An Introduction to the Justification Principle and its Associated Benefits and Challenges within the Mathematics Classroom

By

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Abstract

There is a sizeable portion of secondary mathematics students who resort to memorizing poorly understood procedures in order to score high on their assessments. In the contemporary mathematics education system, there is too much focus on obtaining the correct answer, and simply not enough focus on the underlying mathematical processes involved. As such, individuals who end up studying post-secondary level mathematics courses end up struggling as they discover their previously developed knowledge was superficial, context specific, and heavily reliant on precedence. Thus, my study’s underlying motivation was to determine a way in which students will not only develop deep conceptual and procedural knowledge, but also be deterred to attempt relying on superficial knowledge. My study has turned to justification as a potential solution and examines the following question: how can justification be implemented into one’s math pedagogy and what are its associated benefits and challenges? The participants of this qualitative case study were two exemplary secondary mathematics teachers. My findings suggest that the implementation of justification into one’s math pedagogy provides several benefits to both the instructor and the learner including: the creation of an environment conducive for the growth and development of deep knowledge, heightening the competence of formal math communication skills, and the creation of a framework of authentic assessment for and as learning practices. The main challenge associated with implementing justification into the math classroom is teachers’ lack of content knowledge. Possible changes to eliminate this challenges include the introduction of a math competency test for initial teacher education programs.

Keywords: mathematics, pedagogy, justification, deep knowledge, superficial knowledge, conceptual knowledge, procedural knowledge, authentic, assessment, competency test
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Chapter 1: INTRODUCTION

Introduction to the Research Study

As a mathematician, it is important to understand that mathematics truly is nothing more than the expansion of logic. All mathematical truths stemmed from, and can be inarguably justified with the application of logical principles. That is, every statement that claims truth in mathematics needs to be justified using explicit inarguable evidence. It is one of the very few subjects that places greater emphasis on the ability to justify one’s claims rather than whether or not they are true. Simply put, through the eyes of a mathematician justification and mathematics are inseparable as the science of mathematics cannot exist without justification. Therefore, one would expect that such an important practice would be integrated and modelled within contemporary math education practices. Yet, when one attempts to find research surrounding the need for justification in math education, they are left with almost no relevant studies or articles. This study was created with the intention of bridging that gap of the literature, and to highlight the importance of justification as a pedagogical practice to inform instruction, to perform authentic assessments, and to be used as an instrument of learning.

Purpose of the Study

Rooted in the belief that true mathematical knowledge is contingent on both deep conceptual and procedural knowledge, this study searches for ways to obstruct the development of superficial knowledge. The purpose of this study will be to determine how justification can be integrated into one’s mathematics pedagogical practice, along with its associated benefits and challenges.

This study is of particular importance as for nearly the past thirty years, math education research has predominantly been geared towards differentiating and unravelling the relationships between conceptual and procedural knowledge. Since this time, much has been discovered about
the importance of both knowledge for mathematical successes, as well as the existence of
contceptualized mechanisms by which these knowledge iteratively develop. Yet, there is very
little research outside of the realm of inquiry-based learning that highlights well-defined teaching
mechanisms educators can use to ensure that not only conducive deep learning occurs, but the
emergence and reliance on superficial knowledge is hindered. In mathematics, there is no
concept more valuable than that of justification, so it seems very odd that there is little-to-no
research on its benefits as a pedagogical tool. This study will hopefully shed some light on its
potential benefits through what this study refers to as the justification principle.

Research Topic/Questions

This study is primarily interested in whether or not the justification principle is an
effective pedagogical practice that can be used to not only promote the development of deep
conceptual and procedural knowledge, but also infringe on the potential success of students who
rely on superficial knowledge.

Background of the Researcher

Ever since I entered the school system, I have much enjoyed mathematics, far more than
that of any other subject. I became aware of my love for mathematics as early as the first grade,
and ever since then my educational life has been centralized around the studying of mathematics.
Throughout elementary school, middle school, and high school I was a student who excelled in
all of my mathematics courses, achieving exceptional grades, all the while enjoying my learning
experience. By the time it came around to graduate high school, the decision on what I would
pursue in post-secondary was blatantly clear to me; I would continue to study mathematics.
Having effortlessly succeeded in my high school mathematics courses, and achieving
distinguished awards for my academic successes, I felt as though I was thoroughly prepared to
pursue the learning of high level mathematics in university.
When I arrived in university, I was taken by surprise as I began to notice some very uncomforting properties surrounding my background math knowledge. First of all, I realized that I had no flexibility of approach; that is, when given a question outside of the normal context I was used to seeing it in, I had a lot of trouble finding potential avenues to lead me towards a solution. Secondly, I was suddenly immersed in an environment where justification was held on the highest pedestal, that is, whether or not a statement was true often times did not matter, what mattered was the justification. This was very difficult for me to adjust to, as I never had to justify myself or prove any concepts throughout my whole high school career; I was only ever asked to simply verify, solve, simplify, or graph. The concept of determining truth via definition, disproving via counterexample, and even understanding the notion of propositional/predicate logic statements (i.e. “if and only if” and “if, then” statements) were completely foreign to me. Third, I realized that not only did I have almost no formal reasoning skills, but I also lacked in my formal notation and communication skills. Forth, upon reflecting on the concepts that I “learned” in high school, I discovered that I really did not understand any of what it was, or why I had to perform all of the procedures I had ingrained into my mind. It quite literally felt as though all of the mathematics I had learned in high school was for naught.

I found it extremely troubling that, despite my distinguished academic achievements in high school mathematics, I found university mathematics so difficult to adjust to. One would imagine that, if a student took university preparatory courses, and they scored among the top of their class, that they would be pretty well prepared for university mathematics. What troubled me further was that I was not alone. A majority of my peers who also graduated high school within Ontario struggled in the same manner in which I did.
This was very disheartening, but at the same time motivating. By the time I hit third year of my bachelor degree, I decided to pursue a math specialist degree with a concentration in teaching, and that I would dedicate my educational and professional life towards teaching mathematics and the reform of math education. Upon graduating from U of T St. George with an Honors Bachelor of Science in Math and its Applications, I applied to the Masters of Teaching program at OISE so that I could make a change to the way math was taught, and this research study is a first step towards that cause.

The reason I decided to research the concept of “justification” and its associated benefits is because of the very reason I previously stated. How is it that my peers and I were able to score exceptional grades in high school mathematics, yet when our knowledge was tested by mere introductory post-secondary math courses, it proved to be anything but distinguished? Are assessments and evaluations not to be an accurate tool to measure one’s knowledge or competence in a certain domain? If so, then my high scores should have implied that I possessed an exceptional understanding of the course material. This evidence has lead me to believe that the assessments I was used to seeing up until my post-secondary career were inauthentic; they did not truly test whether or not I truly understood the mathematics. As a result, I managed to get through my secondary studies relying almost entirely on superficial knowledge. In my attempt to discover a solution to the problem of students succeeding with the sole usage of superficial knowledge, I have turned to justification as a possible answer as it is the one major disconnect between my lived experience within both the secondary and post-secondary systems. This has been a main contributing factor as to why I have chosen to pursue math education research and this research study is only but the first step of my intended action towards math education reform.
Overview

Chapter 1 includes the introduction and purpose of the study, the research questions, as well as how I came to be involved in this topic and study. Chapter 2 contains a review of the literature, looking particularly at deep and superficial conceptual and procedural knowledge, justification as a learning and assessment tool, and authentic formative assessments in mathematics education. Chapter 3 provides the methodology and procedure used in this study including information about the sample participants and data collection instruments. Chapter 4 identifies the participants in the study and describes the data as it addresses the research question. Chapter 5 includes limitations of the study, conclusions, recommendations for practice, and further reading and study. References and a list of appendices follow at the end.
Chapter 2: LITERATURE REVIEW

Types of Knowledge

It is of the common belief among math education researchers (MERs) that there are two types of knowledge that contribute to one’s overall mathematical ability: conceptual and procedural. This has been a heavily researched area in mathematical education since the eighties, when many MERs began to conduct rigorous studies to determine what students knew about problem solving. The research yielded some profound results which claimed that most students relied on poorly understood “rules” to blindly solve problems, and that many of the students either did not understand or had major misconceptions regarding key concepts. In other words, the research concluded that a majority of students often memorized poorly understood procedures and rules, and were therefore unable to apply them to problems outside of the specific context in which they were introduced (Schmittau, 2004). These results triggered a math education reform in the late eighties as MERs and educational professionals claimed that students were lacking conceptual knowledge, and that their acquired procedural knowledge was superficial (Schmittau, 2004).

