arccos \left( \frac{q_1^0}{H_0} \right),$$  \hspace{1cm} (5.4) \\
$$\phi_m' = \arctan \left( \frac{h_1^1}{g_1^1} \right),$$  \hspace{1cm} (5.5) \\
$$\lambda_m = \lambda_{GMST}(t) + \phi_m'.$$  \hspace{1cm} (5.6)

The angle $\lambda_{GMST}$ is the right ascension of the Greenwich Meridian at the simulation time. Using the unit dipole vector, the magnetic field vector expressed in the geocentric inertial frame is

$$\mathbf{b}_m(t, e) = H_0 \left( \frac{R_\oplus}{r} \right)^3 \left[ 3 \left( \hat{\mathbf{m}}(t) \cdot \hat{\mathbf{r}}(t) \right) \hat{\mathbf{r}}(t) - \hat{\mathbf{m}}(t) \right],$$  \hspace{1cm} (5.7)

where $\mathbf{r}$ is the position vector of the point of interest in the GCI frame. The magnetic field vector, $\mathbf{b}_m(t, e)$, is written as a function of a spacecraft’s orbital elements, $e$, in order to reflect its dependence on the inertial position of the spacecraft.

### 5.1.1 More On The Lorentz Force

The Lorentz force was briefly introduced in Section 3.1.2. For a charge-carrying spacecraft, the Lorentz force experienced is

$$\mathbf{f}_L(t, e) = \frac{q(t)}{m} \mathbf{v}_{rel} \times \mathbf{b}_m(t, e).$$  \hspace{1cm} (5.8)

As discussed in Section 3.1.1, the electrical field term is negligible and has been omitted. The term $\mathbf{v}_{rel} = \dot{\mathbf{r}}(t) - \mathbf{\omega}_\oplus \times \mathbf{r}(t)$ is the spacecraft’s velocity relative to the Earth’s magnetic field. What follows is a brief discussion of what kind of formation corrections the Lorentz force is suitable for.

As a means of spacecraft actuation, the Lorentz force is quite limited. The behaviour of the Lorentz force is dictated by the orbit that a charged spacecraft is in: the direction of the Lorentz force is determined by the magnetic field vector at the spacecraft’s position and the spacecraft’s velocity relative to the magnetic field. Once in orbit, the magnetic field a spacecraft experiences and its velocity relative to the field are effectively set and do not significantly alter unless a transfer to another orbit is performed. Consequently, the only parameters that can be controlled to effect the behaviour of the Lorentz force are the magnitude of the spacecraft’s charge and its polarity.

Since the Earth’s magnetic field rotates with the angular velocity of the Earth’s rotation, spacecraft
in low orbit, where their velocities relative to the magnetic field are large, are best suited for employing the Lorentz force. Modest specific charges can result in meaningful accelerations for formation-flying applications. In higher orbits, both the magnetic field strength weakens and relative velocity decreases, so a spacecraft needs larger specific charges to realize the same accelerations. In geostationary orbits, the Lorentz force vanishes entirely, since the velocity relative to the Earth’s magnetic field is zero. Fig. 5.1(a) illustrates the RMS magnitude of the acceleration caused by the Lorentz force for a spacecraft with a 10 $\mu$C/kg specific charge for a range of orbit inclinations and semimajor axes. The significant dip in magnitude in the plot corresponds to geostationary orbit, where the relative velocity between the spacecraft and magnetic field is zero.

Differential $J_2$ is the primary perturbation that a spacecraft formation experiences. For a 1 km spacecraft separation, as seen in Fig. 5.1(b), the RMS acceleration due to differential $J_2$ is weaker than the RMS Lorentz acceleration from a 10 $\mu$C/kg specific charge, so it is expected that specific charges of 10–100 $\mu$ C/kg is necessary for formation maintenance, provided the Lorentz acceleration is in the right direction.

This work focuses on using the Lorentz force for formations in LEO. Moderate eccentricities are not feasible in LEO, so this discussion is confined to near-circular orbits. The inclination of the orbit dictates the direction in which the Lorentz force acts in over an orbit. Fig. 5.2 illustrates the magnitude of Lorentz force components expressed in the LVLH frame for orbits with semimajor axes of 6592 km, 7092 km and 8092 km, and inclinations ranging from 0° to 90°. In polar orbits, the Lorentz force primarily acts perpendicular to the orbit plane in the out-of-plane direction; in near equatorial orbits,
its largest component is the radial component. The along-track component is the smallest component over all inclinations, which is not surprising, given that the Lorentz force vector results from the cross product of the spacecraft’s relative velocity and the magnetic field vector. For a fixed inclination and eccentricity and specific charge, an increase in semimajor axis only causes the strength of the Lorentz force to decrease; the time-varying behaviour of the Lorentz force remains the same.

![Figure 5.2: Lorentz force acceleration in the LVLH frame for a circular orbit, $q/m = 100 \mu C/kg$, for three different semimajor axes: 6592 km, 7092 km, 8092 km.](image)

For near-polar orbits, the Lorentz force is well suited for correcting drift in the differential right ascension and argument of perigee — both elements can be altered with an out-of-plane acceleration. In near-equatorial orbits, where the force acts primarily in the radial direction, correcting differential mean anomaly becomes feasible.

### 5.2 Lorentz-Augmented Formation Flying with Continuous Thrust

The relative spacecraft state of interest is the mean differential element error, $\zeta(t) = \delta \vec{e} - \delta \vec{e}_r$, and the state dynamics for an arbitrary acceleration, $u(t)$, are given in Eq. (3.40). Singular orbital elements are used. When a deputy spacecraft is subjected to the Lorentz force, the linearized differential element
error dynamics, Eq. (3.40), become

\[ \dot{\zeta}(t) = \dot{A}(\vec{e}_r)\zeta(t) + B(\vec{e}_r)(v_{rel}(t) \times \vec{b}_m(t, \vec{e}_r)) \frac{q(t)}{m}, \]

where the spacecraft’s position and velocity vectors, and the Earth’s magnetic field and angular velocity vectors are all appropriately expressed in the local spacecraft orbital frame. The state matrix \( \dot{A} \) is defined in Eq. (3.39) and \( B \) corresponds to the GVE, defined in Eq. (3.10). A new input matrix is defined for the Lorentz-augmented dynamics

\[ B_L(\vec{e}_r) := B(\vec{e}_r)(v_{rel}(t) \times \vec{b}_m(t, \vec{e}_r)) \]

such that

\[ \dot{\zeta}(t) = \dot{A}(\vec{e}_r)\zeta(t) + B_L(\vec{e}_r)\frac{q(t)}{m}. \]

One implementation note: for all Lorentz-augmented simulations in this chapter and the next, the differential semimajor axis state is scaled by the chief semimajor axis so that it is the same order of magnitude as the other five orbital elements. The scaled differential element vector is \( \delta e = [\delta a/a \, \delta e \, \delta i \, \delta \Omega \, \delta \omega \, \delta M]^T \). The matrices \( \dot{A} \) and \( B \) are appropriately scaled by \( a \) to be consistent with the new state vector. The scaling imbalance is also evident in the GVE, where the terms for the semimajor axis are typically three orders of magnitude larger than the terms for the other orbital elements. Normalizing the differential semimajor axis avoids numerical difficulties that would otherwise arise when solving time-varying Riccati equations for the spacecraft control laws.

### 5.2.1 Controllability

It would be ideal if completely propellantless actuation could be realized using the Lorentz force. Pollock et al.,\(^58\) show that for circular, equatorial orbits, propellantless fly-around and along-track rendezvous maneuvers are possible using only the Lorentz force. In general, however, due to the constrained nature of the Lorentz force, the full state is not fully controllable when only using the Lorentz force, alone.

The controllability of the Lorentz-augmented relative spacecraft state is determined by calculating the controllability Gramian (see Chapter 2, Eq. (2.12)) for the system described by Eq. (5.11). For the Lorentz-augmented system, \( \Phi(t, \tau) = e^{\hat{A}(t-\tau)} \) and \( B(t) = B_L(\vec{e}_r) \). The Gramian matrix has been calculated for near-circular orbits with a range of possible semimajor axes and inclinations. The titled-dipole magnetic field model was used. The results are presented in Fig. 5.3. For comparison, the second smallest eigenvalue is also plotted. Over the entire range of inclinations and semimajor axes, the minimum eigenvalue of the Gramian matrix, \( W(t, t_0) \), is smaller than \( 4 \times 10^{-10} \), and consistently several
orders in magnitude smaller than the next largest eigenvalue, indicating the system is not completely controllable. This uncontrollability necessitates some additional form of control, such as thruster actuation, to render the formation completely controllable.

![Controllability plot of the differential orbital element system.](image)

Figure 5.3: Controllability plot of the differential orbital element system.

The nature of the uncontrollable subspace appears to stem from the cross product of the relative velocity of the spacecraft with the local magnetic field vector. In LEO, the term $\omega \times \mathbf{r}(t)$ is small, resulting in the spacecraft’s velocity relative to the magnetic field being nearly equal to its total inertial velocity. Consequently, the Lorentz force acts in a direction that is nearly perpendicular to the velocity of the spacecraft. Low-Earth orbits typically have very small eccentricities, so the along-track vector of the LVLH frame is nearly aligned with the spacecraft’s velocity vector. Therefore, there is only a very small component of the Lorentz force acting in the along-track direction. How does this impact the Lorentz force’s ability to affect the orbital elements?

For near-circular low-Earth orbits, the dominant term in Gauss’s variational equation for semimajor axis is in the along-track direction since, when $e \approx 0$, the radial term vanishes and there is no out-of-plane term. Due to the limited amount of Lorentz force acting in the along-track direction, the ability to alter the semimajor axis with the Lorentz is similarly limited. This conclusion corresponds well with numerical results. For near-equatorial orbits, the eigenvector corresponding to the zero eigenvalue of the Gramian is dominated by its $\delta\tilde{a}$ component. For polar orbits, the largest component of uncontrollable eigenvector still belongs to $\delta\tilde{a}$, but in addition to it, there is a small but nonzero $\delta\tilde{t}$ component.

It should be noted that these results are expected to hold even when using a higher-fidelity magnetic field model, such as the IGRF model, is used in the calculation of the controllability Gramian. It is expected because for low-Earth, near circular orbits, the Lorentz force remains nearly perpendicular to the along-track direction, due to the cross product operation, regardless of the magnetic field vector.
5.2.2 Geometric Decomposition

The investigation into hybrid magnetic/thruster formation control begins with adapting a pre-existing differential element control scheme to a hybrid actuation strategy. Note, this technique is suitable for any continuous-thrust formation-keeping strategy. The control strategy we choose is the continuous mean element feedback law developed by Schaub et al., where the control acceleration is given by

\[ \mathbf{u}(t) = -\left( \mathbf{B}^\top (\mathbf{\bar{e}}) \mathbf{B}(\mathbf{\bar{e}}) \right)^{-1} \mathbf{B}^\top (\mathbf{\bar{e}}) \mathbf{P}_S(t) \mathbf{\zeta}(t), \]  

(5.12)

where \( \mathbf{B}(\mathbf{\bar{e}}) \) are GVE. The Lyapunov-based stability proof for the control law can be found in Ref. 63. The time-varying matrix \( \mathbf{P}_S(t) \) is of the form

\[
\mathbf{P}_S(t) = \text{diag} \begin{pmatrix}
P_{a0} + P_{a1} \cos^{12} \frac{f}{2} \\
P_{e0} + P_{e1} \cos^{12} f \\
P_{p0} + P_{p1} \cos^{12} \theta \\
P_{\Omega0} + P_{\Omega1} \sin^{12} \theta \\
P_{I0} + P_{I1} \sin^{12} f \\
P_{M0} + P_{M1} \sin^{12} f 
\end{pmatrix},
\]

(5.13)

where values of the matrix, reproduced from Ref. 63, are given in Table 5.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
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<td>0.010</td>
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</table>

Hybrid control is realized by decomposing the required control effort, \( \mathbf{u}(t) \), that is obtained from Eq. (5.12) into two components: one that is parallel to the Lorentz force vector and one that is perpendicular:

\[ \mathbf{u}(t) = \mathbf{u}_\parallel(t) + \mathbf{u}_\perp(t). \]

(5.14)

In what follows, it is understood that all quantities are time-varying and implicitly dependent on spacecraft position, so the arguments of time and orbital elements are dropped for clarity. The desired
Lorentz force magnitude is obtained by projecting the control effort onto the unit Lorentz force vector:

\[ u_{\parallel} = u^T \hat{f}_L = \frac{u^T (v_{rel}^* b_m)}{\|v_{rel}^* b_m\|}, \tag{5.15} \]

so the parallel control effort \( u_{\parallel}(t) \) is

\[ u_{\parallel} = u_{\parallel} \hat{f}_L, \]

\[ = \left( \frac{u^T (v_{rel}^* b_m)}{\|v_{rel}^* b_m\|^2} \right) v_{rel}^* b_m. \tag{5.16} \]

Vectors \( v_{rel} \) and \( b_m \) are expressed in the local spacecraft orbital frame. Comparing Eq. (5.16) with Eq. (5.8), the charge-to-mass ratio needed to realize the desired Lorentz force on the spacecraft is given by

\[ \frac{q}{m} = \left( \frac{u^T (v_{rel}^* b_m)}{\|v_{rel}^* b_m\|^2} \right). \tag{5.17} \]

The corresponding thruster control effort is obtained by solving Eq. (5.14) for \( u_{\perp}(t) \):

\[ u_{\perp} = u - u_{\parallel} \hat{f}_L. \tag{5.18} \]

This control strategy is inspired by the hybrid magnetorquer/reaction wheel spacecraft attitude control strategy found in Ref. 64 that employs a similar decomposition of control effort. This decomposition strategy is not in any way optimal, but can be computationally simple. The proportion of control effort realized by the Lorentz force is entirely dependent on the orbit geometry. In the next section, the feedback gain matrix is tuned to maximize the usage of the Lorentz force for a particular orbit geometry.

### 5.2.3 Periodic Linear Quadratic Regulation

The decomposition control method offers a relatively simple and computationally inexpensive way of dividing the control effort between Lorentz force and thruster actuation. With the decomposition method, this division is entirely dependent on the orbit geometry. Ideally, however, the Lorentz force would do most of the work in order to conserve fuel. A natural problem that arises is how to obtain a feedback gain matrix \( P(t) \) such that the use of the Lorentz force is maximized. The linear quadratic regulator (LQR) offers the framework to pose such a problem and to obtain a solution, if one exists. For
the hybrid Lorentz force/thruster spacecraft system, the control vector is

\[ u(t) = \begin{bmatrix} q(t)/m & u_r(t) & u_o(t) & u_h(t) \end{bmatrix}^T \]  

(5.19)

and the associated input matrix is a combination of Eq. (3.10) and Eq. (5.10),

\[ \tilde{B}(\tilde{e}_r) := \begin{bmatrix} B(\tilde{e}_r) & \tilde{f}_L(t, \tilde{e}_r) \end{bmatrix}, \]  

(5.20)

where \( \tilde{f}_L(t, \tilde{e}_r) \) is the Lorentz force per specific charge, defined in Eq. (3.28). The solution to the LQR minimizes the cost functional

\[ J(\zeta, u, t) = \zeta^T(t_2)P(t_2)\zeta(t_2) + \int_{t_1}^{t_2} \left( \zeta^T(t)Q(t)\zeta(t) + u^T(t)R(t)u(t) \right) dt, \]  

(5.21)

where matrices \( Q(t) \) and \( R(t) \) are positive semidefinite and positive definite, respectively. The optimal time-varying feedback gain matrix \( P(t) \) can be calculated by integrating the time-varying Riccati equation

\[ -\dot{P}(t) = P(t)\tilde{A}(\tilde{e}_r) + \tilde{A}^T(\tilde{e}_r)P(t) - P(t)\tilde{B}(\tilde{e}_r)R^{-1}(t)\tilde{B}^T(\tilde{e}_r)P(t) + Q(t). \]  

(5.22)

backwards in time over the control interval. The optimal control vector is given by

\[ u^*(t) = -R^{-1}(t)\tilde{B}^T(\tilde{e}_r)P(t)\zeta(t). \]  

(5.23)

Since the spacecraft system is nearly periodic, the solution to the Riccati equation reaches a periodic steady-state and can be conveniently fitted with a Fourier series.

If the thruster control vector is not included, then the uncontrollability of the Lorentz-augmented system manifests itself in the following manner. Some terms of the solution to the time-varying Riccati equation, when it is solved for the dynamics in Eq. (5.11), does not converge to periodic, time-varying solutions. Instead, these terms grow unbounded over the solution time interval.

### 5.2.4 PCO in Polar Orbit Simulation

The proposed Lorentz-augmented, continuous thrust control strategies are evaluated using the 1 km PCO formation for near-circular low-Earth polar orbit that was considered in Section 4.4.1. Chief and deputy initial conditions are given in Table 4.1. The only perturbations included in the simulation environment are the \( f_2 – f_6 \) zonal harmonics and the full IGRF-11 magnetic field model is used to model
the magnetic field. The control laws, however, use the dipole model to approximate the magnetic field vector. Perfect state knowledge is assumed.

The hybrid Lorentz force/thruster LQR was designed with the following state error and control weights:

\[ Q = 10^{10} \cdot 1_{6 \times 6}, \quad R = \text{diag}(10^6 \ 10^{10} \ 10^{10} \ 10^{10}). \]

Several different weighting matrix combinations are considered in this chapter. Later on, this Q and R case is referred to as case ‘C1’. The solution to the Riccati equation Eq. (5.22) was calculated using a fixed-step Runge-Kutta 4-5 integration scheme with a one-second time step. The matrices \( \tilde{A} \) and \( \tilde{B} \) were evaluated using the reference mean orbital elements of the deputy spacecraft. This reference state was obtained by taking the chief spacecraft state expressed in inertial position and velocity, converting it to mean orbital elements and then adding the reference relative deputy spacecraft state, expressed in differential orbital elements, to the chief mean elements. This calculation was done for each time step. The Riccati solution obtained is periodic, but not with a period of the orbital motion. Rather, it repeats with every two rotations of the Earth, i.e., two days. In simulations, a 500-term Fourier series approximated the exact Riccati solution.

Since the classic orbital elements are being used, the reference differential orbital elements are recalculated periodically in order to account for chief argument of perigee drift.

![Figure 5.4: Position error comparison between LQR strategies and the decomposed control.](image)

Figure 5.4: Position error comparison between LQR strategies and the decomposed control.

We begin evaluating the control strategies by comparing the error in the relative position of the deputy spacecraft achieved by the different controllers. The decomposed control strategy keeps relative
position error to below 5 m in each axis, as seen in Fig. 5.4. The low-thrust LQR performs worse, reaching a relative positive error of up to almost 7 m. The corresponding mean differential element errors are presented in Fig. 5.7. The LQR’s inferior in-plane position control can be attributed to the significantly larger error in $\delta c$, $\delta \omega$ and $\delta M$. Although the decomposition strategy regulates these differential elements better than the LQR strategy, this superior performance comes with an increased amount of thruster control effort. Table 5.2 presents a full breakdown of thruster and Lorentz force control efforts for both strategies.

While total $\Delta V$ is similar for the two strategies, the contribution of the Lorentz force differs significantly between the LQR and the decomposition strategy. Nearly 96% of the total control effort is realized with the Lorentz force using the LQR strategy, compared to 88% for the decomposition strategy.

The specific charge for both cases is $O(10^{-5})$ C/kg, which is achievable by current technology; however, as seen in the Fig. 5.5, positive and negative charges are required. As discussed in the introduction to this chapter, this use of both positive and negative charge is not compatible with current hardware architecture.

Table 5.2: Control strategy performance comparison for a polar 1 km PCO formation.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Control Effort</th>
<th>Rel. Pos. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta V_{tot}$</td>
<td>$\Delta V_L$</td>
</tr>
<tr>
<td></td>
<td>[mm/s]</td>
<td>[mm/s]</td>
</tr>
<tr>
<td>Cont. LQR</td>
<td>15.7</td>
<td>15.0</td>
</tr>
<tr>
<td>C1 weights</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decomposed control</td>
<td>16.5</td>
<td>14.5</td>
</tr>
<tr>
<td>Impulsive</td>
<td>11.1</td>
<td>–</td>
</tr>
<tr>
<td>2 thrusts</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5.5: Spacecraft specific charge for different continuous control strategies.

Fig. 5.6 shows the acceleration components in the LVLH frame for both thrusters and the Lorentz force for the low-thrust case. As discussed in Section 3.2.5, the effect of differential $J_2$ in polar orbits...
primarily results in out-of-plane position error due to right ascension drift, with limited effect in radial along-track directions. Out-of-plane acceleration is needed to correct right ascension error (see Eq. (3.10)). The Lorentz force is well suited to correcting this error in near-polar orbits because in near-polar orbits it acts primarily in the out-of-plane direction. This behaviour is exactly what is seen when looking at the acceleration provided by the Lorentz force: a dominant out-of-plane component, with in-plane accelerations an order or two in magnitude smaller. Out-of-plane acceleration provided by the thruster is effectively zero, and in-plane thruster acceleration is primarily needed to correct for the small, unwanted in-plane accelerations caused by the Lorentz force.

![Control accelerations in the LVLH frame for the low-thrust LQR.](image)

**Figure 5.6: Control accelerations in the LVLH frame for the low-thrust LQR.**

### 5.2.5 Performance at Different Inclinations

As seen in Fig. 5.1(a), the behaviour of the Lorentz force varies significantly with the inclination of the spacecraft’s orbit. It shifts from acting primarily in the out-of-plane direction to acting radially. Unsurprisingly, a numerical study of controlled 1 km PCO formations reveals that these combined Lorentz-force/thruster strategies perform significantly differently at different inclinations, in terms of both $∆V$ required for maintenance, and the achieved relative position error.

