A Study on Pre-Service Teachers’ Conceptions of Proofs

By: Hui Jun, Deng

A research paper submitted in conformity with the requirements
For the degree of Master of Teaching
Department of Curriculum, Teaching and Learning
Ontario Institute for Studies in Education of the University of Toronto

APRIL 2015

© Hui Jun, Deng, 2015
Abstract

The study examined pre-service teachers' conceptions of mathematical proofs by describing their perspectives and experiences towards the teaching and learning of proofs. Based on the findings, it is clear that pre-service teachers hold different views on proofs and their visions are shaped by their past experiences. The participants were able to identify the difficulties students experience in learning proofs as well as students' deficiencies in constructing proofs. They also recognized the importance of proofs in mathematics education, and emphasized the importance for teachers to take more responsibility in implementing the lessons and activities to meet students' needs when teaching proofs. Nevertheless, the findings also suggest that pre-service teachers do not possess a clear understanding of what a mathematical proof is, since there lacks an organized and systematic way in the teaching and learning of mathematical proofs.

Key words: Mathematical Proofs, Pre-service Teacher, Conceptions, Experience
Acknowledgement

I would like to acknowledge and thank a number of people for their ongoing support and encouragement throughout the process. Thank you to my research supervisor, Dr. Mary Reid, for all her advices and supports. I would also like to thank my participants who donated their time and shared their experiences with me. They have helped me gain a deeper understanding of the topic. I am grateful to my fellow MT colleagues and PLC group for helping me identify problems and make improvements. I also owe a special debt of gratitude to Frank Qin for editing the paper. To my family and friends, I extend my deepest thanks.
# Table of Contents

Abstract .................................................................................................................. 2  
Acknowledgement ................................................................................................... 3  
Chapter 1 .................................................................................................................. 6  
  Introduction to the Research Study ........................................................................... 6  
  Researcher’s Background ..................................................................................... 7  
  Purpose of the Study ............................................................................................. 9  
  Overview ............................................................................................................. 10  
Chapter 2 Literature Review .................................................................................. 12  
  What constitutes a mathematical proof? ............................................................... 12  
  Roles of Proofs in Mathematics ......................................................................... 13  
  Proofs in Mathematics Education ...................................................................... 15  
    Rigorous/Formal Proof ..................................................................................... 15  
    Evaluating Proofs ............................................................................................ 16  
    Strategic Methods ........................................................................................... 18  
  Teaching and Learning of Mathematical Proof .................................................. 21  
  Teachers' Belief of Proofs .................................................................................... 26  
Chapter 3 Methodology ......................................................................................... 30  
  Nature of the study ............................................................................................. 30  
  Participants ......................................................................................................... 30  
  Data Resources ................................................................................................... 31  
  Interview Procedure ............................................................................................ 31  
  Data Analysis ..................................................................................................... 32  
  Ethical Procedure ............................................................................................... 32  
  Limitation ........................................................................................................... 33  
Chapter 4 Findings ................................................................................................. 34  
  Introduction ........................................................................................................ 34  
  Defining Mathematical Proofs ............................................................................. 34  
  Pre-service Teachers' Conceptions of the Role of Proof .................................... 35  
  Importance ........................................................................................................ 37  
    Competency .................................................................................................... 38  
    Problem Solving Strategies ............................................................................. 39  
  Challenges Confronting in the Teaching and Learning of Mathematical Proofs .... 39  
    Problems ........................................................................................................ 39  
    Introduction of Formal and Informal Proofs .................................................... 40  
    The Teaching and Learning of Proving Techniques ......................................... 41  
    Assessment .................................................................................................... 42  
    Generality ...................................................................................................... 43  
    Positive or Negative Impact .......................................................................... 44  
  Instructional Strategies ....................................................................................... 44  
    Demonstration ................................................................................................ 45  
    Repetition/Practice ......................................................................................... 45  
    "Knowing Why" ............................................................................................. 47  
    Discussion ...................................................................................................... 48  
    Exploration ..................................................................................................... 48
# Chapter 5 Conclusion

- Defining Mathematical Proofs
- Importance
- Challenges Confronting in the Teaching and Learning of Mathematical Proofs
- Instructional Strategies
- Implication
- Future Study

# Appendix

- Appendix A: Interview Questions
- Appendix B: Letter of Consent
- References
Chapter 1 Introduction

A proof is generally defined as “a mathematical argument, a connected sequence of assertions for or against a mathematical claim” (Stylianides, 2007, p. 107). Proof lies in the center of mathematics. It is the “engine that has driven historical developments in mathematics, the vast body of knowledge with roots in the folk traditions and earliest historical records of ancient civilisations” (Gough, 2010, p. 43). Proofs are to “mathematics what spelling is to poetry; mathematical works do consist of proofs, just as poems do consist of characters” (Tall, Cheng, Koichi, Knodrativa, Whitley et al, 2012, p. 14). Proofs distinguish mathematics from other scientific disciplines and enable mathematicians to discover and verify new theorems at the “highest level of study” in mathematics (Tall et al., 2012, p. 14).

Proofs also play an important role in the mathematics curriculum. According to the Ontario Mathematics Curriculum (2007), “reasoning helps students make sense of mathematics. Classroom instruction in mathematics should foster critical thinking – that is, an organized, analytical, well-reasoned approach to learning mathematical concepts and to solving problems” (p. 19). Specifically, it is stated that students should learn to use “counter-examples to disprove conjectures and use deductive reasoning to assess the validity of conjectures and formulate proofs” (Ontario, 2007, p. 19).

However, many researchers have found that a majority of students have difficulties on constructing proofs, especially when they move on to further advanced mathematics courses (e.g. Hemmi, 2010; Jaknke & Wambach, 2013; Stylianides, Stylianides, & Philippou, 2007). For many students, proofs are mysteries. They do not understand how the mechanism of proving works. In order to address this issue, it is suggested that teachers should provide more opportunities for students to practice proof constructions. What's more, proof activities should
provide opportunities for students to “recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs and select varies types of reasoning and methods of proof” (National Council of Teachers of Mathematics [NCTM], 2002, p. 56).

Students learn mathematics through the experiences which teachers provide. To teach students more effectively, teachers must have a solid understanding of the meaning of proofs. However, existing research has shown that in-service teachers hold a limited understanding in the nature of proofs, since they have a lack of experience in learning and teaching proofs. Although teachers recognize the various roles proofs play in mathematics, they are not aware of the main role which proofs play as a tool to improve students’ understandings (Dickerson, 2008).

Few researches have studied pre-service teachers' conceptions of proofs. Due to the vital role, which teachers play in cultivating students' abilities in logical reasoning, and the importance of proofs in mathematics education, in this research paper, I attempt to analyze pre-service teachers' perspectives and experiences on the teaching and learning of mathematical proofs in order to examine pre-service teachers' understanding on proofs.

**Researcher's Background**

I learned my first formal mathematical proof when I was in grade seven. I started with the so called two-column Euclidean geometry proof (i.e. “the formal type of proof in which known or derived statements are written in the left column, and the reason that each statement is known or valid is written next to it in the right column” (Math World, 2014)). At the beginning of learning, I had to confess that I had no idea what a proof was. I did not understand why we needed to learn to construct geometric proofs. I held an opinion that proving was less helpful for mathematics learning. At that time, a proof was more like a mechanical equation to me. In my
mind, when constructing a proof, the only thing I needed to do was to use the same set of theorems repeatedly until I reached the conclusion. Besides, my understanding of proofs was constrained in Euclidean geometry.

Yet, owing to my mathematics teacher, who placed a high value on proof and proving, I got a lot of chances to practice proof constructions. When introduced new theorems, he always demonstrated the proofs, because he believed that proof and proving helped students understand the theorems better. Undoubtedly, I learned a large amount of knowledge about proofs, such as the proving techniques and proof structures during this process unconsciously.

Especially when I was studying for my undergraduate with double majors in mathematics and statistics, the knowledge I accumulated in proving was advantageous for me. I had more opportunities to practice constructing proofs. I learned more strategic knowledge on proving. Also, my understanding in proofs deepened. I expanded my understanding of proofs into different areas, such as number theory and group theory. I also learned how proofs construct from the axiomatic conception to analytical conception (“an analytic proof solves a problem, by making hypotheses and using a mixture of deductive moves and induction to present a solution to the problem” (Goethe & Friend, 2010, p. 273)). The power of proof constructions interested me in different ways, not limited to the traditional ways by which I was taught in middle school. At that time, I realized that a proof was not as simple as equating an identity or unwrapping a set of definitions and theorems to show the truth or falsity of a statement. Proofs required students to organize the knowledge they had learned and choose associated knowledge from the large body of knowledge in order to justify a statement. I deepened my understanding in what a proof was and appreciated the role of proofs in explaining and helping me better understand the mathematics concepts.
Meanwhile, I found that many of my friends were struggling with constructing proofs. More than 70% of the students failed the test of the first year proof course in my math class. Students were even unable to differentiate the meaning of terms such as converse (Given a statement “if P, then Q”, the converse is “if P, then Q” (Math World, 2015)) and contraposition (“If a statement S can be expressed as an implication p → q in symbolic notation, the contraposition of S is ¬ q → ¬ p, where p is the hypothesis and q is the conclusion” (Antonini & Mariotti, 2008, p. 402)). Neither were they aware of the basic techniques used in constructing proofs (e.g. proof by induction, proof by contradiction), nor did they understand the structures they could follow to help them organize their ideas. Their struggles, from my personal experience, were partly correlated to their past learning experiences. For instance, they were not provided with formal instruction in proving or did not have enough practice in constructing proofs.

In my opinion, it is important to prepare students for more advanced mathematics courses. Teachers play a significant role in helping students transit from an intuitive sense to logical reasoning, and learn to construct mathematical proofs for given conjectures. Besides, teachers' conceptions on proofs along with their past learning experiences may affect the way they teach students to construct proofs. Therefore, I decided to conduct a study on pre-service teachers' perspectives and experiences on the teaching and learning of mathematical proofs for this research paper.