Since that time, the study of conceptual and procedural knowledge has been at the forefront of math education research as educators have debated and investigated which knowledge is superior, how to develop these knowledge bases, and how these knowledge bases are interrelated. The current understanding of these knowledge bases will help educators inform their practices on how to properly instruct and assess students to develop their math abilities in an optimal manner. The following section will rigorously define how both conceptual and procedural knowledge will be interpreted throughout this study with reference to the most recent research by leading MERs.
Conceptual Knowledge

Conceptual knowledge is defined as explicit or implicit understanding of the mathematical principles within a given domain, and the ability to draw various meaningful relationships and connections amongst the elements of that framework. Conceptual knowledge is not held to specific types of problems and is flexible in its applications, making it generalizable (Rittle-Johnson, Siegler, & Wagner, 2001). Many educators view conceptual knowledge as the ability to understand why certain mathematical concepts are true. Although this is true, it is important to further distinguish conceptual knowledge in that one’s ability to decipher concepts or justify truths is contingent on a knowledge that is saturated with rich relationships (Baroody, Feil, & Johnson, 2007). From those relationships, students can apply generalizations within a broad scope, apply mathematical logic in a consistent manner, and they are able to apply various principles to comprehend problems. MERs refer to an individual’s knowledge with the previously mentioned qualities as deep conceptual knowledge (Baroody et al., 2007).

Much like how procedural knowledge is overlooked and oversimplified (as will be detailed in the next section), so is that of conceptual knowledge. MERs have recently been interested in studying what is referred to as superficial conceptual knowledge (Baroody et al., 2007; Rittle-Johnson et al., 2001). Superficial conceptual knowledge is defined as being founded on what MERs refer to as weak-schemas. That is, generalizing within a local scope, inconsistent or low standards of logic, heavy reliance on precedence for comprehension, and a lacking of ability to justify prior reasoning (Baroody et al., 2007). For the purpose of this study, when referring to conceptual knowledge, it should be interpreted as deep conceptual knowledge unless otherwise stated.
Procedural Knowledge

MERs usually define procedural knowledge in terms of sequential prescriptions on how to complete tasks, usually referred to as algorithms; it is equated to the knowledge of “how” or simply, “know how to do it” knowledge (Baroody et al., 2007; McCormick, 1997; Rittle-Johnson et al., 2001). Many educators and MERs have been guilty of oversimplifying exactly what procedural knowledge is by blatantly viewing it as knowledge ingrained via rote teaching (Baroody et al., 2007). The notion that procedural knowledge is simply the ability to memorize potentially meaningless sequences via extended hours of repetition is commonly referred to as superficial procedural knowledge. That is, procedural knowledge can either be superficial or deep. Superficial procedural knowledge “permit[s] a recitation of the steps in order”; it is disembodied, unconnected and relies heavily on context-specific mechanical procedures which are often poorly understood. Whereas deep procedural knowledge “empower[s] a cogent explanation of how the steps are interrelated to achieve a goal”; it is contextualized, meaningful, and strategic in nature (Baroody et al., 2007, p. 117).

For the purpose of this study, when referring to procedural knowledge, it should be interpreted as deep procedural knowledge unless otherwise stated. As previously mentioned, many educators immediately dismiss procedural knowledge and its importance by equating it to superficial procedural knowledge. It is important to remember that procedural knowledge is developed with the intention of solving a problem, and that each step of an algorithm follows from a justifiable mathematical principle. When developed properly, procedural knowledge becomes an indispensable tool in the hands of a mathematician as the ability to strategically apply, formulate, and integrate algorithms within unbounded contexts encompasses exceptional problem solving attributes.
Foregrounding Balanced Instruction

There is a consensus amongst MERs that mathematical success depends on both a well-developed conceptual and procedural knowledge (Baroody et al., 2007; Rittle-Johnson et al., 2001, Schmittau, 2004; Star, 2007). Inherently, the core of mathematics is the ability to solve problems, whether one is trying to develop a theorem, prove a conjecture, or determine a solution for a physical or hypothetical problem. Naturally, problem solving can be seen as the underlying motivating factor for the pursuance of mathematical knowledge. It has been extensively researched, argued, and generally accepted that the ability to solve a problem is dependent on one’s problem representation (Rittle-Johnson et al., 2001). Problem representation is simply one’s ability to understand and decipher exactly what it is that a given problem is asking you to solve, and how one goes about understanding and/or visualizing the problem. This seems to be a reasonable conclusion, as any mathematician would agree that one could not solve a problem without understanding what it is that the problem is actually asking. Furthermore, Rittle-Johnson et al. (2001) have convincingly demonstrated to the math education research community that conceptual and procedural knowledge develop in an iterative fashion and improve problem representation. In other words, it has been demonstrated that conceptual (procedural) knowledge predicted gains in procedural (conceptual) knowledge, and that gains in both procedural and conceptual knowledge contribute to one’s problem representation.

Conceptual Knowledge $\rightarrow$ Problem Representation & Procedural Knowledge

It seems natural that, as conceptual knowledge develops, so does one’s problem representation. That is to say, that as individuals are able to draw rich relationships between definitions, principles, and logic, they are able to comprehend mathematical problems and develop potential solution avenues by investigating well understood properties and theorems.
Furthermore, it seems quite reasonable to fundamentally assert that conceptual knowledge predicts gains in procedural knowledge. By the definition of deep procedural knowledge, it seems unclear just how well understood a procedure can be without understanding the conceptual foundation for each of its steps (Baroody et al., 2007; Schmittau, 2004). In fact, for that very reason most MERs, with the exception of Star (2007), would argue that deep procedural knowledge cannot exist without deep conceptual knowledge (Baroody et al., 2007; Rittle-Johnson et al., 2001; Schmittau, 2004). The following is a justification of this statement, as argued by Baroody et al. (2007) in response to Star’s claims.

A logical implication of [Star’s] perspective is that flexible use of procedures develop independently of conceptual knowledge, which would entail demonstrating that appreciable strategy choice or transfer knowledge can be fostered without, for example, understanding the rationale for the steps of procedures. A logical implication of our alternative proposal is that at least a degree of conceptual knowledge is a necessary condition for (relatively) deep procedural knowledge. (p. 126)

In addition, Rittle-Johnson et al.’s (2007) experiments demonstrated that “conceptual knowledge may influence gains in procedural knowledge by improving problem representation, increasing selection of correct procedures, and facilitating adaptation of known procedures to the demands of novel problems (p. 359).” Studies have shown that individuals with uninformed procedural knowledge are more likely to develop error-prone rational and lack procedural flexibility than those who have a more conceptually informed procedural knowledge. Not only that, but there is wide agreement amongst MERs that one of the most crucial skills needed for solving non-routine problems is flexibility of approach (Kilpatrick, Swafford, & Findell, 2001).

Having deep conceptual knowledge is what provides this flexibility as it implies that one is able to understand and make meaningful interactions between all relevant principles and concepts. The high saturation of well understood relationships provides several potential avenues of thought, providing more “tools” to choose from, thereby allowing for the most appropriate and
efficient solution approaches. Whether the individual deduces their own procedure, or whether they are able to appropriately choose from various others, nevertheless, by having this flexibility of approach, a learner’s procedural fluency is bound to improve; that is, one’s procedural flexibility deeply relies on one’s conceptual knowledge. Lastly, conceptual knowledge makes the learning of procedures easier, reduces the amount of errors, and makes forgetting procedures less likely (Baroody et al., 2007).