We saw in Section 5.1.1 that for a fixed inclination, eccentricity and specific charge, the Lorentz force only varies in strength with changes to the semimajor axis; its time-varying behaviour remains the same. In a similar fashion, the effect of $J_2$ varies in strength with changes in semimajor axis. As we consider the performance for the LQR strategies at different inclinations, we expect the trends we observe at the example semimajor axis to hold for different semimajor axes.

The results presented in the following figures are generated from taking the RMS values of var-
ious performance metrics from 30-orbit simulations. A new LQR controller was calculated for each inclination case, using one of the different LQR weight cases. For the results in the previous section, the classical orbital element set was used for the formulation of the control laws; however, for this numerical study, the control laws have all been formulated using the non-singular orbital element set. Recall that when using the classical orbital element set, the reference differential elements require periodic recalculating in order to continue to accurately reflect the desired formation geometry. Although good controller performance using this periodic recalculating approach was shown above, it was found that at low inclinations, relative position control worsens considerably when using the same strategy. Switching to the non-singular orbital element set allows for the chief perigee drift to be included in the differential element dynamics so that the reference elements no longer require recalculation. The result is a much lower relative position error at low inclinations compared to control strategies formulated with the classical orbital elements.

Three different cases of \( Q \) and \( R \) penalty matrices are used to calculate the linear quadratic regulators. For these cases, they are identified by the prefix 'NS'. Each weight combination has been applied to a 1 km PCO formation in inclinations ranging from 0° to 90°. Note the critical \( J_2 \) inclination, 63.4°, is omitted, since mean element theory breaks down at that inclination. The three cases of \( Q \) and \( R \) are:
**Chapter 5. Lorentz-Augmented Formation Control**

NS1: \( Q = 10^8 \times 1_{6 \times 6}, R = \text{diag}(10^3, 10^6, 10^6); \)

NS2: \( Q = 10^6 \times 1_{6 \times 6}, R = \text{diag}(10^3, 10^6, 10^6); \)

NS3: \( Q = 10^{12} \times 1_{6 \times 6}, R = \text{diag}(10^3, 10^6, 10^6). \)

Fig. 5.8 shows the achieved 3DRMS relative position error for the three LQR controllers as well as the 3DRMS error achieved by the decomposed control strategy and the two-impulse control strategy from Section 4.2. All the controllers follow the same general trend for relative position errors. The lowest error is achieved near the \( J_2 \) critical inclination of 63.4° where the differential \( J_2 \) perturbation for the formation is weakest; the largest errors appear at near-polar and near-equatorial orbits. Most of the continuously-applied control effort strategies perform better than the previous chapter’s impulsive control strategy over the majority of orbital inclinations.

The case (NS2) Lorentz-augmented LQR strategy has considerably worse relative position error than the other strategies. This result is because with that choice of state and control penalty weights, the resulting control effort is almost entirely realized through the Lorentz force.

![Figure 5.8: Continuous Lorentz-augmented control strategy comparison of relative position error for a 1 km PCO formation in LEO at inclinations 0° < i ≤ 90°.](image)

Fig. 5.9 breaks down the control effort of the different strategies between amount of thruster \( \Delta V \), total \( \Delta V \) and fraction of \( \Delta V \) realized with the Lorentz force. The figure shows that for the case (NS2) LQR, across all inclinations the Lorentz force realizes 99% of the total control effort. Fig. 5.9(c), in particular, shows the significant decreases in thruster control effort between case (NS2) and the other control strategies. The trade-off of having nearly the entire control effort realized by the Lorentz force is that relative position control suffers.

Much larger specific charges are used by case (NS2) than for the other Lorentz-augmented controllers in order to realize a desired acceleration in one direction, rather than it being realized by thrusters. However, due to the constrained nature of the Lorentz force, this specific charge also induces
a perturbing acceleration in other directions. Since thruster control effort is heavily penalized, little
thruster control effort is expended to correct these perturbative accelerations, resulting in larger posi-
tion errors. This case is a good illustration of the uncontrollable nature of a spacecraft formation using
the Lorentz force alone: although the position errors for the case (2) LQR remains bounded across the
different inclinations it cannot be reduced further without some thruster actuation.

LQR cases (NS1) and (NS3) have similar performance characteristics to the decomposed control
strategy. Both achieve slightly lower RMS positions errors than the decomposed control strategy while
case (NS1) does so while also using less thruster control effort. Interestingly, the decomposed strategy
uses more Lorentz force at low inclinations, achieving nearly 80% of its total control effort with the
Lorentz force in near-equatorial orbit. This behaviour is one that could not be replicated with the LQR
collectors, despite much trial-and-error of penalty weight selections. Except for near the equator,
using the LQR for control results in a control strategy that makes better use of the Lorentz force than
the decomposed control while achieving lower relative position errors.

All the Lorentz-augmented control strategies use less thruster ∆V than the two-impulse control
strategy from Chapter 4. The Lorentz force appears to be least useful for formation keeping, at least for
the 1 km PCO, at orbital inclinations between 15° and 30°. The thruster ∆V of the decomposed control
strategy at 30° is nearly the same as the two-impulse controller. For the LQR cases, thruster ∆V remains
consistently below the two-impulse ∆V.

5.2.6 Charge Saturation

As discussed at the beginning of the chapter, a current limitation of Lorentz-augmented spacecraft
hardware proposed by Streetman and Peck is the restriction to a single charge polarity. For a spacecraft
relying on some sort of particle beam-based charge regulation system, both an ion beam and electron
beam would be needed to realize both charge polarities. However, for a small, formation-flying
spacecraft, having two beam systems may not be feasible. This section considers limiting the formation-
keeping strategies discussed above to single charge polarities.

Results indicate that for formations with out-of-plane separations, a single-polarity restriction can
be accommodated.

Fig. 5.10 shows the fraction of the specific charge history used for the formation keeping of 1 km
PCO formation that is negative, over a 30-orbit simulation. The three LQR cases considered above as
well as the decomposed control are again considered here.

For high orbit inclinations, we see that most of the LQR controllers, as well as the decomposed
control strategy, require a predominantly negative charge to realize the necessary Lorentz force for formation keeping. For LQR case (NS3) in particular, the required specific charge is negative for more than 95% of the control period. This result holds for orbit inclinations greater than 45°. Additional simulations at these inclinations that employ the case (NS3) LQR controller but restrict the specific charge to only a negative charge show that relative position control performance is not impacted by the restriction. LQR case (NS1) and the decomposed control strategy also use a negative charge for over 80% of the control for the same range of inclinations. Case (NS2) LQR requires a much larger fraction of positive specific charge. Recall that case (NS2) LQR also requires a negligible amount of thruster control effort. We conclude then that a single-polarity formation-keeping strategy is feasible for formation in
highly-inclined orbits. Unfortunately, at this time it cannot be determined \textit{a priori} whether an LQR strategy is a predominantly single-polarity strategy. Such a determination must be made via numerical simulation.

5.2.7 Effect of Drag

The continuous Lorentz-augmented control strategies do not use the Lorentz force particularly well to correct for differential drag. Since drag is an unmodeled along-track disturbance, the response of the controller is to simply compensate with an along-track control effort. This control effort is realized by thrusters for all inclinations, since the Lorentz force along-track component is the weakest of the components in any inclination. In order to use the Lorentz force, a large specific charge would be needed to realize the necessary control effort and would also induce unwanted accelerations in radial and out-of-plane directions. Introducing differential drag disturbance into simulations of the PCO formation at any inclination results in an increased along-track thruster control effort and slight worsening of relative position control.

5.3 Lorentz-Augmented Formation Flying with Impulsive Thrust

The continuous-thrust, Lorentz-augmented control strategies still suffer from the use of a continuous thrust effort, which is potentially undesirable for spacecraft formation missions. As noted in Chapter 4, quiet time —that is, vibration free time — on a spacecraft can be particularly important. Also, for spacecraft in close proximity, impulsive thrusting limits the chance of a spacecraft flying through a thruster exhaust plume.
These issues have motivated this investigation into combining continuous Lorentz force actuation with some impulsive thrust control. The approach taken here begins by applying the optimal Lorentz force control component obtained from the LQR solution in Sec 5.2.3 to the orbital element dynamics, then wrapping an impulsive thrust controller around that system. The hybrid impulsive thrust/continuous Lorentz force controller is in a sense “ad hoc” in that the Riccati solution being used to obtain the specific charge command is not designed to work with the impulsive control. The logic behind using it is that the LQR is weighted so that the use of thrusters should be minimal, particularly for near-polar orbit inclinations. Most of the regulation should be achieved via the Lorentz force and impulsive thrusts are applied periodically to maintain overall formation stability.

Using a similar approach to the impulsive control design of Chapter 4, the impulsive control scheme is formulated using the dynamics of the differential orbital elements

\[ \delta \dot{e}(t) = \dot{\hat{A}}(\hat{e}_r) \delta \hat{e}(t) + \mathbf{B}(\hat{e}_r) \mathbf{u}(t) \]  

(5.24)

with the understanding that \( \hat{e}_r \) varies implicitly with time. The control vector is decomposed into Lorentz force and thruster components:

\[ \mathbf{u}(t) = \mathbf{u}_L(t) + \mathbf{u}_T(t) = \frac{q(t)}{m} \hat{f}_L(t, \hat{e}_r) + \sum_{k=1}^{N} \mathbf{u}_T \delta(t - t_k), \]  

(5.25)

where there are \( N \) impulsive thrusts being applied and \( \delta(\cdot) \) is the Dirac Delta function. The optimal specific charge is

\[ \frac{q(t)}{m} = -R_{1,1}^{-1}(t) \hat{f}_L(t, \hat{e}_r) \mathbf{B}^T(\hat{e}_r) \mathbf{P}(t) \left( \delta \hat{e}(t) - \delta \hat{e}_r \right), \]  

(5.26)

where \( \mathbf{P}(t) \) is the solution to the time-varying Riccati for the continuous thrust, Lorentz-augmented LQR. The control vector for the system, then, is

\[ \mathbf{u}(t) = -R_{1,1}^{-1}(t) \hat{f}_L(t, \hat{e}_r) \mathbf{B}^T(\hat{e}_r) \mathbf{P}(t) \left( \delta \hat{e}(t) - \delta \hat{e}_r \right) + \sum_{k=1}^{N} \mathbf{u}_T \delta(t - t_k). \]  

(5.27)

For convenience, the following matrix is defined:

\[ \Xi(t) = R_{1,1}^{-1}(t) \hat{f}_L(t, \hat{e}_r) \mathbf{B}^T(\hat{e}_r) \mathbf{P}(t). \]  

(5.28)
Eq. (5.27) is substituted into Eq. (5.24) to obtain

\[ \delta \dot{\bar{e}}(t) = \left( \dot{\bar{A}}(\bar{e}_c) - \bar{B}(\bar{e}_c) \Xi(t) \right) \delta \bar{e}(t) + \bar{B}(\bar{e}_c) \Xi(t) \delta \bar{e}_r + \sum_{k=1}^{N} \bar{B}(\bar{e}_c(t_k)) u_{t_k} \delta (t - t_k). \]  

(5.29)

Integrating from \( t_0 \) to \( t_f \) yields

\[ \delta \bar{e}(t_f) = \Phi(t_f, t_0) \delta \bar{e}(t_0) + \Psi(t_f, t_0) + \sum_{k=1}^{N} \Phi(t_f, t_k) \bar{B}(\bar{e}_c(t_k)) v_k. \]

(5.30)

The matrix \( \Phi(t_f, t_0) \) is the system’s state transition matrix and must be calculated numerically by integrating

\[ \frac{d}{dt} \Phi(t, t_0) = \left( \dot{\bar{A}}(\bar{e}_c) - \bar{B}(\bar{e}_c) \Xi(t) \right) \Phi(t, t_0), \]

(5.31)

and the matrix \( \Psi(t_f, t_0) \) is the integral

\[ \Psi(t_f, t_0) = \int_{t_0}^{t_f} \Phi(t_f, \tau) \bar{B}(\bar{e}_c(\tau)) \Xi(\tau) \delta \bar{e}_r d\tau. \]

(5.32)

At \( t = t_f \), the deputy’s differential elements should equal the reference elements, \( \delta \bar{e}(t_f) = \delta \bar{e}_r \). Rearranging Eq. (5.29), the following vector constraint equation

\[ \delta \bar{e}(t_f) - \Phi(t_f, t_0) \delta \bar{e}(t_0) = \Psi(t_f, t_0) + \sum_{k=1}^{N} \Phi(t_f, t_k) \bar{B}(\bar{e}_c(t_k)) v_k \]

(5.33)

is obtained and must be satisfied by the \( N \) impulsive thrusts. The left hand side of the equation can be thought of as the remaining error in the differential elements that must be corrected for by the impulsive thrusts. The optimization problem

\[ \text{minimize} \quad J(v_k, t_k) = \sum_{k=1}^{N} v_k^T v_k \]

with respect to \( v_k, t_k, \quad k = 1, \ldots, N \)

subject to \( 0 = \delta \bar{e}(t_f) - \Phi(t_f, t_0) \delta \bar{e}(t_0) - \Psi(t_f, t_0) - \sum_{k=1}^{N} \Phi(t_f, t_k) \bar{B}(\bar{e}_c(t_k)) v_k \)

can be solved numerically to determine the impulsive thrusts and their application times. For \( N = 2 \), the same solution approaches that were used in Chapter 4 are also applicable here. Given two firing
times, the impulsive thrusts can be solved for without resorting to numerical optimization:

\[
\begin{bmatrix}
  v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
  \Phi(t_f, t_1)B(\vec{e}_r(t_1)) & \Phi(t_f, t_2)B(\vec{e}_r(t_2))
\end{bmatrix}^{-1} \left( \delta \vec{e}_r - \Phi(t_f, t_0)\delta \vec{e}_r(t_0) - \Psi(t_f, t_0) \right).
\] (5.34)

### 5.3.1 Two-thrust Stability

When implementing the two-impulse strategy Lorentz-augmented formation-keeping strategy in simulation, choosing two impulse application times that result in a stable formation proved particularly difficult. This difficulty motivated a stability study similar to the analysis found in Section 4.3. We wish to see whether the state error remains bounded, so the mean differential element state error \( \zeta = \delta \vec{e}(t) - \delta \vec{e} \), is considered. The Lorentz-augmented dynamics of the state error are

\[
\dot{\zeta}(t) = \left( \dot{\vec{A}}(\vec{e}_r) - \vec{B}(\vec{e}_r)\Xi(t) \right) \zeta(t) + \vec{B}(\vec{e}_r(t_1))u_{t_1}\delta(t - t_1) + \vec{B}(\vec{e}_r(t_2))u_{t_2}\delta(t - t_2).
\] (5.35)

Integrating Eq. (5.35) over orbit period, \( t_f = T \), yields the discrete-time system

\[
\zeta[k + 1] = \Phi_r(t_f, t_0)\zeta[k] + \Phi_r(t_f, t_1)B(\vec{e}_r(t_1))v_1 + \Phi_r(t_f, t_2)B(\vec{e}_r(t_2))v_2
\] (5.36)

where impulsive thrusts are applied at \( t_1 \) and \( t_2 \), and \( \Phi_r \) is the system’s numerically calculated state transition matrix. The subscript “\( r \)” denotes that when calculating the state transition matrix, the matrices \( \vec{A} \) and \( \vec{B} \) are evaluated with the deputy’s reference mean orbital elements, \( \vec{e}_r(t) = \vec{e}_e(t) + \delta \vec{e}_r \).

This approach differs slightly from Section 5.3, where the state transition matrix, \( \Phi \), is evaluated using chief orbital elements. The matrix \( \Xi(t) \) is defined in Eq. (5.28). The two-impulse control law, Eq. (5.34), is substituted into Eq. (5.36) to yield

\[
\zeta(t_f) = \Phi_r(t_f, t_0)\zeta(t_0) + \left[ \Phi_r(t_f, t_1)B(\vec{e}_r(t_1)) \middle| \Phi_r(t_f, t_2)B(\vec{e}_r(t_2)) \right]^{-1} \left[ \Phi(t_f, t_1)B(\vec{e}_r(t_1)) \middle| \Phi(t_f, t_2)B(\vec{e}_r(t_2)) \right] = \bar{\Xi}
\]

\[
\times \left( \delta \vec{e}_r - \Phi(t_f, t_0)\delta \vec{e}(t_0) - \Psi(t_f, t_0) \right),
\] (5.37)

Letting \( \Phi(t_f, t_0) = 1 + \phi(t_f, t_0) \), Eq. (5.37) is rearranged to obtain

\[
\zeta[k + 1] = \Phi_r(t_f, t_0)\zeta[k] + \bar{\Xi} \left( -\zeta[k] - \phi(t_f, t_0)\delta \vec{e}(t_0) - \Psi(t_f, t_0) \right)
\]

\[
= \left( \Phi_r(t_f, t_0) - \bar{\Xi} \right) \zeta[k] - \left( \phi(t_f, t_0)\delta \vec{e}(t_0) + \Psi(t_f, t_0) \right)
\]
To guarantee bounded-input bounded-output stability of the mean differential element error state, \[ |\lambda_i| < 1, \quad i = 1, \ldots, 6, \]
where \( \lambda_i \) are the eigenvalues of the closed-loop state matrix \( \Phi(t_f, t_0) - \bar{X} \).

### 5.3.2 PCO in Polar Orbit Simulation

To remain consistent with the other simulation results of this chapter, the 1 km PCO formation defined in Table 4.1 is the example considered here for the evaluation of the Lorentz-augmented ad hoc impulsive control strategy. For the continuous feedback control law, around which the ad hoc impulsive control is wrapped, the solution to the Riccati equation from the example in Section 5.2.4 is used. The singular differential elements are used with a once-per-orbit reference element update. A four-impulsive strategy is calculated using the NPSOL numerical optimization package to solve the multi-thrust optimization problem. A two-thrust ad hoc solution was also attempted, but the stability analysis revealed that there were no stable two-thrust solutions for the Riccati solution being used for the Lorentz-force feedback. A 500-term Fourier series approximates the solution to the Riccati equation.

The achieved relative position error is presented in Fig. 5.11. For this particular continuous Riccati

![Figure 5.11: Relative position control error for a 1 km PCO formation in LEO using the ad hoc Lorentz-augmented impulsive thrust strategy. Relative position error using conventional two-impulse control, and the continuous LQR are provided for comparison.](image-url)
solution, the performance achieved by the four-impulse ad hoc strategy is quite good: relative position error is only slightly worse than than the Lorentz-augmented continuous LQR strategy. The total $\Delta V$ and the thruster $\Delta V$ required by the ad hoc strategy is 18.7 mm/s per orbit and 0.9 mm/s per orbit, respectively. This $\Delta V$ is slightly larger than the 15.7 mm/s and 0.64 mm/s required by by the continuous LQR. This is still significantly lower than the 13.2 mm/s per orbit of thruster $\Delta V$ required by Chapter 4’s two-thrust strategy. The additional $\Delta V$ is due to the additional Lorentz force attempting to compensate for the absent continuous thruster acceleration, and also the additional thruster effort needed to compensate for the resulting perturbative effects caused by the excess Lorentz force.

The performance of this ad hoc strategy can vary significantly depending on the $Q$ and $R$ matrices used to design the continuous LQR that the continuous feedback law of the ad hoc strategy uses. For example, when using the Riccati solution for the case NS2 LQR from Section 5.2.5, the ad hoc strategy achieves a significantly larger out-of-plane than its continuous LQR counterpart and requires three times the thruster $\Delta V$.

The prescribed time two-thrust strategy was found to be unstable for many choices of $Q$ and $R$ that yielded good performance in the continuous LQR case. The choice of

$$Q = 10^8 \cdot 1_{6 \times 6}, \quad R = \text{diag}\{10^6, 10^8, 10^8\},$$

which is referred to as case C2, did yield a two-thrust controller that was found to stable for most firing times. For the firing times of $f = \{0, \frac{3}{2} \pi\}$, the ad hoc controller actually improves upon the relative position control of its continuous LQR counterpart. For other firing times, relative position control was significantly worse than the continuous LQR. Referring to Table 5.3, the thruster $\Delta V$ required by the two-thrust ad hoc strategy is significantly larger than that of its continuous LQR counterpart and is also greater than the $\Delta V$ of the conventional two-thrust strategy. This example highlights the drawback of this continuous/impulsive control design methodology: there is no way to ensure that the impulsive and Lorentz force actuation act together in a complementary fashion. The two actuation methods can be combined in this fashion, but considerable trial and error is needed to find a combination of penalty matrices and firing times to make to the two-thrust ad hoc control work well. As shown by the four-impulse example, numerical optimization is the preferred method for applying this approach.
Table 5.3: Control strategy performance comparison for a polar 1 km PCO formation.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Control Effort</th>
<th>Rel. Pos. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta V_{tot}$</td>
<td>$\Delta V_L$</td>
</tr>
<tr>
<td></td>
<td>[mm/s]</td>
<td>[mm/s]</td>
</tr>
<tr>
<td>Cont. LQR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1 weights</td>
<td>15.7</td>
<td>15.0</td>
</tr>
<tr>
<td>C2 weights</td>
<td>15.4</td>
<td>13.1</td>
</tr>
<tr>
<td>ad hoc</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1, 4 thrusts</td>
<td>19.6</td>
<td>18.7</td>
</tr>
<tr>
<td>C2, 2 thrusts</td>
<td>24.0</td>
<td>8.7</td>
</tr>
<tr>
<td>Decomposed control</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>16.5</td>
<td>14.5</td>
</tr>
<tr>
<td>Impulsive</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 thrusts</td>
<td>11.1</td>
<td>–</td>
</tr>
</tbody>
</table>

5.4 Chapter Conclusions

This chapter presented a number of different spacecraft formation-keeping strategies that employed the geomagnetic Lorentz force as a means of actuation for a spacecraft. It was shown that the relative spacecraft state is not completely controllable with the Lorentz force alone, necessitating the need for some combination of thruster and Lorentz force actuation. Three novel Lorentz-augmented differential element control strategies were proposed:

1. First was the adaptation of pre-existing differential element control laws for use with the Lorentz force. It was shown that a given control effort could be decomposed into a Lorentz force component and a thruster component. The method was demonstrated with Schaub and Junkins’ differential element law,$^{63}$ but could be employed with any formation-keeping control law with continuous actuation.