**Purpose of the Study**

Teachers hold different views regarding teaching before entering into the world of education (Canadian Education Association [CEA] & Canadian Teachers Federation [CTE], 2012). It is likely that their views are shaped by their past learning experiences, either positive or
negative. Their visions are likely to have an impact on the ways how they teach students, which include the teaching of mathematical proofs.

The purpose of this paper is to examine pre-service teachers' conceptions of proofs by describing their perspectives and experiences on the teaching and learning of mathematical proofs. Through this research study, I hope to learn more about pre-service teachers’ beliefs on mathematical proofs as well as the influence on their practice. I also hope to learn different instructional strategies which I can use to teach mathematical proofs more effectively in my future class. Finally, I hope to practice and refine the skills I have learned from the program on how to become a reflected researcher.

The main question that will be addressed in my research for the purpose of the study is: What are pre-service teachers' perspectives and experiences on the teaching and learning of mathematical proofs?

The following sub-questions will support the main research question:

1. What constitute mathematical proofs from pre-service teachers' perspectives?
2. What are the roles mathematical proofs play in teaching and learning from pre-service teachers' perspectives?
3. Why or why not, is it necessary to introduce proving techniques?
4. How are students related to proofs from pre-service teachers' perspectives?
5. How do pre-service teachers conceptualize the teaching about mathematical proofs in the mathematics curriculum?

Overview

This research paper is divided into five chapters. The first chapter is introductory in nature. It provides an introduction to the researcher's background, purpose of the study and an
overview for the research paper. Chapter two is the literature review. It discusses the research conducted over time and describes the rationale of the research. Chapter three is the methodology chapter which gives detailed information about procedure for data collections and data analysis. It also goes through the ethical issues needed to be addressed. Chapter four presents the main finding from the data. Chapter five provides conclusions of the research paper and gives some suggestions for possible further work.
Chapter 2 Literature Review

Mathematical proofs play an essential role in mathematics. Under different circumstances, mathematical proofs have different meanings. In this literature review section, I discuss the characteristics of proofs, the roles of proofs playing in mathematics, the strategic methods used to construct proofs as well as the different approaches on the teaching and learning of proofs in a classroom context.

What Constitutes a Mathematical Proof?

For teachers to teach students with proofs, they must first have a solid understanding of the meaning of mathematical proofs. Proofs are the languages which mathematicians use to communicate with each other. Proofs are also the “central disciplines of mathematics” and the "bearers of mathematics knowledge" (Barbeau & Hanna, 2008, p. 345). Although mathematicians can “generally agree on the acceptance of an adequate proof, no explicit general definition of a proof is shared by the entire mathematical community” (Cabassut, Conner, Iscimen, Fruinghetti, Jahnke et al, 2012, p. 169). This makes the situation more complicated to deal with in the teaching of mathematical proofs because teachers cannot consistently provide a clear definition to students (Cabassut et al., 2012, p. 170).

In general, a proof is said to be a “directed tree of statements, connected by implications, whose end point is the conclusion and whose stating points are either in the data or are generally agreed facts or principles” (Bell, 1976, p. 26). It is also defined as “a mathematical argument, a connected sequence of assertions for or against a mathematical claim” (Stylianides, 2007, p. 107). A proof is seen as a “process of logical reasoning, an intellectual and symbolic tool” (Hemmi, 2010, p. 273), and is used for “communicating a system of ideas as one can explore, make conjectures and conviction about the truth and falsity of their conjectures by constructing proofs”
A proof consists of important elements of mathematics, such as strategic methods, and is used to “specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning” (Hanna & de Villiers, 2012, p. 44). Stylianides (2007) summarizes the characteristics of proofs in the classroom community as follows:

Proofs use statements accepted by the classroom community that are true and available without further justifications; proofs employ forms of reasoning that are valid and known to, or within the conceptual reach of the classroom community; proofs are communicated with forms of expression that are appropriate and known to, or within the conceptual reach of the classroom community (p. 107).

This study follows the definition from Stylianides (2007), that is, proofs consist of three major components: “set of accepted statements, modes of arguments and modes of argument representations” (p. 107). These three components are equal in significance (Tabach, Barkai, Dreyfus, Tasmir, & Tirosh 2009). The definition of proofs discussed here is considered from both the mathematical and pedagogical perspectives. It focuses on both of content knowledge (set of accepted statements) and structural knowledge (modes of argumentation and presentation), which students need to know about in order to construct proofs. It also emphasizes the importance of explicit demonstration in teaching these knowledge in order to help students understand the ideas of proof and proving.

**Roles of Proofs in Mathematics**

Teachers need to understand the roles of proofs playing in mathematics in order to motivate students to develop an interest in learning proofs. Proofs have various functions in the practice of mathematics, including “validation, explaining, systematization, communication,
discovery and intellectual challenge” (de Villiers, 1999, p. 5). This section describes the first four functions of proofs.

Validation refers to “constructing reasons to accept a specific statement within an accepted framework, shaped by accepted rules and other previously accepted statement” (Balacheff, 2010, p. 117). Validation is to establish the truth or falsity of a mathematical claim. It enables the consistence in building up mathematics knowledge, and is said to be the most obvious function of proofs (Hanna, 2000), as proofs enable mathematicians to look for absolute certainty for the truth of a statement. To mathematicians, however, “validation is secondary among the roles that proofs play in mathematics. Mathematicians are interested in more than whether a conjecture is correct or not” (Harsh, 1993, p. 390).

The role to explain is considered to be the primary role of proofs. Mathematicians want to know why a mathematical statement holds true because “if a proof only shows the truth or falsity of statement, then this proof provides evidential reason solely” (Hanna et al. 1990, p. 9). In the eyes of mathematicians, the best proof is the one which helps them understand the meaning of the theorems or statements being proved (Ko, 2010). An explained proof shows not only the correctness of a statement, but also brings insight into the related field (Hanna, 2000), as “mathematical proofs offer powerful ways of developing and expressing insights about a wide range of phenomena” (National Council of Teachers of Mathematics [NCTM], 2000, p. 56).

Systematization is defined as the “organization of results into a deductive system of axioms, major concepts and theorems” (Bell, 1976, p. 24). Proofs enable mathematicians to build up new knowledge based on the existed set of axioms, concepts and theorems, and extend the ideas into other domain, for example, the development of number theory from real number to complex number. Proofs also enable mathematicians to find out the interconnections amongst
different mathematical theorems, bridge knowledge in different fields and organize the results into a system of knowledge such as group theory.

Proofs are also communicative in nature. Proofs serve as tools to “transmit mathematical knowledge” (Hanna, 2000, p. 8), and are viewed as “communicative acts made within the mathematical community” (Goethe & Friend, 2010, p. 273). Readers of a proof can follow the proof, make sense of the logical reasoning behind, criticize or challenge the authority by checking the proof. As well, readers learn new methodology from the existing proofs which may lead to the discovery of new knowledge (Weber & Mejia-Ramos, 2010). It is undoubted to say that proofs are communicative by nature.

**Proofs in Mathematics Education**

**Rigorous/formality.** Rigour, “which deepened its root in the 19th century, has a critical position in the development of mathematics” (Kleiner, 1991, p. 304). Rigour is referred to as “depth of analysis and the tolerance for uncertainty” (Goethe & Friend, 2010, p. 285) and is considered to be the “hallmark” of mathematics practice (Poly, 1981, p. 126, cited in Edwards, 1998). However, as the body knowledge of mathematics has been expanded widely, to be specific, with the aid of technology, more and more researchers are challenging the rigour of proofs (e.g. Davis, 1993; Horgan, 1993). The “consistency” on strict deduction has “declined” (Kleiner, 1991, p. 293). Different tools such as computers are introduced and used in verifying the truth of mathematical statements. There is an increase in the tolerance of uncertainty in constructing proofs.

Although “mathematicians are well known for their rigorous presentation of reasoning, they can and do fill in gaps, correct errors, and supply more details and more careful scholarship when they are called on or motivated to do so” (Thurston, 1994, p. 170). As to mathematicians,
the “reliability” of mathematics lies in the “careful” and “critical” thinking behind a statement (Goethe & Friend, 2010, p. 276), not the rigour or formalities. Rigorous and formal presentation of a mathematical claim is not the finishing line of the study, as mathematicians constantly review the existed theories in accordance with the continuous development of the mathematics concepts.

From a pedagogical perspective, it is difficult for students to construct a rigorous proof as well as to understand the significance of rigour in mathematics. Rigour is clearly “secondary in importance to understanding and significance” (Hanna, 1995, p. 42) since proofs are not "instruments of justification but tools of discovery, to be employed in the development of concepts and the refinement of conjectures" (Kleiner, 1991, p. 311). Rigour and formal presentations of proofs are only the “final stage in a much more intuitive and iterative process of exploration and discovery” (Tall et al., 2012, p. 44). Therefore, Herbst (2002) suggests that it is “advantageous for teachers to avoid treating proofs only as formal processes” (p. 176). Aligned with this view, Hemmi (2008) states that there should be a “balance between how much to focus on proof ideas and how much to deal with rigorous detailed presentations of proofs” (p. 416).

The development of proofs should “be set to the direction to seek for solutions to specific problems to stimulate the development of different fields and revive old fields in mathematics” (Kleiner, 1991, p. 309), not to the strictness on rigour.

**Evaluating proofs.** Teachers face different challenges when they evaluate proofs. Firstly, there is “no universal agreement regarding what a valid proof constitutes in the mathematical community” (Inglis, Alcock, Mejia-Ramos, & Weber, 2013, p. 279). The lack of standard or guideline makes the situation more complicated in the classroom. As representatives of the mathematical community in the classroom, proof constructions are evaluated according to
teachers' decision (Tabach et al., 2010). Shaped by their understandings, teachers have their own visions on the evaluations of proofs. This results in an inconsistence in evaluation. Secondly, teachers' content knowledge has an impact on the way how they evaluate the proof constructions. In the study on one secondary school teacher's reaction to students' suggested proofs and justifications in number theory, Tsamir et al. (2009) noted that, when evaluating proofs, the teacher went beyond the correctness of a justification. From the teacher's perspective, knowledge about proofs includes “not only the knowledge of how to validate or refute a statement, but also knowledge about what is sufficient for proving or refuting a given type of statement” (Tsamir et al., 2009, p. 65). In other words, the teacher believes that only minimal proofs reflect students’ understanding in constructing proofs. As a result, when students added some extra information to explain their proofs, the teacher marked the students wrong since the teacher thought that students had lacked understanding in the question as well as the related concepts. Undoubtedly, it can be seen from here that teachers’ understandings on proofs have an impact on the ways how they assess their students.