**Procedural Knowledge → Problem Representation & Conceptual Knowledge**

Some educators believe that the relationship between procedural and conceptual knowledge is unilateral. In other words, that procedural knowledge cannot be used to inform or help the development of conceptual knowledge. Despite these beliefs, it has been demonstrated that the relationship between conceptual and procedural knowledge is bilateral, and that a consequence of improving procedural knowledge is improved conceptual knowledge (Rittle-Johnson et al., 2001). Rittle-Johnson et al. (2001) findings suggest a series of direct developmental consequences as a result of improved procedural knowledge. First, the usage of procedures often times help represent the problem in way that can then be solved; thereby developing their problem representation. A well-documented experiment which highlights this feature is one where the participants procedurally placed decimal numbers on a number line in order to draw comparisons. The implementation of such a procedure allowed the participants to represent the problem in a familiar way, whereby they could reach a solution (Rittle-Johnson et al., 2001).

Second, after repeated experiences of representing and solving problems with any given procedure, students are able to extract new concepts and understandings from them. For instance, using the previously mentioned procedure, students learn to procedurally locate the tenths digits
on the number line before progressing to any other digit thereafter. Forming such problem representations can help deepen one’s understanding of the concept of place value, since the digit’s place in the number is crucial to proper representation. Third, by improving ones procedural knowledge, one can come to highlight and identify their misconceptions. Through the correct usage of procedures, students can observe outcomes which may help unravel their previous misconceptions that founded their prior incorrect procedures. Fourth, by reflecting on why procedures work, students are given the opportunity to conceptually deduce the underlying mathematical logic and principles behind each step (Rittle-Johnson et al., 2001). Lastly, deep conceptual knowledge depends on having the tools to solve problems, and also depends on knowing how to apply those tools to develop and extend mathematical ideas (Baroody, 2007).

Foundation of Math Instruction

It is clear that having both a deep procedural and conceptual knowledge is necessary for success. Furthermore, it has been demonstrated that one cannot separate deep procedural and conceptual knowledge. Not only are they inseparable, but it is also important to note that they develop iteratively. For this reason, educators should not attempt to separate the two when teaching. This notion should be considered when providing instruction to students in order to develop their mathematical knowledge. Educators should be especially aware of the potential existence of superficial procedural knowledge within their classrooms, and provide ample opportunities to ensure deep procedural knowledge is properly fostered. Safeguards should be put in place to ensure that students are unable to successfully complete assessments without demonstrating deep procedural and/or conceptual knowledge. The following section will detail a practice known as the justification principle that educators can utilize to ensure deep knowledge is consistently developed and demonstrated within the classroom.
Justification in Mathematics Teaching

One of the most unique attributes of mathematics is the ability to produce irrefutable statements and arguments. More than any other discipline, mathematics truly is the epitome of rationality. Every statement that claims truth in mathematics needs to be justified using explicit inarguable evidence. It is one of the very few subjects that places greater emphasis on the ability to justify one’s claims than whether or not the claims are true. Consequently, mathematics instills a sensitivity for the importance of justification within students, that is, to expect rigorous, logical explanations backed by evidence before accepting a statement as true. Therefore, a math educator truly has the obligation to train their students to make rational decisions; to realize that although beliefs may have value and be truthful, those beliefs must be based on evidence to truly be rational (D’Amour, 1973). It is important to understand that mathematics truly is nothing more than the expansion of logic. All mathematical truths stemmed from, and can be inarguably justified with the application of logical principles. Students need to know that although it is not always possible to (at least efficiently or effectively) demonstrate how to discover these truths via pure logic, it does not imply that said mathematical truths are not justifiable (D’Amour, 1973). The consistent awareness and reminder of this underlying rationale is the foundation of the justification principle. By enforcing the notion that all concepts are justifiable in mathematics, students are given ample opportunities to develop, engage, and explore both deep conceptual and procedural understandings.

Defining Justification, and the Justification Principle

For the purpose of this study, we will define justification as Staples, Bartlo, & Thanheiser (2012) did in their recent study on justification in middle grade classrooms. Justification will be defined as “an argument that demonstrates (or refutes) the truth of a claim and uses accepted
statements and mathematical forms of reasoning” (p. 448). It is quite clear to mathematicians that as a practice, justification serves many purposes. It is used to validate claims, it provides insight on a particular phenomenon, and helps systematize knowledge (Staples et al., 2012). Aside from just viewing justification as a disciplinary practice, it is also a valuable learning practice. It is a practice by which one’s understanding and proficiency of doing mathematics is enhanced; it is a means by which one comes to learn and do mathematics. Through this practice, learners are constantly forced to expand and construct knowledge (Kidron & Dreyfus, 2010; Staples et al., 2012). As will be explained in the next section, the use of justification as a learning practice promotes several processes of deep knowledge construction.

Lastly, I will define the justification principle as a pedagogical practice whereby educators not only readily enforce their students to get into the habit of always having to justify (or be prepared to justify) their solutions, approaches, and rationale, but also frequently model justification during instruction. Furthermore, the justification principle is defined by weighing the value of justification more so than a correct answer. Whether it be during lessons, homework problems, or assessments, the justification principle is one that applies universally to all areas of the curriculum.

The Justification Principle’s Impact on the Learner

Enforcing the practice of justification within the classroom provides various potential benefits for the development of deep procedural and conceptual knowledge. The first major benefit of consistently upholding the expectation of justification within one’s classroom is that it provides students with authentic opportunities to communicate (either verbally or written) their thought processes and mathematical ideas. Teacher participants in Staples et al.’s (2012) study demonstrated that by engaging students in justification, it greatly helped in developing their
communicational and representational skills. By constantly having to justify their thoughts and procedures, it is only natural that their math communication fluency will develop as a consequence. This was seen as one of the most remarkable improvements found within the participants as “justification created an authentic need to communicate and use mathematical terms in context” (Staples et al., 2012). There are a few immediate corollaries of this constant communication. The first is that I claim that the use of the justification principle greatly develops your students’ deep procedural knowledge. As evidenced by Staples et al. (2012), the implementation of the justification principle greatly increases the classroom’s communication and representation skills. By actively expecting your students to justify their steps and procedures, students will have to critically think and develop an understanding as to why they are applying each step. It also helps in developing procedural fluency and flexibility as students will have to justify their rationale for choosing one procedure over another.

Secondly, I argue that the justification principle predicts gains in deep conceptual knowledge. The increase of communication and representations within the classroom allow students to share various methods and approaches to a problem, providing opportunities to showcase and explore multiple avenues of understanding (Staples et al., 2007). This allows students to increase the saturation of relationships and interactions of concepts and ideas in a specific domain, therefore deepening their conceptual knowledge.

Finally, I argue that the implementation of the justification principle impedes the development of superficial conceptual and procedural knowledge. The justification principle by definition values the justification more than that of the final answer. By engulfing students in such a learning environment where during lessons, in-class activities, homework, and assessments they are required to justify themselves, not only does there leave little room for the
growth of superficial knowledge, but I also predict students will also lack the motivation to pursue such superficial knowledge.

**The Justification Principle’s Impact on Assessment**

The justification principle not only helps in the development of deep knowledge for the students, but also provides many other benefits for the teacher as well. As it will be argued in the following sections, the justification principle provides a very clear picture to the instructor as to whether or not the class truly understands the given topic at hand, and thereby allows the teacher to modify their instruction based off of this evidence. Furthermore, it will be argued that through justification, one can issue the most authentic assessment; that justification is a reliable identifier as to whether or not one has a deep understanding.