2. Second, the design of the linear quadratic regular with the four component control vector

$$u(t) = [q(t)/m \ u_r(t) \ u_\theta(t) \ u_\phi(t)]^T$$

was considered. Through the heavy penalization of the thruster control effort, an LQR could be designed that minimized its use. In the extreme case, less than a 0.1 mm/s per orbit of thruster $\Delta V$ could be used for formation keeping, but at the expense of a comparatively large relative position error.

3. Lastly, the design of an ad hoc continuous Lorentz force/impulsive thruster controller was pro-
posed, which combined the LQR-generated control law for the specific charge with the impulsive
control of Chapter 4. The controller, although capable of maintaining a formation, struggled with
achieving low relative position errors, as well as exhibited stability problems for certain thrust
application times.

The quality considered most important for each controller was its ability to keep relative position
error between chief and deputy spacecraft low while also minimizing the thruster control effort required.
The ideal controller is one that primarily uses the Lorentz force to keep the deputy spacecraft on its
reference trajectory. To that end, it was thought that the combined impulsive thrust/Lorentz force
controller, in addition to the other benefits of impulsive thrusting, would provide the best avenue for
achieving such a controller. The ad hoc control strategy performed well in some cases, but in other
cases the impulsive and Lorentz force actuation worked against each other, eliminating any thruster $\Delta V$
benefits provided by the Lorentz force. We conclude then by asking if there is a method for combining
impulsive and continuous control efforts in some optimal fashion. The answer to this question is
explored thoroughly in the following chapter.
Chapter 6

Extensions to Optimal Hybrid Control

The previous chapter concluded with the development of a spacecraft formation control strategy that combined a continuously applied Lorentz force control effort with periodic impulsive thrusting. This combination of the discrete and continuous control efforts was done in an ad hoc fashion: an impulsive control strategy was designed for a closed-loop differential orbital element system that was using the Lorentz force component of the continuous LQR strategy. While this ad hoc hybrid controller maintains the formation, there is no guarantee that the two types of control efforts perform in a complementary fashion.

This chapter considers the problem of designing a hybrid continuous/impulsive control strategy that minimizes a quadratic performance index. Spacecraft formation flying is put aside for this chapter, as the main focus is on the extension of hybrid control theory and developing a hybrid linear quadratic regulator for linear time-varying (LTV) systems. The problem of determining the optimal application time of the impulsive control is also explored.

6.1 Background

We are interested in obtaining an optimal control law that utilizes both continuous and impulsive control inputs to regulate a system. We refer to this problem as the hybrid control problem due to the combination of continuous and impulsive control inputs. In a general sense, the system is governed by a set of continuous dynamics, but experiences instantaneous changes in its state at certain times.
Systems with states that experience such jumps are common in economics and operations research. Examples concerning optimal well drilling times and machine maintenance and replacement can be found in Ref. 65.

Switched systems are a class of systems closely related to this problem. A switched system is a system whose dynamics change at a specific instance. The states experience a discontinuity, but there is not necessarily a jump in state value. Consider the transmission system of a vehicle: when the engine speed reaches a certain level, the gear changes, altering the behaviour of the system. Switched systems are commonly found in, among others, industrial process control, where logic controllers might switch from one mode of operation to another. The switching can occur autonomously, when the state reaches a specific boundary condition, as in the case of an automatic transmission, or the switch can occur at a prescribed time, in which case the switch itself can be treated as a form of impulsive control.66 Branicky67 provides an elegant classification for the spectrum of these different hybrid systems and discusses some different control approaches. He distinguishes between “hybrid systems,” which have both continuous and discrete states, and “hybrid control systems,” which are hybrid systems that employ both continuous and discrete controls. The problem of spacecraft control with the Lorentz force and thruster actuation is an example of the latter.

The commonality of the states experiencing discontinuities relates all these problems and allows them to be treated with some form of the Impulsive Maximum Principle. Blaqui`ere68,69 extended Pontryagin’s Maximum Principle to the impulsive case and derived necessary and sufficient conditions for the optimal control of systems with state jumps. Various proofs of this extended maximum principle exist.70–72 Separately, Bryson and Ho showed how to derive optimal control laws for systems with state discontinuities at interior points; see Section 3.7 in Ref. 73. Interestingly, Geering74 showed that the solution of an optimal control problem that uses a hybrid penalty function — that is, a penalty function with continuous and discrete components — for a purely continuous system very closely resembles that of systems with state discontinuities.

Alternatively, others have taken to solving the optimal impulsive control problem using the Dynamic Programming Principle. This approach leads to quantifying the solution in terms of quasi-variational inequalities; see Refs. 75–77. Unfortunately, this approach does not lend itself readily to the formulation of a feedback control. Some more recent research does explore how to implement these algorithms in order to solve for the optimal control; see Refs. 67,78.

Linear quadratic control and stability of autonomously switched systems were treated in Ref. 66 and Ref. 79, respectively. Linear quadratic control of switched systems with impulsive inputs was previously treated by Hu et al.80 The solutions presented in this chapter differ from Hu et al. in some
respects and also, for the optimal time problem, go beyond the time-invariant case to consider optimal impulsive application timing for LTV systems.

### 6.2 Optimal Hybrid Control with Prescribed Firing Times

Consider a linear, time-varying continuous/impulsive system, where an impulsive action occurs at \( t = t_k \) for \( k = 1, \ldots, N \), with dynamics that are described by:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) & t \neq t_k \\
x^+_k &= C_k x^-_k + D_k v_k & t = t_k.
\end{align*}
\]

where \( x(\cdot) \in \mathbb{R}^n \) is the state, \( u(\cdot) \in \mathcal{U} \), a bounded set in \( \mathbb{R}^m \), is a continuous control input, and \( v(\cdot) \in \mathcal{V} \), a bounded set in \( \mathbb{R}^l \), is an impulsive control input. Impulsive application times, \( t_k \), are assumed to be prescribed. The superscripts \((\cdot)^{-}\) and \((\cdot)^{+}\) denote, respectively, the instants immediately before and after the discrete dynamics are applied, i.e., \( x^+_k = x(t^+_k) \). The impulsive control introduces a discontinuity in the states of the system. In many examples in the literature concerning hybrid systems, the discontinuity in states arises from a change in the dynamics of the system (see Refs. 66, 79, 80), not an actual impulse. However, the fundamental theory concerning how to treat the discontinuity is the same.

The problem is to determine the controls \( u(t) \) and \( v_k, k = 1, \ldots, N - 1 \), such that the cost functional \( J(x, u, v_k, t_k) \), subject to the constraint \( \dot{x}(t) = f(x(t), u(t), t) \) and \( x^+_k = g(x^-_k, v_k, t_k) \), is minimized over a time interval \( t \in [t_0, t_f] \).

We begin by choosing the following hybrid cost functional for a finite time horizon

\[
J(x, u, v_k, t_k) = \frac{1}{2} x^T(t_f)Sx(t_f) + \frac{1}{2} \sum_{k=1}^{N-1} \left[ x^T_k Q_k x^-_k + v^T_k R_k v_k \right] + \frac{1}{2} \int_{t_0}^{t_f} \left[ x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right] dt,
\]

where \( Q = Q^T \geq 0, Q_k = Q_k^T \geq 0, R = R^T > 0, R_k = R_k^T > 0, S = S^T \geq 0, t_0^+ = t_0 \) and \( t_k^- = t_f \). The continuous and discrete Hamiltonians, respectively, are defined as

\[
\begin{align*}
\mathcal{H}(t, x(t), u(t), A(t)) &= \frac{1}{2} x^T(t)Q(t)x(t) + \frac{1}{2} u^T(t)R(t)u(t) + A(t)x(t) + B(t)u(t) \\
\mathcal{H}_d(t_k, x^-_k, v_k, v_k) &= \frac{1}{2} x^T_k Q_k x^-_k + \frac{1}{2} v^T_k R_k v_k + v^T_k \left( C_k x^-_k + D_k v_k \right),
\end{align*}
\]

where the continuous and impulsive dynamics are adjoined to the cost functional using the continuous
and discrete costates $\lambda(t)$ and $\nu_k$, respectively. The cost functional becomes

$$J(x, u, v, t_k) = \frac{1}{2}x^T(t_f)Sx(t_f) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left[ \mathcal{H}_c(t) - \lambda^T(t)\dot{x}(t) \right] dt + \sum_{k=1}^{N-1} \left[ \mathcal{H}_d(t_k) - v_k^T \dot{x}_k^* \right]. \quad (6.6)$$

As we saw in Chapter 2, a necessary condition for an extremum of a function $J(x)$ is

$$\delta J(x^*) = 0. \quad (6.7)$$

Furthermore, in order for the extremum at $x^*$ to be a minimum, the second variation of the cost functional must satisfy

$$\delta^2 J(x^*) \geq 0. \quad (6.8)$$

The conditions stipulated on the weighting matrices in Eq. (6.3) ensure that Eq. (6.8) is satisfied. What remains to be done is determine under what conditions Eq. (6.7) is satisfied. Several manipulations are performed on Eq. (6.3) in preparation for taking the first variation. First, using integration by parts, we obtain

$$\int_{t_k}^{t_{k+1}} -\lambda^T(t)\dot{x}(t) dt = -\lambda^T(t_0)x(t_0) \bigg|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} \lambda^T(t)x(t) dt, \quad (6.9)$$

so the cost functional becomes

$$J = \frac{1}{2}x^T(t_f)Sx(t_f) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathcal{H}_c(t) + \lambda^T(t)x(t) dt - \lambda^T(t_0)x(t_0) \bigg|_{t_k}^{t_{k+1}} + \sum_{k=1}^{N-1} \left[ \mathcal{H}_d(t_k) - v_k^T \dot{x}_k^* \right]. \quad (6.10)$$

Next, the summations are manipulated, using $t_0^- = t_0^+, t_N^- = t_N^+$

$$\sum_{k=0}^{N-1} \left[ -A_{k+1}^T x_{k+1} + A_k^T x_k^* \right] = \sum_{k=0}^{N-1} \left[ -A_{k+1}^T x_{k+1} + A_k^T x_k^* + \lambda_0^T x_0 - \lambda_N^T x_N \right] = \sum_{k=1}^{N-1} \left[ -A_k^T x_k^- + A_k^T x_k^* + \lambda_0^T x_0 - \lambda_N^T x_N \right]. \quad (6.11)$$

Using Eq. (6.11), the cost functional is

$$J = \frac{1}{2}x^T(t_f)Sx(t_f) + \lambda^T(t_0)x(t_0) - \lambda^T(t_f)x(t_f) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathcal{H}_c(t) + \lambda^T(t)x(t) dt \quad (6.12)$$

$$+ \sum_{k=1}^{N-1} \left[ \mathcal{H}_d(t_k) + \left( \lambda^T(t_k^*) - v_k^T \right) x(t_k) - \lambda^T(t_k^*)x(t_k) \right] \quad (6.13)$$
Now, taking the first variation yields

$$
\delta J = (x^T(t_f)S - \lambda^T(t_f)) \delta x(t_f) + \sum_{k=0}^{N-1} \left[ \int_{t_k^+}^{t_k^-} \left( \frac{\partial \mathcal{H}_c(t)}{\partial x} + \lambda^T \frac{\partial \mathcal{H}_c(t)}{\partial u} \right) \delta x(t) + \frac{\partial \mathcal{H}_c(t)}{\partial u} \delta u(t) dt \right] + \sum_{k=1}^{N-1} \left[ \left( \frac{\partial \mathcal{H}_d(t_k)}{\partial x} - \lambda^T \delta x(t_k^-) + \left( \lambda^T(t_k^+) - v_k^T \right) \delta x(t_k^-) + \frac{\partial \mathcal{H}_d(t_k)}{\partial v_k} \delta v_k \right) \right].
$$

(6.14)

In order for $\delta J = 0$, the continuous and impulsive costates are chosen such that

$$
-\lambda(t) = \frac{\partial \mathcal{H}_c(t)}{\partial x} = Q(t)x(t) + A^T(t)\lambda(t),
$$

(6.15)

$$
\lambda(t_k^-) = \frac{\partial \mathcal{H}_d(t_k)}{\partial x} = Q_kx(t_k^-) + C_k^TV_k,
$$

(6.16)

$$
\lambda(t_f) = Sx(t_f),
$$

(6.17)

$$
\nu_k = \lambda(t_k^+).
$$

(6.18)

The two conditions on the inputs are,

$$
\frac{\partial \mathcal{H}_c}{\partial u} = 0,
$$

(6.19)

$$
\frac{\partial \mathcal{H}_d(t_k)}{\partial v_k} = 0;
$$

(6.20)

however, a controllability argument must be made to justify Eq. (6.19) and Eq. (6.20), since $\delta u(t)$ and $\delta v_k$ are not completely arbitrary. Together, Eqs. (6.15)–(6.20) are the necessary conditions for determining an extremum of $J$. Solving Eq. (6.19) and Eq. (6.20) determines the optimal continuous and impulsive inputs

$$
u^*_k = -R_k^{-1}D_k^T \lambda(t_k^+).
$$

(6.22)

Taking Eq. (6.1), with Eq. (6.21) substituted for $u(t)$, together with Eq. (6.15) forms the continuous regulator two-point boundary value problem for the time interval $t \in [t_k^+, t_{k+1}^-]$

$$
\dot{x}(t) = A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda(t),
$$

(6.23)

$$
\dot{\lambda}(t) = -Q(t)x(t) - A^T(t)\lambda(t)
$$

(6.24)
for which the linear relationship

\[ A(t) = P(t)x(t), \quad (6.25) \]

with the initial condition \( P(t_f) = S \), is well-known. Differentiating Eq. (6.25) yields

\[ \dot{A}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t). \quad (6.26) \]

Substituting Eq. (6.25) and Eq. (6.26) into Eq. (6.15) yields the time-varying matrix Riccati equation

\[ -\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}B^T(t)P(t) + Q(t). \quad (6.27) \]

Therefore, \( P(t) \) can be determined for each time interval \( t \in [t_k^+, t_{k+1}^-] \) by solving Eq. (6.27) backward in time from \( t = t_f \), with the initial condition \( P(t_f) = S \). After each discontinuity, a new boundary condition \( P(t_{k+1}^-) \) is needed to resume integration on the next interval. What remains is to determine how to obtain the new boundary condition \( P(t_{k+1}^-) \). The continuous optimal control can be written as the state feedback law

\[ u^*(t) = -R^{-1}(t)B^T(t)P(t)x(t), \quad (6.28) \]

which is same as the feedback law for the classical continuous LQR problem.

A similar development to that for the continuous dynamics now follows for the discrete dynamics. Substituting Eq. (6.22) into Eq. (6.2) and substituting Eq. (6.18) for \( \nu_k \) into Eq. (6.16) yields a system that is analogous to the well-known discrete-time regulator two-point boundary value problem

\[ x(t_k^+) = C_kx(t_k^-) - D_kR_k^{-1}D_k^TA(t_k^+) \quad (6.29) \]

\[ \lambda(t_k^-) = Q_kx(t_k^-) + C_k^T\lambda(t_k^+) \quad (6.30) \]

for which the relationship

\[ \lambda(t_k^+) = P(t_k^+)x(t_k^+) \quad (6.31) \]

holds. Solving Eq. (6.30) for \( \lambda(t_k^+) \) and using Eq. (6.31) yields

\[ \lambda(t_k^-) = C_k^{-T}(P(t_k^-) - Q_k)x(t_k^-) \quad (6.32) \]
The closed-loop impulsive dynamics become

\[ x(t_k^+) = \left[ C_k - D_k R_k^{-1} D_k^T C_k^{-1} \left( P(t_k^-) - Q_k \right) \right] x(t_k^-) \]  

(6.33)

and the optimal impulsive feedback law is

\[ v_k^* = -R_k^{-1} D_k^T C_k^{-1} \left( P(t_k^-) - Q_k \right) x(t_k^-). \]  

(6.34)

Equating Eq. (6.31) and Eq. (6.32) and using Eq. (6.33), \( P(t_k^-) = P_k^- \) can be solved for as a function of \( P(t_k^+) = P_k^+ \). All other matrices are evaluated at \( t_k \) so the time argument is dropped for clarity.

\[
\begin{align*}
C_k^{-T} (P_k^- - Q_k) &= P_k^+ \left[ C_k - D_k R_k^{-1} D_k^T C_k^{-1} \left( P_k^- - Q_k \right) \right] \\
C_k^{-T} P_k^- &= C_k^{-T} Q_k + P_k^+ C_k - P_k^+ D_k R_k^{-1} D_k^T C_k^{-1} P_k^- + P_k^+ D_k R_k^{-1} D_k^T C_k^{-T} Q_k \\
\left( C_k^{-T} + P_k^+ D_k R_k^{-1} D_k^T C_k^{-1} \right) P_k^- &= \left( C_k^{-T} + P_k^+ D_k R_k^{-1} D_k^T C_k^{-T} \right) Q_k + P_k^+ C_k \\
P_k^- &= Q_k + C_k^T \left( 1 + P_k^+ D_k R_k^{-1} D_k^T \right)^{-1} P_k^+ C_k \\
\end{align*}
\]

(6.35)

Applying the matrix inversion lemma\(^{25}\) and simplifying yields the discrete-time Riccati equation

\[ P_k^- = Q_k + C_k^T P_k^+ C_k - C_k^T P_k^+ D_k \left( R_k + D_k^T P_k^+ D_k \right)^{-1} D_k^T P_k^+ C_k, \]  

(6.36)

but in this context it governs the instantaneous change in \( P(t) \) across an impulse application time. The resulting matrix \( P(t_k^-) \) yields the boundary condition needed to integrate Eq. (6.27) on \([t_{k-1}^+, t_k^-] \).

To summarize, for the continuous/impulsive linear quadratic regulator, the solution to the Riccati equation experiences discontinuities at the impulsive application times. To determine \( P(t) \) for the optimal hybrid impulsive control problem, Eq. (6.27) and Eq. (6.36) must be used in concert. The terminal error weight \( S \) is set as the initial condition and \( \dot{P}(t) \) is integrated backwards using Eq. (6.27) from \( t_f \) to \( t_{N-1}^+ \). Integration is stopped at \( t = t_{N-1}^- \) and \( P(t_{N-1}^-) \), is obtained from Eq. (6.36). \( P(t_{N-1}^-) \) is used as the new boundary condition to resume integrating \( \dot{P}(t) \) until the next impulse time. Should the time-varying system be periodic, the integration of \( P(t) \) backward in time yields a solution that eventually converges to a steady-state periodic solution. We refer to the backward integration of Eq. (6.27) with periodic jumps in it occurring according to Eq. (6.36) as solving the hybrid Riccati equation.
6.3 Optimal Hybrid Control with Optimal Firing Times

The previous section derived an optimal control law for prescribed impulse application times. Rather than choosing an application time, this section considers the minimization of the cost functional

\[ f(x, u, v_k, t_k) = \frac{1}{2} x^T(t_f) S x(t_f) + \sum_{k=0}^{N-1} \int_{t_k^-}^{t_{k+1}^-} \left[ H_c(t) - \Lambda^T(t) \dot{x}(t) \right] dt + \sum_{k=1}^{N-1} \left[ H_d(t_k) - v_k^T x_k^+ \right]. \]  

(6.37)

not only with respect to \( u(t) \) and \( v_k \), but now also with respect to \( t_k \).

As before, a variational approach is taken to determine the necessary conditions for an optimal control. In order to account for the variation of the impulse application time, the differential, \( dJ \), of Eq. (6.37) is taken:

\[
\begin{align*}
    dJ &= x^T(t_f) S dx(t_f) + \sum_{k=0}^{N-1} \left[ (H_c(t) - \Lambda^T(t) \dot{x}(t)) \right]_{t=t_k^-}^{t_{k+1}^-} dt_{k+1}^- - \left[ (H_c(t) - \Lambda^T(t) \dot{x}(t)) \right]_{t=t_k}^{t_k^+} dt_k^+ \\
    &+ \int_{t_k^+}^{t_{k+1}^-} \left( \frac{\partial H_c(t)}{\partial x} + \Lambda^T(t) \right) \delta x(t) + \frac{\partial H_c(t)}{\partial u} \delta u(t) dt - \Lambda^T(t) \delta \dot{x}(t) \bigg|_{t_k}^{t_{k+1}^-} \\
    &+ \sum_{k=1}^{N-1} \left[ \frac{\partial H_d(t_k)}{\partial t} \delta t_k + \frac{\partial H_d(t_k)}{\partial x} dx(t_k^+) - v_k^T dx(t_k^+) + \frac{\partial H_d(t_k)}{\partial v} dv_k \right],
\end{align*}
\]

(6.38)

where Leibniz’s rule of integration,

\[
\frac{d}{dt} \left( \int_{a(t)}^{b(t)} f(x, t) dx \right) = \int_{a(t)}^{b(t)} \frac{df(x, t)}{dt} dx + f(b(t), t) \frac{db(t)}{dt} - f(a(t), t) \frac{da(t)}{dt},
\]

(6.39)

has been used to take the variation of the integral term with respect to the impulse application time. The variation of \( x \) is fixed with respect to time but the differential of \( x \) is not:

\[
dx(t_k^+ - t_k^-) = \delta x(t_k^+) + \delta \dot{x}(t_k^+) dt_k.
\]

(6.40)

This relationship is illustrated in Fig. 6.1.