Finally, studies have also shown that teachers have difficulties in evaluating proofs with different modes of presentation. In the study of teachers' perceptions on verbal justification, Tabach et al. (2010) reported that teachers were more confident to evaluate proofs with symbolic notations. Symbolic modes of presentation were highly valued by teachers. Teachers were less familiar with other modes of presentation, such as verbal justification. Approximately 50% of the teachers rejected correct verbal justifications. Many teachers preferred not to use verbal justifications. Teachers claimed that verbal justifications lacked generality and considered verbal justifications as examples solely. Tabach et al. (2010) concluded that, in order to identify students' deficiencies on constructing proofs, and evaluate students' proofs correctly, teachers
should be aware of students' preferences on the modes of presentation to get students engaged in
the proof activities and encourage students to use different modes of presentation in the
classroom.

**Strategic methods.**

*Strategic methods for proof constructions.* For the same mathematical problem, there
might be various ways for one to solve it. So is a proof. This section describes the strategic
methods commonly used in proving. It also presents studies conducted on pre-service teachers’
methodological knowledge in proving as well as students’ difficulties and deficiencies in
learning these strategic methods.

In general, a statement can be proved both directly and indirectly. Three major methods
are used in proving directly, namely, proof by deduction, proof by construction, and proof by
induction. Proof by deduction is to draw conclusion by unwrapping definitions and theorems (e.g.
two-column Euclidean geometry proof). A constructive proof, which is also called a
demonstrative proofs, is a proof that “directly provides a specific example, or which gives an
algorithm for producing an example” (Math World, 2014) to show that the object exists or the
statement is correct. Proof by induction is a method used when tries to “verify a proposition
whose truth is determined by an integer function and is considered as a rigorous form of
deductive proof” (Ernest 1984, p. 181). Mathematical induction “proceeds in two steps, that is,
the base step, that establishes the equality of the initial value, and the inductive step which
proves the implication from \( P(k) \) to \( P(k + 1) \) to reach the conclusion that the statement is true
for any value in the domain” (Stylianides, Stylianides, & Philippou, 2007, p. 145).

Proving in an indirect way, by contraposition or by contradiction, “is a common practice
in the activity of mathematicians” (Antonini & Mariotti, 2008, p. 401). Nonetheless, it is found
that there is a “minor emphasis” in teaching this technique in the curriculum (Thompson, 1996, p 475). If a statement S can “be expressed as an implication p → q in symbolic notation, a proof by contraposition of S is a direct proof of ¬ q → ¬ p, where p is the hypothesis and q is the conclusion” (Antonini & Mariotti, 2008, p. 402). For example, if we want to prove that the statement “if x^2 is even, then x is also even” is true, we can first prove that its contrapositive statement (i.e. if x is not even, then x^2 is not even) holds. We then conclude that the original statement also holds because of the fact that both statements share the same logical value. A proof by contradiction “establishes the truth of a given proposition by the supposition that it is false and the subsequent drawing of a conclusion that is contradictory to something that is proven to be true” (Math World, 2014).

**Mathematical induction.** A solid understanding of strategic methods of proof is essential for teachers to deliberate the associated content knowledge. National Council of Teachers of Mathematics (NCTM) recommends that “students should learn that certain types of results are proved using the technique of mathematical induction” (NCTM, 2000, p. 345) as students' understanding of mathematical induction may “facilitate their understanding of proof and inductive reasoning” (Palla, Potari, & Spyrou, 2011, p. 1023). It is essential for students to accumulate more knowledge and experience in practicing mathematical induction.

In the study of students' understanding of mathematical induction, Palla and Potari and Spyrou (2011) reported that majority of the students were able to give definition of mathematical induction. Students were aware of the structure of mathematical induction, but at an “operational level” (p. 1041). The study also demonstrated that students have “difficulties in deducing P (k + 1) from P (k) and understanding the general patterns seen from mathematical induction” (Palla, Potari, & Spyrou, 2011, p. 1041). Hence, Palla and Potari and Spyrou (2011) suggested that the
teaching of mathematical induction should be in a “meaningful way to encourage students to focus on the critical properties of mathematical induction” (p. 1042).

Stylianides and Stylianides and Philippou (2007) examined pre-service teachers' ability to identify valid and invalid mathematical induction. The findings suggested that pre-service teachers have difficulties in understanding the importance of the base step of the induction method. Teachers' essential understanding of the base step is limited to the surface structure of mathematical induction. The finding also demonstrated that pre-service teachers are not aware of the meaning of the step in proving the implication $P(k)$ to $P(k + 1)$. Pre-service teachers do not recognize the possibility to use mathematical induction to verify the truth of a mathematical statement outside its domain. Hence, Stylianides and Stylianides and Philippou (2007) suggested that it is important for teacher educator to “develop a sense” of pre-service teachers' understanding of mathematical induction, because it could help teacher educators to “design and implement” the program to equip pre-service teachers better in teaching mathematical induction (pp. 164 – 165).

*Indirect proof.* Existing studies have shown that many high school students have difficulties in understanding indirect proof (e.g. Silver & Carpenter, 1989; Thompson, 1992). Sources of difficulties are proposed as follows: 1) Students do not have a clear understanding of what an indirect proof constitutes and how an indirect proof is structured; 2) Students have difficulties in writing the negation of a statement and understanding the logical reasoning behind; 3) Students lack understanding of the terminology commonly used in indirect proof such as contradiction and contrapositive (Thomson, 1996). In order to address these problems, Thomson (1996) recommends that, before learning indirect proof, students should first learn to classify valid and invalid proofs. Teachers are supposed to give students more opportunities to explain
their reasoning for both valid and invalid proofs. In addition, Thompson (1996) suggests that the learning of indirect proof should be exposed to different content areas, because students may have difficulties in transferring this proving technique when they move to further advanced classes. For instance, teachers can ask students to produce their own indirect proof for statements from different content areas and have students reflect on what they have learned to help students identify their difficulties in constructing indirect proof and make improvements.

**Teaching and Learning of Mathematical Proofs**

**Instructional strategies.**

*Intrinsic motivation.* Proof is not “a thing separable from mathematics, as it appears to be in our curriculum; it is an essential component of doing, communicating, and recording mathematics” (Schoenfeld, 1994, p. 76). It is suggested that “mathematical reasoning and proof should be a consistent part of students' mathematical experiences in pre-kindergarten through grade 12” (NCTM, 2000, p. 56). Despite the recognized importance of proofs, many students lack appreciation in the roles that proofs play in mathematics education. Students hold the opinion that proofs are less helpful in learning, and are isolated from other subject areas. This results in a lack of interest in studying proofs amongst students. Therefore, Balacheff (1999) emphasizes the importance of creating a learning environment for students to experience proofs, recognize and appreciate the value of proofs in mathematics.

*“Asking why”.* It is important for teachers to help students develop a habit to ask why. By creating a climate to cultivate students' habits to ask questions, teachers can guide students to practice thinking, follow students' questions, listen to their inquiries, and provide supports. It helps students strengthen their logical reasoning skills. Through this process, students are able to explore and gain insight into the properties of proof constructions, and develop an interest in
studying proofs. Students could also develop their critical thinking skills, which is essential in helping them understand how the mechanism of proving works. Finally, the student-teacher interactions, which are more likely to contribute to students' development in a positive way, also enable teachers to build high-quality relationships with their students (O’ Donnell, 2013).

**Collaboration.** Researches have shown that cooperative learning is an effective instructional strategy to improve students' achievements both academically and socially (Johnson & Johnson, 2002; Gillies & Boyle, 2010). Under cooperative learning, cooperation is structured by “creating positive interdependence among individuals' goal attainments; individuals perceive that they can reach their goals if and only if the other individuals in the situation also reach their goals” (Johnson & Johnson, 2002, p. 98). It is commonly seen from today's classroom that students are scared to make mistakes in the mathematics class, especially when they move to higher grades. They are more reluctant to ask and answer questions. By integrating cooperative learning, participants empower themselves in the learning community and learn that different opinions are valued and respected. Teachers provide instructions in a small group and differentiate instructions to meet students' needs. Teachers also provide students with more opportunities to share their ideas and explanations with their classmates. This enhances students reasoning skills because students are expected to share their logical reasoning in the discussions. Besides, cooperative learning sets up an environment for students to demonstrate their understandings and learn how to communicate in mathematics language. It enables students to justify their reasoning verbally, as students may have difficulties in writing out their reasoning. It also helps students to clarify the misconceptions students have in the language associated with proofs, because students can support each other’s learning in their groups by providing feedbacks.

**Realistic activities.** Realistic integration is an effective way to get student involved in the
class. Studies have shown that students with more instructions integrated with realistic activities ("application of knowledge in realistic setting", Yang & Wu, 2010, p. 379) are more capable to use different strategies to solve problems than those with traditional instructions (e.g. Yang & Wu, 2010). The realistic activities improve students' performance in different grades. Teachers encourage students to question their peers and share their ideas in order to integrate realistic-activities into mathematics. This is important in teaching proofs because it promotes discussions in the class. It increases students’ participations. It also increases students’ awareness that proofs are applicable to real life situations.

Balacheff (1991) suggests that “if students do not engage in proving process, it is not so much because they are not able to do so, but rather they do not see any reasons or feel anything for it” (as cited in Jahnke & Wambach, 2013, p. 471). This is why Yang and Wu (2010) argue that, when teaching mathematics, it is important for teachers to use the realistic activities to trigger students' interests and inspire students to learn mathematics. The realistic activities enable students to understand the abstract concepts in proving more easily and help increase students' understanding in the reasoning behind the conjectures, as students can test the conjectures with examples and be familiar with the conjectures.