**Embedded Formative Assessment**

As previously explained, mathematics is nothing more than an expansion of logic in its purest form. New concepts stem from previously understood concepts, and newer ones from them. It is because of this logical flow of mathematical progression that mathematics teachers scaffold their students’ knowledge. Scaffolding is a pedagogical practice used by teachers whereby new instruction and knowledge is built off of previously taught skills. This allows students to implement their previously learned understandings to master new knowledge (Cole & Wasburn-Moses, 2010). In the eyes of a mathematician, this notion is well understood. That is, we cannot expand a student’s procedural knowledge of completing the square if they have not already learned how to factor a perfect square trinomial, for instance. If a student lacks understanding of a certain mathematical concept, then we cannot expect them to understand the concept that follows. It is because of this logical progression of mathematical scaffolding that
makes formative assessment uniquely important in mathematics teaching. Ontario’s *Growing Success* (2010) document defines formative assessment in the following manner:

> Assessment that takes place during instruction in order to provide direction for improvement for individual students and for adjustment to instructional programs for individual students and for a whole class. The information gathered is used for the specific purpose of helping students improve while they are still gaining knowledge and practicing skills. (p. 147)

Furthermore, the document goes on to state that formative assessment “occurs frequently and in an ongoing manner during instruction, while students are still gaining knowledge and practicing skills, with support, modelling, and guidance from the teacher” (p. 31). Simply put, formative assessment is a means by which teachers assess to what extent their students have grasped a specific concept in order to inform their instruction before moving forward.

Referring back to the definition of the justification principle, it clearly states that one must constantly have their students justify their reasoning and rationale; whether it be during a lesson, a discussion, homework, or at any time where a student is asked to provide an answer. Therefore, it can be said that one who practices the justification principle is also embedding ongoing assessment within their classroom. Although the learners are benefitting through justification as described in the previous section, the teacher is also receiving ongoing evidence and information from the apparent embedded assessment aspect of the justification principle. From every answer and justification received from students, the teacher is immediately informed of their way of thought and can identify their understandings and misconceptions. The justification principle constantly grants the teacher perspective on their students’ thinking processes and as such, has an informed understanding of how well their classroom understands a specific concept, and as such, can modify their instruction accordingly.
Authenticity

Ensuring that teachers use *authentic assessments* during their mathematics instruction is of great importance. An authentic assessment is an assessment which provides the teacher with the following information: whether or not a student understands a certain concept, how deep their understanding is, and where their misconceptions lie. Consequently, an *inauthentic assessment* is an assessment which is not authentic. Inauthentic assessments include practices such as the “thumbs-up, thumbs-down” strategy, whereby the teacher simply asks the class whether or not they understand a concept and the students reply with either a thumbs-up or thumbs-down. This assessment does not provide accurate data about whether or not the students actually understand the material, as they could very easily lie.

Furthermore, it does not highlight how deep their understanding is, nor where their misconceptions lie, so the teacher is unable to effectively modify their instruction to accommodate. I am arguing that the *most authentic* assessment is one that has implemented the justification principle. By definition, the justification principle requires students to constantly justify their solutions, approaches, and rationale, whether it be during a lesson, activity, or assessment. This claim is supported by a recent study conducted by Staples et al. (2012). The following is an excerpt from their study on justification and its relation to assessment practices in middle grade classrooms.

Justification was seen to play a critical role in creating a venue for students to display their understanding so that teachers could monitor the degree to which students had moved towards desired learning goals. This theme encompassed formative assessment, including self-assessment, and summative assessment. Teachers asked students to justify as a window into students’ reasoning about a particular idea. From this they gleaned information, useful for diagnostic purposes, about how the student, or class, was thinking about a problem. It helped teachers “pinpoint” where students were stuck, or where their misconceptions might lie. Teachers also found value in justification as self-assessment, as justification prompted students to reflect on their own work and identify their own mistakes. (p. 454)
The Current Situation

According to Bieda and Lepak (2012), reasoning and sense making in mathematics is a progression that occurs in three stages: empirical, pre-formal, and formal. Empirical reasoning is reasoning that relies on confirming and generating examples in order to support a conjecture. Several research studies have shown that nearly all K-12 students resort to empirical reasoning when asked to justify even novel problems and/or solutions (Bieda & Lepak, 2012). What we want students to achieve is the ability to formally reason. Formal reasoning is the ability to “logically justify mathematical statements by applying general statements, definitions, and axioms” (Bieda & Lepak, 2012, p. 520). There are many explanations for the cause of this, but the most probable case is that elementary generalist teachers simply lack the knowledge, ability, and confidence required for rich investigations and justifications (McDougall, 2000; Ross, Hogaboam-Gray, & McDougall, 2002).

Educators must realize that there is a distinct difference between verification and justification. As Kidron and Dreyfus (2010) claim, verification is not enough to determine the role of a statement within a theory; it provides no explanation as to the relevance of the statement. They continue, “Every teacher of mathematics [should] know that students will not learn by merely grasping the formal truth of a statement. Students must be given some enlightenment as to the sense of the statement” (Kidron & Dreyfus, 2010, pg. 83). I would argue that it is hard for an educator to appreciate this concept unless they themselves have gone through a rigorous math education themselves. This is supported by research that has shown that teachers often teach mathematics in the same way they were taught. Teachers have preconceptions and assumptions on how to teach based on their own experience as students.
(McDougall, 2000), which may explain why teachers turn to verification as an alternative to justification (or potentially equate them) within their classrooms.

Finally, the second stage of reasoning is known as *pre-formal reasoning*. Pre-formal reasoning is defined as providing intuitive explanations and partial arguments which provide insight into why something is true (Bieda & Lepak, 2012). Research indicates that this stage is often overlooked within the educational system; that students are forced to jump from the empirical to the formal stage of reasoning without being given an opportunity to bridge the two ways of thought (Bieda & Lepak, 2012). A study by Jacobs et al. (2006) showed that out of a random sample of fifty videotaped lessons of eighth grade US classrooms, there were no found examples of lessons that involved justifications. In addition, they found no evidence of teachers attempting to develop logical reasoning, make generalizations, or provide counterexamples. In order to properly bridge the gap between empirical and formal reasoning, Bieda & Lepak (2012) argue that middle school students should be thoroughly developing their pre-formal reasoning abilities. I argue that the implementation of the justification principle provides the perfect platform to carry out such a task.

Research has shown that students’ reliance on verification is linked to their desire to produce a correct answer, that is, they are conditioned to believe that producing a correct answer holds more value than that of communicating their knowledge and understanding (Bieda & Lepak, 2012). This sheds some light on how the manifestation of superficial procedural knowledge fosters within the educational system. If students really do value obtaining the correct answer above all else, then that provides a motive for the pursuance of superficial knowledge. It is understandable why students learn context-specific mechanical procedures which are often poorly understood if their motivational goal is to only find a correct answer.
By implementing the justification principle, I argue that such practices will cease as the success criteria lies within the justification rather than one’s final answer. This argument is supported by an experiment in which justification was implemented as a learning tool on a particular mathematical concept. The study found that the implementation of justification created a motivational need to explain and understand what was happening in order to gain further insight (Kidron & Dreyfus, 2012). I would predict that the implementation of the justification principle would not only yield similar results, but would also obstruct the idea of placing absolute value in obtaining the correct answer, thereby reducing students’ reliance on superficial knowledge and empirical reasoning.
Chapter 3: METHODOLOGY

Research Design

This research paper, focusing on justification’s applications and benefits as a learning, instructional, and assessment practice, is qualitative in nature. Using a qualitative methodology to expand on the ideas and notions grasped from a review of the literature of concepts surrounding the purpose of this study, I collected my data by digitally recording face-to-face semi-structured interviews with teachers who are viewed as exemplary math instructors among the teaching community. Exemplary math teachers will be defined as those who not only have a strong mathematical background, but those who are also well-established with at least five years of teaching experience. These interviews are then transcribed and analyzed for common themes and participant experiences (Creswell, 2013).

The specific type of qualitative study I chose to use is a case study. Yin (2003) states that a case study is an appropriate choice when the researcher’s understanding of specific interests depend on the context in which they occur. The focus of this study was to understand how exemplary mathematics teachers currently implement justification into their pedagogy within the context of attempting to efficiently develop deep knowledge within their students.