Since we start at a fixed initial time \( t_0 \) and state \( x(t_0) \), the corresponding differentials are zero. Likewise, the differential of fixed end time \( t_f = t_N \) is zero. The following summation manipulation is performed:

\[
\sum_{k=0}^{N-1} H_c(t_k^+) dt_k^+ - H_c(t_k^-) dt_k^- = \sum_{k=1}^{N-1} H_c(t_k^+) dt_k^+ - H_c(t_k^-) dt_k^- + H_c(t_0^-) dt_0^+ + H_c(t_f^-) dt_f^0.
\]
As in the prequel, the coefficients multiplying the differentials must vanish in order for the first differential of the cost functional to vanish. The coefficients on the state and control differentials are unchanged from the prescribed time case, so the corresponding necessary conditions are identical to those presented in Eqs. (6.15)-(6.20). An additional necessary condition arises from the coefficient on
the \( dt_k \):

\[
\mathcal{H}_C(t_k^+) - \mathcal{H}_C(t_k^-) + \frac{\partial \mathcal{H}_d}{\partial t_k} = 0, \quad k = 1, \ldots, N-1.
\]  

(6.43)

This new condition is independent of the other necessary conditions. Taking the variation of impulse application time does not change what the continuous and discrete control laws are or the relationships that govern the costates. The new condition makes it possible to identify when the cost is stationary with respect to impulse application time.

In the classic LQR problem, the selection of a quadratic cost functional makes it convex with respect to state and control variables, thus guaranteeing a minimum, provided that the state and control weights are positive semidefinite and positive definite, respectively. In general, the cost functional is not convex with respect to impulse application time. Consequently, Eq. (6.43) does not indicate a minimum in the cost functional, but merely an extremum, which can be a minimum or maximum. Determining whether an application time that satisfies Eq. (6.43) minimizes or maximizes the cost, in a local fashion, is explored in Section 6.3.1.

Let’s examine what Eq. (6.43) looks like for a hybrid LTV system governed by Eqs. (6.1), (6.2). The pre- and post-impulse continuous Hamiltonians can be written in quadratic form:

\[
\mathcal{H}_C(t_k^+) = \frac{1}{2} \begin{bmatrix} x(t_k^+) & \lambda(t_k^+) \end{bmatrix} \begin{bmatrix} Q(t_k^+) & A(t_k^+) \ A(t_k^+) & -B(t_k^-)R^{-1}(t_k^-)B^T(t_k^-) \end{bmatrix} \begin{bmatrix} x(t_k^-) \\ \lambda(t_k^-) \end{bmatrix}.
\]  

(6.44)

The transition of the states and costates from pre- to post-impulse, written in matrix form, is

\[
\begin{bmatrix} x(t_k^+) \\ \lambda(t_k^+) \end{bmatrix} = \begin{bmatrix} C_k + D_kR_k^{-1}D_k^TQ_k & -D_kR_k^{-1}D_k^TC_k^- \\ -Q_k & C_k^- \end{bmatrix} \begin{bmatrix} x(t_k^-) \\ \lambda(t_k^-) \end{bmatrix}.
\]  

(6.45)

Defining the matrices

\[
H_k^+ = \begin{bmatrix} Q(t_k^+) & A(t_k^+) \\ A(t_k^+) & -B(t_k^-)R^{-1}(t_k^-)B^T(t_k^-) \end{bmatrix},
\]  

(6.46)

\[
G_k = \begin{bmatrix} C_k + D_kR_k^{-1}D_k^TQ_k & -D_kR_k^{-1}D_k^TC_k^- \\ -Q_k & C_k^- \end{bmatrix},
\]  

(6.47)

and using Eq. (6.45), the post-impulse Hamiltonian, written in terms of the pre-impulse states and
costates, is
\[ \mathcal{H}_c(t_k^+) = \frac{1}{2} \begin{bmatrix} x^T(t_k^-) & \Lambda^T(t_k^-) \end{bmatrix} G_k^T H_k G_k \begin{bmatrix} x(t_k^-) \\ \Lambda(t_k^-) \end{bmatrix}. \] (6.48)

Lastly, taking the partial derivative of Eq. (6.5) with respect to \( t_k \), we get
\[ \frac{\partial \mathcal{D}_t}{\partial t_k} = \lambda^T(t_k^+) \frac{\partial D_k}{\partial t_k} v_k = -\lambda^T(t_k^+) \frac{\partial D_k}{\partial t_k} R_k^{-1} D_k^T \Lambda(t_k^+). \] (6.49)

assuming that \( Q_k \) and \( R_k \) are constant. Using Eqs. (6.22) and (6.32), and re-writing in matrix form, we obtain
\[ \frac{\partial \mathcal{D}_t}{\partial t_k} = -[x^T(t_k^-) \Lambda^T(t_k^-)] \begin{bmatrix} -Q_k & \frac{\partial D_k}{\partial t_k} R_k^{-1} D_k^T \end{bmatrix} \begin{bmatrix} -Q_k & 1 \end{bmatrix} \begin{bmatrix} x(t_k^-) \\ \Lambda(t_k^-) \end{bmatrix}. \] (6.50)

We define the matrix \( F_k \):
\[ F_k = -\frac{1}{2} \begin{bmatrix} -Q_k & \frac{\partial D_k}{\partial t_k} R_k^{-1} D_k^T \end{bmatrix} \begin{bmatrix} -Q_k & 1 \end{bmatrix}. \] (6.51)

Using \( \Lambda(t_k^-) = P(t_k^-) x(t_k^-) \), the extremal condition can be written in terms of only the pre-impulse state vector
\[ \frac{1}{2} x^T(t_k^-) M_k x(t_k^-) = 0, \] (6.52)

where
\[ M_k = \begin{bmatrix} 1 & P^T(t_k^-) \end{bmatrix} \begin{bmatrix} H_k^- - G_k^T H_k G_k + F_k + F_k^T \end{bmatrix} \begin{bmatrix} 1 \\ P(t_k^-) \end{bmatrix}. \] (6.53)

### 6.3.1 Sufficient Condition for a Minimum

As discussed earlier, satisfying Eq. (6.43) only guarantees that the cost functional is at an extremum with respect to impulse application time. The point could be a local minimum or maximum. Intuitively, this is not unexpected: for systems with time-varying dynamics, there could be both a best and a worst time time to perform an impulsive action. A test for determining whether a point is a minimum or maximum is desirable.

For optimal control problems with no optimization with respect to time, the Legendre-Clebsch condition\(^{25,73}\)
\[ \frac{\partial^2 \mathcal{H}}{\partial u \partial u^T} > 0 \] (6.54)

is typically sufficient for ensuring a minimum in the cost. A similar convexity condition is sought
for the determination of a minimum for optimization with respect to impulse application time. The Legendre-Clebsch condition can be obtained from the more general condition

\[ d^2 J \geq 0, \quad (6.55) \]

which is a sufficient condition for an extremum of \( J \) to be a local minimum of \( J \). We proceed by determining \( d^2 J \) for our hybrid cost functional. Rather than immediately taking the second differential of the cost functional, the first differential is re-calculated, this time including a differential of the costates:

\[
d J = (x^T(t_f)S - \Lambda(t_f)^T)dx(t_f) + \sum_{k=1}^{N-1} \left[ \left( \mathcal{H}_c(t_{k-1}^*) - \mathcal{H}_c(t_{k}^*) + \frac{\partial \mathcal{H}_d}{\partial t_k} \right) dt_k + \left( \Lambda^T(t_{k}^*) - v_k^T \right) dv_k \right. \\
+ \left( \frac{\partial \mathcal{H}_d^T}{\partial x} - \Lambda(t_{k}^*) \right) dx(t_{k}^*) + \frac{\partial \mathcal{H}_d^T}{\partial v} dv_k + \left( \frac{\partial \mathcal{H}_d^T}{\partial \lambda} - x^T(t_{k}^*) \right) d\lambda(t) + \left. \frac{\partial \mathcal{H}_d^T}{\partial u} d\lambda(t) \right] dt_k.
\]

(6.56)

While including the differential of the costates has no effect on results from the first differential of \( J \) — the only difference is that an additional condition is obtained, which recovers the constraint equation — the differential of the costates play an important role in the second differential of \( J \). Indeed, for optimal control problems with no optimization with respect to a specific time, the variation of the costates, \( \delta \lambda \), is not be considered, since the costates are prescribed in order to eliminate the \( \delta x \) term, as was done for Eq. (6.14). The variation of the costate remains zero when optimizing with respect to impulse application time; however, the differential of the costate is nonzero:

\[
d \lambda(t) = \delta \lambda(t)^T + \dot{\lambda}(t) dt_k \\
= \dot{\lambda}(t) dt_k.
\]

(6.58)

Substituting \( v_k = \lambda(t_{k}^*) \) and \( dv_k = d\lambda(t_{k}^*) \), the second differential of the cost functional is:

\[
d^2 J = dx^T(t_f)Sdx(t_f) - d\lambda(t_f)^Td\lambda(t_f) + \\
\sum_{k=1}^{N-1} \left[ \left( \frac{\partial \mathcal{H}_c(t_{k-1}^*)}{\partial t_k} - \frac{\partial \mathcal{H}_c(t_{k}^*)}{\partial t_k} + \frac{\partial \mathcal{H}_d}{\partial t_k} dt_k + \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial x^T} dx(t_{k}^*) + \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial v^T} dv_k + \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial \lambda^T} d\lambda(t_{k}^*) \right) dt_k \right. \\
+ \left( \frac{\partial \mathcal{H}_c(t_{k-1}^*)}{\partial x} dt_k + \frac{\partial \mathcal{H}_d}{\partial t_k \partial x^T} dt_k + dx^T(t_{k}^*) \frac{\partial^2 \mathcal{H}_d}{\partial x^T \partial x} + dv_k^T \frac{\partial^2 \mathcal{H}_d}{\partial v^T \partial x} + d\lambda(t_{k}^*) \frac{\partial^2 \mathcal{H}_d}{\partial \lambda \partial x^T} dx(t_{k}^*) - \left( \frac{\partial \mathcal{H}_c(t_{k}^*)}{\partial \lambda} \right) \right) dt_k + d\lambda(t_{k}^*) dx(t_{k}^*) \\
+ \left( \frac{\partial \mathcal{H}_c(t_{k-1}^*)}{\partial u} \frac{\partial \mathcal{H}_c(t_{k}^*)}{\partial u} \frac{\partial^2 \mathcal{H}_d}{\partial u^T \partial u} \right) dt_k + \frac{\partial \mathcal{H}_d}{\partial t_k \partial v^T} dt_k + dx^T(t_{k}^*) \frac{\partial^2 \mathcal{H}_d}{\partial v^T \partial x} + dv_k^T \frac{\partial^2 \mathcal{H}_d}{\partial v^T \partial v} + d\lambda(t_{k}^*) \frac{\partial^2 \mathcal{H}_d}{\partial \lambda \partial v^T} dv_k.
\]
Also, since the optimal control is applied, 

\[ \frac{\partial \mathcal{H}_d}{\partial u} = 0, \]
\[ \frac{\partial \mathcal{H}_d}{\partial \lambda} = -\dot{\lambda}(t), \]
\[ \frac{\partial \mathcal{H}_d}{\partial \lambda}^T = \dot{x}(t). \]

Lastly, since \( \lambda(t_f) \) and \( t_f \) are fixed, \( d\lambda(t_f) = 0 \). After simplifying, the second differential becomes

\[ d^2 I = d\mathbf{x}^T(t_f) \mathbf{S}d\mathbf{x}(t_f) \]

\[ + \sum_{k=1}^{N-1} \left[ \left( \frac{\partial^2 \mathcal{H}_d}{\partial t_k^2} - \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial t_{k+1}} \right) dt_k + \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial \mathbf{v}_k} d\mathbf{v}_k + \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial \lambda} d\lambda(t_k^*) + \frac{\partial^2 \mathcal{H}_d}{\partial t_k \partial \mathbf{v}_k} d\mathbf{v}_k \right] \]

\[ + \int_{t_k}^{t_{k+1}} \left[ \left( \frac{\partial^2 \mathcal{H}_d}{\partial \mathbf{x}^T \partial \mathbf{x}} - \frac{\partial^2 \mathcal{H}_d}{\partial \mathbf{x} \partial \mathbf{v}_k} d\mathbf{v}_k + \frac{\partial^2 \mathcal{H}_d}{\partial \lambda \partial \mathbf{v}_k} d\mathbf{v}_k \right) d\mathbf{x}(t) \right] dt. \]
The remaining terms are rearranged. For brevity, \((\cdot)^+\) denotes when a function is evaluated at \(t = t_k^+\).

\[
d^2J = \sum_{k=1}^{N-1} \left[ \int \left( \frac{\partial H_c^+}{\partial x} dx + \frac{\partial H_c^-}{\partial u} du + \frac{\partial H_c^-}{\partial \lambda} d\lambda + \frac{\partial H_c^+}{\partial t} dt \right) dt_k \right. \\
- \left. \frac{\partial H_c^+}{\partial x} dt_k + \frac{\partial H_c^-}{\partial u} dt_k + \frac{\partial H_c^-}{\partial \lambda} dt_k \right) dt_k \\
+ \frac{\partial^2 H_d}{\partial t^2} dt_k^2 + 2 \left( \frac{\partial^2 H_d}{\partial t \partial t} \frac{dv_k}{dt} dt_k + \frac{\partial^2 H_d}{\partial t \partial \lambda} d\lambda^+ dt_k + d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial \lambda} dv_k + d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dt_k \right) dx^+
\]

\[
+ dx^T Q_d dx^- + dv_k^T R_d dv_k - d\lambda^+ dx^+ - d\lambda^- dx^-
\]

\[
+ \int_{t_{i+1}}^{t_i} \left( \frac{\partial^2 H_c(t)}{\partial x \partial x} dx(t) + 2 \frac{\partial^2 H_c(t)}{\partial x \partial u} du(t) + \frac{\partial^2 H_c(t)}{\partial u \partial u} du(t) dt \right)
\]

\[
(6.61)
\]

The pre- and post-impulse differentials of the state vector are related in the following manner:

\[
dx^+ = dx^- + D_k dv_k + \frac{\partial D_k}{\partial t_k} v_k dt_k
\]

\[
= dx^- + \frac{\partial^2 H_d}{\partial \lambda \partial t} dv_k + \frac{\partial^2 H_d}{\partial \lambda \partial t} dt_k.
\]

\[
(6.62)
\]

From Eq. (6.16), differentials of the pre- and post-impulse costates are related via:

\[
da^- = d\lambda^+ + Q_d dx^-.
\]

\[
(6.63)
\]

Substituting Eq. (6.62) and Eq. (6.63) into the following portion of Eq. (6.61) yields

\[
2 \left( \frac{\partial^2 H_d}{\partial t \partial t} dv_k dt_k + \frac{\partial^2 H_d}{\partial t \partial \lambda} d\lambda^+ dt_k + d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dv_k + d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dt_k \right) dx^+
\]

\[
+ dx^T Q_d dx^- + dv_k^T R_d dv_k - d\lambda^+ dx^+ - d\lambda^- dx^-
\]

\[
=2 \frac{\partial^2 H_d}{\partial t \partial t} dv_k dt_k + 2 \frac{\partial^2 H_d}{\partial t \partial \lambda} d\lambda^+ dt_k + 2d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dv_k + 2d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dt_k
\]

\[
+ dx^T Q_d dx^- + dv_k^T R_d dv_k - 2d\lambda^+ dx^+ - 2d\lambda^- dx^-
\]

\[
=2 \frac{\partial^2 H_d}{\partial t \partial t} dv_k dt_k + 2 \frac{\partial^2 H_d}{\partial t \partial \lambda} d\lambda^+ dt_k + 2d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dv_k + 2d\lambda^+ \frac{\partial^2 H_d}{\partial \lambda \partial t} dt_k
\]

Next, we define the notation \((\cdot)' = \frac{d(\cdot)}{dt}\). Note that \(\frac{d(\cdot)}{dt} \neq \frac{d(\cdot)}{dt_k}\). The derivative with respect to \(t_k\) is how a function changes with respect to the impulsive dynamics application time. In the case where the impulsive dynamics contain some feedback control, that change is not necessarily the same as how the function changes with respect to time. We have the following relationships for the differentials of the
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states and controls:

\[
\begin{align*}
    dx(t_k^+) &= \delta x(t_k^+) + x'(t_k^+)dt_k, \\
    du(t_k^+) &= \delta u(t_k^+) + u'(t_k^+)dt_k, \\
    dv_k &= \delta v_k + v_k' dt_k.
\end{align*}
\]

We argue that the optimization of \( J \) with respect to \( u, v_k \) and \( t_k \) is equivalent to first optimizing with respect to \( u \) and \( v_k \), and then optimizing that solution with respect to \( t_k \). That is to say

\[
\min_{u, v_k, t_k} J(u, v_k, t_k) = \min_{t_k} \min_{u, v_k} J(u, v_k, t_k) = \min_{t_k} J(u^*, v_k^*, t_k), \tag{6.64}
\]

where \( (\cdot)^* \) denotes the optimal value. In general, the sequential minimization of a cost functional with respect to one variable and then a second variable does not yield the same result as when the minimization of the function was performed simultaneously with respect to both variables. We argue, however, that in the case of optimizing with respect to application time — since the same optimal control laws for \( u \) and \( v_k \) are obtained, regardless of whether the application time is considered or not — sequential optimization of the cost functional is valid. This assumption also follows from the discussion in Section 6.3, where it was noted that the extremal condition with respect to \( t_k \) is independent of the conditions governing the controls and costates. By satisfying the extremal time condition in addition to using the optimal control feedback laws, one is picking the solution that results in an extremum of \( J \) with respect to \( t_k \) out of all possible optimal control solutions corresponding to all different \( t_k \in (t_0, t_f) \).

It follows then that the same sufficient condition for a minimum as for the prescribed time case, i.e., the Legendre-Clebsch condition, is needed, plus an additional condition to ensure a minimum with respect to application time. It is this additional condition that we derive in what follows.

Returning to the second differential of the cost functional, since the optimal control is being applied, the variations of the states, continuous, and discrete controls are all zero. Therefore, we have:

\[
\begin{align*}
    d^2 J &= \sum_{k=1}^{N-1} \left[ \frac{\partial H_c}{\partial t_k} + \frac{\partial H_c^T}{\partial x} x'(t_k^+) + \frac{\partial H_c^T}{\partial u} u'(t_k^+) + \frac{\partial H_c^T}{\partial \lambda} \lambda'(t_k^+) \\
    &\quad - \frac{\partial H_d}{\partial x} x'(t_k^+) + \frac{\partial H_d^T}{\partial u} u'(t_k^+) + \frac{\partial H_d^T}{\partial \lambda} \lambda'(t_k^+) + \frac{\partial^2 H_d}{\partial t_k^2} \\
    &\quad + 2 \frac{\partial^2 H_d}{\partial t_k \partial v^T} v_k' + \frac{\partial^2 H_d}{\partial \lambda \partial v^T} \lambda'(t_k^+) + \lambda^T(t_k^+) \frac{\partial^2 H_d}{\partial \lambda \partial v^T} v_k' + v_k^T R_k v_k' \right] dt_k^2. \tag{6.65}
\end{align*}
\]
We observe that combined, the various partial derivative terms of the continuous Hamiltonian are equal to the full derivative of the continuous Hamiltonian with respect to $t_k$:

$$\frac{d\mathcal{H}}{dt_k} = \frac{\partial \mathcal{H}}{\partial t_k} + \frac{\partial \mathcal{H}}{\partial x} \frac{dx}{dt_k} + \frac{\partial \mathcal{H}}{\partial u} \frac{du}{dt_k} + \frac{\partial \mathcal{H}}{\partial \lambda} \frac{d\lambda}{dt_k}$$

(6.66)

evaluated at both $t = t_k^{-}$ and $t = t_k^{+}$. The remaining terms are now considered. We have

$$\frac{dv_k}{dt_k} = -R_k^{-1} \left( \frac{\partial D_k^T}{\partial t_k} \lambda(t_k^+) + D_k^T \frac{d\lambda(t_k^+)}{dt_k} \right)$$

(6.67)

so

$$\frac{dv_k}{dt_k} R_k \frac{dv_k}{dt_k} = \left( \frac{\partial D_k^T}{\partial t_k} \lambda(t_k^+) + D_k^T \frac{d\lambda(t_k^+)}{dt_k} \right)^T R_k^{-1} \left( \frac{\partial D_k^T}{\partial t_k} \lambda(t_k^+) + D_k^T \frac{d\lambda(t_k^+)}{dt_k} \right)$$

$$= \lambda^T(t_k^+) \frac{\partial D_k}{\partial t_k} R_k^{-1} \frac{\partial D_k^T}{\partial t_k} \lambda(t_k^+) + 2\lambda^T(t_k^+) \frac{\partial D_k}{\partial t_k} R_k^{-1} D_k^T \frac{d\lambda(t_k^+)}{dt_k} + \frac{d\lambda(t_k^+)^T}{dt_k} D_k R_k^{-1} D_k^T \frac{d\lambda(t_k^+)}{dt_k}.$$  

(6.68)

$$\frac{\partial^2 \mathcal{H}}{\partial t_k \partial \lambda} A(t_k^+) = v_k^T \frac{\partial D_k^T}{\partial t_k} \frac{d\lambda(t_k^+)}{dt_k}$$

$$= -\lambda^T(t_k^+) D_k R_k^{-1} \frac{\partial D_k^T}{\partial t_k} \frac{d\lambda(t_k^+)}{dt_k}.$$  

(6.69)

$$\lambda^T(t_k^+) \frac{\partial^2 \mathcal{H}}{\partial \lambda \partial v^T} v_k' = -\frac{d\lambda(t_k^+)^T}{dt_k} D_k R_k^{-1} \left( \frac{\partial D_k^T}{\partial t_k} \lambda(t_k^+) + D_k^T \frac{d\lambda(t_k^+)}{dt_k} \right)$$

$$= -\frac{d\lambda(t_k^+)^T}{dt_k} D_k R_k^{-1} \frac{\partial D_k^T}{\partial t_k} \lambda(t_k^+) - \frac{d\lambda(t_k^+)^T}{dt_k} D_k R_k^{-1} D_k^T \frac{d\lambda(t_k^+)}{dt_k}.$$  

(6.70)