**Explicit instruction on strategic knowledge.** There has been “a trend in mathematics education to employ the authentic ways in which mathematicians work out already in school mathematics” (Hemmi, 2010, p. 274). Students only meet the constructed proofs, and are rarely taught how proofs are structured. In the study on students' conceptions of the methodological knowledge, Heinz and Reiss (2007) reported that majority of the students do not possess clear understanding of the proof methodological knowledge and have difficulties in bridging empirical examples and formal proofs. Heinz and Reiss (2007) stated that students need to learn various
methodological knowledge on constructing proofs, as methodological knowledge is a “component of proof competence” (p. 1). Aligned with this view, Weber (2001) stresses the importance for the instructions of both procedural and strategic knowledge of proofs, such as “knowledge of how to choose which facts and theorems to apply” (p. 101). Different aspects of proof methodological knowledge should be taught in the classroom, proof structures, specifically.

Alibert and Thomas (1991) also state that high school students benefit more if they are taught the proof structures. Every step of a proof needs to be justified using facts, definitions or theorems. More complex structures of proof are formed by integrating smaller pieces of arguments. Instead of letting students discover the structures themselves, students should be taught to understand the structures of proofs. Hemmi (2010) suggests that it can be difficult for students to understand the meaning of proof or learn how to produce own proofs in mathematical practice without explicit focus on it. Hemmi (2010) further explains that “students should not just meet ready proofs and formulae, but should be able to participate in constructing proofs and understand the structures of proofs” (p. 274), because students’ correct understanding of the proof structures is a sufficient condition for students to construct proofs.

In conclusion, the proof methodological knowledge is a sufficient condition for performing proof constructing. A clear understanding in the proof structures helps students clarify their logical reasoning. It is important for teachers to teach students the proof structures explicitly and let students accumulate more knowledge and experience in constructing proofs.

**Experimental mathematics.** The use of different tools, such as dynamics geometry software (DGS), “introduce an experimental dimension into mathematics” (Arzarello, Buss, Leung, Marriott, & Stevenson, 2012, p. 97). In the past decades, the importance of experimentation in mathematics is recognized increasingly. In the 19th International
Commission on Mathematical Instruction (ICMI) Study, Borwein (2012) outlined the characteristics of Experimental Mathematics as follows:

1. Gaining insight and intuition.
2. Discovering new relationships.
3. Visualising math principles.
4. Testing and especially falsifying conjectures.
5. Exploring a proving result to see if it merits (i.e. Context depend) formal proof
6. Computing replacing lengthy hand activities
7. Confirming analytically derived results (pp. 73 - 74)

Experimentation has “revived the old fields in mathematics and led to the discovery and invention of new areas in mathematics” (Polya, 1954, as cited in de Villiers, 2010, p. 206).

Experimentation has also broadened students’ visions of mathematics and helped students developed their intuitive reasoning skill, as researches have shown that “mathematical intuition mostly developed from the regular handling, exploration and manipulation of mathematical objects and ideas” (Davis &Hersh, 1983; Epstein & Levy, 1995, as cited in de Villiers, 2010, p. 207). Aligned with this view, Borwein (2012) suggests that under experimental mathematics, students are enabled to explore and find patterns behind the conjectures, which helps them deepen their understanding and extend their ideas on the conjectures. Students can also use more empirical examples to justify a conjecture and accumulate more experience on practicing it.

Despite the importance of the use of experiments in the teaching and learning of proofs, many students are taught in the traditional way. As de Villiers (2010) describes:

Mathematics education at the school or university level often fails to provide students with a sense of how new results can or could be discovered or invented. Quite often, after
a teacher has carefully presented theorems and their proofs, students are just given exercises with riders of the type “Prove that”. This caricature of mathematics can easily create false impression that mathematics is only a systematic, deductive science (p. 205). Nevertheless, de Villiers (2010) points out that there are some limitations with the use of experiments in teaching proofs. For instance, students might have difficulties distinguishing the differences between “experimentation” and “deduction” (de Villiers, 2010, p. 216), and recognizing that empirical examples only are not sufficient in proving a mathematical conjecture. Lakato (1983) also emphasizes the importance of the “proper balance of conjecturing and adequate experimental exploration” (as cited in de Villiers, p. 219). Teachers need to take more responsibilities in implementing and monitoring the process. It is upon teachers to decide how far to go with the use of technology or other tools to explore, and handle the issues raised with formality in proving.

**Teachers' belief of proofs.** Students are confronting two difficulties when they construct proofs. Firstly, students have a lack of understanding in the nature of proofs. For instance, students are not aware of what a proof constitutes or what a valid proof is. Secondly, students do not have a clear understanding on the mathematical concepts such as theorems (Weber, 2001). Undoubtedly, teachers play a vital role in deliberating and clarifying the content knowledge in proof and proving. Teachers’ beliefs of proofs are likely to contribute to students’ perceptions of proofs. This section presents studies conducted on both pre-service and in-service teachers' beliefs of proofs.

Bike and Iskenderoglu (2011) conducted a quantitative analysis on pre-service elementary mathematics teachers' views on presenting proofs. The study comprised of 187 pre-service elementary mathematics teachers from different years of studies (73 from year one, 35
from year two, 34 from year three and 45 from year four). Bike and Iskenderoglu (2011) found that a majority of the pre-teachers have a positive attitude toward proofs. Pre-service teachers frequently assessed themselves and reviewed what they had done when proving. It is noticeable to find that fourth-grade teachers have higher confidence in presenting proofs. Bike and Iskenderoglu (2011) suggested that fourth-grade teachers are more confident because they have taken more courses at the undergraduate level and have accumulated more knowledge and experience in constructing proofs.

Mingus and Grassl (1999) investigated pre-service teachers’ conceptions of proofs. For the study Mingus and Grassl (1999) interviewed 30 elementary school teachers and 21 high-school teachers. From this study, it was found that all teachers admitted that they had a lack of learning experiences in proving to some extent. About 24% of the teachers claimed that they had no learning experience in proving prior entering to college. Approximately 69% said that they had some learning experiences in constructing proofs at high school; yet, their learning experiences were limited to the two-column Euclid proofs (e.g. trigonometric identity). As a result, many teachers illustrated that the introduction of proofs should begin as early as possible. Some teachers suggested that formal proofs should be introduced before grade 10 so that students could apply the techniques that they had learned to more advanced courses. In addition, teachers suggested that introductions of proofs should not be constrained within geometry. It should go beyond to algebra and trigonometry as well. In the discussion of what constitutes a proof, teachers gave a wide range of definitions. It was also found that teachers’ definitions of proofs were constrained by geometry. Some teachers illustrated that proofs are used to demonstrate and confirm geometric relationship. In terms of the roles of proofs playing in the
teaching and learning of mathematics, most teachers appreciated and valued the role of proofs in explaining why the concepts work and promoting students’ understanding.

Knuth (2002) examined in-service mathematics teachers' conceptions of proofs in United States. Based on the study, Knuth (2002) suggested that teachers recognize different roles that proofs play in mathematics; teachers consider verification as a primary role in mathematics; and teachers also perceive proofs as “means of communicating, and tools to discover and systematize mathematics knowledge” (p. 487). Yet, Knuth (2002) found that teachers do not recognize the role of proofs playing in promoting students' understanding. Besides, Knuth (2002) suggested that “many of the teachers hold limited views of the nature of proofs in mathematics, and demonstrate inadequate understanding of what constitutes a proof” (p. 379). Teachers said that proofs would be more convincing if they tested them with more empirical examples, and had difficulties in recognizing nonproofs. It is also noted that teachers do not have a full understanding in the generality of a proof. Several teachers suggested that a proved statement can be refuted by contradictory examples. Some teachers expressed their doubts in the generality of proofs.

Similarly, Bao and Zhou (2009) conducted a survey on mathematical proofs among teachers. The study examined three aspects of teachers' conceptions: teachers' understanding of what constitute proofs, teachers' ability to construct proofs and teachers' pedagogical knowledge in teaching proofs. From the study, it is seen that majority of the teachers were able to provide a clear definition of proofs. Many teachers were able to distinguish proofs from intuitive reasoning and had a strong foundation in mathematical proofs. Most teachers were able to identify mistakes in the given proofs and recognize generality of proofs. In the discussion of difficulties that students encounter when constructing proofs, teachers gave various answers. For example, some
teachers suggested that students do not understand the structure of proofs. Students are not able to find the assumption of the problem or they have difficulties in transforming words into mathematical language. The results indicated that teachers were relatively familiar with the difficulties and obstacles that students encounter in studying proofs. Meanwhile, it is also found that teachers lacked the pedagogical knowledge to address difficulties that students may have on learning proofs and were not able to differentiate instructions according to students' levels of studies.

In conclusion, existing researches have shown that both pre-service and in-service teachers hold diversified views on the definition of proofs. Pre-service and in-service teachers hold a limited understanding on the roles of proofs playing in mathematics education and their understanding of proofs is limited to geometry. Reflected on their past learning experiences, in-service teachers also suggest the introduction of proofs should begin before grade 10 in order to help students accumulate more learning experience in constructing proofs (Knuth, 2002). In particular, teachers point out the teaching of proofs should be extend to other areas in mathematics.
Chapter 3 Methodology

This chapter discusses the research methods which were used for this study, including nature of the study and selection of participants. It outlines the process of data collection and data analysis. It also addresses the ethical issues of this study. In addition, it describes the possible limitations of this study.

Nature of the study

The purpose of the study is to describe pre-service teachers' perspectives and experiences on the teaching and learning of mathematical proofs in order to examine pre-service teachers' conceptions of proofs. Research reviewed earlier suggests that in-service teachers hold a diverse understanding in the nature of mathematical proofs such as the generality of proofs (Bike & Iskenderoglu, 2011; Knuth, 2002; Mingus & Grassl, 1999). The goal of this study is to extend the study to pre-service teachers' conceptions of proofs.