Participants

There were two participants in this study, and each of them are currently secondary school mathematics teachers. Both participants have been teachers for at least five years thereby having enough experience teaching mathematics to notice trends and changes, implement different teaching strategies and develop observation skills in order to analyze their students’ emotions and behaviors within the classroom.

The most important characteristic of the teachers being interviewed is that they have a mathematical background (university major or specialist in mathematics) and that they are
distinguished math teachers among their peers within the teacher community. It is has already been demonstrated within the literature review that teachers without a thorough mathematical background will simply be unable to as effectively implement the justification principle within their classroom.

**Data Collection**

The informal interviews will be guided by a set of interview questions. The semi-structured interviews will focus on: the teachers’ backgrounds, their approach to developing deep knowledge within their classroom, how they prevent the growth of superficial knowledge, which formative assessments they practice and rely on, how they ensure their assessments are authentic, their current implementations of justification, and their criticisms of the justification principle.

The interview questions were comprised of a combination of open and closed-response questions. I conducted a standardized, open-ended interview which eased the analysis and comparison of the interviews, as it also provided ample opportunity for the participants to answer openly, expand upon their thinking and discuss things that may not have been answered using the specific questions. (O’Leary, 2010). The goal of the interviews was to gather as much information as I can about how the participants were currently implementing justification within their classroom, as well as their opinions on the justification principle as a pedagogical practice used for instruction, learning, and assessment. The interviews took place in an environment of the participants’ choice that ensured confidentiality, focus and comfort at the participants’ convenience (Gill, Stewart, Treasure, & Chadwich, 2008). I will begin with an explanation of the purpose of the interview, remind the participants of their rights according to the ethical review process and an outline of the format the interview will take (McNamara, 2009). I will then proceed with the interview questions.
Data Analysis

After the interviews took place, I transcribed and analyzed the digital recordings for common themes or experiences. Researchers who carry out qualitative studies should immerse themselves in the details of their transcripts, in an attempt to have a general sense of the interview as a whole before attempting to partition it into different parts. As should the researcher search for common experiences and themes of the participants surrounding the phenomenon of interest (Creswell, 2013). After developing a sound understanding of the interview as a whole, I more rigorously dwelled into a traditional approach to coding. The method I chose to follow was similar to that of Creswell’s (2013), where I first identified and highlighted data that was either: immediately relevant to my research question, or data that appeared frequently. Second, I re-evaluated each of my codes and categorized them further by devising a list of sub-codes. Third, I arranged my codes into a data analysis genesis table, whereby I was able to systematically categorize each code and sub-code with an accompanied explanation and analysis as for their significance to my study and lastly, I organized my data under code headings.

Throughout the coding process, I was careful in my analysis to identify common experiences my participants may have surrounding authentic assessments practices and the usage of justification in the mathematics classroom. I was also cognizant in looking for common themes among the two teachers’ answers to discover if, in fact, implementing the justification principle is effective in developing students’ deep conceptual and procedural knowledge.
Ethical Review Procedures

Each voluntary participant in this study was given ample information about this study before they were asked to commit to participate. Prior to the interview, participants received a consent letter to sign thereby giving them time to make an informed decision as to whether or not they wanted to participate. It was made clear to all participants that they were free to ask questions of clarification and/or withdraw from the study at any time.

Participants were informed what the purpose of the study and what topics will be discussed as I provided them with a sample of my question list prior to the interview. Each participant was guaranteed anonymity by the use of pseudonyms throughout the report. Participants were given full knowledge of the interview process and were aware that it was recorded. Interviews took place at a location of the participants’ choice in order to ensure their comfort. After the transcribing of the interviews took place, participants received a copy of their signed consent form along with a transcribed copy of the interview. Participants were given the opportunity to review the transcribed copy of the interview to clarify any points they may wish to exclude. Lastly, participants were given my contact information if they ever need to reach me, and they were given the opportunity to view the final research paper.

Having this openness and prudence was done with each participant in order to increase their comfort level, inform them of what to expect and build trust thereby increasing honesty from each participant (Gill et al., 2008).

Limitations

This study was conducted for the purpose of gaining insight and breadth on which strategies exemplary mathematics teachers implement to help foster and assess the growth of deep procedural and conceptual knowledge through the use of justification. More specifically, I
was interested in the role of justification applications towards learning and assessment practices within the mathematics classroom. The first limitation of this research is the constrained amount of time given for the completion of the work. With more time, more issues could have been addressed, a more thorough data collection method could have been executed, and more grounded conclusions could be made.

A second limitation is the limited number of research participants that were interviewed. By having such a small sample population, it is difficult to draw large generalizations with high confidence or certainty. Lastly, by conducting interviews as my data collection method, the information gathered is merely a collection of opinions. Although the opinions offered are from exemplary math educators, it would have been optimal to have tested these ideas firsthand with a sample group of students. As a result of these limitations, the study has been made more focused and the findings more relevant to the specific sub-themes chosen included within the literature review.
Chapter 4: FINDINGS

Introduction

My aim in this chapter is to narrate the findings of the data while ensuring not to explain, interpret, or evaluate them. The narrative will follow from the methodology and research questions and will set the stage for an in-depth analysis and discussion in Chapter 5. I utilized the collected data from my interviews to create a case study about each of the teachers. I coded the data using the in vivo coding method, and was able to narrow down my prevalent codes into three major themes. The three themes that I will write about for each case are as follows: enhanced focus on the mathematical thinking process, assessment practice, and reshaping students’ perceptions surrounding mathematics. These three themes were the most prevalent within the data and best informed the teachers’ pedagogical practices in relation to justification. Furthermore, in each case study, I will relate the aforementioned themes to justification’s usages within pedagogical practice to establish a picture of how teachers’ attitudes towards justification affect their individual philosophies and practice.

Mary

Mary is a secondary mathematics teacher who has been teaching mathematics exclusively for over thirty years. During my interview with Mary, I asked her to tell me about herself (qualifications, motivations for becoming a teacher, etc.) and the demographics of the students she teaches. She holds both a Bachelors and Masters of Science in mathematics, and obtained both with the intention of one day becoming a mathematics teacher. Her family has a long tradition of being educators as both her parents and grand-parents were professors, so she felt as if becoming a teacher was her calling. Mary is teaching five academic mathematics classes, averaging about twenty students per class. Her classes house students with a wide-spectrum of
ability levels and motivational aspirations in the field of mathematics, as she also has a few
students with IEPs for various reasons.

Mary is teaching in an institution that holds mathematics in the highest esteem where
most of the students would be considered high-achievers within any standard classroom. In
addition, her school offers a lot of funding to develop and enact contemporary and “cutting-
edge” teaching practices in an effort to better their students’ mathematical development. As such,
Mary has a lot of experience with inquiry-based and problem-based learning, amongst many
other methodologies, and has also had the privilege to gain access to a multitude of resources
that most publicly-funded Canadian teachers would otherwise not able to utilize. She frequently
attends math education conferences and professional development sessions to ensure she is
always up-to-date with the most current resources, practices, and research.

Her reasons for implementing justification into her pedagogical practice were many, but
the three major themes that were apparent in her decision making were using justification as a
means to: enhance focus on the mathematical thinking process, embed an ongoing assessment
practice within her curriculum, and reshape her students’ perceptions surrounding mathematics.

**Enhanced focus on the mathematical thinking process**

Mary stated that, throughout grade school, she not only loved mathematics but she
excelled in it. She was particularly interested and focused on the algebraic aspect of the
curriculum and she found herself to be quite proficient in that specific field. She stated that the
way in which she was taught mathematics was very traditional, where the teacher would stand at
the board explaining procedures, then you would have to mimic those procedures. But not only
that, she was taught in this fashion uniformly throughout all of grade school; a “one-size fits all”
type of teaching as she described it. Despite excelling in this environment, she stated during the
interview that once she began university level mathematics, she began to struggle and found the courses to be particularly difficult. She went on to say:

I found the math to be very challenging, and part of the reason I think, was the way it was taught. So I didn’t know how to dig down deeper to engage with the mathematics, which was something I didn’t learn until much later in my mathematics career. And so even the procedures and the mathematical pedagogy I was used to seeing; the procedure approach to learning, was in no way helpful or effective for me in my post-secondary studies. And even though I loved math, as I continued through university I found it very hard, and it continued to get harder as I got to the higher level math courses. It was hard to know how to progress or even how to properly learn because I was not equipped with the proper tools.