Finally, we can write

$$2 \frac{\partial^2 \mathcal{H}}{\partial t_k \partial v} v_k' dt_k + \frac{\partial^2 \mathcal{H}}{\partial t_k \partial \lambda} A^+ dt_k + \lambda^T(t_k^+) \frac{\partial^2 \mathcal{H}}{\partial \lambda \partial v} v_k' + v_k'^T R_k v_k' =$$

$$-2A^+ \frac{\partial D_k}{\partial t_k} R_k^{-1} \frac{\partial D_k^T}{\partial t_k} \lambda^+ - 2A^+ \frac{\partial D_k}{\partial t_k} R_k^{-1} D_k^T \frac{d\lambda}{dt_k} - \lambda^+ A^+ R_k^{-1} D_k^T \frac{d\lambda}{dt_k}.$$
The full second differential can now be written as

\[ d^2 J = \sum_{k=1}^{N-1} \left( \frac{d\mathcal{H}_c(t_k^-)}{dt_k} - \frac{d\mathcal{H}_c(t_k^+)}{dt_k} + \frac{\partial^2 \mathcal{H}_d}{\partial t_k^2} \lambda(t_k^+) \right) dt_k^2. \]

We now show that the last three terms are in fact just the full derivative of \( \frac{\partial \mathcal{H}_d}{\partial t_k} \) with respect to \( t_k \):

\[ \frac{d}{dt_k} \left( \frac{\partial \mathcal{H}_d}{\partial t_k} \right) = \frac{\partial^2 \mathcal{H}_d}{\partial t_k^2} + \frac{d\lambda(t_k^+)}{dt_k} + \frac{\partial^2 \mathcal{H}_d}{\partial t_k^2} \frac{d\mathbf{v}_k}{dt_k} \]

\[ = \lambda^T(t_k^+) \frac{\partial^2 \mathbf{D}_k}{\partial t_k^2} \mathbf{v}_k + \frac{d\lambda(t_k^+)}{dt_k} \frac{\partial \mathbf{D}_k}{\partial t_k} \mathbf{v}_k + \lambda^T(t_k^+) \frac{\partial^2 \mathbf{D}_k}{\partial t_k^2} \frac{d\mathbf{v}_k}{dt_k} \]

\[ = \lambda^T(t_k^+) \frac{\partial^2 \mathbf{D}_k}{\partial t_k^2} \mathbf{v}_k - \lambda^T(t_k^+) \frac{\partial \mathbf{D}_k}{\partial t_k} \mathbf{R}_k^{-1} \mathbf{D}_k^T \lambda(t_k^+) - \lambda^T(t_k^+) \frac{\partial \mathbf{D}_k}{\partial t_k} \mathbf{R}_k^{-1} \mathbf{D}_k^T \lambda(t_k^+)

\]

\[ - \lambda^T(t_k^+) \frac{\partial \mathbf{D}_k}{\partial t_k} \mathbf{R}_k^{-1} \mathbf{D}_k^T \frac{d\lambda(t_k^+)}{dt_k} \]

\[ = \lambda^T(t_k^+) \frac{\partial^2 \mathbf{D}_k}{\partial t_k^2} \mathbf{v}_k - \lambda^T(t_k^+) \frac{\partial \mathbf{D}_k}{\partial t_k} \mathbf{R}_k^{-1} \mathbf{D}_k^T \lambda(t_k^+) - \lambda^T(t_k^+) \frac{d\lambda(t_k^+)}{dt_k} \left( \frac{\partial \mathbf{D}_k}{\partial t_k} \mathbf{R}_k^{-1} \mathbf{D}_k^T + \mathbf{D}_k \frac{\partial \mathbf{D}_k^T}{\partial t_k} \right) \lambda(t_k^+) \]

We arrive at the following result: for a hybrid cost functional where the optimal control laws are being applied for both the continuous control input, \( \mathbf{u}(t) \) and the impulsive control input \( \mathbf{v}_k \), the second differential is equal to:

\[ d^2 J = \sum_{k=1}^{N-1} \frac{d}{dt_k} \left( \frac{\mathcal{H}_c(t_k^-) - \mathcal{H}_c(t_k^+)}{dt_k} + \frac{\partial \mathcal{H}_d}{\partial t_k} \right) dt_k^2. \]  

\[ (6.71) \]

Therefore, a sufficient condition for an extremum of \( J \) with respect to \( t_k \) to be a minimum is

\[ \frac{d}{dt_k} \left( \frac{\mathcal{H}_c(t_k^-) - \mathcal{H}_c(t_k^+)}{dt_k} + \frac{\partial \mathcal{H}_d}{\partial t_k} \right) > 0. \]  

\[ (6.72) \]
Discussion

Originally, when deriving the condition for an extremal in the cost functional with respect to the impulse application time, we used the relationship

\[ dx(t^+_{k}) = \delta x(t^+_{k}) + \dot{x}(t^+_{k}) dt_k, \]  

(6.73)

where

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]  

(6.74)

are the continuous dynamics of the system. The implicit assumption here is that

\[ \frac{dx}{dt} = \frac{dx}{dt_k}. \]  

(6.75)

When taking the second differential of the cost functional, the optimal control laws are prescribed in order to eliminate the variations \( \delta u, \delta v \) and \( \delta x \). However, in doing so, the assumption of Eq. (6.75) no longer holds.

The reason Eq. (6.75) no longer holds lies in the solution of the Riccati equation. To obtain a solution, the continuous time-varying Riccati equation is integrated backwards in time from time \( t_f \) until the impulse application \( t_k \). At time \( t_k \), the solution to the Riccati equation jumps according to the discrete Riccati equation. The remainder of the solution is then obtained by resuming to integrate the continuous Riccati equation backwards in time from \( t_k \) to \( t_0 \), at the new pre-jump values. For two different impulse application times, the solution to the Riccati equation is identical up until the greater of the two applications times. The jump causes the two solutions to diverge. The variations in the solution to the Riccati equation are illustrated in Fig. 6.2. The plot is from a mass-spring example with two masses and an impulse instantaneously affecting each mass’s velocity at a time \( t_k \), considered later on.

The difference between the state at \( t_{k^-} \) and \( t_{k^-} + dt^- \) is due to not only time difference in integrating the state dynamics, \( x(t) \), but also due to the differences in the solution to the Riccati equation that has been used in the continuous feedback law. The resulting differences in state trajectories for different impulse application times are shown in Fig. (6.3), where it is evident that Eq. (6.73) does not hold.
6.3.2 Some Numerical Examples

Before moving forward and applying the theory to the Lorentz-augmented formation maintenance problem, we explore the application of Eq. (6.43) and Eq. (6.72) to several simpler systems: a time-invariant system with only one state; a time-varying, one state system; and a time-varying, two-state system.

Time-invariant Single State System

The first system we consider is a one state, linear, time-varying system:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \neq t_k
\]  

(6.76)
\[ x_k^+ = C_k x^- + D_k v_k, \quad t = t_k \]  
(6.77)

where \( u(t) \) is the continuous control and \( v_k \) is the impulsive control. This generic 1-D system offers some intuition into the behaviour of the optimal timing solution and yields some elegant analytical results. Specifically, conditions are given for when an extremum with respect to application time exists and how to solve for the extremizing \( t_k \).

Consider the time interval \( 0 \leq t \leq t_f \) with a single impulse being applied at a time \( t_k \) contained in that interval. The continuous Riccati equation for the one dimensional system is

\[ -\dot{P}(t) = Q + 2A P(t) - \frac{P(t)^2 B^2}{R}. \]  
(6.78)

The solution to Riccati equation experiences a discontinuity at the impulse time, \( t_k \), from its post-impulse value, \( P(t_k^+) \), to its pre-impulse value, \( P(t_k^-) \), according to

\[ P(t_k^-) = Q_k + \frac{R_k P(t_k^+)}{R_k + P(t_k^+)D_k^2}. \]  
(6.79)

For the time-invariant case, the optimal condition reduces to having continuity in the Hamiltonian across the application of the impulse, \( \mathcal{H}^- - \mathcal{H}^+ = 0 \). Therefore, the sufficient condition for a stationary point with respect to \( t_k \) is

\[ \frac{1}{2} x^2(t_k^-) M_k = \frac{1}{2} x^2(t_k^-) \left( S_k P^2(t_k^+) + 2 \mathcal{A}_k P(t_k^-) + Q_k \right) = 0 \]  
(6.80)

where

\[ S_k = 2A S_k - QS_k^2, \]  
(6.81)

\[ \mathcal{A}_k = QS_k + Q_k QS_k^2 - Q_k S - 2AQ_k S_k, \]  
(6.82)

\[ Q_k = Q_k^2 (S + 2AS_k - QS_k^2) - 2Q_k (QS_k - A), \]  
(6.83)

with

\[ S = \frac{B^2}{R}, \quad S_k = \frac{D_k^2}{R_k}. \]

The optimality condition is satisfied either when \( x(t_k^-) = 0 \), which is a trivial solution, or when \( P(t_k^-) \) is a root of the quadratic expression \( M_k \).

It is well known that the Riccati solution must be positive for it to be a stabilizing solution, so we
have the condition that \( M_k \) must have at least one positive, real root in order for there to be a stationary point with respect to \( t_k \). \( M_k \) having one positive, real root is a necessary condition, but is not sufficient.

Next, we must consider how the Riccati solution changes across the impulse application time. Rearranging Eq. (6.79) gives

\[
P(t^+_k) = \frac{R_k (P(t^-_k) - Q_k)}{R_k - D_k^2 (P(t^-_k) - Q_k)}. \tag{6.84}
\]

In order for a stationary point to exist, any positive root of \( M_k \) must result in a positive \( P(t^+_k) \). Furthermore, the \( P(t^+_k) \) obtained from Eq. (6.84) must be a value that the Riccati solution achieves on the interval \( t^* < t < t_f \). The general solution to Eq. (6.78) on the interval \( 0 \leq t \leq t_f \) and for the terminal condition \( P(t_f) = 0 \) is

\[
P(t) = c_1 + c_2 \left( \frac{1 - \exp(c_3(t - t_0))}{1 + \exp(c_3(t - t_0))} \right) \tag{6.85}
\]

where

\[
c_1 = \frac{AR}{B^2}, \quad c_2 = \frac{\sqrt{A^2R^2 + QRB^2}}{B^2}, \quad c_3 = 2 \sqrt{\frac{A^2R + QB^2}{R}}, \quad t_0 = t_f - \frac{1}{c_3} \log \left( \frac{c_2 + c_1}{c_2 - c_1} \right).
\]

The final requirement for the existence of a stationary point is \( P(t^+_k) \leq P(0) \). Since, for \( c_2 > 0 \), Eq. (6.78) is monotonically decreasing, if \( P(t^+_k) > P(0) \), then the \( P(t^+_k) \) required to obtain a \( P(t^-_k) \) that is a root of \( M_k \) can never be reached during the control interval.

To summarize, the optimal application time for a scalar, LTI can be found by the following algorithm:

1. Determine the two roots of the quadratic expression \( M_k \), \( P(t^-_{k_1}), P(t^-_{k_2}) \);
2. Take any positive roots and calculate the corresponding \( P(t^+_k) \) using Eq. (6.84).
3. For every positive \( P(t^+_k) \), verify that \( P(t^+_k) < P(0) \). If this is the case, the optimal impulse application time, \( t^*_k \) is given by:

\[
t^*_k = t_0 + \frac{1}{c_3} \log \left( \frac{c_2 - P^+_k + c_1}{c_2 + P^+_k + c_1} \right). \tag{6.86}
\]

The one dimensional LTI system case is a peculiar one. For LTI systems, the solution to the Riccati equation converges to a steady solution after some time. It is likely the steady-solution does not satisfy the optimal timing condition, so we must focus our investigation to time intervals during which the solution to the Riccati equation is still converging. For simplicity, only one impulse occurs over the interval. Furthermore, for LTI systems, finding a system that has a stationary point with respect to a single application time is not easy. For many choices of \( A, B, Q, Q_k, R, R_k \), the expression \( M_k \) has no real
roots, and for many choices that do yield a positive, real root, the resulting $P(t^*_k)$ is either negative or exceeds the value of $P(0)$.

After some trial and error, on the time interval $t \in [0, 1]$, it was determined that the following set of system coefficients and weights

$$
A = 0.01 \quad B = 1 \quad Q = 1 \quad R = 4 \\
C = 1 \quad D = 1 \quad Q_k = 1 \quad R_k = 1
$$
does yield a stationary point, with an impulse application time of $t^* = 0.7907$ s. In this case, as is seen in Fig. 6.4(a), the stationary point at time $t^*$ corresponds to a maximum, not a minimum. Nevertheless, the algorithm correctly determines the stationary point.

**Time-varying Single-State System**

For the case of a single-state system where the continuous and impulsive input matrices are time-varying, the analytical algorithm for determining an optimal time is no longer applicable because the solution to the Riccati equation no longer has a closed-form solution. The stationary points of the cost functional are determined by finding the zeros of the scalar polynomial

$$
M_k = \begin{pmatrix} 1 & P(t^*_k) \end{pmatrix} \left( H_k^T - G_k^T G_k + F_k + F_k^T \right) \begin{pmatrix} 1 \\ P(t^*_k) \end{pmatrix}, \tag{6.87}
$$

where $H$, $G$ and $F$ correspond to Eq. (6.46), Eq. (6.47) and Eq. (6.51), respectively. Note that for the single-state case, the stationary points are independent of the state trajectory and are not needed for evaluating the stationary point condition. What is needed, unfortunately, is the full solution to the time-varying Riccati equation over the control interval. If the solution is calculated, the zeros of $M_k$ can be determined numerically using a root-finding method, such as the Newton-Raphson method.

Consider the optimal control of a hybrid linear time-varying system with the following system coefficients and hybrid LQR weights,

$$
A = 0.01 \quad B(t) = \cos \omega t \quad Q = 1 \quad R = 4 \\
C = 1 \quad D(t) = \sin \omega t \quad Q_k = 1 \quad R_k = 1
$$

where $\omega = 2\pi$. A control period of $t \in [0, 1]$ s is considered with a single impulsive action occurring at a time $t_1$. Using a Newton-Raphson root-finding algorithm yields, when different initial guesses are chosen, four extremal points at $t^*_1 = [0.2348 \ 0.4745 \ 0.7393 \ 0.8442]$ s. From the min/max condition,
$t_1 = 0.2348$ s and $t_1 = 0.7393$ s are determined to be minima, while the remaining times are maxima. The cost over the spectrum of possible application times, $t_1 \in (0, 1)$ s, is plotted along with the extremum and min/max conditions in Fig. 6.4(b). Although Eq. (6.43) and Eq. (6.72) succeed in determining all

![Figure 6.4: Cost and extremum and min/max conditions, as they vary with impulse application time.](image)

the local minima within the time interval under consideration, there still remains the possibility of a global minimum at an endpoint. In order to identify the global minimum, the cost functional must be evaluated for all local minima identified as well as at the endpoints of the control interval.

**Time-varying Multi-State System**

For a system dimension greater than one, the stationary point condition can no longer be written as the single scalar function $M_k$ that is independent of the state. Instead, $M_k$ is the symmetric matrix

$$M_k = \begin{bmatrix} \bf{1} & P^T(t_k) \\ \end{bmatrix} \left[ H_k - G_k^T H_k G_k + F_k + F_k^T \right] \begin{bmatrix} \bf{1} \\ P(t_k) \end{bmatrix}. \quad (6.88)$$

and the extremum condition, as stated originally in Eq. (6.52), is

$$\frac{1}{2} x^T(t_k) M_k x(t_k) = 0. \quad (6.89)$$
There are three different cases where the condition is met:

1. \( x(t_k^-) = 0 \);
2. The matrix \( M_k = 0 \);
3. \( x(t_k^-) \) is in the null space of \( M_k \).

The first case is a trivial one. The second case can occur, but in general is too restrictive and considering it alone often results in a failure to determine an extremum. It is the third case that must also be considered in order to determine when Eq. (6.89) is met. Therefore, in the multi-state case, the extremal time is dependent on the state trajectory of the system, and, importantly, on the initial conditions of the system. Intuitively, this result makes some sense: the best time or worst time to perform an action depends on where you start from.

Let’s consider a two-state hybrid time-varying system, defined by the following state and input matrices:

\[
\begin{align*}
A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & B &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
C_k &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & D_k &= \begin{bmatrix} \sin \omega t_k \\ \sin^2 \omega t_k \end{bmatrix}
\end{align*}
\]

(6.90)

where \( \omega = 2\pi \) again. A time interval of \( t \in [0, 1] \) s is considered with a single impulsive action occurring at \( t_1 \in (0, 1) \) s. A hybrid controller is designed with the state penalty matrices of \( Q = 10 \cdot 1_{2\times2} \) and \( Q_k = 2 \cdot 1_{2\times2} \), and control penalty weights of \( R = 5 \) and \( R_k = 1 \). The extremum and min/max conditions cannot be calculated without knowing the state at \( t_0^- \). In order to determine the state at \( t_0^- \), the hybrid Riccati equation must be solved for the entire control interval for each possible \( t_1 \).

However, one does not need to completely recalculate the solution to the hybrid Riccati equation for each new \( t_1 \). As was seen in Fig. 6.2, if one compares the solutions to the solution hybrid Riccati equation for two different impulse application times, \( t_1 \) and \( t_2 \), where \( t_1 < t_2 < t_f \), the solution to the Riccati equation is the same for both application times on the interval of \( t \in [t_1^-, t_f] \). Let’s consider a control interval \( t \in [t_0, t_f] \), divided into \( N \) increments, with an impulse application time at the \( k \)th increment \( t_k = t_f - k\Delta t \), where \( \Delta t = (t_f - t_0)/(N + 1) \), for \( k = 1, \ldots, N \). At each increment, the extremum and min/max conditions is evaluated. By beginning with an impulse application at the end of an
interval, for each subsequent increment, the solution to the Riccati equation needs only to be calculated from the interval \( t \in [t_0, t_{k+1}^+] \), with the impulse occurring at \( t_k \). The solution from the previous increment can be stored and used for the remaining interval, \( t \in [t_{k-1}^+, t_f] \).

Returning to the example, the initial state is \( x(0) = x_0 = [1 - 1]^T \) and the initial condition to the Riccati equation is \( P(t_f) = 1_{2 \times 2} \). Six extrema are found: three local maxima, two local minima and a global minimum at \( t^* = 0.67 \) s. The cost, extremum and min/max conditions are plotted for all possible impulse application times in Fig. 6.5.

![Figure 6.5: Cost with first and secondary order optimality conditions for an example two-state system.](image)

For a second multi-state example, the two mass, two spring, mass-spring system illustrated in Fig. 6.6 is considered over a control interval of \( t \in [0, 5] \) s. It is assumed that an actuator is present on mass 1 to control its velocity. Typically, such a system is not a hybrid system. To make it one, we introduce a single impulsive impact, \( v_1 \), that occurs at \( t = t_1, t_1 \in (0, 1] \) s, and instantaneously changes
the velocity of mass two. The state and input matrices for the dynamics of the system are

\[
\begin{align*}
A &= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{k_1 + k_2}{m_1} & \frac{k_1}{m_1} & 0 & 0 \\
\frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0
\end{bmatrix}, & C_k &= 1 \\
B &= \begin{bmatrix} 0 & 0 & 1/m_1 & 0 \end{bmatrix}^T, & D_k &= \begin{bmatrix} 0 & 0 & \cos(\omega t_k)/m_2 \end{bmatrix}^T
\end{align*}
\]

The masses are \( m_1 = 1 \) kg and \( m_2 = 0.1 \) kg, and the spring constants are \( k_1 = 1 \) N/m and \( k_2 = 1.5 \) N/m. In this example, in addition to calculating the cost for different impulse application times, we also consider the case of no impulse being applied. As Fig. 6.7(a), shows, for \( 0 \leq t < 0.3 \) s, using an impulse results in a lower cost that if one was not used. The global minimum is found on that time interval at \( t^*_1 = 0.14 \) s. Fig. 6.7(b) compares the state trajectories for the case of an optimally-timed impulse and not using an impulse at all. While the hybrid system does not return to rest any faster than the purely continuous system, the application of the impulse results in smaller transient state errors over the control interval.

### 6.4 Chapter Summary

In this chapter, we were concerned with the optimal control, in a linear quadratic sense, of a linear, time-varying system that possesses both continuous and impulsive dynamics. Of particular interest were systems that used both continuous and impulsive control inputs. Optimal continuous and impulsive control laws were derived from hybrid quadratic cost functionals using calculus of variations arguments. Both the case where impulse times are prescribed and the case where impulse applications are determined in an optimal fashion were considered.
The following novel extensions to optimal hybrid control theory were made:

1. For a quadratic cost functional, explicitly determining how the solution changes across the application of the impulsive dynamics. Specifically, Eq. (6.36), showing how \( P(t_k^+) \) transitions to \( P(t_k^-) \)

2. Determining a sufficient condition for a minimum with respect to impulse application time. Specifically, showing that Eq. (6.71) is true and deriving condition Eq. (6.72), which states that the derivative with respect to application time of the extremum time condition must be positive for a minimum.

There was also a strong focus on applying the hybrid optimal control theory to some simple linear systems, both time-invariant and time-varying, in order to gain some intuition into the theory before applying it to the Lorentz-augmented formation problem, particularly the theory for determining the optimal impulse application times. We saw that for single-state, time-invariant systems, if an extremizing time exists on the control interval, it can be determined analytically. For single-state time-varying systems, extremizing times can be determined numerically, using a Newton-Raphson root-
finding algorithm. In both time-varying and time-invariant cases, the extremizing times are independent of the initial conditions.

For multi-state systems, the extremum time condition is no longer independent of the initial state conditions. Two examples have been considered where optimal time determination was performed by sweeping through the possible application times, integrating forward state dynamics after having calculated a Riccati solution and evaluating the extremum condition for each application time. Although numerically intense, a means of reducing computational load was discussed.