This study is qualitative in nature. The study gathered information from various resources which included semi-structural interviews, documents and field notes. For this study, an instrumental case study approach was selected because it enables the “inquirer to focus on the issues or concern raised within a bounded system” (Creswell, 2013, p. 100). It is also a good approach for the researcher who has an “identifiable case with boundaries to seek for and provide in-depth understanding of the case” (Creswell, 2013, p. 100). As in this case, the study was conducted at an educational department at a large university in Southern Ontario with the goal to examine pre-service teachers' conceptions of proofs from their perspective and past experiences.

Participants

Two pre-service teachers, with different educational backgrounds, participated in the
study. Invitations were sent via email to the target population. Selection of participants was based on their willingness to participate in the study. Both participants agreed to meet individually with the researcher for a semi-structured interview.

**Data Resources**

When conducting the study, the procedures of an instrumental case study were followed. In accordance with the recommendations, I collected data from different resources. Literature review was the first resource. I used the conceptual framework to guide my study. In order to obtain detailed information and get insight to the participants' views, I conducted semi-structured interviews with the participants. I also used electronic communication to collect information and clarify the ideas when it was necessary. Additionally, I collected information using the field notes and my reflection journals. These resources enabled me to validate my conclusion as well as to reduce bias of the information which I collected.

**Interview Procedure**

Each pre-service teacher was invited to participate in an interview. Before the interview, they were asked to provide their background information which includes their educational backgrounds and any related teaching experiences. The interviews were video- and audio-taped. Throughout the interview, each participant was encouraged to express their thoughts freely. I learned more about the teachers' conceptions of proofs including their views on the nature of proofs as well as their understanding of the teaching of proof methodology. The length of the interviews varied from 45 minutes to one and half hour, which depended on how much, and how deeply the issues were talked by the participants. During the interview, the two main research questions were addressed:

What constitutes a mathematical proof?
What are the roles mathematical proofs play in the mathematics education?

Besides, the participants were asked to describe the mathematics curriculum, the activities that they would like to use when teaching proofs, their roles in helping students develop logical sense, what their expectations were, including their point of views regarding high-quality instruction in teaching proofs and what they would like to change to better serve their students in the future. In addition, I learned a lot more about how they would assess students' knowledge in constructing a proof and plan their lessons accordingly. All this information was transcribed for data analysis.

In addition, participants were invited for an online discussion to clarify some of their ideas. The discussions were audio-recorded. I facilitated the discussion and took additional notes to learn more about their perspectives on the teaching of mathematical proofs.

**Data Analysis**

Data was transcribed and reviewed carefully after interviews. Initially, I examined each response individually and highlighted the key words or sentences. I also wrote down a word or two to summarize each response. The words were then used to develop a list of codes. I re-read each response and assigned a unique code to each response. For the longer responses, I broke it into shorter one and assigned more codes to find the patterns. I used the codes assigned to compare and explore common responses if there was any. The responses were grouped under different themes emerged and analyzed in terms of the themes.

**Ethical Procedure**

The participants were informed that the data collected would only be used for this study. I explained to them that their life would not be affected. The videos and audios would be deleted once I had finished my data analysis. All participants were asked to sign a consent form (See
Appendix B), which describes the purpose of this study and the procedure of the data collection, when they were fully aware of the purpose and procedure of this study. Also, they were informed that the study would be limited to an education department in one university. Hence, information would be under used to protect the privacy of the participants in the study.

Limitations

In this study, limitations are identified as factors which have led to the insufficiency in the data collection. There are three possible limitations for this study. Firstly, only two pre-service teachers were selected to participate in this study. It was a small-scale research and might not reflect the characteristics of the target population as a whole. Secondly, this study was a voluntary basis. Selection of participants was not based on their subject matter content knowledge. Instead, participants were selected randomly from an education department at a large university in Southern Ontario. The study focused on the teaching of proofs under the Ontario curriculum. It might not reflect pre-service teachers' views on proofs across the country. At last, I could only arrange one interview with each teacher due to the time constraint and the ethical considerations. I did not collect enough information to get an insight into pre-service teachers' conceptions of proofs.
Chapter 4 Findings

Introduction

The purpose of this paper is to examine pre-service teachers' conceptions on mathematical proofs by describing their perspectives and experiences on the teaching and learning of proofs. Data was mainly collected from the semi-structured interviews conducted with the pre-service teachers, Jason and Jeff. This chapter outlines the main findings of the data analysis and presents the results in terms of themes emerged during the interviews. These include the followings: definition of mathematical proofs, pre-service teachers’ views on the roles of proofs, the importance of proofs in mathematics education, challenges confronting in the teaching of proofs, and the effective instructional strategies used in teaching proofs.

Defining Mathematical Proofs

The definition of mathematical proofs discussed here is considered from both the learning and teaching perspective. According to Jason, mathematical proofs are equivalent to identities. He said, “Left side equals to right side. That's my definition of a mathematical proof.” Jeff claimed that mathematical proofs are tools to show how the theory works.

Although the participants responded differently in defining mathematical proofs, they demonstrated their understanding of some of the characteristics of proofs. For instance, both of them recognized that mathematical proofs consist of sets of arguments and can be used to justify the mathematical statements. The participants also recognized that there is no generally accepted definition of what a mathematical proof is in the classroom context. Mathematical proofs have different meanings for different people. This makes the situations more complicated when teachers introduce proofs to students, since teachers cannot consistently provide students with the same definition (Cabassut et al., 2012).
In addition to this, the participants confirmed that there lacks an organized and systematic way in teaching mathematical proofs. The participants admitted that they never had any formal instruction on proving before, even though they had been learning proofs for many years. They did not know the origin of mathematical proofs; neither did they know how proof constructions have developed in history. Their knowledge of proofs accumulated from different parts of the learning from high school to university as the teaching of proofs took place randomly.

One thing I found interesting was that, when I asked them to define mathematical proofs, Jason gave me the answer immediately and his answer was consistent throughout the interview. He made it clear that proofs are identities. Jeff was more hesitated in answering this question and his definition was more general and ambiguous. He suggested that mathematical proofs use evidences to show how the theory works.

Different factors might have influenced their responses. But perhaps their educational backgrounds and past learning experiences are also influencing. Jason studied biology and psychology. It can be seen from Jason's response that his understanding of proofs is limited to Euclid geometry. He considered proofs as identities solely because most of the proof constructions he met were consistently derivations of formulas in this area. Jeff has his educational backgrounds in mathematics and biology. He had more opportunities to encounter various aspects of proofs in different fields of mathematics and get more knowledge in proving. The definition varies under different circumstances. This finding is important to my study because it suggests that pre-service teachers' past learning experiences have an impact on their conceptions towards proofs.

**Pre-service Teachers' Conceptions on the Roles of Proofs**

According to Jason, proofs are communicative in nature. He suggested that when students
look through a proof, they need to think about what is given and how they can use the given information to develop their ideas. Students also need to follow the logical reasoning behind the proof and learn how each step is justified in a cohesive order. Besides, proofs allow students to share their ideas and demonstrate their level of understanding of the newly taught theorems.

Both participants agreed that the most obvious function of proofs in mathematics is to verify the truth or falsity of mathematical statements. Jeff mentioned that a proof can be used to show why the theory is true. Jason suggested that proofs are tools used to “perceive and justify things” or “see if something is true or not”. However, the participants could not explain why mathematicians would use proofs to verify mathematical statements. They perceived it in the way that whenever they are asked to show the truth of a mathematical statement, they have to justify it by proving as taught.

The participants also suggested that the most important role of proofs is to explain why the mathematical statement holds and help students understand the concepts. Not only did they recognize this role, but they also consistently emphasized its importance during the interviews. Jeff pointed out directly that proofs explain why the theory works. Jason confirmed that proofs are the foundation of the mathematics theories and could be used to demonstrate how the theorems could be used. He went on describing it with the hammer example. Here is his response:

Not necessarily we create; the proof just tells me how I got it...
It explains where the tools came from. Think of a hammer, for example, where did it come from? Well, we need a tool to hit something hard. But hits something with your hand hurts. So you got a physical object to do that instead. They built the object. It's kind of explaining where we got it, why we got it, how we use it, how was it useful.

Unfortunately, even though the participants were able to recognize the role which proofs play in explaining why the mathematical statements hold, they did not see how they could optimize its value in improving students' understanding. Jason explained that, when constructing proofs, students are only required to manipulate the given equations and have both sides being equal.
The teaching and learning of mathematical proofs are still on the surface of deriving formulas.

Jason shared his recent experience and reported:

    They [students] have to show a derivation of the cosine law and sine law.  
I show them how they can derive sine law....Tell them that, this is a proof  
of the formula. And the tools they used were the basic equation I taught you  
before. The trigonometric ratio, Pythagorean Theorem. From those equation and formulas.

Apparently, these “evidential” identity-typed of proof constructions would not help students  
understand the theorems or definitions being taught. These proofs show series of steps students  
need to follow in order to justify and complete the proofs. There lacks opportunities for students  
to practice their critical thinking and logical reasoning skill during this proving process. In my  
opinion, when demonstrating proofs, it is important for teachers to know what they want students  
to gain from the lesson. It would be meaningless to show students the proofs which do not help  
students understand the mathematical concepts.

    In conclusion, both participants recognize and value the important roles mathematical  
proofs play in serving as a tool to communicate mathematical knowledge, verifying the truth or  
falsity of a mathematical claim and improving students' understanding. It is worth mentioning  
that even though the participants are aware of that proofs help explain why a statement is true,  
they do not see how they could use proofs to improve students' understanding. Often in time,  
students are only shown the factual proofs.