Mary went on to say that it was because of this experience that she realized that, as a teacher, she had to ensure that her students did not just memorize context-specific procedures superficially, but instead, understood when the appropriate time was to use them, why they should use it, and were able to effectively communicate their thinking processes. Mary claims that one major way in which she uses justification in order to accomplish this goal is through her voiced attitudes and explicit embodiment within the classroom. She states that within her classroom, there is almost never a time when a student will provide an answer without either her or another student saying, “well how did you get that answer?” or, “why do you think that, how did you make that claim?” Mary has integrated this practice of justification into a culture within the classroom and states that it has become a regular part of their learning experience, and that students not only expect it; they simply will not be satisfied without it.

She recalls a recent example to highlight this point, where a student asked her whether or not he had the right answer and she replied, “I don’t really care about your answer, I want to know what you were thinking on your way to the answer.” By modelling and enforcing justification explicitly in this manner throughout each of her classes, she states that her students have come to value the importance of the justification over that of a correct answer, and that
through this learning process, they rely much less on meaningless memorization and are able to solve similar problems far into the future since, “they understand the concepts, so they can rebuild those ideas.”

**Assessment practice**

Mary stated that, through the culture of justification she has created within her classroom, her assessment practices have become more comprehensive and authentic. She states that she uses justification as a means to seamlessly integrate her assessments for and as learning into her lessons. As Mary has shifted away from the traditional rote teaching style of mathematics, she finds herself acting as a facilitator of knowledge, rather than the traditional provider of knowledge. As such, she states that most of her classes end up turning into thoughtful discussions, whether it be in small groups or with the whole class. During this time, she walks around the room and notices what the kids are saying while jotting down notes about who said what, and then asking those students to voice those particular ideas.

She states that during this time, her assessment for learning takes place as she is able to gain insight and perspective into her students’ current understandings and misconceptions surrounding a specific topic. It gives her a general sense of how far along the students are in achieving the desired learning goals, and informs her instruction to properly guide the students’ learning through scaffolded questioning. Furthermore, she claims that assessment as learning is also taking place during this time as the students are forced to reflect on their current understandings and justify and defend their positions during discussions. As Mary put it, “this constant class-wide reflecting, questioning, and building of concepts and solution paths forces the students to re-assess their own understandings during the lesson and learning process.” Due to this, Mary states that she believes that it is through justification that she is able to best assess
for and as learning for her students in a way that provides authentic insight, and that it is deeply ingrained into her pedagogy for these reasons.

**Reshaping students’ perceptions surrounding mathematics**

Mary believes that this is a very important shift that needs to take place in contemporary mathematics education, and it is a major focus of hers as an educator. During our interview she stated that one of the justification principle’s greatest benefits is that it helps students to see that math is a body of knowledge and not just a list of procedures or rules, which is how many students have typically been taught. She went on to say:

They have typically been taught that it is that way, and some of them have actually been pretty successful learning math in that way. But when students come to our school, we essentially pull the rug out from under them because we try to help them understand that there is more to it than just procedures, and that it is important to understand the why of it.

Mary believes that justification is an essential ingredient in ensuring students can make this shift. One of her main strategies towards accomplishing this is through conversations with the students, trying to explain to them that mathematics is a body of knowledge, that things should fit together, and that they should be thinking about connections between the studied topics. Through modelling justification in her classroom, she demonstrates that it is possible to learn on a deeper level; to be able to remember and apply concepts for long periods of time.

Mary says that she does her best to build the concepts together with the students so that they are constantly exploring new things. She believes it is not so much about, “here’s how to do a problem,” or, “here is a theorem,” but instead, posing the right types of questions to push students having to apply their mathematics in new contexts through open ended questioning where there are different solutions depending on the different kinds of assumptions made. It is through this process that Mary believes students will be able to shift their understandings of
mathematics towards that of a body of knowledge; a knowledge that can be used to solve real world problems, and a knowledge that requires consistent justification.

Suzie

Suzie is a secondary mathematics and physics teacher with over twenty years of professional teaching experience. She holds both a Bachelor’s and Master’s in physics with a minor in mathematics, as well as a Master’s in education. She has obtained a teachable in both mathematics and physics at the intermediate/senior level, and has been teaching both subjects throughout her entire teaching career. Suzie has known she wanted to become a teacher from a very young age and has always been very passionate about being a part of a school community, both as a student and a teacher. She is currently teaching three math classes and two physics classes, ranging from grades nine through twelve, and all of her classes are either academic or enhanced classrooms.

Suzie teaches at one of the most prestigious Canadian secondary schools in the country and has been there for practically her entire career. The facilities and resources available to both her and her department are as good as they could possibly be, and as such, she has a wide variety of unique teaching experiences that most teachers simply could not possibly have. She is the head of the math department and leads a team of thirteen math teachers, all of which are math specialists, each holding at least a degree in mathematics along with a master’s in education and/or mathematics. She is a strong believer in team teaching and co-constructs all materials for the math department with every member of the department, along with seasonally hired interns. She has created a system whereby every lesson that is taught, every test that is distributed, and every practice that is enacted, is first thoroughly discussed and worked out with the members of her team via weekly faculty meetings. She is and has been, both the head of the hiring personnel
and the leading teacher evaluator for the school for a number of years and claims to have taken on several student teachers under her wing throughout her career as well.

At Suzie’s school, all students are to take a math course one year ahead of their current level, and the school strictly follows the international bachelorette program, with no other optional routes or streams available for the students. There is a lot of pressure within the culture of the school to excel academically and the students are constantly pushed to achieve heightened understandings of mathematical concepts. By the eleventh grade, students of this school are expected to endeavor into concepts that are typically introduced in first-year university level mathematics courses. As such there is a lot of pressure on the teachers of the math department to ensure the students are properly prepared and are given the tools necessary to succeed in such an environment. The reasons for why and how Suzie integrates justification within her own instruction is highlighted below within each of the central themes.

**Enhanced focus on the mathematical thinking process**

Suzie possesses a deep belief that there is too much of a focus on students obtaining the correct answer. She finds that as new students enter her school, many of them are only concerned about getting the right answer and achieving a good grade, and have no interest in the underlying mathematical thinking processes involved. In an effort to rid their students of this notion, they have completely revitalized their grade ten curriculum into a flipped classroom that has, “literally no explicit teaching,” and relies solely on peer-to-peer problem based learning. The role of the teacher within this environment is to simply facilitate and lead the discussions amongst the students, highlight the importance of specific underlying concepts, and ensure the classroom flows without incident. The math department created a whole booklet full of word problems in such a way where the students develop every major concept on their own, even ones that are
typically just explicitly stated; such as the quadratic formula for example. In addition, the ordering of the questions were done so in a specific fashion so that at that any given point in time, the student should possess the adequate knowledge needed to solve each of the problems.

Suzie explained that it was a very difficult process putting together such a curriculum as the questions not only needed to be scaffolded in a very specific fashion, but it also required a lot of ingenuity to devise such a large quantity of unique and engaging word problems.

At the end of each class, a new page within the booklet is assigned. The students are then to finish all of the questions before the next class, where the solutions to the problems will be thoroughly discussed and worked on until the class reaches a consensus on each problem. Suzie states that through this newly implemented model, students are constantly submerged into the mathematical thinking process. During every single class, the students are having to justify their way of thinking and have to convince the rest of the class with certainty that their solution is correct. She has noticed that, by putting such a large emphasis on the justification aspect of mathematics, students have become less interested in what the final answer is as they learn nothing from just obtaining a final answer. She states that:

Evidence of this shift can be found in the way in which students present their answers to the rest of the class. It is common that during a presented solution a student will say something along the lines of, “assuming all the calculations are correct, if you follow these steps, you will get your answer.” In other words, a lot of the time, students for whatever reason did not determine exactly what the final answer was. Instead, they saw a clear pathway on how to solve the problem and presented that as their solution, and the rest of the class is satisfied with that. This is not something that they would have been okay with at the beginning of the year.