Some of the hybrid control theory presented has been presented previously in Ref. 80 and the extremum time condition can be found in Refs. 73, 80. However, this work is the first to completely present the hybrid linear quadratic regulator theory, specifying explicitly how to determine the solution to the Riccati equation and the second-order minimum time condition. Validation of the theory through its application in numerical simulations is another considerable contribution.
Chapter 7

Optimal Control of Lorentz-Augmented Spacecraft Formations

With the development of hybrid linear quadratic regulator theory, we can return to the problem of control of Lorentz-augmented spacecraft formations. This chapter explores the application of the hybrid LQR to several different formation flying problems: a hybrid LQR with prescribed firing times is applied to the formation-keeping problem; the formation reconfiguration problem is treated using a hybrid LQR with optimally-timed impulses.

7.1 Lorentz-Augmented Formation Keeping

The continuous dynamics of the relative spacecraft state are once again the mean differential element error dynamics with the specific charge as the sole control input, as originally seen in Eq. (5.11). The impulsive dynamics of the system consist of only the application of the impulsive control thrust. The non-singular orbital element set, $\delta e = [\delta a \delta i \delta \Omega \delta q_1 \delta q_2 \delta \lambda]$, is used to avoid having to periodically recalculate the reference orbital elements, as is necessary for the singular elements (see Section 3.2.3, Section 4.4.2).

The dynamical equations are presented below, where $B(\vec{e})$ is now the matrix of Gauss’s variational equations, not a generic input matrix, and $B_L(\vec{e}) = B(\vec{e})\vec{f}_L(t)$, where $\vec{f}_L(t)$ is the local Lorentz force vector.
per specific charge expressed in the LVLH frame.

\[
\dot{\zeta}(t) = \tilde{A}(\bar{e}_r)\zeta(t) + B_L(\bar{e}_r) \frac{q(t)}{m} \quad t \neq t_k, \tag{7.1}
\]

\[
\zeta(t_k^+) = \zeta(t_k^-) + B(\bar{e}_r(t_k))v_k \quad t = t_k. \tag{7.2}
\]

In the context of Lorentz-augmented formation keeping, not only must the control strategy adequately maintain the formation, but it must do so in a manner that minimizes the required thruster control effort. The choices for the state error penalty matrices, \(Q(t)\) and \(Q_k\), and the control effort penalty matrices, \(R(t)\) and \(R_k\), both play roles in reducing the thruster control effort.

In the case of the latter, \(R(t)\) and \(R_k\) are chosen so that the impulsive control effort is penalized more heavily than the continuous control effort.

The selection of appropriate state error penalty matrices also plays a very important role in limiting total thruster control effort. Recall, the original motivation for introducing thruster control effort was because the relative spacecraft state was not controllable with only the Lorentz force. The state error penalty weights are chosen to reflect that lack of controllability.

Consider the controllability Gramian, \(W(t_1, t_0)\), given by Eq. (2.12). For a Lorentz-augmented system without impulsive control (Eq. (7.1) for all \(t\)), the \(W(t_1, t_0)\) calculated for the desired orbit (where \(t_0 = 0, t_1 = T\)) has five nonzero eigenvalues and one zero/near-zero eigenvalue. The eigenvectors of the nonzero eigenvalues, \(\eta_1, \ldots, \eta_5, \eta_k \in \mathbb{R}^6\), represent the linear combinations of the states that are controllable by the Lorentz force. Likewise, the eigenvector corresponding to the zero eigenvalue, \(\eta_0\), denotes the linear combination of states that is uncontrollable. It follows then that the continuous state error penalty matrix, \(Q(t)\) should only penalize the linear combinations of states that are controllable by the Lorentz force. Likewise, the discrete state error penalty weight only targets the linear combination of states that are uncontrollable by the Lorentz force. In this way, the thruster control effort can be considerably reduced.

The two state error penalty matrices, \(Q\) and \(Q_k\) are given by

\[
Q = C_1 \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{bmatrix} \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{bmatrix}^T, \tag{7.3}
\]

\[
Q_k = C_2 \eta_0 \eta_0^T, \tag{7.4}
\]

where \(C_1\) and \(C_2\) are constants that scale the weights in an appropriate fashion. For the results that follow, \(Q\) and \(Q_k\) are kept constant. Having \(Q(t)\) and \(Q_k(t)\) be time-varying so that the weights change to reflect favorable dynamics for correcting orbit elements remains unexplored at this time.
The 1 km PCO formation in polar, low Earth orbit, described by the initial conditions in Table 4.1 is the example considered for the hybrid LQR. For this formation, the unscaled state penalty matrices are as:

\[
Q = C_1 \cdot \begin{bmatrix}
0.0190 & -0.0009 & -0.1364 & 0.0034 & 0.0002 & 0.0001 \\
-0.0009 & 0.9999 & -0.0001 & 0.0 & 0.0 & 0.0 \\
-0.1364 & -0.0001 & 0.9811 & 0.0005 & 0.0 & 0.0 \\
0.0034 & 0.0 & 0.0005 & 0.9999 & 0.0 & 0.0 \\
0.0002 & 0.0 & 0.0 & 0.0 & 1.0000 & 0.0 \\
0.0001 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0000 \\
\end{bmatrix},
\]

\[
Q_k = C_2 \cdot \begin{bmatrix}
0.9810 & 0.0009 & 0.1363 & -0.0034 & -0.0001 & 0.0 \\
0.0009 & 0.0 & 0.0001 & 0.0 & 0.0 & 0.0 \\
0.1363 & 0.0001 & 0.0189 & -0.0004 & 0.0 & 0.0 \\
-0.0034 & 0.0 & -0.0004 & 0.0 & 0.0 & 0.0 \\
-0.0001 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
\end{bmatrix}.
\]

Two different hybrid LQR control strategies are tested: a one-impulse per orbit case and a four-impulse per orbit case. For each case, a terminal state penalty of \( P(t_f) = 10^8 \cdot 1_{6x6} \) is used. The single impulse is applied at a true anomaly of \( f = 1.8 \) radians; for the four-impulse case, impulses occur at \( f = [\pi/2, \pi, 3\pi/2, 2\pi] \) radians.

As with an LQR controller, the selection of control and penalty weights requires some trial and error. For both the one and four impulse cases, a scaling factor of \( C_1 = C_2 = 10^{10} \) combined with control effort penalties of \( R = 10^6 \) and \( R_k = 10^6 \cdot 1_{3x3} \) for continuous and impulsive controls, respectively, was found to yield good performance. As was done in Chapter 5 the tilted-dipole magnetic field model is used to model the Earth’s magnetic field when solving the Riccati equation.

A periodic, near-steady solution for \( P(t) \) is obtained by the method outlined at the conclusion of Section 6.2. Elements of \( P(t) \) for the single-impulse solution are plotted for 75 orbits in Fig. 7.1. Since \( P(t) \) is symmetric, only the top half of the matrix is plotted. The solution to the Riccati equation is not perfectly periodic because the chief spacecraft remains uncontrolled and its orbital elements change due to \( J_2 \) over time. Since the deputy spacecraft is maintaining a relative trajectory to the chief spacecraft,
Figure 7.1: Solution to the hybrid Riccati equation for a LEO formation maintenance problem. The periodicity of the solution is equal to the spacecraft’s synodic period.

its absolute orbit also experiences a secular change. Because the deputy spacecraft’s absolute orbit is changing, albeit slightly, the magnetic field it experiences is no longer completely periodic, resulting in the small, secular changes in the Riccati solution over large time intervals. The dominant period in the solution to the Riccati equation is the synodic period of the spacecraft, which for this example is approximately 15 orbits. This period can be easily see in the plots of Fig. 7.1. What is not immediately evident from Fig. 7.1 is that each element of $P(t)$ has one discontinuity per orbit that corresponds to an impulse application time. This result is more clearly illustrated in Fig. 7.2.

Figure 7.2: Portion of the solution to the Riccati equation for a single element of $P(t)$. Discontinuities are present once every orbit, corresponding to the impulse application time.

The formation-keeping performance of two hybrid LQR controllers is evaluated over two hundred orbits. The simulation gravity model includes the $J_2 - J_6$ zonal harmonics and the IGRF-11 magnetic
field model is used to model the magnetic field. The continuous portions of the control laws use a 500-term Fourier series to approximate the true Riccati solutions. The precise values of the Riccati solution at the jump times, \( P(t_k) \), are stored in a file and are used for the impulsive portions of the control laws. Since we are interested in the mitigation of orbital perturbations, there is no initial differential element error at the beginning of the simulation.

The relative position errors of the deputy spacecraft with respect to the chief spacecraft are plotted in Fig. 7.3. The effect of the additional thrusts per orbit is significant. The relative position error is decreased by almost 50% from the one-impulse to the four-impulse case. This reduction is achieved with only a modest increase in overall thruster \( \Delta V \). For the four-impulse case, a 3DRMS error of 2.59 m is achieved with a max 3D error of 4.82 m. The one-impulse case has a 3DRMS error of 5.47 m and max 3D position error of 16.0 m. Per orbit, the one-impulse control law applies 20.2 mm/s using the Lorentz force and an additional 0.88 mm/s using thrusters. The four-impulse strategy applies 1.3 mm/s with thrusters and 10.9 mm/s with the Lorentz force. In contrast, the two-impulse non-singular element formation-keeping strategy from Chapter 4 requires 10.4 mm/s of \( \Delta V \) per orbit. The one-impulse hybrid LQR only requires 5% of the thruster \( \Delta V \) that the conventional thruster strategy requires; the four-impulse hybrid LQR requires 13%. The trade-off is in position error: the impulse-only strategies in Chapter 4 can achieve sub-metre position control on some components, while these hybrid strategies...
can only achieve metre-level position control.

The corresponding RMS specific charge values for the one- and four- impulse control laws are $2.60 \times 10^{-5}$ C/kg and $0.92 \times 10^{-5}$ C/kg. The maximum absolute specific charge for the one and four impulse cases is $2.11 \times 10^{-4}$ C/kg and $5.44 \times 10^{-5}$ C/kg, respectively. Fig. 7.4 shows both the specific charge (7.4(a)) and thrust magnitude histories (7.4(b)) for 200 orbits of the four-impulse per orbit simulation. Note that individual thrusts are all below 1 mm/s in magnitude.

Similar to what was seen in Section 5.2.6, the appropriate choice of weighting matrices can result in a hybrid LQR controller that uses predominantly one charge polarity. The four-thrust example here is one such example. Clearly from Fig. 7.4(a), a negative specific charge is required for the majority of the control interval. Restricting that controller to only a negative charge has a negligible impact on performance, again illustrating that a single-polarity control is feasible. It was found that this behaviour ceases to be the case for hybrid LQR strategies with larger penalization of the impulsive control.

![Graphs showing specific charge and thrust magnitude history](a) Specific charge history.

![Graphs showing specific charge and thrust magnitude history](b) Impulsive thrust magnitude history.

Figure 7.4: Expended control effort for a polar 1 km PCO formation using one-impulse per orbit hybrid LQR.

How does this optimal continuous/impulsive strategy compare to the ad hoc strategy developed in Section 5.3? The performance of the optimal hybrid four-thruster-per-orbit controller compared to the four-thrust ad hoc controller, using the C1 LQR weights, is considered. The thruster $\Delta V$ for the ad hoc controller in this case is actually better than the four-impulse hybrid LQR controller: the ad hoc
control uses 0.7 mm/s per orbit while the hybrid LQR control uses 1.3 mm/s per orbit. This result is slightly unexpected, but the hybrid LQR actually requires significantly less total $\Delta V$: 12.2 mm/s per orbit is required by the hybrid LQR compared to 19.6 mm/s by the ad hoc control. The hybrid LQR also achieves smaller relative position than the four-thrust ad hoc control with a 3DRMS position error of 2.58 m and maximum position error norm of 4.82 m. Table 7.1 summarizes the performance statistics of the two hybrid LQR strategies discussed and provides a comparison to previously discussed controllers.

Table 7.1: Control strategy performance comparison for a polar 1 km PCO formation.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Control Effort</th>
<th>Rel. Pos. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta V_{\text{tot}}$</td>
<td>$\Delta V_L$</td>
</tr>
<tr>
<td></td>
<td>[mm/s]</td>
<td>[mm/s]</td>
</tr>
<tr>
<td>Cont. LQR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1 weights</td>
<td>15.7</td>
<td>15.0</td>
</tr>
<tr>
<td>C2 weights</td>
<td>15.4</td>
<td>13.1</td>
</tr>
<tr>
<td>ad hoc</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1, 4 thrusts</td>
<td>19.6</td>
<td>18.7</td>
</tr>
<tr>
<td>C2, 2 thrusts</td>
<td>24.0</td>
<td>8.7</td>
</tr>
<tr>
<td>Hybrid LQR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 thrusts</td>
<td>12.2</td>
<td>10.9</td>
</tr>
<tr>
<td>1 thrusts</td>
<td>20.2</td>
<td>19.2</td>
</tr>
<tr>
<td>Decomposed control</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Impulsive</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 thrusts</td>
<td>11.1</td>
<td>–</td>
</tr>
</tbody>
</table>

### 7.1.1 Increasing $R_k$

The effects of the increasing the impulsive control penalty, $R_k$, while holding all other parameters fixed are now discussed, in the context of the 1 km PCO example above. For the one-impulse hybrid LQR, increasing $R_k$ resulted in an unstable controller. The results and trends presented in this section are all observed using the four-impulse controller, for which stable controllers could be computed.

As one might expect, an increase in the impulsive control weight, while maintaining the other LQR weights, results in a reduction in impulsive control effort and a corresponding increase in the continuous control effort. Hybrid controllers were computed for $R_k$ of $10^8 \cdot 1_{3\times3}$, $10^{10} \cdot 1_{3\times3}$ and $10^{12} \cdot 1_{3\times3}$. Each controller successfully maintained a bounded error on the relative position, but the error magnitude increased with a reduction in thruster $\Delta V$. The $10^8$ hybrid LQR still used a meaningful amount of thruster $\Delta V$, requiring 0.25 mm/s per orbit, but the $10^{10}$ and $10^{12}$ controllers required negligible amounts of thruster $\Delta V$: $4.6 \times 10^{-3}$ mm/s and $7.2 \times 10^{-5}$ mm/s per orbit, respectively.
The effect of the decreasing amount of thruster control on the relative position error is illustrated in Fig. 7.5. The position error for the $R_k = I_{3x3} \cdot 10^{12}$ case is nearly identical to that of the $R_k = I_{3x3} \cdot 10^{10}$ and is not shown. The steady-state position error increases with a reduction in thruster control effort, but the error does not grow in an unbounded fashion. In fact, simulations that used the $R_k = I_{3x3} \cdot 10^{10}$ or $R_k = I_{3x3} \cdot 10^{12}$ hybrid LQRs, but did not apply the impulsive control, showed negligible change in position error compared to the analogous simulations that did apply the impulse control effort.

![Figure 7.5: 3D relative position error achieved by four-impulse hybrid LQR strategies, for different impulsive control penalty weights.](image)

We conclude that the relative spacecraft state, while not fully controllable, is stabilizable using the Lorentz force. The uncontrollable component of the state does not cause a growth in position error. It is important to note that, although we have seen that the hybrid LQR can be designed for a Lorentz-augmented formation-keeping problem in a way that eliminates the need for the impulsive thrusting (assuming the increased position error is tolerable), the inclusion of the impulsive control is still necessary for the computation of hybrid LQR in order to obtain a bounded solution to the Riccati equation.

### 7.1.2 Performance at Different Inclinations

As we did with the continuous LQR in Chapter 5, the performance of the hybrid LQR is considered at different orbit inclinations. We also want to determine how much we can get out of the Lorentz force with a minimal amount of thruster control effort. To that end, three hybrid LQR strategies are tested, with the only difference between them being the impulsive control weight, $R_k$. Three different values for $R_k$ are used to illustrate the differences in performance of the hybrid LQR with varying levels of impulsive thrusting. In each case, $R_k$ is the identity matrix, scaled by either $10^6$, $10^8$ or $10^{10}$. Each strategy uses four impulses per orbit, applied at true anomalies $f = [\pi/2, \pi, 3\pi/2, 2\pi]$. The continuous control penalty is $R = 10^6$. 
Six performance metrics are considered: total thruster $\Delta V$; fraction of control effort realized by the Lorentz force; total $\Delta V$ required; RMS position error norm; max absolute specific charge; RMS specific charge. Fig. 7.6 compares these metrics for a 1 km PCO formation with a chief orbit semimajor axis of 7092 km. Inclinations from 0° to 90° are considered, but the critical $J_2$ inclination, 63.4° is again omitted. The simulation for each inclination was 30 orbits in duration. For Fig 7.6(a), Fig 7.6(c) and Fig 7.6(d), the corresponding performance curve for the two-impulse strategy from Chapter 4 is provided for reference.

![Figure 7.6: Performance metrics of four-thrust hybrid LQR performance as it varies across inclinations from 0° to 90°.](image)

From Fig. 7.6(b), we see, as we did for the continuous LQR, that nearly all the the control effort can be realized with the Lorentz force with a sufficiently large impulsive control penalty. The trade-off is that such control comes at the cost of coarser formation keeping, as seen in Fig. 7.6(d) and as discussed in the previous section. Position control is, in general, poorer than what was seen with the continuous LQR, but since thruster actuation only occurs four times per orbit, this result is to be expected.

Regardless of the impulsive control penalty, position error for the hybrid LQR strategies is largest at low inclinations. Differential drift of the right ascension of ascending node can be large at low inclinations for formations with large along-track separations (see Eq. (3.54), (3.56)). Since at low inclinations the Lorentz force is strongest in the radial direction of the LVLH frame, it is not particularly useful for correcting right ascension drift, which requires an out-of-plane acceleration. As a result, for
formations in at low inclinations, any control strategy favoring the use of the Lorentz force struggles with along-track position keeping (see Eq. (3.48)).

The large specific charges required at near-equatorial inclinations, as seen in Fig. 7.6(e) and Fig. 7.6(f), are evidence of the difficulty the control strategies have. Larger specific charges are required to realize the out-of-plane accelerations needed to correct for differential right ascension drift. However, in realizing the out-of-plane acceleration, the Lorentz force also induces a perturbing acceleration in another direction, which will need correcting later. As was originally seen with the results of the case (2) LQR in Section 5.2.5, this additional perturbation contributes considerably to the relative position error and the larger total corrective $\Delta V$ at these inclinations.

7.1.3 Optimal Impulse Application Time

The hybrid linear quadratic regulator theory presented in Chapter 6, strictly speaking, is valid for linear, time-varying plants being driven to zero. To apply it to the formation-keeping problem in Section 7.1, the nonlinear relative dynamics of the deputy spacecraft are approximated by the mean differential element error dynamics. Those error dynamics are used to design the hybrid LQR. A more accurate approximation of the relative spacecraft dynamics are the dynamics of the mean differential elements, not the dynamics of the mean differential element error — see Section 3.2.2.

Using the mean differential element dynamics does, however, transform the problem from a regulation problem to a tracking problem, for which the hybrid regulator theory is not strictly valid. So, as is commonly done, the regulator control law is applied to the tracking problem. The persistent, nonzero relative position error seen in this section’s formation-keeping example results from this incongruous application.

The optimal impulse application time theory has also only been developed for the regulation problem, and is unsuitable for formation-keeping problem. Recall from Section 6.3 that determining the optimal application time on a given interval depended on the initial conditions of the system. In the formation-keeping examples above, the initial state is assumed to have no errors. A system governed by the differential element error dynamics and initialized with zero state error remains at that zero state, which is obviously not representative of the actual formation flying plant. Attempting to use the optimal timing theory for such a case fails to yield a solution.
7.2 Formation Reconfiguration

The spacecraft formation reconfiguration problem is one that, as opposed to the formation-keeping problem, lends itself well to the application of the optimal timing theory. There is a distinct initial error in differential element state: the difference between the spacecraft’s current differential elements and the desired differential elements of the new formation. With the large initial error, the relative dynamics are well represented by the linearized differential element error dynamics.

In order to apply the optimally-timed hybrid LQR to the formation configuration problem, we structure the problem in a certain way, in order to make it computationally tractable. First, we stipulate that there is no time limit on the duration of the reconfiguration, nor is the duration of the reconfiguration penalized. Furthermore, we choose to apply one thrust per orbit and we do not consider the case of no thrust being applied in a specific orbit. Recall that in the formation-keeping examples, the differential element error was never completely zero, but rather, a small, steady-state periodic error was achieved. For the reconfiguration problem, the deputy spacecraft is be considered to be in the desired new formation once the controller can no longer reduce the state error and a similar, steady-state error is observed.

Rather than specifying a duration time and the number of thrusts that occur in this time, and then attempting to numerically determine the optimal timing of those thrusts simultaneously — a problem whose solving does not scale with $N$ in a desirable fashion — the following algorithm optimizes a single thrust timing per orbit, before moving on to the next orbit:

1. For a given initial differential element error, $\zeta(t_0)$, determine an optimal impulse application time $t_k$, and calculate the corresponding solution to the Riccati equation for that time. The optimal application time is determined in the following manner:
   (a) Select an application time, $t_k$ from the control interval, $t_0 < t_k < t_0 + T$;
   (b) Solve the hybrid Riccati equation for $t_k$;
   (c) Simulate the system from $t_0$ to $t_k$ using the Riccati solution and evaluate optimality conditions;
   (d) If $t_k$ is the first or last possible application time, simulate until the end of time interval regardless of optimality conditions and store cost value for comparison.

2. Simulate the formation for orbit, from $t = t_0$ to $t = t_0 + T$, using the optimally-timed hybrid LQR for that orbit;

3. Calculate the new differential element error after one orbit, $\zeta(t_0 + T)$ and return to step 1. Repeat until reconfiguration is complete.
Note, it is possible that on a finite time interval of one orbit, $t_0 \leq t \leq t_0 + T$, there is no time $t_k$ at which the optimality conditions in Eq. (6.43) and Eq. (6.72) are satisfied. If that is the case, then the minimum for that time interval is at one of the end points. This consideration is taken into account when determining the optimal application for each orbit.