**Importance**

    When asked if it is necessary to introduce proving techniques in high school, both  
participants said yes. Jeff described that knowledge in proving is a “must” in the learning of  
mathematics. The participants also suggested that proofs consist of sets of problem solving  
methodologies and help students “gain insight into the theorems”. 
**Competency.** Students' ability to construct proofs is considered as a competency in the learning of mathematics (Weber, 2010). Jeff suggested that knowledge in proving is a “must” for students who are studying math-related disciplines. He described that proofs are the foundation of mathematics. Proofs are the paths that lead students to explore different fields in mathematics and understand how mathematicians build up the theories. Similarly, Jason claimed that knowledge in proving enables students to equip themselves for further advanced studies as students would encounter different types of proof eventually, formal or informal, deductive or inductive. Students cannot avoid learning proofs. Jason also suggested that proving is a flexible type of thinking that requires students to think outside of the box. It helps students expand their ideas and see things from a different perspective. His response is provided below:

I always thought that the reason why we do proofs or learn proofs is to do that type of thinking...Thinking outside of the box...Somehow, we are able to see things from a different perspective.

**Problem solving strategies.** Both participants agreed that students develop a list of problem solving strategies from the existed proofs. They first pointed out that proving is a problem solving process. When students are proving, they need to think about what is given or which route they could follow to reach the final conclusion. This is similar to what we do in problem solving, which indicates that the strategies or techniques used in proving are applicable to other problem solving process. Besides, proving is a logical reasoning process. On the one hand, when students construct proofs, they need to follow their own logical reasoning carefully in order to justify each step. Students are practicing their logical reasoning skill. On the other hand, students learn how others think differently and expand their ideas by reading the constructed proofs. It helps students “think outside the box” as Jason suggested. Finally, a constructed proof itself demonstrates to its readers how they could use a theorem to solve the
problems, since the theorem makes it clearly what would happen if a question satisfies all the given conditions of the theorem. Jeff said:

> When you are solving the problem, you think about the proof...sometimes, it actually follows the proof." In other words, when students see the given conditions match with a theorem, they could probably follow or use the theorem to solve the problem. These proofs are the "tools" available for students to solve the problems.

### Challenges Confronting in the Teaching and Learning of Mathematical Proofs

**Problems.** When asked what would be the biggest problem students have in proving, Jeff responded that students have a lack of an interest. He claimed that students seem not to be engaged in mathematics class because students do not understand why they should study proofs. Jason expressed that the biggest challenge students confront is that students lack both skills and experience in constructing proofs. Students do not possess the methodological knowledge required in proving and are unable to piece up different parts.

In addition, the participants confirmed that students are not aware of what it means by proving. Students have a lack of confidence in constructing proofs. This is consistent with what I have found from the discussion with a group of grade 11 students I was tutoring. When I asked the students why it was difficult for them to learn and construct proofs, they all stated that they are not capable of doing “higher order of thinking” or they do not have the “flexibility to think outside of the box”. One of them also commented that there is no way for a student to construct proofs, even with simple proving questions like identity, if the student does not know what it means by proving. Hence, Jason stressed that it is essential for teachers to spend more time on deconstructing the proofs so that students could follow the ideas, and learn how proofs are structured step by step.

**Introduction of formal and informal proofs.** When asked if students should learn how to construct formal proofs, the participants responded differently. According to Jeff, students
should learn to construct formal proofs starting from grade 10. When students are in grade 11 or 12, they should be able to distinguish between formal and informal proofs as well as to use different methods to prove the mathematical statements. Jason suggested that teachers should introduce the idea of proving to students starting from grade nine. However, no formal proof should be involved until grade 12 because students do not possess sufficient knowledge to distinguish between formal and informal proofs until then. He reported:

> They don't have all the tools to prove something...It's something that can't learn at the beginning ... at the beginning, you have to learn all the tools first... You don't call it a proof, but you can start teaching that at grade 9... Start to give them the idea and strategies to create a proof themselves.

Further, Jason recommended that teachers demonstrate proofs of the theorems in different domains and encourage students to practice more proof constructions, as Jason believed that students need to accumulate more learning experience in constructing proofs. By doing this, it helps students transit from informal proof constructing to formal proof constructing. He commented:

> I still think they should be shown it [proofs of the formulas or theorems]. But maybe in a less complex version of it....Show them the reason why we have that formula. And then, you could frame them like, Oh, this is a proof of the formula. And [remind them that] the tools they used was the basic equation we taught you before.

Apparently, the participants held different points of views on the time to introduce formal proofs. Nonetheless, it is interesting to find that neither of them sees the necessity for students to learn and distinguish between informal and formal proofs. The participants suggested that understanding the central idea behind the proof is the most important part in proving. Teachers need to encourage students to use different ways to construct proofs.

In conclusion, formality is the problem teachers need to deal with when teaching proofs. It is important for teachers to be aware of how much to focus on the formality of proofs while leaving enough space for students to explore. However, different teachers have different views
on when to introduce students to formal proof constructions. This leads to the inconsistency in teaching proofs.

**The teaching and learning of proving techniques.** There are a lot of ways to prove the mathematical statements. This section discusses the participants' past learning experiences on the proving techniques. When asked if it is necessary to introduce proving techniques, both participants said yes. Jason suggested that teachers demonstrate students how they could use different methods to construct proofs. Jeff commented that proving techniques were useful for his learning in university.

In recognizing the importance of learning these proving techniques, the participants admitted that there lacks a systematic way in teaching them. Both participants confirmed that students are not provided enough opportunities to practice these proving techniques. Jason revealed that his teachers would, occasionally, demonstrate proofs of the theorems in class. However, the teachers did not explain how the proofs were constructed; neither did the teachers explicitly talk about the methods used in these proof constructions.

Jeff claimed that there is no smooth transition from high school to university in teaching proofs. He shared:

> [In high school], for those methods [proving techniques], we only did one example for each... In university, the professors didn't really teach us. It was more...I went to the TA... Those proofs the professors wrote on the board, it doesn't make sense to me, because I have never seen it before. So I have to ask the TA...So I learned most of the proving methods by myself.

It is worthwhile to note that this is not an isolated case. Jeff's experiences also reflect the difficulties many students are experiencing nowadays. When I was taking the first year proving course, more than 70% of the students failed their term-test in my class. The majority of the students in my class did not know how proofs are structured in general. They did not know the basic proving techniques that they could use to construct proofs, either. Their struggles indicate
the necessity for teachers to teach these proving techniques explicitly and provide students with more opportunities to learn and practice proof constructions in both high school and university.

**Assessment.** Both participants confirmed that it is difficult to assess students' understanding in proving. Jeff recognized that there is no standard or guideline which teachers can follow to evaluate mathematical proofs. Jason suggested that there is a lot of “grey area” in terms of evaluation because there are many ways to construct proofs. Nonetheless, both participants stressed the importance for teachers to check the proof constructions step by step. Jeff said, “I believe, in our system today, we need to show procedures in order to be correct”. Jason also commented that teachers should “call for the key points and evaluate accordingly”. Besides, they emphasized that it is important for teachers to encourage students to try and prove the mathematical statements using different methods.

In recognizing the importance to follow the steps to evaluate a proof, the participants also discussed that teachers should not be too strict in checking each justified step. Jeff explained that the goal of assessment is to help students be familiar with the proof constructions, identify their defects in proving and deepen their understanding in the proving methodologies, but not to limit students' thinking by marks. Jason suggested that teachers should follow students' thinking carefully as well as to focus on the central ideas of the proof constructions when evaluating proofs. Jeff further added that everyone thinks differently. There are different routes one can follow to go from the initial assumption to the final conclusion. As a result, it is critical for teachers to encourage and provide students with more opportunities to explain their ideas in order to reduce the occurrences of misjudgments.

One interesting point raised through the discussion was that Jason also mentioned teachers should encourage students to find a more effective way to construct proofs. In other
words, Jason believed that proofs should be minimal. He reported:

You can always tell that their reasoning works, but it might be a longer path. So you can encourage them to find a more efficient way. But you have to follow their reasoning carefully.

Finally, the participants emphasized the importance for being flexible when evaluating proofs. Some students are logical thinkers, who might prefer to present their ideas with symbolic notations, whereas some other students are visual learners, who might prefer to demonstrate proofs using diagrams or other visual tools. Therefore, Jason recommended that teachers be aware of students’ preferred modes of presentation in order to modify their expectation to meet students’ needs.

**Generality.** When asked if a proved-statement holds for all the cases, Jason responded that it is true for most cases if not all. He further explained that the truth of a mathematical statement relies on the conditions given. Jason also suggested that teachers use different examples to test and see if a proved-statement is correct before demonstrating the proof. He believed that by doing this it would help students better understand the statement being proved. Students will be more convinced if they can test the statement with more examples.

However, Jason did not recognize the problems raised with the use of examples to test a mathematical statement. For instance, students might be confused with what it means for a mathematical statement to be true. Students might take it for granted that few examples will be sufficient to prove a mathematical statement. It is crucial for teachers to be aware of the fact that the generality of a mathematical statement does not depend on the number of examples being used. It is supported by the "careful" and "critical" reasoning behind (Goethe & Friend, 2010, p. 276).

**Positive or negative impact.** Both participants confirmed that their past learning experiences have an impact on the way how they would teach in the future, positive or negative.
Jason discussed that since he struggled a lot when he was in high school, he is capable to recognize the struggles students facing in learning proofs and be aware of what their needs would be. As a result, he is more willing to adjust his instructional strategies in order to meet students' needs. He shared his experience as follows,

Back in grade 11, 12/ I had a math teacher. He was really good. But I was really lost in class. He would show proofs, he thinks ahead so fast. I couldn't follow along. And because of that, it made me understand when I am tutoring as well, that, sometimes, I skip steps, too. But I realized that students get lost if I skip steps. So I go back and slower.

In the opposite, Jeff commented that even though he recognized that the way he was taught is not as effective in teaching mathematical proofs, it is very likely that he would follow the same way to teach in his future class, because it works for him and he is more comfortable to teach in this way. He responded:

In high school, in university, we all learned...oh...proofs are this, this, this,...you follow that process...right? Because I am so used to it, I feel like I am going to do it. I am pretty sure that it's going to just come up...I will be using the same way that I was taught in high school or university...it's like instinct...you know....It just happens...I will try not to...but I am pretty sure it will.

Undoubtedly, the past learning experiences would have an impact on the way how pre-service teachers would teach in their future class. Nevertheless, it depends on how they perceive it and make the choice that fits oneself.