**Assessment practice**

Justification plays an integral part in Suzie’s assessment practices, and her attitudes towards justification have contributed greatly to how she shapes and integrates assessment into her practice. Suzie states that all of the math classrooms within the school have chalkboards
and/or whiteboards surrounding the perimeter of the room. During each of her lessons, she always integrates some sort of activity that involves students having to apply their knowledge to new contexts in order to solve engaging problems. She says that, during this time, the classroom typically seems to become very chaotic, but as she puts it, “it is a good chaos.” The students are given this time to work out solutions with one another, debate over correct methodologies and share insight with one another. It is during this time that Suzie gets to see whether or not the students understand what is going on and she is able to easily spot their misconceptions. She states that, outside of formative pencil and paper quizzes, this is all she needs to get a grasp of whether or not the students are understanding the specified content.

Furthermore, Suzie states that, on each of her summative assessments, she always includes two specific elements. The first is that she purposefully includes a question on the test that would take the students a very long time to fully solve out. She does this with the expectation that students will simply include a detailed written explanation as to how they would solve the problem, along with a justification of their reasoning and then move on to the next question. She does this to further elude to the fact that it is more important that you understand how approach and solve a problem, than it is to solve out redundant calculations.

The second element she includes on her tests is a major problem solving question that students have not seen before, that typically require students to apply several concepts in order to fully solve out. She makes it very clear to her class that there will always be a question of this nature on the test, and she does this to engage the students in proper studying habits. She says that she makes it very clear in her classroom that it is very important that the students understand the underlying concepts and how to properly apply them, since a major component of their summative evaluations rely on having to justify their rationale. She believes that the usage of
these assessment practices ultimately deter students from resorting to rely on superficial knowledge; as she states: “I do not believe students would be able to achieve a high level of success on my assessments if they were to rely solely on superficial knowledge.”

Reshaping students’ perceptions surrounding mathematics

This theme was particularly prevalent in the data in regards to how Suzie implemented justification into her pedagogy. During my interview with her, she spoke very passionately about how her transition from secondary to post-secondary mathematics was particularly difficult. It was due to these troubles that she has a particular interest in teaching the senior level divisions. Suzie says that the math department she works at has a long-standing tradition of striving towards best preparing their secondary students for post-secondary mathematical studies. Suzie is of the firm belief that the major difference between secondary and post-secondary mathematics is the types of questions being asked, and the rigor in which the students are expected to demonstrate their understanding. She states that the types of questions that should be asked are ones that highlight mathematics as a body of knowledge with infinitely many applications and connections. Moreover, it is of her belief that the reason many students find trouble in mathematics courses after secondary school is because they have not properly developed the proper intensity and rigor needed to strive in university level mathematics courses.

To help ease the transition, her school offers three different streams for the senior students intending to pursue secondary studies; a stream that is targeted for students not intending to study engineering nor the sciences (S-stream), a stream that is targeted for students intending to study finance or life-science (SL-stream), and a stream for students intending to study engineering, mathematics, or any applied science (HL-stream). By dividing the math course into these three streams, Suzie is of the belief that she is able to better prepare each cohort
for their desired outlooks by partitioning them in this way, as she can reshape her students’ perspective of mathematics respective to their goals.

Suzie describes that, within both the SL and HL streams, there is an enhanced focus on viewing mathematics as a body of knowledge. During these courses, students are given a look into mathematical proofs, and a significant portion of their time in the course is spent on developing formal reasoning and communication skills. Suzie states that one of the most noticeable changes that students immediately notice upon entering these classes is that there is an evident shift away from verification, and a move towards having to construct arguments using formal math terminology and notation to either refute or support a claim. She says that by introducing them to formal mathematical justifications, students are given the insight and exposure needed to successfully transition into post-secondary studies, as they have a more comprehensive understanding of what mathematics truly is.
Chapter 5: DISCUSSION

Introduction

This study investigated two mathematics teachers and how they implemented justification into their pedagogical practice to ensure their students developed a deep understanding of the material. The purpose of this study was to examine how justification could be implemented into the secondary mathematics classroom, and to identify the associated benefits and challenges of doing so would be. The teachers in this study shared similar attitudes towards justification’s importance within the mathematics classroom, but the way they went about implementing it within their practice was very different. Despite the differences in their practice, both teachers were in strong agreement that justification was an effective means by which to encourage the growth and development of deep procedural and conceptual knowledge. Mary used justification as a means by which to represent mathematics as a body of knowledge starting at the intermediate levels of secondary studies, and primarily used it in order to embed an ongoing assessment for and as learning framework within her classroom. Suzie, on the other hand, used justification in large to shift the mentality of only caring about the correct answer, towards one that had more of an appreciation for the underlying mathematical processes. Unlike Mary, Suzie utilized justification to explicitly ward off the reliance on superficial knowledge by purposefully assigning a significant weight of summative assessments to problems that required that usage of deep conceptual and procedural knowledge.

In the following sections, I will review the findings in relation to the original purpose of the study. I will then clearly illustrate the significance of my findings and will provide my recommendations for where I think future research could be conducted.
Evaluation and Discussion

The main purpose of this study was to examine the following two questions: how can one implement the justification principle into one’s pedagogical practice, and what benefits and challenges are associated with the justification principle. Within the previous chapter, I highlighted and illustrated all of the main points that fell within the major themes of my data, and I will now take the opportunity to evaluate and discuss the specific relevance and findings of my research in order to answer the underlying intention of this study.

Research Question 1: How can one implement the justification principle into one's pedagogical practice?

In the two cases examined within this study, there were a range of different ways to go about implementing the justification principle into one’s pedagogical practice. The method that was most commonly described by both participants was that first-and-foremost, the teacher should model justification explicitly for the class. Whenever solving a problem for or along with the students, the teacher should provide a justification in the same way in which they would expect their students to. By having the students watch the explicit modelling, they will come to better understand not only how to go about performing a justification, but why it is necessary to justify one’s statements.

The second major way in which the participants implemented justification into their practice was by having students frequently engage in both small group and class-wide discussions on in-depth problems. By having students frequently work on meaningful, in-depth mathematical questions along with their peers, they are given the opportunity to constantly exchange ideas with one another and are given a platform to voice their opinions and justify their reasoning. During this time, the teacher can circulate the classroom, probe the students’ thinking
process, and guide them towards solutions using effective questioning. Debriefing these types of activities with a classroom discussion is how the participants usually performed this specific implementation. By implementing such an activity, the students will: debate with one another, constantly reflect on their own understandings while also building new mathematical connections, and will be regularly be introduced to new solution pathways.

The third way in which the participants implemented justification into their pedagogical practice was by making their attitude towards justification transparent within the classroom. Both of the participants stated very blatantly during the interviews that it is well known within their classrooms that the final answer comes second, whereas how you got there comes first. By constantly highlighting and questioning students’ mathematical thinking process, students will be forced to justify their reasoning. A common example of this is asking a student why they chose to use a particular procedure to solve a problem, or how they knew it was the correct one to use.

Finally, the last way in which the participants integrated justification into their instructional practice was by avoiding explicit instruction whenever possible. The participants stated that, whenever they could, they would utilize investigations to learn new concepts and whenever they did give a lesson, they utilized guided instruction where the students would provide the steps on how to move forward from effective questioning. Through investigations, students would develop theorems and generalizations collectively as a group, allowing them to exchange ideas and apply reasoning along with their mathematical knowledge to reach a conclusion, making the learning more meaningful while also providing a platform for student to justify their reasoning. Via guided instruction, the teacher only moves as fast as the students are able to provide the next step. Through effective questioning, students can move down the correct path towards a solution, and are constantly having to share their thinking process with the class.
Research Question 2: What are the benefits and challenges associated with the justification principle?