The reconfiguration example we consider is a deputy spacecraft transferring from a 10 km projected circular orbit, with a phase angle of $\alpha_0 = \pi/4$ rad to a 1 km PCO with $\alpha_0 = 0$ rad. The chief orbit is a polar, low Earth orbit. Initial and target differential orbital elements are given in Table 7.2.

<table>
<thead>
<tr>
<th>Chief elements</th>
<th>Deputy initial diff. elements</th>
<th>Deputy target diff. elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{a}$ km</td>
<td>$\delta \bar{a}$ m</td>
<td>m</td>
</tr>
<tr>
<td>7092</td>
<td>0.542</td>
<td>-1.836</td>
</tr>
<tr>
<td>$\dot{e}$ deg</td>
<td>$\delta \dot{e}$ deg $10^{-4}$</td>
<td>4.985</td>
</tr>
<tr>
<td>0.05</td>
<td></td>
<td>$10^{-10}$</td>
</tr>
<tr>
<td>$\bar{i}$ deg</td>
<td>$\delta \bar{i}$ deg $10^{-2}$</td>
<td>5.593</td>
</tr>
<tr>
<td>90.0</td>
<td></td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>$\dot{\bar{\Omega}}$ deg</td>
<td>$\delta \dot{\bar{\Omega}}$ deg $10^{-2}$</td>
<td>-5.453</td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>$\dot{\bar{\omega}}$ deg</td>
<td>$\delta \dot{\bar{\omega}}$ deg $10^{-1}$</td>
<td>-5.713</td>
</tr>
<tr>
<td>0.003</td>
<td></td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>$\bar{M}$ deg</td>
<td>$\delta \bar{M}$ deg $10^{-1}$</td>
<td>5.706</td>
</tr>
<tr>
<td>359.997</td>
<td></td>
<td>$10^{-2}$</td>
</tr>
</tbody>
</table>

A hybrid LQR with the following weights is used to realize the reconfiguration. The state penalty weights are calculated according to Eqs. (7.3) and (7.4), with $C_1 = C_2 = 10^6$. The control effort penalties are $R = 10^6$ and $R_k = 10^8 \cdot \text{I}_{3 \times 3}$ for continuous and impulsive controls, respectively. The resulting controller is not a particularly aggressive one: approximately twenty-five orbits are required to perform the desired maneuver. The reconfiguration can be completed in significantly less time by choosing alternate weighting matrices. It was decided to show these results because the required control efforts, while still exceeding what is feasible with current technology, do not exceed those limits significantly, whereas results employing a faster acting regulator do. Fig. 7.7 shows the relative position error of the spacecraft for a fifty orbit simulation and Fig. 7.9 illustrates the trajectory of the deputy spacecraft in the LVLH frame, as it performs the configuration. We see that the position error in all three axes reaches a similar steady-error to what was achieved in the formation-keeping examples. During the simulation, the role of the controller transitions from one of performing the reconfiguration, to one of formation maintenance. After the twenty-fifth orbit, the relative position error is no longer decreasing, but has reached a steady-state, time-varying periodic error; we effectively have formation keeping being performed with a single, optimally-timed impulse. Final relative position error is comparable to that achieved by the four-impulse formation-keeping scheme discussed in Section 7.1.

Although we described the control as not particularly aggressive, the specific charge required to
achieve the reconfiguration is still considerable. For the first seven orbits, the specific charge is $O(10^{-1}) - O(10^{-2})$ C/kg; the RMS specific charge is 0.0148 C/kg. The maximum specific charge occurs at the very beginning of the reconfiguration, and initial simulation results had instantaneous specific charge values, of durations no longer than several seconds, as large as 1 C/kg. Since such an exceptionally large value only occurs very briefly, the specific charge in the results presented below was saturated at an absolute value 0.1 C/kg. Changes in performance were imperceptible, with the only instance of saturation occurring momentarily prior to the application of the first impulsive thrust. Fig. 7.8 illustrates the specific charge history for the fifty orbit reconfiguration.

The magnitude of impulsive thrusting is moderate. For the first five orbits, individual thrust magnitudes are $O(10^{-1})$ m/s, reducing to $O(10^{-3})$ m/s and smaller afterward. Total $\Delta V$ applied by thrusters over the fifty orbits is 0.535 m/s, with the majority of that, as can be seen in Fig. 7.8, being applied in the first ten orbits. After twenty-five orbits, both the magnetic and thruster control efforts are similar to the control efforts required for formation keeping. Let us now compare the control effort required by the hybrid LQR to a more conventional, thruster-only reconfiguration strategy. The first conventional strategy we consider is the formation-keeping scheme proposed in Chapter 4. Although designed for formation keeping, the constraint equations are equally well-suited for performing a desired reconfiguration. When the analytical, decoupled two-thrust strategy (Eq. (4.15) and Eq. (4.16)) is applied, that is, two-thrusts per orbit, with no optimization, then the thruster control effort required is 14.8 m/s. Alternatively, if the two-thrust problem is solved with a numerical optimization package, the required $\Delta V$ is 14.3 m/s. In both cases, the configuration, is completed in a single orbit, rather than
Figure 7.8: Applied control efforts for a reconfiguration maneuver from a 10 km PCO to a 1 km PCO.

The twenty-five required when using the hybrid LQR. If the $N$-thrust impulsive thrust optimization problem is considered with a completion time of twenty-five orbits, with one thrust per orbit, numerical optimization yields a total required $\Delta V$ of 15.1 m/s. In all cases, the thruster $\Delta V$ required by the conventional reconfiguration strategies is two orders of magnitude larger than that required by the hybrid LQR.

The optimal times at which the thrusts are applied for the reconfiguration are shown in Fig. 7.10, each as a fraction of the orbital period; the larger the fraction, the later in the orbit the thrust is applied. Recall, chief argument of perigee is $\omega = 0$, so the beginning of the orbit corresponds to perigee while the middle of the orbit corresponds to apogee. From Fig. 7.10, we see that optimal times for most orbits correspond with perigee: most times are either at the very beginning or very end of the orbital period. In some cases, a thrust at the end of an orbit is followed by the a thrust applied immediately at the start of the next orbit. This results suggests that two thrusts can be combined into a single thrust.

7.3 Chapter Conclusions

The application of the hybrid LQR theory developed in Chapter 6 to the spacecraft formation flying problem has been explored in this chapter.

The prescribed-time hybrid LQR was effectively demonstrated for the formation keeping of a 1 km PCO formation. The choice of $Q$ and $Q_k$ is particularly important in order to specify which states are to be controlled by the continuous control and which should be controlled by the impulsive control. For
Figure 7.9: Trajectory of deputy spacecraft transferring from a 10 km PCO to a 1 km PCO as seen in the LVLH frame. (a) radial along-track plane; (b) radial out-of-plane plane; (c) along-track out-of-plane plane; (d) trajectory in the LVLH frame.

Figure 7.10: Optimal thrust application time for each thrust for a spacecraft transferring from a 10 km PCO to a 1 km PCO, given as a fraction of the orbital period.
the formation-keeping controller, $Q$ and $Q_k$ are chosen to reflect the controllability of the Lorentz-augmented system. The best hybrid LQR position error is very close to that achieved by the best continuous Lorentz-augmented LQR, albeit requiring some more thruster $\Delta V$.

In general, the $Q$ and $Q_k$ weights are a means of choosing which states of a system is regulated by the continuous control and which states is regulated by the impulsive control.

It was shown that for very large impulsive control penalties, the impulsive control input could be virtually eliminated and secular change in the formation be mitigated by the Lorentz force alone. The cost of doing so, however, is a time-varying steady-state relative position error that is considerably larger than when using Lorentz force and thruster actuation in combination.

An optimally-timed hybrid LQR was designed to perform a formation reconfiguration from a 10 km PCO formation to a 1 km PCO, with one impulse occurring per orbit. It was shown that a Lorentz-augmented strategy can considerably reduce the thruster $\Delta V$ for such a maneuver. The algorithm for determining the optimal time for each orbit is an important contribution and highlights that for LTV systems, determining the optimal time can be computationally expensive. Optimizing the number of impulsive thrusts needed for the maneuver is still an open problem.
Chapter 8

Practical Considerations for Lorentz-Augmented Formation Flying

So far in our discussion of Lorentz-augmented formation flight, the storage and regulation of electrical charge on a spacecraft has been taken for granted. It has simply been something that our hypothetical spacecraft is capable of doing. This chapter attempts to address some of the practical issues related to charge storage and control. We review the results of important spacecraft charge control experiments that were performed on orbit with a focus on what they mean for a potential Lorentz-augmented spacecraft as well as review some recently proposed Lorentz-augmented spacecraft architectures before presenting a spacecraft charging model to estimate the power requirements for a beam-controlled Lorentz-augmented spacecraft. Power estimates are presented for a Lorentz-augmented formation-keeping example.

8.1 Spacecraft Charging Missions

The area of charge accumulation on spacecraft has been an active area of study since the 1960s. Numerous experiments have been launched on sounding rockets and spacecraft to study charging phenomena. Ref. 81 provides a detailed summary of many of these experiments. Sounding rocket experiments such as MAIMIK and G-60-S first demonstrated that surface potential could be affected by the emission of electrons. An important result obtained by these tests is that vehicle surfaces could charge to potentials
equal only to that of the emitted beam energy.

The SCATHA (Spacecraft Charging AT High Altitudes) mission was specifically designed to study spacecraft charging and charge mitigation in the geosynchronous plasma environment. It was launched in 1979 into a high, elliptical, near-equatorial orbit with a semimajor axis of $a = 41778$ km, an eccentricity of $e = 0.19$ and inclination of $7.9^\circ$. Unlike the sounding rockets, the plasma environment SCATHA experienced for the majority of its operation was a very different plasma regime than the one that is being considered in this thesis for Lorentz-augmented formation flight. Nevertheless, SCATHA’s beam charging experiments and results are some of the most comprehensive charging experiments performed and warrant some discussion.

The spacecraft’s purpose was to study various aspects of spacecraft charging, the most relevant to this thesis being the effect of the electron and ion beam emission on spacecraft surface potential. To that end, the SCATHA spacecraft was equipped with both ion and electron guns: the former was a xenon-emitting ion gun capable of energies between 1 keV and 2 keV at currents of 0.3 mA to 2 mA; the latter operated with electron currents of 1 $\mu$A to 13 mA and beam energies of 50 eV to 3 keV.\(^8\)

It was found that a 13 mA electron beam charged the spacecraft to a potential of 4 kV.\(^8\) Electron emissions corroborated sounding rocket results, showing that, with sufficient beam current, spacecraft potential could be driven to match the beam energy.\(^8\)

For ion emission, a 1 mA, 1 keV beam drove the spacecraft’s potential to between -600 to -800 V,\(^8\) well below the beam energy. The emission of ions revealed an important, non-monotonic behaviour of spacecraft charging. As beam current increased, charge build up would reach a maximum and then begin to decrease. This was attributed to the formation of a virtual anode, which reflected a portion of the beam back to the spacecraft. This limit is known as the ‘space-charge’ limit and has been predicted in theory.\(^8\) It is possible that for a sufficiently energetic mono-energetic beam that a virtual electrode does not form, so the limit is not reached. Practically speaking, however, any ion beam has an energy distribution and so the less energetic fraction of a beam will return to the spacecraft and limit charging capability.

The SERT-II (Spacecraft Electric Rocket Test) mission is another significant spacecraft charging mission, however, whose primary purpose was not to study spacecraft charging, but rather to perform long-duration tests of one of the first generation of ion thrusters developed by the United States. It was launched in 1970 into a 1000 km altitude, polar orbit and operated for over ten years. The impact of ion engine operation on the spacecraft’s surface potential was controlled through the use of a neutralizing electron emission. For low neutralizer voltages, negative spacecraft potentials were observed.\(^8\) The ion thrusters have two operational levels: with a 15 cm diameter beam, they generated 10 mN of force.
operating at 85 mA of current and 22 mN of force for a 200 mA current.85

8.1.1 Proposed Lorentz-Augmented Spacecraft Architectures

As part of Peck’s original paper proposing Lorentz-augment spacecraft,18 a possible Lorentz-augmented spacecraft architecture was proposed. A spacecraft hub, surrounded by a spherical capacitive shell on which charge would accumulate and be stored was envisioned. It was suggested that a Van der Graaff generator could be used to build up and deposit charge onto the capacitive shell and a plasma contactor would be used to discharge excess charge as needed. A 1 kg spacecraft was proposed with a light-weight capacitive shell, 3 m in radius.

An alternative, more detailed Lorentz-augmented spacecraft architecture was proposed by Streetman and Peck in Ref. 53. It achieves its charge without the use of particle beam emissions. A capacitor constructed of long wire filaments arranged in cylindrical fashion, creating a stocking-like structure, is used to store the spacecraft’s charge. A solar array power supply is used to establish a potential between two exposed spacecraft surfaces. The mobility of ions is much lower than the mobility of electrons, so the positive surface would accumulate a nonzero charge, while the negative end is effectively grounded to the plasma. The total charge stored is then approximately the product of the established potential and the capacitance of the stocking capacitor.

While attractive since it does not use particle beams, this proposed architecture is limited to a negative charge polarity, since charge is built up through the accumulation of the more mobile electrons from the surrounding plasma. Furthermore, such an architecture, at least as envisioned in Ref. 53, results in an extremely large spacecraft: for a proposed spacecraft capable of a specific charge of 0.006 C/kg, a stocking length of 20 km with a diameter of 1 km was required. Such a spacecraft would clearly be a monolithic craft and not one that is suitable for formation flight.

The hub-and-shell spacecraft architecture of Ref. 18, being of a scale more suitable for a possible formation-flying mission, is considered in this chapter for power estimation. Particle beam emission is the means by which charge is accumulated and regulated on the spacecraft.

8.2 Modeling Spacecraft Charging

Understanding spacecraft charging has been important, historically, in order to mitigate adverse affects such as spacecraft surface arc discharging, shifting of spacecraft electrical ground and changes to exposed surface material properties. In the case of Lorentz-augmented spacecraft, spacecraft charging
is no longer a deleterious phenomenon but a desirable, indeed, essential one.

Essential to the spacecraft charging problem is an understanding of the ambient plasma environment surrounding the Earth. Ref. 86 gives a thorough description of the Earth's plasma environment and how it interacts with spacecraft: much of the background material presented here is drawn from there as well as in Ref. 87 and Ref. 84.

An orbiting spacecraft can be thought of as an isolated probe in a plasma, where the plasma in its case is the space plasma surrounding the Earth. The charging of a spacecraft is fully described by the balance of incoming and outgoing currents subject to Poisson's equation, which describes the potential distribution within the spacecraft sheath, from its surface to the sheath boundary, and the time-independent collision-less Boltzman-Vlasov equation, which describes the distribution of particles in the local plasma. Solving the spacecraft charging problem amounts to determining the spacecraft surface potential, \( V_{sc} \), such that the incident currents from the ion and electron fluxes to the spacecraft are in equilibrium.

For the control of the specific charge of a Lorentz-augmented spacecraft, the problem is reversed: the spacecraft's desired surface potential is determined by the specific charge it requires. Given the surface potential, one needs to determine the magnitude of the current beam the spacecraft needs to emit in order to artificially realize that potential, given other incident currents arriving from the spacecraft's environment. The challenge lies in accurately determining the incident currents to spacecraft, given its surface potential. The incident currents themselves are functions of the spacecraft surface potential and are also affected by the structure of the plasma sheath that forms around a charged spacecraft.

A full numerical solution to Poisson's and Vlasov's equations is needed to accurately determine the sheath structure and resulting surface potential. Such a solution is beyond the scope of this work. The goal here is to use reasonable analytical approximations found in the spacecraft charging literature to approximate the various incident currents that a Lorentz-augmented spacecraft in LEO would experience and obtain a rough estimate of spacecraft power requirements.

There are two general regimes to consider in spacecraft charging: the geosynchronous environment and the low Earth orbit environment. The distinguishing feature between the two environments is the size of the Debye length, which dictates on what length scales the plasma can be considered to be charge neutral. In the context of a charged spacecraft, the Debye length, \( \lambda_D \), dictates the length scale over which species of the opposite polarity to the spacecraft’s charge is attracted to the spacecraft.

A sheath of charged particles of a polarity opposite to that of the spacecraft forms around a charged spacecraft. Its thickness is typically on the order of the Debye length of the ambient plasma. The spacecraft’s charging dynamics are intrinsically linked to the size and shape of this sheath. There are
two limiting cases that allow for simplifications to be made and result in analytical expressions for incident currents: a thin-sheath limit, where the sheath thickness $S$ is much smaller than the size of the spacecraft and the thick-sheath limit, where the sheath is much larger than the dimension of the spacecraft. For conventional spacecraft, the thick-sheath assumption is appropriate for geosynchronous orbits, and thin-sheath is appropriate for low Earth orbits. The thin-sheath approximation in LEO is dependent on a low spacecraft surface potential. Large surface potentials, as would be the case for Lorentz-augmented spacecraft, increase the sheath size. In order to precisely determine sheath size and shape, complex numerical computations, such as those presented by Choinière and Gilchrist\textsuperscript{88} for cylindrical geometries, are required.

In LEO, the Debye length is typically $O(10^{-2})$ m, making it, in general, smaller than the radius of a spacecraft. This means that the sheath thickness of a spacecraft with a small surface potential in LEO is on the order of the Debye length. For large surface potentials in LEO, however, sheath thickness may grow to be much larger than the Debye length and can exceed the radius of the spacecraft. As stated before, this can significantly alter the incident currents to the spacecraft.

### 8.2.1 Low-Earth Orbit Charging Model

The low Earth orbit plasma environment is a complicated regime for spacecraft charging. Unfortunately, LEO is also where use of the Lorentz force as a means of propulsion is most viable, as was discussed in Section 5.1.1. The ambient electrons in LEO are not very energetic, so secondary and backscatter electron currents are negligible. The high ambient current density in LEO also means that photoelectric current is also not significant. The magnetic field strength in GEO is sufficiently small as to not significantly affect current collection, however, in LEO, electrons are magnetized by the Earth’s magnetic field, complicating their behaviour near biased surfaces. Furthermore, the spacecraft is moving much faster than the ambient ions, so plasma wake effects in the spacecraft’s sheath can be significant.

The following spacecraft charging model is based primarily on equations from Hastings and Garrett.\textsuperscript{86} $V_{sc}$ is the spacecraft surface potential, $n_{i}$ is the ambient particle density, $T_{i}$ is the thermal energy of charged species, $m_{i}$ is the mass of the species, $j$ and $I$ are current density and current, respectively, and $k$ is the Boltzman constant.

For negative spacecraft surface potentials, $V_{sc} < 0$:

1. Due to the spacecraft’s supersonic velocity with respect to the ambient ions, ion collection is dominated by the ram current

$$I_{i} = A e_{0} n_{i} (v_{rel} + 0.37c_s)$$  \hspace{1cm} (8.1)
where \( c_s = \sqrt{2kT/e} \) is the ion acoustic velocity, \( A \) is the surface area of the spacecraft projected in the direction of the spacecraft velocity and \( v_{rel} \) is the magnitude of the spacecraft velocity relative to the ions.

2. Electrons are repelled at negative potentials; electron distribution is Maxwellian despite the presence of the magnetic field. Current density per unit area is

\[
J_e = en_e(c_e/4)\exp(e_0V_{sc}/kT_e)
\]  

(8.2)

where \( c_e = \sqrt{8kT_e/\pi m_e} \).

For positive spacecraft surface potentials, \( V_{sc} > 0 \):

1. Electrons are now the attracted species. In the limit of large attractive potentials, for a spherical spacecraft, incident electron current in a magnetized plasma is given by

\[
I_e = I_r \left( \frac{1}{2} + \frac{2}{\beta} \sqrt{\frac{eV_{sc}}{kT_e \pi}} + \frac{2}{\pi \beta^2} \right),
\]  

(8.3)

where the random electron current is \( I_r = A_{sc}c_0n_e \sqrt{(kT_e)/(2\pi m_e)} \), \( A_{sc} \) is the total surface area of the spacecraft, \( \beta = R_{sc}|Q_{pe}| \sqrt{(2m_e)/(\pi kT_e)} \) accounts for the magnetization of the electrons, and \( Q_{pe} = (q_eB)/(m_e) \) is the frequency with which the charged particle gyrates due to the Earth’s magnetic field. \( B \) is the magnitude of the local magnetic field in units of Tesla.

2. Ions are now the repelled species and treated the same way as electrons are for negative potentials.

For both positive and negative potentials, the sheath thickness surrounding the spacecraft is roughly approximated by

\[
S(V_{sc}) = \lambda_D \left( \frac{e_0|V_{sc}|}{kT} \right)^{3/4},
\]  

(8.4)

where the Debye length is \( \lambda_D = \sqrt{\epsilon_0 kT/(e_0^2 n)} \), \( \epsilon_0 \) is the permittivity of free space, and \( e_0 \) is the elementary charge of an electron.

The Lorentz-augmented spacecraft concept considered in this study, and illustrated in Fig. 8.1, is a concept proposed by Peck,\(^{18} \) where a spacecraft hub is surrounded by a large, conducting sphere, on which the charge would accumulate. An alternative spacecraft architecture was considered in Ref. 53, but due to the size of the spacecraft envisioned there, it would likely be unsuitable for formation flying.

The capacitance of the spacecraft concept being used here is enhanced by its plasma sheath, such
that the total capacitance of the spacecraft is modeled as that of two concentric spheres

\[ C(V_{sc}) = 4\pi\varepsilon_0 \left( \frac{R_{sc}(R_{sc} + S(V_{sc}))}{S(V_{sc})} \right), \]  

where the outer sphere is the sheath boundary. The charging dynamics of the spacecraft are approximated to first order by those of a capacitor with potential applied to it:

\[ \frac{dV_{sc}}{dt} = \frac{(I_{i}(V_{sc}) + I_{B}(V_{sc}) + I_{beam})}{C(V_{sc})}. \]  

The instantaneous power required by the spacecraft is then \( P = I_{beam}V_{sc} \). It is assumed that the beam energy is sufficiently energetic to escape entirely from the spacecraft.