Instructional Strategies

When asked what would be the effective instructional strategies that could be used in teaching mathematical proofs, Jason and Jeff recommended the followings: demonstration, exploration, questioning, discussion, and motivation.

Demonstration. Jason and Jeff agreed that demonstration is a time-consuming but effective instructional strategy in teaching mathematical proofs. Jason commented that it is not necessary to demonstrate the completed proofs. Teachers should focus more on the main idea of
a proof. He shared his recent experience in demonstrating proofs during his first practicum and revealed:

One thing I did during the practicum was Pythagorean Theorem. You have the three sides a, side b and side c. You are assuming, using the general rule, c side is the hypotenuse. I would actually draw out the right triangle, draw out the squares and measures them and there’s a certain way you can cut out one of the squares, and the idea of the Pythagorean is that, when you add up the areas of the other two sides, to prove to them, not only mathematical with numbers, I showed them with the actual cut of the squares. I actually rearranged it and placed it into the bigger squares, and they saw that they perfectly fit.

Jeff discussed that the direct demonstration of proofs enables students to visualize the procedures they could follow in proving prior to developing a list of strategies to construct proofs. He responded:

There so many different ways to prove something. Students cannot use all the methods. So teachers need to give them the outline and show them the process. Tell them what they need to do for step 1, step 2, etc.

Jeff also pointed out that it is very likely that students would forget the majority of the proof constructions demonstrated on the blackboard. Teachers should avoid demonstrating the meaningless long proofs which are useless for students, and be aware of what they want students to learn at the end of the lesson.

Repetition/practice. Demonstration is definitely important in the teaching of mathematical proofs. Yet, the participants confirmed that it is important for teachers to explain the concept repeatedly to ensure that students understand it. Jeff framed that “the instruction in high school and university is just repetition”. He suggested that students should be constantly exposed to proof constructions and be familiarized with the idea of proving, and accumulate more learning experience in constructing proofs. Indeed, students need more opportunities to practice proof constructions themselves to consolidate their understanding. As often in time, students find it easy to follow a constructed-proof. However, when they are asked to reproduce
the same proof, they do not know where and how to start. Jason also pointed out that it takes time for students to understand the proofs. Teachers should use different examples to help students understand the theorem before demonstrating the proof. He reported:

If you want them to recognize where they can use it, I guess, you need to do a little pattern recognition with them...Not only to show them how they can use it, you can go through example with them as they are going through this... You can't do this with only example. You have to do it with multiple examples.

A complete understanding of a proof consists of two layers of meaning. Firstly, students are able to justify the steps. Secondly, students can apply and use the proof to solve other problems. Obviously, the latter is more important. Practicing helps students deepen their understanding in constructing proofs. It also helps students find their weakness and improve on it. Hence, it is important for students to practice more by themselves.

**Questioning.** Jason believed that questioning is essential in learning mathematical proofs. He explained:

You have to ask questions to prompt them. Do you recognize this? What's missing? What do you need? Prompt them by questions. Kind of guide them. Keep prompting them...When they become more confident, they can try it by themselves. And don't get used to you prompting it, when they all doing the proofs, they should start prompting themselves. If something is missing, they should recognize what they need to add.

According to Jason, questioning functions in two different ways. On the one hand, teachers should ask questions to prompt students and lead them into thinking. This stage happens at the beginning of leaning proof constructions. At this stage, teachers are at the center of teaching. Teachers demonstrate students how students can organize their ideas to produce proofs. On the other hand, teachers should let students prompt themselves in order to cultivate their habits to thinking independently. Students should not be taught with “facts only”. They should also learn to become an independent thinker. After all, proving is a logical reasoning process.
“Knowing why”. The participants agreed that motivation plays an essential role in the teaching and learning of mathematical proofs. Jeff suggested that "knowing why" is the essential motivation that students need to get themselves engaged in the proving process. He reported,

Motivation will be very important for them to learn. Sometimes, you just need to find the motivation... In high school, there are a lot of kids want to know why. Proof will be a good way to show them why.

Jason also emphasized the importance of learning and appreciating the roles that proofs play in explaining and improving students' understanding of the mathematical concepts, as it helps students understand why math works in the way it does. He commented:

Without showing the formula (proofs), we will not understand how we came up with this (the theorems).

Discussion. Jason and Jeff also recommended the use of discussions in teaching proofs. They first pointed out that the use of discussions enables students to find out their weakness and strength in their logical reasoning. Discussions also allow students to demonstrate for teachers what they know and what they want to know. Teachers could assess students' understanding through discussions. Jeff reported:

So far, we have been talking about having group discussions and assessing your own learning. This is very important...We have to assess their thinking process. Through discussion, ask students why they write down a particular step...justify their steps....

Exploration. Both participants stressed the necessity of exploring in learning proofs. Jason suggested that teachers can break the harder question into smaller parts and let students explore to gain insight into the theorems they are working on. Jeff shared his experience with the use of exploration in teaching the properties of triangles. He reported:

I have to prove that the centroid is the balance point of the triangle. So all we did was, everybody make a triangle, we cut it out and we found the centroid by drawing the medians and where they are intersecting. So we have a competition to see who can balance the triangle the best without pencil.
Jeff explained that the exploration made it clearly to students how they could find the centroid of a triangle and allowed students to experience the triangle properties. It helped student understand the properties better. However, as stated by Jeff and Jason, students are rarely provided with opportunities to explore in mathematics, especially in proving, due to the limitations of the textbooks and teaching resources. When students open the mathematics textbook, all they could see are the rigorously justified proofs. There is not enough space left for exploration in proving.

Jeff also pointed out there are some limitations with the use of exploration. For instance, he said, "we can just explore, we can use our experience to show that it is true. Unfortunately, we can't do that in university". Indeed, it is important for teachers to encourage students to explore in learning proofs. Nonetheless, teachers should also be aware of how far to let students explore and deal with the problem of formality when teaching proofs.

There might be several reasons why students do not engage in proving. One reason I can think of, from my experience as a student and as a math tutor, is that students do not really participate in the proving process. Teachers are in the center of teaching. Students are only duplicating the ideas they learn from their teachers. The introduction of exploratory activities encourages students participate in the teaching process and be more engaged in learning. It also allows students to connect mathematics to their experiences. By exploration, students could see that proving is approachable for them.
Chapter 5 Conclusion

Defining Mathematical Proofs

According to Jason, mathematical proofs are equivalent to identities. Jeff claimed that mathematical proofs are important means of communication and show how the mathematics theories work. Apparently, there is a discrepancy in the participants' responses to define mathematical proofs. This echoed the findings from the literature which suggested that teachers hold a wide range of views on the definition of mathematical proofs (Knuth, 2002). Nevertheless, both participants recognized that mathematical proofs consist of sets of arguments and could be used to justify mathematical statements as supported by Stylianides (2007). Additionally, the participants reported that there is no generally accepted definition of what a mathematical proof is in the classroom context. The literature supported this claim, explaining that “although mathematicians can generally agree on the acceptance of an adequate proof, no explicit definition of a proof is shared by the entire mathematical community” (Cabassut et al., 2012, p. 169).

Roles/Functions of Proofs in Mathematics

Both the literature and the participants suggested that pre-service teachers recognize the different roles mathematical proofs play in mathematics. These include: validation, explaining and communication. The participants agreed that the most apparent function of a mathematical proof is to verify the truth or falsity of a mathematical claim. The participants also stated that the most important role of a mathematical proof is to explain why the theorems work. In addition, Jason suggested that proofs serve as tools for students to communicate and demonstrate their understanding. This notion was supported in the literature by Knuth (2002) which suggested that teachers “hold a view of proofs as means of communicating mathematics and convincing others
of one's claims” (p. 387).

Finally, Jeff confirmed that proofs are the foundations of the mathematics theories. Proofs demonstrate students how to use the theory accordingly, thus, strengthening students' understanding of the relevant concepts. This related to the finding of Mingus and Grassl (1999), which suggested that most teachers appreciate and value the role of proofs playing in explaining and promoting students' understanding. However, the participants did not explain how they could optimize its value in improving students' understanding. As seen from the interviews, although the participants were able to recognize that it is meaningless to demonstrate the proofs which would not help students understand the theorems, they confessed that the teaching and learning of mathematical proofs are still on the manipulation level. That is manipulating of the formulas until both sides are equal. Students are only taught with “evidential” proofs (Hanna et al. 1990, p. 9).

Importance

Both the literature and the participants emphasized the importance of constant exposure to mathematical proofs. The participants pointed out that knowledge in proving is essential in learning mathematics. Jeff stated that proofs are the foundations of mathematics, and knowledge in proving is a “must” for students who are studying math-related disciplines. Aligned with this view, Jason explained that knowledge in proving is transferable to other subjects and enables students to equip themselves for further advanced studies. The literature also supported that “mathematical reasoning and proof should be a consistent part of students' mathematical experiences in pre-kindergarten through grade 12” (NCTM, 2000, p. 56). In addition, the participants suggested that students develop a list of problem solving methods from the constructed proofs, as proving is a logical reasoning process and a problem solving process, the
strategies students study from proof constructions are applicable to other problem solving. The literature supported this claim, explaining that the ability to construct proofs is an essential problem solving skill in mathematics.

**Challenges Confronting in the Teaching and Learning of Mathematical Proofs**

As suggested by the findings of Weber (2001), the deficiency students have in constructing proofs is that, students do not understand what a mathematical proof constitutes. Students do not have a clear understanding on the related mathematical concepts, such as definitions and theorems, either. In our conversation relating to the difficulties students confronting in learning mathematical proofs, Jason echoed the literature saying that students do not possess the methodological knowledge in constructing proofs. Jeff suggested that students do not see the necessity to learn proofs and lack an interest in learning proofs. This connected closely to the findings of Jahnke and Wambach (2013) which discussed that “if students do not engage in the proving process, it is not so much because they are not able to do so, but rather they do not see any reason or feel anything for it” (p 471).