There were three main benefits that the participants believe are associated with implementing the justification principle within the mathematics classroom. The first benefit is that it creates an environment within the classroom that is conducive for the development of deep procedural and conceptual knowledge. By shifting the focus of mathematics class away from achieving the correct answer, and towards understanding the mathematical thinking process, students naturally become more engaged in deeper understandings and tend to rely much less on superficial knowledge as a consequence of this.

The second major benefit identified is the students’ heightened development of mathematical communication skills. By having the students constantly justify their approach and rationale both verbally and orally, the ability to communicate one’s thoughts is sure to improve as a consequence. Both participants voiced the opinion during their interview that the communication component of the mathematics curriculum is vital, yet they felt as if it is often overlooked and used as a means to boost students’ marks. Through justification, teachers are able to authentically bring math communication skills back into their classroom and assist their students in drastically improving their competence in formal mathematical communication.

The last major benefit is the authentic assessments for and as learning that justification brings about. By having students constantly justify themselves, the teacher is given a window to observe their current understandings and misconceptions on the subject material at hand. This provides the teacher with authentic evidence as to whether or not the students are ready to move forward, or if you need to re-evaluate your instruction. Furthermore, an authentic assessment as learning practice is also embedded into the applications of the justification principle, as the
students are forced to reflect on their current understandings and justify and defend their positions during discussions. As Mary put it, “this constant class-wide reflecting, questioning, and building of concepts and solution paths forces the students to re-assess their own understandings during the lesson and learning process.”

In terms of the challenges associated with the justification principle, my findings showed that there was one major challenge accompanied by two minor ones. The major challenge with the implementation of the justification principle is that many secondary mathematics teacher simply lack the comfort and background knowledge necessary to effectively utilize it. This major concern was shared by both participants. The other two challenges that were brought up was the fact that it is more time consuming than traditional math teaching, and that classroom management can be an issue as well.

**Implications, Recommendations, and Next Steps**

This study has many implications for me as a researcher. By conducting my literature review, I gained a lot of insight on both the history of mathematics education research, as well as the current research currently being done. I have come to appreciate and understand the amount of work and the level of rigor involved in conducting an educational study such as this, and I have realized the importance for continuing research in the field of mathematics education. Furthermore, I have gained a much more comprehensive understanding of the classifications of knowledge within mathematics, and how these knowledge develop and interact with one another, which leads me to my next implication; that this study has implications for me as a teacher.

It has taught me the value of being a life-long learner and the importance of pedagogy. When I began this study, I noticed that there was little research being done on justification within the mathematics classroom and yet, as my study showed, there are many benefits associated with
implementing justification into one’s pedagogy. This realization has truly showed me that I cannot stop learning throughout my professional career as there is always room for improvement.

Finally and most importantly, this study has implications for the mathematics education community. As my study showed, there are many benefits associated with the implementation of justification within one’s pedagogical practice, and arguably justification offers benefits to students in ways that no other practice can accomplish. Yet the largest challenge preventing such a practice to enter into our schools’ classrooms is a lack in teacher content knowledge. This issue raises many concerns and I feel that the best way in overcoming the challenge is the following:

1. Introducing a mathematics proficiency test to initial teacher education programs for all teachers intending to teach mathematics at the junior, intermediate and senior levels.
2. Ensuring all individuals who wish to gain a math teaching qualification possess at least a minor degree in mathematics as a pre-requisite to registering.
3. There is an increased focus on math pedagogy during initial teacher education programs.

By implementing the aforementioned recommendations, I believe that the main challenge associated with implementing the justification principle into the math classroom will be eliminated.

As for future research, I believe that what should follow from my research is a study that looks at when the most appropriate time would be to begin introducing justification to math students. Further researcher should also be done as to what a potential mathematics competency test for initial teacher education programs should look like and what it would encompass. Finally, I believe there is merit in a study that could look at the possible integration of the justification principle in subjects outside that of just mathematics.
REFERENCES


APPENDICES

Appendix A: Letter of Consent for Interview

Date: ___________________

Dear ___________________,

I am a graduate student at OISE, University of Toronto, and am currently enrolled as a Master of Teaching candidate. I am studying the utilization of justification in the math classroom long with its associated benefits for the purpose of completing a major research paper for my Master’s program. I think that your knowledge and experience will provide insights into this topic.

I am writing a report on this study as a requirement of the Master of Teaching Program. My course instructor who is providing support for the process this year is Dr. Arlo Kempf. My research supervisor is Dr. Douglas McDougall. The purpose of this requirement is to allow us to become familiar with a variety of ways to do research. My data collection consists of a 40 minute interview that will be tape-recorded. I would be grateful if you would allow me to interview you at a place and time convenient to you.

The contents of this interview will be used for my assignment, which will include a final paper, as well as informal presentations to my classmates and/or potentially at a conference or publication. I will not use your name or anything else that might identify you in my written work, oral presentations, or publications. This information remains confidential. The only people who will have access to my assignment work will be my research supervisor and my course instructor. You are free to change your mind at any time, and to withdraw even after you have consented to participate. You may decline to answer any specific questions. I will destroy the tape recording after the paper has been presented and/or published which may take up to five years after the data has been collected. There are no known risks or benefits to you for assisting in the project, and I will share with you a copy of my notes to ensure accuracy.

Please sign the attached form, if you agree to be interviewed. The second copy is for your records. Thank you very much for your help.

Yours sincerely,

Brendon May
Researcher: Brendon May

Phone number: Email: brendon.may@utoronto.ca

Instructor: Dr. Arlo Kempf

Email: arlo.kempf@utoronto.ca

Research Supervisor: Dr. Douglas McDougall

Phone #: Email: doug.mcdougall@utoronto.ca

Consent Form

I acknowledge that the topic of this interview has been explained to me and that any questions that I have asked have been answered to my satisfaction. I understand that I can withdraw at any time without penalty.

I have read the letter provided to me by Brendon May and agree to participate in an interview for the purposes described.

Signature: ______________________________________

Name (printed): ________________________________

Date: ___________________________
Appendix B: Interview Questions

Interview Questions

Section 1: Background
1. For how many years have you been teaching?
2. What specific qualifications do you have? (degrees, teachables, specialists, additional qualifications)
3. How much experience do you have teaching mathematics?
4. Can you tell me a bit about the school you are currently working at?
5. What motivated you to become a mathematics teacher?
6. What were your experiences as a student learning mathematics, positive or negative and why?
7. Do you find that you teach mathematics in the same way in which you were taught yourself?

Section 2: Deep Conceptual and Procedural Knowledge
1. How do you ensure deep conceptual knowledge is fostered within your classroom?
2. How do you ensure deep procedural knowledge and procedural flexibility is developed within your classroom?

Section 3: Superficial Conceptual and Procedural Knowledge
1. Which safeguards, if any, do you implement to ensure the obstruction of superficial knowledge within your classroom?

Section 4: Assessments Practices
1. What current formative assessment practices do you implement in your classroom?
2. For what reason do you choose to use these methods?
3. How do you ensure that the methods listed are practiced in an authentic manner?
4. How do you design your assessments in such a way that one will not be successful strictly relying on superficial knowledge? Why or why not? If you do, how can you be sure your assessment is truly authentic?

Section 5: Current Justification Implementations
1. Do you integrate justification into your mathematics classroom? If so, in what ways and for what reasons? If not, for what reasons?
2. How frequently do you expect your students to justify themselves during: lessons, in class activities, homework assignments, and assessments?
3. What strategies, if any, do you implement in your classroom to ensure your students value the justification of a solution over simply stating a correct solution?

Section 6: Criticism of the Justification Principle
1. After learning about the justification principle, what are your thoughts on its implementation within pedagogical practice?
2. What are your thoughts on its affect towards assessment?
3. Could you see yourself implementing the justification principle within your own practice? Why or why not?
4. What do you think might be some benefits of justification principle?
5. What do you think might be some of the challenges of implementing the justification principle?