The sheath thickness is an important quantity, not only because it dictates the incident currents to the spacecraft (thick-sheath versus thin-sheath) but also because it enhances the spacecraft’s capacitance. The approximation of sheath thickness with Eq. (8.4) and the assumption that the sheath geometry is perfectly spherical (in LEO, the sheath would be elongated due to the spacecraft’s velocity in the plasma) are the biggest limitations of this model.

### 8.2.2 Power Estimation for a 1 km PCO Formation

The formation keeping of the 1 km PCO formation using the four-impulse hybrid LQR from Chapter 7 is considered as an example for the spacecraft charging model. The chief orbit has a semimajor axis of 7092 km and an inclination of 90°. Recall that in Fig. 7.4(a), it was shown that the specific charge
required for formation keeping was predominantly negative. The formation-keeping simulation was re-run with the same controller, but in this case the specific charge was artificially saturated at zero when a positive specific charge was required. As we discussed in Section 5.2.6 and Section 7.1, in cases where the specific charge is predominantly one polarity, implementing this saturation has a negligible effect on the controller’s performance with regards to relative position error. A single specific charge polarity is desirable since it simplifies the spacecraft hardware architecture: for the case under consideration, since only a negative charge is used, the hypothetical spacecraft only requires an ion beam to regulate its charge.

The spacecraft we are considering has the following characteristics: a mass of 1 kg; its capacitive outer spherical shell has a radius of 3 m and is made of low-density Polypropylene Homopolymer as suggested by Peck. The ion beam is assumed to be sufficiently energetic for the beam to escape the spacecraft’s surface potential attraction. Otherwise, the spacecraft’s negative potential would cause the beam ions to return back the spacecraft, negating the effect of their emission.

The International Reference Ionosphere model is used to obtain a realistic description of the ambient plasma environment in the spacecraft’s orbit.

A proportional-integral controller is used to regulate the specific charge of the spacecraft, given the reference specific charge from initial formation-flying simulation. Furthermore, the beam current is limited to a current of 0.5 A. Fifteen orbits of formation keeping are considered.

It will become clear that the driving factor behind the power required for Lorentz-augmented formation keeping is the capacitance of the spacecraft. For this reason, several different values for spacecraft capacitance are considered. We begin by using the sheath-enhanced capacitance described by Eq. (8.5).

Fig. 8.2 presents the power requirements for the 1 km PCO, assuming the concentric shell model for the spacecraft’s capacitance and with a sheath thickness modeled by Eq. (8.4). The maximum instantaneous power required is a 82.7 kW and the RMS power required is 14.5 kW. As can be seen in Fig. 8.2(c), the spacecraft easily discharges from large specific charge magnitudes, requiring little to no beam current to do so. Unfortunately, the power requirements for charging for this case are not particularly practical for the size of spacecraft assumed.

The lower limit of the spacecraft’s capacitance is the capacitance of a sphere in a vacuum, given by $C = 4\pi \varepsilon_0 R_{sc}$. For $R_{sc} = 3$ m, the spacecraft has a ‘base’ capacitance of $C = 3.3 \times 10^{-10}$ F. The plasma sheath effect can augment the spacecraft’s capacitance, but only when the sheath thickness is small compared to the radius of the spacecraft. At large spacecraft surface potentials, sheath thickness can grow to sizes comparable to the spacecraft radius and the enhancing capacitive effect of the sheath is
lost. This is what occurs in this case since, as seen in Fig. 8.2(b), surface potentials for this simulation are on the order of tens to hundreds of kilovolts.

![Graphs](image)

Figure 8.2: Power requirement estimates for 1 km PCO formation keeping, assuming a capacitance modeled by two concentric spheres.

The first two rows in Table 8.1 compare the RMS power and beam current between two runs of the power model for the 1 km PCO: the first row considers the worst case capacitance, that is that of a sphere in vacuum; the second considers the variable capacitance of the sheath-enhanced capacitance. There are only marginal differences between two capacitance cases, largely due to the highly surface potential resulting in a large sheath thickness.
Two additional capacitance cases are also considered in Table 8.1: a fixed capacitance of $1 \times 10^{-9}$ F and $1 \times 10^{-8}$ F. The exact mechanism by which these capacitance values are achieved is not considered here. If the same shell architecture were to be used, the former capacitance would require a shell radius of 9 m, while the latter would require a radius of 90 m. The purpose in including them is to illustrate how the power requirement can drop dramatically with larger spacecraft capacitance.

The power requirement is significantly reduced at these larger capacitance values. The reason for this reduction comes from the definition of capacitance: the amount of charge per unit of potential an object retains. The spacecraft does not need to drive its surface potential to as large of values as it did in the lower capacitance cases to achieve the same specific charge value. As a result, the beam energy can be lower, resulting in a lower power requirement. The higher capacitance cases are also advantageous because, due to the resulting lower surface potentials, the spacecraft experiences lower incident currents from the plasma environment and consequently requiring a lower beam current to affect the spacecraft’s charge.

For comparison, the charge-accumulation method proposed by Streetman and Peck\textsuperscript{53} required a mean power of 53.54 kW to generate a specific charge of $-0.006$ C/kg.

The high surface potential required by the sphere-in-vacuum and concentric sphere capacitance cases is extremely undesirable. These large potentials will cause the particle sheath surrounding the spacecraft to swell in size and enhance beam return, making it more difficult to charge the spacecraft. This result compounds the need for the spacecraft to a higher capacitance than what is being realized by the concentric shell capacitance model.

Table 8.1: Power requirements for the formation keeping of a 1 km PCO formation, for different spacecraft capacitance.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere-only, $3.3 \times 10^{-10}$ F</td>
<td>14789</td>
<td>83972</td>
<td>0.253</td>
<td>0.50</td>
</tr>
<tr>
<td>Concentric shells, variable</td>
<td>14503</td>
<td>82730</td>
<td>0.248</td>
<td>0.50</td>
</tr>
<tr>
<td>$1 \times 10^{-9}$ F</td>
<td>3725.2</td>
<td>26065</td>
<td>0.171</td>
<td>0.50</td>
</tr>
<tr>
<td>$1 \times 10^{-8}$ F</td>
<td>74.9</td>
<td>944.3</td>
<td>0.037</td>
<td>0.50</td>
</tr>
</tbody>
</table>

One issue that has so far been ignored about the usage of an ion beam for charge regulation, but warrants some discussion, is the effect of momentum exchange between the ejected ions and the spacecraft. Ideally, the ion beam current would be sufficiently small that the resultant force would be negligible. Recalling that on the SERT-II mission, a 22 mN force was generated with a 0.22 A Xenon ion current, it is reasonable to expect that at the maximum simulated beam current of 0.5 A used for
formation keeping results in a non-negligible force being applied. Note though that the beam is not continuously operating at 0.5 A. In particular, for the higher capacitance cases, such as the $10^{-8}$ F capacitance case, the RMS value of the current is only 0.037 A. Even in the low capacitance cases, such as in Fig. 8.2(c), the current beam is off for large portions of time and only briefly operates at saturation, meaning that although there are times at which the beam may generate a milliNewton-magnitude force, its duration is likely too small for the force to result in a significant change in velocity.

8.2.3 Model Limitations

The single biggest limitation of the power model presented is the approximation of plasma sheath size with Eq. (8.4). For a more accurate sheath thickness, numerical simulations, like those presented in Ref. 88, of the plasma follow around spacecraft and the charge exchange occurring between the two is required. A more accurate sheath thickness will affect the incident current rates to the spacecraft from the plasma and, in turn, alter the magnitude of the current beam required for current balance at a desired surface potential.

An additional assumption that has been made regarding the current beam from the spacecraft is that there is no beam return to the spacecraft. This assumption stems from the assumption that the current beam is mono-energetic, which in reality, it is not be. Rather, the beam has some energy distribution. As a result, not all the particles in the beam have sufficient energy to escape the attractive surface potential of the spacecraft, causing them to return. The consequence of some of the beam particles returning to the spacecraft is that the spacecraft is not be able to completely charge to the nominal beam energy, limiting the maximum specific charge it can obtain.

8.3 Chapter Conclusions

This chapter presented a first-order power estimation model for a Lorentz-augmented spacecraft using an ion beam to regulate its surface potential. Spacecraft capacitance was found to be a key parameter in the power requirement for such a spacecraft. For a small spacecraft capacitance, very large surface potentials are required to achieve even modest specific charge magnitudes. This results in a large power requirement for two reasons: first, the beam energy needs to be very high in order to drive surface potential to the required amount and, second, incident currents increase as surface potential increases, requiring a larger beam current to charge the spacecraft. Although the model results are not expected to be very accurate, they do provide insight into the charging behaviour of a Lorentz-augmented spacecraft.
Reasonable spacecraft power requirements were found for spacecraft capacitance greater than $1 \times 10^{-8}$ F. However, even with a capacitance of $1 \times 10^{-8}$ F, the proposed Lorentz-augmented architecture is considerably less power-efficient than ion thrusters, like HT-100 Xenon thruster developed by ALTA SpA, which requires about 20 W per milliNewton of force generated. At this time, a beam-charged Lorentz-augmented spacecraft does not appear feasible. In addition to developing an alternative spacecraft charging architecture that is more efficient, further work is needed to better understand and quantify plasma sheath structure and thickness around a Lorentz-augmented spacecraft in LEO. An improved model of a spacecraft’s plasma sheath will result in better estimates of incident currents and ultimately higher accuracy estimate for electrical power.
Chapter 9

Contributions and Future Work

Spacecraft formation flight continues to represent an exciting mission architecture concept for scientific and terrestrial observation applications. The motivation of this thesis has been the minimization of thruster ΔV for spacecraft formation maneuvers. This minimization is achieved by developing control strategies formulated with the mean differential orbital elements. The usage of differential elements was motivated because of their simplicity in representing the relative spacecraft state compared to using a relative Cartesian representation. Recall that a desired formation geometry can be easily described with a single, time-invariant, six-component vector of differential elements, rather than an entire relative position/velocity reference trajectory. Using differential orbital elements considerably simplifies the control problem, changing it from a tracking to a regulation problem. The use of mean elements, in the Brouwer30 sense, allows for the natural inclusion of the secular effect of the $J_2$ zonal harmonic perturbation into the relative formation dynamics. By virtue of consistently using the mean differential element representation of the relative dynamics for a variety of different control strategies, the advantages and disadvantages of the state representation have been implicitly explored.

9.1 Thesis Contributions

The important contributions of this thesis are summarized below.

1. Chapter 4 investigated impulsive formation-keeping strategies for formations in orbits with nonzero eccentricity. Novel contributions in this chapter were:

   1.1. the derivation of the $N$-impulse constraint equations in terms of both classical (Eq. (4.4)–Eq. (4.9)) and non-singular (Eq. (4.22), Eq. (4.25), Eq. (4.26)) orbital sets for differential element
control, valid for both a zero and nonzero chief orbit eccentricity;

1.2. developing a stability test for the two-impulse classical differential orbital element formation-keeping strategy (Section 4.3).

2. Lorentz-augmented formation keeping was proposed in Chapter 5. Specifically,

2.1. the uncontrollable subspace of the Lorentz-augmented differential orbital element dynamical system (Section 5.2.1) was identified;

2.2. the combination of thruster and Lorentz force control was proposed and explored, and led to the development of three novel formation-keeping strategies:

2.2.1. Given a pre-existing control strategy, geometrically decomposing the control effort into a Lorentz force component and thruster component (Section 5.2.2);

2.2.2. Application of the classical LQR to the Lorentz-augmented formation-keeping problem (Section (5.2.3));

2.2.3. Wrapping an impulsive control strategy around the LQR-derived Lorentz-augmented formation-keeping control (Section 5.3).

3. Motivated by the desire to optimally combine impulsive thruster actuation with continuous Lorentz force actuation, Chapter 6 developed new control theory for that purpose. Contributions include:

3.1. the presentation of the necessary conditions for an extremal of a cost function with continuous and impulsive inputs. It was shown that the continuous matrix Riccati equation solves the two-point boundary problem between impulsive actions, and a new boundary condition after an impulsive action is obtained using the discrete-time matrix Riccati equation (Section 6.2);

3.2. the derivation of the sufficient condition for a minimum with respect to an impulse application time (Section 6.3.1, Eq. (6.72));

3.3. developing an analytical optimal impulse application time solution for a single-state LTI system (Section 6.3.2);

3.4. the numerical validation of the necessary and sufficient conditions for an optimal impulse application time for single- and multi-state LTV systems (Section 6.3.2);

4. Chapter 7 applied the control theory developed in Chapter 6 to the formation flying problem. Specific contributions are
4.1. the application of the hybrid linear quadratic regulator for both the formation keeping and formation reconfiguration problems;

4.2. synthesis method for a hybrid linear quadratic regulator with optimal application times;

4.3. identifying that the Lorentz-augmented differential orbital element system is stabilizable with the Lorentz force alone.

5. Chapter 8 explored the power requirements for a Lorentz-augmented spacecraft, specifically

5.1. provided a first-order electrical power estimate for Lorentz-augmented spacecraft formation keeping for spacecraft utilizing a ion/electrical beam for surface charge regulation.

9.2 Future Work

Much of the research presented in this thesis can be continued and developed further. Some topics suitable for further exploration are now discussed.

Control of Systems with Continuous and Impulsive Inputs

With the determination of the necessary and sufficient conditions for a minimum with respect to the impulse application time, the next logical question (and perhaps the most frequently posed one) is “How do you choose the number of impulse applications?” An optimal number, $N^*$, of impulse applications would be desirable, i.e., one that minimizes a cost function similar to Eq. (6.3). At this time, however, a method for determining $N^*$ is an open problem.

The optimal feedback gain for the infinite time-horizon LQR problem applied to a LTI system is obtained by solving the algebraic matrix Riccati equation. An equivalent solution is obtained by integrating the continuous matrix Riccati equation backward in time for the LTI system and retaining only the steady-state portion of the solution. Since an infinite time-horizon is considered, the transient portion of the integration can be discarded. An interesting problem would be to investigate whether there is an equivalent steady-state solution to the hybrid LQR problem and under what conditions it applies.

The duality between control and estimation theory is well known. The linear quadratic regulator minimizes the quadratic cost of state error and control effort for linear systems in an analogous fashion to the Kalman filter minimizing the expected estimation error for a linear system with Gaussian noise on the measurements. A “continuous-discrete” Kalman filter has already been formulated, which combines a continuous-time plant model with discrete-time measurements. What is suggested here is
the investigation of a filter that uses both continuous measurements and periodically-occurring discrete measurements. The application of the optimal impulse timing condition to the question “when is the best time to take a measurement?” is also worth further investigation.

The concepts of the hybrid continuous/impulsive dynamical system are closely related to the concepts governing switched dynamical systems. Indeed, the continuous/impulsive dynamical system can be considered a particular case in a more general switched system formulation. The continuous/impulsive control theory presented here can be generalized to accommodate the broader class of switched systems. Doing so presents some new, interesting aerospace applications, such as the optimal attitude control of a spacecraft that is employing a variety of actuation methods, such as magnetorquers, reaction wheels and reaction control thrusters. The theory stands to address both the optimal combination and optimal times switching between the different actuation modes.

Lorentz-Augmented Formation Flight

There remains considerable work to be done in the area of Lorentz-augmented spacecraft. With regard to the continuous state feedback control applied for formation keeping, recall that knowledge of the local magnetic field is required for the feedback law. An important, outstanding issue is determining the sensitivity of the control law to errors in the magnetic field vector. The Earth’s magnetic field is well-modeled, so for Earth-orbiting Lorentz-augmented missions, errors in the magnetic field vector will stem from errors in the estimation of the spacecraft’s inertial state. A study correlating inertial spacecraft state error to the performance of the Lorentz-augmented feedback control strategy would be interesting.

The concept of Lorentz-augmented spacecraft is still rife with technological and practical issues. If Lorentz-augmented actuation is to be pursued as a means of spacecraft actuation, a high-capacitance means of surface charge retention is required to have a feasible electrical power requirement, as indicated by the results in Chapter 8. An efficient means of charge accumulation and modulation is also required for Lorentz-augmented formation flight. The spacecraft concept proposed by Streetman and Peck achieves a specific charge magnitude on the order $O(10^{-3})$ and is suitable for a monolithic Lorentz-augmented spacecraft, but its dimension are measured in kilometres, making it unsuitable for formation flight.

Beam control presents one feasible way of charge modulation, which is why it was considered in the power requirements study, however beam charging has its own set of issues. For negative surface charges, an ion beam would be used to modulate the charge, however, the same technology
could be used for an ion thruster, which is a more capable actuation method than the Lorentz force. Nevertheless, since the ion beam would be briefly pulsed rather than operating continuously, there may be some advantages to using the Lorentz force over the ion thruster acceleration for formation keeping. Any study investigating such advantages would also need to assess how significant any acceleration resulting from operating a charge-regulating ion beam would be. An electron-beam charge regulation scheme would not suffer from acceleration effects, due to the small mass of the electron, however, it would require a formation-keeping control strategy that requires a predominantly positive charge – recall, many of the results in this work required a predominantly negative charge. In general, more research is required to better develop the spacecraft architecture for a Lorentz-augmented mission.

Regardless of whether a Lorentz-augmented spacecraft uses an ion/electron beam-based charge regulation scheme, there is still considerable improvement to be made in estimating its power consumption. As was emphasized in Chapter 8, the most significant limiting factor of the model is the approximation used for the plasma sheath thickness. To have an accurate understanding of sheath behaviour at the surface potentials that are expected for Lorentz-augmented spacecraft in LEO, intensive computational simulation (similar to simulations presented in Ref. 88 for cylindrical geometry) is required for the spacecraft geometry in LEO plasma conditions. Experimental validation of such simulations would also be needed.
BIBLIOGRAPHY


Appendix A

\section*{J$_2$ No-Drift Condition}

The condition to minimize average along-track drift is:

\[ \delta M + \delta \dot{\omega} + \delta \dot{\Omega} \cos i = 0 \]  
(A.1)

where

\[ \delta M = \frac{\partial M}{\partial a} \delta a + \frac{\partial M}{\partial e} \delta e + \frac{\partial M}{\partial i} \delta i \]  
(A.2)

\[ \delta \dot{\omega} = \frac{\partial \dot{\omega}}{\partial a} \delta a + \frac{\partial \dot{\omega}}{\partial e} \delta e + \frac{\partial \dot{\omega}}{\partial i} \delta i \]  
(A.3)

\[ \delta \dot{\Omega} = \frac{\partial \dot{\Omega}}{\partial a} \delta a + \frac{\partial \dot{\Omega}}{\partial e} \delta e + \frac{\partial \dot{\Omega}}{\partial i} \delta i \]  
(A.4)

Using the partial drift rate from Chapter 3, Section 3.2.3, we obtain

\[ \left( \frac{\partial \dot{M}}{\partial a} + \frac{\partial \dot{\omega}}{\partial e} + \frac{\partial \dot{\Omega}}{\partial e} \cos i \right) \delta a = \left[ -\frac{3n}{2a} - \frac{21}{8} \left( \frac{3 \cos i - 1}{a} \right) \right] G(1 + \eta) \delta a \]  
(A.5)

\[ \left( \frac{\partial \dot{M}}{\partial e} + \frac{\partial \dot{\omega}}{\partial e} + \frac{\partial \dot{\Omega}}{\partial e} \cos i \right) \delta e = \frac{3Ge}{4\eta^2} (3 \cos^2 i - 1)(4 + 3\eta) \delta e \]  
(A.6)

\[ \left( \frac{\partial \dot{M}}{\partial i} + \frac{\partial \dot{\omega}}{\partial i} + \frac{\partial \dot{\Omega}}{\partial i} \cos i \right) \delta i = -\frac{3}{4} G \sin 2i(4 + 3\eta) \delta i \]  
(A.7)

where \( G = J_2 n \left( \frac{R_e}{p} \right)^2 \). For the drift due to \( \delta a \), \( 3n/2a \gg \frac{21}{8} \left( \frac{3 \cos i - 1}{a} \right) G(1 + \eta) \), so we can approximate the drift solely by

\[ \left( \frac{\partial \dot{M}}{\partial a} + \frac{\partial \dot{\omega}}{\partial a} + \frac{\partial \dot{\Omega}}{\partial a} \cos i \right) \delta a \approx \frac{3n}{2a} \delta a \]  
(A.8)

We wish to find a \( \delta a \) such that along-track drift is minimized.

\[ \frac{3n}{2a} \delta a = \left( \frac{\partial \dot{M}}{\partial e} + \frac{\partial \dot{\omega}}{\partial e} + \frac{\partial \dot{\Omega}}{\partial e} \cos i \right) \delta e + \left( \frac{\partial \dot{M}}{\partial i} + \frac{\partial \dot{\omega}}{\partial i} + \frac{\partial \dot{\Omega}}{\partial i} \cos i \right) \delta i \]

\[ = \frac{3Ge}{4\eta^2} (3 \cos^2 i - 1)(4 + 3\eta) \delta e - \frac{3}{4} G \sin 2i(4 + 3\eta) \delta i \]

\[ \delta a = J_2 n \left( \frac{R_e}{p} \right)^2 \frac{(4 + 3\eta)}{2} \frac{a}{n} \left( (3 \cos^2 i - 1) \frac{e}{\eta} \delta e + \sin 2i \delta i \right) \]

\[ = J_2 R_e^2 \frac{(4 + 3\eta)}{2a \eta^4} \left( (3 \cos^2 i - 1) \frac{e}{\eta} \delta e - \sin 2i \delta i \right) \]  
(A.9)
A Note About the Typeface

This thesis is set in the URWPalladioL typeface, a member of Palatino typeface family, of which the original was designed by Hermann Zapf and released in 1948. URWPalladioL is available in \LaTeX through the pxfont package, maintained by Young Ryu.