The generality of a proved statement is also discussed. The literature established evidence that teachers do not have a full understanding in the generality of mathematical proofs. Teachers expressed that proofs would be more convincing if they could test them with more empirical examples. Similarly, in our conversation, Jason suggested that teachers use examples to show that the mathematical claim holds before demonstrating the completed proof. However, Jason did not recognize that students would be confused if teachers constantly verify the mathematical claims using examples. Students could develop a wrong impression that empirical examples are sufficient to validate a mathematical claim.

With respect to assessment, there are also many overlaps with the literature. Jason
suggested that there is a lot of "grey area" in terms of evaluations. Jeff also claimed that there is no standard which teachers can follow to evaluate a mathematical proof. Tabach et al. (2010) supported this claim by saying that as representative of the mathematical community in the classroom, proof constructions are evaluated according to teachers' decisions. This leads to the inconsistency in evaluation. In addition, both the literature and participants agreed on that teachers should be more flexible in evaluating proofs with different modes of presentations such as verbal justification. Teachers should encourage students to use different modes of presentations when constructing proofs.

Another theme from the findings that is also supported by the literature involves the introduction of formal and informal proofs. According to Jeff, students should learn how to construct formal proofs starting from grade 10 and they should be able to prove a mathematical claim both directly and indirectly by grade 12. This related to the literature by Knuth (2002), which suggested that formal proofs should be introduced before grade 10 so that students can apply the techniques they have learned to the advanced mathematics courses. An area of discussion that emerged from the interview but not from the literature is that the participants do not see the necessity for students to learn and distinguish between formal and informal proofs, as they suggested that understanding is the most important portion in proving.

Nonetheless, both the literature and the participants agreed on that students should be provided with more opportunities to practice proof constructions. In particular, students should be taught various ways to construct proofs. According to literature, it was suggested that “proof activities should provide opportunities for students to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs and select varies types of reasoning and
methods of proof” (NCTM, 2002, p. 56). Echoed with the literature, the participants argued that teachers should plan and implement activities to demonstrate how to use different methods to construct proofs, help smooth the transition from the intuitive reasoning stage to the logical reasoning stage, and prepare students for the higher levels of study.

**Instructional Strategies**

**Demonstration.** Both participants agreed that demonstration is an effective instructional strategy in teaching mathematical proofs. They also emphasized the importance of practicing proof constructions. Jason suggested that teachers demonstrate and deconstruct proofs of the theorems students encounter from the textbooks. The literature also stated that students would benefit more if they are taught how to construct proofs step by step. In this way, students are able to accumulate more learning experience in constructing proofs. It also helps students to clarify their misunderstanding in proving.

**Questioning.** Both the literature and participants emphasized the importance of questioning in learning mathematical proofs. Jason suggested that questioning is essential in learning proofs. Jason further explained that there are two stages in this learning process. At stage one, teachers are modeling and guiding students in thinking so that students can see how proof and proving work. At stage two, students learn how to prompt themselves and develop a habit of thinking independently in order to prove or disprove a mathematical claim. The literatures also supported this notion and suggested that by creating a climate to cultivate students’ habits of asking questions, students strengthen their logical reasoning skill and learn to justify their reasoning rigorously.

**“Knowing why”**. Jahnke and Wambach (2013) stated that “if students do not engage in the proving process, it is not so much because they are not able to do so, but rather they do not
see any reason or feel anything for it” (p. 417). Jeff supported this notion by saying that that “knowing why” is the essential motivation students need to get themselves engaged in the proving process. Jason confirmed the significance for students to learn and appreciate the role proofs playing in explaining why. This perspective connected closely to the findings of Balacheff (1999) which emphasized the importance to create a learning environment for students to experience, recognize and appreciate the value of proofs in mathematics.

**Discussion.** The participants highly recommended the use of discussion in teaching mathematical proofs. They pointed out that discussion provides a platform for students to share their ideas and justify their reasoning verbally. It also allows teachers to assess students' learning timely. Existed research has found that a lot of students have difficulties recognizing symbolic notations commonly used in constructing proofs. Students get frustrated easily if they do not know how to use the symbols correctly and lose their interest in learning proofs. Discussion enables students to justify their steps verbally. It also opens up more space for students to express their thoughts freely, since students are not constrained inside the rigid of formatting.

This strategy is not directly addressed in the literature, but it is similar to the notion of integrating cooperation in learning as discussed in the literature. The literature suggested that cooperative learning sets up an environment for students to demonstrate their understanding, empower themselves, and support each other in their learning community. As a result, students improve their achievements both academically and socially.

**Exploration.** The participants agreed that exploration is an integral part of the teaching and learning of mathematical proofs. They pointed out that with the use of exploration, it is easier for students to understand the concepts because students experiment the concepts themselves. The effectiveness of exploration was also supported in the literature which suggested
that the use of experiments enables students to explore and find the pattern behind the conjectures, deepen their understanding and extend their ideas on the conjectures. Nonetheless, both the literature and the participants pointed out that teachers should be aware of how far to go with the use of experiments and how they could help students transit from the intuitive reasoning to logical reasoning so that students would not mix up with these two.

**Implication**

The study examined pre-service teachers' conceptions of mathematical proofs by describing their perspectives and experiences towards the teaching and learning of proofs. Based on the findings, it is clear that pre-service teachers hold different views on proofs and their visions are shaped by their past learning experiences. The participants recognized the importance of proofs in mathematics education and emphasized the importance for teachers to take more responsibility in implementing the lessons and activities to meet students' needs.

The participants also shared the difficulties they experienced in learning and constructing proofs. It is worthwhile to note that the participants’ experiences reflect the difficulties many students are experiencing in today’s class. For instance, students do not know what proofs constitute, how proofs are structured, and how to choose the associated theorems to justify their reasoning. This indicates the necessity for teachers to teach proof constructions explicitly and provide students with more opportunities to learn and construct proofs in both high school and university.

The findings also suggest that pre-service teachers do not possess a clear understanding of what a mathematical proof is, due to the inconsistency in the instructions of proofs. There lacks an organized and systematic way in teaching and learning mathematical proofs. This indicates the necessity for teacher educators to implement the mathematics teachable course to
help pre-service teachers better equip themselves in teaching mathematical proofs.

**Future Study**

As stated before, the purpose of this study was to examine pre-service teachers’ conceptions on mathematical proofs by describing their perspectives and experiences towards the teaching and learning of proofs. Pre-service teachers' subject content knowledge was not examined. It might not reflect their true understanding on mathematical proofs. A further study combining with their perspectives and content knowledge would be beneficial. In addition, only two pre-service teachers participated in this study. A larger scale research is needed to gain a better insight into pre-service teachers' conceptions of proofs.
Appendix A: Interview Questions

1. What is your educational background?
2. How do you define the concept of teaching about mathematical proofs?
3. In your opinion, what are the roles mathematical proofs play in teaching and learning?
4. How do you conceptualize the teaching about mathematical proofs in the mathematics curriculum?
5. From your experience as a student, do you feel that students should learn how to construct mathematical proofs? Why or why not?
6. How were you taught to construct mathematical proofs in high school and/or university?
7. How do you think students should learn about mathematical proofs in a classroom?
8. How do your experiences in learning about mathematical proofs impact the way you would like to teach it?
9. What are the barriers students have in learning mathematical proofs?
10. How would you like to address those problems?
11. What instructional strategies do you find effective for teaching about mathematical proofs?
12. Why or why not, is it necessary to introduce proving techniques?
13. What role does technology play in today’s classroom in teaching about mathematical proofs?
14. In your opinion, how can we, as a teacher, assess and evaluate students’ mathematical proofs?
15. Is there anything you want to add/expand on this topic?
Appendix B:

Date: ___________________

Dear ___________________,

I am a graduate student at OISE, University of Toronto, and am currently enrolled as a Master of Teaching candidate. I am studying on pre-service teachers’ conceptions of proof for the purposes of investigating an educational topic as a major assignment for our program. I think that your knowledge and experience will provide insights into this topic.

I am writing a report on this study as a requirement of the Master of Teaching Program. My course instructor who is providing support for the process this year is Arlo Kempf. My research supervisor is Mary Reid. The purpose of this requirement is to allow us to become familiar with a variety of ways to do research. My data collection consists of a 60 minutes interview that will be tape-recorded and a 45 minutes group discussion that will be audio-recorded. I would be grateful if you would allow me to interview you at a place and time convenient to you. I can conduct the interview at your office or workplace, in a public place, or anywhere else that you might prefer.

The contents of this interview will be used for my assignment, which will include a final paper, as well as informal presentations to my classmates and/or potentially at a conference or publication. I will not use your name or anything else that might identify you in my written work, oral presentations, or publications. This information remains confidential. The only people who will have access to my assignment work will be my research supervisor and my course instructor. You are free to change your mind at any time, and to withdraw even after you have consented to participate. You may decline to answer any specific questions. I will destroy the tape recording after the paper has been presented and/or published which may take up to five years after the data has been collected. There are no known risks or benefits to you for assisting in the project, and I will share with you a copy of my notes to ensure accuracy.

Please sign the attached form, if you agree to be interviewed. The second copy is for your records. Thank you very much for your help.

Yours sincerely,

Researcher name: Hui Jun, Deng

Phone number, email: 647-278-1268, huijun.deng@mail.utoronto.ca

Instructor’s Name: Arlo Kempf
Phone number: Email: arlo.kempf@utoronto.ca

Research Supervisor’s Name: Mary Reid
Phone #: 416-978-0047, Email: mary.reid@utoronto.ca

Consent Form

I acknowledge that the topic of this interview has been explained to me and that any questions that I have asked have been answered to my satisfaction. I understand that I can withdraw at any time without penalty.

I have read the letter provided to me by Hui Jun, Deng and agree to participate in an interview for the purposes described.

Signature: ________________________________

Name (printed): ____________________________

Date: ________________
References


Teaching the way we aspire to teach, now and in the future teachers' vision for teaching and learning in Canada's public schools (2012). Toronto, Ont.: Ottawa, Ont.: Canadian Education Association; Canadian Teachers' Federation, 2012.


